

# CHEM 26100 (Quantum Mechanics) Notes

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## Week 1

# From Classical to Quantum Mechanics

### 1.1 Blackbody Radiation

9/27: • The surface of a hot body emits energy in the form of EM radiation.

• Changes that occur with temperature:

- If less than 500 °C, we have IR Radiation (heat).
- If 500 – 600 °C, we have visible radiation (a glowing body).
- If 5000 °C, we have a “white hot” body (short wavelength).

• As a body gets hotter, it emits shorter wavelength radiation.

• **Stefan-Boltzmann law:** The the total **emissive power**  $R$  (in units of  $\text{J m}^{-2} \text{s}^{-1}$ ) of a blackbody (BB) is given by

$$R(T) = \sigma T^4$$

where  $\sigma \approx 5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$  is **Stefan’s constant**.

– Work done by Stefan and Boltzmann (c. 1870 / 1884, respectively).

• **Wien’s 1st Law:** The wavelength for maximum emissive power obeys the equation

$$\lambda_{\text{max}} T = b$$

where  $b = 2.898 \times 10^{-3} \text{ m K}$  is **Wien’s displacement constant**. *Also known as Wien’s displacement law.*



Figure 1.1: Wien’s 1st Law.

- Area under the curve (found with integration) is the total emissive power.
- We now change variables from emissive power  $R$  to energy density  $\rho$  in the BB cavity.

$$\rho(\lambda, T) = \frac{4}{c} R(\lambda, T)$$

- Wien's 2nd Law (1893): The energy density must have a functional relationship with the following form.

$$\rho(\lambda, T) = \frac{f(\lambda T)}{\lambda^5}$$

- $f(\lambda T)$  cannot be determined from thermodynamics. Thus, something else is needed!

- Lord Rayleigh and his graduate student Jeans (1899) propose a solution.

- EM: The thermal radiation within a cavity must exist in the form of standing waves.
- RJ showed that the number  $n$  of standing waves per unit volume, per wavelength has the following form.

$$n(\lambda) = \frac{8\pi}{\lambda^4}$$

- If  $\bar{\epsilon}$  is the average energy in the mode with wavelength  $\lambda$ , then

$$\rho(\lambda, T) = \frac{8\pi}{\lambda^4} \bar{\epsilon}$$

- Waves come from atoms in the walls of the BB cavity, which act as linear harmonic oscillators at a frequency  $\nu = c/\lambda$ .
- Assuming thermal equilibrium, we obtain

$$\begin{aligned} \bar{\epsilon} &= \frac{\int_0^\infty \epsilon e^{-\epsilon/kT} d\epsilon}{\int_0^\infty e^{-\epsilon/kT} d\epsilon} \\ &= -\frac{\partial}{\partial \beta} \ln \left( \int_0^\infty e^{-\beta \epsilon} d\epsilon \right) \\ &= \frac{1}{\beta} \\ &= kT \end{aligned}$$

where  $k$  is the Boltzmann constant.

- Basically, we sum all energies  $\epsilon$ , weighted by the probability  $e^{-\epsilon/kT}$  of the energy existing, and divide by the total energy.
- The first equation is equivalent to the second with  $\beta = 1/kT$ .
- Therefore,

$$\rho(\lambda, T) = \frac{8\pi kT}{\lambda^4}$$

- UV catastrophe: Rayleigh's formula diverges from the experimental data for short wavelength.
  - The above formula diverges to  $+\infty$ , driven by the  $\lambda^4$  term in the denominator, as  $\lambda \rightarrow 0$ . However, the amount of radiation of shorter wavelengths should decrease past a point, as seen in Figure 1.1.
- Max Planck comes in, proposes an idea to the German academy that's so radical, they think he's insane, but he's actually right and it lays a key idea for quantum mechanics.
- Planck's key insight: The energy levels of the oscillators are not continuous, but are quantized.

- So we can't actually take an integral as Rayleigh did; we have to take an infinite series.
- In reality,

$$\begin{aligned}\bar{\epsilon} &= \frac{\sum_{n=0}^{\infty} n\epsilon_0 e^{-\beta n\epsilon_0}}{\sum_{n=0}^{\infty} e^{-\beta n\epsilon_0}} \\ &= \frac{\epsilon_0}{e^{\beta\epsilon_0} - 1}\end{aligned}$$

- Thus,

$$\rho(\lambda, T) = \frac{8\pi\epsilon_0}{\lambda^4(e^{\epsilon/kT} - 1)}$$

- But to satisfy Wien's 2nd law, we must let  $\epsilon_0 \propto 1/\lambda$ . More specifically,  $\epsilon_0 = hc/\lambda = h\nu$ , where  $h$  is Planck's constant.
  - This setup allowed us to get an accurate value for Planck's constant for the first time in history.
- Planck's theory predicts the data of Figure 1.
- A perfect blackbody absorbs and emits radiation at all frequencies.
  - A star is pretty close to a blackbody. The graphite in a pencil is 97% a blackbody. We are all blackbodies.
  - The entire universe can be viewed as a blackbody.
- Princeton and Bell Labs telescopes find **Cosmic Background Radiation** (A. A. Penzias and R. W. Wilson, 1964).
  - Background radiation from the universe itself.
  - $\lambda_{\text{max}} = 7.35 \text{ cm}$ .
  - **Isotropic** radio signal (comes from everywhere).
  - From this, you can work out the temperature of the universe using Wien's first law.
  - Thus, the whole universe is a blackbody with a temperature of approximately 3 K.

## 1.2 Photoelectric Effect and Bohr Atom

- 9/29:
- In 1887, Hertz shines UV light at electrodes and observes a spark.
    - In 1900, Lenard shows that electrons are ejected from the metal surface of the electrodes.
  - Experimental setup:

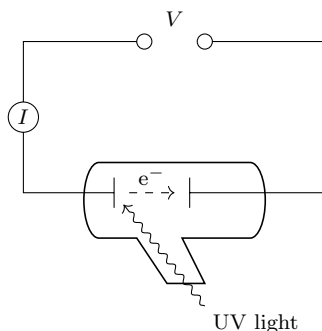


Figure 1.2: Photoelectric effect experiment.



- Shine UV light through a quartz crystal window so that it impinges on the left plate.
- This causes an electron to be ejected from the illuminated plate and cross the potential difference (recall that they didn't know about electrons at the time; they just knew something was happening).
- Increase the external potential until the spark goes away (gives some data about the energy of the electron).
- Odd features:
  1. There is a threshold frequency of radiation required to eject the electrons.
    - You can shine as much light as you want below a certain frequency and nothing will happen.
    - However, as soon as you reach that frequency, you get a spark.
  2. The maximum kinetic energy (KE necessary to overcome the voltage PE) is directly proportional to the frequency and is independent of the intensity.
- Einstein (1906) proposes that light consists of quanta called photons.
- If you assume this, the max KE obeys

$$\frac{1}{2}mv_{\max}^2 = h\nu - W$$

where the work function  $W$  is the energy required to remove the photon from the metal.

- When  $KE \rightarrow 0$ , we obtain the threshold frequency

$$\nu_{\text{th}} = \frac{W}{h}$$

required to remove an electron from the metal.

- Millikan (1914-1917), hot off the success of the oil drop experiment, experimentally corroborates Einstein's theory at UChicago in Ryerson.
  - Noting that we also have  $KE = eV$ , where  $e$  is the charge of an electron and  $V$  is the stopping voltage, Millikan obtains
 
$$V = \frac{h}{e}\nu - \frac{W}{e}$$
  - The slope of this linear data plot is  $h/e$ , and Millikan definitely knows the charge of the electron (!), so he can also measure Planck's constant this way.
  - When Millikan gets the same value Planck got a different way, he corroborates Einstein's theory.
- Thus, this quantization is not just one result, but is fundamental to our understanding of radiation.
- Bohr (1913) makes assumptions.
  1. Circle orbits of electrons about the nucleus.
  2. Only certain stationary orbits are allowed.
  3. The electron radiates energy only during a transition between orbits.
  4. The orbital angular momentum is quantized:  $L = \frac{nh}{2\pi}$  where  $n \in \mathbb{N}$  is a quantum number.
- Assumption 1 is wrong.
- Two equations:

- Equation one: Coulomb attraction of the electron and proton (nucleus) is balanced by a centripetal acceleration.

$$\frac{Ze^2}{4\pi\epsilon_0 r^2} = \frac{mv^2}{r}$$

where  $Z$  is the charge of the nucleus, and  $e$  is the charge of an electron.

- This follows exactly from classical mechanics.

- Equation two: Quantization of the orbital angular momentum:

$$mvr = \frac{nh}{2\pi} = n\hbar$$

where  $\hbar = h/2\pi$ .

- This is a new development from quantum mechanics.

- We now solve the two equations for our two unknowns (the velocity and radius).

$$v = \frac{Ze^2}{4\pi\epsilon_0 \hbar n} \qquad r = \frac{4\pi\epsilon_0 \hbar^2 n^2}{Zme^2}$$

- It follows that the translational kinetic energy  $T$  is given by

$$\begin{aligned} T &= \frac{1}{2}mv^2 \\ &= \frac{m}{2\hbar^2} \left( \frac{Ze^2}{4\pi\epsilon_0} \right)^2 \frac{1}{n^2} \end{aligned}$$

- This is the origin of the  $1/n^2$  in the Bohr model.

- With respect to potential energy, we also have

$$\begin{aligned} V &= -\frac{Ze^2}{4\pi\epsilon_0 r} \\ &= -\frac{m}{\hbar^2} \left( \frac{Ze^2}{4\pi\epsilon_0} \right)^2 \frac{1}{n^2} \end{aligned}$$

- It follows that the total energy  $E$  is given by

$$\begin{aligned} E_n &= T + V \\ &= -\frac{m}{2\hbar^2} \left( \frac{Ze^2}{4\pi\epsilon_0} \right)^2 \frac{1}{n^2} \end{aligned}$$

- Thus, the reason we have discrete transitions is because the atom has discrete energy levels.
- Indeed, energy transitions are described by the following.

$$E_b - E_a = hcR_0 \left( \frac{1}{n_b^2} - \frac{1}{n_a^2} \right)$$

where  $R_0$ , the Rhydborg constant (observed by Rhydborg and his spectral lines far before Bohr, but applicable here), is all of the other constants swept together.

- Note that

$$R_0 = \frac{m \left( \frac{e^2}{4\pi\epsilon_0} \right)^2}{4\pi c \hbar^3}$$

- Thus, quantum mechanics exactly describes the spectral transitions experimentally described earlier.
- Limitations of the Bohr model:
  1. Assumption 1.
  2. Cannot be generalized to many electron atoms and models.
  3. No reliable way to predict the time dependence of events like the electron transitions.
- So the Bohr model brings us to the brink of being able to predict chemistry, but we still need to go a bit further.

### 1.3 Stern-Gerlach Experiment

10/1:

- Measurement of the magnetic dipole moment of atoms.
- Nobel-prize winning experiment done by Otto Stern and W. Gerlach (1922). Stern was Gerlach's grad student.
- **Magnetic dipole moment:** Think about an electron moving in a circle with velocity  $v$ . Then the charge creates a magnetic field  $M$  perpendicular to the plane of the circle.
- Thus, we want to detect the magnetic moment of atoms. They will measure this by measuring the deflection of the atoms by an **inhomogeneous** field.
- **Inhomogeneous** (magnetic field): A magnetic field that does not have equal magnitude and direction everywhere within some space.
- They put the atoms in an "oven" to get them hot and then shoot them out in a beam. The beam passes through a magnet, and if the beam has a magnetic moment, it will break up. There is a collection plate at the far end.
- Effect of the  $\vec{B}$  field:  $PE = -\vec{M} \cdot \vec{B} = W$ .
- It follows that  $F = -\nabla W$ .
- Additionally,
 
$$F_z = M_z \frac{\partial B_z}{\partial z}$$
- Classical expectation: Every value of  $M_z$  would occur, that is,  $|M_z| \leq M$ . Would lead to one continuous pile on the collection plate with a Gaussian proportionality.
- Stern and Gerlach expect it to be discrete/quantized. Focused on Ag atoms. Thought two discrete lines would be formed symmetrically about the center. Thought they would see similar results for Na, Cs, K, H.
- Didn't see anything at first.
- Smoked some cigars, released sulfate, and  $\text{AgSO}_4$  (black) showed up on the collection plate in 2 discrete piles.
- Bohr quantization (varies from  $-\ell$  to  $+\ell$ , where  $\ell$  is orbital angular momentum).  $L = \ell\hbar$  (approximately),  $L_z = m\hbar$  (where  $m$  is the quantum number, not a mass??).
- Actual quantum mechanics gives us  $L = \sqrt{\ell(\ell+1)}\hbar$ .
- But this does not explain the Stern-Gerlach experiment. According to this theory...
  - If  $\ell = 0$  and  $m = 0$ , then we'll observe just 1 spot.

- If  $\ell = 1$  and  $m = -1, 0, +1$ , then we'll observe 3 spots.
- But, of course, they only saw 2 spots.
  - The first case corresponds to silver with  $\ell = 0$ .
  - They were actually seeing electron spin.
- Electron spin is later understood by S. Goudsmit and G. E. Uhlenbeck (1925).
  - Able to show that the splitting of spectral lines when atoms are placed in  $\vec{B}$  fields. The electron must have an intrinsic spin (magnetic moment  $M_1$ ) where two values are allowed:  $M_1 = \pm \frac{1}{2}$ .
  - They postulate that this is a form of intrinsic angular momentum of spin:  $S = \sqrt{s(s+1)}\hbar$ .
- **Total angular momentum:** The vector sum of all partial angular momenta.
  - The angular momentum of the nuclei may be neglected. Addition of the orbital and spin angular momenta of the electrons.
- Stern and Gerlach:
  - The orbital angular momentum of Ag atoms is zero.
  - The net spin angular momentum is  $\frac{1}{2}$ .
  - Thus, the total angular momentum  $m = \pm \frac{1}{2}$ . Thus, we expect two spots on the plate.
- Note that this relates to the Pauli exclusion principle (spin implies no more than 2 electrons together), first posited in 1926.
- Particle-wave duality (by Louis de Broglie): Introduces matter waves (1923-24).
  - Einstein says  $E = h\nu$ . Additionally, momentum of a photon is  $p = h\nu/c = h/\lambda$ . Thus, this formula relates the particle (momentum) and wave (wavelength) natures of the photon.
- de Broglie: Turns in a 4 page thesis, Paris committee will fail him, but they write to Einstein who recognizes this is really important.
  - de Broglie defines a frequency and a wavelength for material particles  $\nu = E/h$ . It follows that  $\lambda = h/p$ . Thus, electrons have a wavelength, too.
- **de Broglie's relationship:** The equation

$$\lambda = \frac{h}{mv}$$

for a nonrelativistic particle.

- Explanation of the Bohr atom:
  - For the electron's orbit to be stable, an integer number of wavelengths must match the circumference of the orbit.
  - This is why the orbits are quantized!
  - Thus,  $n\lambda = 2\pi r$  and  $L = rp$  (from classical physics), so

$$L = \frac{n\lambda p}{2\pi} = \frac{nh}{2\pi} = n\hbar$$

as desired.

## 1.4 Chapter 1: The Dawn of the Quantum Theory

From McQuarrie and Simon (1997).

- 9/28:
- **Blackbody:** A body which absorbs and emits all frequencies. *Also known as ideal body.*
  - “Many theoretical physicists tried to derive expressions consistent with these experimental curves of intensity versus frequency [see Figure 1.1], but they were all unsuccessful. In fact, the expression that is derived according to the laws of nineteenth century physics is” as follows (McQuarrie & Simon, 1997, p. 3).
  - **Rayleigh-Jeans law:** The equation

$$d\rho(\nu, T) = \rho_\nu(T) d\nu = \frac{8\pi k_B T}{c^3} \nu^2 d\nu$$

where  $\rho_\nu(T) d\nu$  is the “radiant energy density between the frequencies  $\nu$  and  $\nu + d\nu$ ” (McQuarrie & Simon, 1997, p. 3).

- The ultraviolet catastrophe is so named because the frequency increases as the radiation enters the ultraviolet region.
- Planck’s solution:
  - Rayleigh and Jeans assumed (as does classical physics) that the energies of the electronic oscillators responsible for the emission of the radiation could have any value whatsoever.
  - However, Planck assumed discrete oscillator energies proportional to an integral multiple of the frequency:  $E = nh\nu$ , where  $n \in \mathbb{Z}$ .
  - Using this quantization energy and ideas from statistical thermodynamics (see Chapter 17), Planck derived the **Planck distribution law for blackbody radiation**.
  - The only undetermined constant in the above equation was  $h$ , and Planck showed that if we let  $h = 6.626 \times 10^{-34}$  Js, then this equation gives excellent agreement with the experimental data for all frequencies and temperatures.
- **Planck distribution law for blackbody radiation:** The equation

$$d\rho(\nu, T) = \rho_\nu(T) d\nu = \frac{8\pi h}{c^3} \frac{\nu^3 d\nu}{e^{h\nu/k_B T} - 1}$$

- Note that for small frequencies, the Planck distribution law and Rayleigh-Jeans law converge, but they diverge for large frequencies, as expected.
  - Because  $\nu$  and  $\lambda$  are related by  $\lambda\nu = c$  (and subsequently by  $d\nu = -c/\lambda^2 d\lambda$ ), we can write the Planck distribution law in terms of wavelength, as well.
- $$d\rho(\lambda, T) = \rho_\lambda(T) d\lambda = \frac{8\pi hc}{\lambda^5} \frac{d\lambda}{e^{hc/\lambda k_B T} - 1}$$
- This is equivalent to the form given in class under the substitution  $hc/\lambda = \epsilon = \epsilon_0$ .
  - Differentiating  $\rho_\lambda(T)$  with respect to  $\lambda$  gives an alternate formulation for  $b$ :

$$\lambda_{\max} T = \frac{hc}{4.965 k_B}$$

- Astronomers use the theory of blackbody radiation to estimate the surface temperatures of stars.
  - We can measure the electromagnetic spectrum of a star (which will follow a curve similar to one of the ones in Figure 1.1).

- Then we can find  $\lambda_{\max}$ . From here, all that's necessary is to plug into Wien's displacement law:

$$T = \frac{b}{\lambda_{\max}}$$

10/3:

- **Photoelectric effect:** The ejection of electrons from the surface of a metal by radiation.
- Classical predictions vs. experimental data.
  - Classical mechanics: Intensity is proportional to the amplitude squared of the incident light. Thus, the electrons at the surface of the metal should oscillate along with the field, and when they are oscillating violently enough, they should break away from the surface with a kinetic energy that depends on the amplitude/intensity (specifically, *not* the frequency).
    - Experimental observation: KE of the ejected electrons is independent of intensity and linearly dependent on the frequency.
  - Classical mechanics: The photoelectric effect should occur for any frequency of light as long as the intensity is sufficiently high.
    - Experimental data: There exists a threshold frequency  $\nu_0$ , characteristic of the metallic surface, below which no electrons are ejected, regardless of intensity.
- **Work function:** The minimum energy required to remove an electron from the surface of the particular metal. *Denoted by  $\phi$ . Units eV.*
  - The work function of the metal is analogous to the ionization energy of an isolated atom.
- Bright line spectra: “For many years, scientists had tried to find a pattern in the wavelengths or frequencies of the lines in the hydrogen atom spectrum. Finally, in 1885, an amateur Swiss scientist, Johann Balmer, showed that a plot of the frequency of the lines versus  $1/n^2$  ( $n = 3, 4, 5, \dots$ ) is linear” (McQuarrie & Simon, 1997, p. 10).
- **Balmer's formula:** The equation

$$\tilde{\nu} = 109\,680 \left( \frac{1}{2^2} - \frac{1}{n^2} \right) \text{cm}^{-1}$$

for  $n = 3, 4, 5, \dots$ , where  $\tilde{\nu}$  denotes wavenumber.

- **Balmer series:** The series of lines predicted by Balmer's formula as  $n$  takes on the values  $3, 4, 5, \dots$ , notably those occurring in the visible and near ultraviolet regions of the hydrogen atomic spectrum.
- **Series limit:** The wavelength of the “last” line in the Balmer series, as  $n \rightarrow \infty$ , of value 364.7 nm.
- **Rydberg formula:** The equation

$$\tilde{\nu} = \frac{1}{\lambda} = 109\,680 \left( \frac{1}{n_1^2} - \frac{1}{n_2^2} \right) \text{cm}^{-1}$$

for  $n_1, n_2 \in \mathbb{N}$  such that  $n_2 > n_1$ .

- The Rydberg formula accounts for all the lines in the hydrogen atomic spectrum.
- **Rydberg constant:** The proportionality constant in the Rydberg formula. *Denoted by  $R_{\text{H}}$ . Given by*

$$R_{\text{H}} = 109\,677.57 \text{cm}^{-1}$$

- The first four series of lines composing the hydrogen atomic spectrum are described in Table 1.1.
- Rydberg found approximate empirical laws for many series of lines in many different atoms throughout the 1890s.

Series name	$n_1$	$n_2$	Region of spectrum
Lyman	1	2, 3, 4, ...	Ultraviolet
Balmer	2	3, 4, 5, ...	Visible
Paschen	3	4, 5, 6, ...	Near infrared
Bracket	4	5, 6, 7, ...	Infrared

Table 1.1: Hydrogen spectral series.

- **Ritz combination rule:** The empirical laws for all the observed lines can be expressed as the difference between terms such as those in the Rydberg formula.
- **de Broglie wavelength:** The wavelength  $\lambda = h/(mv)$  corresponding to a particle of mass  $m$  moving with velocity  $v$ .
- An electron moving at 1.00% the speed of light has a de Broglie wavelength comparable to those of X-rays. Thus, electrons should act like X-rays to an extent.
- **X-ray diffraction:** The scattering of a beam of X-rays directed at a crystalline surface, characteristic of the atomic structure of the crystalline surface.
  - Occurs because the interatomic spacings in the crystal are about the same as the wavelength of the X-rays.
- The scattering of X-rays and electrons at such crystalline surfaces are very similar.
- The wavelike property of electrons is used in electron microscopes.
  - “The wavelength of the electrons can be controlled through an applied voltage, and the small de Broglie wavelengths attainable offer a more precise probe than an ordinary light microscope. In addition, in contrast to electromagnetic radiation of similar wavelengths (X-rays and ultraviolet), the electron beam can be readily focused by using electric and magnetic fields, generating sharper images” (McQuarrie & Simon, 1997, p. 18).
- The Bohr model:
  - The hydrogen atom has a central, rather massive nucleus with one associated electron. Because the nucleus is so much more massive than the electron, we can approximate it as fixed with the electron revolving around it.
  - The force  $f$  holding the electron in a circular orbit is supplied by the Coulombic force of attraction between the proton and the electron:
 
$$f = \frac{e^2}{4\pi\epsilon_0 r^2}$$
  - The Coulombic force is balanced by the centrifugal force
 
$$f = \frac{m_e v^2}{r}$$
  - Since  $\sum f_c = 0$  for a stable circular orbit, we have
 
$$\frac{e^2}{4\pi\epsilon_0 r^2} = \frac{m_e v^2}{r}$$
  - However, according to classical mechanics, an electron under these conditions is constantly accelerating (centripetally), so it should emit EM radiation, lose energy, and spiral into the nucleus. Thus, we make two nonclassical assumptions:

1. Stationary electronic orbits exist.
  2. The de Broglie waves of the orbiting electron must be in phase, as the electron makes one complete revolution.
- For the wave pattern around an orbit to be stable, we must have that the circumference of the orbit is equal to an integral number of wavelengths, i.e.,

$$2\pi r = n\lambda$$

where  $n \in \mathbb{N}$ .

- Substituting the de Broglie wavelength formula into the above gives

$$\begin{aligned} 2\pi r &= n \cdot \frac{h}{m_e v} \\ m_e v r &= \frac{nh}{2\pi} \\ m_e v r &= n\hbar \end{aligned}$$

where  $n \in \mathbb{N}$ .

- Since  $m_e v r$  is the angular momentum of the electron, another interpretation of the above (and the one more commonly attributed to Bohr) is that the angular momentum of the electron about the proton must be quantized.
- Solving for  $r$  by substituting out  $v$  yields

$$r = \frac{\epsilon_0 \hbar^2 n^2}{\pi m_e e^2}$$

- Thus, the radii of the allowed **Bohr orbits** are quantized.
- We now consider the total energy of the electron in an atom:

$$\begin{aligned} E &= \text{KE} + V(r) \\ &= \frac{1}{2} m_e v^2 - \frac{e^2}{4\pi\epsilon_0 r} \\ &= \frac{1}{2} \cdot \frac{e^2 r}{4\pi\epsilon_0 r^2} - \frac{e^2}{4\pi\epsilon_0 r} \\ &= -\frac{e^2}{8\pi\epsilon_0 r} \\ &= -\frac{m_e e^4}{8\epsilon_0^2 \hbar^2} \cdot \frac{1}{n^2} \end{aligned}$$

- The negative sign in this equation indicates that the energy states are bound states; the energies given for finite  $n$  are less than when the proton and electron are infinitely separated ( $n = \infty$  and  $E = 0$ ).
  - From the second to the third line above, we substitute the expression for  $m_e v^2$  obtainable from the original Newton's second law  $\sum f_c = 0$  equation.
- Note that this derivation is entirely analogous to the one presented in class, although it may look quite different on the surface. Key differences:
- The one in class jumps straight to quantization of orbital angular momentum instead of deriving it.
  - This one expands  $\hbar$  and cancels its  $2\pi$  part with the  $4\pi$  coefficient to  $\epsilon_0$ .
  - The one in class uses  $Z$  to be more general; this one just treats hydrogen ( $Z = 1$ ).
  - We can convert the final expression for  $E_n$  in the in-class one to the final expression for  $E$  in this one by substituting  $Z = 1$ ,  $\hbar = h/2\pi$ , and simplifying.



- **First Bohr orbit:** The Bohr orbit corresponding to  $n = 1$ , having radius 52.92 pm. *Denoted by  $a_0$ .*
- **Ground state energy:** The lowest energy, corresponding to  $n = 1$  in the total energy equation.
- **Excited state:** The states of higher energy, i.e., those other than the ground state.
  - Generally unstable with respect to the ground state.
  - At ordinary temperatures, hydrogen atoms as well as other atoms and molecules are found almost exclusively in their ground states.
  - An atom in an excited state will usually relax back to the ground state and give off the energy as electromagnetic radiation.
- With respect to spectral lines, Bohr assumed that the observed spectrum of the hydrogen atom is due to transitions from one allowed energy state to another:

$$\Delta E = \frac{m_e e^4}{8\epsilon_0^2 h^2} \left( \frac{1}{n_1^2} - \frac{1}{n_2^2} \right) = h\nu$$

- **Bohr frequency condition:** The above equation, specifically the relation  $\Delta E = h\nu$ .
  - Implies that as the electron falls from one level to another, the energy evolved is given off as a photon of energy  $E = h\nu$ .
  - Making the substitution  $h\nu = hc\tilde{\nu}$ , we can make the theoretical prediction that spectral lines will be of wavenumber

$$\tilde{\nu} = \frac{m_e e^4}{8\epsilon_0^2 c h^3} \left( \frac{1}{n_1^2} - \frac{1}{n_2^2} \right)$$

- It follows that we should have

$$R_H = \frac{m_e e^4}{8\epsilon_0^2 c h^3}$$

which we indeed do.

## Week 2

# The Schrödinger Equation

## 2.1 Particle-Wave Duality and Uncertainty Relations

- 10/4: • Particle-wave duality (de Broglie's original formulation):

$$\lambda\nu = c \qquad E = h\nu \qquad p = \frac{h}{\lambda}$$

- **Angular frequency:** The quantity  $\omega = 2\pi\nu$ .
- **Wavenumber:** The quantity  $k = 2\pi/\lambda$ .
- We can create a symmetrical formulation of the de Broglie relation using these new quantities:

$$E = \hbar\omega \qquad p = \hbar k$$

- What is the wave that we might associate with a de Broglie particle? Start with a completely general oscillator  $Ae^{iy}$  (are we also using complex coordinates to rigorously describe a circular Bohr orbit??), and substitute in the periodicity conditions from the classical wave equation, i.e.,  $kx - \omega t$  to get

$$\psi(x) = Ae^{i(kx - \omega t)}$$

- Probability:
  - By the classical wave power equation, such a wave might be associated with EM radiation hitting a surface with intensity  $I = |\psi(x)|^2 = \psi(x)\psi^*(x)$ .
  - As soon as we associate a particle (photon) with the wave, the intensity may be re-interpreted as the number of particles reaching the surface or the probability of a particle being at the surface.
  - Thus, the probability of finding a particle at the surface becomes  $|\psi(x)|^2$ , as well.
- Following de Broglie, we also associate waves with particles such as electrons.
  - With the association of light as a particle, the particle wave duality leads to the appearance of probability.
- What is the probability of finding the particle at the origin?

$$\begin{aligned} \text{Prob} &= |Ae^{ik \cdot 0}|^2 \\ &= |A|^2 \end{aligned}$$

- Since the probability is not dependent on position, it is the same everywhere.

- We also run into issues **normalizing** this unbounded wavefunction.
- We know this particle's momentum exactly, but we know nothing about its position.
- **Normalizing** (a wavefunction): Guaranteeing that the integral for the entire wavefunction is equal to 1.
- **Free particle**: A particle that does not have constraints on where it is more likely to be.
- Heisenberg's uncertainty relations are formalized in terms of matrix mechanics.
  - We can Fourier transform the wave function of particle to convert it from a function of position to a function of momentum.
  - The Fourier transform will yield one spike at  $\hbar k$  and will be 0 everywhere else — just like the Dirac delta function.
  - Thus,

$$\psi(p) = \delta(p - \hbar k)$$

- Consider a Gaussian wave packet at  $p = 0$ . Then

$$\phi(p) = Ce^{-\frac{p^2}{2(\Delta p)^2}}$$

- $\Delta p$  is the standard deviation of the Gaussian/width of the distribution. It is a constant such that the probability drops to  $1/e$  of its maximum at  $p = 0$ .
- With the Fourier Transform of  $\psi(p)$ , we obtain

$$\psi(x) = De^{-\frac{(\Delta p)^2 x^2}{2\hbar^2}}$$

- Thus, a Gaussian quantum function produces a Gaussian position function via an FT as well, i.e.,

$$\psi(x) = De^{-\frac{x^2}{2(\Delta x)^2}}$$

- Now if we set the last two equations equal to each other, we get

$$\begin{aligned}\frac{(\Delta p)^2}{2\hbar^2} &= \frac{1}{2(\Delta x)^2} \\ (\Delta p)^2(\Delta x)^2 &= \hbar^2 \\ \Delta p\Delta x &= \hbar = \frac{h}{2\pi}\end{aligned}$$

- This implies that the spread of the Gaussian in momentum times the spread of the Gaussian in position is a constant.
- If we make one Gaussian wave packet more specific, the other gets more spread out, and vice versa.
- Note that the above equality does *satisfy* the Heisenberg uncertainty principle, but it is not it itself.

## 2.2 The Schrödinger Equation and Particle in a Box

10/6:

- Review:
  - de Broglie describes an electron as a free particle.

$$\psi(x) = Ae^{ikx}$$

- We can only observe the real part, but being able to access the complex part is important in quantum mechanics.
- Schrödinger was on vacation in the Swiss Alps with his mistress when he derived the wave equation.
  - Schrödinger realized that

$$\begin{aligned}\frac{d\psi(x)}{dx} &= Aike^{ikx} \\ -i\hbar \frac{d\psi(x)}{dx} &= Ape^{ikx} \\ &= p\psi(x)\end{aligned}$$

- Let's introduce operators in quantum mechanics and let  $\hat{p}$  be an operator that when it acts on  $\psi(x)$ , it gives the above. In other words,

$$\hat{p} = -i\hbar \frac{d}{dx}$$

- Thus,

$$\hat{p}\psi(x) = p\psi(x)$$

- But energy is more important than momentum, so let's introduce an energy operator  $\hat{T}$  related to  $\hat{p}$  by

$$\hat{T} = \frac{\hat{p}^2}{2m} = \frac{-\hbar^2}{2m} \frac{d^2}{dx^2}$$

since  $E = mv^2/2 = p^2/(2m)$ .

- Thus, we have

$$\hat{T}\psi(x) = \frac{p^2}{2m}\psi(x)$$

- It follows from classical physics that the total energy operator  $\hat{H}$  (the Hamiltonian) is the sum of the kinetic and potential energy operators, i.e.,  $\hat{H} = \hat{T} + \hat{V}$ . Therefore, we must have

$$\hat{H}\psi(x) = E\psi(x)$$

and that is the Schrödinger equation.

- The particle in a box is like a single electron in a one-dimensional chamber that runs from  $-a$  to  $a$  with  $L = 2a$  (Schrödinger figured this out a few days later, still in the Swiss Alps).
  - The walls are infinite and have infinite potential.
  - We need the boundary condition, though, to be able to solve a differential equation like the Schrödinger equation.
    - Fortunately, we know that at  $|x| = a$ , we have  $\psi(\pm a) = 0$ .
    - Another important point is that  $d\psi(x)/dx$  at  $a$  is discontinuous.
  - So we have that

$$\begin{aligned}-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi_n(x) &= E_n \psi_n(x) \\ \frac{d^2}{dx^2} \psi(x) &= -k^2 \psi(x)\end{aligned}$$

where

$$k = \sqrt{\frac{2mE}{\hbar^2}}$$

- The solution of the differential equation will be of the form

$$\psi(x) = A \cos(kx) + B \sin(kx)$$

- Boundary conditions 1 and 2, respectively:

$$\begin{aligned} 0 = \psi(a) &= A \cos(ka) + B \sin(ka) & 0 = \psi(-a) &= A \cos(ka) - B \sin(ka) \end{aligned}$$

- Adding/subtracting the two equations yields

$$A \cos(ka) = 0 \qquad B \sin(ka) = 0$$

- We satisfy these equations with either of 2 classes of nontrivial solutions (the trivial solution being  $a = 0$ ).

1.  $B = 0$  and  $\cos(ka) = 0$ , i.e.,  $k_n = \frac{n\pi}{2a}$  for  $n \in 2\mathbb{N} + 1$ .
2.  $A = 0$  and  $\sin(ka) = 0$ , i.e.,  $k_n = \frac{n\pi}{2a}$  for  $n \in 2\mathbb{N}$ .

- Thus, either

$$\psi_n(x) = \frac{1}{\sqrt{a}} \cos\left(\frac{n\pi x}{2a}\right)$$

for  $n \in 2\mathbb{N} + 1$  are the **even solutions** (because cosine is an even function), and

$$\psi_n(x) = \frac{1}{\sqrt{a}} \sin\left(\frac{n\pi x}{2a}\right)$$

for  $n \in 2\mathbb{N}$  are the **odd solutions** (because sine is an odd function).

- Note that we derive the  $1/\sqrt{a}$  coefficient by normalizing  $\psi(x)$  with

$$\int_{-a}^a |\psi(x)|^2 dx = \int_{-a}^a \psi^*(x) \psi(x) dx = 1$$

- The energies come out to

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2}{8m} \cdot \frac{\pi^2 n^2}{a^2}$$

with the substitution  $k_n = \frac{n\pi}{2a}$ .

- Note that this means that the particle becomes more discrete the smaller the box gets (as uncertainty in position goes down, it acts more and more quantum mechanically).

## 2.3 Potential Step

10/8:

- Particle in a box:

- For  $n = 1$ , the potential is defined by one hump of a sine wave.
- For  $n = 2$ , the potential is defined by two humps.
- The number of nodes is equal to the principal quantum number minus 1.
- We have

$$E_n = \frac{\hbar^2}{8m} \frac{\pi^2 n^2}{a^2}$$

for  $n \in \mathbb{N}$ .

- By the Heisenberg uncertainty relationship, we must have  $E_1 > 0$ . In other words, the **zero-point energy** arises from the UR.
- Trend wrt.  $a$ : As  $a \rightarrow \infty$ , all of the energies become degenerate.

- Trend wrt.  $m$ : As  $m \rightarrow \infty$ ,  $E_n \rightarrow 0$  as well.
  - In other words, as  $m \rightarrow \infty$ , the particle behaves more classically!
  - The zero-point energy also disappears as  $a \rightarrow \infty$ .
- **Zero-point energy:** The lowest possible energy a quantum mechanical system may have.
- All of that information comes from the Schrödinger equation, so we now know much more than we used to.
- Free particle vs. particle in a box:
  - For a free particle, we have  $\psi(x) = e^{ikx}$ . Boundary condition was a circle (as per the Bohr model).
  - In the particle in a box, we weed out all of the free particle solutions that don't match the boundary conditions. And the only solutions that match the boundary conditions are the ones that have integers for the quantum number  $n$ .
  - The only constraint is that you can retain more particles the bigger the box gets; this is why the particle gets more quantum mechanical as you shrink the box.
- **Potential step:** Let the energy  $E$  be 0 up until the origin, where it steps up to potential  $V_0$ .

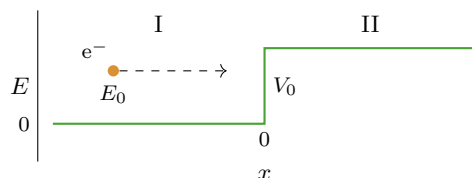


Figure 2.1: Potential step.

- We shoot a particle at a potential wall with energies varying from below the top to above the top.
- In classical mechanics, we have
 
$$E = \frac{p^2}{2m} + V$$
  - In region I, there's no potential, so the total energy is all kinetic. The particle is moving with momentum  $p_I = \sqrt{2mE}$ .
  - In region II, the particle is moving with momentum  $p_{II} = \sqrt{2m(E - V)}$ .
    - If  $E_0 < V$ , the particle *never* passes from region I  $\rightarrow$  II.
    - If  $E_0 > V$ , the particle *always* passes from I  $\rightarrow$  II, but has less KE in II than I<sup>[1]</sup>.
- Quantum particle motion:

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) + V(x)\psi(x) = E\psi(x)$$

$$\frac{d^2}{dx^2} \psi(x) + k^2 \psi(x) = 0$$

where  $k = \sqrt{2m(E - V)/\hbar^2}$ .

- The total wave function will be the sum of the LCAOs that fit the boundary condition.
- Our general solution has two parts:

$$\psi_I(x) = Ae^{i\alpha x} + Be^{-i\alpha x}$$

$$\psi_{II}(x) = Ce^{i\beta x} + De^{-i\beta x}$$

<sup>1</sup>Note that the classical resolution to the case  $E = V_0$  is that the particle never has  $E = V_0$ ; it always has energy  $\epsilon$  above or  $\epsilon$  below  $V$ . However, in some sense, there *is* another answer: Classical mechanics is not an “accurate” reflection of reality, and this is a place where it shows. Indeed, we *need* quantum mechanics to treat this case.

– Two energy cases:  $E > V_0$  and  $E < V_0$ .

–  $E > V_0$ :

- We must maintain the continuity of the  $\psi(x)$  and  $d\psi(x)/dx$  at  $x = 0$ . This yields

$$A + B = C + D \qquad i\alpha(A - B) = i\beta(C - D)$$

- It follows that

$$A = \frac{C(\alpha + \beta)}{2\alpha} + \frac{D(\alpha - \beta)}{2\alpha} \qquad B = \frac{C(\alpha - \beta)}{2\alpha} + \frac{D(\alpha + \beta)}{2\alpha}$$

- Assume that the particles only travel from left to right in II, i.e.,  $D = 0$ .

- The flux of the particle: The probability of the particle going left to right in region I is  $|A|^2$ . Thus, since the incident flux factors in the speed  $v_I$  of the particle, the incident flux is  $v_I|A|^2$ . Similarly, the transmitted flux of the particle is  $v_{II}|C|^2$ .

- Consequently, the reflected flux of the particles is

$$R = \frac{c|B|^2}{c|A|^2} = \frac{|B|^2}{|A|^2} = \frac{(\alpha - \beta)^2}{(\alpha + \beta)^2}$$

Note that the speed of the particle (the speed of light,  $c$ ) is the same in both regions.

- Conclusion: There is a probability of reflection *even when*  $E_0 > V_0$ , disagreeing with classical mechanics.

- Fraction of transmitted particles:

$$T = \frac{v_{II}}{v_I} \frac{|C|^2}{|A|^2} = \frac{4\alpha\beta}{(\alpha + \beta)^2}$$

–  $E < V_0$ :

- The continuity of  $\psi(x)$  and  $\psi'(x)$  at  $x = 0$  again gives us

$$A + B = C + D \qquad i\alpha(A - B) = i\beta(C - D)$$

- But since we have

$$\beta = \frac{\sqrt{2m(E - V_0)}}{\hbar}$$

and  $E - V_0 < 0$ ,  $\beta$  will be an imaginary number.

- Thus, we define  $\beta_2$  to be the real number such that  $\beta = i\beta_2$ .

- Consequently, we may write

$$R = \frac{|B|^2}{|A|^2} = \frac{|\alpha - \beta|^2}{|\alpha + \beta|^2} = \frac{|\alpha - i\beta_2|^2}{|\alpha + i\beta_2|^2} = \frac{\alpha^2 + \beta_2^2}{\alpha^2 + \beta_2^2} = 1$$

- Conclusion: When the energy of the particle is less than the energy of the potential, even quantum mechanics predicts total reflection. However, there's still something subtle happening.

- Let's look at the wave function in region II:

$$\psi_{II} = Ce^{i\beta x} = Ce^{i(i\beta_2)x} = Ce^{-\beta_2 x}$$

where  $\beta_2 > 0$  by definition.

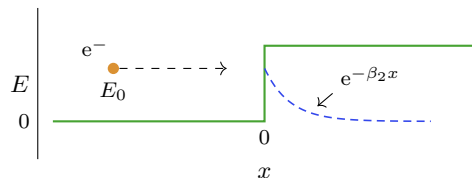


Figure 2.2: Quantum tunneling.

- Thus, even though the particle is reflected 100%, it has some probability of going through the step, namely a probability that decays exponentially the farther you go into the wall.
- This is **quantum tunneling**.
- The particle can't ever get to  $\infty$ , so that's why  $T = 0$ , but it can go into the wall for a little bit, just a sec.

## 2.4 Chapter 2: The Classical Wave Equation

From McQuarrie and Simon (1997).

1/3/23:

- Goal.
  - Ground the study of the Schrödinger equation (a PDE) with an easily visualized and highly related example from classical physics. Introduce the mathematics required for such problems and show how it relates to physical reality.
- Consider a uniform vibrating string stretched between two fixed points at  $x = 0, l$ .
  - At position  $x$  and time  $t$ , we say that the string is displaced  $u(x, t)$  from its equilibrium position.
- **Amplitude** (of a string): The maximum displacement of the string from its equilibrium horizontal position.
- It can be shown — see Figure 15.2 and the associated discussion in Labalme (2021b) — that  $u(x, t)$  satisfies the **classical wave equation**.
- **Classical wave equation**: The following linear partial differential equation. *Given by*

$$\frac{\partial^2 u(x, t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 u(x, t)}{\partial t^2}$$

- **Partial differential equation**: An equation in which the unknown occurs in partial derivatives. *Also known as PDE.*
- **Independent variable**: A variable in a PDE such as  $x$  or  $t$ .
- **Dependent variable**: A variable in a PDE that depends on the independent variables, such as  $u(x, t)$ .
- **Linear differential equation**: A differential equation in which the dependent variable and its derivatives appear only to the first power, and in which there are no cross terms.
- Additional physical conditions  $u(x, t)$  must satisfy.
  - The ends of the string are held fixed. Thus, the amplitude at these two points is always zero. Mathematically,

$$u(0, t) = 0$$

$$u(l, t) = 0$$

for all  $t$ .

- **Boundary condition**: A condition that specifies the behavior of the dependent variable at the boundaries.
  - The above additional physical conditions are boundary conditions.
  - In general, PDEs are solved subject to boundary conditions, the nature of which derives from physical considerations.
- **Separation of variables**: The method needed to solve the classical wave equation, the Schrödinger equation, and many other PDEs that arise in physical chemistry. *Procedure*



1. Assume that the dependent variable factors into the product of functions of its independent variables<sup>[2]</sup>.
  2. Substitute in the **ansatz** and algebraically rearrange the equation to gather all instances of each independent variable into distinct additive terms.
  3. Set each term equal to the **separation constant**.
    - This is allowed because the equality derived in step two must be valid under any variation of the independent variables. But since the *independent* variables can be varied independently, we can move any one of them around while holding the others constant and the equality must be preserved; thus, since the other terms are certainly constant under no variation, the term being varied must be constant as well.
    - This will yield one **ordinary differential equation** for each variable. These can be solved using knowledge from previous calculus courses.
- **Ansatz:** The assumed form of an unknown function.
  - **Separation constant:** A to-be-determined constant.
  - **Ordinary differential equation:** An equation in which the unknown occurs in ordinary derivatives. *Also known as ODE.*
  - Applying the method of separation of variables to the classical wave equation.
    1. Assume  $u(x, t)$  factors into a function  $X(x)$  of  $x$  times a function  $T(t)$  of  $t$ , that is,

$$u(x, t) = X(x)T(t)$$

2. Substitute and rearrange.

$$\begin{aligned}\frac{\partial^2 X(x)T(t)}{\partial x^2} &= \frac{1}{v^2} \frac{\partial^2 X(x)T(t)}{\partial t^2} \\ T(t) \frac{d^2 X(x)}{dx^2} &= \frac{1}{v^2} X(x) \frac{d^2 T(t)}{dt^2} \\ \frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} &= \frac{1}{v^2 T(t)} \frac{d^2 T(t)}{dt^2}\end{aligned}$$

3. Let  $K$  denote the separation constant. Then we obtain the following two ODEs.

$$\begin{aligned}\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} &= K & \frac{1}{v^2 T(t)} \frac{d^2 T(t)}{dt^2} &= K \\ \frac{d^2 X(x)}{dx^2} - KX(x) &= 0 & \frac{d^2 T(t)}{dt^2} - Kv^2 T(t) &= 0\end{aligned}$$

- In particular, these equations are **linear differential equations with constant coefficients**, which can be solved as follows.
- Interrogating the resultant ODEs.
  - $K$  is yet to be determined. Let's see what the solutions look like for different values.
  - $K = 0$ : This yields the solutions

$$X(x) = a_1 x + b_1$$

$$T(t) = a_2 t + b_2$$

where  $a_1, a_2, b_1, b_2$  are constants of integration.

---

<sup>2</sup>In the domain of differential equations, we can always assume that the solution has any form we want, and then solve to find all solutions *of this form*. However, this set of solution may not encapsulate *all* solutions, i.e., it may not be the *general solution*.

- Applying the boundary conditions, we learn that

$$u(0, t) = X(0)T(t) = 0 \qquad u(l, t) = X(l)T(t) = 0$$

- We now divide into two cases.
  - If  $T(t) = 0$  for all  $t$ , then the solution is  $u(x, t) = X(x) \cdot 0 = 0$ . This is referred to as the **trivial solution**, and it corresponds to the string at rest.
  - If  $T(t) \neq 0$  for some  $T$ , then  $X(0) = X(l) = 0$ , which implies that  $a_1 = b_1 = 0$  (think about the functional form algebraically). It follows that  $X(x) = 0 \cdot x + 0 = 0$  for all  $x$ , and we obtain the trivial solution again.
- $K > 0$ : Every (linearly independent) solution to equations of this type is of the form  $e^{\alpha x}$ <sup>[3]</sup>. Substituting into the ODE describing  $X(x)$ , we obtain

$$(\alpha^2 - K^2)X(x) = 0$$

- Assuming  $X(x)$  is nontrivial, we must have  $\alpha^2 - K^2 = 0$ , or  $\alpha = \pm K$ .
- Thus, there are two solutions:  $X(x) = e^{Kx}, e^{-Kx}$ .
- It follows from the fact that the relevant ODE is *linear* that

$$X(x) = c_1 e^{Kx} + c_2 e^{-Kx}$$

is also a solution.

- McQuarrie and Simon (1997) attempts to justify why the above is the general solution; a more rigorous explanation can be found in Labalme (2022).
- McQuarrie and Simon (1997) provides further examples of ODEs that can be solved by the ansatz  $e^{\alpha x}$  and factoring a binomial, as well as solving for  $c_1, c_2$  using initial conditions.

## 2.5 MathChapter B: Probability and Statistics

From McQuarrie and Simon (1997).

10/10:

- “Consider some experiment, such as the tossing of a coin or the rolling of a die, that has  $n$  possible outcomes, each with probability  $p_j$ , where  $j = 1, 2, \dots, n$ ” (McQuarrie & Simon, 1997, p. 63).
- If the experiment is repeated indefinitely, we intuitively expect that for each  $j = 1, \dots, n$

$$p_j = \lim_{N \rightarrow \infty} \frac{N_j}{N}$$

where  $N_j$  is the number of times that the event  $j$  occurs and  $N$  is the total number of repetitions of the experiment.

- The fact that  $0 \leq N_j \leq N$  implies that  $0 \leq p_j \leq 1$  by the above condition.
- **Certainty**: An event  $j$  such that  $p_j = 1$ .
- **Impossibility**: An event  $j$  such that  $p_j = 0$ .
- **Normalization condition**: The result that

$$\sum_{j=1}^n p_j = 1$$

- This follows from the fact that  $\sum_{j=1}^n N_j = N$  and the above.

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<sup>3</sup>See Labalme (2022) for more.

- The normalization condition expresses the idea that “the probability that some event occurs is a certainty” (McQuarrie & Simon, 1997, p. 64).
- **Average** (of  $x$ ): The following quantity, where we associate some number  $x_j$  with each outcome  $j$ . Also known as **mean** (of  $x$ ). Denoted by  $\langle x \rangle$ . Given by

$$\langle x \rangle = \sum_{j=1}^n x_j p_j = \sum_{j=1}^n x_j p(x_j)$$

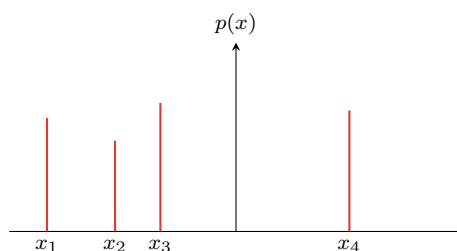


Figure 2.3: The discrete probability frequency function.

- It is helpful to interpret a probability distribution like  $p_j$  as a distribution of a unit mass along the  $x$ -axis in a discrete manner such that  $p_j$  is the fraction of mass located at the point  $x_j$ .
- According to this interpretation, the average value of  $x$  is the center of mass of this system.
- **Second moment** (of the distribution  $\{p_j\}$ ): The following quantity.

$$\langle x^2 \rangle = \sum_{j=1}^n x_j^2 p_j$$

- Note that  $\langle x^2 \rangle \neq \langle x \rangle^2$ .
- Analogous to the moment of inertia.
- The next quantity is physically more interesting than the second moment.
- **Second central moment**: The following quantity. Also known as **variance**. Denoted by  $\sigma_x^2$ . Given by

$$\sigma_x^2 = \langle (x - \langle x \rangle)^2 \rangle = \sum_{j=1}^n (x_j - \langle x \rangle)^2 p_j$$

- $\sigma_x^2 \geq 0$  because it is a sum of positive terms.
- An alternate form of  $\sigma_x^2$ :

$$\begin{aligned} \sigma_x^2 &= \sum_{j=1}^n (x_j - \langle x \rangle)^2 p_j \\ &= \sum_{j=1}^n (x_j^2 - 2\langle x \rangle x_j + \langle x \rangle^2) p_j \\ &= \sum_{j=1}^n x_j^2 p_j - 2\langle x \rangle \sum_{j=1}^n x_j p_j + \langle x \rangle^2 \sum_{j=1}^n p_j \\ &= \langle x^2 \rangle - 2\langle x \rangle \cdot \langle x \rangle + \langle x \rangle^2 \cdot 1 \\ &= \langle x^2 \rangle - \langle x \rangle^2 \end{aligned}$$

- If  $\sigma_x^2 = 0$  or  $\langle x \rangle^2 = \langle x^2 \rangle$ , then we must have  $x_j = \langle x \rangle$  for all  $j$ , i.e., the event is not really probabilistic because the event  $j$  occurs on every trial.

- **Standard deviation:** The positive square root of the variance. Denoted by  $\sigma_x$ .
- Both the standard deviation and the variance are measures of the spread of the distribution about its mean.
- We now step into continuous probability distributions.
- **Linear mass density:** The quantity  $\rho(x)$  defined by

$$dm = \rho(x) dx$$

where  $dm$  is the fraction of the mass lying between  $x$  and  $x + dx$ .

- It follows that the probability that, for example, a particle lies between positions  $x$  and  $x + dx$  in a box is

$$\text{Prob}(x, x + dx) = p(x) dx$$

- Therefore,

$$\text{Prob}(a \leq x \leq b) = \int_a^b p(x) dx$$

- Furthermore, the continuous normalization condition becomes

$$\int_{-\infty}^{\infty} p(x) dx = 1$$

- We may also analogously define

$$\langle x \rangle = \int_{-\infty}^{\infty} xp(x) dx \quad \langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 p(x) dx \quad \sigma_x^2 = \int_{-\infty}^{\infty} (x - \langle x \rangle)^2 p(x) dx$$

- **Gaussian distribution:** The most commonly occurring and the most important continuous probability distribution. Given by

$$p(x) dx = ce^{-x^2/2a^2} dx$$

- Note that the normalization condition implies that

$$c = \frac{1}{\sqrt{2\pi a^2}}$$

- We can also prove that

$$\sigma_x = a$$

- Thus, the standard notation for a normalized Gaussian distribution function is

$$p(x) dx = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-x^2/2\sigma_x^2} dx$$

- Note that as  $\sigma_x$  gets smaller, the bell curves become narrower and taller, and vice versa as it gets larger.
- A more general form (one that accounts for a center at  $x = \langle x \rangle$  as opposed to just  $x = 0$ ) is

$$p(x) dx = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-(x-\langle x \rangle)^2/2\sigma_x^2} dx$$

## 2.6 Chapter 3: The Schrödinger Equation and a Particle In a Box

From McQuarrie and Simon (1997).

- **Schrödinger equation:** The fundamental equation of quantum mechanics. A differential equation whose solution  $\psi(x)$  describes a particle of mass  $m$  moving in a potential field described by  $V(x)$ . Given by

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x)$$

- **Wave function:** A solution to the Schrödinger equation. A measure of the amplitude of the matter wave. Denoted by  $\psi(x)$
- **Stationary-state wave function:** A solution to the time-independent Schrödinger equation.
- **Particle in a box:** A system consisting of a free particle of mass  $m$  that is restricted to lie along a one-dimensional interval of length  $a$ .
- **Spatial amplitude** (of a wave): The function  $\psi(x)$  of position that serves as a coefficient to a time-dependent wave function.
- **Operator:** A symbol that tells you to do something to whatever follows the symbol. Denoted by a capital letter with a carat over it.
- **Linear operator:** An operator  $\hat{A}$  such that

$$\hat{A}[c_1f_1(x) + c_2f_2(x)] = c_1\hat{A}f_1(x) + c_2\hat{A}f_2(x)$$

where  $c_1, c_2$  are (possibly complex) constants.

- **Eigenvalue problem:** The problem of determining  $\phi(x)$  and  $a$  given  $\hat{A}$  such that

$$\hat{A}\phi(x) = a\phi(x)$$

- **Eigenfunction:** The function  $\phi(x)$  in an eigenvalue problem.
- **Eigenvalue:** The constant  $a$  in an eigenvalue problem.
- **Hamiltonian operator:** The following operator. Denoted by  $\hat{H}$ . Given by

$$\hat{H} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)$$

- The substitution of the Hamiltonian operator into the Schrödinger equation allows us to formulate the Schrödinger equation as an eigenvalue problem.
  - The wave function then becomes the eigenfunction and the energy, the eigenvalue of the Hamiltonian operator.

- **Kinetic energy operator:** The following operator. Denoted by  $\hat{K}_x$ . Given by

$$\hat{K}_x = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2}$$

- Defined by taking  $V(x) = 0$  in the Hamiltonian.

- **Momentum operator:** The following operator. Denoted by  $\hat{P}_x$ .

$$\hat{P}_x = -i\hbar \frac{d}{dx}$$

- Defined by applying  $K = p^2/2m$  to the kinetic energy operator to get

$$\hat{P}_x^2 = -\hbar^2 \frac{d^2}{dx^2}$$

and then (noting that we define the square of an operator to be equivalent to applying the same operator successively) factoring the above into

$$\hat{P}_x^2 = \hat{P}_x \hat{P}_x = -\hbar^2 \frac{d^2}{dx^2} = \left(-i\hbar \frac{d}{dx}\right) \left(-i\hbar \frac{d}{dx}\right)$$

- **Free particle:** A particle that experiences no potential energy, i.e., a particle for which  $V(x) = 0$ .
- When solving the particle in a box, we say that  $\psi(x)$  represents the the amplitude of the particle in some sense. More specifically, since the intensity of a wave is the square of the magnitude of the amplitude, we write that the “intensity of the particle” is proportional to  $\psi^*(x)\psi(x)$ .
  - Born, a German physicist working in scattering theory, formalized this by saying that  $\psi^*(x)\psi(x) dx$  is the “probability that the particle is located between  $x$  and  $x + dx$ ” (McQuarrie & Simon, 1997, p. 80).
- Schrödinger’s quantization of energy arises naturally from his equation and the boundary conditions, as opposed to having to be postulated as in Bohr’s model.
- McQuarrie and Simon (1997) use the free-particle model to crudely explain the absorption spectrum of butadiene.
- **Normalized** (wave function): A wave function of the form

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}$$

- “Because the Hamiltonian operator is a linear operator, if  $\psi$  is a solution to  $\hat{H}\psi = E\psi$ , then any constant, say  $A$ , times  $\psi$  is also a solution, and  $A$  can always be chosen to produce a normalized solution to the Schrödinger equation” (McQuarrie & Simon, 1997, p. 84).
- **Correspondence principle:** Quantum mechanical results and classical mechanical results tend to agree in the limit of large quantum numbers.
- Applying the statistical principles to the particle in a box, we can calculate that

$$\begin{aligned} \langle x \rangle &= \frac{a}{2} & \langle x^2 \rangle &= \frac{a^2}{3} - \frac{a^2}{2n^2\pi^2} & \sigma_x^2 &= \langle x^2 \rangle - \langle x \rangle^2 & \sigma_x &= \frac{a}{2\pi n} \sqrt{\frac{\pi^2 n^2}{3} - 2} \\ & & & & & = \left(\frac{a}{2\pi n}\right)^2 \left(\frac{\pi^2 n^2}{3} - 2\right) \end{aligned}$$

- Calculating the average energy or momentum:
  - To calculate the average value of the physical quantity associated with an operator, we sandwich the operator between a wave function  $\psi_n(x)$  and its complex conjugate  $\psi_n^*(x)$ .
  - This will be formalized later, but for now, we assume that

$$\langle s \rangle = \int \psi_n^*(x) \hat{S} \psi_n(x) dx$$

where  $\hat{S}$  is the quantum-mechanical operator associated with the physical quantity  $s$ , and  $\langle s \rangle$  is the average value of  $s$  in the state described for the wave function.

- For example, the average momentum of a particle in a box in the state described by  $\psi_n(x)$  is

$$\langle p \rangle = \int_0^a \left[ \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \right] \left( -i\hbar \frac{d}{dx} \right) \left[ \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} \right] dx$$

- Note that the average momentum of a particle in a box is zero.

10/12:

- Calculating the variance  $\sigma_p^2$  of the momentum of a particle in a box.

$$\begin{aligned} \langle p^2 \rangle &= \int \psi_n^* \hat{P}_x^2 \psi_n(x) dx \\ &= \frac{n^2 \pi^2 \hbar^2}{a^2} \end{aligned}$$

- It follows that  $\sigma_p = n\pi\hbar/a$ .

- **Root-mean-square momentum:** The square root of  $\langle p^2 \rangle$ .
- “Because the variance  $\sigma^2$ , and hence the standard deviation  $\sigma$ , is a measure of the spread of a distribution about its mean value, we can interpret  $\sigma$  as a measure of the uncertainty involved in any measurement” (McQuarrie & Simon, 1997, p. 89).
  - For the simple situation of a particle in a box, we can calculate *exact* uncertainties in position and momentum  $\sigma_x, \sigma_p$ .
    - We can see from these exact formulae that  $\sigma_x$  is directly proportional to the length  $a$  of the box, and  $\sigma_p$  is inversely proportional to the length  $a$  of the box.
    - This means that as the box gets bigger, it becomes harder to know where the particle is but its momentum becomes more certain, and vice versa as the box gets smaller.
  - From the above, it is clear that  $\sigma_x$  and  $\sigma_p$  have a reciprocal relation.
  - Indeed, taking the product  $\sigma_x \sigma_p$  yields the **Heisenberg Uncertainty Principle**

$$\begin{aligned} \sigma_x \sigma_p &= \frac{\hbar}{2} \sqrt{\frac{\pi^2 n^2}{3} - 2} \\ &> \frac{\hbar}{2} \end{aligned}$$

- **Free particle:** A particle that is allowed to range over the entire  $x$ -axis.
  - “A free particle has a definite momentum, but its position is completely indefinite” (McQuarrie & Simon, 1997, p. 90).
- The Uncertainty Principle also says that the minimum product of the two uncertainties is on the order of Planck’s constant.
- The particle in a three-dimensional box:
  - If the box (a rectangular parallelepiped) has sides of length  $a, b, c$ , then the Schrödinger equation for this case is

$$-\frac{\hbar^2}{2m} \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right) = E\psi(x, y, z)$$

where  $0 \leq x \leq a$ ,  $0 \leq y \leq b$ , and  $0 \leq z \leq c$ .

- An alternate form is

$$-\frac{\hbar^2}{2m} \nabla^2 \psi = E\psi$$

- Boundary conditions:

- $\psi(0, y, z) = \psi(a, y, z) = 0$  for all  $y, z$ .
  - $\psi(x, 0, z) = \psi(x, b, z) = 0$  for all  $x, z$ .
  - $\psi(x, y, 0) = \psi(x, y, c) = 0$  for all  $x, y$ .
- Invoke the method of separation of variables, i.e., suppose

$$\psi(x, y, z) = X(x)Y(y)Z(z)$$

- Then

$$-\frac{\hbar^2}{2m} \frac{1}{X(x)} \frac{d^2 X}{dx^2} - \frac{\hbar^2}{2m} \frac{1}{Y(y)} \frac{d^2 Y}{dy^2} - \frac{\hbar^2}{2m} \frac{1}{Z(z)} \frac{d^2 Z}{dz^2} = E$$

- It follows since each of the three terms contains only one of the variables and hence each of the terms can be varied independently that each term must equal a constant. The sum of the three constants will be  $E$ .
- But dividing the above equations into three smaller equations gives us cases entirely analogous to the one-dimensional particle in a box, meaning that

$$X(x) = A_x \sin \frac{n_x \pi x}{a} \quad Y(y) = A_y \sin \frac{n_y \pi y}{b} \quad Z(z) = A_z \sin \frac{n_z \pi z}{c}$$

for  $n = 1, 2, 3, \dots$

- Therefore, the solution is

$$\psi(x, y, z) = A_x A_y A_z \sin \frac{n_x \pi x}{a} \sin \frac{n_y \pi y}{b} \sin \frac{n_z \pi z}{c}$$

- The normalization constant then turns out to be

$$1 = \int_0^a \int_0^b \int_0^c \psi^*(x, y, z) \psi(x, y, z) dx dy dz$$

$$A_x A_y A_z = \sqrt{\frac{8}{abc}}$$

- We can now also obtain the following formula for the energies by plugging the full solution back into the original Schrödinger equation.

$$E_{n_x n_y n_z} = \frac{\hbar^2}{8m} \left( \frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2} \right)$$

- McQuarrie and Simon (1997) does calculations to show that  $\langle x \rangle = (a/2, b/2, c/2)$  and  $\langle p \rangle = 0$ .
- Note that we have the following 3D operators:

- Position operator:

$$\hat{\mathbf{R}} = \hat{X}\mathbf{i} + \hat{Y}\mathbf{j} + \hat{Z}\mathbf{k}$$

- Momentum operator:

$$\hat{\mathbf{P}} = -i\hbar \left( \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right)$$

- Consider the special case where  $a = b = c$ . Then only one set of values  $n_x, n_y, n_z$  corresponds to the lowest energy level, but *three degenerate ones* correspond to the second energy level (211, 121, 112).

- Note that the degeneracy is introduced by the symmetry of the box and is lifted when the box becomes no longer symmetric.

- **Laplacian operator:** The operator  $\nabla^2$ .
- **Separable (operator):** An operator that is the sum of multiple variably independent terms.



## Week 3

# Vibrational Motion and the Harmonic Oscillator

### 3.1 Vibrational Motion and the Harmonic Oscillator

- 10/11: • Suppose we have an attractive force  $F$  proportional to the displacement  $x$  from the center of a system

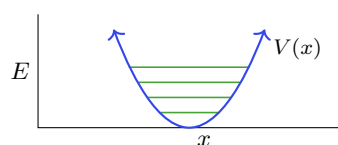
$$F = -kx$$

- Then we also have an associated potential energy

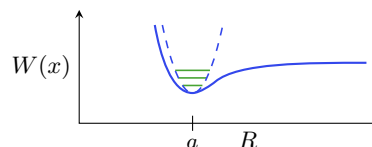
$$V(x) = \frac{1}{2}kx^2$$

- Recall that  $F = -\partial V/\partial x$ .

- Thus, we have a harmonic (or parabolic) potential well.



(a) Harmonic (parabolic) well.



(b) Approximating a potential well.

Figure 3.1: Parabolic potential wells.

- However, because this is a quantum system, the attainable energy levels will be quantized (see Figure 3.1a).
- We can use a parabolic well to approximate the minimum of the potential well (see Figure 3.1b).
- **Reduced mass:** The following quantity, which corresponds to two objects of mass  $m_A, m_B$ . Denoted by  $\mu$ . Given by

$$\mu = \frac{m_A m_B}{m_A + m_B}$$

- We can map the two body problem of two atoms being drawn together and pulled apart onto the one body problem of a single harmonic oscillator of reduced mass  $\mu$  at the center of mass of the diatomic system.

- The Taylor series expansion of the Moiré Potential about  $x = a$  where  $a$  is the minimum potential:

$$\begin{aligned} W(x) &= W(a) + (x - a)W'(a) + \frac{1}{2!}(x - a)^2W''(a) + \dots \\ &= W(a) + \frac{1}{2}(x - a)^2W''(a) \\ &= \frac{1}{2}kx^2 \end{aligned}$$

- We reduce by noting that  $W'(a) = 0$  at the minimum of the potential well, we can let  $W(a) = 0$ , we can set  $a = 0$  to be the origin of our coordinate system, and we can let  $k = W''(a)$ .

- The Schrödinger equation describing this system is<sup>[1]</sup>

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) + \frac{1}{2} m \omega^2 x^2 \psi(x) = E \psi(x)$$

- Note that since  $\omega = \sqrt{k/m}$ , we substituted  $k = m\omega^2$ .

- If we change coordinates via  $x = y\sqrt{\hbar/m\omega}$ ,  $\psi(x) = \psi(y)$ , and  $E = \omega\hbar\epsilon/2$ , then we can simplify to

$$\frac{d^2}{dy^2} \psi(y) + (\epsilon - y^2) \psi(y) = 0$$

- Asymptotic solution: In the limit of large  $y$ , the finite value of  $\epsilon$  becomes negligible, that is

$$\frac{d^2}{dy^2} \psi(y) - y^2 \psi(y) = 0$$

- Linearly independent solutions:

$$\psi(y) = y^p e^{-y^2/2}$$

- This is the Gaussian exponential.
- $p$  is any integer; each value of  $p$  gives rise to a linearly independent solution.

- Since the general solution is the linear combination of all linearly independent solutions, it has the form

$$\psi(y) = e^{-y^2/2} H(y)$$

where  $H$  is a polynomial in  $y$ .

- Substituting the above back into the equation before the asymptotic solution, we learn that the polynomials  $H$  must satisfy the **Hermite equation** in order to yield a full solution.

$$\frac{d^2}{dy^2} H(y) - 2y \frac{d}{dy} H(y) + (\epsilon - 1) H(y) = 0$$

- What are the correct polynomials?

- The polynomials that are even about the origin will give us the even solutions, and vice versa for the odd ones.
- Even solution: Because the potential has a definite parity (even-ness or odd-ness), we know that the solution to the polynomials must be even or odd.
- Expanding in an infinite power series:

$$H(y) = \sum_{j=0}^{\infty} c_j y^{2j}$$

---

<sup>1</sup>All aspects of this derivation are covered in greater detail in the notes on Dudik (2004).

- This is the power series solution to differential equations. We have to plug into the differential equation and get a recursion relation.

- Upon substitution,

$$\sum_{j=0}^{\infty} (2j(2j-1)c_j y^{2j-2} + (\epsilon - 1 - 4j)c_j y^{2j}) = 0$$

- The indices are arbitrary, so

$$\sum_{j=0}^{\infty} (2(j+1)(2j+1)c_{j+1} + (\epsilon - 1 - 4j)c_j) y^{2j} = 0$$

- Recursion relation: The whole coefficient above must equal 0 for all  $j$ , but that gives us a relationship between  $c_j$  and  $c_{j+1}$ ! Explicitly,

$$c_{j+1} = \frac{4j + 1 - \epsilon}{2(j+1)(2j+1)} c_j$$

- How do we know when to stop?

- If the recursion never stops, then the ratio is approximately equal to

$$\frac{c_{j+1}}{c_j} = \frac{1}{j}$$

- But this means that asymptotically, the boundary conditions will be violated because it will keep expanding. The probability of finding the particle will actually diverge (infinite probability at infinite distances). Thus, the expansion procedure *must* terminate.
- The truncation of this expansion requires us to pick a particular energy  $\epsilon$  (in particular, one such that  $\epsilon = 4j + 1$ ).

- Test on Friday:

- 5 questions. Each question is approximately 20 points.
- There will be a formula page in the back with formulas and constants.
- There will be a periodic table provided.
- Topics: Everything in problem sets 1-2. BB radiation, Photoelectric effect, SG experiment, particle-wave duality, Heisenberg uncertainty relations, Gaussian wave packets and their role in the Heisenberg uncertainty, de Broglie formula, free particles, particle in a box, potential step, and a bit of the harmonic oscillator.
- Study for it by going back to the problem sets and seeing which ones might be doable in a 50 minute test.
- Go back to your notes and review some of the key highlights of each of the topics in the topic list.
- You're allowed to use a calculator. The test won't be too calculator-heavy though.
- Deriving vs. understanding and applying: Emphasis on applying and getting answers.

## 3.2 Harmonic Oscillator (cont.)

10/13:

- Assume that the  $(n+1)^{\text{th}}$  coefficient vanishes by the recurrence relation; this causes the energy to be quantized.
  - Therefore,  $\epsilon = 4n + 1$ .

- For each value of  $n$ , there is an even Hermite polynomial<sup>[2]</sup>.
  - Thus, our even solutions include  $H_0(y) = 1$ ,  $H_2(y) = 4y^2 - 2$ ,  $H_4(y) = 16y^4 - 48y^2 + 2$ , for example.
- Odd solutions:
  - Let

$$H(y) = \sum_{j=0}^{\infty} d_j y^{2j+1}$$

- Our recurrence relation works out to be

$$d_{j+1} = \frac{4j+3-\epsilon}{2(j+1)(2j+3)} d_j$$

- Again, if it does not terminate,  $d_{j+1}/d_j \approx 1/j$ , so the solutions will blow up at the edges due to the high powers of  $y$ . Therefore, the series must truncate.
- If the coefficient at  $n$  exists but the next one will vanish, it must be true that  $4n+3=\epsilon$ .
- Example odd solutions:  $H_1(y) = 2y$ ,  $H_3(y) = 8y^3 - 12y$ ,  $H_5(y) = 32y^5 - 160y^3 + 120y$ .
  - Note that you can make the coefficients pretty much of any scale because they will be normalized later as part of the wave function.

- Energies and wave functions:

- The energy levels are  $\epsilon_n = 2n + 1$  for  $n = 0, 1, 2, \dots$  or  $E_n = (n + \frac{1}{2})\hbar\omega$  for  $n = 0, 1, 2, \dots$
- It follows that if  $\psi(y) = N_y e^{-y^2/2} H_n(y)$ , where

$$N_y = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}}$$

- In terms of  $x$ , we get

$$N_x = \sqrt[4]{\frac{\omega m}{\hbar}} N_y$$

- Observations:
  1. The energy levels are quantized (discrete).
  2. The energy levels are equally spaced apart where  $\Delta E = \hbar\omega$ .
  3. The energy levels are non-degenerate.
  4. The zero-point energy is equal to  $\hbar\omega/2$ . (Recall that the finite energy is there due to the Uncertainty Relation.)
  5. Similar to the levels discovered by Max Planck.

- Classical harmonic oscillator:

- Position:  $x(t) = x_0 \sin(\omega t)$ .
- Velocity:  $v(x) = \omega x_0 \cos(\omega t) = \omega x_0 \cos(\sin^{-1}(x/x_0)) = \omega \sqrt{x_0^2 - x^2}$ .
- Energy:  $E = mv_{\max}^2/2 = m\omega^2 x_0^2/2$ . Alternatively, turning point:  $x_0 = \sqrt{2E/m\omega^2}$ .
- Period:  $T = 2\pi/\omega$ .
- Probability of the oscillator being at  $x$ :

$$P(x) dx = \frac{2 dt}{T} = \frac{\frac{2 dx}{v(x)}}{T} = \frac{dx}{\pi \sqrt{x_0^2 - x^2}}$$

---

<sup>2</sup>So named because they were studied by the mathematician Charles Hermite before they were utilized in Quantum Mechanics.

- Thus, classically, the oscillator spends most of its time at the turning points (this makes intuitive sense because a pendulum slows down at the turning points, spending more time there).
- In quantum mechanics, we don't have a hard turning point the way we do classically.
- Classical limit of quantum theory: In higher and higher order Hermite polynomials, the probability gets pushed to the edges.

### 3.3 Chapter 4: Some Postulates and General Principles of Quantum Mechanics

From McQuarrie and Simon (1997).

- 10/19:
- **Dynamical variable:** A quantity dealt with by classical mechanics, e.g., position, momentum, angular momentum, and energy.
  - **Observable:** A measurable dynamical variable.
  - “The classical-mechanical state of a particle at any particular time is specified completely by the three position coordinates  $(x, y, z)$  and the three momenta  $(p_x, p_y, p_z)$  or velocities  $(v_x, v_y, v_z)$  at that time. The time evolution of the system is governed by Newton's equation”  $F = ma$  applied separately in each dimension (McQuarrie & Simon, 1997, p. 115).
  - **Trajectory** (of a classical particle): The three-dimensional path of the particle described by the initial conditions and Newton's laws.

– A particle's trajectory completely describes its state.

- However, since the Uncertainty Principle decrees that we cannot specify the position and momentum of a *microscopic*<sup>[3]</sup> particle to any desired precision, we have our first postulate of quantum mechanics.

**Postulate 1.** *The state of a quantum-mechanical system is completely specified by a function  $\psi(x)$  that depends upon the coordinate of the particle. All possible information about the system can be derived from  $\psi(x)$ . This function, called the wave function or the state function, has the important property that  $\psi^*(x)\psi(x) dx$  is the probability that the particle lies in the interval  $dx$ , located at the position  $x$ .*

– We will use one-dimensional notation, but note that what is stated in the postulates is equally applicable to two or three dimensions.

– “If there is more than one particle, say two, then  $\psi^*(x_1, x_2)\psi(x_1, x_2) dx_1 dx_2$  is the probability that particle 1 lies in the interval  $dx_1$  located at  $x_1$ , and that particle 2 lies in the interval  $dx_2$  located at  $x_2$ ” (McQuarrie & Simon, 1997, p. 116).

- **Normalizable** (function): A function  $\psi(x)$  for which the following integral converges.

$$\int_{\text{all space}} \psi^*(x)\psi(x) dx$$

- **Well-behaved** (function): A function  $\psi(x)$  such that it and its first derivative are single-valued, continuous, and finite.
- Every state function must be normalizable and well-behaved.
- We now formalize the notion that classical mechanical quantities have analogous linear operators in quantum mechanics.

**Postulate 2.** *To every observable in classical mechanics, there corresponds a linear Hermitian operator in quantum mechanics.*

OBSERVABLE		OPERATOR	
Name	Symbol	Symbol	Operation
Position	$x$	$\hat{X}$	Multiply by $x$
	$\mathbf{r}$	$\hat{\mathbf{R}}$	Multiply by $\mathbf{r}$
Momentum	$p_x$	$\hat{P}_x$	$-i\hbar \frac{\partial}{\partial x}$
	$\mathbf{p}$	$\hat{\mathbf{P}}$	$-i\hbar \left( \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right)$
Kinetic energy	$K_x$	$\hat{K}_x$	$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$
	$K$	$\hat{K}$	$-\frac{\hbar^2}{2m} \nabla^2$
Potential energy	$V(x)$	$\hat{V}(\hat{x})$	Multiply by $V(x)$
	$V(x, y, z)$	$\hat{V}(\hat{x}, \hat{y}, \hat{z})$	Multiply by $V(x, y, z)$
Total energy	$E$	$\hat{H}$	$-\frac{\hbar^2}{2m} \nabla^2 + V(x, y, z)$
Angular momentum	$L_x = yp_z - zp_y$	$\hat{L}_x$	$-i\hbar \left( y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right)$
	$L_y = zp_x - xp_z$	$\hat{L}_y$	$-i\hbar \left( z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right)$
	$L_z = xp_y - yp_x$	$\hat{L}_z$	$-i\hbar \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$

Table 3.1: Classical-mechanical observables and their corresponding quantum-mechanical operators.

– Note that we derive the expressions for  $L_x, L_y, L_z$  in Table 3.1 from  $\mathbf{r} \times \mathbf{p}$ .

- If an eigenvalue problem is two-fold degenerate with linearly independent solutions  $\phi_1, \phi_2$ , then any linear combination of  $\phi_1, \phi_2$  is also a solution.
- Although each observable has a corresponding operator by Postulate 2, only eigenvalues of our operators are actually observable.

**Postulate 3.** *In any measurement of the observable associated with the operator  $\hat{A}$ , the only values that will ever be observed are the eigenvalues  $a_n$ , which satisfy the eigenvalue equation*

$$\hat{A}\psi_n = a_n\psi_n$$

- We now formally introduce the method for finding the average values of observables.

**Postulate 4.** *If a system is in a state described by a normalized wave function  $\psi$ , then the average value of the observable corresponding to  $\hat{A}$  is given by*

$$\langle a \rangle = \int_{\text{all space}} \psi^*(x) \hat{A} \psi(x) dx$$

- Note that we can prove from Postulate 4 that  $\sigma$  for any observable is zero, i.e., that we *can* only observe the eigenvalues.
- We now postulate the time-dependent Schrödinger equation<sup>[4]</sup>.

<sup>3</sup>Note that classical mechanics still describes *macroscopic* bodies perfectly well.

<sup>4</sup>Note that just like Newton's laws, we cannot derive this — we can only show that data is consistent with it.

**Postulate 5.** *The wave function (or state function) of a system evolves in time according to the time-dependent Schrödinger equation*

$$\hat{H}\psi(x,t) = i\hbar \frac{\partial \psi(x,t)}{\partial t}$$

- Relating the time-dependent and time-independent Schrödinger equations.

- Consider a system where  $\hat{H}$  does not explicitly contain time.
- In such a case, we may use separation of variables and let

$$\psi(x,t) = \psi(x)f(t)$$

- Substituting in the above and dividing by  $\psi(x)f(t)$  allows us to fully separate the variables:

$$\begin{aligned}\hat{H}\psi(x)f(t) &= i\hbar \frac{d\psi(x)f(t)}{dt} \\ f(t)\hat{H}\psi(x) &= i\hbar \psi(x) \frac{df(t)}{dt} \\ \frac{1}{\psi(x)}\hat{H}\psi(x) &= \frac{i\hbar}{f(t)} \frac{df(t)}{dt}\end{aligned}$$

- But if two functions of entirely different variables are equal, the individual functions must both be constant. If we let the constant they are both equal to be  $E$ , then we obtain

$$\begin{aligned}\frac{1}{\psi(x)}\hat{H}\psi(x) &= E & \frac{i\hbar}{f(t)} \frac{df(t)}{dt} &= E \\ \hat{H}\psi(x) &= E\psi(x) & \frac{df(t)}{dt} &= \frac{1}{-i^{-1}\hbar} Ef(t) \\ & & &= -\frac{i}{\hbar} Ef(t) \\ & & f(t) &= e^{-iEt/\hbar}\end{aligned}$$

- The left equation above is the time-independent Schrödinger equation, and the right equation allows us to define the general form of a time-dependent wave function as follows.

$$\begin{aligned}\psi(x,t) &= \psi(x)f(t) \\ &= \psi(x)e^{-iEt/\hbar}\end{aligned}$$

- Note, however, that even the time-dependent Schrödinger equation gives stationary-state probability densities and averages:

$$\begin{aligned}\psi_n^*(x,t)\psi_n(x,t) dx &= \psi_n^*(x)e^{iE_nt/\hbar}\psi_n(x)e^{-iE_nt/\hbar} dx \\ &= \psi_n^*(x)\psi_n(x) dx\end{aligned}$$

- **Stationary-state** (wave function): The  $\psi_n(x)$  in the above equation.
- It follows from postulate 3 that the eigenvalues of quantum mechanical operators must be real (not complex) if they are to, indeed, to be observable.
  - A consequence of this is that any two eigenfunctions  $\psi_m(x), \psi_n(x)$  where  $m \neq n$  of quantum-mechanical operators are **orthogonal**.
- **Orthogonal** (wave functions): Two wave functions  $\psi_m(x), \psi_n(x)$  such that

$$\int_{-\infty}^{\infty} \psi_m^*(x)\psi_n(x) dx = 0$$

- **Orthonormal** (set of wave functions): A set of wave functions that are both normalized and mutually orthogonal.

– In other words, a set  $\psi_1, \dots, \psi_n$  of wave functions is orthonormal iff

$$\int_{-\infty}^{\infty} \psi_i^* \psi_j \, dx = \delta_{ij}$$

for all  $1 \leq i, j \leq n$ .

- Note that if a quantum-mechanical operator  $\hat{A}$  is to have real eigenvalues, it must satisfy the equation

$$\int_{\text{all space}} f^*(x) \hat{A} g(x) \, dx = \int_{\text{all space}} g(x) [\hat{A} f]^*(x) \, dx$$

where  $f, g$  are any two state functions.

– McQuarrie and Simon (1997) does a worked example of applying the above equation to the momentum operator in the context of the solutions to the one-dimensional harmonic oscillator problem.

- **Hermitian** (operator): An operator that satisfies the previous equation.
- **Commutator** (of  $\hat{A}, \hat{B}$ ): The operator defined as follows. Denoted by  $[\hat{A}, \hat{B}]$ . Given by

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

– We can show that  $\hat{A}$  and  $\hat{B}$  commute iff the commutator is equal to the zero operator.

- Relating the Uncertainty Principle and the commutator (without proof):
  - Consider two operators  $\hat{A}, \hat{B}$  and the standard deviations  $\sigma_a, \sigma_b$  corresponding to the uncertainties in the observed values of their respective physical quantities.
  - It follows from the fact that

$$\sigma_a^2 = \langle A^2 \rangle - \langle A \rangle^2 = \int \psi^*(x) \hat{A}^2 \psi(x) \, dx - \left[ \int \psi^*(x) \hat{A} \psi(x) \, dx \right]^2$$

(and similarly for  $\sigma_b$ ) that

$$\sigma_a \sigma_b \geq \frac{1}{2} \left| \int \psi^*(x) [\hat{A}, \hat{B}] \psi(x) \, dx \right|$$

- Results from the above.
  - If  $\hat{A}, \hat{B}$  commute, there is no restriction on the uncertainties in the measurements of  $a$  and  $b$ .
  - If  $\hat{A}, \hat{B}$  do not commute, there will be a restriction on the precision to which we can measure  $a$  and  $b$ , namely that as our measurements of one get more precise, our measurements of the other necessarily get less precise.
  - In particular, if we consider the case of  $\hat{X}, \hat{P}_x$  (the operators corresponding to the standard uncertainty relation), we have that

$$\begin{aligned} \hat{P}_x \hat{X} \psi(x) &= \left( -i\hbar \frac{d}{dx} \right) x \psi(x) & \hat{X} \hat{P}_x \psi(x) &= x \left( -i\hbar \frac{d}{dx} \right) \psi(x) \\ &= -i\hbar x \psi(x) - i\hbar \frac{d\psi}{dx} & &= -i\hbar x \frac{d\psi}{dx} \end{aligned}$$



so the corresponding commutator is

$$\begin{aligned} [\hat{P}_x, \hat{X}] &= \hat{P}_x \hat{X} - \hat{X} \hat{P}_x \\ &= \left[ -i\hbar \frac{d}{dx} - i\hbar x \frac{d}{dx} \right] - \left[ -i\hbar x \frac{d}{dx} \right] \\ &= -i\hbar \end{aligned}$$

so

$$\begin{aligned} \sigma_p \sigma_x &\geq \frac{1}{2} \left| \int \psi^*(x) (-i\hbar) \psi(x) dx \right| \\ &= \frac{1}{2} \left| -i\hbar \int \psi^*(x) \psi(x) dx \right| \\ &= \frac{1}{2} |-i\hbar \cdot 1| \\ &= \frac{\hbar}{2} \end{aligned}$$

which is the typical formulation of the Heisenberg Uncertainty Relation.

### 3.4 MathChapter D: Spherical Coordinates

*From McQuarrie and Simon (1997).*

- We can integrate products of multiple variables one at a time and multiply the results instead of taking a traditional triple integral.
- Example:
  - To find the volume of a sphere of radius  $a$  with a triple integral, we'd traditionally do the following.

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^\pi \int_0^a r^2 \sin \theta \, dr \, d\theta \, d\phi \\ &= \int_0^{2\pi} \int_0^\pi \frac{a^3}{3} \sin \theta \, d\theta \, d\phi \\ &= \int_0^{2\pi} \frac{a^3}{3} \cdot 2 \, d\phi \\ &= \frac{4}{3} \pi a^3 \end{aligned}$$

- However, we can alternatively do the following.

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^\pi \int_0^a r^2 \sin \theta \, dr \, d\theta \, d\phi \\ &= \frac{a^3}{3} \int_0^{2\pi} \int_0^\pi \sin \theta \, d\theta \, d\phi \\ &= \frac{a^3}{3} \cdot 2 \int_0^{2\pi} d\phi \\ &= \frac{a^3}{3} \cdot 2 \cdot 2\pi \\ &= \left( \int_0^a r^2 \, dr \right) \left( \int_0^\pi \sin \theta \, d\theta \right) \left( \int_0^{2\pi} d\phi \right) \end{aligned}$$

- In fact, to generalize, we can express an integral of the form

$$I = \int_0^\infty \int_0^\pi \int_0^{2\pi} F(r, \theta, \phi) r^2 \sin \theta \, dr \, d\theta \, d\phi$$

as an operator

$$I = \int_0^\infty dr \, r^2 \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi F(r, \theta, \phi)$$

where we understand each integral to act on everything that lies to its right, i.e., “we first integrate  $F(r, \theta, \phi)$  over  $\phi$  from 0 to  $2\pi$ , then multiply the result by  $\sin \theta$  and integrate over  $\theta$  from 0 to  $\pi$ , and finally multiply the result by  $r^2$  and integrate over  $r$  from 0 to  $\infty$ ” (McQuarrie & Simon, 1997, p. 150).

- If our integrand is spherically symmetric, i.e.,  $F(r, \theta, \phi) = f(r)$ , then

$$I = \int_0^\infty dr \, r^2 \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi f(r) = \int_0^\infty f(r) 4\pi r^2 \, dr$$

- In other words, we take a one-dimensional integral, but multiply the integrand by a factor of  $4\pi r^2 \, dr$ , which is the volume of a spherical shell of radius  $r$  and thickness  $dr$ .
- On the other hand, we can also hold  $r$  constant and integrate over the surface of a sphere of unit radius.
  - In this case, we sum pieces of differential surface area

$$dA = \sin \theta \, d\theta \, d\phi$$

- **Solid angle:** The solid enclosed by the surface that connects the origin and the area  $dA$ .
  - In the same way that a complete angle of the unit circle is  $2\pi$ , a complete solid angle is  $4\pi$ .
  - We often denote a complete solid angle by  $d\Omega$ . Thus, we can write

$$\int_{\text{sphere}} d\Omega = 4\pi$$

- The quantum theory of the hydrogen atom frequently involves angular integrals of the form

$$I = \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi F(\theta, \phi)$$

### 3.5 Chapter 5: The Harmonic Oscillator and the Rigid Rotator — Two Spectroscopic Models

From McQuarrie and Simon (1997).

- **Hooke’s law:** The equation describing the motion of a spring whose restoring force is directly proportional to its displacement from equilibrium. *Given by*

$$f = -kx$$

- **Force constant** (of a spring): The positive proportionality constant  $k$  in the Hooke’s law description of the spring.
- The general solution to  $ma = f = -kx$  is

$$x(t) = c_1 \sin \omega t + c_2 \cos \omega t$$

where  $\omega = \sqrt{k/m}$ .

- This solution can be rewritten in the form

$$x(t) = A \sin(\omega t + \phi)$$

- Thus, the displacement oscillates sinusoidally, or harmonically, with a natural frequency  $\omega$ , an amplitude  $A$ , and a phase angle  $\phi$ .
- Additionally, since  $f = -dV/dx$  where  $V$  is the potential energy of the system, we have that

$$-\frac{dV}{dx} = -kx$$

$$V(x) = \frac{1}{2}kx^2 + C$$

where  $C$  is an arbitrary constant used to fix the zero energy.

- Kinetic and potential energy:

- If  $x(t) = A \cos \omega t$ , then

$$K = \frac{1}{2}m \left( \frac{dx}{dt} \right)^2 \qquad V = \frac{1}{2}k[x(t)]^2$$

$$= \frac{1}{2}m\omega^2 A^2 \sin^2 \omega t \qquad = \frac{1}{2}kA^2 \cos^2 \omega t$$

- It follows that the total energy is

$$E = K + V$$

$$= \frac{1}{2}m\omega^2 A^2 \sin^2 \omega t + \frac{1}{2}kA^2 \cos^2 \omega t$$

$$= \frac{1}{2}kA^2 (\sin^2 \omega t + \cos^2 \omega t)$$

$$= \frac{1}{2}kA^2$$

- Since the total energy is constant, the system is **conservative**.

- **Conservative system:** A system in which the total energy is conserved.
- We now analyze a harmonic oscillator classically.

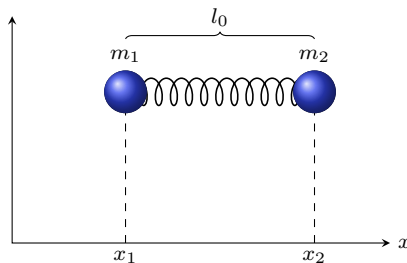


Figure 3.2: Harmonically oscillating diatomic molecule.

- We have two equations of motion, one for each mass.

$$m_1 \frac{d^2 x_1}{dt^2} = k(x_2 - x_1 - l_0) \qquad m_2 \frac{d^2 x_2}{dt^2} = -k(x_2 - x_1 - l_0)$$

- If we add them, we get the equation

$$\frac{d^2}{dt^2}(m_1x_1 + m_2x_2) = 0$$

which tells us that the center of mass of the system is not accelerating.

- Thus, the motion of the two-body system must depend only on the relative separation of the two masses, or on the **relative coordinate**

$$x = x_2 - x_1 - l_0$$

- Indeed, subtracting the initial two equations and substituting for both the relative coordinate and **reduced mass**

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

gives us

$$\begin{aligned}\frac{d^2x_2}{dt^2} - \frac{d^2x_1}{dt^2} &= -\frac{k}{m_2}(x_2 - x_1 - l_0) - \frac{k}{m_1}(x_2 - x_1 - l_0) \\ \frac{d^2}{dt^2}(x_2 - x_1) &= -k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)(x_2 - x_1 - l_0) \\ \mu \frac{d^2x}{dt^2} + kx &= 0\end{aligned}$$

- Conclusion: The kinematics of a diatomic harmonic oscillator are identical to those of a single ball on a spring except with the reduced mass and relative coordinate.
- “Generally, if the potential energy depends upon only the *relative* distances between two bodies, then we can introduce relative coordinates such as  $x_2 - x_1$  and reduce a two-body problem to a one-body problem” (McQuarrie & Simon, 1997, p. 163).
- Note that while the harmonic oscillator potential may be a terrible approximation of the exponential curve of bond energy overall, it is a very good approximation in the region of the minimum, which is the physically important region for many molecules at room temperature. In particular, the approximation works well for oscillators of small amplitude.
- **Equilibrium bond length:** The bond length at the minimum potential energy of a molecule.
- McQuarrie and Simon (1997) goes through the Taylor series derivation of Hooke’s law for diatomic molecules.
  - Additional result: The force constant is equal to the curvature of  $V(l)$  at the minimum.
- **Anharmonic term:** A term in the Taylor series expansion of a potential energy curve about the minimum of degree greater than two.
- **Morse potential:** An approximation of an intermolecular potential energy curve. *Given by*

$$V(l) = D(1 - e^{-\beta(l-l_0)})^2$$

- $D$  is the dissociation energy of the molecule measured from the minimum of  $V(l)$ .
- $\beta$  is a measure of the curvature of  $V(l)$  at its minimum.
- McQuarrie and Simon (1997) derives from the Taylor series expansion of the Morse potential about  $V(l_0)$  that the force constant  $k = 2D\beta^2$ .
- We are now ready to discuss a quantum-mechanical harmonic oscillator.

- Because of our prior simplification of a two-body harmonic oscillator to a single particle of reduced mass  $\mu$  moving in a potential field described by  $V(x) = kx^2/2$ , we have that the Schrödinger equation for a one-dimensional harmonic oscillator is

$$-\frac{\hbar^2}{2\mu} \frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x)$$

$$\frac{d^2\psi}{dx^2} + \frac{2\mu}{\hbar^2} \left( E - \frac{1}{2}kx^2 \right) \psi(x) = 0$$

- This differential equation does not have constant coefficients, so we need a new, custom method to solve it.
- Indeed, when solved, well-behaved solutions can be obtained only if the energy is restricted to the quantized values

$$E_v = \hbar\omega \left( v + \frac{1}{2} \right) = h\nu \left( v + \frac{1}{2} \right)$$

where  $\omega = \sqrt{k/\mu}$ ,  $\nu = \omega/2\pi$ , and  $v = 0, 1, 2, \dots$

- Note that these energy levels are equally spaced apart (see Figure 3.1a).
- Note the nonzero zero-point energy — since  $E = p^2/2\mu + kx^2/2$ , we cannot have  $E = 0$  as this would imply that  $0 = 0 \cdot 0 = (\Delta x)(\Delta p) < \hbar/2$ , violating the Uncertainty Principle.
- **Selection rule:** A criterion that narrows the states between which a quantum system can move.
- McQuarrie and Simon (1997) will later prove that the harmonic oscillator model allows transitions only between adjacent energy states.
  - Thus, the only frequency absorbed or emitted by a diatomic harmonic oscillator should be

$$\begin{aligned} \nu_{\text{obs}} &= \frac{\Delta E}{h} \\ &= \frac{E_{v+1} - E_v}{h} \\ &= \frac{h\nu}{h} \\ &= \frac{1}{2\pi} \sqrt{\frac{k}{\mu}} \end{aligned}$$

- We can convert to wavenumbers by dividing the above by  $c$ .
- **Fundamental vibration frequency:** The single frequency absorbed or emitted by a diatomic molecule.
  - We can use the fundamental vibration frequency and the above equation to calculate the force constant of the attraction.
- McQuarrie and Simon (1997) will later prove that the dipole moment of the molecule must change as the molecule vibrates in order for it to absorb infrared radiation.
- As a last note, be aware that although deviations from the prediction do crop up, they can be systematically corrected for.
- Harmonic oscillator wave functions:

$$\psi_v(x) = N_v H_v(\sqrt{\alpha}x) e^{-\alpha x^2/2}$$

- We have

$$\alpha = \frac{\sqrt{k\mu}}{\hbar} \qquad N_v = \frac{1}{\sqrt{2^v v!}} \sqrt{\frac{\alpha}{\pi}}$$

where  $N_v$  is a normalization constant and  $H_v(\sqrt{\alpha}x)$  is the  $v^{\text{th}}$ -degree **Hermite polynomial** in  $\sqrt{\alpha}x$ .

- **Hermite polynomials:** A classical orthogonal polynomial sequence. *Examples*

$H_0(\xi) = 1$	$H_1(\xi) = 2\xi$
$H_2(\xi) = 4\xi^2 - 2$	$H_3(\xi) = 8\xi^3 - 12\xi$
$H_4(\xi) = 16\xi^4 - 48\xi^2 + 12$	$H_5(\xi) = 32\xi^5 - 160\xi^3 + 120\xi$

Table 3.2: The first few Hermite polynomials.

- The polynomials in Table 3.2 allow us to generate the first few wave functions.

$\psi_0(x) = \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2/2}$	$\psi_1(x) = \sqrt{\frac{4\alpha^3}{\pi}} x e^{-\alpha x^2/2}$
$\psi_2(x) = \sqrt{\frac{\alpha}{4\pi}} (2\alpha x^2 - 1) e^{-\alpha x^2/2}$	$\psi_3(x) = \sqrt{\frac{\alpha^3}{9\pi}} (2\alpha x^3 - 3x) e^{-\alpha x^2/2}$

Table 3.3: The first few harmonic-oscillator wave functions.

- When graphed, the above wave functions and their corresponding probabilities look like the following.

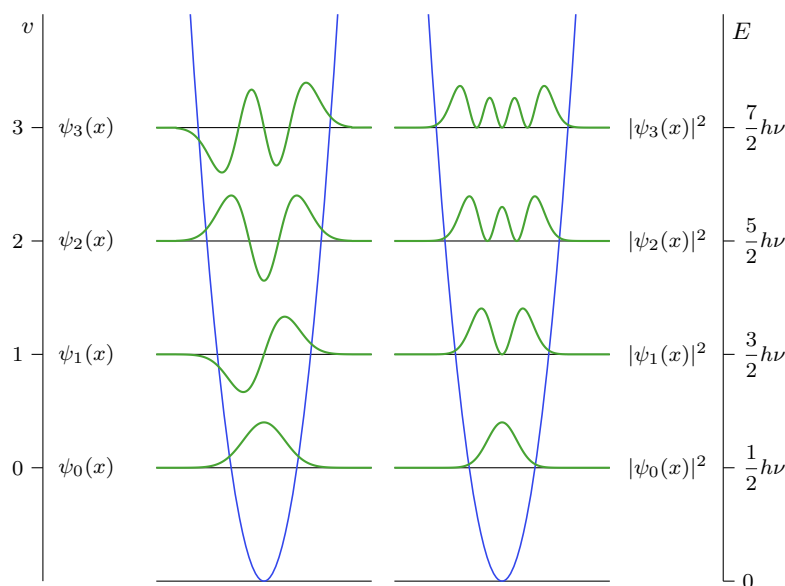


Figure 3.3: Harmonic-oscillator wave functions and probability densities.

- Notice how the probability distributions are slowly converging to the classical limit of a parabola of peaks.

- McQuarrie and Simon (1997) proves that a couple of the given wave functions satisfy the relevant Schrödinger equation, are normalized, and are orthogonal.
- Hermite polynomials are even functions if  $v$  is even and odd functions if  $v$  is odd.
  - Thus,  $\psi_v$  is even when  $v$  is even, and odd when  $v$  is odd.
  - It follows that  $\psi_v^2$  is even for any  $v$  (since the square of either an odd or even function is even).
- Consequently, since  $x\psi_v^2(x)$  is odd, we have that

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi_v(x) x \psi_v(x) dx = 0$$

for a harmonic oscillator.

- Similarly, since the derivative of an even function is odd and the product of an even and odd function is odd, we have that

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi_v(x) \left( -i\hbar \frac{d}{dx} \right) \psi_v(x) dx = 0$$

for a harmonic oscillator.

## Problems

10/26: **3-28.** The Schrödinger equation for a particle of mass  $m$  constrained to move on a circle of radius  $a$  is

$$-\frac{\hbar^2}{2I} \frac{d^2\psi}{d\theta^2} = E\psi(\theta)$$

where  $I = ma^2$  is the moment of inertia and  $0 \leq \theta \leq 2\pi$  is the angle that describes the position of the particle around the ring. Show by direct substitution that the solutions to this equation are

$$\psi(\theta) = Ae^{in\theta}$$

where  $n = \pm\sqrt{2IE}/\hbar$ . Argue that the appropriate boundary condition is  $\psi(\theta) = \psi(\theta + 2\pi)$  and use this condition to show that

$$E = \frac{n^2\hbar^2}{2I}$$

where  $n = 0, \pm 1, \pm 2, \dots$ . Show that the normalization constant  $A$  is  $1/\sqrt{2\pi}$ . Discuss how you might use these results for a free-electron model of benzene.

*Answer.* We have that

$$\begin{aligned} E(Ae^{in\theta}) &= -\frac{\hbar^2}{2I} \frac{d^2}{d\theta^2} (Ae^{in\theta}) \\ &= \frac{An^2\hbar^2}{2I} e^{in\theta} \\ &= \frac{2IEA\hbar^2}{2\hbar^2 I} e^{in\theta} \\ &= EAe^{in\theta} \end{aligned}$$

so the given wave function is a solution to the given Schrödinger equation, as desired.

Since the free space is a 2D ring into which all wave functions must “fit,” i.e., align in phase, it is necessary that  $\psi(\theta) = \psi(\theta + 2\pi)$ . Substituting in our solutions, we have that

$$\begin{aligned} Ae^{in\theta} &= Ae^{in(\theta+2\pi)} \\ &= Ae^{in\theta} e^{2\pi in} \\ e^{2\pi in} &= 1 \\ n &= 0, \pm 1, \pm 2, \dots \end{aligned}$$

as desired.

As for the normalization constant, we have that

$$\begin{aligned} 1 &= \int_0^{2\pi} \psi^*(\theta)\psi(\theta) d\theta \\ &= \int_0^{2\pi} (Ae^{-in\theta})(Ae^{in\theta}) d\theta \\ &= A^2 \int_0^{2\pi} 1 d\theta \\ A &= \frac{1}{\sqrt{2\pi}} \end{aligned}$$

as desired.

This could potentially be used for a free-electron model of benzene by looking at the energy levels and associated probability densities and overlaying them on the benzene ring.  $\square$

### 3.6 Harmonic Oscillator Energy Derivation

From Dudik (2004).

- 11/17: • The one-dimensional time-independent Schrödinger equation governing the motion of the harmonic oscillator is

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi = E\psi$$

as derived from the sum of the quantum kinetic energy operator and classical potential for a harmonic oscillator.

- To find the wave functions and eigenenergies of this second-order differential equation, we use the power series method.
- To begin, introduce the dimensionless variable

$$y = \sqrt{\frac{m\omega}{\hbar}} x$$

and perform a change of variables as follows.

$$\begin{aligned} 0 &= \frac{d^2\psi(x)}{dx^2} + \left( \frac{2mE}{\hbar^2} - \frac{m^2\omega^2}{\hbar^2} x^2 \right) \psi(x) \\ &= \frac{d}{dy} \left( \frac{d\psi(y)}{dy} \cdot \frac{dy}{dx} \right) \cdot \frac{dy}{dx} + \left[ \frac{2mE}{\hbar^2} - \frac{m^2\omega^2}{\hbar^2} \left( \sqrt{\frac{\hbar}{m\omega}} y \right)^2 \right] \psi(y) \\ &= \frac{d}{dy} \left( \frac{d\psi(y)}{dy} \cdot \sqrt{\frac{m\omega}{\hbar}} \right) \cdot \sqrt{\frac{m\omega}{\hbar}} + \left( \frac{2mE}{\hbar^2} - \frac{m\omega}{\hbar} y^2 \right) \psi(y) \\ &= \frac{d^2\psi(y)}{dy^2} + \left( \frac{2E}{\hbar\omega} - y^2 \right) \psi(y) \end{aligned}$$

- If we let  $\epsilon = 2E/\hbar\omega$  as well, we obtain

$$\frac{d^2\psi(y)}{dy^2} + (\epsilon - y^2)\psi(y) = 0$$

as a fully simplified version of the original differential equation.



- We now perform an asymptotic analysis.
- In the limit as  $y \rightarrow \infty$ , the constant  $\epsilon$  becomes negligible in comparison to the  $y^2$  term. Thus, our differential equation becomes

$$\frac{d^2\psi(y)}{dy^2} - y^2\psi(y) = 0$$

11/18:

- The general solution is

$$\psi(y) = H(y)e^{-y^2/2}$$

where  $H(y)$  is some function.

- Substituting this general solution back into the simplified differential equation yields a condition that the function(s)  $H(y)$  must satisfy.

$$\begin{aligned} 0 &= \frac{d^2}{dy^2}(\psi(y)) + (\epsilon - y^2)\psi(y) \\ &= \frac{d}{dy} \left( \frac{d}{dy} \left( H(y)e^{-y^2/2} \right) \right) + (\epsilon - y^2)H(y)e^{-y^2/2} \\ &= \frac{d}{dy} \left( \frac{dH(y)}{dy} e^{-y^2/2} - yH(y)e^{-y^2/2} \right) + (\epsilon - y^2)H(y)e^{-y^2/2} \\ &= \frac{d}{dy} \left( \frac{dH(y)}{dy} e^{-y^2/2} \right) + \frac{d}{dy} \left( -yH(y)e^{-y^2/2} \right) + (\epsilon - y^2)H(y)e^{-y^2/2} \\ &= \left( \frac{d^2H(y)}{dy^2} e^{-y^2/2} - y \frac{dH(y)}{dy} e^{-y^2/2} \right) + \left( -H(y)e^{-y^2/2} - y \frac{dH(y)}{dy} e^{-y^2/2} + y^2 H(y)e^{-y^2/2} \right) + (\epsilon - y^2)H(y)e^{-y^2/2} \\ &= \frac{d^2H(y)}{dy^2} e^{-y^2/2} - 2y \frac{dH(y)}{dy} e^{-y^2/2} + (\epsilon - 1)H(y)e^{-y^2/2} \\ &= \frac{d^2H(y)}{dy^2} - 2y \frac{dH(y)}{dy} + (\epsilon - 1)H(y) \end{aligned}$$

– Note that the last line above is the Hermite equation.

- To solve the second-order differential equation above, suppose

$$H(y) = \sum_{n=0}^{\infty} c_n y^n$$

- Then we have

$$\begin{aligned} 0 &= \frac{d^2}{dy^2}(H(y)) - 2y \frac{d}{dy}(H(y)) + (\epsilon - 1)H(y) \\ &= \frac{d^2}{dy^2} \left( \sum_{n=0}^{\infty} c_n y^n \right) - 2y \frac{d}{dy} \left( \sum_{n=0}^{\infty} c_n y^n \right) + (\epsilon - 1) \sum_{n=0}^{\infty} c_n y^n \\ &= \sum_{n=2}^{\infty} c_n n(n-1) y^{n-2} - 2y \sum_{n=1}^{\infty} c_n n y^{n-1} + (\epsilon - 1) \sum_{n=0}^{\infty} c_n y^n \\ &= \sum_{n=0}^{\infty} c_{n+2} (n+2)(n+1) y^n - 2y \sum_{n=0}^{\infty} c_{n+1} (n+1) y^n + (\epsilon - 1) \sum_{n=0}^{\infty} c_n y^n \\ &= \sum_{n=0}^{\infty} (c_{n+2} (n+2)(n+1) y^n - 2c_{n+1} (n+1) y^{n+1} + (\epsilon - 1)c_n y^n) \\ &= \sum_{n=0}^{\infty} (c_{n+2} (n+2)(n+1) y^n - 2c_n n y^n + (\epsilon - 1)c_n y^n) \end{aligned}$$

$$= \sum_{n=0}^{\infty} ((n+2)(n+1)c_{n+2} + (\epsilon - 1 - 2n)c_n)y^n$$

- For this equality to hold, the coefficient of each power of  $y$  on the right hand side of the above equation must equal zero, i.e., we must have

$$\begin{aligned}(n+2)(n+1)c_{n+2} + (\epsilon - 1 - 2n)c_n &= 0 \\ (n+2)(n+1)c_{n+2} &= (2n+1-\epsilon)c_n \\ c_{n+2} &= \frac{2n+1-\epsilon}{(n+1)(n+1)}c_n\end{aligned}$$

- The above equation is a “series representation of all the expansion coefficients in terms of  $c_0$  for the power series solution” to the Hermite equation (Dudik, 2004).
- However, there is a problem: For large values of  $y$ , if  $n$  keeps increasing indefinitely,  $H(y)$  will keep increasing indefinitely. Thus, to have a physically meaningful solution over the whole number line, the Taylor series must contain only a finite number of terms.
- It follows that at some point,  $a_{n+2} = 0$ . But this happens if and only if the numerator of the recursion relation equals zero, i.e., if

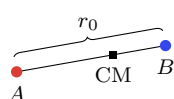
$$\begin{aligned}0 &= 2n+1-\epsilon \\ \frac{2E}{\hbar\omega} &= 2n+1 \\ E &= \hbar\omega \left( n + \frac{1}{2} \right)\end{aligned}$$

## Week 4

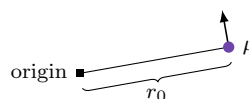
# The Hydrogen Atom and Angular Momentum

### 4.1 Rotational Motion

- 10/18: • Consider the diatomic molecule AB at a distance  $r_0$  apart rotating about its center of mass.



(a) Diatomic.



(b) An equivalent system.

Figure 4.1: Diatomic rotation.

- To simplify the problem, replace the two particles rotating about the center of mass with one particle of reduced mass  $\mu$  rotating about the center of mass with lever arm  $r_0$ .
- Classically, the kinetic energy of the translational motion is

$$T = \frac{L^2}{2I}$$

where  $I = \mu r_0^2$  and  $L = p \times r_0 = pr_0$  (for this kind of rotation; see Figure 4.1b).

- To further talk about this problem, we should introduce **spherical coordinates**.
- **Spherical coordinates:** The coordinate system  $(r, \theta, \phi)$  related to the Cartesian coordinates  $(x, y, z)$  by

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

- Classically, we will have

$$H = \frac{1}{2\mu}(p_x^2 + p_y^2 + p_z^2) + V(x, y, z)$$

in Cartesian coordinates.

- In spherical coordinates, this becomes

$$H = \frac{1}{2\mu} \left( p_r^2 + \frac{L^2}{r^2} \right) + V(r)$$

- Thus, in quantum mechanics, we get

$$\begin{aligned}\hat{H} &= \frac{1}{2\mu}(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2) + V(x, y, z) \\ &= \frac{1}{2\mu} \left( \hat{p}_r^2 + \frac{\hat{L}^2}{r^2} \right) + V(r)\end{aligned}$$

- Thus, we have in spherical coordinates that

$$\hat{T}\psi = -\frac{\hbar^2}{2\mu} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \psi \right) + \frac{1}{r^2 \sin \theta} \left( \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \psi \right) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \psi \right]$$

- 2D rigid rotor:

- Let the system from Figure 4.1b be confined to rotating in two dimensions.
- This simplifies the problem since both the  $\partial/\partial r$  and  $\partial/\partial \theta$  terms in the kinetic energy operator disappear (since, respectively, the particle is at a fixed distance from the center of mass and it cannot move out of the 2D plane).
- Thus, our Schrödinger equation for this system is

$$-\frac{\hbar^2}{2\mu r_0^2} \frac{\partial^2}{\partial \phi^2} \psi(\phi) = E\psi(\phi)$$

- Solution: Let  $\psi(\phi) = e^{im\phi}$ ; then

$$E_m = \frac{\hbar^2 m^2}{2\mu r_0^2}$$

- $m = 0, 1, 2, \dots$  is a new quantum number.
  - $m$  doesn't go to infinity because  $|m|$  is bounded by  $l$  (the total angular momentum).
- Remembering our original restriction, we have that this math describes the system from Figure 4.1a but confined to rotate in the  $xy$  plane with angular momentum in the  $z$  direction.
  - Thus, for example, the energies of the system from Figure 4.1a are dependent on  $m$  and  $I = \mu r_0^2$ .
- Such a system occurs in physical reality when we put the diatomic in an external field, or attach to it a big functional group.
- Zero point energy:  $m = 0$  does not violate the UR since we still have  $\Delta L \Delta \theta \geq \hbar/2$  (as everything is still rotating in the sense that we have equal probability of the particle being everywhere [as opposed to more localized/normal rotation with higher values of  $m$ ]).

- 3D rigid rotor:

- Assume that the potential energy is zero on the surface of the sphere (so we basically have a particle on a sphere).
- Then

$$\hat{H} = \frac{\hat{L}^2}{2\mu r_0^2} = \frac{\hat{L}^2}{2I}$$

- Solving  $\hat{H}\psi = E\psi$  asserts that the eigenfunctions of the Hamiltonian are the spherical harmonics  $Y_{lm}(\theta, \phi)$ .
- Energy:

$$E_l = \frac{\hbar^2}{2I} l(l+1)$$

where  $l = 0, 1, 2, \dots$

- Recall that  $m$  corresponds to the projection of angular momentum onto the  $z$ -axis, so that

$$m = -l, \dots, +l$$

## 4.2 Hydrogen Atom

10/20:

- Microwaves (for food) excite the rotational motion of water molecules.
- Spherical harmonics: The solution of  $\psi_{lm}(\theta, \phi) = Y_{lm}(\theta, \phi)$ , where  $l, m$  are quantum numbers.
- $E_l = \hbar^2/2I \cdot l(l+1)$  for  $l = 0, 1, 2, \dots$

- Form of the spherical harmonics:

$$Y(\theta, \phi) = P_{lm}(\cos \theta) e^{im\phi}$$

where  $P_{lm}(\cos \theta)$  is a polynomial.

- The polynomials  $P_{lm}(\cos \theta)$  are the associated Legendre polynomials.
- When  $m = 0$ , we have the Legendre polynomials.

- The differential equation describing these is

$$\frac{d}{dx} \left( (1-x^2) \frac{d}{dx} P_l(x) + l(l+1) P_l(x) \right) = 0$$

- The Legendre polynomials converge very quickly to functions on  $[-1, 1]$ .
- People map these polynomials onto other domains, too, to solve a variety of problems.
- Legendre polynomials have more of their roots at the boundaries — since the boundary conditions are the most important part of solving a differential equation, it makes sense that accurate representations would sample near the boundary more.
- Examples:

$$P_0(x) = 1$$

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$

$$P_1(x) = x$$

$$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$$

- Consider HCl.

- For it,

$$\frac{\hbar^2}{2I} = 1.3 \times 10^{-3} \text{ eV}$$

- Rotational spectral lines may arise from different values of the quantum number  $l$ .
- The molecule vibrates in harmonic oscillation with spacings  $\approx 0.1 \text{ eV}$  (it's not strictly rigid).
- Rovibrational spectra includes both forms of movement.
  - Very high precision.
  - Very big in the 90s.
- Electronic spectra: A few electron volts.

- The hydrogen atom.

- Two generalizations of the 3D rigid rotor combine to treat the hydrogen atom:
  1. An addition of the kinetic energy in the radial direction  $\hat{r}$ .
  2. An addition of the Coulomb potential.

- Schrödinger equation:

$$\hat{H}\psi(r, \theta, \phi) = E\psi(r, \theta, \phi)$$

where

$$\begin{aligned}\hat{H} &= \frac{1}{2\mu} \left( \hat{p}_r^2 + \frac{\hat{L}^2}{r^2} \right) + V(r) \\ \hat{p}_r^2 &= -\frac{\hbar^2}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \psi(r) \right) \\ V(r) &= -\frac{e(eZ)}{4\pi\epsilon_0 r}\end{aligned}$$

- Because the particle has spherical symmetry (that it, does not depend on  $\theta$  or  $\phi$ ), the wave function is separable, that is, it may be written as a product

$$\psi(r, \theta, \phi) = R_n(r)Y_{lm}(\theta, \phi)$$

- Note that there is no analytic solution to the Schrödinger equation in Cartesian coordinates — we need spherical coordinates to take advantage of the spherical symmetry.
- Substitution into the Schrödinger equation yields

$$\left( \frac{1}{2\mu} \left( \hat{p}_r^2 + \frac{\hat{L}^2}{r^2} \right) + V(r) \right) R(r)Y(\theta, \phi) = ER(r)Y(\theta, \phi)$$

where  $\mu$  is the reduced mass of the electron and the nucleus (which is approximately the mass of the electron).

- But since

$$\frac{1}{2\mu r^2} \hat{L}^2 Y_{lm}(\theta, \phi) = \frac{l(l+1)}{2\mu r^2} \hbar^2 Y_{lm}(\theta, \phi)$$

we have that

$$\left( \frac{1}{2\mu} \left( \hat{p}_r^2 + \frac{l(l+1)\hbar^2}{r^2} \right) + V(r) \right) R_n(r) = E_n R_n(r)$$

- We have reduced the three-dimensional case of the hydrogen atom to a one-dimensional differential equation.

### 4.3 Hydrogen Atom (cont.)

10/22:

- We want to solve the SE  $\hat{H}\psi_n = E\psi_n$  for the hydrogen atom.
- To exploit the spherical symmetry of the hydrogen atom, we use  $(r, \theta, \phi)$ .
- Thus, our Hamiltonian is equal to

$$\hat{H} = \frac{1}{2\mu} \left( \hat{p}_r^2 + \frac{\hat{L}^2}{r^2} \right) + V(r)$$

- Since  $V$  is just a function of  $r$ , the wave function is separable:  $\psi(r, \theta, \phi) = R_n(r)Y_{l,m}(\theta, \phi)$ .
- “Whenever you have a separation of variables additively in the Hamiltonian, you have a separation of variables multiplicatively in the wave function.”

- Substitute the wave function into the Schrödinger equation:

$$\begin{aligned}\left(\frac{1}{2\mu}\left(\hat{p}_r^2 + \frac{\hat{L}^2}{r^2}\right) + V(r)\right) R_n(r) Y_{lm}(\theta, \phi) &= E R_n(r) Y_{lm}(\theta, \phi) \\ \left(\frac{1}{2\mu}\left(\hat{p}_r^2 + \frac{l(l+1)\hbar^2}{r^2}\right) + V(r)\right) R_n(r) Y_{lm}(\theta, \phi) &= E R_n(r) Y_{lm}(\theta, \phi) \\ \left(\frac{1}{2\mu}\left(\hat{p}_r^2 + \frac{l(l+1)\hbar^2}{r^2}\right) + V(r)\right) R_n(r) &= E_{nl} R_n(r)\end{aligned}$$

- Noting that we only care about the behavior of the differential equation on  $[0, \infty)$ , specifically at really large distances, we perform an asymptotic analysis.

$$\lim_{r \rightarrow \infty} \left(\frac{1}{2\mu}\hat{p}_r^2\right) R(r) = E R(r)$$

- Plugging in the value of the momentum operator, we have that

$$\begin{aligned}-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} R(r) &= E R(r) \\ R(r) &= e^{-\alpha r}\end{aligned}$$

where  $\alpha = i\hbar/\sqrt{2\mu E}$ .

- But we want to multiply  $e^{-\alpha r}$  by some type of polynomial. Namely, **Laguerre polynomials**.

- These polynomials may be found by expanding a power series.
- Let

$$L\left(\frac{\alpha r}{n}\right) = \sum_{j=0}^{\infty} c_j \left(\frac{\alpha r}{n}\right)^j$$

- A recursion relation may be found.
- The quantization of energy in the hydrogen atom again arises from the truncation of the polynomials.
- Thus, the general solution of  $R_{nl}(r)$  is

$$R_{nl}(r) = \left(\frac{\alpha r}{n}\right)^l L_{n+1, 2l+1}\left(\frac{\alpha r}{n}\right) e^{-\alpha r/2n}$$

- Energy levels:

$$E_n = -\frac{\mu}{2\hbar^2} \left(\frac{(Ze)e}{4\pi\epsilon_0}\right)^2 \frac{1}{n^2}$$

- These are the discrete energies of Bohr!
- But now we also have the electron's probability distributions, i.e., the ground state wave function is

$$\psi(r, \theta, \phi) = \frac{1}{\sqrt{\pi}} \left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0}$$

- Thus, we can now take a look at the probabilities.

- The probability is

$$\text{Prob} = |\psi(r, \theta, \phi)|^2 r^2 \sin \theta$$

- The radial probability is

$$\text{Prob}(r) = r^2 |R_{nl}(r)|^2$$

- The radial probability peaks at  $a_0$ , the **Bohr radius**.
  - Thus, the Bohr radius (the radius of the circular orbit of Bohr's hydrogen electron) is just the most probable distance from the nucleus!
- The average distance from the nucleus (the expectation value of  $r$ ) is

$$\begin{aligned}\langle r \rangle &= \int_0^\infty \text{Prob}(r) r \, dr \\ &= \int_0^\infty \psi^*(r) r \psi(r) \, dr\end{aligned}$$

- Note we also find that  $\langle \hat{H} \rangle = E$ , so there is no uncertainty in the energy.
- Note that in some ways, quantum mechanics is more certain than classical mechanics since, for instance, in quantum we know the energy exactly.

## 4.4 Chapter 5: The Harmonic Oscillator and the Rigid Rotator — Two Spectroscopic Models

*From McQuarrie and Simon (1997).*

- 10/19:
- **Rigid-rotator model:** Two point masses  $m_1$  and  $m_2$  at fixed distances  $r_1$  and  $r_2$  from their center of mass.
    - Since the vibrational amplitude of a rotating molecule is small compared to its amplitude, this is a good model.
  - Kinetic energy of the rigid rotator:

$$\begin{aligned}K &= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \\ &= \frac{1}{2} (m_1 r_1^2 + m_2 r_2^2) \omega^2 \\ &= \frac{1}{2} I \omega^2\end{aligned}$$

- Note that

$$\begin{aligned}I &= m_1 r_1^2 + m_2 r_2^2 \\ &= \mu r^2\end{aligned}$$

(see Problem 5-29).

- It follows that we the two-body problem of the rigid rotator is equivalent to the one-body problem of a single body of mass  $\mu$  rotating at a distance  $r$  from a fixed center.
- Since there are no external forces on the rigid rotator (we're not applying any electric or magnetic fields), the energy of the molecule is solely kinetic (i.e., there is no potential energy term in the Hamiltonian).
  - Thus, for a rigid rotator,

$$\hat{H} = \hat{K} = -\frac{\hbar^2}{2\mu} \nabla^2$$



- Since this particle has a natural center of spherical symmetry, we opt for spherical coordinates. However, this necessitates expressing  $\nabla^2$  as the following.

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right)_{\theta, \phi} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right)_{r, \phi} + \frac{1}{r^2 \sin^2 \theta} \left( \frac{\partial^2}{\partial \phi^2} \right)_{r, \theta}$$

– See Problem 5-32 for a derivation.

- With respect to the rigid rotator,  $r$  is constant. Thus,

$$\begin{aligned}\nabla^2 &= \frac{1}{r^2} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2} \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \\ \hat{H} &= -\frac{\hbar^2}{2I} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \left( \frac{\partial^2}{\partial \phi^2} \right) \right] \\ \hat{L}^2 &= -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \left( \frac{\partial^2}{\partial \phi^2} \right) \right]\end{aligned}$$

– Note that since both  $\theta$  and  $\phi$  are unitless, the units of angular momentum for quantum system are  $\hbar$ .

- Rigid-rotator wave functions are customarily denoted by  $Y(\theta, \phi)$ .
- In solving  $\hat{H}Y(\theta, \phi) = EY(\theta, \phi)$ , it will be useful to multiply the original Schrödinger equation by  $\sin^2 \theta$  and let  $\beta = 2IE/\hbar^2$  to obtain the partial differential equation

$$\sin \theta \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y}{\partial \theta} \right) + \frac{\partial^2 Y}{\partial \phi^2} + (\beta \sin^2 \theta)Y = 0$$

- The solutions to the above equation are intimately linked to those for the hydrogen atom.
- Solving the above equation yields the condition that  $\beta = J(J+1)$  for  $J = 0, 1, 2, \dots$ . Therefore,

$$E_J = \frac{\hbar^2}{2I} J(J+1)$$

for  $J = 0, 1, 2, \dots$

■ Each energy level has a degeneracy  $g_J = 2J + 1$  as well.

- Once again, electromagnetic radiation can cause a rigid rotator to undergo transitions from one state to another subject to the selection rules that only transitions between adjacent states are allowed and the molecule must possess a permanent dipole moment.
- As before, we can calculate

$$\Delta E = \frac{h^2}{4\pi^2 I} (J+1)$$

and the frequencies at which absorption transitions occur are

$$\nu = \frac{h}{4\pi^2 I} (J+1)$$

for  $J = 0, 1, 2, \dots$

- It follows from reduced mass, bond length, and moment of inertia data that the frequencies typically lie in the microwave region.
- **Microwave spectroscopy:** The direct study of rotational transitions.
- **Rotational constant** (of a molecule): The following quantity. *Given by*

$$B = \frac{h}{8\pi^2 I}$$

- We often write the absorption frequencies as  $\nu = 2B(J + 1)$ .
- The spacing of lines in a microwave spectrum is  $2B$ .
- Like IR spectroscopy can be used to determine the force constants of molecular attractions in diatomics, microwave spectroscopy can be used to determine the bond lengths of diatomics.

## Problems

12/1: **5-24.** There are a number of general relations between the Hermite polynomials and their derivatives (which we will not derive). Some of these are

$$\begin{aligned}\frac{dH_v(\xi)}{d\xi} &= 2\xi H_v(\xi) - H_{v+1}(\xi) \\ H_{v+1}(\xi) - 2\xi H_v(\xi) + 2v H_{v-1}(\xi) &= 0 \\ \frac{dH_v(\xi)}{d\xi} &= 2v H_{v-1}(\xi)\end{aligned}$$

Such connecting relations are called **recursion formulas**. Verify these formulas explicitly using the first few Hermite polynomials given in Table 3.2.

10/19: **5-29.** Show that the moment of inertia for a rigid rotator can be written as  $I = \mu r^2$  where  $r = r_1 + r_2$  (the fixed separation of the two masses) and  $\mu$  is the reduced mass.

*Answer.* First, note that  $m_1 r_1 = m_2 r_2$  for such a rotation about the center of mass. Then

$$\begin{aligned}I &= \mu r^2 \\ &= \frac{m_1 m_2}{m_1 + m_2} (r_1 + r_2)^2 \\ &= \frac{m_1 m_2 r_1}{m_1 r_1 + m_2 r_1} (r_1 + r_2)^2 \\ &= \frac{m_1 m_2 r_1}{m_2 r_2 + m_2 r_1} (r_1 + r_2)^2 \\ &= \frac{m_1 r_1}{r_2 + r_1} (r_1 + r_2)^2 \\ &= m_1 r_1 (r_1 + r_2) \\ &= m_1 r_1^2 + m_1 r_1 r_2 \\ &= m_1 r_1^2 + m_2 r_2^2\end{aligned}$$

as desired. □

**5-30.** Consider the transformation from Cartesian coordinates to plane polar coordinates where

$$\begin{aligned}x &= r \cos \theta & r &= \sqrt{x^2 + y^2} \\ y &= r \sin \theta & \theta &= \tan^{-1} \left( \frac{y}{x} \right)\end{aligned}$$

If a function  $f(r, \theta)$  depends upon the polar coordinates  $r$  and  $\theta$ , then the chain rule of partial differentiation says that<sup>1</sup>

$$\left( \frac{\partial f}{\partial x} \right)_y = \left( \frac{\partial f}{\partial r} \right)_\theta \left( \frac{\partial r}{\partial x} \right)_y + \left( \frac{\partial f}{\partial \theta} \right)_r \left( \frac{\partial \theta}{\partial x} \right)_y$$

and that

$$\left( \frac{\partial f}{\partial y} \right)_x = \left( \frac{\partial f}{\partial r} \right)_\theta \left( \frac{\partial r}{\partial y} \right)_x + \left( \frac{\partial f}{\partial \theta} \right)_r \left( \frac{\partial \theta}{\partial y} \right)_x$$

---

<sup>1</sup>Note that the subscript means that the subscripted variable is held constant.

For simplicity, we will assume that  $r$  is constant so that we can ignore terms involving derivatives with respect to  $r$ . In other words, we will consider a particle that is constrained to move on the circumference of a circle. This system is sometimes called a **particle on a ring**. Using the above equations, show that

$$\left(\frac{\partial f}{\partial x}\right)_y = -\frac{\sin \theta}{r} \left(\frac{\partial f}{\partial \theta}\right)_r \quad \left(\frac{\partial f}{\partial y}\right)_x = \frac{\cos \theta}{r} \left(\frac{\partial f}{\partial \theta}\right)_r$$

for  $r$  fixed. Now apply the above equations again to show that

$$\begin{aligned} \left(\frac{\partial^2 f}{\partial x^2}\right)_y &= \left[\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x}\right)_y\right]_r \\ &= \left[\frac{\partial}{\partial \theta} \left(\frac{\partial f}{\partial x}\right)_y\right]_r \left(\frac{\partial \theta}{\partial x}\right)_y \\ &= \left\{ \frac{\partial}{\partial \theta} \left[ -\frac{\sin \theta}{r} \left(\frac{\partial f}{\partial \theta}\right)_r \right] \right\}_r \left(-\frac{\sin \theta}{r}\right) \\ &= \frac{\sin \theta \cos \theta}{r^2} \left(\frac{\partial f}{\partial \theta}\right)_r + \frac{\sin^2 \theta}{r^2} \left(\frac{\partial^2 f}{\partial \theta^2}\right)_r \end{aligned}$$

for  $r$  fixed. Similarly, show that

$$\left(\frac{\partial^2 f}{\partial y^2}\right)_x = -\frac{\sin \theta \cos \theta}{r^2} \left(\frac{\partial f}{\partial \theta}\right)_r + \frac{\cos^2 \theta}{r^2} \left(\frac{\partial^2 f}{\partial \theta^2}\right)_r$$

and that

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \longrightarrow \frac{1}{r^2} \left(\frac{\partial^2 f}{\partial \theta^2}\right)_r$$

both for  $r$  fixed. Now show that the Schrödinger equation for a particle of mass  $m$  constrained to move on a circle of radius  $r$  (see Problem 3-28) is

$$-\frac{\hbar^2}{2I} \frac{\partial^2 \psi(\theta)}{\partial \theta^2} = E\psi(\theta)$$

where  $I = mr^2$  is the moment of inertia and  $0 \leq \theta \leq 2\pi$ .

*Answer.* We have that

$$\begin{aligned} \left(\frac{\partial \theta}{\partial x}\right)_y &= \frac{\partial}{\partial x} \left(\tan^{-1} \left(\frac{y}{x}\right)\right) & \left(\frac{\partial \theta}{\partial y}\right)_x &= \frac{\partial}{\partial y} \left(\tan^{-1} \left(\frac{y}{x}\right)\right) \\ &= \frac{1}{\left(\frac{y}{x}\right)^2 + 1} \cdot y \cdot -\frac{1}{x^2} & &= \frac{1}{\left(\frac{y}{x}\right)^2 + 1} \cdot \frac{1}{x} \\ &= -\frac{y}{x^2 + y^2} & &= \frac{x}{x^2 + y^2} \\ &= -\frac{r \sin \theta}{r^2} & &= \frac{r \cos \theta}{r^2} \\ &= -\frac{\sin \theta}{r} & &= \frac{\cos \theta}{r} \end{aligned}$$

This combined with the fact that  $r$  is constant yields

$$\begin{aligned} \left(\frac{\partial f}{\partial x}\right)_y &= \left(\frac{\partial f}{\partial r}\right)_\theta \left(\frac{\partial r}{\partial x}\right)_y + \left(\frac{\partial f}{\partial \theta}\right)_r \left(\frac{\partial \theta}{\partial x}\right)_y & \left(\frac{\partial f}{\partial y}\right)_x &= \left(\frac{\partial f}{\partial r}\right)_\theta \left(\frac{\partial r}{\partial y}\right)_x + \left(\frac{\partial f}{\partial \theta}\right)_r \left(\frac{\partial \theta}{\partial y}\right)_x \\ &= 0 \cdot \left(\frac{\partial r}{\partial x}\right)_y + \left(\frac{\partial f}{\partial \theta}\right)_r \cdot -\frac{\sin \theta}{r} & &= 0 \cdot \left(\frac{\partial r}{\partial y}\right)_x + \left(\frac{\partial f}{\partial \theta}\right)_r \frac{\cos \theta}{r} \\ &= -\frac{\sin \theta}{r} \left(\frac{\partial f}{\partial \theta}\right)_r & &= \frac{\cos \theta}{r} \left(\frac{\partial f}{\partial \theta}\right)_r \end{aligned}$$

as desired.

The rest of the differential equation derivations follow fairly easy with simple calculus rules.

As for the Schrödinger equation, we know that there is no potential field for this particle. Additionally, although we could choose to express  $\psi$  as a function of  $x, y$ , owing to the radial symmetry, we choose to express it as a function of  $r, \theta$ . In fact, we need only express  $\psi$  as a function of  $\theta$  since  $r$  is invariant throughout the chosen free space. Thus, expanding from the general form, we get

$$\begin{aligned}\hat{H}\psi(\theta) &= E\psi(\theta) \\ \left(-\frac{\hbar^2}{2m}\nabla^2\right)\psi(\theta) &= E\psi(\theta) \\ -\frac{\hbar^2}{2m} \cdot \frac{1}{r^2} \left(\frac{\partial^2\psi(\theta)}{\partial\theta^2}\right)_r &= E\psi(\theta) \\ -\frac{\hbar^2}{2I} \frac{\partial^2\psi(\theta)}{\partial\theta^2} &= E\psi(\theta)\end{aligned}$$

as desired. □

- 5-31.** Generalize Problem 5-30 to the case of a particle moving in a plane under the influence of a central force; in other words, convert

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

to plane polar coordinates, this time without assuming that  $r$  is constant. Use the method of separation of variables to separate the equation for this problem. Solve the angular equation.

*Answer.* We have

$$\begin{aligned}\frac{\partial r}{\partial x} &= \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x & \frac{\partial r}{\partial y} &= \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y \\ &= \frac{x}{r} & &= \frac{y}{r} \\ &= \cos\theta & &= \sin\theta\end{aligned}$$

Thus

$$\frac{\partial f}{\partial x} = \cos\theta \frac{\partial f}{\partial r} - \frac{\sin\theta}{r} \frac{\partial f}{\partial\theta} \qquad \frac{\partial f}{\partial y} = \sin\theta \frac{\partial f}{\partial r} + \frac{\cos\theta}{r} \frac{\partial f}{\partial\theta}$$

Also note that

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) \\ &= \frac{\partial}{\partial\theta} \left( \frac{\partial f}{\partial x} \right) \frac{\partial\theta}{\partial x} + \frac{\partial}{\partial r} \left( \frac{\partial f}{\partial x} \right) \frac{\partial r}{\partial x}\end{aligned}$$

The rest of the expansions are routine, leading to

$$\nabla^2 f = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial\theta^2}$$

□

- 5-32.** Using Problems 5-30 and 5-31 as a guide, convert  $\nabla^2$  from three-dimensional Cartesian coordinates to spherical coordinates.

## 4.5 Chapter 6: The Hydrogen Atom

From McQuarrie and Simon (1997).

- 10/29: • Idealize the hydrogen atom to a proton fixed at the origin and an electron of mass  $m_e$  interacting with the proton through a Coulombic potential

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r}$$

- Because of the spherical geometry of the model, we opt for spherical coordinates.
- The appropriate Hamiltonian is thus

$$\hat{H} = -\frac{\hbar}{2m_e}\nabla^2 - \frac{e^2}{4\pi\epsilon_0 r}$$

- Expressing the Laplacian in spherical coordinates gives us

$$\left\{ -\frac{\hbar^2}{2m_e} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} \right] - \frac{e^2}{4\pi\epsilon_0 r} \right\} \psi(r, \theta, \phi) = E\psi(r, \theta, \phi)$$

as the Schrödinger equation for the hydrogen atom.

- Multiply through by  $2m_e r^2$  and move all terms over to the lefthand side of the equality to get the above equation into the following separable form.

$$\left\{ -\hbar^2 \left( \frac{\partial}{\partial r} r^2 \frac{\partial \psi}{\partial r} \right) - \hbar^2 \left[ \frac{1}{\sin \theta} \left( \frac{\partial}{\partial \theta} \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} \right] - 2m_e r^2 \left[ \frac{e^2}{4\pi\epsilon_0 r} + E \right] \right\} \psi(r, \theta, \phi) = 0$$

- Since all the  $\theta, \phi$  dependence occurs within the large square brackets, assume that

$$\psi(r, \theta, \phi) = R(r)Y(\theta, \phi)$$

- Substituting into the differential equation and separating terms (dividing by  $R(r)Y(\theta, \phi)$ ) yields

$$-\frac{\hbar^2}{R(r)} \left[ \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{2m_e r^2}{\hbar^2} \left( \frac{e^2}{4\pi\epsilon_0 r} + E \right) R(r) \right] - \frac{\hbar^2}{Y(\theta, \phi)} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \phi^2} \right] = 0$$

- All terms containing  $r$  are in the first set of square brackets, and all terms containing  $\theta, \phi$  are in the second set of square brackets. Since the above equation must hold for all  $r, \theta, \phi$ , and  $r, \theta, \phi$  are independent variables, we must have that varying  $r$  while holding  $\theta, \phi$  constant does not change the  $r$  term of the above equation (as this would violate the equality). Similarly, changing  $\theta, \phi$  must not change the  $\theta, \phi$  term. Therefore, we may let

$$\begin{aligned} -\beta &= -\frac{1}{R(r)} \left[ \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{2m_e r^2}{\hbar^2} \left( \frac{e^2}{4\pi\epsilon_0 r} + E \right) R(r) \right] \\ \beta &= -\frac{1}{Y(\theta, \phi)} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \phi^2} \right] \end{aligned}$$

where  $\beta$  is a **separation constant** into which we have incorporated  $\hbar^2$ .

- **Radial equation:** The first equation above.
- **Angular equation:** The second equation above after having been multiplied by  $\sin^2 \theta Y(\theta, \phi)$ . *Given by*

$$\sin \theta \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y}{\partial \theta} + \frac{\partial^2 Y}{\partial \phi^2} + \beta \sin^2 \theta Y \right) = 0$$

- Note that this is identical to the partial differential equation derived in Chapter 5 for the rigid rotator.
- Thus, the angular parts of hydrogen atomic orbitals are also rigid-rotator wave functions.

- **Spherical harmonic:** A wave function of the rigid rotator.
- We now solve the angular equation.
- Invoking separation of variables again with  $Y(\theta, \phi) = \Theta(\theta)\Phi(\phi)$  yields

$$\frac{\sin \theta}{\Theta(\theta)} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \beta \sin \theta + \frac{1}{\Phi(\phi)} \frac{d^2 \Phi}{d\phi^2} = 0$$

- We now introduce a new separation constant (written in a form that predicts future algebraic manipulations) to get

$$\begin{aligned} \frac{\sin \theta}{\Theta(\theta)} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \beta \sin^2 \theta &= m^2 \\ \frac{1}{\Phi(\phi)} \frac{d^2 \Phi}{d\phi^2} &= -m^2 \end{aligned}$$

- Solving the bottom equation above gives solutions

$$\Phi(\phi) = A_m e^{im\phi} \qquad \Phi(\phi) = A_{-m} e^{-im\phi}$$

where the constants  $A_m, A_{-m}$  may depend on the value of  $m$ .

- From the domain of  $\phi$ , we have the boundary condition  $\Phi(\phi + 2\pi) = \Phi(\phi)$ .
- This yields

$$\begin{aligned} A_m e^{im(\phi+2\pi)} &= A_m e^{im\phi} & A_{-m} e^{-im(\phi+2\pi)} &= A_{-m} e^{-im\phi} \\ e^{i2\pi m} &= 1 & e^{-i2\pi m} &= 1 \end{aligned}$$

from which it follows that we must have

$$\begin{aligned} 1 + 0i &= 1 \\ &= e^{\pm i2\pi m} \\ &= \cos(2\pi m) \pm i \sin(2\pi m) \end{aligned}$$

- Therefore,  $m = 0, \pm 1, \pm 2, \dots$
- It follows that we can express the solutions to the bottom angular equation in the more compact form

$$\Phi_m(\phi) = A_m e^{im\phi}$$

where  $m = 0, \pm 1, \pm 2, \dots$

- Normalizing gives us the coefficient  $A_m$ :

$$\begin{aligned} 1 &= \int_0^{2\pi} \Phi_m^*(\phi) \Phi_m(\phi) d\phi \\ &= \int_0^{2\pi} (A_m e^{-im\phi})(A_m e^{im\phi}) d\phi \\ &= A_m^2 \int_0^{2\pi} 1 d\phi \\ A_m &= \frac{1}{\sqrt{2\pi}} \end{aligned}$$

- Thus, the normalized wave functions of  $\Phi(\phi)$  are

$$\Phi(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$$

for  $m = 0, \pm 1, \pm 2, \dots$

- We now direct our attention to solving the differential equation containing  $\Theta(\theta)$ .
- Let  $x = \cos \theta$  and  $P(x) = \Theta(\theta)$ . Substituting, we have (see Problem 6-2)

$$(1-x^2) \frac{d^2 P}{dx^2} - 2x \frac{dP}{dx} + \left[ \beta - \frac{m^2}{1-x^2} \right] P(x) = 0$$

where  $m = 0, \pm 1, \pm 2, \dots$

– Note that since  $x = \cos \theta$ ,  $0 \leq \theta \leq \pi$ , the range of  $x$  is  $-1 \leq x \leq 1$ .

- Solving the above equation gives **Legendre's equation**.
- **Legendre's equation:** The following equation. *Given by*

$$(1-x^2) \frac{d^2 P}{dx^2} - 2x \frac{dP}{dx} + \left[ l(l+1) - \frac{m^2}{1-x^2} \right] P(x) = 0$$

– The key difference between Legendre's equation and the previous one is that solving the previous one finds  $\beta$  equal to  $l(l+1)$  with  $l = 0, 1, 2, \dots$  and  $|m| \leq l$  if the solutions are to remain finite.

- **Legendre polynomials:** The solutions to Legendre's equation when  $m = 0$ . *Denoted by  $P_l(x)$ .*

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$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2 - 1) \\ P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\ P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3) \end{aligned}$$


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Table 4.1: The first few Legendre polynomials.

- Notice that  $P_l(x)$  is an even function if  $l$  is even and an odd function if  $l$  is odd.
- The factors in front of  $P_l(x)$  are chosen such that  $P_l(1) = 1$ .
- The Legendre polynomials are orthogonal: If  $l \neq n$ , then

$$\int_{-1}^1 P_l(x) P_n(x) dx = 0$$

- We have that

$$\int_{-1}^1 [P_l(x)]^2 dx = \frac{2}{2l+1}$$

for each  $l$ , so the normalization constant of  $P_l(x)$  is

$$\sqrt{\frac{2l+1}{2}}$$

- **Associated Legendre functions:** The solutions to Legendre's equation when  $m \neq 0$ . *Denoted by  $P_l^{|m|}(x)$ . Given by*

$$P_l^{|m|}(x) = (1-x^2)^{|m|/2} \frac{d^{|m|}}{dx^{|m|}} P_l(x)$$

10/30:

Function	$x$ -coordinates	$\theta$ -coordinates
$P_0^0(x)$	1	1
$P_1^0(x)$	$x$	$\cos \theta$
$P_1^1(x)$	$\sqrt{1-x^2}$	$\sin \theta$
$P_2^0(x)$	$\frac{1}{2}(3x^2-1)$	$\frac{1}{2}(3\cos^2 \theta - 1)$
$P_2^1(x)$	$3x\sqrt{1-x^2}$	$3\cos \theta \sin \theta$
$P_2^2(x)$	$3(1-x^2)$	$3\sin^2 \theta$
$P_3^0(x)$	$\frac{1}{2}(5x^2-3x)$	$\frac{1}{2}(5\cos^2 \theta - 3\cos \theta)$
$P_3^1(x)$	$\frac{3}{2}(5x^2-1)\sqrt{1-x^2}$	$\frac{3}{2}(5\cos^2 \theta - 1)\sin \theta$
$P_3^2(x)$	$15x(1-x^2)$	$15\cos \theta \sin^2 \theta$
$P_3^3(x)$	$15\sqrt{1-x^2}$	$15\sin^2 \theta$

Table 4.2: The first few associated Legendre functions.

- Note that only the magnitude of  $m$  is present because  $m^2$  is the only form of  $m$  present in Legendre's equation.
- Notice how the  $\theta$ -coordinates functions mirror those in Table 0.2 of Labalme (2021a).
- Relating the normalization and orthogonality conditions back to the variable of interest ( $\theta$ ), we have (for the Legendre polynomials) that

$$\int_0^\pi P_l(\cos \theta) P_n(\cos \theta) \sin \theta \, d\theta = \frac{2\delta_{ln}}{2l+1}$$

- Similarly, for the associated Legendre functions, we have that

$$\int_0^\pi P_l^{|m|}(\cos \theta) P_n^{|m|}(\cos \theta) \sin \theta \, d\theta = \frac{2}{2l+1} \frac{(l+|m|)!}{(l-|m|)!} \delta_{ln}$$

- Thus, the normalization constant for the associated Legendre functions is

$$N_{lm} = \sqrt{\frac{2l+1}{2} \frac{(l-|m|)!}{(l+|m|)!}}$$

- Thus, the normalized wave functions of  $\Theta(\theta)$  are

$$\Theta(\theta) = N_{lm} P_l^{|m|}(\cos \theta)$$

for  $l = 0, 1, 2, \dots$  and  $m = 0, \pm 1, \dots, \pm l$ .

- **Spherical harmonics:** The orthonormal set of angular wave functions of the hydrogen atom. *Given by*

$$Y_l^m(\theta, \phi) = \Theta(\theta) \Phi(\phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} P_l^{|m|}(\cos \theta) e^{im\phi}$$

- Note that  $l = 0, 1, 2, \dots$  and  $m = 0, \pm 1, \dots, \pm l$ .
- The orthonormality condition:

$$\int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi Y_l^m(\theta, \phi)^* Y_n^k(\theta, \phi) = \delta_{ln} \delta_{mk}$$



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$Y_0^0 = \frac{1}{\sqrt{4\pi}}$	$Y_2^1 = \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi}$
$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos \theta$	$Y_2^{-1} = \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{-i\phi}$
$Y_1^1 = \sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}$	$Y_2^2 = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{2i\phi}$
$Y_1^{-1} = \sqrt{\frac{3}{8\pi}} \sin \theta e^{-i\phi}$	$Y_2^{-2} = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{-2i\phi}$
$Y_2^0 = \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1)$	

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Table 4.3: The first few spherical harmonics.

- 10/31: • Recalling the definition of  $\hat{L}^2$  from Chapter 5, we have that

$$\beta = -\frac{1}{Y_l^m(\theta, \phi)} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y_l^m(\theta, \phi)}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 Y_l^m(\theta, \phi)}{\partial \phi^2} \right]$$

$$\hbar^2 l(l+1) Y_l^m(\theta, \phi) = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] Y_l^m(\theta, \phi)$$

$$\hat{L}^2 Y_l^m(\theta, \phi) = \hbar^2 l(l+1) Y_l^m(\theta, \phi)$$

- Implication: The spherical harmonics are eigenfunctions of  $\hat{L}^2$ .
- Additionally, the square of the angular momentum  $L^2$  can only have the values given by  $\hbar^2 l(l+1)$  for  $l = 0, 1, 2, \dots$
- We can also adapt the above to reflect energy with  $\hat{H} = \hat{L}^2/2I$ .

- 11/1: • We can apply partial differentiation to convert the expressions for  $\hat{L}_x, \hat{L}_y, \hat{L}_z$  in Table 3.1 to spherical coordinates.

- Doing so yields complex expressions for  $\hat{L}_x$  and  $\hat{L}_y$ , but a simple expression for  $\hat{L}_z$ :

$$\hat{L}_z = -i\hbar \frac{d}{d\phi}$$

- The spherical harmonics are eigenfunctions of  $\hat{L}_z$ :

$$\begin{aligned} \hat{L}_z Y_l^m(\theta, \phi) &= -i\hbar \frac{d}{d\phi} \left( N_{lm} P_l^{|m|}(\cos \theta) e^{im\phi} \right) \\ &= -i\hbar N_{lm} P_l^{|m|}(\cos \theta) \frac{d}{d\phi} (e^{im\phi}) \\ &= m\hbar N_{lm} P_l^{|m|}(\cos \theta) e^{im\phi} \\ &= m\hbar Y_l^m(\theta, \phi) \end{aligned}$$

- It follows that  $L_z$  is quantized by  $m\hbar$ .
- This implies that  $\hbar$  is a fundamental measure of the angular momentum of a quantum mechanical system.
- Note that the spherical harmonics are not eigenfunctions of  $\hat{L}_x$  or  $\hat{L}_y$ .
- The last two results imply that we can determine precise values of  $L^2$  and  $L_z$  simultaneously, and that  $\hat{L}^2, \hat{L}_z$  commute.

- Proving that  $|m| < l$ .

– We have that

$$\hat{L}_z^2 Y_l^m(\theta, \phi) = m^2 \hbar^2 Y_l^m(\theta, \phi) \qquad \hat{L}^2 Y_l^m(\theta, \phi) = l(l+1) \hbar^2 Y_l^m(\theta, \phi)$$

– Subtracting and substituting  $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$  yields

$$\begin{aligned} (\hat{L}^2 - \hat{L}_z^2) Y_l^m(\theta, \phi) &= [l(l+1) - m^2] \hbar^2 Y_l^m(\theta, \phi) \\ (\hat{L}_x^2 + \hat{L}_y^2) Y_l^m(\theta, \phi) &= [l(l+1) - m^2] \hbar^2 Y_l^m(\theta, \phi) \end{aligned}$$

– It follows since  $m, l \in \mathbb{Z}$  that

$$\begin{aligned} 0 &\leq L_x^2 + L_y^2 = [l(l+1) - m^2] \hbar^2 \\ 0 &\leq l(l+1) - m^2 \\ m^2 &\leq l(l+1) \\ |m| &\leq l \end{aligned}$$

as desired.

- Consider the case  $l = 1$ .

– In this case,  $L^2 = 2\hbar^2$  and  $L_z = -\hbar, 0, \hbar$ .

– It follows from the first condition that  $|L| = \sqrt{L^2} = \sqrt{2}\hbar$ , i.e., the magnitude of the angular momentum vector is  $\sqrt{2}\hbar$ .

– Since  $L$  and  $L_z$  have differing magnitudes, their vectors cannot point in the same direction.

– Since  $\hat{L}^2$  commutes with each of  $\hat{L}_x, \hat{L}_y, \hat{L}_z$  individually but the three directional components don't commute among themselves, it is possible to measure the values of  $L^2$  and any directional component simultaneously to any precision, but this leads to complete uncertainty in the other two directional components.

– This combined with the fact that  $\langle L_x \rangle = \langle L_y \rangle = 0$  leads to this nice interpretation: Imagine  $\mathbf{L}$  precessing around a cone whose axis is the  $z$ -axis. We know that the height of  $\mathbf{L}$  (its  $z$ -coordinate) never changes, but the  $x, y$ -coordinates could be anywhere.

- Solving the radial equation.

– Energy is quantized by

$$E_n = -\frac{m_e e^4}{8\epsilon_0^2 \hbar^2 n^2} = -\frac{m_e e^4}{32\pi^2 \epsilon_0^2 \hbar^2 n^2} = -\frac{e^2}{8\pi\epsilon_0 a_0 n^2}$$

for  $n = 1, 2, \dots$  where  $a_0$  is the Bohr radius.

- Note that these energies are identical to those predicted by the Bohr model, even though the electron here is not restricted to sharply defined circular orbits but rather is described by  $\psi(r, \theta, \phi)$ .

– We also have the condition

$$0 \leq l \leq n - 1$$

on the quantum number  $n$ .

– Thus, the radial wave functions depend on the quantum numbers  $n$  and  $l$  as follows.

$$R_{nl}(r) = -\sqrt{\frac{(n-l-1)!}{2n[(n+l)!]^3}} \left(\frac{2}{na_0}\right)^{l+3/2} r^l e^{-r/na_0} L_{n+l}^{2l+1} \left(\frac{2r}{na_0}\right)$$

11/2:

$n = 1$	$l = 0$	$L_1^1(x) = -1$
$n = 2$	$l = 0$	$L_2^1(x) = -2!(2 - x)$
	$l = 1$	$L_3^3(x) = -3!$
$n = 3$	$l = 0$	$L_3^1(x) = -3!(3 - 3x + \frac{1}{2}x^2)$
	$l = 1$	$L_4^3(x) = -4!(4 - x)$
	$l = 2$	$L_5^5(x) = -5!$
$n = 4$	$l = 0$	$L_4^1(x) = -4!(4 - 6x + 2x^2 - \frac{1}{6}x^3)$
	$l = 1$	$L_5^3(x) = -5!(10 - 5x + \frac{1}{2}x^2)$
	$l = 2$	$L_6^5(x) = -6!(6 - x)$
	$l = 3$	$L_7^7(x) = -7!$

Table 4.4: The first few associated Laguerre polynomials.

- The  $L_{n+l}^{2l+1}(2r/na_0)$  are the **associated Laguerre polynomials**.
- The combinatorial factors in Table 4.4 just guarantee that the  $R_{nl}(r)$  are normalized.
- We can now construct the complete hydrogen atomic wave functions as

$$\psi_{nlm}(r, \theta, \phi) = R_{nl}(r)Y_l^m(\theta, \phi)$$

$n = 1$	$l = 0$	$m = 0$	$\psi_{100} = \frac{1}{\sqrt{\pi}} \left(\frac{Z}{a_0}\right)^{3/2} e^{-\sigma}$
$n = 2$	$l = 0$	$m = 0$	$\psi_{200} = \frac{1}{\sqrt{32\pi}} \left(\frac{Z}{a_0}\right)^{3/2} (2 - \sigma)e^{-\sigma/2}$
	$l = 1$	$m = 0$	$\psi_{210} = \frac{1}{\sqrt{32\pi}} \left(\frac{Z}{a_0}\right)^{3/2} \sigma e^{-\sigma/2} \cos \theta$
	$l = 1$	$m = \pm 1$	$\psi_{21\pm 1} = \frac{1}{\sqrt{64\pi}} \left(\frac{Z}{a_0}\right)^{3/2} \sigma e^{-\sigma/2} \sin \theta e^{\pm i\phi}$
$n = 3$	$l = 0$	$m = 0$	$\psi_{300} = \frac{1}{81\sqrt{3\pi}} \left(\frac{Z}{a_0}\right)^{3/2} (27 - 18\sigma + 2\sigma^2)e^{-\sigma/3}$
	$l = 1$	$m = 0$	$\psi_{310} = \frac{\sqrt{2}}{81\sqrt{\pi}} \left(\frac{Z}{a_0}\right)^{3/2} (6\sigma - \sigma^2)e^{-\sigma/3} \cos \theta$
	$l = 1$	$m = \pm 1$	$\psi_{31\pm 1} = \frac{1}{81\sqrt{\pi}} \left(\frac{Z}{a_0}\right)^{3/2} (6\sigma - \sigma^2)e^{-\sigma/3} \sin \theta e^{\pm i\phi}$
	$l = 2$	$m = 0$	$\psi_{320} = \frac{1}{81\sqrt{6\pi}} \left(\frac{Z}{a_0}\right)^{3/2} \sigma^2 e^{-\sigma/3} (3 \cos^2 \theta - 1)$
	$l = 2$	$m = \pm 1$	$\psi_{32\pm 1} = \frac{1}{81\sqrt{\pi}} \left(\frac{Z}{a_0}\right)^{3/2} \sigma^2 e^{-\sigma/3} \sin \theta \cos \theta e^{\pm i\phi}$
	$l = 2$	$m = \pm 2$	$\psi_{32\pm 2} = \frac{1}{162\sqrt{\pi}} \left(\frac{Z}{a_0}\right)^{3/2} \sigma^2 e^{-\sigma/3} \sin^2 \theta e^{\pm 2i\phi}$

Table 4.5: The first few complete hydrogenlike atomic wave functions.

- Note that in Table 4.5,  $Z$  is the atomic number of the nucleus and  $\sigma = Zr/a_0$  where  $a_0$  is the Bohr radius.

- The orthonormality condition is

$$\int_0^\infty dr r^2 \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi \psi_{n'l'm'}^*(r, \theta, \phi) \psi_{nlm}(r, \theta, \phi) = \delta_{nn'} \delta_{ll'} \delta_{mm'}$$

- **Principal quantum number:** The quantum number  $n$ .

- $n = 1, 2, \dots$
- The energy of the hydrogen atom depends on  $n$ .

- **Angular momentum quantum number:** The quantum number  $l$ .

- $l = 0, 1, \dots, n - 1$ .
- The angular momentum of the electron about the proton is determined by  $l$  through  $|L| = \hbar \sqrt{l(l+1)}$ .
- When  $l = 0, 1, 2, 3$ , it is customary to denote the value of  $l$  with the letters  $s, p, d, f$ , respectively.
  - After 3, we continue alphabetically past  $f$ .
  - The origin of  $s, p, d, f$  is historic from the designation of the observed spectral lines of atomic sodium;  $s, p, d, f$  stand for sharp, principal, diffuse, and fundamental, respectively.
- A wave function with  $n = 1$  and  $l = 0$  is called a  $1s$  wave function; a wave function with  $n = 2$  and  $l = 0$  is called a  $2s$  wave function, and so on and so forth.

- **Magnetic quantum number:** The quantum number  $m$ .

- $m = 0, \pm 1, \pm 2, \dots, \pm l$ .
- So named because “the energy of a hydrogen atom in a magnetic field depends on  $m$ ” (McQuarrie & Simon, 1997, p. 209).

- **Zeeman effect:** In the absence of a magnetic field, energy levels are  $2l + 1$ -fold degenerate; in the presence of a magnetic field, however, the energy levels split.

- In the magnetic field case,  $E$  is a function of both  $n$  and  $m$ .

- Since the radial wave functions are normalized, we have that

$$\int_0^\infty [R_{nl}(r)]^2 r^2 dr = 1$$

- It follows that the probability that the  $nl$  electron lies between  $r$  and  $r + dr$  is

$$\text{Prob} = [R_{nl}(r)]^2 r^2 dr$$

- Alternatively, we can consider the full wave functions to learn that

$$\text{Prob} = r^2 dr \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi \psi_{nlm}^*(r, \theta, \phi) \psi_{nlm}(r, \theta, \phi)$$

- For any specific example, these two equations will be in agreement.

- The number of radial nodes is equal to  $n - l - 1$ .

- **Orbital:** An atomic wave function.

- The above radial probability functions can be used to calculate average values of  $r$  via

$$\langle r \rangle_{nl} = \int_0^\infty r R_{nl}^*(r) R_{nl}(r) dr$$

- Using general properties of the associated Laguerre polynomials, we can show that

$$\langle r \rangle_{ns} = \frac{3}{2} a_0 n^2$$

- We can calculate the most probable value of  $r$  for an  $nl$  wave function by differentiating the probability function and setting it equal to zero and then maximizing over the possible values.
- $p$  orbitals.
  - Considers the  $p_z$  orbital corresponding to  $Y_1^0$ , which is real.
  - Covers tangent sphere picture, probability density plot, and contour map representations of atomic orbitals.
  - When  $m \neq 0$ , the angular wave functions  $Y_1^{\pm 1}$  are complex.
  - To make them real, note that  $|Y_1^1|^2 = |Y_1^{-1}|^2$ , so the probability densities and hence energies are the same.
  - It follows that any linear combination of these two wave functions is another energy eigenfunction with the same energy, so take

$$p_x = \frac{1}{\sqrt{2}}(Y_1^1 + Y_1^{-1}) \qquad p_y = \frac{1}{\sqrt{2}i}(Y_1^1 - Y_1^{-1})$$

- $d$  orbitals.
  - We can take analogous linear combination to find real solutions (see Table 4.6).
- “There is no fundamental reason to choose these linear combinations of spherical harmonics over the spherical harmonic themselves, but most chemists use the five  $d$  orbitals given... because the functions... are real and have directional properties consistent with molecular structures” (McQuarrie & Simon, 1997, p. 218).
- The Schrödinger equation of the helium atom is

$$\left( -\frac{\hbar^2}{2M} \nabla^2 - \frac{\hbar^2}{2m_e} \nabla_1^2 - \frac{\hbar^2}{2m_e} \nabla_2^2 \right) \psi(\mathbf{R}, \mathbf{r}_1, \mathbf{r}_2) + \left( -\frac{2e^2}{4\pi\epsilon_0|\mathbf{R} - \mathbf{r}_1|} - \frac{2e^2}{4\pi\epsilon_0|\mathbf{R} - \mathbf{r}_2|} + \frac{e^2}{4\pi\epsilon_0|\mathbf{r}_1 - \mathbf{r}_2|} \right) \psi(\mathbf{R}, \mathbf{r}_1, \mathbf{r}_2) = E\psi(\mathbf{R}, \mathbf{r}_1, \mathbf{r}_2)$$

- $\mathbf{R}$  is the position of the helium nucleus.
- $\mathbf{r}_1, \mathbf{r}_2$  are the positions of the two electrons.
- $M$  is the mass of the nucleus.
- $m_e$  is the electronic mass.
- $\nabla^2$  is the Laplacian operator with respect to the position of the nucleus.
- $\nabla_1^2, \nabla_2^2$  are the Laplacian operators with respect to the positions of the electrons.
- The above Schrödinger equation describes a three body problem and thus is not solveable.
- By fixing the nucleus at the center of a spherical coordinate system (as is justifiable since  $M \gg m_e$ ), we can simplify the Schrödinger equation to

$$-\frac{\hbar^2}{2m_e}(\nabla_1^2 + \nabla_2^2)\psi(\mathbf{r}_1, \mathbf{r}_2) - \frac{2e^2}{4\pi\epsilon_0} \left( \frac{1}{r_1} + \frac{1}{r_2} \right) \psi(\mathbf{r}_1, \mathbf{r}_2) + \frac{e^2}{4\pi\epsilon_0|\mathbf{r}_1 - \mathbf{r}_2|} \psi(\mathbf{r}_1, \mathbf{r}_2) = E\psi(\mathbf{r}_1, \mathbf{r}_2)$$

- **Interelectronic repulsion term:** The  $e^2/4\pi\epsilon_0|\mathbf{r}_1 - \mathbf{r}_2|$  term in the Schrödinger equation of the helium atom.
  - This term makes it so that even this equation is still not solveable.

---

$n = 1$	$l = 0$	$m = 0$	$\psi_{1s} = \frac{1}{\sqrt{\pi}} \left( \frac{Z}{a_0} \right)^{3/2} e^{-\sigma}$
$n = 2$	$l = 0$	$m = 0$	$\psi_{2s} = \frac{1}{4\sqrt{2\pi}} \left( \frac{Z}{a_0} \right)^{3/2} (2 - \sigma) e^{-\sigma/2}$
	$l = 1$	$m = 0$	$\psi_{2p_z} = \frac{1}{4\sqrt{2\pi}} \left( \frac{Z}{a_0} \right)^{3/2} \sigma e^{-\sigma/2} \cos \theta$
	$l = 1$	$m = \pm 1$	$\psi_{2p_x} = \frac{1}{4\sqrt{2\pi}} \left( \frac{Z}{a_0} \right)^{3/2} \sigma e^{-\sigma/2} \sin \theta \cos \phi$
			$\psi_{2p_y} = \frac{1}{4\sqrt{2\pi}} \left( \frac{Z}{a_0} \right)^{3/2} \sigma e^{-\sigma/2} \sin \theta \sin \phi$
$n = 3$	$l = 0$	$m = 0$	$\psi_{3s} = \frac{1}{81\sqrt{3\pi}} \left( \frac{Z}{a_0} \right)^{3/2} (27 - 18\sigma + 2\sigma^2) e^{-\sigma/3}$
	$l = 1$	$m = 0$	$\psi_{3p_z} = \frac{\sqrt{2}}{81\sqrt{\pi}} \left( \frac{Z}{a_0} \right)^{3/2} (6\sigma - \sigma^2) e^{-\sigma/3} \cos \theta$
	$l = 1$	$m = \pm 1$	$\psi_{3p_x} = \frac{1}{81\sqrt{\pi}} \left( \frac{Z}{a_0} \right)^{3/2} (6\sigma - \sigma^2) e^{-\sigma/3} \sin \theta \cos \phi$
			$\psi_{3p_y} = \frac{1}{81\sqrt{\pi}} \left( \frac{Z}{a_0} \right)^{3/2} (6\sigma - \sigma^2) e^{-\sigma/3} \sin \theta \sin \phi$
	$l = 2$	$m = 0$	$\psi_{3d_{z^2}} = \frac{1}{81\sqrt{6\pi}} \left( \frac{Z}{a_0} \right)^{3/2} \sigma^2 e^{-\sigma/3} (3 \cos^2 \theta - 1)$
	$l = 2$	$m = \pm 1$	$\psi_{3d_{xz}} = \frac{\sqrt{2}}{81\sqrt{\pi}} \left( \frac{Z}{a_0} \right)^{3/2} \sigma^2 e^{-\sigma/3} \sin \theta \cos \theta \cos \phi$
			$\psi_{3d_{yz}} = \frac{\sqrt{2}}{81\sqrt{\pi}} \left( \frac{Z}{a_0} \right)^{3/2} \sigma^2 e^{-\sigma/3} \sin \theta \cos \theta \sin \phi$
	$l = 2$	$m = \pm 2$	$\psi_{3d_{x^2-y^2}} = \frac{1}{81\sqrt{2\pi}} \left( \frac{Z}{a_0} \right)^{3/2} \sigma^2 e^{-\sigma/3} \sin^2 \theta \cos 2\phi$
			$\psi_{3d_{xy}} = \frac{1}{81\sqrt{2\pi}} \left( \frac{Z}{a_0} \right)^{3/2} \sigma^2 e^{-\sigma/3} \sin^2 \theta \sin 2\phi$

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Table 4.6: The first few complete real hydrogenlike atomic wave functions.

## Problems

10/29: **6-2.** In terms of the variable  $\theta$ , Legendre's equation is

$$\sin \theta \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta(\theta)}{d\theta} \right) + (\beta \sin^2 \theta - m^2) \Theta(\theta) = 0$$

Let  $x = \cos \theta$  and  $P(x) = \Theta(\theta)$ . Show that

$$(1 - x^2) \frac{d^2 P(x)}{dx^2} - 2x \frac{dP(x)}{dx} + \left[ \beta - \frac{m^2}{1 - x^2} \right] P(x) = 0$$

*Answer.* If  $\cos \theta = \frac{x}{1}$ , then

$$\sin \theta = \sqrt{1 - x^2}$$

Additionally,

$$\frac{d\Theta}{d\theta} = \frac{dP}{d\theta} = \frac{dP}{dx} \frac{dx}{d\theta} = -\sin \theta \frac{dP}{dx} = -\sqrt{1 - x^2} \frac{dP}{dx}$$

Therefore, we have by substitution and simplification that

$$\begin{aligned}
 0 &= \sqrt{1-x^2} \frac{d}{d\theta} \left( \sqrt{1-x^2} \cdot -\sqrt{1-x^2} \frac{dP}{dx} \right) + (\beta^2(1-x^2) - m^2)P(x) \\
 &= \sqrt{1-x^2} \frac{d}{dx} \left( -(1-x^2) \frac{dP}{dx} \right) \frac{dx}{d\theta} + (\beta^2(1-x^2) - m^2)P(x) \\
 &= (1-x^2) \frac{d}{dx} \left( (1-x^2) \frac{dP}{dx} \right) + (\beta^2(1-x^2) - m^2)P(x) \\
 &= (1-x^2) \left[ (1-x^2) \frac{d^2P}{dx^2} - 2x \frac{dP}{dx} \right] + (\beta^2(1-x^2) - m^2)P(x) \\
 &= (1-x^2) \frac{d^2P}{dx^2} - 2x \frac{dP}{dx} + \left[ \beta^2 - \frac{m^2}{1-x^2} \right] P(x)
 \end{aligned}$$

□

**6-13.** Prove that  $\hat{L}^2$  commutes with  $\hat{L}_x, \hat{L}_y, \hat{L}_z$  but that

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z \qquad [\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x \qquad [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$$

(Hint: Use Cartesian coordinates.) Do you see a pattern in these formulas?

*Proof.* Expand from the definition of the commutator and Table 3.1, and then substitute and reduce.

□

## Week 5

# Approximate Solutions of the Schrödinger Equation

## 5.1 Approximation Methods

10/25:

- The **variational method** and **perturbation theory** are two methods of approximating solutions to Schrödinger equations describing systems more complex than the hydrogen atom.

- To begin our investigation of the variational method, we will look at the particle in a box.

- Consider a Hamiltonian for an electron in a box of length  $L = 2$  a.u. centered around  $x = 0$ .

- Note that we take the electron as the fundamental mass,  $\hbar$  as the fundamental unit of energy time, and the charge of the electron as the fundamental unit of charge, and the Bohr radius as the fundamental unit of length.

- Our Hamiltonian is

$$H\psi(x) = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x)$$

or, in atomic units,

$$H\psi(x) = -\frac{1}{2} \frac{d^2}{dx^2} \psi(x)$$

- **Variational theorem:** The expectation value of our Hamiltonian with respect to a trial wave function produces an approximate energy. Moreover<sup>[1]</sup>,

$$E_{\text{approx}} \geq E_{\text{gr. st.}}$$

- **Variational method:** Take  $\psi_{\text{trial}} = \sum_n a_n |\psi_n\rangle$  where  $\psi_n$  is a trial wave function and the  $a_j$ 's are parameters of the wave function which we want to optimize to lower  $E_{\text{trial}}$ .

- Dirac's ket describes an abstract state of the particle (possibly position, possibly its Fourier transform, momentum).

- Back to the particle in a box:

- A possible trial wave function (that satisfies the boundary conditions) is

$$\psi_{\text{tr}} = (1+x)(1-x) = 1-x^2$$

---

<sup>1</sup>We will prove that the approximate energy is an upper bound on the ground state energy in the homework.



- The energy of  $\psi_{\text{tr}}$  may be evaluated as follows.

$$\begin{aligned}
 E &= \frac{\int \psi_{\text{tr}}^*(x) \hat{H} \psi_{\text{tr}}(x) dx}{\int \psi_{\text{tr}}^*(x) \psi_{\text{tr}}(x) dx} \\
 &= \frac{\int_{-1}^1 (1-x^2) \left(-\frac{1}{2} \frac{d^2}{dx^2}\right) (1-x^2) dx}{\int_{-1}^1 (1-x^2)(1-x^2) dx} \\
 &= \frac{\int_{-1}^1 (1-x^2) dx}{\int_{-1}^1 (1-x^2)(1-x^2) dx} \\
 &= \frac{4/3}{16/15} \\
 &= \frac{5}{4} \\
 &= 1.25 \text{ a.u.}
 \end{aligned}$$

- From the exact solution to the particle in a box

$$E_1 = 1.23370055 < 1.25 = E_{\text{trial}}$$

so the variational theorem is satisfied.

- Next step: Trial wave function as a linear combination is  $\psi_{\text{tr}}(x) = c_1 \psi_1(x) + c_2 \psi_2(x)$ .
- Plugging this into the SE yields

$$c_1(\hat{H} - E)\psi_1(x) + c_2(\hat{H} - E)\psi_2(x) = 0$$

■  $\psi_1, \psi_2$  span the (Hilbert) space of solutions.

- To solve the above equation, multiply by  $\psi_1^*(x)$  and integrate to obtain

$$c_1 \int_{-1}^1 \psi_1^*(x)(\hat{H} - E)\psi_1(x) dx + c_2 \int_{-1}^1 \psi_1^*(x)(\hat{H} - E)\psi_2(x) dx = 0$$

and multiply by  $\psi_2^*(x)$  and integrate to obtain

$$c_1 \int_{-1}^1 \psi_2^*(x)(\hat{H} - E)\psi_1(x) dx + c_2 \int_{-1}^1 \psi_2^*(x)(\hat{H} - E)\psi_2(x) dx = 0$$

- Substituting, we have

$$c_1(H_{11} - ES_{11}) + c_2(H_{12} - ES_{12}) = 0 \quad c_1(H_{21} - ES_{21}) + c_2(H_{22} - ES_{22}) = 0$$

- In matrix form, the above two equations become

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} - E \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\mathbb{H}\vec{c} - E\mathbb{S}\vec{c} = 0$$

- We get a matrix that has the same dimension as the size of the expansion (in the first case, we had a  $1 \times 1$  matrix).
- $\mathbb{S}$  is the overlap matrix because the wave functions aren't normalized.

## 5.2 Variational Method

- 10/27: • Approximating the ground state energy with some trial wave function and applying

$$E_{\text{approx}} = \frac{\int \psi_{\text{tr}}^* \hat{H} \psi_{\text{tr}} \, dx}{\int \psi_{\text{tr}}^* \psi_{\text{tr}} \, dx}$$

where

$$\psi_{\text{tr}} = \sum_n c_n \psi_n(x)$$

- Example 2:
- For our second term, we need another even function (since the ground state wavefunction is even). Thus, choose

$$\psi_{\text{tr}}(x) = c_1(1 - x^2) + c_2(1 - x^2)x^2$$

- Think about this in the context of power series — we have  $(1 - x^2)$  times an even power series expansion  $(c_1 + c_2x^2)$ .
- To find  $c_1, c_2$ , we could plug into the approximation integral and minimize.
- Alternatively, we can use matrices. We essentially project the Schrödinger equation onto the space of the two wave functions.
- Take  $\hat{H}\psi = E\psi$  and expand it to  $\hat{H}(c_1\psi_1 + c_2\psi_2) = E(c_1\psi_1 + c_2\psi_2)$ . In matrix form,  $\mathbb{H}\vec{c} = E\mathbb{S}\vec{c}$ .
- We have an overlap matrix  $\mathbb{S}$  because our wave functions aren't normalized. If the basis *is* orthonormal,  $\mathbb{S}$  collapses to the identity matrix.
  - Each  $s_{ij}$  equals

$$s_{ij} = \int \psi_i^* \psi_j \, dx$$

- If  $\psi_1, \psi_2$  is orthonormal, then  $s_{ij} = \delta_{ij}$ .
- The elements of the Hamiltonian matrix:

$$\begin{aligned} H_{11} &= \int \psi_1^*(x) \hat{H} \psi_1(x) \, dx \\ &= \frac{4}{3} \end{aligned}$$

$$\begin{aligned} H_{12} &= \int \psi_1^*(x) \hat{H} \psi_2(x) \, dx \\ &= \frac{8}{15} \end{aligned}$$

$$H_{21} = \frac{8}{15}$$

$$H_{22} = \frac{44}{105}$$

- Notice that  $\mathbb{H}$  is symmetric with  $H_{12} = H_{21}$ .
- Elements of the overlap matrix:

$$S_{11} = \frac{16}{15}$$

$$S_{12} = \frac{32}{105}$$

$$S_{21} = \frac{32}{105}$$

$$S_{22} = \frac{16}{315}$$

- Notice that  $\mathbb{S}$  is symmetric with  $S_{12} = S_{21}$ .
- Note that there are multiple ways to solve  $\mathbb{H}\vec{c} = E\mathbb{S}\vec{c}$ ; McQuarrie and Simon (1997) only teaches one. Thus, you can get computers to do the math and solve far bigger systems than you could by hand.

- Solving  $\mathbb{H}\vec{c} = E\mathbb{S}\vec{c}$  with the textbook method:
  - Rewrite as  $(\mathbb{H} - E\mathbb{S})\vec{c} = 0$ . Find the null space of  $\mathbb{H} - E\mathbb{S}$ .
  - Since the determinant is the product of the eigenvalues,  $\det(\mathbb{H} - E\mathbb{S}) = (E_1 - E)(E_2 - E)$ .
  - This determinant is equal to zero only when  $E = E_1$  or  $E = E_2$ .
    - The energy is becoming quantized because of the linear algebra!
  - Now taking  $\det(\mathbb{H} - E\mathbb{S})$  gives a characteristic polynomial in  $E$ .

$$0 = \begin{vmatrix} \frac{4}{3} - \frac{16}{15}E & \frac{4}{15} - \frac{16}{105}E \\ \frac{4}{15} - \frac{16}{105}E & \frac{44}{105} - \frac{16}{315}E \end{vmatrix}$$

$$= \frac{256}{525} - \frac{2048}{4725}E + \frac{1024}{33075}E^2$$

- Solving the quadratic gives us

$$E = 7 \pm \frac{\sqrt{133}}{2}$$

- Thus,

$$E_1 = 1.233\,718\,705 \text{ a.u.}$$

$$E_2 = 12.766 \text{ a.u.}$$

- Notice that the  $E_1$  we found is only *marginally* greater than the real value of  $E_1$ . Our value is accurate to four decimal places!
- Solving for  $\vec{c}$  with  $E_1$  gives us

$$\vec{c}_1 = -0.9764$$

$$\vec{c}_2 = 0.2156$$

## 5.3 Perturbation Theory

10/29:

- Consider the Hamiltonian

$$\hat{H} = \hat{H}_0 + \lambda \hat{V}$$

where  $\hat{H}_0$  is the reference hamiltonian,  $\hat{V}$  is the perturbation, and  $\lambda$  is the perturbation parameter.

- The energy may be expressed as a Taylor series expansion in  $\lambda$ :

$$E(\lambda) = E(0) + \lambda \left. \frac{dE}{d\lambda} \right|_0 + \frac{\lambda^2}{2} \left. \frac{d^2E}{d\lambda^2} \right|_0 + \dots$$

- If  $\lambda$  is sufficiently small, we can get good approximations without resorting to higher order derivatives.
- It follows that our reference energy is

$$E(0) = \int \psi_0^* \hat{H}_0 \psi_0 \, dx$$

- We now have that

$$E(\lambda) = \int \psi^*(\lambda) \hat{H}(\lambda) \psi(\lambda) \, dx$$

- We also have from differentiating that

$$\frac{dE}{d\lambda} = \int \frac{d\psi}{d\lambda} \hat{H} \psi(\lambda) \, dx + \int \psi^*(\lambda) \hat{H}^* \frac{d\psi}{d\lambda} \, dx + \int \psi^* \frac{d\hat{H}}{d\lambda} \psi(\lambda) \, dx$$

$$\begin{aligned}
&= E \int \frac{d\psi^*}{d\lambda} \psi(\lambda) dx + E \int \psi^*(\lambda) \frac{d\psi}{d\lambda} dx + \int \psi^* \frac{d\hat{H}}{d\lambda} \psi(\lambda) dx \\
&= E \frac{d}{d\lambda} \left( \int \psi^*(\lambda) \psi(\lambda) dx \right) + \int \psi^* \hat{H} \psi dx \\
&= \int \psi^*(\lambda) \frac{d\hat{H}}{d\lambda} \psi(\lambda) dx \\
&= \int \psi^*(\lambda) \hat{V} \psi(\lambda) dx
\end{aligned}$$

– Note that the commutativity of  $\hat{H}$  follows from the fact that it's a Hermitian operator.

- It follows that

$$\left. \frac{dE}{d\lambda} \right|_{\lambda=0} = \int \psi_0^* V \psi_0 dx$$

- Richard Feynman worked this out for his undergraduate thesis at MIT. This laid the foundation of quantum electrodynamics, for which he would eventually win the Nobel prize.
- This is known as the **Hellmann-Feynman theorem** (1939).
- Note that the second derivative of  $E(\lambda)$  unfortunately depends on  $d\psi/d\lambda$ .

- Many electron molecules: The Helium atom.

– We have  $\hat{H}\psi = E\psi$  where

$$\hat{H} = -\frac{1}{2}\hat{\nabla}_1^2 - \frac{1}{2}\hat{\nabla}_2^2 - \frac{Z}{r_1} - \frac{Z}{r_2} + \frac{1}{r_{12}}$$

- Note that the  $\nabla$ 's are Laplacians.
- This equation takes into account the kinetic and potential energy of two electrons, plus the electron-electron repulsion.
- Solve using perturbation theory. Our reference Hamiltonian is

$$\hat{H}_0 = \hat{H}_{\text{He}_1^+} + \hat{H}_{\text{He}_2^+} = \underbrace{-\frac{1}{2}\hat{\nabla}_1^2 - \frac{Z}{r_1}}_{\text{He}_1^+} + \underbrace{-\frac{1}{2}\hat{\nabla}_2^2 - \frac{Z}{r_2}}_{\text{He}_2^+}$$

- I.e., it's the sum of the Hamiltonians of two helium ions (one-electron systems like hydrogen).
- Since  $\hat{V} = +1/r_{12}$ , we have that

$$\hat{H}(1) = \hat{H}_0 + \hat{V}$$

is the Hamiltonian of the atom.

- Now we look for the solution to  $\hat{H}_0\psi_0 + E_0\psi_0$ .
- We know that

$$\psi_0 = \psi_{1s}(r_1\theta_1\phi_1)\psi_{1s}(r_2\theta_2\phi_2)$$

- The fact that only two electrons fit in an orbital emerges naturally from the quantum mechanics!
- We also know that

$$E_0 = -\frac{Z^2}{2n^2} - \frac{Z^2}{2n^2} = -4 \text{ a.u.}$$

- Thus, by perturbation theory,

$$\begin{aligned}
\left. \frac{dE}{d\lambda} \right|_{\lambda=0} &= \int \psi_0^* \hat{V} \psi_0 d\vec{r}_1 d\vec{r}_2 \\
&= \int 1s^*(1)1s^*(2) \hat{V} 1s(1)1s(2) d1 d2 \\
&= \frac{5}{8}Z
\end{aligned}$$

## 5.4 Chapter 7: Approximation Methods

From McQuarrie and Simon (1997).

- 11/2:
- Although the Schrödinger equation cannot be solved exactly for any system more complicated than the hydrogen atom, we can use either the **variational method** or **perturbation theory** to approximate solutions to Schrödinger equations to almost any desired accuracy.
  - We begin with the variational method.
  - If the ground-state wave function and corresponding energy of some arbitrary system are  $\psi_0$  and  $E_0$ , respectively, then

$$\begin{aligned}\hat{H}\psi_0 &= E_0\psi_0 \\ \psi_0^*\hat{H}\psi_0 &= \psi_0^*E_0\psi_0 \\ \int \psi_0^*\hat{H}\psi_0 d\tau &= E_0 \int \psi_0^*\psi_0 d\tau \\ E_0 &= \frac{\int \psi_0^*\hat{H}\psi_0 d\tau}{\int \psi_0^*\psi_0 d\tau}\end{aligned}$$

where  $d\tau$  represents the appropriate volume element for integrating over all space.

- **Variational principle:** If  $\phi$  is any function, then

$$\frac{\int \phi^*\hat{H}\phi d\tau}{\int \phi^*\phi d\tau} = E_\phi \geq E_0 = \frac{\int \psi_0^*\hat{H}\psi_0 d\tau}{\int \psi_0^*\psi_0 d\tau}$$

and  $E_\phi = E_0$  iff  $\phi = \psi_0$ <sup>[2]</sup>.

- Essentially, the variational method consists of choosing a trial function  $\phi$  that depends on **variational parameters** (numbers) so that  $E_\phi$  depends on the variational parameters, too, and minimizing  $E_\phi$  over the parameters.
- McQuarrie and Simon (1997) does a worked example with the ground state of the hydrogen atom, comparing answers with the exact solution.
  - Uses the formula for  $E_\phi$  to directly express it in terms of the variational parameter and then differentiates to find the maxima.
  - Applied to a case that can be solved exactly, the variational method with a trial wave function of the appropriate form will give the exact solution.
- Estimating the ground state energy of a helium atom, and deriving effective nuclear charge.
  - Consider the Hamiltonian for helium from the end of Chapter 6.
  - It can be written in the form

$$\hat{H} = \hat{H}_H(1) + \hat{H}_H(2) + \frac{e^2}{4\pi\epsilon_0} \frac{1}{r_{12}}$$

where  $\hat{H}_H(j)$ ,  $j = 1, 2$ , is the Hamiltonian operator for a single electron around a helium nucleus.

- Ignoring the interelectron repulsion term gives us a separable wave function of the form

$$\phi_0(\mathbf{r}_1, \mathbf{r}_2) = \psi_{1s}(\mathbf{r}_1)\psi_{1s}(\mathbf{r}_2)$$

- We use the above as our trial wave function with  $Z$  (see Table 4.6) as our variational parameter.

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<sup>2</sup>See Problem ?? for a proof of the variational principle.

- Minimizing gives us  $Z_{\min} = 27/16$  and a fairly good  $E_{\min}$  value.
- Note that  $Z_{\min}$  can naturally be interpreted as the **effective nuclear charge**.
- Since  $Z_{\min} < 2$ , we have that each electron partially **screens** the nucleus from the other.
- McQuarrie and Simon (1997) arrives at the two linear algebraic equations Mazziotti did by expanding  $\int \phi \hat{H} \phi d\tau$  with  $\phi = c_1 f_1 + c_2 f_2$ , applying the condition that  $H$  is Hermitian from Chapter 4 to show that  $H_{ij} = H_{ji}$  and  $S_{ij} = S_{ji}$ , and differentiating the resulting energy function with respect to  $c_1$  and then with respect to  $c_2$ .
- **Matrix elements:** The quantities  $H_{ij}$  and  $S_{ij}$ .
- **Secular determinant:** A determinant of the form

$$\begin{vmatrix} H_{11} - ES_{11} & \cdots & H_{1n} - ES_{1n} \\ \vdots & & \vdots \\ H_{n1} - ES_{n1} & \cdots & H_{nn} - ES_{nn} \end{vmatrix}$$

- There exists a solution to the linear algebraic equations iff the secular determinant vanishes (equals 0).
- **Secular equation:** The equation obtained by expanding the secular determinant.
  - The smaller root of a second-degree secular equation is an upper bound on the ground state energy.
  - The larger root of a second-degree secular equation is an upper bound on the energy of the first excited state (albeit usually a crude one; we do not investigate methods of approximating energies other than that of the ground state here).
- Normalizing the trial wave function.
  - Since the linear system is homogenous, all of the  $c_1$  terms (for example) can be expressed as a linear combination of all of the other terms.
  - From here, we can find the ratios  $c_2/c_1, c_3/c_1, \dots, c_N/c_1$ .
  - Thus, we simply integrate the modulus squared of our trial wave function with  $c_1$  free and set the result equal to 1. Solving for  $c_1$  normalizes the equation.
- Note that trial wave functions can be linear combinations of functions that also contain variational parameters, but minimization here must be done numerically as the system is no longer linear.
- We now discuss perturbation theory.
- Suppose we cannot solve the Schrödinger equation  $\hat{H}\psi = E\psi$  but we can solve  $\hat{H}^{(0)}\psi^{(0)} = E^{(0)}\psi^{(0)}$ , which corresponds to a system that is in some sense similar to the unsolvable system. Then let  $\hat{H} = \hat{H}^{(0)} + \hat{H}^{(1)}$ .
- **Unperturbed Hamiltonian operator:** The Hamiltonian operator corresponding to the system we can solve exactly. Denoted by  $\hat{H}^{(0)}$ .
- **Perturbation:** The difference between the Hamiltonian operator we cannot solve and the unperturbed Hamiltonian operator. Denoted by  $\hat{H}^{(1)}$ .
- In perturbation theory, we let

$$\psi = \psi^{(0)} + \psi^{(1)} + \psi^{(2)} + \dots \qquad E = E^{(0)} + E^{(1)} + E^{(2)} + \dots$$

where  $\psi^{(1)}, \psi^{(2)}, \dots$  and  $E^{(1)}, E^{(2)}, \dots$  are successive corrections to  $\psi^{(0)}$  and  $E^{(0)}$ , respectively.

- We can show (see Problem ??) that

$$E^{(1)} = \int \psi^{(0)*} \hat{H}^{(1)} \psi^{(0)} d\tau$$

- Note that we can (although we will not here) derive explicit expressions for higher order corrections. These do get messy though.
- Uses perturbation theory to derive the energy of a particle in a one-dimensional slanted box, and separately the energy of a Helium electron.

## Week 6

# Multi-Electron Atoms and Molecules

## 6.1 Many-Electron Atoms and Molecules

- 11/1: • Picking up from last time, since  $Z = 2$  for helium,

$$\left. \frac{dE}{d\lambda} \right|_{\lambda=0} = \frac{5}{8}Z = \frac{5}{4}$$

- Additionally,

$$E_{\text{He}} = -Z^2 + \frac{5}{8}Z = -4 + \frac{5}{4} = -2.75 \text{ a.u.}$$

- The first equality is the ground state energy of two-electron atoms or ions.

- Summary:

- Zeroth-order Perturbation Theory approximation:  $E_0 = -4 \text{ a.u.}$
- First-order approximation:  $E_0 + E' = -2.75 \text{ a.u.}$
- Second-order:  $E_0 + E' + \frac{1}{2}E'' = -2.9077 \text{ a.u.}$
- Thirteenth-order:  $-2.903\,724\,33 \text{ a.u.}$

- Variational calculation of Helium:

- Simplest:  $\psi_0(12) = 1s(1)1s(2)$ . Gives  $E = -2.75 \text{ a.u.}$
- Trial  $\psi$  with one parameter:  $\psi_0(1,2) = \tilde{1}s(1)\tilde{1}s(2)$ , where  $\tilde{1}s(1) = \sqrt{Z^3/\pi}e^{-Zr_1}$ . Energy:  $E = -2.8477 \text{ a.u.}$

- Ionization energy.

- Simplest approximation:

$$\begin{aligned} \text{IE} &= E_{\text{He}^+} - E_{\text{He}} \\ &= -2 - (-2.75) \\ &= 0.75 \text{ a.u.} \\ &= 1969 \text{ kJ/mol} \end{aligned}$$

- Exact number:

$$\begin{aligned} \text{IE} &= E_{\text{He}^+} - E_{\text{He}} \\ &= -2 - (-2.903\,724) \\ &= 0.9033 \text{ a.u.} \\ &= 2372 \text{ kJ/mol} \end{aligned}$$



- Optimal orbitals (Hartree-Fock): Optimizing the orbitals to lower the energy as much as possible.

- $\psi(12) = \phi(r_1)\phi(r_2)$ .
- The orbital energies converge to

$$E_{\text{HF}} = -2.8617 \text{ a.u.}$$

- Allow the  $\psi(12)$  to move beyond a simple product of orbitals (Hylleras (1930)).

- $\psi(r_1, r_2, r_{12}) = e^{-Zr_1}e^{-Zr_2}(1 + cr_{12})$ .
- Accounting for the electron-electron repulsion (the **electron correlation energy**) gives us an energy much better than the Hartree-Fock calculation:

$$E = -2.8913 \text{ a.u.}$$

- Pekeris (1959): Did a variational calculation with 1078 parameters. Was working at IBM, who told him to do something with their newest computer that would be impressive to the world. Pekeris tackled this, and got

$$E = -2.903\,724\,375 \text{ a.u.}$$

which is even more accurate than 13th order perturbation theory.

- This value cannot be accurately measured to this precision in the laboratory. Additionally, relativistic quantum mechanics (using the Dirac equation of which the Schrödinger equation is only a part) predicts a value that diverges from this one around the fifth decimal point, and this is the experimentally verifiable value.
- This is important because scientists want to figure out how accurately can we account for the electron cusp.
- To summarize, the improvement tiers are
  1. H orbitals.
  2. MO picture (Hartree-Fock).
  3. Solution of the Schrödinger equation (electron correlation).
- Recall that the electron has spin (from the Stern-Gerlach and Uhlenbeck-Goudsmit experiments).

- This is analogous to orbital angular momentum:

$$\hat{L}^2 Y_{lm} = l(l+1)\hbar^2 Y_{lm} \qquad \hat{L}_z Y_{lm} = m\hbar Y_{lm}$$

- These two operators give rise to

$$\hat{S}^2 \sigma = s(s+1)\hbar^2 \sigma \qquad \hat{S}_z \sigma = m_s \hbar \sigma$$

where  $\sigma$  is the spin eigenfunction,  $s$  is the total spin angular momentum quantum number, and  $m_s$  is the spin quantum number.

- We have  $\sigma = \alpha$  or  $\sigma = \beta$  where  $\alpha$  represents “up” and  $\beta$  represents “down.”
- $m_s = \pm 1/2$  where  $s = 1/2$ , so  $m_s = -s, s$ .
- Spin eigenfunctions are orthonormal, i.e.,

$$\int \alpha^* \alpha = \int \beta^* \beta = 1 \qquad \int \alpha^* \beta = \int \beta^* \alpha = 0$$

- Hence, each electron has four degrees of freedom (three spatial and one spin).
- This gives rise to a **spin orbital**  $\psi(x, y, z, \sigma) = \phi(x, y, z)\sigma$ , which is the product of the spin and spatial orbitals.
- In fact, our ability to calculate the energy of the helium atom was complete luck — we cannot calculate the energy of any other element on the periodic table without accounting for spin.

## 6.2 Many-Electron Atoms and Molecules / Spin

- 11/3: • **Spin orbital:** A spatial orbital augmented with a spin eigenfunction. *Given by*

$$\psi(x, y, z, \sigma) = \phi(x, y, z)\sigma$$

- Key idea: Electrons are indistinguishable, i.e., if  $\psi(1, 2, \dots, N)$  where each number represents the three spatial coordinates of an electron, the  $N$  electrons in  $\psi$  must be indistinguishable.
- There are two possible ways to achieve *mathematical* indistinguishability.

1. We can require that

$$\psi(1, 2, 3, \dots, N) = \psi(2, 1, 3, \dots, N)$$

- Essentially, this is a symmetric permutation of all pairs of particles.

2. We can alternatively require that

$$\psi(1, 2, 3, \dots, N) = -\psi(2, 1, 3, \dots, N)$$

- Because the probability is invariant to the sign (think about taking the modulus squared and how that eliminates sign considerations).
- Essentially, this is an antisymmetric permutation of all pairs of particles.

- Experimental observation shows two types of particles in nature.

1. Bosons (symmetric permutations).
2. Fermions (antisymmetric permutations).

- Electrons are Fermions.

- Prob =  $\psi^*(1, 2, \dots, N)\psi(1, 2, \dots, N)$  tells us the probability of finding the  $N$  electrons at the positions specified by 1 through  $N$ .

- Example (helium):

- We had  $\psi(1, 2) = 1s\alpha(1)1s\beta(2)$ .
- But since the electrons are distinguishable here, we have to make them indistinguishable by letting

$$\begin{aligned}\psi(1, 2) &= 1s\alpha(1)1s\beta(2) - 1s\alpha(2)1s\beta(1) \\ &= 1s(1)1s(2)[\alpha(1)\beta(2) - \alpha(2)\beta(1)]\end{aligned}$$

- Here, we've separated out the spatial and spin parts.

- How to represent antisymmetry.

- Grassmann Wedge Product (1850s): Herman de Grassmann was studying linear algebra and would learn about higher dimensional spaces in his dreams.
- de Grassman says

$$\begin{aligned}\psi(1, 2) &= 1s\alpha(1) \wedge 1s\beta(2) \\ &= \frac{1}{2!}(1s\alpha(1)1s\beta(2) - 1s\alpha(2)1s\beta(1))\end{aligned}$$

where  $\wedge$  is the wedge product, a tensor product.

- We can normalize the above by substituting  $\sqrt{2!}$  for  $2!$ .

- Slater determinant (1930): Took

$$\psi(1, 2) = \frac{1}{\sqrt{2!}} \begin{vmatrix} 1s\alpha(1) & 1s\beta(1) \\ 1s\alpha(2) & 1s\beta(2) \end{vmatrix}$$

- We now consider lithium.

- Let's put 3 electrons in a single orbital:

$$\psi(1, 2, 3) = 1s\alpha(1) \wedge 1s\beta(2) \wedge 1s\beta(3)$$

- $1s\beta(2) \wedge 1s\beta(3)$  will vanish as per Grassmann algebra since the electrons have the same spin state. In general,  $\phi \wedge \phi = 0$ .

- Thus, in general,

$$\psi(1, 2, 3) = \sqrt{3!}(1s\alpha(1) \wedge 1s\beta(2) \wedge 2s\alpha(3))$$

or, using the Slater determinant,

$$\psi(1, 2, 3) = \frac{1}{\sqrt{3!}} \begin{vmatrix} 1s\alpha(1) & 1s\beta(1) & 2s\alpha(1) \\ 1s\alpha(2) & 1s\beta(2) & 2s\alpha(2) \\ 1s\alpha(3) & 1s\beta(3) & 2s\alpha(3) \end{vmatrix}$$

- **Pauli exclusion principle:** Each electron must occupy a distinct spin orbital, that is, it must have a distinct set of 4 quantum numbers  $n, l, m, m_s$ .

- Emerges naturally from the quantum mechanics as per the above.

- What if electrons were bozons?

- Then

$$\psi(1, 2, 3) = 1s\alpha(1) \vee 1s\beta(2) \vee 2s\alpha(3)$$

where  $\vee$  is the bozonic operator, i.e., the symmetric wedge product, which denotes the positive sum of all permutations.

- We have

$$\begin{aligned} \phi(1) \vee \phi(2) &= \phi(1)\phi(2) + \phi(2)\phi(1) \\ &= 2\phi(1)\phi(2) \end{aligned}$$

- Thus, we could put an infinite number of electrons in the same orbital (all electrons could occupy the same orbital) if electrons were bozons, and our shell structure would disintegrate.

- Everything would also merge; we could not form matter as we know it.

- Sodium atoms are bozons.

## 6.3 Chapter 8: Multielectron Atoms

From McQuarrie and Simon (1997).

11/7:

- **Atomic units:** A system of units widely adopted for atomic and molecular calculations to simplify the equations.

Property	Atomic unit	SI Equivalent
mass	mass of an electron, $m_e$	$9.1094 \times 10^{-31}$ kg
charge	charge on a proton, $e$	$1.6022 \times 10^{-19}$ C
angular momentum	reduced Planck constant, $\hbar$	$1.0546 \times 10^{-34}$ J s
distance	Bohr radius, $a_0 = 4\pi\epsilon_0\hbar^2/m_e e^2$	$5.2918 \times 10^{-11}$ m
energy	$e^2/4\pi\epsilon_0 a_0 = E_h$	$4.3597 \times 10^{-18}$ J
permittivity	$4\pi\epsilon_0$	$1.1127 \times 10^{-10}$ C <sup>2</sup> J <sup>-1</sup> m <sup>-1</sup>

Table 6.1: Atomic units and their SI equivalents.

- **Hartree:** The atomic unit of energy. Denoted by  $E_h$ .
  - Note that in atomic units, the ground-state energy of a hydrogen atom (in the fixed nucleus approximation) is  $-E_h/2$ .
- Another good reason to use atomic units is that we are still refining the values of  $m_e$ ,  $e$ ,  $\hbar$ , etc., so results computed in atomic units will hold even as these constants evolve.
- Both perturbation theory and the variational method can yield excellent results for helium.
- **Slater orbital**<sup>[1]</sup>: An orbital of the following form. Denoted by  $S_{nlm}(\mathbf{r}, \theta, \phi)$ . Given by

$$S_{nlm}(r, \theta, \phi) = N_{nl} r^{n-1} e^{-\zeta r} Y_l^m(\theta, \phi)^{[2]}$$

where  $N_{nl} = (2\zeta)^{n+1/2} / \sqrt{(2n)!}$  and the  $Y_l^m(\theta, \phi)$  are the spherical harmonics.

- The parameter  $\zeta$  is arbitrary, i.e., not necessarily  $Z/n$  as in hydrogenlike orbitals.
  - The radial components of the Slater orbitals do not have nodes like the hydrogenlike orbitals.
- **Orbital:** A one-electron wave function.
- **Hartree-Fock limit:** The best value of the energy that can be obtained using a trial function of the form of a product of orbitals.
- As such, it is advantageous to generalize our trial wave functions past simple products of orbitals.
  - Better wave functions typically include the interelectronic distance.
- This realization led scientists to abandon the orbital concept altogether in favor of finding Hartree-Fock orbitals (because they are still a useful model) and correcting them using an approach such as perturbation theory.
- Self-consistent field method for finding the Hartree-Fock orbitals (of helium).
  - Write  $\psi(\mathbf{r}_1, \mathbf{r}_2)$  as a product of orbitals  $\phi(\mathbf{r}_1)\phi(\mathbf{r}_2)$ .
  - It follows that the potential energy that electron 1 experiences at point  $\mathbf{r}_1$  due to electron 2 is

$$V_1^{\text{eff}}(\mathbf{r}_1) = \int \phi^*(\mathbf{r}_2) \frac{1}{r_{12}} \phi(\mathbf{r}_2) d\mathbf{r}_2$$

where the superscript “eff” emphasizes that  $V_1^{\text{eff}}(\mathbf{r}_1)$  is an effective, or average, potential.

- This allows us to define the effective one-electron Hamiltonian for electron 1 by
- $$\hat{H}_1^{\text{eff}}(\mathbf{r}_1) = -\frac{1}{2}\nabla_1^2 - \frac{2}{r_1} + V_1^{\text{eff}}(\mathbf{r}_1)$$
- Using this Hamiltonian and the corresponding Schrödinger equation, we can determine  $\phi(\mathbf{r}_1)$ , given an estimate for  $\phi(\mathbf{r}_2)$ . Since there is analogous Schrödinger equation for  $\phi(\mathbf{r}_2)$ , the self-consistent field method consists of starting with a guess for  $\phi(\mathbf{r}_2)$ , calculating from that  $\phi(\mathbf{r}_1)$ , calculating from that a better guess for  $\phi(\mathbf{r}_2)$ , and so on and so forth until the two wave equations are reasonably close, or **self-consistent**.
    - In practice, linear combinations of Slater orbitals are used for each  $\phi(\mathbf{r})$ .
  - **Uncorrelated** (electrons): A set of electrons that are taken to be independent of each other, or at least to interact only through some average (or effective) potential.

- Uncorrelated electrons occupy mathematically separable orbitals, as in  $\psi(\mathbf{r}_1, \mathbf{r}_2) = \phi(\mathbf{r}_1)\phi(\mathbf{r}_2)$ .

<sup>1</sup>First introduced by American physicist John Slater in the 1930s.

<sup>2</sup> $\zeta$  is the Greek “zeta.”

- **Correlation energy:** The difference between the exact energy and the Hartree-Fock approximation energy. *Denoted by CE.*
- **Spin quantum number:** The fourth quantum number, which represents the  $z$ -component of the electron spin angular momentum. *Denoted by  $m_s$ . Given by*

$$m_s = \pm \frac{1}{2} \text{ a.u.}$$

- Uhlenbeck and Goudsmit first suggested in 1925 that electrons behave like spinning tops having  $z$ -components of spin angular momentum  $\pm \hbar/2$  to explain the phenomena of sodium having a doublet in its atomic spectrum where quantum theory predicts it should have a singlet.
- We graft spin onto quantum theory in an ad hoc manner here. This will suit our purposes.
  - Note, however, that Dirac developed in the 1930s a relativistic extension of quantum mechanics in which spin arises in a perfectly natural way.
- As part of this grafting, we *define* the **spin operators**  $\hat{S}^2$  and  $\hat{S}_z$  in an analogous manner to how we defined the angular momentum operators  $\hat{L}^2$  and  $\hat{L}_z$ , except that we introduce half-integral angular momentum for electron spin: Indeed, let

$$\hat{S}^2 \alpha = \frac{1}{2} \left( \frac{1}{2} + 1 \right) \hbar^2 \alpha \qquad \hat{S}^2 \beta = \frac{1}{2} \left( \frac{1}{2} + 1 \right) \hbar^2 \beta$$

and let

$$\hat{S}_z \alpha = m_s \alpha = \frac{1}{2} \hbar \alpha \qquad \hat{S}_z \beta = m_s \beta = -\frac{1}{2} \hbar \beta$$

- We can formally associate  $\alpha = Y_{1/2}^{1/2}$  and  $\beta = Y_{1/2}^{-1/2}$  in analogy with  $\hat{L}^2$  and  $\hat{L}_z$ , but we do not have to.
- To continue the analogy, we can actually say that the square of the *spin* angular momentum is  $S^2 = \hbar^2 s(s+1)$ , where  $s = \pm 1/2$ .
- Since  $s$  cannot assume large values like  $l$  and thus cannot approach a classical limit, spin is strictly nonclassical.
- **Spin eigenfunctions:** The functions  $\alpha$  and  $\beta$ .
- The spin operators are Hermitian, so  $\alpha, \beta$  are orthonormal.
  - Formally,

$$\begin{aligned} \int \alpha^*(\sigma) \alpha(\sigma) d\sigma &= \int \beta^*(\sigma) \beta(\sigma) d\sigma = 1 \\ \int \alpha^*(\sigma) \beta(\sigma) d\sigma &= \int \beta^*(\sigma) \alpha(\sigma) d\sigma = 0 \end{aligned}$$

where  $\sigma$  is the **spin variable**, an object with no classical analog.

- We now include the spin function with the spatial wave function and postulate that the two are independent so that one of the following two cases holds.

$$\psi(x, y, z, \sigma) = \psi(x, y, z) \alpha(\sigma) \qquad \psi(x, y, z, \sigma) = \psi(x, y, z) \beta(\sigma)$$

- **Spin orbital:** The complete one-electron wave function  $\psi$  dependent on  $x, y, z, \sigma$ .
- For example, the first two spin orbitals of a hydrogenlike atom are

$$\psi_{100\frac{1}{2}} = \sqrt{\frac{Z^3}{\pi}} e^{-Zr} \alpha \qquad \psi_{100-\frac{1}{2}} = \sqrt{\frac{Z^3}{\pi}} e^{-Zr} \beta$$

- Note that these orbitals are orthonormal since

$$\int \psi_{100\frac{1}{2}}^*(\mathbf{r}, \sigma) \psi_{100\frac{1}{2}}(\mathbf{r}, \sigma) 4\pi r^2 dr d\sigma = \int_0^\infty \frac{Z^3}{\pi} e^{-2Zr} 4\pi r^2 dr \int \alpha^* \alpha d\sigma = 1$$

and similarly for  $\psi_{100-\frac{1}{2}}$ , and since

$$\int \psi_{100\frac{1}{2}}^*(\mathbf{r}, \sigma) \psi_{100-\frac{1}{2}}(\mathbf{r}, \sigma) 4\pi r^2 dr d\sigma = \int_0^\infty \frac{Z^3}{\pi} e^{-2Zr} 4\pi r^2 dr \int \alpha^* \beta d\sigma = 0$$

- **Pauli Exclusion Principle** (elementary): No two electrons in an atom can have the same values of all four quantum numbers.
- Consider the helium atom.
  - Let  $\psi(1, 2) = 1s\alpha(1)1s\beta(2)$  where  $1s\alpha$  and  $1s\beta$  represent  $\psi_{100\frac{1}{2}}$  and  $\psi_{100-\frac{1}{2}}$ , respectively, and 1 and 2 represent all four coordinates (3 spatial plus 1 spin) of electrons 1 and 2, respectively.
  - Similarly,  $\psi(2, 1) = 1s\alpha(2)1s\beta(1)$ .
  - Since no known experiment can distinguish one electron from another,  $\psi(1, 2) = \psi(2, 1)$ .
  - Formally, we must consider the two possible states

$$\psi_1 = \psi(1, 2) + \psi(2, 1) \qquad \psi_2 = \psi(1, 2) - \psi(2, 1)$$

of two indistinguishable electrons.

- Experimentally,  $\psi_2$  describes the ground state of helium since  $\psi_2(1, 2) = -\psi_2(2, 1)$ , so  $\psi_2$  is **antisymmetric**.
- Note that the normalization constant of  $\psi_2$  is  $1/\sqrt{2}$ .
- **Antisymmetric wave function:** A wave function that changes sign when two electrons are interchanged.
- The importance of antisymmetry is captured by the following more fundamental statement of the Pauli Exclusion Principle.

**Postulate 6.** *All electronic wave functions must be antisymmetric under the interchange of any two electrons.*

- We were able to ignore spin in previous treatments of helium since  $\psi_2$  can be factored into a spatial and a spin part, and any contributions from spin cancel under a Hamiltonian that does not contain any spin operators.
  - Note that factorization of  $\psi^* \hat{H} \psi$  and  $\psi^* \psi$  does not occur in general, but does occur for two-electron systems.
- **Determinantal wave function:** A wave function given by the determinant of a matrix whose entries are individual spin orbitals.
  - Since the determinant representation changes signs when two rows are interchanged (representing interchanging two electrons) and is equal to zero under repeated rows/columns (representing multiple electrons in the same spin orbital), the determinant representation is antisymmetric and satisfies the Pauli Exclusion Principle, respectively.
- **Normalized  $N$ -electron determinantal wave function:** The normalized determinantal wave function

$$\psi(1, \dots, N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} u_1(1) & \cdots & u_N(1) \\ \vdots & \ddots & \vdots \\ u_1(N) & \cdots & u_N(N) \end{vmatrix}$$

where the  $u$ 's are orthonormal spin orbitals.

- **Fock operator:** The more general effective Hamiltonian operator used for self-consistent field solutions to systems more complex than helium. *Denoted by  $\hat{F}_i$ .*
- **Hartree-Fock orbital:** An eigenfunction of the Fock operator. *Denoted by  $\phi_i$ .*
- **Orbital energy:** An eigenvalue of the Fock operator. *Denoted by  $\epsilon_i$ .*
  - Alternatively, the energy of a Hartree-Fock orbital.
  - $\epsilon_i$  approximates the ionization energy of an electron from the  $i^{\text{th}}$  orbital.

11/13:

- **Russell-Saunders coupling:** The result of determining the total angular momentum  $\mathbf{L}$  and the total spin angular momentum  $\mathbf{S}$  and then adding  $\mathbf{L}$  and  $\mathbf{S}$  together vectorially to obtain the total angular momentum  $\mathbf{J}$ .
- **Atomic term symbol:** A scheme denoting an electronic configuration of a specific energy, as determined by Russell-Saunders coupling. *Also known as **spectroscopic term symbol**. Denoted by  $^{2S+1}L_J$ .*
- **$L$ :** The total orbital angular momentum quantum number. *Given by*

$$\mathbf{L} = \sum_i \mathbf{l}_i$$

- Each  $\mathbf{l}$  is the angular momentum of a single electron in the atom at hand.
- **$S$ :** The total spin quantum number.
  - Each  $\mathbf{s}$  is the spin of a single electron in the atom at hand.
- **$J$ :** The total angular momentum quantum number.
- **Spin multiplicity:** The quantity  $2S + 1$ .
- “Just as the  $z$ -component of  $\mathbf{l}$  can assume the  $2l + 1$  values  $m_l = l, l - 1, \dots, 0, \dots, -l$ , the  $z$ -component of  $\mathbf{L}$  can assume the  $2L + 1$  value  $M_L = L, L - 1, \dots, 0, \dots, -L$ ” (McQuarrie & Simon, 1997, p. 293).
- “ $M_S$  can take on the  $2S + 1$  values  $S, S - 1, \dots, -S + 1, -S$ . Thus, the spin multiplicity is simply the  $2S + 1$  projections that the  $z$ -component of  $\mathbf{S}$  can assume” (McQuarrie & Simon, 1997, p. 294).
- Example ( $ns^2$  electron configuration):
  - The only possible set of quantum numbers is

$$m_{l_1} = 0 \qquad m_{s_1} = +\frac{1}{2} \qquad m_{l_2} = 0 \qquad m_{s_2} = -\frac{1}{2}$$

$$M_L = 0$$

$$M_S = 0$$

- Since  $M_L = 0$ ,  $L = 0$ . Similarly,  $S = 0$ .
- Thus,  $J = 0$ .
- Therefore, our term symbol is  $^1S_0$  (“singlet S zero”).
- Every filled subshell is in the  $^1S_0$  state because each all spin and orbital angular momentum cancels, as in the above example.
  - This allows us to completely ignore the contributions of all filled orbitals in any electron configuration, for example, the  $1s^2 2s^2$  in  $1s^2 2s^2 2p^2$  wrt. the carbon atom.
- Covers microstates and microstate tables from Labalme (2021a) (see Module 41).

- Shows that the electron configuration  $1s^1 2s^1$  of a helium atom has two term symbols (the triplet  $S$  and the singlet  $S$  states).
- Derives the term symbols for a carbon atom in a generalizable procedure also done in Labalme (2021a).
- Each state designated by a term symbol corresponds to a determinantal wave function that is an eigenfunction of  $\hat{L}^2$  and  $\hat{S}^2$ , and each state corresponds to a certain energy. These energies *could* be calculated from the quantum mechanics, but instead we'll use **Hund's rules**.
- **Hund's rules:** Three empirical rules formulated by German spectroscopist Friedrich Hund. *Given by*
  1. The state with the largest value of  $S$  is the most stable (has the lowest energy) and stability decreases with decreasing  $S$ .
  2. For states with the same value of  $S$ , the state with the largest value of  $L$  is the most stable.
  3. If the states have the same value of  $L$  and  $S$ , then for a subshell that is less than half filled, the state with the smallest value of  $J$  is the most stable, and vice versa for a subshell that is more than half filled.
- The Schrödinger equation implies that the electronic energy of the hydrogen atom depends only upon the principal quantum number.
  - However, **spin-orbit coupling** implies that the various  $n$  levels are split into sets of closely lying energy levels.
- **Spin-orbit interaction:** The interaction of the magnetic moment associated with the spin of an electron with the magnetic field generated by the electric current produced by the electron's own orbital motion.
  - There also exist **spin-spin interaction** and **orbit-orbit interaction**.
- A quantum mechanical treatment of spin-orbit coupling.
  - Essentially, consider the Hamiltonian operator for a multielectron atom

$$\hat{H} = -\frac{1}{2} \sum_j \delta_j^2 - \sum_j \frac{Z}{r_j} + \sum_{i < j} \frac{1}{r_{ij}} + \sum_j \xi(r_j) \mathbf{l}_j \cdot \mathbf{s}_j$$

where  $\xi(r_j)$  is a scalar function of  $r$ .

- We've treated the first three terms previously, and for light elements, we can treat the fourth term as a sufficiently small perturbation. Applying perturbation theory then predicts the observed orbital splitting.

- **Fine structure:** The increased spectral complexity caused by spin-orbit coupling.



## Week 7

# Molecular Spectroscopy and Group Theory

## 7.1 Hydrogen Molecule

- 11/8: • Reviews what would happen if electrons were bosons.

– The wave function  $\psi$  of the lithium atom would be

$$\psi(123) = 1s\alpha(1) \vee 1s\alpha(2) \vee 1s\alpha(3)$$

– Reviews wedge products.

– Conclusion: All electrons could occupy the same orbital.

- A sodium atom (the nucleus and the electrons jointly) acts like a boson.

– At temperatures on the order of microkelvin,  $10^{11}$  atoms have been placed in the same ground-state orbital.

– These substances are known as **Bose-Einstein condensates**.

– We use a magnetic field to confine the atoms to a harmonic potential. Since the atoms form a Gaussian curve at the bottom of said potential, they are all in the ground state (see Figure 3.3).

– Evaporative cooling and laser cooling allow you to reach such temperatures.

– Fermionic atoms cannot condense in such a way (because of the Pauli Exclusion Principle).

– Superconductivity is a condensation phenomena.

– Bose-Einstein condensates were predicted by Einstein in the 1930s but not experimentally verified until the 1990s.

– A very dilute gas was used here. In such a condition, the atoms feel the statistics force of the wedge product which forces them into such a state.

- Consider the Boron atom:

– It has 5 electrons.

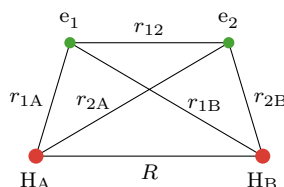
– It's electron configuration is  $1s^2 2s^2 sp^1$ .

– It's wave function is

$$\psi(12345) = 1s\alpha(1) \wedge 1s\beta(2) \wedge 2s\alpha(3) \wedge 2s\beta(4) \wedge 2p\alpha(5)$$

- Recall that this is not the exact wave function; this is still a product of hydrogenlike orbitals at the Hartree-Fock level.

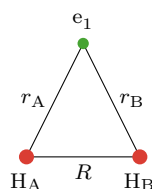
- More general wave functions can be used to obtain more accurate results.
- Consider the diatomic molecule  $\text{H}_2$ .

Figure 7.1:  $\text{H}_2$  distances.

- We have hydrogen atoms  $\text{H}_A$  and  $\text{H}_B$ , separated by a distance  $R$ .
- We have electrons  $e_1$  and  $e_2$ .
- The distance from object  $i$  to  $j$  where  $i, j = A, B, 1, 2$  and  $i \neq j$  is  $r_{ij}$ .
- Hamiltonian:

$$\hat{H} = -\frac{1}{2}(\nabla_1^2 + \nabla_2^2) - \frac{1}{r_{1A}} - \frac{1}{r_{1B}} - \frac{1}{r_{2A}} - \frac{1}{r_{2B}} + \frac{1}{r_{12}} + \frac{1}{R}$$

- This is pretty complicated.
- Thus, let's start with the hydrogen molecular ion ( $\text{H}_2^+$ ).

Figure 7.2:  $\text{H}_2^+$  distances.

- Hamiltonian:

$$\hat{H} = -\frac{1}{2}\nabla^2 - \frac{1}{r_A} - \frac{1}{r_B} + \frac{1}{R}$$

- This *can* be solved exactly in cylindrical coordinates, but it's nasty.
- Thus, let's approximate with the following variational wave function (originally by Heitler and London in the 1960s).

$$\psi(1) = c_1 1s_A + c_2 1s_B$$

- Albeit simple, this wave function gives pretty good results.
- By the variational principle,  $\mathbb{H}\vec{c} = E\mathbb{S}\vec{c}$ , or

$$\begin{vmatrix} H_{AA} - ES_{AA} & H_{AB} - ES_{AB} \\ H_{BA} - ES_{BA} & H_{BB} - ES_{BB} \end{vmatrix} = 0$$

solves for  $E$ .

– We have that

$$\begin{aligned}H_{AA} &= H_{BB} = \int d\vec{r} 1s_A^* \hat{H} 1s_A = \int d\vec{r} 1s_B^* \hat{H} 1s_B \\H_{AB} &= H_{BA} = \int d\vec{r} 1s_A^* \hat{H} 1s_B = \int d\vec{r} 1s_B^* \hat{H} 1s_A \\S_{AA} &= S_{BB} = \int d\vec{r} 1s_A^* 1s_A = \int d\vec{r} 1s_B^* 1s_B \\S_{AB} &= S_{BA} = \int d\vec{r} 1s_A^* 1s_B = \int d\vec{r} 1s_B^* 1s_A\end{aligned}$$

– We can show that

$$\begin{aligned}H_{AA} &= H_{BB} = E_{1s} + J \\H_{AB} &= H_{BA} = E_{1s}S + K\end{aligned}$$

where  $E_{1s}$  is the energy of the  $1s$  orbital of the hydrogen atom,  $J$  is the **Coulomb integral**

$$J = \int d\vec{r} 1s_A^* \left( -\frac{1}{r_B} + \frac{1}{R} \right) 1s_A$$

and  $K$  is the **exchange integral**

$$K = \int d\vec{r} 1s_B^* \left( -\frac{1}{r_B} + \frac{1}{R} \right) 1s_A$$

## 7.2 The Hydrogen Molecular Ion

11/10: • Continuing from last time, the determinant for the equation  $\mathbb{H}\vec{c}_n = E_n\mathbb{S}\vec{c}_n$  is

$$\begin{vmatrix} E_{1s} + J - E & E_{1s}S + K \\ E_{1s}S + K & E_{1s} + J - E \end{vmatrix}$$

- Therefore, the characteristic polynomial is a quadratic equation in  $E$ .
- Solving said quadratic gives us

$$E_{\pm} = E_{1s} + \frac{J \pm K}{1 \pm S}$$

where

$$J(R) = e^{-2R} \left( 1 + \frac{1}{R} \right) \quad S(R) = e^{-R} \left( 1 + R + \frac{R^2}{3} \right) \quad K(R) = \frac{S(R)}{R} - e^{-R}(1 + R)$$

- We then determine  $\vec{c}_n$  in the two different cases. But this yields

$$|c_1| = |c_2| = k$$

- Therefore,

$$\psi_{\pm} = k(1s_A \pm 1s_B)$$

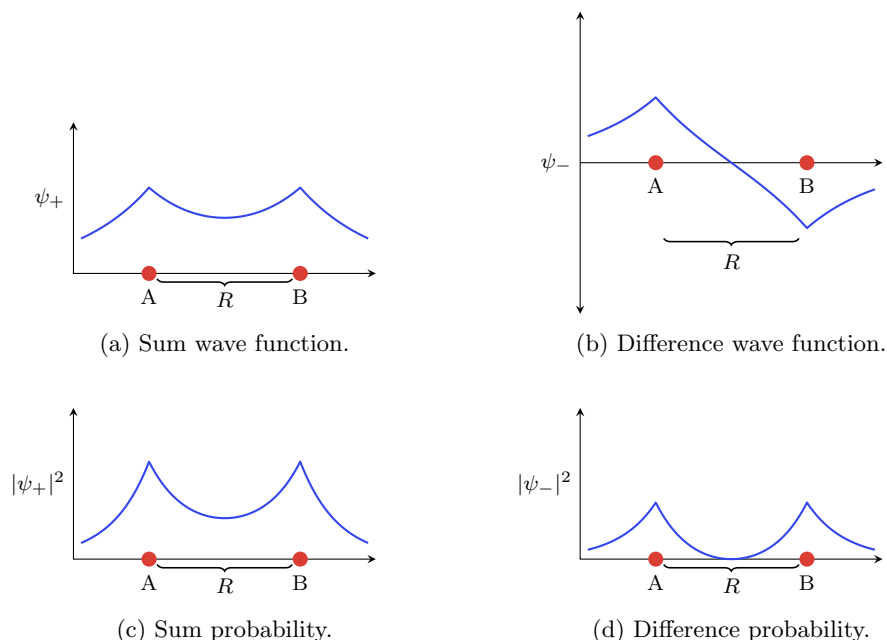


Figure 7.3: Hydrogen ion bonding.

- Note that in  $\psi_-$ , a node arises naturally from the quantum mechanics.
- Thus,  $\psi_+$  is a bonding orbital and  $\psi_-$  is an antibonding orbital.
- We now consider the **potential energy surface** or **PES** of the molecule.

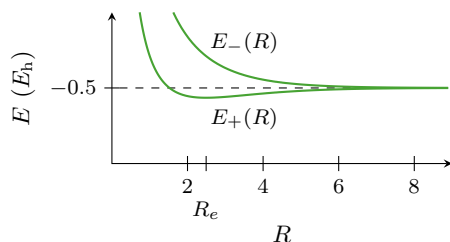


Figure 7.4: Potential energy surface of the hydrogen molecular ion.

- The  $x, y$ -axis units are Bohrs and Hartrees, respectively.
- The bound state only occurs in the bonding orbital; if the electron is excited to the antibonding orbital, the atoms will drift apart to  $\infty$  to minimize energy.
- **Born-Oppenheimer approximation.**
  - Throughout this derivation, we neglected the kinetic energy of the nuclei.
  - Thus, technically the total Hamiltonian is

$$\hat{H}_{\text{tot}} = -\frac{\hbar^2}{2M}(\hat{\nabla}_A^2 + \hat{\nabla}_B^2) + \hat{H}_{\text{electr}}$$

where  $\hat{H}_{\text{electr}}$  is the Hamiltonian associated with Figure 7.2.

- We have assumed that the nuclei are fixed relative to the motion of the electrons. We can do this because  $m_e/M \approx 10^{-3}$ , i.e., the electrons travel much faster than the nuclei. Therefore, the kinetic energy of the electrons is more important.

- The wave functions of the nuclei (which do exist) are very sharp peaks, so the nuclei don't move much, so we may regard them as fixed.
- **Molecular orbital:** A linear combination of atomic orbitals. *Also known as MO.*
- Example ( $\text{H}_2$ ):
  - $\psi_{\pm} = 1s_A \pm 1s_B$  (bonding and antibonding).
  - $\phi_{\text{MO}} = \phi_{1s_A} + \phi_{1s_B}$ .
  - $\phi(12) = \phi_{\text{MO}}\alpha(1)\wedge\phi_{\text{MO}}\beta(2)$ ; thus, the MO diagram is connected back to the rigorous mathematics of Schrödinger.
- Filling rules: Fill the MOs that are lower in energy first.
- Example ( $\text{C}_2$ ):
  - The MO diagram is identical to Figure III.17 from Labalme (2021a) except that the  $\sigma_g$  corresponding to the  $2p$  orbitals has higher energy than the  $\pi_u$ 's due to mixing.

## 7.3 Huckel Theory

11/12: • Reviews that diatomics with more electrons will have less mixing.

- **Bond order:** A measure of stability. *Given by*

$$BO = \frac{1}{2}[(\# \text{ of electrons in } BO_1) - (\# \text{ of electrons in } ABO_1)]$$

- For example, for  $\text{C}_2$ , we will have  $BO = \frac{1}{2}(8 - 4) = 2$ .
- Huckel was a physicist/chemist from the 1930s who worked out the MO theory for conjugated molecules.
- Huckel's strategy.
  1.  $\mathbb{H}\vec{c} = E\vec{S}\vec{c}$  becomes  $\mathbb{H}\vec{c} = E\vec{c}$ , where the only orbitals considered are the  $p_z$  orbitals.
  2. We let  $\mathbb{S} = \mathbb{I}$ , i.e., we assume all orbitals are orthonormal.  $S_{ij} = \delta_{ij}$ .
  3.  $H_{ii} = \alpha$  (the energy of the electrons in  $p_z$ ).
    - $H_{i,i+1} = H_{i+1,i} = \beta$  (interaction between adjacent  $p_z$  orbitals).
    - Everything else is zero!
- Example: Ethene.

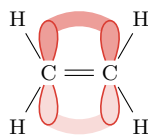


Figure 7.5: Ethene orbital diagram.

- The Hamiltonian matrix is

$$\mathbb{H} = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$$

- If we let  $(\alpha - E)/\beta = x$ , then

$$\begin{aligned} 0 &= \begin{vmatrix} \alpha - E & \beta \\ \beta & \alpha - E \end{vmatrix} \\ &= \frac{1}{\beta^2} \begin{vmatrix} \alpha - E & \beta \\ \beta & \alpha - E \end{vmatrix} \\ &= \begin{vmatrix} x & 1 \\ 1 & x \end{vmatrix} \end{aligned}$$

- Thus, our characteristic polynomial is

$$\begin{aligned} 0 &= x^2 - 1 \\ &= (x + 1)(x - 1) \end{aligned}$$

so  $x = \pm 1$ .

- It follows from returning the substitution that  $\alpha \pm \beta = E_{\mp}$ .
- This yields the following energy diagram.

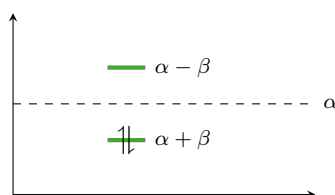


Figure 7.6: Ethene energy diagram.

- Note that the ground state MO is more stable than the 1 electron in each  $p_z$  orbital scenario; thus bonding occurs.
- It follows that

$$\psi_{\alpha+\beta} = 2p_{zA} + 2p_{zB} \qquad \psi_{\alpha-\beta} = 2p_{zA} - 2p_{zB}$$

■ Thus,  $\psi_{\alpha+\beta}$  has  $\pi_u$  symmetry and  $\psi_{\alpha-\beta}$  has  $\pi_g$  symmetry.

- Example: 1,3-butadiene.

- Our determinant here will be

$$\begin{aligned} 0 &= \begin{vmatrix} \alpha - E & \beta & 0 & 0 \\ \beta & \alpha - E & \beta & 0 \\ 0 & \beta & \alpha - E & \beta \\ 0 & 0 & \beta & \alpha - E \end{vmatrix} \\ &= \begin{vmatrix} x & 1 & 0 & 0 \\ 1 & x & 1 & 0 \\ 0 & 1 & x & 1 \\ 0 & 0 & 1 & x \end{vmatrix} \\ &= x^4 - 3x^2 + 1 \\ &= y^2 - 3y + 1 \\ y &= \frac{3 \pm \sqrt{5}}{2} \\ x &= \pm 1.618, \pm 0.618 \end{aligned}$$

- Thus, our energy diagram will be

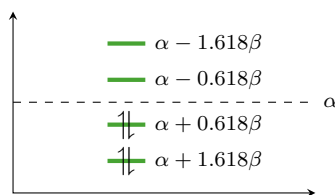


Figure 7.7: 1,3-butadiene energy diagram.

- This yields the bond order  $BO = \frac{1}{2}(4 - 0) = 2$ .
- Solving for the possible  $\vec{c}$ 's yields four wave function LCAOs analogous to the particle in a box wave functions (even function, odd function with one node, even function with two nodes, odd function with three nodes).
  - Thus, the Huckel approximation agrees with the approximation of four particles in a box.
- Example 3: 1,3,5-hexatriene.
  - Working out the determinant yields

$$\begin{aligned} 0 &= x^6 - 5x^4 + 6x^2 - 1 \\ &= y^3 - 5y^2 + 6y - 1 \end{aligned}$$

- Our energy diagram is as one might expect with 3 bonding orbitals and 3 antibonding orbitals.

## 7.4 Office Hours (Mazziotti)

- How does the asymptotic analysis guarantee us the  $n = 1$  solution, from Problem Set 5.1e?
  - $n = 1$  dominates in the asymptotic analysis, especially among the ground states.
- Math that we're using?
  - We're doing a lot of linear algebra, ordinary differential equations, and partial differential equations (boundary condition differential equations). Typically elliptic boundary conditions.
  - Eigenvalue equations, determinants, etc.
  - We're in a Hilbert space. Hilbert spaces are subsets of the Banach spaces, so all of that linear algebra applies.
  - Differential equations in a subspace can be rewritten as eigenvalue equations?
  - Ties into real analysis and complex analysis with  $e^{i\phi}$ .
  - The Grassmann algebra is a piece of classical mathematics that plays in.
  - Point groups and spin groups fall under abstract algebra; more specifically, Lie algebra.

## 7.5 Chapter 9: The Chemical Bond — Diatomic Molecules

From McQuarrie and Simon (1997).

- 11/8:
- Quantum mechanics was the first theory to explain why atoms combined to form a stable bond.
  - Since  $\text{H}_2^+$  has the simplest chemical bond, we will discuss it in detail.
    - The ideas developed will be applicable to more complex molecules, motivating molecular orbitals.
  - Describes the Hamiltonian for  $\text{H}_2$ , as in the discussion associated with Figure 7.1.

- **Born-Oppenheimer approximation:** The approximation of neglecting the nuclear motion, allowing us to ignore  $\nabla_{A,B}$  terms.
  - We can correct for the BO approximation using perturbation theory, but realistically we don't really need to (corrections are on the order of the mass ratio  $10^{-3}$ ).
- **Molecular-orbital theory:** The method we will use to describe the bonding properties of molecules.
- **Molecular orbital:** A single-electron wave function corresponding to a molecule.
- Like we constructed atomic wave functions in terms of determinants involving atomic orbitals, we will construct molecular wave functions in terms of determinants involving molecular orbitals.
- Note that  $\text{H}_2^+$  is a stable, well-studied species in real life.
- Although the Schrödinger equation for  $\text{H}_2^+$  can be solved exactly within the BO approximation, the solutions are not easy to use and their mathematical form gives little physical insight into how and why bonding occurs.
  - Thus, we use approximate solutions that provide good physical insight and are in good agreement with experimental observations.
- As a first trial wave function  $\psi(r_A, r_B; R)$ , use

$$\psi_{\pm} = c_1 1s_A \pm c_2 1s_B$$

where  $1s_{A,B}$  are the hydrogen atomic orbitals centered on nuclei A and B, respectively.

- By symmetry,  $c_1 = c_2$  for  $\text{H}_2^+$ .
- **LCAO molecular orbital:** A molecular orbital that is a linear combination of atomic orbitals.
- We seek to find  $E_{\pm}$  by taking the quotient of the integrals  $\int d\mathbf{r} \psi_{\pm}^* \hat{H} \psi_{\pm}$  and  $\int d\mathbf{r} \psi_{\pm}^* \psi_{\pm}$ .
- Considering  $\int d\mathbf{r} \psi_{\pm}^* \psi_{\pm}$  first, we are led to define the **overlap integral**.
- **Overlap integral:** The following integral, which is only significant where there is a large overlap between the two hydrogenlike atomic orbitals. *Denoted by  $S$ . Given by*

$$S = \int d\mathbf{r} n l_A^* n l_B = \int d\mathbf{r} n l_B^* n l_A = \int d\mathbf{r} n l_A n l_B$$



Figure 7.8: Overlap integral vs. internuclear distance.

- As  $R \rightarrow 0$ ,  $S \rightarrow 1$ . As  $R \rightarrow \infty$ ,  $S \rightarrow 0$ .
- The overlap integral when  $nl = 1s$  can be evaluated analytically, giving

$$S(R) = e^{-R} \left( 1 + R + \frac{R^2}{3} \right)$$



- The overlap integral allows us to express  $\int d\mathbf{r} \psi_{\pm}^* \psi_{\pm}$  in the following convenient form.

$$\begin{aligned} \int d\mathbf{r} \psi_{+}^* \psi_{+} &= \int d\mathbf{r} (1s_A^* + 1s_B^*)(1s_A + 1s_B) \\ &= \int d\mathbf{r} 1s_A^* 1s_A + \int d\mathbf{r} 1s_A^* 1s_B + \int d\mathbf{r} 1s_B^* 1s_A + \int d\mathbf{r} 1s_B^* 1s_B \\ &= 1 + S + S + 1 \\ &= 2(1 + S) \end{aligned}$$

- It follows that the normalized  $\psi_{\pm}$  are

$$\psi_{\pm} = \frac{1}{\sqrt{2(1 \pm S)}} (1s_A \pm 1s_B)$$

11/14:

- We now look to evaluate the integral  $\int d\mathbf{r} \psi_{+}^* \hat{H} \psi_{+}$ .
- To begin, we have that

$$\begin{aligned} \int d\mathbf{r} \psi_{+}^* \hat{H} \psi_{+} &= \int d\mathbf{r} (1s_A^* + 1s_B^*) \left( -\frac{1}{2} \nabla^2 - \frac{1}{r_A} - \frac{1}{r_B} + \frac{1}{R} \right) (1s_A + 1s_B) \\ &= \int d\mathbf{r} (1s_A^* + 1s_B^*) \left( -\frac{1}{2} \nabla^2 - \frac{1}{r_A} - \frac{1}{r_B} + \frac{1}{R} \right) 1s_A \\ &\quad + \int d\mathbf{r} (1s_A^* + 1s_B^*) \left( -\frac{1}{2} \nabla^2 - \frac{1}{r_A} - \frac{1}{r_B} + \frac{1}{R} \right) 1s_B \\ &= \int d\mathbf{r} (1s_A^* + 1s_B^*) \left( E_{1s} - \frac{1}{r_B} + \frac{1}{R} \right) 1s_A \\ &\quad + \int d\mathbf{r} (1s_A^* + 1s_B^*) \left( E_{1s} - \frac{1}{r_A} + \frac{1}{R} \right) 1s_B \\ &= \int d\mathbf{r} \left( E_{1s} - \frac{1}{r_B} + \frac{1}{R} \right) 1s_A^* 1s_A + \int d\mathbf{r} \left( E_{1s} - \frac{1}{r_B} + \frac{1}{R} \right) 1s_B^* 1s_A \\ &\quad + \int d\mathbf{r} \left( E_{1s} - \frac{1}{r_B} + \frac{1}{R} \right) 1s_A^* 1s_B + \int d\mathbf{r} \left( E_{1s} - \frac{1}{r_B} + \frac{1}{R} \right) 1s_B^* 1s_B \\ &= E_{1s} \cdot 1 + \int d\mathbf{r} \left( -\frac{1}{r_B} + \frac{1}{R} \right) 1s_A^* 1s_A + E_{1s} \cdot S + \int d\mathbf{r} \left( -\frac{1}{r_B} + \frac{1}{R} \right) 1s_B^* 1s_A \\ &\quad + E_{1s} \cdot 1 + \int d\mathbf{r} \left( -\frac{1}{r_B} + \frac{1}{R} \right) 1s_A^* 1s_B + E_{1s} \cdot S + \int d\mathbf{r} \left( -\frac{1}{r_B} + \frac{1}{R} \right) 1s_B^* 1s_B \\ &= 2E_{1s}(1 + S) \\ &\quad + \int d\mathbf{r} \left( -\frac{1}{r_B} + \frac{1}{R} \right) 1s_A^* 1s_A + \int d\mathbf{r} \left( -\frac{1}{r_B} + \frac{1}{R} \right) 1s_B^* 1s_A \\ &\quad + \int d\mathbf{r} \left( -\frac{1}{r_B} + \frac{1}{R} \right) 1s_A^* 1s_B + \int d\mathbf{r} \left( -\frac{1}{r_B} + \frac{1}{R} \right) 1s_B^* 1s_B \end{aligned}$$

where  $E_{1s} = -E_h/2$  is the ground-state energy of the hydrogen atom (the energy corresponding to the  $1s_A$  wave function).

- **Coulomb integral:** The following integral, which reflects both the charge density of the electron about nucleus A interacting with nucleus B via the Coulomb potential and the internuclear repulsion. Denoted by  $J$ . Given by

$$J = \int d\mathbf{r} 1s_A^* \left( -\frac{1}{r_B} + \frac{1}{R} \right) 1s_A$$

- **Exchange integral:** The following integral. *Also known as resonance integral. Denoted by  $K$ .* Given by

$$K = \int d\mathbf{r} 1s_B^* \left( -\frac{1}{r_B} + \frac{1}{R} \right) 1s_A$$

- The exchange integral describes a purely quantum-mechanical effect; it has no analogy in classical mechanics.
- Using the definitions of the Coulomb and exchange integrals, we have that

$$\int d\mathbf{r} \psi_+^* \hat{H} \psi_+ = 2E_{1s}(1 + S) + 2J + 2K$$

- We can now determine a final expression for the energy  $E_+$  corresponding to  $\psi_+$ .

$$\begin{aligned} E_+ &= \frac{\int d\mathbf{r} \psi_+^* \hat{H} \psi_+}{\int d\mathbf{r} \psi_+^* \psi_+} \\ &= \frac{2E_{1s}(1 + S) + 2J + 2K}{2(1 + S)} \\ &= E_{1s} + \frac{J + K}{1 + S} \end{aligned}$$

- The Coulomb and exchange integrals can be evaluated analytically, giving

$$J = e^{-2R} \left( 1 + \frac{1}{R} \right) \qquad K = \frac{S}{R} - e^{-R}(1 + R)$$

- Note that since the Coulomb integral is always positive, the exchange integral is entirely responsible for the existence of the chemical bond in  $H_2^+$ .
- This highlights the importance of the quantum-mechanical nature of the chemical bond.
- **Bonding orbital:** A state that exhibits a stable chemical bond.
  - Example:  $\psi_+$ .
- **Antibonding orbital:** A state that leads to a repulsive interaction between the two nuclei for all internuclear distances.
  - Example:  $\psi_-$ .
- The bonding orbital  $\psi_b = \psi_+$  describes the ground state of  $H_2^+$ , while the antibonding orbital  $\psi_a = \psi_-$  describes an excited state.
- Note that we have only found two molecular orbitals (as opposed to the infinitely many hydrogenlike atomic orbitals) because we used a trial wave function that was the linear combination of only two atomic orbitals. We could use longer linear combinations to find more molecular orbitals, but for pedagogical reasons, we will limit ourselves to two for now.
- “Because  $\psi_b$  is the molecular orbital corresponding to the ground-state energy of  $H_2^+$ , we can describe the ground state of  $H_2$  by placing two electrons with opposite spins in  $\psi_b$ , just as we placed two electrons in a  $1s$  atomic orbital to describe the helium atom” (McQuarrie & Simon, 1997, p. 336).
- This leads to the Slater determinantal wave function

$$\psi = \frac{1}{\sqrt{2!}} \begin{vmatrix} \psi_b\alpha(1) & \psi_b\beta(1) \\ \psi_b\alpha(2) & \psi_b\beta(2) \end{vmatrix}$$

for  $H_2$ .

- As before, the spatial and spin parts of this two-electron wave function separate.
- It follows that the spatial molecular wave function is

$$\begin{aligned}\psi_{\text{MO}} &= \psi_{\text{b}}(1)\psi_{\text{b}}(2) \\ &= \frac{1}{2(1+S)}[1s_{\text{A}}(1) + 1s_{\text{B}}(1)][1s_{\text{A}}(2) + 1s_{\text{B}}(2)]\end{aligned}$$

- **Linear combination of atomic orbitals-molecular orbitals method:** The method of constructing molecular wave functions by taking the product of molecular orbitals, which are in turn linear combinations of atomic orbitals. *Also known as LCAO-MO method.*

- This is how we constructed  $\psi_{\text{MO}}$ , above.
- Knowing that  $\psi_{\text{MO}}$  is normalized, we have that

$$E_{\text{MO}} = \int d\mathbf{r}_1 d\mathbf{r}_2 \psi_{\text{MO}}^*(1,2) \hat{H} \psi_{\text{MO}}(1,2)$$

- **$\sigma$  orbital:** An orbital that is symmetric about the internuclear axis.
  - Both  $\psi_{\pm}$  are  $\sigma$  orbitals.
  - Molecular orbitals constructed from  $1s$  orbitals are denoted by  $\sigma 1s$ .
- Distinguishing between  $\sigma 1s$  bonding and antibonding orbitals.
  - We may let  $\sigma 1s$  refer to  $\psi_+$  and  $\sigma^* 1s$  refer to  $\psi_-$ .
  - We may let  $\sigma_g 1s$  refer to  $\psi_+$  ( $g$  being short for “gerade” [the German word for even] since  $\psi_+$  does not change sign under inversion) and  $\sigma_u 1s$  refer to  $\psi_-$  ( $u$  being short for “ungerade” [the German word for odd] since  $\psi_-$  does change sign under inversion).
  - McQuarrie and Simon (1997) will favor the latter notation.
- $\sigma 2s$  molecular orbitals.
  - Of the form  $2s_{\text{A}} \pm 2s_{\text{B}}$ .
  - Have radial nodes about each nuclei (owing to the structure of the  $2s$  hydrogenlike atomic orbital) in addition to the possible planar node in the middle of the molecule.
  - Have higher energies than  $\sigma 1s$  MOs since they’re associated with higher energy atomic orbitals. This can be demonstrated rigorously the same way we calculated the energies for the  $\sigma 1s$  MOs.
  - In terms of energy,  $\sigma_g 1s < \sigma_u 1s < \sigma_g 2s < \sigma_u 2s$ .
- $2p$  molecular orbitals.
  - Choose the internuclear axis to be the  $z$ -axis.
  - $2p_{z_{\text{A}}} \pm 2p_{z_{\text{B}}}$  are cylindrically symmetric about the internuclear axis and are therefore  $\sigma_u 2p_z$  and  $\sigma_g 2p_z$  orbitals, respectively.
  - $2p_{x,y_{\text{A}}} \pm 2p_{x,y_{\text{B}}}$  are  **$\pi$  orbitals**, denoted  $\pi_u 2p_{x,y}$  and  $\pi_g 2p_{x,y}$ , respectively.
  - $\pi_u 2p_x, \pi_u 2p_y$  and  $\pi_g 2p_x, \pi_g 2p_y$  are degenerate pairs of orbitals since they are identical except for their spatial orientation.
- **$\pi$  orbital:** A molecular orbital with one nodal plane that contains the internuclear axis.
- From  $\text{Li}_2$  through  $\text{N}_2$ , the energy order is

$$\sigma_g 2s < \sigma_u 2s < \pi_u 2p_{x,y} < \sigma_g 2p_z < \pi_g 2p_{x,y} < \sigma_u 2p_z$$

- From  $O_2$  through  $F_2$ , the energy order is

$$\sigma_g 2s < \sigma_u 2s < \sigma_g 2p_z < \pi_u 2p_{x,y} < \pi_g 2p_{x,y} < \sigma_u 2p_z$$

- Diatomic helium does not exist since the bonding and antibonding electrons cancel each other out, leaving only repulsive nuclei.
  - Bond order formalizes this notion.
  - Note that in 1993, Gentry and co. reported spectroscopic observations of  $He_2$  in a gas-phase sample of helium of temperature 0.001 K, possessing by far the weakest chemical bond known.
- Note that since there is little difference between the electron densities of core electron-derived molecular orbitals, we need only consider electrons in the valence shell when discussing chemical bonding.
- We denote the filled  $n = 1$  shell by  $K$ , e.g., the ground state electron configuration of  $Li_2$  is  $KK(\sigma_g 2s)^2$ .
- MO theory accurately predicts that  $O_2$  is paramagnetic.
  - MO theory also predicts bond orders of 2.5, 2, 1.5, 1 for  $O_2^+$ ,  $O_2$ ,  $O_2^-$ ,  $O_2^{2-}$ , respectively. This would suggest that bond length increases and bond energy decreases down this list, as we do indeed observe experimentally.
- McQuarrie and Simon (1997) provides data on the ground-state electron configuration, bond order, bond length, and bond energy for the homonuclear diatomics of lithium through neon.
- **Ionization energy:** The energy required to eject an electron from a molecule.
- **Photoelectron spectroscopy:** The measurement of the energies of the electrons ejected by radiation incident on gaseous molecules.
  - Goes through how PES supports MO theory, as on Labalme (2021a, p. 56).
- **Heteronuclear diatomic molecule:** A diatomic molecule in which the two nuclei are different.
- For heteronuclear diatomics whose constituent atoms' atomic numbers differ by 1 or 2, the scheme we used for homonuclear diatomics is good enough.
- For heteronuclear diatomics whose constituent atoms' atomic numbers differ greatly (e.g., HF), we need a new scheme.
  - Noting that the energy of the  $1s$  atomic orbital of H is similar to the energy of the  $2p$  atomic orbitals of F, we look for constructive and destructive interference and find that  $1s_H$  and  $2p_{zF}$  interact nicely.
  - Thus, a first approximation the molecular orbital would be  $\psi = c_1 1s_H + c_2 2p_{zF}$ .
  - Note that both MOs have  $\sigma$  symmetry.
- The scheme presented thus far is the simplest possible.
- More generally, we may create trial wave functions that include more and orbitals, minimizing the coefficients with the variational method along the way, until we reach the Hartree-Fock limit.
- **SCF-LCAO-MO wave function:** A wave function obtained from molecular orbitals that are linear combinations of atomic orbitals whose coefficients are determined by a self-consistent field method.
  - Such orbitals, obtained from many atomic orbitals, cannot be meaningfully classified as  $\sigma 2s$  or  $\pi 2p$ , for example, so we classify them as the first  $\sigma_g$  orbital ( $1\sigma_g$ ), the first  $\sigma_u$  orbital ( $1\sigma_u$ ), and so on and so forth.
  - An SCF-LCAO-MO is only the same as a Hartree-Fock orbital if the SCF-LCAO-MO contains enough terms to reach the Hartree-Fock limit.

- **Hartree-Fock-Roothaan method:** The method for determining a SCF-LCAO-MO wave function.
- McQuarrie and Simon (1997) lists data on the Hartree-Fock limit vs. experimental calculations.
- Molecular term symbols.
- Excited states of molecules.
  - Consider  $\text{H}_2$  with one electron in the  $\sigma_g 1s$  orbital and one electron in the  $\sigma_g 2s$  orbital. This species has a slightly longer bond length than ground state  $\text{H}_2$ .

## 7.6 Chapter 10: Bonding in Polyatomic Molecules

*From McQuarrie and Simon (1997).*

11/15:

- **$\pi$ -electron approximation:** The approximation of the delocalized  $\pi$  electrons of a conjugated polyene or benzene as moving in some fixed, effective, electrostatic potential due to the electrons in the  $\sigma$  framework.

- Note that this can be rigorously developed from the Schrödinger equation.
- We may write the wave function of the  $\pi$  orbital of ethene as

$$\psi_{\pi} = c_1 2p_{z_A} + c_2 2p_{z_B}$$

- The diagonal entries in the corresponding secular determinant are **Coulomb integrals** and the off-diagonal entries are **resonance integrals**.
- To evaluate the determinant, we either need to specify the Hamiltonian or come up with another approximation for the integrals.
- **Hückel molecular-orbital theory:** An approximation composed of the following three assertions.
  1. The overlap integrals  $S_{ij} = \delta_{ij}$ .
  2. All of the Coulomb integrals are assumed to be the same for all carbon atoms and are commonly denoted by  $\alpha$ .
  3. The resonance integrals involving nearest-neighbor carbon atoms are assumed to be the same and are denoted by  $\beta$ .
- Once we calculate  $E$  from the reworked secular determinant, we plug in experimental values for  $\alpha$  and  $\beta$  and go from there (no effective Hamiltonian needed).
- McQuarrie and Simon (1997) covers the same butadiene derivation from class.
  - Addendum: The total  $\pi$ -electronic energy of butadiene is

$$\begin{aligned} E_{\pi} &= 2(\alpha + 1.618\beta) + 2(\alpha + 0.618\beta) \\ &= 4\alpha + 4.472\beta \end{aligned}$$

- Comparing this energy to the energy of two ethene molecules gives us the **delocalization energy**.
- The molecular orbitals are given and their schematic diagrams drawn.
- McQuarrie and Simon (1997) treats benzene the same way. See Nocera Lectures 6-7 from Labalme (2021a).

## Week 8

# Electronic Structure of Atoms and Molecules

### 8.1 Hybridization and Time-Dependent Quantum Mechanics

11/15:

- Consider BeH<sub>2</sub>.
  - Then we can form the *sp* molecular orbital

$$\psi_{sp} = \frac{1}{\sqrt{2}}(2s \pm 2p_z)$$

- Draws an orbital diagram of BeH<sub>2</sub>.
- BH<sub>3</sub>.
  - Lists *sp*<sup>2</sup> hybrid orbital wave functions.
- CH<sub>4</sub>.
  - Lists *sp*<sup>3</sup> hybrid orbital wave functions.
- The probability of finding a  $\psi_1$  electron in the  $2p_z$  orbital.
  - The absolute value of the coefficient of  $2p_z$  in the wave function, squared.
  - Alternatively,

$$\text{Prob} = \left| \int 2p_z^* \psi_1 \, d\tau \right|^2$$

- Time-dependent Schrödinger equation:

$$\hat{H}\psi = i\hbar \frac{d\psi}{dt}$$

- Let  $\psi_n(r, t) = \phi_n(r)e^{-iE_n t/\hbar}$ . Plugging this into the time-dependent Schrödinger equation yields

$$\begin{aligned}\hat{H}\psi &= i\hbar \frac{d}{dt} \left( \phi_n(r) e^{-iE_n t/\hbar} \right) \\ &= i\hbar \cdot -\frac{iE_n}{\hbar} \cdot \phi_n(r) e^{-iE_n t/\hbar} \\ &= E_n \psi_n\end{aligned}$$

- Thus, to this point, we've been solving the spatial part of the time-dependent Schrödinger equation.

- The probability density of the time-dependent solution is equal to the probability density of the time-independent solution.

$$\begin{aligned}
 |\psi(r, t)|^2 &= \left| \phi(r) e^{-iEt/\hbar} \right|^2 \\
 &= \psi^*(r) \psi(r) e^{iEt/\hbar} e^{-iEt/\hbar} \\
 &= \psi^*(r) \psi(r) \\
 &= |\psi(r)|^2
 \end{aligned}$$

- The time-dependent Schrödinger equation allows us to make predictions for how long it takes an electron to relax back to a lower energy state after excitation, for instance.
- Using the time-dependent Schrödinger equation to describe the interaction of the system with electromagnetic radiation.

1. Solution:  $\vec{E} = \vec{E}_0 \cos(\omega t)$ <sup>[1]</sup>.

2. Dipole moment of the molecule ( $\mu$ ).

- We let  $\hat{V} = -\vec{\mu} \cdot \vec{E}$ . This is how radiation interacts with matter.
- This leads to the overall Hamiltonian for the molecule interacting with the electric field as  $\hat{H} = \hat{H}_0 + \hat{V}$ , where  $\hat{H}_0$  is the Hamiltonian of the molecule and  $\hat{V}$  is the dipole-electric field perturbation defined above.
- If the electron is transitioning from the ground to the first-excited state, we let

$$\psi(t) = a_1(t)\phi_1 e^{-iE_1 t/\hbar} + a_2(t)\phi_2 e^{-iE_2 t/\hbar}$$

where the first part refers to the ground state, the second part refers to the excited state, and  $a_1, a_2$  are time-dependent expansion coefficients that we are solving for.

- Substitution into the time-dependent Schrödinger equation yields

$$\begin{aligned}
 i\hbar \left( \psi_1 \frac{da_1}{dt} + \psi_2 \frac{da_2}{dt} \right) &= a_1(t)\hat{V}\psi_1 + a_2(t)\hat{V}\psi_2 \\
 i\hbar e^{-iE_2 t/\hbar} \frac{da_2}{dt} &= a_1(t) \int \phi_2^* \hat{V} \psi_1 d\vec{r} + a_2(t) \int \phi_2^* \hat{V} \psi_2 d\vec{r}
 \end{aligned}$$

## 8.2 Time-Dependent Schrödinger Equation

11/17:

- The TD SE is an **initial value differential equation** since we know  $\psi_0$  and can propagate that over time.
- We will approximate the solutions to this with both an expansion (variational method) and perturbation theory.
- Now we continue from before (finding the time-dependence of the interaction of a EM radiation with a molecule).
  - Let

$$\hat{H} = \hat{H}_0 + \hat{V}$$

where  $\hat{H}_0$  is the Hamiltonian of the molecule and  $\hat{V}$  describes the interaction of the photon and the molecule.

---

<sup>1</sup>This is the definition of a classical electric radiation field. In Advanced Quantum Mechanics, we can turn this into a photon field for quantum electrodynamics.

- Let

$$\psi(t) = a_1(t) \underbrace{\phi_1 e^{-iE_1 t/\hbar}}_{\psi_1} + a_2(t) \underbrace{\phi_2 e^{-iE_2 t/\hbar}}_{\psi_2}$$

- Substituting into the TD SE, multiplying through by  $\phi_2^*$ , and integrating and simplifying yields (continuing from before)

$$i\hbar \frac{da_2}{dt} = a_1(t) e^{-i(E_1-E_2)t/\hbar} \int \phi_2^* \hat{V} \phi_1 d\vec{r} + a_2(t) \int \phi_2^* \hat{V} \phi_2 d\vec{r}$$

- Computing  $a_2(t)$  as a function of time.
- Initial values:

$$a_1(0) = 1$$

$$a_2(0) = 0$$

i.e., all of the electron density is in the ground state, and none is in the excited state.

- Assume a small perturbation and use the initial conditions on the right hand side of the above equation.
- This yields

$$i\hbar \frac{da_2}{dt} = e^{-i(E_1-E_2)t/\hbar} \int \phi_2^* \hat{V} \phi_1 d\vec{r}$$

- But since

$$\hat{V} = -\frac{\mu_z}{2} E_{0z} (e^{i2\pi\nu t} + e^{-i2\pi\nu t})$$

where we focus on the component of the electric field along the internuclear axis, we get

$$\frac{da_2}{dt} \propto \underbrace{\left( \int \phi_2 \mu_z \phi_1 d\vec{r} \right)}_{(\mu_z)_{12}} E_{0z} \left( e^{i(E_2-E_1+h\nu)t/\hbar} + e^{i(E_2-E_1-h\nu)t/\hbar} \right)$$

where  $(\mu_z)_{12}$  is the **transition dipole moment** between states 1 and 2.

■  $(\mu_z)_{12}$  must be nonzero to get a transition.

- Integrating from 0 to  $\tau$  gives

$$a_1(t) = (\mu_z)_{12} E_{0z} \left( \frac{1 - e^{i(E_2-E_1+h\nu)t/\hbar}}{E_2 - E_1 + h\nu} + \frac{1 - e^{i(E_2-E_1-h\nu)t/\hbar}}{E_2 - E_1 - h\nu} \right)$$

- The second term becomes large as  $E_2 - E_1 - h\nu \rightarrow 0$ .
- Thus, the probability in state 2 at  $\tau$  is

$$a_2^*(\tau) a_2(\tau) \propto \frac{\sin^2[(E_2 - E_1 - \hbar\omega)\tau/2\hbar]}{(E_2 - E_1 - \hbar\omega)^2}$$

- The right term above is a function of the form  $\sin^2(xt/2\hbar)/x^2$ , which has a peak right at the resonance frequency.

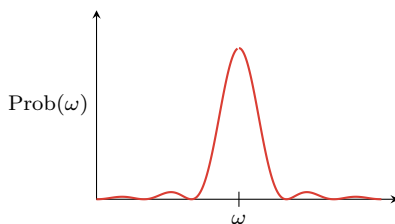


Figure 8.1: The sinc function.



■ This is the **sinc function**.

- Test 3:
  - Problem sets 6-7.
  - Helium from a perturbation theory/variational principle perspective.
  - Chemical bonding (MOs for  $\text{H}_2^+$ , bonding and antibonding orbitals).
  - MO diagrams for diatomics ( $\text{N}_2$ ,  $\text{O}_2$ , etc.)
  - Grassmann wedge notation.
  - Huckel theory (the MOs for conjugated molecules). There WILL be one of these on the test.
  - Hybridization.
  - Time-dependent perturbation theory and the TD SE (probably not a question on this, but if you have expansion coefficients evolving with time, how do you find the probability [modulus squared]).
  - 5 parts, formula page, calculator allowed.

## 8.3 Chapter 10: Bonding in Polyatomic Molecules

*From McQuarrie and Simon (1997).*

11/15:

- Consider  $\text{BeH}_2$ .
  - The two Be–H bonds are equivalent with an angle of  $180^\circ$  between them
  - MOs are formed from orbitals that are similar in energy, so in addition to the  $\text{H}(1s)$  and  $\text{Be}(2s)$ , throw in the energetically close  $\text{Be}(2p)$ .
  - We also add in a  $2p$  orbital because we need the resultant molecular orbitals to point in opposite directions so as to explain the linear structure of the molecule, and directionality is not something the spherically symmetric  $2s$  orbital can provide.
  - Thus, choose

$$\psi_{\text{Be-H}} = c_1\psi_{\text{Be}(2s)} + c_2\psi_{\text{Be}(2p)} + c_3\psi_{\text{H}(1s)}$$

- The first two terms in the above linear combination can be thought of as representing a new **hybrid** “orbital” on beryllium.
- The normalized  $sp$  hybrid orbitals are given by

$$\psi_{sp} = \frac{1}{\sqrt{2}}(2s \pm 2p_z)$$

where the  $z$ -axis is defined to be the internuclear H–Be–H axis.

- Note that  $sp$  hybrid orbitals concentrate electron density in one specific direction since the sign of the  $2p$  wave function is different in the two direction ( $\pm z$ ) but the sign of the  $2s$  wave function is everywhere positive.
- **Hybrid orbital:** A linear combination of atomic orbitals on the same atom.
- The normalized  $sp^2$  hybrid orbitals are

$$\begin{aligned}\psi_1 &= \frac{1}{\sqrt{3}}2s + \sqrt{\frac{2}{3}}2p_z & \psi_2 &= \frac{1}{\sqrt{3}}2s - \frac{1}{\sqrt{6}}2p_z + \frac{1}{\sqrt{2}}2p_x \\ \psi_3 &= \frac{1}{\sqrt{3}}2s - \frac{1}{\sqrt{6}}2p_z - \frac{1}{\sqrt{2}}2p_x\end{aligned}$$

- The normalized  $sp^3$  hybrid orbitals are

$$\begin{aligned}\psi_1 &= \frac{1}{2}(2s + 2p_x + 2p_y + 2p_z) & \psi_2 &= \frac{1}{2}(2s - 2p_x - 2p_y + 2p_z) \\ \psi_3 &= \frac{1}{2}(2s + 2p_x - 2p_y - 2p_z) & \psi_4 &= \frac{1}{2}(2s - 2p_x + 2p_y - 2p_z)\end{aligned}$$

- Bonding vs. lone pair electrons:  $H_2O$ .
  - We could form bond orbitals out of the singly occupied  $p_{y,z}$  orbitals of O, but this would predict a bond angle of  $90^\circ$ . Thus, we need hybrid orbitals.
  - The bond angle in water is between that predicted using  $sp^2$  hybrid orbitals and  $2p$  orbitals.
  - Since oxygen has two bonding and four lone pair electrons, we let

$$\psi = c_1 2s + c_2 2p_y + c_3 2p_z$$

- Determining the coefficients such that two orthogonal orbitals directed at an angle of  $104.5^\circ$  are generated yields

$$\psi_1 = 0.45 \cdot 2s + 0.71 \cdot 2p_y + 0.55 \cdot 2p_z \quad \psi_2 = 0.45 \cdot 2s - 0.71 \cdot 2p_y + 0.55 \cdot 2p_z$$

- Physical interpretation of the normalization constants:  $c_i^2$  is the fractional character of the  $i^{\text{th}}$  orbital<sup>[2]</sup>.
- $\psi_1, \psi_2$  accommodate the bonding electrons. Our unused hybrid orbital and unused  $2p_x$  orbital accommodate the lone pair electrons as a linear combination (since symmetry necessitates that the two lone pair orbitals be equivalent).
- Why  $BeH_2$  is linear but  $H_2O$  is bent.
  - Using a more general LCAO-MO with all relevant orbitals and solving yields a set of molecular orbitals.
  - Upon bending, the degeneracy of the  $\pi$  orbitals is lifted as the hydrogen orbitals move to intersect more with one specific side of the molecule.
  - Such changes are summarized by a **Walsh correlation diagram** (see Labalme (2021a, p. 65)).
  - We want the minimal energy for however many electrons we have, and with the additional electrons of  $H_2O$ , it is more energetically favorable to bend than to remain linear.
- **Walsh correlation diagram:** A plot of the energy of a molecular orbital as a function of a systematic change in molecular geometry.
- **$\sigma$ -bond framework:** The collection of all  $\sigma$  bonds in a molecule.

## Problems

12/1:10-47. The dipole moment of a polyatomic molecule is defined by

$$\boldsymbol{\mu} = e \sum_j z_j \mathbf{r}_j$$

where  $z_j e$  is the magnitude of a charge located at the point given by  $\mathbf{r}_j$ . Show that the value of  $\boldsymbol{\mu}$  is independent of the origin chosen for  $\mathbf{r}_j$  if the net charge is zero. Show that  $\boldsymbol{\mu} = 0$  for  $SO_3$  (trigonal planar),  $CCl_4$  (tetrahedral),  $SF_6$  (octahedral),  $XeF_4$  (square planar), and  $PF_5$  (trigonal bipyramidal).

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<sup>2</sup>Think  $s$ -character,  $p$ -character.

## 8.4 Chapter 12: Group Theory — The Exploitation of Symmetry

From McQuarrie and Simon (1997).

11/30:

- **Dihedral** (plane of symmetry): A plane of symmetry that bisects the angle between two-fold axes that are perpendicular to the principal axis.
- Deriving the two-dimensional irreducible representation  $E$  corresponding to the  $\mathbf{C}_{3v}$  point group.
  - Consider an arbitrary vector  $\mathbf{u}_1 = u_{1x}\mathbf{i} + u_{1y}\mathbf{j} = (\cos \alpha)\mathbf{i} + (\sin \alpha)\mathbf{j}$ .
  - Applying  $\hat{C}_3$  to  $\mathbf{u}_1$  yields

$$\begin{aligned}
 \mathbf{u}_2 &= \hat{C}_3 \mathbf{u}_1 \\
 &= \cos(120^\circ + \alpha)\mathbf{i} + \sin(120^\circ + \alpha)\mathbf{j} \\
 &= \left(-\frac{1}{2}\cos\alpha - \frac{\sqrt{3}}{2}\sin\alpha\right)\mathbf{i} + \left(\frac{\sqrt{3}}{2}\cos\alpha - \frac{1}{2}\sin\alpha\right)\mathbf{j} \\
 &= \left(-\frac{1}{2}u_{1x} - \frac{\sqrt{3}}{2}u_{1y}\right)\mathbf{i} + \left(\frac{\sqrt{3}}{2}u_{1x} - \frac{1}{2}u_{1y}\right)\mathbf{j} \\
 &= \underbrace{\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}}_{\hat{C}_3} \underbrace{\begin{pmatrix} u_{1x} \\ u_{1y} \end{pmatrix}}_{\mathbf{u}_1}
 \end{aligned}$$

- The other matrices can be derived similarly.
- This derivation also illustrates why  $(x, y)$  are coupled in this point group.
- **Basis** (of a representation): The components of the vectors used to generate the representation.
  - For example,  $u_x, u_y$  form a basis of  $E$ . Alternatively, we can say that they **belong** to  $E$ .
  - As another example, the basis vectors for  $R_z$  corresponding to the  $\mathbf{C}_{3v}$  are depicted in the following figure.

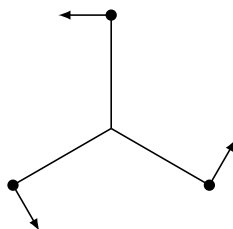


Figure 8.2: Rotational basis vectors.

## Week 9

# Quantum Dynamics and Control

### 9.1 Time-Dependent Schrödinger Equation and Spectroscopy

11/29:

- From last time:

- The probability of a two-level system being in state 2 at time  $\tau$  is

$$|a_2(\tau)|^2 \equiv a_2^*(\tau)a_2(\tau) \propto \frac{\sin^2[(E_2 - E_1 - \hbar\omega)t/2\hbar]}{(E_2 - E_1 - \hbar\omega)^2}$$

- When  $\hbar\omega = E_2 - E_1$ , you'll have the maximum resonance/strongest absorption and the peak in the graph of the sinc function.
- Leads to selection rules: Only certain states directly couple with a weak perturbation.

- Now consider the transition dipole moment:

$$(\mu_z)_{12} = \int \phi_2^* \mu_z \phi_1 \, dx$$

- The integral will vanish in the rigid rotor approximation.

- Up to the approximation of the rigid rotor,  $l$  can only change by  $\pm 1$  and  $m$  stays the same.

- Taking the harmonic oscillator as the paradigm for vibrational motion, we find that the quantum number must be  $\Delta n = \pm 1$ .

- The harmonic oscillator wave functions are of the form

$$\psi_n(x) = N_n H_n(\sqrt{\alpha}x) e^{-\alpha x^2/2}$$

where  $\alpha = \sqrt{k\mu}/\hbar$ .

- Assume  $\alpha = 1$ . Then  $\psi_n = N_n H_n(x) e^{-x^2/2}$ .
- We want to evaluate

$$(\mu_z)_{12} = \int \psi_{n'}(q) \mu_z(q) \psi_n(q) \, dq$$

where

$$\mu_z(q) = \mu_0 + \left( \frac{d\mu}{dq} \right)_{q=0} q + \dots$$

- This means that the transition dipole moment changes as a function of the bond length.
- When  $q = 0$  (at equilibrium), the transition dipole moment is the equilibrium one ( $\mu_0$ ).
- Then we Taylor series expand to correct the transition dipole moment away from equilibrium.

- Therefore,

$$(\mu_z)_{nn'} \equiv \mu_0 \int_{-\infty}^{\infty} H_{n'}(q) H_n(q) e^{-q^2} dq + \left( \frac{d\mu}{dq} \right)_{q=0} \int_{-\infty}^{\infty} H_{n'}(q) q H_n(q) e^{-q^2} dq + \dots$$

■ The first term goes to zero as long as  $n \neq n'$  because the Hermite polynomials are orthonormal.

- Recall that we used a recurrence relation to define the Hermite polynomials.
- In addition to the one we used previously, we have the recurrence relation

$$qH_n(q) = nH_{n-1}(q) + \frac{1}{2}H_{n+1}(q)$$

for all  $n$ .

- Thus,

$$\int_{-\infty}^{\infty} H_{n'}(q) q H_n(q) e^{-q^2} dq = \int_{-\infty}^{\infty} H_{n'}(q) \left[ nH_{n-1}(q) + \frac{1}{2}H_{n+1}(q) \right] e^{-q^2} dq$$

- Thus, unless  $n' = n \pm 1$ , the integral vanishes.
- This is the selection rule!
- Note that we're just tickling the molecule with a bit of radiation to get this to happen — if we hit it with too hard of a hammer, the selection rule will no longer hold.
- Spectroscopy.
  - $\Delta E = E_n - E_l = h\nu$ .
  - Regions of EM:

Region	Microwave	Far IR	IR	Vis/UV
Wavenumber (cm <sup>-1</sup> )	0.033 – 3.3	3.3 – 330	330 – 14 500	14 500 – 500 000
Molecular property	Rotation of polyatomics	Rotation of small molecules	Vibration	Electron transitions

Table 9.1: Spectroscopy in various regions of the electromagnetic spectrum.

- The Schrödinger equation within the Born-Oppenheimer approximation:

$$\left[ \sum_A \left( -\frac{\nabla_A^2}{2} \right) + \hat{H}_{\text{elec}} \right] \psi_{\text{elec}} \psi_{\text{nucl}} = (E_{\text{elec}} + E_{\text{nucl}}) \psi_{\text{elec}} \psi_{\text{nucl}}$$

- We invoke the Born-Oppenheimer approximation to split the wavefunction into electronic and nuclear components.
- We then split this equation into two parts. The first of which is the electronic Schrödinger equation

$$\hat{H}_{\text{elec}} \psi_{\text{elec}} = E_{\text{elec}} \psi_{\text{elec}}$$

- The BO approximation is what allows us to split the original Schrödinger equation in two.
- We now multiply by  $\psi_{\text{elec}}$  and integrate over all of the electrons.

$$\left[ \sum_A \left( -\frac{\nabla_A^2}{2} \right) + V(R) \right] \psi_{\text{nucl}} = (E_{\text{elec}} + E_{\text{nucl}}) \psi_{\text{nucl}}$$

- $V(R) = E_{\text{elec}}(R) = \int \psi_{\text{elec}}^* \hat{H}_{\text{elec}} \psi_{\text{elec}} dx$  is the potential energy surface (PES).

## 9.2 Spectroscopy (cont.)

- 12/1: • From last time, we have that

$$\left[ \sum_A \left( -\frac{\nabla_A^2}{2\mu_A} \right) + V(R) \right] \psi_{\text{nucl}} = (E_{\text{elec}} + E_{\text{nucl}}) \psi_{\text{nucl}}$$

- The PES is  $V(R) = \int \psi_{\text{elec}}^* \hat{H}_{\text{elec}} \psi_{\text{elec}} dx$ .
- This begins to break down where we have **conical intersections**.
- **Conical intersection:** An intersection between potential energy surfaces, where electrons can jump from one to the other.
  - Allows for radiation-less transitions from one surface to another.
- Potential energy surface<sup>[1]</sup> for the ground electronic wave function and its surface.
  - Think Figure 3.1b with its zero-point energy in the HO approximation.
  - The energies of the vibrational levels get closer and closer together until they're continuous forming a continuum at the level of the asymptote, reaching the classical limit.
- We have that

$$\begin{aligned} E_{\text{nucl}} &\approx E_{\text{HO}} + E_{\text{RR}} \\ &\approx \hbar\omega\left(n + \frac{1}{2}\right) + \frac{\hbar^2}{2I}l(l+1) \end{aligned}$$

- Since harmonic oscillator transitions are in the IR region and rigid rotor transitions are in the microwave region, vibrational transitions are of higher energy ( $\hbar\omega > \hbar^2/2I$ ).
- Rotational levels.

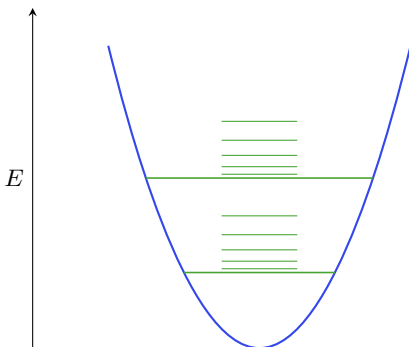


Figure 9.1: Rotational and vibrational energy levels.

- Each vibrational energy level has a number of smaller rotational states that are part of it.
- Changes in rotational levels often accompany changes in vibrational levels.
  - If we change the vibrational level by  $\pm 1$ , we have to change the rotational level (by  $\pm 1$ ) as well.

<sup>1</sup>Note that we technically have a potential energy curve at this point; we will only have a surface in higher-dimensional systems.

- IR/vibrational spectroscopy.

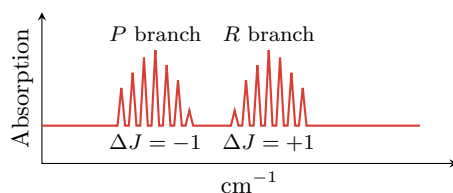


Figure 9.2: IR/vibrational spectrum.

- The whole spectrum represents a rovibrational transition.
- The *P* branch corresponds to when you go from a lower vibrational state to a higher vibrational state, *but* you go from a higher rotational state to a lower rotational state.
- The *R* branch corresponds to when you go from a lower vibrational state to a higher vibrational state, *and* you go from a lower rotational state to a higher rotational state.
- The number of peaks in the *R* and *P* branches reveals which rotational states are occupied (although we could theoretically go on forever, realistically, only lower energy rotational states are occupied).

- Electronic states.

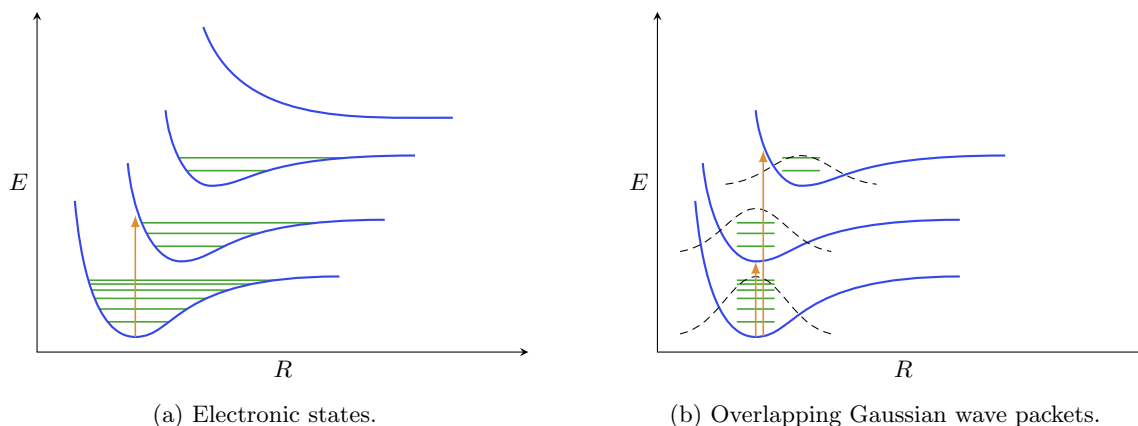


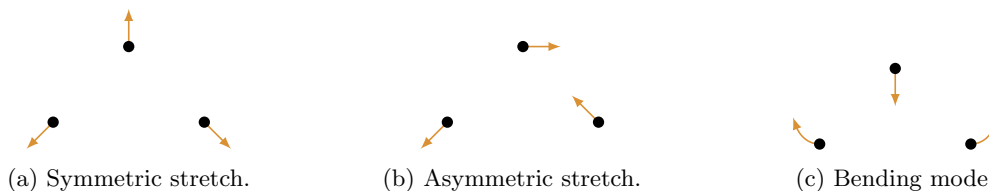
Figure 9.3: Exciting between electronic states.

- You can also do excitations from one potential energy surface to another.
- Recall that we have a Gaussian wave packet around the minimum energy on the PES.
- It's important that our Gaussian wave packets have some overlap for us to jump from one PES to the other.
- In Figure 9.3b, we'd expect a much higher probability of electronic transitions to the PES's with significant orbital overlap than from either to the third, where there is much less overlap.
- **Franck-Condon Principle**<sup>[2]</sup>: The intensity of the transitions is proportional to the product of two harmonic oscillator wave functions —  $\psi_1$  from the one vibrational state and  $\psi_2$  from the other vibrational state.
- In a polyatomic molecule, any bond can be approximated as a spring, and any spring can be approximated as a harmonic oscillator.

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<sup>2</sup>Developed at UChicago!

- Thus, any polyatomic molecule can be thought of as a bunch of harmonic oscillators linked together.
- Normal modes and normal coordinates:
  - Degrees of freedom:  $3N$  ( $x, y, z$  coordinates).
    - There are translational, vibrational, and rotational DOFs.
    - We always have 3 translational DOFs, 2 or 3 rotational DOFs (depending on whether or not the molecule is linear), and  $3N - 5$  or  $3N - 6$  vibrational DOFs (depending on whether or not the molecule is linear once again).
  - Taking the right linear combinations of coupled oscillator stretches gives normal modes that are orthogonal to each other.
    - Diagonalizing gives normal modes that are decoupled from each other.
  - Example ( $\text{H}_2\text{O}$ ):

Figure 9.4: Normal modes of  $\text{H}_2\text{O}$ .

- 3 vibrational modes.
- Symmetric stretch, asymmetric stretch, and bending mode.
- Their respective frequencies are  $\nu_1 = 3650 \text{ cm}^{-1}$ ,  $\nu_2 = 3760 \text{ cm}^{-1}$ , and  $\nu_3 = 1600 \text{ cm}^{-1}$ .

## 9.3 Lasers

- 12/3: • Consider a 2-level system.

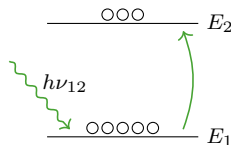


Figure 9.5: Two-level system.

- Hit it with a photon of frequency  $\nu_{12} = (E_2 - E_1)/h$ .
- Population conditions:

$$\frac{dN_1}{dt} + \frac{dN_2}{dt} = 0 \qquad N_1 + N_2 = N_{\text{tot}}$$

- Radiation:
  - Energy density  $\rho$  in units of  $\text{J m}^{-3}$ .
  - Spectral energy density  $\rho_\nu(\nu_{12})$  in units of energy density per unit frequency (i.e.,  $\text{J s m}^{-3}$ ).
- Three types of changes that can occur to our population.



1. Absorption: A photon comes in, gets absorbed by a molecule in the ground state, and gets excited to the excited state.

$$-\frac{dN_1}{dt} = B_{12}\rho_\nu(\nu_{12})N_1(t)$$

- This is very similar to chemical kinetics!
- We have a rate dependent on quantity. Our rate constant is  $B_{12}$  and  $\rho_\nu(\nu_{12})$  is like a photonic reagent interacting with the number of molecules in the state of interest.
- The whole thing is essentially a binary reaction.
- The  $B$  is Einstein's coefficient.

2. Stimulated emission:

$$-\frac{dN_2}{dt} = B_{21}\rho_\nu(\nu_{12})N_2(t)$$

- The photon field can cause the molecule to emit a photon as well.

3. Spontaneous emission:

$$-\frac{dN_2}{dt} = A_{21}N_2(t)$$

- This looks like a unimolecular reaction.

- This yields the overall rate equation

$$-\frac{dN_1}{dt} = \frac{dN_2}{dt} = B_{12}\rho_\nu(\nu_{12})N_1(t) - A_{12}N_2(t) - B_{21}\rho_\nu(\nu_{12})N_2(t)$$

- We now get to lasers.
- Lasers work by **population inversion**.
- Consider a 3-level system.

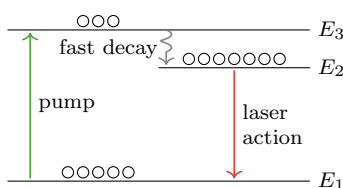


Figure 9.6: Three-level system.

- We first pump molecules from  $E_1$  into  $E_3$  (this is called the **pump phase**).
- Then there's a **fast decay** to the second excited state, followed by a delay before moving into the first state. Thus, you can get a lot of molecules in  $E_2$ .
- The **laser action** between states 2 and 1 occurs in the subsequent relaxation from the second to the first excited state.
- Pump phase:
  - Sending in a photon with frequency  $\nu_{13} = (E_3 - E_1)/h$ .
- Note that we can't get a laser built in only two levels.
  - The same photons that build our population inversion eat away at it through stimulated emission.
  - From the differential equation: Once the levels are equally occupied, the  $B$  terms cancel; only spontaneous emission is left (which will also keep us below an inversion).
    - $dN_2/dt = 0$  when  $N_1(t) = N_2(t)$ .
- As an aside, note that

$$B_{12} = B_{21} = \frac{\pi}{3\hbar^3\epsilon_0} |\mu_{12}|^2$$

- I.e., the rates of absorption depend on the transition dipole moment!
- Example: Ruby laser.
  - We have  $\text{Al}_2\text{O}_3$  with  $\text{Cr}^{2+}$  impurities absorbing radiation at  $\lambda = 5500 \text{ \AA}$  (green is absorbed for red to be emitted; opposite colors on the color wheel).
  - We pump with  $h\nu_{13}$  (green photons) and emit with  $h\nu_{12}$  (ruby photons).
    - $\nu_{12}$  leads to  $\lambda_{12} = 6943 \text{ \AA}$ .
  - The ruby photons, as they're emitted can stimulate emission from more atoms in  $E_2$  in an exponentially increasing (amplified) pattern.
    - This is why the population inversion is key, i.e., to create an avalanche.
  - Lasers are important to create coherent radiation, which allow us to really carefully interrogate our molecules as chemists.
- For a laser pointer, you can set up the photons to all come out in the same direction, but that's more of an engineering question.
- Relating  $A$  and  $B$ .
  - Solve our differential equation for

$$\rho_\nu(\nu_{12}) = \frac{A_{21}}{(N_2/N_1)B_{12} - B_{21}}$$

- Invoking the Boltzmann distribution and setting this equal to the Planck blackbody distribution yields

$$\frac{8\pi h}{c^3} \frac{\nu_{12}^3}{e^{h\nu_{12}/k_B T} - 1} = \frac{A_{21}}{B_{12}e^{h\nu_{12}/k_B T} - B_{21}}$$

- So our first notions of quantum mechanics connect back to our most modern advances. The first line of the story is also the last.

## 9.4 Chapter 13: Molecular Spectroscopy

From McQuarrie and Simon (1997).

- 11/30:
- **Spectroscopy:** The study of the interaction of electromagnetic radiation with atoms and molecules.
  - “Electromagnetic radiation is customarily divided into different energy regions reflecting the different types of molecular processes that can be caused by such radiation” (McQuarrie & Simon, 1997, pp. 495–96).
  - **Vibrational selection rule:** Transitions among vibrational levels resulting from the absorption of radiation have  $\Delta v = \pm 1$  and have a dipole moment that varies during the vibration.
    - For a harmonic oscillator, the spectrum consists of one line in the infrared region at the frequency  $\nu_{\text{obs}} = \sqrt{k/\mu}/2\pi$ .
  - **Vibrational term:** The vibrational energy of a molecule. Denoted by  $G(v)$ . Units  $\text{cm}^{-1}$ . Given by

$$G(v) = \frac{E_v}{hc}$$

where  $E_v = (v + 1/2)h\nu$  and  $\nu = \sqrt{k/\mu}/2\pi$ .

- Each energy  $E_J = \hbar^2/(2I) \cdot J(J + 1)$  of the rigid rotator is associated with degeneracy  $g_J = 2J + 1$ .

- 12/1: • **Rotational term:** The rotational energy of a molecule. Denoted by  $F(J)$ . Units  $\text{cm}^{-1}$ . Given by

$$F(J) = \frac{E_J}{hc}$$

- Rotational selection rule: Transitions among rotational levels resulting from the absorption of radiation have  $\Delta J = \pm 1$  and have a permanent dipole moment.
- The rotational and vibrational energy of a diatomic molecule within the rigid rotator-harmonic oscillator approximation is

$$\tilde{E}_{v,J} = G(v) + F(J) = (v + \frac{1}{2})\tilde{\nu} + \tilde{B}J(J+1)$$

where  $\tilde{\nu} = \sqrt{k/\mu}/2\pi c$ ,  $\tilde{B} = h/8\pi^2 cI$ , and  $v, J = 0, 1, 2, \dots$

- Since  $\tilde{\nu}$  and  $\tilde{B}$  are on the order of  $10^3 \text{ cm}^{-1}$  and  $1 \text{ cm}^{-1}$ , respectively, vibrational energy levels are usually spaced about 100 to 1000 times farther apart than rotational energy levels.
- Rovibrational selection rule:  $\Delta v = \pm 1$  and  $\Delta J = \pm 1$ .
- Observed frequencies for rovibrational transitions.

- $\Delta J = +1$ .

$$\tilde{\nu}_{\text{obs}} = \tilde{E}_{v+1,J+1} - \tilde{E}_{v,J} = \tilde{\nu} + 2\tilde{B}(J+1)$$

- $\Delta J = -1$ .

$$\tilde{\nu}_{\text{obs}} = \tilde{E}_{v+1,J-1} - \tilde{E}_{v,J} = \tilde{\nu} - 2\tilde{B}J$$

- Note that  $J$  is the initial quantum number in both of the above equations.

- **R branch:** The series toward the high-frequency side of the rotational-vibrational spectrum, due to rotational transitions with  $\Delta J = +1$ .
- **P branch:** The series toward the low frequency side of the rotational-vibrational spectrum, due to rotational transitions with  $\Delta J = -1$ .
- **Vibration-rotation interaction:** The decrease in  $\tilde{B}$  as  $v$  increases.
  - Cause: As  $v$  increases,  $R_e$  (the equilibrium bond length) increases (Figure 3.1a). Thus  $\tilde{B} \propto 1/R_e$  decreases.
- Thus, for the  $v = 0 \rightarrow 1$  transition, the frequencies of the  $R$  and  $P$  branches are truly given by

$$\tilde{\nu}_R = E_{1,J+1} - E_{0,J} = \tilde{\nu} + 2\tilde{B}_1 + (3\tilde{B}_1 - \tilde{B}_0)J + (\tilde{B}_1 - \tilde{B}_0)J^2$$

$$\tilde{\nu}_P = E_{1,J-1} - E_{0,J} = \tilde{\nu} - (\tilde{B}_1 + \tilde{B}_0)J + (\tilde{B}_1 - \tilde{B}_0)J^2$$

for  $J = 0, 1, 2, \dots$  and  $J = 1, 2, 3, \dots$ , respectively.

- $\tilde{B}_v$  denotes the value of  $\tilde{B}$  at vibrational energy level  $v$ .
- The above equations reduce to the original ones for  $\tilde{B}_1 = \tilde{B}_0$ .
- Since  $\tilde{B}_1 < \tilde{B}_0$ , “the spacing between the lines in the  $R$  branch decreases and the spacing between the lines in the  $P$  branch increases with increasing  $J$ ” (McQuarrie & Simon, 1997, p. 502).
- The dependence of  $\tilde{B}$  on  $v$  is usually expressed as

$$\tilde{B}_v = \tilde{B}_e - \tilde{\alpha}_e(v + \frac{1}{2})$$

- Rotational lines are not exactly equally spaced since a chemical bond stretches due to the centrifugal force as a molecule rotates more and more energetically.

- This small deviation from the rigid rotator approximation can be treated by perturbation theory, resulting in

$$F(J) = \tilde{B}J(J+1) - \tilde{D}J^2(J+1)^2$$

where  $\tilde{D}$  is the **centrifugal distortion constant**.

- Modified frequencies of the absorption:

$$\tilde{\nu} = F(J+1) - F(J) = 2\tilde{B}(J+1) - 4\tilde{D}(J+1)^2$$

- Rotational energy levels of a nonrigid rotator are spaced closer together than those of a rigid rotator.
- The harmonic-oscillator approximation of a vibrating diatomic predicts only one line in its vibrational spectrum.
  - However, while there is an experimental **fundamental**, there are also **overtones**.
- **Fundamental**: The dominant line in the vibrational spectrum of a diatomic molecule.
- **Overtone**: A line in the vibrational spectrum of a diatomic molecules of weaker intensity than the fundamental, appearing at an almost integral multiple of the fundamental.
- Terms past the quadratic in the Taylor series expansion of the potential energy well  $V(R)$  about the equilibrium bond length  $R_e$  can be accounted for by applying perturbation theory to the harmonic oscillator approximation.
  - Doing so gives rise to the vibrational term

$$G(v) = \tilde{\nu}_e(v + \frac{1}{2}) - \tilde{x}_e\tilde{\nu}_e(v + \frac{1}{2})^2 + \dots$$

for  $v = 0, 1, 2, \dots$ , where  $\tilde{x}_e$  is the **anharmonicity constant**.

- The separation between the energy levels of an anharmonic oscillator decreases with increasing  $v$ .
- “The selection rule for an anharmonic oscillator is that  $\Delta v$  can have any integral value, although the intensities of the  $\Delta v = \pm 2, \pm 3, \dots$  transitions are much less than for the  $\Delta v = \pm 1$  transitions” (McQuarrie & Simon, 1997, p. 506).
- Since most diatomics are in the ground vibrational state at room temperature, the frequencies of the observed  $0 \rightarrow v$  transitions will be

$$\tilde{\nu}_{\text{obs}} = G(v) - G(0) = \tilde{\nu}_e v - \tilde{x}_e\tilde{\nu}_e v(v+1)$$

for  $v = 1, 2, \dots$

- “Just as rotational transitions accompany vibrational transitions, both rotational and vibrational transitions accompany electronic transitions” (McQuarrie & Simon, 1997, p. 507).
- The BO approximation allows us to separate the electronic energy from the vibrational-rotational energy since vibration and rotation, overall, are nuclear-motion phenomena.
- **Vibronic transition**: A vibrational transition in an electronic spectrum.
- Vibronic selection rule:  $\Delta v$  may take on any integral value.
  - Rotational transitions in electronic spectra are not considered because they are so much smaller.
- Vibronic transitions usually originate from the  $v = 0$  vibrational state, yielding predicted frequencies

$$\tilde{\nu}_{\text{obs}} = \tilde{T}_e + (\frac{1}{2}\tilde{\nu}'_e - \frac{1}{4}\tilde{x}'_e\tilde{\nu}'_e) - (\frac{1}{2}\tilde{\nu}''_e - \frac{1}{4}\tilde{x}''_e\tilde{\nu}''_e) + \tilde{\nu}'_e v' - \tilde{x}'_e\tilde{\nu}'_e v'(v'+1)$$

where  $\tilde{T}_e$  is the difference in energies of the minima of the two electronic potential energy curves in wave numbers, and the single and double primes indicate the upper and lower energy states, respectively.

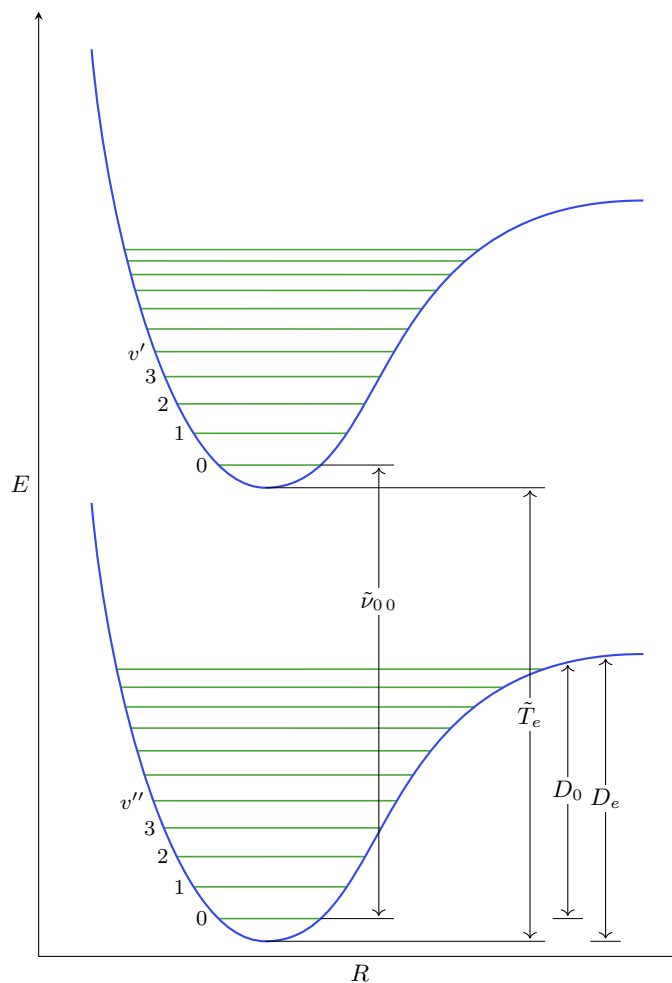


Figure 9.7: The quantities describing an electronic transition.

- $D_e$ : The difference in energy between the minimum of the potential energy curve and the dissociated atoms.
- $D_0$ : The corresponding dissociation energy from the ground-vibrational level.
- We thus have that  $D_e = D_0 + \frac{1}{2}h\nu$  in the harmonic oscillator approximation and  $D_e = D_0 + \frac{1}{2}h(\nu_e - \frac{1}{2}x_e\nu_e)$  in the anharmonic oscillator approximation.
- Since the nuclei do not move appreciably during an electronic transition, such transitions can be depicted as vertical lines in the energy diagram, as we have done thus far.
- McQuarrie and Simon (1997) covers the Franck-Condon principle.
- Consider a rigid body, a polyatomic molecule modeled as a rigid network of  $N$  atoms.
- **Moments of inertia** (of a rigid body): Given a Cartesian coordinate system, the following values. Denoted by  $I_{xx}$ ,  $I_{yy}$ ,  $I_{zz}$ . Given by

$$I_{xx} = \sum_{j=1}^N m_j [(y_j - y_{cm})^2 + (z_j - z_{cm})^2]$$

$$I_{yy} = \sum_{j=1}^N m_j [(x_j - x_{cm})^2 + (z_j - z_{cm})^2]$$

$$I_{zz} = \sum_{j=1}^N m_j [(x_j - x_{cm})^2 + (y_j - y_{cm})^2]$$

where  $m_j$  is the mass of the  $j^{\text{th}}$  atom situated at the point  $(x_j, y_j, z_j)$  and  $(x_{cm}, y_{cm}, z_{cm})$  are the coordinates of the center of mass of the rigid body.

- **Products of inertia** (of a rigid body): The quantities of the form

$$I_{xy} = - \sum_{j=1}^N m_j (x_j - x_{cm})(y_j - y_{cm})$$

- **Principal axes** (of a rigid body): The particular set of Cartesian coordinates  $X, Y, Z$  passing through the center of mass of a rigid body such that all of the products of inertia vanish.
- **Principal moments of inertia** (of a rigid body): The moments of inertia of a rigid body about the principal axes. Denoted by  $I_A, I_B, I_C$  where  $I_A \leq I_B \leq I_C$ .
  - Usually given in terms of rotational constants in units of reciprocal centimeters, e.g.,  $\tilde{A} = h/8\pi^2 c I_A$ .
- **Spherical top**: A rigid body with all three principal moments of inertia equal.
  - For example,  $\text{CH}_4$  and  $\text{SF}_6$ .
- **Symmetric top**: A rigid body with two principal moments of inertia equal.
  - For example,  $\text{NH}_3$  and  $\text{C}_6\text{H}_6$ .
- **Asymmetric top**: A rigid body with all three principal moments of inertia unequal.
  - For example,  $\text{H}_2\text{O}$ .
- The quantum-mechanical problem of a spherical top can be solved exactly, yielding energy levels  $F(J) = \tilde{B}J(J+1)$  for  $J = 0, 1, 2, \dots$  (the same as for a linear molecule) with respective degeneracies  $g_J = (2J+1)^2$ .
- Since spherical tops cannot have a permanent dipole moment, they do not have pure rotational spectra.
- The quantum-mechanical problem of a symmetric top can also be solved exactly.
- **Oblate symmetric top**: A symmetric top with unique moment of inertia larger than the two equal ones.
  - For example,  $\text{C}_6\text{H}_6$  or an O-ring.
  - Energy levels  $F(J, K) = \tilde{B}J(J+1) + (\tilde{C} - \tilde{B})K^2$  for  $J = 0, 1, 2, \dots$ ,  $K = 0, \pm 1, \pm 2, \dots, \pm J$ , and degeneracy  $g_{JK} = 2J+1$ .
    - $J$  is a measure of the total rotational angular momentum of the molecule.
    - $K$  is a measure of the component of the rotational angular momentum along the unique axis of the symmetric top.
- **Prolate symmetric top**: A symmetric top with unique moment of inertia smaller than the two equal ones.
  - For example,  $\text{CH}_3\text{Cl}$  or a cigar.

- Energy levels  $F(J, K) = \tilde{B}J(J+1) + (\tilde{A} - \tilde{B})K^2$  with the same quantum numbers and degeneracy as in the oblate case.
- Symmetric top molecule selection rule: The dipole moment must be directed along the axis of symmetry, and then we may have  $\Delta J = 0, \pm 1$ ,  $\Delta K = 0$  for  $K \neq 0$  and  $\Delta J = \pm 1$ ,  $\Delta K = 0$  for  $K = 0$ .
- Centrifugal distortion effects are larger for larger molecules.
- To understand the vibrational spectra of polyatomic molecules in terms of the harmonic-oscillator approximation, we introduce normal coordinates.
- McQuarrie and Simon (1997) reviews degrees of freedom.
  - A complete specification of a molecule containing  $N$  nuclei in space requires  $3N$  coordinates.
  - However, using some to specify the overall molecules position and orientation in space leaves the rest of the degrees of freedom available to describe vibration.
- “In the absence of external fields, the energy of a molecule does not depend upon the position of its center of mass or its orientation” (McQuarrie & Simon, 1997, p. 519).
- $N_{\text{vib}}$ : The number of vibrational degrees of freedom.
- Thus, the potential energy is solely a function of the  $N_{\text{vib}}$  vibrational coordinates. Letting the displacements about the equilibrium values of these coordinates be  $q_1, \dots, q_{N_{\text{vib}}}$ , we have that

$$\Delta V = V(q_1, \dots, q_{N_{\text{vib}}}) - V(0, \dots, 0) = \frac{1}{2} \sum_{i=1}^{N_{\text{vib}}} \sum_{j=1}^{N_{\text{vib}}} \left( \frac{\partial^2 V}{\partial q_i \partial q_j} \right) q_i q_j + \dots$$

where the  $\dots$  terms are anharmonic.

- The cross terms in the above expression make the solution to the corresponding Schrödinger equation quite tedious, but a theorem of classical mechanics and a straightforward procedure using matrix algebra allows us to find a new set of coordinates  $\{Q_j\}$  called **normal coordinates** or **normal modes** such that

$$\Delta V = \frac{1}{2} \sum_{j=1}^{N_{\text{vib}}} F_j Q_j^2$$

- It follows that the vibrational Schrödinger equation  $\hat{H}_{\text{vib}} \psi_{\text{vib}} = E_{\text{vib}} \psi_{\text{vib}}$  is separable and defined by

$$\begin{aligned} \hat{H}_{\text{vib}} &= \sum_{j=1}^{N_{\text{vib}}} \left( -\frac{\hbar^2}{2\mu_j} \frac{d^2}{dQ_j^2} + \frac{1}{2} F_j Q_j^2 \right) \\ \psi_{\text{vib}}(Q_1, \dots, Q_{N_{\text{vib}}}) &= \psi_{\text{vib},1}(Q_1) \cdots \psi_{\text{vib},N_{\text{vib}}}(Q_{N_{\text{vib}}}) \\ E_{\text{vib}} &= \sum_{j=1}^{N_{\text{vib}}} h\nu_j \left( v_j + \frac{1}{2} \right) \end{aligned}$$

- Thus, the vibrational motion of a polyatomic molecule appears as  $N_{\text{vib}}$  independent harmonic oscillators, each with their own characteristic fundamental frequency  $\nu_j$ .
- Vibrational absorption spectroscopy selection rule: The dipole moment of the molecule must vary during the normal mode.
- **Infrared active** (normal mode): A normal mode such that the dipole moment of the molecule does vary during the prescribed motion.
- **Infrared inactive** (normal mode): A normal mode such that the dipole moment of the molecule does not vary during the prescribed motion.

- McQuarrie and Simon (1997) analyzes the normal modes of  $\text{CO}_2$ ,  $\text{H}_2\text{CO}$ , and  $\text{CH}_3\text{Cl}$ .
- **Parallel band:** An absorption band corresponding to a normal mode with dipole moment oscillating parallel to the molecular axis.
  - Governed by the selection rule  $\Delta v = +1$ ,  $\Delta J = \pm 1$ , just like a diatomic molecules.
  - Generates a vibration-rotation spectrum consisting of a *P* branch and an *R* branch.
- The case of a dipole moment oscillating perpendicular to the molecular axis.
  - Governed by the selection rule  $\Delta v = +1$ ,  $\Delta J = 0, \pm 1$ .
  - The band due to  $\Delta J = 0$  is called the **Q branch**, and is centered between the *P* and *R* branches.
- Identifying normal coordinates with irreducible representations.

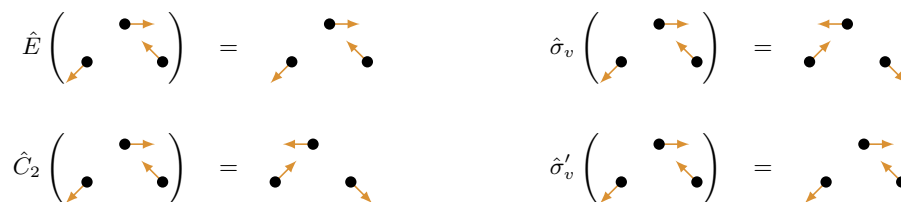


Figure 9.8: The asymmetric stretch normal mode under the operations of the  $\text{C}_{2v}$  point group.

- Consider, for example,  $\text{H}_2\text{O}$ , belonging to the  $\text{C}_{2v}$  point group.
- Let the asymmetric stretch normal coordinate be  $Q_{as}$ .
- Then by Figure 9.8, we have that

$$\hat{E}Q_{as} = Q_{as} \quad \hat{C}_2Q_{as} = -Q_{as} \quad \hat{\sigma}_vQ_{as} = -Q_{as} \quad \hat{\sigma}'_vQ_{as} = Q_{as}$$

so  $Q_{as}$  belongs to  $B_2$ .

- McQuarrie and Simon (1997) also derives the relevant irreducible representations as in Labalme (2021a, p. 40).
- We now begin the derivation of selection rules.
- Since we are considering *transitions*, we will need the time-dependent Schrödinger equation.
  - Stationary states, such as those wave functions we have considered thus far, only pertain to isolated systems with Hamiltonians that do not depend on time.
  - As such, to consider transitions between states, we will be working not just with the time-dependent Schrödinger equation but also with a time-dependent Hamiltonian.
- In particular, let the molecule interact with electromagnetic field

$$\mathbf{E} = \mathbf{E}_0 \cos 2\pi\nu t$$

where  $\nu$  is the frequency of the radiation incident on the molecule and  $\mathbf{E}_0$  is the electric field vector.

- It follows by Problem 13-49 that the Hamiltonian operator for the interaction of the electric field with the molecule is

$$\hat{H}^{(1)} = -\boldsymbol{\mu} \cdot \mathbf{E} = -\boldsymbol{\mu} \cdot \mathbf{E}_0 \cos 2\pi\nu t$$



- Thus, the overall problem is to solve

$$\hat{H}\Psi = i\hbar \frac{d\Psi}{dt}$$

where  $\hat{H} = \hat{H}^{(0)} + \hat{H}^{(1)} = \hat{H}^{(0)} - \boldsymbol{\mu} \cdot \mathbf{E}_0 \cos 2\pi\nu t$  and  $\hat{H}^{(0)}$  is the Hamiltonian of the isolated molecule.

- Treating  $\hat{H}^{(1)}$  as a small perturbation, we can solve the above with **time-dependent perturbation theory**, an extension of time-independent perturbation theory worked through as follows.
  - For simplicity, we consider only a two-state system (in spite of the fact that an isolated molecule generally has an infinite number of stationary states).

- In such a system, the solutions to

$$\hat{H}^{(0)}\Psi = i\hbar \frac{d\Psi}{dt}$$

are

$$\Psi_1(t) = \psi_1 e^{-iE_1 t/\hbar} \quad \Psi_2(t) = \psi_2 e^{-iE_2 t/\hbar}$$

where  $\psi_1, \psi_2$  are the two stationary states.

- Let the system initially be in state 1, and assume that the overall solution  $\Psi(t)$  is a linear combination of  $\Psi_1(t), \Psi_2(t)$  that evolves over time. In particular, assume

$$\Psi(t) = a_1(t)\Psi_1(t) + a_2(t)\Psi_2(t)$$

and let our initial conditions be  $a_1(t) = 1$  and  $a_2(t) = 0$ .

- Recall that  $a_i^* a_i$  gives the probability that the molecule is in state  $i$ .
- Substituting our wave function into the full time-dependent Schrödinger equation gives

$$\begin{aligned} (\hat{H}^{(0)} + \hat{H}^{(1)})\Psi(t) &= i\hbar \frac{d\Psi}{dt} \\ a_1(t)\hat{H}^{(0)}\Psi_1 + a_2(t)\hat{H}^{(0)}\Psi_2 + a_1(t)\hat{H}^{(1)}\Psi_1 + a_2(t)\hat{H}^{(1)}\Psi_2 &= a_1(t)i\hbar \frac{d\Psi_1}{dt} + i\hbar\Psi_1 \frac{da_1}{dt} + a_2(t)i\hbar \frac{d\Psi_2}{dt} + i\hbar\Psi_2 \frac{da_2}{dt} \\ a_1(t)\hat{H}^{(1)}\Psi_1 + a_2(t)\hat{H}^{(1)}\Psi_2 &= i\hbar\Psi_1 \frac{da_1}{dt} + i\hbar\Psi_2 \frac{da_2}{dt} \end{aligned}$$

where we cancel terms that are equal by the TD SE in going from the second to the third equality.

- We now multiply through by  $\psi_2^*$ , integrate over all space, and simplify.

$$\begin{aligned} \psi_2^* a_1(t) \hat{H}^{(1)} \Psi_1 + \psi_2^* a_2(t) \hat{H}^{(1)} \Psi_2 &= \psi_2^* i\hbar \Psi_1 \frac{da_1}{dt} + \psi_2^* i\hbar \Psi_2 \frac{da_2}{dt} \\ a_1(t) \int \psi_2^* \hat{H}^{(1)} \Psi_1 d\tau + a_2(t) \int \psi_2^* \hat{H}^{(1)} \Psi_2 d\tau &= i\hbar \frac{da_1}{dt} \int \psi_2^* \psi_1 e^{-iE_1 t/\hbar} d\tau + i\hbar \frac{da_2}{dt} \int \psi_2^* \psi_2 e^{-iE_2 t/\hbar} d\tau \\ a_1(t) \int \psi_2^* \hat{H}^{(1)} \Psi_1 d\tau + a_2(t) \int \psi_2^* \hat{H}^{(1)} \Psi_2 d\tau &= i\hbar e^{-iE_1 t/\hbar} \frac{da_1}{dt} \int \psi_2^* \psi_1 d\tau + i\hbar e^{-iE_2 t/\hbar} \frac{da_2}{dt} \int \psi_2^* \psi_2 d\tau \\ a_1(t) \int \psi_2^* \hat{H}^{(1)} \Psi_1 d\tau + a_2(t) \int \psi_2^* \hat{H}^{(1)} \Psi_2 d\tau &= i\hbar e^{-iE_2 t/\hbar} \frac{da_2}{dt} \end{aligned}$$

- Solve for  $i\hbar da_2/dt$  and simplify.

$$\begin{aligned} i\hbar \frac{da_2}{dt} &= a_1(t) e^{iE_2 t/\hbar} \int \psi_2^* \hat{H}^{(1)} \psi_1 e^{-iE_1 t/\hbar} d\tau + a_2(t) e^{iE_2 t/\hbar} \int \psi_2^* \hat{H}^{(1)} \psi_2 e^{-iE_2 t/\hbar} d\tau \\ &= a_1(t) e^{-i(E_1 - E_2)t/\hbar} \int \psi_2^* \hat{H}^{(1)} \psi_1 d\tau + a_2(t) \int \psi_2^* \hat{H}^{(1)} \psi_2 d\tau \end{aligned}$$

- Since  $\hat{H}^{(1)}$  is a small perturbation, the initial values corresponding to the TD SE may be approximated as identical to those corresponding to the TI SE. As such, substitute  $a_1(0) = 1$  and  $a_2(0) = 0$  into the above to obtain

$$i\hbar \frac{da_2}{dt} = e^{-i(E_1-E_2)t/\hbar} \int \psi_2^* \hat{H}^{(1)} \psi_1 d\tau$$

- Take the electric field to be in the  $z$ -direction. Then

$$\begin{aligned} \hat{H}^{(1)} &= -\mu_z E_{0z} \cos 2\pi\nu t \\ &= -\frac{\mu_z E_{0z}}{2} (e^{i2\pi\nu t} + e^{-i2\pi\nu t}) \end{aligned}$$

- Substituting this into the above then yields

$$\begin{aligned} i\hbar \frac{da_2}{dt} &= e^{-i(E_1-E_2)t/\hbar} \int \psi_2^* \cdot -\frac{\mu_z E_{0z}}{2} (e^{i2\pi\nu t} + e^{-i2\pi\nu t}) \psi_1 d\tau \\ &= -\frac{E_{0z}}{2} e^{i(E_2-E_1)t/\hbar} (e^{ih\nu t 2\pi/\hbar} + e^{-ih\nu t 2\pi/\hbar}) \int \psi_2^* \mu_z \psi_1 d\tau \\ \frac{da_2}{dt} &= -\frac{1}{2i\hbar} \int \psi_2^* \mu_z \psi_1 d\tau E_{0z} (e^{i(E_2-E_1+h\nu)t/\hbar} + e^{i(E_2-E_1-h\nu)t/\hbar}) \\ &\propto (\mu_z)_{12} E_{0z} (e^{i(E_2-E_1+h\nu)t/\hbar} + e^{i(E_2-E_1-h\nu)t/\hbar}) \end{aligned}$$

where we define

$$(\mu_z)_{12} = \int \psi_2^* \mu_z \psi_1 d\tau$$

to be the  $z$ -component of the **transition dipole moment** between states 1 and 2.

- Note that if  $(\mu_z)_{12} = 0$ , then  $da_2/dt = 0$  and there will be no transition out of state 1 and into state 2.
- In other words, the dipole must change during the transition, and we have derived the first part of the selection rule!
- Before finishing our derivation of explicit selection rules, integrate the above.

$$\begin{aligned} a_2(t) &\propto \int_0^t (\mu_z)_{12} E_{0z} (e^{i(E_2-E_1+h\nu)t/\hbar} + e^{i(E_2-E_1-h\nu)t/\hbar}) dt \\ &\propto (\mu_z)_{12} E_{0z} \left( \frac{1 - e^{i(E_2-E_1+h\nu)t/\hbar}}{E_2 - E_1 + h\nu} + \frac{1 - e^{i(E_2-E_1-h\nu)t/\hbar}}{E_2 - E_1 - h\nu} \right) \end{aligned}$$

- Note that since  $E_2 > E_1$ , when  $E_2 - E_1 \approx h\nu$ , the **resonance denominators** cause the second term above to be of major importance in determining  $a_2(t)$ .
- Thus, the Bohr frequency condition arises naturally from the quantum mechanics!
- **Resonance denominators:** The two denominators in the above equation for  $a_2(t)$ .
- Given the above function, we can now determine the probability of observing the molecule in state 2.
  - This quantity is proportional to both the probability of absorption and the intensity of absorption.
  - It is given by  $a_2^* a_2$ . In calculating this quantity, though, we need only take into account the second term in the definition of  $a_2$  as per the above discussion of resonance denominators.
  - This gives us (see Problem 13-40)

$$a_2^*(t) a_2(t) \propto \frac{\sin^2[(E_2 - E_1 - \hbar\omega)t/2\hbar]}{(E_2 - E_1 - \hbar\omega)^2}$$

which has its largest peak (largest probability of a transition) at  $\hbar\omega = E_2 - E_1$  (see Figure 8.1).

- We now derive the selection rule in the rigid-rotator approximation.
- We require that

$$0 \neq (\mu_z)_{J,M;J',M'} = \int_0^{2\pi} \int_0^\pi Y_{J'}^{M'}(\theta, \phi)^* \mu_z Y_J^M(\theta, \phi) \sin \theta \, d\theta \, d\phi$$

- Invoke  $\mu_z = \mu \cos \theta$ .

$$(\mu_z)_{J,M;J',M'} = \mu \int_0^{2\pi} \int_0^\pi Y_{J'}^{M'}(\theta, \phi)^* Y_J^M(\theta, \phi) \cos \theta \sin \theta \, d\theta \, d\phi$$

- Thus,  $\mu \neq 0$  for a nonzero transition dipole moment.
- In other words, the molecule must have a permanent dipole moment for it to have a pure rotational spectrum, as asserted earlier!
- As to the rest of the selection rule, substitute the spherical harmonics into the above equation and simplify. Also substitute  $x = \cos \theta$ .

$$\begin{aligned} (\mu_z)_{J,M;J',M'} &= \mu \int_0^{2\pi} \int_0^\pi (N_{J'M'} P_{J'}^{|M'|}(\cos \theta) e^{-iM'\phi}) (N_{JM} P_J^{|M|}(\cos \theta) e^{iM\phi}) \cos \theta \sin \theta \, d\theta \, d\phi \\ &= \mu N_{JM} N_{J'M'} \int_0^{2\pi} d\phi e^{i(M-M')\phi} \int_0^\pi d\theta P_{J'}^{|M'|}(\cos \theta) P_J^{|M|}(\cos \theta) \cos \theta \sin \theta \\ &= \mu N_{JM} N_{J'M'} \int_0^{2\pi} d\phi e^{i(M-M')\phi} \int_{-1}^1 dx x P_{J'}^{|M'|}(x) P_J^{|M|}(x) \end{aligned}$$

- For the leftmost integral above to be nonzero, we must have  $M = M'$ .
- This is the  $\Delta M = 0$  selection rule!
- Evaluating the leftmost integral yields

$$(\mu_z)_{J,M;J',M'} = 2\pi \mu N_{JM} N_{J'M'} \int_{-1}^1 dx x P_{J'}^{|M'|}(x) P_J^{|M|}(x)$$

- Given the recursion rule

$$(2J+1)xP_J^{|M|}(x) = (J-|M|+1)P_{J+1}^{|M|}(x) + (J+|M|)P_{J-1}^{|M|}(x)$$

pertaining to the associated Legendre functions, we have

$$(\mu_z)_{J,M;J',M} = 2\pi \mu N_{JM} N_{J'M} \int_{-1}^1 dx P_{J'}^{|M|}(x) \left[ \frac{J-|M|+1}{2J+1} P_{J+1}^{|M|}(x) + \frac{J+|M|}{2J+1} P_{J-1}^{|M|}(x) \right]$$

- Thus, by the orthogonality of the associated Legendre functions, we must have  $J' = J+1$  or  $J' = J-1$  for the above integral not to vanish.
- This is the  $\Delta J = \pm 1$  selection rule!
- We now derive the selection rule in the harmonic-oscillator approximation.
- As before, let

$$(\mu_z)_{v,v'} = \int_{-\infty}^{\infty} [N_{v'} H_{v'}(\sqrt{\alpha}q) e^{-\alpha q^2/2}] \mu_z(q) [N_v H_v(\sqrt{\alpha}q) e^{-\alpha q^2/2}] dq$$

- Expand  $\mu_z(q)$  about the equilibrium nuclear separation to two terms

$$\mu_z(q) = \mu_0 + \left( \frac{d\mu}{dq} \right)_0 q$$

where  $\mu_0$  is the dipole moment at the equilibrium bond length and  $q$  is the displacement from that equilibrium value.

- Substituting this expansion into the original equation yields

$$\begin{aligned} (\mu_z)_{v,v'} &= \int_{-\infty}^{\infty} [N_{v'} H_{v'}(\sqrt{\alpha}q) e^{-\alpha q^2/2}] \left[ \mu_0 + \left( \frac{d\mu}{dq} \right)_0 q \right] [N_v H_v(\sqrt{\alpha}q) e^{-\alpha q^2/2}] dq \\ &= N_v N_{v'} \mu_0 \int_{-\infty}^{\infty} H_{v'}(\sqrt{\alpha}q) H_v(\sqrt{\alpha}q) e^{-\alpha q^2} dq + N_v N_{v'} \left( \frac{d\mu}{dq} \right)_0 \int_{-\infty}^{\infty} H_{v'}(\sqrt{\alpha}q) q H_v(\sqrt{\alpha}q) e^{-\alpha q^2} dq \end{aligned}$$

- The first integral above vanishes if  $v \neq v'$  by the orthogonality of the Hermite polynomials with respect to a Gaussian weighting function.
- As to the second integral, invoke the Hermite polynomial recursion formula

$$\xi H_v(\xi) = v H_{v-1}(\xi) + \frac{1}{2} H_{v+1}(\xi)$$

from Problem 5-24 to get

$$(\mu_z)_{v,v'} = \frac{N_v N_{v'}}{\alpha} \left( \frac{d\mu}{dq} \right)_0 \int_{-\infty}^{\infty} H_{v'}(\xi) [v H_{v-1}(\xi) + \frac{1}{2} H_{v+1}(\xi)] e^{-\xi^2} d\xi$$

- Note that we substitute  $\xi = \sqrt{\alpha}q$ .
- Thus, by the orthogonality of the Hermite polynomials, we must have  $v' = v \pm 1$  for the above integral not to vanish.
- This is the  $\Delta v = \pm 1$  selection rule!
- Additionally, the  $(d\mu/dq)_0$  term indicates that the dipole moment must vary during the vibration.
- A normal mode will have a nonzero transition only if it belongs to the same irreducible representation as one of the  $\mu_{x,y,z}$ <sup>[3]</sup>.

- We have

$$\psi_0(Q_1, \dots, Q_{N_{\text{vib}}}) = c e^{-\alpha_1 Q_1^2 - \dots - \alpha_{N_{\text{vib}}} Q_{N_{\text{vib}}}^2}$$

where  $c$  is a normalization constant and  $\alpha_j = \sqrt{\mu_j k_j} / 2\hbar$ .

- Since the normal modes belong to the irreducible representations of the relevant molecular point group, the effect of an arbitrary symmetry operation  $\hat{R}$  on  $Q_j$  gives  $\pm Q_j$ .

- Thus,  $\psi_0$  as a function of quadratic terms of  $Q_j$  is wholly invariant under any  $\hat{R}$ .

- We also have

$$\psi_1(Q_1, \dots, Q_{N_{\text{vib}}}) = c' Q_j e^{-\alpha_1 Q_1^2 - \dots - \alpha_{N_{\text{vib}}} Q_{N_{\text{vib}}}^2}$$

since an excited state necessitates exciting one and only one normal mode.

- Thus,  $\psi_1$  transforms as  $Q_j$  under the symmetry operations of the group.

- Since  $I_{0 \rightarrow 1}$  is invariant under all operations of the group,

$$I_{0 \rightarrow 1} = \hat{R} I_{0 \rightarrow 1} = \int (\hat{R} \psi_0)(\hat{R} \mu_x)(\hat{R} \psi_1) dQ_1 \cdots dQ_{N_{\text{vib}}} = \chi_{A_1}(\hat{R}) \chi_{\mu_x}(\hat{R}) \chi_{Q_j}(\hat{R}) I_{0,1}$$

- Thus, the product of the characters equals 1. More specifically, since  $\chi_{A_1}(\hat{R})$  always equals 1, the excited state and dipole must transform together, i.e., must belong to the same irreducible representation for the transition to be nonzero.

<sup>3</sup>Related to Module 12 on Labalme (2021a, pp. 49–50).

## Problems

**13-40.** Derive Equation 13.57 from Equation 13.55 in McQuarrie and Simon (1997).

*Answer.* Since we are considering proportionality and ignoring the first term, we will work with the following version of Equation 13.55.

$$a_2(t) \propto \frac{1 - e^{i(E_2 - E_1 - h\nu)t/\hbar}}{E_2 - E_1 - h\nu}$$

Let

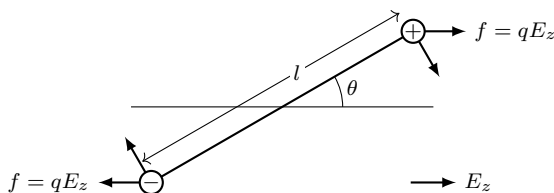
$$x = E_2 - E_1 - h\nu = E_2 - E_1 - \frac{h}{2\pi} \cdot 2\pi\nu = E_2 - E_1 - \hbar\omega$$

Then

$$\begin{aligned} a_2^*(t)a_2(t) &= \frac{1 - e^{-ixt/\hbar}}{x} \cdot \frac{1 - e^{ixt/\hbar}}{x} \\ &= \frac{1 - e^{ixt/\hbar} - e^{-ixt/\hbar} + e^0}{x^2} \\ &= \frac{2 - (e^{ixt/\hbar} + e^{-ixt/\hbar})}{x^2} \\ &= \frac{2 - 2\cos(xt/\hbar)}{x^2} \\ &= \frac{4}{x^2} \cdot \frac{1 - \cos(2(xt/2\hbar))}{2} \\ &= \frac{4}{x^2} \cdot \sin^2(xt/2\hbar) \\ &\propto \frac{\sin^2(xt/2\hbar)}{x^2} \end{aligned}$$

Returning the substitution yields the desired result.  $\square$

**13-49.** Consider a molecule with a dipole moment  $\mu$  in an electric field  $\mathbf{E}$ . We picture the dipole moment as a positive charge and a negative charge of magnitude  $q$  separated by a vector  $\mathbf{l}$ .



The field  $\mathbf{E}$  causes the dipole to rotate into a direction parallel to  $\mathbf{E}$ . Therefore, work is required to rotate the dipole to an angle  $\theta$  to  $\mathbf{E}$ . The force causing the molecule to rotate is actually a torque (torque is the angular analog of force) and is given by  $l/2$  times the force perpendicular to  $\mathbf{l}$  at each end of the vector  $\mathbf{l}$ . Show that this torque is equal to  $\mu E \sin \theta$  and that the energy required to rotate the dipole from some initial angle  $\theta_0$  to some arbitrary angle  $\theta$  is

$$V = \int_{\theta_0}^{\theta} \mu E \sin \theta' d\theta'$$

Given that  $\theta_0$  is customarily taken to be  $\pi/2$ , show that

$$V = -\mu E \cos \theta = -\boldsymbol{\mu} \cdot \mathbf{E}$$

The magnetic analog of this result will be given in Chapter 14.

## 9.5 Chapter 14: Nuclear Magnetic Resonance Spectroscopy

From McQuarrie and Simon (1997).

- 1/3:
- The intrinsic spin of the electron, as a charged particle, makes it behave like a magnetic dipole when placed in a magnetic field.
  - The intrinsic spin angular momentum of a nucleus makes it behave similarly.
    - We denote the intrinsic spin angular momentum of a nucleus by  $\mathbf{I}$ .
    - The nuclear spin eigenvalue equations, as follows, are analogous to those for electrons.

$$\begin{aligned}\hat{I}^2\alpha &= \frac{1}{2}(\frac{1}{2} + 1)\hbar^2\alpha & \hat{I}^2\beta &= \frac{1}{2}(\frac{1}{2} + 1)\hbar^2\beta \\ \hat{I}_z\alpha &= \frac{1}{2}\hbar\alpha & \hat{I}_z\beta &= -\frac{1}{2}\hbar\beta\end{aligned}$$

■ Analogous orthonormality conditions are also satisfied.

- Nuclear spins are not limited to  $\pm 1/2$  (see Table 9.2).
- Why a charged particle with nonzero spin acts like a magnetic dipole.
  - The motion of an electric charge around a closed loop generates a magnetic dipole  $\boldsymbol{\mu}$ , with magnitude

$$\mu = iA$$

where  $i = qv/2\pi r$  is the current generated by a charge of magnitude  $q$  moving at velocity  $v$  around the circumference of a circle of radius  $r$ , and  $A = \pi r^2$  is the area of the circular loop.

- Making the above substitutions yields  $\mu = qrv/2$  in the circular loop case, or

$$\boldsymbol{\mu} = \frac{q(\mathbf{r} \times \mathbf{v})}{2}$$

if the orbit is not circular.

- Noting that  $\mathbf{L} = \mathbf{r} \times \mathbf{p}$  and  $\mathbf{p} = m\mathbf{v}$ , we can express  $\boldsymbol{\mu}$  in terms of the angular momentum, as follows.

$$\boldsymbol{\mu} = \frac{q}{2m}\mathbf{L}$$

- From here, we need only replace the classical angular momentum  $\mathbf{L}$  with the spin angular momentum  $\mathbf{I}$ .

$$\boldsymbol{\mu} = g_N \frac{q}{2m_N} \mathbf{I} = g_N \beta_N \mathbf{I} = \gamma \mathbf{I}$$

- **Nuclear g factor:** A unitless constant whose magnitude is on the order of unity and is characteristic of each nucleus. Denoted by  $g_N$ .
- **Nuclear magneton:** The following quantity. Denoted by  $\beta_N$ . Given by

$$\beta_N = \frac{q}{2m_N}$$

- **Magnetogyric ratio:** A characteristic quantity for each nucleus upon which the detection sensitivity of a particular type of nucleus in an NMR experiment depends. Units  $\text{rad T}^{-1} \text{s}^{-1}$ . Denoted by  $\lambda$ . Given by

$$\lambda = g_N \beta_N$$

- The larger the value, the easier it is to observe the nucleus.

1/5:

Nucleus	Spin	Nuclear g factor	Magnetic moment (in nuclear magnetons)	Magnetogyric ratio $\gamma/10^7 \text{ rad T}^{-1} \text{ s}^{-1}$
$^1\text{H}$	1/2	5.5854	2.7928	26.7522
$^2\text{H}$	1	0.8574	0.8574	4.1066
$^{13}\text{C}$	1/2	1.4042	0.7021	6.7283
$^{14}\text{N}$	1	0.4036	0.4036	1.9338
$^{31}\text{P}$	1/2	2.2610	1.1305	10.841

Table 9.2: Properties of nuclei commonly studied in NMR experiments.

- The SI unit of magnetic field strength is the tesla.

– However, we commonly use the gauss:

$$1 \text{ G} = 10^{-4} \text{ T}$$

- The interaction of a quantum magnetic moment  $\boldsymbol{\mu}$  with a magnetic field  $\mathbf{B}$ .

– Let the magnetic field  $\mathbf{B}$  be oriented in the  $z$ -direction, so that

$$V = -\mu_z B_z$$

(from Problem 13.49 in the magnetic field case).

– Expressing the magnetic moment in terms of the spin angular momentum yields

$$V = -\gamma B_z I_z$$

– As an expression for the energy of a magnetic moment in terms of an observable, the above tells us that the Hamiltonian for this quantum magnetic moment is

$$\hat{H} = -\gamma B_z \hat{I}_z$$

– Thus, we may solve the Schrödinger equation describing this system as follows.

$$\hat{H}\psi = E\psi$$

$$-\gamma B_z \hat{I}_z \psi = E\psi$$

$$-\gamma B_z \hbar m_I \psi_I = E\psi_I$$

where  $\psi_I$  represents the spin eigenfunctions and  $m_I = I, I-1, \dots, -I$ .

– This yields

$$E = -\hbar\gamma m_I B_z$$

as the equation describing the allowed energies of the system.

- We can use the above result to calculate the difference in energy between a proton aligned with a magnetic field ( $m_I = +1/2$ ) and one aligned against it ( $m_I = -1/2$ ).

$$\Delta E = [-\hbar\gamma \cdot -\frac{1}{2} \cdot B_z] - [-\hbar\gamma \cdot \frac{1}{2} \cdot B_z] = \hbar\gamma B_z$$

- Note that the above result shows that  $\Delta E$  depends linearly on magnetic field strength.
- If a proton in an applied magnetic field is hit by a photon of energy  $\Delta E$ , it will transition from the lower energy state to the higher energy state.
  - For a field of 2.11 T, the photon must be in the radiofrequency region.

- The frequency associated with a transition of one assigned nuclear spin state to another for a spin  $+1/2$  nucleus is

$$\begin{aligned}h\nu &= \hbar\gamma B_z \\ 2\pi\nu &= \gamma B_z \\ \nu &= \frac{\gamma B_z}{2\pi}\end{aligned}$$

- 1/6:
- **Resonance frequency** (of a proton): The frequency at which a spin-state transition will occur.
  - The linear dependence of resonance frequency on magnetic field strength implies that to record a spectrum, we can either fix magnetic field strength and vary the frequency of the applied radiation, or fix the frequency and vary the magnetic field strength.
    - We choose to do the latter.
    - We also present spectra with the strength of the magnetic field increasing from left to right.
  - Setting the frequency higher and using more powerful electromagnets gives greater resolution.
    - Proton NMR spectrometers operate at frequencies of 60 – 750 MHz.
  - Since hydrogen nuclei are most commonly surrounded by electrons, we often have to take shielding into account.
    - Applying a magnetic field to a hydrogen nucleus in a molecule causes the electrons surrounding the nucleus to move in a circular motion, generating a magnetic field that (in most substances) opposes the external magnetic field.
    - Since it turns out that the magnitude of the electrically generated magnetic field  $B_{\text{elec}}$  is proportional to the applied field  $B_0$ , we can write

$$B_{\text{elec}} = -\sigma B_0$$

where  $\sigma$  is the **shielding constant** and the negative sign accounts for the opposing directions (if  $B_0$  is oriented in the positive  $z$ -direction,  $B_{\text{elec}}$  will be oriented in the negative  $z$ -direction).

- **Shielding constant:** The unitless proportionality constant relating the magnitudes of the magnetic field applied to a molecule and the electrically generated shielding magnetic field. *Denoted by  $\sigma$ .*
    - On the order of  $10^{-5}$ .
- 1/7:
- Depends on the local electrochemical environment of the hydrogen nuclei, i.e., different hydrogen nuclei within the same molecule can experience different local fields.
  - If the total magnetic field experienced by a hydrogen nucleus is  $B_z = (1 - \sigma)B_0$ , then the field strength (at fixed frequency  $\nu$ ) at which the nucleus will undergo a spin transition is

$$B_0 = \frac{2\pi\nu}{\gamma(1 - \sigma)}$$

- Characteristics of NMR spectra.
  - Spectra have a peak at zero corresponding to a small amount of added tetramethylsilane (TMS;  $\text{Si}(\text{CH}_3)_4$ ) used for calibration purposes.
    - TMS is ideal because it has 12 equivalent hydrogen atoms (thus yields only one peak), is relatively unreactive, and most organic compounds absorb **downfield** from it.
  - Spectra have two scales.
    - The top one is in hertz and runs from 0 Hz on the right to some higher number of hertz (perhaps around 500 Hz) on the left side.



■ The bottom one is derived from the top as follows.

- **Downfield** (of a spike): At smaller fields; to the left of said spike on the spectrum.
- **Chemical shift**: A spectrometer-independent measure of the frequency required to cause a nuclear spin-state transition relative to TMS. *Units ppm. Denoted by  $\delta_{\text{H}}$ . Given by*

$$\begin{aligned}\delta_{\text{H}} &= \frac{\text{resonance frequency of H nucleus relative to TMS}}{\text{spectrometer frequency}} \times 10^6 \\ &= \frac{\nu_{\text{H}} - \nu_{\text{TMS}}}{\nu_{\text{spectrometer}}} \times 10^6\end{aligned}$$

- As we have established above, the field strength  $B_0$  needed for a nucleus to absorb radiation at a fixed frequency  $\nu$  is directly proportional to that frequency.
  - For example, it will take 1.5 times stronger magnetic field for a nucleus to absorb the same amount of radiation on a 90 MHz-spectrometer as a 60 MHz-spectrometer.
  - This change means that peaks separated by  $x$  hertz on a 60 MHz-spectrometer will be separated by  $1.5x$  hertz on a 90 MHz-spectrometer.
  - Note that the unit of measurement “hertz” is proportional to and interconvertible with joules of energy and teslas of magnetic field strength.
- Mathematically, we can also see that  $\delta_{\text{H}}$  is independent of  $\nu_{\text{spectrometer}}$ .

$$\begin{aligned}\delta_{\text{H}} &= \frac{\nu_{\text{H}} - \nu_{\text{TMS}}}{\nu_{\text{spectrometer}}} \times 10^6 \\ &= \frac{\frac{\gamma B_0}{2\pi}(1 - \sigma_{\text{H}}) - \frac{\gamma B_0}{2\pi}(1 - \sigma_{\text{TMS}})}{\nu_{\text{spectrometer}}} \times 10^6 \\ &= \frac{1}{\nu_{\text{spectrometer}}} \cdot \frac{\gamma B_0}{2\pi}(\sigma_{\text{TMS}} - \sigma_{\text{H}}) \times 10^6 \\ &\approx \frac{2\pi}{\gamma B_0} \cdot \frac{\gamma B_0}{2\pi}(\sigma_{\text{TMS}} - \sigma_{\text{H}}) \times 10^6 \\ &= (\sigma_{\text{TMS}} - \sigma_{\text{H}}) \times 10^6\end{aligned}$$

- 1/8:
- The degree of shielding increases with increasing electron density around the nucleus.
    - The greater the electron density, the more **upfield** the spike will be.
    - Hence, hydrogens with many EDGs nearby (such as alkanes) will have small chemical shifts, and hydrogens with many EWGs nearby (such as haloalkanes) will have large chemical shifts.
  - **Upfield** (of a spike): At smaller fields; to the right of said spike on the spectrum.
  - McQuarrie and Simon (1997) list typical chemical shifts for hydrogen nuclei in various electrochemical environments.
  - The relative areas of peaks in an NMR spectrum indicate the number of equivalent hydrogen atoms in each set.
  - **Split** (signal): A signal in an NMR spectrum corresponding to a single set of hydrogen atoms that appears as two or more closely spaced spikes. *Also known as **multiplet**.*
  - **Doublet**: A split signal with two spikes.
  - **Triplet**: A split signal with three spikes.
  - Split signals occur because in addition to the external magnetic field and the electrically generated magnetic field, neighboring hydrogen nuclei on adjacent carbon atoms act like tiny bar magnets in their own right.

- **Spin-spin interaction:** The interaction between the nuclear spins of neighboring hydrogen nuclei on adjacent carbon atoms.
- Consider a molecule with two hydrogen atoms in different electronic environments.
  - The spin Hamiltonian describing this molecule is

$$\hat{H} = -\gamma B_0(1 - \sigma_1)\hat{I}_{z1} - \gamma B_0(1 - \sigma_2)\hat{I}_{z2} + \frac{hJ_{12}}{\hbar^2}\hat{\mathbf{I}}_1 \cdot \hat{\mathbf{I}}_2$$

- This Hamiltonian includes one term for each hydrogen nucleus independently, and one interaction term derived from the classical expression for the interaction between two magnetic dipole moments.
  - $J_{12}$  is the **spin-spin coupling constant**.
- We now apply first-order perturbation theory to

$$\hat{H}^{(0)} = -\gamma B_0(1 - \sigma_1)\hat{I}_{z1} - \gamma B_0(1 - \sigma_2)\hat{I}_{z2} \qquad \hat{H}^{(1)} = \frac{hJ_{12}}{\hbar^2}\hat{\mathbf{I}}_1 \cdot \hat{\mathbf{I}}_2$$

- The unperturbed wave functions are

$$\begin{aligned} \psi_1 &= \alpha(1)\alpha(2) & \psi_2 &= \beta(1)\alpha(2) \\ \psi_3 &= \alpha(1)\beta(2) & \psi_4 &= \beta(1)\beta(2) \end{aligned}$$

- The energy corresponding to  $\psi_j$  is

$$E_j = E_j^{(0)} + \int d\tau_1 d\tau_2 \psi_j^* \hat{H}^{(1)} \psi_j$$

where  $\tau_1, \tau_2$  are spin variables (to avoid confusion with the shielding constant, we avoid using the conventional  $\sigma_1, \sigma_2$ ).

- We can determine  $E_1^{(0)}$  as follows.

$$\begin{aligned} \hat{H}^{(0)}\psi_1 &= \hat{H}^{(0)}\alpha(1)\alpha(2) \\ &= -\gamma B_0(1 - \sigma_1)\hat{I}_{z1}\alpha(1)\alpha(2) - \gamma B_0(1 - \sigma_2)\hat{I}_{z2}\alpha(1)\alpha(2) \\ &= -\frac{\hbar\gamma B_0(1 - \sigma_1)}{2}\alpha(1)\alpha(2) - \frac{\hbar\gamma B_0(1 - \sigma_2)}{2}\alpha(1)\alpha(2) \\ &= -\hbar\gamma B_0\left(\frac{1 - \sigma_1}{2} + \frac{1 - \sigma_2}{2}\right)\alpha(1)\alpha(2) \\ &= -\hbar\gamma B_0\left(1 - \frac{\sigma_1 + \sigma_2}{2}\right)\alpha(1)\alpha(2) \\ &= E_1^{(0)}\alpha(1)\alpha(2) \\ &= E_1^{(0)}\psi_1 \end{aligned}$$

- The other  $E_j^{(0)}$  are determined analogously to yield

$$\begin{aligned} E_1^{(0)} &= -\hbar\gamma B_0\left(1 - \frac{\sigma_1 + \sigma_2}{2}\right) & E_2^{(0)} &= -\frac{\hbar\gamma B_0}{2}(\sigma_1 - \sigma_2) \\ E_3^{(0)} &= \frac{\hbar\gamma B_0}{2}(\sigma_1 - \sigma_2) & E_4^{(0)} &= \hbar\gamma B_0\left(1 - \frac{\sigma_1 + \sigma_2}{2}\right) \end{aligned}$$

- As to the first-order corrections, we must evaluate integrals of the form

$$H_{ii} = \frac{hJ_{12}}{\hbar^2} \int d\tau_1 d\tau_2 \psi_i^* \hat{\mathbf{I}}_1 \cdot \hat{\mathbf{I}}_2 \psi_i$$

- Since we know that

$$\hat{\mathbf{I}}_1 \cdot \hat{\mathbf{I}}_2 = \hat{I}_{x1}\hat{I}_{x2} + \hat{I}_{y1}\hat{I}_{y2} + \hat{I}_{z1}\hat{I}_{z2}$$

we will have to evaluate three such integrals for each wavefunction.

- Evaluating the integrals with  $\hat{I}_{z1}\hat{I}_{z2}$  is fairly easy. For example,

$$\begin{aligned} H_{z,11} &= \frac{hJ_{12}}{\hbar^2} \int d\tau_1 d\tau_2 \psi_1^* \hat{I}_{z1} \hat{I}_{z2} \psi_1 \\ &= \frac{hJ_{12}}{\hbar^2} \int d\tau_1 d\tau_2 \alpha^*(1) \alpha^*(2) \hat{I}_{z1} \hat{I}_{z2} \alpha(1) \alpha(2) \\ &= \frac{hJ_{12}}{\hbar^2} \int d\tau_1 d\tau_2 \alpha^*(1) \alpha^*(2) [\hat{I}_{z1} \alpha(1)] [\hat{I}_{z2} \alpha(2)] \\ &= \frac{hJ_{12}}{\hbar^2} \int d\tau_1 d\tau_2 \alpha^*(1) \alpha^*(2) \left[\frac{\hbar}{2} \alpha(1)\right] \left[\frac{\hbar}{2} \alpha(2)\right] \\ &= \frac{hJ_{12}}{4} \int d\tau_1 d\tau_2 \alpha^*(1) \alpha^*(2) \alpha(1) \alpha(2) \\ &= \frac{hJ_{12}}{4} \int d\tau_1 \alpha^*(1) \alpha(1) \int d\tau_2 \alpha^*(2) \alpha(2) \\ &= \frac{hJ_{12}}{4} \end{aligned}$$

1/11:

- Evaluating the other three yields

$$H_{z,22} = H_{z,33} = -\frac{hJ_{12}}{4} \qquad H_{z,44} = \frac{hJ_{12}}{4}$$

- Evaluating the integrals with  $\hat{I}_{x1}\hat{I}_{x2}$  and  $\hat{I}_{y1}\hat{I}_{y2}$  is much more complicated: Problems 14-18 through 14-21 conclude that

$$\begin{aligned} \hat{I}_x \alpha &= \frac{\hbar}{2} \beta & \hat{I}_y \alpha &= \frac{i\hbar}{2} \beta \\ \hat{I}_x \beta &= \frac{\hbar}{2} \alpha & \hat{I}_y \beta &= -\frac{i\hbar}{2} \alpha \end{aligned}$$

from which it follows that the  $x, y$  terms of  $\hat{\mathbf{I}}_1 \cdot \hat{\mathbf{I}}_2$  do not contribute to any of the first-order energies.

- It follows that the energy of each level to first-order is

$$\begin{aligned} E_1 &= -h\nu_0 \left(1 - \frac{\sigma_1 + \sigma_2}{2}\right) + \frac{hJ_{12}}{4} & E_2 &= -\frac{h\nu_0}{2}(\sigma_1 - \sigma_2) - \frac{hJ_{12}}{4} \\ E_3 &= \frac{h\nu_0}{2}(\sigma_1 - \sigma_2) - \frac{hJ_{12}}{4} & E_4 &= h\nu_0 \left(1 - \frac{\sigma_1 + \sigma_2}{2}\right) + \frac{hJ_{12}}{4} \end{aligned}$$

where  $\nu_0 = \gamma B_0 / 2\pi$ .

- Nuclear spin selection rule: Only one type of nucleus at a time can undergo a transition.
- Thus, the only allowed transitions are

$$\begin{aligned} \alpha(1)\alpha(2) &\longrightarrow \beta(1)\alpha(2) & (\psi_1 \rightarrow \psi_2) \\ \alpha(1)\alpha(2) &\longrightarrow \alpha(1)\beta(2) & (\psi_1 \rightarrow \psi_3) \\ \beta(1)\alpha(2) &\longrightarrow \beta(1)\beta(2) & (\psi_2 \rightarrow \psi_4) \\ \alpha(1)\beta(2) &\longrightarrow \beta(1)\beta(2) & (\psi_3 \rightarrow \psi_4) \end{aligned}$$

and the corresponding frequencies are

$$\begin{aligned}\nu_{1 \rightarrow 2} &= \nu_0(1 - \sigma_1) - \frac{J_{12}}{2} \\ \nu_{1 \rightarrow 3} &= \nu_0(1 - \sigma_2) - \frac{J_{12}}{2} \\ \nu_{2 \rightarrow 4} &= \nu_0(1 - \sigma_2) + \frac{J_{12}}{2} \\ \nu_{3 \rightarrow 4} &= \nu_0(1 - \sigma_1) + \frac{J_{12}}{2}\end{aligned}$$

- Since, by hypothesis,  $J_{12}$  is sufficiently small for first-order perturbation theory to be applicable, we know that the four resonance frequencies appear as a pair of doublets.

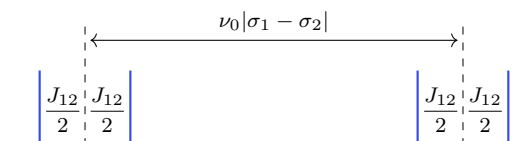


Figure 9.9: Split peak distances.

- **AX system:** A molecule in which the two hydrogen nuclei are in very different chemical environments, i.e.,  $\nu_0|\sigma_1 - \sigma_2| \gg J_{12}$ .
  - The notation arises because every nonequivalent hydrogen is assigned a letter, and nonequivalent hydrogens whose chemical shifts differ by a relatively large amount are assigned letters that are relatively far apart in the alphabet (and vice versa for those with similar chemical shifts).
  - Furthermore, if there is more than one hydrogen of a given type, we add a subscript to the letter (for example, 1,1,2-trichloroethane is a  $A_2X$  system).
- Note that since  $J_{12}$  is independent of  $\nu_0$  (the frequency of the spectrometer), doublet spacing does not change between spectra from different spectrometers.
- **First-order spectrum:** A spectrum for which  $J_{12} \ll \nu_0|\sigma_1 - \sigma_2|$ , i.e., for which first-order perturbation theory is applicable.
- While spin-spin coupling between nonequivalent hydrogens splits signals, no spin-spin coupling is observed between equivalent hydrogens.
- Consider an  $A_2$  system (such as dichloromethane).
  - The spin Hamiltonian describing this molecule is identical to the last one except that the shielding constants are now equal.

$$\hat{H} = \underbrace{-\gamma B_0(1 - \sigma_A)\hat{I}_{z1} - \gamma B_0(1 - \sigma_A)\hat{I}_{z2}}_{\hat{H}^{(0)}} + \underbrace{\frac{hJ_{AA}}{\hbar^2}\hat{\mathbf{I}}_1 \cdot \hat{\mathbf{I}}_2}_{\hat{H}^{(1)}}$$

- As before, we apply first-order perturbation theory.
- However, we now arrive at a difference from last time, specifically concerning the unperturbed wavefunctions.
  - Indeed, because the two nuclei are equivalent (i.e., indistinguishable), we must use combinations of  $\alpha$  and  $\beta$  that are either symmetric or antisymmetric.

- The four acceptable ones are

$$\begin{aligned}\phi_1 &= \alpha(1)\alpha(2) & \phi_2 &= \frac{1}{\sqrt{2}}[\alpha(1)\beta(2) - \beta(1)\alpha(2)] \\ \phi_3 &= \frac{1}{\sqrt{2}}[\alpha(1)\beta(2) + \beta(1)\alpha(2)] & \phi_4 &= \beta(1)\beta(2)\end{aligned}$$

- We will now calculate  $E_1$  as an example (the procedure for the others is symmetric).

- First off, we have that

$$\begin{aligned}E_1 &= E_1^{(0)} + E_1^{(1)} \\ &= \iint d\tau_1 d\tau_2 \alpha^*(1)\alpha^*(2) \left[ -\gamma B_0(1 - \sigma_A)(\hat{I}_{z1} + \hat{I}_{z2}) \right] \alpha(1)\alpha(2) \\ &\quad + \iint d\tau_1 d\tau_2 \alpha^*(1)\alpha^*(2) \frac{hJ_{AA}}{\hbar^2} (\hat{I}_{x1}\hat{I}_{x2} + \hat{I}_{y1}\hat{I}_{y2} + \hat{I}_{z1}\hat{I}_{z2}) \alpha(1)\alpha(2)\end{aligned}$$

- The first integral is readily evaluated by noting that

$$(\hat{I}_{z1} + \hat{I}_{z2})\alpha(1)\alpha(2) = \left(\frac{\hbar}{2} + \frac{\hbar}{2}\right)\alpha(1)\alpha(2) = \hbar\alpha(1)\alpha(2)$$

- The second integral is readily evaluated by noting that

$$\begin{aligned}(\hat{I}_{x1}\hat{I}_{x2} + \hat{I}_{y1}\hat{I}_{y2} + \hat{I}_{z1}\hat{I}_{z2})\alpha(1)\alpha(2) &= \frac{\hbar^2}{4}\beta(1)\beta(2) - \frac{\hbar^2}{4}\beta(1)\beta(2) + \frac{\hbar^2}{4}\alpha(1)\alpha(2) \\ &= \frac{\hbar^2}{4}\alpha(1)\alpha(2)\end{aligned}$$

- Therefore,

$$\begin{aligned}E_1 &= -\hbar\gamma B_0(1 - \sigma_A) \iint d\tau_1 d\tau_2 \alpha^*(1)\alpha^*(2)\alpha(1)\alpha(2) \\ &\quad + \frac{hJ_{AA}}{\hbar^2} \frac{\hbar^2}{4} \iint d\tau_1 d\tau_2 \alpha^*(1)\alpha^*(2)\alpha(1)\alpha(2) \\ &= -\hbar\gamma B_0(1 - \sigma_A) + \frac{hJ_{AA}}{4}\end{aligned}$$

- The energy of the remaining levels to first-order is

$$E_2 = -\frac{3hJ_{AA}}{4} \quad E_3 = \frac{hJ_{AA}}{4} \quad E_4 = \hbar\gamma B_0(1 - \sigma_A) + \frac{hJ_{AA}}{4}$$

- Nuclear spin symmetry selection rule: Only transitions between states of the same spin symmetry are allowed.
- Based on the two selection rules, the only allowed transitions are  $1 \rightarrow 3$  and  $3 \rightarrow 4$ , with corresponding frequencies

$$\nu_{1 \rightarrow 3} = \nu_{3 \rightarrow 4} = \nu_0(1 - \sigma_A)$$

- “Therefore, although the spin-spin coupling between equivalent protons alters the energy levels, the selection rules are such that the spin-spin coupling constant effect cancels in the transition frequencies, so only a single proton resonance is observed” (McQuarrie & Simon, 1997, p. 573).
- **$n + 1$  rule:** If a proton has  $n$  equivalent neighboring protons, then its NMR signal will be split into  $n + 1$  closely spaced peaks.
  - Rationalized qualitatively by considering the possible spin states of the neighboring nuclei. Relative intensities of the multiplets are discussed, too.

- When  $J$  and  $\nu_0|\sigma_1 - \sigma_2|$  are closer in magnitude, we can no longer use perturbation theory; instead, we must resort to a variational calculation.
- Example: Variational calculation of a molecule with two nonequivalent hydrogen atoms.
  - We use as our trial wavefunction

$$\Psi = c_1\psi_1 + c_2\psi_2 + c_3\psi_3 + c_4\psi_4$$

- $\Psi$  has the interesting property that minimizing its variational parameters will result in an *exact* solution. This is because it represents *all* possible two-proton spin functions, hence is as general as possible, allowing a variational calculation to yield an exact result.
  - To minimize  $\Psi$ , employ the secular determinant

$$\begin{vmatrix} H_{11} - E & H_{12} & H_{13} & H_{14} \\ H_{12} & H_{22} - E & H_{23} & H_{24} \\ H_{13} & H_{23} & H_{33} - E & H_{34} \\ H_{14} & H_{24} & H_{34} & H_{44} - E \end{vmatrix} = 0$$

where  $H_{ij} = \iint d\tau_1 d\tau_2 \psi_i^* \hat{H} \psi_j$ .

- Evaluating the matrix elements significantly simplifies the secular determinant.

$$\begin{vmatrix} -d_1 - d_2 + \frac{hJ}{4} - E & 0 & 0 & 0 \\ 0 & -d_1 + d_2 - \frac{hJ}{4} - E & \frac{hJ}{2} & 0 \\ 0 & \frac{hJ}{2} & d_1 - d_2 - \frac{hJ}{4} - E & 0 \\ 0 & 0 & 0 & d_1 + d_2 + \frac{hJ}{4} - E \end{vmatrix} = 0$$

- Note that in the following,  $d_1 = \frac{1}{2}h\nu_0(1 - \sigma_1)$ ,  $d_2 = \frac{1}{2}h\nu_0(1 - \sigma_2)$ , and we drop the 12 subscript on  $J$  for convenience.
  - Expanding yields the following energy levels.

$$\begin{aligned} E_1 &= -h\nu_0 \left( 1 - \frac{\sigma_1 + \sigma_2}{2} \right) + \frac{hJ}{4} & E_2 &= -\frac{hJ}{4} - \frac{h}{2} \sqrt{\nu_0^2(\sigma_1 - \sigma_2)^2 + J^2} \\ E_3 &= -\frac{hJ}{4} + \frac{h}{2} \sqrt{\nu_0^2(\sigma_1 - \sigma_2)^2 + J^2} & E_4 &= h\nu_0 \left( 1 - \frac{\sigma_1 + \sigma_2}{2} \right) + \frac{hJ}{4} \end{aligned}$$

- It follows that the frequencies of the allowed transitions are

$$\begin{aligned} \nu_{1 \rightarrow 2} &= \frac{\nu_0}{2}(2 - \sigma_1 - \sigma_2) - \frac{J}{2} - \frac{1}{2} \sqrt{\nu_0^2(\sigma_1 - \sigma_2)^2 + J^2} \\ \nu_{1 \rightarrow 3} &= \frac{\nu_0}{2}(2 - \sigma_1 - \sigma_2) - \frac{J}{2} + \frac{1}{2} \sqrt{\nu_0^2(\sigma_1 - \sigma_2)^2 + J^2} \\ \nu_{2 \rightarrow 4} &= \frac{\nu_0}{2}(2 - \sigma_1 - \sigma_2) + \frac{J}{2} + \frac{1}{2} \sqrt{\nu_0^2(\sigma_1 - \sigma_2)^2 + J^2} \\ \nu_{3 \rightarrow 4} &= \frac{\nu_0}{2}(2 - \sigma_1 - \sigma_2) + \frac{J}{2} - \frac{1}{2} \sqrt{\nu_0^2(\sigma_1 - \sigma_2)^2 + J^2} \end{aligned}$$

- The relative intensities of the signals are, respectively,

$$\frac{(r-1)^2}{(r+1)^2} : 1 : \frac{(r-1)^2}{(r+1)^2} : 1$$

where

$$r = \sqrt{\frac{\sqrt{\Delta^2 + J^2} + \Delta}{\sqrt{\Delta^2 + J^2} - \Delta}} \quad \Delta = \nu_0(\sigma_1 - \sigma_2)$$

- Thus, the relative intensities of the peaks depend upon the relative values of  $\nu_0|\sigma_1 - \sigma_2|$  and  $J$ .
- Special cases:
  - $J = 0$ : Two separate singlets, as per two distinct but uncoupled hydrogen atoms.
  - $\sigma_1 = \sigma_2$ : One singlet, as per two chemically equivalent hydrogen atoms.
- **Second-order spectrum:** A spectrum in between the two extremes of  $J \ll \nu_0|\sigma_1 - \sigma_2|$  and  $J = 0$  /  $\sigma_1 = \sigma_2$ .
  - The  $n + 1$  rule does not apply here.
- Note that in the case where  $J \ll \nu_0|\sigma_1 - \sigma_2|$ , the second-order frequencies approach their respective first-order limits.
- Resolution increases with spectrometer frequency because as  $\nu_0$  (and hence  $\nu_0|\sigma_1 - \sigma_2|$ ) increase, higher-order spectra converge to the first-order limit.

## Problems

- 14-18.** The nuclear spin operators  $\hat{I}_x, \hat{I}_y, \hat{I}_z$ , like all angular momentum operators, obey the commutation relations

$$[\hat{I}_x, \hat{I}_y] = i\hbar\hat{I}_z \quad [\hat{I}_y, \hat{I}_z] = i\hbar\hat{I}_x \quad [\hat{I}_z, \hat{I}_x] = i\hbar\hat{I}_y$$

(see Problem 6-13). Define the (non-Hermitian) operators

$$\hat{I}_+ = \hat{I}_x + i\hat{I}_y \quad \hat{I}_- = \hat{I}_x - i\hat{I}_y \quad (1)$$

and show that

$$\hat{I}_z\hat{I}_+ = \hat{I}_+\hat{I}_z + \hbar\hat{I}_+ \quad (2)$$

and

$$\hat{I}_z\hat{I}_- = \hat{I}_-\hat{I}_z - \hbar\hat{I}_- \quad (3)$$

- 14-19.** Using the definitions of  $\hat{I}_+$  and  $\hat{I}_-$  from the previous problem, verify the following two equations where  $\hat{I}^2 = \hat{I}_x^2 + \hat{I}_y^2 + \hat{I}_z^2$ .

$$\hat{I}_+\hat{I}_- = \hat{I}^2 - \hat{I}_z^2 + \hbar\hat{I}_z \quad \hat{I}_-\hat{I}_+ = \hat{I}^2 - \hat{I}_z^2 - \hbar\hat{I}_z$$

- 14-20.** Use Equation 2 from Problem 14-18 and the fact that  $\hat{I}_z\beta = -\frac{\hbar}{2}\beta$  to show that

$$\hat{I}_z\hat{I}_+\beta = \hat{I}_+\left(-\frac{\hbar}{2}\beta + \hbar\beta\right) = \frac{\hbar}{2}\hat{I}_+\beta$$

Because  $\hat{I}_z\alpha = \frac{\hbar}{2}\alpha$ , this result shows that

$$\hat{I}_+\beta \propto \alpha = c\alpha$$

where  $c$  is a proportionality constant. Problem 14-21 shows that  $c = \hbar$ , so we have

$$\hat{I}_+\beta = \hbar\alpha \quad (1)$$

Now use Equation 3 from Problem 14-18 and the fact that  $\hat{I}_z\alpha = \frac{\hbar}{2}\alpha$  to show that

$$\hat{I}_-\alpha = c\beta$$

where  $c$  is a proportionality constant. Problem 14-21 shows that  $c = \hbar$ , so we have

$$\hat{I}_-\alpha = \hbar\beta \quad (2)$$

Notice that  $\hat{I}_+$  “raises” the spin function from  $\beta$  to  $\alpha$ , whereas  $\hat{I}_-$  “lowers” the spin function from  $\alpha$  to  $\beta$ . The two operators  $\hat{I}_+$  and  $\hat{I}_-$  are called the **raising operator** and **lowering operator**, respectively.

Now argue that a consequence of the raising and lowering properties of  $\hat{I}_+$  and  $\hat{I}_-$  is that

$$\hat{I}_+\alpha = 0 \qquad \hat{I}_-\beta = 0 \qquad (3)$$

Now use Equations 1-3 to show that

$$\begin{aligned} \hat{I}_x\alpha &= \frac{\hbar}{2}\beta & \hat{I}_y\alpha &= \frac{i\hbar}{2}\beta \\ \hat{I}_x\beta &= \frac{\hbar}{2}\alpha & \hat{I}_y\beta &= -\frac{i\hbar}{2}\alpha \end{aligned}$$

**14-21.** This problem shows that the proportionality constant  $c$  in

$$\hat{I}_+\beta = c\alpha \qquad \hat{I}_-\alpha = c\beta$$

is equal to  $\hbar$ . Start with

$$\int \alpha^* \alpha \, d\tau = 1 = \frac{1}{c^2} \int (\hat{I}_+\beta)^* (\hat{I}_+\beta) \, d\tau$$

Let  $\hat{I}_+ = \hat{I}_x + i\hat{I}_y$  in the second factor in the above integral and use the fact that  $\hat{I}_x$  and  $\hat{I}_y$  are Hermitian to get

$$\int (\hat{I}_x\hat{I}_+\beta)^* \beta \, d\tau + i \int (\hat{I}_y\hat{I}_+\beta)^* \beta \, d\tau = c^2$$

Now take the complex conjugate of both sides to get

$$\begin{aligned} \int \beta^* \hat{I}_x \hat{I}_+ \beta \, d\tau - i \int \beta^* \hat{I}_y \hat{I}_+ \beta \, d\tau &= c^2 \\ &= \int \beta^* \hat{I}_- \hat{I}_+ \beta \, d\tau \end{aligned}$$

Now use the result in Problem 14-19 to show that

$$\begin{aligned} c^2 &= \int \beta^* \hat{I}_- \hat{I}_+ \beta \, d\tau \\ &= \int \beta^* (\hat{I}^2 - \hat{I}_z^2 - \hbar \hat{I}_z) \beta \, d\tau \\ &= \int \beta^* \left( \frac{3}{4}\hbar^2 - \frac{1}{4}\hbar^2 + \frac{\hbar^2}{2} \right) \beta \, d\tau \\ &= \hbar^2 \end{aligned}$$

or that  $c = \hbar$ .

## 9.6 Chapter 15: Lasers, Laser Spectroscopy, and Photochemistry

From McQuarrie and Simon (1997).

- 12/2:
- “Laser is an acronym for light amplification by stimulated emission of radiation” (McQuarrie & Simon, 1997, p. 591).
  - **Photochemistry:** The study of light-initiated reactions.
  - **Photodissociation:** Light-induced dissociation.
  - **Radiative transition:** A transition between energy levels that involves either the absorption or the emission of radiation.



- **Nonradiative transition:** A transition between energy levels that occurs without the absorption or the emission of radiation.
- **Vibrational relaxation:** A collision between an excited molecule and another molecule in the sample resulting in an exchange of energy that removes some of the excess vibrational energy.
  - Causes an excited molecule to quickly relax to the lowest vibrational state of the electronic excited state.
- **Fluorescence:** The radiative decay process involving a transition between states of the same spin multiplicity.
- **Internal conversion:** The nonradiative decay process involving a transition between states of the same spin multiplicity.
- **Intersystem crossing:** A nonradiative transition between states of the same spin multiplicity enabled by overlap between vibrational and rotational states.
  - Requires a change in the spin of the electron.
- **Phosphorescence:** The radiative decay process involving a transition between states of differing spin multiplicities.
- Einstein's approach to the dynamics of atomic spectroscopic transitions.
  - Einstein makes several assumptions that can be justified via time-dependent quantum mechanics.
  - No quantum mechanics is required for Einstein's approach except that the energy levels of the atom are assumed to be quantized.
- The approach (absorption only).
  - Consider a sample of  $N_{\text{total}}$  identical atoms having only two nondegenerate electronic energy levels  $E_1 < E_2$ .
  - Let number of atoms in stated  $i$  be denoted by  $N_i$ .
  - Let the sample of atoms have insufficient thermal energy to occupy state 2, i.e.,  $k_B T \ll E_2 - E_1$ .
  - Expose the sample to EM radiation of frequency  $\nu_{12}$  where  $h\nu_{12} = E_2 - E_1$ .
  - The energy density of said EM radiation is described by the **radiant energy density** and the **spectral radiant energy density**.
  - Einstein proposes that the rate  $-dN_1/dt = dN_2/dt$  of excitation from the ground electronic state to the excited electronic state is proportional to  $\rho_\nu(\nu_{12})$  and  $N_1(t)$ . In particular,

$$-\frac{dN_1}{dt} = B_{12}\rho_\nu(\nu_{12})N_1(t)$$

where  $B_{12}$  is a proportionality constant called an **Einstein coefficient**.

- **Radiant energy density:** The radiant energy per unit volume. *Denoted by  $\rho$ . Units  $\text{J m}^{-3}$ .*
  - The intensity  $I$  of light is of the form  $I = \rho c$ .
- **Spectral radiant energy density:** The radiant energy density per unit frequency. *Denoted by  $\rho_\nu$ . Units  $\text{J m}^{-3} \text{s}$ . Given by*

$$\rho_\nu = \frac{d\rho}{d\nu}$$

- The approach (relaxation).
  - Einstein proposes **spontaneous emission** and **stimulated emission** as two pathways by which atoms atoms relax back into the ground state.

- **Spontaneous emission:** The process by which atoms spontaneously emit a photon of energy  $h\nu_{12} = E_2 - E_1$  at some time after the excitation.

- The rate can be described by  $-dN_2/dt$ , which we assume is simply proportional to the number of atoms  $N_2(t)$  in the excited state at time  $t$ . In particular,

$$-\frac{dN_2}{dt} = A_{21}N_2(t)$$

where  $A_{21}$  is another Einstein coefficient.

- **Stimulated emission:** The process by which exposure of an atom in an excited electronic state to electromagnetic radiation of energy  $h\nu_{12} = E_2 - E_1$  could stimulate the emission of a photon and thereby regenerate the ground state.

- The rate, we assume, is proportional to  $N_2(t)$  and  $\rho_\nu(\nu_{12})$ . In particular,

$$-\frac{dN_2}{dt} = B_{21}\rho_\nu(\nu_{12})N_2(t)$$

where  $B_{21}$  is a third Einstein coefficient.

- In this process, one photon at frequency  $\nu_{12}$  stimulates an atom to emit another. These two photons can then go on to cause more excited atoms to release photons, and on and on resulting in a substantial amplification of an incident light beam at frequency  $\nu_{12}$ .

- Lasers (light amplification by stimulated emission of radiation devices) exploit this phenomenon.

- Since a sample of atoms exposed to light undergoes absorption, spontaneous emission, and stimulated emission, the true rate of change in the population is

$$-\frac{dN_1}{dt} = \frac{dN_2}{dt} = B_{12}\rho_\nu(\nu_{12})N_1(t) - A_{21}N_2(t) - B_{21}\rho_\nu(\nu_{12})N_2(t)$$

- Deriving a relation between the three Einstein coefficients.

- Consider the limit at which the two energy states are in thermal equilibrium, i.e.,  $-dN_1/dt = dN_2/dt = 0$ . Also let  $\rho_\nu(\nu_{12})$  be the equilibrium spectral radiant energy density, which we assume comes from a thermal blackbody radiation source.

- Solving the above overall population rate of change equation for  $\rho_\nu(\nu_{12})$  yields

$$\rho_\nu(\nu_{12}) = \frac{A_{21}}{(N_1/N_2)B_{12} - B_{21}}$$

- Applying the fact (see Chapter 17) that for a system in equilibrium at temperature  $T$ , the number of atoms or molecules in the state  $j$  with energy  $E_j$  is  $N_j = ce^{-E_j/k_B T}$ , we have that

$$\frac{N_2}{N_1} = e^{-(E_2-E_1)/k_B T} = e^{-h\nu_{12}/k_B T}$$

- Plug this into the above equation and set it equal to Planck's blackbody distribution law.

$$\frac{8\pi h}{c^3} \frac{\nu_{12}^3}{e^{h\nu_{12}/k_B T} - 1} = \frac{A_{21}}{B_{12}e^{h\nu_{12}/k_B T} - B_{21}}$$

- From the above equation it is clear that

$$B_{12} = B_{21}$$

and

$$A_{21} = \frac{8h\pi\nu_{12}^3}{c^3} B_{21}$$

- This allows us to write the rate equation for a nondegenerate two-level system as

$$-\frac{dN_1}{dt} = \frac{dN_2}{dt} = B\rho_\nu(\nu_{12})[N_1(t) - N_2(t)] - AN_2(t)$$

- For a laser to amplify light, a photon that passes through the sample must have a greater probability of stimulating emission from an excited atom than of being absorbed by an atom in its ground state.

- Symbolically, we must have

$$\text{rate stimulated emission} > \text{rate absorption}$$

$$B_{21}\rho_\nu(\nu_{12})N_2 > B_{12}\rho_\nu(\nu_{12})N_1$$

$$N_2 > N_1$$

- **Population inversion:** A situation where the population of the excited state is greater than that of the lower state.

- By the above equation for  $N_2/N_1$ , we must have  $N_2/N_1 < 1$  when the system is at equilibrium since  $h\nu_{12}/k_B T$  is a strictly positive quantity.
- Thus, population inversions constitute nonequilibrium situations.

- A population inversion cannot occur in a nondegenerate two-level system.

- Integrating the rate equation for a such a system yields

$$N_2(t) = \frac{B\rho_\nu(\nu_{12})N_{\text{total}}}{A + 2B\rho_\nu(\nu_{12})} \{1 - e^{-(A+2B\rho_\nu(\nu_{12}))t}\}$$

- This equation's behavior as  $t \rightarrow \infty$  converges to

$$\frac{N_2}{N_{\text{total}}} = \frac{B\rho_\nu(\nu_{12})}{A + 2B\rho_\nu(\nu_{12})}$$

which, since  $A > 0$ , will always be strictly less than 1/2.

- Therefore, no such population inversion can be achieved.

- Relaxing back to equilibrium after an incident light source is turned off.

- The rate equation simplifies to just spontaneous emission, i.e.,

$$\begin{aligned} \frac{dN_2}{dt} &= -AN_2 \\ N_2(t) &= N_2(0)e^{-At} \end{aligned}$$

- **Fluorescence lifetime:** The reciprocal of  $A$ . Also known as **radiative lifetime**. Denoted by  $\tau_R$ .

12/8:

- We now consider a three-level system, which can achieve a population inversion.
- **Pump source:** A light beam that is used to create excited state populations.
- Although we can derive rate equations for every energy level of the three-level system, only the rate equation of the second, intermediate level is needed to give a useful result.

$$\frac{dN_2}{dt} = A_{32}N_3 - A_{21}N_2 + \rho_\nu(\nu_{32})B_{32}N_3 - \rho_\nu(\nu_{32})B_{32}N_2$$

- At equilibrium (e.g.,  $dN_2/dt = 0$ ), the above equation can be rearranged to the form

$$\frac{N_3}{N_2} = \frac{A_{21} + B_{32}\rho_\nu(\nu_{32})}{A_{32} + B_{32}\rho_\nu(\nu_{32})}$$

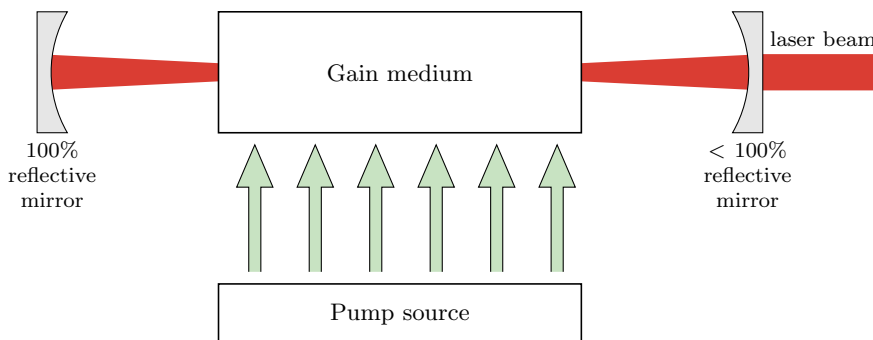
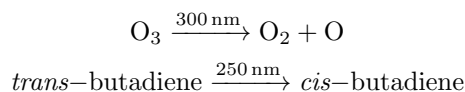


Figure 9.10: The insides of a laser.

- This implies that  $N_3 > N_2$  if  $A_{21} > A_{32}$ , i.e., if it takes a while for atoms in state 3 to decay to state 2, but atoms in state 2 decay quickly to state 1.
- **Gain medium:** A system of atoms that has a population inversion between states 2 and 3.
- The insides of a laser.
- **Laser cavity:** The area containing the components in Figure 9.10.
- Gain medium.
  - A solid-state material, liquid solution, or a gas mixture.
  - The first laser used a solid synthetically grown ruby rod.
    - Ruby is a crystal of corundum ( $\text{Al}_2\text{O}_3$ ) where some  $\text{Al}^{3+}$  ions are replaced by  $\text{Cr}^{3+}$  ions.
    - The ruby must be synthetic since naturally occurring crystals contain natural strains and defects. We use  $\text{Al}_2\text{O}_3$  doped at about 0.05 mass percent with  $\text{Cr}^{3+}$ .
    - The photophysical properties of the **active ions** ( $\text{Cr}^{3+}$ ) in the **host material** ( $\text{Al}_2\text{O}_3$ ) are suitable for achieving a population inversion.
  - Depending on the gain media, laser output can be a continuous light beam or a short burst of light.
  - McQuarrie and Simon (1997) lists data on various solid-state and gas phase gain media.
  - Gas phase lasers produce light in the ultraviolet, visible, and infrared spectra. Some are even capable of producing multiple wavelengths.
  - Since the stimulated-emission process requires that the phases of the incident light wave and stimulated light wave have the same phase, lasers emit **coherent** light.
- **Coherent** (light): Light the constituent waves of which are all in phase.
- **Radiant power:** Radiant energy per unit time. *Units* W.
- **Average radiant power:** Total energy emitted per second. *Units* W.
- Pump source.
  - Gain media are pumped either via **optical excitation** or **electrical excitation**.
- **Optical excitation:** Using a high-intensity light source such as a continuous lamp, flash lamp, or laser to excite the gain medium.
  - Since only light with the right frequency is absorbed, lasers tend to be inefficient devices.
  - Gas phase lasers convert only 0.001 – 0.1 % of input energy to laser light.

- Solid state lasers convert a few percent.
- The CO<sub>2</sub> laser and some semiconductor lasers convert 50 – 70 %.
- This is why some lasers use other lasers as pump sources.
- **Electrical excitation:** Using intense electrical discharges to excite the gain medium.
  - Commonly used in gas lasers.
- Laser cavity design.
  - To sufficiently amplify the light, the light must be directed back and forth through the gain medium a number of times using a **resonator**.
  - The less than 100% reflectivity of one of the mirrors allows some light to escape.
- **Resonator:** An optical cavity including a pair of mirrors that direct light back and forth through the gain medium.
- He-Ne laser.
  - The first continuous-wave laser.
  - Can be made to produce light at a number of wavelengths, but most are red or infrared.
  - Very common (supermarket scanners, range finders, etc.).
  - Uses 1.0 torr of helium and 0.1 torr neon in a glass gas chamber as a gain medium with a high current DC pump source as electrical excitation.
  - The constant current delivered leads to continuous lasing.
  - How it works:
    - Electrons collide with helium atoms, generating a number of excited states but notably the long lived  $2s^3S_1$  and  $2s^1S_0$  excited states, with respective lifetimes  $10^{-4}$  s and  $5 \times 10^{-6}$  s.
    - On average, within  $10^{-7}$  s, the excited helium atom will collide with a neon atom, initiating a nonradiative energy transfer to neon's fortuitously near equivalent energy levels.
 
$$\text{He}^*(2s^3S_1) + \text{Ne}_{(g)} \longrightarrow \text{He}_{(g)} + \text{Ne}^*(2p^5 4s)$$

$$\text{He}^*(2s^1S_0) + \text{Ne}_{(g)} \longrightarrow \text{He}_{(g)} + \text{Ne}^*(2p^5 5s)$$
    - The lifetime of the excited neon states are such that a population inversion can be set up and maintained.
- **Spectral resolution:** The range over which a spectrophotometer cannot distinguish a difference in wavenumbers.
- **Hyperfine interaction:** Small changes in the energies of the rotational state that are caused by the interaction of the electron spins with the nuclear spins.
  - It is possible to include such interactions in the Hamiltonian operator and predict the spacings.
- Time-resolved laser spectroscopy can be used to study the dynamics of chemical reactions.
- **Photochemical reaction:** A chemical reaction that results from the absorption of light.
- Examples:



- The top reaction above is **photodissociation**, while the bottom one is **photoisomerization**.

- **Quantum yield** (of a photochemical reaction): The number of molecules that undergo a reaction per photon absorbed. *Denoted by  $\Phi$ .*
- Time-resolved laser apparatuses split short pulses into two time-separated pulses.
- **Pump pulse**: The laser pulse that initiates the photochemistry.
- **Probe pulse**: The laser pulse that is used to record changes in the sample since the pump pulse arrived.
- McQuarrie and Simon (1997) does a worked example with ICN.
- **Laser-induced fluorescence**: A form of detection that relies on using a laser to cause the product molecules to fluoresce.

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