

## Week 2

# Partition Functions and Ideal Gases

## 2.1 System Partition Functions

1/19:

- Decomposing the partition function of a molecule into the product of separate sums as partitioned by degrees of freedom (e.g., translation, rotation, vibration, and electronic).
- The partition functions of **independent**, distinguishable/indistinguishable molecules.
  - We should not double count the same states.
  - The  $N!$  in  $Q = q^N/N!$  is not important when calculating energy (because of the properties of the natural log), but it is very important when calculating quantities such as entropy.
- **Independent** (particles): A set of particles that do not interact with one another.
- Discusses bosons and fermions.
  - We can have a two fermions in the state  $|1, 1\rangle$  because it is a symmetric state.
- Recall the Fermi level, the boundary between the filled and unfilled electronic states in a solid.
  - If  $T$  is small, this level is a hard boundary.
  - If  $T$  is large, electrons can easily be excited and the Fermi level is a soft boundary.
- Does the 3D particle in a box derivation for the translation molecular partition function.
  - Note that since the de Broglie wavelength  $\lambda_{\text{DB}} = \sqrt{h^2/2mk_{\text{B}}T}$ , we may write

$$q_x = \sum_{n_x} e^{-h^2/8mk_{\text{B}}TL_x^2} = \sum_{n_x} e^{-\lambda_{\text{DB}}^2 n_x^2/4L_x^2}$$

- The number of states are occupied/have energy within  $k_{\text{B}}T$  of the ground state.
  - $-\lambda_{\text{DB}}^2 n_x^2/4L_x^2$  is on the order of 1, implying that  $n_x$  is on the order of  $2L/\lambda_{\text{DB}}$ .
  - It follows if  $L$  is on a macroscopic scale (e.g.,  $L \approx 1$  m) and  $\lambda_{\text{DB}}$  is on a sub-angstrom scale that  $n_x$  is on the order of  $10^{10}$ . When  $n_x$  is at such a scale,  $e^{-\lambda_{\text{DB}}^2 n_x^2/4L_x^2} \approx 1/e$ .
  - It follows that in a  $1 \text{ m}^3$  box, we will have about  $10^{30}$  states, so we really are in a regime where the number of states is larger than the number of molecules.
- More precisely, we want

$$N \ll n_x n_y n_z = \left( \frac{8mk_{\text{B}}T}{h^2} \right)^{3/2} L_x L_y L_z$$

where the middle term approximates the number of states so that

$$\frac{N}{V} \ll \left( \frac{8mk_{\text{B}}T}{h^2} \right)^{3/2}$$

- Approximating the translational energy with an integral.
  - Concludes with the translational partition function.
  - Since we can approach this problem from a classical perspective (as we did last Friday) or quantum mechanically (as we did today) to achieve the same result, this system again demonstrates the relation between quantum and classical mechanics.

## 2.2 Molecular Partition Functions

- 1/21: • We approximate the total molecular energy as

$$q = q_{\text{elec}} q_{\text{trans}} q_{\text{vib}} q_{\text{rot}}$$

- The heat capacity in the very high temperature limit where translations, rotations, and vibrations are classical.
  - Translational:  $\frac{3}{2}k_B$ .
  - Vibrational: Each degree of freedom ( $3N - 5$  for a linear molecule and  $3N - 6$  for a nonlinear molecule) contributes  $k_B$ .
  - Rotational: Each degree of freedom (2 for a linear molecule and 3 for a nonlinear molecule) contributes  $\frac{1}{2}k_B$ .
- We can use the above to calculate the heat capacity of various molecules at very high temperatures (note, however, that at such temperatures, molecules would likely dissociate; we're simply theoretically considering the classical limit here).
  - Ne:  $\frac{3}{2}k_B$ .
  - H<sub>2</sub>O:  $\frac{3}{2}k_B + 3 \cdot k_B + 3 \cdot \frac{1}{2}k_B = 6k_B$ .
  - O<sub>2</sub>:  $\frac{3}{2} + 1 \cdot k_B + 2 \cdot \frac{1}{2}k_B = \frac{7}{2}k_B$ .
  - CO<sub>2</sub>:  $\frac{3}{2}k_B + 4 \cdot k_B + 2 \cdot \frac{1}{2}k_B = \frac{13}{2}k_B$ .
  - CHCl<sub>3</sub>:  $\frac{3}{2}k_B + 9 \cdot k_B + 3 \cdot \frac{1}{2}k_B = 12k_B$ .
- Electronic partition function.
  - Consider the bottom  $D_e$  of the potential well of a diatomic.
  - $D_0$  is the ionization energy from the bottom state ( $D_e \neq D_0$ , but relations can be obtained via spectroscopy).
  - It follows that
 
$$q_{\text{elec}} = g_1 e^{-(D_e/k_B T)} + g_2 e^{-E_2/k_B T}$$
  - If  $dT \ll (E_2 + D_e)$ , then  $q_{\text{elec}} = g_1 e^{D_e/k_B T}$ .
- Vibrational partition function.
  - As before with the law of Dulong and Petit.
  - It's a special point where  $T = h\nu/k_B$ .
- Rotational partition function.
  - Almost always classical.
  - The rotational energy of a polyatomic molecule will almost always be  $\frac{3}{2}k_B$ .
  - Let's look at a heteronuclear diatomic, such as CO. Derives

$$q_{\text{rot}} = \sum_{J=0}^{\infty} (2J+1) e^{-\hbar^2 J(J+1)/2Ik_B T}$$

- The **rotational temperature** leads to

$$q_{\text{rot}} = \sum_{J=0}^{\infty} (2J+1) e^{\Theta_{\text{rot}}/T} = \frac{T}{\Theta_{\text{rot}}}$$

- Thus, at the temperature at which we exist, rotation is equivalent classically to quantum mechanically.

- **Rotational temperature:** The following quantity. Denoted by  $\Theta_{\text{rot}}$ . Given by

$$\Theta_{\text{rot}} = \hbar^2 / 2Ik_{\text{B}}$$

- PGS will not specify whether we need a quantum vs. classical model.

- Homonuclear diatomic (e.g.,  $\text{H}_2$ ).

- The vibrational differences in energy become visible with spectroscopy.

- $q_{\text{rot}} = T/2\Theta_{\text{rot}}$ .

- Partition functions:

- If the molecule is linear, it's of the form  $T/\Theta_{\text{rot}}$ .

- If the molecule is nonlinear, it's of the form  $T/2\Theta_{\text{rot}}$ .

- Spherical top (e.g.,  $\text{CH}_4$ ):

$$\frac{\sqrt{\pi}}{\sigma} \left( \frac{T}{\Theta_{\text{rot}}} \right)^{3/2}$$

- Symmetric top (e.g.,  $\text{NH}_3$ ):

$$\frac{\sqrt{\pi}}{\sigma} \sqrt{\frac{T^3}{\Theta_{\text{rot},a}^2 \Theta_{\text{rot},b}}}$$

- $a$  and  $b$  are the two different symmetry axes.

- Asymmetric top (e.g.,  $\text{H}_2\text{O}$ ):

$$\frac{\sqrt{\pi}}{\sigma} \sqrt{\frac{T^3}{\Theta_{\text{rot},a} \Theta_{\text{rot},b} \Theta_{\text{rot},c}}}$$

- Application to total energy and heat capacity of a molecule.

- We have that

$$q = \left( \frac{2\pi m k_{\text{B}} T}{h^2} \right)^{3/2} \cdot V \cdot \sqrt{\frac{T^2}{\sigma \Theta_{\text{rot},a} \Theta_{\text{rot},b} \Theta_{\text{rot},c}}} \cdot \sum_1^{3N-6} \frac{e^{-\Theta_{\text{vib}}/2T}}{1 - e^{-\Theta_{\text{vib}}/T}} \cdot g_1 e^{D_e/k_{\text{B}}T}$$

- Thus,

$$\langle E \rangle = k_{\text{B}} T^2 \frac{\partial \ln q}{\partial T} = k_{\text{B}} T^2 \frac{\partial}{\partial T} \left( \frac{3}{2} \ln T + \text{constant} + \frac{3}{2} \ln T + \text{vibration} - D_e \right)$$

- The energy of the vibration is  $E = k_{\text{B}} \Theta_{\text{vib}} / (e^{\Theta_{\text{vib}}/k_{\text{B}}T} - 1) + k_{\text{B}} \Theta_{\text{vib}}/2$ . It follows that

$$C = \frac{\partial E}{\partial T} = k_{\text{B}} \frac{\Theta_{\text{vib}}^2}{T^2} \frac{e^{-\Theta_{\text{vib}}/T}}{(1 - e^{-\Theta_{\text{vib}}/T})}$$

## 2.3 Chapter 18: Partition Functions and Ideal Gases

From McQuarrie and Simon (1997).

- 1/23:
- Herein, we will calculate the partition functions and heat capacities of ideal gases.
    - We heavily rely on the expression of the partition function for a system of independent, indistinguishable particles, which ideal gases are likely to satisfy because of their low density.
  - Deriving the translational molecular partition function of an atom in a monatomic ideal gas.
    - As mentioned in Chapter 17, if we let the container be cubic, then

$$\varepsilon(n_x, n_y, n_z) = \frac{h^2}{8ma^2}(n_x^2 + n_y^2 + n_z^2)$$

- It follows that

$$\begin{aligned} q_{\text{trans}} &= \sum_{n_x, n_y, n_z=1}^{\infty} e^{-\beta \varepsilon(n_x, n_y, n_z)} \\ &= \sum_{n_x=1}^{\infty} \exp\left(-\frac{\beta h^2 n_x^2}{8ma^2}\right) \sum_{n_y=1}^{\infty} \exp\left(-\frac{\beta h^2 n_y^2}{8ma^2}\right) \sum_{n_z=1}^{\infty} \exp\left(-\frac{\beta h^2 n_z^2}{8ma^2}\right) \\ &= \left[ \sum_{n=1}^{\infty} \exp\left(-\frac{\beta h^2 n^2}{8ma^2}\right) \right]^3 \end{aligned}$$

- The above sum cannot be evaluated in closed form. However, since later terms in the summation get very small, it is an excellent approximation to replace the summation with an integral, i.e.,

$$\begin{aligned} q_{\text{trans}} &= \left( \int_0^{\infty} e^{-\beta h^2 n^2 / 8ma^2} dn \right)^3 \\ &= \left( \sqrt{\frac{\pi}{4\beta h^2 / 8ma^2}} \right)^3 \\ &= \left( \sqrt{\frac{2\pi m}{\beta h^2}} \right)^3 a^3 \\ &= \left( \frac{2\pi m k_B T}{h^2} \right)^{3/2} V \end{aligned}$$

- 1/24:
- Deriving the electronic molecular partition function of an atom in a monatomic ideal gas.
    - We express the partition function here in terms of levels, i.e., by

$$q_{\text{elec}} = \sum_i g_{ei} e^{-\beta \varepsilon_{ei}}$$

where  $g_{ei}$  is the degeneracy and  $\varepsilon_{ei}$  is the energy of the  $i^{\text{th}}$  electronic level.

- Taking  $\varepsilon_{e1} = 0$  to be the zero of energy yields

$$q_{\text{elec}} = g_{e1} + g_{e2} e^{-\beta \varepsilon_{e2}} + \dots$$

- Note that since  $\varepsilon$ 's are usually on the order of tens of thousands of wavenumbers,  $e^{-\beta \varepsilon_{e2}}$  is around  $10^{-5}$  for most atoms at ordinary temperatures, so only the first term in the summation is significantly different from zero.
- For some gases such as halogens, other terms may be important, but even there the sum converges very rapidly.

- Using spectroscopic data, we can show that the fraction of atoms of most gases in the first excited state is very small.
  - For example, the fraction of helium atoms at 300 K in the first excited state is  $10^{-334}$ .
  - For fluorine, however, the fraction is on the order of  $10^{-2}$ , which is significant. In this case, we need to approximate  $q_{\text{elec}}$  with more than one or two terms.
- McQuarrie and Simon (1997) recalculates the average energy, heat capacity, and pressure of a monatomic ideal gas using the above results.
- Diatomics.
  - The translational partition function is

$$q_{\text{trans}}(V, T) = \left[ \frac{2\pi(m_1 + m_2)k_{\text{B}}T}{h^2} \right]^{3/2} V$$

- We take the zero of rotational energy to be the  $J = 0$  state.
- We take the zero of vibrational energy to be the bottom of the internuclear potential well of the lowest electronic state (so that the energy of the ground vibrational state is  $h\nu/2$ ).
- We take the zero of electronic energy to be the energy of the separated atoms at rest in their ground electronic state (so that the energy of the ground electronic state is  $-D_e^{[1]}$ ).
- **Vibrational temperature:** The following quantity. Denoted by  $\Theta_{\text{vib}}$ . Given by

$$\Theta_{\text{vib}} = \frac{h\nu}{k_{\text{B}}}$$

- Deriving the vibrational molecular partition function of a molecule in a diatomic ideal gas.

$$\begin{aligned} q_{\text{vib}}(T) &= \sum_{v=0}^{\infty} e^{-\beta(v+1/2)h\nu} \\ &= e^{-\beta h\nu/2} \sum_{v=0}^{\infty} e^{-\beta h\nu v} \\ &= e^{-\beta h\nu/2} \frac{1}{1 - e^{-\beta h\nu}} \\ &= \frac{e^{-\beta h\nu/2}}{1 - e^{-\beta h\nu}} \end{aligned}$$

- In terms of  $\Theta_{\text{vib}}$ ,

$$\begin{aligned} q_{\text{vib}}(T) &= \frac{e^{-\Theta_{\text{vib}}/2T}}{1 - e^{-\Theta_{\text{vib}}/T}} \\ \langle E_{\text{vib}} \rangle &= Nk_{\text{B}} \left( \frac{\Theta_{\text{vib}}}{2} + \frac{\Theta_{\text{vib}}}{e^{\Theta_{\text{vib}}/T} - 1} \right) \\ \bar{C}_{V,\text{vib}} &= R \left( \frac{\Theta_{\text{vib}}}{T} \right)^2 \frac{e^{-\Theta_{\text{vib}}/T}}{(1 - e^{-\Theta_{\text{vib}}/T})^2} \end{aligned}$$

- Note that the high temperature limit of  $\bar{C}_{V,\text{vib}}$  is  $R$ , and  $\bar{C}_{V,\text{vib}}$  attains  $R/2$  at  $T = 0.34 \Theta_{\text{vib}}$ .
- Calculating the fraction of molecules in the ground vibrational state reveals that generally, most molecules are in the ground vibrational state.

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<sup>1</sup>See Figure 9.7 of Labalme (2021).

- Exceptions include Br<sub>2</sub>, the smaller force constant and larger mass of which lead to a smaller value of  $\Theta_{\text{vib}}$ .

- **Rotational temperature:** The following quantity. Denoted by  $\Theta_{\text{rot}}$ . Given by

$$\Theta_{\text{rot}} = \frac{\hbar^2}{2Ik_{\text{B}}} = \frac{hB}{k_{\text{B}}}$$

- $B$  is the rotational constant (see Chapter 5) in the above equation.

- Deriving the rotational molecular partition function of a *heteronuclear* molecule in a diatomic ideal gas.

- We have

$$q_{\text{rot}}(T) = \sum_{J=0}^{\infty} (2J+1)e^{-\Theta_{\text{rot}}J(J+1)/T}$$

- As with the translational partition function, for  $\Theta_{\text{rot}} \ll T$  (which is true for normal temperatures), we may approximate the above sum via an integral. This approximation is known as the high-temperature limit, and under it,

$$\begin{aligned} q_{\text{rot}}(T) &= \int_0^{\infty} (2J+1)e^{-\Theta_{\text{rot}}J(J+1)/T} dJ \\ &= \int_0^{\infty} e^{-\Theta_{\text{rot}}x/T} dx \\ &= \frac{T}{\Theta_{\text{rot}}} = \frac{8\pi^2 I k_{\text{B}} T}{h^2} \end{aligned}$$

- For low temperatures or molecules with large values of  $\Theta_{\text{rot}}$  we evaluate some number of terms of the sum directly, but we will not consider these cases further.

- It follows from the above that

$$\langle E_{\text{rot}} \rangle = Nk_{\text{B}}T \quad \quad \quad \overline{C}_{V,\text{rot}} = R$$

- Each of the two rotational degrees of freedom of a diatomic contributes  $R/2$  to  $\overline{C}_{V,\text{rot}}$ .

- Contrary to the other component parts of energy, higher energy rotational states are significantly occupied.

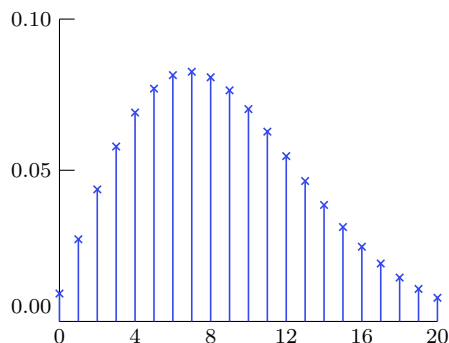


Figure 2.1: The fraction of molecules in the  $J^{\text{th}}$  rotational level for CO at 300 K.

- We have that the fraction  $f_J$  of molecules in the  $J^{\text{th}}$  vibrational state is

$$f_J = \frac{(2J+1)e^{-\Theta_{\text{rot}}J(J+1)/T}}{q_{\text{rot}}} = (2J+1)\left(\frac{\Theta_{\text{rot}}}{T}\right)e^{-\Theta_{\text{rot}}J(J+1)/T}$$

- We can estimate the most probable value of  $J$  by solving  $\partial f_J / \partial J = 0$ , which gives  $J = 7$  in agreement with Figure 2.1.

1/25:

- We now address the rotational molecular partition function for a *homonuclear* diatomic ideal gas molecule.
  - Because of the additional perpendicular  $C_2$  axes of symmetry in a homonuclear diatomic compared to a heteronuclear diatomic, the diatomic's constituent atoms are 'more' indistinguishable, i.e., only nuclear spin can distinguish them.
  - "In particular, if the two nuclei have integral spins (bosons), the molecular wave function must be symmetric with respect to an interchange of the two nuclei; if the nuclei have half odd integer spin (fermions), the molecular wave function must be antisymmetric" (McQuarrie & Simon, 1997, p. 747).
  - This symmetry affects the population of the rotational energy levels in a way that *can* be derived, but we will just state the important result, which is that for a homonuclear diatomic molecule,

$$q_{\text{rot}}(T) = \frac{T}{2\Theta_{\text{rot}}}$$

- To unify the two rotational molecular partition functions, we let

$$q_{\text{rot}}(T) = \frac{T}{\sigma\Theta_{\text{rot}}}$$

in general, where  $\sigma$  is the **symmetry number**.

- **Symmetry number:** The number of different ways a given molecule can be rotated into a configuration indistinguishable from the original. *Denoted by  $\sigma$ . Given by*

$$\sigma = \begin{cases} 1 & \text{heteronuclear} \\ 2 & \text{homonuclear} \end{cases}$$

- Taking the energy of an ideal diatomic gas molecule to be a simple sum of its translational, rotational, vibrational, and electronic energies yields the molecular partition function

$$q(V, T) = \left( \frac{2\pi M k_B T}{h^2} \right)^{3/2} V \cdot \frac{T}{\sigma\Theta_{\text{rot}}} \cdot \frac{e^{-\Theta_{\text{vib}}/2T}}{1 - e^{-\Theta_{\text{vib}}/T}} \cdot g_e e^{D_e/k_B T}$$

where we require  $\Theta_{\text{rot}} \ll T$ , that the only populated electronic state is the ground state, that the zero of electronic energy is the separated atoms at rest in their ground electronic states, and that the zero of vibrational energy is the bottom of the internuclear potential well of the lowest electronic state.

- McQuarrie and Simon (1997) derives the molar energy and heat capacity of a diatomic ideal gas one more time using the above equation.
  - The only difference is that the newly added electronic factor in the partition function adds a term of  $-N_A D_e$  to the Chapter 17 formula for  $\bar{U}$ .
  - Also note that we can greatly improve the agreement of the harmonic oscillator-rigid rotator model with even small first-order corrections, such as including centrifugal distortion and anharmonicity.
- The translational and electronic molecular partition functions of an ideal polyatomic molecule are the same as those of an ideal monatomic or diatomic molecule.
- On the vibrational molecular partition function of an ideal polyatomic molecule.
  - Recall from Chapter 13 that the vibrational motion of a polyatomic molecule can be expressed in terms of normal coordinates.

- Thus, the vibrational energy of a polyatomic molecule in state  $v_j = 0, 1, 2, \dots$  is

$$\varepsilon_{\text{vib}} = \sum_{j=1}^{N_{\text{vib}}} \left(v_j + \frac{1}{2}\right) h\nu_j$$

where  $\nu_j$  is the frequency of the  $j^{\text{th}}$  normal mode.

- It follows that for a polyatomic molecule,

$$q_{\text{vib}} = \prod_{j=1}^{N_{\text{vib}}} \frac{e^{-\Theta_{\text{vib},j}/2T}}{1 - e^{-\Theta_{\text{vib},j}/T}}$$

$$E_{\text{vib}} = Nk_{\text{B}} \sum_{j=1}^{N_{\text{vib}}} \left( \frac{\Theta_{\text{vib},j}}{2} + \frac{\Theta_{\text{vib},j} e^{-\Theta_{\text{vib},j}/T}}{1 - e^{-\Theta_{\text{vib},j}/T}} \right)$$

$$C_{V,\text{vib}} = Nk_{\text{B}} \sum_{j=1}^{N_{\text{vib}}} \left[ \left( \frac{\Theta_{\text{vib},j}}{T} \right)^2 \frac{e^{-\Theta_{\text{vib},j}/T}}{(1 - e^{-\Theta_{\text{vib},j}/T})^2} \right]$$

- Rotational molecular partition functions for linear molecules.

- We can still apply the rigid-rotator approximation, but with

$$I = \sum_{j=1}^n m_j d_j^2$$

where  $d_j$  is the distance of the  $j^{\text{th}}$  nucleus from the center of mass of the molecule.

- Doing so yields

$$q_{\text{rot}}(T) = \frac{T}{\sigma \Theta_{\text{rot}}}$$

where  $\sigma = 1$  for unsymmetrical molecules such as  $\text{N}_2\text{O}$  and  $\text{COS}$  and  $\sigma = 2$  for symmetrical molecules such as  $\text{CO}_2$  and  $\text{C}_2\text{H}_2$ .

- Note that the symmetry number of  $\text{NH}_3$  is three.

- Rotational molecular partition functions for nonlinear molecules.

- Recall the discussion surrounding the principal moments of inertia in Chapter 13.
- We define three characteristic rotational temperatures, namely  $\Theta_{\text{rot},j} = \hbar^2/2I_j k_{\text{B}}$  for  $j = A, B, C$ .
- Spherical top.
  - In this case,  $\Theta_{\text{rot},A} = \Theta_{\text{rot},B} = \Theta_{\text{rot},C} = \Theta_{\text{rot}}$ .
  - The quantum-mechanical spherical top can be solved exactly to give

$$\varepsilon_J = \frac{\hbar^2}{2I} J(J+1) \qquad g_J = (2J+1)^2$$

- Now  $\Theta_{\text{rot}} \ll T$  for almost all spherical top molecules at ordinary temperatures, and this has two important consequence. First, we can approximate the partition function with an integral. Second, we can neglect 1 in comparison with  $J$  since the important values of  $J$  are large. Thus, we have that

$$q_{\text{rot}}(T) = \frac{1}{\sigma} \sum_{J=0}^{\infty} (2J+1)^2 e^{-\hbar^2 J(J+1)/2Ik_{\text{B}}T}$$

$$= \frac{1}{\sigma} \int_0^{\infty} (2J+1)^2 e^{-\Theta_{\text{rot}} J(J+1)/T} dJ$$



$$\begin{aligned}
&= \frac{1}{\sigma} \int_0^\infty 4J^2 e^{-\Theta_{\text{rot}} J^2 / T} dJ \\
&= \frac{4}{\sigma} \int_0^\infty J^2 e^{-aJ^2} dJ \\
&= \frac{4}{\sigma} \cdot \frac{1}{4a} \sqrt{\frac{\pi}{a}} \\
q_{\text{rot}}(T) &= \frac{\sqrt{\pi}}{\sigma} \left( \frac{T}{\Theta_{\text{rot}}} \right)^{3/2}
\end{aligned}$$

– Similarly, we have respectively for a symmetric top and an asymmetric top that

$$q_{\text{rot}}(T) = \frac{\sqrt{\pi}}{\sigma} \left( \frac{T}{\Theta_{\text{rot},A}} \right) \sqrt{\frac{T}{\Theta_{\text{rot},C}}} \quad q_{\text{rot}}(T) = \frac{\sqrt{\pi}}{\sigma} \sqrt{\frac{T}{\Theta_{\text{rot},A} \Theta_{\text{rot},B} \Theta_{\text{rot},C}}}$$

- It follows that

$$\bar{U}_{\text{rot}} = \frac{3RT}{2} \quad \bar{C}_{V,\text{rot}} = \frac{3R}{2}$$

- Linear molecule equations.

$$\begin{aligned}
q(V, T) &= \left( \frac{2\pi M k_B T}{h^2} \right)^{3/2} V \cdot \frac{T}{\sigma \Theta_{\text{rot}}} \cdot \prod_{j=1}^{3n-5} \frac{e^{-\Theta_{\text{vib},j}/2T}}{1 - e^{-\Theta_{\text{vib},j}/T}} \cdot g_{e1} e^{D_e/k_B T} \\
\frac{U}{N k_B T} &= \frac{3}{2} + \frac{2}{2} + \sum_{j=1}^{3n-5} \left( \frac{\Theta_{\text{vib},j}}{2T} + \frac{\Theta_{\text{vib},j}/T}{e^{\Theta_{\text{vib},j}/T} - 1} \right) - \frac{D_e}{k_B T} \\
\frac{C_V}{N k_B} &= \frac{3}{2} + \frac{2}{2} + \sum_{j=1}^{3n-5} \left( \frac{\Theta_{\text{vib},j}}{T} \right)^2 \frac{e^{-\Theta_{\text{vib},j}/T}}{(1 - e^{-\Theta_{\text{vib},j}/T})^2}
\end{aligned}$$

- Nonlinear molecule equations.

$$\begin{aligned}
q(V, T) &= \left( \frac{2\pi M k_B T}{h^2} \right)^{3/2} V \cdot \frac{\sqrt{\pi}}{\sigma} \sqrt{\frac{T^3}{\Theta_{\text{rot},A} \Theta_{\text{rot},B} \Theta_{\text{rot},C}}} \cdot \prod_{j=1}^{3n-6} \frac{e^{-\Theta_{\text{vib},j}/2T}}{1 - e^{-\Theta_{\text{vib},j}/T}} \cdot g_{e1} e^{D_e/k_B T} \\
\frac{U}{N k_B T} &= \frac{3}{2} + \frac{3}{2} + \sum_{j=1}^{3n-6} \left( \frac{\Theta_{\text{vib},j}}{2T} + \frac{\Theta_{\text{vib},j}/T}{e^{\Theta_{\text{vib},j}/T} - 1} \right) - \frac{D_e}{k_B T} \\
\frac{C_V}{N k_B} &= \frac{3}{2} + \frac{3}{2} + \sum_{j=1}^{3n-6} \left( \frac{\Theta_{\text{vib},j}}{T} \right)^2 \frac{e^{-\Theta_{\text{vib},j}/T}}{(1 - e^{-\Theta_{\text{vib},j}/T})^2}
\end{aligned}$$