The Knot Book

Notes

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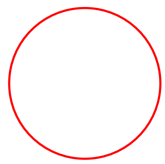
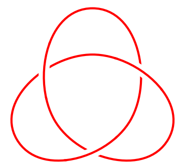
1 Introduction

1.1 Introduction

- **Knot**: "A knotted loop of string, except that we think of the string as having no thickness, its cross-section being a single point" (2).
- Do not distinguish between a 'nice, even' knot and one that has been deformed through space.
- **Unknot**: "The simplest knot of all...the unknoted circle" (2). *Also known as* **trivial knot**. See Figure 1.1a.
- **Trefoil knot**: "The next simplest knot" (2). See Figure 1.1b.



(a) Trivial knot.



(b) Trefoil knot.

Figure 1.1: Projections of the two simplest knots.

- **Projection**: A picture of a knot, such as those in Figure 1.1.
  - The same knots can have multiple projections (as they are deformed in space).
- **Crossings**: The places in a projection where a knot crosses itself.
  - The trefoil knot in Figure 1.1b is a three-crossing knot because it crosses itself 3 times.
  - Any one-crossing knot is trivial.
  - *Exercise 1.2*: Any two-crossing knot must be trivial because the simplest nontrivial knot is the trefoil knot, which has three crossings.
- Atoms were originally thought to be tangles (knots) in the ether of the universe, but when chemists moved on, mathematicians took up knot theory. In the 1980s, biochemists began to see applications of knot theory in their research (see Section **??**).
- **Topology**: "The study of the properties of geometric objects that are preserved under deformations" (6).
  - Knot theory is a subfield of topology (see Section **??**).

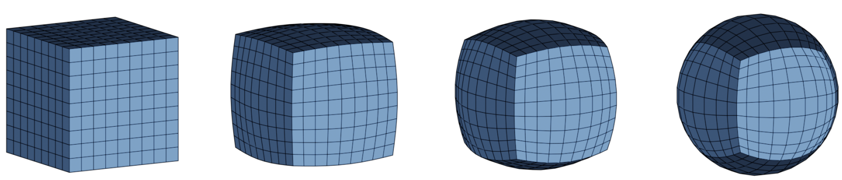


Figure 1.2: Deformation of a cube into a sphere.

- Any knot can have a projection with as many crossings as desired.

- **Alternating knot**: "A knot with a projection that has crossings that alternate between over and under as one travels around the knot in a fixed direction" (7).
  - The trefoil is such a knot.
- *Exercise 1.7**: By changing some of the crossings from over to under or vice versa, any projection of a knot can be made into a projection of the unknot[1]. See Figure 1.3.

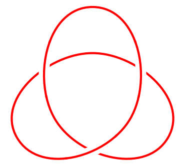


Figure 1.3: A projection of the unknot evoking the trefoil knot.

1.2 Composition of Knots

- **Composition** (of two knots): "A new knot obtained by removing a small arc from two knot projections and then connecting the four endpoints by two new arcs" (7).
  - If two knots are designated J and K, then their composition is denoted $J\#K$.
  - Do not overlap the projections and choose two arcs that are on the outside to avoid new crossings.
  - Make sure that the new arcs do not cross any of the the original knot projections or each other.
- **Composite knot**: A knot that "can be expressed as the composition of two knots, neither of which is the trivial knot" (8).
  - This definition is analogous to composite integers, where an integer is composite if it is the product of positive integers, neither of which is 1.
  - Similarly, if we compose any knot with the unknot, we get the same knot back.
- **Factor knots**: "The knots that make up the composite knot" (8).
- **Prime knot**: "A knot [that] is not the composition of any two nontrivial knots" (9).
- The unknot, trefoil knot, and figure-eight knots are all prime (see Section **??**).
  - The unknot is not composite for the same reason that 1 is not the product of two integers greater than 1.

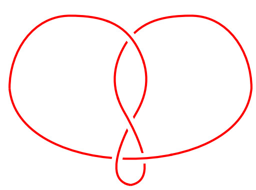


Figure 1.4: The figure-eight knot.

- Similar to integers, "a composite knot factors into a unique set of prime knots" (10).
- *Exercise 1.8*: Using the appendix table, identify the factor knots that make up the composite knot in Figure 1.5.
  - The knot in Figure 1.5 is the composition of two projections of 5_2.

[1]How can I *show* something? How can I do these proofs? What kind of logic solves one of these?

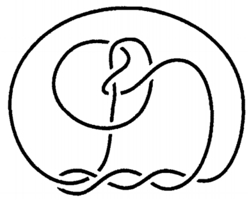


Figure 1.5: The composite knot.

- *Exercise 1.9*: Show that the knot in Figure 1.6a is composite.

(a) The composite knot. (b) Factors.

Figure 1.6: Factorization of a 'double trefoil.'

- There is more than one way to take the composition of two knots (by removing different arcs).
  - This is not analogous to multiplication — a break in the pattern.
- **Orientation**: A direction to travel around the knot. Denoted by placing "coherently directed arrows along the projection of the knot in the direction of our choice" (10). A knot with such arrows is **oriented**.
  - All compositions $J\#K$ where the orientations of J and K <u>do</u> match up will yield the same composite knot.
    - J can be 'slid around' $J\#K$ until it reaches the second position where the composition was taken.
  - All compositions $J\#K$ where the orientations of J and K <u>do not</u> match up will yield the same composite knot.
  - These two compositions can be distinct.

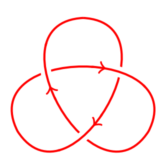
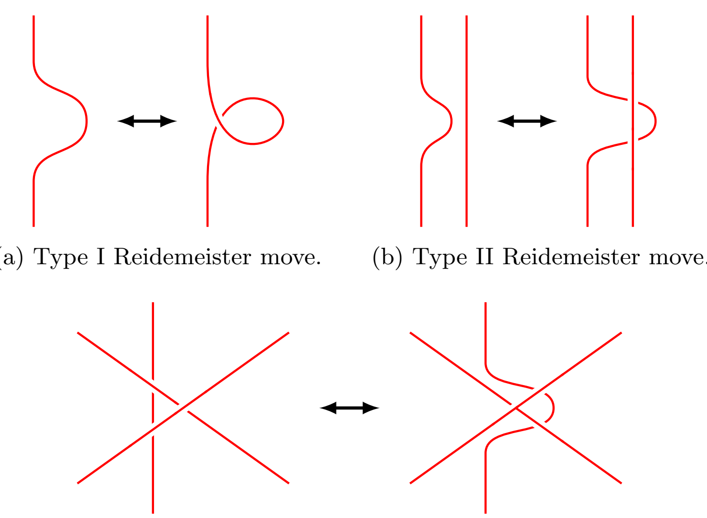


Figure 1.7: Orientation notation.

- **Invertible**: A knot that can be deformed back to itself so that an orientation on it is sent to the opposite orientation.
  - "In the case that one of the two knots is invertible, say J, we can always deform the composite knot so that the orientation on K is reversed, and hence so that the orientations of J and K always match. Therefore, there is only one composite knot that we can construct from the two knots" (11).
- To determine the possible compositions of knots, it is necessary to know which knots are invertible, but no general technique has yet been discovered.

1.3 Reidemeister Moves

- **Ambient isotropy**: "The movement of the string through three-dimensional space without letting it pass through itself" (12).
- **Planar isotropy**: A deformation of "the projection plane as if it were made of ruber with the projection drawn upon it" (12).
  - Stretching, squeezing, rotating, bending single arcs, etc.
- **Reidemeister move**: "One of three ways to change a projection of the knot that *will* change the relation between the crossings" (13).
- **First Reidemeister move**: "Put in or take out a twist in the knot" (13). See Figure 1.8a. *Also known as* **type I Reidemeister move**.
- **Second Reidemeister move**: "Either add two crossings or remove two crossings" (13). See Figure 1.8b. *Also known as* **type II Reidemeister move**.
- **Third Reidemeister move**: "Slide a strand of the knot from one side of a crossing to the other side of the crossing" (13). See Figure 1.8c. *Also known as* **type III Reidemeister move**.
  - Note that the crossings in Figure 1.8 can be reversed and the move will still be classified under the same category.



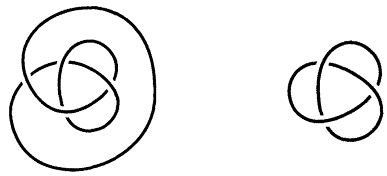
(a) Type I Reidemeister move. (b) Type II Reidemeister move.

(c) Type III Reidemeister move.

Figure 1.8: Reidemeister moves.

- All Reidemeister moves are ambient isotropies.
- "If we have two distinct projections of the same knot, we can get from the one projection to the other by a series of Reidemeister moves and planar isotropies" (14).
- **Amphicheiral**: A knot that "is equivalent to its mirror image, that is, the knot obtained by changing every crossing. . . to the opposite crossing" (14-15). *Also known as* **achiral** *by chemists.*
  - A knot and its mirror image are distinct unless the knot is amphicheiral.
  - See Section **??** for more on amphicheirality.

- *Exercise 1.10*: Show that the two projections in Figure 1.9 represent the same knot by finding a series of Reidemeister moves from one to the other.



(a) Initial projection. (b) Final projection.

Figure 1.9: Finding Reidemeister moves.

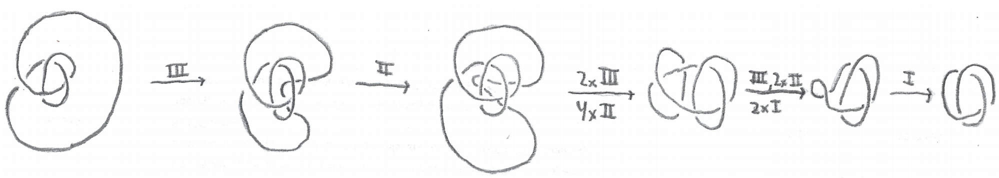


Figure 1.10: Solution to *Exercise 1.10.*

- *Exercise 1.11**: Find a sequence of Reidemeister moves to untangle the unknot shown in Figure 1.11.

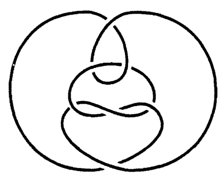


Figure 1.11: Unknot to be untangled.

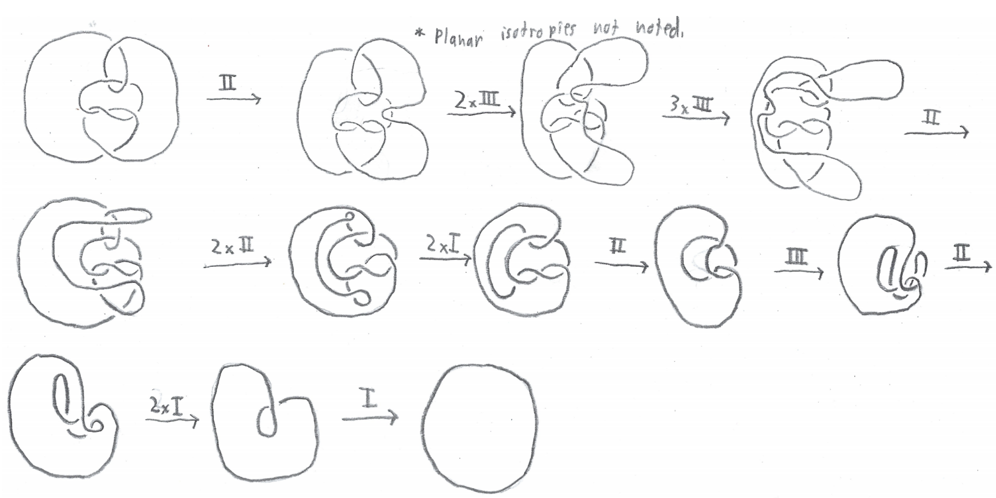
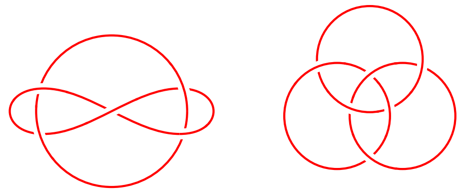


Figure 1.12: Solution to *Exercise 1.11**.

- The bounds on the increase in crossings generated by Reidemeister moves from one projection to another are unknown.

1.4 Links

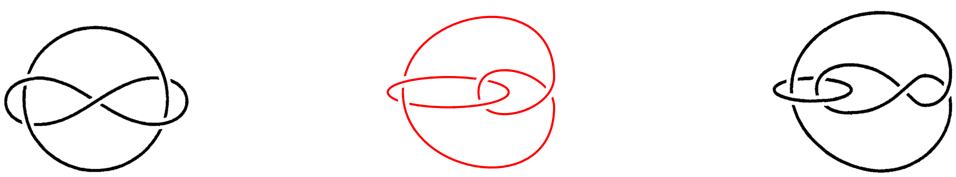
- **Link**: “A set of knotted loops all tangled up together” (17).
- “Two links are considered to be the same if we can deform the one link to the other link without ever having any one of the loops intersect itself or any of the other loops in the process” (17).



(a) Whitehead link. (b) Borromean rings.

Figure 1.13: Projections of two common links.

- *Exercise 1.13*: Show that the two projections in Figure 1.14a and 1.14c represent the same link.
  - Untwist the right side of the figure-eight loop, twisting the right side of the circle at the same time. Then add a futher twist to the circle loop and flip (rotate along the horizontal axis) the figure-eight loop (which now looks like an ellipse).



(a) Initial projection. (b) Intermediate projection. (c) Final projection.

Figure 1.14: Ambient isotropies of the Whitehead link.

- **Link of *n* components**: A link made up of n loops knotted with each other.
  - The Whitehead link (Figure 1.13a) is a link of 2 components.
  - The Borromean rings (Figure 1.13b) are a link of 3 components.
  - A knot is a link of 1 component.
  - If the number of components in two links differ, then the links are clearly distinct.
- **Splittable**: A link whose components “can be deformed so that they lie on different sides of a plane in three-space” (17).
- *Exercise 1.14*: Show that the link in Figure 1.15 is splittable.

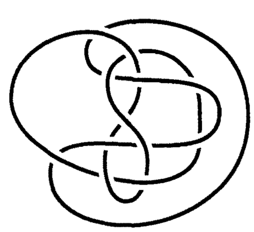


Figure 1.15: Link to be split.

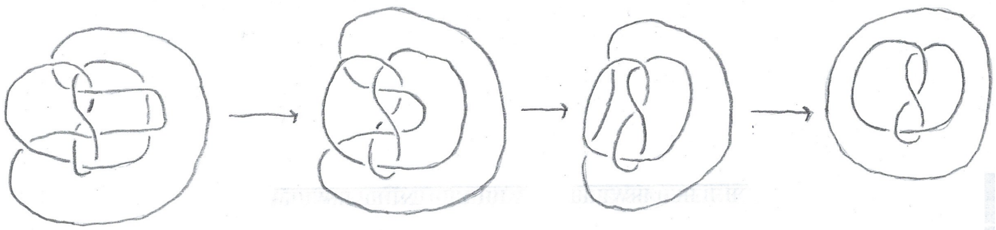


Figure 1.16: Solution to *Exercise 1.14*.

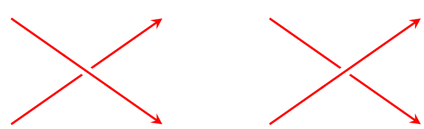
- **Unlink**: One of the two simplest links of two components and the simplest splittable link of two components. *Also known as* **trivial link**. See Figure 1.17a.
- **Hopf link** The other of the two simplest links of two components and the simplest nonsplittable link of two components. See Figure 1.17b.

(a) Trivial link.

(b) Hopf link.

Figure 1.17: Projections of the two simplest links of two components.

- **Linking number**: A quantity that numerically measures how linked up two components are.
  - If M and N are two components in a link, begin by orienting both of them.
  - At each crossing, count a $+1$ if Figure 1.18a holds or count a -1 if Figure 1.18b holds.
    - Note that if you are unsure, do the following: If rotating the bottom strand clockwise lines up the arrows (correlates the two orientations), count $+1$.
    - In the same vein, if rotating the bottom strand counterclockwise lines up the arrows, count -1.
    - If a component crosses itself, do not count it.



(a) Count $+1$.

(b) Count -1.

Figure 1.18: Computing linking numbers per crossing.

  - Sum all of the $+1$s and -1s and divide this sum by 2 to yield the linking number.
    - For example, the linking number for the Hopf link (Figure 1.17b) is ± 1 depending on the chosen orientation.
    - Note that reversing the orientation for one of the two links is equivalent to multiplying the linking number by -1.
    - As such, the absolute value of the linking number remains constant whatever orientation is chosen.
- *Exercise 1.15*: Compute the linking number of the link pictured in Figure 1.19. Now reverse the direction on one of the components and recompute it.

- Add up all of the numbers in the second row of Table 1.1 to yield 2. Divide by 2 to yield the linking number, 1.

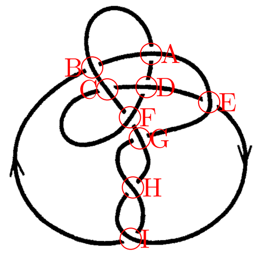


Figure 1.19: Link of linking number n.

A	B	C	D	E	F	G	H	I
-1	-1	0	0	$+1$	0	$+1$	$+1$	$+1$

Table 1.1: Counting crossings.

- Although a linking number is computed using a single projection of a link, the linking number will be the same for any projection of the link.
  - This can be proven by demonstrating that the Redemeister moves do not change the linking number. "Since we can get from any one projection of a link to any other via a sequence of Reidemeister moves, none of which will change the linking number, it must be that two different projections of the same link yield the same linking number" (20).
  - Let's take this case by case.
    - A Type I Reidemeister move generates a self-intersection. Because self-crossings do not count toward the linking number by definition, the first Reidemeister move does not affect the linking number.
    - A Type II Reidemeister move generates two crossings. There are now two cases: Either the new crossings are self-intersections or they are not. If the new crossings are self-intersections, then there is no change to the linking number. If the new crossings are not self-intersections, they will always have opposite linking numbers, and, thus, the two new crossings cancel each other out. Therefore, the second Reidemeister move does not affect the linking number.
    - A Type III Reidemeister move removes two crossings and generates two crossings. There are now two cases: Either the new crossings are self-intersections or they are not. If the new crossings are self-intersections, then there is no change to the linking number. If the new crossings are not self-intersections, both pairs of crossings will always have opposite linking numbers and, thus, both pairs of crossings cancel each other out. Therefore, the third Reidemeister move does not affect the linking number.
  - This logic can be visually confirmed by assigning orientations and counting crossings in Figure 1.8.
- **Invariant**: A quality of a knot or link that, once orientations are chosen, is unchanged by ambient isotropy.
  - Both the linking number and the number of components are <u>link invariants</u>.
- *Exercise 1.16*: Explain why the linking number of a splittable two-component link will always be 0, no matter what projection is used to compute it.

- Solution 1: By definition, the linking number numerically measures how linked up two components are. Since splittable links are not joined (or "linked up") in any way, the linking number must be 0.
- Solution 2: If a two-component link is splittable, then there exists a projection of it with no crossings that are not self-intersections (a projection of the split link, as in Figure 1.17b). If all crossings are self-intersections, then the linking number computed for this projection must be 0. Now that it is known that said link has a linking number of 0, any combination of Reidemeister moves can be used to manipulate the link into any other projection. But because Reidemeister moves do not affect the linking number, the linking number will remain 0.

- The absolute value of the linking number, as a link invariant, can be used to distinguish certain distinct links, regardless of orientation.
  - "Any two links with two components that have distinct absolute values of their linking numbers have to be different links" (21).
  - For example, the difference in linking number between the unlink (0; Figure 1.17a) and the Hopf link (1; Figure 1.17b) distinguishes them.
- However, the linking number cannot distinguish between all links.
  - For instance, both the Whitehead link (Figure 1.13a) and the unlink (Figure 1.17a) have linking number 0.
  - One such distinction is discussed in Section 1.5.
- **Brunnian link**: A nontrivial link where the removal of any one component leaves behind a set of trivial unlinked circles.
  - The Borromean rings (Figure 1.13b) are Brunnian[2].

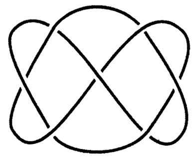
1.5 Tricolorability

- How do we *prove* that every knot is not just a projection of the unknot?
- **Strand**: "A piece of the link that goes from one undercrossing to another with only overcrossings in between" (23).
- **Tricolorable**: A projection of a knot or link in which each strand "can be colored one of three different colors, so that at each crossing, either three different colors come together or all the same color comes together" (23).

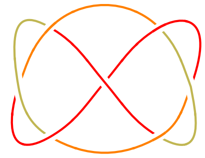
Figure 1.20: Tricolored trefoil.

- *Exercise 1.21*: Determine which of the projections of the three six-crossing knots 6_1, 6_2, and 6_3 are tricolorable.
  - 6_1 is tricolorable and the other two knots are not. I determined this through trial and error.
- *Exercise 1.22*: Show that the projection of the knot 7_4 in Figure 1.21a is tricolorable.

[2]How can I draw a Brunnian link with four components? Or more?



(a) Initial projection.



(b) Tricolored projection.

Figure 1.21: Tricolored 7_4.

- Reidemeister moves do not affect tricolorability.
  - A Type I Reidemeister move generates a self-intersection. At such a crossing (made out of one original strand), every color will be the same; only one color meets at the crossing.
  - A Type II Reidemeister move generates two crossings. If the strands are the same color, then everything stays the same color. If the strands are different colors, than newly created loop takes on the third color.
  - Type III has many cases (*Exercise 1.23*).
- Since the unknot is not tricolorable and the trefoil knot is, we have just proven that there is at least one other knot besides the unknot.
  - Tricolorability is a knot invariant.
- For links, tricolorability is a bit different — the unlink, for example, is tricolorable.

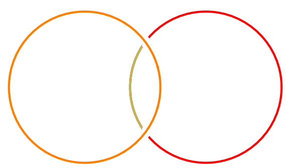


Figure 1.22: Tricolored unlink.

1.6 Knots and Sticks

- We can also consider knots made out of straight sticks glued to each other at each end.
- The first nontrivial knot made out of sticks is the trefoil, with six sticks.
  - We know that the stick trefoil in Figure 1.23 could be made in the real world because, if two vertices lie in the xy-plane, then two lie above and two lie below.
  - This P(lanar), L(ow), H(igh) notation is commonly used.

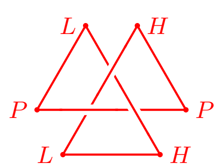


Figure 1.23: A trefoil knot made of sticks.

- **Stick number**: "The least number of straight sticks necessary to make a knot K" (29). *Also known as* **s(K)**.

- The stick number for the composition of n trefoil knots is $2n + 4$.
- Many problems listed[3].

[3]These could make good topics for a larger paper. Alternatively, some I am not ready to solve and need to learn more math first (those that I will come back to at the end of the book). I do not believe I'm missing anything essential by skipping these for now, but we'll see.