Chapter 3

Linear Maps

3.A The Vector Space of Linear Maps

9/5:
- **Linear map** (from V to W): A function $T : V \to W$ with the following properties. *Also known as* **linear transformation**.

  additivity
  $T(u + v) = Tu + Tv$ for all $u, v \in V$.

  homogeneity
  $T(\lambda v) = \lambda(Tv)$ for all $\lambda \in \mathbb{F}$ and all $v \in V$.

  - Note that for linear maps, Tv means the same as the more standard functional notation $T(v)$.
- $\boldsymbol{\mathcal{L}(V, W)}$: The set of all linear maps from V to W.
- **Zero map**: The function $0 \in \mathcal{L}(V, W)$ that takes each element of some vector space to the additive identity of another vector space. *Defined by*

$$0v = 0$$

- **Identity map**: The function $I \in \mathcal{L}(V, V)$ on some vector space that takes each element to itself. *Defined by*

$$Iv = v$$

- We can also formalize more esoteric linear processes as linear maps.
  - For example, $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ can be thought of as the differentiation map $Dp = p'$. This formalizes the fact that $(f + g)' = f' + g'$ and $(\lambda f)' = \lambda f'$.
  - We can do the same with integration: Let $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$ be described by $Tp = \int_0^1 p(x)\,\mathrm{d}x$. This formalizes the fact that integrals are additive and homogenous.
  - Axler (2015) gives a number more examples.
- We now prove that there exists a unique linear map from a vector space of dimension n to any n vectors in another vector space.

  Theorem 3.1. *Suppose $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_n \in W$. Then there exists a unique linear map $T : V \to W$ such that $Tv_j = w_j$ for each $j = 1, \dots, n$.*

  Proof. First, we define a function $T : V \to W$. We then show that T satisfies the specified property. After that, we show that it is a linear map. Lastly, we show that it is unique. Let's begin.

Let $T : V \to W$ be defined by

$$T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n$$

for all $c_1v_1 + \cdots + c_nv_n \in V$. Note that this definition is valid since, by Theorem 2.5, each $v \in V$ can be written in the form $c_1v_1 + \cdots + c_nv_n$ where $c_1, \dots, c_n \in \mathbb{F}$.

To prove that $Tv_j = w_j$ for all $j = 1, \dots, n$, let each c_i in the above definition equal 0 save c_j, which we set equal to 1. Then we have

$$\begin{aligned} T(0v_1 + \cdots + 0v_{j-1} + 1v_j + 0v_{j+1} + \cdots + 0v_n) &= 0w_1 + \cdots + 0w_{j-1} + 1w_j + 0w_{j+1} + \cdots + 0w_n \\ T(v_j) &= w_j \end{aligned}$$

as desired.

To prove that T is a linear map, it will suffice to verify additivity and homogeneity, which we may do as follows. Let $u, v \in V$ with $u = a_1v_1 + \cdots + a_nv_n$ and $v = c_1v_1 + \cdots + c_nv_n$, and let $\lambda \in \mathbb{F}$. Then

$$\begin{aligned} T(u+v) &= T((a_1+c_1)v_1 + \cdots + (a_n+c_n)v_n) \\ &= (a_1+c_1)w_1 + \cdots + (a_n+c_n)w_n \\ &= Tu + Tv \end{aligned}$$

and

$$\begin{aligned} T(\lambda v) &= T(\lambda c_1v_1 + \cdots + \lambda c_nv_n) \\ &= \lambda c_1w_1 + \cdots + \lambda c_nw_n \\ &= \lambda Tv \end{aligned}$$

as desired.

Now suppose $\tilde{T} \in \mathcal{L}(V, W)$ satisfies $\tilde{T}v_j = w_j$ for all $j = 1, \dots, n$. To prove that $T = \tilde{T}$, it will suffice to show that $\tilde{T}(c_1v_1 + \cdots + c_nv_n) = T(c_1v_1 + \cdots + c_nv_n)$ for all $c_1v_1 + \cdots + c_nv_n \in V$. Let $c_1v_1 + \cdots + c_nv_n \in V$ be arbitrary. We know that $\tilde{T}(v_j) = w_j$ for all $j = 1, \dots, n$. It follows since $\tilde{T}$ is a linear map (specifically, since it's homogenous) that $c_jw_j = c_j\tilde{T}(v_j) = \tilde{T}(c_jv_j)$ for all $j = 1, \dots, n$. Similarly, the additivity of $\tilde{T}$ implies that

$$\begin{aligned} T(c_1v_1 + \cdots + c_nv_n) &= c_1w_1 + \cdots + c_nw_n \\ &= \tilde{T}(c_1v_1) + \cdots + \tilde{T}(c_nv_n) \\ &= \tilde{T}(c_1v_1 + \cdots + c_nv_n) \end{aligned}$$

as desired. ■

- **Sum** (of $S, T \in \mathcal{L}(V, W)$): The linear map $(S+T) \in \mathcal{L}(V, W)$ defined by $(S+T)(v) = Sv + Tv$ for all $v \in V$.
- **Product** (of $T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$): The linear map $(\lambda T) \in \mathcal{L}(V, W)$ defined by $(\lambda T)(v) = \lambda(Tv)$ for all $v \in V$.
- It follows that, under these definitions of addition and multiplication, $\mathcal{L}(V, W)$ is a vector space.
- **Product** (of $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$): The linear map $ST \in \mathcal{L}(U, W)$ defined by $(ST)(u) = S(Tu)$ for all $u \in U$.
  - Note that the product is just function composition, but most mathematicians do write ST instead of $S \circ T$.
- Linear maps (with the correct corresponding domains) satisfy the associativity, identity, and distributive properties. However, multiplication of linear maps is not necessarily commutative.

- $(T_1T_2)T_3 = T_1(T_2T_3)$.
- $TI_V = I_WT = T$ (note that if $T \in \mathcal{L}(V,W)$, $I_V \in \mathcal{L}(V,V)$ and $I_W \in \mathcal{L}(W,W)$).
- $(S_1 + S_2)T = S_1T + S_2T$ and $S(T_1 + T_2) = ST_1 + ST_2$.

- Linear maps send 0 to 0.

Theorem 3.2. *Suppose $T \in \mathcal{L}(V,W)$. Then $T(0) = 0$.*

Proof. By additivity, we have

$$\begin{aligned} T(0) &= T(0+0) = T(0) + T(0) \\ 0 &= T(0) \end{aligned}$$

as desired. ■

3.B Null Spaces and Ranges

- **Null space** (of $T \in \mathcal{L}(V,W)$): The subset of V consisting of those vectors that T maps to 0. *Also known as* **kernel**. *Denoted by* $\textbf{null}\,\boldsymbol{T}$. *Given by*

$$\operatorname{null} T = \{v \in V : Tv = 0\}$$

- The null space is a subspace.

Theorem 3.3. *Suppose $T \in \mathcal{L}(V,W)$. Then* $\operatorname{null} T$ *is a subspace of V.*

Proof. To prove that $\operatorname{null} T$ is a subspace of V, it will suffice to show that $0 \in \operatorname{null} T$, $u, v \in \operatorname{null} T$ implies that $u + v \in \operatorname{null} T$, and $u \in \operatorname{null} T$ and $\lambda \in \mathbb{F}$ imply $\lambda u \in \operatorname{null} T$. Let's begin.

By Theorem 3.2, $T(0) = 0$. Therefore, $0 \in \operatorname{null} T$, as desired.

Let $u, v \in \operatorname{null} T$ be arbitrary. Then by additivity

$$T(u+v) = Tu + Tv = 0 + 0 = 0$$

so $u + v \in \operatorname{null} T$, as desired.

Let $u \in \operatorname{null} T$ and $\lambda \in \mathbb{F}$ be arbitrary. Then by homogeneity,

$$T(\lambda u) = \lambda Tu = \lambda 0 = 0$$

so $\lambda u \in \operatorname{null} T$, as desired. ■

- **Injective** (function): A function $T : V \to W$ such that $Tu = Tv$ implies $u = v$. *Also known as* **one-to-one**.

- If 0 is the only vector that gets mapped to 0, then T is injective.

Theorem 3.4. *Let $T \in \mathcal{L}(V,W)$. Then T is injective if and only if* $\operatorname{null} T = \{0\}$.

Proof. Suppose first that T is injective. To prove that $\operatorname{null} T = \{0\}$, it will suffice to show that $0 \in \operatorname{null} T$ and for every $v \in \operatorname{null} T$, $v = 0$. By Theorem 3.3, $0 \in \operatorname{null} T$. Now let $v \in \operatorname{null} T$ be arbitrary. By the definition of the null space, we have $Tv = 0$. By Theorem 3.2, we have $T(0) = 0$. Thus, by transitivity, we have that $Tv = T(0)$. It follows by injectivity that $v = 0$, as desired.

Now suppose that $\operatorname{null} T = \{0\}$. To prove that T is injective, it will suffice to show that if $Tu = Tv$, then $u = v$. Suppose $u, v \in V$ satisfy $Tu = Tv$. Then

$$0 = Tu - Tv = T(u - v)$$

so $(u - v) \in \operatorname{null} T = \{0\}$. It follows that $u - v = 0$, i.e., that $u = v$, as desired. ■

- **Range** (of $T \in \mathcal{L}(V,W)$): The subset of W consisting of those vectors that are of the form Tv for some $v \in V$. *Also known as* **image**. *Denoted by* $\textbf{range}\,\boldsymbol{T}$. *Given by*

$$\operatorname{range} T = \{Tv : v \in V\}$$

- The range is a subspace.

Theorem 3.5. *Suppose $T \in \mathcal{L}(V,W)$. Then* $\operatorname{range} T$ *is a subspace of W.*

Proof. To prove that $\operatorname{range} T$ is a subspace of W, it will suffice to show that $0 \in \operatorname{range} T$, $w_1, w_2 \in \operatorname{range} T$ implies that $(w_1 + w_2) \in \operatorname{range} T$, and $w \in \operatorname{range} T$ and $\lambda \in \mathbb{F}$ imply $\lambda w \in \operatorname{range} T$. Let's begin.

By the definition of a vector space, $0 \in V$. By Theorem 3.2, $T(0) = 0$. Therefore, $0 \in \operatorname{range} T$, as desired.

Let $w_1, w_2 \in \operatorname{range} T$ be arbitrary. Then there exist $v_1, v_2 \in V$ such that $Tv_1 = w_1$ and $Tv_2 = w_2$. It follows by additivity that

$$T(v_1 + v_2) = Tv_1 + Tv_2 = w_1 + w_2$$

Therefore, since $v_1 + v_2 \in V$, we have that $(w_1 + w_2) \in \operatorname{range} T$, as desired.

Let $w \in \operatorname{range} T$ and $\lambda \in \mathbb{F}$ be arbitrary. Then there exists $v \in V$ such that $Tv = w$. It follows by homogeneity that

$$T(\lambda v) = \lambda Tv = \lambda w$$

Therefore, since $\lambda v \in V$, we have that $\lambda w \in \operatorname{range} T$, as desired. ∎

- **Surjective** (function): A function $T : V \to W$ such that $\operatorname{range} T = W$. *Also known as* **onto**.

- We now prove a very important theorem.

Theorem 3.6 (Fundamental Theorem of Linear Maps)**.** *Suppose V is finite-dimensional and $T \in \mathcal{L}(V,W)$. Then* $\operatorname{range} T$ *is finite-dimensional and*

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

Proof. By Theorem 3.3, $\operatorname{null} T$ is a subspace of V finite-dimensional. Thus, by Theorem 2.4, $\operatorname{null} T$ is finite-dimensional. It follows by Theorem 2.7 that we may let $u_1, \dots, u_m$ be a basis of $\operatorname{null} T$. As a basis of a subspace of V, $u_1, \dots, u_m$ is a linearly independent list of vectors in V. Consequently, by Theorem 2.8, we may extend it to a basis $u_1, \dots, u_m, v_1, \dots, v_n$ of V.

Having established this terminology, we can now see that to prove that $\operatorname{range} T$ is finite-dimensional, it will suffice to show that $Tv_1, \dots, Tv_n$ spans it. To show that $\operatorname{span}(Tv_1, \dots, Tv_n) = \operatorname{range} T$, it will suffice to show that every $b_1Tv_1 + \cdots + b_nTv_n \in \operatorname{span}(Tv_1, \dots, Tv_n)$ is an element of $\operatorname{range} T$ and that every $Tv \in \operatorname{range} T$ is an element of $\operatorname{span}(Tv_1, \dots, Tv_n)$. Let $b_1Tv_1 + \cdots + b_nTv_n \in \operatorname{span}(Tv_1, \dots, Tv_n)$ be arbitrary. Then

$$\begin{aligned}
b_1Tv_1 + \cdots + b_nTv_n &= T(b_1v_1 + \cdots + b_nv_n) \\
&= T(0u_1 + \cdots + 0u_m + b_1v_1 + \cdots + b_nv_n)
\end{aligned}$$

Therefore, since $0u_1 + \cdots + 0u_m + b_1v_1 + \cdots + b_nv_n \in V$ by V's closure under addition and scalar multiplication, we have that $b_1Tv_1 + \cdots + b_nTv_n \in \operatorname{range} T$, as desired. Now let $Tv \in \operatorname{range} T$ be arbitrary. Since $v \in V$ and $u_1, \dots, u_m, v_1, \dots, v_n$ is a basis of V, Theorem 2.5 implies that $v = a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n$ for some $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{F}$. Therefore,

$$\begin{aligned}
Tv &= T(a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_nv_n) \\
&= T(a_1u_1 + \cdots + a_mu_m) + T(b_1v_1 + \cdots + b_nv_n) \\
&= a_1Tu_1 + \cdots + a_mTu_m + b_1Tv_1 + \cdots + v_nTv_n \\
&= a_10 + \cdots + a_m0 + b_1Tv_1 + \cdots + v_nTv_n \\
&= b_1Tv_1 + \cdots + v_nTv_n
\end{aligned}$$

where each $Tu_j = 0$ because each $u_j \in \operatorname{null} T$, so $Tv \in \operatorname{span}(Tv_1, \dots, Tv_n)$, as desired.

Before we can verify the equation from the theorem, we need to establish one last fact: that $Tv_1, \dots, Tv_n$ is linearly independent. Suppose $c_1, \dots, c_n \in \mathbb{F}$ make

$$\begin{aligned} c_1 Tv_1 + \cdots + c_n Tv_n &= 0 \\ T(c_1 v_1 + \cdots + c_n v_n) &= 0 \end{aligned}$$

It follows that $c_1 v_1 + \cdots + c_n v_n \in \operatorname{null} T$. Thus, since $u_1, \dots, u_m$ is a basis of $\operatorname{null} T$ by Theorem 2.5, we have that

$$\begin{aligned} c_1 v_1 + \cdots + c_n v_n &= d_1 u_1 + \cdots + d_m u_m \\ 0 &= d_1 u_1 + \cdots + d_m u_m - c_1 v_1 - \cdots - c_n v_n \end{aligned}$$

for some $d_1, \dots, d_m \in \mathbb{F}$. But since $u_1, \dots, u_m, v_1, \dots, v_n$ is linearly independent as the basis of V, the above equation implies that $c_1 = \cdots = c_n = 0$, as desired.

Having established that $u_1, \dots, u_m, v_1, \dots, v_n$ is a basis of V, $u_1, \dots, u_m$ is a basis of $\operatorname{null} T$, and $Tv_1, \dots, Tv_n$ spans $\operatorname{range} T$ and is linearly independent in $\operatorname{range} T$ (i.e., is a basis of $\operatorname{range} T$), we have that

$$\begin{aligned} \dim V &= m + n \\ &= \dim \operatorname{null} T + \dim \operatorname{range} T \end{aligned}$$

as desired. ∎

- We can now prove that a linear map to a "smaller" vector space cannot be injective.

Theorem 3.7. *Suppose V and W are finite-dimensional vector spaces such that $\dim V > \dim W$. Then no linear map from V to W is injective.*

Proof. Let $T \in \mathcal{L}(V, W)$. Then

$$\begin{aligned} \dim \operatorname{null} T &= \dim V - \dim \operatorname{range} T && \text{Fundamental Theorem of Linear Maps} \\ &\geq \dim V - \dim \operatorname{range} T && \text{Theorem 2.11} \\ &> 0 \end{aligned}$$

It follows that $\operatorname{null} T$ has a basis consisting of a list of one or more vectors. As a linearly independent list, naturally none of these vectors will be equal to the zero vector. Thus, since $\operatorname{null} T$ contains vectors other than 0, Theorem 3.4 implies that T is not injective. ∎

- Similarly, we can prove that a linear map to a "bigger" vector space cannot be surjective.

Theorem 3.8. *Suppose V and W are finite-dimensional vector spaces such that $\dim V < \dim W$. Then no linear map from V to W is surjective.*

Proof. Let $T \in \mathcal{L}(V, W)$. Then

$$\begin{aligned} \dim \operatorname{range} T &= \dim V - \dim \operatorname{null} T && \text{Fundamental Theorem of Linear Maps} \\ &\leq \dim V && < \dim W \end{aligned}$$

Therefore, $\operatorname{range} T \neq W$, so T cannot be surjective. ∎

- Theorems 3.7 and 3.8 allow us to express questions about systems of linear equations in terms of linear maps.

- For example, a question about a **homogenous** system of linear equations could be, "does there exist a nonzero solution to the homogenous system $\sum_{k=1}^{n} A_{1,k}x_k = 0, \dots, \sum_{k=1}^{n} A_{m,k}x_k = 0$?"
- If we define $T : \mathbb{F}^n \to \mathbb{F}^m$ by

$$T(x_1, \dots, x_n) = \left(\sum_{k=1}^{n} A_{1,k}x_k, \dots, \sum_{k=1}^{n} A_{m,k}x_k \right)$$

we can express the system of equations as $T(x_1, \dots, x_n) = 0$ and ask instead, "is $\dim \operatorname{null} T > 0$?"

- **Homogenous** (system of linear equations): A system of m linear equations $\sum_{k=1}^{n} A_{1,k}x_k = c_1$ through $\sum_{k=1}^{n} A_{m,k}x_k = c_m$ such that the constant term $c_j = 0$ for all $j = 1, \dots, m$.
- Continuing with the linear equations example, we can rigorously show the following.

Theorem 3.9. *A homogenous system of linear equations with more variables than equations has nonzero solutions.*

Proof. In terms of the above, $T : \mathbb{F}^n \to \mathbb{F}^m$ where $n > m$. Thus, by Theorem 3.7, T is not injective. Consequently, by Theorem 3.4, $\dim \operatorname{null} T > 0$. Therefore, the system has nonzero solutions. ∎

Theorem 3.10. *An inhomogenous system of linear equations with more equations than variables has no solution for some choice of constant terms.*

Proof. In terms of the above, $T : \mathbb{F}^n \to \mathbb{F}^m$ where $m > n$. We want to know if there exists $(c_1, \dots, c_m) \in \mathbb{F}^m$ such that $T(x_1, \dots, x_n) \neq (c_1, \dots, c_m)$ for any $(x_1, \dots, x_n) \in \mathbb{F}^n$. In other words, we want to know if there exists $(c_1, \dots, c_m) \in \mathbb{F}^m$ such that $(c_1, \dots, c_m) \notin \operatorname{range} T$, i.e., if $\operatorname{range} T \neq \mathbb{F}^m$. But since $n < m$, Theorem 3.8 asserts that T is not surjective, meaning that $\operatorname{range} T \neq W$, as desired. ∎

- Note that while the past two results are typically proven with Gaussian elimination, the abstract approach taken here leads to cleaner proofs.

3.C Matrices

- ***m*-by-*n* matrix**: A rectangular array A of elements of $\mathbb{F}$ with m rows and n columns

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$$

where m and n are positive integers.

- The notation $A_{j,k}$ denotes the entry in row j, column k of A. In other words, the first index refers to the row nunber and the second index refers to the column number.

- **Matrix** (of $T \in \mathcal{L}(V, W)$ with respect to the bases $v_1, \dots, v_n$ of V and $w_1, \dots, w_m$ of W): The m-by-n matrix $\mathcal{M}(T)$ whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m$$

- If the bases are not clear from context, then the notation $\mathcal{M}(T, (v_1, \dots, v_n), (w_1, \dots, w_m))$ is used.
- Another way of wording the definition states that the k^{th} column of $\mathcal{M}(T)$ consists of the scalars needed to write Tv_k as a linear combination of $w_1, \dots, w_m$.
- Assuming standard bases, we "can think of the k^{th} column of $\mathcal{M}(T)$ as the T applied to the k^{th} standard basis vector" (Axler, 2015, p. 71).

- **Sum** (of two m-by-n matrices A, C): The m-by-n matrix $A+C$ defined by $(A+C)_{j,k} = A_{j,k} + C_{j,k}$.
  - Symbolically,

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} + \begin{pmatrix} C_{1,1} & \cdots & C_{1,n} \\ \vdots & & \vdots \\ C_{m,1} & \cdots & C_{m,n} \end{pmatrix} = \begin{pmatrix} A_{1,1}+C_{1,1} & \cdots & A_{1,n}+C_{1,n} \\ \vdots & & \vdots \\ A_{m,1}+C_{m,1} & \cdots & A_{m,n}+C_{m,n} \end{pmatrix}$$

- Suppose $S, T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$.
- **Product** (of an m-by-n matrix A and $\lambda \in \mathbb{F}$): The m-by-n matrix λA defined by $(\lambda A)_{j,k} = \lambda A_{j,k}$.
  - Symbolically,

$$\lambda \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} = \begin{pmatrix} \lambda A_{1,1} & \cdots & \lambda A_{1,n} \\ \vdots & & \vdots \\ \lambda A_{m,1} & \cdots & \lambda A_{m,n} \end{pmatrix}$$

- Suppose $\lambda \in \mathbb{F}$ and $T \in \mathcal{L}(V, W)$. Then $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$.
- $\boldsymbol{\mathbb{F}^{m,n}}$: The set of all m-by-n matrices with entries in $\mathbb{F}$, where m and n are positive integers.
- We have that $\dim \mathbb{F}^{m,n} = mn$.
  - Note that a basis of $\mathbb{F}^{m,n}$ is the set of all m-by-n matrices that have 0s everywhere save a 1 in a single place.
- **Product** (of an m-by-n matrix A and an n-by-p matrix C): The m-by-p matrix AC defined by $(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k}$.
  - We may derive this by noting that if $\mathcal{M}(S) = A$ and $\mathcal{M}(T) = C$, $T : U \to V$ and $S : V \to W$, and $u_1, \dots, u_p$, $v_1, \dots, v_n$, and $w_1, \dots, w_m$ are bases, then

$$\begin{aligned}
(ST)u_k &= S\left(\sum_{r=1}^{n} C_{r,k} v_r\right) \\
&= \sum_{r=1}^{n} C_{r,k} S v_r \\
&= \sum_{r=1}^{n} C_{r,k} \sum_{j=1}^{m} A_{j,r} w_j \\
&= \sum_{j=1}^{m} \left(\sum_{r=1}^{n} A_{j,r} C_{r,k}\right) w_j
\end{aligned}$$

  - Matrix multiplication is not commutative, but is distributive and associative.
- Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$. Then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.
- If A is an m-by-n matrix, then...
  - We let $A_{j,\cdot}$ denote the 1-by-n matrix consisting of row j of A;
  - We let $A_{\cdot,k}$ denote the m-by-1 matrix consisting of column k of A.
- Thus, if A is an m-by-n matrix and C is an n-by-p matrix, then $(AC)_{j,k} = A_{j,\cdot} C_{\cdot,k}$ for all $1 \leq j \leq m$ and $1 \leq k \leq p$.
- Similarly, $(AC)_{\cdot,k} = AC_{\cdot,k}$.

- Lastly, suppose A is an m-by-n matrix and $c = (c_1, \ldots, c_n)$ is an n-by-1 matrix. Then $Ac = c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}$.
  - In other words, Ac is a linear combination of the columns of A, with the scalars that multiply the columns coming from c.