Chapter 4

Polynomials

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- **Real part** (of $a+bi \in \mathbb{C}$): The number a. *Denoted by* $\mathbf{Re}\, \boldsymbol{z}$.
- **Imaginary part** (of $a+bi \in \mathbb{C}$): The number b. *Denoted by* $\mathbf{Im}\, \boldsymbol{z}$.
- **Complex conjugate** (of $z \in \mathbb{C}$): The number $\operatorname{Re} z - (\operatorname{Im} z)i$. *Denoted by* $\bar{\boldsymbol{z}}$.
- **Absolute value** (of $z \in \mathbb{C}$): The number $\sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}$. *Denoted by* $|\boldsymbol{z}|$.
- $z = \bar{z}$ if and only if $z \in \mathbb{R}$.
- Properties of complex numbers.

  Theorem 4.1. *Suppose $w, z \in \mathbb{C}$. Then*

  sum of z and $\bar{z}$
  $z + \bar{z} = 2 \operatorname{Re} z$.

  difference of z and $\bar{z}$
  $z - \bar{z} = 2(\operatorname{Im} z)i$.

  product of z and $\bar{z}$
  $z\bar{z} = |z|^2$.

  additivity and multiplicativity of the complex conjugate
  $\overline{w+z} = \bar{w} + \bar{z}$ *and* $\overline{wz} = \bar{w}\bar{z}$.

  conjugate of conjugate
  $\bar{\bar{z}} = z$.

  real and imaginary parts are bounded by $|z|$
  $|\operatorname{Re} z| \leq |z|$ *and* $|\operatorname{Im} z| \leq |z|$.

  absolute value of the complex conjugate
  $|\bar{z}| = |z|$.

  multiplicativity of absolute value
  $|wz| = |w|\,|z|$.

  Triangle Inequality
  $|w + z| \leq |w| + |z|$.

- If a polynomial is the zero function, then all coefficients are 0.
  - It follows that the coefficients of a polynomial are uniquely determined.
- **Division Algorithm** (for integers): If p, s are nonnegative integers with $s \neq 0$, then there exist nonnegative integers q, r such that $r < s$ and

$$p = sq + r$$

- Analogously,

  Theorem 4.2 (Division Algorithm for Polynomials). *Suppose that $p, s \in \mathcal{P}(\mathbb{F})$, with $s \neq 0$. Then there exist unique polynomials $q, r \in \mathcal{P}(\mathbb{F})$ such that*

  $$p = sq + r$$

  and $\deg r < \deg s$.

  Proof. Let $n = \deg p$ and $m = \deg s$. We divide into two cases ($n < m$ and $n \geq m$). If $n < m$, then take $q = 0$ and $r = p$.

  On the other hand, if $n \geq P$, then let $T : \mathcal{P}_{n-m}(\mathbb{F}) \times \mathcal{P}_{m-1}(\mathbb{F}) \to \mathcal{P}_n(()\mathbb{F})$ be defined by

  $$T(q, r) = sq + r$$

  We can easily confirm that T is a linear map.

  We now seek to prove that $\operatorname{null} T = \{(0, 0)\}$. Let $(q, r) \in \operatorname{null} T$ be arbitrary. Then $sq + r = 0$. It follows that all coefficients of the polynomial $sq+r$ are zero. Consequently, $q = 0$ and $r = 0$, as desired. Therefore, $\dim \operatorname{null} T = 0$. Additionally, Theorem 3.19 implies that

  $$\dim(\mathcal{P}_{n-m}(\mathbb{F}) \times \mathcal{P}_{m-1}(\mathbb{F})) = (n - m + 1) + (m - 1 + 1) = n + 1$$

  It follows by the Fundamental Theorem of Linear Maps that

  $$\begin{aligned} \dim \operatorname{range} T &= \dim(\mathcal{P}_{n-m}(\mathbb{F}) \times \mathcal{P}_{m-1}(\mathbb{F})) - \dim \operatorname{null} T \\ &= n + 1 \\ &= \dim \mathcal{P}_n(\mathbb{F}) \end{aligned}$$

  Thus, by Exercise 2.C.1, $\operatorname{range} T = \mathcal{P}_n(\mathbb{F})$. Therefore, since $p \in \mathcal{P}_n(\mathbb{F})$, we know that there exists $q \in \mathcal{P}_{n-m}(\mathbb{F})$ and $r \in \mathcal{P}_{m-1}(\mathbb{F})$ such that $p = T(q, r) = sq + r$.

  Additionally, we know that q, r are unique: If there exist q', r' such that $T(q', r') = p$, then $T(q - q', r - r') = p - p = 0$, implying since $\operatorname{null} T = \{(0, 0)\}$ that $q - q' = 0$ and $r - r' = 0$, i.e., that $q = q'$ and $r = r'$. ∎

- **Zero** (of $p \in \mathcal{P}(\mathbb{F})$): A number $\lambda \in \mathbb{F}$ such that $p(\lambda) = 0$. *Also known as* **root**.

- **Factor** (of $p \in \mathcal{P}(\mathbb{F})$): A polynomial $s \in \mathcal{P}(\mathbb{F})$ such that there exists a polynomial $q \in \mathcal{P}(\mathbb{F})$ satisfying $p = sq$.

- We now relate zeroes and factors.

  Theorem 4.3. *Suppose $p \in \mathcal{P}(\mathbb{F})$ and $\lambda \in \mathbb{F}$. Then $p(\lambda) = 0$ if and only if there is a polynomial $q \in \mathcal{P}(\mathbb{F})$ such that*

  $$p(z) = (z - \lambda)q(z)$$

  for every $z \in \mathbb{F}$.

- Putting bounds on the number of zeroes a polynomial can have.

  Theorem 4.4. *Suppose $p \in \mathcal{P}(\mathbb{F})$ is a polynomial with degree $m \geq 0$. Then p has at most m distinct zeros in $\mathbb{F}$.*

- We cannot prove the following without complex analysis, but we will state it, regardless.

  Theorem 4.5 (Fundamental Theorem of Algebra). *Every nonconstant polynomial with complex coefficients has a zero.*

- The following proceeds immediately from the Fundamental Theorem of Algebra.

  Theorem 4.6. *If $p \in \mathcal{P}(\mathbb{C})$ is a nonconstant polynomial, then p has a unique factorization (except for the order of the factors) of the form*

  $$p(z) = c(z - \lambda_1) \cdots (z - \lambda_m)$$

  where $c, \lambda_1, \dots, \lambda_m \in \mathbb{C}$.

- We now explore some of the differences between $\mathbb{R}$ and $\mathbb{C}$.

  Theorem 4.7. *Suppose $p \in \mathcal{P}(\mathbb{C})$ is a polynomial with real coefficients. If $\lambda \in \mathbb{C}$ is a zero of p, then so is $\bar{\lambda}$.*

  Theorem 4.8. *Suppose $b, c \in \mathbb{R}$. Then there is a polynomial factorization of the form*

  $$x^2 + bx + c = (x - \lambda_1)(x - \lambda_2)$$

  with $\lambda_1, \lambda_2 \in \mathbb{R}$ if and only if $b^2 \geq 4c$.

  Theorem 4.9. *Suppose $p \in \mathcal{P}(\mathbb{R})$ is a nonconstant polynomial. Then p has a unique factorization (except for the order of factors) of the form*

  $$p(x) = c(x - \lambda_1) \cdots (x - \lambda_m)(x^2 + b_1 x + c_1) \cdots (x^2 + b_M x + c_M)$$

  where $c, \lambda_1, \dots, \lambda_m, b_1, \dots, b_M, c_1, \dots, c_M \in \mathbb{R}$, with $b_j^2 < 4c_j$ for each j.