Chapter 5

Eigenvalues, Eigenvectors, and Invariant Subspaces

5.A Invariant Subspaces

9/8: - Let $T \in \mathcal{L}(V)$, and let V be decomposable into a direct sum of proper subspaces as follows.

$$V = U_1 \oplus \cdots \oplus U_m$$

  - To understand T, we need only understand each each **restriction** of T to a U_j.
  - Since $T|_{U_j}$ may not map U_j onto itself in every case, to use operator-based tools, we need to consider only direct sum decompositions into subspaces that T maps onto themselves, or **invariant subspace**.
- **Invariant subspace** (of V under T): A subspace U of V such that $u \in U$ implies $Tu \in U$, where $T \in \mathcal{L}(V)$.
  - In other words, U is invariant under T iff $T|_U \in \mathcal{L}(U)$.
- Some invariant subspaces under $T \in \mathcal{L}(V)$: $\{0\}$, V, null T, and range T.
- **Invariant subspace problem**: The most famous unsolved problem in functional analysis, dealing with invariant subspaces of operators on infinite-dimensional vector spaces.
- To begin our study of invariant subspaces, we consider the simplest possible type of invariant subspace: those with dimension 1.
- Every 1-dimensional subspace of V is of the form span(v) for some $v \in V$.
  - If span(v) is invariant under $T \in \mathcal{L}(V)$, then $Tv \in \text{span}(v)$.
  - If $Tv \in \text{span}(v)$, then there exists $\lambda \in \mathbb{F}$ such that $Tv = \lambda v$.
- **Eigenvalue** (of T): A number $\lambda \in \mathbb{F}$ such that there exists a nonzero vector $v \in V$ satisfying the equation $Tv = \lambda v$. *Also known as* **characteristic value**.
- "T has a 1-dimensional invariant subspace if and only if T has an eigenvalue" (Axler, 2015, p. 134).
- We now give some conditions λ can satisfy to be deemed an eigenvalue.

  Theorem 5.1. *Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, $I \in \mathcal{L}(V)$ is the identity operator on V, and $\lambda \in \mathbb{F}$. Then the following are equivalent.*

  (a) *λ is an eigenvalue of T.*

(*b*) *$T - \lambda I$ is not injective.*

(*c*) *$T - \lambda I$ is not surjective.*

(*d*) *$T - \lambda I$ is not invertible.*

Proof. Suppose first that λ is an eigenvalue of T. Then

$$\begin{aligned} Tv &= \lambda v \\ Tv &= \lambda I v \\ Tv - \lambda I v &= 0 \\ (T - \lambda I)v &= 0 \end{aligned}$$

for some $v \in V$ such that $v \neq 0$. It follows that $v \in \text{null}(T - \lambda I)$, so by Theorem 3.4, $T - \lambda I$ is not injective, as desired. The proof is symmetric in the other direction. Therefore, conditions (a) and (b) are equivalent.

To prove that (a), (b), (c), and (d) are equivalent at this point, it will suffice to show that (b), (c), and (d) are equivalent. But we have this by Theorem 3.12, as desired. ■

- **Eigenvector** (of T): A nonzero vector $v \in V$ such that there exists a $\lambda \in \mathbb{F}$ satisfying the equation $Tv = \lambda v$.
- Since $Tv = \lambda v$ iff $(T - \lambda I)v = 0$, "a vector $v \in V$ with $v \neq 0$ is an eigenvector of T corresponding to λ if and only if $v \in \text{null}(T - \lambda I)$" (Axler, 2015, p. 135).
- Eigenvectors corresponding to distinct eigenvalues are linearly independent.

Theorem 5.2. *Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of T and $v_1, \dots, v_m$ are corresponding eigenvectors. Then $v_1, \dots, v_m$ is linearly independent.*

Proof. Suppose for the sake of contradiction that $v_1, \dots, v_m$ is linearly dependent. Then by the Linear Dependence Lemma, we may let k be the smallest positive integer such that $v_k \in \text{span}(v_1, \dots, v_{k-1})$. It follows that

$$v_k = a_1 v_1 + \cdots + a_{k-1} v_{k-1}$$

for some $a_1, \dots, a_{k-1} \in \mathbb{F}$. Thus, applying T, we have that

$$\begin{aligned} Tv_k &= a_1 T v_1 + \cdots + a_{k-1} T v_{k-1} \\ \lambda_k v_k &= a_1 \lambda_1 v_1 + \cdots + a_{k-1} \lambda_{k-1} v_{k-1} \end{aligned}$$

If we multiply the first equation by λ_k and subtract the above equation from it, we get that

$$0 = a_1(\lambda_k - \lambda_1)v_1 + \cdots + a_{k-1}(\lambda_k - \lambda_{k-1})v_{k-1}$$

But since k is the smallest positive integer j such that $v_j \in \text{span}(v_1, \dots, v_{j-1})$, we know that $v_1, \dots, v_{k-1}$ are linearly independent. Thus, $a_1(\lambda_k - \lambda_1) = \cdots = a_{k-1}(\lambda_k - \lambda_{k-1}) = 0$. More specifically, since all eigenvalues are distinct (i.e., $\lambda_k - \lambda_j \neq 0$ for any $j = 1, \dots, k-1$), we must have that $a_1 = \cdots = a_{k-1} = 0$. But this implies that

$$\begin{aligned} v_k &= a_1 v_1 + \cdots + a_{k-1} v_{k-1} \\ &= 0 \end{aligned}$$

contradicting the fact that v_k, as an eigenvector, is nonzero. ■

- We now put a bound on the number of eigenvalues.

Theorem 5.3. *Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.*

Proof. Let $T \in \mathcal{L}(V)$ have distinct eigenvalues $\lambda_1, \dots, \lambda_m$ and corresponding eigenvectors $v_1, \dots, v_m$. Then by Theorem 5.2, $v_1, \dots, v_m$ is linearly independent. It follows by Theorem 2.3 that $m \leq \dim V$ ■

- **Restriction operator** (of $T : V \to W$ to $U \subset V$): The function $T|_U : U \to W$ defined by $T|_U(u) = Tu$ for all $u \in U$. *Denoted by* $\boldsymbol{T|_U}$.
  - The fact that U is invariant under T is what allows us to consider $T|_U$ to be in $\mathcal{L}(U)$ as opposed to just $\mathcal{L}(V)$.
- **Quotient operator**: The operator $T/U \in \mathcal{L}(V/U)$ defined by $(T/U)(v + U) = Tv + U$ for all $v \in V$.
- Axler (2015) verifies that the restriction operator and the quotient operator actually *are* operators, in general, as defined.