Chapter 7

Operators on Inner Product Spaces

7.A Self-Adjoints and Normal Operators

• **Adjoint** (of $T \in \mathcal{L}(V, W)$): The function $T^* : W \to V$ that satisfies

$$\langle Tv, w \rangle = \langle v, T^*w \rangle$$

for all $v \in V$ and $w \in W$[1].

  – Calculating T^*w: Consider the linear functional $\varphi : V \to \mathbb{F}$ defined by $\varphi(v) = \langle Tv, w \rangle$ for all $v \in V$. By the Riesz Representation Theorem, there exists a unique vector $T^*w \in V$ such that $\varphi(v) = \langle v, T^*w \rangle$ for all $v \in V$. This vector in V will guarantee that $\langle Tv, w \rangle = \varphi(v) = \langle v, T^*w \rangle$ for all $v \in V$, and we can find vectors $T^*w \in V$ for all $w \in W$.

• The adjoint is a linear map.

Theorem 7.1. *If $T \in \mathcal{L}(V, W)$, then $T^* \in \mathcal{L}(W, V)$.*

Proof. Let $T \in \mathcal{L}(V, W)$, let $w_1, w_2 \in W$, and let $\lambda \in \mathbb{F}$. By the definition of T^*, we have that for any $v \in V$,

$$\begin{aligned}
\langle v, T^*(w_1 + w_2) \rangle &= \langle Tv, w_1 + w_2 \rangle & \langle v, T^*(\lambda w_1) \rangle &= \langle Tv, \lambda w_1 \rangle \\
&= \langle Tv, w_1 \rangle + \langle Tv, w_2 \rangle & &= \bar{\lambda} \langle Tv, w_1 \rangle \\
&= \langle v, T^*w_1 \rangle + \langle v, T^*w_2 \rangle & &= \bar{\lambda} \langle v, T^*w_1 \rangle \\
&= \langle v, T^*w_1 + T^*w_2 \rangle & &= \langle v, \lambda T^*w_1 \rangle
\end{aligned}$$

Thus, by the definition of T^*,

$$T^*(w_1 + w_2) = T^*w_1 + T^*w_2 \qquad\qquad T^*(\lambda w_1) = \lambda T^*w$$

so T^* is a linear map, as desired. ∎

• Properties of the adjoint.

Theorem 7.2.

(a) $(S + T)^* = S^* + T^*$ for all $S < T \in \mathcal{L}(V, W)$.

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[1]Note that the word adjoint has another, unrelated meaning in algebra. Fortunately, this other meaning will not be covered in Axler (2015).

Proof. Suppose $S, T \in \mathcal{L}(V, W)$. If $v \in V$ and $w \in W$, then

$$\begin{aligned}
\langle v, (S+T)^* w \rangle &= \langle (S+T)v, w \rangle \\
&= \langle Sv, w \rangle + \langle Tv, w \rangle \\
&= \langle v, S^* w \rangle + \langle v, T^* w \rangle \\
&= \langle v, S^* w + T^* w \rangle
\end{aligned}$$

Thus, $(S+T)^* w = S^* w + T^* w$, as desired. $\blacksquare$

(b) $(\lambda T)^* = \bar{\lambda} T^*$ *for all* $\lambda \in \mathbb{F}$ *and* $T \in \mathcal{L}(V, W)$.

Proof. Suppose $T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$. If $v \in V$ and $w \in W$, then

$$\begin{aligned}
\langle v, (\lambda T)^* w \rangle &= \langle \lambda Tv, w \rangle \\
&= \lambda \langle Tv, w \rangle \\
&= \lambda \langle v, T^* w \rangle \\
&= \langle v, \bar{\lambda} T^* w \rangle
\end{aligned}$$

Thus, $(\lambda T)^* w = \bar{\lambda} T^* w$, as desired. $\blacksquare$

(c) $(T^*)^* = T$ *for all* $T \in \mathcal{L}(V, W)$.

Proof. Suppose $T \in \mathcal{L}(V, W)$. If $v \in V$ and $w \in W$, then

$$\begin{aligned}
\langle w, (T^*)^* v \rangle &= \langle T^* w, v \rangle \\
&= \overline{\langle v, T^* w \rangle} \\
&= \overline{\langle Tv, w \rangle} \\
&= \langle w, Tv \rangle
\end{aligned}$$

Thus, $(T^*)^* v = Tv$, as desired. $\blacksquare$

(d) $I^* = I$, *where* I *is the identity operator on* V.

Proof. If $v, u \in V$, then
$$\langle v, I^* u \rangle = \langle Iv, u \rangle = \langle v, Iu \rangle$$

Thus, $I^* u = Iu$, as desired. $\blacksquare$

(e) $(ST)^* = T^* S^*$ *for all* $T \in \mathcal{L}(V, W)$ *and* $S \in \mathcal{L}(W, U)$. *Here* U *is an inner product space over* $\mathbb{F}$.

Proof. Suppose $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, U)$. If $v \in V$ and $u \in U$, then

$$\begin{aligned}
\langle v, (ST)^* u \rangle &= \langle STv, u \rangle \\
&= \langle Tv, S^* u \rangle \\
&= \langle v, T^* S^* u \rangle
\end{aligned}$$

Thus, $(ST)^* u = T^* S^* u$, as desired. $\blacksquare$

- Null space and range of T^*.

Theorem 7.3. *Suppose* $T \in \mathcal{L}(V, W)$. *Then*

(a) $\operatorname{null} T^* = (\operatorname{range} T)^{\perp}$.

Proof. Let $w \in W$ be an arbitrary element of $\operatorname{null} T^*$. Then $T^* w = 0$ by definition. It follows by Theorem 6.13 that $\langle v, T^* w \rangle = 0$ for all $v \in V$. Thus, by the definition of the adjoint, $\langle Tv, w \rangle = 0$ for all $v \in V$. But this implies that w is orthogonal to every vector in $\operatorname{range} T$ (i.e., the set of all Tv), meaning that $w \in (\operatorname{range} T)^{\perp}$.

The proof is symmetric in the other direction. $\blacksquare$

(b) range $T^* = (\text{null}\, T)^\perp$.

    Proof. We have that

$$\text{range}\, T^* = ((\text{range}\, T^*)^\perp)^\perp \qquad\qquad \text{Theorem 6.22}$$
$$= (\text{null}(T^*)^*)^\perp \qquad\qquad \text{Theorem 7.3a}$$
$$= (\text{null}\, T)^\perp \qquad\qquad \text{Theorem 7.2c}$$

    as desired. ∎

(c) null $T = (\text{range}\, T^*)^\perp$.

    Proof. We have that

$$\text{null}\, T = \text{null}(T^*)^* \qquad\qquad \text{Theorem 7.2c}$$
$$= (\text{range}\, T^*)^\perp \qquad\qquad \text{Theorem 7.3a}$$

    as desired. ∎

(d) range $T = (\text{null}\, T^*)^\perp$.

    Proof. We have that

$$\text{range}\, T = ((\text{range}\, T)^\perp)^\perp \qquad\qquad \text{Theorem 6.22}$$
$$= (\text{null}\, T^*)^\perp \qquad\qquad \text{Theorem 7.3a}$$

    as desired. ∎

- **Conjugate transpose** (of an m-by-n matrix): The n-by-m matrix obtained by interchanging the rows and columns and then taking the complex conjugate of each entry.

  - "If $\mathbb{F} = \mathbb{R}$, then the conjugate transpose of a matrix is the same as its transpose" (Axler, 2015, p. 207).

- The next result shows how to compute the matrix of T^* from the matrix of T. Note, however, that if $\mathcal{M}(T)$ is with respect to nonorthonormal bases, $\mathcal{M}(T^*)$ does not necessarily equal the conjugate transpose of $\mathcal{M}(T)$.

  Theorem 7.4. *Let $T \in \mathcal{L}(V,W)$. Suppose $e_1,\ldots,e_n$ is an orthonormal basis of V and $f_1,\ldots,f_m$ is an orthonormal basis of W. Then*

  $$\mathcal{M}(T^*, (f_1,\ldots,f_m),(e_1,\ldots,e_n))$$

  is the conjugate transpose of

  $$\mathcal{M}(T, (e_1,\ldots,e_n),(f_1,\ldots,f_m))$$

  Proof. Recall that the kth column of $\mathcal{M}(T)$ is given by writing Te_k as a linear combination of the f_j's. Since $f_1,\ldots,f_m$ is an orthonormal basis of W, Theorem 6.12 implies that

  $$Te_k = \langle Te_k, f_1 \rangle\, f_1 + \cdots + \langle Te_k, f_m \rangle\, f_m$$

  Thus, the entry in row j column k of $\mathcal{M}(T)$ is $\langle Te_k, f_j \rangle$. On the other hand, since

  $$T^* f_k = \langle T^* f_k, e_1 \rangle\, e_1 + \cdots + \langle T^* f_k, e_n \rangle\, e_n$$

  we have that the entry in row j column k of $\mathcal{M}(T^*)$ is

  $$\langle T^* f_k, e_j \rangle = \langle f_k, Te_j \rangle$$
  $$= \overline{\langle Te_j, f_k \rangle}$$

  Therefore, the entry in row k column j of $\mathcal{M}(T^*)$ is the complex conjugate of the entry in row j column k of $\mathcal{M}(T)$, as desired. ∎

- **Self-adjoint** (operator $T \in \mathcal{L}(V)$): An operator T such that $T = T^*$. *Also known as* **Hermitian**.

  - In other words, $T \in \mathcal{L}(V)$ is self-adjoint if and only if

$$\langle Tv, w \rangle = \langle v, Tw \rangle$$

  for all $v, w \in V$.

- The sum of two self-adjoint operators is self-adjoint, and the product of a real scalar and a self-adjoint operator is self-adjoint.

- Note the analogy between self-adjoint operator and complex numbers: A complex number z is real iff $z = \bar{z}$, and thus a self-adjoint operator ($T = T^*$) is analogous to a real number.

- Eigenvalues of self-adjoint operators.

  Theorem 7.5. *Every eigenvalue of a self-adjoint operator is real.*

  Proof. Let T be a self-adjoint operator on V, let λ be an eigenvalue of T, and let v be a nonzero vector in V such that $Tv = \lambda v$. Then

$$\lambda \|v\|^2 = \langle \lambda v, v \rangle = \langle Tv, v \rangle = \langle v, Tv \rangle = \langle v, \lambda v \rangle = \bar{\lambda} \|v\|^2$$

  so $\lambda = \bar{\lambda}$, which implies that λ is real, as desired. ∎

- The next result is false for real inner product spaces (consider a rotation matrix), but true for complex ones.

  Theorem 7.6. *Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Suppose $\langle Tv, v \rangle = 0$ for all $v \in V$. Then $T = 0$.*

  Proof. Let $u \in V$ be arbitrary. By inner product algebra, we have that

$$\langle Tu, w \rangle = \frac{\langle T(u + w), u + w \rangle - \langle T(u - w), u - w \rangle}{4} + \frac{\langle T(u + iw), u + iw \rangle - \langle T(u - iw), u - iw \rangle}{4} i$$

  for all $w \in V$. Since each term on the right-hand side of the above equation is of the form $\langle Tv, v \rangle$ and we know by hypothesis that $\langle Tv, v \rangle = 0$ for all $v \in V$, we have that $\langle Tu, w \rangle = 0$ for all $w \in V$. In particular, if we let $w = Tu$, we learn that $\langle Tu, Tu \rangle = 0$, which implies that $Tu = 0$. But this implies that $Tu = 0$ for all $u \in V$, i.e., that $T = 0$. ∎

- The next result provides another example of how self-adjoint operators behave like real numbers, and is also false for real inner product spaces (consider a operator on such a space that is not self-adjoint).

  Theorem 7.7. *Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Then T is self-adjoint if and only if $\langle Tv, v \rangle \in \mathbb{R}$ for every $v \in V$.*

  Proof. Suppose first that T is self-adjoint. Let $v \in V$ be arbitrary. Then

$$\langle Tv, v \rangle - \overline{\langle Tv, v \rangle} = \langle Tv, v \rangle - \langle v, Tv \rangle = \langle Tv, v \rangle - \langle T^*v, v \rangle = \langle (T - T^*)v, v \rangle = \langle 0v, v \rangle = 0$$

  so $\langle Tv, v \rangle = \overline{\langle Tv, v \rangle}$. Therefore, $\langle Tv, v \rangle \in \mathbb{R}$, as desired.

  Now suppose that $\langle Tv, v \rangle \in \mathbb{R}$ for every $v \in V$. Let $v \in V$ be arbitrary. Then

$$\langle (T - T^*)v, v \rangle = \langle Tv, v \rangle - \langle T^*v, v \rangle = \langle Tv, v \rangle - \langle v, Tv \rangle = \langle Tv, v \rangle - \overline{\langle Tv, v \rangle} = 0$$

  Therefore, by Theorem 7.6, $T - T^* = 0$, or $T = T^*$, as desired. ∎

- We now show that on complex *or* real vector spaces, self-adjoint operators that satisfy $\langle Tv, v \rangle = 0$ *must* be the zero operator.

  Theorem 7.8. *Suppose T is a self-adjoint operator on V such that*

  $$\langle Tv, v \rangle = 0$$

  for all $v \in V$. Then $T = 0$.

  Proof. We divide into two cases. If V is complex, invoke Theorem 7.6. If V is real, we continue.

  Let $u \in V$ be arbitrary. By inner product algebra, we have that

  $$\langle Tu, w \rangle = \frac{\langle T(u + w), u + w \rangle - \langle T(u - w), u - w \rangle}{4}$$

  By a symmetric argument to that used in the later part of the proof of Theorem 7.6, we can confirm that $T = 0$. ∎

- **Normal** (operator): An operator that commutes with its adjoint.

  - In other words, $T \in \mathcal{L}(V)$ is normal if
  $$TT^* = T^*T$$

- Every self-adjoint operator is normal.

- We now characterize normal operators.

  Theorem 7.9. *An operator is normal if and only if*

  $$\|Tv\| = \|T^*v\|$$

  for all $v \in V$.

  Proof. Let $T \in \mathcal{L}(V)$.

  Suppose first that T is normal. Then $T^*T - TT^* = 0$. Thus, by Theorem 6.12, $\langle (T^*T - TT^*)v, v \rangle = 0$ for all $v \in V$. It follows that

  $$\langle T^*Tv, v \rangle = \langle TT^*v, v \rangle$$
  $$\langle Tv, Tv \rangle = \langle T^*v, T^*v \rangle$$
  $$\|Tv\|^2 = \|T^*v\|^2$$
  $$\|Tv\| = \|T^*v\|$$

  for all $v \in V$, as desired.

  Now suppose that $\|Tv\| = \|T^*v\|$ for all $v \in V$. Then following the reverse of the procedure for the forward direction, we can easily show that $\langle (T^*T - TT^*)v, v \rangle = 0$ for all $v \in V$. Additionally, by consecutive applications of Theorem 7.2, we have that

  $$(T^*T - TT^*)^* = (T^*T)^* - (TT^*)^*$$
  $$= T^*(T^*)^* - (T^*)^*T^*$$
  $$= T^*T - TT^*$$

  It follows that $T^*T - TT^*$ is self-adjoint. This combined with the previous result implies by Theorem 7.8 that $T^*T - TT^* = 0$. It follows that $T^*T = TT^*$, so T is normal, as desired. ∎

- While an operator and its adjoint may have different eigenvectors, a normal operator and its adjoint have the same eigenvectors.

Theorem 7.10. *Suppose $T \in \mathcal{L}(V)$ is normal and $v \in V$ is an eigenvector of T with eigenvalue λ. Then v is also an eigenvector of T^* with eigenvalue $\bar{\lambda}$.*

Proof. By consecutive applications of Theorem 7.2, we have that

$$
\begin{aligned}
(T - \lambda I)(T - \lambda I)^* &= (T - \lambda I)(T^* - \bar{\lambda} I) \\
&= TT^* - \bar{\lambda} T - \lambda T^* + \lambda \bar{\lambda} I \\
&= T^* T - \lambda T^* - \bar{\lambda} T + \bar{\lambda} \lambda I \\
&= (T^* - \bar{\lambda} I)(T - \lambda I) \\
&= (T - \lambda I)^*(T - \lambda I)
\end{aligned}
$$

Thus, $T - \lambda I$ is self-adjoint. It follows by Theorem 7.9 that

$$
0 = \|(T - \lambda I)v\| = \|(T - \lambda I)^* v\| = \|(T^* - \bar{\lambda} I)v\|
$$

Hence v is an eigenvector of T^* with eigenvalue $\bar{\lambda}$, as desired. ∎

- Normal operators have orthogonal eigenvectors.

Theorem 7.11. *Suppose $T \in \mathcal{L}(V)$ is normal. Then the eigenvectors of T corresponding to distinct eigenvalues are orthogonal.*

Proof. Let α, β be distinct eigenvalues of T, and let u, v be their corresponding eigenvectors. Thus, we have that

$$
\begin{aligned}
(\alpha - \beta) \langle u, v \rangle &= \langle \alpha u, v \rangle - \langle u, \bar{\beta} v \rangle \\
&= \langle Tu, v \rangle - \langle u, T^* v \rangle \qquad \text{Theorem 7.10} \\
&= 0
\end{aligned}
$$

Since $\alpha \neq \beta$ by hypothesis, we must have that $\langle u, v \rangle = 0$. Therefore, u, v are orthogonal, as desired. ∎

7.B   The Spectral Theorem

- Diagonal operators are nice operators.

  - An operator has a diagonal matrix with respect to some basis iff the basis consists of eigenvectors of the operator (see Theorem 5.11).

- The nicest operators are those for which there is an orthonormal basis of V with respect to which the operator has a diagonal matrix.

  - The Spectral Theorem characterizes the operators $T \in \mathcal{L}(V)$ for which there exists an orthonormal basis of V consisting of eigenvectors of T.

  - In particular, it characterizes them as the normal operators when $\mathbb{F} = \mathbb{C}$ and the self-adjoint operators when $\mathbb{F} = \mathbb{R}$.

  - "The Spectral Theorem is probably the most useful tool in the study of operators on inner product spaces" (Axler, 2015, p. 217).

- For the purposes of proving the Spectral Theorem, we will break it into a Complex Spectral Theorem and a Real Spectral Theorem.

- The complex portion is simpler, so we begin with it.

Theorem 7.12 (Complex Spectral Theorem)**.** *Suppoe $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent.*

(a) *T is normal.*

(b) *V has an orthonormal basis consisting of eigenvectors of T.*

(c) *T has a diagonal matrix with respect to some orthonormal basis of V.*

Proof. We have by Theorem 5.11 that (b) and (c) are equivalent, so we will focus on proving the equivalence of (a) and (c).

Suppose first that (c) holds. Since $\mathcal{M}(T)$ is diagonal and $\mathcal{M}(T^*)$ is the conjugate transpose of $\mathcal{M}(T)$, $\mathcal{M}(T^*)$ is diagonal. Therefore, since any two diagonal matrices commute, T is normal, so (a) holds.

Now suppose that (a) holds. By Schur's Theorem, there exists an orthonormal basis $e_1, \ldots, e_n$ of V with respect to which T has an upper triangular matrix. We will show that this matrix is actually diagonal. To begin, since $\mathcal{M}(T)$ is upper triangular, we know that

$$\|Te_1\|^2 = |a_{1,1}|^2$$

Similarly, since T^* is the conjugate *transpose*, we have that

$$\|T^*e_1\|^2 = |a_{1,1}|^2 + \cdots + |a_{1,n}|^2$$

But since $\|Te_1\| = \|T^*e_1\|$ by Theorem 7.9, the two equations above imply that

$$0 = |a_{1,2}|^2 + \cdots + |a_{1,n}|^2$$

Therefore, we know that all entries in row 1 save the first are zero. We may repeat this procedure for every row to finish the proof. ∎

- The next result continues to build on the likeness of normal matrices and real numbers. Specifically, it plays off the fact that if $b, c \in \mathbb{R}$ with $b^2 < 4c$, then $x^2 + bx + c > 0$, i.e., $x^2 + bx + c$ nonzero is an "invertible" real number.

Theorem 7.13. *Suppose $T \in \mathcal{L}(V)$ is self-adjoint and $b, c \in \mathbb{R}$ are such that $b^2 < 4c$. Then*

$$T^2 + bT + cI$$

is invertible.

Proof. To prove that $T^2 + bT + cI$ is invertible, Theorem 3.18 tells us that it will suffice to show that T is injective. To do this, Theorem 3.4 tells us that we must verify that $\text{null}(T^2 + bT + cI) \subset \{0\}$, i.e., that if $v \in V$ is nonzero, then $(T^2 + bT + cI)v \neq 0$. Let's begin.

Let $v \in V$ be arbitrary. Then we have that

$$
\begin{aligned}
\langle (T^2 + bT + cI)v, v \rangle &= \langle T^2 v, v \rangle + b \langle Tv, v \rangle + c \langle v, v \rangle \\
&= \langle Tv, Tv \rangle + b \langle Tv, v \rangle + c\|v\|^2 \\
&\geq \|Tv\|^2 - |b|\|Tv\|\|v\| + c\|v\|^2 \qquad \text{Cauchy-Schwarz Inequality} \\
&= \left(\|Tv\| - \frac{|b|\|v\|}{2} \right)^2 + \left(c - \frac{b^2}{4} \right) \|v\|^2 \\
&> 0
\end{aligned}
$$

The overall strict inequality implies by the contrapositive of Theorem 6.12 that $(T^2 + bT + cI)v \neq 0$, as desired. ∎

- Like Theorem 5.5 told us that operators on *finite-dimensional nonzero complex* vector spaces have eigenvalues, the following tells us that *self-adjoint* operators on *any nonzero* vector space have eigenvalues.

Theorem 7.14. *Suppose $V \neq \{0\}$ and $T \in \mathcal{L}(V)$ is a self-adjoint operator. Then T has an eigenvalue.*

Proof. Let V be a real inner product space, let $n = \dim V$, and let $v \in V$ be arbitrary and nonzero. Since $v, Tv, T^2v, \ldots, T^nv$ has length $n + 1 > \dim V$, it is linearly dependent. Thus, there exist $a_0, \ldots, a_n \in \mathbb{F}$ such that

$$0 = a_0v + a_1Tv + \cdots + a_nT^nv$$

If we let the a's be the coefficients of a degree n polynomial, then we have by Theorem 4.9 that

$$a_0 + a_1x + \cdots + a_nx^n = c(x^2 + b_1x + c_1)\cdots(x^2 + b_Mx + c_M)(x - \lambda_1)\cdots(x - \lambda_m)$$

where $c \in \mathbb{R}$ is nonzero, each $b_j, c_j, \lambda_j \in \mathbb{R}$, each $b_j^2 < 4c_j$, $m + M \geq 1$, and the equation holds for all $x \in \mathbb{R}$. It follows that

$$
\begin{aligned}
0 &= a_0v + a_1Tv + \cdots + a_nT^nv \\
&= (a_0I + a_1T + \cdots + a_nT^n)v \\
&= c(T^2 + b_1T + c_1I)\cdots(T_2 + b_MT + c_MI)(T - \lambda_1I)\cdots(T - \lambda_mI)v
\end{aligned}
$$

Since T is self-adjoint and $b_j, c_j \in \mathbb{R}$ satisfy $b_j^2 < 4c_j$ for each j, we have by consecutive applications of Theorem 7.13 that each $T^2 + b_jT + c_jI$ is invertible. Thus, if we multiply both sides of the above equation by $1/c$ (recall that $c \neq 0$) and $(T^2 + b_jT + c_jI)^{-1}$ for each j, we obtain

$$0 = (T - \lambda I)\cdots(T - \lambda_mI)v$$

Therefore, by an argument symmetric to that used in the last paragraph of the proof of Theorem 5.5, we have that T has an eigenvalue, as desired. ∎

- Invariant subspaces and self-adjoint operators.

Theorem 7.15. *Suppose $T \in \mathcal{L}(V)$ is self-adjoint and U is a subspace of V that is invariant under T. Then*

(a) *$U^\perp$ is invariant under T.*

    Proof. Let $v \in U^\perp$ be arbitrary, and let u be any element of U. Then

$$\langle Tv, u \rangle = \langle v, Tu \rangle = 0$$

    where the first equality holds because T is self-adjoint and the second equality holds because U is invariant under T (so $Tu \in U$, and we know that the inner product of an element of $U^\perp$ with an element of U is 0). Thus, since $\langle Tv, u \rangle = 0$ for all $u \in U$, $Tv \in U^\perp$, as desired. ∎

(b) *$T|_U \in \mathcal{L}(U)$ is self-adjoint.*

    Proof. If $u, v \in U$, then

$$\langle (T|_U)u, v \rangle = \langle Tu, v \rangle = \langle u, Tv \rangle = \langle u, (T|_U)v \rangle$$

    as desired. ∎

(c) *$T|_{U^\perp} \in \mathcal{L}(U^\perp)$ is self-adjoint.*

    Proof. The proof is symmetric to that of Theorem 7.15b. ∎

- We can now prove the real portion of the spectral theorem.

Theorem 7.16 (Real Spectral Theorem). *Suppose $\mathbb{F} = \mathbb{R}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent.*

(a) T is self-adjoint.

(b) V has an orthonormal basis consisting of eigenvectors of T.

(c) T has a diagonal matrix with respect to some orthonormal basis of V.

Proof. We will prove that (a) implies (b), (b) implies (c), and (c) implies (a). Let's begin.

First, suppose that T is self-adjoint. We induct on $\dim V$. For the base case $\dim V = 1$, we must have $Tv = \lambda v$ for any $v \in V$. Thus, take $e = v/\|v\|$ as an orthonormal basis of V consisting of eigenvectors of T. Now suppose inductively that (a) implies (b) for all real inner product spaces of dimension less than $\dim V > 1$. Suppose $T \in \mathcal{L}(V)$ is self-adjoint. By Theorem 7.14, we may let v be an eigenvector of T. It follows that $u = v/\|v\|$ is a normal eigenvector of T. Let $U = \text{span}(u)$. Then U is a subspace of V that is invariant under T, so we have by Theorem 7.15c that $T|_{U^\perp} \in \mathcal{L}(U^\perp)$ is self-adjoint. But since $\dim U^\perp = \dim V - \dim U = \dim V - 1$, we have by the inductive hypothesis that there is an orthonormal basis of $U^\perp$ consisting of eigenvectors of $T|_{U^\perp}$. Adjoining u to this list gives an orthonormal basis of V consisting of eigenvectors of T, as desired.

Second, suppose that V has an orthonormal basis $e_1, \ldots, e_n$ consisting of eigenvectors of T. Then since

$$Te_j = 0e_1 + \cdots + 0e_{j-1} + \lambda_j e_j + 0e_{j+1} + \cdots + 0e_n$$

for all j, we have by the definition that $\mathcal{M}(T, (e_1, \ldots, e_n))$ is diagonal, as desired.

Third, suppose that T has a diagonal matrix $\mathcal{M}(T)$ with respect to some orthonormal basis of V. In a real inner product space, $\overline{\mathcal{M}(T)} = \mathcal{M}(T)$. Additionally, any diagonal matrix is equal to its transpose. Thus, $T = T^*$, so T is self-adjoint, as desired. ∎

7.C   Positive Operators and Isometries

10/11:

- **Positive** ($T \in \mathcal{L}(V)$): A self-adjoint operator $T \in \mathcal{L}(V)$ such that

$$\langle Tv, v \rangle \geq 0$$

for all $v \in V$. *Also known as* **positive semidefinite** (operator).

     – Note that if V is complex, Theorem 7.7 implies based on the condition that $\langle Tv, v \rangle \geq 0$ for all $v \in V$ that T is self-adjoint. Therefore, in this case, we need not explicitly postulate that T is self-adjoint.

- **Square root** (of $T \in \mathcal{L}(V)$): An operator R such that $R^2 = T$.

- The following characterization of positive operators is directly analogous to the characterization of nonnegative complex numbers.

Theorem 7.17. *Let $T \in \mathcal{L}(V)$. Then the following are equivalent.*

(a) *T is positive.*

(b) *T is self-adjoint and all the eigenvalues of T are nonnegative.*

(c) *T has a positive square root.*

(d) *T has a self-adjoint square root.*

(e) *There exists an operator $R \in \mathcal{L}(V)$ such that $T = R^*R$.*

Proof. We will prove that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (a). Let's begin.

First, suppose that T is positive. Then by definition, T is self-adjoint. Additionally, let $\lambda \in \mathbb{F}$ be an eigenvalue of T. It follows by the definition of positive operators and by the positivity of the inner product that

$$0 \leq \langle Tv, v \rangle = \langle \lambda v, v \rangle = \lambda \langle v, v \rangle$$
$$0 \leq \lambda$$

as desired.

Second, suppose that T is self-adjoint and all the eigenvalues of T are nonnegative. Since T is self-adjoint, the Real and Complex Spectral Theorems imply that there exists an orthonormal basis $e_1, \ldots, e_n$ of V consisting of eigenvectors of T. Let $\lambda_1, \ldots, \lambda_n$ be the corresponding eigenvalues; it follows by hypothesis that $\lambda_j \geq 0$ for all j. We now define $R \in \mathcal{L}(V)$ by

$$Re_j = \sqrt{\lambda_j} e_j$$

for all j. To prove that R is positive, let $v \in V$ be arbitrary. Suppose $v = a_1 e_1 + \cdots + a_n e_n$ where $a_1, \ldots, a_n \in \mathbb{F}$. Then

$$
\begin{aligned}
\langle Rv, v \rangle &= \langle R(a_1 e_1 + \cdots + a_n e_n), a_1 e_1 + \cdots + a_n e_n \rangle \\
&= \left\langle \sqrt{\lambda_1} a_1 e_1 + \cdots + \sqrt{\lambda_n} a_n e_n, a_1 e_1 + \cdots + a_n e_n \right\rangle \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \left\langle \sqrt{\lambda_i} a_i e_i, a_j e_j \right\rangle \\
&= \sum_{i=1}^{n} \left\langle \sqrt{\lambda_i} a_i e_i, a_i e_i \right\rangle \\
&= \sum_{i=1}^{n} \sqrt{\lambda_i} \\
&\geq 0
\end{aligned}
$$

as desired. Furthermore, $R^2 e_j = \lambda_j e_j = T e_j$ for each j, so by Theorem 3.1, $R^2 = T$, as desired.

Third, suppose that T has a positive square root R. Then by the definition of a positive operator, R is self-adjoint as well, as desired.

Fourth, suppose that T has a self-adjoint square root R. Since R is self-adjoint, $R = R^*$. Therefore,

$$T = R^2 = R^* R$$

as desired.

Fifth, suppose that there exists an operator $R \in \mathcal{L}(V)$ such that $T = R^* R$. To prove that T is positive, it will suffice to show that it is self-adjoint and that $\langle Tv, v \rangle \geq 0$ for all $v \in V$. First off, T is self-adjoint since

$$T^* = (R^* R)^* = R^* (R^*)^* = R^* R = T$$

Second, we have that

$$\langle Tv, v \rangle = \langle R^* Rv, v \rangle = \langle Rv, Rv \rangle \geq 0$$

for all $v \in V$. Therefore, T is positive, as desired. $\blacksquare$

- Since each nonnegative number has a unique nonnegative square root, the next result makes sense by analogy.

Theorem 7.18. *Every positive operator on V has a unique positive square root.*

Proof. Let T be a positive operator on V, let $v \in V$ be an eigenvector of T, let $\lambda \in \mathbb{F}$ be the corresponding eigenvalue, and let R be a positive square root of T (Theorem 7.17 guarantees that at least one such operator exists). Since T is positive, Theorem 7.17 implies that $\lambda \geq 0$. Thus, to prove that R is unique, we will prove that $Rv = \sqrt{\lambda} v$. This will imply that the behavior of R on the eigenvectors of T is uniquely determined. It will follows since there is a basis of V consisting of the eigenvectors of T (by the Real and Complex Spectral Theorems), the behavior of R on V (and hence R) is uniquely determined. Let's begin.

Since R is positive (hence self-adjoint), the Real and Complex Spectral Theorems assert that there exists an orthonormal basis $e_1, \ldots, e_n$ of V consisting of eigenvectors of R. Additionally, because R is positive, the corresponding eigenvalues $\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}$ are nonnegative.

Now let

$$v = a_1 e_1 + \cdots + a_n e_n$$

for $a_1, \ldots, a_n \in \mathbb{F}$. Then

$$a_1 \lambda e_1 + \cdots + a_n \lambda e_n = Tv = R^2 v = a_1 \lambda_1 e_1 + \cdots + a_n \lambda_n e_n$$

so since $e_1, \ldots, e_n$ is linearly independent, $a_j(\lambda - \lambda_j) = 0$ for all j. It follows that

$$v = \sum_{\{j : \lambda_j = \lambda\}} a_j e_j$$

so that

$$
\begin{aligned}
Rv &= \sum_{j=1}^{n} a_j \sqrt{\lambda_j} e_j \\
&= \sum_{\{j : \lambda_j = \lambda\}} a_j \sqrt{\lambda_j} e_j \\
&= \sum_{\{j : \lambda_j = \lambda\}} a_j \sqrt{\lambda} e_j \\
&= \sqrt{\lambda} v
\end{aligned}
$$

as desired. ∎

- **Isometry**: An operator $S \in \mathcal{L}(V)$ such that

$$\|Sv\| = \|v\|$$

for all $v \in V$.

  – In other words, an isometry is an operator that preserves norms.

- **Orthogonal** (operator): An isometry on a real inner product space.

- **Unitary** (operator): An isometry on a complex inner product space.

- Characterizing isometries.

  Theorem 7.19. *Suppose $S \in \mathcal{L}(V)$. Then the following are equivalent.*

  (a) *S is an isometry.*

  (b) *$\langle Su, Sv \rangle = \langle u, v \rangle$ for all $u, v \in V$.*

  (c) *$Se_1, \ldots, Se_n$ is orthonormal for every orthonormal list of vectors $e_1, \ldots, e_n$ in V.*

  (d) *There exists an orthonormal basis $e_1, \ldots, e_n$ of V such that $Se_1, \ldots, Se_n$ is orthonormal.*

  (e) *$S^* S = I$.*

  (f) *$SS^* = I$.*

  (g) *S^* is an isometry.*

  (h) *S is invertible and $S^{-1} = S^*$.*

Proof. We will prove that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (g) $\Rightarrow$ (h) $\Rightarrow$ (a). Let's begin.

First, suppose that S is an isometry. Let $u, v \in V$ be arbitrary. We divide into two cases (V is a real inner product space and V is a complex inner product space). If V is a real inner product space, then

$$\langle Su, Sv \rangle = \frac{\|Su + Sv\|^2 - \|Su - Sv\|^2}{4} \qquad \text{Exercise 6.A.19}$$
$$= \frac{\|S(u + v)\|^2 - \|S(u - v)\|^2}{4}$$
$$= \frac{\|u + v\|^2 - \|u - v\|^2}{4}$$
$$= \langle u, v \rangle \qquad \text{Exercise 6.A.19}$$

as desired. On the other hand, if V is a complex vector space, then the proof is symmetric to the above except with the use of Exercise 6.A.20 instead of Exercise 6.A.19.

Second, suppose that $\langle Su, Sv \rangle = \langle u, v \rangle$ for all $u, v \in V$. Let $e_1, \ldots, e_n$ be an orthonormal list of vectors in V. Then by hypothesis,

$$\langle Se_i, Se_j \rangle = \langle e_i, e_j \rangle = \delta_{ij}$$

for all $1 \leq i, j \leq n$, proving that $Se_1, \ldots, Se_n$ is orthonormal, as desired.

Third, suppose that $Se_1, \ldots, Se_n$ is orthonormal for every orthonormal list of vectors $e_1, \ldots, e_n$ in V. By Theorem 6.14, there exists an orthonormal basis $e_1, \ldots, e_n$ of V. It follows by hypothesis that $Se_1, \ldots, Se_n$ is orthonormal, as desired.

Fourth, suppose that there exists an orthonormal basis $e_1, \ldots, e_n$ of V such that $Se_1, \ldots, Se_n$ is orthonormal. Then

$$\langle S^* Se_j, e_k \rangle = \langle Se_j, Se_k \rangle = \delta_{jk} = \langle e_j, e_k \rangle$$

for all $1 \leq j, k \leq n$. It follows that if $u, v \in V$, then

$$\langle S^* Su, v \rangle = \langle S^* S(a_1 e_1 + \cdots + a_n e_n), b_1 e_1 + \cdots + b_n e_n \rangle$$
$$= \langle S^* Sa_1 e_1, b_1 e_1 \rangle + \cdots + \langle S^* Sa_n e_n, b_n e_n \rangle$$
$$= \langle a_1 e_1, b_1 e_1 \rangle + \cdots + \langle a_n e_n, b_n e_n \rangle$$
$$= \langle a_1 e_1 + \cdots + a_n e_n, b_1 e_1 + \cdots + b_n e_n \rangle$$
$$= \langle u, v \rangle$$

Therefore, $S^* S = I$, as desired.

Fifth, suppose that $S^* S = I$. Then by Exercise 3.D.10, $SS^* = I$, as desired.

Sixth, suppose that $SS^* = I$. To prove that S^* is an isometry, it will suffice to show that $\|S^* v\| = \|v\|$ for all $v \in V$. Let $v \in V$ be arbitrary. Then

$$\|S^* v\|^2 = \langle S^* v, S^* v \rangle = \langle SS^* v, v \rangle = \langle v, v \rangle = \|v\|^2$$

Taking square roots yields the desired equality.

Seventh, suppose that S^* is an isometry. It follows by our previous chain of proofs that $(S^*)^* S^* = SS^* = I$ and $S^*(S^*)^* = S^* S = I$. Therefore, S is invertible with inverse $S^{-1} = S^*$, as desired.

Eighth, suppose that S is invertible and $S^{-1} = S^*$. Then if $v \in V$, we have that

$$\|Sv\|^2 = \langle Sv, Sv \rangle = \langle S^* Sv, v \rangle = \langle S^{-1} Sv, v \rangle = \langle v, v \rangle = \|v\|^2$$

Taking square roots yields the desired equality. $\blacksquare$

- It follows from (e) and (f) that every isometry is normal.

- Thus, characterizations of normal operators (e.g., the Complex Spectral Theorem) can be used to give descriptions of isometries.

Theorem 7.20. *Suppose V is a complex inner product space and $S \in \mathcal{L}(V)$. Then the following are equivalent.*

(a) *S is an isometry.*

(b) *There is an orthonormal basis of V consisting of eigenvectors of S whose corresponding eigenvalues all have absolute value 1.*

Proof. Suppose first that S is an isometry. Then by the Complex Spectral Theorem, there is an orthonormal basis $e_1, \ldots, e_n$ of V consisting of the eigenvectors of S. Let $\lambda_1, \ldots, \lambda_n$ be the corresponding eigenvalues. Then for each $j = 1, \ldots, n$, we have that

$$|\lambda_j| = \|\lambda_j e_j\| = \|S e_j\| = \|e_j\| = 1$$

as desired.

Now suppose that there is an orthonormal basis $e_1, \ldots, e_n$ of V consisting of eigenvectors of S whose corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ all have absolute value 1. Let $v \in V$ be arbitrary. Then by Theorem 6.12, we have that

$$v = \langle v, e_1 \rangle \, e_1 + \cdots + \langle v, e_n \rangle \, e_n \qquad\qquad \|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2$$

It follows that

$$\begin{aligned}
Sv &= \langle v, e_1 \rangle \, S e_1 + \cdots + \langle v, e_n \rangle \, S e_n \\
&= \langle v, e_1 \rangle \, \lambda_1 e_1 + \cdots + \langle v, e_n \rangle \, \lambda_n e_n
\end{aligned}$$

Thus, we have that

$$\begin{aligned}
\|Sv\|^2 &= \langle \langle v, e_1 \rangle \, \lambda_1 e_1 + \cdots + \langle v, e_n \rangle \, \lambda_n e_n, \langle v, e_1 \rangle \, \lambda_1 e_1 + \cdots + \langle v, e_n \rangle \, \lambda_n e_n \rangle \\
&= \langle \langle v, e_1 \rangle \, \lambda_1 e_1, \langle v, e_1 \rangle \, \lambda_1 e_1 \rangle + \cdots + \langle \langle v, e_n \rangle \, \lambda_n e_n, \langle v, e_n \rangle \, \lambda_n e_n \rangle \\
&= \langle v, e_1 \rangle \, \lambda_1 \cdot \overline{\langle v, e_1 \rangle \, \lambda_1} \cdot \langle e_1, e_1 \rangle + \cdots + \langle v, e_n \rangle \, \lambda_n \cdot \overline{\langle v, e_n \rangle \, \lambda_n} \cdot \langle e_n, e_n \rangle \\
&= \langle v, e_1 \rangle \, \lambda_1 \cdot \overline{\langle v, e_1 \rangle} \bar{\lambda}_1 \cdot 1 + \cdots + \langle v, e_n \rangle \, \lambda_n \cdot \overline{\langle v, e_n \rangle} \bar{\lambda}_n \cdot 1 \\
&= |\langle v, e_1 \rangle|^2 |\bar{\lambda}_1|^2 + \cdots + |\langle v, e_n \rangle|^2 |\bar{\lambda}_n|^2 \\
&= |\langle v, e_1 \rangle|^2 \cdot 1 + \cdots + |\langle v, e_n \rangle|^2 \cdot 1 \\
&= \|v\|^2
\end{aligned}$$

Taking square roots yields the desired equality.   ∎