Chapter 7

Operators on Inner Product Spaces

7.A Self-Adjoints and Normal Operators

10/7:

- **Adjoint** (of $T \in \mathcal{L}(V,W)$): The function $T^* : W \to V$ that satisfies

$$\langle Tv, w \rangle = \langle v, T^*w \rangle$$

for all $v \in V$ and $w \in W$[1].

  - Calculating T^*w: Consider the linear functional $\varphi : V \to \mathbb{F}$ defined by $\varphi(v) = \langle Tv, w \rangle$ for all $v \in V$. By the Riesz Representation Theorem, there exists a unique vector $T^*w \in V$ such that $\varphi(v) = \langle v, T^*w \rangle$ for all $v \in V$. This vector in V will guarantee that $\langle Tv, w \rangle = \varphi(v) = \langle v, T^*w \rangle$ for all $v \in V$, and we can find vectors $T^*w \in V$ for all $w \in W$.

- The adjoint is a linear map.

Theorem 7.1. *If $T \in \mathcal{L}(V,W)$, then $T^* \in \mathcal{L}(W,V)$.*

Proof. Let $T \in \mathcal{L}(V,W)$, let $w_1, w_2 \in W$, and let $\lambda \in \mathbb{F}$. By the definition of T^*, we have that for any $v \in V$,

$$\begin{aligned} \langle v, T^*(w_1 + w_2) \rangle &= \langle Tv, w_1 + w_2 \rangle \\ &= \langle Tv, w_1 \rangle + \langle Tv, w_2 \rangle \\ &= \langle v, T^*w_1 \rangle + \langle v, T^*w_2 \rangle \\ &= \langle v, T^*w_1 + T^*w_2 \rangle \end{aligned} \qquad \begin{aligned} \langle v, T^*(\lambda w_1) \rangle &= \langle Tv, \lambda w_1 \rangle \\ &= \bar{\lambda} \langle Tv, w_1 \rangle \\ &= \bar{\lambda} \langle v, T^*w_1 \rangle \\ &= \langle v, \lambda T^*w_1 \rangle \end{aligned}$$

Thus, by the definition of T^*,

$$T^*(w_1 + w_2) = T^*w_1 + T^*w_2 \qquad T^*(\lambda w_1) = \lambda T^*w$$

so T^* is a linear map, as desired. ■

- Properties of the adjoint.

Theorem 7.2.

(a) $(S+T)^* = S^* + T^*$ *for all* $S < T \in \mathcal{L}(V,W)$.

[1] Note that the word adjoint has another, unrelated meaning in algebra. Fortunately, this other meaning will not be covered in Axler (2015).

Proof. Suppose $S, T \in \mathcal{L}(V, W)$. If $v \in V$ and $w \in W$, then

$$\begin{aligned}\langle v, (S+T)^* w\rangle &= \langle (S+T)v, w\rangle \\ &= \langle Sv, w\rangle + \langle Tv, w\rangle \\ &= \langle v, S^*w\rangle + \langle v, T^*w\rangle \\ &= \langle v, S^*w + T^*w\rangle\end{aligned}$$

Thus, $(S+T)^*w = S^*w + T^*w$, as desired. ∎

(b) $(\lambda T)^* = \bar{\lambda} T^*$ *for all* $\lambda \in \mathbb{F}$ *and* $T \in \mathcal{L}(V, W)$.

Proof. Suppose $T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$. If $v \in V$ and $w \in W$, then

$$\begin{aligned}\langle v, (\lambda T)^* w\rangle &= \langle \lambda T v, w\rangle \\ &= \lambda \langle Tv, w\rangle \\ &= \lambda \langle v, T^* w\rangle \\ &= \left\langle v, \bar{\lambda} T^* w\right\rangle\end{aligned}$$

Thus, $(\lambda T)^* w = \bar{\lambda} T^* w$, as desired. ∎

(c) $(T^*)^* = T$ *for all* $T \in \mathcal{L}(V, W)$.

Proof. Suppose $T \in \mathcal{L}(V, W)$. If $v \in V$ and $w \in W$, then

$$\begin{aligned}\langle w, (T^*)^* v\rangle &= \langle T^* w, v\rangle \\ &= \overline{\langle v, T^* w\rangle} \\ &= \overline{\langle Tv, w\rangle} \\ &= \langle w, Tv\rangle\end{aligned}$$

Thus, $(T^*)^* v = Tv$, as desired. ∎

(d) $I^* = I$*, where* I *is the identity operator on* V.

Proof. If $v, u \in V$, then

$$\langle v, I^* u\rangle = \langle Iv, u\rangle = \langle v, Iu\rangle$$

Thus, $I^* u = Iu$, as desired. ∎

(e) $(ST)^* = T^* S^*$ *for all* $T \in \mathcal{L}(V, W)$ *and* $S \in \mathcal{L}(W, U)$. *Here* U *is an inner product space over* $\mathbb{F}$.

Proof. Suppose $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, U)$. If $v \in V$ and $u \in U$, then

$$\begin{aligned}\langle v, (ST)^* u\rangle &= \langle STv, u\rangle \\ &= \langle Tv, S^* u\rangle \\ &= \langle v, T^* S^* u\rangle\end{aligned}$$

Thus, $(ST)^* u = T^* S^* u$, as desired. ∎

- Null space and range of T^*.

Theorem 7.3. *Suppose* $T \in \mathcal{L}(V, W)$. *Then*

(a) $\operatorname{null} T^* = (\operatorname{range} T)^\perp$.

Proof. Let $w \in W$ be an arbitrary element of $\operatorname{null} T^*$. Then $T^* w = 0$ by definition. It follows by Theorem 6.13 that $\langle v, T^* w\rangle = 0$ for all $v \in V$. Thus, by the definition of the adjoint, $\langle Tv, w\rangle = 0$ for all $v \in V$. But this implies that w is orthogonal to every vector in $\operatorname{range} T$ (i.e., the set of all Tv), meaning that $w \in (\operatorname{range} T)^\perp$.
The proof is symmetric in the other direction. ∎

(*b*) $\operatorname{range} T^* = (\operatorname{null} T)^\perp$.

Proof. We have that

$$\begin{aligned}
\operatorname{range} T^* &= ((\operatorname{range} T^*)^\perp)^\perp && \text{Theorem 6.22}\\
&= (\operatorname{null}(T^*)^*)^\perp && \text{Theorem 7.3a}\\
&= (\operatorname{null} T)^\perp && \text{Theorem 7.2c}
\end{aligned}$$

as desired. ■

(*c*) $\operatorname{null} T = (\operatorname{range} T^*)^\perp$.

Proof. We have that

$$\begin{aligned}
\operatorname{null} T &= \operatorname{null}(T^*)^* && \text{Theorem 7.2c}\\
&= (\operatorname{range} T^*)^\perp && \text{Theorem 7.3a}
\end{aligned}$$

as desired. ■

(*d*) $\operatorname{range} T = (\operatorname{null} T^*)^\perp$.

Proof. We have that

$$\begin{aligned}
\operatorname{range} T &= ((\operatorname{range} T)^\perp)^\perp && \text{Theorem 6.22}\\
&= (\operatorname{null} T^*)^\perp && \text{Theorem 7.3a}
\end{aligned}$$

as desired. ■

- **Conjugate transpose** (of an m-by-n matrix): The n-by-m matrix obtained by interchanging the rows and columns and then taking the complex conjugate of each entry.
  - "If $\mathbb{F} = \mathbb{R}$, then the conjugate transpose of a matrix is the same as its transpose" (Axler, 2015, p. 207).
- The next result shows how to compute the matrix of T^* from the matrix of T. Note, however, that if $\mathcal{M}(T)$ is with respect to nonorthonormal bases, $\mathcal{M}(T^*)$ does not necessarily equal the conjugate transpose of $\mathcal{M}(T)$.

Theorem 7.4. *Let $T \in \mathcal{L}(V, W)$. Suppose $e_1, \dots, e_n$ is an orthonormal basis of V and $f_1, \dots, f_m$ is an orthonormal basis of W. Then*

$$\mathcal{M}(T^*, (f_1, \dots, f_m), (e_1, \dots, e_n))$$

is the conjugate transpose of

$$\mathcal{M}(T, (e_1, \dots, e_n), (f_1, \dots, f_m))$$

Proof. Recall that the k^{th} column of $\mathcal{M}(T)$ is given by writing Te_k as a linear combination of the f_j's. Since $f_1, \dots, f_m$ is an orthonormal basis of W, Theorem 6.12 implies that

$$Te_k = \langle Te_k, f_1 \rangle f_1 + \cdots + \langle Te_k, f_m \rangle f_m$$

Thus, the entry in row j column k of $\mathcal{M}(T)$ is $\langle Te_k, f_j \rangle$. On the other hand, since

$$T^* f_k = \langle T^* f_k, e_1 \rangle e_1 + \cdots + \langle T^* f_k, e_n \rangle e_n$$

we have that the entry in row j column k of $\mathcal{M}(T^*)$ is

$$\begin{aligned}
\langle T^* f_k, e_j \rangle &= \langle f_k, Te_j \rangle\\
&= \overline{\langle Te_j, f_k \rangle}
\end{aligned}$$

Therefore, the entry in row k column j of $\mathcal{M}(T^*)$ is the complex conjugate of the entry in row j column k of $\mathcal{M}(T)$, as desired. ■

- **Self-adjoint** (operator $T \in \mathcal{L}(V)$): An operator T such that $T = T^*$. *Also known as* **Hermitian**.
  - In other words, $T \in \mathcal{L}(V)$ is self-adjoint if and only if
  $$\langle Tv, w \rangle = \langle v, Tw \rangle$$
  for all $v, w \in V$.
- The sum of two self-adjoint operators is self-adjoint, and the product of a real scalar and a self-adjoint operator is self-adjoint.
- Note the analogy between self-adjoint operator and complex numbers: A complex number z is real iff $z = \bar{z}$, and thus a self-adjoint operator ($T = T^*$) is analogous to a real number.
- Eigenvalues of self-adjoint operators.

  Theorem 7.5. *Every eigenvalue of a self-adjoint operator is real.*

  Proof. Let T be a self-adjoint operator on V, let λ be an eigenvalue of T, and let v be a nonzero vector in V such that $Tv = \lambda v$. Then
  $$\lambda \|v\|^2 = \langle \lambda v, v \rangle = \langle Tv, v \rangle = \langle v, Tv \rangle = \langle v, \lambda v \rangle = \bar{\lambda} \|v\|^2$$
  so $\lambda = \bar{\lambda}$, which implies that λ is real, as desired. ■

- The next result is false for real inner product spaces (consider a rotation matrix), but true for complex ones.

  Theorem 7.6. *Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Suppose $\langle Tv, v \rangle = 0$ for all $v \in V$. Then $T = 0$.*

  Proof. Let $u \in V$ be arbitrary. By inner product algebra, we have that
  $$\langle Tu, w \rangle = \frac{\langle T(u+w), u+w \rangle - \langle T(u-w), u-w \rangle}{4} + \frac{\langle T(u+iw), u+iw \rangle - \langle T(u-iw), u-iw \rangle}{4} i$$
  for all $w \in V$. Since each term on the right-hand side of the above equation is of the form $\langle Tv, v \rangle$ and we know by hypothesis that $\langle Tv, v \rangle = 0$ for all $v \in V$, we have that $\langle Tu, w \rangle = 0$ for all $w \in V$. In particular, if we let $w = Tu$, we learn that $\langle Tu, Tu \rangle = 0$, which implies that $Tu = 0$. But this implies that $Tu = 0$ for all $u \in V$, i.e., that $T = 0$. ■

- The next result provides another example of how self-adjoint operators behave like real numbers, and is also false for real inner product spaces (consider a operator on such a space that is not self-adjoint).

  Theorem 7.7. *Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Then T is self-adjoint if and only if $\langle Tv, v \rangle \in \mathbb{R}$ for every $v \in V$.*

  Proof. Suppose first that T is self-adjoint. Let $v \in V$ be arbitrary. Then
  $$\langle Tv, v \rangle - \overline{\langle Tv, v \rangle} = \langle Tv, v \rangle - \langle v, Tv \rangle = \langle Tv, v \rangle - \langle T^* v, v \rangle = \langle (T - T^*)v, v \rangle = \langle 0v, v \rangle = 0$$
  so $\langle Tv, v \rangle = \overline{\langle Tv, v \rangle}$. Therefore, $\langle Tv, v \rangle \in \mathbb{R}$, as desired.

  Now suppose that $\langle Tv, v \rangle \in \mathbb{R}$ for every $v \in V$. Let $v \in V$ be arbitrary. Then
  $$\langle (T - T^*)v, v \rangle = \langle Tv, v \rangle - \langle T^* v, v \rangle = \langle Tv, v \rangle - \langle v, Tv \rangle = \langle Tv, v \rangle - \overline{\langle Tv, v \rangle} = 0$$
  Therefore, by Theorem 7.6, $T - T^* = 0$, or $T = T^*$, as desired. ■

- We now show that on complex *or* real vector spaces, self-adjoint operators that satisfy $\langle Tv, v\rangle = 0$ *must* be the zero operator.

  Theorem 7.8. *Suppose T is a self-adjoint operator on V such that*

  $$\langle Tv, v\rangle = 0$$

  for all $v \in V$. Then $T = 0$.

  Proof. We divide into two cases. If V is complex, invoke Theorem 7.6. If V is real, we continue.

  Let $u \in V$ be arbitrary. By inner product algebra, we have that

  $$\langle Tu, w\rangle = \frac{\langle T(u+w), u+w\rangle - \langle T(u-w), u-w\rangle}{4}$$

  By a symmetric argument to that used in the later part of the proof of Theorem 7.6, we can confirm that $T = 0$. ■

- **Normal** (operator): An operator that commutes with its adjoint.
  - In other words, $T \in \mathcal{L}(V)$ is normal if
    $$TT^* = T^*T$$

- Every self-adjoint operator is normal.

- We now characterize normal operators.

  Theorem 7.9. *An operator is normal if and only if*

  $$\|Tv\| = \|T^*v\|$$

  for all $v \in V$.

  Proof. Let $T \in \mathcal{L}(V)$.

  Suppose first that T is normal. Then $T^*T - TT^* = 0$. Thus, by Theorem 6.12, $\langle (T^*T - TT^*)v, v\rangle = 0$ for all $v \in V$. It follows that

  $$\begin{aligned}\langle T^*Tv, v\rangle &= \langle TT^*v, v\rangle\\ \langle Tv, Tv\rangle &= \langle T^*v, T^*v\rangle\\ \|Tv\|^2 &= \|T^*v\|^2\\ \|Tv\| &= \|T^*v\|\end{aligned}$$

  for all $v \in V$, as desired.

  Now suppose that $\|Tv\| = \|T^*v\|$ for all $v \in V$. Then following the reverse of the procedure for the forward direction, we can easily show that $\langle (T^*T - TT^*)v, v\rangle = 0$ for all $v \in V$. Additionally, by consecutive applications of Theorem 7.2, we have that

  $$\begin{aligned}(T^*T - TT^*)^* &= (T^*T)^* - (TT^*)^*\\ &= T^*(T^*)^* - (T^*)^*T^*\\ &= T^*T - TT^*\end{aligned}$$

  It follows that $T^*T - TT^*$ is self-adjoint. This combined with the previous result implies by Theorem 7.8 that $T^*T - TT^* = 0$. It follows that $T^*T = TT^*$, so T is normal, as desired. ■

- While an operator and its adjoint may have different eigenvectors, a normal operator and its adjoint have the same eigenvectors.

Theorem 7.10. *Suppose $T \in \mathcal{L}(V)$ is normal and $v \in V$ is an eigenvector of T with eigenvalue λ. Then v is also an eigenvector of T^* with eigenvalue $\bar{\lambda}$.*

Proof. By consecutive applications of Theorem 7.2, we have that

$$\begin{aligned}(T-\lambda I)(T-\lambda I)^* &= (T-\lambda I)(T^*-\bar{\lambda}I) \\ &= TT^* - \bar{\lambda}T - \lambda T^* + \lambda\bar{\lambda}I \\ &= T^*T - \lambda T^* - \bar{\lambda}T + \bar{\lambda}\lambda I \\ &= (T^*-\bar{\lambda}I)(T-\lambda I) \\ &= (T-\lambda I)^*(T-\lambda I)\end{aligned}$$

Thus, $T-\lambda I$ is self-adjoint. It follows by Theorem 7.9 that

$$0 = \|(T-\lambda I)v\| = \|(T-\lambda I)^*v\| = \left\|(T^*-\bar{\lambda}I)v\right\|$$

Hence v is an eigenvector of T^* with eigenvalue $\bar{\lambda}$, as desired. ■

- Normal operators have orthogonal eigenvectors.

  Theorem 7.11. *Suppose $T \in \mathcal{L}(V)$ is normal. Then the eigenvectors of T corresponding to distinct eigenvalues are orthogonal.*

  Proof. Let α, β be distinct eigenvalues of T, and let u, v be their corresponding eigenvectors. Thus, we have that

  $$\begin{aligned}(\alpha-\beta)\langle u,v\rangle &= \langle \alpha u, v\rangle - \langle u, \bar{\beta}v\rangle \\ &= \langle Tu, v\rangle - \langle u, T^*v\rangle && \text{Theorem 7.10} \\ &= 0\end{aligned}$$

  Since $\alpha \neq \beta$ by hypothesis, we must have that $\langle u, v\rangle = 0$. Therefore, u, v are orthogonal, as desired. ■

7.B The Spectral Theorem

- Diagonal operators are nice operators.
  - An operator has a diagonal matrix with respect to some basis iff the basis consists of eigenvectors of the operator (see Theorem 5.11).
- The nicest operators are those for which there is an orthonormal basis of V with respect to which the operator has a diagonal matrix.
  - The Spectral Theorem characterizes the operators $T \in \mathcal{L}(V)$ for which there exists an orthonormal basis of V consisting of eigenvectors of T.
  - In particular, it characterizes them as the normal operators when $\mathbb{F} = \mathbb{C}$ and the self-adjoint operators when $\mathbb{F} = \mathbb{R}$.
  - "The Spectral Theorem is probably the most useful tool in the study of operators on inner product spaces" (Axler, 2015, p. 217).
- For the purposes of proving the Spectral Theorem, we will break it into a Complex Spectral Theorem and a Real Spectral Theorem.
- The complex portion is simpler, so we begin with it.

  Theorem 7.12 (Complex Spectral Theorem)**.** *Suppoe $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent.*

(a) *T is normal.*

(b) *V has an orthonormal basis consisting of eigenvectors of T.*

(c) *T has a diagonal matrix with respect to some orthonormal basis of V.*

Proof. We have by Theorem 5.11 that (b) and (c) are equivalent, so we will focus on proving the equivalence of (a) and (c).

Suppose first that (c) holds. Since $\mathcal{M}(T)$ is diagonal and $\mathcal{M}(T^*)$ is the conjugate transpose of $\mathcal{M}(T)$, $\mathcal{M}(T^*)$ is diagonal. Therefore, since any two diagonal matrices commute, T is normal, so (a) holds.

Now suppose that (a) holds. By Schur's Theorem, there exists an orthonormal basis $e_1, \dots, e_n$ of V with respect to which T has an upper triangular matrix. We will show that this matrix is actually diagonal. To begin, since $\mathcal{M}(T)$ is upper triangular, we know that

$$\|Te_1\|^2 = |a_{1,1}|^2$$

Similarly, since T^* is the conjugate *transpose*, we have that

$$\|T^*e_1\|^2 = |a_{1,1}|^2 + \cdots + |a_{1,n}|^2$$

But since $\|Te_1\| = \|T^*e_1\|$ by Theorem 7.9, the two equations above imply that

$$0 = |a_{1,2}|^2 + \cdots + |a_{1,n}|^2$$

Therefore, we know that all entries in row 1 save the first are zero. We may repeat this procedure for every row to finish the proof. ∎

- The next result continues to build on the likeness of normal matrices and real numbers. Specifically, it plays off the fact that if $b, c \in \mathbb{R}$ with $b^2 < 4c$, then $x^2 + bx + c > 0$, i.e., $x^2 + bx + c$ nonzero is an "invertible" real number.

Theorem 7.13. *Suppose $T \in \mathcal{L}(V)$ is self-adjoint and $b, c \in \mathbb{R}$ are such that $b^2 < 4c$. Then*

$$T^2 + bT + cI$$

is invertible.

Proof. To prove that $T^2 + bT + cI$ is invertible, Theorem 3.18 tells us that it will suffice to show that T is injective. To do this, Theorem 3.4 tells us that we must verify that $\operatorname{null}(T^2 + bT + cI) \subset \{0\}$, i.e., that if $v \in V$ is nonzero, then $(T^2 + bT + cI)v \neq 0$. Let's begin.

Let $v \in V$ be arbitrary. Then we have that

$$\begin{aligned}
\left\langle (T^2 + bT + cI)v, v \right\rangle &= \left\langle T^2 v, v \right\rangle + b\left\langle Tv, v\right\rangle + c\left\langle v, v\right\rangle \\
&= \langle Tv, Tv\rangle + b\langle Tv, v\rangle + c\|v\|^2 \\
&\geq \|Tv\|^2 - |b|\|Tv\|\|v\| + c\|v\|^2 && \text{Cauchy-Schwarz Inequality} \\
&= \left(\|Tv\| - \frac{|b|\|v\|}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)\|v\|^2 \\
&> 0
\end{aligned}$$

The overall strict inequality implies by the contrapositive of Theorem 6.12 that $(T^2 + bT + cI)v \neq 0$, as desired. ∎

- Like Theorem 5.5 told us that operators on *finite-dimensional nonzero complex* vector spaces have eigenvalues, the following tells us that *self-adjoint* operators on *any nonzero* vector space have eigenvalues.

Theorem 7.14. *Suppose $V \neq \{0\}$ and $T \in \mathcal{L}(V)$ is a self-adjoint operator. Then T has an eigenvalue.*

Proof. Let V be a real inner product space, let $n = \dim V$, and let $v \in V$ be arbitrary and nonzero. Since $v, Tv, T^2v, \dots, T^nv$ has length $n+1 > \dim V$, it is linearly dependent. Thus, there exist $a_0, \dots, a_n \in \mathbb{F}$ such that

$$0 = a_0v + a_1Tv + \cdots + a_nT^nv$$

If we let the a's be the coefficients of a degree n polynomial, then we have by Theorem 4.9 that

$$a_0 + a_1x + \cdots + a_nx^n = c(x^2 + b_1x + c_1)\cdots(x^2 + b_Mx + c_M)(x - \lambda_1)\cdots(x - \lambda_m)$$

where $c \in \mathbb{R}$ is nonzero, each $b_j, c_j, \lambda_j \in \mathbb{R}$, each $b_j^2 < 4c_j$, $m + M \geq 1$, and the equation holds for all $x \in \mathbb{R}$. It follows that

$$\begin{aligned}
0 &= a_0v + a_1Tv + \cdots + a_nT^nv \\
&= (a_0I + a_1T + \cdots + a_nT^n)v \\
&= c(T^2 + b_1T + c_1I)\cdots(T_2 + b_MT + c_MI)(T - \lambda_1I)\cdots(T - \lambda_mI)v
\end{aligned}$$

Since T is self-adjoint and $b_j, c_j \in \mathbb{R}$ satisfy $b_j^2 < 4c_j$ for each j, we have by consecutive applications of Theorem 7.13 that each $T^2 + b_jT + c_jI$ is invertible. Thus, if we multiply both sides of the above equation by $1/c$ (recall that $c \neq 0$) and $(T^2 + b_jT + c_jI)^{-1}$ for each j, we obtain

$$0 = (T - \lambda I)\cdots(T - \lambda_mI)v$$

Therefore, by an argument symmetric to that used in the last paragraph of the proof of Theorem 5.5, we have that T has an eigenvalue, as desired. ∎

- Invariant subspaces and self-adjoint operators.

Theorem 7.15. *Suppose $T \in \mathcal{L}(V)$ is self-adjoint and U is a subspace of V that is invariant under T. Then*

(*a*) *$U^\perp$ is invariant under T.*

Proof. Let $v \in U^\perp$ be arbitrary, and let u be any element of U. Then

$$\langle Tv, u\rangle = \langle v, Tu\rangle = 0$$

where the first equality holds because T is self-adjoint and the second equality holds because U is invariant under T (so $Tu \in U$, and we know that the inner product of an element of $U^\perp$ with an element of U is 0). Thus, since $\langle Tv, u\rangle = 0$ for all $u \in U$, $Tv \in U^\perp$, as desired. ∎

(*b*) *$T|_U \in \mathcal{L}(U)$ is self-adjoint.*

Proof. If $u, v \in U$, then

$$\langle (T|_U)u, v\rangle = \langle Tu, v\rangle = \langle u, Tv\rangle = \langle u, (T|_U)v\rangle$$

as desired. ∎

(*c*) *$T|_{U^\perp} \in \mathcal{L}(U^\perp)$ is self-adjoint.*

Proof. The proof is symmetric to that of Theorem 7.15b. ∎

- We can now prove the real portion of the spectral theorem.

Theorem 7.16 (Real Spectral Theorem)**.** *Suppose $\mathbb{F} = \mathbb{R}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent.*

(*a*) *T is self-adjoint.*

(*b*) *V has an orthonormal basis consisting of eigenvectors of T.*

(*c*) *T has a diagonal matrix with respect to some orthonormal basis of V.*

Proof. We will prove that (a) implies (b), (b) implies (c), and (c) implies (a). Let's begin.

First, suppose that T is self-adjoint. We induct on $\dim V$. For the base case $\dim V = 1$, we must have $Tv = \lambda v$ for any $v \in V$. Thus, take $e = v/\|v\|$ as an orthonormal basis of V consisting of eigenvectors of T. Now suppose inductively that (a) implies (b) for all real inner product spaces of dimension less than $\dim V > 1$. Suppose $T \in \mathcal{L}(V)$ is self-adjoint. By Theorem 7.14, we may let v be an eigenvector of T. It follows that $u = v/\|v\|$ is a normal eigenvector of T. Let $U = \operatorname{span}(u)$. Then U is a subspace of V that is invariant under T, so we have by Theorem 7.15c that $T|_{U^\perp} \in \mathcal{L}(U^\perp)$ is self-adjoint. But since $\dim U^\perp = \dim V - \dim U = \dim V - 1$, we have by the inductive hypothesis that there is an orthonormal basis of $U^\perp$ consisting of eigenvectors of $T|_{U^\perp}$. Adjoining u to this list gives an orthonormal basis of V consisting of eigenvectors of T, as desired.

Second, suppose that V has an orthonormal basis $e_1, \dots, e_n$ consisting of eigenvectors of T. Then since

$$Te_j = 0e_1 + \cdots + 0e_{j-1} + \lambda_j e_j + 0e_{j+1} + \cdots + 0e_n$$

for all j, we have by the definition that $\mathcal{M}(T, (e_1, \dots, e_n))$ is diagonal, as desired.

Third, suppose that T has a diagonal matrix $\mathcal{M}(T)$ with respect to some orthonormal basis of V. In a real inner product space, $\overline{\mathcal{M}(T)} = \mathcal{M}(T)$. Additionally, any diagonal matrix is equal to its transpose. Thus, $T = T^*$, so T is self-adjoint, as desired. ∎