

# Chapter 3

## Linear Maps

### 3.A The Vector Space of Linear Maps

- 9/5: • **Linear map** (from  $V$  to  $W$ ): A function  $T : V \rightarrow W$  with the following properties. *Also known as linear transformation.*

**additivity**

$$T(u + v) = Tu + Tv \text{ for all } u, v \in V.$$

**homogeneity**

$$T(\lambda v) = \lambda(Tv) \text{ for all } \lambda \in \mathbb{F} \text{ and all } v \in V.$$

- Note that for linear maps,  $Tv$  means the same as the more standard functional notation  $T(v)$ .

- $\mathcal{L}(V, W)$ : The set of all linear maps from  $V$  to  $W$ .
- **Zero map**: The function  $0 \in \mathcal{L}(V, W)$  that takes each element of some vector space to the additive identity of another vector space. *Defined by*

$$0v = 0$$

- **Identity map**: The function  $I \in \mathcal{L}(V, V)$  on some vector space that takes each element to itself. *Defined by*

$$Iv = v$$

- We can also formalize more esoteric linear processes as linear maps.
  - For example,  $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$  can be thought of as the differentiation map  $Dp = p'$ . This formalizes the fact that  $(f + g)' = f' + g'$  and  $(\lambda f)' = \lambda f'$ .
  - We can do the same with integration: Let  $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathbb{R})$  be described by  $Tp = \int_0^1 p(x) dx$ . This formalizes the fact that integrals are additive and homogenous.
  - Axler (2015) gives a number more examples.
- We now prove that there exists a unique linear map from a vector space of dimension  $n$  to any  $n$  vectors in another vector space.

**Theorem 3.1.** *Suppose  $v_1, \dots, v_n$  is a basis of  $V$  and  $w_1, \dots, w_n \in W$ . Then there exists a unique linear map  $T : V \rightarrow W$  such that  $Tv_j = w_j$  for each  $j = 1, \dots, n$ .*

*Proof.* First, we define a function  $T : V \rightarrow W$ . We then show that  $T$  satisfies the specified property. After that, we show that it is a linear map. Lastly, we show that it is unique. Let's begin.

Let  $T : V \rightarrow W$  be defined by

$$T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + \cdots + c_nw_n$$

for all  $c_1v_1 + \cdots + c_nv_n \in V$ . Note that this definition is valid since, by Theorem 2.5, each  $v \in V$  can be written in the form  $c_1v_1 + \cdots + c_nv_n$  where  $c_1, \dots, c_n \in \mathbb{F}$ .

To prove that  $Tv_j = w_j$  for all  $j = 1, \dots, n$ , let each  $c_i$  in the above definition equal 0 save  $c_j$ , which we set equal to 1. Then we have

$$\begin{aligned} T(0v_1 + \cdots + 0v_{j-1} + 1v_j + 0v_{j+1} + \cdots + 0v_n) &= 0w_1 + \cdots + 0w_{j-1} + 1w_j + 0w_{j+1} + \cdots + 0w_n \\ T(v_j) &= w_j \end{aligned}$$

as desired.

To prove that  $T$  is a linear map, it will suffice to verify additivity and homogeneity, which we may do as follows. Let  $u, v \in V$  with  $u = a_1v_1 + \cdots + a_nv_n$  and  $v = c_1v_1 + \cdots + c_nv_n$ , and let  $\lambda \in \mathbb{F}$ . Then

$$\begin{aligned} T(u + v) &= T((a_1 + c_1)v_1 + \cdots + (a_n + c_n)v_n) \\ &= (a_1 + c_1)w_1 + \cdots + (a_n + c_n)w_n \\ &= Tu + Tv \end{aligned}$$

and

$$\begin{aligned} T(\lambda v) &= T(\lambda c_1v_1 + \cdots + \lambda c_nv_n) \\ &= \lambda c_1w_1 + \cdots + \lambda c_nw_n \\ &= \lambda Tv \end{aligned}$$

as desired.

Now suppose  $\tilde{T} \in \mathcal{L}(V, W)$  satisfies  $\tilde{T}v_j = w_j$  for all  $j = 1, \dots, n$ . To prove that  $T = \tilde{T}$ , it will suffice to show that  $\tilde{T}(c_1v_1 + \cdots + c_nv_n) = T(c_1v_1 + \cdots + c_nv_n)$  for all  $c_1v_1 + \cdots + c_nv_n \in V$ . Let  $c_1v_1 + \cdots + c_nv_n \in V$  be arbitrary. We know that  $\tilde{T}(v_j) = w_j$  for all  $j = 1, \dots, n$ . It follows since  $\tilde{T}$  is a linear map (specifically, since it's homogenous) that  $c_jw_j = c_j\tilde{T}(v_j) = \tilde{T}(c_jv_j)$  for all  $j = 1, \dots, n$ . Similarly, the additivity of  $\tilde{T}$  implies that

$$\begin{aligned} T(c_1v_1 + \cdots + c_nv_n) &= c_1w_1 + \cdots + c_nw_n \\ &= \tilde{T}(c_1v_1) + \cdots + \tilde{T}(c_nv_n) \\ &= \tilde{T}(c_1v_1 + \cdots + c_nv_n) \end{aligned}$$

as desired. ■

- **Sum** (of  $S, T \in \mathcal{L}(V, W)$ ): The linear map  $(S + T) \in \mathcal{L}(V, W)$  defined by  $(S + T)(v) = Sv + Tv$  for all  $v \in V$ .
- **Product** (of  $T \in \mathcal{L}(V, W)$  and  $\lambda \in \mathbb{F}$ ): The linear map  $(\lambda T) \in \mathcal{L}(V, W)$  defined by  $(\lambda T)(v) = \lambda(Tv)$  for all  $v \in V$ .
- It follows that, under these definitions of addition and multiplication,  $\mathcal{L}(V, W)$  is a vector space.
- **Product** (of  $T \in \mathcal{L}(U, V)$  and  $S \in \mathcal{L}(V, W)$ ): The linear map  $ST \in \mathcal{L}(U, W)$  defined by  $(ST)(u) = S(Tu)$  for all  $u \in U$ .
  - Note that the product is just function composition, but most mathematicians do write  $ST$  instead of  $S \circ T$ .
- Linear maps (with the correct corresponding domains) satisfy the associativity, identity, and distributive properties. However, multiplication of linear maps is not necessarily commutative.

- $(T_1 T_2) T_3 = T_1 (T_2 T_3)$ .
- $T I_V = I_W T = T$  (note that if  $T \in \mathcal{L}(V, W)$ ,  $I_V \in \mathcal{L}(V, V)$  and  $I_W \in \mathcal{L}(W, W)$ ).
- $(S_1 + S_2) T = S_1 T + S_2 T$  and  $S(T_1 + T_2) = S T_1 + S T_2$ .

- Linear maps send 0 to 0.

**Theorem 3.2.** Suppose  $T \in \mathcal{L}(V, W)$ . Then  $T(0) = 0$ .

*Proof.* By additivity, we have

$$\begin{aligned} T(0) &= T(0 + 0) = T(0) + T(0) \\ 0 &= T(0) \end{aligned}$$

as desired. ■

### 3.B Null Spaces and Ranges

- **Null space** (of  $T \in \mathcal{L}(V, W)$ ): The subset of  $V$  consisting of those vectors that  $T$  maps to 0. Also known as **kernel**. Denoted by **null  $T$** . Given by

$$\text{null } T = \{v \in V : Tv = 0\}$$

- The null space is a subspace.

**Theorem 3.3.** Suppose  $T \in \mathcal{L}(V, W)$ . Then  $\text{null } T$  is a subspace of  $V$ .

*Proof.* To prove that  $\text{null } T$  is a subspace of  $V$ , it will suffice to show that  $0 \in \text{null } T$ ,  $u, v \in \text{null } T$  implies that  $u + v \in \text{null } T$ , and  $u \in \text{null } T$  and  $\lambda \in \mathbb{F}$  imply  $\lambda u \in \text{null } T$ . Let's begin.

By Theorem 3.2,  $T(0) = 0$ . Therefore,  $0 \in \text{null } T$ , as desired.

Let  $u, v \in \text{null } T$  be arbitrary. Then by additivity

$$T(u + v) = Tu + Tv = 0 + 0 = 0$$

so  $u + v \in \text{null } T$ , as desired.

Let  $u \in \text{null } T$  and  $\lambda \in \mathbb{F}$  be arbitrary. Then by homogeneity,

$$T(\lambda u) = \lambda Tu = \lambda 0 = 0$$

so  $\lambda u \in \text{null } T$ , as desired. ■

- **Injective** (function): A function  $T : V \rightarrow W$  such that  $Tu = Tv$  implies  $u = v$ . Also known as **one-to-one**.
- If 0 is the only vector that gets mapped to 0, then  $T$  is injective.

**Theorem 3.4.** Let  $T \in \mathcal{L}(V, W)$ . Then  $T$  is injective if and only if  $\text{null } T = \{0\}$ .

*Proof.* Suppose first that  $T$  is injective. To prove that  $\text{null } T = \{0\}$ , it will suffice to show that  $0 \in \text{null } T$  and for every  $v \in \text{null } T$ ,  $v = 0$ . By Theorem 3.3,  $0 \in \text{null } T$ . Now let  $v \in \text{null } T$  be arbitrary. By the definition of the null space, we have  $Tv = 0$ . By Theorem 3.2, we have  $T(0) = 0$ . Thus, by transitivity, we have that  $Tv = T(0)$ . It follows by injectivity that  $v = 0$ , as desired.

Now suppose that  $\text{null } T = \{0\}$ . To prove that  $T$  is injective, it will suffice to show that if  $Tu = Tv$ , then  $u = v$ . Suppose  $u, v \in V$  satisfy  $Tu = Tv$ . Then

$$0 = Tu - Tv = T(u - v)$$

so  $(u - v) \in \text{null } T = \{0\}$ . It follows that  $u - v = 0$ , i.e., that  $u = v$ , as desired. ■

- **Range** (of  $T \in \mathcal{L}(V, W)$ ): The subset of  $W$  consisting of those vectors that are of the form  $Tv$  for some  $v \in V$ . Also known as **image**. Denoted by **range** $T$ . Given by

$$\text{range } T = \{Tv : v \in V\}$$

- The range is a subspace.

**Theorem 3.5.** Suppose  $T \in \mathcal{L}(V, W)$ . Then  $\text{range } T$  is a subspace of  $W$ .

*Proof.* To prove that  $\text{range } T$  is a subspace of  $W$ , it will suffice to show that  $0 \in \text{range } T$ ,  $w_1, w_2 \in \text{range } T$  implies that  $(w_1 + w_2) \in \text{range } T$ , and  $w \in \text{range } T$  and  $\lambda \in \mathbb{F}$  imply  $\lambda w \in \text{range } T$ . Let's begin.

By the definition of a vector space,  $0 \in V$ . By Theorem 3.2,  $T(0) = 0$ . Therefore,  $0 \in \text{range } T$ , as desired.

Let  $w_1, w_2 \in \text{range } T$  be arbitrary. Then there exist  $v_1, v_2 \in V$  such that  $Tv_1 = w_1$  and  $Tv_2 = w_2$ . It follows by additivity that

$$T(v_1 + v_2) = Tv_1 + Tv_2 = w_1 + w_2$$

Therefore, since  $v_1 + v_2 \in V$ , we have that  $(w_1 + w_2) \in \text{range } T$ , as desired.

Let  $w \in \text{range } T$  and  $\lambda \in \mathbb{F}$  be arbitrary. Then there exists  $v \in V$  such that  $Tv = w$ . It follows by homogeneity that

$$T(\lambda v) = \lambda Tv = \lambda w$$

Therefore, since  $\lambda v \in V$ , we have that  $\lambda w \in \text{range } T$ , as desired. ■

- **Surjective** (function): A function  $T : V \rightarrow W$  such that  $\text{range } T = W$ . Also known as **onto**.
- We now prove a very important theorem.

**Theorem 3.6** (Fundamental Theorem of Linear Maps). Suppose  $V$  is finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Then  $\text{range } T$  is finite-dimensional and

$$\dim V = \dim \text{null } T + \dim \text{range } T$$

*Proof.* By Theorem 3.3,  $\text{null } T$  is a subspace of  $V$  finite-dimensional. Thus, by Theorem 2.4,  $\text{null } T$  is finite-dimensional. It follows by Theorem 2.7 that we may let  $u_1, \dots, u_m$  be a basis of  $\text{null } T$ . As a basis of a subspace of  $V$ ,  $u_1, \dots, u_m$  is a linearly independent list of vectors in  $V$ . Consequently, by Theorem 2.8, we may extend it to a basis  $u_1, \dots, u_m, v_1, \dots, v_n$  of  $V$ .

Having established this terminology, we can now see that to prove that  $\text{range } T$  is finite-dimensional, it will suffice to show that  $Tv_1, \dots, Tv_n$  spans it. To show that  $\text{span}(Tv_1, \dots, Tv_n) = \text{range } T$ , it will suffice to show that every  $b_1Tv_1 + \dots + b_nTv_n \in \text{span}(Tv_1, \dots, Tv_n)$  is an element of  $\text{range } T$  and that every  $Tv \in \text{range } T$  is an element of  $\text{span}(Tv_1, \dots, Tv_n)$ . Let  $b_1Tv_1 + \dots + b_nTv_n \in \text{span}(Tv_1, \dots, Tv_n)$  be arbitrary. Then

$$\begin{aligned} b_1Tv_1 + \dots + b_nTv_n &= T(b_1v_1 + \dots + b_nv_n) \\ &= T(0u_1 + \dots + 0u_m + b_1v_1 + \dots + b_nv_n) \end{aligned}$$

Therefore, since  $0u_1 + \dots + 0u_m + b_1v_1 + \dots + b_nv_n \in V$  by  $V$ 's closure under addition and scalar multiplication, we have that  $b_1Tv_1 + \dots + b_nTv_n \in \text{range } T$ , as desired. Now let  $Tv \in \text{range } T$  be arbitrary. Since  $v \in V$  and  $u_1, \dots, u_m, v_1, \dots, v_n$  is a basis of  $V$ , Theorem 2.5 implies that  $v = a_1u_1 + \dots + a_mu_m + b_1v_1 + \dots + b_nv_n$  for some  $a_1, \dots, a_m, b_1, \dots, b_n \in \mathbb{F}$ . Therefore,

$$\begin{aligned} Tv &= T(a_1u_1 + \dots + a_mu_m + b_1v_1 + \dots + b_nv_n) \\ &= T(a_1u_1 + \dots + a_mu_m) + T(b_1v_1 + \dots + b_nv_n) \\ &= a_1Tu_1 + \dots + a_mTu_m + b_1Tv_1 + \dots + b_nTv_n \\ &= a_10 + \dots + a_m0 + b_1Tv_1 + \dots + b_nTv_n \\ &= b_1Tv_1 + \dots + b_nTv_n \end{aligned}$$

where each  $Tu_j = 0$  because each  $u_j \in \text{null } T$ , so  $Tv \in \text{span}(Tv_1, \dots, Tv_n)$ , as desired.

Before we can verify the equation from the theorem, we need to establish one last fact: that  $Tv_1, \dots, Tv_n$  is linearly independent. Suppose  $c_1, \dots, c_n \in \mathbb{F}$  make

$$\begin{aligned} c_1Tv_1 + \dots + c_nTv_n &= 0 \\ T(c_1v_1 + \dots + c_nv_n) &= 0 \end{aligned}$$

It follows that  $c_1v_1 + \dots + c_nv_n \in \text{null } T$ . Thus, since  $u_1, \dots, u_m$  is a basis of  $\text{null } T$  by Theorem 2.5, we have that

$$\begin{aligned} c_1v_1 + \dots + c_nv_n &= d_1u_1 + \dots + d_mu_m \\ 0 &= d_1u_1 + \dots + d_mu_m - c_1v_1 - \dots - c_nv_n \end{aligned}$$

for some  $d_1, \dots, d_m \in \mathbb{F}$ . But since  $u_1, \dots, u_m, v_1, \dots, v_n$  is linearly independent as the basis of  $V$ , the above equation implies that  $c_1 = \dots = c_n = 0$ , as desired.

Having established that  $u_1, \dots, u_m, v_1, \dots, v_n$  is a basis of  $V$ ,  $u_1, \dots, u_m$  is a basis of  $\text{null } T$ , and  $Tv_1, \dots, Tv_n$  spans  $\text{range } T$  and is linearly independent in  $\text{range } T$  (i.e., is a basis of  $\text{range } T$ ), we have that

$$\begin{aligned} \dim V &= m + n \\ &= \dim \text{null } T + \dim \text{range } T \end{aligned}$$

as desired. ■

- We can now prove that a linear map to a “smaller” vector space cannot be injective.

**Theorem 3.7.** *Suppose  $V$  and  $W$  are finite-dimensional vector spaces such that  $\dim V > \dim W$ . Then no linear map from  $V$  to  $W$  is injective.*

*Proof.* Let  $T \in \mathcal{L}(V, W)$ . Then

$$\begin{aligned} \dim \text{null } T &= \dim V - \dim \text{range } T && \text{Fundamental Theorem of Linear Maps} \\ &\geq \dim V - \dim W && \text{Theorem 2.11} \\ &> 0 \end{aligned}$$

It follows that  $\text{null } T$  has a basis consisting of a list of one or more vectors. As a linearly independent list, naturally none of these vectors will be equal to the zero vector. Thus, since  $\text{null } T$  contains vectors other than 0, Theorem 3.4 implies that  $T$  is not injective. ■

- Similarly, we can prove that a linear map to a “bigger” vector space cannot be surjective.

**Theorem 3.8.** *Suppose  $V$  and  $W$  are finite-dimensional vector spaces such that  $\dim V < \dim W$ . Then no linear map from  $V$  to  $W$  is surjective.*

*Proof.* Let  $T \in \mathcal{L}(V, W)$ . Then

$$\begin{aligned} \dim \text{range } T &= \dim V - \dim \text{null } T && \text{Fundamental Theorem of Linear Maps} \\ &\leq \dim V && < \dim W \end{aligned}$$

Therefore,  $\text{range } T \neq W$ , so  $T$  cannot be surjective. ■

- Theorems 3.7 and 3.8 allow us to express questions about systems of linear equations in terms of linear maps.

- For example, a question about a **homogenous** system of linear equations could be, “does there exist a nonzero solution to the homogenous system  $\sum_{k=1}^n A_{1,k}x_k = 0, \dots, \sum_{k=1}^n A_{m,k}x_k = 0$ ?”
- If we define  $T : \mathbb{F}^n \rightarrow \mathbb{F}^m$  by

$$T(x_1, \dots, x_n) = \left( \sum_{k=1}^n A_{1,k}x_k, \dots, \sum_{k=1}^n A_{m,k}x_k \right)$$

we can express the system of equations as  $T(x_1, \dots, x_n) = 0$  and ask instead, “is  $\dim \text{null } T > 0$ ?”

- **Homogenous** (system of linear equations): A system of  $m$  linear equations  $\sum_{k=1}^n A_{1,k}x_k = c_1$  through  $\sum_{k=1}^n A_{m,k}x_k = c_m$  such that the constant term  $c_j = 0$  for all  $j = 1, \dots, m$ .
- Continuing with the linear equations example, we can rigorously show the following.

**Theorem 3.9.** *A homogenous system of linear equations with more variables than equations has nonzero solutions.*

*Proof.* In terms of the above,  $T : \mathbb{F}^n \rightarrow \mathbb{F}^m$  where  $n > m$ . Thus, by Theorem 3.7,  $T$  is not injective. Consequently, by Theorem 3.4,  $\dim \text{null } T > 0$ . Therefore, the system has nonzero solutions. ■

**Theorem 3.10.** *An inhomogenous system of linear equations with more equations than variables has no solution for some choice of constant terms.*

*Proof.* In terms of the above,  $T : \mathbb{F}^n \rightarrow \mathbb{F}^m$  where  $m > n$ . We want to know if there exists  $(c_1, \dots, c_m) \in \mathbb{F}^m$  such that  $T(x_1, \dots, x_n) \neq (c_1, \dots, c_m)$  for any  $(x_1, \dots, x_n) \in \mathbb{F}^n$ . In other words, we want to know if there exists  $(c_1, \dots, c_m) \in \mathbb{F}^m$  such that  $(c_1, \dots, c_m) \notin \text{range } T$ , i.e., if  $\text{range } T \neq \mathbb{F}^m$ . But since  $n < m$ , Theorem 3.8 asserts that  $T$  is not surjective, meaning that  $\text{range } T \neq W$ , as desired. ■

- Note that while the past two results are typically proven with Gaussian elimination, the abstract approach taken here leads to cleaner proofs.

### 3.C Matrices

- **$m$ -by- $n$  matrix:** A rectangular array  $A$  of elements of  $\mathbb{F}$  with  $m$  rows and  $n$  columns

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$$

where  $m$  and  $n$  are positive integers.

- The notation  $A_{j,k}$  denotes the entry in row  $j$ , column  $k$  of  $A$ . In other words, the first index refers to the row number and the second index refers to the column number.
- **Matrix** (of  $T \in \mathcal{L}(V, W)$  with respect to the bases  $v_1, \dots, v_n$  of  $V$  and  $w_1, \dots, w_m$  of  $W$ ): The  $m$ -by- $n$  matrix  $\mathcal{M}(T)$  whose entries  $A_{j,k}$  are defined by

$$Tv_k = A_{1,k}w_1 + \cdots + A_{m,k}w_m$$

- If the bases are not clear from context, then the notation  $\mathcal{M}(T, (v_1, \dots, v_n), (w_1, \dots, w_m))$  is used.
- Another way of wording the definition states that the  $k^{\text{th}}$  column of  $\mathcal{M}(T)$  consists of the scalars needed to write  $Tv_k$  as a linear combination of  $w_1, \dots, w_m$ .
- Assuming standard bases, we “can think of the  $k^{\text{th}}$  column of  $\mathcal{M}(T)$  as the  $T$  applied to the  $k^{\text{th}}$  standard basis vector” (Axler, 2015, p. 71).

- **Sum** (of two  $m$ -by- $n$  matrices  $A, C$ ): The  $m$ -by- $n$  matrix  $A + C$  defined by  $(A + C)_{j,k} = A_{j,k} + C_{j,k}$ .  
– Symbolically,

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} + \begin{pmatrix} C_{1,1} & \cdots & C_{1,n} \\ \vdots & & \vdots \\ C_{m,1} & \cdots & C_{m,n} \end{pmatrix} = \begin{pmatrix} A_{1,1} + C_{1,1} & \cdots & A_{1,n} + C_{1,n} \\ \vdots & & \vdots \\ A_{m,1} + C_{m,1} & \cdots & A_{m,n} + C_{m,n} \end{pmatrix}$$

- Suppose  $S, T \in \mathcal{L}(V, W)$ . Then  $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$ .
- **Product** (of an  $m$ -by- $n$  matrix  $A$  and  $\lambda \in \mathbb{F}$ ): The  $m$ -by- $n$  matrix  $\lambda A$  defined by  $(\lambda A)_{j,k} = \lambda A_{j,k}$ .  
– Symbolically,

$$\lambda \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} = \begin{pmatrix} \lambda A_{1,1} & \cdots & \lambda A_{1,n} \\ \vdots & & \vdots \\ \lambda A_{m,1} & \cdots & \lambda A_{m,n} \end{pmatrix}$$

- Suppose  $\lambda \in \mathbb{F}$  and  $T \in \mathcal{L}(V, W)$ . Then  $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$ .
- $\mathbb{F}^{m,n}$ : The set of all  $m$ -by- $n$  matrices with entries in  $\mathbb{F}$ , where  $m$  and  $n$  are positive integers.
- We have that  $\dim \mathbb{F}^{m,n} = mn$ .  
– Note that a basis of  $\mathbb{F}^{m,n}$  is the set of all  $m$ -by- $n$  matrices that have 0s everywhere save a 1 in a single place.
- **Product** (of an  $m$ -by- $n$  matrix  $A$  and an  $n$ -by- $p$  matrix  $C$ ): The  $m$ -by- $p$  matrix  $AC$  defined by  $(AC)_{j,k} = \sum_{r=1}^n A_{j,r} C_{r,k}$ .  
– We may derive this by noting that if  $\mathcal{M}(S) = A$  and  $\mathcal{M}(T) = C$ ,  $T : U \rightarrow V$  and  $S : V \rightarrow W$ , and  $u_1, \dots, u_p$ ,  $v_1, \dots, v_n$ , and  $w_1, \dots, w_m$  are bases, then

$$\begin{aligned} (ST)u_k &= S \left( \sum_{r=1}^n C_{r,k} v_r \right) \\ &= \sum_{r=1}^n C_{r,k} S v_r \\ &= \sum_{r=1}^n C_{r,k} \sum_{j=1}^m A_{j,r} w_j \\ &= \sum_{j=1}^m \left( \sum_{r=1}^n A_{j,r} C_{r,k} \right) w_j \end{aligned}$$

- Matrix multiplication is not commutative, but is distributive and associative.
- Suppose  $T \in \mathcal{L}(U, V)$  and  $S \in \mathcal{L}(V, W)$ . Then  $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$ .
- If  $A$  is an  $m$ -by- $n$  matrix, then...  
– We let  $A_{j,\cdot}$  denote the 1-by- $n$  matrix consisting of row  $j$  of  $A$ ;  
– We let  $A_{\cdot,k}$  denote the  $m$ -by-1 matrix consisting of column  $k$  of  $A$ .
- Thus, if  $A$  is an  $m$ -by- $n$  matrix and  $C$  is an  $n$ -by- $p$  matrix, then  $(AC)_{j,k} = A_{j,\cdot} C_{\cdot,k}$  for all  $1 \leq j \leq m$  and  $1 \leq k \leq p$ .
- Similarly,  $(AC)_{\cdot,k} = AC_{\cdot,k}$ .
- Lastly, suppose  $A$  is an  $m$ -by- $n$  matrix and  $c = (c_1, \dots, c_n)$  is an  $n$ -by-1 matrix. Then  $Ac = c_1 A_{\cdot,1} + \cdots + c_n A_{\cdot,n}$ .  
– In other words,  $Ac$  is a linear combination of the columns of  $A$ , with the scalars that multiply the columns coming from  $c$ .

### 3.D Invertibility and Isomorphic Vector Spaces

- 9/6:
- **Invertible** (linear map): A linear map  $T \in \mathcal{L}(V, W)$  such that there exists a linear map  $S \in \mathcal{L}(V, W)$  such that  $ST$  equals the identity map on  $V$  and  $TS$  equals the identity map on  $W$ .
  - **Inverse** (of  $T \in \mathcal{L}(V, W)$ ): The linear map  $S \in \mathcal{L}(W, V)$  satisfying  $ST = I_V$  and  $TS = I_W$ . Denoted by  $T^{-1}$ .
  - We now justify the use of the word “the” in the definition of the inverse.

**Theorem 3.11.** *An invertible linear map has a unique inverse.*

*Proof.* Suppose  $T \in \mathcal{L}(V, W)$  is invertible and  $S_1, S_2$  are inverses of  $T$ . Then

$$S_1 = S_1 I = S_1 (TS_2) = (S_1 T) S_2 = I S_2 = S_2$$

as desired. ■

- We now give a criterion for invertibility.

**Theorem 3.12.** *A linear map is invertible if and only if it is injective and surjective.*

*Proof.* Let  $T \in \mathcal{L}(V, W)$ . Suppose first that  $T$  is invertible.

To prove that  $T$  is injective, it will suffice to show that for all  $u, v \in V$ ,  $Tu = Tv$  implies that  $u = v$ . Let  $u, v$  be arbitrary elements of  $V$  that satisfy  $Tu = Tv$ . Then

$$u = T^{-1}(Tu) = T^{-1}(Tv) = v$$

as desired.

To prove that  $T$  is surjective, it will suffice to show that  $\text{range } T = W$ . Since  $\text{range } T \subset W$ , we need only show that  $W \subset \text{range } T$ . Let  $w \in W$  be arbitrary. Since  $w = T(T^{-1}w)$  where  $T^{-1}w \in V$ , we have that  $w \in \text{range } T$ , as desired.

Now suppose that  $T$  is injective and surjective. To prove that  $T$  is invertible, we will define a function  $S : W \rightarrow V$ , prove that it is a linear map, prove that  $TS = I_W$ , and prove that  $ST = I_V$ . Let  $Sw$  be the unique element of  $V$  such that  $T(Sw) = w$  (the surjectivity of  $T$  guarantees that there exists an element of  $V$  that  $T$  maps to  $w$ , and the injectivity of  $T$  guarantees the uniqueness of said element).

To prove that  $S$  is a linear map, it will suffice to show that  $S$  is additive and homogenous. To verify additivity, first note that the additivity of  $T$  implies that

$$T(Sw_1 + Sw_2) = T(Sw_1) + T(Sw_2) = w_1 + w_2$$

But since the above equation implies that  $Sw_1 + Sw_2$  is the unique element of  $V$  that  $T$  maps to  $w_1 + w_2$ , we have by the definition of  $S$  that  $S(w_1 + w_2) = Sw_1 + Sw_2$ . The proof is symmetric for homogeneity.

To prove that  $TS = I_W$ , we need only appeal to the definition of  $S$ , which states that  $(TS)w = T(Sw) = w$  for all  $w \in W$ . It immediately follows that  $TS = I_W$ .

To prove that  $ST = I_V$ , first note that for all  $v \in V$ ,

$$T((ST)v) = (TS)(Tv) = I(Tv) = Tv$$

It follows by the injectivity of  $T$  that  $(ST)v = v$ , i.e., that  $ST = I_V$ , as desired. ■

- **Isomorphism:** An invertible linear map.
- **Isomorphic** (vector spaces): Two vector spaces such that there exists an isomorphism from one vector space onto the other one.



- Isomorphic vector spaces have the same dimension.

**Theorem 3.13.** *Two finite-dimensional vector spaces over  $\mathbb{F}$  are isomorphic if and only if they have the same dimension.*

*Proof.* Suppose  $V, W$  are isomorphic finite-dimensional vector spaces over  $\mathbb{F}$ . Then there exists an isomorphism  $T : V \rightarrow W$ . By the definition of isomorphism,  $T$  is an invertible linear map, meaning by Theorem 3.12 that  $T$  is injective and surjective. Thus, since there exists an injective linear map  $T : V \rightarrow W$ , the contrapositive of Theorem 3.7 asserts that  $\dim V \leq \dim W$ . Additionally, since there exists a surjective linear map  $T : V \rightarrow W$ , the contrapositive of Theorem 3.8 asserts that  $\dim V \geq \dim W$ . Therefore, we have that  $\dim V = \dim W$ , as desired.

Now suppose that  $\dim V = \dim W$ . Let  $v_1, \dots, v_n$  be a basis of  $V$ , and let  $w_1, \dots, w_n$  be a basis of  $W$ . By Theorem 3.1, there exists a unique linear map  $T : V \rightarrow W$  such that  $Tv_j = w_j$  for each  $j = 1, \dots, n$ . To prove that  $T$  is an isomorphism, Theorem 3.12 tells us that it will suffice to show that it is injective and surjective. To show that  $T$  is surjective, it will suffice to show that  $\text{range } T = W = \text{span}(w_1, \dots, w_n)$ . But since  $Tv_j = w_j \in \text{range } T$  for all  $j = 1, \dots, n$ ,  $\text{range } T \subset W$ , and  $\text{range } T$  is a vector space (see Theorem 3.5), we have that  $\text{range } T = \text{span}(w_1, \dots, w_n) = W$ , as desired. To prove that  $T$  is injective, Theorem 3.4 tells us that it will suffice to show that  $\text{null } T = \{0\}$ , i.e., that  $\dim \text{null } T = 0$ . But since  $\dim \text{range } T = \dim W = \dim V$ , we have by the Fundamental Theorem of Linear Maps that

$$\begin{aligned} \dim \text{null } T + \dim \text{range } T &= \dim V \\ &= \dim W \\ &= \dim \text{range } T \\ \dim \text{null } T &= 0 \end{aligned}$$

as desired. ■

- This result implies that every finite-dimensional vector space of dimension  $n$  is isomorphic to  $\mathbb{F}^n$ .
- It also allows us to formalize the link between linear maps from  $V$  to  $W$  and matrices in  $\mathbb{F}^{m,n}$ .

**Theorem 3.14.** *Suppose  $v_1, \dots, v_n$  is a basis of  $V$  and  $w_1, \dots, w_m$  is a basis of  $W$ . Then  $\mathcal{M}$  is an isomorphism between  $\mathcal{L}(V, W)$  and  $\mathbb{F}^{m,n}$ .*

*Proof.* We have already established that  $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$  and that  $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$ , so we already know that  $\mathcal{M}$  is a linear map. To prove that it is invertible, Theorem 3.12 tells us that it will suffice to show that  $\mathcal{M}$  is injective and surjective.

To show that  $\mathcal{M}$  is injective, Theorem 3.4 tells us that it will suffice to verify that  $\text{null } \mathcal{M} = \{0\}$ . Let  $T \in \mathcal{L}(V, W)$  be arbitrary. If  $\mathcal{M}(T) = 0$  (0 denoting the zero matrix), then

$$Tv_k = 0w_1 + \dots + 0w_m = 0$$

for all  $k = 1, \dots, n$ . But since  $v_1, \dots, v_n$  is a basis of  $V$ , this implies that  $T = 0$  (0 denoting the zero transformation), as desired.

To show that  $\mathcal{M}$  is surjective, it will suffice to verify that  $\text{range } \mathcal{M} = \mathbb{F}^{m,n}$ . Clearly  $\text{range } \mathcal{M} \subset \mathbb{F}^{m,n}$ , so we focus on the other direction. Let  $A \in \mathbb{F}^{m,n}$  be arbitrary. Define  $T \in \mathcal{L}(V, W)$  by

$$Tv_k = \sum_{j=1}^m A_{j,k} w_j$$

for  $k = 1, \dots, n$ . It follows by the definition of a matrix of a linear transformation that  $\mathcal{M}(T) = A$ , as desired. ■

- We can now determine the dimension of the vector space of linear maps from one finite-dimensional vector space to another.

**Theorem 3.15.** *Suppose  $V$  and  $W$  are finite-dimensional. Then  $\mathcal{L}(V, W)$  is finite-dimensional and*

$$\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$$

*Proof.* By Theorem 3.14,  $\mathcal{L}(V, W)$  and  $\mathbb{F}^{m,n}$  are isomorphic. Thus, by Theorem 3.13,  $\mathcal{L}(V, W)$  and  $\mathbb{F}^{m,n}$  have the same dimension. Therefore, we have that

$$\begin{aligned} \dim \mathcal{L}(V, W) &= \dim \mathbb{F}^{m,n} \\ &= mn \\ &= (\dim V)(\dim W) \end{aligned}$$

as desired. ■

- **Matrix** (of  $v \in V$  with respect to the basis  $v_1, \dots, v_n$  of  $V$ ): The  $n$ -by-1 matrix  $\mathcal{M}(v)$  whose entries  $A_{j,1}$  are defined by

$$v = A_{1,1}v_1 + \dots + A_{n,1}v_n$$

- We now show that the columns of the matrix of  $T$  are directly related to the effect  $T$  has on basis vectors.

**Theorem 3.16.** *Suppose  $T \in \mathcal{L}(V, W)$ ,  $v_1, \dots, v_n$  is a basis of  $V$ , and  $w_1, \dots, w_m$  is a basis of  $W$ . Let  $1 \leq k \leq n$ . Then*

$$\mathcal{M}(T)_{\cdot, k} = \mathcal{M}(Tv_k)$$

*Proof.* As an element of  $W$ ,  $Tv_k = c_1w_1 + \dots + c_mw_m$  for some  $c_1, \dots, c_m \in \mathbb{F}$ . By the definition of the matrix of  $T$ , the values in column  $k$  are  $c_1, \dots, c_m$ . Similarly, by the definition of the matrix of  $Tv_k$ , the values in its one column are  $c_1, \dots, c_m$ , as desired. ■

- Linear maps act like matrix multiplication.

**Theorem 3.17.** *Suppose  $T \in \mathcal{L}(V, W)$ ,  $v \in V$ ,  $v_1, \dots, v_n$  is a basis of  $V$ , and  $w_1, \dots, w_m$  is a basis of  $W$ . Then*

$$\mathcal{M}(Tv) = \mathcal{M}(T)\mathcal{M}(v)$$

*Proof.* Let  $v = c_1v_1 + \dots + c_nv_n$ . Then by the linearity of  $T$ ,  $Tv = c_1Tv_1 + \dots + c_nTv_n$ . It follows by the linearity of  $\mathcal{M}$ , Theorem 3.16, and the fact that a matrix times a vector is equal to a linear combination of columns that

$$\begin{aligned} \mathcal{M}(Tv) &= c_1\mathcal{M}(Tv_1) + \dots + c_n\mathcal{M}(Tv_n) \\ &= c_1\mathcal{M}(T)_{\cdot, 1} + \dots + c_n\mathcal{M}(T)_{\cdot, n} \\ &= \mathcal{M}(T)\mathcal{M}(v) \end{aligned}$$

as desired. ■

- “Each  $m$ -by- $n$  matrix  $A$  induces a linear map from  $\mathbb{F}^{n,1}$  to  $\mathbb{F}^{m,1}$ , namely the matrix multiplication function that takes  $x \in \mathbb{F}^{n,1}$  to  $Ax \in \mathbb{F}^{m,1}$ ” (Axler, 2015, p. 85).
- **Operator:** A linear map from a vector space to itself.
- **$\mathcal{L}(V)$ :** The set of all operators on  $V$ .
  - Mathematically,  $\mathcal{L}(V) = \mathcal{L}(V, V)$ .

- On infinite-dimensional vector spaces, neither injectivity nor surjectivity, alone, implies invertibility.
- However, only one does on finite-dimensional vector spaces.

**Theorem 3.18.** *Suppose  $V$  is finite-dimensional and  $T \in \mathcal{L}(V)$ . Then the following are equivalent.*

- (a)  $T$  is invertible;
- (b)  $T$  is injective;
- (c)  $T$  is surjective.

*Proof.* First, suppose that  $T$  is invertible. Then by Theorem 3.12,  $T$  is injective, as desired.

Second, suppose that  $T$  is injective. Then by Theorem 3.4,  $\text{null } T = \{0\}$ . It follows by the Fundamental Theorem of Linear Maps that

$$\begin{aligned}\dim \text{range } T &= \dim V - \dim \text{null } T \\ &= \dim V\end{aligned}$$

Thus, since  $\text{range } T$  has the same dimension as  $V$  and is a subspace of  $V$  (by Theorem 3.5),  $\text{range } T = V$ . Therefore,  $T$  is surjective, as desired.

Third, suppose that  $T$  is surjective. Then  $\text{range } T = V$ . It follows that  $\dim \text{range } T = \dim V$ . Thus, by the Fundamental Theorem of Linear Maps, we have that

$$\begin{aligned}\dim \text{null } T &= \dim V - \dim \text{range } T \\ &= 0\end{aligned}$$

Consequently, by Theorem 3.4,  $T$  is injective. Therefore, by Theorem 3.12,  $T$  is invertible, as desired. ■

### 3.E Products and Quotients of Vector Spaces

- **Product** (of  $V_1, \dots, V_m$ ): The set

$$V_1 \times \cdots \times V_m = \{(v_1, \dots, v_m) : v_1 \in V_1, \dots, v_m \in V_m\}$$

- Addition on  $V_1 \times \cdots \times V_m$  is defined by

$$(u_1, \dots, u_m) + (v_1, \dots, v_m) = (u_1 + v_1, \dots, u_m + v_m)$$

- Scalar multiplication on  $V_1 \times \cdots \times V_m$  is defined by

$$\lambda(v_1, \dots, v_m) = (\lambda v_1, \dots, \lambda v_m)$$

- The product of  $n$  vector spaces over  $\mathbb{F}$  is a vector space over  $\mathbb{F}$ , with addition and scalar multiplication defined as above.
- We can, for example, identify  $\mathbb{R}^2 \times \mathbb{R}^3$  with  $\mathbb{R}^5$  by constructing an isomorphism from every vector  $((x_1, x_2), (x_3, x_4, x_5)) \in \mathbb{R}^2 \times \mathbb{R}^3$  to the vector  $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ .
- The dimension of the product of vector spaces is equal to the sum of the dimensions of the component vector spaces.

**Theorem 3.19.** *Suppose  $V_1, \dots, V_m$  are finite-dimensional vector spaces. Then  $V_1 \times \cdots \times V_m$  is finite-dimensional and*

$$\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V_m$$

*Proof.* Choose a basis of each  $V_j$ . For each basis vector of each  $V_j$ , consider the element of  $V_1 \times \cdots \times V_m$  that equals the basis vector in the  $j^{\text{th}}$  slot and 0 in the other slots. The list of all such vectors is linearly independent and spans  $V_1 \times \cdots \times V_m$ . Thus, it is a basis of  $V_1 \times \cdots \times V_m$ . The length of this basis is  $\dim V_1 + \cdots + \dim V_m$ , as desired. ■

- We now relate products and direct sums.

**Theorem 3.20.** Suppose that  $U_1, \dots, U_m$  are subspaces of  $V$ . Define a linear map  $\Gamma : U_1 \times \cdots \times U_m \rightarrow U_1 + \cdots + U_m$  by

$$\Gamma(u_1, \dots, u_m) = u_1 + \cdots + u_m$$

Then  $U_1 + \cdots + U_m$  is a direct sum if and only if  $\Gamma$  is injective.

*Proof.* Suppose first that  $\Gamma$  is injective. Then the only way to write 0 as a sum  $u_1 + \cdots + u_m$ , where each  $u_j \in U_j$ , is by taking each  $u_j = 0$ . It follows by the condition on direct sums that  $U_1 + \cdots + U_m$  is a direct sum. The proof is symmetric in the reverse direction. ■

- Note that since  $\Gamma$  is surjective by the definition of  $U_1 + \cdots + U_m$ , the condition that  $\Gamma$  is injective could be changed to the condition that  $\Gamma$  is invertible.

- We can now prove that the dimensions add up in a direct sum.

**Theorem 3.21.** Suppose  $V$  is finite-dimensional and  $U_1, \dots, U_m$  are subspaces of  $V$ . Then  $U_1 + \cdots + U_m$  is a direct sum if and only if

$$\dim(U_1 + \cdots + U_m) = \dim U_1 + \cdots + \dim U_m$$

*Proof.* Suppose first that  $U_1 + \cdots + U_m$  is a direct sum. Then by Theorem 3.20, there exists an invertible linear map  $\Gamma$  from  $U_1 \times \cdots \times U_m$  to  $U_1 + \cdots + U_m$ . Thus, by Theorem 3.13,  $U_1 \times \cdots \times U_m$  and  $U_1 + \cdots + U_m$  have the same dimension. Therefore,

$$\begin{aligned} \dim(U_1 + \cdots + U_m) &= \dim(U_1 \times \cdots \times U_m) \\ &= \dim U_1 + \cdots + \dim U_m \end{aligned} \quad \text{Theorem 3.19}$$

as desired.

The proof is symmetric in the other direction. ■

- **Sum** (of  $v \in V$  and  $U$  a subspace of  $V$ ): The subset of  $V$  defined by

$$v + U = \{v + u : u \in U\}$$

- **Affine subset** (of  $V$ ): A subset of  $V$  of the form  $v + U$  for some  $v \in V$  and some subspace  $U$  of  $V$ .
- **Parallel** (subset to  $U$ ): An affine subset  $v + U$  of  $V$ .
- **Quotient space**: The set of all affine subsets of  $V$  parallel to  $U$ .

- Symbolically,

$$V/U = \{v + U : v \in V\}$$

- Two affine subsets parallel to  $U$  are equal or disjoint.

**Theorem 3.22.** Suppose  $U$  is a subspace of  $V$  and  $v, w \in V$ . Then the following are equivalent.

- $v - w \in U$ ;
- $v + U = w + U$ ;
- $(v + U) \cap (w + U) \neq \emptyset$ .

*Proof.* First, suppose that  $v - w \in U$ . Let  $x \in v + U$  be arbitrary. Then  $x = v + u$  for some  $u \in U$ . Now since  $v - w \in U$ ,  $u \in U$ , and  $U$  is a subspace, we have that  $v - w + u \in U$ . Thus,  $x = w - w + v + u = w + (v - w + u) \in w + U$ . The proof is symmetric in the other direction. Therefore,  $v + U = w + U$ , as desired.

Second, suppose that  $v + U = w + U$ . Since  $U$  is nonempty ( $0 \in U$  by definition), we know that  $v + U \neq \emptyset \neq w + U$ . Therefore,  $(v + U) \cap (w + U) \supset \{0\} \neq \emptyset$ , as desired.

Third, suppose that  $(v + U) \cap (w + U) \neq \emptyset$ . Then there exists  $x$  such that  $x \in v + U$  and  $x \in w + U$ . It follows that  $x = v + u_1$  and  $x = w + u_2$  for some  $u_1, u_2 \in U$ . Thus, by transitivity,  $v + u_1 = w + u_2$ . Therefore,  $v - w = u_2 - u_1 \in U$ , as desired. ■

- **Sum** (of  $v + U, w + U \in V/U$ ): The affine subset  $(v + w) + U$ .
- **Product** (of  $v + U \in V/U$  and  $\lambda \in \mathbb{F}$ ): The affine subset  $(\lambda v) + U$ .
- We now verify that the above operations are well-defined and prove that the quotient space is a vector space.

**Theorem 3.23.** *Suppose  $U$  is a subspace of  $V$ . Then  $V/U$ , with the operations of addition and scalar multiplication as defined above, is a vector space.*

*Proof.* The way affine subsets are defined, we may have  $v + U = \hat{v} + U$  and yet have  $v \neq \hat{v}$ . Thus, we must first guarantee that the operations of addition and scalar multiplication, as defined above, are well-defined, i.e., that if  $v + U = \hat{v} + U$  and  $w + U = \hat{w} + U$ , then  $(v + w) + U = (\hat{v} + \hat{w}) + U$  and  $(\lambda v) + U = (\lambda \hat{v}) + U$ . Let's begin.

To confirm that addition as defined above is a well-defined operation, let  $v, \hat{v}, w, \hat{w} \in V$  be such that  $v + U = \hat{v} + U$  and  $w + U = \hat{w} + U$ . Then by Theorem 3.22,  $v - \hat{v} \in U$  and  $w - \hat{w} \in U$ . It follows since  $U$  is a subspace that  $(v - \hat{v}) + (w - \hat{w}) \in U$ . Consequently,  $(v + w) - (\hat{v} + \hat{w}) \in U$ , so by Theorem 3.22 again,  $(v + w) + U = (\hat{v} + \hat{w}) + U$ , as desired.

Similarly,  $v + U = \hat{v} + U$  implies  $v - \hat{v} \in U$ , implies  $\lambda(v - \hat{v}) = \lambda v - \lambda \hat{v} \in U$ , implies  $(\lambda v) + U = (\lambda \hat{v}) + U$ , as desired.

The remaining proof that  $V/U$  is a vector space is straightforward; note that  $0 + U$  is the identity element and  $(-v) + U$  is the additive inverse. ■

- **Quotient map:** The linear map  $\pi : V \rightarrow V/U$  defined by  $\pi(v) = v + U$  for all  $v \in V$ .
- We now give a formula for the dimension of a quotient space.

**Theorem 3.24.** *Suppose  $V$  is finite-dimensional and  $U$  is a subspace of  $V$ . Then*

$$\dim V/U = \dim V - \dim U$$

*Proof.* Let  $\pi$  be the quotient map from  $V$  to  $V/U$ . From Theorem 3.22, we know that in order for  $w + U = 0 + U$ , we must have  $w - 0 = w \in U$ . Thus,  $\pi(u) = 0$  if and only if  $u \in U$ , meaning  $\text{null } \pi = U$ . Additionally, we clearly have that  $\text{range } \pi = V/U$ . Therefore, by the Fundamental Theorem of Linear Maps, we have that

$$\begin{aligned} \dim V &= \dim \text{null } \pi + \dim \text{range } \pi \\ &= \dim U + \dim V/U \\ \dim V/U &= \dim V - \dim U \end{aligned}$$

as desired. ■

- Lastly, consider the fact that we can add any vector in the null space of a linear map  $T$  to an argument passed to  $T$  without changing its output. In other words, if  $T \in \mathcal{L}(V, W)$ ,  $v \in V$ , and  $u \in \text{null } T$ , then  $T(v + u) = Tv + Tu = Tv$ . We formalize this concept with the following definition.

- $\tilde{T}$ : The function from  $V/(\text{null } T)$  to  $W$  defined by  $\tilde{T}(v + \text{null } T) = Tv$ , where  $T \in \mathcal{L}(V, W)$ .
- We now state a few basic results about  $\tilde{T}$ .

**Theorem 3.25.** *Suppose  $T \in \mathcal{L}(V, W)$ . Then*

- (a)  $\tilde{T}$  is a linear map from  $V/(\text{null } T)$  to  $W$ ;
- (b)  $\tilde{T}$  is injective;
- (c)  $\text{range } \tilde{T} = \text{range } T$ ;
- (d)  $V/(\text{null } T)$  is isomorphic to  $\text{range } T$ .