Chapter 2

Finite-Dimensional Vector Spaces

2.A Span and Linear Independence

9/3:

- **Linear combination** (of a list $v_1, \dots, v_m$ of vectors in V): A vector of the form $a_1 v_1 + \cdots + a_m v_m$, where $a_1, \dots, a_m \in \mathbb{F}$.
- **Span** (of $v_1, \dots, v_m$): The set of all linear combinations of a list of vectors $v_1, \dots, v_m$ in V. *Also known as* **linear span**. *Denoted by* $\operatorname{span}(v_1, \dots, v_m)$. *Given by*

$$\operatorname{span}(v_1, \dots, v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_1, \dots, a_m \in \mathbb{F}\}$$

  - We define $\operatorname{span}() = \{0\}$.
- Span as a subspace.

Theorem 2.1. *The span of a list of vectors in V is the smallest subspace of V containing all the vectors in the list.*

Proof. Let $v_1, \dots, v_m \in V$ be a list of vectors. We will first prove that $\operatorname{span}(v_1, \dots, v_m)$ is a subspace of V. We will then prove that it contains all of the vectors in the list. Lastly, we will prove that any proper subset of $\operatorname{span}(v_1, \dots, v_m)$ either doesn't contain all the vectors in the list or is not a subspace of V. Let's begin.

To prove that $\operatorname{span}(v_1, \dots, v_m)$ is a subspace of V, it will suffice to show that $\operatorname{span}(v_1, \dots, v_m)$ contains the additive identity, $\operatorname{span}(v_1, \dots, v_m)$ is closed under addition, and $\operatorname{span}(v_1, \dots, v_m)$ is closed under scalar multiplication. By the definition of $\operatorname{span}(v_1, \dots, v_m)$, we know that $0v_1 + \cdots + 0v_m = 0 \in \operatorname{span}(v_1, \dots, v_m)$. If $a_1 v_1 + \cdots + a_m v_m \in \operatorname{span}(v_1, \dots, v_m)$ and $b_1 v_1 + \cdots + b_m v_m \in \operatorname{span}(v_1, \dots, v_m)$, then naturally $(a_1 v_1 + \cdots + a_m v_m) + (b_1 v_1 + \cdots + b_m v_m) = (a_1 + b_1) v_1 + \cdots + (a_m + b_m) v_m \in \operatorname{span}(v_1, \dots, v_m)$. Lastly, if $a_1 v_1 + \cdots + a_m v_m \in \operatorname{span}(v_1, \dots, v_m)$ and $\lambda \in \mathbb{F}$, then naturally $\lambda(a_1 v_1 + \cdots + a_m v_m) = (\lambda a_1) v_1 + \cdots + (\lambda a_m) v_m \in \operatorname{span}(v_1, \dots, v_m)$.

By setting every $a_i = 0$ except $a_j = 1$, we can guarantee that $v_j \in \operatorname{span}(v_1, \dots, v_m)$ for all $j \in [m]$.

Suppose for the sake of contradiction that there exists a smaller subspace U of V that contains $v_1, \dots, v_m$. It follows that there exists a vector $u \in \operatorname{span}(v_1, \dots, v_m)$ such that $u \notin U$. Since $u \in \operatorname{span}(v_1, \dots, v_m)$, $u = a_1 v_1 + \cdots + a_m v_m$ for some $a_1, \dots, a_m \in \mathbb{F}$. However, by definition, $v_1, \dots, v_m \in U$, so since U is closed under addition and scalar multiplication, their linear combination $a_1 v_1 + \cdots + a_m v_m = u \in U$, a contradiction. ∎

- If $\operatorname{span}(v_1, \dots, v_m) = V$, we say that $v_1, \dots, v_m$ **spans** V.
- **Finite-dimensional vector space**: A vector space such that some list of vectors in it spans the space.

  - Recall that by definition, every list has finite length.
- **Polynomial** (with coefficients in $\mathbb{F}$): A function $p : \mathbb{F} \to \mathbb{F}$ such that there exist $a_0, \dots, a_m \in \mathbb{F}$ such that
$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$
for all $z \in \mathbb{F}$.
- $\boldsymbol{\mathcal{P}(\mathbb{F})}$: The set of all polynomials with coefficients in $\mathbb{F}$.
  - $\mathcal{P}(\mathbb{F})$, under the usual addition and scalar multiplication, is a vector space over $\mathbb{F}$.
  - Thus, $\mathcal{P}(\mathbb{F})$ is a subspace of $\mathbb{F}^{\mathbb{F}}$.
- We will later prove that the coefficients of a polynomial uniquely determine it.
- **Degree** (of a polynomial p): The number m, where $p = a_0 + a_1 z + \cdots + a_m z^m$ and $a_m \neq 0$. *Denoted by* $\deg p = m$.
  - The polynomial $p(z) = 0$ is said to have degree $-\infty$.
- $\boldsymbol{\mathcal{P}_m(\mathbb{F})}$: The set of all polynomials with coefficients in $\mathbb{F}$ and degree at most m, where m is a nonnegative integer.
  - $\mathcal{P}_m(\mathbb{F}) = \operatorname{span}(1, z, \dots, z^m)$ is a finite-dimensional vector space for all nonnegative integers m.
- **Infinite-dimensional vector space**: A vector space that is not finite dimensional.
  - $\mathcal{P}(\mathbb{F})$ is infinite-dimensional.
- **Linearly independent** (list $v_1, \dots, v_m$): A list $v_1, \dots, v_m$ of vectors in V such that the only choice of $a_1, \dots, a_m \in \mathbb{F}$ that makes $a_1 v_1 + \cdots + a_m v_m = 0$ is $a_1 = \cdots = a_m = 0$.
  - We also let the empty list be linearly independent.
- $v_1, \dots, v_m$ is linearly independent if and only if each vector in $\operatorname{span}(v_1, \dots, v_m)$ has only one representation as a linear combination of $v_1, \dots, v_m$.
- If some vectors are removed from a linearly independent list, the remaining list is also linearly independent.
  - Suppose $v_1, \dots, v_m$ is linearly independent. Suppose $v_1, \dots, v_n$ is not linearly independent, with $n < m$. Then $a_1 v_1 + \cdots + a_n v_n = 0$ for some $a_1, \dots, a_n \in \mathbb{F}$ such that $a_i \neq 0$ for all $i \in [n]$. But then $a_1 v_1 + \cdots + a_n v_n + 0 v_{n+1} + \cdots + 0 v_m = 0$, a contradiction.
- **Linearly dependent** (list $v_1, \dots, v_m$): A list $v_1, \dots, v_m$ of vectors in V that is not linearly independent.
  - In other words, $v_1, \dots, v_m$ are linearly dependent if there exist $a_1, \dots, a_m \in \mathbb{F}$, not all 0, such that $a_1 v_1 + \cdots + a_m v_m = 0$.
- The following is an important and oft-used lemma.

  Lemma 2.2 (Linear Dependence Lemma)**.** *Suppose $v_1, \dots, v_m$ is a linearly dependent list in V. Then there exists $j \in \{1, \dots, m\}$ such that the following hold:*

  (a) $v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$*;*

  (b) if the j^{th} term is removed from $v_1, \dots, v_m$, the span of the remaining list equals $\operatorname{span}(v_1, \dots, v_m)$.

Proof. We divide into two cases (the list is $v_1 = 0$, and the list is $v_1, \dots, v_m$).

If the list is $v_1 = 0$, then the list is linearly dependent. Choose $j = 1$. Clearly, $v_1 \in \text{span}() = \{0\}$ by definition. Additionally, $\text{span}() = \{0\} = \{a_1 0 : a_1 \in \mathbb{F}\} = \text{span}(v_1)$, as desired.

Since $v_1, \dots, v_m$ is linearly dependent, there exist $a_1, \dots, a_m \in \mathbb{F}$, not all 0, such that $a_1v_1 + \cdots + a_mv_m = 0$. Let j be the largest element of $\{1, \dots, m\}$ such that $a_j \neq 0$. Then

$$\begin{aligned} 0 &= a_1v_1 + \cdots + a_jv_j + 0v_{j+1} + \cdots + 0v_m \\ -a_jv_j &= a_1v_1 + \cdots + a_{j-1}v_{j-1} \\ v_j &= -\frac{a_1}{a_j}v_1 - \cdots - \frac{a_{j-1}}{a_j}v_{j-1} \end{aligned}$$

It follows that $v_j \in \text{span}(v_1, \dots, v_{j-1})$, as desired.

Now clearly $\text{span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_m) \subset \text{span}(v_1, \dots, v_m)$. In the other direction, suppose $u = c_1v_1 + \cdots + c_mv_m \in \text{span}(v_1, \dots, v_m)$. Then

$$u = c_1v_1 + \cdots + c_{j-1}v_{j-1} + c_j\left(-\frac{a_1}{a_j}v_1 - \cdots - \frac{a_{j-1}}{a_j}v_{j-1}\right) + c_{j+1}v_{j+1} + \cdots + c_mv_m \in \text{span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_m)$$

as desired. ■

- We next prove an immediate consequence of the Linear Dependence Lemma.

Theorem 2.3. *In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.*

Proof. Suppose that $u_1, \dots, u_m$ is linearly independent in V, and that $w_1, \dots, w_n$ spans V. We must prove that $m \leq n$. To do so, it will suffice to use the following m-step process.

Step 1: Let $B = w_1, \dots, w_n$. Adding any $v \in V$ to this list produces a linearly dependent list (because the newly adjoined vector can be written as a linear combination of the other vectors). In particular, the list $u_1, w_1, \dots, w_n$ is linearly dependent. Thus, since $u_1 \neq 0$ (it's part of a linearly independent list, and thus cannot be written as $0u_i$ for any u_i), the Linear Dependence Lemma asserts that we can remove one of the w_i's such that the new list B consisting of u_1 and the remaining w_i's spans V.

Step j: The list B from step $j-1$ spans V. Thus, as before, adjoin vector u_j to B, placing it just after $u_1, \dots, u_{j-1}$. It follows by the Linear Dependence Lemma that one of the following vectors (i.e., one of the w_i's) is in $\text{span}(u_1, \dots, u_j)$, so we can remove it and know that the list comprised of $u_1, \dots, u_j$ followed by the remaining w_i's spans V.

After step m, we have added all of the u's and the process stops. At each step, as we add a u to B, the Linear Dependence Lemma implies that there is some w to remove. Thus, there are at least as many w's as u's.[1] ■

  - This theorem allows us to prove results such as no list of 4 or more vectors is linearly independent in $\mathbb{R}^3$ (since $(1,0,0), (0,1,0), (0,0,1)$ spans $\mathbb{R}^3$), and no list of fewer than 4 vectors spans $\mathbb{R}^4$ (since $(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$ is linearly independent in $\mathbb{R}^4$).

- We now can rigorously prove that subspaces of finite-dimensional vector spaces are also finite-dimensional.

Theorem 2.4. *Every subspace of a finite-dimensional vector space is finite-dimensional.*

Proof. Let V be finite-dimensional, and suppose for the sake of contradiction that U is infinite-dimensional subspace of V. Since V is finite-dimensional, there exists a list of vectors $v_1, \dots, v_m$ such that $\text{span}(v_1, \dots, v_m) = V$. To arrive at a contradiction, we will construct a linearly independent list of vectors in U of length $m+1$, contradicting Theorem 2.3.

[1] We should be able to do this more rigorously via induction on m.

Since U is infinite-dimensional, there is no list of vectors in U spans it. Thus, if we choose $u_1 \in U$, we know that $\text{span}(u_1) \neq U$. It follows since $\text{span}(u_1) \subset U$ (as we know from the closure of U) that there exists $u_2 \in U$ such that $u_2 \notin \text{span}(u_1)$. However, we will still have that $\text{span}(u_1, u_2) \neq U$. More importantly, though, since $u_2 \notin \text{span}(u_1)$ and $u_1 \notin \text{span}()$, the Linear Dependence Lemma implies that u_1, u_2 is linearly independent. We can clearly continue in this fashion up to $u_1, \dots, u_{m+1}$, as desired. ■

2.B Bases

- **Basis** (of V): A list of vectors in V that is linearly independent and spans V.
- **Standard basis** (of $\mathbb{F}^n$): The list $(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$.
- Determining whether a list of vectors is a basis:

Theorem 2.5. *A list $v_1, \dots, v_n$ of vectors in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form*

$$v = a_1 v_1 + \cdots + a_n v_n$$

where $a_1, \dots, a_n \in \mathbb{F}$.

Proof. Suppose first that $v_1, \dots, v_n$ is a basis of V. Let $v \in V$ be arbitrary. We will first show that v can be written in the desired form, and then we will show that this writing is unique. Let's begin. By the definition of a basis, $v_1, \dots, v_n$ spans V. Thus, $\text{span}(v_1, \dots, v_n) = V$. It follows that $v \in \text{span}(v_1, \dots, v_n)$, which implies by the definition of span that $v = a_1 v_1 + \cdots + a_n v_n$ where $a_1, \dots, a_n \in \mathbb{F}$, as desired. Now suppose for the sake of contradiction $v = c_1 v_1 + \cdots + c_n v_n$ as well, where $c_1, \dots, c_n \in \mathbb{F}$ and $c_j \neq a_j$ for some $i \in [n]$. Then

$$\begin{aligned} 0 &= v - v \\ &= (a_1 v_1 + \cdots + a_n v_n) - (c_1 v_1 + \cdots + c_n v_n) \\ &= (a_1 - c_1) v_1 + \cdots + (a_n - c_n) v_n \end{aligned}$$

Since at least $a_j - c_j \neq 0$ but the above sum still does equal 0, we have that $v_1, \dots, v_n$ are not linearly independent, a contradiction.

Now suppose that every $v \in V$ can be written uniquely in the form $v = a_1 v_1 + \cdots + a_n v_n$. To prove that $v_1, \dots, v_n$ is a basis of V, it will suffice to show that $v_1, \dots, v_n$ spans V and is linearly independent. Let's start with the first claim. Clearly, $\text{span}(v_1, \dots, v_n) \subset V$, and since every $v \in V$ may be written as a linear combination of $v_1, \dots, v_n$, we know that every $v \in V$ is an element of $\text{span}(v_1, \dots, v_n)$, as desired. On the other hand, we know that $0 = 0v_1 + \cdots + 0v_n$ and 0 can only be written in this unique form. Thus, the only choice of $a_1, \dots, a_m \in \mathbb{F}$ that makes $a_1 v_1 + \cdots + a_n v_n = 0$ is $a_1 = \cdots = a_n = 0$, proving that $v_1, \dots, v_n$ is linearly independent. ■

- Finding the basis in a spanning list.

Theorem 2.6. *Every spanning list in a vector space can be reduced to a basis of the vector space.*

Proof. Let $v_1, \dots, v_n$ span V. We induct on n. For the base case $n = 0$, if $()$ spans V, then since $()$ is linearly independent by definition, we are done. Now suppose inductively that we have proven that every spanning list in V of length n can be reduced to a basis of V; we wish to prove that every spanning list in V of length $n + 1$ can be reduced to a basis of V. Let $v_1, \dots, v_{n+1}$ span V. If $v_1, \dots, v_{n+1}$ is linearly independent, we are done. If $v_1, \dots, v_{n+1}$ is linearly dependent, then by the Linear Dependence Lemma, we can remove a vector from the list without changing its span. Our new list only has length n, so by the inductive hypothesis, it will reduce to a basis of V. ■

- Proving the existence of a basis in a finite-dimensional vector space.

Theorem 2.7. *Every finite-dimensional vector space has a basis.*

Proof. Let V be finite-dimensional. As such, there exists a list $v_1, \dots, v_n$ of vectors in V that spans V. It follows by Theorem 2.6 that some sublist of $v_1, \dots, v_n$ is a basis of V, as desired. ■

- Extending a linearly independent list into a basis (basically the dual of Theorem 2.6).

Theorem 2.8. *Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.*

Proof. Let $u_1, \dots, u_m$ be a linearly independent list of vectors in V. By Theorem 2.7, V has a basis $w_1, \dots, w_n$. It follows that $u_1, \dots, u_m, w_1, \dots, w_n$ spans V. Thus, by Theorem 2.6, which removes the first linearly dependent vector in $u_1, \dots, u_m, w_1, \dots, w_n$ (necessarily one of the w_i's since $u_1, \dots, u_m$ are linearly independent) via the Linear Dependence Lemma, there exists a sublist of $u_1, \dots, u_m, w_1, \dots, w_n$ containing $u_1, \dots, u_m$ that is a basis of V. ■

- Finding orthogonal complements.

Theorem 2.9. *Suppose V is finite-dimensional and U is a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$.*

Proof. Since V is finite-dimensional, Theorem 2.4 asserts that U is finite-dimensional. Thus, by Theorem 2.7, U has a basis $u_1, \dots, u_m$. It follows by Theorem 2.8 that there exist $w_1, \dots, w_n \in V$ such that $u_1, \dots, u_m, w_1, \dots, w_n$ is a basis of V. Let $W = \operatorname{span}(w_1, \dots, w_n)$.

To prove that $U \oplus W = V$, it will suffice to show that

$$U + W = V \qquad\qquad U \cap W = \{0\}$$

To prove the first equation, it will suffice to show that for any vector $v \in V$, $v = u + w$ for $u \in U$ and $w \in W$. But since $u_1, \dots, u_m, w_1, \dots, w_n$ is a basis of V, we have by the definition of span that

$$v = \underbrace{a_1u_1 + \cdots + a_mu_m}_{u} + \underbrace{b_1w_1 + \cdots + b_nw_n}_{w}$$

as desired.

To prove the second equation, let $v \in U \cap W$ be arbitrary. Then since $v \in U$ and $u_1, \dots, u_m$ is a basis of U, we have that $v = a_1u_1 + \cdots + a_mu_m$ where $a_1, \dots, a_m \in \mathbb{F}$. Similarly, we have that $v = b_1w_1 + \cdots + b_mw_n$ where $b_1, \dots, b_n \in \mathbb{F}$. It follows that

$$\begin{aligned} 0 &= v - v \\ &= a_1u_1 + \cdots + a_mu_m - b_1w_1 - \cdots - b_nw_n \end{aligned}$$

But since $u_1, \dots, u_m, w_1, \dots, w_n$ is a basis of V, $u_1, \dots, u_m, w_1, \dots, w_n$ is linearly independent. It follows that $a_1 = \cdots = a_m = b_1 = \cdots = b_n = 0$. Therefore, $v = a_1u_1 + \cdots + a_mu_m = 0$, as desired. ■

  - Note that this same basic idea extends to the infinite-dimensional case, but that proof requires considerably more advanced tools.

2.C Dimension

- It would be nice to define the dimension of a vector space as the length of a basis, but in order to do this we must first show that all bases have the same length.

Theorem 2.10. *Any two bases of a finite-dimensional vector space have the same length.*

Proof. Let B_1, B_2 be two arbitrary bases of V. Since B_1 is linearly independent in V and B_2 spans V, Theorem 2.3 asserts that $\operatorname{len} B_1 \leq \operatorname{len} B_2$. Similarly, since B_2 is linearly independent in V and B_1 spans V, Theorem 2.3 asserts that $\operatorname{len} B_2 \leq \operatorname{len} B_1$. Therefore, $\operatorname{len} B_1 = \operatorname{len} B_2$, as desired. ■

- **Dimension** (of V finite-dimensional): The length of any basis of V. *Denoted by* $\mathbf{dim}\,\boldsymbol{V}$.
- We can now give the expected inequality regarding the dimension of subspaces with respect to the dimension of the vector space.

Theorem 2.11. *If V is finite-dimensional, and U is a subspace of V, then* $\dim U \leq \dim V$.

Proof. Since V is finite dimensional, Theorem 2.4 asserts that U is finite-dimensional. Thus, Theorem 2.7 implies that they have bases $B_U = u_1, \dots, u_m$ and $B_V = v_1, \dots, v_n$. Therefore, since B_U is linearly independent in V and B_V spans V, Theorem 2.3 asserts that $\dim U = \operatorname{len} B_U \leq \operatorname{len} B_V = \dim V$, as desired. ■

- Note that while there exists an order-preserving bijection over addition and scalar multiplication by elements of $\mathbb{R}$ between $\mathbb{R}^2$ and $\mathbb{C}$, $\dim \mathbb{R}^2 = 2 \neq 1 = \dim \mathbb{C}$. Thus, when we talk about the dimension of a vector space, the role played by the choice of $\mathbb{F}$ cannot be neglected.
- Consider a list of vectors in a vector space that has the same length as a basis of the vector space. To check that this list is a basis, we can show that it suffices to show that the list is linearly independent *or* spans the vector space, as we will now prove.

Theorem 2.12. *Suppose V is finite-dimensional. Then every linearly independent list of vectors in V with length* $\dim V$ *is a basis of V.*

Proof. Let $\dim V = n$, and let $v_1, \dots, v_n$ be linearly independent. By Theorem 2.8, we can extend $v_1, \dots, v_n$ to a basis of V. However, since every basis of V has length n by Theorem 2.10, we need not add any vectors to $v_1, \dots, v_n$ to make it a basis; in other words, $v_1, \dots, v_n$ already is a basis. ■

Theorem 2.13. *Suppose V is finite-dimensional. Then every spanning list of vectors in V with length* $\dim V$ *is a basis of V.*

Proof. The proof is symmetric to the proof of Theorem 2.12. ■

- Lastly, we tackle the dimension of a sum of subspaces, which bears a resemblance to the principle of inclusion-exclusion.

Theorem 2.14. *If U_1 and U_2 are subspaces of a finite-dimensional vector space, then*

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

Proof. By Theorem 2.7, $U_1 \cap U_2$ (which we can prove is a subspace in its own right) has a basis, which we may denote $u_1, \dots, u_m$. Since $u_1, \dots, u_m$ is linearly independent in U_1, Theorem 2.8 asserts that it can be extended to a basis $u_1, \dots, u_m, v_1, \dots, v_j$ of U_1. Similarly, it can be extended to a basis $u_1, \dots, u_m, w_1, \dots, w_k$ of U_2.

To prove that $u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k$ is a basis of $U_1 + U_2$, it will suffice to show that it is linearly independent and spans $U_1 + U_2$.

To show that $u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k$ is linearly independent, it will suffice to verify that

$$\begin{aligned} a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_jv_j + c_1w_1 + \cdots + c_kw_k = 0 \\ c_1w_1 + \cdots + c_kw_k = -a_1u_1 - \cdots - a_mu_m - b_1v_1 - \cdots - b_jv_j \end{aligned}$$

implies that all of the coefficients equal 0. Suppose that the above equation holds. Then since $c_1w_1 + \cdots + c_kw_k$ can be written as a linear combination of the basis vectors of U_1, $c_1w_1 + \cdots + c_kw_k \in U_1$.

Additionally, since $c_1w_1 + \cdots + c_kw_k$ is a linear combination of vectors in U_2, $c_1w_1 + \cdots + c_kw_k \in U_2$. Thus, $c_1w_1 + \cdots + c_kw_k \in U_1 \cap U_2$. It follows that $c_1w_1 + \cdots + c_kw_k$ can be written as a linear combination of $u_1, \dots, u_m$, i.e.,

$$\begin{aligned} c_1w_1 + \cdots + c_kw_k &= d_1u_1 + \cdots + d_mu_m \\ 0 &= d_1u_1 + \cdots + d_mu_m - c_1w_1 - \cdots - c_kw_k \end{aligned}$$

for some $d_1, \dots, d_m \in \mathbb{F}$. But since $u_1, \dots, u_m, w_1, \dots, w_k$ is linearly independent as the basis of U_2, the above equation implies that $c_1 = \cdots = c_k = 0$. This implies that

$$0 = -a_1u_1 - \cdots - a_mu_m - b_1v_1 - \cdots - b_jv_j$$

meaning since $u_1, \dots, u_m, v_1, \dots, v_j$ is linearly independent as the basis of U_1, the above equation implies that $a_1 = \cdots = a_m = b_1 = \cdots = b_j = 0$, as desired.

To show that $u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k$ spans $U_1 + U_2$, it will suffice to show that all vectors in the list are elements of $U_1 + U_2$ (i.e., $\operatorname{span}(u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k) \subset U_1 + U_2$), and that every vector in $U_1 + U_2$ can be written as a linear combination of $u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k$ (i.e., that $U_1 + U_2 \subset \operatorname{span}(u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k)$). Since every vector in the list is an element of U_1 or U_2, we can show that it is an element of $U_1 + U_2$ by adding it to the additive identity of the other space. On the other hand, let $x \in U_1 + U_2$. Then $x = x_1 + x_2$, where $x_1 \in U_1$ and $x_2 \in U_2$. It follows that $x_1 = a_1u_1 + \cdots + a_mu_m + b_1v_1 + \cdots + b_jv_j$ and $x_2 = a_1'u_1 + \cdots + a_m'u_m + c_1w_1 + \cdots + c_kw_k$. Therefore, $x = (a_1 + a_1')u_1 + \cdots + (a_m + a_m')u_m + b_1v_1 + \cdots + b_jv_j + c_1w_1 + \cdots + c_kw_k$, as desired.

Having established that $u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k$ is a basis of $U_1 + U_2$, we have that

$$\begin{aligned} \dim(U_1 + U_2) &= m + j + k \\ &= (m + j) + (m + k) - m \\ &= \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2) \end{aligned}$$

as desired. ∎

Exercises

1 Suppose V is finite-dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$.

Proof. Since V is finite dimensional and U is a subspace of V, we have by Theorem 2.9 that there is a subspace W of V such that $V = U \oplus W$. It follows by Theorem 2.14 that

$$\dim W = \dim V - \dim U + \dim\{0\} = 0$$

Thus, $W = \{0\}$. Therefore,

$$\begin{aligned} V &= U \oplus W \\ &= \{u + w : u \in U, w \in W\} \\ &= \{u + 0 : u \in U\} \\ &= U \end{aligned}$$

as desired. ∎