Chapter 6

Inner Product Spaces

6.A Inner Products and Norms

9/30:

- **Norm** (of $x \in \mathbb{R}^n$): The length of x. *Denoted by* $\|x\|$. *Given by*

$$\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

- **Dot product** (of $x, y \in \mathbb{R}^n$): The quantity

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n$$

- Properties of the dot product:
  - $x \cdot x = \|x\|^2$.
  - $x \cdot x \geq 0$.
  - $x \cdot x = 0$ iff $x = 0$.
  - Let $y \in \mathbb{R}^n$. Then $T : \mathbb{R}^n \to \mathbb{R}$ defined by $Tx = x \cdot y$ is linear.
  - $x \cdot y = y \cdot x$.
- **Norm** (of $z \in \mathbb{C}^n$): The quantity

$$\|z\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$$

  - Note that $\|z\|^2 = z \cdot \bar{z}$.
- **Inner product** (on V): A function that takes each ordered pair $(u, v) \in V$ to a number $\langle u, v \rangle \in \mathbb{F}$ and has the following properties.

  positivity
  : $\langle v, v \rangle \geq 0$ for all $v \in V$.

  definiteness
  : $\langle v, v \rangle = 0$ iff $v = 0$.

  additivity in first slot
  : $\langle u + v, v \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$.

  homogeneity in first slot
  : $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$ for all $\lambda \in \mathbb{F}$ and all $u, v \in V$.

  conjugate symmetry
  : $\langle u, v \rangle = \overline{\langle v, u \rangle}$ for all $u, v \in V$.

  - Since every real number equals its complex conjugate, if V is real, we can dispense with the conjugacy condition in the conjugate symmetry condition and just have $\langle u,v\rangle = \langle v,u\rangle$.
  - "Although most mathematicians define an inner product as above, many physicists use a definition that requires homogeneity in the second slot instead of the first" (Axler, 2015, p. 166).

- **Euclidean inner product** (on $\mathbb{F}^n$): The function defined by

$$\langle w,z\rangle = w_1\bar{z_1} + \cdots + w_n\bar{z_n}$$

- **Inner product space**: A vector space V along with an inner product on V.
  - When $\mathbb{F}^n$ is referred to as an inner product space, assume that the inner product is the Euclidean inner product unless explicitly stated otherwise.

- Basic properties of an inner product.

Theorem 6.1.

(a) *For each fixed $u \in V$, the function that takes v to $\langle v,u\rangle$ is a linear map from V to $\mathbb{F}$.*

Proof. Let $u \in V$ be arbitrary, and let $T : V \to \mathbb{F}$ be defined by $Tv = \langle v,u\rangle$. Let $v,w \in V$ be arbitrary, and let $\lambda \in \mathbb{F}$. Then

$$\begin{aligned} T(v+w) &= \langle v+w,u\rangle \\ &= \langle v,u\rangle + \langle w,u\rangle \\ &= Tv + Tw \end{aligned} \qquad\qquad \begin{aligned} T(\lambda v) &= \langle \lambda v,u\rangle \\ &= \lambda\langle v,u\rangle \\ &= \lambda Tv \end{aligned}$$

as desired. ∎

(b) *$\langle 0,u\rangle = 0$ for every $u \in V$.*

Proof. Let $u \in V$ be arbitrary. Since T as defined above is linear, Theorem 3.2 implies that $0 = T(0) = \langle 0,u\rangle$, as desired. ∎

(c) *$\langle u,0\rangle = 0$ for every $u \in V$.*

Proof. Let $u \in V$ be arbitrary. By the conjugate symmetry property and the above, $\langle u,0\rangle = \overline{\langle 0,u\rangle} = \bar{0} = 0$, as desired. ∎

(d) *$\langle u,v+w\rangle = \langle u,v\rangle + \langle u,w\rangle$ for all $u,v,w \in V$.*

Proof. Let $u,v,w \in V$ be arbitrary. Then

$$\begin{aligned} \langle u,v+w\rangle &= \overline{\langle v+w,u\rangle} \\ &= \overline{\langle v,u\rangle + \langle w,u\rangle} \\ &= \overline{\langle v,u\rangle} + \overline{\langle w,u\rangle} && \text{Theorem 4.1} \\ &= \langle u,v\rangle + \langle v,w\rangle \end{aligned}$$

as desired. ∎

(e) *$\langle u,\lambda v\rangle = \bar{\lambda}\langle u,v\rangle$ for all $\lambda \in \mathbb{F}$ and $u,v \in V$.*

Proof. Let $u,v \in V$, and let $\lambda \in \mathbb{F}$. Then

$$\begin{aligned} \langle u,\lambda v\rangle &= \overline{\langle \lambda v,u\rangle} \\ &= \overline{\lambda\langle v,u\rangle} \\ &= \bar{\lambda}\overline{\langle v,u\rangle} && \text{Theorem 4.1} \\ &= \bar{\lambda}\langle u,v\rangle \end{aligned}$$

∎

- **Norm** (of $v \in V$): The quantity
$$\|v\| = \sqrt{\langle v, v\rangle}$$
- Basic properties of the norm.

Theorem 6.2. *Suppose $v \in V$. Then*

(a) $\|v\| = 0$ *iff* $v = 0$.

Proof. Suppose first that $\|v\| = 0$. Then $0 = \sqrt{\langle v, v\rangle} = \langle v, v\rangle$. Thus, $v = 0$. The proof is symmetric in the reverse direction. ∎

(b) $\|\lambda v\| = |\lambda|\|v\|$ *for all* $\lambda \in \mathbb{F}$.

Proof. Let $\lambda \in \mathbb{F}$ be arbitrary. Then
$$\begin{aligned} \|\lambda v\|^2 &= \langle \lambda v, \lambda v\rangle \\ &= \lambda\bar{\lambda}\langle v, v\rangle \\ &= |\lambda|^2\|v\|^2 && \text{Theorem 4.1} \end{aligned}$$
Taking square roots of the above gives the desired equality.[1] ∎

- **Orthogonal** (vectors $u, v \in V$): Two vectors $u, v \in V$ such that $\langle u, v\rangle = 0$.[2]
- If $u, v \in \mathbb{R}^2$ are nonzero, then $\langle u, v\rangle = \|u\|\|v\|\cos\theta$.
  - "Thus, two vectors in $\mathbb{R}^2$ are orthogonal (with respect to the usual Euclidean inner product) if and only if the cosine of the angle between them is 0, which happens if and only if the vectors are perpendicular in the usual sense of plane geometry. Thus, you can think of the word *orthogonal* as a fancy word meaning *perpendicular*." (Axler, 2015, p. 169).
- Orthogonality and zero.

Theorem 6.3.

(a) 0 *is orthogonal to every vector in* V.

Proof. Let $u \in V$ be arbitrary. Then Theorem 6.12 implies that $\langle 0, u\rangle = 0$. Thus, u and 0 are orthogonal, as desired. ∎

(b) 0 *is the only vector in* V *that is orthogonal to itself.*

Proof. Let $v \in V$ be such that v is orthogonal to itself. Then $\langle v, v\rangle = 0$. But by the property of definiteness, it follows that $v = 0$, as desired. ∎

- The special case where $V = \mathbb{R}^2$ of the following is over 2 500 years old, although the following is not the original proof.

Theorem 6.4 (Pythagorean Theorem)**.** *Suppose $u, v \in V$ are orthogonal. Then*
$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$

Proof. We have
$$\begin{aligned} \|u + v\|^2 &= \langle u + v, u + v\rangle \\ &= (\langle u, u\rangle + \langle u, v\rangle) + (\langle v, u\rangle + \langle v, v\rangle) \\ &= \|u\|^2 + 0 + 0 + \|v\|^2 \\ &= \|u\|^2 + \|v\|^2 \end{aligned}$$
as desired. ∎

[1] Notice this technique: Working with norms squared is usually easier than working directly with norms.
[2] The word *orthogonal* derives from the Greek word *orthogonios*, which means right-angled.

- Note that we can prove the converse of the Pythagorean Theorem in *real* inner product spaces as follows: If $\|u+v\|^2 = \|u\|^2 + \|v\|^2$ for u, v real, then

$$\begin{aligned} 0 &= \langle u, v\rangle + \langle v, u\rangle \\ &= \langle u, v\rangle + \overline{\langle u, v\rangle} \\ &= 2\,\mathrm{Re}\langle u, v\rangle \\ &= \langle u, v\rangle \end{aligned}$$

  as desired.

- Let $u, v \in V$ with $v \neq 0$. We are now equipped to consider how to write u as the sum of v plus a vector w orthogonal to v, as in Figure 6.1 below.

Theorem 6.5. *Suppose $u, v \in V$ with $v \neq 0$. Set $c = \frac{\langle u,v\rangle}{\|v\|^2}$ and $w = u - \frac{\langle u,v\rangle}{\|v\|^2}v$. Then $\langle w, v\rangle = 0$ and $u = cv + w$.*

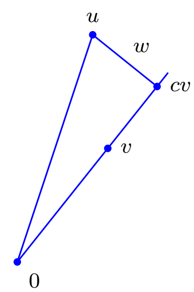

Figure 6.1: An orthogonal decomposition.

Proof. We want to write u in the form $u = cv + w$ where w is orthogonal to v. We know that

$$u = cv + (u - cv)$$

so we need only choose c such that v is orthogonal to $u - cv$. In other words, we want

$$\begin{aligned} 0 &= \langle u - cv, v\rangle \\ &= \langle u, v\rangle + \langle -cv, v\rangle \\ &= \langle u, v\rangle - c\langle v, v\rangle \\ &= \langle u, v\rangle - c\|v\|^2 c \qquad\qquad = \frac{\langle u, v\rangle}{\|v\|^2} \end{aligned}$$

But this gives the values we want for c and w, as desired. ■

- This allows for the proof of a very important result.

Theorem 6.6 (Cauchy-Schwarz Inequality)**.** *Suppose $u, v \in V$. Then*

$$|\langle u, v\rangle| \leq \|u\|\|v\|$$

This inequality is an equality if and only if one of u, v is a scalar multiple of the other.

Proof. We divide into two cases ($v = 0$ and $v \neq 0$). If $v = 0$, then

$$|\langle u, v\rangle| = 0 \leq 0 = \|u\|\sqrt{0} = \|u\|\sqrt{\langle u, v\rangle} = \|u\|\|v\|$$

and we also have that the equality holds since $v = 0 = 0u$, $0 \in \mathbb{F}$. Now let $v \neq 0$. Then by Theorem 6.5,

$$u = \frac{\langle u, v\rangle}{\|v\|^2} v + w$$

where $\langle v, w\rangle = 0$. It follows by the Pythagorean Theorem that

$$\begin{aligned}
\|u\|^2 &= \left\| \frac{\langle u, v\rangle}{\|v\|^2} v \right\|^2 + \|w\|^2 \\
&= \langle \frac{\langle u, v\rangle}{\|v\|^2} v, \frac{\langle u, v\rangle}{\|v\|^2} v\rangle + \|w\|^2 \\
&= \frac{\langle u, v\rangle}{\|v\|^2} \overline{\frac{\langle u, v\rangle}{\|v\|^2}} \langle v, v\rangle + \|w\|^2 \\
&= \frac{\langle u, v\rangle \overline{\langle u, v\rangle}}{\|v\|^4} \|v\|^2 + \|w\|^2 \\
&= \frac{|\langle u, v\rangle|^2}{\|v\|^2} + \|w\|^2 \\
&\geq \frac{|\langle u, v\rangle|^2}{\|v\|^2}
\end{aligned}$$

Multiplying both sides by $\|v\|^2$ and taking square roots gives the desired inequality.

Also note that the Cauchy-Schwarz inequality is an equality iff the last line is an equality, which happens iff $w = 0$. But $w = 0$ iff u is a scalar multiple of v, as desired. ■

- Note that the Cauchy-Schwarz is known as such because the French mathematician Augustin-Louis Cauchy proved the top inequality below in 1821, and the German mathematician Hermann Schwarz proved the bottom inequality below in 1886; both are special cases of the above.

$$|x_1 y_1 + \cdots + x_n y_n|^2 \leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2)$$
$$\left| \int_{-1}^{1} f(x) g(x) \,\mathrm{d}x \right|^2 \leq \left(\int_{-1}^{1} (f(x))^2 \,\mathrm{d}x \right) \left(\int_{-1}^{1} (g(x))^2 \,\mathrm{d}x \right)$$

  - For the top one, we let $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}$.
  - For the bottom one, we let f, g be continuous real-valued functions on $[-1, 1]$.

- We now prove another important inequality.

Theorem 6.7 (Triangle Inequality)**.** *Suppose $u, v \in V$. Then*

$$\|u + v\| \leq \|u\| + \|v\|$$

This inequality is an equality if and only if one of u, v is a nonnegative multiple of the other.

Proof. We have

$$\begin{aligned}
\|u + v\|^2 &= \langle u + v, u + v\rangle \\
&= \langle u, u\rangle + \langle v, v\rangle + \langle u, v\rangle + \langle v, u\rangle \\
&= \langle u, u\rangle + \langle v, v\rangle + \langle u, v\rangle + \overline{\langle u, v\rangle} \\
&= \|u\|^2 + \|v\|^2 + 2\operatorname{Re}\langle u, v\rangle \\
&\leq \|u\|^2 + \|v\|^2 + 2|\langle u, v\rangle| \\
&\leq \|u\|^2 + \|v\|^2 + 2\|u\|\|v\| && \text{Cauchy-Schwarz Inequality} \\
&= (\|u\| + \|v\|)^2
\end{aligned}$$

Taking square roots of both sides gives the desired inequality.

This inequality is an equality iff $\langle u, v\rangle = \|u\|\|v\|$. Now suppose $u = cv$ where $c \in \mathbb{F}$ is positive. Then

$$\begin{aligned}
\langle u, v\rangle &= \langle cv, v\rangle \\
&= c\langle v, v\rangle \\
&= c\|v\|^2 \\
&= c\sqrt{\langle v, v\rangle}\|v\| \\
&= \sqrt{c^2\langle v, v\rangle}\|v\| \\
&= \sqrt{\langle cv, cv\rangle}\|v\| \\
&= \|u\|\|v\|
\end{aligned}$$

The proof is the same in the reverse direction. ■

- One last equality.

Theorem 6.8 (Parallelogram Equality). *Suppose $u, v \in V$. Then*

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

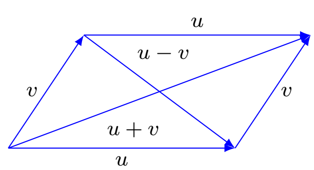

Figure 6.2: The parallelogram equality.

Proof. We have

$$\begin{aligned}
\|u+v\|^2 + \|u-v\|^2 &= \langle u+v, u+v\rangle + \langle u-v, u-v\rangle \\
&= \|u\|^2 + \|v\|^2 + \langle u, v\rangle + \langle v, u\rangle + \|u\|^2 + \|v\|^2 - \langle u, v\rangle - \langle v, u\rangle \\
&= 2(\|u\|^2 + \|v\|^2)
\end{aligned}$$

as desired. ■