Chapter 1

Vector Spaces

1.A $\mathbb{R}^n$ and $\mathbb{C}^n$

10/27:
- Assumed familiarity with the set $\mathbb{R}$ of real numbers.
- **Complex number**: An ordered pair (a, b), where $a, b \in \mathbb{R}$, but we will write this as $a + bi$.
  - The set of all complex number is denoted by $\mathbb{C}$:

$$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}^{[1]}$$

  - Definitions of **addition** and **multiplication** on $\mathbb{C}$ are given, but I know these.
- Properties of complex arithmetic:
  - **Commutativity**: $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbb{C}$.
  - **Associativity**: $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ and $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.
  - **Identities**: $\lambda + 0 = \lambda$ and $\lambda 1 = \lambda$ for all $\lambda \in \mathbb{C}$.
  - **Additive inverse**: For every $\alpha \in \mathbb{C}$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$.
  - **Multiplicative inverse**: For every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$.
  - **Distributive property**: $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbb{C}$.
- "The properties above are proved using the familiar properties of real numbers and the definitions of complex addition and multiplication" (Axler, 2015, p. 3).
- $\mathbb{F}$ stands for $\mathbb{R}$ or $\mathbb{C}$.
  - Any theorem proved with $\mathbb{F}$ holds when $\mathbb{F}$ is replaced with $\mathbb{R}$ and when $\mathbb{F}$ is replaced with $\mathbb{C}$.
- **Scalar**: A number or magnitude. This word is commonly used to differentiate a quantity from a **vector** quantity.
- Subtraction and division are defined.
- Properties of exponents are defined.
- The set $\mathbb{R}^2$, which can be conceived as a plane, is the set of all **ordered pairs** of real numbers:

$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$$

[1] The complex numbers equal the set of numbers $a + bi$ such that a and b are elements of the real numbers.

- The set $\mathbb{R}^3$, which can be conceived as ordinary space, is the set of all **ordered triples** of real numbers:

$$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$$

- "Suppose n is a nonnegative integer. A **list** of **length** n is an ordered collection of n elements (which might be numbers, other lists, or more abstract entities) separated by commas and surrounded by parentheses. A list of length n looks like this:

$$(x_1, \dots, x_n)$$

  Two lists are equal if and only if they have the same length and the same elements in the same order" (Axler, 2015, p. 5).

- **Ordered pair**: A list of length 2.
- **Ordered triple**: A list of length 3.
- ***n*-tuple**: A list of length n.
- Although lists are sometimes discussed without specifying their length, a list must, by definition, have a finite length, i.e. $(x_1, x_2, \dots)$ is not a list.
- A list of length 0 looks like this: ().
  - Such an object is defined to avoid trivial exceptions to theorems.
- Lists vs. **sets**: In lists, order matters and repetitions have meaning. In sets, order and repetitions are irrelevant.
- "$\mathbb{F}^n$ is the set of all lists of length n of elements of $\mathbb{F}$:

$$\mathbb{F}^n = \{(x_1, \dots, x_n) : x_j \in \mathbb{F} \text{ for } j = 1, \dots, n)\}$$

  For $(x_1, \dots, x_n) \in \mathbb{F}^n$ and $j \in \{1, \dots, n\}$, we say that x_j is the j^{th} **coordinate** of $(x_1, \dots, x_n)$" (Axler, 2015, p. 6).

- For help in conceiving higher dimensional spaces, consider reading Abbott (1952). This is an amusing account of how $\mathbb{R}^3$ would be perceived by creatures living in $\mathbb{R}^2$.
- **Addition** (in $\mathbb{F}^n$): Add corresponding coordinates:

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

- For a simpler notation, use a single letter to denote a list of n numbers.
  - **Commutativity** (of addition in $\mathbb{F}^n$): If $x, y \in \mathbb{F}^n$, then $x + y = y + x$.
  - However, the proof still requires the more formal, cumbersome list notation.
- **0**: The list of length n whose coordinates are all 0:

$$0 = (0, \dots, 0)$$

  - Although the ambiguity in the use of "0" on the left vs. right side of the equation may seem confusing, context can always differentiate between which definition is needed.
- A picture can help visualize $\mathbb{R}^2$ because $\mathbb{R}^2$ can be sketched on 2-dimensional surfaces such as paper.
  - A typical element of $\mathbb{R}^2$ is a point $x = (x_1, x_2)$.
  - However, points are generally though of as an arrow starting at the origin and ending at x, as shown below.

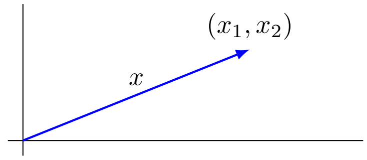

Figure 1.1: $x \in \mathbb{R}^2$ can be conceived as a point or a vector.

- When thought of as an arrow, x is called a **vector**.
- When translated without varying length or direction, it is still the same vector.
- Remember that these pictures are aids — although we cannot visualize higher dimensional vector spaces, the algebraic elements are as rigorously defined as those of $\mathbb{R}^2$.

- Addition has a simple geometric interpretation in $\mathbb{R}^2$.
- If we want to add $x + y$, slide y so that its initial point coincides with the terminal point of x. The sum is the vector from the tail of x to the head of y.

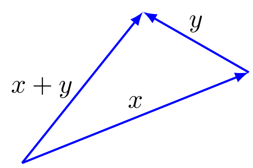

Figure 1.2: Vector addition.

- "For $x \in \mathbb{F}^n$, the **additive inverse** of x, denoted $-x$, is the vector $-x \in \mathbb{F}^n$ such that

$$x + (-x) = 0$$

In other words, if $x = (x_1, \ldots, x_n)$, then $-x = (-x_1, \ldots, -x_n)$" (Axler, 2015, p. 9).

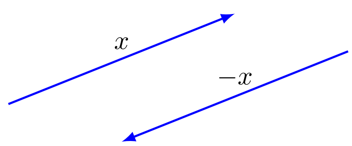

Figure 1.3: A vector and its additive inverse.

  - For $x \in \mathbb{R}^2$, $-x$ is the vector parallel to x with the same length but in the opposite direction.

- **Product (scalar multiplication)**: When multiplying $\lambda \in \mathbb{F}$ and $x \in \mathbb{F}^n$, multiply each coordinate of x by λ:

$$\lambda\left(x_1, \ldots, x_n\right) = \left(\lambda x_1, \ldots, \lambda x_n\right)$$

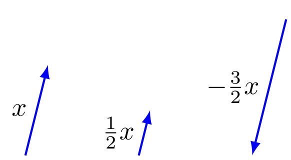

Figure 1.4: Scalar multiplication.

- **Field**: A "set containing at least two distinct elements called 0 and 1, along with operations of addition and multiplication satisfying all the properties" of complex arithmetic (see earlier in this section) (Axler, 2015, p. 10).

1.B Definition of Vector Space

- **Addition (on a set V)**: "A function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$" (Axler, 2015, p. 12).
- **Scalar multiplication (on a set V)**: "A function that assigns an element $\lambda v \in V$ to each $\lambda \in \mathbb{F}$ and each $v \in V$" (Axler, 2015, p. 12).
- **Vector space**: "A set V along with an addition and a scalar multiplication on V such that the following properties hold:" (Axler, 2015, p. 12).

  commutativity
  $u + v = v + u$ for all $u, v \in V$

  associativity
  $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and all $a, b \in \mathbb{F}$

  additive identity
  There exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$

  additive inverse
  For every $v \in V$, there exists $w \in V$ such that $v + w = 0$

  multiplicative identity
  $1v = v$ for all $v \in V$

  distributive properties
  $a(u + v) = au + av$ and $(a + b)v = av + bv$ for all $a, b \in \mathbb{F}$ and all $u, v \in V$

- To be more precise, V depends on $\mathbb{F}$, so sometimes we say V is a **vector space over** $\mathbb{F}$.
  - For example, $\mathbb{R}^n$ is only a vector space over $\mathbb{R}$, not $\mathbb{C}$.
- **Real vector space**: A vector space over $\mathbb{R}$.
- **Complex vector space**: A vector space over $\mathbb{C}$.
- $\mathbb{F}^\infty$ is a vector space.
- $\mathbb{F}^S$ denotes the set of functions from S to $\mathbb{F}$.
  - For example, $\mathbb{R}^{[0,1]}$ is the "set of real-valued functions on the interval $[0, 1]$" (Axler, 2015, p. 14).
  - You can think of $\mathbb{F}^n$ as $\mathbb{F}^{\{1,2,\ldots,n\}}$.
- Elementary properties of vector spaces:

  Theorem 1.1 (Unique additive identity)**.** *A vector space has a unique additive identity.*

  Proof. Suppose 0 and $0'$ are both additive identities in V. Then

  $$0' = 0' + 0 = 0 + 0' = 0$$

  The first equality holds due to 0 being an additive identity. The second holds due to commutativity. The third holds due to $0'$ being an additive identity. Thus, $0 = 0'$, and V has only one additive identity. ■

  Theorem 1.2 (Unique additive inverse)**.** *Each element $v \in V$ has a unique additive inverse.*

Proof. Same idea:

$$w = w + 0 = w + (v + w') = (w + v) + w' = 0 + w' = w'$$

■

Theorem 1.3 (The number 0 times a vector)**.** *$0v = 0\ \forall\ v \in V$, where 0 on the left side is a scalar and 0 on the right side is a vector (the additive identity of V).*

Proof. Since this property asserts something about both scalar multiplication and the additive identity, the distributive property (the only part of the definition of a vector space that connects scalar multiplication and vector addition) must be used in the proof.

$$\begin{aligned} 0v &= (0 + 0)v \\ 0v &= 0v + 0v \\ 0v - 0v &= 0v + 0v - 0v \\ 0 &= 0v \end{aligned}$$

■

Theorem 1.4 (A number times the vector 0)**.** *$a0 = 0\ \forall\ a \in \mathbb{F}$, where 0 is a vector.*

Proof. Same as above. ■

Theorem 1.5 (The number -1 times a vector)**.** *$(-1)v = -v\ \forall\ v \in V$, where -1 is a scalar and $-v$ is the additive inverse of v.*

Proof.

$$v + (-1)v = 1v + (-1)v = (1 + (-1))v = 0v = 0$$

■

1.C Subspaces

- **Subspace**: A subset U of V that is a vector space under the same definition of addition and scalar multiplication as on V, e.g., satisfies the following three conditions.

  additive identity
  $0 \in U$

  closed under addition
  $u, w \in U$ implies $u + w \in U$

  closed under scalar multiplication
  $a \in \mathbb{F}$ and $u \in U$ implies $au \in U$

- The other conditions can be derived from the above 3.
- When we look at subspaces within the differentiable functions, the logical foundation of calculus appears.
- The subspaces of $\mathbb{R}^2$ are $\{0\}$, $\mathbb{R}^2$, and any straight line through the origin.
- The subspaces of $\mathbb{R}^3$ are $\{0\}$, $\mathbb{R}^3$, any straight line through the origin, and any flat plane through the origin.
- **Sum of subsets**: If $U_1, \dots, U_n$ are subsets of V, their **sum** (denoted $U_1 + \cdots + U_n$) is the set of all possible sums of elements of $U_1, \dots, U_n$:

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m : u_1 \in U_1, \dots, u_m \in U_m\}$$

- The sum of subspaces is the smallest containing subspace.
  - Clearly, the sum of subspaces is a subspace (satisfies 3 tenets).
  - The sum of subspaces contains every original element (u_1 plus the 0 from u_2, etc.). Any subspace containing all of these elements must contain every finite sum of them (by definition). Thus, no smaller subspace can be created than that of the sum of every element.
- **Direct sum**: A sum of subspaces where each element of $U_1 + \cdots + U_m$ can be written in only one way as a sum $u_1 + \cdots + u_m$.
  - $U_1 \oplus \cdots \oplus U_m$ denotes $U_1 + \cdots + U_m$ if $U_1 + \cdots + U_m$ is a direct sum.
- A sum of subspaces is a direct sum if and only if the only way to write 0 as a sum of elements is by summing the 0 of each subset.
- A sum of subspaces U and W is a direct sum if and only if $U \cap W = \{0\}$.