GIEP Notes

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1 Vector Spaces

1.1 $\mathbb{R}^n$ and $\mathbb{C}^n$

From [Axler, 2015].

- Assumed familiarity with the set $\mathbb{R}$ of real numbers.
- **Complex number**: An ordered pair (a, b), where $a, b \in \mathbb{R}$, but we will write this as $a + bi$.
  - The set of all complex number is denoted by $\mathbb{C}$:

    $$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}^{[1]}$$

  - Definitions of **addition** and **multiplication** on $\mathbb{C}$ are given, but I know these.
- Properties of complex arithmetic:
  - **Commutativity**: $\alpha + \beta = \beta + \alpha$ and $\alpha\beta = \beta\alpha$ for all $\alpha, \beta \in \mathbb{C}$.
  - **Associativity**: $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ and $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbb{C}$.
  - **Identities**: $\lambda + 0 = \lambda$ and $\lambda 1 = \lambda$ for all $\lambda \in \mathbb{C}$.
  - **Additive inverse**: For every $\alpha \in \mathbb{C}$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$.
  - **Multiplicative inverse**: For every $\alpha \in \mathbb{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha\beta = 1$.
  - **Distributive property**: $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbb{C}$.
- "The properties above are proved using the familiar properties of real numbers and the definitions of complex addition and multiplication" [Axler, 2015, 3].
- $\mathbb{F}$ stands for $\mathbb{R}$ or $\mathbb{C}$.
  - Any theorem proved with $\mathbb{F}$ holds when $\mathbb{F}$ is replaced with $\mathbb{R}$ and when $\mathbb{F}$ is replaced with $\mathbb{C}$.
- **Scalar**: A number or magnitude. This word is commonly used to differentiate a quantity from a **vector** quantity.
- Subtraction and division are defined.
- Properties of exponents are defined.
- The set $\mathbb{R}^2$, which can be conceived as a plane, is the set of all **ordered pairs** of real numbers:

  $$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$$

- The set $\mathbb{R}^3$, which can be conceived as ordinary space, is the set of all **ordered triples** of real numbers:

  $$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$$

- "Suppose n is a nonnegative integer. A **list** of **length** n is an ordered collection of n elements (which might be numbers, other lists, or more abstract entities) separated by commas and surrounded by parentheses. A list of length n looks like this:

  $$(x_1, \ldots, x_n)$$

  Two lists are equal if and only if they have the same length and the same elements in the same order" [Axler, 2015, 5].

[1] The complex numbers equal the set of numbers $a + bi$ such that a and b are elements of the real numbers.

- **Ordered pair**: A list of length 2.
- **Ordered triple**: A list of length 3.
- n**-tuple**: A list of length n.
- Although lists are sometimes discussed without specifying their length, a list must, by definition, have a finite length, i.e. $(x_1, x_2, \ldots)$ is not a list.
- A list of length 0 looks like this: ().
  - Such an object is defined to avoid trivial exceptions to theorems.
- Lists vs. **sets**: In lists, order matters and repetitions have meaning. In sets, order and repetitions are irrelevant.
- "$\mathbb{F}^{\mathbf{n}}$ is the set of all lists of length n of elements of $\mathbb{F}$:

  $$\mathbb{F}^n = \{(x_1, \ldots, x_n) : x_j \in \mathbb{F} \text{ for } j = 1, \ldots, n)\}$$

  For $(x_1, \ldots, x_n) \in \mathbb{F}^n$ and $j \in \{1, \ldots, n\}$, we say that x_j is the j^{th} **coordinate** of $(x_1, \ldots, x_n)$" [Axler, 2015, 6].
- For help in conceiving higher dimensional spaces, consider reading *Flatland: A Romance of Many Dimensions* by Edwin A. Abbot. This is an amusing account of how $\mathbb{R}^3$ would be perceived by creatures living in $\mathbb{R}^2$.
- **Addition** (in $\mathbb{F}^n$): Add corresponding coordinates:

  $$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

- For a simpler notation, use a single letter to denote a list of n numbers.
  - **Commutativity** (of addition in $\mathbb{F}^n$): If $x, y \in \mathbb{F}^n$, then $x + y = y + x$.
  - However, the proof still requires the more formal, cumbersome list notation[2].
- **0**: The list of length n whose coordinates are all 0:

  $$0 = (0, \ldots, 0)$$

  - Although the ambiguity in the use of "0" on the left vs. right side of the equation may seem confusing, context can always differentiate between which definition is needed.
- A picture can help visualize $\mathbb{R}^2$ because $\mathbb{R}^2$ can be sketched on 2-dimensional surfaces such as paper.

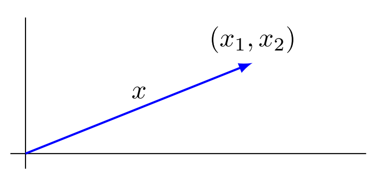


Figure 1.1: $x \in \mathbb{R}^2$ can be conceived as a point or a vector.

  - A typical element of $\mathbb{R}^2$ is a point $x = (x_1, x_2)$.
  - However, points are generally though of as an arrow starting at the origin and ending at x, as shown below.
  - When thought of as an arrow, x is called a **vector**.

[2]Note that ■ means "end of the proof."

- When translated without varying length or direction, it is still the same vector.
- Remember that these pictures are aids — although we cannot visualize higher dimensional vector spaces, the algebraic elements are as rigorously defined as those of $\mathbb{R}^2$.

- Addition has a simple geometric interpretation in $\mathbb{R}^2$.
- If we want to add $x + y$, slide y so that its initial point coincides with the terminal point of x. The sum is the vector from the tail of x to the head of y.

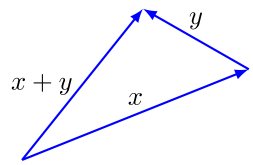


Figure 1.2: Vector addition.

- "For $x \in \mathbb{F}^n$, the **additive inverse** of x, denoted $-x$, is the vector $-x \in \mathbb{F}^n$ such that

$$x + (-x) = 0$$

In other words, if $x = (x_1, \ldots, x_n)$, then $-x = (-x_1, \ldots, -x_n)$" [Axler, 2015, 9].

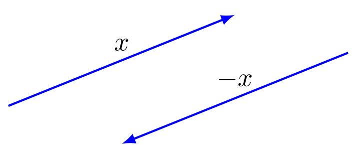


Figure 1.3: A vector and its additive inverse.

  - For $x \in \mathbb{R}^2$, $-x$ is the vector parallel to x with the same length but in the opposite direction.

- **Product (scalar multiplication)**: When multiplying $\lambda \in \mathbb{F}$ and $x \in \mathbb{F}^n$, multiply each coordinate of x by λ:

$$\lambda (x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

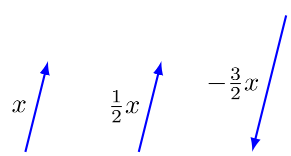


Figure 1.4: Scalar multiplication.

- **Field**: A "set containing at least two distinct elements called 0 and 1, along with operations of addition and multiplication satisfying all the properties" of complex arithmetic (see earlier in this section) [Axler, 2015, 10].

1.2 Definition of Vector Space

- **Addition (on a set V)**: "A function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$" [Axler, 2015, 12].

- **Scalar multiplication (on a set V)**: "A function that assigns an element $\lambda v \in V$ to each $\lambda \in \mathbb{F}$ and each $v \in V$" [Axler, 2015, 12].
- **Vector space**: "A set V along with an addition and a scalar multiplication on V such that the following properties hold:" [Axler, 2015, 12].

  commutativity
  : $u + v = v + u$ for all $u, v \in V$

  associativity
  : $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and all $a, b \in \mathbb{F}$

  additive identity
  : There exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$

  additive inverse
  : For every $v \in V$, there exists $w \in V$ such that $v + w = 0$

  multiplicative identity
  : $1v = v$ for all $v \in V$

  distributive properties
  : $a(u + v) = au + av$ and $(a + b)v = av + bv$ for all $a, b \in \mathbb{F}$ and all $u, v \in V$
- To be more precise, V depends on $\mathbb{F}$, so sometimes we say V is a **vector space over** $\mathbb{F}$.
  - For example, $\mathbb{R}^n$ is only a vector space over $\mathbb{R}$, not $\mathbb{C}$.
- **Real vector space**: A vector space over $\mathbb{R}$.
- **Complex vector space**: A vector space over $\mathbb{C}$.
- $\mathbb{F}^\infty$ is a vector space.
- $\mathbb{F}^S$ denotes the set of functions from S to $\mathbb{F}$.
  - For example, $\mathbb{R}^{[0,1]}$ is the "set of real-valued functions on the interval $[0, 1]$" [Axler, 2015, 14].
  - You can think of $\mathbb{F}^n$ as $\mathbb{F}^{\{1,2,...,n\}}$.
- Elementary properties of vector spaces:
  - A vector space has a unique additive identity.
    - Suppose 0 and $0'$ are both additive identities in V. Then

      $$0' = 0' + 0 = 0 + 0' = 0$$

      The first equality holds due to 0 being an additive identity. The second holds due to commutativity. The third holds due to $0'$ being an additive identity. Thus, $0 = 0'$, and V has only one additive identity.
  - Each element $v \in V$ has a unique additive inverse.
    - Same idea:

      $$w = w + 0 = w + (v + w') = (w + v) + w' = 0 + w' = w'$$
  - $0v = 0 \; \forall \; v \in V$, where 0 on the left side is a scalar and 0 on the right side is a vector (the additive identity of V).
    - Since this property asserts something about both scalar multiplication and the additive identity, the distributive property (the only part of the definition of a vector space that connects scalar multiplication and vector addition) must be used in the proof.

      $$\begin{aligned} 0v &= (0 + 0)v \\ 0v &= 0v + 0v \\ 0v - 0v &= 0v + 0v - 0v \\ 0 &= 0v \end{aligned}$$

- $a0 = 0 \; \forall \; a \in \mathbb{F}$, where 0 is a vector.
  - Same as above.
- $(-1)v = -v \; \forall \; v \in V$, where -1 is a scalar and $-v$ is the additive inverse of v.

$$v + (-1)v = 1v + (-1)v = (1 + (-1))v = 0v = 0$$

1.3 Subspaces

- **Subspace**: A subset U of V that is a vector space under the same definition of addition and scalar multiplication as on V, e.g., satisfies the following three conditions.

  additive identity
  $0 \in U$

  closed under addition
  $u, w \in U$ implies $u + w \in U$

  closed under scalar multiplication
  $a \in \mathbb{F}$ and $u \in U$ implies $au \in U$

- The other conditions can be derived from the above 3.
- When we look at subspaces within the differentiable functions, the logical foundation of calculus appears.
- The subspaces of $\mathbb{R}^2$ are $\{0\}$, $\mathbb{R}^2$, and any straight line through the origin.
- The subspaces of $\mathbb{R}^3$ are $\{0\}$, $\mathbb{R}^3$, any straight line through the origin, and any flat plane through the origin.
- **Sum of subsets**: If $U_1, \dots, U_n$ are subsets of V, their **sum** (denoted $U_1 + \cdots + U_n$) is the set of all possible sums of elements of $U_1, \dots, U_n$:

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m : u_1 \in U_1, \dots, u_m \in U_m\}$$

- The sum of subspaces is the smallest containing subspace.
  - Clearly, the sum of subspaces is a subspace (satisfies 3 tenets).
  - The sum of subspaces contains every original element (u_1 plus the 0 from u_2, etc.). Any subspace containing all of these elements must contain every finite sum of them (by definition). Thus, no smaller subspace can be created than that of the sum of every element.
- **Direct sum**: A sum of subspaces where each element of $U_1 + \cdots + U_m$ can be written in only one way as a sum $u_1 + \cdots + u_m$.
  - $U_1 \oplus \cdots \oplus U_m$ denotes $U_1 + \cdots + U_m$ if $U_1 + \cdots + U_m$ is a direct sum.
- A sum of subspaces is a direct sum if and only if the only way to write 0 as a sum of elements is by summing the 0 of each subset.
- A sum of subspaces U and W is a direct sum if and only if $U \cap W = \{0\}$.

2 Graphs

2.1 A Gentle Introduction

From [Goodaire and Parmenter, 2002].

- Begins with the Königsberg Bridge Problem and Euler's solution by reducing the land masses to a mathematical **graph**.
- Finding an abstract mathematical model of a concrete problem requires ingenuity and experience. "The primary aim of this chapter is to provide the reader with some of this experience by presenting several real-world problems and showing how they can be formulated in mathematical terms" [Goodaire and Parmenter, 2002, 277-78].
- Considers the Three Houses–Three Utilities Problem.
- Deeply considers Instant Insanity (stack four cubes, each with one of four colors on each face, in such a way that every color is represented on every side of the column):

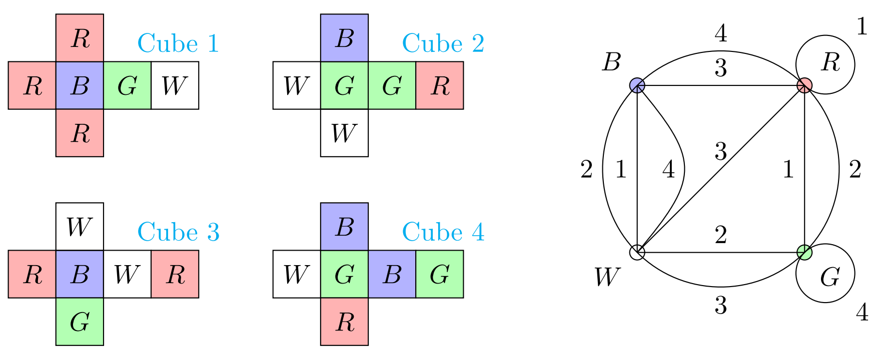


Figure 2.1: Four colored cubes and a graphical representation.

- Each vertex in Figure 2.1 represents one of the colors. Each edge connects a color on one face of a cube to the color on the opposite face (the cube to which this relationship pertains is identified by the number along the edge).
  - For example, B is connected to R by a line with 3 above it because on Cube 3, the blue face and the red face are on opposite sides of the cube.
- Let's take a look at a possible stack and see what we can learn from it and its **subgraph** (see Figure 2.2).

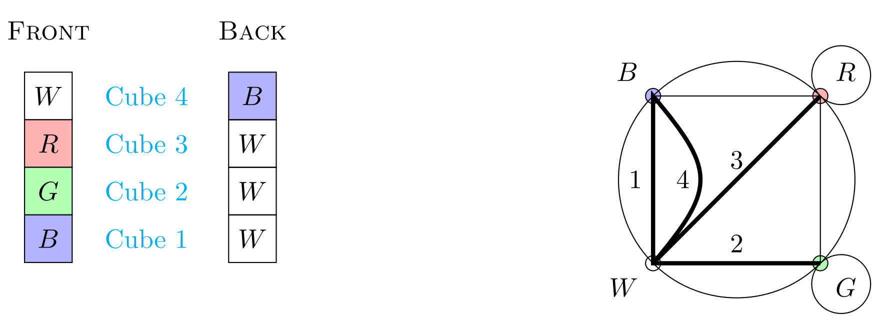


Figure 2.2: A possible stacking and graph.

- The reason this stack fails to provide a correct column, front and back, is because too many edges touch white and not enough touch red or green.
- Also note that this subgraph represents a feasible stack because each cube is represented once (each number appears once in the subgraph).
- Therefore, we can conjecture conditions of the subgraph that will lead to a stack that is solved front-to-back, namely:
  - The subgraph will contain all four vertices.
  - The subgraph will consist of four edges, one from each cube.
  - The subgraph will have exactly two edges meeting at each vertex.
- Following these strictures, several graphs can be easily drawn. One such graph is shown in Figure 2.3 in correspondence with two columns (this is also the solution to Pause 1).

FRONT		BACK
B	Cube 4	R
G	Cube 3	W
R	Cube 2	G
W	Cube 1	B

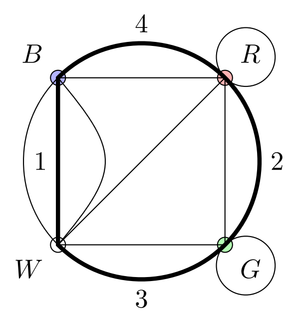


Figure 2.3: A front-to-back-solved stacking and graph.

- Getting the front and back correct is comparably easy to getting the sides correct when playing with the toy.
- Graphically, there must be a second subgraph that satisfies the above conditions and is **edge disjoint** from the first.
- **Edge disjoint** (subgraphs): Two subgraphs that share no edges between them.
- Figure 2.4 shows two edge disjoint subgraphs superimposed on the same graph.

	FRONT	BACK	RIGHT	LEFT
Cube 4	B	R	W	B
Cube 3	G	W	B	R
Cube 2	R	G	G	W
Cube 1	W	B	R	G

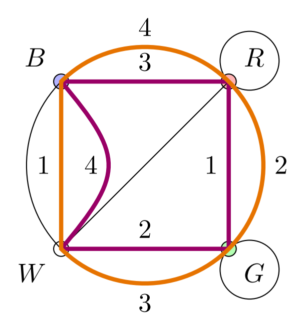


Figure 2.4: A solution to Instant Insanity.

  - These subgraphs correspond to the columns on the left.
  - Note that the orange subgraph corresponds to the front/back solution while the purple subgraph corresponds to the left/right solution.

- **Connected** (graph): A graph where "any two vertices are joined by a sequence of edges" [Goodaire and Parmenter, 2002, 282].

2.2 Definitions and Basic Properties

- **Graph**: "A pair $(\mathcal{V}, \mathcal{E})$ of sets, $\mathcal{V}$ nonempty and each element of $\mathcal{E}$ a set of two distinct elements of $\mathcal{V}$" [Goodaire and Parmenter, 2002, 286].
- **Vertex**: An element of $\mathcal{V}$.
- **Edge**: An element of $\mathcal{E}$.
- **End vertices** (of e): v and w where $v, w \in \mathcal{V}$ such that if e is an edge, then $e = \{v, w\}$. *Also known as* **ends**.
  - Colloquially, edge e **joins** vertices v and w.
  - Set notation is often set aside so that edge e can be referred to as edge vw or wv.
- **Incident** (vertices): The vertices v and w and the ends of edge vw.
- **Incident** (edge): The edge vw connecting vertices v and w.
- **Adjacent** (vertices): Two vertices that are the end vertices of an edge.
- **Adjacent** (edges): Two edges that share a vertex.
- **Degree** (of v): The number of edges incident with a vertex v. *Also known as* $\deg v$.
- **Even** (vertex): A vertex such that $\deg v$ is an even number.
- **Odd** (vertex): A vertex such that $\deg v$ is an odd number.
- **Isolated** (vertex): A vertex such that $\deg v = 0$.
- **Finite** (graph): A graph such that both sets $\mathcal{V}$ and $\mathcal{E}$ are finite.
  - All graphs in this text will be finite.
- $\mathcal{G}(\mathcal{V}, \mathcal{E})$ denotes a graph $\mathcal{G}$ with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$.

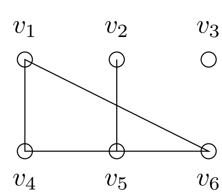


Figure 2.5: The graph $\mathcal{G}$.

  - Normally, a graph is represented by a picture as opposed to its formal set definition.
  - For example, the graph $\mathcal{G}$ with vertex set

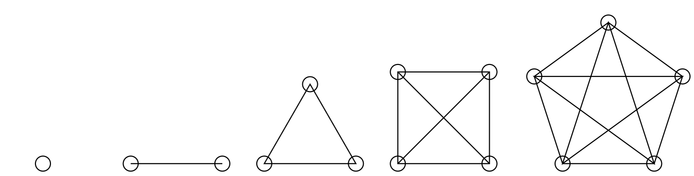
$$\mathcal{V} = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$

and edge set

$$\mathcal{E} = \{v_1v_4, v_1v_6, v_2v_5, v_4v_5, v_5v_6\}$$

can be represented by Figure 2.5.

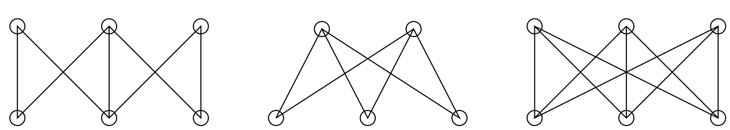
- The above definition of a graph does not allow for **multiple edges** or **loops**, such as those in Figure 2.1.
  - This is because most graphs of interest do not have these features.
- **Multiple edges**: "Several edges incident with the same two vertices" [Goodaire and Parmenter, 2002, 286].
- **Loop**: "An edge which is incident with only one vertex" [Goodaire and Parmenter, 2002, 286].
- **Pseudograph**: A graph that may contain loops and/or multiple edges.
- Note that loops are counted twice when calculating degree — for instance, vertex G in Figure 2.1 has $\deg G = 6$.
- There is no standard set of definitions of terms and symbols in graph theory, so make sure to check the glossary from book to book.
- **Subgraph** (of $\mathcal{G}$): A graph $\mathcal{G}_1$ such that its vertex and edge sets are, respectively, subsets of the vertex and edge sets of $\mathcal{G}$.
  - Subgraphs do not have to be drawn in the same manner as the original graph.
- Denoting deletions:
  - For the graph $\mathcal{G}$ containing edge e, the subgraph $\mathcal{G}_1$ without e will be denoted $\mathcal{G} \setminus \{e\}$ herein.
  - For the graph $\mathcal{G}$ containing vertex v, the subgraph $\mathcal{G}_1$ without v will be denoted $\mathcal{G} \setminus \{v\}$ herein.
  - Note that if the vertex v is deleted, all edges incident with v must also be deleted.



(a) $\mathcal{K}_1$ (b) $\mathcal{K}_2$ (c) $\mathcal{K}_3$ (d) $\mathcal{K}_4$ (e) $\mathcal{K}_5$

Figure 2.6: The first five complete graphs.

- **Complete** (graph of n vertices): The graph with n vertices where any two vertices are adjacent. *Also known as* $\mathcal{K}_n$.



(a) A bipartite graph. (b) $\mathcal{K}_{2,3}$. (c) $\mathcal{K}_{3,3}$.

Figure 2.7: Three bipartite graphs, two of which are complete bipartite.

- **Bipartite** (graph): A graph whose vertices can be partitioned into two (disjoint) sets $\mathcal{V}_1$ and $\mathcal{V}_2$ such that every edge joins a vertex in $\mathcal{V}_1$ with a vertex in $\mathcal{V}_2$.
  - These sets are called **bipartition sets**.
  - A graph that can be drawn such that no two top vertices are adjacent and no two bottom vertices are adjacent.

- **Complete bipartite** (graph): A bipartite graph in which every vertex $v \in \mathcal{V}_1$ is incident with every vertex $w \in \mathcal{V}_2$.
  - The complete bipartite graph with m and n vertices in each respective bipartition set $\mathcal{V}_1$ and $\mathcal{V}_2$ is denoted $\mathcal{K}_{m,n}$.
  - A bipartite graph that can be drawn such that every top vertex is adjacent to every bottom vertex.
- Note that $\mathcal{K}_1$ is technically bipartite since bipartition sets are not required to be nonempty.
- Note that a graph is bipartite "if and only if its vertices can be colored with two colors such that every edge has ends of different colors" [Goodaire and Parmenter, 2002, 289].
- A bipartite graph can contain no **triangles**.
- **Triangle** (in a graph): A set of three vertices with an edge joining each pair.
- "The sum of the degrees of the vertices of a pseudograph is an even number equal to twice the number of edges" [Goodaire and Parmenter, 2002, 290]. Symbolically, if $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is a pseudograph, then the following holds.

$$\sum_{v \in \mathcal{V}} \deg v = 2|\mathcal{E}|$$

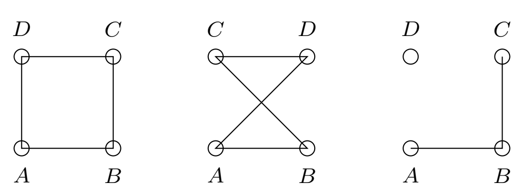
  - Because each edge gets counted twice — once at each vertex it touches. A loop gets counted twice for the same vertex.
  - Example: How many edges does $\mathcal{K}_{m,n}$ have?
    - $\mathcal{K}_{m,n}$ has m vertices of degree n and n vertices of degree m. Therefore,

$$\sum_{v \in \mathcal{V}} \deg v = mn + nm = 2mn = 2|\mathcal{E}|$$

      so the number of edges is equal to $m \times n$.
- **Degree sequence** (of $\mathcal{G}$): The degrees $d_1, \ldots, d_n$ of the vertices $v_1, \ldots, v_n$ of a graph (or pseudograph) $\mathcal{G}$ ordered such that $d_1 \geq \cdots \geq d_n$.
- "The number of odd vertices in a pseudograph is even" [Goodaire and Parmenter, 2002, 290].
  - The sum of the degrees of the vertices of a pseudograph is an even number, and the sum of the degrees of the even vertices is an even number. Thus, the sum of the degrees of the odd vertices must be even. Since only the sum of an even number of odd numbers is even, there must be an even number of odd vertices.

2.3 Isomorphism

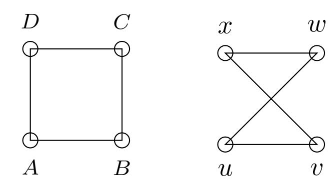
- There is a distinction between a graph and its picture because a graph is pair of sets whereas its picture can be drawn many different ways.



(a) $\mathcal{G}_1$. (b) $\mathcal{G}_2$. (c) $\mathcal{G}_3$.

Figure 2.8: $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic, but neither is isomorphic to $\mathcal{G}_3$.

- **Isomorphic** (graphs): Two graphs $\mathcal{G}_1 = \mathcal{G}_1(\mathcal{V}_1, \mathcal{E}_1)$ and $\mathcal{G}_2 = \mathcal{G}_2(\mathcal{V}_2, \mathcal{E}_2)$ such that there exists a one-to-one function φ from $\mathcal{V}_1$ onto $\mathcal{V}_2$ such that...
  - if vw is an edge in $\mathcal{E}_1$, then $\varphi(v)\varphi(w)$ is an edge in $\mathcal{E}_2$, and
  - every edge in $\mathcal{E}_2$ has the form $\varphi(v)\varphi(w)$ for some edge $vw \in \mathcal{E}_1$.
- **Isomorphism**: A one-to-one function φ from $\mathcal{G}_1$ to $\mathcal{G}_2$.
- Two isomorphic graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ are denoted $\mathcal{G}_1 \cong \mathcal{G}_2$.
- An isomorphism from $\mathcal{G}_1$ to $\mathcal{G}_2$ is denoted (herein) $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$.
- The definition of isomorphism is naturally symmetric — $\mathcal{G}_1 \cong \mathcal{G}_2 \Rightarrow \mathcal{G}_2 \cong \mathcal{G}_1$.
- Likewise, $\varphi : \mathcal{G}_1 \to \mathcal{G}_2 \Rightarrow \varphi^{-1} : \mathcal{G}_2 \to \mathcal{G}_1$.
- To avoid ambiguity, say that two graphs "are isomorphic."
- Isomorphisms relabel vertices without changing any incidence relations, i.e., two graphs are isomorphic if and only if there exists a **bijection** between their sets that "preserves incidence relations."
- Isomorphisms can be written down explicitly.



(a) $\mathcal{G}_1$. (b) $\mathcal{G}_2$.

Figure 2.9: Isomorphisms as explicit functions.

  - The isomorphism between the vertices in Figures 2.9a and 2.9b (recall Figures 2.8a and 2.8b) can be denoted as follows.
$$\varphi(u) = A,\ \varphi(v) = B,\ \varphi(w) = D,\ \varphi(x) = C$$
- Isomorphisms are very important in mathematics.
  - Although the term "isomorphism" may be new, the concept is not — 0.5 and $\frac{2}{4}$ are isomorphic objects.
  - They are **symmetric**, as previously mentioned.
  - They are also **reflexive**: $\mathcal{G} \cong \mathcal{G}$ for any graph $\mathcal{G}$.
    - Because the map $\mathcal{G} \to \mathcal{G}$ (an identity) is an isomorphism.
  - They are also **transitive**: $\mathcal{G}_1 \cong \mathcal{G}_2$ AND $\mathcal{G}_2 \cong \mathcal{G}_3 \Rightarrow \mathcal{G}_1 \cong \mathcal{G}_3$.
    - Because if $\varphi_1 : \mathcal{G}_1 \to \mathcal{G}_2$ and $\varphi_2 : \mathcal{G}_2 \to \mathcal{G}_3$ are isomorphisms, then so is the composition $\varphi_1 \circ \varphi_2 : \mathcal{G}_1 \to \mathcal{G}_3$.
- The set of all graphs is partitioned into disjoint equivalence classes known as **isomorphism classes**.
- **Isomorphism class**: The set of all graphs $\mathcal{G}$ that are isomorphic to one another.
  - In Figure 2.8, $\mathcal{G}_1$ and $\mathcal{G}_2$ are in the same isomorphism class while $\mathcal{G}_1$ and $\mathcal{G}_3$, and $\mathcal{G}_2$ and $\mathcal{G}_3$ are not.
- It is often difficult to prove that graphs are isomorphic, but easy to prove that they are not.

- “If $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic graphs, then $\mathcal{G}_1$ and $\mathcal{G}_2$ have the
  - same number of vertices,
  - same number of edges, and
  - same degree sequences” [Goodaire and Parmenter, 2002, 297].
- Note that the converses of the above qualities are not necessarily true — two graphs with the same number of vertices are not necessarily isomorphic.

References

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[Goodaire and Parmenter, 2002] Goodaire, E. G. and Parmenter, M. M. (2002). *Discrete Mathematics with Graph Theory.* Prentice-Hall, Upper Saddle River, NJ 07458, second edition.