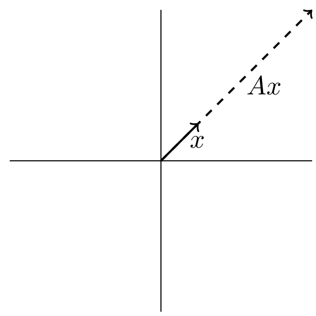
1/28: Introduction to Eigenvalues and Eigenvectors

- $Ax = b = \lambda x$
- $Ax = \lambda x$, $\lambda \in \mathbb{F}$, $x \in \mathbb{R}^n$
- λ is an eigenvalue. λx is an eigenvector.

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

- $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with corresponding eigenvalue of 4.
- $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



$$\begin{aligned} Ax &= \lambda x \\ Ax - \lambda x &= 0 \\ Ax - \lambda I x &= 0 \\ (A - \lambda I)x &= 0 \end{aligned}$$

- $(A - \lambda I)x = 0 \Rightarrow x \in N(A - \lambda I)$[1] $\Rightarrow |A - \lambda I| = 0$
- $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$
- $\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$$\begin{aligned} 0 &= (3-\lambda)^2 - 1^2 \\ &= 3^2 - 6\lambda + \lambda^2 - 1 \\ &= \lambda^2 - 6\lambda + 8 \\ &= (\lambda - 4)(\lambda - 2) \end{aligned}$$

- $\lambda = 4, 2.$
- $\lambda^2 - 6\lambda + 8$ is the **characteristic polynomial** of A.
- $A - 2I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in N(A - 2I)$.
- $A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in N(A - 4I)$.

[1] To have a null space, $A - \lambda I$ has free columns.

- "Eigenspace" is not $\mathbb{R}^2$, but two lines in $\mathbb{R}^2$, specifically $y = \pm x$.
  - $y = \pm x$ comes from $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

1/29:

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\begin{aligned} P(\lambda) &= |A - \lambda I| \\ &= \begin{vmatrix} 2-\lambda & -2 & 3 \\ 0 & 3-\lambda & -2 \\ 0 & -1 & 2-\lambda \end{vmatrix} \\ &= -1 \begin{vmatrix} 2-\lambda & 3 \\ 0 & -2 \end{vmatrix} (-1)^{3+2} + (2-\lambda) \begin{vmatrix} 2-\lambda & -2 \\ 0 & 3-\lambda \end{vmatrix} (-1)^{3+3} \\ &= ((2-\lambda)(-2)) + (2-\lambda)((2-\lambda)(3-\lambda)) \\ &= -4 + 2\lambda + (2-\lambda)^2(3-\lambda) \\ &= -4 + 2\lambda + (4 - 4\lambda + \lambda^2)(3-\lambda) \\ &= -4 + 2\lambda + 12 - 4\lambda - 12\lambda + 4\lambda^2 + 3\lambda^2 - \lambda^3 \\ &= -\lambda^3 + 7\lambda^2 - 14\lambda + 8 \\ &= -(\lambda-1)(\lambda-2)(\lambda-4) \end{aligned}$$

$$A - I = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -2 \\ 0 & -1 & 1 \end{bmatrix} \qquad A - 2I = \begin{bmatrix} 0 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & -1 & 0 \end{bmatrix} \qquad A - 4I = \begin{bmatrix} -2 & -2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix}$$

- $P(\lambda)$ is positive when $n \in 2\mathbb{N}$, negative otherwise.
  - Signs flip term to term (think about binomial expansion).
- Coefficients of the $n-1$ degree term is the sum of the diagonal entries.
- Coefficient of the 0^{th} degree term is $|A|$.
  - $P_\lambda(0) = |A - 0 \cdot I| = |A|$.
- Product of the eigenvalues is $|A|$.
  - Think about expanding the factorization.
- Eigenvalues of U are the diagonal values.
  - $\lambda_1 \lambda_2 \cdots \lambda_n = |A|$, which is the product of the diagonal entries.
  - $\lambda_1 + \cdots + \lambda_n = \text{trace}(A)$, which is the sum of the diagonal entries.
- $Ax = \lambda x$
  - $A^2 x = AAx = A\lambda x = \lambda Ax = \lambda\lambda x = \lambda^2 x$

Similarity

1/30: • $A \sim B$[2] iff $\exists\, S : A = SBS^{-1},\ B = S^{-1}AS$.

1. If $A \sim B$, then $|A| = |B|$.

$$\begin{aligned} B &= S^{-1}AS \\ |B| &= |S^{-1}AS| \\ |B| &= |S^{-1}||A||S| \\ |B| &= \frac{1}{|S|}|A||S| \\ |B| &= |A| \end{aligned}$$

2. If $A \sim B$, then they share the same characteristic polynomial.

$$\begin{aligned} B &= S^{-1}AS \\ |B - \lambda I| &= |S^{-1}AS - \lambda I| \\ &= |S^{-1}AS - \lambda S^{-1}IS| \\ &= |S^{-1}S(A - \lambda I)| \\ &= |I(A - \lambda I)| \\ |B - \lambda I| &= |A - \lambda I| \end{aligned}$$

   - If they have the same characteristic polynomial, $\therefore$ A and B have the same eigenvalues.

- What is the best possible B if $A \sim B$?
  - Sparse.
  - Diagonal.
  - $A = [\text{ugly}] \quad \to \quad B = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = \Lambda$

- **Diagonalization**:

$$\begin{aligned} A &= S\Lambda S^{-1} \\ AS &= S\Lambda \\ \Lambda &= S^{-1}AS \end{aligned}$$

- $A = S\Lambda S^{-1}$
  - $A^2 = AA = S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda\Lambda S^{-1} = S\Lambda^2 S^{-1}$
  - $A^k = S\Lambda^k S^{-1}$
  - $A^k = S \begin{bmatrix} {\lambda_1}^k & & 0 \\ & \ddots & \\ 0 & & {\lambda_n}^k \end{bmatrix} S^{-1}$

- Diagonalize the following matrix A.

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

[2] A "is similar to" B

- Find the characteristic polynomial.

$$\begin{aligned}|A-\lambda I| &= \begin{vmatrix}-1-\lambda & 0 & 1\\ 3 & 0-\lambda & -3\\ 1 & 0 & -1-\lambda\end{vmatrix}\\ &= (-1-\lambda)((-\lambda)(-1-\lambda))+(-1)(-\lambda)\\ &= -\lambda(-1-\lambda)^2+\lambda\\ &= -\lambda(1+2\lambda+\lambda^2)+\lambda\\ &= -\lambda^3-2\lambda^2\\ &= -\lambda^2(\lambda+2)\end{aligned}$$

- Find the eigenvalues: $\lambda_1=\lambda_2=0$, $\lambda_3=-2$
- **Algebraic multiplicity** of λ_1, λ_2 is 2.
- A.M. of λ_3 is 1.
- $A-0I=\begin{bmatrix}-1 & 0 & 1\\ 3 & 0 & -3\\ 1 & 0 & -1\end{bmatrix}$
- $\text{rank}(A-0I)=1 \Rightarrow \dim(N(A-0I))=2$
- The 2 directly above is the **geometric multiplicity**.
- A is diagonalizable iff A.M. of λ_i = G.M.

1/31: - Eigenvectors are $x_1=\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $x_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$.
- $A+2I=\begin{bmatrix}1 & 0 & 1\\ 3 & 2 & -3\\ 1 & 0 & 1\end{bmatrix}$
- Eigenvector is $x_3=\begin{bmatrix}1\\-3\\-1\end{bmatrix}$
- Use an S matrix of eigenvectors.
- $A=S\Lambda S^{-1}=\frac{1}{2}\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & -3\\ 0 & 1 & -1\end{bmatrix}\begin{bmatrix}0 & & \\ & 0 & \\ & & -2\end{bmatrix}\begin{bmatrix}3 & 2 & -3\\ 1 & 0 & 1\\ 1 & 0 & -1\end{bmatrix}$
- Note that $A^{9752}=\frac{1}{2}\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & -3\\ 0 & 1 & -1\end{bmatrix}\begin{bmatrix}0 & & \\ & 0 & \\ & & (-2)^{9752}\end{bmatrix}\begin{bmatrix}3 & 2 & -3\\ 1 & 0 & 1\\ 1 & 0 & -1\end{bmatrix}$

- **Algebraic multiplicity**: The number of repeated roots to a polynomial. For all of the roots, it adds up to n (n-square matrix). *Also known as* **A.M.**
- **Geometric multiplicity**: The number of eigenvectors produced from each root. For all of the roots, it may not add up to n (n-square matrix). $\dim(N(A-\lambda I))$. *Also known as* **G.M.**
- A nondiagonalizable example:

$$A=\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 4\end{bmatrix}$$

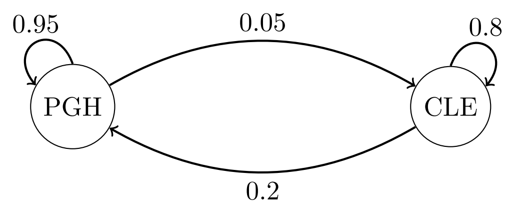
- $\lambda_1=\lambda_2=1$ and $\lambda_3=4$.
- λ_1 and λ_2 have A.M. = 2.

- λ_3 has A.M. = 1.
- $A - I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}$
- $\text{rank}(A - I) = 2 \Rightarrow \dim(N(A - I)) = 1 \Rightarrow$ G.M. = 1.
- $x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
- $A - 4I = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
- $x_2 = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$
- S would be 3×2 and, thus, not square, so $\nexists\ S^{-1}$[3].

- **Canonical** (form): An accepted way of expressing something.

Markov Chains

2/3:



- $u_0 = \begin{bmatrix} 500,000 \\ 500,000 \end{bmatrix}$
- $Au_0 = u_1$.
- $Au_1 = u_2$, $A(Au_0) = u_2$, $A^2 u_0 = u_2$, $A^k u_0 = u_k$, $(S\Lambda S^{-1})^k u_0 = u_k$, $S\Lambda^k S^{-1} u_0 = u_k$.

$$A = \begin{bmatrix} 0.95 & 0.20 \\ 0.05 & 0.80 \end{bmatrix} \qquad u_k = \begin{bmatrix} \text{PGH} \\ \text{CLE} \end{bmatrix}$$

- A is a **Markov matrix**, where all columns and rows add to 1.
- $Au_0 = \begin{bmatrix} 0.95 & 0.20 \\ 0.05 & 0.80 \end{bmatrix} \begin{bmatrix} 500,000 \\ 500,000 \end{bmatrix} = u_1$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 0.95 - \lambda & 0.20 \\ 0.05 & 0.80 - \lambda \end{vmatrix} \\ &= (0.95 - \lambda)(0.8 - \lambda) - (0.2)(0.05) \\ &= (\lambda - 1)(\lambda - 0.75) \end{aligned}$$

- $\lambda_1 = 1,\ \lambda_2 = 0.75$.

[3] At a later date, we will look at an analogy of projections to diagonalization that finds the "best possible" diagonalization (which may not be perfectly diagonal).

- $A - I = \begin{bmatrix} -0.05 & 0.2 \\ 0.05 & -0.2 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$
- $A - 0.75I = \begin{bmatrix} 0.2 & 0.2 \\ 0.05 & 0.05 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\begin{aligned} A^k u_0 &= \frac{1}{5}\begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.75 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 500,000 \\ 500,000 \end{bmatrix} \\ &= \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.75^k \end{bmatrix}\begin{bmatrix} 200,000 \\ 300,000 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 1 \end{bmatrix}(200,000) + \begin{bmatrix} -1 \\ 1 \end{bmatrix}(0.75)^k(300,000) \end{aligned}$$

- $\begin{bmatrix} 800,000 \\ 200,000 \end{bmatrix}$ is the steady-state vector.
- $\begin{bmatrix} -(0.75)^k(300,000) \\ (0.75)^k(300,000) \end{bmatrix}$ is the dynamically changing vector.
- $\lim_{k\to\infty} A^k u_0 = \begin{bmatrix} 800,000 \\ 200,000 \end{bmatrix} = \begin{bmatrix} \text{PGH} \\ \text{CLE} \end{bmatrix}$

Explicit Formula for the Fibonacci Sequence

2/4:

- 1, 1, 2, 3, 5, 8, ...
- Recursively defined formula: F_n[4]$= F_{n-1} + F_{n-2}$.

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} \\ F_{n-1} &= F_{n-1} \\ \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix} \end{aligned}$$

- $u_n = A^n u_0 = S\Lambda^n S^{-1} u_0$.

$$\begin{aligned} 0 &= |A - \lambda I| \\ &= \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} \\ &= -\lambda(1-\lambda) - 1 \\ &= \lambda^2 - \lambda - 1 \end{aligned}$$

- $\lambda = \frac{1\pm\sqrt{5}}{2}$[5].
- $\lambda_1 = \frac{1+\sqrt{5}}{2}$.

$$\begin{aligned} N(A - \lambda_1 I) &= N\left(\begin{bmatrix} 1 - \frac{1+\sqrt{5}}{2} & 1 \\ 1 & -\frac{1+\sqrt{5}}{2} \end{bmatrix}\right) \\ &= N\left(\begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{-1-\sqrt{5}}{2} \end{bmatrix}\right) \end{aligned}$$

[4]The n-th Fibonacci number.
[5]This is the Golden ratio!

- $\begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{-1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- Let $x_2 = 1$.

$$\begin{aligned} \frac{1-\sqrt{5}}{2}x_1 + 1 &= 0 \\ \frac{1-\sqrt{5}}{2}x_1 &= -\frac{2}{2} \\ x_1 &= \frac{-2}{1-\sqrt{5}} \times \frac{1+\sqrt{5}}{1+\sqrt{5}} \\ &= \frac{-2-2\sqrt{5}}{-4} \\ &= \frac{1+\sqrt{5}}{2} \end{aligned}$$

- $s_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$, $s_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$.
- $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} S^{-1}$
- $S^{-1} = \frac{1}{|S|}C_S^{\mathrm{T}} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}$
- $u_k = A^k u_0 = S\Lambda^k S^{-1} u_0$.

$$\begin{aligned} S^{-1}u_0 &= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ \frac{-1+\sqrt{5}}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1+\sqrt{5}}{2\sqrt{5}} \\ \frac{-1+\sqrt{5}}{2\sqrt{5}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{5+\sqrt{5}}{10} \\ \frac{5-\sqrt{5}}{10} \end{bmatrix} \end{aligned}$$

- $u_k = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^k \left(\frac{5+\sqrt{5}}{10}\right) + \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} \left(\frac{1-\sqrt{5}}{2}\right)^k \left(\frac{5-\sqrt{5}}{10}\right)$