Complex Linear Independence: Decomplexification

4/7:
- When given a complex system of equations, it is necessary to **decomplexify** it.
- **Decomplexify**: To model a complex system of equations with a strictly real system for the purpose of applying the tenets of real linear algebra to it.
- Consider the following complex system of equations.

$$\begin{aligned}(2+i)x_1 + (1+i)x_2 &= 3+6i \\ (3-i)x_1 + (2-2i)x_2 &= 7-i\end{aligned}$$

  - The solutions will be complex numbers: $x_1 = a_1 + ib_1$ and $x_2 = a_2 + ib_2$, where $a_1, a_2, b_1, b_2 \in \mathbb{R}$.
- Transform it into a matrix system of equations. Separate the real and complex parts, and factor out all instances of the imaginary number i so that it is a coefficient to any complex matrix.

$$\begin{bmatrix}2+i & 1+i \\ 3-i & 2-2i\end{bmatrix}\begin{bmatrix}a_1+ib_1 \\ a_2+ib_2\end{bmatrix} = \begin{bmatrix}3+6i \\ 7-i\end{bmatrix}$$
$$\left(\begin{bmatrix}2 & 1 \\ 3 & 2\end{bmatrix} + \begin{bmatrix}i & i \\ -i & -2i\end{bmatrix}\right)\left(\begin{bmatrix}a_1 \\ a_2\end{bmatrix} + \begin{bmatrix}ib_1 \\ ib_2\end{bmatrix}\right) = \left(\begin{bmatrix}3 \\ 7\end{bmatrix} + \begin{bmatrix}6i \\ -i\end{bmatrix}\right)$$
$$\underbrace{\left(\begin{bmatrix}2 & 1 \\ 3 & 2\end{bmatrix} + i\begin{bmatrix}1 & 1 \\ -1 & -2\end{bmatrix}\right)}_{A}\underbrace{\left(\begin{bmatrix}a_1 \\ a_2\end{bmatrix} + i\begin{bmatrix}b_1 \\ b_2\end{bmatrix}\right)}_{x} = \underbrace{\left(\begin{bmatrix}3 \\ 7\end{bmatrix} + i\begin{bmatrix}6 \\ -1\end{bmatrix}\right)}_{b}$$

- Foil the left side of the above equation[1].

$$\left(\begin{bmatrix}2 & 1 \\ 3 & 2\end{bmatrix}\begin{bmatrix}a_1 \\ a_2\end{bmatrix} - \begin{bmatrix}1 & 1 \\ -1 & -2\end{bmatrix}\begin{bmatrix}b_1 \\ b_2\end{bmatrix}\right) + i\left(\begin{bmatrix}2 & 1 \\ 3 & 2\end{bmatrix}\begin{bmatrix}b_1 \\ b_2\end{bmatrix} + \begin{bmatrix}1 & 1 \\ -1 & -2\end{bmatrix}\begin{bmatrix}a_1 \\ a_2\end{bmatrix}\right) = \begin{bmatrix}3 \\ 7\end{bmatrix} + i\begin{bmatrix}6 \\ -1\end{bmatrix}$$

- Split the above system of equations into a real system of equations and a complex system of equations by setting equal to each other the real components of each side and the imaginary components of each side.

$$\begin{bmatrix}2 & 1 \\ 3 & 2\end{bmatrix}\begin{bmatrix}a_1 \\ a_2\end{bmatrix} - \begin{bmatrix}1 & 1 \\ -1 & -2\end{bmatrix}\begin{bmatrix}b_1 \\ b_2\end{bmatrix} = \begin{bmatrix}3 \\ 7\end{bmatrix}$$
$$\begin{bmatrix}2 & 1 \\ 3 & 2\end{bmatrix}\begin{bmatrix}b_1 \\ b_2\end{bmatrix} + \begin{bmatrix}1 & 1 \\ -1 & -2\end{bmatrix}\begin{bmatrix}a_1 \\ a_2\end{bmatrix} = \begin{bmatrix}6 \\ -1\end{bmatrix}$$

- Multiply out the matrices above to yield a system of four equations.

$$\begin{aligned}2a_1 + a_2 - b_1 - b_2 &= 3 \\ 3a_1 + 2a_2 + b_1 + 2b_2 &= 7 \\ a_1 + a_2 + 2b_1 + b_2 &= 6 \\ -a_1 - 2a_2 + 3b_1 + 2b_2 &= -1\end{aligned}$$

- Condense the above system of equations into a single matrix system of equations.

$$\begin{bmatrix}2 & 1 & -1 & -1 \\ 3 & 2 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ -1 & -2 & 3 & 2\end{bmatrix}\begin{bmatrix}a_1 \\ a_2 \\ b_1 \\ b_2\end{bmatrix} = \begin{bmatrix}3 \\ 7 \\ 6 \\ -1\end{bmatrix}$$

[1]Note that the minus sign appears in the real component because, when multiplying the two "last" parts, $i^2 = -1$.

- Solve for a_1, a_2, b_1, and b_2 using an augmented matrix and Gauss-Jordan elimination.

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & -1 & 3 \\ 3 & 2 & 1 & 2 & 7 \\ 1 & 1 & 2 & 1 & 6 \\ -1 & -2 & 3 & 2 & -1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}$$

- From these four values, the original solutions $x_1 = a_1 + ib_1$ and $x_2 = a_2 + ib_2$ can be found.

$$x_1 = 1 + 2i$$
$$x_2 = 2 - i$$

Hermitian, Unitary, and Normal Matrices

4/13:

- What necessitates different categorizations of complex vectors and matrices?
- Consider a vector v.

$$v = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

- If you want to find $||v||$, you typically evaluate $\sqrt{v^{\mathrm{T}}v}$. However, this equals to 0 (see the following), which is clearly not the magnitude of v.

$$\begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 - 1 = 0$$

  - Note that $||v||$ must be an element of $\mathbb{R}$ because it measures a distance.

- With complex vectors, it is necessary to evaluate $\sqrt{\bar{v}^{\mathrm{T}}v}$ to find $||v||$.

$$\begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 + 1 = 2$$

$$||v|| = \sqrt{2}$$

  - This makes sense because $\begin{bmatrix} 1 \\ i \end{bmatrix}$ extends one unit into $\mathbb{R}^1$ and one unit into $\mathbb{C}^2$.
  - If $z\bar{z} = |z|^2$ and $\bar{v}^{\mathrm{T}}v = v \cdot \bar{v}$, it stands to reason that $\bar{v}^{\mathrm{T}}v = ||v||^2$. Essentially, the dot product multiplies every element of v by its complex conjugate and sums them.

- Instead of writing $\bar{v}^{\mathrm{T}}$[2] every time, mathematicians shorthand to v^{H}[3].
  - v^{H} works for all vectors, but it is necessary for complex ones.

- **Hermitian** (matrix): A matrix A such that $A = A^{\mathrm{H}}$.
  - Typically defined for $A \in \mathbb{C}^n$, but holds for $A \in \mathbb{R}^n$, too.
  - Parallel to how if $A \in \mathbb{R}^n$ and $A = A^{\mathrm{T}}$, A is symmetrical.
  - Also note that if $A^{\mathrm{H}}A = A^2 = AA^{\mathrm{H}}$, A is Hermitian.

[2] "v conjugate transpose"
[3] "v Hermitian" after French mathematician Charles Hermite.

  - A Hermitian matrix has to have real values on the principal diagonal. When A is transposed and conjugated, the diagonal entries are the only values that don't move. Thus, their conjugates must equal themselves, so they must be real[4].
- **Unitary** (matrix): A matrix A such that $A^{-1} = A^{\mathrm{H}}$.
  - Typically defined for $A \in \mathbb{C}^n$, but holds for $A \in \mathbb{R}^n$, too.
  - Parallel to how if $A \in \mathbb{R}^n$ and $A^{-1} = A^{\mathrm{T}}$, A is orthonormal.
  - Also note that if $A^{\mathrm{H}}A = I = AA^{\mathrm{H}}$, A is unitary.
- **Normal** (matrix): A matrix that is unitarily diagonalizable.
  - Typically defined for $A \in \mathbb{C}^n$, but holds for $A \in \mathbb{R}^n$, too.
  - Parallel to matrices $A \in \mathbb{R}^n$ such that A is orthonormally diagonalizable.
- Note that not every complex matrix has to be one of these three types.
- When $A^{\mathrm{H}}A = AA^{\mathrm{H}}$, $A = U\Lambda U^{\mathrm{H}}$.

$$\begin{aligned} AA^{\mathrm{H}} &= \left(U\Lambda U^{\mathrm{H}}\right)\left(U\Lambda U^{\mathrm{H}}\right)^{\mathrm{H}} \\ &= U\Lambda U^{\mathrm{H}}U\Lambda^{\mathrm{H}}U^{\mathrm{H}} \\ &= U\Lambda\Lambda^{\mathrm{H}}U^{\mathrm{H}} \\ &= U\Lambda^{\mathrm{H}}\Lambda U^{\mathrm{H}\,[5]} \\ &= U\Lambda^{\mathrm{H}}U^{\mathrm{H}}U\Lambda U^{\mathrm{H}} \\ &= \left(U\Lambda U^{\mathrm{H}}\right)^{\mathrm{H}}\left(U\Lambda U^{\mathrm{H}}\right) \\ &= A^{\mathrm{H}}A \end{aligned}$$

- When $A = A^{\mathrm{H}}$, all eigenvalues are elements of $\mathbb{R}$ (similar to spectral theorem).

$$v^{\mathrm{H}}Av = \left(v^{\mathrm{H}}Av\right)^{\mathrm{H}} = v^{\mathrm{H}}Av$$

  - The above proves that $v^{\mathrm{H}}Av \in \mathbb{R}$ because it's its own conjugate[4].

$$\begin{aligned} Av &= \lambda v \\ v^{\mathrm{H}}Av &= \lambda v^{\mathrm{H}}v \end{aligned}$$

  - $\lambda = \frac{v^{\mathrm{H}}Av}{v^{\mathrm{H}}v} \to \frac{\mathbb{R}}{\mathbb{R}} = \mathbb{R}$[6].
- When $A = A^{\mathrm{H}}$ and $Ax = \lambda x$, all x's can be chosen orthonormally (also similar to spectral theorem).
  - Normality is implied because any eigenvector can be scaled to any version (including a normal version) and still be an eigenvector.

$$x_i = \begin{bmatrix} x_{i_1} \\ x_{i_2} \\ \vdots \\ x_{i_n} \end{bmatrix} \qquad x_i^{\mathrm{H}} = \begin{bmatrix} \bar{x}_{i_1} & \bar{x}_{i_2} & \cdots & \bar{x}_{i_n} \end{bmatrix}$$

[4] Recall that only real quantities can be their own conjugates because $a + 0i = a - 0i$.
[5] Since $\Lambda = \Lambda^{\mathrm{H}}$.
[6] Note that the denominator is real because it's how one finds $||v||$, and $||v||$ must be real, as discussed above.

$$A=\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix} \qquad A^{\mathrm{H}}=\begin{bmatrix}a_{11} & \bar{a}_{21} & \cdots & \bar{a}_{n1}\\ \bar{a}_{12} & a_{22} & \cdots & \bar{a}_{n2}\\ \vdots & \vdots & \ddots & \vdots\\ \bar{a}_{1n} & \bar{a}_{2n} & \cdots & a_{nn}\end{bmatrix}$$

- Define an arbitrary vector x_i and matrix A, along with their conjugate transposes (or Hermitian versions). Note that the diagonal entries of A^{H} aren't shown as conjugated because their conjugates equal themselves.

$$\begin{aligned}Ax_1 &= \lambda_1 x_1\\ x_2^{\mathrm{H}}Ax_1 &= \lambda_1 x_2^{\mathrm{H}}x_1\end{aligned}$$

$$\begin{aligned}Ax_2 &= \lambda_2 x_2\\ (Ax_2)^{\mathrm{H}} &= (\lambda_2 x_2)^{\mathrm{H}}\\ x_2^{\mathrm{H}}A^{\mathrm{H}} &= \lambda_2 x_2^{\mathrm{H}}\\ x_2^{\mathrm{H}}Ax_1 &= \lambda_2 x_2^{\mathrm{H}}x_1\end{aligned}$$

- $\lambda_1 x_2^{\mathrm{H}}x_1 = \lambda_2 x_2^{\mathrm{H}}x_1$ implies that, since $\lambda_1 \neq \lambda_2$, $x_2^{\mathrm{H}}x_1$ must equal 0, proving orthogonality.

Complex Diagonalization

4/15: • Diagonalize the following matrix A.

$$A=\begin{bmatrix}0.9 & -0.4\\ 0.1 & 0.9\end{bmatrix}$$

- Find the characteristic polynomial.

$$\begin{aligned}0 &= \begin{vmatrix}0.9-\lambda & -0.4\\ 0.1 & 0.9-\lambda\end{vmatrix}\\ &= (0.9-\lambda)^2-(-0.4)(0.1)\\ &= 0.81-1.8\lambda+\lambda^2+0.04\\ &= \lambda^2-1.8\lambda+0.85\end{aligned}$$

- Find the eigenvalues[7].

$$\begin{aligned}\lambda &= \frac{-(-1.8)\pm\sqrt{(-1.8)^2-4(1)(0.85)}}{2(1)}\\ &= 0.9\pm\frac{\sqrt{-0.16}}{2}\\ &= 0.9\pm\frac{\sqrt{-1}\sqrt{0.16}}{2}\\ &= 0.9\pm\frac{0.4i}{2}\\ &= 0.9\pm 0.2i\end{aligned}$$

$$\lambda_1 = 0.9+0.2i \qquad \lambda_2 = 0.9-0.2i$$

[7] It is interesting that the eigenvalues are complex conjugates of each other.

– Find the eigenvectors[8].

$$\begin{aligned}(A-(0.9+0.2i))x_1 &= \begin{bmatrix}0.9-(0.9+0.2i) & -0.4\\ 0.1 & 0.9-(0.9+0.2i)\end{bmatrix}\begin{bmatrix}x_{1_1}\\ x_{1_2}\end{bmatrix}\\ &= \begin{bmatrix}-0.2i & -0.4\\ 0.1 & -0.2i\end{bmatrix}\begin{bmatrix}2i\\ 1\end{bmatrix}\\ &= \begin{bmatrix}0\\ 0\end{bmatrix}\end{aligned}$$

$$\begin{aligned}(A-(0.9-0.2i))x_1 &= \begin{bmatrix}0.9-(0.9-0.2i) & -0.4\\ 0.1 & 0.9-(0.9-0.2i)\end{bmatrix}\begin{bmatrix}x_{1_1}\\ x_{1_2}\end{bmatrix}\\ &= \begin{bmatrix}0.2i & -0.4\\ 0.1 & 0.2i\end{bmatrix}\begin{bmatrix}-2i\\ 1\end{bmatrix}\\ &= \begin{bmatrix}0\\ 0\end{bmatrix}\end{aligned}$$

$$x_1 = \begin{bmatrix}2i\\ 1\end{bmatrix} \qquad x_2 = \begin{bmatrix}-2i\\ 1\end{bmatrix}$$

– Compile the diagonalization.

$$A = \frac{1}{4i}\begin{bmatrix}2i & -2i\\ 1 & 1\end{bmatrix}\begin{bmatrix}0.9+0.2i & 0\\ 0 & 0.9-0.2i\end{bmatrix}\begin{bmatrix}1 & 2i\\ -1 & 2i\end{bmatrix}$$

Real versus Complex

4/16:

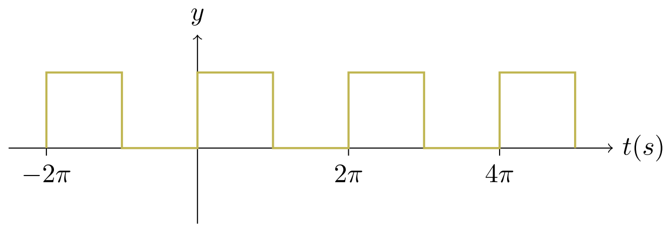
Real	**Complex**
$\mathbb{R}^n$: vectors with n real components	$\mathbb{C}^n$: vectors with n complex components
length: $\|\|x\|\|^2 = x_1^2 + \cdots + x_n^2$	length: $\|\|z\|\|^2 = \|z_1\|^2 + \cdots + \|z_n\|^2$
transpose: $\left(A^{\mathrm{T}}\right)_{ij} = A_{ji}$	conjugate transpose: $\left(A^{\mathrm{H}}\right)_{ij} = \overline{A_{ji}}$
product rule: $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$	product rule: $(AB)^{\mathrm{H}} = B^{\mathrm{H}}A^{\mathrm{H}}$
dot product: $x^{\mathrm{T}}y = x_1y_1 + \cdots + x_ny_n$	inner product: $u^{\mathrm{H}}v = \bar{u}_1u_1 + \cdots + \bar{u}_nv_n$
reason for A^{T}: $(Ax)^{\mathrm{T}}y = x^{\mathrm{T}}(A^{\mathrm{T}}y)$	reason for A^{H}: $(Au)^{\mathrm{H}}v = u^{\mathrm{H}}(A^{\mathrm{H}}v)$
orthogonality: $x^{\mathrm{T}}y = 0$	orthogonality: $u^{\mathrm{H}}v = 0$.
symmetric matrices: $A = A^{\mathrm{T}}$	Hermitian matrices: $A = A^{\mathrm{H}}$
$A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}}$ (real Λ)	$A = U\Lambda U^{-1} = U\Lambda U^{\mathrm{H}}$ (real Λ)
skew-symmetric matrices: $k^{\mathrm{T}} = -K$	skew-Hermitian matrices: $K^{\mathrm{H}} = -K$
orthogonal matrices: $Q^{\mathrm{T}} = Q^{-1}$	unitary matrices: $U^{\mathrm{H}} = U^{-1}$
orthonormal columns: $Q^{\mathrm{T}}Q = I$	orthonormal columns: $U^{\mathrm{H}} = U^{-1}$
$(Qx)^{\mathrm{T}}(Qy) = x^{\mathrm{T}}y$ and $\|\|Qx\|\| = \|\|x\|\|$	$(Ux)^{\mathrm{H}}(Uy) = x^{\mathrm{H}}y$ and $\|\|Uz\|\| = \|\|z\|\|$

- Note that the columns and eigenvectors of Q and U are orthonormal, and all of their eigenvalues λ satisfy $|\lambda| = 1$.

[8]It is interesting that the eigenvectors are *also* complex conjugates of each other.

Fourier Series

4/21: • Consider the square wave $f(t)$.

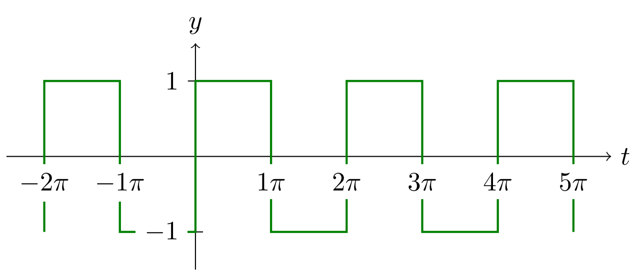


- Its period is $2\pi \frac{\text{sec}}{\text{cycle}}$, and its frequency is $\frac{1}{2\pi}$Hz.
- Can we write $f(t)$ as a sum of sines and cosines?

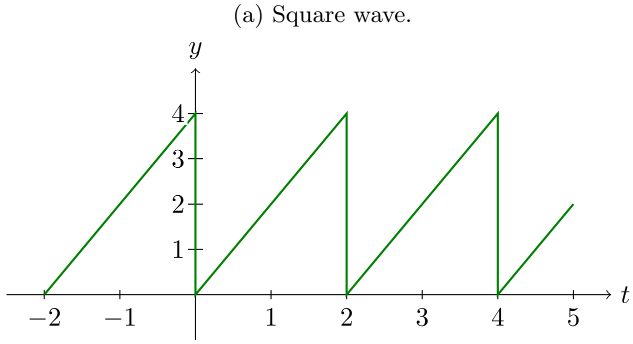
$$f(t) = a_0 + a_1 \cos(t) + b_1 \sin(t) + a_2 \cos(2t) + b_2 \sin(2t) + a_3 \cos(3t) + b_3 \sin(3t) + \cdots$$

  - Be general.
  - Since $T = 2\pi$, it makes sense to use some functions with $T = 2\pi$ to model it.
  - The weighting coefficients account for how much each function contributes to the whole.
- Historically studied by Fourier, who studied differential equations. Differential equations were often easy to solve for sines and cosines, so if a function could be modeled by a sum of sines and cosines, a related differential equation would be easier to solve.
- Fourier series, transforms, and analysis also tell us how much of each frequency a function contains (as measured by the weight coefficients).

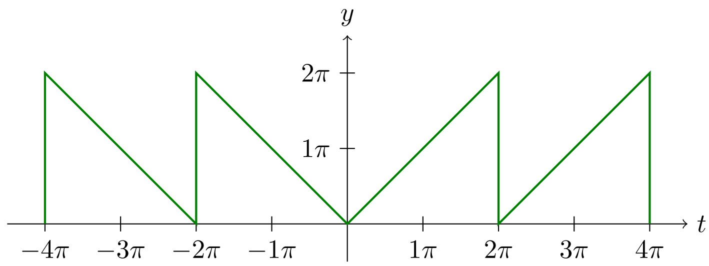
Wave Type Sketches



(a) Square wave.



(b) Sawtooth wave.



(c) Triangular wave.

- Square wave: $f(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & \pi < t < 2\pi \end{cases}$ where periodicity is defined by $f(t + 2\pi) = f(t)$.
- Sawtooth wave: $f(t) = 2t \qquad 0 < t < 2$ where periodicity is defined by $f(t + 2) = f(t)$.
- Triangular wave: $f(t) = |t| \qquad 0 < t < 2\pi$ where periodicity is defined by $f(t + 2\pi) = f(t)$.