COMPOSITION OF LINEAR TRANSFORMATIONS

Let $T:U\to V$ and $S:V\to W$ be linear transformations. The composition of S with T is the mapping $S\circ T$ definite by $(S\circ T)(u)=S(T(u))$

EXAMPLE:

Let $T:\mathbb{R}^2\to F_1$ **and** $S:F_1\to F_2$ be the linear transformations defined by

$T\begin{bmatrix} a \\ b \end{bmatrix}=a+(a+b)x$ and $S(f(x))=xf(x)$

- Find $(S\circ T)\begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$$(S\circ T)\begin{bmatrix} 3 \\ -2 \end{bmatrix}=S\left(T\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right)\right)=S(3+(3-2)x)=S(3+x)=x(3+x)=3x+x^2$$

- Find $(S\circ T)\begin{bmatrix} a \\ b \end{bmatrix}$

$$(S\circ T)\begin{bmatrix} a \\ b \end{bmatrix}=S\left(T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)\right)=S(a+(a+b)x)=x(a+(a+b)x)=ax+(a+b)x^2$$

If $T:U\to V$ and $S:V\to W$ are linear transformation, then $S\circ T:U\to V$ is a linear transformation.

$$\begin{aligned}(S\circ T)(u+v)&=S(T(u+v))\\&=S(T(u)+T(v))\\&=S(T(u))+S(T(v))\\&=(S\circ T)(u)+(S\circ T)(v)\end{aligned}$$

$$\begin{aligned}(S\circ T)(cu)&=S(T(cu))\\&=S(cT(u))\\&=cS(T(u))\\&=c(S\circ T)(u)\end{aligned}$$

INVERSES OF LINEAR TRANSFORMATION

A linear transformation $T:V \to W$ **is invertible if there is a linear transformation** $T':W \to V$ **such that** $T' \circ T = I_V$ **and** $T \circ T' = I_W$

EXAMPLE:

Verify that the linear mapping $T:\mathbb{R}^2 \to F_1 : T\begin{bmatrix} a \\ b \end{bmatrix} = a + (a+b)x$ **and** $T':F_1 \to \mathbb{R}^2 : T'(c+dx) = \begin{bmatrix} c \\ d-c \end{bmatrix}$ **are inverses.**

$$(T' \circ T)\begin{bmatrix} a \\ b \end{bmatrix} = T'\left(T\begin{bmatrix} a \\ b \end{bmatrix} \right) = T'(a+(a+b)x) = \begin{bmatrix} a \\ (a+b)-a \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$(T \circ T')(c+dx) = T(T'(c+dx)) = T\begin{bmatrix} c \\ d-c \end{bmatrix} = c + (c+(d-c))x = c+dx$$

Therefore they are inverses because $T' \circ T = I_{\mathbb{R}^2}$ **and** $T \circ T' = I_{F_1}$ **.**