Complex Linear Independence: Decomplexification

4/7:
- When given a complex system of equations, it is necessary to **decomplexify** it.
- **Decomplexify**: To model a complex system of equations with a strictly real system for the purpose of applying the tenets of real linear algebra to it.
- Consider the following complex system of equations.

$$\begin{aligned}(2+i)x_1 + (1+i)x_2 &= 3+6i\\ (3-i)x_1 + (2-2i)x_2 &= 7-i\end{aligned}$$

  - The solutions will be complex numbers: $x_1 = a_1 + ib_1$ and $x_2 = a_2 + ib_2$, where $a_1, a_2, b_1, b_2 \in \mathbb{R}$.
- Transform it into a matrix system of equations. Separate the real and complex parts, and factor out all instances of the imaginary number i so that it is a coefficient to any complex matrix.

$$\begin{bmatrix}2+i & 1+i\\ 3-i & 2-2i\end{bmatrix}\begin{bmatrix}a_1+ib_1\\ a_2+ib_2\end{bmatrix} = \begin{bmatrix}3+6i\\ 7-i\end{bmatrix}$$
$$\left(\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix} + \begin{bmatrix}i & i\\ -i & -2i\end{bmatrix}\right)\left(\begin{bmatrix}a_1\\ a_2\end{bmatrix} + \begin{bmatrix}ib_1\\ ib_2\end{bmatrix}\right) = \left(\begin{bmatrix}3\\ 7\end{bmatrix} + \begin{bmatrix}6i\\ -i\end{bmatrix}\right)$$
$$\underbrace{\left(\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix} + i\begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}\right)}_{A}\underbrace{\left(\begin{bmatrix}a_1\\ a_2\end{bmatrix} + i\begin{bmatrix}b_1\\ b_2\end{bmatrix}\right)}_{x} = \underbrace{\left(\begin{bmatrix}3\\ 7\end{bmatrix} + i\begin{bmatrix}6\\ -1\end{bmatrix}\right)}_{b}$$

- Foil the left side of the above equation[1].

$$\left(\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}\begin{bmatrix}a_1\\ a_2\end{bmatrix} - \begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}b_1\\ b_2\end{bmatrix}\right) + i\left(\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}\begin{bmatrix}b_1\\ b_2\end{bmatrix} + \begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}a_1\\ a_2\end{bmatrix}\right) = \begin{bmatrix}3\\ 7\end{bmatrix} + i\begin{bmatrix}6\\ -1\end{bmatrix}$$

- Split the above system of equations into a real system of equations and a complex system of equations by setting equal to each other the real components of each side and the imaginary components of each side.

$$\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}\begin{bmatrix}a_1\\ a_2\end{bmatrix} - \begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}b_1\\ b_2\end{bmatrix} = \begin{bmatrix}3\\ 7\end{bmatrix}$$
$$\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}\begin{bmatrix}b_1\\ b_2\end{bmatrix} + \begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}a_1\\ a_2\end{bmatrix} = \begin{bmatrix}6\\ -1\end{bmatrix}$$

- Multiply out the matrices above to yield a system of four equations.

$$\begin{aligned}2a_1 + a_2 - b_1 - b_2 &= 3\\ 3a_1 + 2a_2 + b_1 + 2b_2 &= 7\\ a_1 + a_2 + 2b_1 + b_2 &= 6\\ -a_1 - 2a_2 + 3b_1 + 2b_2 &= -1\end{aligned}$$

- Condense the above system of equations into a single matrix system of equations.

$$\begin{bmatrix}2 & 1 & -1 & -1\\ 3 & 2 & 1 & 2\\ 1 & 1 & 2 & 1\\ -1 & -2 & 3 & 2\end{bmatrix}\begin{bmatrix}a_1\\ a_2\\ b_1\\ b_2\end{bmatrix} = \begin{bmatrix}3\\ 7\\ 6\\ -1\end{bmatrix}$$

[1] Note that the minus sign appears in the real component because, when multiplying the two "last" parts, $i^2 = -1$.

- Solve for a_1, a_2, b_1, and b_2 using an augmented matrix and Gauss-Jordan elimination.

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & -1 & 3 \\ 3 & 2 & 1 & 2 & 7 \\ 1 & 1 & 2 & 1 & 6 \\ -1 & -2 & 3 & 2 & -1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}$$

- From these four values, the original solutions $x_1 = a_1 + ib_1$ and $x_2 = a_2 + ib_2$ can be found.

$$\begin{aligned} x_1 &= 1 + 2i \\ x_2 &= 2 - i \end{aligned}$$

Hermitian, Unitary, and Normal Matrices

4/13:

- What necessitates different categorizations of complex vectors and matrices?
- Consider a vector v.

$$v = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

- If you want to find $||v||$, you typically evaluate $\sqrt{v^{\mathrm{T}} v}$. However, this equals to 0 (see the following), which is clearly not the magnitude of v.

$$\begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 - 1 = 0$$

  - Note that $||v||$ must be an element of $\mathbb{R}$ because it measures a distance.
- With complex vectors, it is necessary to evaluate $\sqrt{\bar{v}^{\mathrm{T}} v}$ to find $||v||$.

$$\begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 + 1 = 2$$

$$||v|| = \sqrt{2}$$

  - This makes sense because $\begin{bmatrix} 1 \\ i \end{bmatrix}$ extends one unit into $\mathbb{R}^1$ and one unit into $\mathbb{C}^2$.
  - If $z\bar{z} = |z|^2$ and $\bar{v}^{\mathrm{T}} v = v \cdot \bar{v}$, it stands to reason that $\bar{v}^{\mathrm{T}} v = ||v||^2$. Essentially, the dot product multiplies every element of v by its complex conjugate and sums them.
- Instead of writing $\bar{v}^{\mathrm{T}}$[2] every time, mathematicians shorthand to v^{H}[3].
  - v^{H} works for all vectors, but it is necessary for complex ones.
- **Hermitian** (matrix): A matrix A such that $A = A^{\mathrm{H}}$.
  - Typically defined for $A \in \mathbb{C}^n$, but holds for $A \in \mathbb{R}^n$, too.
  - Parallel to how if $A \in \mathbb{R}^n$ and $A = A^{\mathrm{T}}$, A is symmetrical.
  - Also note that if $A^{\mathrm{H}} A = A^2 = AA^{\mathrm{H}}$, A is Hermitian.

[2] "v conjugate transpose"
[3] "v Hermitian" after French mathematician Charles Hermite.

- **Unitary** (matrix): A matrix A such that $A^{-1} = A^{\mathrm{H}}$.
  - Typically defined for $A \in \mathbb{C}^n$, but holds for $A \in \mathbb{R}^n$, too.
  - Parallel to how if $A \in \mathbb{R}^n$ and $A^{-1} = A^{\mathrm{T}}$, A is orthonormal.
  - Also note that if $A^{\mathrm{H}}A = I = AA^{\mathrm{H}}$, A is unitary.
- **Normal** (matrix): A matrix that is unitarily diagonalizable.
  - Typically defined for $A \in \mathbb{C}^n$, but holds for $A \in \mathbb{R}^n$, too.
  - Parallel to matrices $A \in \mathbb{R}^n$ such that A is orthonormally diagonalizable.
- Note that not every complex matrix has to be one of these three types.
- When $A^{\mathrm{H}}A = AA^{\mathrm{H}}$, $A = U\Lambda U^{\mathrm{H}}$.

$$\begin{aligned}
AA^{\mathrm{H}} &= \left(U\Lambda U^{\mathrm{H}}\right)\left(U\Lambda U^{\mathrm{H}}\right)^{\mathrm{H}} \\
&= U\Lambda U^{\mathrm{H}}U\Lambda^{\mathrm{H}}U^{\mathrm{H}} \\
&= U\Lambda\Lambda^{\mathrm{H}}U^{\mathrm{H}} \\
&= U\Lambda^{\mathrm{H}}\Lambda U^{\mathrm{H}[4]} \\
&= U\Lambda^{\mathrm{H}}U^{\mathrm{H}}U\Lambda U^{\mathrm{H}} \\
&= \left(U\Lambda U^{\mathrm{H}}\right)^{\mathrm{H}}\left(U\Lambda U^{\mathrm{H}}\right) \\
&= A^{\mathrm{H}}A
\end{aligned}$$

- When $A = A^{\mathrm{H}}$, all eigenvalues are elements of $\mathbb{R}$ (similar to spectral theorem).

$$v^{\mathrm{H}}Av = \left(v^{\mathrm{H}}Av\right)^{\mathrm{H}} = v^{\mathrm{H}}Av$$

  - The above proves that $v^{\mathrm{H}}Av \in \mathbb{R}$ because it's its own conjugate[5].

$$\begin{aligned}
Av &= \lambda v \\
v^{\mathrm{H}}Av &= \lambda v^{\mathrm{H}}v
\end{aligned}$$

  - $\lambda = \frac{v^{\mathrm{H}}Av}{v^{\mathrm{H}}v} \to \frac{\mathbb{R}}{\mathbb{R}} = \mathbb{R}$[6].
- When $A = A^{\mathrm{H}}$ and $Ax = \lambda x$, all x's can be chosen orthonormally (also similar to spectral theorem).
  - Normality is implied because any eigenvector can be scaled to any version (including a normal version) and still be an eigenvector.

$$x_i = \begin{bmatrix} x_{i_1} \\ x_{i_2} \\ \vdots \\ x_{i_n} \end{bmatrix} \qquad x_i^{\mathrm{H}} = \begin{bmatrix} \bar{x}_{i_1} & \bar{x}_{i_2} & \cdots & \bar{x}_{i_n} \end{bmatrix}$$

$$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \qquad A^{\mathrm{H}} = \begin{bmatrix} a_1 & \bar{a}_3 \\ \bar{a}_2 & a_4 \end{bmatrix}$$

[4] Since $\Lambda = \Lambda^{\mathrm{H}}$.

[5] Recall that only real quantities can be their own conjugates because $a + 0i = a - 0i$.

[6] Note that the denominator is real because it's how one finds $||v||$, and $||v||$ must be real, as discussed above.

- Define an arbitrary vector x_i and matrix A, along with their conjugate transposes (or Hermitian versions).

$$
\begin{aligned}
Ax_1 &= \lambda_1 x_1 \\
x_2^{\mathrm{H}} A x_1 &= \lambda_1 x_2^{\mathrm{H}} x_1
\end{aligned}
$$

$$
\begin{aligned}
Ax_2 &= \lambda_2 x_2 \\
(Ax_2)^{\mathrm{H}} &= (\lambda_2 x_2)^{\mathrm{H}} \\
x_2^{\mathrm{H}} A^{\mathrm{H}} &= \lambda_2 x_2^{\mathrm{H}} \\
x_2^{\mathrm{H}} A x_1 &= \lambda_2 x_2^{\mathrm{H}} x_1
\end{aligned}
$$

- $\lambda_1 x_2^{\mathrm{H}} x_1 = \lambda_2 x_2^{\mathrm{H}} x_1$ implies that, since $\lambda_1 \neq \lambda_2$, $x_2^{\mathrm{H}} x_1$ must equal 0, proving orthogonality.