

## Complex Linear Independence: Decomplexification

4/7:

- When given a complex system of equations, it is necessary to **decomplexify** it.
- **Decomplexify:** To model a complex system of equations with a strictly real system for the purpose of applying the tenets of real linear algebra to it.
- Consider the following complex system of equations.

$$\begin{aligned}(2+i)x_1 + (1+i)x_2 &= 3+6i \\ (3-i)x_1 + (2-2i)x_2 &= 7-i\end{aligned}$$

– The solutions will be complex numbers:  $x_1 = a_1 + ib_1$  and  $x_2 = a_2 + ib_2$ , where  $a_1, a_2, b_1, b_2 \in \mathbb{R}$ .

- Transform it into a matrix system of equations. Separate the real and complex parts, and factor out all instances of the imaginary number  $i$  so that it is a coefficient to any complex matrix.

$$\begin{aligned}\begin{bmatrix} 2+i & 1+i \\ 3-i & 2-2i \end{bmatrix} \begin{bmatrix} a_1+ib_1 \\ a_2+ib_2 \end{bmatrix} &= \begin{bmatrix} 3+6i \\ 7-i \end{bmatrix} \\ \left( \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} i & i \\ -i & -2i \end{bmatrix} \right) \left( \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} ib_1 \\ ib_2 \end{bmatrix} \right) &= \left( \begin{bmatrix} 3 \\ 7 \end{bmatrix} + \begin{bmatrix} 6i \\ -i \end{bmatrix} \right) \\ \underbrace{\left( \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + i \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \right)}_A \underbrace{\left( \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + i \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right)}_x &= \underbrace{\left( \begin{bmatrix} 3 \\ 7 \end{bmatrix} + i \begin{bmatrix} 6 \\ -1 \end{bmatrix} \right)}_b\end{aligned}$$

- Foil the left side of the above equation<sup>[1]</sup>.

$$\left( \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) + i \left( \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 7 \end{bmatrix} + i \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

- Split the above system of equations into a real system of equations and a complex system of equations by setting equal to each other the real components of each side and the imaginary components of each side.

$$\begin{aligned}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} &= \begin{bmatrix} 3 \\ 7 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} &= \begin{bmatrix} 6 \\ -1 \end{bmatrix}\end{aligned}$$

- Multiply out the matrices above to yield a system of four equations.

$$\begin{aligned}2a_1 + a_2 - b_1 - b_2 &= 3 \\ 3a_1 + 2a_2 + b_1 + 2b_2 &= 7 \\ a_1 + a_2 + 2b_1 + b_2 &= 6 \\ -a_1 - 2a_2 + 3b_1 + 2b_2 &= -1\end{aligned}$$

- Condense the above system of equations into a single matrix system of equations.

$$\begin{bmatrix} 2 & 1 & -1 & -1 \\ 3 & 2 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ -1 & -2 & 3 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 6 \\ -1 \end{bmatrix}$$

<sup>1</sup>Note that the minus sign appears in the real component because, when multiplying the two “last” parts,  $i^2 = -1$ .

- Solve for  $a_1$ ,  $a_2$ ,  $b_1$ , and  $b_2$  using an augmented matrix and Gauss-Jordan elimination.

$$\left[ \begin{array}{cccc|c} 2 & 1 & -1 & -1 & 3 \\ 3 & 2 & 1 & 2 & 7 \\ 1 & 1 & 2 & 1 & 6 \\ -1 & -2 & 3 & 2 & -1 \end{array} \right] \rightarrow \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{array} \right]$$

$$\begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}$$

- From these four values, the original solutions  $x_1 = a_1 + ib_1$  and  $x_2 = a_2 + ib_2$  can be found.

$$x_1 = 1 + 2i$$

$$x_2 = 2 - i$$

## Hermitian, Unitary, and Normal Matrices

4/13:

- What necessitates different categorizations of complex vectors and matrices?
- Consider a vector  $v$ .

$$v = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

- If you want to find  $\|v\|$ , you typically evaluate  $\sqrt{v^T v}$ . However, this equals to 0 (see the following), which is clearly not the magnitude of  $v$ .

$$\begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 - 1 = 0$$

– Note that  $\|v\|$  must be an element of  $\mathbb{R}$  because it measures a distance.

- With complex vectors, it is necessary to evaluate  $\sqrt{\bar{v}^T v}$  to find  $\|v\|$ .

$$\begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 + 1 = 2$$

$$\|v\| = \sqrt{2}$$

- This makes sense because  $\begin{bmatrix} 1 \\ i \end{bmatrix}$  extends one unit into  $\mathbb{R}^1$  and one unit into  $\mathbb{C}^2$ .
- If  $z\bar{z} = |z|^2$  and  $\bar{v}^T v = v \cdot \bar{v}$ , it stands to reason that  $\bar{v}^T v = \|v\|^2$ . Essentially, the dot product multiplies every element of  $v$  by its complex conjugate and sums them.
- Instead of writing  $\bar{v}^T$ <sup>[2]</sup> every time, mathematicians shorthand to  $v^H$ <sup>[3]</sup>.
  - $v^H$  works for all vectors, but it is necessary for complex ones.
- **Hermitian** (matrix): A matrix  $A$  such that  $A = A^H$ .
  - Typically defined for  $A \in \mathbb{C}^n$ , but holds for  $A \in \mathbb{R}^n$ , too.
  - Parallel to how if  $A \in \mathbb{R}^n$  and  $A = A^T$ ,  $A$  is symmetrical.
  - Also note that if  $A^H A = A^2 = A A^H$ ,  $A$  is Hermitian.

<sup>2</sup>“ $v$  conjugate transpose”

<sup>3</sup>“ $v$  Hermitian” after French mathematician Charles Hermite.

- A Hermitian matrix has to have real values on the principal diagonal. When  $A$  is transposed and conjugated, the diagonal entries are the only values that don't move. Thus, their conjugates must equal themselves, so they must be real<sup>[4]</sup>.
- **Unitary** (matrix): A matrix  $A$  such that  $A^{-1} = A^H$ .
  - Typically defined for  $A \in \mathbb{C}^n$ , but holds for  $A \in \mathbb{R}^n$ , too.
  - Parallel to how if  $A \in \mathbb{R}^n$  and  $A^{-1} = A^T$ ,  $A$  is orthonormal.
  - Also note that if  $A^H A = I = A A^H$ ,  $A$  is unitary.
- **Normal** (matrix): A matrix that is unitarily diagonalizable.
  - Typically defined for  $A \in \mathbb{C}^n$ , but holds for  $A \in \mathbb{R}^n$ , too.
  - Parallel to matrices  $A \in \mathbb{R}^n$  such that  $A$  is orthonormally diagonalizable.
- Note that not every complex matrix has to be one of these three types.
- When  $A^H A = A A^H$ ,  $A = U \Lambda U^H$ .

$$\begin{aligned}
 A A^H &= (U \Lambda U^H) (U \Lambda U^H)^H \\
 &= U \Lambda U^H U \Lambda^H U^H \\
 &= U \Lambda \Lambda^H U^H \\
 &= U \Lambda^H \Lambda U^H \text{ [5]} \\
 &= U \Lambda^H U^H U \Lambda U^H \\
 &= (U \Lambda U^H)^H (U \Lambda U^H) \\
 &= A^H A
 \end{aligned}$$

- When  $A = A^H$ , all eigenvalues are elements of  $\mathbb{R}$  (similar to spectral theorem).

$$v^H A v = (v^H A v)^H = v^H A v$$

- The above proves that  $v^H A v \in \mathbb{R}$  because it's its own conjugate?? .

$$\begin{aligned}
 A v &= \lambda v \\
 v^H A v &= \lambda v^H v
 \end{aligned}$$

- $\lambda = \frac{v^H A v}{v^H v} \rightarrow \frac{\mathbb{R}}{\mathbb{R}} = \mathbb{R}$  [6].
- When  $A = A^H$  and  $A x = \lambda x$ , all  $x$ 's can be chosen orthonormally (also similar to spectral theorem).
  - Normality is implied because any eigenvector can be scaled to any version (including a normal version) and still be an eigenvector.

$$x_i = \begin{bmatrix} x_{i_1} \\ x_{i_2} \\ \vdots \\ x_{i_n} \end{bmatrix} \qquad x_i^H = [\bar{x}_{i_1} \quad \bar{x}_{i_2} \quad \cdots \quad \bar{x}_{i_n}]$$

<sup>4</sup>Recall that only real quantities can be their own conjugates because  $a + 0i = a - 0i$ .

<sup>5</sup>Since  $\Lambda = \Lambda^H$ .

<sup>6</sup>Note that the denominator is real because it's how one finds  $\|v\|$ , and  $\|v\|$  must be real, as discussed above.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad A^H = \begin{bmatrix} a_{11} & \bar{a}_{21} & \cdots & \bar{a}_{n1} \\ \bar{a}_{12} & a_{22} & \cdots & \bar{a}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{1n} & \bar{a}_{2n} & \cdots & a_{nn} \end{bmatrix}$$

- Define an arbitrary vector  $x_i$  and matrix  $A$ , along with their conjugate transposes (or Hermitian versions). Note that the diagonal entries of  $A^H$  aren't shown as conjugated because their conjugates equal themselves.

$$Ax_1 = \lambda_1 x_1$$

$$x_2^H Ax_1 = \lambda_1 x_2^H x_1$$

$$Ax_2 = \lambda_2 x_2$$

$$(Ax_2)^H = (\lambda_2 x_2)^H$$

$$x_2^H A^H = \lambda_2 x_2^H$$

$$x_2^H Ax_1 = \lambda_2 x_2^H x_1$$

- $\lambda_1 x_2^H x_1 = \lambda_2 x_2^H x_1$  implies that, since  $\lambda_1 \neq \lambda_2$ ,  $x_2^H x_1$  must equal 0, proving orthogonality.

## Complex Diagonalization

- 4/15:      • Diagonalize the following matrix  $A$ .

$$\begin{bmatrix} 0.9 & -0.4 \\ 0.1 & 0.9 \end{bmatrix}$$

- Find the characteristic polynomial.

$$\begin{aligned} 0 &= \begin{vmatrix} 0.9 - \lambda & -0.4 \\ 0.1 & 0.9 - \lambda \end{vmatrix} \\ &= (0.9 - \lambda)^2 - (-0.4)(0.1) \\ &= 0.81 - 1.8\lambda + \lambda^2 + 0.04 \\ &= \lambda^2 - 1.8\lambda + 0.85 \end{aligned}$$

- Find the eigenvalues.

$$\begin{aligned} \lambda &= \frac{-(-1.8) \pm \sqrt{(-1.8)^2 - 4(1)(0.85)}}{2(1)} \\ &= 0.9 \pm \frac{\sqrt{-0.16}}{2} \\ &= 0.9 \pm \frac{\sqrt{-1}\sqrt{0.16}}{2} \\ &= 0.9 \pm \frac{0.4i}{2} \\ &= 0.9 \pm 0.2i \end{aligned}$$

$$\lambda_1 = 0.9 + 0.2i$$

$$\lambda_2 = 0.9 - 0.2i$$

– Find the eigenvectors.

$$\begin{aligned}(A - (0.9 + 0.2i))x_1 &= \begin{bmatrix} 0.9 - (0.9 + 0.2i) & -0.4 \\ 0.1 & 0.9 - (0.9 + 0.2i) \end{bmatrix} \begin{bmatrix} x_{1_1} \\ x_{1_2} \end{bmatrix} \\ &= \begin{bmatrix} -0.2i & -0.4 \\ 0.1 & -0.2i \end{bmatrix} \begin{bmatrix} 2i \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}(A - (0.9 - 0.2i))x_1 &= \begin{bmatrix} 0.9 - (0.9 - 0.2i) & -0.4 \\ 0.1 & 0.9 - (0.9 - 0.2i) \end{bmatrix} \begin{bmatrix} x_{1_1} \\ x_{1_2} \end{bmatrix} \\ &= \begin{bmatrix} 0.2i & -0.4 \\ 0.1 & 0.2i \end{bmatrix} \begin{bmatrix} -2i \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}\end{aligned}$$

$$x_1 = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} -2i \\ 1 \end{bmatrix}$$

– Compile the diagonalization.

$$A = \frac{1}{4i} \begin{bmatrix} 2i & -2i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0.9 + 0.2i & 0 \\ 0 & 0.9 - 0.2i \end{bmatrix} \begin{bmatrix} 1 & 2i \\ -1 & 2i \end{bmatrix}$$

## Real versus Complex

4/16:

Real	Complex
$\mathbb{R}^n$ : vectors with $n$ real components	$\mathbb{C}^n$ : vectors with $n$ complex components
length: $\ x\ ^2 = x_1^2 + \cdots + x_n^2$	length: $\ z\ ^2 =  z_1 ^2 + \cdots +  z_n ^2$
transpose: $(A^T)_{ij} = A_{ji}$	conjugate transpose: $(A^H)_{ij} = \overline{A_{ji}}$
product rule: $(AB)^T = B^T A^T$	product rule: $(AB)^H = B^H A^H$
dot product: $x^T y = x_1 y_1 + \cdots + x_n y_n$	inner product: $u^H v = \bar{u}_1 v_1 + \cdots + \bar{u}_n v_n$
reason for $A^T$ : $(Ax)^T y = x^T (A^T y)$	reason for $A^H$ : $(Au)^H v = u^H (A^H v)$
orthogonality: $x^T y = 0$	orthogonality: $u^H v = 0$ .
symmetric matrices: $A = A^T$	Hermitian matrices: $A = A^H$
$A = Q\Lambda Q^{-1} = Q\Lambda Q^T$ (real $\Lambda$ )	$A = U\Lambda U^{-1} = U\Lambda U^H$ (real $\Lambda$ )
skew-symmetric matrices: $k^T = -K$	skew-Hermitian matrices: $K^H = -K$
orthogonal matrices: $Q^T = Q^{-1}$	unitary matrices: $U^H = U^{-1}$
orthonormal columns: $Q^T Q = I$	orthonormal columns: $U^H = U^{-1}$
$(Qx)^T (Qy) = x^T y$ and $\ Qx\  = \ x\ $	$(Ux)^H (Uy) = x^H y$ and $\ Uz\  = \ z\ $

- Note that the columns and eigenvectors of  $Q$  and  $U$  are orthonormal, and all of their eigenvalues  $\lambda$  satisfy  $|\lambda| = 1$ .