LINEAR TRANSFORMATIONS

- We have already seen repeatedly that a matrix can be used to transform vectors. Let T be a matrix, and v be a vector in $\mathbb{R}^2$ and w be a vector in $\mathbb{R}^3$. Let $w = T(v)$:

$$T = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix} \quad v = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad w = T(v) = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

- T will transform any vector in $\mathbb{R}^2$ into a vector in $\mathbb{R}^3$: $\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x - y \\ 3x + 4y \end{bmatrix}$
- The **transformation** or **mapping** T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns each vector v in $\mathbb{R}^n$ to a unique vector $T(v)$ in $\mathbb{R}^m$
- The **domain** of T is $\mathbb{R}^n$.
- The **codomain** of T is $\mathbb{R}^m$.
- Thus the transformation would be described as $T : \mathbb{R}^n \to \mathbb{R}^m$
- The vector $T(v)$ is the **image** of v.
- The set of all possible images is known as the **range** of T.
- In our example $\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x - y \\ 3x + 4y \end{bmatrix}$, the range of T would be all linear combinations,

  $x\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$, thus the column space of T.

WHEN IS A TRANFORMATION "LINEAR?"

- $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a linear transformation if and only if :

  $T(u + v) = T(u) + T(v)$ for all u and v in $\mathbb{R}^n$

  $T(cv) = cT(v)$ for all v in $\mathbb{R}^n$ and all scalars c
- Steps to verify that our example transformation is linear:

$T : \mathbb{R}^n \to \mathbb{R}^m = T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x - y \\ 3x + 4y \end{bmatrix}$, let $u = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $v = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$$T(u+v)=T\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}+\begin{bmatrix}x_2\\y_2\end{bmatrix}\right)=T\left(\begin{bmatrix}x_1+x_2\\y_1+y_2\end{bmatrix}\right)=\begin{bmatrix}x_1+x_2\\2(x_1+x_2)-(y_1+y_2)\\3(x_1+x_2)+4(y_1+y_2)\end{bmatrix}=$$

$$=\begin{bmatrix}x_1+x_2\\2x_1+2x_2-y_1-y_2\\3x_1+3x_2+4y_1+4y_2\end{bmatrix}=\begin{bmatrix}x_1+x_2\\(2x_1-y_1)+(2x_2-y_2)\\(3x_1+4y_1)+(3x_2+4y_2)\end{bmatrix}=$$

$$=\begin{bmatrix}x_1\\2x_1-y_1\\3x_1+4y_1\end{bmatrix}+\begin{bmatrix}x_2\\2x_2-y_2\\3x_2+4y_2\end{bmatrix}=T\begin{bmatrix}x_1\\y_1\end{bmatrix}+T\begin{bmatrix}x_2\\y_2\end{bmatrix}=T(u)+T(v)$$

$$T(cv)=T\left(c\begin{bmatrix}x\\y\end{bmatrix}\right)=T\left(\begin{bmatrix}cx\\cy\end{bmatrix}\right)=$$

$$=\begin{bmatrix}cx\\2(cx)-(cy)\\3(cx)+4(cy)\end{bmatrix}=\begin{bmatrix}cx\\c(2x-y)\\c(3x+4y)\end{bmatrix}=$$

$$=c\begin{bmatrix}x\\2x-y\\3x+4y\end{bmatrix}=cT\begin{bmatrix}x\\y\end{bmatrix}=cT(v)$$

- Therefore, our example transformation was indeed linear.
- This is true for any $m\times n$ matrix A. Let $T_A:\mathbb{R}^n\to\mathbb{R}^m$ be defined by $T_A(x)=Ax$ for any x in $\mathbb{R}^n$. Let u and v be vectors in $\mathbb{R}^n$ and c be a scalar:

$T_A(u+v)=A(u+v)=Au+Av=T_A(u)+T_A(v)$ and $T_A(cv)=A(cv)=c(Av)=cT_A(v)$

ROTATIONS AS LINEAR TRANSFORMATIONS

- A rotation about the origin through an angle θ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$.
- Let R_θ be the rotation, and u and v be vectors in R^2. Provided that the vectors are not parallel, we know that $\|u+v\|$ is the diagonal of a parallelogram formed by the two vectors. Applying R_θ would rotate the entire parallelogram through angle θ, and thus the diagonal of the parallelogram must be $R_\theta(u)+R_\theta(v)$ and therefore $R_\theta(u+v)=R_\theta(u)+R_\theta(v)$.

  Applying R_θ to v and cv would obtain $R_\theta(v)$ and $R_\theta(cv)$. Since a rotation does not affect lengths, $R_\theta(cv)=cR_\theta(v)$

- The matrix of this linear transformation can be found by determining the effects on standard basis vectors: $R_\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $R_\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$(which is orthogonal, so $R_\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$)
- $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

PROJECTIONS AS LINEAR TRANSFORMATIONS

- As an example, let ℓ be a line through the origin in $\mathbb{R}^2$. The linear transformation $P_\ell : \mathbb{R}^2 \to \mathbb{R}^2$ projects a vector in $\mathbb{R}^2$ onto ℓ.
- ℓ has a directional vector d onto which we will project an arbitrary vector v.
- The projection of v onto d is given by $\left(\frac{d^T v}{d^T d} \right) d$
- Therefore, the transformation $P_\ell : \mathbb{R}^2 \to \mathbb{R}^2$ is linear because:

$$P_\ell(u+v) = \left(\frac{d^T(u+v)}{d^T d} \right) d = \left(\frac{d^T u + d^T v}{d^T d} \right) d =$$
$$= \left(\frac{d^T u}{d^T d} + \frac{d^T v}{d^T d} \right) d = \left(\frac{d^T u}{d^T d} \right) d + \left(\frac{d^T v}{d^T d} \right) d =$$
$$P_\ell(u) + P_l(v)$$

Similarly $P_\ell(cv) = cP_\ell(v)$

TRANSPOSITION AS A LINEAR TRANSFORMATION

- Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined as $T_A = A^T$
- $T(A+B) = (A+B)^T = A^T + B^T = T(A) + T(B)$
- $T(cA) = (cA)^T = cA^T = cT(A)$

DIFFERENTIATION AS A LINEAR TRANSFORMATION

- Let D be the differential operator $D : \Delta \to \Phi$ defined by $D(f) = f'$
- Let f and g be differentiable functions.
- D is a linear transformation because $D(f+g) = (f+g)' = f' + g' = D(f) + D(g)$ and $D(cf) = (cf)' = cf' = cD(f)$.b

INTEGRATION AS A LINEAR TRANSFORMATION

- Let $S : \partial[a,b] \to \mathbb{R}$ by $S(f) = \int_a^b f(x)dx$
- Let f and g be functions in $\partial[a,b]$.
- S is a linear transformation because

  $S(f+g) = \int_a^b (f+g)(x)dx = \int_a^b (f(x)+g(x))dx =$

  $= \int_a^b f(x)dx + \int_a^b g(x)dx = S(f) + S(g)$
- Similarly, $S(cf) = cS(f)$

EXAMPLES OF NON-LINEAR TRANSFORMATIONS

- $T : \mathbb{R} \to \mathbb{R} : T(x) = 2^x$

  Let $x = 1$ and $y = 2$

  $T(x+y) = T(3) = 2^3 = 8 \neq 6 = 2^1 + 2^2 = T(x) + T(y)$
- $T : \mathbb{R} \to \mathbb{R} : T(x) = x+1$

  Let $x = 1$ and $y = 2$

  $T(x+y) = T(3) = 3+1 = 4 \neq 5 = (1+1)+(2+1) = T(x)+T(y)$
- $T : M_{22} \to \mathbb{R} : T(A) = |A|$

  Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

  $T(A+B) = |A+B| = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 = \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = T(A) + T(B)$

$T:V \to W$ and T's effect on a basis for V

Example:

$$T:\mathbb{R}^2 \to F : T\begin{bmatrix}1\\1\end{bmatrix} = 2-3x+x^2 \text{ and } T\begin{bmatrix}2\\3\end{bmatrix} = 1-x^2$$

- Find $T\begin{bmatrix}-1\\2\end{bmatrix}$.

$\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}2\\3\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^2$, and therefore $\begin{bmatrix}-1\\2\end{bmatrix}$ is in its span. Solve $\begin{bmatrix}1 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix}$ to find the appropriate scalars $c_1 = -7$ and $c_2 = 3$.

Therefore $T\begin{bmatrix}-1\\2\end{bmatrix} = T\left(-7\begin{bmatrix}1\\1\end{bmatrix}+3\begin{bmatrix}2\\3\end{bmatrix}\right) = -7T\begin{bmatrix}1\\1\end{bmatrix}+3T\begin{bmatrix}2\\3\end{bmatrix}$

$$= -7(2-3x-10x^2)+3(1-x^2) = -11+21x-10x^2$$

- Find $T\begin{bmatrix}a\\b\end{bmatrix}$.

$= (5a-3b)+(-9a+6b)x+(4a-3b)x^2$

Therefore, let $T:V \to W$ be a linear transformation, and let $B = \ v_1,\ldots,v_n\ $ be a basis that spans V. $T(B) = \ T(v_1),\ldots T(v_n)\ $ is a basis that spans the range of T.