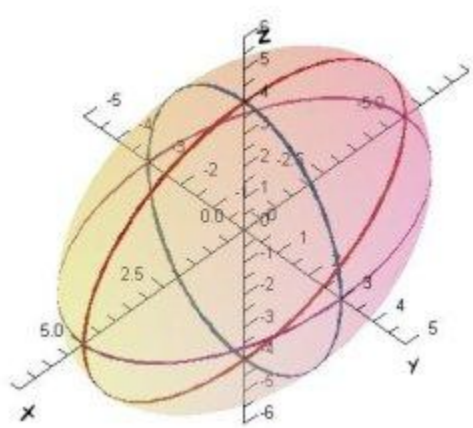
QUADRIC SURFACES

- A **quadric surface** is the graph of a second-degree equation in three variables: x, y, z
- Its **general form** equation is $Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$ for which $A, B, C, \ldots, J$ are constants
- Through appropriate translation and rotation the general form may be reduced to either $Ax^2 + By^2 + Cz^2 + J = 0$ or $Ax^2 + By^2 + Iz = 0$
- Quadric surfaces are analogous in three dimensions to the conic sections in two dimensions

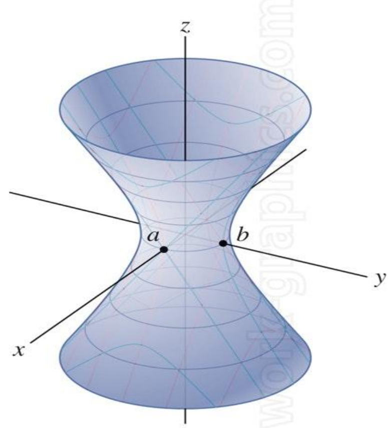
- In order to sketch the graph of a quadric surface (or any surface for that matter) it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes. These are called the **traces**

ELLIPSOIDS

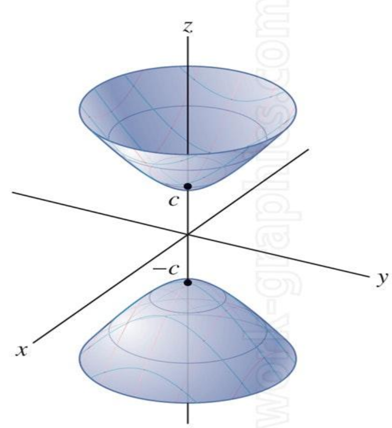


- **Ellipsoids** have elliptical traces
- Equations are in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (Equation 1)

HYPERBOLOIDS



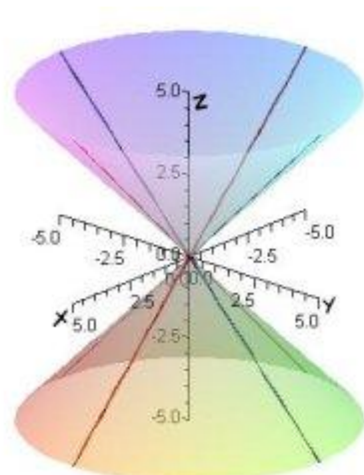
(A) Hyperboloid of one sheet



(B) Hyperboloid of two sheets

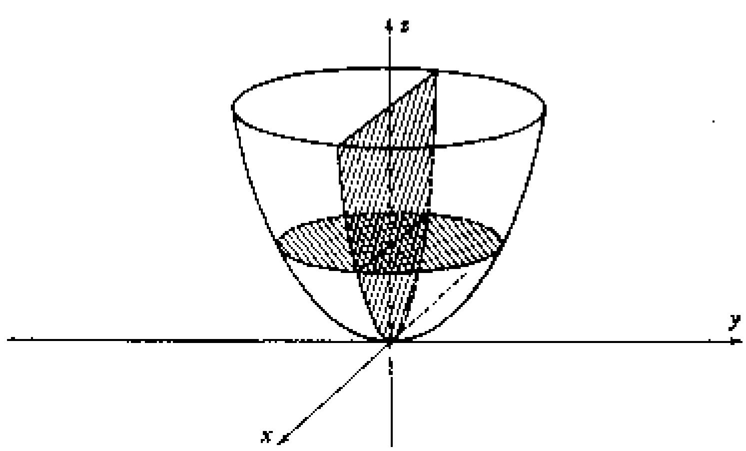
- Equations for (A) are of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ (Equation 2)
- Horizontal traces are ellipses
- Vertical traces are hyperbolas
- The axis of symmetry corresponds to the variables whose coefficient is negatives
- Equations for (B) are of the form $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ (Equation 3)
- Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$
- Vertical traces are hyperbolas
- The two negative signs indicate the two sheets

CONES



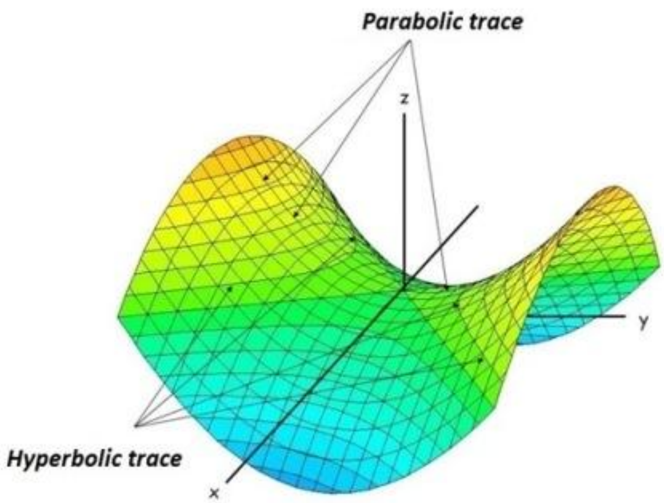
- Equations of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c}$ (Equation 4)
- Horizontal traces are ellipses
- Vertical lines in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$

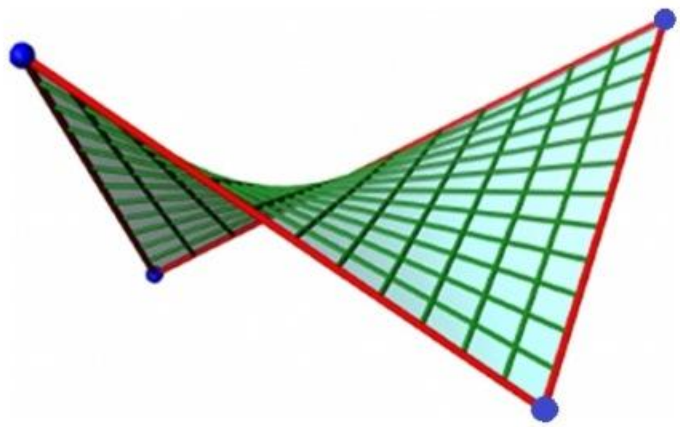
ELLIPTIC PARABOLOIDS



- Equations of the form $\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ (Equation 5)
- Horizontal traces are ellipses
- Vertical traces are parabolas
- The variables raised to the first power indicates the axis of the paraboloid

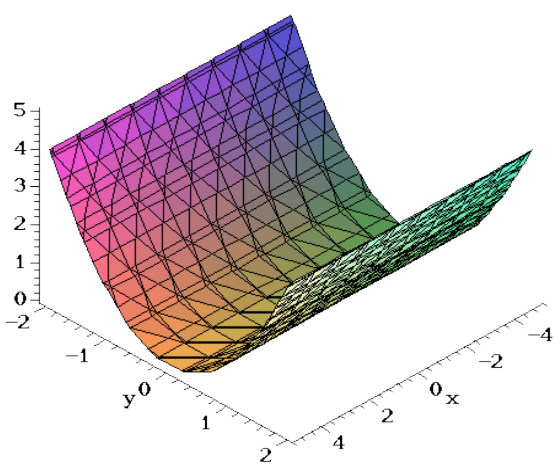
HYPERBOLIC PARABOLOIDS

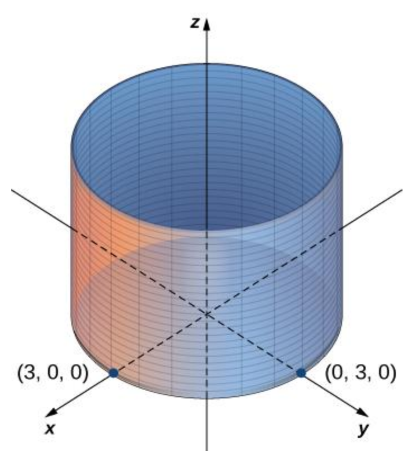




- Equations of the form $\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ (Equation 6)
- Horizontal traces are hyperbolas
- Vertical traces are parabolas

QUADRIC CYLINDERS





Parabolic Cylinder **Elliptic Cylinder**

- Equations of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (Equation 7)

Example: Use traces to sketch the quadric surface $x^2+\frac{y^2}{9}+\frac{z^2}{4}=1$

- Let $z=0$ to see that the trace in the $xy-$plane is $x^2+\frac{y^2}{9}=1$, an ellipse.
- Generally speaking, the horizontal trace in plane $z=k$ is $x^2+\frac{y^2}{9}=1-\frac{k^2}{4}$ which is an ellipse provided that $k^2<4$, or $-2<k<2$.
- Vertical traces are also ellipses: $$\frac{y^2}{9}+\frac{z^2}{4}=1-k^2, x=k, -1<k<1$$ $$x^2+\frac{z^2}{4}=1-\frac{k^2}{9}, y=k, -3<k<3$$

Example: Use traces to sketch $z=4x^2+y^2$

- Let $x=0$ to see $z=y^2$, indicating that the $xy-$plane intersects the surface in a parabola
- Let $x=k$ to yield $z=y^2+4k^2$, indicating that if we were to slice the graph with any plane parallel to the $yz-$plane we obtain a parabola that opens upward
- Let $y=k$ to yield $z=4x^2+k^2$, indicating another parabola opening upward
- Let $z=k$ to yield horizontal traces $4x^2+y^2=k$, which are ellipse.

Example: Sketch the surface $z=y^2-x^2$

- The traces in vertical planes $x=k$ are parabolas $z=y^2-k^2$,which open upward
- The traces in $y=k$ are parabolas $z=-x^2+k^2$, which open downward
- Horizontal traces are $y^2-x^2=k$, a family of hyperbolas
- Fitting together all these traces, we have a hyperbolic paraoloid

Example: Sketch the surface $\frac{x^2}{4}+y^2-\frac{z^2}{4}=1$

- The trace in any horizontal plane $z=k$ is the ellipse $\frac{x^2}{4}+y^2=1+\frac{k^2}{4}, z=k$
- Traces in $xz-$and $yz-$planes are the hyperbolas $\frac{x^2}{4}-\frac{z^2}{4}=1, y=0$ and $y^2-\frac{z^2}{4}=1, z=0$

Example : Identify and sketch the surface $4x^2 - y^2 + 2z^2 + 4 = 0$.

- First divide the equation by -4 to put it in its standard form: $-x^2 + \frac{y^2}{4} - \frac{z^2}{2} = 1$
- Next match its form with one of the seven equations above. Notice its standard form resembles Equation 3: $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$; this is a hyperboloid of two sheets. Notice the difference is that since the y^2 term of our equation is positive, this hyperboloid will be formed along the y-axis. Equation 3 has a positive z^2 term meaning it would be formed along the z-axis
- This means we will need to sketch the traces of the hyperboloid along the xy-plane and the yz-plane
- For the xy-plane set $z = 0$ and sketch the hyperbola $-x^2 + \frac{y^2}{4} = 1$
- For the yz-plane set $x = 0$ and sketch the hyperbola $\frac{y^2}{4} - \frac{z^2}{2} = 1$
- Since this hyperboloid is formed along the y-axis, there will be no xz-plane traces
- Set $y = k$
- $-x^2 + \frac{k^2}{4} - \frac{z^2}{2} = 1 \rightarrow x^2 + \frac{z^2}{4} = \frac{k^2}{4} - 1$
- Dividing each term in the equation above yields $\frac{x^2}{\frac{k^2}{4} - 1} + \frac{z^2}{2\left(\frac{k^2}{4} - 1\right)} = 1$; This is the trace the plane $y = k$ and

  we can see it is in the form of an ellipse

Try sketching the following surfaces: (a) $x^2 + 2z^2 - 6x - y + 10 = 0$ (Hint: complete the square)

(b) $z^2 = 4x^2 + 9y^2 + 36$ (c) $4x^2 + y^2 + 4z^2 - 4y - 24z + 36 = 0$ (d) $x^2 - y^2 + z^2 - 2x + 2y + 4z + 2 = 0$