KERNEL AND RANGE

Let $T:V \to W$ be a linear transformation

- The **kernel** of T [ker(T)] is the set of all vectors in V that are mapped by T to 0 in W.

  ker(T) ={v in V : T(v) =0}

- The **range** of T [range(T)] is the set of all vectors in W that are images of vectors in V under T.

EXAMPLES:

Find the kernel and range of the differential operator $D: P_3 \to P_2: D(p(x)) = p'(x)$

$$D(a+bx+cx^2+dx^3) = b+2cx+3dx^2$$

$$\ker(D) = \{a+bx+cx^2+dx^3 : D(a+bx+cx^2+dx^3) = 0\}$$

$$= \{a+bx+cx^2+dx^3 : b+2cx+3dx^2 = 0\}$$

$b+2cx+3dx^2 = 0$ iff $b = 2c = 3d = 0$, implying $b = c = d = 0$, therefore

$$\ker(D) = \{a+bx+cx^2+dx^3 : b = c = d = 0\}$$

$$= \{a : a \text{ in } \mathbb{R}\}$$

Therefore, the kernel of D is the set of all constant polynomials. The range of D is all of P_2 because every polynomial in P_2 is the image under D of some polynomial in P_3.

Find the kernel and range of $S: P_1 \to \mathbb{R} : S(p(x)) = \int_0^1 p(x)dx$

$$S(a+bx) = \int_0^1 (a+bx)dx$$

$$\left[ax + \frac{b}{2}x^2\right]_0^1 = \left(a + \frac{b}{2}\right) - 0 = a + \frac{b}{2}$$

$$\ker(S) = \{a + bx : S(a + bx) = 0\}$$

$$= \left\{a + bx : a + \frac{b}{2} = 0\right\}$$
$$= \left\{a + bx : a = -\frac{b}{2}\right\}$$
$$= \left\{-\frac{b}{2} + bx\right\}$$

Therefore, the kernel consists of all linear polynomials whose graphs have the property that the area between the line and the x-axis is equally distributed above and below the axis.

The range of S is $\mathbb{R}$ since every real number can be obtained as the image under S of a first degree polynomial.

Find the kernel and range of $T : M_{22} \to M_{22} : T(A) = A^T$

$$\ker(T) = \{A \text{ in } M_{22} : T(A) = 0\}$$

$$= \{A \text{ in } M_{22} : A^T = 0\}$$

$\ker(T) = 0$ **because if** $A^T = 0$**, then** $(A^T)^T = 0^T = 0$**. Also range** $(T) = M_{22}$ **because** $A = (A^T)^T = T(A^T)$

In each of the examples, the kernel of T is a subspace of V, the range of T is a subspace of W.

RANK AND NULLITY

- The rank of T is the dimension of the range of T.
- The nullity of T is the dimension of the kernel of T.

FROM OUR EXAMPLES:

$$D : P_3 \to P_2 :\ D(p(x)) = p'(x)$$

Rank(D)=3; nullity (D)=1

$$S : P_1 \to \mathbb{R} : S(p(x)) = \int_0^1 p(x)dx$$

Rank(S)=1; nullity (S)=1

$$T : M_{22} \to M_{22} : T(A) = A^T$$

Rank (T)=4; Nullity(T)=0

RANK THEOREM:

Rank(T) + Nullity(T) = dim (V) when $T : V \to W$

Let W be the vector space of all 2×2 matrices for which $A = A^T$. Let $T : W \to P_2 : T\begin{bmatrix} a & b \\ b & c \end{bmatrix} = (a-b)+(b-c)x+(c-a)x^2$. Find the rank on the nullity of T.

$$\ker(T) = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : T\begin{bmatrix} a & b \\ b & c \end{bmatrix} = 0 \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : (a-b)+(b-c)x+(c-a)x^2 = 0 \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : (a-b)=(b-c)=(c-a)=0 \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : a=b=c \right\}$$

$$= \left\{ \begin{bmatrix} c & c \\ c & c \end{bmatrix} \right\} = span\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right)$$

$\ker(T) = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ **and thus the nullity** $= \dim(\ker(T)) = 1$**. Therefore the rank** $= \dim W - nullity(T) = 3-1=2$