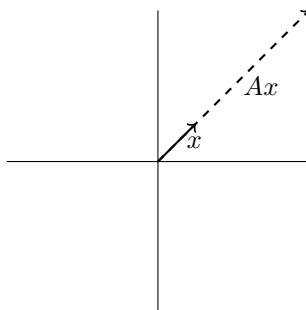


1/28: Introduction to Eigenvalues and Eigenvectors

- $Ax = b = \lambda x$
- $Ax = \lambda x, \lambda \in \mathbb{F}, x \in \mathbb{R}^n$
- λ is an eigenvalue. λx is an eigenvector.

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

- $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with corresponding eigenvalue of 4.
- $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



$$Ax = \lambda x$$

$$Ax - \lambda x = 0$$

$$Ax - \lambda Ix = 0$$

$$(A - \lambda I)x = 0$$

- $(A - \lambda I)x = 0 \Rightarrow x \in N(A - \lambda I)^{[1]} \Rightarrow |A - \lambda I| = 0$

- $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{bmatrix}$

- $\begin{vmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{vmatrix} = 0$

$$\begin{aligned} 0 &= (3 - \lambda)^2 - 1^2 \\ &= 3^2 - 6\lambda + \lambda^2 - 1 \\ &= \lambda^2 - 6\lambda + 8 \\ &= (\lambda - 4)(\lambda - 2) \end{aligned}$$

- $\lambda = 4, 2$.
- $\lambda^2 - 6\lambda + 8$ is the **characteristic polynomial** of A .
- $A - 2I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, x = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in N(A - 2I)$.
- $A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in N(A - 4I)$.

¹To have a null space, $A - \lambda I$ has free columns.

- “Eigenspace” is not \mathbb{R}^2 , but two lines in \mathbb{R}^2 , specifically $y = \pm x$.

$$- y = \pm x \text{ comes from } c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

1/29:

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\begin{aligned} P(\lambda) &= |A - \lambda I| \\ &= \begin{vmatrix} 2-\lambda & -2 & 3 \\ 0 & 3-\lambda & -2 \\ 0 & -1 & 2-\lambda \end{vmatrix} \\ &= -1 \begin{vmatrix} 2-\lambda & 3 \\ 0 & -2 \end{vmatrix} (-1)^{3+2} + (2-\lambda) \begin{vmatrix} 2-\lambda & -2 \\ 0 & 3-\lambda \end{vmatrix} (-1)^{3+3} \\ &= ((2-\lambda)(-2)) + (2-\lambda)((2-\lambda)(3-\lambda)) \\ &= -4 + 2\lambda + (2-\lambda)^2(3-\lambda) \\ &= -4 + 2\lambda + (4 - 4\lambda + \lambda^2)(3-\lambda) \\ &= -4 + 2\lambda + 12 - 4\lambda - 12\lambda + 4\lambda^2 + 3\lambda^2 - \lambda^3 \\ &= -\lambda^3 + 7\lambda^2 - 14\lambda + 8 \\ &= -(\lambda - 1)(\lambda - 2)(\lambda - 4) \end{aligned}$$

$$A - I = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -2 \\ 0 & -1 & 1 \end{bmatrix} \quad A - 2I = \begin{bmatrix} 0 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & -1 & 0 \end{bmatrix} \quad A - 4I = \begin{bmatrix} -2 & -2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix}$$

- $P(\lambda)$ is positive when $n \in 2\mathbb{N}$, negative otherwise.
 - Signs flip term to term (think about binomial expansion).
- Coefficients of the $n - 1$ degree term is the sum of the diagonal entries.
- Coefficient of the 0th degree term is $|A|$.
 - $P_\lambda(0) = |A - 0 \cdot I| = |A|$.
- Product of the eigenvalues is $|A|$.
 - Think about expanding the factorization.
- Eigenvalues of U are the diagonal values.
 - $\lambda_1 \lambda_2 \cdots \lambda_n = |A|$, which is the product of the diagonal entries.
 - $\lambda_1 + \cdots + \lambda_n = \text{trace}(A)$, which is the sum of the diagonal entries.
- $Ax = \lambda x$
 - $A^2x = AAx = A\lambda x = \lambda Ax = \lambda \lambda x = \lambda^2 x$

Similarity

1/30: • $A \sim B^{[2]}$ iff $\exists S : A = SBS^{-1}, B = S^{-1}AS$.

1. If $A \sim B$, then $|A| = |B|$.

$$\begin{aligned} B &= S^{-1}AS \\ |B| &= |S^{-1}AS| \\ |B| &= |S^{-1}||A||S| \\ |B| &= \frac{1}{|S|}|A||S| \\ |B| &= |A| \end{aligned}$$

2. If $A \sim B$, then they share the same characteristic polynomial.

$$\begin{aligned} B &= S^{-1}AS \\ |B - \lambda I| &= |S^{-1}AS - \lambda I| \\ &= |S^{-1}AS - \lambda S^{-1}IS| \\ &= |S^{-1}S(A - \lambda I)| \\ &= |I(A - \lambda I)| \\ |B - \lambda I| &= |A - \lambda I| \end{aligned}$$

– If they have the same characteristic polynomial, $\therefore A$ and B have the same eigenvalues.

• What is the best possible B if $A \sim B$?

- Sparse.
- Diagonal.

$$- A = [\text{ugly}] \rightarrow B = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = \Lambda$$

• **Diagonalization:**

$$\begin{aligned} A &= S\Lambda S^{-1} \\ AS &= S\Lambda \\ \Lambda &= S^{-1}AS \end{aligned}$$

• $A = S\Lambda S^{-1}$

$$- A^2 = AA = S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda\Lambda S^{-1} = S\Lambda^2 S^{-1}$$

$$- A^k = S\Lambda^k S^{-1}$$

$$- A^k = S \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix} S^{-1}$$

• Diagonalize the following matrix A .

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

² A “is similar to” B

- Find the characteristic polynomial.

$$\begin{aligned}
 |A - \lambda I| &= \begin{vmatrix} -1 - \lambda & 0 & 1 \\ 3 & 0 - \lambda & -3 \\ 1 & 0 & -1 - \lambda \end{vmatrix} \\
 &= (-1 - \lambda)((-\lambda)(-1 - \lambda)) + (-1)(-\lambda) \\
 &= -\lambda(-1 - \lambda)^2 + \lambda \\
 &= -\lambda(1 + 2\lambda + \lambda^2) + \lambda \\
 &= -\lambda^3 - 2\lambda^2 \\
 &= -\lambda^2(\lambda + 2)
 \end{aligned}$$

- Find the eigenvalues: $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -2$

- **Algebraic multiplicity** of λ_1, λ_2 is 2.

- A.M. of λ_3 is 1.

$$- A - 0I = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

- $\text{rank}(A - 0I) = 1 \Rightarrow \dim(N(A - 0I)) = 2$

- The 2 directly above is the **geometric multiplicity**.

- A is diagonalizable iff A.M. of $\lambda_i = \text{G.M.}$

1/31:

$$- \text{Eigenvectors are } x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$- A + 2I = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & -3 \\ 1 & 0 & 1 \end{bmatrix}$$

$$- \text{Eigenvector is } x_3 = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$$

- Use an S matrix of eigenvectors.

$$- A = SAS^{-1} = \frac{1}{2} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & & \\ & 0 & \\ & & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 & -3 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$- \text{Note that } A^{9752} = \frac{1}{2} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & & \\ & 0 & \\ & & (-2)^{9752} \end{bmatrix} \begin{bmatrix} 3 & 2 & -3 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

- **Algebraic multiplicity:** The number of repeated roots to a polynomial. For all of the roots, it adds up to n (n -square matrix). *Also known as A.M.*
- **Geometric multiplicity:** The number of eigenvectors produced from each root. For all of the roots, it may not add up to n (n -square matrix). $\dim(N(A - \lambda I))$. *Also known as G.M.*
- A nondiagonalizable example:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

- $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 4$.

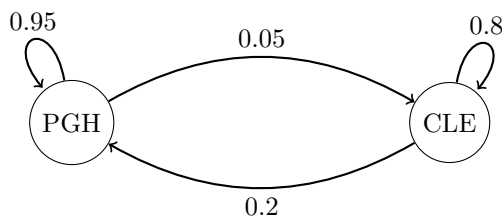
- λ_1 and λ_2 have A.M. = 2.

- λ_3 has A.M. = 1.
- $A - I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}$
- $\text{rank}(A - I) = 2 \Rightarrow \dim(N(A - I)) = 1 \Rightarrow \text{G.M.} = 1$.
- $x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
- $A - 4I = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
- $x_2 = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$
- S would be 3×2 and, thus, not square, so $\nexists S^{-1}$ ^[3].

- **Canonical** (form): An accepted way of expressing something.

Markov Chains

2/3:



- $u_0 = \begin{bmatrix} 500,000 \\ 500,000 \end{bmatrix}$
- $Au_0 = u_1$.
- $Au_1 = u_2$, $A(Au_0) = u_2$, $A^2u_0 = u_2$, $A^ku_0 = u_k$, $(S\Lambda S^{-1})^ku_0 = u_k$, $S\Lambda^kS^{-1}u_0 = u_k$.

$$A = \begin{bmatrix} 0.95 & 0.20 \\ 0.05 & 0.80 \end{bmatrix} \quad u_k = \begin{bmatrix} \text{PGH} \\ \text{CLE} \end{bmatrix}$$

- A is a **Markov matrix**, where all columns and rows add to 1.
- $Au_0 = \begin{bmatrix} 0.95 & 0.20 \\ 0.05 & 0.80 \end{bmatrix} \begin{bmatrix} 500,000 \\ 500,000 \end{bmatrix} = u_1$

$$\begin{aligned}
 |A - \lambda I| &= \begin{vmatrix} 0.95 - \lambda & 0.20 \\ 0.05 & 0.80 - \lambda \end{vmatrix} \\
 &= (0.95 - \lambda)(0.80 - \lambda) - (0.2)(0.05) \\
 &= (\lambda - 1)(\lambda - 0.75)
 \end{aligned}$$

- $\lambda_1 = 1$, $\lambda_2 = 0.75$.

³At a later date, we will look at an analogy of projections to diagonalization that finds the “best possible” diagonalization (which may not be perfectly diagonal).

- $A - I = \begin{bmatrix} -0.05 & 0.2 \\ 0.05 & -0.2 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$
- $A - 0.75I = \begin{bmatrix} 0.2 & 0.2 \\ 0.05 & 0.05 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\begin{aligned}
 A^k u_0 &= \frac{1}{5} \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.75 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 500,000 \\ 500,000 \end{bmatrix} \\
 &= \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.75^k \end{bmatrix} \begin{bmatrix} 200,000 \\ 300,000 \end{bmatrix} \\
 &= \begin{bmatrix} 4 \\ 1 \end{bmatrix} (200,000) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} (0.75)^k (300,000)
 \end{aligned}$$

- $\begin{bmatrix} 800,000 \\ 200,000 \end{bmatrix}$ is the steady-state vector.
- $\begin{bmatrix} -(0.75)^k (300,000) \\ (0.75)^k (300,000) \end{bmatrix}$ is the dynamically changing vector.
- $\lim_{k \rightarrow \infty} A^k u_0 = \begin{bmatrix} 800,000 \\ 200,000 \end{bmatrix} = \begin{bmatrix} \text{PGH} \\ \text{CLE} \end{bmatrix}$

Explicit Formula for the Fibonacci Sequence

2/4: 1, 1, 2, 3, 5, 8, ...

- Recursively defined formula: $F_n^{[4]} = F_{n-1} + F_{n-2}$.

$$\begin{aligned}
 F_n &= F_{n-1} + F_{n-2} \\
 F_{n-1} &= F_{n-1} \\
 \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix}
 \end{aligned}$$

- $u_n = A^n u_0 = S \Lambda^n S^{-1} u_0$.

$$\begin{aligned}
 0 &= |A - \lambda I| \\
 &= \begin{vmatrix} 1 - \lambda & 1 \\ 1 & -\lambda \end{vmatrix} \\
 &= -\lambda(1 - \lambda) - 1 \\
 &= \lambda^2 - \lambda - 1
 \end{aligned}$$

- $\lambda = \frac{1 \pm \sqrt{5}}{2}$ [5].
- $\lambda_1 = \frac{1 + \sqrt{5}}{2}$.

$$\begin{aligned}
 N(A - \lambda_1 I) &= N \left(\begin{bmatrix} 1 - \frac{1 + \sqrt{5}}{2} & 1 \\ 1 & -\frac{1 + \sqrt{5}}{2} \end{bmatrix} \right) \\
 &= N \left(\begin{bmatrix} \frac{1 - \sqrt{5}}{2} & 1 \\ 1 & \frac{-1 - \sqrt{5}}{2} \end{bmatrix} \right)
 \end{aligned}$$

⁴The n -th Fibonacci number.

⁵This is the Golden ratio!

- $\begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{-1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- Let $x_2 = 1$.

$$\begin{aligned} \frac{1-\sqrt{5}}{2}x_1 + 1 &= 0 \\ \frac{1-\sqrt{5}}{2}x_1 &= -\frac{2}{2} \\ x_1 &= \frac{-2}{1-\sqrt{5}} \times \frac{1+\sqrt{5}}{1+\sqrt{5}} \\ &= \frac{-2-2\sqrt{5}}{-4} \\ &= \frac{1+\sqrt{5}}{2} \end{aligned}$$

- $s_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}, s_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$.
- $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} S^{-1}$
- $S^{-1} = \frac{1}{|S|} C_S^T = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}$
- $u_k = A^k u_0 = S \Lambda^k S^{-1} u_0$.

$$\begin{aligned} S^{-1} u_0 &= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ \frac{-1+\sqrt{5}}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1+\sqrt{5}}{2\sqrt{5}} \\ \frac{-1+\sqrt{5}}{2\sqrt{5}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{5+\sqrt{5}}{10} \\ \frac{5-\sqrt{5}}{10} \end{bmatrix} \end{aligned}$$

- $u_k = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} \left(\frac{1+\sqrt{5}}{2} \right)^k \left(\frac{5+\sqrt{5}}{10} \right) + \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} \left(\frac{1-\sqrt{5}}{2} \right)^k \left(\frac{5-\sqrt{5}}{10} \right)$

Systems of First-Order Ordinary Differential Equations

2/11: • Let $f(x) = y$ and $a, c, K \in \mathbb{F}$.

$$\begin{aligned}\frac{dy}{dx} &= ay \\ \frac{1}{y} \frac{dy}{dx} &= a \\ \frac{1}{y} \frac{dy}{dx} dx &= a dx \\ \frac{1}{y} dy &= a dx \\ \int \frac{1}{y} dy &= \int a dx \\ \ln y &= ax + c \\ y &= e^{ax+c} \\ &= e^{ax} e^c \\ &= Ke^{ax}\end{aligned}$$

- Let $\frac{dy}{dx} = y'$.
 - $y'_1 = a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n$.
 - $y'_2 = a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n$.
 - \vdots
 - $y'_n = a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n$.

- This is a **square system** of equations.

- Rewrite as $y' = Ay$.

$$\begin{bmatrix} y'_1 \\ y'_2 \\ \vdots \\ y'_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

- Solve the following system of differential equations.

$$\begin{aligned}y'_1 &= 3y_1 \\ y'_2 &= -2y_2 \\ y'_3 &= 5y_3\end{aligned}$$

$$\begin{bmatrix} y'_1 \\ y'_2 \\ y'_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

- General Solution:

$$\begin{aligned}y_1 &= k_1 e^{3x} \\ y_2 &= k_2 e^{-2x} \\ y_3 &= k_3 e^{5x}\end{aligned}$$

- Particular Solution (where $y_1(0) = 2$, $y_2(0) = -1$, and $y_3(0) = 7$ are the initial conditions):

$$\begin{aligned}y_1 &= 2e^{3x} \\y_2 &= -e^{-2x} \\y_3 &= 7e^{5x}\end{aligned}$$

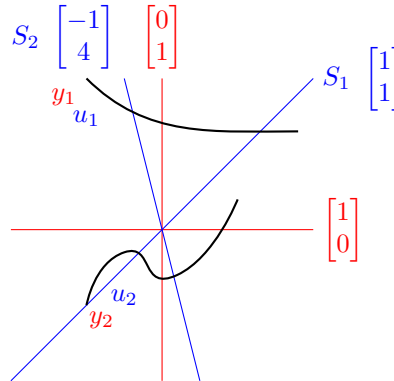
- Consider a different system. Remember throughout that we are solving for y .

$$\begin{aligned}y_1' &= y_1 + y_2 \\y_2' &= 4y_1 - 2y_2\end{aligned}$$

- The previous system was so easy to solve because the matrix was diagonal. This one (as follows) will not be. Therefore, we should diagonalize it.

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

- Start with $y' = Ay$.
- Substitute $y = Su$.
 - Note that $y = Su \Rightarrow y' = Su'^{[6]}$.
 - If we can find u' in terms of a diagonal matrix and u , we can solve for y .



- We seek to find a new basis S such that the matrix scaling u will be diagonal.

$$\begin{aligned}Su' &= Ay \\Su' &= ASu \\u' &= S^{-1}ASu \\u' &= \Lambda u\end{aligned}$$

- The last substitution above is legal because if $A = SAS^{-1}$, then $\Lambda = S^{-1}AS$.

$$\begin{aligned}0 &= \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} \\&= (1-\lambda)(-2-\lambda) - 4 \\&= -2 - \lambda + 2\lambda + \lambda^2 - 4 \\&= \lambda^2 + \lambda - 6 \\&= (\lambda - 2)(\lambda + 3)\end{aligned}$$

$$\lambda_1 = 2$$

$$\lambda_2 = -3$$

⁶Think about differentiating both sides: $y \rightarrow y'$ is obvious, S will be unchanged because it's just coefficients, and the functions of u will be differentiated.

$$- A - 2I = \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$- A + 3I = \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$u' = \Lambda u$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$u_1 = k_1 e^{2x}$$

$$u_2 = k_2 e^{-3x}$$

$$y = Su$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} k_1 e^{2x} \\ k_2 e^{-3x} \end{bmatrix}$$

$$y_1 = k_1 e^{2x} - k_2 e^{-3x}$$

$$y_2 = k_1 e^{2x} + 4k_2 e^{-3x}$$

2/12: • Initial conditions: $y_1(0) = 1$ and $y_2(0) = 6$.

– Use augmented matrices to solve a system of equations.

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 4 & 6 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array} \right]$$

• Particular solution:

$$y_1 = 2e^{2x} - e^{-3x}$$

$$y_2 = 2e^{2x} + 4e^{-3x}$$

Matrix Exponentiation

• e^A is a matrix defined as the infinite sum of a power series.

• $f(t) = e^t$.

Differential Equations	Power Series
$f'(t) = f(t)$	$f(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \dots$
$f(0) = 1$	$\frac{d}{dt}(t) = 1, \frac{d}{dt}\left(\frac{t^2}{2}\right) = t, \dots$
	$f(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}$

• $f(t) = e^{at}$.

Differential Equations	Power Series
$f'(t) = af(t)$	$f(t) = \sum_{n=0}^{\infty} \frac{a^n t^n}{n!}$
$f(0) = 1$	

- $F(t) = e^{At}$.
 - A matrix-valued function.
 - Ex. $F(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$
 - $F(\theta)A$ rotates points (arrows) of A by θ .

Differential Equations	Power Series
$F'(t) = Ae^{At}$	$F(t) = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \dots$
$F(0) = I$	$F(t) = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}$

Diagonalization of e^{At}

- Find an alternate form for e^{At} by manipulating its power series definition:

$$\begin{aligned}
 e^{At} &= \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \frac{S \Lambda^n S^{-1} t^n}{n!} \\
 &= \sum_{n=0}^{\infty} S \left(\frac{\Lambda^n t^n}{n!} \right) S^{-1} \\
 &= S \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \Lambda^n \right) S^{-1} \\
 &= S \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix}^n \right) S^{-1} \\
 &= S \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} \begin{bmatrix} \lambda_1^n & & \\ & \ddots & \\ & & \lambda_k^n \end{bmatrix} \right) S^{-1} \\
 &= S \left(\sum_{n=0}^{\infty} \begin{bmatrix} \frac{t^n}{n!} \lambda_1^n & & \\ & \ddots & \\ & & \frac{t^n}{n!} \lambda_k^n \end{bmatrix} \right) S^{-1} \\
 &= S \left(\sum_{n=0}^{\infty} \begin{bmatrix} \frac{\lambda_1^n t^n}{n!} & & \\ & \ddots & \\ & & \frac{\lambda_k^n t^n}{n!} \end{bmatrix} \right) S^{-1} \\
 &= S \begin{bmatrix} \sum_{n=0}^{\infty} \frac{\lambda_1^n t^n}{n!} & & \\ & \ddots & \\ & & \sum_{n=0}^{\infty} \frac{\lambda_k^n t^n}{n!} \end{bmatrix} S^{-1} \\
 &= S \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_k t} \end{bmatrix} S^{-1} \\
 &= S e^{\Lambda t} S^{-1} \\
 &= F(t)
 \end{aligned}$$

2/13:

- Prove, using the above result, that $F'(t)$ can be defined in terms of $F(t)$:

$$\begin{aligned}
 F(t) &= e^{At} \\
 &= S \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_k t} \end{bmatrix} S^{-1} \\
 F'(t) &= \frac{d}{dt} \left(S \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_k t} \end{bmatrix} S^{-1} \right) \\
 &= S \frac{d}{dt} \left(\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_k t} \end{bmatrix} \right) S^{-1} \\
 &= S \begin{bmatrix} \frac{d}{dt} e^{\lambda_1 t} & & \\ & \ddots & \\ & & \frac{d}{dt} e^{\lambda_k t} \end{bmatrix} S^{-1} \\
 &= S \begin{bmatrix} \lambda_1 e^{\lambda_1 t} & & \\ & \ddots & \\ & & \lambda_k e^{\lambda_k t} \end{bmatrix} S^{-1} \\
 &= S \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_k t} \end{bmatrix} S^{-1} \\
 &= S \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} I_k \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_k t} \end{bmatrix} S^{-1} \\
 &= S \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} S^{-1} S \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_k t} \end{bmatrix} S^{-1} \\
 &= S \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} S^{-1} S \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_k t} \end{bmatrix} S^{-1} \\
 &= A F(t) \\
 &= A e^{At}
 \end{aligned}$$

- In other words, $y'(t) = Ay(t)$ and $y(0) = y_0$. The solution is $y = e^{At}y_0$.
- Example:

$$\begin{aligned}
 y_1' &= 5y_1 + y_2 & y_1(0) &= -3 \\
 y_2' &= -2y_1 + 2y_2 & y_2(0) &= 8
 \end{aligned}$$

$$\begin{aligned}
 y(t) &= e^{At}y(0) \\
 &= S e^{At} S^{-1} y(0)
 \end{aligned}$$

$$\begin{aligned}
 0 &= |A - \lambda I| \\
 &= \begin{vmatrix} 5 - \lambda & 1 \\ -2 & 2 - \lambda \end{vmatrix} \\
 &= (\lambda - 3)(\lambda - 4)
 \end{aligned}$$

$$\begin{aligned}
- A - 3I &= \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \\
- N(A - 3I) &= \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\} \\
- A - 4I &= \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \\
- N(A - 4I) &= \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}
\end{aligned}$$

$$\begin{aligned}
y(t) &= Se^{\Lambda t} S^{-1} y(0) \\
&= \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{4t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 8 \end{bmatrix} \\
&= \begin{bmatrix} e^{3t} & -e^{4t} \\ -2e^{3t} & e^{4t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 8 \end{bmatrix} \\
&= \begin{bmatrix} -e^{3t} + 2e^{4t} & -e^{3t} + e^{4t} \\ 2e^{3t} - 2e^{4t} & 2e^{3t} - e^{4t} \end{bmatrix} \begin{bmatrix} -3 \\ 8 \end{bmatrix} \\
&= \begin{bmatrix} 3e^{3t} - 6e^{4t} - 8e^{3t} + 8e^{4t} \\ -6e^{3t} + 6e^{4t} + 16e^{3t} - 8e^{4t} \end{bmatrix} \\
&= \begin{bmatrix} -5e^{3t} + 2e^{4t} \\ 10e^{3t} - 2e^{4t} \end{bmatrix} \\
&= \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}
\end{aligned}$$