Complex Linear Independence: Decomplexification

4/7:

- When given a complex system of equations, it is necessary to **decomplexify** it.
- **Decomplexify**: To model a complex system of equations with a strictly real system for the purpose of applying the tenets of real linear algebra to it.
- Consider the following complex system of equations.

$$\begin{aligned}(2+i)x_1+(1+i)x_2 &= 3+6i\\(3-i)x_1+(2-2i)x_2 &= 7-i\end{aligned}$$

  - The solutions will be complex numbers: $x_1 = a_1 + ib_1$ and $x_2 = a_2 + ib_2$, where $a_1, a_2, b_1, b_2 \in \mathbb{R}$.

- Transform it into a matrix system of equations. Separate the real and complex parts, and factor out all instances of the imaginary number i so that it is a coefficient to any complex matrix.

$$\begin{bmatrix}2+i & 1+i\\3-i & 2-2i\end{bmatrix}\begin{bmatrix}a_1+ib_1\\a_2+ib_2\end{bmatrix}=\begin{bmatrix}3+6i\\7-i\end{bmatrix}$$

$$\left(\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}+\begin{bmatrix}i & i\\-i & -2i\end{bmatrix}\right)\left(\begin{bmatrix}a_1\\a_2\end{bmatrix}+\begin{bmatrix}ib_1\\ib_2\end{bmatrix}\right)=\left(\begin{bmatrix}3\\7\end{bmatrix}+\begin{bmatrix}6i\\-i\end{bmatrix}\right)$$

$$\underbrace{\left(\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}+i\begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}\right)}_{A}\underbrace{\left(\begin{bmatrix}a_1\\a_2\end{bmatrix}+i\begin{bmatrix}b_1\\b_2\end{bmatrix}\right)}_{x}=\underbrace{\left(\begin{bmatrix}3\\7\end{bmatrix}+i\begin{bmatrix}6\\-1\end{bmatrix}\right)}_{b}$$

- Foil the left side of the above equation[1].

$$\left(\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix}-\begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}\begin{bmatrix}b_1\\b_2\end{bmatrix}\right)+i\left(\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}\begin{bmatrix}b_1\\b_2\end{bmatrix}+\begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix}\right)=\begin{bmatrix}3\\7\end{bmatrix}+i\begin{bmatrix}6\\-1\end{bmatrix}$$

- Split the above system of equations into a real system of equations and a complex system of equations by setting equal to each other the real components of each side and the imaginary components of each side.

$$\begin{aligned}\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix}-\begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}\begin{bmatrix}b_1\\b_2\end{bmatrix}&=\begin{bmatrix}3\\7\end{bmatrix}\\\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}\begin{bmatrix}b_1\\b_2\end{bmatrix}+\begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix}&=\begin{bmatrix}6\\-1\end{bmatrix}\end{aligned}$$

- Multiply out the matrices above to yield a system of four equations.

$$\begin{aligned}2a_1+a_2-b_1-b_2&=3\\3a_1+2a_2+b_1+2b_2&=7\\a_1+a_2+2b_1+b_2&=6\\-a_1-2a_2+3B_1+2B_2&=-1\end{aligned}$$

- Condense the above system of equations into a single matrix system of equations.

$$\begin{bmatrix}2 & 1 & -1 & -1\\3 & 2 & 1 & 2\\1 & 1 & 2 & 1\\-1 & -2 & 3 & 2\end{bmatrix}\begin{bmatrix}a_1\\a_2\\b_1\\b_2\end{bmatrix}=\begin{bmatrix}3\\7\\6\\-1\end{bmatrix}$$

[1]Note that the minus sign appears in the real component because, when multiplying the two “last” parts, $i^2=-1$.Note that the minus sign appears in the real component because, when multiplying the two “last” parts, $i^2=-1$.

- Solve for a_1, a_2, b_1, and b_2 using an augmented matrix and Gauss-Jordan elimination.

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & -1 & 3 \\ 3 & 2 & 1 & 2 & 7 \\ 1 & 1 & 2 & 1 & 6 \\ -1 & -2 & 3 & 2 & -1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}$$

- From these four values, the original solutions $x_1 = a_1 + ib_1$ and $x_2 = a_2 + ib_2$ can be found.

$$x_1 = 1 + 2i$$
$$x_2 = 2 - i$$