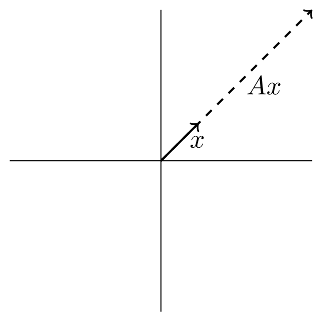
1/28: Introduction to Eigenvalues and Eigenvectors

- $Ax = b = \lambda x$
- $Ax = \lambda x$, $\lambda \in \mathbb{F}$, $x \in \mathbb{R}^n$
- λ is an eigenvalue. λx is an eigenvector.

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

- $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with corresponding eigenvalue of 4.
- $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



$$\begin{aligned} Ax &= \lambda x \\ Ax - \lambda x &= 0 \\ Ax - \lambda I x &= 0 \\ (A - \lambda I)x &= 0 \end{aligned}$$

- $(A - \lambda I)x = 0 \Rightarrow x \in N(A - \lambda I)$[1] $\Rightarrow |A - \lambda I| = 0$
- $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{bmatrix}$
- $\begin{vmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{vmatrix} = 0$

$$\begin{aligned} 0 &= (3 - \lambda)^2 - 1^2 \\ &= 3^2 - 6\lambda + \lambda^2 - 1 \\ &= \lambda^2 - 6\lambda + 8 \\ &= (\lambda - 4)(\lambda - 2) \end{aligned}$$

- $\lambda = 4, 2$.
- $\lambda^2 - 6\lambda + 8$ is the **characteristic polynomial** of A.
- $A - 2I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in N(A - 2I)$.
- $A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in N(A - 4I)$.

[1] To have a null space, $A - \lambda I$ has free columns.

- "Eigenspace" is not $\mathbb{R}^2$, but two lines in $\mathbb{R}^2$, specifically $y = \pm x$.
  - $y = \pm x$ comes from $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

1/29:

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\begin{aligned} P(\lambda) &= |A - \lambda I| \\ &= \begin{vmatrix} 2-\lambda & -2 & 3 \\ 0 & 3-\lambda & -2 \\ 0 & -1 & 2-\lambda \end{vmatrix} \\ &= -1 \begin{vmatrix} 2-\lambda & 3 \\ 0 & -2 \end{vmatrix} (-1)^{3+2} + (2-\lambda) \begin{vmatrix} 2-\lambda & -2 \\ 0 & 3-\lambda \end{vmatrix} (-1)^{3+3} \\ &= ((2-\lambda)(-2)) + (2-\lambda)((2-\lambda)(3-\lambda)) \\ &= -4 + 2\lambda + (2-\lambda)^2(3-\lambda) \\ &= -4 + 2\lambda + (4 - 4\lambda + \lambda^2)(3-\lambda) \\ &= -4 + 2\lambda + 12 - 4\lambda - 12\lambda + 4\lambda^2 + 3\lambda^2 - \lambda^3 \\ &= -\lambda^3 + 7\lambda^2 - 14\lambda + 8 \\ &= -(\lambda-1)(\lambda-2)(\lambda-4) \end{aligned}$$

$$A - I = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -2 \\ 0 & -1 & 1 \end{bmatrix} \qquad A - 2I = \begin{bmatrix} 0 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & -1 & 0 \end{bmatrix} \qquad A - 4I = \begin{bmatrix} -2 & -2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix}$$

- $P(\lambda)$ is positive when $n \in 2\mathbb{N}$, negative otherwise.
  - Signs flip term to term (think about binomial expansion).
- Coefficients of the $n-1$ degree term is the sum of the diagonal entries.
- Coefficient of the 0^{th} degree term is $|A|$.
  - $P_\lambda(0) = |A - 0 \cdot I| = |A|$.
- Product of the eigenvalues is $|A|$.
  - Think about expanding the factorization.
- Eigenvalues of U are the diagonal values.
  - $\lambda_1 \lambda_2 \cdots \lambda_n = |A|$, which is the product of the diagonal entries.
  - $\lambda_1 + \cdots + \lambda_n = \text{trace}(A)$, which is the sum of the diagonal entries.
- $Ax = \lambda x$
  - $A^2 x = AAx = A\lambda x = \lambda Ax = \lambda\lambda x = \lambda^2 x$

Similarity

1/30:

- $A \sim B$[2] iff $\exists\, S : A = SBS^{-1},\ B = S^{-1}AS$.
  1. If $A \sim B$, then $|A| = |B|$.

$$\begin{aligned} B &= S^{-1}AS \\ |B| &= |S^{-1}AS| \\ |B| &= |S^{-1}||A||S| \\ |B| &= \frac{1}{|S|}|A||S| \\ |B| &= |A| \end{aligned}$$

  2. If $A \sim B$, then they share the same characteristic polynomial.

$$\begin{aligned} B &= S^{-1}AS \\ |B - \lambda I| &= |S^{-1}AS - \lambda I| \\ &= |S^{-1}AS - \lambda S^{-1}IS| \\ &= |S^{-1}S(A - \lambda I)| \\ &= |I(A - \lambda I)| \\ |B - \lambda I| &= |A - \lambda I| \end{aligned}$$

     - If they have the same characteristic polynomial, $\therefore$ A and B have the same eigenvalues.
- What is the best possible B if $A \sim B$?
  - Sparse.
  - Diagonal.
  - $A = [\text{ugly}] \quad \rightarrow \quad B = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = \Lambda$
- **Diagonalization**:

$$\begin{aligned} A &= S\Lambda S^{-1} \\ AS &= S\Lambda \\ \Lambda &= S^{-1}AS \end{aligned}$$

- $A = S\Lambda S^{-1}$
  - $A^2 = AA = S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda\Lambda S^{-1} = S\Lambda^2 S^{-1}$
  - $A^k = S\Lambda^k S^{-1}$
  - $A^k = S \begin{bmatrix} {\lambda_1}^k & & 0 \\ & \ddots & \\ 0 & & {\lambda_n}^k \end{bmatrix} S^{-1}$
- Diagonalize the following matrix A.

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

[2] A "is similar to" B

- Find the characteristic polynomial.

$$\begin{aligned}|A-\lambda I| &= \begin{vmatrix}-1-\lambda & 0 & 1\\ 3 & 0-\lambda & -3\\ 1 & 0 & -1-\lambda\end{vmatrix}\\ &= (-1-\lambda)((-\lambda)(-1-\lambda))+(-1)(-\lambda)\\ &= -\lambda(-1-\lambda)^2+\lambda\\ &= -\lambda(1+2\lambda+\lambda^2)+\lambda\\ &= -\lambda^3-2\lambda^2\\ &= -\lambda^2(\lambda+2)\end{aligned}$$

- Find the eigenvalues: $\lambda_1=\lambda_2=0$, $\lambda_3=-2$
- **Algebraic multiplicity** of λ_1,λ_2 is 2.
- A.M. of λ_3 is 1.
- $A-0I=\begin{bmatrix}-1 & 0 & 1\\ 3 & 0 & -3\\ 1 & 0 & -1\end{bmatrix}$
- $\text{rank}(A-0I)=1 \Rightarrow \dim(N(A-0I))=2$
- The 2 directly above is the **geometric multiplicity**.
- A is diagonalizable iff A.M. of λ_i = G.M.

1/31: - Eigenvectors are $x_1=\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $x_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$.
- $A+2I=\begin{bmatrix}1 & 0 & 1\\ 3 & 2 & -3\\ 1 & 0 & 1\end{bmatrix}$
- Eigenvector is $x_3=\begin{bmatrix}1\\-3\\-1\end{bmatrix}$
- Use an S matrix of eigenvectors.
- $A=S\Lambda S^{-1}=\frac{1}{2}\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & -3\\ 0 & 1 & -1\end{bmatrix}\begin{bmatrix}0 & & \\ & 0 & \\ & & -2\end{bmatrix}\begin{bmatrix}3 & 2 & -3\\ 1 & 0 & 1\\ 1 & 0 & -1\end{bmatrix}$
- Note that $A^{9752}=\frac{1}{2}\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & -3\\ 0 & 1 & -1\end{bmatrix}\begin{bmatrix}0 & & \\ & 0 & \\ & & (-2)^{9752}\end{bmatrix}\begin{bmatrix}3 & 2 & -3\\ 1 & 0 & 1\\ 1 & 0 & -1\end{bmatrix}$

- **Algebraic multiplicity**: The number of repeated roots to a polynomial. For all of the roots, it adds up to n (n-square matrix). *Also known as* **A.M.**
- **Geometric multiplicity**: The number of eigenvectors produced from each root. For all of the roots, it may not add up to n (n-square matrix). $\dim(N(A-\lambda I))$. *Also known as* **G.M.**
- A nondiagonalizable example:

$$A=\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 4\end{bmatrix}$$

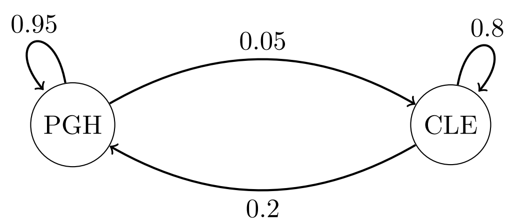
  - $\lambda_1=\lambda_2=1$ and $\lambda_3=4$.
  - λ_1 and λ_2 have A.M. = 2.

- λ_3 has A.M. $= 1$.
- $A - I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}$
- $\text{rank}(A - I) = 2 \Rightarrow \dim(N(A - I)) = 1 \Rightarrow$ G.M. $= 1$.
- $x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
- $A - 4I = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
- $x_2 = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$
- S would be 3×2 and, thus, not square, so $\nexists\ S^{-1}$[3].

- **Canonical** (form): An accepted way of expressing something.

Markov Chains

2/3:



- $u_0 = \begin{bmatrix} 500,000 \\ 500,000 \end{bmatrix}$
- $Au_0 = u_1$.
- $Au_1 = u_2$, $A(Au_0) = u_2$, $A^2u_0 = u_2$, $A^ku_0 = u_k$, $(S\Lambda S^{-1})^ku_0 = u_k$, $S\Lambda^kS^{-1}u_0 = u_k$.

$$A = \begin{bmatrix} 0.95 & 0.20 \\ 0.05 & 0.80 \end{bmatrix} \qquad\qquad u_k = \begin{bmatrix} \text{PGH} \\ \text{CLE} \end{bmatrix}$$

- A is a **Markov matrix**, where all columns and rows add to 1.
- $Au_0 = \begin{bmatrix} 0.95 & 0.20 \\ 0.05 & 0.80 \end{bmatrix} \begin{bmatrix} 500,000 \\ 500,000 \end{bmatrix} = u_1$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 0.95 - \lambda & 0.20 \\ 0.05 & 0.80 - \lambda \end{vmatrix} \\ &= (0.95 - \lambda)(0.8 - \lambda) - (0.2)(0.05) \\ &= (\lambda - 1)(\lambda - 0.75) \end{aligned}$$

- $\lambda_1 = 1,\ \lambda_2 = 0.75$.

[3]At a later date, we will look at an analogy of projections to diagonalization that finds the "best possible" diagonalization (which may not be perfectly diagonal).

- $A - I = \begin{bmatrix} -0.05 & 0.2 \\ 0.05 & -0.2 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$
- $A - 0.75I = \begin{bmatrix} 0.2 & 0.2 \\ 0.05 & 0.05 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\begin{aligned} A^k u_0 &= \frac{1}{5} \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.75 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 500,000 \\ 500,000 \end{bmatrix} \\ &= \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.75^k \end{bmatrix} \begin{bmatrix} 200,000 \\ 300,000 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 1 \end{bmatrix} (200,000) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} (0.75)^k (300,000) \end{aligned}$$

- $\begin{bmatrix} 800,000 \\ 200,000 \end{bmatrix}$ is the steady-state vector.
- $\begin{bmatrix} -(0.75)^k(300,000) \\ (0.75)^k(300,000) \end{bmatrix}$ is the dynamically changing vector.
- $\lim_{k\to\infty} A^k u_0 = \begin{bmatrix} 800,000 \\ 200,000 \end{bmatrix} = \begin{bmatrix} \text{PGH} \\ \text{CLE} \end{bmatrix}$

Explicit Formula for the Fibonacci Sequence

2/4:

- 1, 1, 2, 3, 5, 8, ...
- Recursively defined formula: F_n[4]$= F_{n-1} + F_{n-2}$.

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} \\ F_{n-1} &= F_{n-1} \\ \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix} \end{aligned}$$

- $u_n = A^n u_0 = S\Lambda^n S^{-1} u_0$.

$$\begin{aligned} 0 &= |A - \lambda I| \\ &= \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} \\ &= -\lambda(1-\lambda) - 1 \\ &= \lambda^2 - \lambda - 1 \end{aligned}$$

- $\lambda = \frac{1\pm\sqrt{5}}{2}$[5].
- $\lambda_1 = \frac{1+\sqrt{5}}{2}$.

$$\begin{aligned} N(A - \lambda_1 I) &= N\left(\begin{bmatrix} 1 - \frac{1+\sqrt{5}}{2} & 1 \\ 1 & -\frac{1+\sqrt{5}}{2} \end{bmatrix} \right) \\ &= N\left(\begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{-1-\sqrt{5}}{2} \end{bmatrix} \right) \end{aligned}$$

[4] The n-th Fibonacci number.
[5] This is the Golden ratio!

- $\begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{-1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- Let $x_2 = 1$.

$$\begin{aligned} \frac{1-\sqrt{5}}{2} x_1 + 1 &= 0 \\ \frac{1-\sqrt{5}}{2} x_1 &= -\frac{2}{2} \\ x_1 &= \frac{-2}{1-\sqrt{5}} \times \frac{1+\sqrt{5}}{1+\sqrt{5}} \\ &= \frac{-2-2\sqrt{5}}{-4} \\ &= \frac{1+\sqrt{5}}{2} \end{aligned}$$

- $s_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$, $s_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$.
- $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} S^{-1}$
- $S^{-1} = \frac{1}{|S|} C_S^{\mathrm{T}} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}$
- $u_k = A^k u_0 = S \Lambda^k S^{-1} u_0$.

$$\begin{aligned} S^{-1} u_0 &= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ \frac{-1+\sqrt{5}}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1+\sqrt{5}}{2\sqrt{5}} \\ \frac{-1+\sqrt{5}}{2\sqrt{5}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{5+\sqrt{5}}{10} \\ \frac{5-\sqrt{5}}{10} \end{bmatrix} \end{aligned}$$

- $u_k = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^k \left(\frac{5+\sqrt{5}}{10}\right) + \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} \left(\frac{1-\sqrt{5}}{2}\right)^k \left(\frac{5-\sqrt{5}}{10}\right)$

Systems of First-Order Ordinary Differential Equations

2/11:

- Let $f(x) = y$ and $a, c, K \in \mathbb{F}$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= ay \\
\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= a \\
\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x &= a\,\mathrm{d}x \\
\frac{1}{y}\,\mathrm{d}y &= a\,\mathrm{d}x \\
\int \frac{1}{y}\,\mathrm{d}y &= \int a\,\mathrm{d}x \\
\ln y &= ax + c \\
y &= \mathrm{e}^{ax+c} \\
&= \mathrm{e}^{ax}\mathrm{e}^{c} \\
&= K\mathrm{e}^{ax}
\end{aligned}$$

- Let $\frac{\mathrm{d}y}{\mathrm{d}x} = y'$.
  - $y_1' = a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n$.
  - $y_2' = a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n$.
  - $\vdots$
  - $y_n' = a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n$.
- This is a **square system** of equations.
- Rewrite as $y' = Ay$.

$$\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

- Solve the following system of differential equations.

$$\begin{aligned}
y_1' &= 3y_1 \\
y_2' &= -2y_2 \\
y_3' &= 5y_3
\end{aligned}$$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

  - General Solution:

$$\begin{aligned}
y_1 &= k_1\mathrm{e}^{3x} \\
y_2 &= k_2\mathrm{e}^{-2x} \\
y_3 &= k_3\mathrm{e}^{5x}
\end{aligned}$$

  - Particular Solution (where $y_1(0) = 2$, $y_2(0) = -1$, and $y_3(0) = 7$ are the initial conditions):

$$\begin{aligned} y_1 &= 2\mathrm{e}^{3x} \\ y_2 &= -\mathrm{e}^{-2x} \\ y_3 &= 7\mathrm{e}^{5x} \end{aligned}$$

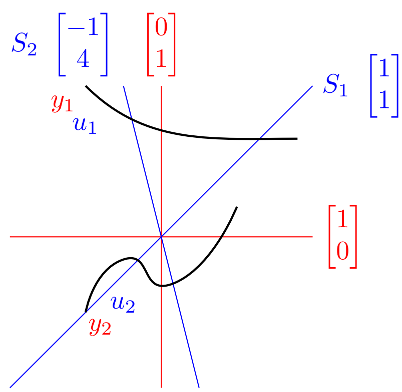
- Consider a different system. Remember throughout that we are solving for y.

$$\begin{aligned} y_1' &= y_1 + y_2 \\ y_2' &= 4y_1 - 2y_2 \end{aligned}$$

  - The previous system was so easy to solve because the matrix was diagonal. This one (as follows) will not be. Therefore, we should diagonalize it.

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

  - Start with $y' = Ay$.
  - Substitute $y = Su$.
    - Note that $y = Su \Rightarrow y' = Su'$[6].
    - If we can find u' in terms of a diagonal matrix and u, we can solve for y.



  - We seek to find a new basis S such that the matrix scaling u will be diagonal.

$$\begin{aligned} Su' &= Ay \\ Su' &= ASu \\ u' &= S^{-1}ASu \\ u' &= \Lambda u \end{aligned}$$

  - The last substitution above is legal because if $A = S\Lambda S^{-1}$, then $\Lambda = S^{-1}AS$.

$$\begin{aligned} 0 &= \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} \\ &= (1-\lambda)(-2-\lambda) - 4 \\ &= -2 - \lambda + 2\lambda + \lambda^2 - 4 \\ &= \lambda^2 + \lambda - 6 \\ &= (\lambda - 2)(\lambda + 3) \end{aligned}$$

$\lambda_1 = 2$ $\lambda_2 = -3$

[6]Think about differentiating both sides: $y \to y'$ is obvious, S will be unchanged because it's just coefficients, and the functions of u will be differentiated.

- $A - 2I = \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- $A + 3I = \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$u' = \Lambda u$$
$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$u_1 = k_1 \mathrm{e}^{2x}$$
$$u_2 = k_2 \mathrm{e}^{-3x}$$

$$y = Su$$
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} k_1 \mathrm{e}^{2x} \\ k_2 \mathrm{e}^{-3x} \end{bmatrix}$$

$$y_1 = k_1 \mathrm{e}^{2x} - k_2 \mathrm{e}^{-3x}$$
$$y_2 = k_1 \mathrm{e}^{2x} + 4k_2 \mathrm{e}^{-3x}$$

2/12:

- Initial conditions: $y_1(0) = 1$ and $y_2(0) = 6$.
  - Use augmented matrices to solve a system of equations.

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 4 & 6 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

- Particular solution:

$$y_1 = 2\mathrm{e}^{2x} - \mathrm{e}^{-3x}$$
$$y_2 = 2\mathrm{e}^{2x} + 4\mathrm{e}^{-3x}$$

Matrix Exponentiation

- e^A is a matrix defined as the infinite sum of a power series.
- $f(t) = \mathrm{e}^t$.

Differential Equations	**Power Series**
$f'(t) = f(t)$	$f(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \cdots$
$f(0) = 1$	$\frac{\mathrm{d}}{\mathrm{d}t}(t) = 1,\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^2}{2}\right) = t,\ \ldots$
	$f(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}$

- $f(t) = \mathrm{e}^{at}$.

Differential Equations	**Power Series**
$f'(t) = af(t)$	$f(t) = \sum_{n=0}^{\infty} \frac{a^n t^n}{n!}$
$f(0) = 1$	

- $F(t) = \mathrm{e}^{At}$.
  - A matrix-valued function.
  - Ex. $F(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$
  - $F(\theta)A$ rotates points (arrows) of A by θ.

Differential Equations	**Power Series**
$F'(t) = A\mathrm{e}^{At}$	$F(t) = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$
$F(0) = I$	$F(t) = \sum_{n=0}^{\infty} \frac{A^nt^n}{n!}$

Diagonalization of e^{At}

- Find an alternate form for e^{At} by manipulating its power series definition:

$$\begin{aligned}
\mathrm{e}^{At} &= \sum_{n=0}^{\infty} \frac{A^nt^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{S\Lambda^nS^{-1}t^n}{n!} \\
&= \sum_{n=0}^{\infty} S\left(\frac{\Lambda^nt^n}{n!}\right)S^{-1} \\
&= S\left(\sum_{n=0}^{\infty} \frac{t^n}{n!}\Lambda^n\right)S^{-1} \\
&= S\left(\sum_{n=0}^{\infty} \frac{t^n}{n!}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix}^n\right)S^{-1} \\
&= S\left(\sum_{n=0}^{\infty} \frac{t^n}{n!}\begin{bmatrix} \lambda_1^n & & \\ & \ddots & \\ & & \lambda_k^n \end{bmatrix}\right)S^{-1} \\
&= S\left(\sum_{n=0}^{\infty} \begin{bmatrix} \frac{t^n}{n!}\lambda_1^n & & \\ & \ddots & \\ & & \frac{t^n}{n!}\lambda_k^n \end{bmatrix}\right)S^{-1} \\
&= S\left(\sum_{n=0}^{\infty} \begin{bmatrix} \frac{\lambda_1^nt^n}{n!} & & \\ & \ddots & \\ & & \frac{\lambda_k^nt^n}{n!} \end{bmatrix}\right)S^{-1} \\
&= S\begin{bmatrix} \sum_{n=0}^{\infty} \frac{\lambda_1^nt^n}{n!} & & \\ & \ddots & \\ & & \sum_{n=0}^{\infty} \frac{\lambda_k^nt^n}{n!} \end{bmatrix}S^{-1} \\
&= S\begin{bmatrix} \mathrm{e}^{\lambda_1t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_kt} \end{bmatrix}S^{-1} \\
&= S\mathrm{e}^{\Lambda t}S^{-1} \\
&= F(t)
\end{aligned}$$

2/13:

- Prove, using the above result, that $F'(t)$ can be defined in terms of $F(t)$:

$$\begin{aligned} F(t) &= \mathrm{e}^{At} \\ &= S \begin{bmatrix} \mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_k t} \end{bmatrix} S^{-1} \end{aligned}$$

$$\begin{aligned} F'(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \left(S \begin{bmatrix} \mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_k t} \end{bmatrix} S^{-1} \right) \\ &= S \frac{\mathrm{d}}{\mathrm{d}t} \left(\begin{bmatrix} \mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_k t} \end{bmatrix} \right) S^{-1} \\ &= S \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\lambda_k t} \end{bmatrix} S^{-1} \\ &= S \begin{bmatrix} \lambda_1 \mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \lambda_k \mathrm{e}^{\lambda_k t} \end{bmatrix} S^{-1} \\ &= S \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} \begin{bmatrix} \mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_k t} \end{bmatrix} S^{-1} \\ &= S \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} I_k \begin{bmatrix} \mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_k t} \end{bmatrix} S^{-1} \\ &= {\color{red} S \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} S^{-1}} {\color{blue} S \begin{bmatrix} \mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_k t} \end{bmatrix} S^{-1}} \\ &= {\color{red} A} {\color{blue} F(t)} \\ &= A\mathrm{e}^{At} \end{aligned}$$

- In other words, $y'(t) = Ay(t)$ and $y(0) = y_0$. The solution is $y = \mathrm{e}^{At} y_0$.
- Example:

$$\begin{aligned} y_1' &= 5y_1 + y_2 & y_1(0) &= -3 \\ y_2' &= -2y_1 + 2y_2 & y_2(0) &= 8 \end{aligned}$$

$$\begin{aligned} y(t) &= \mathrm{e}^{At} y(0) \\ &= S\mathrm{e}^{\Lambda t} S^{-1} y(0) \end{aligned}$$

$$\begin{aligned} 0 &= |A - \lambda I| \\ &= \begin{vmatrix} 5-\lambda & 1 \\ -2 & 2-\lambda \end{vmatrix} \\ &= (\lambda - 3)(\lambda - 4) \end{aligned}$$

- $A - 3I = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$
- $N(A - 3I) = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$
- $A - 4I = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}$
- $N(A - 4I) = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

$$\begin{aligned}
y(t) &= S\mathrm{e}^{\Lambda t}S^{-1}y(0) \\
&= \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{e}^{3t} & 0 \\ 0 & \mathrm{e}^{4t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 8 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{3t} & -\mathrm{e}^{4t} \\ -2\mathrm{e}^{3t} & \mathrm{e}^{4t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 8 \end{bmatrix} \\
&= \begin{bmatrix} -\mathrm{e}^{3t} + 2\mathrm{e}^{4t} & -\mathrm{e}^{3t} + \mathrm{e}^{4t} \\ 2\mathrm{e}^{3t} - 2\mathrm{e}^{4t} & 2\mathrm{e}^{3t} - \mathrm{e}^{4t} \end{bmatrix} \begin{bmatrix} -3 \\ 8 \end{bmatrix} \\
&= \begin{bmatrix} 3\mathrm{e}^{3t} - 6\mathrm{e}^{4t} - 8\mathrm{e}^{3t} + 8\mathrm{e}^{4t} \\ -6\mathrm{e}^{3t} + 6\mathrm{e}^{4t} + 16\mathrm{e}^{3t} - 8\mathrm{e}^{4t} \end{bmatrix} \\
&= \begin{bmatrix} -5\mathrm{e}^{3t} + 2\mathrm{e}^{4t} \\ 10\mathrm{e}^{3t} - 2\mathrm{e}^{4t} \end{bmatrix} \\
&= \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}
\end{aligned}$$

Orthonormally Diagonalizable Matrices

2/19:

- $A = Q\Lambda Q^{\mathrm{T}}$.
  - Eigenvectors are orthonormal.
- $A^{\mathrm{T}} = \left(Q\Lambda Q^{\mathrm{T}}\right)^{\mathrm{T}} = Q^{\mathrm{TT}}\Lambda^{\mathrm{T}}Q^{\mathrm{T}} = Q\Lambda Q^{\mathrm{T}} = A$.
- Prove that the symmetric matrices are exactly those that are orthonormally diagonalizable.
  - Let $A = A^{\mathrm{T}}$.

$$Ax_1 = \lambda_1 x_1 \tag{1}$$
$$Ax_2 = \lambda_2 x_2 \tag{2}$$

  - Multiply Equation 1 by x_2^{T} from Equation 2.
    - We have to relate the two equations.
    - Later, we transpose, because we have to specifically target the properties of symmetric matrices.

$$\begin{aligned}
\lambda_1 x_2^{\mathrm{T}} x_1 &= x_2^{\mathrm{T}} A x_1 \\
&= (x_2^{\mathrm{T}} A) x_1 \\
&= \left(A^{\mathrm{T}} x_2\right)^{\mathrm{T}} x_1 \\
&= (Ax_2)^{\mathrm{T}} x_1 \\
\lambda_1 x_2^{\mathrm{T}} x_1 &= \lambda_2 x_2^{T} x_1 \\
\lambda_1 x_2^{\mathrm{T}} x_1 - \lambda_2 x_2^{T} x_1 &= 0 \\
x_2^{\mathrm{T}} x_1 (\lambda_1 - \lambda_2) &= 0
\end{aligned}$$

- The last line above implies that $x_2^{\mathrm{T}} x_1 = 0$ iff $\lambda_1 \neq \lambda_2$.

- The only matrices that we can guarantee will never have complex eigenvalues are symmetric matrices.
- On complex numbers/vectors:
  - $z = a + bi$ and $\bar{z} = a - bi$, where $a, b \in \mathbb{R}$, $i = \sqrt{-1}$.
  - $z\bar{z} = a^2 + b^2$.
  - $x = \begin{bmatrix} a_1 + b_1 i \\ \vdots \\ a_n + b_n i \end{bmatrix}$ and $\bar{x} = \begin{bmatrix} a_1 - b_1 i \\ \vdots \\ a_n - b_n i \end{bmatrix}$.
- Prove that when $A = A^{\mathrm{T}}$, $\lambda_n \in \mathbb{R}$.
  - Let $A = A^{\mathrm{T}}$, $A \in \mathbb{R}^n$.

$$\begin{aligned} Ax &= \lambda x \qquad (3) \\ \bar{A}\bar{x} &= \bar{\lambda}\bar{x} \end{aligned}$$

  - If $A \in \mathbb{R}^n$, then $A = \bar{A}$.

$$\begin{aligned} A\bar{x} &= \bar{\lambda}\bar{x} \\ (A\bar{x})^{\mathrm{T}} &= \left(\bar{\lambda}\bar{x}\right)^{\mathrm{T}} \\ \bar{x}^{\mathrm{T}} A^{\mathrm{T}} &= \bar{\lambda}\bar{x}^{\mathrm{T}} \\ \bar{x}^{\mathrm{T}} A &= \bar{\lambda}\bar{x}^{\mathrm{T}} \qquad (4) \end{aligned}$$

  - Multiply Equation 3 by $\bar{x}^{\mathrm{T}}$ from the left.
    - $\bar{x}^{\mathrm{T}} A x = \lambda \bar{x}^{\mathrm{T}} x$.
  - Multiply Equation 4 by x from the right.
    - $\bar{x}^{\mathrm{T}} A x = \bar{\lambda} \bar{x}^{\mathrm{T}} x$.
  - $\lambda \bar{x}^{\mathrm{T}} x = \bar{\lambda} \bar{x}^{\mathrm{T}} x \Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}$.

Spectral Decomposition

2/20:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

- $\lambda_1 = 4$, $\lambda_2 = \lambda_3 = 1$.
- $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $x_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.
  - $x_1^{\mathrm{T}} x_2 = 0$, $x_1^{\mathrm{T}} x_3 = 0$, $x_2^{\mathrm{T}} x_3 = -1$.
- Orthogonalize by Gram-Schmidt, inspection, put the vectors in a matrix and find the null space (the null vector will be orthogonal by the fundamental theorem).
- $x_3' = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$.
  - x_3' is not scaled along its line by A, but it is scaled in the plane of x_2 and x_3 by A.

  - $x_1^{\mathrm{T}} x_3' = 0$, $x_2^{\mathrm{T}} x_3' = 0$.

- $q_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $q_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$.

$$\begin{aligned} A &= Q\Lambda Q^{\mathrm{T}} \\ &= \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} — & q_1^{\mathrm{T}} & — \\ & \vdots & \\ — & q_n^{\mathrm{T}} & — \end{bmatrix} \\ &= \begin{bmatrix} | & & | \\ q_1\lambda_1 & \cdots & q_n\lambda_n \\ | & & | \end{bmatrix} \begin{bmatrix} — & q_1^{\mathrm{T}} & — \\ & \vdots & \\ — & q_n^{\mathrm{T}} & — \end{bmatrix} \\ &= \lambda_1 q_1 q_1^{\mathrm{T}} + \cdots + \lambda_n q_n q_n^{\mathrm{T}} \end{aligned}$$

- $q_1 q_1^{\mathrm{T}} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$

- $q_2 q_2^{\mathrm{T}} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$

- $q_3 q_3^{\mathrm{T}} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{bmatrix}$

- $A = 4 \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{bmatrix}$

- $A = 4 \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} + \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$ is the spectral decomposition of A.