NUMERICAL LINEAR ALGEBRA

SPEED VS. ACCURACY IN FINDING SOLUTIONS WHEN SOLVING “REAL” PROBLEMS

Roundoff Error

Computers and calculators store real numbers in “floating-point form.”

- 2001 would be stored as 0.2001×10^4
- -0.0063 would be stored as -0.63×10^{-3}
- Generally the floating-point form of a number is $\pm M \times 10^k$ where k is an integar and M (mantissa) satisfies $0.1 \le M < 1$
- Depending upon the computer, calculator, or CAS, the number of significant digits (number of decimal places that can be stored) may vary between 8 and 12.
- For example, $\pi \approx 3.141592654$. If a calculator can store 5 significant digits, π would be stored as 0.31416×10^1.
- This truncation of π will introduce roundoff error.

EXAMPLES

$$x + y = 0$$
$$x + \frac{801}{800} y = 1$$

$$4.552x + 7.083y = 1.931$$
$$1.731x + 2.693y = 2.001$$

Experiment with exact solutions, then rounding to five digits, then four, then three.

These systems that are sensitive to roundoff error are called ***ill-conditioned.***

Partial Pivoting: A Way to Reduce/Eliminate Roundoff Error

EXAMPLE:

- Solve for $x: 0.00021x = 1$
- Then suppose a calculator can only carry for significant digits. You would be solving $0.0002x = 1$
- The difference between these answers is the effect of an error of 0.00001 on the solution of an equation.

IN A SYSTEM OF EQUATIONS:

- Solve $\begin{aligned} 0.400x + 99.6y &= 100 \\ 75.3x - 45.3y &= 30.0 \end{aligned}$ using three significant digits in each calculation.
- The result is $\begin{bmatrix} -1.00 \\ 1.01 \end{bmatrix}$ but the actual result should be $\begin{bmatrix} 1.00 \\ 1.00 \end{bmatrix}$!!!
- Resolve $\begin{aligned} 0.400x + 99.6y &= 100 \\ 75.3x - 45.3y &= 30.0 \end{aligned}$ by interchanging the two rows of the augmented matrix and take each solution to three significant digits again.

Choosing pivots matters!!! At each pivoting step, choose from among all possible pivots in a column the entry with the largest absolute value. Use row interchanges to bring this element into the correct position and use it to create zeros where needed in the column. This is ***partial pivoting***.

Use partial pivoting to solve the following systems using three significant digits.

$0.001x + 0.995y = 1.00$ $-10.2x + 1.00y = -50.0$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5.00 \\ 1.00 \end{bmatrix}$ Exact solutions listed above	$10x - 7y = 7$ $-3x + 2.09 + 6z = 3.91$ $5x - y + 5z = 6$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.00 \\ -1.00 \\ 1.00 \end{bmatrix}$ Exact solutions listed above

Counting Operations

Consider the augmented matrix $\left[A \mid b\right]=\begin{bmatrix} 2 & 4 & 6 & 8 \\ 3 & 9 & 6 & 12 \\ -1 & 1 & -1 & 1 \end{bmatrix}$.

- How many operations (multiplications or divisions) are required to use Gaussian Elimination to bring the matrix to its row-echelon form $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$?
- How many additional operations are required to perform the back substitution part of Gaussian elimination?
- How many operations are required to perform Gauss-Jordan elimination to bring the matrix to its reduced row-echelon form $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$?
- Which appears to be a more efficient algorithm?

<u>A generalized approach</u>

Let $\left[A \mid b\right]=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{bmatrix}$ and note that $\left[A \mid b\right]$ is $n\times(n\times 1)$.

- n operations would be required create the first leading 1 and then adjust all other entries in row 1.
- n operations would be required to obtain the first zero in column one and adjust all other entries in row 2.
- This will continue through row n, bringing the total number of operations up to the point that the entire first column is zeroed out to $n+(n-1)n$.
- This would have to continue with row 2 and column 2. The total number of operations needed to reach row-echelon form is
  $[n+(n-1)n]+[(n-1)+(n-2)(n-1)]+[(n-2)+(n-3)(n-2)]+\cdots+[2+1\ 2]+1$
  $=n^2+(n-1)^2+\cdots+2^2+1^2$
  - Then it's time for back substitution. The number of operations required for this would be $1+2+\cdots+(n-1)$

Therefore, using summation, the total number of operations performed by Guassian Elmination is $S(n)=\frac{1}{3}n^3+n^2-\frac{1}{3}n$

(Gauss-Jordan, by comparison $S(n)\approx\frac{1}{2}n^3$ and using the inverse requires n^3)