Steven Labalme

(H) Linear Algebra, 3

Chapter 2: Solving Linear Equations (Notes 1) Forsyth

9/9: Gaussian Elimination

- $Ax = b$ — Capital letter denotes matrix; lowercase denotes vector.

$$\begin{array}{c} R_1 \\ -R_2 \\ R_3 \end{array}\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -9 \end{bmatrix}$$

$$- \begin{cases} 2x_1 + x_2 + x_3 = 5 & R_1 \cdot x = 5 \\ 4x_1 - 6x_2 = -2 & R_2 \cdot x = -2 \\ -2x_1 + 7x_2 + 2x_3 = -9 & R_3 \cdot x = -9 \end{cases}$$

$$- x_1 \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -9 \end{bmatrix}$$

- Three elementary row operations
  - Scaling an entire row by a scalar multiple.
  - Adding a multiple of one row to another.
  - Swapping two rows.
- A is an $m \times n$ matrix: m rows and n columns.
  - An entry has the form $a_{i,j}$, where i is the row number and j is the column number.

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,j} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,j} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i,1} & a_{i,2} & \cdots & a_{i,j} \end{bmatrix}$$

- $A \xrightarrow[\text{using E.R.O.'s}]{\text{row reduce}} u$ then back-substitution
  - u is an upper triangular matrix.
- Upper triangular matrix: Every entry below the top-left to bottom-right diagonal is 0.

- $$A|b = \left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & -9 \end{array}\right]$$
  - An augmented matrix.
  - $A|b \longrightarrow u|c$, then back substitution

- $\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & -9 \end{array}\right] \xrightarrow[R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & -4 \end{array}\right] \xrightarrow{R_2+R_3} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & -16 \end{array}\right] = u|c$
  - $x_3 = -16$
  - $-8x_2 - 2x_3 = -12, \quad x_2 = \frac{44}{8} = 5.5$
  - $2x_1 + x_2 + x_3 = 5, \quad x_1 = \frac{-21}{4}$

9/10:

- **Pivots** are 2, -8, and 1.
  - Use pivots to get 0's
- ref = Row Echelon Form of a matrix.
- rref = Reduced-Row Echelon Form of a matrix.

$$A = \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 2 & 4 & 0 & 0 & 2 \\ 2 & 3 & 2 & 1 & 5 \\ -1 & -1 & 3 & 6 & 5 \end{bmatrix}$$

- $A \xrightarrow[\substack{-2R_1+R_3 \\ R_1+R_4}]{-2R_1+R_2} \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 1 & -1 & 2 & 10 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 1 & -1 & 2 & 10 \end{bmatrix}$

- $\xrightarrow{R_2+R_4} \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 0 & 9 & 11 & 5 \end{bmatrix} \xrightarrow{-\frac{9}{8}R_3+R_4} \begin{bmatrix} 1 & 2 & -4 & -4 & 5 \\ 0 & -1 & 10 & 9 & -5 \\ 0 & 0 & 8 & 8 & -8 \\ 0 & 0 & 0 & 2 & 14 \end{bmatrix} = \text{ref}(A)$

- ref - leading entry in every row is first nonzero entry, and leading entry is to the right of the row above it.
  - $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 6 & 7 & 9 & 9 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is in ref.

Chapter 2: Solving Linear Equations (Notes 2) Forsyth

Gauss-Jordan Elimination

- $A|b \to u|c \to R|d$, when $R = rref(A)$
  - ref(A)
  - Leading entries scaled to 1.
  - All entries above the leading entries are 0.

9/11:

- $\begin{cases} x_1 + x_2 + x_3 = 4 \\ 3x_1 + 3x_2 - x_3 = 16 \\ x_1 - x_2 + x_3 = 2 \end{cases}$

- $Ax = b \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 16 \\ 2 \end{bmatrix} \longrightarrow x_1\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + x_3\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 16 \\ 2 \end{bmatrix}$

- $A|b = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 3 & 3 & -1 & 16 \\ 1 & -1 & 1 & 2 \end{bmatrix} \xrightarrow[-R_1+R_3]{-3R_1+R_2} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 0 & -4 & 4 \\ 0 & -2 & 0 & -2 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -4 & 4 \end{bmatrix} = u|c$

  - $ref(A) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

  - Gaussian:
    - $-4x_3 = 4$, $x_3 = -1$
    - $-2x_2 = -2$, $x_2 = 1$
    - $x_1 + (1) + (-1) = 4$, $x_1 = 4$
  - Gauss-Jordan:
    - $u|c \xrightarrow[-\frac{1}{4}R_3]{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{(-R_2 + -R_3) + R_1} \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} = R|d$
    - $x_1 = 4$, $x_2 = 1$, $x_3 = -1$

- $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

- $\underline{I_3}$: 3x3 multiplicative (dot product) identity matrix.

  - $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

9/12: <u>Matrix Operations</u>

- $A = \begin{bmatrix} 1 & 2 & 0 & -3 \\ -1 & 4 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 2 \\ -4 & -2 & 1 \\ 0 & 4 & 5 \\ -3 & 6 & 1 \end{bmatrix}$
  - $A + B$ = undefined; $\mathbb{R}^2$ & $\mathbb{R}^4$ cannot add.
    - Have to have same number of rows and columns; unless $A_{m \times n}$, $B_{m \times n}$.
  - $cA = \begin{bmatrix} c & 2c & 0 & -3c \\ -c & 4c & c & 0 \end{bmatrix}$
  - $A \cdot B = \begin{bmatrix} 7 & -21 & 1 \\ -4 & -5 & 7 \end{bmatrix}$, $B \cdot A$ = undefined
    - $\underbrace{2 \times \underbrace{4 \cdot 4}_{\text{possible}} \times 3}_{\text{final dimensions}}$ $\quad$ $4 \times \underbrace{3 \cdot 2}_{\text{impossible}} \times 4$

<u>Elementary Matrices</u>

- Those that produce elementary row operations.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

- Reduce A to u.

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- $A \xrightarrow{-3R_1+R_2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$

  This is an elementary matrix

  - $(E_{2,1}A) = E_{2,1} \cdot A$

- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \cdot (E_{2,1}A) = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}$

9/13:

  - $E_{3,2} \cdot (E_{2,1}A) = E_{3,2}(E_{2,1}A) = u$

- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

  - $E_{3,2} \cdot E_{2,1} = (E_{3,2}E_{2,1})$

Block Multiplication

$$A = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 4 & 0 \\ \hline 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 7 & 2 \end{array}\right] \qquad B = \left[\begin{array}{cccc|c} 4 & 3 & 1 & 2 & 1 \\ -1 & 2 & 2 & 1 & 1 \\ 1 & -5 & 3 & 3 & 1 \\ \hline 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \end{array}\right]$$

- Sparse matrix: A matrix with many zero entries.

- $A = \left[\begin{array}{c|c} I_3 & C \\ \hline 0 & D \end{array}\right], \quad B = \begin{bmatrix} E & F & G \\ I_2 & 0 & H \end{bmatrix}$

- $AB = \left[\begin{array}{c|c|c} E+C & F & G+CH \\ \hline D & 0 & DH \end{array}\right]$

$$AB = \left[\begin{array}{cc|cc|c} 6 & 2 & 1 & 2 & 2 \\ 0 & 5 & 2 & 1 & 12 \\ 5 & -5 & 3 & 3 & 9 \\ \hline 1 & 7 & 0 & 0 & 23 \\ 7 & 2 & 0 & 0 & 20 \end{array}\right]$$

9/16: Proof by Induction

- Prove that the sum of consecutive odd integers is always a perfect square.
  - Basis Step: $1 \overset{\checkmark}{=} 1$
  - Induction hypothesis: $1 + 3 + 5 + \cdots + (2k-1) = k^2$
  - Induction step: $1+3+5+\cdots+(2(k+1)-1) \overset{?}{=} (k+1)^2$ (1)
    - $1+3+5+\cdots+(2k+1) \overset{?}{=} (k+1)^2$ (2)
    - $1+3+5+\cdots+(2k-1)+(2k+1) \overset{?}{=} (k+1)^2$ (3)
    - $k^2+2k+1 \overset{?}{=} (k+1)^2$ (4)
    - $(k+1)^2 = (k+1)^2$ (5)
      - By the induction hypothesis, $k^2 = 1+3+5+\cdots+(2k-1)$, allowing the transition from step 3 to 4.
- Prove that the sum of consecutive integers is half the greatest integer in the sequence times one greater than that integer.
  - Basis step: $1 \overset{\checkmark}{=} \frac{1}{2}(1+1)$
  - Induction hypothesis: $1+2+3+\cdots+k = \frac{k}{2}(k+1)$
  - Induction step: $1+2+3+\cdots+(k+1) \overset{?}{=} \frac{k+1}{2}((k+1)+1)$
    - $1+2+3+\cdots+k+(k+1) \overset{?}{=} \frac{k+1}{2}(k+2)$
    - $\frac{k}{2}(k+1) + \frac{2}{2}(k+1) \overset{?}{=} \frac{(k+1)(k+2)}{2}$
    - $\frac{(k+1)(k+2)}{2} = \frac{(k+1)(k+2)}{2}$
- $a^m a^k = a^{m+k}$, $k \geq 0$
  - Basis step: $a^m a^0 \overset{\checkmark}{=} a^{m+0}$
  - Induction hypothesis: $a^m a^k = a^{m+k}$
  - " step: $a^m a^{k+1} \overset{?}{=} a^{m+k+1}$
    - $a^m a^k a \overset{?}{=} a^{m+k} a$
    - $a^{m+k} a = a^{m+k} a$

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- $k! \geq 2^k$, $k \geq 4$
  - $4! \geq 2^4$, $24 \overset{\checkmark}{\geq} 16$
  - $(k)\cdots(4)(3)(2)(1) \geq 2^k 2^4$
  - $(k+1)\cdots(4)(3)(2)(1) \overset{?}{\geq} 2^{k+1} 2^4$
    - $(k+1)(k)\cdots(4)(3)(2)(1) \overset{?}{\geq} 2 \cdot 2^k 2^4$
    - $(k+1)\, 2^k 2^4 \geq 2 \cdot 2^k 2^4$

      $k+1 \geq 2$, $k \geq 4$

9/17:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

- $A^2 = AA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$
- $A^3 = A^2 A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$
- $A^n = \begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix}$
  - Basis step:
  - Induction hypothesis:
  - Induction step: $A^{n+1} \overset{?}{=} \begin{bmatrix} 2^{(n+1)-1} & 2^{(n+1)-1} \\ 2^{(n+1)-1} & 2^{(n+1)-1} \end{bmatrix}$

$$A^n A \overset{?}{=} \begin{bmatrix} 2^n & 2^n \\ 2^n & 2^n \end{bmatrix}$$

$$\begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 2^n & 2^n \\ 2^n & 2^n \end{bmatrix}$$

$$\begin{bmatrix} 2^{n-1}+2^{n+1} & 2^{n-1}+2^{n-1} \\ 2^{n-1}+2^{n-1} & 2^{n-1}+2^{n-1} \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 2^n & 2^n \\ 2^n & 2^n \end{bmatrix}$$

$$\begin{bmatrix} 2^1 \cdot 2^{n-1} & 2^1 \cdot 2^{n-1} \\ 2^1 \cdot 2^{n-1} & 2^1 \cdot 2^{n-1} \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 2^n & 2^n \\ 2^n & 2^n \end{bmatrix}$$

$$\begin{bmatrix} 2^n & 2^n \\ 2^n & 2^n \end{bmatrix} = \begin{bmatrix} 2^n & 2^n \\ 2^n & 2^n \end{bmatrix}$$

9/18 **Matrix Inverses**

- Inverse: A map to an identity element.
- Additive inverse: A value that, when added to a number, yields 0.
- Additive identity: Zero.
- Multiplicative inverse: A value that, when multiplied by a number, yields 1. Also known as reciprocal.
- Multiplicative identity: One.
- Inverse matrix: A matrix that, when dotted with a matrix, yields an identity matrix.
  - Must be square.
  - A matrix multiplied by its inverse is commutative.
- Invertible matrix: A matrix that has an inverse.
  - $\exists A^{-1}$ s.t. $AA^{-1} = I = A^{-1}A$
  - "There exists a matrix, A^{-1}, such that $A \cdot A^{-1} = I = A^{-1} \cdot A$."
- Singular matrix: A matrix that does not have an inverse.
  - An $m \times n$ matrix where $m \neq n$.
    - Rectangular matrix multiplication is never commutative.
    - $(m \times n) \cdot (n \times m) = (m \times m)$ but $(n \times m) \cdot (m \times n) = (n \times n)$.
  - A zero matrix.
    - Any matrix multiplied by a zero matrix is a zero matrix (never an identity matrix).
  - A matrix in the form, $\begin{bmatrix} a & b \\ na & nb \end{bmatrix}$, where $n \in [1, \infty)$.
    - $\begin{bmatrix} aa_{11} + ba_{12} = 1 & \cdots \\ naa_{11} + nbb_{12} = 0 & \cdots \end{bmatrix}$. If $na \geq a$ and $nb \geq b$, the lower row (0), must be greater than 1 – contradiction!

Chapter 2: Solving Linear Equations (Notes 5) Forsyth

- Column vectors are dependent; we can't escape their line via a linear combination.

- $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix} \begin{bmatrix} | & | & | \\ x_1 & x_2 & x_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

9/19:

- $A_{x_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad A_{x_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad A_{x_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

- $A|I = \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & -3 & 0 & 0 & 1 \end{array}\right]$

  - $A|I = R|A^{-1}$

  - $A|I \xrightarrow[-R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 6 & -2 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right] \xrightarrow{-\frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right]$

$\xrightarrow[-R_2+R_3]{-2R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & -1 & 1 & 0 \\ 0 & 1 & -3 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & 1 \end{array}\right] \xrightarrow[3R_3+R_2]{-5R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -\frac{3}{2} & -5 \\ 0 & 1 & 0 & -5 & 1 & 3 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & 1 \end{array}\right]$

  - Test by: $AA^{-1} = I = A^{-1}A$

9/23:

- $A = LU$ decomposition

"lower triangular" "upper triangular"

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix}$$

- $E_{2,1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad$ $E_{2,1}A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ -2 & 5 & 5 \end{bmatrix}$

- $E_{3,1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $\quad$ $E_{3,1}(E_{2,1}A) = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 6 & 8 \end{bmatrix}$

- $E_{3,2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ $\quad$ $E_{3,2}(E_{3,1}(E_{2,1}A)) = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix}$

- Matrix multiplication is associative, but not commutative.
  - $(E_{3,2}E_{3,1}E_{2,1})A = U$
  - $(E_{3,2}E_{3,1}E_{2,1})^{-1}(E_{3,2}E_{3,1}E_{2,1})A = (E_{3,2}E_{3,1}E_{2,1})^{-1}U$
  - $IA = (E_{3,2}E_{3,1}E_{2,1})^{-1}U$
  - $A = (E_{3,2}E_{3,1}E_{2,1})^{-1}U$
  - $A = LU$
- A side note on finding the inverse of a matrix product:
  - $AB(AB)^{-1} = I$
    - No commutative property, so switch order when expanding.
  - $ABB^{-1}A^{-1} = I$
    - NOT $ABA^{-1}B^{-1} = I$
  - $A(BB^{-1})A^{-1} = I$
    - Associative property.
  - $AIA^{-1} = I$
  - $AA^{-1} = I$
  - $I = I$
- Therefore,
  - $L = (E_{3,2}E_{3,1}E_{2,1})^{-1} = E_{2,1}^{-1}E_{3,1}^{-1}E_{3,2}^{-1}$
    - To find $E_{2,1}^{-1}$, $E_{2,1}E_{2,1}^{-1} = I$, or $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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- Just a sign change.

- Likewise, $E_{3,1}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ and $E_{3,2}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

- Although one could now multiply the inverse elementary matrices formally, this can be accomplished by combining all nonzero entries into one matrix as follows.

  - $E_{2,1}^{-1} E_{3,1}^{-1} E_{3,2}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix}$

  - Therefore, $A = LU \Rightarrow \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ -2 & 5 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix}$

"for every"

- Applicability: Solve $Ax = b$ $\forall b$, given $A = LU$.

  - Example: $b = \begin{bmatrix} 1 \\ -4 \\ 9 \end{bmatrix}$

    - Solve $Ly = b$: $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 9 \end{bmatrix}$; $y_1 = 1$, $2y_1 + y_2 = -4$, $y_2 = -6$, $-y_1 - 2y_2 + y_3 = 9$, $y_3 = -2$

    - Solve $Ux = y$: $\begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & -3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ -2 \end{bmatrix}$; $2x_3 = -2$, $x_3 = -1$, $-3x_2 - 3x_3 = -6$, $x_2 = 3$, $2x_1 + x_2 + 3x_3 = 1$, $x_1 = \frac{1}{2}$

9/24:

— Alternatively, $L^{-1}b=y$, or $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -4 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 2 \end{bmatrix}$

9/25: • $PA = LU$ factorization
↑ "permutation matrix"

$$A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$$

— $\underset{P_{1\leftrightarrow 2}}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}\underset{A}{\begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}} = \underset{P_{1\leftrightarrow 2}A}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 6 \\ 2 & 1 & 4 \end{bmatrix}}$

— $\underset{P_{2\leftrightarrow 3}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}\underset{P_{1\leftrightarrow 2}A}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 6 \\ 2 & 1 & 4 \end{bmatrix}} = \underset{P_{2\leftrightarrow 3}(P_{1\leftrightarrow 2}A)}{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 0 & 0 & 6 \end{bmatrix}}$

— $(P_{2\leftrightarrow 3}P_{1\leftrightarrow 2})A$, $\underset{P_{2\leftrightarrow 3}}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}\underset{P_{1\leftrightarrow 2}}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} = \underset{P}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}$

— $\underset{P}{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}\underset{A}{\begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}\underset{U}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & 6 \end{bmatrix}}$

Matrix Transposition

- A^T = "A transpose"
- Rows of A are columns of A^T.
- $[A]_{i,j} = [A]^T_{j,i}$
- If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 8 & 5 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 0 \\ 2 & 8 \\ 3 & 5 \end{bmatrix}$

Steven Labalme (H) Linear Algebra, 3

Chapter 2: Solving Linear Equations (Notes 7) Forsyth

- Symmetric matrix: A matrix such that $A = A^T$.
  - Necessarily square.
  - If A is a 3x3 symmetric matrix, $A = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$
- Properties of transposition
  - $(A^T)^T = A$
    - i,j entry of A^T is the j,i entry of A, so the i,j entry of $(A^T)^T$ is the j,i entry of A^T, which is the i,j entry of A.
  - $(A+B)^T = A^T + B^T$
  - $(cA)^T = cA^T$
  - $(AB)^T = B^T A^T$
    - $\overset{A}{\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}} \overset{B}{\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}} = \begin{bmatrix} b_{1j} & b_{2j} & \cdots & b_{nj} \end{bmatrix} \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix}$
    - $a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj} = b_{1j} a_{i1} + b_{2j} a_{i2} + \cdots + b_{nj} a_{in}$

9/26:
    - Note that the above represents any arbitrary i,j entry in AB. When transposed, it becomes the j,i entry in $(AB)^T$.
  - $(A^{-1})^T = (A^T)^{-1}$
    - $A^T (A^{-1})^T = (A^{-1}A)^T = I^T = I$
    - $(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$
    - $\therefore$ the inverse of A^T is $(A^{-1})^T$ using property 4.
      "therefore"
- $PA = LU$, $P^{-1}PA = P^{-1}LU$, $IA = P^{-1}LU$, $A = P^{-1}LU$, $P^{-1} = P^T$ $\therefore$ $A = P^T LU$
  - $P^{-1} = P^T \Rightarrow PP^T = I$
  - If $i=j$, 1; If $i \neq j$, 0
  - Ex.: $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A = LDU$ Decomposition

"Diagonal matrix"

- Diagonal matrix: A matrix when only entries along the NW→SE line are non zero.

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$$

- $A = LU = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & -5 \end{bmatrix}$

- $A = LDU = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

  - D gives the pivots of A.

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Cryptography Forsyth

10/7: Hill Cypher

- $53 \equiv 3 \pmod{10}$
  - $\frac{53}{10} = 5\text{R}3 \rightarrow 53 = 5(10) + 3$
- $27 \equiv 3 \pmod{6}$
  - $\frac{27}{6} = 4\text{R}3 \rightarrow 27 = 4(6) + 3$
- $n = qm + R$
  - n: a number
  - q: the quotient
  - m: the modulo
  - R: the remainder
- mod (x): the modulo function
- $\equiv$: "is congruent to"
- $-51 \equiv 9 \bmod(10)$
  - $-51 = {}^{-}6(10) + 9$
- $-37 \equiv 3 \bmod(5)$

- Encoding matrix: $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$

  K C C XQLKPU U
  - I AM HIDING
  - 9 1 13 8 9 4 9 14 7 7

  - $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 9 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 3 \end{bmatrix}$

  - $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 13 \\ 8 \end{bmatrix} = \begin{bmatrix} 29 \\ 24 \end{bmatrix} \bmod(26) = \begin{bmatrix} 3 \\ 24 \end{bmatrix}$

  - $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 9 \\ 4 \end{bmatrix} = \begin{bmatrix} 17 \\ 12 \end{bmatrix}$

  - $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 9 \\ 14 \end{bmatrix} = \begin{bmatrix} 37 \\ 42 \end{bmatrix} \bmod(26) = \begin{bmatrix} 11 \\ 16 \end{bmatrix}$

$$-\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 7 \\ 7 \end{bmatrix} = \begin{bmatrix} 21 \\ 21 \end{bmatrix}$$

- Reciprocal of a modulus:
  - $a \equiv b \pmod{m}$
  - $aa^{-1} = a^{-1}a \equiv 1 \pmod{m}$
  - Reciprocal of $3 \pmod{26}$
    - $3x = 1 \bmod 26$
    - $x = 9 \rightarrow 27 = 1 \bmod 26$
- Find the inverse of $\begin{bmatrix} 5 & 6 \\ 2 & 3 \end{bmatrix} \bmod 26$.

$$-A^{-1} = \begin{bmatrix} 1 & -2 \\ -\frac{2}{3} & \frac{5}{3} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & -6 \\ -2 & 5 \end{bmatrix} \bmod 26 = 3^{-1}\begin{bmatrix} 3 & -6 \\ -2 & 5 \end{bmatrix} \bmod 26 = 9\begin{bmatrix} 3 & -6 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 27 & -54 \\ -18 & 45 \end{bmatrix} \bmod 26 = \begin{bmatrix} 1 & 24 \\ 8 & 19 \end{bmatrix}$$

- GTNKGKDUSK

$$-\begin{bmatrix} 1 & 24 \\ 8 & 19 \end{bmatrix}\begin{bmatrix} 7 \\ 20 \end{bmatrix} = \begin{bmatrix} 487 \\ 436 \end{bmatrix} \bmod 26 = \Big[$$