Steven Labalme (H) Linear Algebra, 3

Chapter 5: Determinants (Notes 1) Forsyth

12/12: **Properties of Determinants**

- $A = \{1, 2, 3, 4, 5, 6\}$
  - $6! = 720$ possible permutations
  - $B = \{6, 1, 4, 2, 5, 3\}$
  - $5+0+2+0+1+0 = 8$ inversions.
  - Makes B an even permutation
- Inversion: The number of elements out of order following an element.
- Even permutation: Σ (Inversions) = an even number.
- Determinant $(A) = \det(A) = |A|$: Sum of signed elementary products of A.
- Elementary product: Exactly one entry from each row/column of A.

- $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} + \begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix}$

E.P.	Permutation	Inversion	Parity	Sign
$a_{11}a_{22}$	1,2	0	even	+
$a_{12}a_{21}$	2,1	1	odd	−

- $|A| = a_{11}a_{22} - a_{12}a_{21}$

• Properties:

1. $|I| = 1$
   - There is only one nonzero elementary product.
2. Row swaps change the sign of $|A|$.
   - $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ vs. $\begin{vmatrix} c & d \\ a & b \end{vmatrix} = bc - ad = -(ad - bc)$
   - Use induction for higher dimensions.
3. $|A|$ is a linear operation one row at a time.
   - $\begin{vmatrix} xa & xb \\ c & d \end{vmatrix} = x(ad - bc)$
   - $\begin{vmatrix} a+x & b+y \\ c & d \end{vmatrix} = ad - bc + xd - yc = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} x & y \\ c & d \end{vmatrix}$
4. If A has 2 equal rows, $|A| = 0$
   - $\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab = 0$
5. Adding a multiple of a row of A to another row preserves $|A|$.
   - $\begin{vmatrix} a & b \\ c - xa & d - xb \end{vmatrix} = ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$
6. Row of zeros $\Rightarrow |A| = 0$
   - $\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0$

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7. U is the upper triangular form of A, then $|U| = u_{11} u_{22} \cdots u_{nn}$

- $U = \begin{bmatrix} u_{12} & u_{12} & \cdots & u_{1n} \\ & u_{22} & \cdots & u_{2n} \\ & 0 & \ddots & \vdots \\ & & & u_{nn} \end{bmatrix}$

- $|U| = |A|$ because of 5.

8. If A is singular, then $|A| = 0$.

12/13: **Method of Cofactor Expansion**

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- $$|A| = \underset{(1,2,3)}{\begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix}} + \underset{(1,3,2)}{\begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix}} + \underset{(2,1,3)}{\begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix}} + \cdots$$

$$\cdots + \underset{(3,1,2)}{\begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix}} + \underset{(2,3,1)}{\begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix}} + \underset{(3,2,1)}{\begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix}}$$

$$= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

- Minor $= M_{1,1} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$

- Cofactor $= C_{1,1} = \pm(a_{22}a_{33} - a_{23}a_{32})$

$$A = \begin{bmatrix} 5 & -2 & 2 & 7 \\ 1 & 0 & 0 & 3 \\ -3 & 1 & 5 & 0 \\ -3 & -1 & -9 & 4 \end{bmatrix}$$

- Find $|A|$.

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

$$= -1 \begin{vmatrix} -2 & 2 & 7 \\ 1 & 5 & 0 \\ -1 & -9 & 4 \end{vmatrix} + 3 \begin{vmatrix} 5 & -2 & 2 \\ -3 & 1 & 5 \\ 3 & -1 & -9 \end{vmatrix}$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= -1\left(-1 \begin{vmatrix} 2 & 7 \\ -9 & 4 \end{vmatrix} + 5 \begin{vmatrix} -2 & 7 \\ -1 & 4 \end{vmatrix}\right) + 3\left(5 \begin{vmatrix} 1 & 5 \\ -1 & -9 \end{vmatrix} + 2 \begin{vmatrix} -3 & 5 \\ 3 & -9 \end{vmatrix} + 2 \begin{vmatrix} -3 & 1 \\ 3 & -1 \end{vmatrix}\right)$$

$$= -1\left(-1 \cdot 71 + 5 \cdot -1\right) + 3\left(5 \cdot -4 + 2 \cdot 12 + 0\right)$$

$$= 71 + 5 - 60 + 72$$

$$= 88$$

12/17: Cramer's Rule

- A is an $n \times n$ invertible matrix, $b \in \mathbb{R}^n$.
- $Ax = b$ is solved by $x_i = \dfrac{|A_i(b)|}{|A|}$ for $i = 1, \ldots, n$.

$$Ax = b = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

- $x_2 = \begin{vmatrix} 1 & 0 \\ 2 & 6 \end{vmatrix} \div \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = \dfrac{6}{-2} = -3$

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- Proof: Let the columns of I_n be noted as $e_1, \dots, e_n$.
  - If $Ax = b$, $AI_n x = b$.
  - $AI_i(x) = A[e_1 \cdots x \cdots e_n] = [Ae_1 \cdots Ax \cdots Ae_n] = [a_1 \cdots b \cdots a_n] = A_i(b)$
  - $|A||I_i(x)| = |AI_i(x)| = |A_i(b)|$
  - $|I_i(x)| = x_i$ (think about the matrix)
  - $|A|x_i = |A_i(b)|$, $x_i = \frac{|A_i(b)|}{|A|}$

12/18: <u>Inverses</u>

- A is an $n \times n$ matrix.
- C is a matrix of the cofactors of A.
  - c_{ij} is the cofactor of a_{ij}

- $$AC^T = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} |A| & & & 0 \\ & |A| & & \\ & & \ddots & \\ 0 & & & |A| \end{bmatrix} = |A| I_n$$

- $AC^T = |A|I$, $A^{-1}AC^T = |A|A^{-1}I$, $C^T = |A|A^{-1}$, $\frac{1}{|A|}C^T = A^{-1}$

- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $C = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$, $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

- C^T is called the <u>adjoint</u> of A, or adj(A).

PROPRETY IX OF DETERMINANTS: $|AB| = |A||B|$

Let E be an $n \times n$ elementary matrix.

a) If E results from a row swap I_n (an $n \times n$ identity matrix), then $|E| = -1$ by property II

b) If E results from multiplying one row of I_n by $k \in F$, then $|E| = k$ by property III

c) If E results from adding a multiple of one row of I_n to another row, then $|E| = 1$ by property V

LEMMA:

Recall that multiplying a matrix B by an elementary matrix on the left performs the corresponding row operation on B. Because of this AND statements a-c above we can say that $|EB| = |E||B|$.

Now, let A be an arbitrary $n \times n$ matrix whose reduced row echelon form is R. Let $E_1 E_2 \ldots E_r$ be the elementary matrices corresponding to the elementary row operations that reduce A to R such that $E_r \cdots E_2 E_1 A = R$

Taking determinants of both sides and repeatedly applying the Lemma, we obtain $|E_r| \cdots |E_2||E_1||A| = |R|$.

Since the determinants of all elementary matrices are nonzero, as shown in the top box above, we can conclude that $|A| \neq 0$ if and only if $|R| \neq 0$.

Suppose A is invertible, then $R = I_n$, so $|R| = 1 \neq 0$. Therefore $|A| \neq 0$ also. Conversely if $|A| \neq 0$ then $|R| \neq 0$, so R cannot contain a zero row by property VI and it follows that R must be I_n so A is invertible.

To show that $|AB| = |A||B|$ for any arbitrary matrices A and B of size $n \times n$, consider two cases: when A is invertible and when A is not invertible.

Invertible Case: If A is invertible it can be written as a product of elementary matrices $A = E_1 E_2 \cdots E_k$, so then $AB = E_1 E_2 \cdots E_k B$, and k applications of the Lemma eventually give
$|AB| = |E_1 E_2 \cdots E_k B| = |E_1||E_2| \cdots |E_k||B|$.
Continuing to apply the Lemma, we obtain $|AB| = |E_1 E_2 \cdots E_k||B| = |A||B|$.

Singular Case: On the other hand, if A is not invertible, then neither is AB. $|AB| = |A| = 0$