FOURIER SERIES FOR PERIODIC FUNCTIONS WHEN PERIOD $\neq 2\pi$

Given:

$$\frac{a_0}{2} = \int_{-\pi}^{\pi} \frac{1}{2\pi} f(t)dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt \text{ , } n \in \mathbb{Z} \text{ and } n > 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt \text{ , } n \in \mathbb{Z} \text{ and } n > 0$$

Change the variable t to $x = \frac{2\pi}{P}t$. In this case $x = \pi$ corresponds to $t = \frac{P}{2}$ and $x = -\pi$ corresponds to $t = -\frac{P}{2}$.

Therefore regarded as a function of t, this is a function with period P. When we make the substitution $x = \frac{2\pi}{P}t$ and $dx = \frac{2\pi}{P}dt$ into the expressions for a_n and b_n we arrive at:

$$\boxed{\begin{aligned} a_n &= \frac{2}{P}\int_{-\frac{P}{2}}^{\frac{P}{2}} f(t)\cos\left(\frac{2n\pi t}{P}\right)dt, n \in \mathbb{Z}, n \geq 0 \\ b_n &= \frac{2}{P}\int_{-\frac{P}{2}}^{\frac{P}{2}} f(t)\sin\left(\frac{2n\pi t}{P}\right)dt, n \in \mathbb{Z}, n > 0 \end{aligned}}$$

These integrals will give the Fourier coefficients from a function of period P whose Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2n\pi t}{P}\right) + b_n \sin\left(\frac{2n\pi t}{P}\right)\right)$$

Note: In Differential Equations it is often convenient to write the period P as 2ℓ and in Physics and Engineering it is often written in terms of angular frequency ω as $P = \frac{2\pi}{\omega}$. Those substitutions would result in the following formulas:

$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(t)\cos\left(\frac{n\pi t}{\ell}\right)dt, n \in \mathbb{Z}, n \geq 0$ for Fourier Series $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi t}{\ell}\right) + b_n \sin\left(\frac{n\pi t}{\ell}\right)\right)$

And

$a_n = \frac{\omega}{\pi}\int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} f(t)\cos(n\omega t)dt, n \in \mathbb{Z}, n \geq 0$ for Fourier Series $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos(n\omega t) + b_n \sin(n\omega t)\right)$

Any convenient integration range of length P, 2ℓ, or $\frac{2\pi}{\omega}$ can be used, and formulas for b_n would follow similarly for those for a_n as shown above.

*Why divide the constant term by 2? To account for the fact that the formula for a_n could be true for all $n \in \mathbb{Z}, n \geq 0$ (which would include the constant term) depending upon how the formula for the Fourier coefficients are written. Recall from problems 6 and 7 for the integration problems combined that $\int_{-\pi}^{\pi} \cos^2(nt)dt = \pi$ when $n \neq 0$ but $\int_{-\pi}^{\pi} \cos^2(nt)dt = 2\pi$ when $n = 0$. Using the formula $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt$ we could write the Fourier Series as $f(t) = \sum_{n=0}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt)$ but in this case $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt = \begin{Bmatrix} 2a_0, n = 0 \\ a_n, n \neq 0 \end{Bmatrix}$. To compensate for this the constant term is customarily written as $\frac{a_0}{2}$.