Steven Labalme (H) Linear Algebra, 3

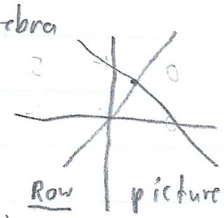
Intro to Linear Algebra Forsyth

8/26: R_1 $2x - y = 0$

R_2 $-x + 2y = 3$

$3y = 6$

$x = 2$ $(1,2)$



Steps:

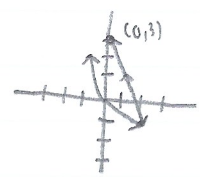
(1) Scale R_2 by 2. } Elimination
(2) Add $R_1 + R_2$.

Scaling & adding produce linearity: $y = ax + b$

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \end{bmatrix} \qquad Ax = b;\ x = A^{-1}b;\ x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

What about a column picture? Vectors!

$$\underline{1}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \underline{2}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$



8/27: $$A = \begin{bmatrix} \boxed{\begin{matrix} 2 \\ -1 \end{matrix}} & \begin{matrix} -1 \\ 2 \end{matrix} \end{bmatrix} \quad x = \begin{bmatrix} \boxed{1} \\ 2 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

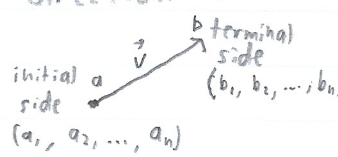
$\boxed{1}$ of $\boxed{\text{this}}$ vector plus 2 of this vector makes $\langle 0,3 \rangle$.

- Makes sense b/c matrix multiplication

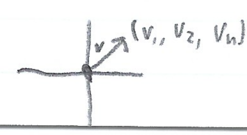
Vector Review

- Vector: A representation of magnitude and direction. A measurement of displacement.

$$\vec{v} = \begin{bmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_n - a_n \end{bmatrix}$$

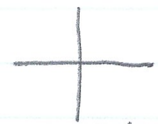


- Position vector: A vector from the origin.



$$V = \begin{bmatrix} v_1 \\ v_2 \\ v_n \end{bmatrix}$$

- Zero vector: $v = \begin{bmatrix}0\\0\end{bmatrix}$, $v = \begin{bmatrix}0\\0\\ \vdots \\0\end{bmatrix}$



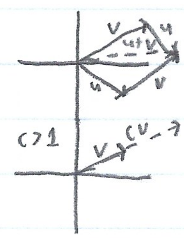
- Vector spaces: $\mathbb{R}$ real numbers, $\mathbb{Z}$ integers, $\mathbb{C}$ complex numbers, $\mathbb{Q}$ rational numbers, $\mathbb{N}$ counting numbers

set of all

  - $\mathbb{R}^2$ = "are-two"
  - $\vec{v} = \begin{bmatrix}0\\0\end{bmatrix}$, $\vec{v} \in \mathbb{R}^2$

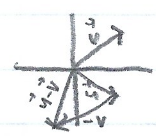
<u>Vector Operations</u>

- Addition: $\vec{u} = \begin{bmatrix}u_1\\u_2\end{bmatrix}$ $\vec{v} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$ $\vec{u} + \vec{v} = \begin{bmatrix}u_1 + v_1\\u_2 + v_2\end{bmatrix}$



- Scaling: $\vec{v} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$ $c = c$ $c\vec{v} = \begin{bmatrix}cv_1\\cv_2\end{bmatrix}$
  - $\vec{v}$ and $c\vec{v}$ are <u>parallel vectors.</u>
  - if $c\vec{v}$ was opposite $\vec{v}$ ($c<0$), $\vec{v}$ & $c\vec{v}$ are <u>antiparallel vectors</u>

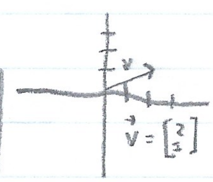
8/28:
- Subtraction: $\vec{u} = \begin{bmatrix}u_1\\u_2\end{bmatrix}$ $\vec{v} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$ $\vec{u} - \vec{v} = \begin{bmatrix}u_1 - v_1\\u_2 - v_2\end{bmatrix}$



<u>Basis</u> $\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}$ – <u>Standard basis</u> for $\mathbb{R}^2$

coordinate system

$$\frac{2}{}\begin{bmatrix}1\\0\end{bmatrix} + \frac{1}{}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

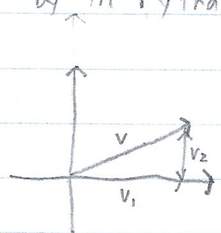


Linear combination (adding + scaling):
- $\begin{bmatrix}2\\1\end{bmatrix}$ is a linear combination of two basis vectors!
- Zero, parallel, and antiparallel vectors cannot be a basis in $\mathbb{R}^2$.
  - Dependent vectors cannot form a basis. One dependent vector is a linear combination of the other.
    - $\begin{bmatrix}3\\2\end{bmatrix}$ and $\begin{bmatrix}6\\4\end{bmatrix}$ are dependent because $\begin{bmatrix}6\\4\end{bmatrix} = 2\begin{bmatrix}3\\2\end{bmatrix}$.
- Basis vectors can, when linearly combined, create any and all vectors in $\mathbb{R}^n$.
- $\mathbb{R}^n$ needs n basis vectors.

Chapter 1 Notes (2) Forsyth

9/4: **Vector Length**

- In $\mathbb{R}^2$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, so, by the Pythagorean theorem, $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2}$.



- In $\mathbb{R}^3$, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, so $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ by analogy.
- In $\mathbb{R}^n$, $\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$.
- Unit vector: A vector of length 1.
  - $\vec{u}$ is a unit vector iff $\|\vec{u}\| = 1$
- To normalize a vector, divide it by its magnitude.
  - Proof:

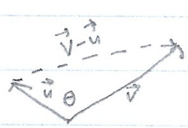
$$\|\vec{u}\| = \left\| \frac{v}{\|v\|} \right\|$$

$$= \frac{1}{\|v\|} \|v\|$$

$$= 1$$

  - $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\|\vec{v}\| = \sqrt{5}$, $\vec{u} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$
- Normalize: Reduce the magnitude of a vector to 1 while preserving its direction.
- To find the magnitude of a scaled vector, multiply the magnitude of the vector by the absolute value of the scalar.
  - $\|c\vec{v}\| = |c| \|\vec{v}\|$

Angle Between Vectors



$u, v \in \mathbb{R}^n$

- Use the law of cosines.
  - $c^2 = a^2 + b^2 - 2ab\cos(C)$
  - $\|\vec{v} - \vec{u}\|^2 = \|\vec{v}\|^2 + \|\vec{u}\|^2 - 2\|\vec{v}\|\|\vec{u}\| \cos\theta$
  - $(v_1 - u_1)^2 + (v_2 - u_2)^2 = v_1^2 + v_2^2 + u_1^2 + u_2^2 - 2\sqrt{v_1^2 + v_2^2}\sqrt{u_1^2 u_2^2} \cos\theta$

— $\cancel{v_1^2} - \cancel{2}u_1v_1 + \cancel{u_1^2} + \cancel{V_2^2} - 2u_2v_2 + \cancel{u_2^2} = \cancel{v_1^2} + \cancel{v_2^2} + \cancel{u_1^2} + \cancel{u_2^2} - \cancel{2}\sqrt{v_1^2+v_2^2}\sqrt{u_1^2+u_2^2}\cos\theta$

— $\dfrac{u_1 v_1 + u_2 v_2}{\sqrt{v_1^2+v_2^2}\sqrt{u_1^2+u_2^2}} = \cos\theta$

— $\theta = \cos^{-1}\left(\dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right)$

- If $\vec{u}\cdot\vec{v} = 0$, then $\theta = 90° = \frac{\pi}{2}$, $\vec{u} \perp \vec{v}$.

- $|\vec{u}\cdot\vec{v}| \leq \|\vec{u}\|\,\|\vec{v}\|$ – Cauchy-Schwarz inequality