LINEAR TRANSFORMATIONS

- We have already seen repeatedly that a matrix can be used to transform vectors. Let T be a matrix, and v be a vector in $\mathbb{R}^2$ and w be a vector in $\mathbb{R}^3$. Let $w = T(v)$:

$$T = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix} \quad v = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad w = T(v) = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

- T will transform any vector in $\mathbb{R}^2$ into a vector in $\mathbb{R}^3$: $\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x-y \\ 3x+4y \end{bmatrix}$
- The **transformation** or **mapping** T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns each vector v in $\mathbb{R}^n$ to a unique vector $T(v)$ in $\mathbb{R}^m$
- The **domain** of T is $\mathbb{R}^n$.
- The **codomain** of T is $\mathbb{R}^m$.
- Thus the transformation would be described as $T:\mathbb{R}^n \to \mathbb{R}^m$
- The vector $T(v)$ is the **image** of v.
- The set of all possible images is known as the **range** of T.
- In our example $\begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x-y \\ 3x+4y \end{bmatrix}$, the range of T would be all linear combinations,

  $x\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$, thus the column space of T.

WHEN IS A TRANFORMATION "LINEAR?"

- $T:\mathbb{R}^n \to \mathbb{R}^m$ is called a linear transformation if and only if :

  $T(u+v) = T(u) + T(v)$ for all u and v in $\mathbb{R}^n$

  $T(cv) = cT(v)$ for all v in $\mathbb{R}^n$ and all scalars c
- Steps to verify that our example transformation is linear:

$T:\mathbb{R}^n \to \mathbb{R}^m = T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x-y \\ 3x+4y \end{bmatrix}$, let $u = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $v = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$$T(u+v)=T\left(\begin{bmatrix}x_1\\y_1\end{bmatrix}+\begin{bmatrix}x_2\\y_2\end{bmatrix}\right)=T\left(\begin{bmatrix}x_1+x_2\\y_1+y_2\end{bmatrix}\right)=\begin{bmatrix}x_1+x_2\\2(x_1+x_2)-(y_1+y_2)\\3(x_1+x_2)+4(y_1+y_2)\end{bmatrix}=$$

$$=\begin{bmatrix}x_1+x_2\\2x_1+2x_2-y_1-y_2\\3x_1+3x_2+4y_1+4y_2\end{bmatrix}=\begin{bmatrix}x_1+x_2\\(2x_1-y_1)+(2x_2-y_2)\\(3x_1+4y_1)+(3x_2+4y_2\end{bmatrix}=$$

$$=\begin{bmatrix}x_1\\2x_1-y_1\\3x_1+4y_1\end{bmatrix}+\begin{bmatrix}x_2\\2x_2-y_2\\3x_2+4y_2\end{bmatrix}=T\begin{bmatrix}x_1\\y_1\end{bmatrix}+T\begin{bmatrix}x_2\\y_2\end{bmatrix}=T(u)+T(v)$$

$$T(cv)=T\left(c\begin{bmatrix}x\\y\end{bmatrix}\right)=T\left(\begin{bmatrix}cx\\cy\end{bmatrix}\right)=$$

$$=\begin{bmatrix}cx\\2(cx)-(cy)\\3(cx)+4(cy)\end{bmatrix}=\begin{bmatrix}cx\\c(2x-y)\\c(3x+4y)\end{bmatrix}=$$

$$=c\begin{bmatrix}x\\2x-y\\3x+4y\end{bmatrix}=cT\begin{bmatrix}x\\y\end{bmatrix}=cT(v)$$

- Therefore, our example transformation was indeed linear.
- This is true for any $m\times n$ matrix A. Let $T_A:\mathbb{R}^n\to\mathbb{R}^m$ be defined by $T_A(x)=Ax$ for any x in $\mathbb{R}^n$. Let u and v be vectors in $\mathbb{R}^n$ and c be a scalar:

$T_A(u+v)=A(u+v)=Au+Av=T_A(u)+T_A(v)$ and $T_A(cv)=A(cv)=c(Av)=cT_A(v)$

ROTATIONS AS LINEAR TRANSFORMATIONS

- A rotation about the origin through an angle θ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$.
- Let R_θ be the rotation, and u and v be vectors in R^2. Provided that the vectors are not parallel, we know that $\|u+v\|$ is the diagonal of a parallelogram formed by the two vectors. Applying R_θ would rotate the entire parallelogram through angle θ, and thus the diagonal of the parallelogram must be $R_\theta(u)+R_\theta(v)$ and therefore $R_\theta(u+v)=R_\theta(u)+R_\theta(v)$.

  Applying R_θ to v and cv would obtain $R_\theta(v)$ and $R_\theta(cv)$. Since a rotation does not affect lengths, $R_\theta(cv)=cR_\theta(v)$

- The matrix of this linear transformation can be found by determining the effects on standard basis vectors: $R_\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $R_\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (which is orthogonal, so $R_\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$)
- $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

PROJECTIONS AS LINEAR TRANSFORMATIONS

- As an example, let ℓ be a line through the origin in $\mathbb{R}^2$. The linear transformation $P_\ell : \mathbb{R}^2 \to \mathbb{R}^2$ projects a vector in $\mathbb{R}^2$ onto ℓ.
- ℓ has a directional vector d onto which we will project an arbitrary vector v.
- The projection of v onto d is given by $\left(\dfrac{d^T v}{d^T d} \right) d$
- Therefore, the transformation $P_\ell : \mathbb{R}^2 \to \mathbb{R}^2$ is linear because:

$$P_\ell(u+v) = \left(\frac{d^T(u+v)}{d^T d} \right) d = \left(\frac{d^T u + d^T v}{d^T d} \right) d =$$
$$= \left(\frac{d^T u}{d^T d} + \frac{d^T v}{d^T d} \right) d = \left(\frac{d^T u}{d^T d} \right) d + \left(\frac{d^T v}{d^T d} \right) d =$$
$$P_\ell(u) + P_l(v)$$

Similarly $P_\ell(cv) = cP_\ell(v)$

TRANSPOSITION AS A LINEAR TRANSFORMATION

- Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined as $T_A = A^T$
- $T(A+B) = (A+B)^T = A^T + B^T = T(A) + T(B)$
- $T(cA) = (cA)^T = cA^T = cT(A)$

DIFFERENTIATION AS A LINEAR TRANSFORMATION

- Let D be the differential operator $D : \Delta \to \Phi$ defined by $D(f) = f'$
- Let f and g be differentiable functions.
- D is a linear transformation because $D(f+g) = (f+g)' = f' + g' = D(f) + D(g)$ and $D(cf) = (cf)' = cf' = cD(f)$.b

INTEGRATION AS A LINEAR TRANSFORMATION

- Let $S:\partial[a,b]\to\mathbb{R}$ by $S(f)=\int_a^b f(x)dx$
- Let f and g be functions in $\partial[a,b]$.
- S is a linear transformation because

  $S(f+g)=\int_a^b (f+g)(x)dx=\int_a^b (f(x)+g(x))dx=$

  $=\int_a^b f(x)dx+\int_a^b g(x)dx=S(f)+S(g)$
- Similarly, $S(cf)=cS(f)$

EXAMPLES OF NON-LINEAR TRANSFORMATIONS

- $T:\mathbb{R}\to\mathbb{R}:T(x)=2^x$

  Let $x=1$ and $y=2$

  $T(x+y)=T(3)=2^3=8\neq 6=2^1+2^2=T(x)+T(y)$
- $T:\mathbb{R}\to\mathbb{R}:T(x)=x+1$

  Let $x=1$ and $y=2$

  $T(x+y)=T(3)=3+1=4\neq 5=(1+1)+(2+1)=T(x)+T(y)$
- $T:M_{22}\to\mathbb{R}:T(A)=|A|$

  Let $A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ and $B=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$

  $T(A+B)=|A+B|=\begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix}=1\neq 0=\begin{vmatrix}1 & 0\\ 0 & 0\end{vmatrix}+\begin{vmatrix}0 & 0\\ 0 & 1\end{vmatrix}=T(A)+T(B)$

$T:V \to W$ **and T's effect on a basis for V**

Example:

$$T:\mathbb{R}^2 \to F : T\begin{bmatrix}1\\1\end{bmatrix} = 2-3x+x^2 \text{ and } T\begin{bmatrix}2\\3\end{bmatrix} = 1-x^2$$

- Find $T\begin{bmatrix}-1\\2\end{bmatrix}$.

$\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}2\\3\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^2$, and therefore $\begin{bmatrix}-1\\2\end{bmatrix}$ is in its span. Solve $\begin{bmatrix}1 & 2\\1 & 3\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}-1\\2\end{bmatrix}$ to find the appropriate scalars $c_1 = -7$ and $c_2 = 3$.

Therefore $T\begin{bmatrix}-1\\2\end{bmatrix} = T\left(-7\begin{bmatrix}1\\1\end{bmatrix} + 3\begin{bmatrix}2\\3\end{bmatrix}\right) = -7T\begin{bmatrix}1\\1\end{bmatrix} + 3T\begin{bmatrix}2\\3\end{bmatrix}$

$$= -7(2-3x-10x^2) + 3(1-x^2) = -11+21x-10x^2$$

- Find $T\begin{bmatrix}a\\b\end{bmatrix}$.

$= (5a-3b) + (-9a+6b)x + (4a-3b)x^2$

Therefore, let $T:V \to W$ be a linear transformation, and let $B = v_1,\ldots,v_n$ be a basis that spans V. $T(B) = T(v_1),\ldots T(v_n)$ is a basis that spans the range of T.

COMPOSITION OF LINEAR TRANSFORMATIONS

Let $T:U \to V$ and $S:V \to W$ be linear transformations. The composition of S with T is the mapping $S \circ T$ definite by $(S \circ T)(u) = S(T(u))$

EXAMPLE:

Let $T:\mathbb{R}^2 \to F_1$ **and** $S:F_1 \to F_2$ be the linear transformations defined by

$T\begin{bmatrix} a \\ b \end{bmatrix} = a+(a+b)x$ and $S(f(x)) = xf(x)$

- Find $(S \circ T)\begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$$(S \circ T)\begin{bmatrix} 3 \\ -2 \end{bmatrix} = S\left(T\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right)\right) = S(3+(3-2)x) = S(3+x) = x(3+x) = 3x+x^2$$

- Find $(S \circ T)\begin{bmatrix} a \\ b \end{bmatrix}$

$$(S \circ T)\begin{bmatrix} a \\ b \end{bmatrix} = S\left(T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)\right) = S(a+(a+b)x) = x(a+(a+b)x) = ax+(a+b)x^2$$

If $T:U \to V$ and $S:V \to W$ are linear transformation, then $S \circ T:U \to V$ is a linear transformation.

$$\begin{aligned}(S \circ T)(u+v) &= S(T(u+v)) \\ &= S(T(u)+T(v)) \\ &= S(T(u))+S(T(v)) \\ &= (S \circ T)(u)+(S \circ T)(v)\end{aligned}$$

$$\begin{aligned}(S \circ T)(cu) &= S(T(cu)) \\ &= S(cT(u)) \\ &= cS(T(u)) \\ &= c(S \circ T)(u)\end{aligned}$$

INVERSES OF LINEAR TRANSFORMATION

A linear transformation $T:V\to W$ is invertible if there is a linear transformation $T':W\to V$ such that $T'\circ T=I_V$ and $T\circ T'=I_W$

EXAMPLE:

Verify that the linear mapping $T:\mathbb{R}^2\to F_1:T\begin{bmatrix}a\\b\end{bmatrix}=a+(a+b)x$ and $T':F_1\to\mathbb{R}^2:T'(c+dx)=\begin{bmatrix}c\\d-c\end{bmatrix}$ are inverses.

$$(T'\circ T)\begin{bmatrix}a\\b\end{bmatrix}=T'\left(T\begin{bmatrix}a\\b\end{bmatrix}\right)=T'(a+(a+b)x)=\begin{bmatrix}a\\(a+b)-a\end{bmatrix}=\begin{bmatrix}a\\b\end{bmatrix}$$

$$(T\circ T')(c+dx)=T(T'(c+dx))=T\begin{bmatrix}c\\d-c\end{bmatrix}=c+(c+(d-c))x=c+dx$$

Therefore they are inverses because $T'\circ T=I_{\mathbb{R}^2}$ and $T\circ T'=I_{F_1}$.

KERNEL AND RANGE

Let $T : V \to W$ be a linear transformation

- The **kernel** of T [ker(T)] is the set of all vectors in V that are mapped by T to 0 in W.

  ker(T) ={v in V : T(v) =0}

- The **range** of T [range(T)] is the set of all vectors in W that are images of vectors in V under T.

EXAMPLES:

Find the kernel and range of the differential operator $D : P_3 \to P_2 :\ D(p(x)) = p'(x)$

$$D(a+bx+cx^2+dx^3) = b+2cx+3dx^2$$

$$\ker(D) = \{a+bx+cx^2+dx^3 : D(a+bx+cx^2+dx^3) = 0\}$$

$$= \{a+bx+cx^2+dx^3 : b+2cx+3dx^2 = 0\}$$

$b+2cx+3dx^2 = 0$ iff $b = 2c = 3d = 0$, implying $b = c = d = 0$, therefore

$$\ker(D) = \{a+bx+cx^2+dx^3 : b = c = d = 0\}$$

$$= \{a : a \text{ in } \mathbb{R}\}$$

Therefore, the kernel of D is the set of all constant polynomials. The range of D is all of P_2 because every polynomial in P_2 is the image under D of some polynomial in P_3.

Find the kernel and range of $S : P_1 \to \mathbb{R} : S(p(x)) = \int_0^1 p(x)dx$

$$S(a+bx) = \int_0^1 (a+bx)dx$$

$$\left[ax+\frac{b}{2}x^2\right]_0^1=\left(a+\frac{b}{2}\right)-0=a+\frac{b}{2}$$

$$\ker(S)=\{a+bx:S(a+bx)=0\}$$

$$=\left\{a+bx:a+\frac{b}{2}=0\right\}$$
$$=\left\{a+bx:a=-\frac{b}{2}\right\}$$
$$=\left\{-\frac{b}{2}+bx\right\}$$

Therefore, the kernel consists of all linear polynomials whose graphs have the property that the area between the line and the x-axis is equally distributed above and below the axis.

The range of S is $\mathbb{R}$ since every real number can be obtained as the image under S of a first degree polynomial.

Find the kernel and range of $T:M_{22}\to M_{22}:T(A)=A^T$

$$\ker(T)=\{A\text{ in }M_{22}:T(A)=0\}$$

$$=\{A\text{ in }M_{22}:A^T=0\}$$

$\ker(T)=\ 0$ **because if** $A^T=0$**, then** $(A^T)^T=0^T=0$**. Also range** $(T)=M_{22}$ **because** $A=(A^T)^T=T(A^T)$

In each of the examples, the kernel of T is a subspace of V, the range of T is a subspace of W.

RANK AND NULLITY

- The rank of T is the dimension of the range of T.
- The nullity of T is the dimension of the kernel of T.

FROM OUR EXAMPLES:

$$D: P_3 \to P_2 : \ D(p(x)) = p'(x)$$

Rank(D)=3; nullity (D)=1

$$S: P_1 \to \mathbb{R} : S(p(x)) = \int_0^1 p(x)dx$$

Rank(S)=1; nullity (S)=1

$$T: M_{22} \to M_{22} : T(A) = A^T$$

Rank (T)=4; Nullity(T)=0

RANK THEOREM:

Rank(T) + Nullity(T) = dim (V) when $T: V \to W$

Let W be the vector space of all 2×2 matrices for which $A = A^T$. Let $T: W \to P_2 : T\begin{bmatrix} a & b \\ b & c \end{bmatrix} = (a-b)+(b-c)x+(c-a)x^2$. Find the rank on the nullity of T.

$$\ker(T) = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : T\begin{bmatrix} a & b \\ b & c \end{bmatrix} = 0 \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : (a-b)+(b-c)x+(c-a)x^2 = 0 \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : (a-b)=(b-c)=(c-a)=0 \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : a=b=c \right\}$$

$$= \left\{ \begin{bmatrix} c & c \\ c & c \end{bmatrix} \right\} = span\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right)$$

$\ker(T) = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ **and thus the nullity** $= \dim(\ker(T)) = 1$. **Therefore the rank** $= \dim W - nullity(T) = 3 - 1 = 2$

ONE-TO-ONE and ONTO LINEAR TRANSFORMATIONS

- $T:V \to W$ **is called one-to-one if** T **maps distinct vectors in** V **to distinct vectors in** W**.**
- T **is called onto if** $range(T) = W$**.**
  - $T:V \to W$ **is one-to-one if for all vectors in** V**,** $u \neq v$ **implies that** $T(u) \neq T(v)$ **and** $T(u) = T(v)$ **implies that** $u = v$**.**

EXAMPLES:

$$T:\mathbb{R}^2 \to \mathbb{R}^3 : T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ x-y \\ 0 \end{bmatrix}$$

- T **is one-to-one because Let** $T\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = T\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$**. Then** $\begin{bmatrix} 2x_1 \\ x_1 - y_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 - y_2 \\ 0 \end{bmatrix}$**, which implies** $2x_1 = 2x_2$ **and** $x_1 - y_1 = x_2 - y_2$ **so** $x_1 = x_2$ **and** $y_1 = y_2$**.**
- **T is NOT onto because its range is not all of** $\mathbb{R}^3$**. There is no vector such that** $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$**.**

$$D: P_3 \to P_2 : D(p(x)) = p'(x)$$

- **D is not one-to-one because distinct polynomials in** P_3 **can have the same derivative.**
- **D is onto because** $range(D) = P_2$**.**

$T: V \to W$ **is one-to one if and only if** $\ker(T) = 0$

$T\begin{bmatrix} a \\ b \end{bmatrix} = a + (a+b)x$ **is one-to-one because** $\ker(T) = 0$ **since if** $T(A) = 0$**, then** $a = 0$ **and** $a + b = 0$ **(therefore** $b = 0$**).** $T\begin{bmatrix} a \\ b \end{bmatrix} = a + (a+b)x$ **is also onto because if** $\ker(T) = 0$ **then it is full rank:**

$$rank(T) = \dim(\mathbb{R}^2) - nullity(T) = 2 - 0 = 2$$

THE MATRIX OF A LINEAR TRANSFORMATION

Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ **be defined by** $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-2y \\ x+y-3z \end{bmatrix}$ **with bases** $B = \ e_1, e_2, e_3$ **for** $\mathbb{R}^3$ **and** $C = \ e_2, e_1$ **for** $\mathbb{R}^2$**. Find the matrix of** T **with respect to** B **and** C**.**

- **Compute** $T(e_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $T(e_2) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $T(e_3) = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$
- **Find their coordinate vectors with respect to** C

  Since $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = e_2 + e_1, \begin{bmatrix} -2 \\ 1 \end{bmatrix} = e_2 - 2e_1, \begin{bmatrix} -3 \\ 0 \end{bmatrix} = -3e_2 + 0e_1$**, we have**

  $$T(e_1)_C = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ T(e_2)_C = \begin{bmatrix} 1 \\ -2 \end{bmatrix},\ T(e_3)_C = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

- **Therefore, the matrix of** T **with respect to** B **and** C **is** $\begin{bmatrix} 1 & 1 & -3 \\ 1 & -2 & 0 \end{bmatrix}$**.**

Let $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$

$$T(v) = T\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ 10 \end{bmatrix}$$

$v_B = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ **and** $T(v)_C = \begin{bmatrix} -5 \\ 10 \end{bmatrix}_C = \begin{bmatrix} 10 \\ -5 \end{bmatrix}$

$$A\ v_B = \begin{bmatrix} 1 & 1 & -3 \\ 1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \end{bmatrix} = T(v)_C$$

Let $D: P_3 \to P_2$ be the differential operator $D(p(x)) = p'(x)$. A basis for P_3 would be $B = \ 1, x, x^2, x^3\ $ and a basis for P_2 would be $C = \ 1, x, x^2\ $.

Find the matrix of D with respect to B and C.

The images of B **under** D **are** $D(1) = 0, D(x) = 1, D(x^2) = 2x, D(x^3) = 3x^2$ **and therefore their coordinate vectors with respect to** C **are:**

$$D(1)\ _C = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ D(x)\ _C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \left[D(x^2)\right]_C = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \left[D(x^3)\right]_C = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

Therefore $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Find the matrix A' of D with respect to B' and C where $B' = \ x^3, x^2, x, 1$

The order of the vectors in a basis will affect the matrix of a transformation with respect to the basis. Since basis B' is simply B in reverse order, we see that

$$A' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}$$

Use matrix A to compute $D(5 - x + 2x^3)$

$$A\left[5 - x + 2x^3\right]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix} = \left[D(5 - x + 2x^3\right]_C$$

Let $T : P_2 \to P_2 : T(p(x) = p(2x-1)$

Find the matrix of the linear transformation with respect to $B = \; 1, x, x^2$

$$T(1) = 1, T(x) = 2x-1, T(x^2) = (2x-1)^2 = 1-4x+4x^2$$

$$T(1)_{\;B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\; T(x)_{\;B} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \left[T(x^2)\right]_B = \begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix}$$

Therefore $T_{\;B} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}$

Compute $T(3+2x-x^2)$

MATRICES OF COMPOSITE AND INVERSE TRANSFORMATIONS

$T:\mathbb{R}^2 \to P_1 : T\begin{bmatrix} a \\ b \end{bmatrix} = a+(a+b)x$ **and** $S : P_1 \to P_2 : S(a+bx) = ax+bx^2 = xp(x)$

FIND A MATRIX FOR $S \circ T$

- **Standard Basis for** $\mathbb{R}^2$ **is** $B = \ e_1, e_2$
- **Standard Basis for** P_1 **is** $C = \ 1, x$
- **Standard Basis for** P_2 **is** $D = \ 1, x, x^2$

$$T(e_1) = 1+(1+0)x = 1+x \quad T(e_2) = 0+(0+1)x = x$$

$$T \ _C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$S\begin{bmatrix} 1 \\ 1 \end{bmatrix} = x+x^2 \quad S\begin{bmatrix} 0 \\ 1 \end{bmatrix} = x^2$$

$$S \ _D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S \circ T \ = \ S \quad T \ = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$T:\mathbb{R}^2 \to P_1 : T\begin{bmatrix} a \\ b \end{bmatrix} = a+(a+b)x$$

FIND T^{-1}

Because the transformation is both one-to-one and onto, it is invertible.

In the above example we found $T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$$\left[T^{-1}\right] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

This implies $\left[T^{-1}(a+bx)\right] = \left[T^{-1}\right] \ a+bx \ = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b-a \end{bmatrix}$

This is the same as $T^{-1}(a+bx) = ae_1 + (b-a)e_2 = \begin{bmatrix} a \\ b-a \end{bmatrix}$

AN EXAMPLE FROM CALCULUS

Using the inverse of a differential operator, find $\int x^2 e^{3x} dx$ **(which usually requires two applications of integration by parts.)**

From the example problem given in class, we found $D_B = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ **for which**

$$B = \ e^{3x}, xe^{3x}, x^2e^{3x}$$

$$\left[D^{-1}\right]_B = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{9} & \frac{2}{27} \\ 0 & \frac{1}{3} & -\frac{2}{9} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$\left[x^2e^{3x}\right]_B=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\left[\int x^2e^{3x}dx\right]_B=\left[D^{-1}(x^2e^{3x})\right]B$$

$$=\left[D^{-1}\right]_B\left[x^2e^{3x}\right]_B$$

$$=\begin{bmatrix}\frac{1}{3} & -\frac{1}{9} & \frac{2}{27}\\0 & \frac{1}{3} & -\frac{2}{9}\\0 & 0 & \frac{1}{3}\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}\frac{2}{27}\\-\frac{2}{9}\\\frac{1}{3}\end{bmatrix}$$

$$\int x^2e^{3x}dx=\frac{2}{27}e^{3x}-\frac{2}{9}xe^{3x}+\frac{1}{3}x^2e^{3x}+C$$

CHANGE OF BASIS AND SIMILARITY

$$T:\mathbb{R}^2 \to \mathbb{R}^2 : T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+3y \\ 2x+2y \end{bmatrix}$$

Is it possible to find a basis B for $\mathbb{R}^2$ such that the transformation matrix T is diagonal with respect to B?

With respect to a standard basis, $[T]_E = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$.

We can show that $[T]_E = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ **is diagonalizable into** $[T]_E = S\Lambda S^{-1}$**, in which** Λ **is a diagonal matrix.**

$$\Lambda = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \quad \textbf{and} \quad S = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}$$

If $[T]_E = S\Lambda S^{-1}$**, then** $\Lambda = S^{-1}[T]_E S$**. Let** B **be the basis in** $\mathbb{R}^2$ **consisting of the columns of** S **, then** S **is the change-of-basis matrix from** B **to** E**. Then**

$$[T]_B = S^{-1}[T]_E S = \Lambda$$

Therefore the transformation matrix T **with respect to the basis** $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right\}$ **is diagonal.**

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad \textbf{and} \quad T\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

So the vectors that form the columns of $[T]_B$ **are**

$$\left[T\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]_B = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \quad \textbf{and} \quad \left[T\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right]_B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$T_{E} \sim T_{B}$ **because when $T: V \to W$, changing a basis for V of transformation matrix A would require multiplication AM (multiply by M on the right to come first). Chaging the basis for W would change A to $M^{-1}A$ (to come last). Therefore, to change both bases the same way, the new matrix is $B = M^{-1}AM$. The good basis vectors are thus the eigenvectors of A, and $B = M^{-1}AM$ becomes $B = S^{-1}AS$.**

WORKING WITHIN A NON-STANDARD OR NON-EIGEN BASIS

$$T: P_2 \to P_2 : T(p(x)) = p(2x-1)$$

Find T with respect to basis $B = 1+x, 1-x, x^2$

With respect to a standard basis :

$$T(1) = 1, T(x) = 2x-1, T(x^2) = (2x-1)^2 = 1-4x+4x^2$$

$$T(1)_{E} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ T(x)_{E} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \left[T(x^2)\right]_E = \begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix}$$

Therefore $T_{E} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}$

The change of basis matrix from B to E is $\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Therefore it follows that $[T]_B = M^{-1}[T]_E M$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ -1 & 2 & \frac{5}{2} \\ 0 & 0 & 4 \end{bmatrix}$$

Find a basis C **for** P_2 **such that** $[T]_C$ **is a diagonal matrix.**

$[T]_E$ **has eigenvalues 1, 2, and 4 with eigenvectors** $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$S = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$ **and** $\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$S^{-1}[T]_E S = \Lambda$**, and therefore** S **is a change of basis matrix from a basis** C **to** E**.**

Therefore $C = \{1, -1+x, 1-2x+x^2\}$

INVERSES: LEFT/RIGHT/PSEUDOINVERSE

2-SIDED INVERSES

- **A is full row rank.**
- **A is full column rank.**
- **1 solution to $Ax = b$**
- $n = m = r$
- $A^{-1}A = I = AA^{-1}$

LEFT INVERSES

- **A is full column rank.**
- **A has independent columns.**
- $N(A) = 0$
- **0 or 1 solution to $Ax = b$**
- $r = n < m$
- **$A^T A$ is symmetric and invertible**
- **Therefore $\left[(A^T A)^{-1} A^T\right]$ is a "left inverse" because $\left[(A^T A)^{-1} A^T\right] A = I$**
- **If we multiply $\left[(A^T A)^{-1} A^T\right]$ on the right, we get $A\left[(A^T A)^{-1} A^T\right]$ which is P, a projection onto the column space.**

RIGHT INVERSES

- **A is full row rank.**
- **A has independent rows.**
- $N(A^T) = 0$
- **Infinite solutions to $Ax = b$**
- $r = m < n$
- **$n - m$ free variables.**
- **AA^T is invertible**
- **Therefore $\left[A^T (AA^T)^{-1}\right]$ is a "right inverse" because $A\left[A^T (AA^T)^{-1}\right] = I$**
- **If we multiply $\left[A^T (AA^T)^{-1}\right]$ on the left we get $\left[A^T (AA^T)^{-1}\right] A$, which is a projection onto the row space.**

PSEUDOINVERSES

- **A is neither full row nor full column rank.**
- **A has dependent columns and dependent rows**
- **$r < n$ and $r < m$**
- **Nonetheless, there is a one-to-one and onto relationship between a row vector x and a column vector Ax, ignoring the left and right null spaces of the matrix.**

To Calculate A Pseudoinverse:

- **Because every matrix has a singular value decomposition, the pseudoinverse is calculated from the SVD.**
- **Let $A = U\Sigma V^T$ in which U and V^T are orthogonal square invertible matrices, and Σ is an $m \times n$ matrix of singular values:** $\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0 \end{bmatrix}$.
- **The pseudoinverse, A^+, can then be found through the inverse of $A = U\Sigma V^T$,** $A^+ = V\Sigma^+ U^T$.
- **Because U and V^T are invertible, there is no need to find their pseudoinverse. Because Σ is a purely diagonal matrix, it's inverse is easy to calculate:**

$$\Sigma^+ = \begin{bmatrix} \frac{1}{\sigma_1} & & & \\ & \ddots & & \\ & & \frac{1}{\sigma_r} & \\ & & & 0 \end{bmatrix}.$$

- **Note that** $\Sigma\Sigma^+ = \Sigma^+\Sigma = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{bmatrix} \neq I$

Let A be an $n \times n$ matrix. Let $T: V \to W$ be a linear transformation whose

transformation matrix T is A.

THE FOLLOWING ARE ALL EQUIVALENT STATEMENTS:

- A is invertible.
- $Ax = b$ has a unique solution for every b in $\mathbb{R}^n$.
- $Ax = 0$ has only the trivial solution.
- The reduced row echelon form of A is I.
- A is a product of elementary matrices.
- rank (A) = n.
- nullity (A)=0.
- The column vectors of A are linearly independent.
- The column vectors of A span $\mathbb{R}^n$.
- The column vectors of A form a basis for $\mathbb{R}^n$.
- The row vectors of A are linearly independent.
- The row vectors of A span $\mathbb{R}^n$.
- The row vectors of A form a basis for $\mathbb{R}^n$.
- $|A| \neq 0$
- 0 is not an eigenvalue of A.
- 0 is not a singular value of A.
- T is invertible.
- T is one-to-one and onto.
- ker $(T) =$ 0
- range $(T) = W$