

## Complex Linear Independence: Decomplexification

4/7:

- When given a complex system of equations, it is necessary to **decomplexify** it.
- **Decomplexify:** To model a complex system of equations with a strictly real system for the purpose of applying the tenets of real linear algebra to it.
- Consider the following complex system of equations.

$$\begin{aligned}(2+i)x_1 + (1+i)x_2 &= 3+6i \\ (3-i)x_1 + (2-2i)x_2 &= 7-i\end{aligned}$$

– The solutions will be complex numbers:  $x_1 = a_1 + ib_1$  and  $x_2 = a_2 + ib_2$ , where  $a_1, a_2, b_1, b_2 \in \mathbb{R}$ .

- Transform it into a matrix system of equations. Separate the real and complex parts, and factor out all instances of the imaginary number  $i$  so that it is a coefficient to any complex matrix.

$$\begin{aligned}\begin{bmatrix} 2+i & 1+i \\ 3-i & 2-2i \end{bmatrix} \begin{bmatrix} a_1+ib_1 \\ a_2+ib_2 \end{bmatrix} &= \begin{bmatrix} 3+6i \\ 7-i \end{bmatrix} \\ \left( \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} i & i \\ -i & -2i \end{bmatrix} \right) \left( \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} ib_1 \\ ib_2 \end{bmatrix} \right) &= \left( \begin{bmatrix} 3 \\ 7 \end{bmatrix} + \begin{bmatrix} 6i \\ -i \end{bmatrix} \right) \\ \underbrace{\left( \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + i \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \right)}_A \underbrace{\left( \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + i \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right)}_x &= \underbrace{\left( \begin{bmatrix} 3 \\ 7 \end{bmatrix} + i \begin{bmatrix} 6 \\ -1 \end{bmatrix} \right)}_b\end{aligned}$$

- Foil the left side of the above equation<sup>[1]</sup>.

$$\left( \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) + i \left( \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 7 \end{bmatrix} + i \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

- Split the above system of equations into a real system of equations and a complex system of equations by setting equal to each other the real components of each side and the imaginary components of each side.

$$\begin{aligned}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} &= \begin{bmatrix} 3 \\ 7 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} &= \begin{bmatrix} 6 \\ -1 \end{bmatrix}\end{aligned}$$

- Multiply out the matrices above to yield a system of four equations.

$$\begin{aligned}2a_1 + a_2 - b_1 - b_2 &= 3 \\ 3a_1 + 2a_2 + b_1 + 2b_2 &= 7 \\ a_1 + a_2 + 2b_1 + b_2 &= 6 \\ -a_1 - 2a_2 + 3b_1 + 2b_2 &= -1\end{aligned}$$

- Condense the above system of equations into a single matrix system of equations.

$$\begin{bmatrix} 2 & 1 & -1 & -1 \\ 3 & 2 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ -1 & -2 & 3 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 6 \\ -1 \end{bmatrix}$$

<sup>1</sup>Note that the minus sign appears in the real component because, when multiplying the two “last” parts,  $i^2 = -1$ .

- Solve for  $a_1, a_2, b_1$ , and  $b_2$  using an augmented matrix and Gauss-Jordan elimination.

$$\left[ \begin{array}{cccc|c} 2 & 1 & -1 & -1 & 3 \\ 3 & 2 & 1 & 2 & 7 \\ 1 & 1 & 2 & 1 & 6 \\ -1 & -2 & 3 & 2 & -1 \end{array} \right] \rightarrow \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{array} \right]$$

$$\begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}$$

- From these four values, the original solutions  $x_1 = a_1 + ib_1$  and  $x_2 = a_2 + ib_2$  can be found.

$$x_1 = 1 + 2i$$

$$x_2 = 2 - i$$

## Hermitian, Unitary, and Normal Matrices

4/13:

- What necessitates different categorizations of complex vectors and matrices?
- Consider a vector  $v$ .

$$v = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

- If you want to find  $\|v\|$ , you typically evaluate  $\sqrt{v^T v}$ . However, this equals to 0 (see the following), which is clearly not the magnitude of  $v$ .

$$\begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 - 1 = 0$$

– Note that  $\|v\|$  must be an element of  $\mathbb{R}$  because it measures a distance.

- With complex vectors, it is necessary to evaluate  $\sqrt{\bar{v}^T v}$  to find  $\|v\|$ .

$$\begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 + 1 = 2$$

$$\|v\| = \sqrt{2}$$

- This makes sense because  $\begin{bmatrix} 1 \\ i \end{bmatrix}$  extends one unit into  $\mathbb{R}^1$  and one unit into  $\mathbb{C}^2$ .
- If  $z\bar{z} = |z|^2$  and  $\bar{v}^T v = v \cdot \bar{v}$ , it stands to reason that  $\bar{v}^T v = \|v\|^2$ . Essentially, the dot product multiplies every element of  $v$  by its complex conjugate and sums them.
- Instead of writing  $\bar{v}^T$ <sup>[2]</sup> every time, mathematicians shorthand to  $v^H$ <sup>[3]</sup>.
  - $v^H$  works for all vectors, but it is necessary for complex ones.
- **Hermitian** (matrix): A matrix  $A$  such that  $A = A^H$ .
  - Typically defined for  $A \in \mathbb{C}^n$ , but holds for  $A \in \mathbb{R}^n$ , too.
  - Parallel to how if  $A \in \mathbb{R}^n$  and  $A = A^T$ ,  $A$  is symmetrical.
  - Also note that if  $A^H A = A^2 = A A^H$ ,  $A$  is Hermitian.

<sup>2</sup>“ $v$  conjugate transpose”

<sup>3</sup>“ $v$  Hermitian” after French mathematician Charles Hermite.

- A Hermitian matrix has to have real values on the principal diagonal. When  $A$  is transposed and conjugated, the diagonal entries are the only values that don't move. Thus, their conjugates must equal themselves, so they must be real<sup>[4]</sup>.
- **Unitary** (matrix): A matrix  $A$  such that  $A^{-1} = A^H$ .
  - Typically defined for  $A \in \mathbb{C}^n$ , but holds for  $A \in \mathbb{R}^n$ , too.
  - Parallel to how if  $A \in \mathbb{R}^n$  and  $A^{-1} = A^T$ ,  $A$  is orthonormal.
  - Also note that if  $A^H A = I = A A^H$ ,  $A$  is unitary.
- **Normal** (matrix): A matrix that is unitarily diagonalizable.
  - Typically defined for  $A \in \mathbb{C}^n$ , but holds for  $A \in \mathbb{R}^n$ , too.
  - Parallel to matrices  $A \in \mathbb{R}^n$  such that  $A$  is orthonormally diagonalizable.
- Note that not every complex matrix has to be one of these three types.
- When  $A^H A = A A^H$ ,  $A = U \Lambda U^H$ .

$$\begin{aligned}
 A A^H &= (U \Lambda U^H) (U \Lambda U^H)^H \\
 &= U \Lambda U^H U \Lambda^H U^H \\
 &= U \Lambda \Lambda^H U^H \\
 &= U \Lambda^H \Lambda U^H \text{ [5]} \\
 &= U \Lambda^H U^H U \Lambda U^H \\
 &= (U \Lambda U^H)^H (U \Lambda U^H) \\
 &= A^H A
 \end{aligned}$$

- When  $A = A^H$ , all eigenvalues are elements of  $\mathbb{R}$  (similar to spectral theorem).

$$v^H A v = (v^H A v)^H = v^H A v$$

- The above proves that  $v^H A v \in \mathbb{R}$  because it's its own conjugate<sup>[4]</sup>.

$$\begin{aligned}
 A v &= \lambda v \\
 v^H A v &= \lambda v^H v
 \end{aligned}$$

- $\lambda = \frac{v^H A v}{v^H v} \rightarrow \frac{\mathbb{R}}{\mathbb{R}} = \mathbb{R}$ <sup>[6]</sup>.
- When  $A = A^H$  and  $A x = \lambda x$ , all  $x$ 's can be chosen orthonormally (also similar to spectral theorem).
  - Normality is implied because any eigenvector can be scaled to any version (including a normal version) and still be an eigenvector.

$$x_i = \begin{bmatrix} x_{i_1} \\ x_{i_2} \\ \vdots \\ x_{i_n} \end{bmatrix} \qquad x_i^H = [\bar{x}_{i_1} \quad \bar{x}_{i_2} \quad \cdots \quad \bar{x}_{i_n}]$$

<sup>4</sup>Recall that only real quantities can be their own conjugates because  $a + 0i = a - 0i$ .

<sup>5</sup>Since  $\Lambda = \Lambda^H$ .

<sup>6</sup>Note that the denominator is real because it's how one finds  $\|v\|$ , and  $\|v\|$  must be real, as discussed above.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad A^H = \begin{bmatrix} a_{11} & \bar{a}_{21} & \cdots & \bar{a}_{n1} \\ \bar{a}_{12} & a_{22} & \cdots & \bar{a}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{1n} & \bar{a}_{2n} & \cdots & a_{nn} \end{bmatrix}$$

- Define an arbitrary vector  $x_i$  and matrix  $A$ , along with their conjugate transposes (or Hermitian versions). Note that the diagonal entries of  $A^H$  aren't shown as conjugated because their conjugates equal themselves.

$$Ax_1 = \lambda_1 x_1$$

$$x_2^H Ax_1 = \lambda_1 x_2^H x_1$$

$$Ax_2 = \lambda_2 x_2$$

$$(Ax_2)^H = (\lambda_2 x_2)^H$$

$$x_2^H A^H = \lambda_2 x_2^H$$

$$x_2^H Ax_1 = \lambda_2 x_2^H x_1$$

- $\lambda_1 x_2^H x_1 = \lambda_2 x_2^H x_1$  implies that, since  $\lambda_1 \neq \lambda_2$ ,  $x_2^H x_1$  must equal 0, proving orthogonality.

## Complex Diagonalization

- 4/15:      • Diagonalize the following matrix  $A$ .

$$A = \begin{bmatrix} 0.9 & -0.4 \\ 0.1 & 0.9 \end{bmatrix}$$

- Find the characteristic polynomial.

$$\begin{aligned} 0 &= \begin{vmatrix} 0.9 - \lambda & -0.4 \\ 0.1 & 0.9 - \lambda \end{vmatrix} \\ &= (0.9 - \lambda)^2 - (-0.4)(0.1) \\ &= 0.81 - 1.8\lambda + \lambda^2 + 0.04 \\ &= \lambda^2 - 1.8\lambda + 0.85 \end{aligned}$$

- Find the eigenvalues<sup>[7]</sup>.

$$\begin{aligned} \lambda &= \frac{-(-1.8) \pm \sqrt{(-1.8)^2 - 4(1)(0.85)}}{2(1)} \\ &= 0.9 \pm \frac{\sqrt{-0.16}}{2} \\ &= 0.9 \pm \frac{\sqrt{-1}\sqrt{0.16}}{2} \\ &= 0.9 \pm \frac{0.4i}{2} \\ &= 0.9 \pm 0.2i \end{aligned}$$

$$\lambda_1 = 0.9 + 0.2i$$

$$\lambda_2 = 0.9 - 0.2i$$

<sup>7</sup>It is interesting that the eigenvalues are complex conjugates of each other.

- Find the eigenvectors<sup>[8]</sup>.

$$\begin{aligned}(A - (0.9 + 0.2i))x_1 &= \begin{bmatrix} 0.9 - (0.9 + 0.2i) & -0.4 \\ 0.1 & 0.9 - (0.9 + 0.2i) \end{bmatrix} \begin{bmatrix} x_{1_1} \\ x_{1_2} \end{bmatrix} \\ &= \begin{bmatrix} -0.2i & -0.4 \\ 0.1 & -0.2i \end{bmatrix} \begin{bmatrix} 2i \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}(A - (0.9 - 0.2i))x_1 &= \begin{bmatrix} 0.9 - (0.9 - 0.2i) & -0.4 \\ 0.1 & 0.9 - (0.9 - 0.2i) \end{bmatrix} \begin{bmatrix} x_{1_1} \\ x_{1_2} \end{bmatrix} \\ &= \begin{bmatrix} 0.2i & -0.4 \\ 0.1 & 0.2i \end{bmatrix} \begin{bmatrix} -2i \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}\end{aligned}$$

$$x_1 = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} -2i \\ 1 \end{bmatrix}$$

- Compile the diagonalization.

$$A = \frac{1}{4i} \begin{bmatrix} 2i & -2i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0.9 + 0.2i & 0 \\ 0 & 0.9 - 0.2i \end{bmatrix} \begin{bmatrix} 1 & 2i \\ -1 & 2i \end{bmatrix}$$

## Real versus Complex

4/16:

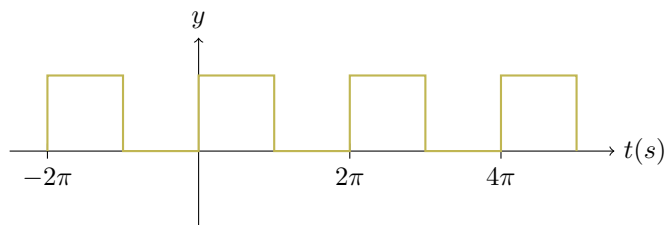
Real	Complex
$\mathbb{R}^n$ : vectors with $n$ real components	$\mathbb{C}^n$ : vectors with $n$ complex components
length: $\ x\ ^2 = x_1^2 + \cdots + x_n^2$	length: $\ z\ ^2 =  z_1 ^2 + \cdots +  z_n ^2$
transpose: $(A^T)_{ij} = A_{ji}$	conjugate transpose: $(A^H)_{ij} = \overline{A_{ji}}$
product rule: $(AB)^T = B^T A^T$	product rule: $(AB)^H = B^H A^H$
dot product: $x^T y = x_1 y_1 + \cdots + x_n y_n$	inner product: $u^H v = \bar{u}_1 v_1 + \cdots + \bar{u}_n v_n$
reason for $A^T$ : $(Ax)^T y = x^T (A^T y)$	reason for $A^H$ : $(Au)^H v = u^H (A^H v)$
orthogonality: $x^T y = 0$	orthogonality: $u^H v = 0$ .
symmetric matrices: $A = A^T$	Hermitian matrices: $A = A^H$
$A = Q\Lambda Q^{-1} = Q\Lambda Q^T$ (real $\Lambda$ )	$A = U\Lambda U^{-1} = U\Lambda U^H$ (real $\Lambda$ )
skew-symmetric matrices: $k^T = -K$	skew-Hermitian matrices: $K^H = -K$
orthogonal matrices: $Q^T = Q^{-1}$	unitary matrices: $U^H = U^{-1}$
orthonormal columns: $Q^T Q = I$	orthonormal columns: $U^H = U^{-1}$
$(Qx)^T (Qy) = x^T y$ and $\ Qx\  = \ x\ $	$(Ux)^H (Uy) = x^H y$ and $\ Uz\  = \ z\ $

- Note that the columns and eigenvectors of  $Q$  and  $U$  are orthonormal, and all of their eigenvalues  $\lambda$  satisfy  $|\lambda| = 1$ .

<sup>8</sup>It is interesting that the eigenvectors are *also* complex conjugates of each other.

## Real Fourier Series

- 4/21: • Consider the square wave  $f(t)$ .

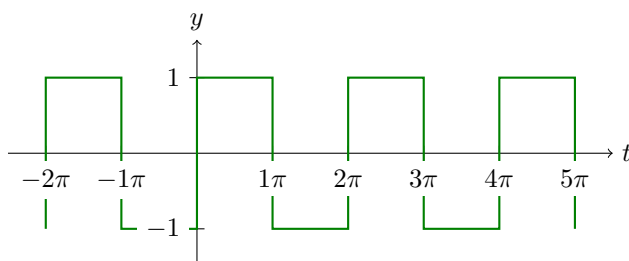


- Its period is  $2\pi \frac{\text{sec}}{\text{cycle}}$ , and its frequency is  $\frac{1}{2\pi}$  Hz.
- Can we write  $f(t)$  as a sum of sines and cosines?

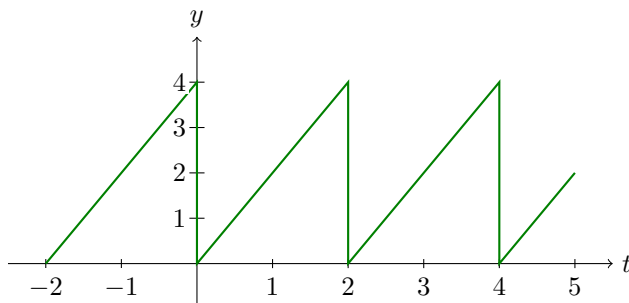
$$f(t) = a_0 + a_1 \cos(t) + b_1 \sin(t) + a_2 \cos(2t) + b_2 \sin(2t) + a_3 \cos(3t) + b_3 \sin(3t) + \dots$$

- Be general.
- Since  $T = 2\pi$ , it makes sense to use some functions with  $T = 2\pi$  to model it.
- The weighting coefficients account for how much each function contributes to the whole.
- Historically studied by Fourier, who studied differential equations. Differential equations were often easy to solve for sines and cosines, so if a function could be modeled by a sum of sines and cosines, a related differential equation would be easier to solve.
- Fourier series, transforms, and analysis also tell us how much of each frequency a function contains (as measured by the weight coefficients).

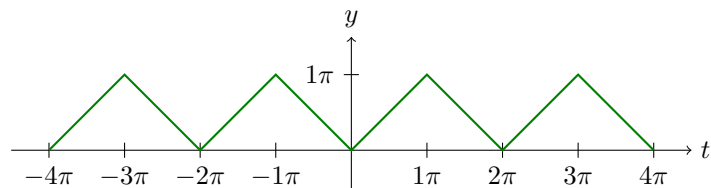
## Wave Type Sketches



(a) Square wave.



(b) Sawtooth wave.



(c) Triangular wave.

- Square wave:  $f(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & \pi < t < 2\pi \end{cases}$  where periodicity is defined by  $f(t + 2\pi) = f(t)$ .
- Sawtooth wave:  $f(t) = 2t$   $0 < t < 2$  where periodicity is defined by  $f(t + 2) = f(t)$ .
- Triangular wave:  $f(t) = |t|$   $-\pi < t < \pi$  where periodicity is defined by  $f(t + 2\pi) = f(t)$ .

1) Evaluate  $\int_{-\pi}^{\pi} \sin(nt) dt$ , where  $n$  is an integer.

2) Evaluate  $\int_{-\pi}^{\pi} \cos(nt) dt$ , where  $n$  is an integer.

3) Using the results from problem 1 and problem 2, integrate both sides of the equation

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)) \text{ from } -\pi \text{ to } \pi. \text{ Then simplify the result in terms of } a_0. \text{ Divide the result in half and simplify in terms of } \frac{a_0}{2}.$$

4) Using trigonometric identity  $\sin(nt)\cos(mt) = \frac{1}{2}(\sin(n+m)t + \sin(n-m)t)$ , evaluate  $\int_{-\pi}^{\pi} \sin(nt)\cos(mt) dt$  where  $n$  and  $m$  are any integers.

5) Evaluate  $\int_{-\pi}^{\pi} \cos(nt)\cos(mt) dt$  where  $n$  and  $m$  are any integers and  $n \neq m$ .

6) Evaluate  $\int_{-\pi}^{\pi} \cos(nt)\cos(mt) dt$  where  $n$  and  $m$  are any integers and  $n = m$  and  $n \neq 0$ , in other words evaluate  $\int_{-\pi}^{\pi} \cos^2(nt) dt$ .

7) Evaluate  $\int_{-\pi}^{\pi} \cos(nt)\cos(mt) dt$  when  $n = m = 0$

8) In a similar way to 5- 7, evaluate  $\int_{-\pi}^{\pi} \sin(nt)\sin(mt) dt$  for the cases where  $n \neq m, n = m \neq 0, n = m = 0$ ,

Hints: Use identity  $\sin(nt)\sin(mt) = \frac{1}{2}(\cos(n-m)t - \cos(n+m)t)$  for the case when  $n \neq m$  and use identity  $\cos(2\theta) = 1 - 2\sin^2 \theta$  for which  $\theta = nt$  for the case when  $n = m \neq 0$ .



$$1) \int_{-\pi}^{\pi} \sin(nt) dt = \left[ -\frac{1}{n} \cos(nt) \right]_{-\pi}^{\pi} = \frac{1}{n} (-\cos(n\pi) + \cos(n\pi)) = 0, n \neq 0$$

$$2) \int_{-\pi}^{\pi} \cos(nt) dt = \left[ \frac{1}{n} \sin(nt) \right]_{-\pi}^{\pi} = 0, n \neq 0$$

$$3) \int_{-\pi}^{\pi} f(t) dt = \frac{1}{2} \int_{-\pi}^{\pi} a_0 dt + \sum_{n=1}^{\infty} \left( \int_{-\pi}^{\pi} a_n \cos(nt) dt + \int_{-\pi}^{\pi} b_n \sin(nt) dt \right) = \frac{1}{2} [a_0 t]_{-\pi}^{\pi} + \sum_{n=1}^{\infty} (0+0)$$

$$\therefore \int_{-\pi}^{\pi} f(t) dt = \frac{1}{2} (2a_0 \pi) \rightarrow a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt \rightarrow \frac{a_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$$

$$4) \int_{-\pi}^{\pi} \sin(nt) \cos(mt) dt = \frac{1}{2} \left( \int_{-\pi}^{\pi} \sin(n+m)t dt + \int_{-\pi}^{\pi} \sin(n-m)t dt \right) = \frac{1}{2} (0+0) = 0 \text{ using the results from problems 1 and}$$

2 since if  $n$  is an integer and  $m$  is an integer then  $n+m$  and  $n-m$  are also integers.

$$5) \int_{-\pi}^{\pi} \cos(nt) \cos(mt) dt = \frac{1}{2} \left( \int_{-\pi}^{\pi} \cos(n+m)t dt + \int_{-\pi}^{\pi} \cos(n-m)t dt \right) = \frac{1}{2} (0+0) = 0 \text{ when } n \neq m$$

$$6) \int_{-\pi}^{\pi} \cos^2(nt) dt = \frac{1}{2} \int_{-\pi}^{\pi} (1 + \cos(2n)t) dt = \frac{1}{2} \left[ t + \frac{1}{2n} \sin(2n)t \right]_{-\pi}^{\pi} = \pi \text{ when } n \neq 0$$

$$7) \pi - (-\pi) = 2\pi$$

$$8) a) \int_{-\pi}^{\pi} \sin(nt) \sin(mt) dt = 0$$

$$b) \int_{-\pi}^{\pi} \sin^2(nt) dt = \frac{1}{2} \int_{-\pi}^{\pi} (1 - \cos(2nt)) dt = \pi$$

$$c) \int_{-\pi}^{\pi} \sin(nt) \sin(mt) dt = 0$$

For integers  $m, n$ :

$$\int_{-\pi}^{\pi} \sin(nt) \cos(mt) dt = 0$$

$$\int_{-\pi}^{\pi} \cos(nt) \cos(mt) dt = \begin{cases} 0, n \neq m \\ \pi, n = m \neq 0 \\ 2\pi, n = m = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(nt) \sin(mt) dt = \begin{cases} 0, n \neq m, n = m = 0 \\ \pi, n = m \end{cases}$$

## FOURIER SERIES

- We have shown that when  $v^T w = 0$ , then vectors  $v$  and  $w$  are orthogonal.
- Fourier series ask us to think of continuous functions as vectors.
- Let  $f(t) = \cos(t)$  and let  $g(t) = \sin(t)$
- To find  $f \cdot g$  of these two function “vectors” would be asking us to take a dot product that has infinitely many terms. All vectors we have used have had a finite number of components. Because these “vectors” would have infinitely many terms, it would be like asking us to take an integral!
- $f \cdot g = \int_0^{2\pi} \cos(t) \sin(t) dt$  (We limit the domain because these are sinusoidal periodic functions and thus they repeat after a period of  $2\pi$ .)

$$\int_0^{2\pi} \cos(t) \sin(t) dt$$

Let  $u = \cos(t)$  and  $du = -\sin(t)$

$$-\int_1^1 u du$$

$$\left[ -\frac{u^2}{2} \right]_1^1$$

0

OR

$$\left[ -\frac{1}{2} \cos^2(t) \right]_0^{2\pi}$$

$$-\frac{1}{2} (\cos^2 2\pi - \cos^2 0)$$

$$-\frac{1}{2} (1 - 1) =$$

0

- Therefore  $\cos t \cdot \sin t = 0$ , therefore these function “vectors” are orthogonal, therefore they serve as basis vectors for the function space of continuous periodic functions in  $F^2$ , therefore any continuous function also in this space can be written as a linear combination of  $\cos(t)$  and  $\sin(t)$  !!!

$$f(t) = \frac{a_0}{2} + a_1 \cos(t) + b_1 \sin(t) + a_2 \cos(2t) + b_2 \sin(2t) + \dots$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt))$$

\*Why divide the constant term by 2?

## CALCULATING THE FOURIER COEFFICIENTS

As sine and cosine can serve as an orthogonal basis for periodic functions, consider the Fourier Series for a function

$$f(t) \text{ of period } 2\pi \text{ to be } f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt))$$

In problem 3 from your handout you already found a formula for  $a_0$ , the constant term.

$$\frac{a_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$$

To obtain the coefficients  $a_n, n \in \mathbb{Z}$ , multiply both sides of the equation by  $\cos(mt)$  where  $m \in \mathbb{Z}$  and  $m > 0$  and integrate both sides from  $-\pi$  to  $\pi$  since this is a periodic function of period  $2\pi$ .

$$\int_{-\pi}^{\pi} f(t) \cos(mt) dt = \frac{a_0}{2} \int_{-\pi}^{\pi} \cos(mt) dt + \sum_{n=1}^{\infty} \left( a_n \int_{-\pi}^{\pi} \cos(nt) \cos(mt) dt + b_n \int_{-\pi}^{\pi} \sin(nt) \cos(mt) dt \right)$$

Simplify this result using the integrals on the right side of this equation from the handout problems 5-7. The only nonzero integral results from  $\int_{-\pi}^{\pi} \cos^2(mt) dt = \pi$  in the case where  $n = m$ .

$$\therefore \int_{-\pi}^{\pi} \cos(mt) dt = a_m \pi$$

Since this is the case where  $n = m$ , replacing  $m$  with  $n$  and solving for  $a_n$  we obtain:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt, n \in \mathbb{Z} \text{ and } n > 0$$

To obtain the coefficients  $b_n, n \in \mathbb{Z}, n > 0$ , multiply both sides of the equation by  $\sin(mt)$  where  $m \in \mathbb{Z}$  and  $m > 0$  and integrate both sides from  $-\pi$  to  $\pi$  since this is a periodic function of period  $2\pi$ .

$$\int_{-\pi}^{\pi} f(t) \sin(mt) dt = \frac{a_0}{2} \int_{-\pi}^{\pi} \sin(mt) dt + \sum_{n=1}^{\infty} \left( a_n \int_{-\pi}^{\pi} \cos(nt) \sin(mt) dt + b_n \int_{-\pi}^{\pi} \sin(nt) \sin(mt) dt \right)$$

Simplify this result using the integrals on the right side of this equation using the handout problems you did for 8. The only nonzero integral results from  $\int_{-\pi}^{\pi} b_m \sin^2(mt) dt = b_m \pi$  in the case where  $n = m$ . Relabeling  $m$  as  $n$  and solving for  $b_n$  we obtain:

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt, n \in \mathbb{Z} \text{ and } n > 0$$

## Square Wave

4/29: • **Hilbert space:** An infinite-dimensional vector space — extends a lot of the ideas of linear algebra to functions in infinite dimensions.

– Recall that functions behave with linearity ( $\alpha f(t) + \beta g(t)$  is still a function).

• Square wave (period  $2\pi$ ):  $f(t) = \begin{cases} 0 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases}$  and  $f(t + 2\pi) = f(t)$ .

• Find  $\frac{a_0}{2}$  term:

$$\begin{aligned} \frac{a_0}{2} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^0 (0) dt + \frac{1}{2\pi} \int_0^{\pi} (1) dt \\ &= 0 + \left[ \frac{t}{2\pi} \right]_0^{\pi} \\ &= \frac{\pi}{2\pi} \\ \boxed{\frac{a_0}{2} = \frac{1}{2}} \end{aligned}$$

– Makes sense because  $\frac{a_0}{2}$  is like the sinusoidal axis and  $\frac{1}{2}$  is half way between 0 and 1.

• Find  $a_n$  terms:

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt \\ &= \frac{1}{\pi} \int_{-\pi}^0 (0) \cos(nt) dt + \frac{1}{\pi} \int_0^{\pi} (1) \cos(nt) dt \\ &= 0 + \left[ \frac{1}{n\pi} \sin(nt) \right]_0^{\pi} \\ \boxed{a_n = 0} \end{aligned}$$

• Find  $b_n$  terms:

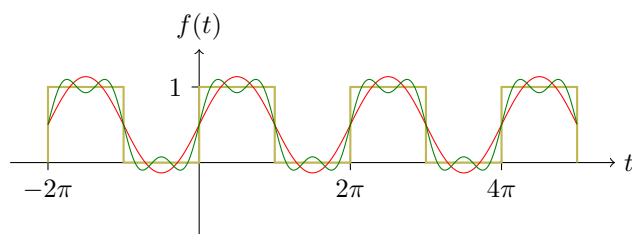
$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt \\ &= \frac{1}{\pi} \int_{-\pi}^0 (0) \sin(nt) dt + \frac{1}{\pi} \int_0^{\pi} (1) \sin(nt) dt \\ &= 0 - \frac{1}{n\pi} [\cos(nt)]_0^{\pi} \\ &= -\frac{1}{n\pi} [\cos(n\pi) - \cos(0)] \\ &= -\frac{1}{n\pi} [\cos(n\pi) - 1]^{[9]} \\ \boxed{b_n = \begin{cases} 0 & n = 2, 4, 6, \dots \\ \frac{2}{n\pi} & n = 1, 3, 5, \dots \end{cases}} \end{aligned}$$

---

<sup>9</sup> $\cos(n\pi)$  equals 1 when  $n$  is even and  $-1$  when  $n$  is odd.

- Assemble the Fourier series for the square wave function:

$$\begin{aligned} f(t) &= \frac{1}{2} + \frac{2}{\pi} \sin(t) + \frac{2}{3\pi} \sin(3t) + \frac{2}{5\pi} \sin(5t) + \cdots \\ &= \frac{1}{2} + \frac{2}{\pi} \left( \frac{1}{1} \sin(t) + \frac{1}{3} \sin(3t) + \frac{1}{5} \sin(5t) + \cdots \right) \\ &= \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin((2n-1)t) \end{aligned}$$



## FOURIER SERIES FOR PERIODIC FUNCTIONS WHEN PERIOD $\neq 2\pi$

Given:

$$\frac{a_0}{2} = \int_{-\pi}^{\pi} \frac{1}{2\pi} f(t) dt$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt, n \in \mathbb{Z} \text{ and } n > 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt, n \in \mathbb{Z} \text{ and } n > 0$$

Change the variable  $t$  to  $x = \frac{2\pi}{P}t$ . In this case  $x = \pi$  corresponds to  $t = \frac{P}{2}$  and  $x = -\pi$  corresponds to  $t = -\frac{P}{2}$ .

Therefore regarded as a function of  $t$ , this is a function with period  $P$ . When we make the substitution  $x = \frac{2\pi}{P}t$  and

$dx = \frac{2\pi}{P} dt$  into the expressions for  $a_n$  and  $b_n$  we arrive at:

$$a_n = \frac{2}{P} \int_{-\frac{P}{2}}^{\frac{P}{2}} f(t) \cos\left(\frac{2n\pi t}{P}\right) dt, n \in \mathbb{Z}, n \geq 0$$

$$b_n = \frac{2}{P} \int_{-\frac{P}{2}}^{\frac{P}{2}} f(t) \sin\left(\frac{2n\pi t}{P}\right) dt, n \in \mathbb{Z}, n > 0$$

These integrals will give the Fourier coefficients from a function of period  $P$  whose Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{2n\pi t}{P}\right) + b_n \sin\left(\frac{2n\pi t}{P}\right) \right)$$

Note: In Differential Equations it is often convenient to write the period  $P$  as  $2\ell$  and in Physics and Engineering it is often written in terms of angular frequency  $\omega$  as  $P = \frac{2\pi}{\omega}$ . Those substitutions would result in the following formulas:

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(t) \cos\left(\frac{n\pi t}{\ell}\right) dt, n \in \mathbb{Z}, n \geq 0 \text{ for Fourier Series } f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi t}{\ell}\right) + b_n \sin\left(\frac{n\pi t}{\ell}\right) \right)$$

And

$$a_n = \frac{\omega}{\pi} \int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} f(t) \cos(n\omega t) dt, n \in \mathbb{Z}, n \geq 0 \text{ for Fourier Series } f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\omega t) + b_n \sin(n\omega t))$$

Any convenient integration range of length  $P$ ,  $2\ell$ , or  $\frac{2\pi}{\omega}$  can be used, and formulas for  $b_n$  would follow similarly for those for  $a_n$  as shown above.

\*Why divide the constant term by 2? To account for the fact that the formula for  $a_n$  could be true for all  $n \in \mathbb{Z}, n \geq 0$  (which would include the constant term) depending upon how the formula for the Fourier coefficients are written.

Recall from problems 6 and 7 for the integration problems combined that  $\int_{-\pi}^{\pi} \cos^2(nt) dt = \pi$  when  $n \neq 0$  but

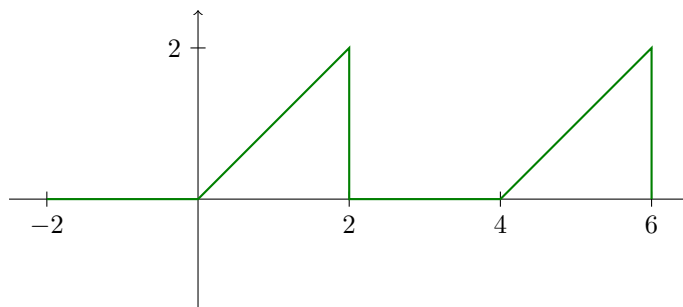
$\int_{-\pi}^{\pi} \cos^2(nt) dt = 2\pi$  when  $n = 0$ . Using the formula  $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt$  we could write the Fourier Series as

$f(t) = \sum_{n=0}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt)$  but in this case  $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt = \begin{cases} 2a_0, n=0 \\ a_n, n \neq 0 \end{cases}$ . To compensate for this

the constant term is customarily written as  $\frac{a_0}{2}$ .

## Modified Sawtooth Wave

- 4/30: • Modified sawtooth wave (period 4):  $f(t) = \begin{cases} 0 & -2 < t < 0 \\ t & 0 < t < 2 \end{cases}$  and  $f(t+4) = f(t)$ .



– Period is 4:  $P = 2\ell \Rightarrow \ell = 2$ .

- Use this Fourier model:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi t}{\ell}\right) + b_n \sin\left(\frac{n\pi t}{\ell}\right) \right)$$

- $\frac{a_0}{2} = \frac{1}{2}$  (think of it as a weighted average of where the function spends the most time — note that this is actually exactly what the integral computes).
- Find  $a_n$  terms:

$$\begin{aligned} a_n &= \frac{1}{2} \int_{-2}^2 f(t) \cos\left(\frac{n\pi t}{2}\right) dt \\ &= 0 + \frac{1}{2} \int_0^2 t \cos\left(\frac{n\pi t}{2}\right) dt \end{aligned}$$

$$\begin{aligned} u &= t & dv &= \cos\left(\frac{n\pi t}{2}\right) dt \\ du &= dt & v &= \frac{2}{n\pi} \sin\left(\frac{n\pi t}{2}\right) \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{2} \left( \left[ \left( t \right) \left( \frac{2}{n\pi} \sin\left(\frac{n\pi t}{2}\right) \right) \right]_0^2 - \int_0^2 \frac{2}{n\pi} \sin\left(\frac{n\pi t}{2}\right) dt \right) \\ &= \frac{1}{2} \left( \left( \frac{4}{n\pi} \sin(n\pi) - 0 \right) + \frac{4}{n^2\pi^2} \left[ \cos\left(\frac{n\pi t}{2}\right) \right]_0^2 \right) \\ &= \frac{1}{2} \left( 0 + \frac{4}{n^2\pi^2} (\cos(n\pi) - \cos(0)) \right) \\ &= \frac{2}{n^2\pi^2} (\cos(n\pi) - 1) \end{aligned}$$

$$a_n = \begin{cases} 0 & n = 2, 4, 6, \dots \\ -\frac{4}{n^2\pi^2} & n = 1, 3, 5, \dots \end{cases}$$



- Find  $b_n$  terms:

$$\begin{aligned} b_n &= \frac{1}{2} \int_{-2}^2 f(t) \sin\left(\frac{n\pi t}{2}\right) dt \\ &= 0 + \frac{1}{2} \int_0^2 t \sin\left(\frac{n\pi t}{2}\right) dt \end{aligned}$$

$$\begin{aligned} u &= t & dv &= \sin\left(\frac{n\pi t}{2}\right) dt \\ du &= dt & v &= -\frac{2}{n\pi} \cos\left(\frac{n\pi t}{2}\right) dt \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{2} \left( \left[ \left( t \right) \left( -\frac{2}{n\pi} \cos\left(\frac{n\pi t}{2}\right) \right) \right]_0^2 + \frac{2}{n\pi} \int_0^2 \cos\left(\frac{n\pi t}{2}\right) dt \right) \\ &= \frac{1}{2} \left( \left( -\frac{4}{n\pi} \cos(n\pi) - 0 \right) + 0 \right) \end{aligned}$$

$$b_n = \begin{cases} -\frac{2}{n\pi} & n = 2, 4, 6, \dots \\ \frac{2}{n\pi} & n = 1, 3, 5, \dots \end{cases}$$

- Assemble the Fourier series for this modified sawtooth wave function:

$$\begin{aligned} a_n &= -\frac{2}{n^2\pi^2} + (-1)^n \frac{2}{n^2\pi^2} \\ b_n &= (-1)^{n+1} \frac{2}{n\pi} \\ f(t) &= \frac{1}{2} + \sum_{n=1}^{\infty} \left( \left( -\frac{2}{n^2\pi^2} + (-1)^n \frac{2}{n^2\pi^2} \right) \cos\left(\frac{n\pi t}{2}\right) + \left( (-1)^{n+1} \frac{2}{n\pi} \right) \sin\left(\frac{n\pi t}{2}\right) \right) \end{aligned}$$

## Complex Fourier Series

5/4: • **Euler's formula:**  $e^{i\theta} = \cos \theta + i \sin \theta$ .

- Using Euler's Formula, we can replace the trigonometric functions in Fourier series with complex exponential functions. By combining the Fourier coefficients  $a_n$  and  $b_n$  into one complex coefficient  $c_n$ , we find that, for a given periodic signal, both sets of constants can be found in one operation.
- Let's derive sine and cosine in terms of complex exponentials.
  - For this to work, we will need Euler's formula, and a second, negative version of Euler's formula:  $e^{-i\theta} = \cos \theta - i \sin \theta$ .

$$\begin{aligned} (\cos \theta + i \sin \theta) + (\cos \theta - i \sin \theta) &= e^{i\theta} + e^{-i\theta} & (\cos \theta + i \sin \theta) - (\cos \theta - i \sin \theta) &= e^{i\theta} - e^{-i\theta} \\ 2 \cos \theta &= e^{i\theta} + e^{-i\theta} & 2i \sin \theta &= e^{i\theta} - e^{-i\theta} \end{aligned}$$

$$\cos \theta = \frac{1}{2} (e^{i\theta} + e^{-i\theta})$$

$$\sin \theta = \frac{1}{2i} (e^{i\theta} - e^{-i\theta})$$

- We can use the above results to express  $a_n \cos(n\omega_0\theta) + b_n \sin(n\omega_0\theta)$ , where  $\omega_0 = \frac{2\pi}{P}$ , in terms of complex exponentials.

$$\begin{aligned}
a_n \cos(n\omega_0\theta) + b_n \sin(n\omega_0\theta) &= \frac{a_n}{2} (e^{in\omega_0\theta} + e^{-in\omega_0\theta}) + \frac{b_n}{2i} (e^{in\omega_0\theta} - e^{-in\omega_0\theta}) \\
&= \frac{a_n}{2} e^{in\omega_0\theta} + \frac{a_n}{2} e^{-in\omega_0\theta} + \frac{b_n}{2i} e^{in\omega_0\theta} - \frac{b_n}{2i} e^{-in\omega_0\theta} \\
&= \left( \frac{a_n}{2} + \frac{b_n}{2i} \right) e^{in\omega_0\theta} + \left( \frac{a_n}{2} - \frac{b_n}{2i} \right) e^{-in\omega_0\theta} \\
&= \frac{1}{2} \left( a_n + \frac{b_n}{i} \right) e^{in\omega_0\theta} + \frac{1}{2} \left( a_n - \frac{b_n}{i} \right) e^{-in\omega_0\theta} \\
&= \frac{1}{2} \left( a_n + \left( \frac{b_n}{i} \right) (1) \right) e^{in\omega_0\theta} + \frac{1}{2} \left( a_n - \left( \frac{b_n}{i} \right) (1) \right) e^{-in\omega_0\theta} \\
&= \frac{1}{2} \left( a_n + \left( \frac{b_n}{i} \right) (i^4) \right) e^{in\omega_0\theta} + \frac{1}{2} \left( a_n - \left( \frac{b_n}{i} \right) (i^4) \right) e^{-in\omega_0\theta} \\
&= \frac{1}{2} (a_n + i^3 b_n) e^{in\omega_0\theta} + \frac{1}{2} (a_n - i^3 b_n) e^{-in\omega_0\theta} \\
&= \frac{1}{2} (a_n + (-i)b_n) e^{in\omega_0\theta} + \frac{1}{2} (a_n - (-i)b_n) e^{-in\omega_0\theta} \\
&= \frac{1}{2} (a_n - ib_n) e^{in\omega_0\theta} + \frac{1}{2} (a_n + ib_n) e^{-in\omega_0\theta}
\end{aligned}$$

- Define  $c_n = \frac{1}{2} (a_n - ib_n)$  and complex conjugate  $\bar{c}_n = \frac{1}{2} (a_n + ib_n)$ . Now we have the following.

$$a_n \cos(n\omega_0\theta) + b_n \sin(n\omega_0\theta) = c_n e^{in\omega_0\theta} + \bar{c}_n e^{-in\omega_0\theta}$$

- Substitution into the Fourier series sum gives the following.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (c_n e^{in\omega_0\theta} + \bar{c}_n e^{-in\omega_0\theta}) \quad (1)$$

- Equation 1 can become still neater and more concise through the following steps.

1. Define  $c_0 = \frac{a_0}{2}$ <sup>[10]</sup>.
2. Define  $c_{-n} = \bar{c}_n$ . This permits the following.

$$\sum_{n=1}^{\infty} \bar{c}_n e^{-in\omega_0 t} = \bar{c}_1 e^{-i\omega_0 t} + \bar{c}_2 e^{-2i\omega_0 t} + \dots = c_{-1} e^{-i\omega_0 t} + c_{-2} e^{-2i\omega_0 t} + \dots = \sum_{n=-1}^{-\infty} c_n e^{in\omega_0 t}$$

3. Using the new definitions of  $c_n$  for  $n \in (-\infty, 0]$ , it is possible to write Equation 1 as follows.

$$\begin{aligned}
f(t) &= c_0 + \sum_{n=1}^{\infty} c_n e^{in\omega_0 t} + \sum_{n=1}^{\infty} c_{-n} e^{-in\omega_0 t} \\
&= c_0 + \sum_{n=1}^{\infty} c_n e^{in\omega_0 t} + \sum_{n=-\infty}^{-1} c_n e^{in\omega_0 t}
\end{aligned}$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}$$

<sup>10</sup>Note that this is consistent with the general definition of  $c_n$  since  $b_0 = 0$ .

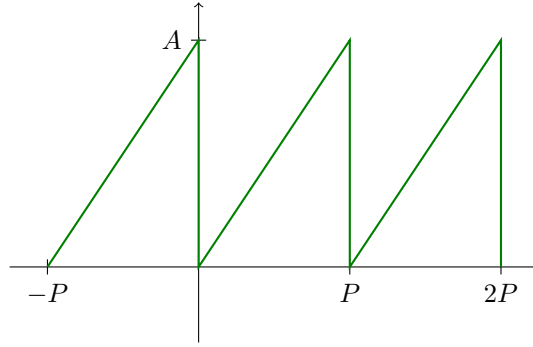
- We now tackle how to solve for the complex coefficients  $c_n$ <sup>[11]</sup>.

1. For  $n = 0$ ,  $c_0 = \frac{a_0}{2} = \frac{1}{P} \int_{-P/2}^{P/2} f(t) dt$ .
2. For  $n \in \mathbb{Z}^+$ ,  $c_n = \frac{1}{2} (a_n - ib_n) = \frac{1}{P} \int_{-P/2}^{P/2} f(t) (\cos(n\omega_0 t) - i \sin(n\omega_0 t)) dt = \frac{1}{P} \int_{-P/2}^{P/2} f(t) e^{-in\omega_0 t} dt$ .
3. For  $n \in \mathbb{Z}^-$ ,  $c_n = \frac{1}{2} (a_n + ib_n) = \frac{1}{P} \int_{-P/2}^{P/2} f(t) e^{-in\omega_0 t} dt$ <sup>[12]</sup>.
4. The above three results can be condensed into the following expression for all  $n \in \mathbb{Z}$ .

$$c_n = \frac{1}{P} \int_{-P/2}^{P/2} f(t) e^{-in\omega_0 t} dt$$

### Generalized Sawtooth Wave

- 5/6: • Generalized sawtooth wave (period  $P$ , amplitude  $A$ ):  $f(t) = \frac{At}{P}$  and  $f(t+P) = f(t)$ .



- $c_0 = \frac{A}{2}$ .
- $\omega_0 = \frac{2\pi}{P} \Rightarrow \omega_0 P = 2\pi$ .
- Find  $c_n$  terms:

$$\begin{aligned}
 c_n &= \frac{1}{P} \int_0^P \frac{At}{P} e^{-in\omega_0 t} dt \\
 &= \frac{A}{P^2} \int_0^P t e^{-in\omega_0 t} dt \\
 &\quad \begin{aligned} u &= t & dv &= e^{-in\omega_0 t} \\ du &= dt & v &= -\frac{1}{in\omega_0} e^{-in\omega_0 t} \end{aligned} \\
 &= \frac{A}{P^2} \left( \left[ -\frac{te^{-in\omega_0 t}}{in\omega_0} \right]_0^P + \frac{1}{in\omega_0} \int_0^P e^{-in\omega_0 t} dt \right) \\
 &= \frac{A}{P^2} \left( \left( -\frac{Pe^{-in\omega_0 P}}{in\omega_0} - 0 \right) - \frac{1}{(in\omega_0)^2} [e^{-in\omega_0 t}]_0^P \right) \\
 &= \frac{A}{P^2} \left( -\frac{Pe^{-in\omega_0 P}}{in\omega_0} - \frac{1}{(in\omega_0)^2} (e^{-in\omega_0 P} - 1) \right) \\
 &= \frac{A}{P^2} \left( -\frac{Pe^{-i2\pi n}}{in\omega_0} - \frac{1}{(in\omega_0)^2} (e^{-i2\pi n} - 1) \right)
 \end{aligned}$$

<sup>11</sup>Note that this can also be derived in an analogous method to how the original  $a_n$  and  $b_n$  expressions were derived.

<sup>12</sup>The negative exponential, when multiplied by a negative  $n$ , generates the + expansion of Euler's formula, as desired.

$$\begin{aligned}
&= \frac{A}{P^2} \left( -\frac{P(1)}{in\omega_0} - \frac{1}{(in\omega_0)^2} ((1) - 1) \right) \\
&= \frac{A}{P^2} \left( -\frac{P}{in\omega_0} \right) \\
&= -\frac{A}{in\omega_0 P} \\
&= \frac{A}{2\pi n(-i)} \\
\boxed{c_n} &= \frac{Ai}{2\pi n}
\end{aligned}$$

- Assemble the Fourier series for this generalized sawtooth wave function.

$$f(t) = \frac{Ai}{2\pi} \sum_{n=-\infty}^{\infty} \frac{e^{in\omega_0 t}}{n}$$

## Continuous Fourier Transform

- 5/8: • The complex Fourier series can be summarized as one entity as follows.

$$f(t) = \sum_{n=-\infty}^{\infty} \left( \frac{1}{P} \int_{t_0}^{t_0+P} f(t) e^{-in\omega_0 t} dt \right) e^{in\omega_0 t}$$

- $t_0$  is an arbitrary  $t$ , and the above reflects that it is important that we integrate over a full period  $P$ , but it does not matter what we define to be a single iteration/period of  $f(t)$ .

- Substitute  $\omega_0 = \frac{2\pi}{P}$ .

$$f(t) = \sum_{n=-\infty}^{\infty} \left( \frac{1}{P} \int_{t_0}^{t_0+P} f(t) e^{-in2\pi \frac{1}{P} t} dt \right) e^{in2\pi \frac{1}{P} t}$$

- Let  $t_0 = -\frac{P}{2}$  and let  $\frac{1}{P} = \Delta f$ .

$$f(t) = \sum_{n=-\infty}^{\infty} \left( \int_{-P/2}^{P/2} f(t) e^{-in2\pi \Delta f t} dt \right) e^{in2\pi \Delta f t} \Delta f$$

- What happens as  $P \rightarrow \infty$ ?
  - Since  $\frac{1}{P} = \Delta f$ ,  $\Delta f \rightarrow df^{[13]}$  (think  $\frac{1}{\infty}$ ).
  - Also, define a continuous variable  $f$  equivalent to  $n\Delta f = \frac{n}{P}^{[14]}$ .
  - Now that we have a continuous variable and a differential, the formula's summation is integration!

$$\begin{aligned}
f(t) &= \lim_{P \rightarrow \infty} \left( \sum_{n=-\infty}^{\infty} \left( \int_{-P/2}^{P/2} f(t) e^{-i2\pi n \Delta f t} dt \right) e^{i2\pi n \Delta f t} \Delta f \right) \\
&= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(t) e^{-i2\pi f t} dt \right) e^{i2\pi f t} df
\end{aligned}$$

<sup>13</sup>A teeny-tiny differential.

<sup>14</sup>As  $P$  gets smaller, changes in  $\frac{n}{P}$  become less discrete (less like increments) and more continuous.

- Let  $2\pi f = \omega \Rightarrow df = d\omega$ .

$$f(t) = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \right) e^{i\omega t} d\omega \quad (2)$$

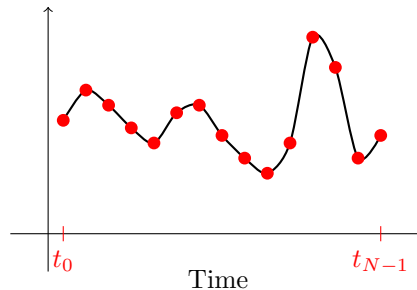
$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt \quad (3)$$

- Equation 3 is the Fourier Transform and Equation 2 is the Inverse Fourier Transform.

## Discrete Fourier Transform

5/11:

- Consider Equation 3, the integral of the product of two functions.
  - $f(t)$  is the function or periodic signal whose domain we are trying to transform.
  - $e^{-i\omega t}$  is the analyzing function, which, as we know from Euler's formula, represents sinusoids.
  - Whenever  $f(t)$  and  $e^{-i\omega t}$  are *similar*, they will multiply and sum to a *large* coefficient<sup>[15]</sup>.
  - Whenever  $f(t)$  and  $e^{-i\omega t}$  are *dissimilar*, they will multiply and sum to a *small* coefficient.
- Infinite integral bounds: In the real practice of signal analysis, they doesn't make much sense because you would only be interested in collecting data on a signal within a finite time frame, and sampling can only be done discretely. In reality, a signal sample might look like this:



- A discrete number ( $N = 15$ ) of evenly spaced samples over a discrete quantity of time.
  - Note:  $t_0$  to  $t_{N-1}$  because computer science canonically counts starting at 0.
- Change notation (too many  $f$ 's) and substitute  $\omega = 2\pi F$ :

$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi F t} dt$$

- Make the above continuous definition discrete:

$$X_k = \sum_{n=0}^{N-1} x_n \cdot e^{-i2\pi \frac{k}{N} n}$$

- $k^{\text{th}}$  frequency.
  - Discrete summation to evaluate  $N$  samples.
- With the Discrete Fourier Transform (DFT), we can no longer look at frequency and time continuously; we instead look at the  $k^{\text{th}}$  frequency and the  $n^{\text{th}}$  sample.
  - Essentially,  $\frac{k}{N}$  corresponds to frequency and  $n$  corresponds to time.

<sup>15</sup>We're trying to find the most common or largest frequencies in this periodic signal; when  $\omega$  (independent variable) gets very close to the frequency of a component sinusoid in  $f(t)$ ,  $F(\omega)$  (dependent variable) will get quite big.

- What happens when you expand the summation?

– Let  $b_n = \frac{2\pi kn}{N}$ .

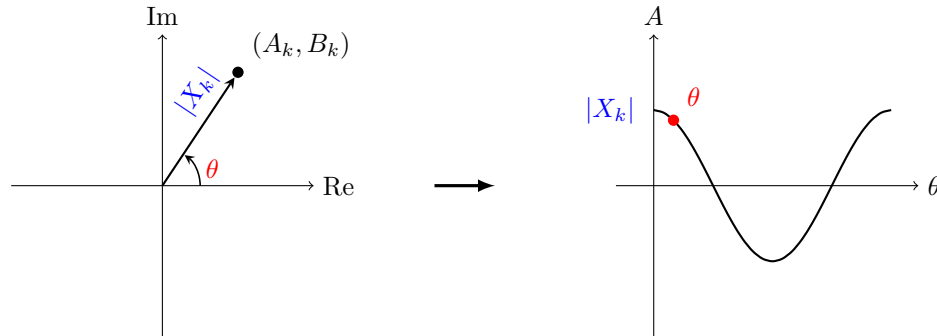
$$X_k = x_0 e^{-b_0 i} + x_1 e^{-b_1 i} + \cdots + x_{N-1} e^{-b_{N-1} i}$$

– Since  $e^{ix} = \cos(x) + i \sin(x)$ ,

$$X_k = x_0 (\cos(-b_0) + i \sin(-b_0)) + \cdots + x_{N-1} (\cos(-b_{N-1}) + i \sin(-b_{N-1}))$$

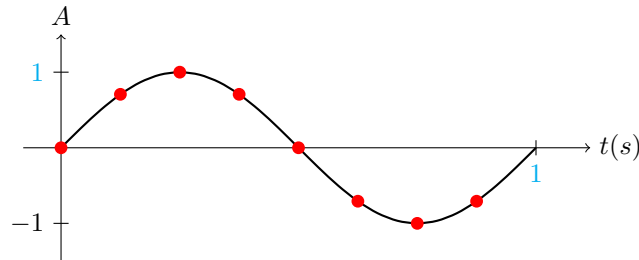
– Thus, the sum is just some complex number  $X_k = A_k + B_k i$ .

– Note that the magnitude and angle just correspond to an amplitude and phase displacement from a linear combination of the sine and cosine functions:



### Simple DFT Example

- 1 Hz sine wave of amplitude 1.
- Sampling frequency of 8 Hz, so  $N = 8$ .



- Sampled values:

$x_n$	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$x_7$
val	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$

- Now apply the formula to find the Fourier coefficients in the eight “frequency bins.”

$$X_k = \sum_{n=0}^{8-1} x_n \cdot e^{-\frac{i2\pi kn}{N}}$$

$$X_0 = (0)e^{-\frac{i2\pi(0)(0)}{8}} + \left(\frac{\sqrt{2}}{2}\right)e^{-\frac{i2\pi(0)(1)}{8}} + \cdots + \left(-\frac{\sqrt{2}}{2}\right)e^{-\frac{i2\pi(0)(7)}{8}}$$

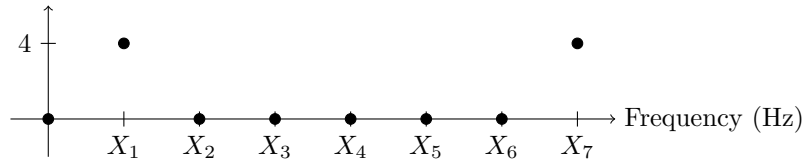
$$\vdots$$

$$X_7 = (0)e^{-\frac{i2\pi(7)(0)}{8}} + \left(\frac{\sqrt{2}}{2}\right)e^{-\frac{i2\pi(7)(1)}{8}} + \cdots + \left(-\frac{\sqrt{2}}{2}\right)e^{-\frac{i2\pi(7)(7)}{8}}$$
(4)

- 5/12: • The sums come out to be the following.

Sum	$X_0$	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$	$X_6$	$X_7$
Value	0	$-4i$	0	0	0	0	0	$4i$

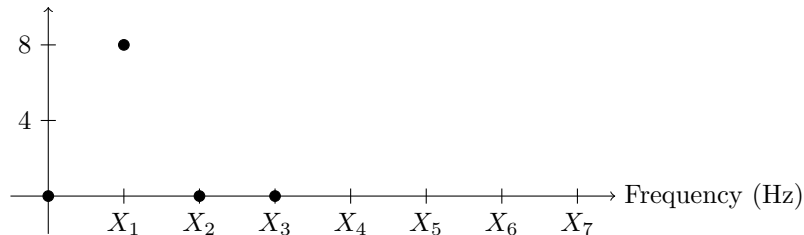
- Plot the magnitudes of each sum in the following **Spectrum Plot**.



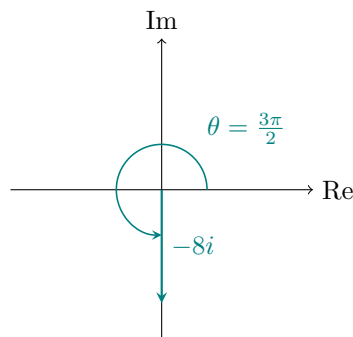
- Info:
- Frequency Resolution =  $\frac{\text{Sampling Frequency}}{N} = \frac{8 \text{ Hz}}{8} = 1 \text{ Hz}$ .
  - A non-zero value at  $X_1$  corresponds to 1 Hz, and this is a 1 Hz sine wave!
- **Nyquist frequency:** Remove values at or above  $\frac{\text{Sampling frequency}}{2}$  and double values below.
- Thus, discount values above  $\frac{8}{2} = 4$  (i.e.,  $X_4, X_5, X_6$ , and  $X_7$ ) and double the values for  $X_0$  through  $X_3$ .

Sum	$X_0$	$X_1$	$X_2$	$X_3$
Value	0	$-8i$	0	0

- Plot the magnitudes below.



- amplitude =  $\frac{8 \text{ (from plot)}}{8 \text{ samples}} = 1$ .



$$1 \cos\left(\theta - \frac{3\pi}{2}\right) = \sin \theta$$

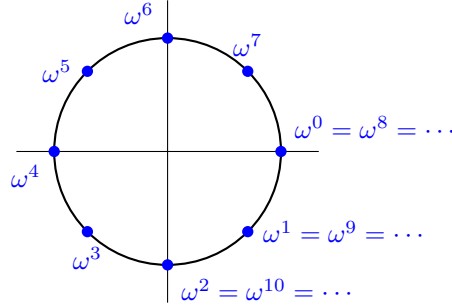
## Condensing the Bins into a Matrix

5/13:

- Look at what is a constant and what is a variable in Equations 4.
  - In each  $x_n \cdot e^{-\frac{i2\pi kn}{N}}$  term,  $e^{-\frac{i2\pi kn}{N}}$  is constant,  $k$  and  $n$  are variables (that happen to correspond to the “row and column location” of the term), and  $x_n$  is a scalar.
- Thus, let  $\omega = e^{-\frac{i2\pi}{N}}$ .
  - Under this definition,  $e^{-\frac{i2\pi kn}{N}}$  terms become  $\omega^{kn}$  terms.
- Notice that the  $X_k$  calculations are just scaling & adding — scaling by  $x_n$  and adding the terms of the summation!
- Therefore, with the help of our new definition of  $\omega$ , we can rewrite Equations 4 in the form  $X_k = Fx_n$ .

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \end{bmatrix} = \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} & \omega^{14} \\ \omega^0 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} & \omega^{21} \\ \omega^0 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} & \omega^{28} \\ \omega^0 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} & \omega^{35} \\ \omega^0 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} & \omega^{42} \\ \omega^0 & \omega^7 & \omega^{14} & \omega^{21} & \omega^{28} & \omega^{35} & \omega^{42} & \omega^{49} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}$$

- Keep in mind — there are really only 8 distinct entries in this matrix because these powers are periodic.



- Thus,  $F$  can be written as follows.

$$F = \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^0 & \omega^2 & \omega^4 & \omega^6 \\ \omega^0 & \omega^3 & \omega^6 & \omega^1 & \omega^4 & \omega^7 & \omega^2 & \omega^5 \\ \omega^0 & \omega^4 & \omega^0 & \omega^4 & \omega^0 & \omega^4 & \omega^0 & \omega^4 \\ \omega^0 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega^1 & \omega^6 & \omega^3 \\ \omega^0 & \omega^6 & \omega^4 & \omega^2 & \omega^0 & \omega^6 & \omega^4 & \omega^2 \\ \omega^0 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega^1 \end{bmatrix}$$

- Note that  $F$  is symmetric but not Hermitian because the diagonal is not real.