NUMERICAL LINEAR ALGEBRA

SPEED VS. ACCURACY IN FINDING SOLUTIONS WHEN SOLVING “REAL” PROBLEMS

Roundoff Error

Computers and calculators store real numbers in “floating-point form.”

- 2001 would be stored as 0.2001×10^4
- -0.0063 would be stored as -0.63×10^{-3}
- Generally the floating-point form of a number is $\pm M \times 10^k$ where k is an integar and M (mantissa) satisfies $0.1 \le M < 1$
- Depending upon the computer, calculator, or CAS, the number of significant digits (number of decimal places that can be stored) may vary between 8 and 12.
- For example, $\pi \approx 3.141592654$. If a calculator can store 5 significant digits, π would be stored as 0.31416×10^1.
- This truncation of π will introduce roundoff error.

EXAMPLES

$$x + y = 0$$
$$x + \frac{801}{800} y = 1$$

$$4.552x + 7.083y = 1.931$$
$$1.731x + 2.693y = 2.001$$

Experiment with exact solutions, then rounding to five digits, then four, then three.

These systems that are sensitive to roundoff error are called ***ill-conditioned.***

Partial Pivoting: A Way to Reduce/Eliminate Roundoff Error

EXAMPLE:

- Solve for $x: 0.00021x = 1$
- Then suppose a calculator can only carry for significant digits. You would be solving $0.0002x = 1$
- The difference between these answers is the effect of an error of 0.00001 on the solution of an equation.

IN A SYSTEM OF EQUATIONS:

- Solve $\begin{array}{l} 0.400x + 99.6y = 100 \\ 75.3x - 45.3y = 30.0 \end{array}$ using three significant digits in each calculation.
- The result is $\begin{bmatrix} -1.00 \\ 1.01 \end{bmatrix}$ but the actual result should be $\begin{bmatrix} 1.00 \\ 1.00 \end{bmatrix}$!!!
- Resolve $\begin{array}{l} 0.400x + 99.6y = 100 \\ 75.3x - 45.3y = 30.0 \end{array}$ by interchanging the two rows of the augmented matrix and take each solution to three significant digits again.

Choosing pivots matters!!! At each pivoting step, choose from among all possible pivots in a column the entry with the largest absolute value. Use row interchanges to bring this element into the correct position and use it to create zeros where needed in the column. This is ***partial pivoting***.

Use partial pivoting to solve the following systems using three significant digits.

$0.001x + 0.995y = 1.00$ $-10.2x + 1.00y = -50.0$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5.00 \\ 1.00 \end{bmatrix}$ Exact solutions listed above	$10x - 7y = 7$ $-3x + 2.09 + 6z = 3.91$ $5x - y + 5z = 6$ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.00 \\ -1.00 \\ 1.00 \end{bmatrix}$ Exact solutions listed above

Counting Operations

Consider the augmented matrix $\left[A \mid b\right]=\begin{bmatrix} 2 & 4 & 6 & 8 \\ 3 & 9 & 6 & 12 \\ -1 & 1 & -1 & 1 \end{bmatrix}$.

- How many operations (multiplications or divisions) are required to use Gaussian Elimination to bring the matrix to its row-echelon form $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$?
- How many additional operations are required to perform the back substitution part of Gaussian elimination?
- How many operations are required to perform Gauss-Jordan elimination to bring the matrix to its reduced row-echelon form $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$?
- Which appears to be a more efficient algorithm?

<u>A generalized approach</u>

Let $\left[A \mid b\right]=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{bmatrix}$ and note that $\left[A \mid b\right]$ is $n\times(n\times 1)$.

- n operations would be required create the first leading 1 and then adjust all other entries in row 1.
- n operations would be required to obtain the first zero in column one and adjust all other entries in row 2.
- This will continue through row n, bringing the total number of operations up to the point that the entire first column is zeroed out to $n+(n-1)n$.
- This would have to continue with row 2 and column 2. The total number of operations needed to reach row-echelon form is
  $[n+(n-1)n]+[(n-1)+(n-2)(n-1)]+[(n-2)+(n-3)(n-2)]+\cdots+[2+1\ 2]+1$
  $=n^2+(n-1)^2+\cdots+2^2+1^2$
  - Then it's time for back substitution. The number of operations required for this would be $1+2+\cdots+(n-1)$

Therefore, using summation, the total number of operations performed by Guassian Elmination is $S(n)=\frac{1}{3}n^3+n^2-\frac{1}{3}n$

(Gauss-Jordan, by comparison $S(n)\approx\frac{1}{2}n^3$ and using the inverse requires n^3)

THE NORM OF A MATRIX

Let $A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & -1 & -2 \\ -5 & 1 & 3 \end{bmatrix}$

Find $\|A\|_1$ and $\|A\|_\infty$

THE CONDITION NUMBER OF A MATRIX

Show that $A = \begin{bmatrix} 1 & 1 \\ 1 & 1.0005 \end{bmatrix}$ is ill-conditioned.

Let $b = \begin{bmatrix} 3 \\ 3.0010 \end{bmatrix}$. The solution to Ax = b is $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. If A changes to $A' = \begin{bmatrix} 1 & 1 \\ 1 & 1.0010 \end{bmatrix}$, then $x' = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

A relative change of $0.0005/1.0005 \approx 0.0005$ or about 0.05% causes a change of (2-1)/1=1 or 100% in x_1 and (1-2)/2=-0.5 or -50% in x_2. A is ill-conditioned.

Let the change from A to A' be an error ΔA that introduces an error Δx in the solution x to Ax =b. Then $A' = A + \Delta A$ and $x' = x + \Delta x$.

$$\Delta A = \begin{bmatrix} 0 & 0 \\ 0 & 0.0005 \end{bmatrix} \quad \text{and} \quad \Delta x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Since Ax = b and A'x' = b, we have $(A + \Delta A)(x + \Delta x) = b$

$$A(\Delta x) + (\Delta A)x + (\Delta A)(\Delta x) = 0 \quad \text{or} \quad A(\Delta x) = -\Delta A(x + \Delta x)$$

A is invertible if it has a solution, so

$$\Delta x = -A^{-1}(\Delta A)(x + \Delta x) = -A^{-1}(\Delta A)x'$$

Taking norms of all sides

$$\|\Delta x\| = \|-A^{-1}(\Delta A)(x + \Delta x)\| = \|-A^{-1}(\Delta A)x'\|$$

$$\le \|A^{-1}(\Delta A)\|\|x'\|$$
$$\le \|A^{-1}\|\|\Delta A\|\|x'\|$$

Therefore $\dfrac{\|\Delta x\|}{\|x'\|} \le \|A^{-1}\|\|\Delta A\| = \left(\|A^{-1}\|\|A\|\right)\dfrac{\|\Delta A\|}{\|A\|}$

$\|A^{-1}\|\|A\|$ is the CONDITION NUMBER. Cond(A)

If A is not invertible, cond(A) = ∞

$\frac{\|\Delta A\|}{\|A\|}$ is a measure of the relative change in the matrix A, which we assume to be small.

$\frac{\|\Delta x\|}{\|x'\|}$ is a measure of the relative error created in the solution to Ax = b. In this case, the error is measured relative to the new solution x'.

$\frac{\|\Delta x\|}{\|x'\|} \leq cond(A)\frac{\|\Delta A\|}{\|A\|}$ gives an upper bound on how large the relative error in the solution can be in terms of the relative error in the coefficient matrix. The larger the condition number, the more ill-conditioned the matrix, since there is more "room" for the error to be large relative to the solution.

CHAPTER 9.3 Iterative Methods

Iterative Methods of Solving Linear Systems

Roundoff error can actually accelerate the convergence of an iteraitive method towards a solution

JACOBI’S METHOD

$$\begin{aligned} 7x_1 - x_2 &= 5 \\ 3x_1 - 5x_2 &= -7 \end{aligned}$$

Solve the first equation for x_1 and the second equation for x_2

$$\begin{aligned} x_1 &= \frac{5 + x_2}{7} \\ x_2 &= \frac{7 + 3x_1}{5} \end{aligned}$$

Then use $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as an initial approximation.

$$\begin{aligned} x_1 &= \frac{5+0}{7} = \frac{5}{7} \approx 0.714 \\ x_2 &= \frac{7+3\ 0}{5} = \frac{7}{5} = 1.400 \end{aligned}$$

Substitute these values into the second iteration

$$\begin{aligned} x_1 &= \frac{5+1.4}{7} \approx 0.914 \\ x_2 &= \frac{7+3\ \frac{5}{7}}{5} \approx 1.829 \end{aligned}$$

After several iterations we can see a convergence to $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

n	0	1	2	3	4	5	6
x_1	0	0.714	0.914	0.976	0.993	0.998	0.999
x_2	0	1.400	1.829	1.949	1.985	1.996	1.999

GAUSS-SEIDEL METHOD

The method is the same as Jacobi's except that we use each new value as soon as we can. In other words, calculate x_1 as we did before, but then use the result ($\frac{5}{7}$ in this case) to get the next value of x_2.

$$x_2 = \frac{7 + 3\,\frac{5}{7}}{5} \approx 1.829$$

This provides a quicker convergence

n	0	1	2	3	4	5
x_1	0	0.714	0.976	0.998	1.000	1.000
x_2	0	1.829	1.985	1.999	2.000	2.000

ITERATIVE METHODS FOR EIGENVALUES

The Power Method

This is for a matrix with a DOMINANT EIGENVALUE: one that is larger in absolute values than all of the other eigenvalues.

The power method proceeds iteratively to produce a sequence of scalars that converge to the dominant eigenvalue and a sequence of vectors that converges to the corresponding eigenvector.

For simplicity, let A be diagonalizable with a dominant eigenvalue. There exists a nonzero vector x_0 such that the sequence of vectors x_k defined by

$$x_1 = Ax_0; x_2 = Ax_1; x_k = Ax_{k-1} \ldots$$

approaches the dominant eigenvector of A.

Proof:

Eigenvalues of A would be listed as $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$. Corresponding eigenvectors would be independent (A is diagonalizable) and would correspond $v_1, v_2, \cdots, v_n$. Since there are a basis they can be expressed as a linear combination of eigenvectors: $x_0 = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$

Now $x_1 = Ax_0, x_2 = Ax_1 = A(Ax_0) = A^2 x_0$ etc.

So in general $x_k = A^k x_0$ for $k \geq 1$

$$A^k x_0 = c_1 \lambda_1^k v_1 + c_2 \lambda_2^k v_2 + \cdots + c_n \lambda_n^k v_n$$

$$= \lambda_1^k \left(c_1 v_1 + c_2 \left(\frac{\lambda_2}{\lambda_1} \right)^k v_2 + \cdots + c_n \left(\frac{\lambda_n}{\lambda_1} \right)^k v_n \right)$$

Since λ_1 is the dominant eigenvalue, each of these fractions go to zero as $k \to \infty$

Therefore $x_k = A^k x_0 \to \lambda_1^k c_1 v_1$

Example:

Approximate the dominant eigenvalue of $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$

Take $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as the initial vector.

Then
$$x_1 = Ax_0 = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$x_2 = Ax_1 = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

k	0	1	2	3	4	5	6	7	8
x_k	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 5 \\ 6 \end{bmatrix}$	$\begin{bmatrix} 11 \\ 10 \end{bmatrix}$	$\begin{bmatrix} 21 \\ 22 \end{bmatrix}$	$\begin{bmatrix} 43 \\ 42 \end{bmatrix}$	$\begin{bmatrix} 85 \\ 86 \end{bmatrix}$	$\begin{bmatrix} 171 \\ 170 \end{bmatrix}$

The iterates are clearly converging on $c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, the dominant eigenvector.

$$Ax = \lambda x \qquad \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

If , then $\lambda = 2$

Practice:

Use the power method to approximate the dominant eigenvalue of $A = \begin{bmatrix} 0 & 5 & -6 \\ -4 & 12 & -12 \\ -2 & -2 & 10 \end{bmatrix}$ with

an initial vector $x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

THE SHIFTED POWER METHOD

Since the power method finds the dominant eigenvalue, the shifted power method understands that if λ is an eigenvalue of A, then $\lambda - \alpha A$ is an eigenvalue of $A - \alpha I$.

Continue with the previous example where $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ and we discovered that $\lambda = 2$

Then $A - 2I = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}$. Let $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

k	0	1	2	3	4
x_k	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} -1 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 1.5 \\ -3 \end{bmatrix}$	$\begin{bmatrix} 1.5 \\ -3 \end{bmatrix}$	$\begin{bmatrix} 1.5 \\ -3 \end{bmatrix}$

After 2 iterations, we find $v_2 = c\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$Ax = \lambda x$$

$$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} -3 \\ 6 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\lambda = -3$$

INVERSE POWER METHOD

Recall that if A is invertible with eigenvalue λ, then A^{-1} has an eigenvalue $\frac{1}{\lambda}$. So if we apply the power method to A^{-1}, its dominant eigenvalue will be the reciprocal of the smallest eigenvalue of A.

Example:

We will use the inverse power method to find the second eigenvalue of $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$.

Compute $x_k = A^{-1} y_{k-1}$. Rather than computing an inverse (which takes a lot of time), we can solve the equivalent equation $Ax_k = y_{k-1}$.

Let $x_0 = y_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Solve $Ax_1 = y_0$

$$[A \mid y_0] = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Therefore $x_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so $y_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and we get x_2 from $Ax_2 = y_1$.

$x_2 = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$, and by scaling we can get $y_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

k	0	1	2	3	4	5	6	7	8
x_k	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ -0.83 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ -1.1 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ -0.95 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ -1.02 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ -0.99 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ -1.01 \end{bmatrix}$
y_k	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	$\begin{bmatrix} -0.33 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.6 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.45 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.52 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.49 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.51 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.50 \\ 1 \end{bmatrix}$

$$Ay = \lambda y$$

$$\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \lambda \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \lambda \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\lambda = -1$$

THE SHIFTED INVERSE POWER METHOD

This can be used to find an approximation of any eigenvalue

If λ is an eigenvalue of A and $\alpha \neq \lambda$, then $A-\alpha I$ is invertible if α is not an eigenvalue of A and if $\frac{1}{(\lambda-\alpha)}$ is an eigenvalue of $(A-\alpha I)^{-1}$.

If α is close to λ, then $\frac{1}{(\lambda-\alpha)}$ will be a dominant eigenvalue of $(A-\alpha I)^{-1}$. If α is very close to λ then $\frac{1}{(\lambda-\alpha)}$ will be much bigger in magnitude than next eigenvalue, so the convergence will be very rapid.

Example: Use the shifted Power Method to find the eigenvalue closest to 5 when

$$A=\begin{bmatrix} 0 & 5 & -6 \\ -4 & 12 & -12 \\ -2 & -2 & 10 \end{bmatrix}$$

Shifting, we would have $A-5I=\begin{bmatrix} -5 & 5 & -6 \\ -4 & 7 & -12 \\ -2 & -2 & 5 \end{bmatrix}$.

Applying the inverse power method, let $x_0 = y_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$$\left[A-5I \mid y_0\right]=\left[\begin{array}{ccc|c} -5 & 5 & -6 & 1 \\ -4 & 7 & 12 & 1 \\ -2 & -2 & 5 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -0.61 \\ 0 & 1 & 0 & -0.88 \\ 0 & 0 & 1 & -0.39 \end{array}\right]$$

k	0	1	2	3	4	5	6	7
x_k	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} -0.61 \\ -0.88 \\ -0.39 \end{bmatrix}$	$\begin{bmatrix} -0.41 \\ -0.69 \\ -0.35 \end{bmatrix}$	$\begin{bmatrix} -0.47 \\ -0.89 \\ -0.44 \end{bmatrix}$	$\begin{bmatrix} -0.49 \\ -0.95 \\ -0.48 \end{bmatrix}$	$\begin{bmatrix} -0.50 \\ -0.98 \\ -0.49 \end{bmatrix}$	$\begin{bmatrix} -0.50 \\ -0.99 \\ -0.50 \end{bmatrix}$	$\begin{bmatrix} -0.50 \\ -1.00 \\ -0.50 \end{bmatrix}$
y_k	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0.69 \\ 1.00 \\ 0.45 \end{bmatrix}$	$\begin{bmatrix} 0.59 \\ 1.00 \\ 0.51 \end{bmatrix}$	$\begin{bmatrix} 0.53 \\ 1.00 \\ 0.50 \end{bmatrix}$	$\begin{bmatrix} 0.51 \\ 1.00 \\ 0.50 \end{bmatrix}$	$\begin{bmatrix} 0.50 \\ 1.00 \\ 0.50 \end{bmatrix}$	$\begin{bmatrix} 0.50 \\ 1.00 \\ 0.50 \end{bmatrix}$	$\begin{bmatrix} 0.50 \\ 1.00 \\ 0.50 \end{bmatrix}$

$$Ax = \lambda x$$

$$\begin{bmatrix} 0 & 5 & -6 \\ -4 & 12 & -12 \\ -2 & -2 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 8 \\ 4 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\lambda = 4$$

JACOBI AND GAUSS-SEIDEL METHODS

1) Apply the Jacobi method to the following system of linear equations using initial approximation *x* = (0, 0, 0). Round to three significant digits.

$$\begin{aligned} 2x_1 - x_2 &= 2 \\ x_1 - 3x_2 + x_3 &= -2 \\ -x_1 + x_2 - 3x_3 &= -6 \end{aligned}$$

2) Show that the Gauss-Seidel method diverges for the following system given initial approximation *x* = (0, 0)

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ 2x_1 + x_2 &= 3 \end{aligned}$$

3) Which of the following matrices are strictly diagonally dominant?

a) $\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$ b) $\begin{bmatrix} -1 & -2 \\ 0 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 12 & 6 & 0 \\ 2 & -3 & 2 \\ 0 & 6 & 13 \end{bmatrix}$ d) $\begin{bmatrix} 7 & 5 & -1 \\ 1 & -4 & 1 \\ 0 & 2 & -3 \end{bmatrix}$

4) Interchange the rows of the following system of linear equations to obtain a system with a strictly diagonally dominant coefficient matrix. Then apply the Gauss-Seidel method to approximate the solution to two significant digits.

$$\begin{aligned} 2x_1 - 3x_2 &= -7 \\ x_1 + 3x_2 - 10x_3 &= 9 \\ 3x_1 + x_3 &= 13 \end{aligned}$$