Complex Linear Independence: Decomplexification

4/7:

- When given a complex system of equations, it is necessary to **decomplexify** it.
- **Decomplexify**: To model a complex system of equations with a strictly real system for the purpose of applying the tenets of real linear algebra to it.
- Consider the following complex system of equations.

$$\begin{aligned}(2+i)x_1+(1+i)x_2&=3+6i\\(3-i)x_1+(2-2i)x_2&=7-i\end{aligned}$$

  - The solutions will be complex numbers: $x_1=a_1+ib_1$ and $x_2=a_2+ib_2$, where $a_1,a_2,b_1,b_2\in\mathbb{R}$.

- Transform it into a matrix system of equations. Separate the real and complex parts, and factor out all instances of the imaginary number i so that it is a coefficient to any complex matrix.

$$\begin{gathered}\begin{bmatrix}2+i & 1+i\\3-i & 2-2i\end{bmatrix}\begin{bmatrix}a_1+ib_1\\a_2+ib_2\end{bmatrix}=\begin{bmatrix}3+6i\\7-i\end{bmatrix}\\
\left(\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}+\begin{bmatrix}i & i\\-i & -2i\end{bmatrix}\right)\left(\begin{bmatrix}a_1\\a_2\end{bmatrix}+\begin{bmatrix}ib_1\\ib_2\end{bmatrix}\right)=\left(\begin{bmatrix}3\\7\end{bmatrix}+\begin{bmatrix}6i\\-i\end{bmatrix}\right)\\
\underbrace{\left(\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}+i\begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}\right)}_{A}\underbrace{\left(\begin{bmatrix}a_1\\a_2\end{bmatrix}+i\begin{bmatrix}b_1\\b_2\end{bmatrix}\right)}_{x}=\underbrace{\left(\begin{bmatrix}3\\7\end{bmatrix}+i\begin{bmatrix}6\\-1\end{bmatrix}\right)}_{b}\end{gathered}$$

- Foil the left side of the above equation[1].

$$\left(\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix}-\begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}\begin{bmatrix}b_1\\b_2\end{bmatrix}\right)+i\left(\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}\begin{bmatrix}b_1\\b_2\end{bmatrix}+\begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix}\right)=\begin{bmatrix}3\\7\end{bmatrix}+i\begin{bmatrix}6\\-1\end{bmatrix}$$

- Split the above system of equations into a real system of equations and a complex system of equations by setting equal to each other the real components of each side and the imaginary components of each side.

$$\begin{gathered}\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix}-\begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}\begin{bmatrix}b_1\\b_2\end{bmatrix}=\begin{bmatrix}3\\7\end{bmatrix}\\
\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}\begin{bmatrix}b_1\\b_2\end{bmatrix}+\begin{bmatrix}1 & 1\\-1 & -2\end{bmatrix}\begin{bmatrix}a_1\\a_2\end{bmatrix}=\begin{bmatrix}6\\-1\end{bmatrix}\end{gathered}$$

- Multiply out the matrices above to yield a system of four equations.

$$\begin{aligned}2a_1+a_2-b_1-b_2&=3\\3a_1+2a_2+b_1+2b_2&=7\\a_1+a_2+2b_1+b_2&=6\\-a_1-2a_2+3b_1+2b_2&=-1\end{aligned}$$

- Condense the above system of equations into a single matrix system of equations.

$$\begin{bmatrix}2 & 1 & -1 & -1\\3 & 2 & 1 & 2\\1 & 1 & 2 & 1\\-1 & -2 & 3 & 2\end{bmatrix}\begin{bmatrix}a_1\\a_2\\b_1\\b_2\end{bmatrix}=\begin{bmatrix}3\\7\\6\\-1\end{bmatrix}$$

[1] Note that the minus sign appears in the real component because, when multiplying the two "last" parts, $i^2=-1$.

- Solve for a_1, a_2, b_1, and b_2 using an augmented matrix and Gauss-Jordan elimination.

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & -1 & 3 \\ 3 & 2 & 1 & 2 & 7 \\ 1 & 1 & 2 & 1 & 6 \\ -1 & -2 & 3 & 2 & -1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}$$

- From these four values, the original solutions $x_1 = a_1 + ib_1$ and $x_2 = a_2 + ib_2$ can be found.

$$x_1 = 1 + 2i$$
$$x_2 = 2 - i$$

Hermitian, Unitary, and Normal Matrices

4/13:
- What necessitates different categorizations of complex vectors and matrices?
- Consider a vector v.

$$v = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

- If you want to find $||v||$, you typically evaluate $\sqrt{v^{\mathrm{T}}v}$. However, this equals to 0 (see the following), which is clearly not the magnitude of v.

$$\begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 - 1 = 0$$

  - Note that $||v||$ must be an element of $\mathbb{R}$ because it measures a distance.
- With complex vectors, it is necessary to evaluate $\sqrt{\bar{v}^{\mathrm{T}}v}$ to find $||v||$.

$$\begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 + 1 = 2$$

$$||v|| = \sqrt{2}$$

  - This makes sense because $\begin{bmatrix} 1 \\ i \end{bmatrix}$ extends one unit into $\mathbb{R}^1$ and one unit into $\mathbb{C}^2$.
  - If $z\bar{z} = |z|^2$ and $\bar{v}^{\mathrm{T}}v = v \cdot \bar{v}$, it stands to reason that $\bar{v}^{\mathrm{T}}v = ||v||^2$. Essentially, the dot product multiplies every element of v by its complex conjugate and sums them.
- Instead of writing $\bar{v}^{\mathrm{T}}$[2] every time, mathematicians shorthand to v^{H}[3].
  - v^{H} works for all vectors, but it is necessary for complex ones.
- **Hermitian** (matrix): A matrix A such that $A = A^{\mathrm{H}}$.
  - Typically defined for $A \in \mathbb{C}^n$, but holds for $A \in \mathbb{R}^n$, too.
  - Parallel to how if $A \in \mathbb{R}^n$ and $A = A^{\mathrm{T}}$, A is symmetrical.
  - Also note that if $A^{\mathrm{H}}A = A^2 = AA^{\mathrm{H}}$, A is Hermitian.

[2] "v conjugate transpose"
[3] "v Hermitian" after French mathematician Charles Hermite.

  - A Hermitian matrix has to have real values on the principal diagonal. When A is transposed and conjugated, the diagonal entries are the only values that don't move. Thus, their conjugates must equal themselves, so they must be real[4].

- **Unitary** (matrix): A matrix A such that $A^{-1} = A^{\mathrm{H}}$.
  - Typically defined for $A \in \mathbb{C}^n$, but holds for $A \in \mathbb{R}^n$, too.
  - Parallel to how if $A \in \mathbb{R}^n$ and $A^{-1} = A^{\mathrm{T}}$, A is orthonormal.
  - Also note that if $A^{\mathrm{H}}A = I = AA^{\mathrm{H}}$, A is unitary.
- **Normal** (matrix): A matrix that is unitarily diagonalizable.
  - Typically defined for $A \in \mathbb{C}^n$, but holds for $A \in \mathbb{R}^n$, too.
  - Parallel to matrices $A \in \mathbb{R}^n$ such that A is orthonormally diagonalizable.
- Note that not every complex matrix has to be one of these three types.
- When $A^{\mathrm{H}}A = AA^{\mathrm{H}}$, $A = U\Lambda U^{\mathrm{H}}$.

$$\begin{aligned} AA^{\mathrm{H}} &= \left(U\Lambda U^{\mathrm{H}}\right)\left(U\Lambda U^{\mathrm{H}}\right)^{\mathrm{H}} \\ &= U\Lambda U^{\mathrm{H}}U\Lambda^{\mathrm{H}}U^{\mathrm{H}} \\ &= U\Lambda\Lambda^{\mathrm{H}}U^{\mathrm{H}} \\ &= U\Lambda^{\mathrm{H}}\Lambda U^{\mathrm{H}\,[5]} \\ &= U\Lambda^{\mathrm{H}}U^{\mathrm{H}}U\Lambda U^{\mathrm{H}} \\ &= \left(U\Lambda U^{\mathrm{H}}\right)^{\mathrm{H}}\left(U\Lambda U^{\mathrm{H}}\right) \\ &= A^{\mathrm{H}}A \end{aligned}$$

- When $A = A^{\mathrm{H}}$, all eigenvalues are elements of $\mathbb{R}$ (similar to spectral theorem).

$$v^{\mathrm{H}}Av = \left(v^{\mathrm{H}}Av\right)^{\mathrm{H}} = v^{\mathrm{H}}Av$$

  - The above proves that $v^{\mathrm{H}}Av \in \mathbb{R}$ because it's its own conjugate[4].

$$\begin{aligned} Av &= \lambda v \\ v^{\mathrm{H}}Av &= \lambda v^{\mathrm{H}}v \end{aligned}$$

  - $\lambda = \frac{v^{\mathrm{H}}Av}{v^{\mathrm{H}}v} \to \frac{\mathbb{R}}{\mathbb{R}} = \mathbb{R}$[6].
- When $A = A^{\mathrm{H}}$ and $Ax = \lambda x$, all x's can be chosen orthonormally (also similar to spectral theorem).
  - Normality is implied because any eigenvector can be scaled to any version (including a normal version) and still be an eigenvector.

$$x_i = \begin{bmatrix} x_{i_1} \\ x_{i_2} \\ \vdots \\ x_{i_n} \end{bmatrix} \qquad\qquad x_i^{\mathrm{H}} = \begin{bmatrix} \bar{x}_{i_1} & \bar{x}_{i_2} & \cdots & \bar{x}_{i_n} \end{bmatrix}$$

[4]Recall that only real quantities can be their own conjugates because $a + 0i = a - 0i$.
[5]Since $\Lambda = \Lambda^{\mathrm{H}}$.
[6]Note that the denominator is real because it's how one finds $||v||$, and $||v||$ must be real, as discussed above.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad A^{\mathrm{H}} = \begin{bmatrix} a_{11} & \bar{a}_{21} & \cdots & \bar{a}_{n1} \\ \bar{a}_{12} & a_{22} & \cdots & \bar{a}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{1n} & \bar{a}_{2n} & \cdots & a_{nn} \end{bmatrix}$$

- Define an arbitrary vector x_i and matrix A, along with their conjugate transposes (or Hermitian versions). Note that the diagonal entries of A^{H} aren't shown as conjugated because their conjugates equal themselves.

$$\begin{aligned} Ax_1 &= \lambda_1 x_1 \\ x_2^{\mathrm{H}} A x_1 &= \lambda_1 x_2^{\mathrm{H}} x_1 \end{aligned}$$

$$\begin{aligned} Ax_2 &= \lambda_2 x_2 \\ (Ax_2)^{\mathrm{H}} &= (\lambda_2 x_2)^{\mathrm{H}} \\ x_2^{\mathrm{H}} A^{\mathrm{H}} &= \lambda_2 x_2^{\mathrm{H}} \\ x_2^{\mathrm{H}} A x_1 &= \lambda_2 x_2^{\mathrm{H}} x_1 \end{aligned}$$

- $\lambda_1 x_2^{\mathrm{H}} x_1 = \lambda_2 x_2^{\mathrm{H}} x_1$ implies that, since $\lambda_1 \neq \lambda_2$, $x_2^{\mathrm{H}} x_1$ must equal 0, proving orthogonality.

Complex Diagonalization

4/15: • Diagonalize the following matrix A.

$$A = \begin{bmatrix} 0.9 & -0.4 \\ 0.1 & 0.9 \end{bmatrix}$$

- Find the characteristic polynomial.

$$\begin{aligned} 0 &= \begin{vmatrix} 0.9 - \lambda & -0.4 \\ 0.1 & 0.9 - \lambda \end{vmatrix} \\ &= (0.9 - \lambda)^2 - (-0.4)(0.1) \\ &= 0.81 - 1.8\lambda + \lambda^2 + 0.04 \\ &= \lambda^2 - 1.8\lambda + 0.85 \end{aligned}$$

- Find the eigenvalues[7].

$$\begin{aligned} \lambda &= \frac{-(-1.8) \pm \sqrt{(-1.8)^2 - 4(1)(0.85)}}{2(1)} \\ &= 0.9 \pm \frac{\sqrt{-0.16}}{2} \\ &= 0.9 \pm \frac{\sqrt{-1}\sqrt{0.16}}{2} \\ &= 0.9 \pm \frac{0.4i}{2} \\ &= 0.9 \pm 0.2i \end{aligned}$$

$$\lambda_1 = 0.9 + 0.2i \qquad \lambda_2 = 0.9 - 0.2i$$

[7] It is interesting that the eigenvalues are complex conjugates of each other.

- Find the eigenvectors[8].

$$(A-(0.9+0.2i))x_1 = \begin{bmatrix} 0.9-(0.9+0.2i) & -0.4 \\ 0.1 & 0.9-(0.9+0.2i) \end{bmatrix} \begin{bmatrix} x_{1_1} \\ x_{1_2} \end{bmatrix}$$
$$= \begin{bmatrix} -0.2i & -0.4 \\ 0.1 & -0.2i \end{bmatrix} \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(A-(0.9-0.2i))x_1 = \begin{bmatrix} 0.9-(0.9-0.2i) & -0.4 \\ 0.1 & 0.9-(0.9-0.2i) \end{bmatrix} \begin{bmatrix} x_{1_1} \\ x_{1_2} \end{bmatrix}$$
$$= \begin{bmatrix} 0.2i & -0.4 \\ 0.1 & 0.2i \end{bmatrix} \begin{bmatrix} -2i \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 2i \\ 1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} -2i \\ 1 \end{bmatrix}$$

- Compile the diagonalization.

$$A = \frac{1}{4i} \begin{bmatrix} 2i & -2i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0.9+0.2i & 0 \\ 0 & 0.9-0.2i \end{bmatrix} \begin{bmatrix} 1 & 2i \\ -1 & 2i \end{bmatrix}$$

Real versus Complex

4/16:

Real	**Complex**
$\mathbb{R}^n$: vectors with n real components	$\mathbb{C}^n$: vectors with n complex components
length: $\|\|x\|\|^2 = x_1^2 + \cdots + x_n^2$	length: $\|\|z\|\|^2 = \|z_1\|^2 + \cdots + \|z_n\|^2$
transpose: $\left(A^{\mathrm{T}}\right)_{ij} = A_{ji}$	conjugate transpose: $\left(A^{\mathrm{H}}\right)_{ij} = \overline{A_{ji}}$
product rule: $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$	product rule: $(AB)^{\mathrm{H}} = B^{\mathrm{H}}A^{\mathrm{H}}$
dot product: $x^{\mathrm{T}}y = x_1y_1 + \cdots + x_ny_n$	inner product: $u^{\mathrm{H}}v = \bar{u}_1u_1 + \cdots + \bar{u}_nv_n$
reason for A^{T}: $(Ax)^{\mathrm{T}}y = x^{\mathrm{T}}(A^{\mathrm{T}}y)$	reason for A^{H}: $(Au)^{\mathrm{H}}v = u^{\mathrm{H}}(A^{\mathrm{H}}v)$
orthogonality: $x^{\mathrm{T}}y = 0$	orthogonality: $u^{\mathrm{H}}v = 0$.
symmetric matrices: $A = A^{\mathrm{T}}$	Hermitian matrices: $A = A^{\mathrm{H}}$
$A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}}$ (real Λ)	$A = U\Lambda U^{-1} = U\Lambda U^{\mathrm{H}}$ (real Λ)
skew-symmetric matrices: $k^{\mathrm{T}} = -K$	skew-Hermitian matrices: $K^{\mathrm{H}} = -K$
orthogonal matrices: $Q^{\mathrm{T}} = Q^{-1}$	unitary matrices: $U^{\mathrm{H}} = U^{-1}$
orthonormal columns: $Q^{\mathrm{T}}Q = I$	orthonormal columns: $U^{\mathrm{H}} = U^{-1}$
$(Qx)^{\mathrm{T}}(Qy) = x^{\mathrm{T}}y$ and $\|\|Qx\|\| = \|\|x\|\|$	$(Ux)^{\mathrm{H}}(Uy) = x^{\mathrm{H}}y$ and $\|\|Uz\|\| = \|\|z\|\|$

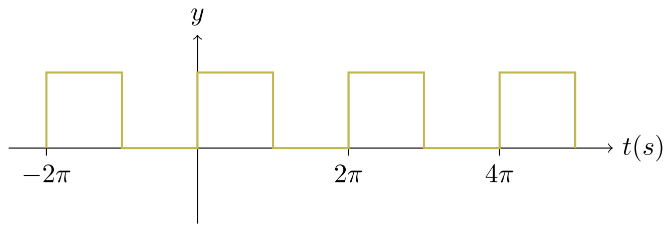
- Note that the columns and eigenvectors of Q and U are orthonormal, and all of their eigenvalues λ satisfy $|\lambda| = 1$.

[8]It is interesting that the eigenvectors are *also* complex conjugates of each other.

Real Fourier Series

4/21:

- Consider the square wave $f(t)$.

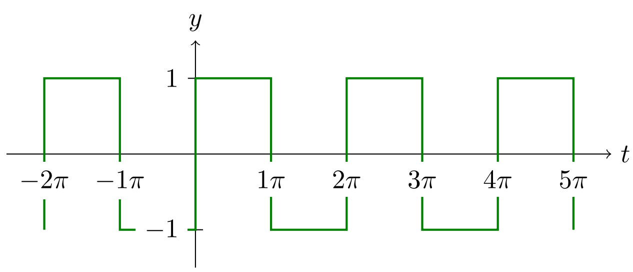


- Its period is $2\pi\,\frac{\text{sec}}{\text{cycle}}$, and its frequency is $\frac{1}{2\pi}$Hz.
- Can we write $f(t)$ as a sum of sines and cosines?

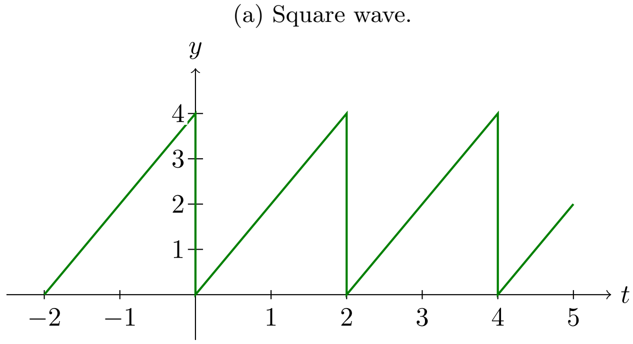
$$f(t) = a_0 + a_1\cos(t) + b_1\sin(t) + a_2\cos(2t) + b_2\sin(2t) + a_3\cos(3t) + b_3\sin(3t) + \cdots$$

  - Be general.
  - Since $T = 2\pi$, it makes sense to use some functions with $T = 2\pi$ to model it.
  - The weighting coefficients account for how much each function contributes to the whole.
- Historically studied by Fourier, who studied differential equations. Differential equations were often easy to solve for sines and cosines, so if a function could be modeled by a sum of sines and cosines, a related differential equation would be easier to solve.
- Fourier series, transforms, and analysis also tell us how much of each frequency a function contains (as measured by the weight coefficients).

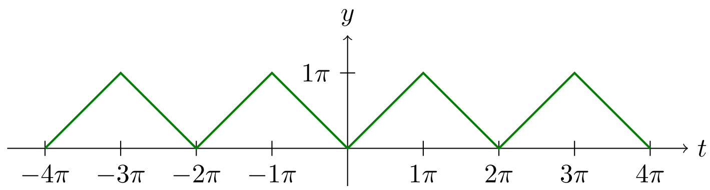
Wave Type Sketches



(a) Square wave.



(b) Sawtooth wave.



(c) Triangular wave.

- Square wave: $f(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & \pi < t < 2\pi \end{cases}$ where periodicity is defined by $f(t + 2\pi) = f(t)$.
- Sawtooth wave: $f(t) = 2t \qquad 0 < t < 2$ where periodicity is defined by $f(t + 2) = f(t)$.
- Triangular wave: $f(t) = |t| \qquad -\pi < t < \pi$ where periodicity is defined by $f(t + 2\pi) = f(t)$.

1) Evaluate $\int_{-\pi}^{\pi} \sin(nt)dt$, where n is an integer.

2) Evaluate $\int_{-\pi}^{\pi} \cos(nt)dt$, where n is an integer.

3) Using the results from problem 1 and problem 2, integrate both sides of the equation $f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt) \right)$ from $-\pi$ to π. Then simplify the result in terms of a_0. Divide the result in half and simplify in terms of $\frac{a_0}{2}$.

4) Using trigonometric identity $\sin(nt)\cos(mt) = \frac{1}{2}\left(\sin(n+m)t + \sin(n-m)t \right)$, evaluate $\int_{-\pi}^{\pi} \sin(nt)\cos(mt)dt$ where n and m are any integers.

5) Evaluate $\int_{-\pi}^{\pi} \cos(nt)\cos(mt)dt$ where n and m are any integers and $n \neq m$.

6) Evaluate $\int_{-\pi}^{\pi} \cos(nt)\cos(mt)dt$ where n and m are any integers and $n = m$ and $n \neq 0$, in other words evaluate $\int_{-\pi}^{\pi} \cos^2(nt)dt$.

7) Evaluate $\int_{-\pi}^{\pi} \cos(nt)\cos(mt)dt$ when $n = m = 0$

8) In a similar way to 5- 7, evaluate $\int_{-\pi}^{\pi} \sin(nt)\sin(mt)dt$ for the cases where $n \neq m, n = m \neq 0, n = m = 0$,

Hints: Use identity $\sin(nt)\sin(mt) = \frac{1}{2}\left(\cos(n-m)t - \cos(n+m)t \right)$ for the case when $n \neq m$ and use identity $\cos(2\theta) = 1 - 2\sin^2\theta$ for which $\theta = nt$ for the case when $n = m \neq 0$.

1) $\int_{-\pi}^{\pi} \sin(nt)dt = \left[-\frac{1}{n}\cos(nt)\right]_{-\pi}^{\pi} = \frac{1}{n}\left(-\cos(n\pi)+\cos(n\pi)\right) = 0, n \neq 0$

2) $\int_{-\pi}^{\pi} \cos(nt)dt = \left[\frac{1}{n}\sin(nt)\right]_{-\pi}^{\pi} = 0, n \neq 0$

3) $\int_{-\pi}^{\pi} f(t)dt = \frac{1}{2}\int_{-\pi}^{\pi} a_0 dt + \sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} a_n \cos(nt)dt + \int_{-\pi}^{\pi} b_n \sin(nt)dt\right) = \frac{1}{2}\left[a_0 t\right]_{-\pi}^{\pi} + \sum_{n=1}^{\infty}(0+0)$

$\therefore \int_{-\pi}^{\pi} f(t)dt = \frac{1}{2}(2a_0\pi) \to a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt \to \frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt$

4) $\int_{-\pi}^{\pi} \sin(nt)\cos(mt)dt = \frac{1}{2}\left(\int_{-\pi}^{\pi} \sin(n+m)tdt + \int_{-\pi}^{\pi} \sin(n-m)tdt\right) = \frac{1}{2}(0+0) = 0$ using the results from problems 1 and 2 since if n is an integer and m is an integer then $n+m$ and $n-m$ are also integers.

5) $\int_{-\pi}^{\pi} \cos(nt)\cos(mt)dt = \frac{1}{2}\left(\int_{-\pi}^{\pi} \cos(n+m)tdt + \int_{-\pi}^{\pi} \cos(n-m)tdt\right) = \frac{1}{2}(0+0) = 0$ when $n \neq m$

6) $\int_{-\pi}^{\pi} \cos^2(nt)dt = \frac{1}{2}\int_{-\pi}^{\pi} \left(1+\cos(2n)t\right)dt = \frac{1}{2}\left[t + \frac{1}{2n}\sin(2n)t\right]_{-\pi}^{\pi} = \pi$ when $n \neq 0$

7) $\pi - (-\pi) = 2\pi$

8) a) $\int_{-\pi}^{\pi} \sin(nt)\sin(mt)dt = 0$

b) $\int_{-\pi}^{\pi} \sin^2(nt)dt = \frac{1}{2}\int_{-\pi}^{\pi} \left(1-\cos(2nt)\right)dt = \pi$

c) $\int_{-\pi}^{\pi} \sin(nt)\sin(mt)dt = 0$

For integers m, n:

$$\int_{-\pi}^{\pi} \sin(nt)\cos(mt)dt = 0$$

$$\int_{-\pi}^{\pi} \cos(nt)\cos(mt)dt = \begin{Bmatrix} 0, n \neq m \\ \pi, n = m \neq 0 \\ 2\pi, n = m = 0 \end{Bmatrix}$$

$$\int_{-\pi}^{\pi} \sin(nt)\sin(mt)dt = \begin{Bmatrix} 0, n \neq m, n = m = 0 \\ \pi, n = m \end{Bmatrix}$$

FOURIER SERIES

- We have shown that when $v^T w = 0$, then vectors *v* and *w* are orthogonal.
- Fourier series ask us to think of continuous functions as vectors.
- Let $f(t) = \cos(t)$ and let $g(t) = \sin(t)$
- To find $f \bullet g$ of these two function "vectors" would be asking us to take a dot product that has infinitely many terms. All vectors we have used have had a finite number of components. Because these "vectors" would have infinitely many terms, it would be like asking us to take an integral!
- $f \bullet g = \int_0^{2\pi} \cos(t)\sin(t)dt$ (We limit the domain because these are sinusoidal periodic functions and thus they repeat after a period of 2π.)

$$\int_0^{2\pi} \cos(t)\sin(t)dt$$

Let $u = \cos(t)$ and $du = -\sin(t)$

$$-\int_1^1 u du$$

$$\left[-\frac{u^2}{2}\right]_1^1$$

$$0$$

OR

$$\left[-\frac{1}{2}\cos^2(t)\right]_0^{2\pi}$$

$$-\frac{1}{2}(\cos^2 2\pi - \cos^2 0)$$

$$-\frac{1}{2}(1-1) =$$

$$0$$

- Therefore $\cos t \bullet \sin t = 0$, therefore these function "vectors" are orthogonal, therefore they serve as basis vectors for the function space of continuous periodic functions in F^2, therefore any continuous function also in this space can be written as a linear combination of $\cos(t)$ and $\sin(t)$!!!

$$f(t) = \frac{a_0}{2} + a_1\cos(t) + b_1\sin(t) + a_2\cos(2t) + b_2\sin(2t) + \cdots$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nt) + b_n\sin(nt)\right)$$

*Why divide the constant term by 2?

CALCULATING THE FOURIER COEFFICENTS

As sine and cosine can serve as an orthogonal basis for periodic functions, consider the Fourier Series for a function $f(t)$ of period 2π to be $f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos(nt)+b_n\sin(nt)\right)$

In problem 3 from your handout you already found a formula for a_0, the constant term.

$$\boxed{\frac{a_0}{2}=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)dt}$$

To obtain the coefficients $a_n, n\in\mathbb{Z}$, multiply both sides of the equation by $\cos(mt)$ where $m\in\mathbb{Z}$ and $m>0$ and integrate both sides from $-\pi$ to π since this is a periodic function of period 2π.

$$\int_{-\pi}^{\pi}f(t)\cos(mt)dt=\frac{a_0}{2}\int_{-\pi}^{\pi}\cos(mt)dt+\sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos(nt)\cos(mt)dt+b_n\int_{-\pi}^{\pi}\sin(nt)\cos(mt)dt\right)$$

Simplify this result using the integrals on the right side of this equation from the handout problems 5-7. The only nonzero integral results from $\int_{-\pi}^{\pi}\cos^2(mt)dt=\pi$ in the case where $n=m$.

$$\therefore\int_{-\pi}^{\pi}\cos(mt)dt=a_m\pi$$

Sine this is the case where $n=m$, replacing m with n and solving for a_n we obtain:

$$\boxed{a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos(nt)dt\text{ , }n\in\mathbb{Z}\text{ and }n>0}$$

To obtain the coefficients $b_{n,}b\in\mathbb{Z},n>0$, multiply both sides of the equation by $\sin(mt)$ where $m\in\mathbb{Z}$ and $m>0$ and integrate both sides from $-\pi$ to π since this is a periodic function of period 2π.

$$\int_{-\pi}^{\pi}f(t)\sin(mt)dt=\frac{a_0}{2}\int_{-\pi}^{\pi}\sin(mt)dt+\sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos(nt)\sin(mt)dt+b_n\int_{-\pi}^{\pi}\sin(nt)\sin(mt)dt\right)$$

Simplify this result using the integrals on the right side of this equation using the handout problems you did for 8. The only nonzero integral results from $\int_{-\pi}^{\pi}b_m\sin^2(mt)dt=b_m\pi$ in the case where $n=m$. Relabeling m as n and solving for b_n we obtain:

$$\boxed{b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin(nt)dt\text{ , }n\in\mathbb{Z}\text{ and }n>0}$$

Square Wave

4/29:
- **Hilbert space**: An infinite-dimensional vector space — extends a lot of the ideas of linear algebra to functions in infinite dimensions.
  - Recall that functions behave with linearity ($\alpha f(t) + \beta g(t)$ is still a function).
- Square wave (period 2π): $f(t) = \begin{cases} 0 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases}$ and $f(t + 2\pi) = f(t)$.
- Find $\frac{a_0}{2}$ term:

$$\begin{aligned} \frac{a_0}{2} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\,\mathrm{d}t \\ &= \frac{1}{2\pi} \int_{-\pi}^{0} (0)\,\mathrm{d}t + \frac{1}{2\pi} \int_{0}^{\pi} (1)\,\mathrm{d}t \\ &= 0 + \left[\frac{t}{2\pi}\right]_0^{\pi} \\ &= \frac{\pi}{2\pi} \\ \boxed{\frac{a_0}{2}} &\boxed{= \frac{1}{2}} \end{aligned}$$

  - Makes sense because $\frac{a_0}{2}$ is like the sinusoidal axis and $\frac{1}{2}$ is half way between 0 and 1.
- Find a_n terms:

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\,\mathrm{d}t \\ &= \frac{1}{\pi} \int_{-\pi}^{0} (0) \cos(nt)\,\mathrm{d}t + \frac{1}{\pi} \int_{0}^{\pi} (1) \cos(nt)\,\mathrm{d}t \\ &= 0 + \left[\frac{1}{n\pi} \sin(nt)\right]_0^{\pi} \\ \boxed{a_n} &\boxed{= 0} \end{aligned}$$

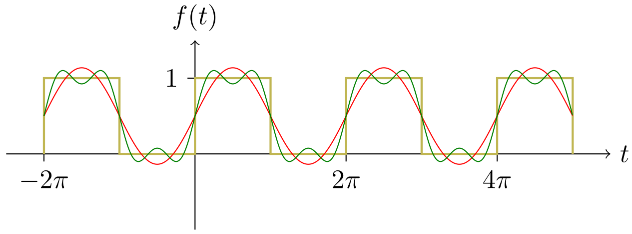
- Find b_n terms:

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt)\,\mathrm{d}t \\ &= \frac{1}{\pi} \int_{-\pi}^{0} (0) \sin(nt)\,\mathrm{d}t + \frac{1}{\pi} \int_{0}^{\pi} (1) \sin(nt)\,\mathrm{d}t \\ &= 0 - \frac{1}{n\pi} [\cos(nt)]_0^{\pi} \\ &= -\frac{1}{n\pi} [\cos(n\pi) - \cos(0)] \\ &= -\frac{1}{n\pi} [\cos(n\pi) - 1]^{[9]} \\ \boxed{b_n} &\boxed{= \begin{cases} 0 & n = 2, 4, 6, \dots \\ \frac{2}{n\pi} & n = 1, 3, 5, \dots \end{cases}} \end{aligned}$$

[9] $\cos(n\pi)$ equals 1 when n is even and -1 when n is odd.

- Assemble the Fourier series for the square wave function:

$$\begin{aligned} f(t) &= \frac{1}{2} + \frac{2}{\pi}\sin(t) + \frac{2}{3\pi}\sin(3t) + \frac{2}{5\pi}\sin(5t) + \cdots \\ &= \frac{1}{2} + \frac{2}{\pi}\left(\frac{1}{1}\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots\right) \\ &= \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((2n-1)t) \end{aligned}$$



FOURIER SERIES FOR PERIODIC FUNCTIONS WHEN PERIOD $\neq 2\pi$

Given:

$$\frac{a_0}{2} = \int_{-\pi}^{\pi} \frac{1}{2\pi} f(t)dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt \text{ , } n \in \mathbb{Z} \text{ and } n > 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt \text{ , } n \in \mathbb{Z} \text{ and } n > 0$$

Change the variable t to $x = \frac{2\pi}{P}t$. In this case $x = \pi$ corresponds to $t = \frac{P}{2}$ and $x = -\pi$ corresponds to $t = -\frac{P}{2}$.

Therefore regarded as a function of t, this is a function with period P. When we make the substitution $x = \frac{2\pi}{P}t$ and $dx = \frac{2\pi}{P}dt$ into the expressions for a_n and b_n we arrive at:

$$\boxed{\begin{aligned} a_n &= \frac{2}{P}\int_{-\frac{P}{2}}^{\frac{P}{2}} f(t)\cos\left(\frac{2n\pi t}{P}\right)dt, n \in \mathbb{Z}, n \geq 0 \\ b_n &= \frac{2}{P}\int_{-\frac{P}{2}}^{\frac{P}{2}} f(t)\sin\left(\frac{2n\pi t}{P}\right)dt, n \in \mathbb{Z}, n > 0 \end{aligned}}$$

These integrals will give the Fourier coefficients from a function of period P whose Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2n\pi t}{P}\right) + b_n \sin\left(\frac{2n\pi t}{P}\right)\right)$$

Note: In Differential Equations it is often convenient to write the period P as 2ℓ and in Physics and Engineering it is often written in terms of angular frequency ω as $P = \frac{2\pi}{\omega}$. Those substitutions would result in the following formulas:

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(t)\cos\left(\frac{n\pi t}{\ell}\right)dt, n \in \mathbb{Z}, n \geq 0 \text{ for Fourier Series } f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi t}{\ell}\right) + b_n \sin\left(\frac{n\pi t}{\ell}\right)\right)$$

And

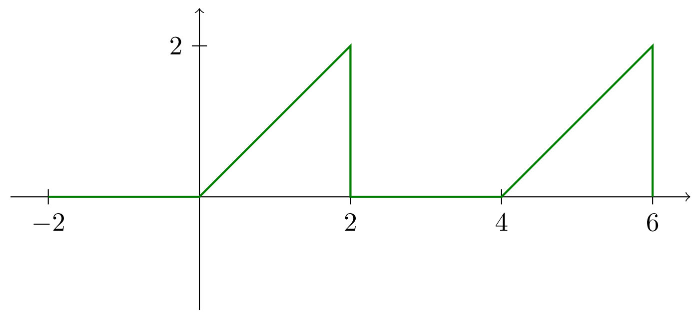
$$a_n = \frac{\omega}{\pi}\int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} f(t)\cos(n\omega t)\,dt, n \in \mathbb{Z}, n \geq 0 \text{ for Fourier Series } f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos(n\omega t) + b_n \sin(n\omega t)\right)$$

Any convenient integration range of length P, 2ℓ, or $\frac{2\pi}{\omega}$ can be used, and formulas for b_n would follow similarly for those for a_n as shown above.

*Why divide the constant term by 2? To account for the fact that the formula for a_n could be true for all $n \in \mathbb{Z}, n \geq 0$ (which would include the constant term) depending upon how the formula for the Fourier coefficients are written. Recall from problems 6 and 7 for the integration problems combined that $\int_{-\pi}^{\pi} \cos^2(nt)dt = \pi$ when $n \neq 0$ but $\int_{-\pi}^{\pi} \cos^2(nt)dt = 2\pi$ when $n = 0$. Using the formula $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt$ we could write the Fourier Series as $f(t) = \sum_{n=0}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt)$ but in this case $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt = \begin{Bmatrix} 2a_0, n = 0 \\ a_n, n \neq 0 \end{Bmatrix}$. To compensate for this the constant term is customarily written as $\frac{a_0}{2}$.

Modified Sawtooth Wave

4/30: • Modified sawtooth wave (period 4): $f(t) = \begin{cases} 0 & -2 < t < 0 \\ t & 0 < t < 2 \end{cases}$ and $f(t+4) = f(t)$.



- Period is 4: $P = 2\ell \Rightarrow \ell = 2$.

• Use this Fourier model:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi t}{\ell}\right) + b_n \sin\left(\frac{n\pi t}{\ell}\right)\right)$$

• $\frac{a_0}{2} = \frac{1}{2}$ (think of it as a weighted average of where the function spends the most time — note that this is actually exactly what the integral computes).

• Find a_n terms:

$$\begin{aligned} a_n &= \frac{1}{2}\int_{-2}^{2} f(t)\cos\left(\frac{n\pi t}{2}\right)\,\mathrm{d}t \\ &= 0 + \frac{1}{2}\int_{0}^{2} t\cos\left(\frac{n\pi t}{2}\right)\,\mathrm{d}t \end{aligned}$$

$$\begin{aligned} u &= t & \mathrm{d}v &= \cos\left(\frac{n\pi t}{2}\right)\,\mathrm{d}t \\ \mathrm{d}u &= \mathrm{d}t & v &= \frac{2}{n\pi}\sin\left(\frac{n\pi t}{2}\right) \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{2}\left(\left[(t)\left(\frac{2}{n\pi}\sin\left(\frac{n\pi t}{2}\right)\right)\right]_0^2 - \int_0^2 \frac{2}{n\pi}\sin\left(\frac{n\pi t}{2}\right)\,\mathrm{d}t\right) \\ &= \frac{1}{2}\left(\left(\frac{4}{n\pi}\sin(n\pi) - 0\right) + \frac{4}{n^2\pi^2}\left[\cos\left(\frac{n\pi t}{2}\right)\right]_0^2\right) \\ &= \frac{1}{2}\left(0 + \frac{4}{n^2\pi^2}(\cos(n\pi) - \cos(0))\right) \\ &= \frac{2}{n^2\pi^2}(\cos(n\pi) - 1)^{[9]} \end{aligned}$$

$$\boxed{a_n = \begin{cases} 0 & n = 2,4,6,\dots \\ -\frac{4}{n^2\pi^2} & n = 1,3,5,\dots \end{cases}}$$

- Find b_n terms:

$$\begin{aligned} b_n &= \frac{1}{2}\int_{-2}^{2} f(t)\sin\left(\frac{n\pi t}{2}\right)\,\mathrm{d}t \\ &= 0 + \frac{1}{2}\int_{0}^{2} t\sin\left(\frac{n\pi t}{2}\right)\,\mathrm{d}t \end{aligned}$$

$$u = t \qquad \mathrm{d}v = \sin\left(\frac{n\pi t}{2}\right)\,\mathrm{d}t$$

$$\mathrm{d}u = \mathrm{d}t \qquad v = -\frac{2}{n\pi}\cos\left(\frac{n\pi t}{2}\right)\,\mathrm{d}t$$

$$\begin{aligned} b_n &= \frac{1}{2}\left(\left[(t)\left(-\frac{2}{n\pi}\cos\left(\frac{n\pi t}{2}\right)\right)\right]_0^2 + \frac{2}{n\pi}\int_0^2 \cos\left(\frac{n\pi t}{2}\right)\,\mathrm{d}t\right) \\ &= \frac{1}{2}\left(\left(-\frac{4}{n\pi}\cos(n\pi) - 0\right) + 0\right) \\ &\boxed{b_n = \begin{cases} -\frac{2}{n\pi} & n = 2,4,6,\dots \\ \frac{2}{n\pi} & n = 1,3,5,\dots \end{cases}} \end{aligned}$$

- Assemble the Fourier series for this modified sawtooth wave function:

$$a_n = -\frac{2}{n^2\pi^2} + (-1)^n\frac{2}{n^2\pi^2}$$

$$b_n = (-1)^{n+1}\frac{2}{n\pi}$$

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(\left(-\frac{2}{n^2\pi^2} + (-1)^n\frac{2}{n^2\pi^2}\right)\cos\left(\frac{n\pi t}{2}\right) + \left((-1)^{n+1}\frac{2}{n\pi}\right)\sin\left(\frac{n\pi t}{2}\right)\right)$$

Complex Fourier Series

5/4:

- **Euler's formula**: $\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$.
- Using Euler's Formula, we can replace the trigonometric functions in Fourier series with complex exponential functions. By combining the Fourier coefficients a_n and b_n into one complex coefficient c_n, we find that, for a given periodic signal, both sets of constants can be found in one operation.
- Let's derive sine and cosine in terms of complex exponentials.
  - For this to work, we will need Euler's formula, and a second, negative version of Euler's formula: $\mathrm{e}^{-i\theta} = \cos\theta - i\sin\theta$.

$$\begin{aligned} (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta) &= \mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta} \\ 2\cos\theta &= \mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta} \\ &\boxed{\cos\theta = \frac{1}{2}\left(\mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta}\right)} \end{aligned}$$

$$\begin{aligned} (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta) &= \mathrm{e}^{i\theta} - \mathrm{e}^{-i\theta} \\ 2i\sin\theta &= \mathrm{e}^{i\theta} - \mathrm{e}^{-i\theta} \\ &\boxed{\sin\theta = \frac{1}{2i}\left(\mathrm{e}^{i\theta} - \mathrm{e}^{-i\theta}\right)} \end{aligned}$$

- We can use the above results to express $a_n\cos(n\omega_0\theta) + b_n\sin(n\omega_0\theta)$, where $\omega_0 = \frac{2\pi}{P}$, in terms of complex exponentials.

$$\begin{aligned}
a_n\cos(n\omega_0\theta) + b_n\sin(n\omega_0\theta) &= \frac{a_n}{2}\left(\mathrm{e}^{in\omega_0\theta} + \mathrm{e}^{-in\omega_0\theta}\right) + \frac{b_n}{2i}\left(\mathrm{e}^{in\omega_0\theta} - \mathrm{e}^{-in\omega_0\theta}\right) \\
&= \frac{a_n}{2}\mathrm{e}^{in\omega_0\theta} + \frac{a_n}{2}\mathrm{e}^{-in\omega_0\theta} + \frac{b_n}{2i}\mathrm{e}^{in\omega_0\theta} - \frac{b_n}{2i}\mathrm{e}^{-in\omega_0\theta} \\
&= \left(\frac{a_n}{2} + \frac{b_n}{2i}\right)\mathrm{e}^{in\omega_0\theta} + \left(\frac{a_n}{2} - \frac{b_n}{2i}\right)\mathrm{e}^{-in\omega_0\theta} \\
&= \frac{1}{2}\left(a_n + \frac{b_n}{i}\right)\mathrm{e}^{in\omega_0\theta} + \frac{1}{2}\left(a_n - \frac{b_n}{i}\right)\mathrm{e}^{-in\omega_0\theta} \\
&= \frac{1}{2}\left(a_n + \left(\frac{b_n}{i}\right)(1)\right)\mathrm{e}^{in\omega_0\theta} + \frac{1}{2}\left(a_n - \left(\frac{b_n}{i}\right)(1)\right)\mathrm{e}^{-in\omega_0\theta} \\
&= \frac{1}{2}\left(a_n + \left(\frac{b_n}{i}\right)(i^4)\right)\mathrm{e}^{in\omega_0\theta} + \frac{1}{2}\left(a_n - \left(\frac{b_n}{i}\right)(i^4)\right)\mathrm{e}^{-in\omega_0\theta} \\
&= \frac{1}{2}\left(a_n + i^3 b_n\right)\mathrm{e}^{in\omega_0\theta} + \frac{1}{2}\left(a_n - i^3 b_n\right)\mathrm{e}^{-in\omega_0\theta} \\
&= \frac{1}{2}(a_n + (-i)b_n)\mathrm{e}^{in\omega_0\theta} + \frac{1}{2}(a_n - (-i)b_n)\mathrm{e}^{-in\omega_0\theta} \\
&= \frac{1}{2}(a_n - ib_n)\mathrm{e}^{in\omega_0\theta} + \frac{1}{2}(a_n + ib_n)\mathrm{e}^{-in\omega_0\theta}
\end{aligned}$$

  - Define $c_n = \frac{1}{2}(a_n - ib_n)$ and complex conjugate $\bar{c}_n = \frac{1}{2}(a_n + ib_n)$. Now we have the following.

$$a_n\cos(n\omega_0\theta) + b_n\sin(n\omega_0\theta) = c_n\mathrm{e}^{in\omega_0\theta} + \bar{c}_n\mathrm{e}^{-in\omega_0\theta}$$

  - Substitution into the Fourier series sum gives the following.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(c_n\mathrm{e}^{in\omega_0\theta} + \bar{c}_n\mathrm{e}^{-in\omega_0\theta}\right) \tag{1}$$

- Equation 1 can become still neater and more concise through the following steps.

  1. Define $c_0 = \frac{a_0}{2}$[10].
  2. Define $c_{-n} = \bar{c}_n$. This permits the following.

$$\sum_{n=1}^{\infty}\bar{c}_n\mathrm{e}^{-in\omega_0 t} = \bar{c}_1\mathrm{e}^{-i\omega_0 t} + \bar{c}_2\mathrm{e}^{-2i\omega_0 t} + \cdots = c_{-1}\mathrm{e}^{-i\omega_0 t} + c_{-2}\mathrm{e}^{-2i\omega_0 t} + \cdots = \sum_{n=-1}^{-\infty} c_n\mathrm{e}^{in\omega_0 t}$$

  3. Using the new definitions of c_n for $n \in (-\infty, 0]$, it is possible to write Equation 1 as follows.

$$\begin{aligned}
f(t) &= c_0 + \sum_{n=1}^{\infty} c_n\mathrm{e}^{in\omega_0 t} + \sum_{n=1}^{\infty} c_{-n}\mathrm{e}^{-in\omega_0 t} \\
&= c_0 + \sum_{n=1}^{\infty} c_n\mathrm{e}^{in\omega_0 t} + \sum_{n=-\infty}^{-1} c_n\mathrm{e}^{in\omega_0 t}
\end{aligned}$$

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} c_n\mathrm{e}^{in\omega_0 t}}$$

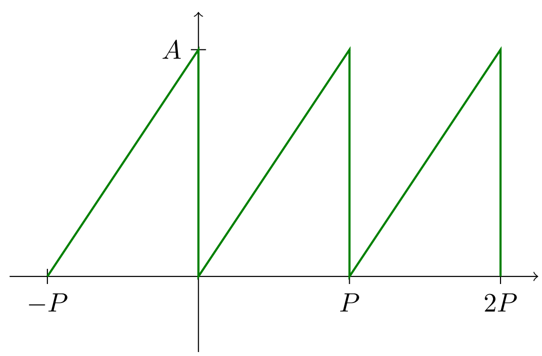
[10] Note that this is consistent with the general definition of c_n since $b_0 = 0$.

- We now tackle how to solve for the complex coefficients c_n[11].
  1. For $n = 0$, $c_0 = \frac{a_0}{2} = \frac{1}{P}\int_{-P/2}^{P/2} f(t)\,\mathrm{d}t$.
  2. For $n \in \mathbb{Z}^+$, $c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{P}\int_{-P/2}^{P/2} f(t)\left(\cos(n\omega_0 t) - i\sin(n\omega_0 t)\right)\,\mathrm{d}t = \frac{1}{P}\int_{-P/2}^{P/2} f(t)\mathrm{e}^{-in\omega_0 t}\,\mathrm{d}t$.
  3. For $n \in \mathbb{Z}^-$, $c_n = \frac{1}{2}(a_n + ib_n) = \frac{1}{P}\int_{-P/2}^{P/2} f(t)\mathrm{e}^{-in\omega_0 t}\,\mathrm{d}t$[12].
  4. The above three results can be condensed into the following expression for all $n \in \mathbb{Z}$.

$$\boxed{c_n = \frac{1}{P}\int_{-P/2}^{P/2} f(t)\mathrm{e}^{-in\omega_0 t}\,\mathrm{d}t}$$

Generalized Sawtooth Wave

5/6: • Generalized sawtooth wave (period P, amplitude A): $f(t) = \frac{At}{P}$ and $f(t+P) = f(t)$.



- $c_0 = \frac{A}{2}$.
- $\omega_0 = \frac{2\pi}{P} \Rightarrow \omega_0 P = 2\pi$.
- Find c_n terms:

$$\begin{aligned} c_n &= \frac{1}{P}\int_0^P \frac{At}{P}\mathrm{e}^{-in\omega_0 t}\,\mathrm{d}t \\ &= \frac{A}{P^2}\int_0^P t\mathrm{e}^{-in\omega_0 t}\,\mathrm{d}t \end{aligned}$$

$$\begin{aligned} u &= t \qquad & \mathrm{d}v &= \mathrm{e}^{-in\omega_0 t}\,\mathrm{d}t \\ \mathrm{d}u &= \mathrm{d}t \qquad & v &= -\frac{1}{in\omega_0}\mathrm{e}^{-in\omega_0 t} \end{aligned}$$

$$\begin{aligned} &= \frac{A}{P^2}\left(\left[-\frac{t\mathrm{e}^{-in\omega_0 t}}{in\omega_0}\right]_0^P + \frac{1}{in\omega_0}\int_0^P \mathrm{e}^{-in\omega_0 t}\,\mathrm{d}t\right) \\ &= \frac{A}{P^2}\left(\left(-\frac{P\mathrm{e}^{-in\omega_0 P}}{in\omega_0} - 0\right) - \frac{1}{(in\omega_0)^2}\left[\mathrm{e}^{-in\omega_0 t}\right]_0^P\right) \\ &= \frac{A}{P^2}\left(-\frac{P\mathrm{e}^{-in\omega_0 P}}{in\omega_0} - \frac{1}{(in\omega_0)^2}\left(\mathrm{e}^{-in\omega_0 P} - 1\right)\right) \\ &= \frac{A}{P^2}\left(-\frac{P\mathrm{e}^{-i2\pi n}}{in\omega_0} - \frac{1}{(in\omega_0)^2}\left(\mathrm{e}^{-i2\pi n} - 1\right)\right) \end{aligned}$$

[11] Note that this can also be derived in an analogous method to how the original a_n and b_n expressions were derived.
[12] The negative exponential, when multiplied by a negative n, generates the + expansion of Euler's formula, as desired.

$$
\begin{aligned}
&= \frac{A}{P^2}\left(-\frac{P(1)}{in\omega_0} - \frac{1}{(in\omega_0)^2}((1)-1)\right) \\
&= \frac{A}{P^2}\left(-\frac{P}{in\omega_0}\right) \\
&= -\frac{A}{in\omega_0 P} \\
&= \frac{A}{2\pi n(-i)} \\
&\boxed{c_n = \frac{Ai}{2\pi n}}
\end{aligned}
$$

- Assemble the Fourier series for this generalized sawtooth wave function.

$$f(t) = \frac{Ai}{2\pi}\sum_{n=-\infty}^{\infty}\frac{\mathrm{e}^{in\omega_0 t}}{n}$$

Continuous Fourier Transform

5/8:
- The complex Fourier series can be summarized as one entity as follows.

$$f(t) = \sum_{n=-\infty}^{\infty}\left(\frac{1}{P}\int_{t_0}^{t_0+P} f(t)\mathrm{e}^{-in\omega_0 t}\,\mathrm{d}t\right)\mathrm{e}^{in\omega_0 t}$$

  - t_0 is an arbitrary t, and the above reflects that it is important that we integrate over a full period P, but it does not matter what we define to be a single iteration/period of $f(t)$.

- Substitute $\omega_0 = \frac{2\pi}{P}$.

$$f(t) = \sum_{n=-\infty}^{\infty}\left(\frac{1}{P}\int_{t_0}^{t_0+P} f(t)\mathrm{e}^{-in2\pi\frac{1}{P}t}\,\mathrm{d}t\right)\mathrm{e}^{in2\pi\frac{1}{P}t}$$

- Let $t_0 = -\frac{P}{2}$ and let $\frac{1}{P} = \Delta f$.

$$f(t) = \sum_{n=-\infty}^{\infty}\left(\int_{-P/2}^{P/2} f(t)\mathrm{e}^{-in2\pi\Delta f t}\,\mathrm{d}t\right)\mathrm{e}^{in2\pi\Delta f t}\Delta f$$

- What happens as $P \to \infty$?
  - Since $\frac{1}{P} = \Delta f$, $\Delta f \to \mathrm{d}f$[13] (think $\frac{1}{\infty}$).
  - Also, define a continuous variable f equivalent to $n\Delta f = \frac{n}{P}$[14].
  - Now that we have a continuous variable and a differential, the formula's summation is integration!

$$
\begin{aligned}
f(t) &= \lim_{P\to\infty}\left(\sum_{n=-\infty}^{\infty}\left(\int_{-P/2}^{P/2} f(t)\mathrm{e}^{-i2\pi n\Delta f t}\,\mathrm{d}t\right)\mathrm{e}^{i2\pi n\Delta f t}\Delta f\right) \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-i2\pi f t}\,\mathrm{d}t\right)\mathrm{e}^{i2\pi f t}\,\mathrm{d}f
\end{aligned}
$$

[13] A teeny-tiny differential.
[14] As P gets smaller, changes in $\frac{n}{P}$ become less discrete (less like increments) and more continuous.

- Let $2\pi f = \omega \Rightarrow \mathrm{d}f = \mathrm{d}\omega$.

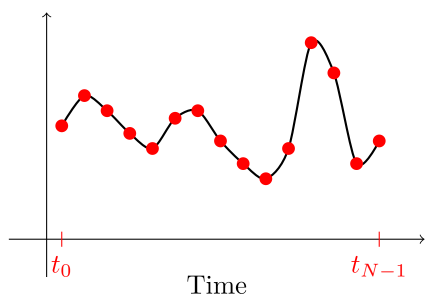
$$f(t) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t)\mathrm{e}^{-i\omega t}\,\mathrm{d}t \right) \mathrm{e}^{i\omega t}\,\mathrm{d}\omega \tag{2}$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-i\omega t}\,\mathrm{d}t \tag{3}$$

- Equation 3 is the Fourier Transform and Equation 2 is the Inverse Fourier Transform.

Discrete Fourier Transform

5/11: - Consider Equation 3, the integral of the product of two functions.
  - $f(t)$ is the function or periodic signal whose domain we are trying to transform.
  - $\mathrm{e}^{-i\omega t}$ is the analyzing function, which, as we know from Euler's formula, represents sinusoids.
  - Whenever $f(t)$ and $\mathrm{e}^{-i\omega t}$ are *similar*, they will multiply and sum to a *large* coefficient[15].
  - Whenever $f(t)$ and $\mathrm{e}^{-i\omega t}$ are *dissimilar*, they will multiply and sum to a *small* coefficient.
- Infinite integral bounds: In the real practice of signal analysis, they doesn't make much sense because you would only be interested in collecting data on a signal within a finite time frame, and sampling can only be done discretely. In reality, a signal sample might look like this:



  - A discrete number ($N = 15$) of evenly spaced samples over a discrete quantity of time.
  - Note: t_0 to t_{N-1} because computer science canonically counts starting at 0.
- Change notation (too many f's) and substitute $\omega = 2\pi F$:

$$X(F) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-i2\pi F t}\,\mathrm{d}t$$

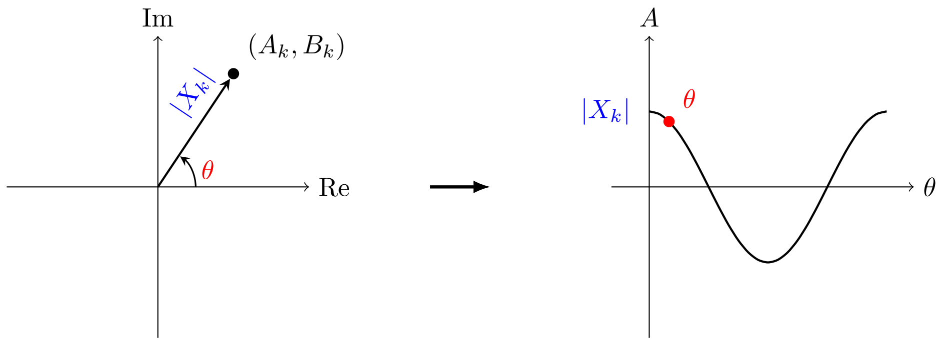
- Make the above continuous definition discrete:

$$X_k = \sum_{n=0}^{N-1} x_n \cdot \mathrm{e}^{-\frac{i2\pi kn}{N}}$$

  - k^{th} frequency.
  - Discrete summation to evaluate N samples.
- With the Discrete Fourier Transform (DFT), we can no longer look at frequency and time continuously; we instead look at the k^{th} frequency and the n^{th} sample.
  - Essentially, $\frac{k}{N}$ corresponds to frequency and n corresponds to time.

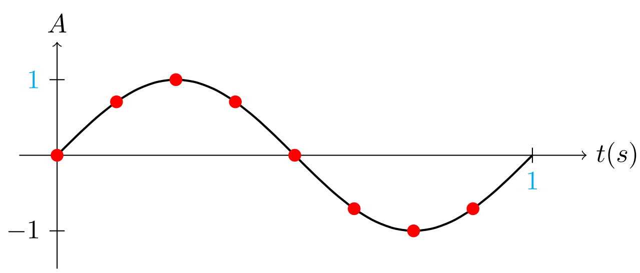
[15]We're trying to find the most common or largest frequencies in this periodic signal; when ω (independent variable) gets very close to the frequency of a component sinusoid in $f(t)$, $F(\omega)$ (dependent variable) will get quite big.

- What happens when you expand the summation?
  - Let $b_n = \frac{2\pi k n}{N}$.
  $$X_k = x_0 \mathrm{e}^{-b_0 i} + x_1 \mathrm{e}^{-b_1 i} + \cdots + x_{N-1} \mathrm{e}^{-b_{N-1} i}$$
  - Since $\mathrm{e}^{ix} = \cos(x) + i\sin(x)$,
  $$X_k = x_0\left(\cos(-b_0) + i\sin(-b_0)\right) + \cdots + x_{N-1}\left(\cos(-b_{N-1}) + i\sin(-b_{N-1})\right)$$
  - Thus, the sum is just some complex number $X_k = A_k + B_k i$.
  - Note that the magnitude and angle just correspond to an amplitude and phase displacement from a linear combination of the sine and cosine functions:



Simple DFT Example

- 1 Hz sine wave of amplitude 1.
- Sampling frequency of 8 Hz, so $N = 8$.



- Sampled values:

$\boldsymbol{x_n}$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7
val	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{2}}{2}$	0	$-\frac{\sqrt{2}}{2}$	-1	$-\frac{\sqrt{2}}{2}$

- Now apply the formula to find the Fourier coefficients in the eight "frequency bins."

$$X_k = \sum_{n=0}^{8-1} x_n \cdot \mathrm{e}^{-\frac{i2\pi k n}{N}}$$

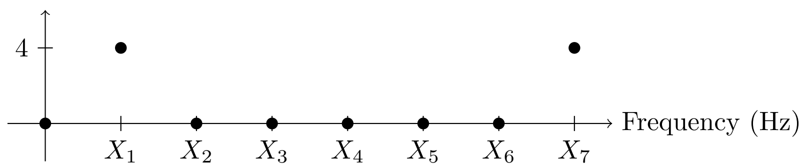
$$\begin{aligned} X_0 &= (0)\mathrm{e}^{-\frac{i2\pi(0)(0)}{8}} + \left(\frac{\sqrt{2}}{2}\right)\mathrm{e}^{-\frac{i2\pi(0)(1)}{8}} + \cdots + \left(-\frac{\sqrt{2}}{2}\right)\mathrm{e}^{-\frac{i2\pi(0)(7)}{8}} \\ &\vdots \\ X_7 &= (0)\mathrm{e}^{-\frac{i2\pi(7)(0)}{8}} + \left(\frac{\sqrt{2}}{2}\right)\mathrm{e}^{-\frac{i2\pi(7)(1)}{8}} + \cdots + \left(-\frac{\sqrt{2}}{2}\right)\mathrm{e}^{-\frac{i2\pi(7)(7)}{8}} \end{aligned} \tag{4}$$

5/12:

- The sums come out to be the following.

Sum	X_0	X_1	X_2	X_3	X_4	X_5	X_6	X_7
Value	0	$-4i$	0	0	0	0	0	$4i$

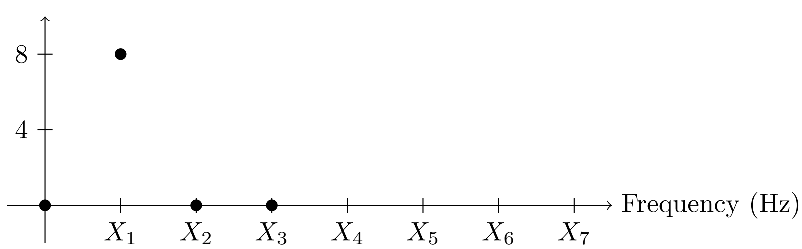
- Plot the magnitudes of each sum in the following **Spectrum Plot**.



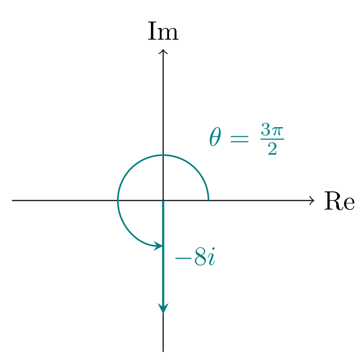
- Info:
  - Frequency Resolution $= \frac{\text{Sampling Frequency}}{N} = \frac{8 \text{ Hz}}{8} = 1$ Hz.
  - A non-zero value at X_1 corresponds to 1 Hz, and this is a 1 Hz sine wave!
- **Nyquist frequency**: Remove values at or above $\frac{\text{Sampling frequency}}{2}$ and double values below.
- Thus, discount values above $\frac{8}{2} = 4$ (i.e., X_4, X_5, X_6, and X_7) and double the values for X_0 through X_3.

Sum	X_0	X_1	X_2	X_3
Value	0	$-8i$	0	0

- Plot the magnitudes below.



- amplitude $= \frac{8 \text{ (from plot)}}{8 \text{ samples}} = 1$.



$$1\cos\left(\theta - \frac{3\pi}{2}\right) = \sin\theta$$

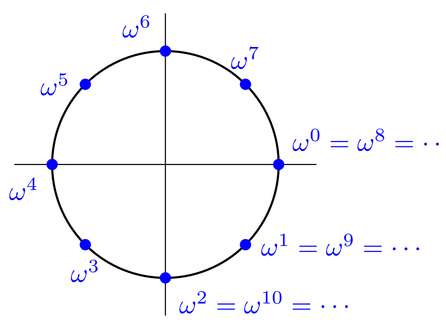
Condensing the Bins into a Matrix

5/13:

- Look at what is a constant and what is a variable in Equations 4.
  - In each $x_n \cdot \mathrm{e}^{-\frac{i2\pi kn}{N}}$ term, $\mathrm{e}^{-\frac{i2\pi}{8}}$ is constant, k and n are variables (that happen to correspond to the "row and column location" of the term), and x_n is a scalar.
- Thus, let $\omega = \mathrm{e}^{-\frac{i2\pi}{8}}$.
  - Under this definition, $\mathrm{e}^{-\frac{i2\pi kn}{8}}$ terms become ω^{kn} terms.
- Notice that the X_k calculations are just scaling & adding — scaling by x_n and adding the terms of the summation!
- Therefore, with the help of our new definition of ω, we can rewrite Equations 4 in the form $X_k = Fx_n$.

$$\begin{bmatrix}X_0\\X_1\\X_2\\X_3\\X_4\\X_5\\X_6\\X_7\end{bmatrix} = \begin{bmatrix}\omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0\\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7\\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} & \omega^{14}\\ \omega^0 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} & \omega^{21}\\ \omega^0 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} & \omega^{28}\\ \omega^0 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} & \omega^{35}\\ \omega^0 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} & \omega^{42}\\ \omega^0 & \omega^7 & \omega^{14} & \omega^{21} & \omega^{28} & \omega^{35} & \omega^{42} & \omega^{49}\end{bmatrix}\begin{bmatrix}x_0\\x_1\\x_2\\x_3\\x_4\\x_5\\x_6\\x_7\end{bmatrix}$$

- Keep in mind — there are really only 8 distinct entries in this matrix because these powers are periodic.



  - Thus, F can be written as follows.

$$F = \begin{bmatrix}\omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0\\ \omega^0 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7\\ \omega^0 & \omega^2 & \omega^4 & \omega^6 & \omega^0 & \omega^2 & \omega^4 & \omega^6\\ \omega^0 & \omega^3 & \omega^6 & \omega^1 & \omega^4 & \omega^7 & \omega^2 & \omega^5\\ \omega^0 & \omega^4 & \omega^0 & \omega^4 & \omega^0 & \omega^4 & \omega^0 & \omega^4\\ \omega^0 & \omega^5 & \omega^2 & \omega^7 & \omega^4 & \omega^1 & \omega^6 & \omega^3\\ \omega^0 & \omega^6 & \omega^4 & \omega^2 & \omega^0 & \omega^6 & \omega^4 & \omega^2\\ \omega^0 & \omega^7 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega^1\end{bmatrix}$$

- Note that F is symmetric but not Hermitian because the diagonal is not real.

Fast Fourier Transform

5/15:

- Let F_8 be an 8×8 Fourier matrix, similar to the first one in the last section. Permute its columns by multiplying by a specific permutation matrix P from the back to yield F_8P.

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^7 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^8 & \omega^{10} & \omega^{12} & \omega^{14} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^{15} & \omega^{18} & \omega^{21} \\ 1 & \omega^4 & \omega^8 & \omega^{12} & \omega^{16} & \omega^{20} & \omega^{24} & \omega^{28} \\ 1 & \omega^5 & \omega^{10} & \omega^{15} & \omega^{20} & \omega^{25} & \omega^{30} & \omega^{35} \\ 1 & \omega^6 & \omega^{12} & \omega^{18} & \omega^{24} & \omega^{30} & \omega^{36} & \omega^{42} \\ 1 & \omega^7 & \omega^{14} & \omega^{21} & \omega^{28} & \omega^{35} & \omega^{42} & \omega^{49} \end{bmatrix}}_{F_8} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{P} =$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega^1 & \omega^3 & \omega^5 & \omega^7 \\ 1 & \omega^4 & \omega^8 & \omega^{12} & \omega^2 & \omega^6 & \omega^{10} & \omega^{14} \\ 1 & \omega^6 & \omega^{12} & \omega^{18} & \omega^3 & \omega^9 & \omega^{15} & \omega^{21} \\ 1 & \omega^8 & \omega^{16} & \omega^{24} & \omega^4 & \omega^{12} & \omega^{20} & \omega^{28} \\ 1 & \omega^{10} & \omega^{20} & \omega^{30} & \omega^5 & \omega^{15} & \omega^{25} & \omega^{35} \\ 1 & \omega^{12} & \omega^{24} & \omega^{36} & \omega^6 & \omega^{18} & \omega^{30} & \omega^{42} \\ 1 & \omega^{14} & \omega^{28} & \omega^{42} & \omega^7 & \omega^{21} & \omega^{35} & \omega^{49} \end{bmatrix}}_{F_8P}$$

- Partition F_8P into four 4×4 blocks $\begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}$, where, for example, f_1 is defined as follows.

$$f_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^4 & \omega^8 & \omega^{12} \\ 1 & \omega^6 & \omega^{12} & \omega^{18} \end{bmatrix}$$

- Although it may not appear as such at first, f_1 is a 4×4 Fourier matrix in $\omega^2 = \mathrm{e}^{-\frac{i2\pi}{4}}$:

$$f_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & (\omega^2)^1 & (\omega^2)^2 & (\omega^2)^3 \\ 1 & (\omega^2)^2 & (\omega^2)^4 & (\omega^2)^6 \\ 1 & (\omega^2)^3 & (\omega^2)^6 & (\omega^2)^9 \end{bmatrix}$$

  - In fact, f_1 is the Fourier matrix that would have resulted if only 4 evenly spaced samples had been used instead of 8. Thus, we can rightfully call it F_4. This notation will occasionally be used in the following to keep with the canonical notation.

- In the same way that the powers of ω were cyclic, the powers of ω^2 are cyclic. Thus, f_3 can be shown to be equal to f_1.

$$f_3 = \begin{bmatrix} 1 & \omega^8 & \omega^{16} & \omega^{24} \\ 1 & \omega^{10} & \omega^{20} & \omega^{30} \\ 1 & \omega^{12} & \omega^{24} & \omega^{36} \\ 1 & \omega^{14} & \omega^{28} & \omega^{42} \end{bmatrix} = \begin{bmatrix} 1 & (\omega^2)^4 & (\omega^2)^8 & (\omega^2)^{12} \\ 1 & (\omega^2)^5 & (\omega^2)^{10} & (\omega^2)^{15} \\ 1 & (\omega^2)^6 & (\omega^2)^{12} & (\omega^2)^{18} \\ 1 & (\omega^2)^7 & (\omega^2)^{14} & (\omega^2)^{21} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & (\omega^2)^1 & (\omega^2)^2 & (\omega^2)^3 \\ 1 & (\omega^2)^2 & (\omega^2)^4 & (\omega^2)^6 \\ 1 & (\omega^2)^3 & (\omega^2)^6 & (\omega^2)^9 \end{bmatrix} = f_1$$

- Define diagonal matrix D as follows.

$$f_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega^1 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & \omega^3 \end{bmatrix}$$

- Thus, we can see observe the following relationships.

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ \omega^1 & \omega^3 & \omega^5 & \omega^7 \\ \omega^2 & \omega^6 & \omega^{10} & \omega^{14} \\ \omega^3 & \omega^9 & \omega^{15} & \omega^{21} \end{bmatrix}}_{f_2} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega^1 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & \omega^3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^4 & \omega^8 & \omega^{12} \\ 1 & \omega^6 & \omega^{12} & \omega^{18} \end{bmatrix}}_{f_1}$$

$$\underbrace{\begin{bmatrix} \omega^4 & \omega^{12} & \omega^{20} & \omega^{28} \\ \omega^5 & \omega^{15} & \omega^{25} & \omega^{35} \\ \omega^6 & \omega^{18} & \omega^{30} & \omega^{42} \\ \omega^7 & \omega^{21} & \omega^{35} & \omega^{49} \end{bmatrix}}_{f_4} = \underbrace{\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -\omega^1 & 0 & 0 \\ 0 & 0 & -\omega^2 & 0 \\ 0 & 0 & 0 & -\omega^3 \end{bmatrix}}_{-D} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^4 & \omega^8 & \omega^{12} \\ 1 & \omega^6 & \omega^{12} & \omega^{18} \end{bmatrix}}_{f_1}$$

- In summary, $F_8 P = \begin{bmatrix} f_1 & Df_1 \\ f_1 & -Df_1 \end{bmatrix} = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_4 & 0 \\ 0 & F_4 \end{bmatrix}$.
- This 8-sample case can be generalized to an n sample case in the following, final FFT factorization[16].

$$F_n = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \begin{bmatrix} F_{n/2} & 0 \\ 0 & F_{n/2} \end{bmatrix} P^{\mathrm{T}}$$

  – Where I is the $\frac{n}{2}$ identity matrix, D is the $\frac{n}{2}$ identity matrix where the elements of row j have been multiplied by ω^j (the top row being row 0), and P is a permutation matrix that separates even and odd columns.

[16]Note that P is removed from the left side of the equation and P^{T} is added to the right because $P^{-1} = P^{\mathrm{T}}$ is multiplied from the back to both sides of the equation (the left case reduces to $PP^{\mathrm{T}} = I$).