INVERSES: LEFT/RIGHT/PSEUDOINVERSE

2-SIDED INVERSES

- **A is full row rank.**
- **A is full column rank.**
- **1 solution to $Ax = b$**
- $n = m = r$
- $A^{-1}A = I = AA^{-1}$

LEFT INVERSES

- **A is full column rank.**
- **A has independent columns.**
- $N(A) = 0$
- **0 or 1 solution to $Ax = b$**
- $r = n < m$
- **$A^T A$ is symmetric and invertible**
- **Therefore $\left[(A^T A)^{-1} A^T\right]$ is a "left inverse" because $\left[(A^T A)^{-1} A^T\right] A = I$**
- **If we multiply $\left[(A^T A)^{-1} A^T\right]$ on the right, we get $A\left[(A^T A)^{-1} A^T\right]$ which is P, a projection onto the column space.**

RIGHT INVERSES

- **A is full row rank.**
- **A has independent rows.**
- $N(A^T) = 0$
- **Infinite solutions to $Ax = b$**
- $r = m < n$
- **$n - m$ free variables.**
- **AA^T is invertible**
- **Therefore $\left[A^T (AA^T)^{-1}\right]$ is a "right inverse" because $A\left[A^T (AA^T)^{-1}\right] = I$**
- **If we multiply $\left[A^T (AA^T)^{-1}\right]$ on the left we get $\left[A^T (AA^T)^{-1}\right] A$, which is a projection onto the row space.**

PSEUDOINVERSES

- A **is neither full row nor full column rank.**
- A **has dependent columns and dependent rows**
- $r < n$ **and** $r < m$
- **Nonetheless, there is a one-to-one and onto relationship between a row vector** x **and a column vector** Ax**, ignoring the left and right null spaces of the matrix.**

To Calculate A Pseudoinverse:

- **Because every matrix has a singular value decomposition, the pseudoinverse is calculated from the SVD.**
- **Let** $A = U\Sigma V^T$ **in which** U **and** V^T **are orthogonal square invertible matrices, and** Σ **is an** $m \times n$ **matrix of singular values:** $\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0 \end{bmatrix}$**.**
- **The pseudoinverse,** A^+**, can then be found through the inverse of** $A = U\Sigma V^T$**,** $A^+ = V\Sigma^+ U^T$**.**
- **Because** U **and** V^T **are invertible, there is no need to find their pseudoinverse. Because** Σ **is a purely diagonal matrix, it's inverse is easy to calculate:**

$$\Sigma^+ = \begin{bmatrix} \frac{1}{\sigma_1} & & & \\ & \ddots & & \\ & & \frac{1}{\sigma_r} & \\ & & & 0 \end{bmatrix}.$$

- **Note that** $\Sigma\Sigma^+ = \Sigma^+\Sigma = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{bmatrix} \neq I$