

Complex Linear Independence: Decomplexification

4/7:

- When given a complex system of equations, it is necessary to **decomplexify** it.
- **Decomplexify:** To model a complex system of equations with a strictly real system for the purpose of applying the tenets of real linear algebra to it.
- Consider the following complex system of equations.

$$\begin{aligned}(2+i)x_1 + (1+i)x_2 &= 3+6i \\ (3-i)x_1 + (2-2i)x_2 &= 7-i\end{aligned}$$

– The solutions will be complex numbers: $x_1 = a_1 + ib_1$ and $x_2 = a_2 + ib_2$, where $a_1, a_2, b_1, b_2 \in \mathbb{R}$.

- Transform it into a matrix system of equations. Separate the real and complex parts, and factor out all instances of the imaginary number i so that it is a coefficient to any complex matrix.

$$\begin{aligned}\begin{bmatrix} 2+i & 1+i \\ 3-i & 2-2i \end{bmatrix} \begin{bmatrix} a_1+ib_1 \\ a_2+ib_2 \end{bmatrix} &= \begin{bmatrix} 3+6i \\ 7-i \end{bmatrix} \\ \left(\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} i & i \\ -i & -2i \end{bmatrix} \right) \left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} ib_1 \\ ib_2 \end{bmatrix} \right) &= \left(\begin{bmatrix} 3 \\ 7 \end{bmatrix} + \begin{bmatrix} 6i \\ -i \end{bmatrix} \right) \\ \underbrace{\left(\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + i \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \right)}_A \underbrace{\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + i \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right)}_x &= \underbrace{\left(\begin{bmatrix} 3 \\ 7 \end{bmatrix} + i \begin{bmatrix} 6 \\ -1 \end{bmatrix} \right)}_b\end{aligned}$$

- Foil the left side of the above equation^[1].

$$\left(\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) + i \left(\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 7 \end{bmatrix} + i \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

- Split the above system of equations into a real system of equations and a complex system of equations by setting equal to each other the real components of each side and the imaginary components of each side.

$$\begin{aligned}\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} &= \begin{bmatrix} 3 \\ 7 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} &= \begin{bmatrix} 6 \\ -1 \end{bmatrix}\end{aligned}$$

- Multiply out the matrices above to yield a system of four equations.

$$\begin{aligned}2a_1 + a_2 - b_1 - b_2 &= 3 \\ 3a_1 + 2a_2 + b_1 + 2b_2 &= 7 \\ a_1 + a_2 + 2b_1 + b_2 &= 6 \\ -a_1 - 2a_2 + 3b_1 + 2b_2 &= -1\end{aligned}$$

- Condense the above system of equations into a single matrix system of equations.

$$\begin{bmatrix} 2 & 1 & -1 & -1 \\ 3 & 2 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ -1 & -2 & 3 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 6 \\ -1 \end{bmatrix}$$

¹Note that the minus sign appears in the real component because, when multiplying the two “last” parts, $i^2 = -1$.

- Solve for a_1, a_2, b_1 , and b_2 using an augmented matrix and Gauss-Jordan elimination.

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & -1 & 3 \\ 3 & 2 & 1 & 2 & 7 \\ 1 & 1 & 2 & 1 & 6 \\ -1 & -2 & 3 & 2 & -1 \end{array} \right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{array} \right]$$

$$\begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}$$

- From these four values, the original solutions $x_1 = a_1 + ib_1$ and $x_2 = a_2 + ib_2$ can be found.

$$x_1 = 1 + 2i$$

$$x_2 = 2 - i$$

Hermitian, Unitary, and Normal Matrices

4/13:

- What necessitates different categorizations of complex vectors and matrices?
- Consider a vector v .

$$v = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

- If you want to find $\|v\|$, you typically evaluate $\sqrt{v^T v}$. However, this equals to 0 (see the following), which is clearly not the magnitude of v .

$$\begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 - 1 = 0$$

– Note that $\|v\|$ must be an element of \mathbb{R} because it measures a distance.

- With complex vectors, it is necessary to evaluate $\sqrt{\bar{v}^T v}$ to find $\|v\|$.

$$\begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 + 1 = 2$$

$$\|v\| = \sqrt{2}$$

- This makes sense because $\begin{bmatrix} 1 \\ i \end{bmatrix}$ extends one unit into \mathbb{R}^1 and one unit into \mathbb{C}^2 .
- If $z\bar{z} = |z|^2$ and $\bar{v}^T v = v \cdot \bar{v}$, it stands to reason that $\bar{v}^T v = \|v\|^2$. Essentially, the dot product multiplies every element of v by its complex conjugate and sums them.
- Instead of writing \bar{v}^T ^[2] every time, mathematicians shorthand to v^H ^[3].
 - v^H works for all vectors, but it is necessary for complex ones.
- **Hermitian** (matrix): A matrix A such that $A = A^H$.
 - Typically defined for $A \in \mathbb{C}^n$, but holds for $A \in \mathbb{R}^n$, too.
 - Parallel to how if $A \in \mathbb{R}^n$ and $A = A^T$, A is symmetrical.
 - Also note that if $A^H A = A^2 = A A^H$, A is Hermitian.

²“ v conjugate transpose”

³“ v Hermitian” after French mathematician Charles Hermite.

- A Hermitian matrix has to have real values on the principal diagonal. When A is transposed and conjugated, the diagonal entries are the only values that don't move. Thus, their conjugates must equal themselves, so they must be real^[4].
- **Unitary** (matrix): A matrix A such that $A^{-1} = A^H$.
 - Typically defined for $A \in \mathbb{C}^n$, but holds for $A \in \mathbb{R}^n$, too.
 - Parallel to how if $A \in \mathbb{R}^n$ and $A^{-1} = A^T$, A is orthonormal.
 - Also note that if $A^H A = I = A A^H$, A is unitary.
- **Normal** (matrix): A matrix that is unitarily diagonalizable.
 - Typically defined for $A \in \mathbb{C}^n$, but holds for $A \in \mathbb{R}^n$, too.
 - Parallel to matrices $A \in \mathbb{R}^n$ such that A is orthonormally diagonalizable.
- Note that not every complex matrix has to be one of these three types.
- When $A^H A = A A^H$, $A = U \Lambda U^H$.

$$\begin{aligned}
 A A^H &= (U \Lambda U^H) (U \Lambda U^H)^H \\
 &= U \Lambda U^H U \Lambda^H U^H \\
 &= U \Lambda \Lambda^H U^H \\
 &= U \Lambda^H \Lambda U^H \text{ [5]} \\
 &= U \Lambda^H U^H U \Lambda U^H \\
 &= (U \Lambda U^H)^H (U \Lambda U^H) \\
 &= A^H A
 \end{aligned}$$

- When $A = A^H$, all eigenvalues are elements of \mathbb{R} (similar to spectral theorem).

$$v^H A v = (v^H A v)^H = v^H A v$$

- The above proves that $v^H A v \in \mathbb{R}$ because it's its own conjugate^[4].

$$\begin{aligned}
 A v &= \lambda v \\
 v^H A v &= \lambda v^H v
 \end{aligned}$$

- $\lambda = \frac{v^H A v}{v^H v} \rightarrow \frac{\mathbb{R}}{\mathbb{R}} = \mathbb{R}$ ^[6].
- When $A = A^H$ and $A x = \lambda x$, all x 's can be chosen orthonormally (also similar to spectral theorem).
 - Normality is implied because any eigenvector can be scaled to any version (including a normal version) and still be an eigenvector.

$$x_i = \begin{bmatrix} x_{i_1} \\ x_{i_2} \\ \vdots \\ x_{i_n} \end{bmatrix} \qquad x_i^H = [\bar{x}_{i_1} \quad \bar{x}_{i_2} \quad \cdots \quad \bar{x}_{i_n}]$$

⁴Recall that only real quantities can be their own conjugates because $a + 0i = a - 0i$.

⁵Since $\Lambda = \Lambda^H$.

⁶Note that the denominator is real because it's how one finds $\|v\|$, and $\|v\|$ must be real, as discussed above.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad A^H = \begin{bmatrix} a_{11} & \bar{a}_{21} & \cdots & \bar{a}_{n1} \\ \bar{a}_{12} & a_{22} & \cdots & \bar{a}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{1n} & \bar{a}_{2n} & \cdots & a_{nn} \end{bmatrix}$$

- Define an arbitrary vector x_i and matrix A , along with their conjugate transposes (or Hermitian versions). Note that the diagonal entries of A^H aren't shown as conjugated because their conjugates equal themselves.

$$Ax_1 = \lambda_1 x_1$$

$$x_2^H Ax_1 = \lambda_1 x_2^H x_1$$

$$Ax_2 = \lambda_2 x_2$$

$$(Ax_2)^H = (\lambda_2 x_2)^H$$

$$x_2^H A^H = \lambda_2 x_2^H$$

$$x_2^H Ax_1 = \lambda_2 x_2^H x_1$$

- $\lambda_1 x_2^H x_1 = \lambda_2 x_2^H x_1$ implies that, since $\lambda_1 \neq \lambda_2$, $x_2^H x_1$ must equal 0, proving orthogonality.

Complex Diagonalization

- 4/15: • Diagonalize the following matrix A .

$$A = \begin{bmatrix} 0.9 & -0.4 \\ 0.1 & 0.9 \end{bmatrix}$$

- Find the characteristic polynomial.

$$\begin{aligned} 0 &= \begin{vmatrix} 0.9 - \lambda & -0.4 \\ 0.1 & 0.9 - \lambda \end{vmatrix} \\ &= (0.9 - \lambda)^2 - (-0.4)(0.1) \\ &= 0.81 - 1.8\lambda + \lambda^2 + 0.04 \\ &= \lambda^2 - 1.8\lambda + 0.85 \end{aligned}$$

- Find the eigenvalues^[7].

$$\begin{aligned} \lambda &= \frac{-(-1.8) \pm \sqrt{(-1.8)^2 - 4(1)(0.85)}}{2(1)} \\ &= 0.9 \pm \frac{\sqrt{-0.16}}{2} \\ &= 0.9 \pm \frac{\sqrt{-1}\sqrt{0.16}}{2} \\ &= 0.9 \pm \frac{0.4i}{2} \\ &= 0.9 \pm 0.2i \end{aligned}$$

$$\lambda_1 = 0.9 + 0.2i$$

$$\lambda_2 = 0.9 - 0.2i$$

⁷It is interesting that the eigenvalues are complex conjugates of each other.

- Find the eigenvectors^[8].

$$\begin{aligned}(A - (0.9 + 0.2i))x_1 &= \begin{bmatrix} 0.9 - (0.9 + 0.2i) & -0.4 \\ 0.1 & 0.9 - (0.9 + 0.2i) \end{bmatrix} \begin{bmatrix} x_{1_1} \\ x_{1_2} \end{bmatrix} \\ &= \begin{bmatrix} -0.2i & -0.4 \\ 0.1 & -0.2i \end{bmatrix} \begin{bmatrix} 2i \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}(A - (0.9 - 0.2i))x_1 &= \begin{bmatrix} 0.9 - (0.9 - 0.2i) & -0.4 \\ 0.1 & 0.9 - (0.9 - 0.2i) \end{bmatrix} \begin{bmatrix} x_{1_1} \\ x_{1_2} \end{bmatrix} \\ &= \begin{bmatrix} 0.2i & -0.4 \\ 0.1 & 0.2i \end{bmatrix} \begin{bmatrix} -2i \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}\end{aligned}$$

$$x_1 = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} -2i \\ 1 \end{bmatrix}$$

- Compile the diagonalization.

$$A = \frac{1}{4i} \begin{bmatrix} 2i & -2i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0.9 + 0.2i & 0 \\ 0 & 0.9 - 0.2i \end{bmatrix} \begin{bmatrix} 1 & 2i \\ -1 & 2i \end{bmatrix}$$

Real versus Complex

4/16:

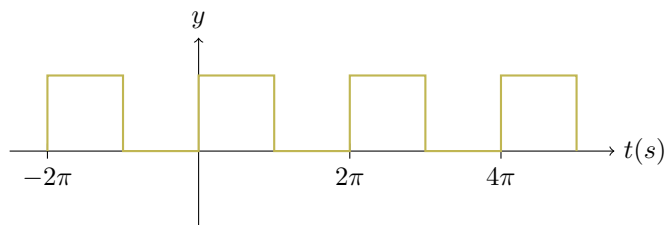
Real	Complex
\mathbb{R}^n : vectors with n real components	\mathbb{C}^n : vectors with n complex components
length: $\ x\ ^2 = x_1^2 + \cdots + x_n^2$	length: $\ z\ ^2 = z_1 ^2 + \cdots + z_n ^2$
transpose: $(A^T)_{ij} = A_{ji}$	conjugate transpose: $(A^H)_{ij} = \overline{A_{ji}}$
product rule: $(AB)^T = B^T A^T$	product rule: $(AB)^H = B^H A^H$
dot product: $x^T y = x_1 y_1 + \cdots + x_n y_n$	inner product: $u^H v = \bar{u}_1 v_1 + \cdots + \bar{u}_n v_n$
reason for A^T : $(Ax)^T y = x^T (A^T y)$	reason for A^H : $(Au)^H v = u^H (A^H v)$
orthogonality: $x^T y = 0$	orthogonality: $u^H v = 0$.
symmetric matrices: $A = A^T$	Hermitian matrices: $A = A^H$
$A = Q\Lambda Q^{-1} = Q\Lambda Q^T$ (real Λ)	$A = U\Lambda U^{-1} = U\Lambda U^H$ (real Λ)
skew-symmetric matrices: $k^T = -K$	skew-Hermitian matrices: $K^H = -K$
orthogonal matrices: $Q^T = Q^{-1}$	unitary matrices: $U^H = U^{-1}$
orthonormal columns: $Q^T Q = I$	orthonormal columns: $U^H = U^{-1}$
$(Qx)^T (Qy) = x^T y$ and $\ Qx\ = \ x\ $	$(Ux)^H (Uy) = x^H y$ and $\ Uz\ = \ z\ $

- Note that the columns and eigenvectors of Q and U are orthonormal, and all of their eigenvalues λ satisfy $|\lambda| = 1$.

⁸It is interesting that the eigenvectors are *also* complex conjugates of each other.

Real Fourier Series

- 4/21: • Consider the square wave $f(t)$.

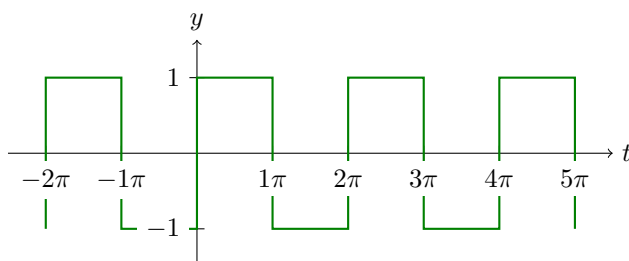


- Its period is $2\pi \frac{\text{sec}}{\text{cycle}}$, and its frequency is $\frac{1}{2\pi}$ Hz.
- Can we write $f(t)$ as a sum of sines and cosines?

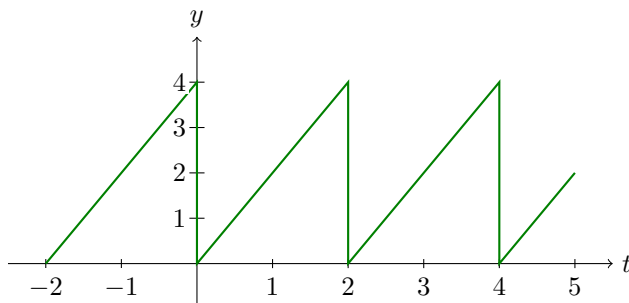
$$f(t) = a_0 + a_1 \cos(t) + b_1 \sin(t) + a_2 \cos(2t) + b_2 \sin(2t) + a_3 \cos(3t) + b_3 \sin(3t) + \dots$$

- Be general.
- Since $T = 2\pi$, it makes sense to use some functions with $T = 2\pi$ to model it.
- The weighting coefficients account for how much each function contributes to the whole.
- Historically studied by Fourier, who studied differential equations. Differential equations were often easy to solve for sines and cosines, so if a function could be modeled by a sum of sines and cosines, a related differential equation would be easier to solve.
- Fourier series, transforms, and analysis also tell us how much of each frequency a function contains (as measured by the weight coefficients).

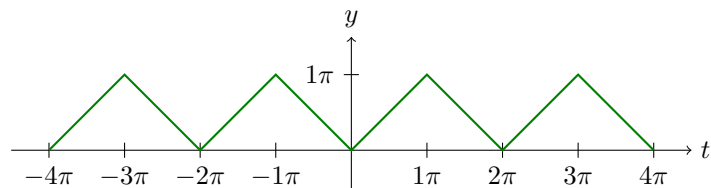
Wave Type Sketches



(a) Square wave.



(b) Sawtooth wave.



(c) Triangular wave.

- Square wave: $f(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & \pi < t < 2\pi \end{cases}$ where periodicity is defined by $f(t + 2\pi) = f(t)$.
- Sawtooth wave: $f(t) = 2t$ $0 < t < 2$ where periodicity is defined by $f(t + 2) = f(t)$.
- Triangular wave: $f(t) = |t|$ $-\pi < t < \pi$ where periodicity is defined by $f(t + 2\pi) = f(t)$.

1) Evaluate $\int_{-\pi}^{\pi} \sin(nt) dt$, where n is an integer.

2) Evaluate $\int_{-\pi}^{\pi} \cos(nt) dt$, where n is an integer.

3) Using the results from problem 1 and problem 2, integrate both sides of the equation

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt)) \text{ from } -\pi \text{ to } \pi. \text{ Then simplify the result in terms of } a_0. \text{ Divide the result in half and simplify in terms of } \frac{a_0}{2}.$$

4) Using trigonometric identity $\sin(nt)\cos(mt) = \frac{1}{2}(\sin(n+m)t + \sin(n-m)t)$, evaluate $\int_{-\pi}^{\pi} \sin(nt)\cos(mt) dt$ where n and m are any integers.

5) Evaluate $\int_{-\pi}^{\pi} \cos(nt)\cos(mt) dt$ where n and m are any integers and $n \neq m$.

6) Evaluate $\int_{-\pi}^{\pi} \cos(nt)\cos(mt) dt$ where n and m are any integers and $n = m$ and $n \neq 0$, in other words evaluate $\int_{-\pi}^{\pi} \cos^2(nt) dt$.

7) Evaluate $\int_{-\pi}^{\pi} \cos(nt)\cos(mt) dt$ when $n = m = 0$

8) In a similar way to 5- 7, evaluate $\int_{-\pi}^{\pi} \sin(nt)\sin(mt) dt$ for the cases where $n \neq m, n = m \neq 0, n = m = 0$,

Hints: Use identity $\sin(nt)\sin(mt) = \frac{1}{2}(\cos(n-m)t - \cos(n+m)t)$ for the case when $n \neq m$ and use identity $\cos(2\theta) = 1 - 2\sin^2 \theta$ for which $\theta = nt$ for the case when $n = m \neq 0$.

$$1) \int_{-\pi}^{\pi} \sin(nt) dt = \left[-\frac{1}{n} \cos(nt) \right]_{-\pi}^{\pi} = \frac{1}{n} (-\cos(n\pi) + \cos(n\pi)) = 0, n \neq 0$$

$$2) \int_{-\pi}^{\pi} \cos(nt) dt = \left[\frac{1}{n} \sin(nt) \right]_{-\pi}^{\pi} = 0, n \neq 0$$

$$3) \int_{-\pi}^{\pi} f(t) dt = \frac{1}{2} \int_{-\pi}^{\pi} a_0 dt + \sum_{n=1}^{\infty} \left(\int_{-\pi}^{\pi} a_n \cos(nt) dt + \int_{-\pi}^{\pi} b_n \sin(nt) dt \right) = \frac{1}{2} [a_0 t]_{-\pi}^{\pi} + \sum_{n=1}^{\infty} (0+0)$$

$$\therefore \int_{-\pi}^{\pi} f(t) dt = \frac{1}{2} (2a_0 \pi) \rightarrow a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt \rightarrow \frac{a_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$$

$$4) \int_{-\pi}^{\pi} \sin(nt) \cos(mt) dt = \frac{1}{2} \left(\int_{-\pi}^{\pi} \sin(n+m)t dt + \int_{-\pi}^{\pi} \sin(n-m)t dt \right) = \frac{1}{2} (0+0) = 0 \text{ using the results from problems 1 and}$$

2 since if n is an integer and m is an integer then $n+m$ and $n-m$ are also integers.

$$5) \int_{-\pi}^{\pi} \cos(nt) \cos(mt) dt = \frac{1}{2} \left(\int_{-\pi}^{\pi} \cos(n+m)t dt + \int_{-\pi}^{\pi} \cos(n-m)t dt \right) = \frac{1}{2} (0+0) = 0 \text{ when } n \neq m$$

$$6) \int_{-\pi}^{\pi} \cos^2(nt) dt = \frac{1}{2} \int_{-\pi}^{\pi} (1 + \cos(2n)t) dt = \frac{1}{2} \left[t + \frac{1}{2n} \sin(2n)t \right]_{-\pi}^{\pi} = \pi \text{ when } n \neq 0$$

$$7) \pi - (-\pi) = 2\pi$$

$$8) a) \int_{-\pi}^{\pi} \sin(nt) \sin(mt) dt = 0$$

$$b) \int_{-\pi}^{\pi} \sin^2(nt) dt = \frac{1}{2} \int_{-\pi}^{\pi} (1 - \cos(2nt)) dt = \pi$$

$$c) \int_{-\pi}^{\pi} \sin(nt) \sin(mt) dt = 0$$

For integers m, n :

$$\int_{-\pi}^{\pi} \sin(nt) \cos(mt) dt = 0$$

$$\int_{-\pi}^{\pi} \cos(nt) \cos(mt) dt = \begin{cases} 0, n \neq m \\ \pi, n = m \neq 0 \\ 2\pi, n = m = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(nt) \sin(mt) dt = \begin{cases} 0, n \neq m, n = m = 0 \\ \pi, n = m \end{cases}$$

FOURIER SERIES

- We have shown that when $v^T w = 0$, then vectors v and w are orthogonal.
- Fourier series ask us to think of continuous functions as vectors.
- Let $f(t) = \cos(t)$ and let $g(t) = \sin(t)$
- To find $f \cdot g$ of these two function “vectors” would be asking us to take a dot product that has infinitely many terms. All vectors we have used have had a finite number of components. Because these “vectors” would have infinitely many terms, it would be like asking us to take an integral!
- $f \cdot g = \int_0^{2\pi} \cos(t) \sin(t) dt$ (We limit the domain because these are sinusoidal periodic functions and thus they repeat after a period of 2π .)

$$\int_0^{2\pi} \cos(t) \sin(t) dt$$

Let $u = \cos(t)$ and $du = -\sin(t)$

$$-\int_1^1 u du$$

$$\left[-\frac{u^2}{2} \right]_1^1$$

0

OR

$$\left[-\frac{1}{2} \cos^2(t) \right]_0^{2\pi}$$

$$-\frac{1}{2} (\cos^2 2\pi - \cos^2 0)$$

$$-\frac{1}{2} (1 - 1) =$$

0

- Therefore $\cos t \cdot \sin t = 0$, therefore these function “vectors” are orthogonal, therefore they serve as basis vectors for the function space of continuous periodic functions in F^2 , therefore any continuous function also in this space can be written as a linear combination of $\cos(t)$ and $\sin(t)$!!!

$$f(t) = \frac{a_0}{2} + a_1 \cos(t) + b_1 \sin(t) + a_2 \cos(2t) + b_2 \sin(2t) + \dots$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt))$$

*Why divide the constant term by 2?

CALCULATING THE FOURIER COEFFICIENTS

As sine and cosine can serve as an orthogonal basis for periodic functions, consider the Fourier Series for a function

$$f(t) \text{ of period } 2\pi \text{ to be } f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin(nt))$$

In problem 3 from your handout you already found a formula for a_0 , the constant term.

$$\frac{a_0}{2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$$

To obtain the coefficients $a_n, n \in \mathbb{Z}$, multiply both sides of the equation by $\cos(mt)$ where $m \in \mathbb{Z}$ and $m > 0$ and integrate both sides from $-\pi$ to π since this is a periodic function of period 2π .

$$\int_{-\pi}^{\pi} f(t) \cos(mt) dt = \frac{a_0}{2} \int_{-\pi}^{\pi} \cos(mt) dt + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos(nt) \cos(mt) dt + b_n \int_{-\pi}^{\pi} \sin(nt) \cos(mt) dt \right)$$

Simplify this result using the integrals on the right side of this equation from the handout problems 5-7. The only nonzero integral results from $\int_{-\pi}^{\pi} \cos^2(mt) dt = \pi$ in the case where $n = m$.

$$\therefore \int_{-\pi}^{\pi} \cos(mt) dt = a_m \pi$$

Since this is the case where $n = m$, replacing m with n and solving for a_n we obtain:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt, n \in \mathbb{Z} \text{ and } n > 0$$

To obtain the coefficients $b_n, n \in \mathbb{Z}, n > 0$, multiply both sides of the equation by $\sin(mt)$ where $m \in \mathbb{Z}$ and $m > 0$ and integrate both sides from $-\pi$ to π since this is a periodic function of period 2π .

$$\int_{-\pi}^{\pi} f(t) \sin(mt) dt = \frac{a_0}{2} \int_{-\pi}^{\pi} \sin(mt) dt + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos(nt) \sin(mt) dt + b_n \int_{-\pi}^{\pi} \sin(nt) \sin(mt) dt \right)$$

Simplify this result using the integrals on the right side of this equation using the handout problems you did for 8. The only nonzero integral results from $\int_{-\pi}^{\pi} b_m \sin^2(mt) dt = b_m \pi$ in the case where $n = m$. Relabeling m as n and solving for b_n we obtain:

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt, n \in \mathbb{Z} \text{ and } n > 0$$

Square Wave

4/29: • **Hilbert space:** An infinite-dimensional vector space — extends a lot of the ideas of linear algebra to functions in infinite dimensions.

– Recall that functions behave with linearity ($\alpha f(t) + \beta g(t)$ is still a function).

• Square wave (period 2π): $f(t) = \begin{cases} 0 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases}$ and $f(t + 2\pi) = f(t)$.

• Find $\frac{a_0}{2}$ term:

$$\begin{aligned} \frac{a_0}{2} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt \\ &= \frac{1}{2\pi} \int_{-\pi}^0 (0) dt + \frac{1}{2\pi} \int_0^{\pi} (1) dt \\ &= 0 + \left[\frac{t}{2\pi} \right]_0^{\pi} \\ &= \frac{\pi}{2\pi} \\ \boxed{\frac{a_0}{2} = \frac{1}{2}} \end{aligned}$$

– Makes sense because $\frac{a_0}{2}$ is like the sinusoidal axis and $\frac{1}{2}$ is half way between 0 and 1.

• Find a_n terms:

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt \\ &= \frac{1}{\pi} \int_{-\pi}^0 (0) \cos(nt) dt + \frac{1}{\pi} \int_0^{\pi} (1) \cos(nt) dt \\ &= 0 + \left[\frac{1}{n\pi} \sin(nt) \right]_0^{\pi} \\ \boxed{a_n = 0} \end{aligned}$$

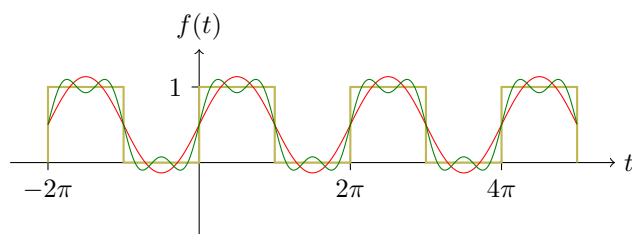
• Find b_n terms:

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt \\ &= \frac{1}{\pi} \int_{-\pi}^0 (0) \sin(nt) dt + \frac{1}{\pi} \int_0^{\pi} (1) \sin(nt) dt \\ &= 0 - \frac{1}{n\pi} [\cos(nt)]_0^{\pi} \\ &= -\frac{1}{n\pi} [\cos(n\pi) - \cos(0)] \\ &= -\frac{1}{n\pi} [\cos(n\pi) - 1]^{[9]} \\ \boxed{b_n = \begin{cases} 0 & n = 2, 4, 6, \dots \\ \frac{2}{n\pi} & n = 1, 3, 5, \dots \end{cases}} \end{aligned}$$

⁹ $\cos(n\pi)$ equals 1 when n is even and -1 when n is odd.

- Assemble the Fourier series for the square wave function:

$$\begin{aligned} f(t) &= \frac{1}{2} + \frac{2}{\pi} \sin(t) + \frac{2}{3\pi} \sin(3t) + \frac{2}{5\pi} \sin(5t) + \cdots \\ &= \frac{1}{2} + \frac{2}{\pi} \left(\frac{1}{1} \sin(t) + \frac{1}{3} \sin(3t) + \frac{1}{5} \sin(5t) + \cdots \right) \\ &= \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin((2n-1)t) \end{aligned}$$



FOURIER SERIES FOR PERIODIC FUNCTIONS WHEN PERIOD $\neq 2\pi$

Given:

$$\frac{a_0}{2} = \int_{-\pi}^{\pi} \frac{1}{2\pi} f(t) dt$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt, n \in \mathbb{Z} \text{ and } n > 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt, n \in \mathbb{Z} \text{ and } n > 0$$

Change the variable t to $x = \frac{2\pi}{P}t$. In this case $x = \pi$ corresponds to $t = \frac{P}{2}$ and $x = -\pi$ corresponds to $t = -\frac{P}{2}$.

Therefore regarded as a function of t , this is a function with period P . When we make the substitution $x = \frac{2\pi}{P}t$ and

$dx = \frac{2\pi}{P} dt$ into the expressions for a_n and b_n we arrive at:

$$a_n = \frac{2}{P} \int_{-\frac{P}{2}}^{\frac{P}{2}} f(t) \cos\left(\frac{2n\pi t}{P}\right) dt, n \in \mathbb{Z}, n \geq 0$$

$$b_n = \frac{2}{P} \int_{-\frac{P}{2}}^{\frac{P}{2}} f(t) \sin\left(\frac{2n\pi t}{P}\right) dt, n \in \mathbb{Z}, n > 0$$

These integrals will give the Fourier coefficients from a function of period P whose Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{2n\pi t}{P}\right) + b_n \sin\left(\frac{2n\pi t}{P}\right) \right)$$

Note: In Differential Equations it is often convenient to write the period P as 2ℓ and in Physics and Engineering it is often written in terms of angular frequency ω as $P = \frac{2\pi}{\omega}$. Those substitutions would result in the following formulas:

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(t) \cos\left(\frac{n\pi t}{\ell}\right) dt, n \in \mathbb{Z}, n \geq 0 \text{ for Fourier Series } f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{\ell}\right) + b_n \sin\left(\frac{n\pi t}{\ell}\right) \right)$$

And

$$a_n = \frac{\omega}{\pi} \int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} f(t) \cos(n\omega t) dt, n \in \mathbb{Z}, n \geq 0 \text{ for Fourier Series } f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\omega t) + b_n \sin(n\omega t))$$

Any convenient integration range of length P , 2ℓ , or $\frac{2\pi}{\omega}$ can be used, and formulas for b_n would follow similarly for those for a_n as shown above.

*Why divide the constant term by 2? To account for the fact that the formula for a_n could be true for all $n \in \mathbb{Z}, n \geq 0$ (which would include the constant term) depending upon how the formula for the Fourier coefficients are written.

Recall from problems 6 and 7 for the integration problems combined that $\int_{-\pi}^{\pi} \cos^2(nt) dt = \pi$ when $n \neq 0$ but

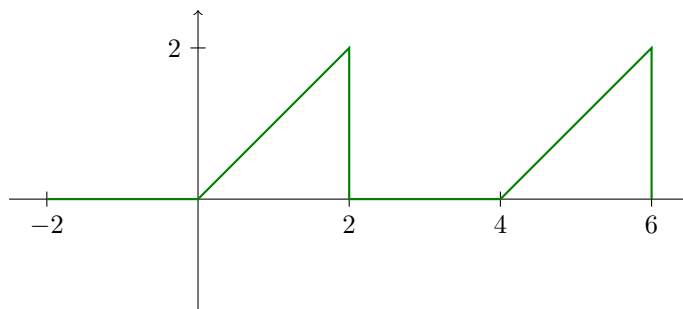
$\int_{-\pi}^{\pi} \cos^2(nt) dt = 2\pi$ when $n = 0$. Using the formula $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt$ we could write the Fourier Series as

$f(t) = \sum_{n=0}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt)$ but in this case $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt = \begin{cases} 2a_0, n=0 \\ a_n, n \neq 0 \end{cases}$. To compensate for this

the constant term is customarily written as $\frac{a_0}{2}$.

Modified Sawtooth Wave

- 4/30: • Modified sawtooth wave (period 4): $f(t) = \begin{cases} 0 & -2 < t < 0 \\ t & 0 < t < 2 \end{cases}$ and $f(t+4) = f(t)$.



– Period is 4: $P = 2\ell \Rightarrow \ell = 2$.

- Use this Fourier model:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{\ell}\right) + b_n \sin\left(\frac{n\pi t}{\ell}\right) \right)$$

- $\frac{a_0}{2} = \frac{1}{2}$ (think of it as a weighted average of where the function spends the most time — note that this is actually exactly what the integral computes).
- Find a_n terms:

$$\begin{aligned} a_n &= \frac{1}{2} \int_{-2}^2 f(t) \cos\left(\frac{n\pi t}{2}\right) dt \\ &= 0 + \frac{1}{2} \int_0^2 t \cos\left(\frac{n\pi t}{2}\right) dt \end{aligned}$$

$$\begin{aligned} u &= t & dv &= \cos\left(\frac{n\pi t}{2}\right) dt \\ du &= dt & v &= \frac{2}{n\pi} \sin\left(\frac{n\pi t}{2}\right) \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{2} \left(\left[\left(t \right) \left(\frac{2}{n\pi} \sin\left(\frac{n\pi t}{2}\right) \right) \right]_0^2 - \int_0^2 \frac{2}{n\pi} \sin\left(\frac{n\pi t}{2}\right) dt \right) \\ &= \frac{1}{2} \left(\left(\frac{4}{n\pi} \sin(n\pi) - 0 \right) + \frac{4}{n^2\pi^2} \left[\cos\left(\frac{n\pi t}{2}\right) \right]_0^2 \right) \\ &= \frac{1}{2} \left(0 + \frac{4}{n^2\pi^2} (\cos(n\pi) - \cos(0)) \right) \\ &= \frac{2}{n^2\pi^2} (\cos(n\pi) - 1) \end{aligned}$$

$$a_n = \begin{cases} 0 & n = 2, 4, 6, \dots \\ -\frac{4}{n^2\pi^2} & n = 1, 3, 5, \dots \end{cases}$$

- Find b_n terms:

$$\begin{aligned} b_n &= \frac{1}{2} \int_{-2}^2 f(t) \sin\left(\frac{n\pi t}{2}\right) dt \\ &= 0 + \frac{1}{2} \int_0^2 t \sin\left(\frac{n\pi t}{2}\right) dt \end{aligned}$$

$$\begin{aligned} u &= t & dv &= \sin\left(\frac{n\pi t}{2}\right) dt \\ du &= dt & v &= -\frac{2}{n\pi} \cos\left(\frac{n\pi t}{2}\right) dt \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{2} \left(\left[\left(t \right) \left(-\frac{2}{n\pi} \cos\left(\frac{n\pi t}{2}\right) \right) \right]_0^2 + \frac{2}{n\pi} \int_0^2 \cos\left(\frac{n\pi t}{2}\right) dt \right) \\ &= \frac{1}{2} \left(\left(-\frac{4}{n\pi} \cos(n\pi) - 0 \right) + 0 \right) \end{aligned}$$

$$b_n = \begin{cases} -\frac{2}{n\pi} & n = 2, 4, 6, \dots \\ \frac{2}{n\pi} & n = 1, 3, 5, \dots \end{cases}$$

- Assemble the Fourier series for this modified sawtooth wave function:

$$\begin{aligned} a_n &= -\frac{2}{n^2\pi^2} + (-1)^n \frac{2}{n^2\pi^2} \\ b_n &= (-1)^{n+1} \frac{2}{n\pi} \\ f(t) &= \frac{1}{2} + \sum_{n=1}^{\infty} \left(\left(-\frac{2}{n^2\pi^2} + (-1)^n \frac{2}{n^2\pi^2} \right) \cos\left(\frac{n\pi t}{2}\right) + \left((-1)^{n+1} \frac{2}{n\pi} \right) \sin\left(\frac{n\pi t}{2}\right) \right) \end{aligned}$$

Complex Fourier Series

5/4: • **Euler's formula:** $e^{i\theta} = \cos \theta + i \sin \theta$.

- Using Euler's Formula, we can replace the trigonometric functions in Fourier series with complex exponential functions. By combining the Fourier coefficients a_n and b_n into one complex coefficient c_n , we find that, for a given periodic signal, both sets of constants can be found in one operation.
- Let's derive sine and cosine in terms of complex exponentials.
 - For this to work, we will need Euler's formula, and a second, negative version of Euler's formula: $e^{-i\theta} = \cos \theta - i \sin \theta$.

$$\begin{aligned} (\cos \theta + i \sin \theta) + (\cos \theta - i \sin \theta) &= e^{i\theta} + e^{-i\theta} & (\cos \theta + i \sin \theta) - (\cos \theta - i \sin \theta) &= e^{i\theta} - e^{-i\theta} \\ 2 \cos \theta &= e^{i\theta} + e^{-i\theta} & 2i \sin \theta &= e^{i\theta} - e^{-i\theta} \end{aligned}$$

$$\cos \theta = \frac{1}{2} (e^{i\theta} + e^{-i\theta})$$

$$\sin \theta = \frac{1}{2i} (e^{i\theta} - e^{-i\theta})$$

- We can use the above results to express $a_n \cos(n\omega_0\theta) + b_n \sin(n\omega_0\theta)$, where $\omega_0 = \frac{2\pi}{P}$, in terms of complex exponentials.

$$\begin{aligned}
a_n \cos(n\omega_0\theta) + b_n \sin(n\omega_0\theta) &= \frac{a_n}{2} (e^{in\omega_0\theta} + e^{-in\omega_0\theta}) + \frac{b_n}{2i} (e^{in\omega_0\theta} - e^{-in\omega_0\theta}) \\
&= \frac{a_n}{2} e^{in\omega_0\theta} + \frac{a_n}{2} e^{-in\omega_0\theta} + \frac{b_n}{2i} e^{in\omega_0\theta} - \frac{b_n}{2i} e^{-in\omega_0\theta} \\
&= \left(\frac{a_n}{2} + \frac{b_n}{2i} \right) e^{in\omega_0\theta} + \left(\frac{a_n}{2} - \frac{b_n}{2i} \right) e^{-in\omega_0\theta} \\
&= \frac{1}{2} \left(a_n + \frac{b_n}{i} \right) e^{in\omega_0\theta} + \frac{1}{2} \left(a_n - \frac{b_n}{i} \right) e^{-in\omega_0\theta} \\
&= \frac{1}{2} \left(a_n + \left(\frac{b_n}{i} \right) (1) \right) e^{in\omega_0\theta} + \frac{1}{2} \left(a_n - \left(\frac{b_n}{i} \right) (1) \right) e^{-in\omega_0\theta} \\
&= \frac{1}{2} \left(a_n + \left(\frac{b_n}{i} \right) (i^4) \right) e^{in\omega_0\theta} + \frac{1}{2} \left(a_n - \left(\frac{b_n}{i} \right) (i^4) \right) e^{-in\omega_0\theta} \\
&= \frac{1}{2} (a_n + i^3 b_n) e^{in\omega_0\theta} + \frac{1}{2} (a_n - i^3 b_n) e^{-in\omega_0\theta} \\
&= \frac{1}{2} (a_n + (-i)b_n) e^{in\omega_0\theta} + \frac{1}{2} (a_n - (-i)b_n) e^{-in\omega_0\theta} \\
&= \frac{1}{2} (a_n - ib_n) e^{in\omega_0\theta} + \frac{1}{2} (a_n + ib_n) e^{-in\omega_0\theta}
\end{aligned}$$

- Define $c_n = \frac{1}{2} (a_n - ib_n)$ and complex conjugate $\bar{c}_n = \frac{1}{2} (a_n + ib_n)$. Now we have the following.

$$a_n \cos(n\omega_0\theta) + b_n \sin(n\omega_0\theta) = c_n e^{in\omega_0\theta} + \bar{c}_n e^{-in\omega_0\theta}$$

- Substitution into the Fourier series sum gives the following.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (c_n e^{in\omega_0\theta} + \bar{c}_n e^{-in\omega_0\theta}) \quad (1)$$

- Equation 1 can become still neater and more concise through the following steps.

1. Define $c_0 = \frac{a_0}{2}$ ^[10].
2. Define $c_{-n} = \bar{c}_n$. This permits the following.

$$\sum_{n=1}^{\infty} \bar{c}_n e^{-in\omega_0 t} = \bar{c}_1 e^{-i\omega_0 t} + \bar{c}_2 e^{-2i\omega_0 t} + \dots = c_{-1} e^{-i\omega_0 t} + c_{-2} e^{-2i\omega_0 t} + \dots = \sum_{n=-1}^{-\infty} c_n e^{in\omega_0 t}$$

3. Using the new definitions of c_n for $n \in (-\infty, 0]$, it is possible to write Equation 1 as follows.

$$\begin{aligned}
f(t) &= c_0 + \sum_{n=1}^{\infty} c_n e^{in\omega_0 t} + \sum_{n=1}^{\infty} c_{-n} e^{-in\omega_0 t} \\
&= c_0 + \sum_{n=1}^{\infty} c_n e^{in\omega_0 t} + \sum_{n=-\infty}^{-1} c_n e^{in\omega_0 t}
\end{aligned}$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t}$$

¹⁰Note that this is consistent with the general definition of c_n since $b_0 = 0$.

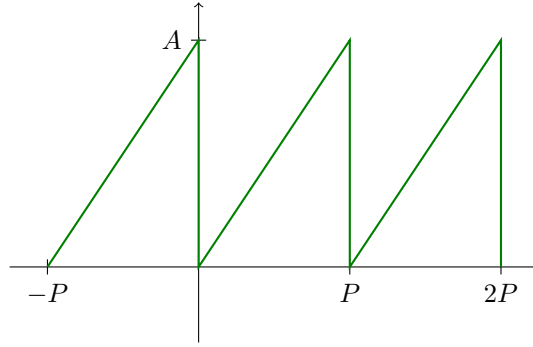
- We now tackle how to solve for the complex coefficients c_n ^[11].

1. For $n = 0$, $c_0 = \frac{a_0}{2} = \frac{1}{P} \int_{-P/2}^{P/2} f(t) dt$.
2. For $n \in \mathbb{Z}^+$, $c_n = \frac{1}{2} (a_n - ib_n) = \frac{1}{P} \int_{-P/2}^{P/2} f(t) (\cos(n\omega_0 t) - i \sin(n\omega_0 t)) dt = \frac{1}{P} \int_{-P/2}^{P/2} f(t) e^{-in\omega_0 t} dt$.
3. For $n \in \mathbb{Z}^-$, $c_n = \frac{1}{2} (a_n + ib_n) = \frac{1}{P} \int_{-P/2}^{P/2} f(t) e^{-in\omega_0 t} dt$ ^[12].
4. The above three results can be condensed into the following expression for all $n \in \mathbb{Z}$.

$$c_n = \frac{1}{P} \int_{-P/2}^{P/2} f(t) e^{-in\omega_0 t} dt$$

Generalized Sawtooth Wave

- 5/6: • Generalized sawtooth wave (period P , amplitude A): $f(t) = \frac{At}{P}$ and $f(t + P) = f(t)$.



- $c_0 = \frac{A}{2}$.
- $\omega_0 = \frac{2\pi}{P} \Rightarrow \omega_0 P = 2\pi$.
- Find c_n terms:

$$\begin{aligned}
 c_n &= \frac{1}{P} \int_0^P \frac{At}{P} e^{-in\omega_0 t} dt \\
 &= \frac{A}{P^2} \int_0^P t e^{-in\omega_0 t} dt \\
 &\quad \begin{aligned} u &= t & dv &= e^{-in\omega_0 t} \\ du &= dt & v &= -\frac{1}{in\omega_0} e^{-in\omega_0 t} \end{aligned} \\
 &= \frac{A}{P^2} \left(\left[-\frac{t e^{-in\omega_0 t}}{in\omega_0} \right]_0^P + \frac{1}{in\omega_0} \int_0^P e^{-in\omega_0 t} dt \right) \\
 &= \frac{A}{P^2} \left(\left(-\frac{P e^{-in\omega_0 P}}{in\omega_0} - 0 \right) - \frac{1}{(in\omega_0)^2} [e^{-in\omega_0 t}]_0^P \right) \\
 &= \frac{A}{P^2} \left(-\frac{P e^{-in\omega_0 P}}{in\omega_0} - \frac{1}{(in\omega_0)^2} (e^{-in\omega_0 P} - 1) \right) \\
 &= \frac{A}{P^2} \left(-\frac{P e^{-i2\pi n}}{in\omega_0} - \frac{1}{(in\omega_0)^2} (e^{-i2\pi n} - 1) \right)
 \end{aligned}$$

¹¹Note that this can also be derived in an analogous method to how the original a_n and b_n expressions were derived.

¹²The negative exponential, when multiplied by a negative n , generates the $+$ expansion of Euler's formula, as desired.

$$\begin{aligned}
&= \frac{A}{P^2} \left(-\frac{P(1)}{in\omega_0} - \frac{1}{(in\omega_0)^2} ((1) - 1) \right) \\
&= \frac{A}{P^2} \left(-\frac{P}{in\omega_0} \right) \\
&= -\frac{A}{in\omega_0 P} \\
&= \frac{A}{2\pi n(-i)} \\
\boxed{c_n} &= \frac{Ai}{2\pi n}
\end{aligned}$$

- Assemble the Fourier series for this generalized sawtooth wave function.

$$f(t) = \frac{Ai}{2\pi} \sum_{n=-\infty}^{\infty} \frac{e^{in\omega_0 t}}{n}$$