# Complex Linear Independence: Decomplexification

4/7:
- When given a complex system of equations, it is necessary to **decomplexify** it.

- **Decomplexify**: To model a complex system of equations with a strictly real system for the purpose of applying the tenets of real linear algebra to it.

- Consider the following complex system of equations.

$$(2+i)x_1 + (1+i)x_2 = 3 + 6i$$
$$(3-i)x_1 + (2-2i)x_2 = 7 - i$$

  - The solutions will be complex numbers: $x_1 = a_1 + ib_1$ and $x_2 = a_2 + ib_2$, where $a_1, a_2, b_1, b_2 \in \mathbb{R}$.

- Transform it into a matrix system of equations. Separate the real and complex parts, and factor out all instances of the imaginary number $i$ so that it is a coefficient to any complex matrix.

$$\begin{bmatrix} 2+i & 1+i \\ 3-i & 2-2i \end{bmatrix} \begin{bmatrix} a_1 + ib_1 \\ a_2 + ib_2 \end{bmatrix} = \begin{bmatrix} 3+6i \\ 7-i \end{bmatrix}$$

$$\left( \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} i & i \\ -i & -2i \end{bmatrix} \right) \left( \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} ib_1 \\ ib_2 \end{bmatrix} \right) = \left( \begin{bmatrix} 3 \\ 7 \end{bmatrix} + \begin{bmatrix} 6i \\ -i \end{bmatrix} \right)$$

$$\underbrace{\left( \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + i \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \right)}_{A} \underbrace{\left( \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + i \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right)}_{x} = \underbrace{\left( \begin{bmatrix} 3 \\ 7 \end{bmatrix} + i \begin{bmatrix} 6 \\ -1 \end{bmatrix} \right)}_{b}$$

- Foil the left side of the above equation[1].

$$\left( \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right) + i \left( \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 7 \end{bmatrix} + i \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

- Split the above system of equations into a real system of equations and a complex system of equations by setting equal to each other the real components of each side and the imaginary components of each side.

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

- Multiply out the matrices above to yield a system of four equations.

$$2a_1 + a_2 - b_1 - b_2 = 3$$
$$3a_1 + 2a_2 + b_1 + 2b_2 = 7$$
$$a_1 + a_2 + 2b_1 + b_2 = 6$$
$$-a_1 - 2a_2 + 3b_1 + 2b_2 = -1$$

- Condense the above system of equations into a single matrix system of equations.

$$\begin{bmatrix} 2 & 1 & -1 & -1 \\ 3 & 2 & 1 & 2 \\ 1 & 1 & 2 & 1 \\ -1 & -2 & 3 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 6 \\ -1 \end{bmatrix}$$

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[1]Note that the minus sign appears in the real component because, when multiplying the two "last" parts, $i^2 = -1$.

- Solve for $a_1$, $a_2$, $b_1$, and $b_2$ using an augmented matrix and Gauss-Jordan elimination.

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & -1 & 3 \\ 3 & 2 & 1 & 2 & 7 \\ 1 & 1 & 2 & 1 & 6 \\ -1 & -2 & 3 & 2 & -1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}$$

- From these four values, the original solutions $x_1 = a_1 + ib_1$ and $x_2 = a_2 + ib_2$ can be found.

$$x_1 = 1 + 2i$$
$$x_2 = 2 - i$$

# Hermitian, Unitary, and Normal Matrices

- What necessitates different categorizations of complex vectors and matrices?

- Consider a vector $v$.

$$v = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

- If you want to find $||v||$, you typically evaluate $\sqrt{v^\mathrm{T} v}$. However, this equals to 0 (see the following), which is clearly not the magnitude of $v$.

$$\begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 - 1 = 0$$

  - Note that $||v||$ must be an element of $\mathbb{R}$ because it measures a distance.

- With complex vectors, it is necessary to evaluate $\sqrt{\bar{v}^\mathrm{T} v}$ to find $||v||$.

$$\begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 + 1 = 2$$

$$||v|| = \sqrt{2}$$

  - This makes sense because $\begin{bmatrix} 1 \\ i \end{bmatrix}$ extends one unit into $\mathbb{R}^1$ and one unit into $\mathbb{C}^2$.

  - If $z\bar{z} = |z|^2$ and $\bar{v}^\mathrm{T} v = v \cdot \bar{v}$, it stands to reason that $\bar{v}^\mathrm{T} v = ||v||^2$. Essentially, the dot product multiplies every element of $v$ by its complex conjugate and sums them.

- Instead of writing $\bar{v}^{\mathrm{T}\,[2]}$ every time, mathematicians shorthand to $v^{\mathrm{H}\,[3]}$.

  - $v^\mathrm{H}$ works for all vectors, but it is necessary for complex ones.

- **Hermitian** (matrix): A matrix $A$ such that $A = A^\mathrm{H}$.

  - Typically defined for $A \in \mathbb{C}^n$, but holds for $A \in \mathbb{R}^n$, too.
  - Parallel to how if $A \in \mathbb{R}^n$ and $A = A^\mathrm{T}$, $A$ is symmetrical.
  - Also note that if $A^\mathrm{H} A = A^2 = A A^\mathrm{H}$, $A$ is Hermitian.

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[2] "$v$ conjugate transpose"
[3] "$v$ Hermitian" after French mathematician Charles Hermite.

– A Hermitian matrix has to have real values on the principal diagonal. When $A$ is transposed and conjugated, the diagonal entries are the only values that don't move. Thus, their conjugates must equal themselves, so they must be real[4].

- **Unitary** (matrix): A matrix $A$ such that $A^{-1} = A^{\mathrm{H}}$.

   – Typically defined for $A \in \mathbb{C}^n$, but holds for $A \in \mathbb{R}^n$, too.
   – Parallel to how if $A \in \mathbb{R}^n$ and $A^{-1} = A^{\mathrm{T}}$, $A$ is orthonormal.
   – Also note that if $A^{\mathrm{H}}A = I = AA^{\mathrm{H}}$, $A$ is unitary.

- **Normal** (matrix): A matrix that is unitarily diagonalizable.

   – Typically defined for $A \in \mathbb{C}^n$, but holds for $A \in \mathbb{R}^n$, too.
   – Parallel to matrices $A \in \mathbb{R}^n$ such that $A$ is orthonormally diagonalizable.

- Note that not every complex matrix has to be one of these three types.

- When $A^{\mathrm{H}}A = AA^{\mathrm{H}}$, $A = U\Lambda U^{\mathrm{H}}$.

$$
\begin{aligned}
AA^{\mathrm{H}} &= \left(U\Lambda U^{\mathrm{H}}\right)\left(U\Lambda U^{\mathrm{H}}\right)^{\mathrm{H}} \\
&= U\Lambda U^{\mathrm{H}} U\Lambda^{\mathrm{H}} U^{\mathrm{H}} \\
&= U\Lambda\Lambda^{\mathrm{H}} U^{\mathrm{H}} \\
&= U\Lambda^{\mathrm{H}}\Lambda U^{\mathrm{H}[5]} \\
&= U\Lambda^{\mathrm{H}} U^{\mathrm{H}} U\Lambda U^{\mathrm{H}} \\
&= \left(U\Lambda U^{\mathrm{H}}\right)^{\mathrm{H}}\left(U\Lambda U^{\mathrm{H}}\right) \\
&= A^{\mathrm{H}}A
\end{aligned}
$$

- When $A = A^{\mathrm{H}}$, all eigenvalues are elements of $\mathbb{R}$ (similar to spectral theorem).

$$
v^{\mathrm{H}}Av = \left(v^{\mathrm{H}}Av\right)^{\mathrm{H}} = v^{\mathrm{H}}Av
$$

   – The above proves that $v^{\mathrm{H}}Av \in \mathbb{R}$ because it's its own conjugate[4].

$$
\begin{aligned}
Av &= \lambda v \\
v^{\mathrm{H}}Av &= \lambda v^{\mathrm{H}}v
\end{aligned}
$$

   – $\lambda = \frac{v^{\mathrm{H}}Av}{v^{\mathrm{H}}v} \to \frac{\mathbb{R}}{\mathbb{R}} = \mathbb{R}^{[6]}$.

- When $A = A^{\mathrm{H}}$ and $Ax = \lambda x$, all $x$'s can be chosen orthonormally (also similar to spectral theorem).

   – Normality is implied because any eigenvector can be scaled to any version (including a normal version) and still be an eigenvector.

$$
x_i = \begin{bmatrix} x_{i_1} \\ x_{i_2} \\ \vdots \\ x_{i_n} \end{bmatrix}
\qquad\qquad
x_i^{\mathrm{H}} = \begin{bmatrix} \bar{x}_{i_1} & \bar{x}_{i_2} & \cdots & \bar{x}_{i_n} \end{bmatrix}
$$

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[4]Recall that only real quantities can be their own conjugates because $a + 0i = a - 0i$.
[5]Since $\Lambda = \Lambda^{\mathrm{H}}$.
[6]Note that the denominator is real because it's how one finds $||v||$, and $||v||$ must be real, as discussed above.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad A^{\mathrm{H}} = \begin{bmatrix} a_{11} & \bar{a}_{21} & \cdots & \bar{a}_{n1} \\ \bar{a}_{12} & a_{22} & \cdots & \bar{a}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{1n} & \bar{a}_{2n} & \cdots & a_{nn} \end{bmatrix}$$

– Define an arbitrary vector $x_i$ and matrix $A$, along with their conjugate transposes (or Hermitian versions). Note that the diagonal entries of $A^{\mathrm{H}}$ aren't shown as conjugated because their conjugates equal themselves.

$$Ax_1 = \lambda_1 x_1$$
$$x_2^{\mathrm{H}} A x_1 = \lambda_1 x_2^{\mathrm{H}} x_1$$

$$Ax_2 = \lambda_2 x_2$$
$$(Ax_2)^{\mathrm{H}} = (\lambda_2 x_2)^{\mathrm{H}}$$
$$x_2^{\mathrm{H}} A^{\mathrm{H}} = \lambda_2 x_2^{\mathrm{H}}$$
$$x_2^{\mathrm{H}} A x_1 = \lambda_2 x_2^{\mathrm{H}} x_1$$

– $\lambda_1 x_2^{\mathrm{H}} x_1 = \lambda_2 x_2^{\mathrm{H}} x_1$ implies that, since $\lambda_1 \neq \lambda_2$, $x_2^{\mathrm{H}} x_1$ must equal 0, proving orthogonality.

## Complex Diagonalization

4/15: • Diagonalize the following matrix $A$.
$$A = \begin{bmatrix} 0.9 & -0.4 \\ 0.1 & 0.9 \end{bmatrix}$$

– Find the characteristic polynomial.

$$0 = \begin{vmatrix} 0.9 - \lambda & -0.4 \\ 0.1 & 0.9 - \lambda \end{vmatrix}$$
$$= (0.9 - \lambda)^2 - (-0.4)(0.1)$$
$$= 0.81 - 1.8\lambda + \lambda^2 + 0.04$$
$$= \lambda^2 - 1.8\lambda + 0.85$$

– Find the eigenvalues[7].

$$\lambda = \frac{-(-1.8) \pm \sqrt{(-1.8)^2 - 4(1)(0.85)}}{2(1)}$$
$$= 0.9 \pm \frac{\sqrt{-0.16}}{2}$$
$$= 0.9 \pm \frac{\sqrt{-1}\sqrt{0.16}}{2}$$
$$= 0.9 \pm \frac{0.4i}{2}$$
$$= 0.9 \pm 0.2i$$

$$\lambda_1 = 0.9 + 0.2i \qquad\qquad \lambda_2 = 0.9 - 0.2i$$

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[7]It is interesting that the eigenvalues are complex conjugates of each other.

– Find the eigenvectors[8].

$$(A - (0.9 + 0.2i))x_1 = \begin{bmatrix} 0.9 - (0.9 + 0.2i) & -0.4 \\ 0.1 & 0.9 - (0.9 + 0.2i) \end{bmatrix} \begin{bmatrix} x_{1_1} \\ x_{1_2} \end{bmatrix}$$

$$= \begin{bmatrix} -0.2i & -0.4 \\ 0.1 & -0.2i \end{bmatrix} \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(A - (0.9 - 0.2i))x_1 = \begin{bmatrix} 0.9 - (0.9 - 0.2i) & -0.4 \\ 0.1 & 0.9 - (0.9 - 0.2i) \end{bmatrix} \begin{bmatrix} x_{1_1} \\ x_{1_2} \end{bmatrix}$$

$$= \begin{bmatrix} 0.2i & -0.4 \\ 0.1 & 0.2i \end{bmatrix} \begin{bmatrix} -2i \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 2i \\ 1 \end{bmatrix} \qquad\qquad x_2 = \begin{bmatrix} -2i \\ 1 \end{bmatrix}$$

– Compile the diagonalization.

$$A = \frac{1}{4i} \begin{bmatrix} 2i & -2i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0.9 + 0.2i & 0 \\ 0 & 0.9 - 0.2i \end{bmatrix} \begin{bmatrix} 1 & 2i \\ -1 & 2i \end{bmatrix}$$

## Real versus Complex

4/16:

| Real | Complex |
|---|---|
| $\mathbb{R}^n$: vectors with $n$ real components | $\mathbb{C}^n$: vectors with $n$ complex components |
| length: $\|x\|^2 = x_1^2 + \cdots + x_n^2$ | length: $\|z\|^2 = |z_1|^2 + \cdots + |z_n|^2$ |
| transpose: $\left(A^{\mathrm{T}}\right)_{ij} = A_{ji}$ | conjugate transpose: $\left(A^{\mathrm{H}}\right)_{ij} = \overline{A_{ji}}$ |
| product rule: $(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}}$ | product rule: $(AB)^{\mathrm{H}} = B^{\mathrm{H}} A^{\mathrm{H}}$ |
| | |
| dot product: $x^{\mathrm{T}}y = x_1 y_1 + \cdots + x_n y_n$ | inner product: $u^{\mathrm{H}}v = \bar{u}_1 u_1 + \cdots + \bar{u}_n v_n$ |
| reason for $A^{\mathrm{T}}$: $(Ax)^{\mathrm{T}}y = x^{\mathrm{T}}(A^{\mathrm{T}}y)$ | reason for $A^{\mathrm{H}}$: $(Au)^{\mathrm{H}}v = u^{\mathrm{H}}(A^{\mathrm{H}}v)$ |
| orthogonality: $x^{\mathrm{T}}y = 0$ | orthogonality: $u^{\mathrm{H}}v = 0$. |
| symmetric matrices: $A = A^{\mathrm{T}}$ | Hermitian matrices: $A = A^{\mathrm{H}}$ |
| $A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}}$ (real $\Lambda$) | $A = U\Lambda U^{-1} = U\Lambda U^{\mathrm{H}}$ (real $\Lambda$) |
| skew-symmetric matrices: $k^{\mathrm{T}} = -K$ | skew-Hermitian matrices: $K^{\mathrm{H}} = -K$ |
| orthogonal matrices: $Q^{\mathrm{T}} = Q^{-1}$ | unitary matrices: $U^{\mathrm{H}} = U^{-1}$ |
| orthonormal columns: $Q^{\mathrm{T}}Q = I$ | orthonormal columns: $U^{\mathrm{H}} = U^{-1}$ |
| $(Qx)^{\mathrm{T}}(Qy) = x^{\mathrm{T}}y$ and $\|Qx\| = \|x\|$ | $(Ux)^{\mathrm{H}}(Uy) = x^{\mathrm{H}}y$ and $\|Uz\| = \|z\|$ |

- Note that the columns and eigenvectors of $Q$ and $U$ are orthonormal, and all of their eigenvalues $\lambda$ satisfy $|\lambda| = 1$.

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[8]It is interesting that the eigenvectors are *also* complex conjugates of each other.

# Fourier Series

4/21:   • Consider the square wave $f(t)$.
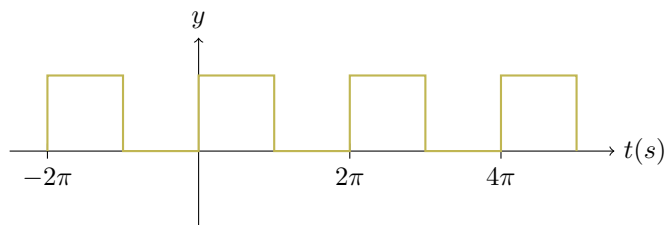


• Its period is $2\pi\frac{\text{sec}}{\text{cycle}}$, and its frequency is $\frac{1}{2\pi}$ Hz.

• Can we write $f(t)$ as a sum of sines and cosines?

$$f(t) = a_0 + a_1\cos(t) + b_1\sin(t) + a_2\cos(2t) + b_2\sin(2t) + a_3\cos(3t) + b_3\sin(3t) + \cdots$$

  – Be general.

  – Since $T = 2\pi$, it makes sense to use some functions with $T = 2\pi$ to model it.

  – The weighting coefficients account for how much each function contributes to the whole.

• Historically studied by Fourier, who studied differential equations. Differential equations were often easy to solve for sines and cosines, so if a function could be modeled by a sum of sines and cosines, a related differential equation would be easier to solve.

• Fourier series, transforms, and analysis also tell us how much of each frequency a function contains (as measured by the weight coefficients).
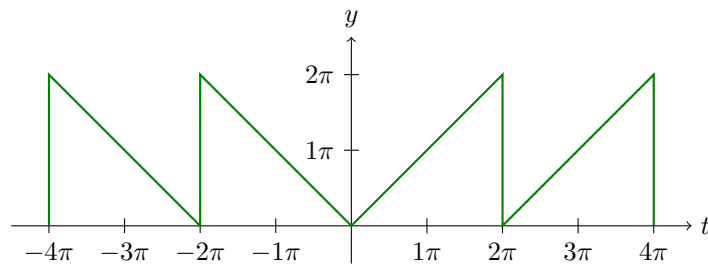
## Wave Type Sketches
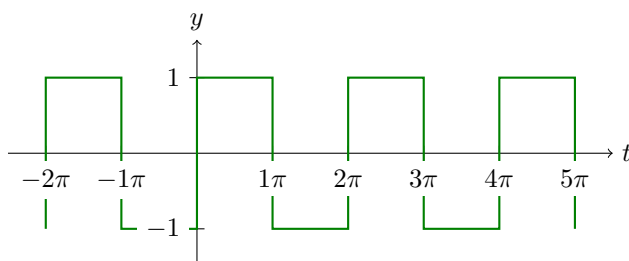


(a) Square wave.
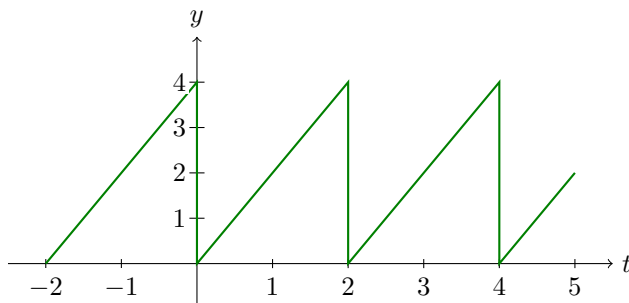


(b) Sawtooth wave.

(c) Triangular wave.

- Square wave: $f(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & \pi < t < 2\pi \end{cases}$ where periodicity is defined by $f(t + 2\pi) = f(t)$.

- Sawtooth wave: $f(t) = 2t \quad 0 < t < 2$ where periodicity is defined by $f(t + 2) = f(t)$.

- Triangular wave: $f(t) = |t| \quad 0 < t < 2\pi$ where periodicity is defined by $f(t + 2\pi) = f(t)$.

1) $\int\limits_{-\pi}^{\pi} \sin(nt)dt = \left[-\frac{1}{n}\cos(nt)\right]_{-\pi}^{\pi} = \frac{1}{n}\left(-\cos(n\pi) + \cos(n\pi)\right) = 0, n \neq 0$

2) $\int\limits_{-\pi}^{\pi} \cos(nt)dt = \left[\frac{1}{n}\sin(nt)\right]_{-\pi}^{\pi} = 0, n \neq 0$

3) $\int\limits_{-\pi}^{\pi} f(t)dt = \frac{1}{2}\int\limits_{-\pi}^{\pi} a_0 dt + \sum\limits_{n=1}^{\infty}\left(\int\limits_{-\pi}^{\pi} a_n \cos(nt)dt + \int\limits_{-\pi}^{\pi} b_n \sin(nt)dt\right) = \frac{1}{2}[a_0 t]_{-\pi}^{\pi} + \sum\limits_{n=1}^{\infty}(0+0)$

$\therefore \int\limits_{-\pi}^{\pi} f(t)dt = \frac{1}{2}(2a_0\pi) \rightarrow a_0 = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(t)dt \rightarrow \frac{a_0}{2} = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f(t)dt$

4) $\int\limits_{-\pi}^{\pi} \sin(nt)\cos(mt)dt = \frac{1}{2}\left(\int\limits_{-\pi}^{\pi} \sin(n+m)t\,dt + \int\limits_{-\pi}^{\pi} \sin(n-m)t\,dt\right) = \frac{1}{2}(0+0) = 0$ using the results from problems 1 and

2 since if $n$ is an integer and $m$ is an integer then $n+m$ and $n-m$ are also integers.

5) $\int\limits_{-\pi}^{\pi} \cos(nt)\cos(mt)dt = \frac{1}{2}\left(\int\limits_{-\pi}^{\pi} \cos(n+m)t\,dt + \int\limits_{-\pi}^{\pi} \cos(n-m)t\,dt\right) = \frac{1}{2}(0+0) = 0$ when $n \neq m$

6) $\int\limits_{-\pi}^{\pi} \cos^2(nt)dt = \frac{1}{2}\int\limits_{-\pi}^{\pi}\left(1+\cos(2n)t\right)dt = \frac{1}{2}\left[t + \frac{1}{2n}\sin(2n)t\right]_{-\pi}^{\pi} = \pi$ when $n \neq 0$

7) $\pi - (-\pi) = 2\pi$

8) a) $\int\limits_{-\pi}^{\pi} \sin(nt)\sin(mt)dt = 0$

b) $\int\limits_{-\pi}^{\pi} \sin^2(nt)dt = \frac{1}{2}\int\limits_{-\pi}^{\pi}\left(1-\cos(2nt)\right)dt = \pi$

c) $\int\limits_{-\pi}^{\pi} \sin(nt)\sin(mt)dt = 0$

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For integers $m,n$:

$\int\limits_{-\pi}^{\pi} \sin(nt)\cos(mt)dt = 0$

$\int\limits_{-\pi}^{\pi} \cos(nt)\cos(mt)dt = \begin{cases} 0, n \neq m \\ \pi, n = m \neq 0 \\ 2\pi, n = m = 0 \end{cases}$

$\int\limits_{-\pi}^{\pi} \sin(nt)\sin(mt)dt = \begin{cases} 0, n \neq m, n = m = 0 \\ \pi, n = m \end{cases}$