CHANGE OF BASIS AND SIMILARITY

$$T:\mathbb{R}^2 \to \mathbb{R}^2 : T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+3y \\ 2x+2y \end{bmatrix}$$

Is it possible to find a basis B for $\mathbb{R}^2$ such that the transformation matrix T is diagonal with respect to B?

With respect to a standard basis, $\left[T\right]_E = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$.

We can show that $\left[T\right]_E = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ **is diagonalizable into** $\left[T\right]_E = S\Lambda S^{-1}$**, in which** Λ **is a diagonal matrix.**

$$\Lambda = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix} \quad \textbf{and} \quad S = \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}$$

If $\left[T\right]_E = S\Lambda S^{-1}$**, then** $\Lambda = S^{-1}\left[T\right]_E S$**. Let** B **be the basis in** $\mathbb{R}^2$ **consisting of the columns of** S**, then** S **is the change-of-basis matrix from** B **to** E**. Then**

$$\left[T\right]_B = S^{-1}\left[T\right]_E S = \Lambda$$

Therefore the transformation matrix T **with respect to the basis** $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix} \right\}$ **is diagonal.**

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad \textbf{and} \quad T\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

So the vectors that form the columns of $\left[T\right]_B$ **are**

$$\left[T\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]_B = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \quad \textbf{and} \quad \left[T\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right]_B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$T_E \sim T_B$ **because when** $T:V \to W$ **, changing a basis for** V **of transformation matrix** A **would require multiplication** AM **(multiply by** M **on the right to come first). Chaging the basis for** W **would change** A **to** $M^{-1}A$ **(to come last). Therefore, to change both bases the same way, the new matrix is** $B = M^{-1}AM$ **. The good basis vectors are thus the eigenvectors of** A**, and** $B = M^{-1}AM$ **becomes** $B = S^{-1}AS$ **.**

WORKING WITHIN A NON-STANDARD OR NON-EIGEN BASIS

$$T : P_2 \to P_2 : T(p(x)) = p(2x-1)$$

Find T **with respect to basis** $B = \ 1+x, 1-x, x^2$

With respect to a standard basis :

$$T(1) = 1, T(x) = 2x-1, T(x^2) = (2x-1)^2 = 1-4x+4x^2$$

$$T(1)_E = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ T(x)_E = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \left[T(x^2)\right]_E = \begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix}$$

Therefore $T_E = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}$

The change of basis matrix from B **to** E **is** $\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Therefore it follows that $[T]_B = M^{-1}[T]_E M$

$$= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ -1 & 2 & \frac{5}{2} \\ 0 & 0 & 4 \end{bmatrix}$$

Find a basis C **for** P_2 **such that** $[T]_C$ **is a diagonal matrix.**

$[T]_E$ **has eigenvalues 1, 2, and 4 with eigenvectors** $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$$S = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \quad \textbf{and} \quad \Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$S^{-1}[T]_E S = \Lambda$**, and therefore** S **is a change of basis matrix from a basis** C **to** E**.**

Therefore $C = \{1, -1+x, 1-2x+x^2\}$