FOURIER SERIES

- We have shown that when $v^T w = 0$, then vectors *v* and *w* are orthogonal.
- Fourier series ask us to think of continuous functions as vectors.
- Let $f(t) = \cos(t)$ and let $g(t) = \sin(t)$
- To find $f \bullet g$ of these two function "vectors" would be asking us to take a dot product that has infinitely many terms. All vectors we have used have had a finite number of components. Because these "vectors" would have infinitely many terms, it would be like asking us to take an integral!
- $f \bullet g = \int_0^{2\pi} \cos(t)\sin(t)dt$ (We limit the domain because these are sinusoidal periodic functions and thus they repeat after a period of 2π.)

$$\int_0^{2\pi} \cos(t)\sin(t)dt$$

Let $u = \cos(t)$ and $du = -\sin(t)$

$$-\int_1^1 udu$$

$$\left[-\frac{u^2}{2}\right]_1^1$$

$$0$$

OR

$$\left[-\frac{1}{2}\cos^2(t)\right]_0^{2\pi}$$

$$-\frac{1}{2}(\cos^2 2\pi - \cos^2 0)$$

$$-\frac{1}{2}(1-1) =$$

$$0$$

- Therefore $\cos t \bullet \sin t = 0$, therefore these function "vectors" are orthogonal, therefore they serve as basis vectors for the function space of continuous periodic functions in F^2, therefore any continuous function also in this space can be written as a linear combination of $\cos(t)$ and $\sin(t)$!!!

$$f(t) = \frac{a_0}{2} + a_1\cos(t) + b_1\sin(t) + a_2\cos(2t) + b_2\sin(2t) + \cdots$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nt) + b_n\sin(nt)\right)$$

*Why divide the constant term by 2?

CALCULATING THE FOURIER COEFFICENTS

As sine and cosine can serve as an orthogonal basis for periodic functions, consider the Fourier Series for a function $f(t)$ of period 2π to be $f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos(nt)+b_n\sin(nt)\right)$

In problem 3 from your handout you already found a formula for a_0, the constant term.

$$\boxed{\frac{a_0}{2}=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)dt}$$

To obtain the coefficients $a_n, n\in\mathbb{Z}$, multiply both sides of the equation by $\cos(mt)$ where $m\in\mathbb{Z}$ and $m>0$ and integrate both sides from $-\pi$ to π since this is a periodic function of period 2π.

$$\int_{-\pi}^{\pi}f(t)\cos(mt)dt=\frac{a_0}{2}\int_{-\pi}^{\pi}\cos(mt)dt+\sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos(nt)\cos(mt)dt+b_n\int_{-\pi}^{\pi}\sin(nt)\cos(mt)dt\right)$$

Simplify this result using the integrals on the right side of this equation from the handout problems 5-7. The only nonzero integral results from $\int_{-\pi}^{\pi}\cos^2(mt)dt=\pi$ in the case where $n=m$.

$$\therefore\int_{-\pi}^{\pi}\cos(mt)dt=a_m\pi$$

Sine this is the case where $n=m$, replacing m with n and solving for a_n we obtain:

$$\boxed{a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos(nt)dt\text{ , }n\in\mathbb{Z}\text{ and }n>0}$$

To obtain the coefficients $b_{n,}b\in\mathbb{Z},n>0$, multiply both sides of the equation by $\sin(mt)$ where $m\in\mathbb{Z}$ and $m>0$ and integrate both sides from $-\pi$ to π since this is a periodic function of period 2π.

$$\int_{-\pi}^{\pi}f(t)\sin(mt)dt=\frac{a_0}{2}\int_{-\pi}^{\pi}\sin(mt)dt+\sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos(nt)\sin(mt)dt+b_n\int_{-\pi}^{\pi}\sin(nt)\sin(mt)dt\right)$$

Simplify this result using the integrals on the right side of this equation using the handout problems you did for 8. The only nonzero integral results from $\int_{-\pi}^{\pi}b_m\sin^2(mt)dt=b_m\pi$ in the case where $n=m$. Relabeling m as n and solving for b_n we obtain:

$$\boxed{b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin(nt)dt\text{ , }n\in\mathbb{Z}\text{ and }n>0}$$