MATRICES OF COMPOSITE AND INVERSE TRANSFORMATIONS

$T:\mathbb{R}^2 \to P_1 : T\begin{bmatrix} a \\ b \end{bmatrix} = a + (a+b)x$ **and** $S : P_1 \to P_2 : S(a+bx) = ax + bx^2 = xp(x)$

FIND A MATRIX FOR $S \circ T$

- **Standard Basis for** $\mathbb{R}^2$ **is** $B = \ e_1, e_2$
- **Standard Basis for** P_1 **is** $C = \ 1, x$
- **Standard Basis for** P_2 **is** $D = \ 1, x, x^2$

$$T(e_1) = 1 + (1+0)x = 1 + x \quad T(e_2) = 0 + (0+1)x = x$$

$$T \;_C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$S\begin{bmatrix} 1 \\ 1 \end{bmatrix} = x + x^2 \qquad S\begin{bmatrix} 0 \\ 1 \end{bmatrix} = x^2$$

$$S \;_D = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S \circ T \ = \ S \quad T \ = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$T:\mathbb{R}^2 \to P_1 : T\begin{bmatrix} a \\ b \end{bmatrix} = a + (a+b)x$$

FIND T^{-1}

Because the transformation is both one-to-one and onto, it is invertible.

In the above example we found $T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$$\left[T^{-1}\right] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

This implies $\left[T^{-1}(a+bx)\right] = \left[T^{-1}\right]\ a+bx\ = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b-a \end{bmatrix}$

This is the same as $T^{-1}(a+bx) = ae_1 + (b-a)e_2 = \begin{bmatrix} a \\ b-a \end{bmatrix}$

AN EXAMPLE FROM CALCULUS

Using the inverse of a differential operator, find $\int x^2 e^{3x}\,dx$ **(which usually requires two applications of integration by parts.)**

From the example problem given in class, we found $D_B = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ **for which**

$$B = \ e^{3x}, xe^{3x}, x^2e^{3x}$$

$$\left[D^{-1}\right]_B = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{9} & \frac{2}{27} \\ 0 & \frac{1}{3} & -\frac{2}{9} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$\left[x^2e^{3x}\right]_B = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\left[\int x^2e^{3x}dx\right]_B = \left[D^{-1}(x^2e^{3x})\right]B$$

$$= \left[D^{-1}\right]_B\left[x^2e^{3x}\right]_B$$

$$= \begin{bmatrix}\frac{1}{3} & -\frac{1}{9} & \frac{2}{27}\\ 0 & \frac{1}{3} & -\frac{2}{9}\\ 0 & 0 & \frac{1}{3}\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}\frac{2}{27}\\ -\frac{2}{9}\\ \frac{1}{3}\end{bmatrix}$$

$$\int x^2e^{3x}dx = \frac{2}{27}e^{3x} - \frac{2}{9}xe^{3x} + \frac{1}{3}x^2e^{3x} + C$$