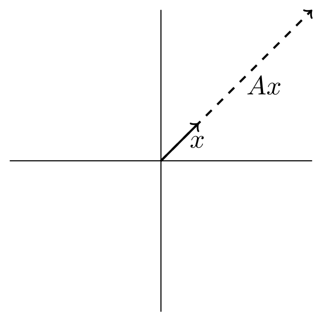
1/28:
Introduction to Eigenvalues and Eigenvectors

- $Ax = b = \lambda x$
- $Ax = \lambda x$, $\lambda \in \mathbb{F}$, $x \in \mathbb{R}^n$
- λ is an eigenvalue. λx is an eigenvector.

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

- $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with corresponding eigenvalue of 4.
- $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



$$\begin{aligned} Ax &= \lambda x \\ Ax - \lambda x &= 0 \\ Ax - \lambda I x &= 0 \\ (A - \lambda I)x &= 0 \end{aligned}$$

- $(A - \lambda I)x = 0 \Rightarrow x \in N(A - \lambda I)$[1] $\Rightarrow |A - \lambda I| = 0$
- $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$
- $\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$$\begin{aligned} 0 &= (3-\lambda)^2 - 1^2 \\ &= 3^2 - 6\lambda + \lambda^2 - 1 \\ &= \lambda^2 - 6\lambda + 8 \\ &= (\lambda - 4)(\lambda - 2) \end{aligned}$$

- $\lambda = 4, 2$.
- $\lambda^2 - 6\lambda + 8$ is the **characteristic polynomial** of A.
- $A - 2I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in N(A - 2I)$.
- $A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in N(A - 4I)$.

[1] To have a null space, $A - \lambda I$ has free columns.

- "Eigenspace" is not $\mathbb{R}^2$, but two lines in $\mathbb{R}^2$, specifically $y = \pm x$.
  - $y = \pm x$ comes from $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

1/29:

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\begin{aligned}
P(\lambda) &= |A - \lambda I| \\
&= \begin{vmatrix} 2-\lambda & -2 & 3 \\ 0 & 3-\lambda & -2 \\ 0 & -1 & 2-\lambda \end{vmatrix} \\
&= -1 \begin{vmatrix} 2-\lambda & 3 \\ 0 & -2 \end{vmatrix} (-1)^{3+2} + (2-\lambda) \begin{vmatrix} 2-\lambda & -2 \\ 0 & 3-\lambda \end{vmatrix} (-1)^{3+3} \\
&= ((2-\lambda)(-2)) + (2-\lambda)((2-\lambda)(3-\lambda)) \\
&= -4 + 2\lambda + (2-\lambda)^2(3-\lambda) \\
&= -4 + 2\lambda + (4 - 4\lambda + \lambda^2)(3-\lambda) \\
&= -4 + 2\lambda + 12 - 4\lambda - 12\lambda + 4\lambda^2 + 3\lambda^2 - \lambda^3 \\
&= -\lambda^3 + 7\lambda^2 - 14\lambda + 8 \\
&= -(\lambda-1)(\lambda-2)(\lambda-4)
\end{aligned}$$

$$A - I = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -2 \\ 0 & -1 & 1 \end{bmatrix} \qquad A - 2I = \begin{bmatrix} 0 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & -1 & 0 \end{bmatrix} \qquad A - 4I = \begin{bmatrix} -2 & -2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix}$$

- $P(\lambda)$ is positive when $n \in 2\mathbb{N}$, negative otherwise.
  - Signs flip term to term (think about binomial expansion).
- Coefficient of the $n-1$ degree term is the sum of the diagonal entries.
- Coefficient of the 0^{th} degree term is $|A|$.
  - $P_\lambda(0) = |A - 0 \cdot I| = |A|$.
- Product of the eigenvalues is $|A|$.
  - Think about expanding the factorization of $P(\lambda)$.
- Eigenvalues of U are the diagonal values.
  - $\lambda_1 \lambda_2 \cdots \lambda_n = |A|$, which is the product of the diagonal entries.
  - $\lambda_1 + \cdots + \lambda_n = \text{trace}(A)$, which is the sum of the diagonal entries.
- $Ax = \lambda x$
  - $A^2 x = AAx = A\lambda x = \lambda Ax = \lambda\lambda x = \lambda^2 x$

Similarity

1/30: • $A \sim B$[2] iff $\exists\, S : A = SBS^{-1},\ B = S^{-1}AS$.

1. If $A \sim B$, then $|A| = |B|$.

$$\begin{aligned} B &= S^{-1}AS \\ |B| &= |S^{-1}AS| \\ |B| &= |S^{-1}||A||S| \\ |B| &= \frac{1}{|S|}|A||S| \\ |B| &= |A| \end{aligned}$$

2. If $A \sim B$, then they share the same characteristic polynomial.

$$\begin{aligned} B &= S^{-1}AS \\ |B - \lambda I| &= |S^{-1}AS - \lambda I| \\ &= |S^{-1}AS - \lambda S^{-1}IS| \\ &= |S^{-1}(A - \lambda I)S| \\ &= \frac{1}{|S|}|A - \lambda I||S| \\ |B - \lambda I| &= |A - \lambda I| \end{aligned}$$

   - If they have the same characteristic polynomial, $\therefore$ A and B have the same eigenvalues.

- What is the best possible B if $A \sim B$?
  - Sparse.
  - Diagonal.
  - $A = [\text{ugly}] \quad \to \quad B = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = \Lambda$

- **Diagonalization**:

$$\begin{aligned} A &= S\Lambda S^{-1} \\ AS &= S\Lambda \\ \Lambda &= S^{-1}AS \end{aligned}$$

- $A = S\Lambda S^{-1}$
  - $A^2 = AA = S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda\Lambda S^{-1} = S\Lambda^2 S^{-1}$
  - $A^k = S\Lambda^k S^{-1}$
  - $A^k = S \begin{bmatrix} {\lambda_1}^k & & 0 \\ & \ddots & \\ 0 & & {\lambda_n}^k \end{bmatrix} S^{-1}$

- Diagonalize the following matrix A.

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

[2] A "is similar to" B

- Find the characteristic polynomial.

$$\begin{aligned}
|A-\lambda I| &= \begin{vmatrix} -1-\lambda & 0 & 1 \\ 3 & 0-\lambda & -3 \\ 1 & 0 & -1-\lambda \end{vmatrix} \\
&= (-1-\lambda)((-\lambda)(-1-\lambda)) + (-1)(-\lambda) \\
&= -\lambda(-1-\lambda)^2 + \lambda \\
&= -\lambda(1+2\lambda+\lambda^2) + \lambda \\
&= -\lambda^3 - 2\lambda^2 \\
&= -\lambda^2(\lambda+2)
\end{aligned}$$

- Find the eigenvalues: $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = -2$
- **Algebraic multiplicity** of λ_1, λ_2 is 2.
- A.M. of λ_3 is 1.
- $A - 0I = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$
- $\operatorname{rank}(A-0I) = 1 \Rightarrow \dim(N(A-0I)) = 2$
- The 2 directly above is the **geometric multiplicity**.
- A is diagonalizable iff A.M. of λ_i = G.M.

1/31: - Eigenvectors are $x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

- $A + 2I = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & -3 \\ 1 & 0 & 1 \end{bmatrix}$
- Eigenvector is $x_3 = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$.
- Use an S matrix of eigenvectors.
- $A = S\Lambda S^{-1} = \frac{1}{2} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & & \\ & 0 & \\ & & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 & -3 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$
- Note that $A^{9752} = \frac{1}{2} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -3 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & & \\ & 0 & \\ & & (-2)^{9752} \end{bmatrix} \begin{bmatrix} 3 & 2 & -3 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$

• **Algebraic multiplicity**: The number of repeated roots to a polynomial. For all of the roots, it adds up to n (n-square matrix). *Also known as* **A.M.**

• **Geometric multiplicity**: The number of eigenvectors produced from each root. For all of the roots, it may not add up to n (n-square matrix). $\dim(N(A-\lambda I))$. *Also known as* **G.M.**

• A nondiagonalizable example:

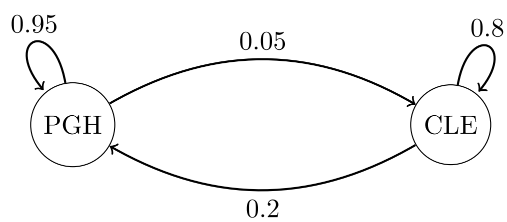
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

- $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 4$.
- λ_1 and λ_2 have A.M. = 2.

  - λ_3 has A.M. = 1.
  - $A - I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}$
  - $\text{rank}(A - I) = 2 \Rightarrow \dim(N(A - I)) = 1 \Rightarrow$ G.M. = 1.
  - $x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
  - $A - 4I = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
  - $x_2 = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$
  - S would be 3×2 and, thus, not square, so $\nexists\ S^{-1}$[3].
- **Canonical** (form): An accepted way of expressing something.

Markov Chains

2/3:



- $u_0 = \begin{bmatrix} 500,000 \\ 500,000 \end{bmatrix}$
- $Au_0 = u_1$.
- $Au_1 = u_2$, $A(Au_0) = u_2$, $A^2u_0 = u_2$, $A^ku_0 = u_k$, $(S\Lambda S^{-1})^k u_0 = u_k$, $S\Lambda^k S^{-1} u_0 = u_k$.

$$A = \begin{bmatrix} 0.95 & 0.20 \\ 0.05 & 0.80 \end{bmatrix} \qquad u_k = \begin{bmatrix} \text{PGH} \\ \text{CLE} \end{bmatrix}$$

- A is a **Markov matrix**, where all columns add to 1.
- $Au_0 = \begin{bmatrix} 0.95 & 0.20 \\ 0.05 & 0.80 \end{bmatrix} \begin{bmatrix} 500,000 \\ 500,000 \end{bmatrix} = u_1$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 0.95 - \lambda & 0.20 \\ 0.05 & 0.80 - \lambda \end{vmatrix} \\ &= (0.95 - \lambda)(0.8 - \lambda) - (0.2)(0.05) \\ &= (\lambda - 1)(\lambda - 0.75) \end{aligned}$$

- $\lambda_1 = 1,\ \lambda_2 = 0.75$.

[3]At a later date, we will look at an analogy of projections to diagonalization that finds the "best possible" diagonalization (which may not be perfectly diagonal).

- $A - I = \begin{bmatrix} -0.05 & 0.2 \\ 0.05 & -0.2 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$
- $A - 0.75I = \begin{bmatrix} 0.2 & 0.2 \\ 0.05 & 0.05 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\begin{aligned} A^k u_0 &= \frac{1}{5}\begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.75 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 500,000 \\ 500,000 \end{bmatrix} \\ &= \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0.75^k \end{bmatrix}\begin{bmatrix} 200,000 \\ 300,000 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 1 \end{bmatrix}(200,000) + \begin{bmatrix} -1 \\ 1 \end{bmatrix}(0.75)^k(300,000) \end{aligned}$$

- $\begin{bmatrix} 800,000 \\ 200,000 \end{bmatrix}$ is the steady-state vector.
- $\begin{bmatrix} -(0.75)^k(300,000) \\ (0.75)^k(300,000) \end{bmatrix}$ is the dynamically changing vector.
- $\lim_{k\to\infty} A^k u_0 = \begin{bmatrix} 800,000 \\ 200,000 \end{bmatrix} = \begin{bmatrix} \text{PGH} \\ \text{CLE} \end{bmatrix}$

Explicit Formula for the Fibonacci Sequence

2/4:

- 1, 1, 2, 3, 5, 8, ...
- Recursively defined formula: F_n[4] $= F_{n-1} + F_{n-2}$.

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} \\ F_{n-1} &= F_{n-1} \\ \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix} \end{aligned}$$

- $u_n = A^n u_0 = S\Lambda^n S^{-1} u_0$.

$$\begin{aligned} 0 &= |A - \lambda I| \\ &= \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} \\ &= -\lambda(1-\lambda) - 1 \\ &= \lambda^2 - \lambda - 1 \end{aligned}$$

- $\lambda = \frac{1\pm\sqrt{5}}{2}$[5].
- $\lambda_1 = \frac{1+\sqrt{5}}{2}$.

$$\begin{aligned} N(A - \lambda_1 I) &= N\left(\begin{bmatrix} 1 - \frac{1+\sqrt{5}}{2} & 1 \\ 1 & -\frac{1+\sqrt{5}}{2} \end{bmatrix}\right) \\ &= N\left(\begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{-1-\sqrt{5}}{2} \end{bmatrix}\right) \end{aligned}$$

[4]The n-th Fibonacci number.
[5]This is the Golden ratio!

- $\begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{-1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- Let $x_2 = 1$.

$$\begin{aligned} \frac{1-\sqrt{5}}{2}x_1 + 1 &= 0 \\ \frac{1-\sqrt{5}}{2}x_1 &= -\frac{2}{2} \\ x_1 &= \frac{-2}{1-\sqrt{5}} \times \frac{1+\sqrt{5}}{1+\sqrt{5}} \\ &= \frac{-2-2\sqrt{5}}{-4} \\ &= \frac{1+\sqrt{5}}{2} \end{aligned}$$

- $s_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$, $s_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$.
- $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} S^{-1}$
- $S^{-1} = \frac{1}{|S|}C_S^{\mathrm{T}} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}$
- $u_k = A^k u_0 = S\Lambda^k S^{-1} u_0$.

$$\begin{aligned} S^{-1}u_0 &= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ \frac{-1+\sqrt{5}}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1+\sqrt{5}}{2\sqrt{5}} \\ \frac{-1+\sqrt{5}}{2\sqrt{5}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{5+\sqrt{5}}{10} \\ \frac{5-\sqrt{5}}{10} \end{bmatrix} \end{aligned}$$

- $u_k = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^k \left(\frac{5+\sqrt{5}}{10}\right) + \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} \left(\frac{1-\sqrt{5}}{2}\right)^k \left(\frac{5-\sqrt{5}}{10}\right)$

Systems of First-Order Ordinary Differential Equations

2/11:

- Let $f(x) = y$ and $a, c, K \in \mathbb{F}$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= ay \\
\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= a \\
\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x &= a\,\mathrm{d}x \\
\frac{1}{y}\,\mathrm{d}y &= a\,\mathrm{d}x \\
\int \frac{1}{y}\,\mathrm{d}y &= \int a\,\mathrm{d}x \\
\ln y &= ax + c \\
y &= \mathrm{e}^{ax+c} \\
&= \mathrm{e}^{ax}\mathrm{e}^{c} \\
&= K\mathrm{e}^{ax}
\end{aligned}$$

- Let $\frac{\mathrm{d}y}{\mathrm{d}x} = y'$.
  - $y_1' = a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n$.
  - $y_2' = a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n$.
  - $\vdots$
  - $y_n' = a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n$.
- This is a **square system** of equations.
- Rewrite as $y' = Ay$.

$$\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

- Solve the following system of differential equations.

$$\begin{aligned}
y_1' &= 3y_1 \\
y_2' &= -2y_2 \\
y_3' &= 5y_3
\end{aligned}$$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

  - General Solution:

$$\begin{aligned}
y_1 &= k_1\mathrm{e}^{3x} \\
y_2 &= k_2\mathrm{e}^{-2x} \\
y_3 &= k_3\mathrm{e}^{5x}
\end{aligned}$$

  - Particular Solution (where $y_1(0) = 2$, $y_2(0) = -1$, and $y_3(0) = 7$ are the initial conditions):

$$y_1 = 2e^{3x}$$
$$y_2 = -e^{-2x}$$
$$y_3 = 7e^{5x}$$

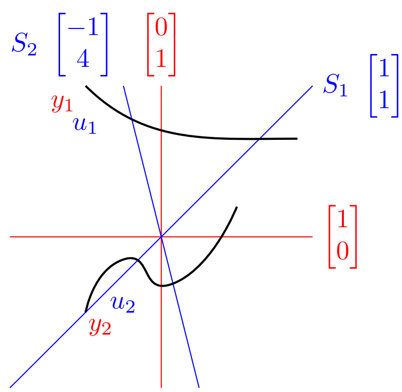
- Consider a different system. Remember throughout that we are solving for y.

$$y_1' = y_1 + y_2$$
$$y_2' = 4y_1 - 2y_2$$

  - The previous system was so easy to solve because the matrix was diagonal. This one (as follows) will not be. Therefore, we should diagonalize it.

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

  - Start with $y' = Ay$.
  - Substitute $y = Su$.
    - Note that $y = Su \Rightarrow y' = Su'$[6].
    - If we can find u' in terms of a diagonal matrix and u, we can solve for y.



  - We seek to find a new basis S such that the matrix scaling u will be diagonal.

$$\begin{aligned} Su' &= Ay \\ Su' &= ASu \\ u' &= S^{-1}ASu \\ u' &= \Lambda u \end{aligned}$$

  - The last substitution above is legal because if $A = S\Lambda S^{-1}$, then $\Lambda = S^{-1}AS$.

$$\begin{aligned} 0 &= \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} \\ &= (1-\lambda)(-2-\lambda) - 4 \\ &= -2 - \lambda + 2\lambda + \lambda^2 - 4 \\ &= \lambda^2 + \lambda - 6 \\ &= (\lambda - 2)(\lambda + 3) \end{aligned}$$

$$\lambda_1 = 2 \qquad\qquad \lambda_2 = -3$$

[6] Think about differentiating both sides: $y \to y'$ is obvious, S will be unchanged because it's just coefficients, and the functions of u will be differentiated.

- $A - 2I = \begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- $A + 3I = \begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$u' = \Lambda u$$
$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$u_1 = k_1 \mathrm{e}^{2x}$$
$$u_2 = k_2 \mathrm{e}^{-3x}$$

$$y = Su$$
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} k_1 \mathrm{e}^{2x} \\ k_2 \mathrm{e}^{-3x} \end{bmatrix}$$

$$y_1 = k_1 \mathrm{e}^{2x} - k_2 \mathrm{e}^{-3x}$$
$$y_2 = k_1 \mathrm{e}^{2x} + 4k_2 \mathrm{e}^{-3x}$$

2/12:
- Initial conditions: $y_1(0) = 1$ and $y_2(0) = 6$.
  - Use augmented matrices to solve a system of equations.

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & 4 & 6 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

- Particular solution:

$$y_1 = 2\mathrm{e}^{2x} - \mathrm{e}^{-3x}$$
$$y_2 = 2\mathrm{e}^{2x} + 4\mathrm{e}^{-3x}$$

Matrix Exponentiation

- e^A is a matrix defined as the infinite sum of a power series.
- $f(t) = \mathrm{e}^t$.

Differential Equations	**Power Series**
$f'(t) = f(t)$	$f(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \cdots$
$f(0) = 1$	$\frac{\mathrm{d}}{\mathrm{d}t}(t) = 1,\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^2}{2}\right) = t,\ \ldots$
	$f(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}$

- $f(t) = \mathrm{e}^{at}$.

Differential Equations	**Power Series**
$f'(t) = af(t)$	$f(t) = \sum_{n=0}^{\infty} \frac{a^n t^n}{n!}$
$f(0) = 1$	

- $F(t) = \mathrm{e}^{At}$.
  - A matrix-valued function.
  - Ex. $F(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$
  - $F(\theta)A$ rotates points (arrows) of A by θ.

Differential Equations	**Power Series**
$F'(t) = A\mathrm{e}^{At}$	$F(t) = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$
$F(0) = I$	$F(t) = \sum_{n=0}^{\infty} \frac{A^nt^n}{n!}$

Diagonalization of e^{At}

- Find an alternate form for e^{At} by manipulating its power series definition:

2/13:

$$\begin{aligned}
\mathrm{e}^{At} &= \sum_{n=0}^{\infty} \frac{A^nt^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{S\Lambda^nS^{-1}t^n}{n!} \\
&= \sum_{n=0}^{\infty} S\left(\frac{\Lambda^nt^n}{n!}\right)S^{-1} \\
&= S\left(\sum_{n=0}^{\infty} \frac{t^n}{n!}\Lambda^n\right)S^{-1} \\
&= S\left(\sum_{n=0}^{\infty} \frac{t^n}{n!}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix}^n\right)S^{-1} \\
&= S\left(\sum_{n=0}^{\infty} \frac{t^n}{n!}\begin{bmatrix} \lambda_1^n & & \\ & \ddots & \\ & & \lambda_k^n \end{bmatrix}\right)S^{-1} \\
&= S\left(\sum_{n=0}^{\infty} \begin{bmatrix} \frac{t^n}{n!}\lambda_1^n & & \\ & \ddots & \\ & & \frac{t^n}{n!}\lambda_k^n \end{bmatrix}\right)S^{-1} \\
&= S\left(\sum_{n=0}^{\infty} \begin{bmatrix} \frac{\lambda_1^nt^n}{n!} & & \\ & \ddots & \\ & & \frac{\lambda_k^nt^n}{n!} \end{bmatrix}\right)S^{-1} \\
&= S\begin{bmatrix} \sum_{n=0}^{\infty} \frac{\lambda_1^nt^n}{n!} & & \\ & \ddots & \\ & & \sum_{n=0}^{\infty} \frac{\lambda_k^nt^n}{n!} \end{bmatrix}S^{-1} \\
&= S\begin{bmatrix} \mathrm{e}^{\lambda_1t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_kt} \end{bmatrix}S^{-1} \\
&= S\mathrm{e}^{\Lambda t}S^{-1} \\
&= F(t)
\end{aligned}$$

- Prove, using the above result, that $F'(t)$ can be defined in terms of $F(t)$:

$$\begin{aligned}F(t) &= \mathrm{e}^{At}\\ &= S\begin{bmatrix}\mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_k t}\end{bmatrix} S^{-1}\end{aligned}$$

$$\begin{aligned}F'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(S\begin{bmatrix}\mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_k t}\end{bmatrix} S^{-1}\right)\\
&= S\frac{\mathrm{d}}{\mathrm{d}t}\left(\begin{bmatrix}\mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_k t}\end{bmatrix}\right) S^{-1}\\
&= S\begin{bmatrix}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\lambda_k t}\end{bmatrix} S^{-1}\\
&= S\begin{bmatrix}\lambda_1\mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \lambda_k\mathrm{e}^{\lambda_k t}\end{bmatrix} S^{-1}\\
&= S\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_k\end{bmatrix}\begin{bmatrix}\mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_k t}\end{bmatrix} S^{-1}\\
&= S\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_k\end{bmatrix} I_k \begin{bmatrix}\mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_k t}\end{bmatrix} S^{-1}\\
&= {\color{red}S\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_k\end{bmatrix} S^{-1}}{\color{blue}S\begin{bmatrix}\mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_k t}\end{bmatrix} S^{-1}}\\
&= {\color{red}A}{\color{blue}F(t)}\\
&= A\mathrm{e}^{At}\end{aligned}$$

- In other words, $y'(t) = Ay(t)$ and $y(0) = y_0$. The solution is $y = \mathrm{e}^{At}y_0$.
- Example:

$$\begin{aligned}y_1' &= 5y_1 + y_2 & y_1(0) &= -3\\ y_2' &= -2y_1 + 2y_2 & y_2(0) &= 8\end{aligned}$$

$$\begin{aligned}y(t) &= \mathrm{e}^{At}y(0)\\ &= S\mathrm{e}^{\Lambda t}S^{-1}y(0)\end{aligned}$$

$$\begin{aligned}0 &= |A - \lambda I|\\ &= \begin{vmatrix}5-\lambda & 1\\ -2 & 2-\lambda\end{vmatrix}\\ &= (\lambda-3)(\lambda-4)\end{aligned}$$

- $A - 3I = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$
- $N(A - 3I) = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$
- $A - 4I = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}$
- $N(A - 4I) = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

$$
\begin{aligned}
y(t) &= S\mathrm{e}^{\Lambda t}S^{-1}y(0) \\
&= \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{e}^{3t} & 0 \\ 0 & \mathrm{e}^{4t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 8 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{3t} & -\mathrm{e}^{4t} \\ -2\mathrm{e}^{3t} & \mathrm{e}^{4t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 8 \end{bmatrix} \\
&= \begin{bmatrix} -\mathrm{e}^{3t} + 2\mathrm{e}^{4t} & -\mathrm{e}^{3t} + \mathrm{e}^{4t} \\ 2\mathrm{e}^{3t} - 2\mathrm{e}^{4t} & 2\mathrm{e}^{3t} - \mathrm{e}^{4t} \end{bmatrix} \begin{bmatrix} -3 \\ 8 \end{bmatrix} \\
&= \begin{bmatrix} 3\mathrm{e}^{3t} - 6\mathrm{e}^{4t} - 8\mathrm{e}^{3t} + 8\mathrm{e}^{4t} \\ -6\mathrm{e}^{3t} + 6\mathrm{e}^{4t} + 16\mathrm{e}^{3t} - 8\mathrm{e}^{4t} \end{bmatrix} \\
&= \begin{bmatrix} -5\mathrm{e}^{3t} + 2\mathrm{e}^{4t} \\ 10\mathrm{e}^{3t} - 2\mathrm{e}^{4t} \end{bmatrix} \\
&= \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}
\end{aligned}
$$

Orthonormally Diagonalizable Matrices

2/19:

- $A = Q\Lambda Q^{\mathrm{T}}$.
  - Eigenvectors are orthonormal.
- $A^{\mathrm{T}} = \left(Q\Lambda Q^{\mathrm{T}}\right)^{\mathrm{T}} = Q^{\mathrm{TT}}\Lambda^{\mathrm{T}}Q^{\mathrm{T}} = Q\Lambda Q^{\mathrm{T}} = A$.
- Prove that the symmetric matrices are exactly those that are orthonormally diagonalizable.
  - Let $A = A^{\mathrm{T}}$.

$$
Ax_1 = \lambda_1 x_1 \tag{1}
$$

$$
Ax_2 = \lambda_2 x_2 \tag{2}
$$

  - Multiply Equation 1 by x_2^{T} from Equation 2.
    - We have to relate the two equations.
    - Later, we transpose, because we have to specifically target the properties of symmetric matrices.

$$
\begin{aligned}
\lambda_1 x_2^{\mathrm{T}} x_1 &= x_2^{\mathrm{T}} A x_1 \\
&= (x_2^{\mathrm{T}} A) x_1 \\
&= \left(A^{\mathrm{T}} x_2\right)^{\mathrm{T}} x_1 \\
&= (Ax_2)^{\mathrm{T}} x_1 \\
\lambda_1 x_2^{\mathrm{T}} x_1 &= \lambda_2 x_2^{T} x_1 \\
\lambda_1 x_2^{\mathrm{T}} x_1 - \lambda_2 x_2^{T} x_1 &= 0 \\
x_2^{\mathrm{T}} x_1 (\lambda_1 - \lambda_2) &= 0
\end{aligned}
$$

- The last line above implies that $x_2^{\mathrm{T}} x_1 = 0$ iff $\lambda_1 \neq \lambda_2$.

- The only matrices that we can guarantee will never have complex eigenvalues are symmetric matrices.
- On complex numbers/vectors:
  - $z = a + bi$ and $\bar{z} = a - bi$, where $a, b \in \mathbb{R}$, $i = \sqrt{-1}$. Note that $\bar{z}$ is the **complex conjugate** of z.
  - $z\bar{z} = a^2 + b^2$.
  - $x = \begin{bmatrix} a_1 + b_1 i \\ \vdots \\ a_n + b_n i \end{bmatrix}$ and $\bar{x} = \begin{bmatrix} a_1 - b_1 i \\ \vdots \\ a_n - b_n i \end{bmatrix}$.
- Prove that when $A = A^{\mathrm{T}}$, $\lambda_n \in \mathbb{R}$.
  - Let $A = A^{\mathrm{T}}$, $A \in \mathbb{R}^n$.

$$\begin{aligned} Ax &= \lambda x \qquad (3) \\ \bar{A}\bar{x} &= \bar{\lambda}\bar{x} \end{aligned}$$

  - If $A \in \mathbb{R}^n$, then $A = \bar{A}$.

$$\begin{aligned} A\bar{x} &= \bar{\lambda}\bar{x} \\ (A\bar{x})^{\mathrm{T}} &= \left(\bar{\lambda}\bar{x}\right)^{\mathrm{T}} \\ \bar{x}^{\mathrm{T}} A^{\mathrm{T}} &= \bar{\lambda}\bar{x}^{\mathrm{T}} \\ \bar{x}^{\mathrm{T}} A &= \bar{\lambda}\bar{x}^{\mathrm{T}} \qquad (4) \end{aligned}$$

  - Multiply Equation 3 by $\bar{x}^{\mathrm{T}}$ from the left.
    - $\bar{x}^{\mathrm{T}} A x = \lambda \bar{x}^{\mathrm{T}} x$.
  - Multiply Equation 4 by x from the right.
    - $\bar{x}^{\mathrm{T}} A x = \bar{\lambda} \bar{x}^{\mathrm{T}} x$.
  - $\lambda \bar{x}^{\mathrm{T}} x = \bar{\lambda} \bar{x}^{\mathrm{T}} x \Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}$.

Spectral Decomposition

2/20:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

- $\lambda_1 = 4,\ \lambda_2 = \lambda_3 = 1$.
- $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $x_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.
  - $x_1^{\mathrm{T}} x_2 = 0$, $x_1^{\mathrm{T}} x_3 = 0$, $x_2^{\mathrm{T}} x_3 = -1$.
- Orthogonalize by Gram-Schmidt, inspection, put the vectors in a matrix and find the null space (the null vector will be orthogonal by the fundamental theorem).
- $x_3' = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$.
  - x_3' is not scaled along its line by A, but it is scaled in the plane of x_2 and x_3 by A.

  - $x_1^{\mathrm{T}} x_3' = 0$, $x_2^{\mathrm{T}} x_3' = 0$.
- $q_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $q_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$.

$$\begin{aligned} A &= Q\Lambda Q^{\mathrm{T}} \\ &= \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} — & q_1^{\mathrm{T}} & — \\ & \vdots & \\ — & q_n^{\mathrm{T}} & — \end{bmatrix} \\ &= \begin{bmatrix} | & & | \\ q_1\lambda_1 & \cdots & q_n\lambda_n \\ | & & | \end{bmatrix} \begin{bmatrix} — & q_1^{\mathrm{T}} & — \\ & \vdots & \\ — & q_n^{\mathrm{T}} & — \end{bmatrix} \\ &= \lambda_1 q_1 q_1^{\mathrm{T}} + \cdots + \lambda_n q_n q_n^{\mathrm{T}} \end{aligned}$$

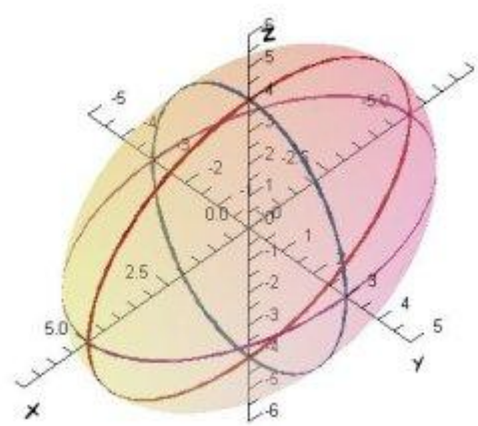
- $q_1 q_1^{\mathrm{T}} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac13 & \frac13 & \frac13 \\ \frac13 & \frac13 & \frac13 \\ \frac13 & \frac13 & \frac13 \end{bmatrix}$
- $q_2 q_2^{\mathrm{T}} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac12 & 0 & -\frac12 \\ 0 & 0 & 0 \\ -\frac12 & 0 & \frac12 \end{bmatrix}$
- $q_3 q_3^{\mathrm{T}} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac16 & -\frac13 & \frac16 \\ -\frac13 & \frac23 & -\frac13 \\ \frac16 & -\frac13 & \frac16 \end{bmatrix}$
- $A = 4 \begin{bmatrix} \frac13 & \frac13 & \frac13 \\ \frac13 & \frac13 & \frac13 \\ \frac13 & \frac13 & \frac13 \end{bmatrix} + \begin{bmatrix} \frac12 & 0 & -\frac12 \\ 0 & 0 & 0 \\ -\frac12 & 0 & \frac12 \end{bmatrix} + \begin{bmatrix} \frac16 & -\frac13 & \frac16 \\ -\frac13 & \frac23 & -\frac13 \\ \frac16 & -\frac13 & \frac16 \end{bmatrix}$
- $A = 4 \begin{bmatrix} \frac13 & \frac13 & \frac13 \\ \frac13 & \frac13 & \frac13 \\ \frac13 & \frac13 & \frac13 \end{bmatrix} + \begin{bmatrix} \frac23 & -\frac13 & -\frac13 \\ -\frac13 & \frac23 & -\frac13 \\ -\frac13 & -\frac13 & \frac23 \end{bmatrix}$ is the spectral decomposition of A.

QUADRIC SURFACES

- A **quadric surface** is the graph of a second-degree equation in three variables: x, y, z
- Its **general form** equation is $Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$ for which $A, B, C, \ldots, J$ are constants
- Through appropriate translation and rotation the general form may be reduced to either $Ax^2 + By^2 + Cz^2 + J = 0$ or $Ax^2 + By^2 + Iz = 0$
- Quadric surfaces are analogous in three dimensions to the conic sections in two dimensions

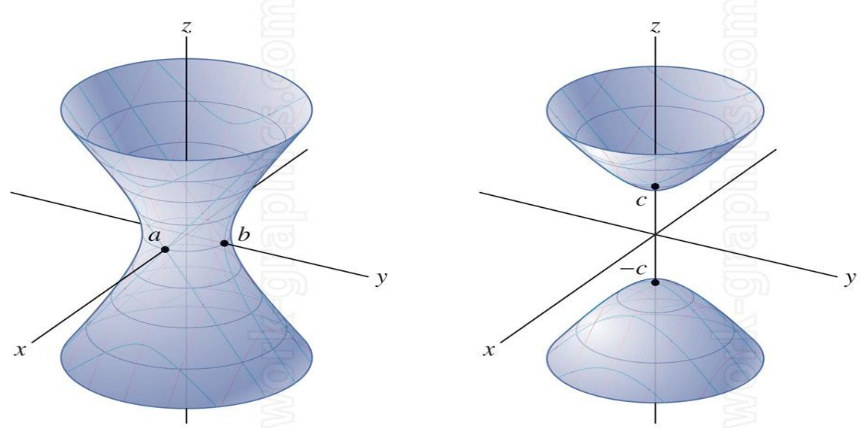
- In order to sketch the graph of a quadric surface (or any surface for that matter) it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes. These are called the **traces**

ELLIPSOIDS



- **Ellipsoids** have elliptical traces
- Equations are in the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ (Equation 1)

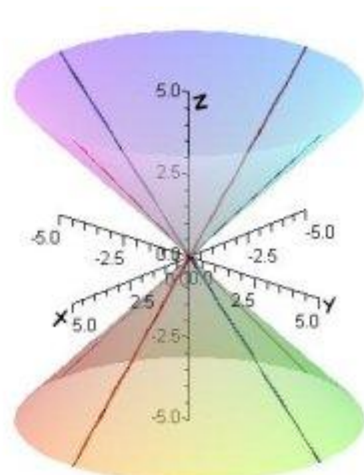
HYPERBOLOIDS



(A) Hyperboloid of one sheet (B) Hyperboloid of two sheets

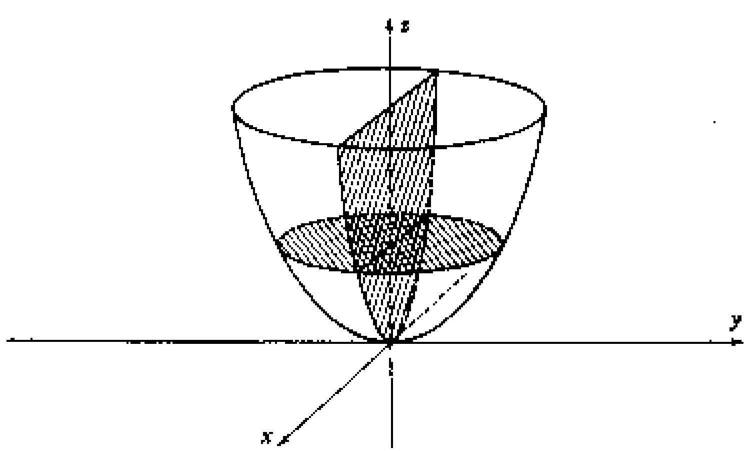
- Equations for (A) are of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ (Equation 2)
- Horizontal traces are ellipses
- Vertical traces are hyperbolas
- The axis of symmetry corresponds to the variables whose coefficient is negatives
- Equations for (B) are of the form $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ (Equation 3)
- Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$
- Vertical traces are hyperbolas
- The two negative signs indicate the two sheets

CONES



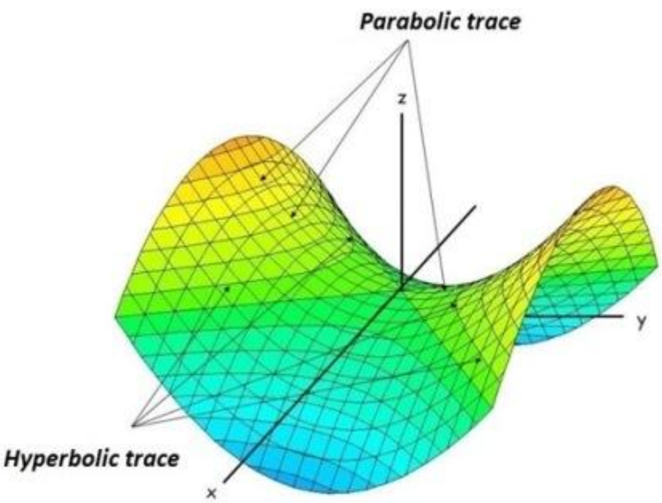
- Equations of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c}$ (Equation 4)
- Horizontal traces are ellipses
- Vertical lines in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$

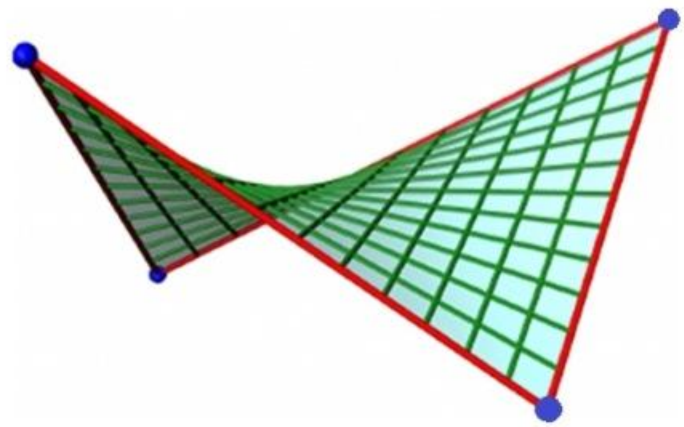
ELLIPTIC PARABOLOIDS



- Equations of the form $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ (Equation 5)
- Horizontal traces are ellipses
- Vertical traces are parabolas
- The variables raised to the first power indicates the axis of the paraboloid

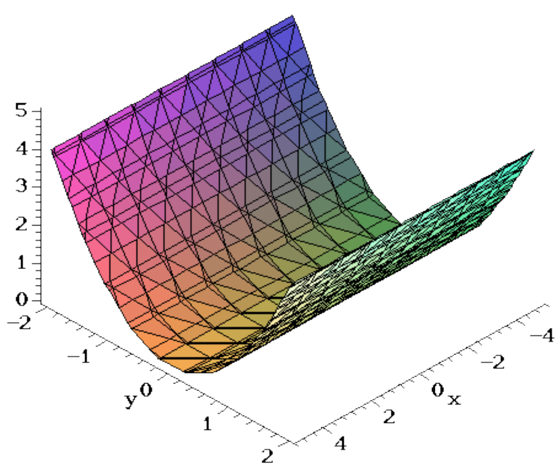
HYPERBOLIC PARABOLOIDS

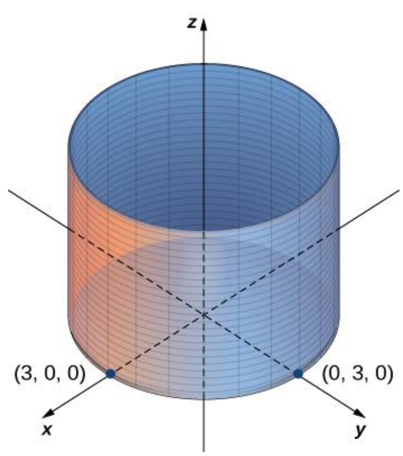




- Equations of the form $\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ (Equation 6)
- Horizontal traces are hyperbolas
- Vertical traces are parabolas

QUADRIC CYLINDERS





Parabolic Cylinder **Elliptic Cylinder**

- Equations of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (Equation 7)

Example: Use traces to sketch the quadric surface $x^2+\frac{y^2}{9}+\frac{z^2}{4}=1$

- Let $z=0$ to see that the trace in the $xy-$plane is $x^2+\frac{y^2}{9}=1$, an ellipse.
- Generally speaking, the horizontal trace in plane $z=k$ is $x^2+\frac{y^2}{9}=1-\frac{k^2}{4}$ which is an ellipse provided that $k^2<4$, or $-2<k<2$.
- Vertical traces are also ellipses: $$\frac{y^2}{9}+\frac{z^2}{4}=1-k^2, x=k, -1<k<1$$ $$x^2+\frac{z^2}{4}=1-\frac{k^2}{9}, y=k, -3<k<3$$

Example: Use traces to sketch $z=4x^2+y^2$

- Let $x=0$ to see $z=y^2$, indicating that the $xy-$plane intersects the surface in a parabola
- Let $x=k$ to yield $z=y^2+4k^2$, indicating that if we were to slice the graph with any plane parallel to the $yz-$plane we obtain a parabola that opens upward
- Let $y=k$ to yield $z=4x^2+k^2$, indicating another parabola opening upward
- Let $z=k$ to yield horizontal traces $4x^2+y^2=k$, which are ellipse.

Example: Sketch the surface $z=y^2-x^2$

- The traces in vertical planes $x=k$ are parabolas $z=y^2-k^2$,which open upward
- The traces in $y=k$ are parabolas $z=-x^2+k^2$, which open downward
- Horizontal traces are $y^2-x^2=k$, a family of hyperbolas
- Fitting together all these traces, we have a hyperbolic paraoloid

Example: Sketch the surface $\frac{x^2}{4}+y^2-\frac{z^2}{4}=1$

- The trace in any horizontal plane $z=k$ is the ellipse $\frac{x^2}{4}+y^2=1+\frac{k^2}{4}, z=k$
- Traces in $xz-$and $yz-$planes are the hyperbolas $\frac{x^2}{4}-\frac{z^2}{4}=1, y=0$ and $y^2-\frac{z^2}{4}=1, z=0$

Example : Identify and sketch the surface $4x^2 - y^2 + 2z^2 + 4 = 0$.

- First divide the equation by -4 to put it in its standard form: $-x^2 + \frac{y^2}{4} - \frac{z^2}{2} = 1$
- Next match its form with one of the seven equations above. Notice its standard form resembles Equation 3: $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$; this is a hyperboloid of two sheets. Notice the difference is that since the y^2 term of our equation is positive, this hyperboloid will be formed along the y-axis. Equation 3 has a positive z^2 term meaning it would be formed along the z-axis
- This means we will need to sketch the traces of the hyperboloid along the xy-plane and the yz-plane
- For the xy-plane set $z = 0$ and sketch the hyperbola $-x^2 + \frac{y^2}{4} = 1$
- For the yz-plane set $x = 0$ and sketch the hyperbola $\frac{y^2}{4} - \frac{z^2}{2} = 1$
- Since this hyperboloid is formed along the y-axis, there will be no xz-plane traces
- Set $y = k$
- $-x^2 + \frac{k^2}{4} - \frac{z^2}{2} = 1 \to x^2 + \frac{z^2}{4} = \frac{k^2}{4} - 1$
- Dividing each term in the equation above yields $\frac{x^2}{\frac{k^2}{4} - 1} + \frac{z^2}{2\left(\frac{k^2}{4} - 1\right)} = 1$; This is the trace the plane $y = k$ and

  we can see it is in the form of an ellipse

Try sketching the following surfaces: (a) $x^2+2z^2-6x-y+10=0$ (Hint: complete the square)

(b) $z^2=4x^2+9y^2+36$ (c) $4x^2+y^2+4z^2-4y-24z+36=0$ (d) $x^2-y^2+z^2-2x+2y+4z+2=0$

Quadric Forms / Positive Definite Matrices

2/24:

- Motivation: Find bases for conic sections that eliminate rotation terms.
- Let $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $A = \begin{bmatrix} a & \frac{d}{2} & \frac{e}{2} \\ \frac{d}{2} & b & \frac{f}{2} \\ \frac{e}{2} & \frac{f}{2} & c \end{bmatrix}$.

$$\begin{aligned}
x^{\mathrm{T}}Ax &= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & \frac{d}{2} & \frac{e}{2} \\ \frac{d}{2} & b & \frac{f}{2} \\ \frac{e}{2} & \frac{f}{2} & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\
&= \begin{bmatrix} ax + \frac{d}{2}y + \frac{e}{2}z & \frac{d}{2}x + by + \frac{f}{2}z & \frac{e}{2}x + \frac{f}{2}y + zc \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\
&= x\left(ax + \frac{d}{2}y + \frac{e}{2}z\right) + y\left(\frac{d}{2}x + by + \frac{f}{2}z\right) + z\left(\frac{e}{2}x + \frac{f}{2}y + zc\right) \\
&= ax^2 + \frac{d}{2}xy + \frac{e}{2}xz + \frac{d}{2}xy + by^2 + \frac{f}{2}yz + \frac{e}{2}xz + \frac{f}{2}yz + cz^2 \\
&= ax^2 + by^2 + cz^2 + dxy + exz + fyz
\end{aligned}$$

- Example:

$$f(x, y) = z = 5x^2 + 4xy + 2y^2$$

  - An elliptic paraboloid (but not necessary to know this).

$$\begin{aligned}
x^{\mathrm{T}}Ax &= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 5 & \frac{4}{2} \\ \frac{4}{2} & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}
\end{aligned}$$

  - Let $x = Qy \Rightarrow y = Q^{\mathrm{T}}x$.
  - If $A = A^{\mathrm{T}}$, then $A = Q\Lambda Q^{\mathrm{T}} \Rightarrow \Lambda = Q^{\mathrm{T}}AQ$.

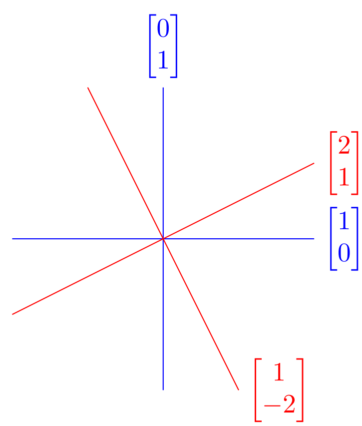
$$\begin{aligned}
x^{\mathrm{T}}Ax &= (Qy)^{\mathrm{T}}A(Qy) \\
&= y^{\mathrm{T}}Q^{\mathrm{T}}AQy \\
&= y^{\mathrm{T}}\Lambda y
\end{aligned}$$

  - Diagonalize A.

$$\begin{aligned}
0 &= |A - \lambda I| \\
&= \begin{vmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} \\
&= (5-\lambda)(2-\lambda) - 4 \\
&= \lambda^2 - 7\lambda + 6
\end{aligned}$$

$$\lambda_1 = 1 \qquad\qquad \lambda_2 = 6$$

- $N(A-I) = N\left(\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\right) = \left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}$
- $N(A-6I) = N\left(\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}\right) = \left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$
- $q_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix}$
- $q_2 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$
- On standard basis: $f(x,y) = 5x^2 + 4xy + 4y^2$.
- On basis Q: $f(x,y) = x^2 + 6y^2$.
- We're rotating the x and y axes by the same angle and keeping them orthogonal:



- How are they equivalent?
  - Let $x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$
  - $x^{\mathrm{T}}Ax = 5(1)^2 + 4(1)(3) + 2(3^2) = 35$.
  - $y = Q^{\mathrm{T}}x = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{5}{\sqrt{5}} \\ \frac{5}{\sqrt{5}} \end{bmatrix} = \begin{bmatrix} -\sqrt{5} \\ \sqrt{5} \end{bmatrix}$
  - $y^{\mathrm{T}}\Lambda y = \left(-\sqrt{5}\right)^2 + 6\left(\sqrt{5}\right)^2 = 35$.

Sketching a Rotated Conic

2/25:

- Principal Axes Theorem: $x^{\mathrm{T}}Ax \to y^{\mathrm{T}}\Lambda y$.
- Consider $13x^2 - 10xy + 13y^2 - 72 = 0$.
  - We want to reexpress this using the Principal Axes Theorem to deal with the rotation term.
  - $x^{\mathrm{T}}Ax = 72 = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 13 & -5 \\ -5 & 13 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$
  - $0 = |A - \lambda I| = (\lambda - 8)(\lambda - 18)$.

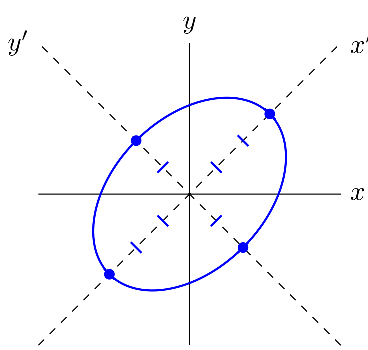
$$\lambda_1 = 8 \qquad\qquad \lambda_2 = 18$$

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad\qquad x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

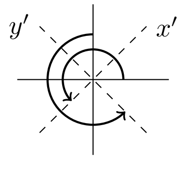
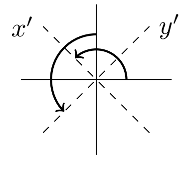
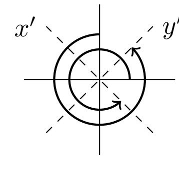
- Let $y = \begin{bmatrix} x' \\ y' \end{bmatrix}$, $y^{\mathrm{T}}\Lambda y = \begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 18 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 72$.
- $8(x')^2 + 18(y')^2 = 72$.
- $\left(\frac{x'}{3}\right)^2 + \left(\frac{y'}{2}\right)^2 = 1$.

- Sidenote on rotation matrices:
  - No rotation (rotates $\begin{bmatrix} 0^\circ & 0^\circ \\ 0^\circ & 0^\circ \end{bmatrix}$): $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
    - Identity matrix is special case of rotation matrix when $\theta = 0^\circ$.
  - Any rotation matrix has $|R| = 1$ because otherwise, it would do something to the size.

- Back with the example:
  - $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$
  - $|Q| = 1$.
  - When Q is set equivalent to the general rotation matrix (above), it is shown that $\theta = \frac{\pi}{4}$.



  - If the sign of $|Q|$ is wrong, flip the columns (flips the sign of the determinant).

λ's	$\lambda_1 = 8,\ \lambda_2 = 18$	$\lambda_1 = 18,\ \lambda_2 = 8$	$\lambda_1 = 18,\ \lambda_2 = 8$
Q	$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$	$\begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$
θ	$\frac{5\pi}{4}$	$\frac{3\pi}{4}$	$\frac{7\pi}{4}$
Graph			
Eq.	$\left(\frac{x'}{3}\right)^2 + \left(\frac{y'}{2}\right)^2 = 1$	$\left(\frac{x'}{2}\right)^2 + \left(\frac{y'}{3}\right)^2 = 1$	$\left(\frac{x'}{2}\right)^2 + \left(\frac{y'}{3}\right)^2 = 1$

- Classification of symmetric matrices:

Classification	$x^{\mathrm{T}}Ax$	λ's
Positive definite	$f = x^{\mathrm{T}}Ax,\ f > 0$	All λ's > 0
Positive semidefinite	$f \geq 0$	All λ's ≥ 0
Negative definite	$f < 0$	All λ's < 0
Negative semidefinite	$f \leq 0$	All λ's ≤ 0
Indefinite	f is positive and negative	λ's $\in \mathbb{R}$

Nondiagonalizable Matrices

Perturbations

2/26: • If it's nondiagonalizable, how can we get it to as close to diagonalizable as possible? If that doesn't help, how can we get it into a useful form?

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

- A is nondiagonalizable because the eigenvalues are sitting on the principal diagonal already, and they're the same. We want distinct eigenvalues.
- Let ϵ be a very small, approaching zero quantity.

$$A_\epsilon = \begin{bmatrix} 1 & 1 \\ \epsilon^2 & 1 \end{bmatrix}$$

  - Choice of ϵ^2 vs. ϵ helps when diagonalizing.

$$\begin{aligned} 0 &= |A_\epsilon - \lambda I| \\ &= \begin{vmatrix} 1-\lambda & 1 \\ \epsilon^2 & 1-\lambda \end{vmatrix} \\ &= (1-\lambda)^2 - \epsilon^2 \\ &= 1 - 2\lambda + \lambda^2 - \epsilon^2 \\ &= \lambda^2 - 2\lambda + (1-\epsilon)^2 \\ &= (\lambda - (1+\epsilon))(\lambda - (1-\epsilon)) \end{aligned}$$

  - Quadratic formula may be helpful for factoring.
  - Alternatively, think of it as a kind of difference of squares with a middle component.

$$\lambda_1 = 1 + \epsilon \qquad\qquad \lambda_2 = 1 - \epsilon$$

- $A_\epsilon - \lambda_1 I = \begin{bmatrix} -\epsilon & 1 \\ \epsilon^2 & -\epsilon \end{bmatrix}$, $x_1 = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix}$
- $A_\epsilon - \lambda_2 I = \begin{bmatrix} \epsilon & 1 \\ \epsilon^2 & \epsilon \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ -\epsilon \end{bmatrix}$
- Find S^{-1}.
  - $S^{-1} = \frac{1}{|S|}C^{\mathrm{T}} = -\frac{1}{2\epsilon}\begin{bmatrix} -\epsilon & -1 \\ -\epsilon & 1 \end{bmatrix}$
- $A_\epsilon = S\Lambda S^{-1} = -\frac{1}{2\epsilon}\begin{bmatrix} 1 & 1 \\ \epsilon & -\epsilon \end{bmatrix}\begin{bmatrix} 1+\epsilon & 0 \\ 0 & 1-\epsilon \end{bmatrix}\begin{bmatrix} -\epsilon & -1 \\ -\epsilon & 1 \end{bmatrix}$

- $A_\epsilon^n = -\frac{1}{2\epsilon}\begin{bmatrix}1 & 1\\ \epsilon & -\epsilon\end{bmatrix}\begin{bmatrix}(1+\epsilon)^n & 0\\ 0 & (1-\epsilon)^n\end{bmatrix}\begin{bmatrix}-\epsilon & -1\\ -\epsilon & 1\end{bmatrix}$
- Goal: In A, $\epsilon^2 = 0$. So we want $\lim_{\epsilon\to 0} A_\epsilon^n$.

$$\begin{aligned}
\lim_{\epsilon\to 0} A_\epsilon^n &= \lim_{\epsilon\to 0} -\frac{1}{2\epsilon}\begin{bmatrix}1 & 1\\ \epsilon & -\epsilon\end{bmatrix}\begin{bmatrix}(1+\epsilon)^n & 0\\ 0 & (1-\epsilon)^n\end{bmatrix}\begin{bmatrix}-\epsilon & -1\\ -\epsilon & 1\end{bmatrix}\\
&= \lim_{\epsilon\to 0} -\frac{1}{2\epsilon}\begin{bmatrix}1 & 1\\ \epsilon & -\epsilon\end{bmatrix}\begin{bmatrix}-\epsilon(1+\epsilon)^n & -(1+\epsilon)^n\\ -\epsilon(1-\epsilon)^n & (1-\epsilon)^n\end{bmatrix}\\
&= \lim_{\epsilon\to 0} -\frac{1}{2\epsilon}\begin{bmatrix}-\epsilon(1+\epsilon)^n - \epsilon(1-\epsilon)^n & -(1+\epsilon)^n + (1-\epsilon)^n\\ -\epsilon^2(1+\epsilon)^n + \epsilon^2(1-\epsilon)^n & -\epsilon(1+\epsilon)^n - \epsilon(1-\epsilon)^n\end{bmatrix}\\
&= \lim_{\epsilon\to 0}\begin{bmatrix}\frac{(1+\epsilon)^n+(1-\epsilon)^n}{2} & \frac{(1+\epsilon)^n-(1-\epsilon)^n}{2\epsilon}\\ \frac{\epsilon(1+\epsilon)^n-\epsilon(1-\epsilon)^n}{2} & \frac{(1+\epsilon)^n+(1-\epsilon)^n}{2}\end{bmatrix}\\
&= \begin{bmatrix}1 & \lim_{\epsilon\to 0}\frac{(1+\epsilon)^n-(1-\epsilon)^n}{2\epsilon}\\ 0 & 1\end{bmatrix}
\end{aligned}$$

  - Use L'Hôpital's rule for the last limit.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left((1+\epsilon)^n - (1-\epsilon)^n\right) &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}(1+\epsilon)^n - \frac{\mathrm{d}}{\mathrm{d}\epsilon}(1-\epsilon)^n\\
&= n(1+\epsilon)^{n-1}\frac{\mathrm{d}}{\mathrm{d}\epsilon}(1+\epsilon) - n(1-\epsilon)^{n-1}\frac{\mathrm{d}}{\mathrm{d}\epsilon}(1-\epsilon)\\
&= n(1+\epsilon)^{n-1}(1) - n(1-\epsilon)^{n-1}(-1)\\
&= n(1+\epsilon)^{n-1} + n(1-\epsilon)^{n-1}\\
&= n\left((1+\epsilon)^{n-1} + (1-\epsilon)^{n-1}\right)
\end{aligned}$$

$$\begin{aligned}
\lim_{\epsilon\to 0}\frac{(1+\epsilon)^n - (1-\epsilon)^n}{2\epsilon} &= \lim_{\epsilon\to 0}\frac{n\left((1+\epsilon)^{n-1} + (1-\epsilon)^{n-1}\right)}{2}\\
&= \frac{2n}{2}\\
&= n
\end{aligned}$$

  - Thus, we find the following final formula for A^n.

$$A^n = \begin{bmatrix}1 & n\\ 0 & 1\end{bmatrix}$$

Jordan Canonical Form

2/27: - If A is diagonalizable, $A = S\Lambda S^{-1}$.

  - $\Lambda = \begin{bmatrix}\lambda_1 & & & 0\\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n\end{bmatrix}$

- If A is nondiagonalizable, $A = MJM^{-1}$[7].

  - $J = \begin{bmatrix}\lambda_1 & 1 & & 0\\ & \lambda_2 & & \\ & & \ddots & 1\\ 0 & & & \lambda_n\end{bmatrix}$

[7] J for Jordan matrix.

- Has some 1s directly above the principal diagonal, notably above missing eigenvectors.
- Example: $J = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{bmatrix}$
- Example: $J = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_3 \end{bmatrix}$
- Note that if a Jordan form factorization is necessary, at least two λ_1, λ_2, and λ_3 are the same value.

Eigenvectors	**Generalized Eigenvectors / Power Vectors**
$Ax = \lambda x$	$(A - \lambda I)^p x_m = 0$
$(A - \lambda I)x_s = 0$	
x_s is an eigenvector.	

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

- $\lambda_1 = \lambda_2 = 2$.
- $A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, so $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. This is an eigenvector.
- Now, note that $A - \lambda I$ raised to a power p will eventually have a null space different from that of $A - \lambda I$ for some $p > 1$ and $p \in \mathbb{N}$.
  - Also note that $p \leq n$ for an n-square matrix.
  - This implies that there exists a power vector distinct from the eigenvector(s) for some power p.
- Let's find that power vector.
- $(A - 2I)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so we can define a power vector $x_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This P.V. has degree 2.
- Note that A is already a Jordan matrix, so it makes sense that the eigenvectors/power vectors form the identity matrix. In other words, $M = M^{-1} = I_2$.
- Let's try another, less immediately clear example.

$$A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$$

- Since A is neither upper triangular nor diagonal, we must derive the characteristic polynomial to find the eigenvectors.

$$\begin{aligned} 0 &= |A - \lambda I| \\ &= (3 - \lambda)(1 - \lambda) + 1 \\ &= \lambda^2 - 4\lambda + 4 \\ &= (\lambda - 2)^2 \end{aligned}$$

$$\lambda_1 = \lambda_2 = 2$$

- Since the algebraic multiplicity is 2 and the geometric multiplicity is only 1, find the one possible eigenvector.
  - $A - 2I = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$, so $x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. This is an E.V.

- We can also find a power vector.
  - $(A-2I)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. This is a P.V. of degree 2.
- We know that $Ax_1 = 2x_1$. However, in order to finish finding J, we need to (the reason will soon become clear) find Ax_2 in terms of x_2 and/or x_1. This can be done as follows.

$$\begin{aligned}
(A-2I)^2 x_2 &= 0 \\
(A-2I)(A-2I)x_2 &= (A-2I)x_1 \\
(A-2I)x_2 &= x_1 \\
Ax_2 - 2Ix_2 &= x_1 \\
Ax_2 - 2x_2 &= x_1 \\
Ax_2 &= x_1 + 2x_2
\end{aligned}$$

  - Alternatively, we can think of starting from $(A-2I)x_2 = x_1$ because we need to introduce the coefficient of 1 for one of the eigenvectors somehow.
  - What combination of $A - \lambda I$ gives us another eigenvector?
- Combine this result with $Ax_1 = 2x_1$ to yield the following set of equations.

$$\begin{aligned}
Ax_1 &= 2x_1 + 0x_2 \\
Ax_2 &= x_1 + 2x_2
\end{aligned}$$

- Express the set of equations above explicitly in terms of the matrices that the variables represent.

$$\begin{aligned}
\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} &= 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= 1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\end{aligned}$$

- Rewrite the right side of the above set of equations to condense linear operations into matrix multiplication.

$$\begin{aligned}
\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\end{aligned}$$

- Combine the two equations into one, giving $AM = MJ$.

$$\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

- Now, we have the Jordan matrix (the rightmost matrix above).
- Note that the following algebra would give us the MJM^{-1} factorization of A.

$$\begin{aligned}
AM &= MJ \\
AMM^{-1} &= MJM^{-1} \\
A &= MJM^{-1}
\end{aligned}$$

- Also note that if we find a coefficient matrix from the original set of equations and transpose it, we will get the Jordan matrix:

$$\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}^{\mathrm{T}} = J = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

- Let's try a bigger example.

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

- $\lambda_1 = \lambda_2 = \lambda_3 = 3$
- A.M. = 3, G.M. = 2.
- $A - 3I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$
- $x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$
- Because $(A - 3I)^2 = 0_3$ and because of the equivalence outlined in the previous example, we can use one of the following to find an x_3 that will suit out needs.
  - $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} | \\ x_3 \\ | \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} | \\ x_3 \\ | \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$
- The first option will not work because no x_3 vector can get a 1 in the top position of x_1.
- For the second one, use inspection: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$
  - x_3 must be a power vector of $A - 3I$ and must be independent of x_1 and x_2, so that M is invertible.
- Find Ax_3 in terms of x_1, x_2, and x_3.
  - $(A - 3I)x_3 = x_2 \Rightarrow Ax_3 - 3x_3 = x_2 \Rightarrow Ax_3 = x_2 + 3x_3$.
- Write the three equations as a set.

$$\begin{aligned} Ax_1 &= 3x_1 + 0x_2 + 0x_3 \\ Ax_2 &= 0x_1 + 3x_2 + 0x_3 \\ Ax_3 &= 0x_1 + 1x_2 + 3x_3 \end{aligned}$$

- Instead of going through converting this set of equations to matrix equations, simply take a coefficient matrix and transpose it to find J:
  - $J = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$
- $A = MJM^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$
- Raising J^n is easier than raising A^n, but it's still not easy.
- Note that $A^n = MJ^nM^{-1}$ and $J^n = \begin{bmatrix} 3^n & 0 & 0 \\ 0 & 3^n & n3^{n-1} \\ 0 & 0 & 3^n \end{bmatrix}$, as can be proved via induction.
  - In other words, this factorization is useful.
  - The MJ^nM^{-1} factorization could also be condensed into a single matrix.

Computational Complexity of the Jordan Canonical Form

3/2:

- Recall:
  - Eigenvector: $Ax - \lambda x \Rightarrow A^n x_i = \lambda_i^n x_i \Rightarrow (A - \lambda I)x = 0$.
  - Power vector: $(A - \lambda I)^n x = 0$.
- Binomial expansion:
  - Let $\binom{n}{r} = {}_nC_r = \frac{n!}{r!(n-r)!}$.
  - Theorem:

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

- Apply binomial expansion to finding A^n.

$$\begin{aligned}A^n &= (A - \lambda I + \lambda I)^n \\ &= ((A - \lambda I) + (\lambda I))^n \\ &= \binom{n}{0}(A - \lambda I)^n + \binom{n}{1}(A - \lambda I)^{n-1}(\lambda I) + \binom{n}{2}(A - \lambda I)^{n-2}(\lambda I)^2 + \cdots + \binom{n}{n}(\lambda I)^n\end{aligned}$$

- Now consider each term in the above expansion of A^n.
- Consider a degree 2 power vector.
  - Let x be a power vector of degree 2.
    - Note that we are doing the binomial expansion in the reverse order in the following equation.

$$(A - \lambda I)^n x = \lambda^n I x + n\lambda^{n-1}(A - \lambda I)x + \binom{n}{2}\lambda^{n-1}(A - \lambda I)^2 x + \cdots + \binom{n}{n}(A - \lambda I)^n x$$

  - Because x is a second-degree power vector and the blue term includes $(A - \lambda I)^2 x$, the blue term equals zero.
  - Furthermore, every subsequent term (the green terms) also equals zero. This is because the $(A - \lambda I)^k x$, $k \in [3, n] \cap \mathbb{N}$, term can always be factored into $(A - \lambda I)^{k-2}(A - \lambda I)^2 x$, and, as we know, $(A - \lambda I)^2 x = 0$. If one part of the term is equal to zero, the whole term must be equal to zero as well.
  - The implication is that only the red term remains. More succinctly:

$$(A - \lambda I)^n x = \lambda^n I x + n\lambda^{n-1}(A - \lambda I)x$$

- Let's generalize this to a degree p power vector.
  - Let x be a power vector of degree p.

$$(A - \lambda I)^n x = \lambda^n I x + n\lambda^{n-1}(A - \lambda I)x + \cdots + \binom{n}{p-1}\lambda^{n-(p-1)}(A - \lambda I)^{p-1}x$$

- Basically, the higher the degree of the power vector, the harder it is to compute the JCF.
- Note: $\mathrm{e}^{(A - \lambda I + \lambda I)t} = \mathrm{e}^{\lambda I t}\mathrm{e}^{(A - \lambda I)t} = \mathrm{e}^{\lambda t}\left(\sum_{n=0}^{\infty}\frac{(A - \lambda I)^n t^n}{n!}\right)$.

Singular Value Decomposition

3/5:
- Let's deal with rectangular matrices.
- What we know:
  - Regardless of the shape of $A_{m\times n}$, $A^\mathrm{T}A$ will be n-square and symmetric.
    - From several chapters ago: $(A^\mathrm{T}A)^\mathrm{T} = A^\mathrm{T}A^\mathrm{TT} = A^\mathrm{T}A$.
  - A symmetric matrix has a full set of real eigenvectors (guaranteed by spectral theorem).
  - If $A = A^\mathrm{T}$, then $A = Q\Lambda Q^\mathrm{T}$ (A is orthonormally diagonalizable).
- We will show:
  - Any $A_{m\times n} = QD^\dagger Q'^\mathrm{T}$.
    - $D^\dagger$ is pseudo-diagonal (close to diagonal).
    - Q and Q'^T[[8]] are slightly different sets of eigenvectors.
- Singular values:
  - $A^\mathrm{T}A$ is symmetric.
  - All λ's are real.
  - All eigenvectors ($(A-\lambda I)x = 0$) can be chosen orthonormally.
  - All λ's are positive (shown by the following).
- Let x be a unit[9] eigenvector of $A^\mathrm{T}A$ with corresponding eigenvalue λ.
  - Symbolically, $A^\mathrm{T}Ax = \lambda x$.

$$\begin{aligned}
0 &\leq ||Ax||^2 \\
&= (Ax)^\mathrm{T}(Ax) \\
&= x^\mathrm{T}A^\mathrm{T}Ax \\
&= x^\mathrm{T}\lambda x \\
&= \lambda x^\mathrm{T}x \\
&= \lambda(1) \\
&= \lambda
\end{aligned}$$

- Singular values: $\sigma_i = \sqrt{\lambda_i}$.
- By convention, we list the singular values in descending order: $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$.
- Let's try an example.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  - Find $A^\mathrm{T}A$.

$$A^\mathrm{T}A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  - Find the eigenvectors of $A^\mathrm{T}A$.

$$\lambda_1 = 2 \qquad\qquad \lambda_2 = 1 \qquad\qquad \lambda_3 = 0$$

[8] Que prime transpose.

[9] Note that we are guaranteed that x can have $||x|| = 1$ because we are guaranteed orthonormal eigenvectors.

- Find the singular values of $A^{\mathrm{T}}A$.

$$\sigma_1 = \sqrt{2} \qquad\qquad \sigma_2 = 1 \qquad\qquad \sigma_3 = 0$$

3/6:

- Let's step back for a second.
- The SVD is canonically written $A = U\Sigma V^{\mathrm{T}}$.
  - If A is $m \times n$:
    - U is $m \times m$ and orthonormal.
    - Σ is $m \times n$.
    - V is $n \times n$ and orthonormal.
- Σ has the following form.

$$\Sigma = \begin{array}{c} \begin{array}{cc} r & n-r \end{array} \\ \left[\begin{array}{ccc|c} \sigma_1 & & 0 & \\ & \sigma_2 & & 0 \\ 0 & & \ddots & \\ \hline & 0 & & 0 \end{array}\right] \end{array} \begin{array}{l} r \\ \\ m-r \end{array}$$

  - All of the nonzero singular values go diagonally in the upper-left block.
  - Zeroes everywhere else.
- Continuing with the previous example:
  - For V, form an orthonormal basis of eigenvectors of $A^{\mathrm{T}}A$.
  - $x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $x_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $x_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$
  - $V = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}$
  - Note that we put the eigenvectors in V (not U) because the eigenvectors are n-dimensional and V is the only n-square matrix (recall U is m-square).
  - For U, manipulate the eigenvectors by using them to rearrange $C(A)$.
  - Note that $\{Av_1, Av_2, \dots, Av_n\}$ is still an orthogonal basis for $\mathbb{R}^m$.
    - This is because A scales each orthogonal vector by the same amount, but brings them into $C(A) \in \mathbb{R}^m$.
    - Furthermore, the following proves that orthogonality is maintained.

$$\begin{aligned} (Av_i)^{\mathrm{T}}(Av_j) &= v_i^{\mathrm{T}}A^{\mathrm{T}}Av_j \\ &= v_i^{\mathrm{T}}\lambda v_j \\ &= \lambda v_i^{\mathrm{T}}v_j \\ &= \lambda(0) \\ &= 0 \end{aligned}$$

  - Since $\sigma_i = ||Av_i||$, set $u_i = \frac{1}{\sigma_i}Av_i$ to normalize Av_i into the correct dimension.

- Essentially, the singular values allow us to define U as follows.

$$Av_i = \sigma_i u_i$$
$$AV = U\Sigma$$

$$\begin{aligned}
A\begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} &= \begin{bmatrix} | & & | \\ Av_1 & \cdots & Av_n \\ | & & | \end{bmatrix} \\
&= \begin{bmatrix} | & & | \\ \sigma_1 u_1 & \cdots & \sigma_n u_n \\ | & & | \end{bmatrix} \\
&= \begin{bmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & & 0 \\ & & \ddots & \\ & 0 & & 0 \end{bmatrix}
\end{aligned}$$

- Note: If $A = U\Sigma V^{\mathrm{T}}$, then the following holds.

$$\begin{aligned}
A^{\mathrm{T}}A &= (U\Sigma V^{\mathrm{T}})^{\mathrm{T}}(U\Sigma V^{\mathrm{T}}) \\
&= V\sigma^{\mathrm{T}}U^{\mathrm{T}}U\Sigma V^{\mathrm{T}} \\
&= V\Sigma\Sigma V^{\mathrm{T}} \\
&= V\Sigma^2 V^{\mathrm{T}} \\
&= V\Lambda V^{\mathrm{T}}
\end{aligned}$$

- In the opposite direction:

$$\begin{aligned}
AA^{\mathrm{T}} &= (U\Sigma V^{\mathrm{T}})(U\Sigma V^{\mathrm{T}})^{\mathrm{T}} \\
&= \vdots \\
&= U\Lambda U^{\mathrm{T}}
\end{aligned}$$

- Back to the example to finish it.
- The following finds u_1. This process can be iterated for u_i.
- $u_1 = \frac{1}{\sigma_1}Av_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- Assemble the final factorization.
- $A = U\Sigma V^{\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\left[\begin{array}{cc|c} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}^{\mathrm{T}}$

Matrix Approximations / Correlations

3/10: • The importance of SVD:

- $A = U\Sigma V^{\mathrm{T}}$

$$\begin{bmatrix} | & & | & & | \\ u_1 & \cdots & u_m & \cdots & u_n \\ | & & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_m & \\ & 0 & & 0 \end{bmatrix} \begin{bmatrix} — & v_1^{\mathrm{T}} & — \\ & \vdots & \\ — & v_m^{\mathrm{T}} & — \end{bmatrix}$$

- $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m$, so the principal eigenvector corresponds to the largest singular value, and then on and on down the list.

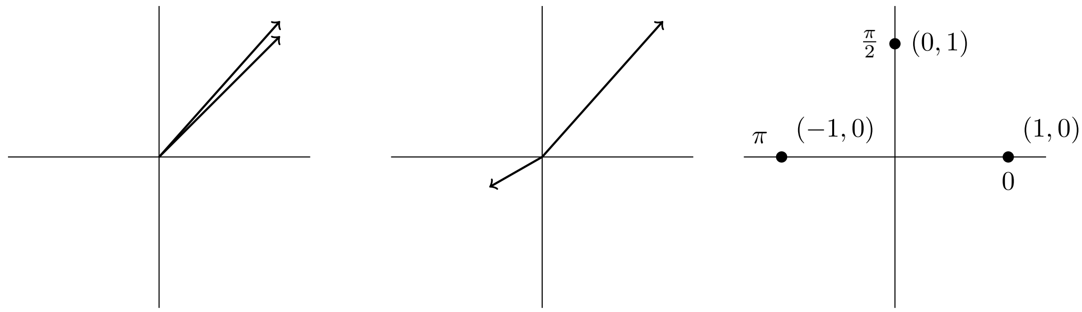
$$\begin{aligned} A &= \sigma_1 u_1 v_1^{\mathrm{T}} + \sigma_2 u_2 v_2^{\mathrm{T}} + \cdots + \sigma_m u_m v_m^{\mathrm{T}} + 0 + \cdots + 0 \\ &= \sigma_1 \begin{bmatrix} | \\ u_1 \\ | \end{bmatrix} \begin{bmatrix} — & v_1^{\mathrm{T}} & — \end{bmatrix} + \sigma_2 \begin{bmatrix} | \\ u_2 \\ | \end{bmatrix} \begin{bmatrix} — & v_2^{\mathrm{T}} & — \end{bmatrix} + \cdots \end{aligned}$$

- From a data science perspective:
- Each term above is a rank 1 matrix.
- The SVD is nothing more than a sum of rank 1 matrices.
- We order it $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m$ because this way, we add on less and less the farther along the sum we go. In other words, we can slowly build an approximation of A from rank 1 matrices via a sum. The terms at the end are not of any real importance. You can truncate the sum at a certain point, yielding an approximation from just a couple of sums.
- From an SVD, you can see the most important and strongest building blocks of A.
- This approximation is denoted $A = \hat{U}\hat{\Sigma}V^{\mathrm{T}}$ — an economy SVD.
- **Economy SVD**s ignore the extra eigenvectors in U.
  - First m columns of U.
  - First m singular values.
  - All of V^{T}
  - Not an approximation (all other eigenvectors in u are multiplied by 0s in Σ initially).
- To approximate, let $A \approx \tilde{U}\tilde{\Sigma}\tilde{V}^{\mathrm{T}}$.
  - Choose r singular values based upon dominance.

- Consider the correlation between the vectors of A:
  - A is tall and skinny.
  - Let $A = U\Sigma V^{\mathrm{T}}$.

$$\begin{aligned} A^{\mathrm{T}}A &= V\Sigma U^{\mathrm{T}}U\Sigma V^{\mathrm{T}} \\ &= V\Sigma^2 V^{\mathrm{T}} \\ A^{\mathrm{T}}A &= V\Lambda V^{\mathrm{T}} \\ A^{\mathrm{T}}AV &= V\Lambda \end{aligned}$$

  - $A^{\mathrm{T}}A = \begin{bmatrix} — & a_1^{\mathrm{T}} & — \\ & \vdots & \\ — & a_m^{\mathrm{T}} & — \end{bmatrix} \begin{bmatrix} | & & | \\ a_1 & \cdots & a_m \\ | & & | \end{bmatrix} = \begin{bmatrix} a_1^{\mathrm{T}}a_1 & \cdots & a_1^{\mathrm{T}}a_m \\ \vdots & \ddots & \\ a_m^{\mathrm{T}}a_1 & & a_m^{\mathrm{T}}a_m \end{bmatrix}$
  - $u^{\mathrm{T}}v = \cos(\theta) \cdot ||u|| \cdot ||v||$
  - The larger the dot product, the greater the correlation.



- U tells us how related all vectors in A are.
- V tells us how unrelated all vectors in A are.

- Consider a real world example. Let's imagine Netflix users 1-7 have ranked 5 films on a 0-5 scale, producing the following matrix.

$$\begin{array}{cc} & \begin{array}{ccccc} \text{LOTR} & \text{HP} & \text{Hobbit} & \text{Spiderman} & \text{Avengers} \end{array} \\ M = & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 0 & 0 \\ 4 & 4 & 4 & 0 & 0 \\ 5 & 5 & 5 & 0 & 0 \\ 0 & 2 & 0 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 1 & 0 & 2 & 2 \end{bmatrix} \begin{array}{l} \text{User 1} \\ \text{User 2} \\ \text{User 3} \\ \text{User 4} \\ \text{User 5} \\ \text{User 6} \\ \text{User 7} \end{array} \end{array}$$

  - The matrices U, Σ, and V for the SVD of M can be found to be approximately the following.

$$U = \begin{bmatrix} 0.137599 & -0.0236115 & 0.0108085 & 0 & -0.980591 & -0.121046 & -0.0649204 \\ 0.412696 & -0.0708344 & 0.0324254 & 0 & 0 & 0 & 0.907495 \\ 0.550397 & -0.0944458 & 0.0432339 & 0 & 0 & 0.786796 & -0.259281 \\ 0.687996 & -0.118057 & 0.0540424 & 0 & 0.196116 & -0.605228 & -0.324102 \\ 0.152775 & 0.591101 & -0.653651 & -0.447214 & 0 & 0 & 0 \\ 0.0722165 & 0.731312 & 0.678209 & 0 & 0 & 0 & 0 \\ 0.763875 & 0.29555 & -0.326825 & 0.894427 & 0 & 0 & 0 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 12.481 & 0 & 0 & 0 & 0 \\ 0 & 9.50861 & 0 & 0 & 0 \\ 0 & 0 & 1.34556 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 0.562258 & -0.126641 & 0.40967 & 0 & -0.707107 \\ 0.59286 & 0.0287706 & -0.804792 & 0 & 0 \\ 0.562258 & -0.126641 & 0.409667 & 0 & 0.707107 \\ 0.0901335 & 0.695376 & 0.0912571 & -0.707107 & 0 \\ 0.0901335 & 0.695376 & 0.0912571 & 0.707107 & 0 \end{bmatrix}$$

- The red entries in U show that users 1-4 liked the fantasy genre (Lord of the Rings, Harry Potter, and The Hobbit) more. On the other hand, the blue entries in U show that users 5-7 liked the superhero genre (Spiderman and The Avengers) more.
  - U correlates users with genres.
- The red entries in V show that LOTR, HP, and Hobbit represent the fantasy genre better than the others, with HP being the most well received representative. On the other hand, the blue entries in V show that Spiderman and Avengers represent the superhero genre better than the others; both do so equally well.
  - V correlates films with genres.
- The red entry in Σ represents the fantasy genre (which is liked the most). The blue entry in Σ represents the superhero genre (which is still well liked). The last nonzero entry in Σ is low enough, comparatively, to be insignificant, and is just a meaningless leftover of the SVD process.