Chapter 3: Vector Spaces (Notes 1) Forsyth

10/8: Definitions

- **Vector Space**: A set V along with addition and scalar multiplication on V such that...
- **Set**: A collection of objects, which is an object itself.
- **Symbols**:
  - $\in$: "is an element of"
  - $\forall$: "for all"/"for every"
  - $\ni$: "such that"
  - $\exists$: "there exists"
  - F: "a field"
- **Field**: A fundamental algebraic structure containing at least $0, 1, +,$ and $\times$.
- **Addition** (on a set V): A function that assigns an element $u+v \in V$ to each pair of elements $u, v \in V$.
- **Scalar multiplication** (on a set V): A function that assigns $cv \in V$ to each $c \in F$ to each $v \in V$.
- **Commutivity**:
  1. $u+v = v+u \quad \forall\, u, v \in V$
- **Associativity**:
  2. $(u+v)+w = u+(v+w) \quad \forall\, u, v, w \in V$
  3. $(ab)v = a(bv) \quad \forall\, v \in V$ AND $\forall\, a, b \in F$
- **Additive identity**:
  4. $\exists\, 0 \in V \ni v+0 = v \quad \forall\, v \in V$
- **Additive inverse**:
  5. $\forall\, v \in V, \exists\, w \in V \ni v+w = 0$
- **Multiplicative identity**:
  6. $1v = v \quad \forall\, v \in V$
- **Distributivity**:
  7. $a(v+w) = av + aw \quad \forall\, a \in F$ AND $\forall\, v, w \in V$
  8. $(a+b)v = av + bv \quad \forall\, a, b \in F$ AND $\forall\, v \in V$
- Is $\mathbb{R}^n$ a vector space? Prove all 8 are true or at least one is false.
  - $u, v \in V \quad u_i, v_i \in F \quad u+v = v+u$: $\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \vdots \\ u_n+v_n \end{bmatrix} = \begin{bmatrix} v_1+u_1 \\ v_2+u_2 \\ \vdots \\ v_n+u_n \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$
  - Based on commutativity of addition.

— $(ab)v = a(bv)$ $a, b, v_i \in F$ $v \in V$

— $a\begin{bmatrix} bv_1 \\ bv_2 \\ \vdots \\ bv_n \end{bmatrix} = \begin{bmatrix} abv_1 \\ abv_2 \\ \vdots \\ abv_n \end{bmatrix} = ab\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

- Is $M_{2,3}$ a vector space, where $M_{2,3}$ is the set of all 2x3 matrices?

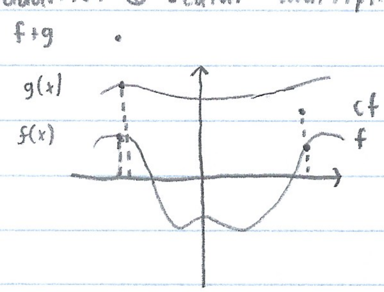
10/10: — Yes!

- $\mathcal{P}$ is the set of all polynomial functions.
  - — It is a vector space! All addition & scalar multiplication tennants hold. It is closed under addition and under scalar multiplication.
  - — $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$

    $q(x) = b_0 + b_1x + b_2x^2 + \dots + b_nx^n$

    $p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \dots + (a_n + b_n)x^n$

    $cp(x) = ca_0 + ca_1x + ca_2x^2 + \dots + ca_nx^n$
- $\mathcal{F}$ is the set of all real-valued functions.
  - — Yes - closed under addition & scalar multiplication.



- $\mathbb{Z}$ is the set of all integers.
  - — No if $c \notin \mathbb{Z}$, e.g. $\frac{1}{2}$
- Let $V = \mathbb{R}^2$ with "normal" addition but scalar multiplication defined as:

$$c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ 0 \end{bmatrix}$$

— No - fails 6: $1\begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} x \\ 0 \end{bmatrix}$ in a vector space.

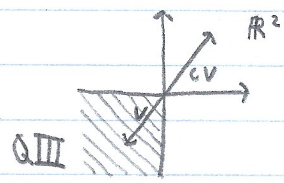
Steven Labalme

(H) Linear Algebra, 3

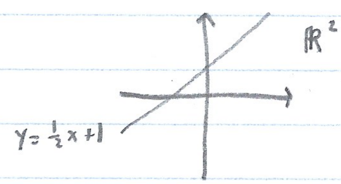
Chapter 3: Vector Spaces (Notes 2)

Forsyth

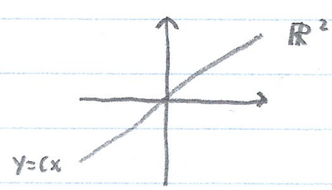
- Let $S \subset V$. S is a subspace $\leftrightarrow$
  1. $0 \in S$
  2. $u, v, u+v \in S$
  3. $c \in F$, u, $cu \in S$



- Is QIII $\subset \mathbb{R}^2$?
  - No - let $c<0$, then cv is in QI



- Is $y=\frac{1}{2}x+1 \subset \mathbb{R}^2$?
  - No - $0 \notin y=\frac{1}{2}x+1$



- Is $y=cx \subset \mathbb{R}^2$?
  - Yes - passes all 3 tests.
- Possible subspaces for $\mathbb{R}^2$
  1. $\mathbb{R}^2$
  2. Line through $(0,0)$
  3. $\begin{bmatrix}0\\0\end{bmatrix}$

10/11: • Symbols 2:
  - $A \subset B$: A "is a subset of" B
  - $\leftrightarrow$: "if and only if"
  - $\longrightarrow$: "implies"

Column Space

- Closed under addition and scalar multiplication.
  - $c, d \in F$, $u, v \in \mathbb{R}^n$, $cu + dv \in \mathbb{R}^n$
- A vector space consisting of all possible linear combinations of column vectors.
- $C(A) = \{ v \in \mathbb{R}^n : c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \}$

"the set of"

"column space of matrix A"

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

- $C(A) = c \begin{bmatrix} 2 \\ 1 \end{bmatrix} + d \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2c + d \\ c + 3d \end{bmatrix}$

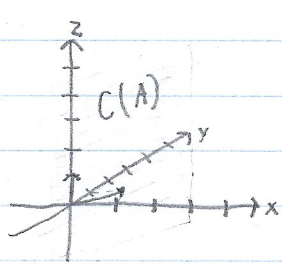
$$= \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$$ Basis for $C(A)$

$$= \mathbb{R}^2$$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

- $C(A) = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$



- $C(A) = \begin{bmatrix} c \\ c \\ d \end{bmatrix}$

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$$A=\begin{bmatrix}1&5&6\\2&3&5\\3&1&4\end{bmatrix} \quad C_1+C_2=C_3$$

- $A \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \begin{bmatrix}1&5&6\\0&-7&-7\\0&-14&-14\end{bmatrix} \xrightarrow{-2R_2+R_3} \begin{bmatrix}1&5&6\\0&-7&-7\\0&0&0\end{bmatrix} = \underset{L}{\begin{bmatrix}1&0&0\\2&1&0\\3&2&1\end{bmatrix}}\underset{U}{\begin{bmatrix}1&5&6\\0&-7&-7\\0&0&0\end{bmatrix}}$

- $Ax=b$, $L^{-1}b=\begin{bmatrix}1&0&0\\-2&1&0\\-3&-2&1\end{bmatrix}\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}=\begin{bmatrix}b_1\\-2b_1+b_2\\-3b_1-2b_2+b_3\end{bmatrix}=c$

- $u|c=\left[\begin{array}{ccc|c}1&5&6&b_1\\0&-7&-7&-2b_1+b_2\\0&0&0&-3b_1-2b_2+b_3\end{array}\right]$

— $-3b_1-2b_2+b_3=0$

10/15:

$$\begin{array}{c}\text{"pivot column"}\rightarrow \text{P} \quad \text{F} \quad \text{P} \quad \text{F} \leftarrow \text{"free column"}\\ A=\begin{bmatrix}1&1&2&3\\2&2&8&10\\3&3&10&13\end{bmatrix}\end{array}$$

- Find a basis for $C(A)$.

- $A \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \begin{bmatrix}1&1&2&3\\0&0&4&4\\0&0&4&4\end{bmatrix} \xrightarrow{-R_2+R_3} \begin{bmatrix}1&1&2&3\\0&0&4&4\\0&0&0&0\end{bmatrix} = \text{ref}(A) \xrightarrow{\frac{1}{4}R_2}$

$\begin{bmatrix}1&1&2&3\\0&0&1&1\\0&0&0&0\end{bmatrix} \xrightarrow{-2R_2+R_1} \begin{bmatrix}1&1&0&1\\0&0&1&1\\0&0&0&0\end{bmatrix} = \text{rref}(A)$

- $C(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} \right\}$

<u>Null Space</u>

- $N(A) = \{ v \in V : Av = 0 \}$

- $A \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, or $v_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + v_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + v_3 \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} + v_4 \begin{bmatrix} 3 \\ 10 \\ 13 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

free columns — don't matter.

- Find a basis for $N(A)$

- $\begin{array}{c} P \\ F \\ P \\ F \end{array} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \longrightarrow N(A) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

— $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is a trivial solution.

- $\text{rref}(A) \xrightarrow{C_2 \leftrightarrow C_3} \left[\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 \end{array} \right] = \left[\begin{array}{c|c} I & F \\ \hline 0 & 0 \end{array} \right]$

- $\underset{A}{\left[\begin{array}{c|c} I & F \\ \hline 0 & 0 \end{array} \right]} \underset{x}{\begin{bmatrix} -F \\ I \end{bmatrix}} = \underset{0}{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}$

- $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \begin{array}{c} -F \\ \\ I \\ \end{array}$

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10/21: Conditions on Solvability

$$A|b = \begin{array}{cccccc|c} P & & & & & & \\ \hline \end{array}$$
$$A|b = \left[\begin{array}{cccccc|c} 1 & -3 & 4 & -2 & 5 & 4 & b_1 \\ 2 & -6 & 9 & -1 & 8 & 2 & b_2 \\ 2 & -6 & 9 & -1 & 9 & 7 & b_3 \\ -1 & 3 & -4 & 2 & -5 & -4 & b_4 \end{array}\right]$$

- $$A|b \xrightarrow[\substack{-2R_1+R_3 \\ R_1+R_4}]{-2R_1+R_2} \begin{array}{cccccc} & F & P & & & \\ \end{array} \left[\begin{array}{cccccc|c} 1 & -3 & 4 & -2 & 5 & 4 & b_1 \\ 0 & 0 & 1 & 3 & -2 & -6 & -2b_1+b_2 \\ 0 & 0 & 1 & 3 & -1 & -1 & -2b_1+b_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_1+b_4 \end{array}\right] \xrightarrow{-R_2+R_3}$$

- $$\begin{array}{cccccc} P & F & P & F & P & F \end{array}$$
$$\left[\begin{array}{cccccc|c} 1 & -3 & 4 & -2 & 5 & 4 & b_1 \\ 0 & 0 & 1 & 3 & -2 & 6 & -2b_1+b_2 \\ 0 & 0 & 0 & 0 & 1 & 5 & -b_2+b_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_1+b_4 \end{array}\right]$$

— $b_1+b_4=0$, $b_1=-b_4$

— Infinite solutions when $b \in C(A)$

- $$C(A) = \left\{ \begin{array}{c} P \\ \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} \end{array}, \begin{array}{c} P \\ \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix} \end{array}, \begin{array}{c} P \\ \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix} \end{array} \right\}$$

- $$N(A) = \left\{ \begin{array}{c} P \\ F \\ P \\ F \\ P \\ F \end{array} \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 14 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 37 \\ 0 \\ -4 \\ 0 \\ -5 \\ 1 \end{bmatrix} \right\}$$

10/22:

$$-\ \text{rref}(A) = \begin{bmatrix} 1 & -3 & 0 & -14 & 0 & -37 \\ 0 & 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[C_5 \leftrightarrow C_3]{C_2 \leftrightarrow C_5}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -14 & -3 & -37 \\ 0 & 1 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0 & 5 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{ccc} 14 & 3 & 37 \\ -3 & 0 & -4 \\ 0 & 0 & -5 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] = 0 \xrightarrow[R_2 \leftrightarrow R_3]{R_5 \leftrightarrow R_3} \begin{bmatrix} 14 & 3 & 37 \\ 0 & 1 & 0 \\ -3 & 0 & -4 \\ 1 & 0 & 0 \\ 0 & 0 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

10/23:

Complete Solution to $Ax = b$

- $Ax_p = b$
  - x_p is a "particular solution"
- $Ax_n = 0$
  - x_n is a "null solution"
- $Ax_p + Ax_n = b + 0$
- $A(x_p + x_n) = b$
  - $x_p + x_n$ is the "complete solution"

$$A|b = \begin{array}{c} \begin{array}{cccc} P & F & P & F \end{array} \\ \left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 \\ 0 & 0 & 1 & 4 & b_3 \end{array}\right] \end{array}$$

- $$A|b \xrightarrow{-R_2 + R_3} \begin{array}{c} \begin{array}{cccc} & & & F \end{array} \\ \left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & b_1 \\ 0 & 0 & 1 & 4 & b_2 \\ 0 & 0 & 0 & 0 & -b_2 + b_3 \end{array}\right] \end{array}$$

  - $-b_2 + b_3 = 0, \quad b_3 = b_2$

- $$C(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad N(A) = \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$$

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- $A|b = \left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 4 & 6 \end{array}\right]$, $R|d = \left[\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$, solve for x

- $x_p = \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix}$, $R|d' = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 6 \\ \hline 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} I & b' \\ 0 & 0 \end{bmatrix}$

- $x = x_p + x_n = \begin{bmatrix} 1 \\ 0 \\ 6 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}$, $c_i \in F$

10/24:

Rank Theorem

- $\text{rank}(A) = \dim(C(A))$
  - The dimension of the column space of A.
- $\text{nullity}(A) = \dim(N(A))$
- A is an $m \times n$ matrix.

I. A is full rank $\to r = m = n \to A$ is $n \times n$
- 1 solution to $Ax = b$
- $A(x_p + x_n) = b \to x_n = 0 \to N(A) = \{0\}$
- $C(A) = \{v_j \in A : j = 1, \dots, n\}$
- $\text{rank}(A) = n$
- $\text{nullity}(A) = 0$
- A is invertible; $A|I \to I|A^{-1}$

II. A is full row rank $\to r = m < n$
- ∞ solutions to $Ax = b$
- $\text{rank}(A) = m$
- $\text{nullity}(A) = n - m$

III. A is full column rank $\to r = n < m$
- 1 or 0 solutions to $Ax = b$
  - $b \in C(A) \to 1$, $b \notin C(A) \to 0$

- $\text{rank}(A) = n$
- $\text{nullity}(A) = 0 \longrightarrow N(A) = \{0\}$
- $C(A) = \{v_j \in A : j = 1, \dots, n\}$

IV. $m > r$ AND $n > r$

- ∞ solutions to $Ax = b$

10/28:

Definitions

- Let $S \subset V$. $S = \{v_1, v_2, \dots, v_n\}$ is dependent if $\exists\ c_i \in \mathbb{R} \ni c_1 v_1 + \dots + c_n v_n = 0$ AND $c_i \neq 0$.
  - Dependent,
- Independent: $c_1 v_1 + \dots + c_n v_n = 0 \leftrightarrow c_i = 0\ \forall\ i = 1, \dots, n$.
- Span: Let $v_1, \dots, v_n \in V$. $\text{span}(v_1, \dots, v_n) = \{\sum_{i=1}^{n} c_i v_i : c_i \in \mathbb{R}\}$. Set of all linear combinations of $v \in S$, $S \subset V$.

  - Is $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ in the span of $\begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} 13 \\ 8 \\ 9 \end{bmatrix}$?

  - $\left[\begin{array}{cc|c} 2 & 13 & 1 \\ 4 & 8 & 3 \\ 7 & 9 & 5 \end{array}\right]$: Yes: $\text{rref}(A) = \begin{bmatrix} 1 & 0 & a_1 \\ 0 & 1 & a_2 \\ 0 & 0 & \boxed{0} \end{bmatrix}$, No! $\text{rref}(A) = \begin{bmatrix} I \end{bmatrix}$

- Basis: B is a basis for V. B is a linearly independent subset of V that spans V. $B \subset V$.
- Dimension: $\dim(V)$ is the cardinality of the basis B.
  - Cardinality: The number of elements in a set.

10/29:

Fundamental Theorems

$$A = \begin{bmatrix} 1 & 3 & 0 & 5 \\ 2 & 6 & 1 & 16 \\ 5 & 15 & 0 & 25 \end{bmatrix}$$

Chapter 3: Vector Spaces (Notes 6)

Forsyth

- $\text{rref}(A) = \begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

- $C(A) \in \mathbb{R}^3$, $C(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$, $\dim(C(A)) = 2$

- $N(A) \in \mathbb{R}^4$, $N(A) = \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -6 \\ 1 \end{bmatrix} \right\}$, $\dim(N(A)) = 2$

- Row space: $C(A^T) \in \mathbb{R}^4$, $C(A^T) = \left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 1 \\ 16 \end{bmatrix} \right\}$, $\dim(C(A^T)) = 2$

- Left null space: $N(A^T) \in \mathbb{R}^3$, $N(A^T) = \left\{ \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \right\}$, $\dim(N(A^T)) = 1$
  - $A^T y = 0$, $(A^T y)^T = 0^T$, $y^T A^{TT} = 0^T$, $y^T A = 0^T$
  - $\begin{bmatrix} -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 & 5 \\ 2 & 6 & 1 & 16 \\ 5 & 15 & 0 & 25 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$

- Fundamental Theorem I:
  - $C(A) \oplus N(A^T) = \mathbb{R}^3$
  - $C(A^T) \oplus N(A) = \mathbb{R}^4$
  - $A \in A^{3\times 4}$
- $\oplus$: "direct sum"

— $u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

— sum $u + v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

— $\text{span}(v) + \text{span}(w) = \text{span}(v)$
  - Not a direct sum.
    - $\text{span}(v) \neq \mathbb{R}^2$ "intersect"
    - $\text{span}(v) \cap \text{span}(w) \neq \{0\}$

— $\text{span}(u) \oplus \text{span}(v) = \mathbb{R}^2$
  - Is a direct sum.
    - $\text{span}(u) + \text{span}(v) = \mathbb{R}^2$
    - $\text{span}(u) \cap \text{span}(v) = \{0\}$

10/30:

$$\left.\begin{array}{l} \bullet\ C(A) \oplus N(A^T) = \mathbb{R}^m \\ \bullet\ C(A^T) \oplus N(A) = \mathbb{R}^n \end{array}\right\} \text{Fundamental Theorem I}$$

$$\begin{array}{ll} \bullet\ C(A), N(A^T) \leq \mathbb{R}^m & \\ \bullet\ C(A^T), N(A) \leq \mathbb{R}^n & \end{array} \quad \left.\begin{array}{l} C(A) \cap N(A^T) = \{0\} \\ C(A^T) \cap N(A) = \{0\} \end{array}\right\} \text{Direct Sum } (\oplus)$$

- A is $m \times n$ — m rows, n columns
- $C(A) = \{b \in \mathbb{R}^m \mid \exists\, x \in \mathbb{R}^n : b = Ax\} \subset \mathbb{R}^m$
- $N(A) = \{x \in \mathbb{R}^n \mid Ax = 0\} \subset \mathbb{R}^n$
- $C(A^T) = \{c \in \mathbb{R}^n \mid \exists\, y \in \mathbb{R}^m : c = Ay\} \subset \mathbb{R}^n$
- $N(A^T) = \{y \in \mathbb{R}^m \mid Ay = 0\} \subset \mathbb{R}^m$
- Show $C(A) \leq \mathbb{R}^m$
  - Let $u, v \in C(A)$ and $c, d \in F$, show $cu + dv \in C(A)$
  - $u = Ax$, $v = Ay$
  - $c(Ax) + d(Ay) = A(cx + dy)$
  - $\therefore cx + dy \in C(A)$
- Show $N(A^T) \leq \mathbb{R}^m$
  - $x, y \in N(A^T)$ $\quad c, d \in F$
  - $A^T x = 0$, $A^T y = 0$
  - $A^T(cx + dy) = c(A^T x) + d(A^T y) = c(0) + d(0) = 0$

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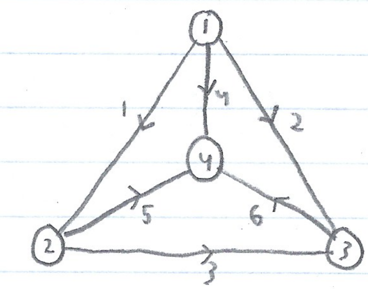
- Show $C(A) \cap N(A^T) = \{0\}$
  - By contradiction, if $Ax=b$, assume $b \in C(A)$, $b \in N(A^T)$, $b \neq 0$
  - Since $b \in C(A)$, $\exists\, x \in \mathbb{R}^n : b = Ax$
  - Since $b \in N(A^T)$, $A^T b = A^T(Ax) = 0$ BUT $x \neq 0$ since $x=0 \rightarrow b=0$
  - $A^T(Ax)=0$, $x^T A^T(Ax) = x^T 0$, $(Ax)^T(Ax) = 0$, $b^T b = 0$, $\|b\|^2 = 0, b = 0$

10/31:

Networks and Graphs

- Incidence Matrix:

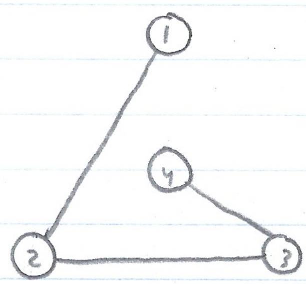
$$\begin{array}{r c}
 & \begin{matrix} ① & ② & ③ & ④ \end{matrix} \\
\begin{matrix} \checkmark\ 1 \\ 2 \\ \checkmark\ 3 \\ 4 \\ 5 \\ \checkmark\ 6 \end{matrix} & \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}
\end{array}$$



Complete Graph of Electric Flow

- ref(A) =

$$\begin{array}{r c}
 & \begin{matrix} ① & ② & ③ & ④ \end{matrix} \\
\begin{matrix} 1 \\ 3 \\ 6 \\ 2 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}$$



Tree

- $A \quad x \quad = \quad b$

potentials

difference of potentials / Voltage

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \qquad b = \begin{bmatrix} x_2 - x_1 \\ x_3 - x_1 \\ x_3 - x_2 \\ x_4 - x_1 \\ x_4 - x_2 \\ x_4 - x_3 \end{bmatrix}$$

$$\bullet\ N(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

— No current flows if all potentials are the same!

11/4:

$$\bullet\ C(A^T) = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

— Basis for the row space: How to get through the nodes w/o creating a loop.

— After 3 edges are added, loops are formed. $\dim C(A^T) = 3$

$$\bullet\ C(A) = \left\{ \begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}$$

— How do you tell when $b \in C(A)$?

• Kirchoff's voltage law: Algebraic sum of voltage around any loop $= 0$.

— $(x_2 - x_1) + (x_3 - x_2) - (x_3 - x_1) = 0$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} -1 & -1 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Steven Labalme

(H) Linear Algebra, 3

Chapter 3: Vector Spaces (Notes 8)

Forsyth

- Kirchoff's current law: flow in = flow out

- Forward 1, forward 3, backward 2: $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

- $\dim N(A^T) = 3$ (Fun. theorem)