Complex Linear Independence: Decomplexification

4/7:

- When given a complex system of equations, it is necessary to **decomplexify** it.
- **Decomplexify**: To model a complex system of equations with a strictly real system for the purpose of applying the tenets of real linear algebra to it.
- Consider the following complex system of equations.

$$\begin{aligned}(2+i)x_1 + (1+i)x_2 &= 3+6i\\ (3-i)x_1 + (2-2i)x_2 &= 7-i\end{aligned}$$

  - The solutions will be complex numbers: $x_1 = a_1 + ib_1$ and $x_2 = a_2 + ib_2$, where $a_1, a_2, b_1, b_2 \in \mathbb{R}$.
- Transform it into a matrix system of equations. Separate the real and complex parts, and factor out all instances of the imaginary number i so that it is a coefficient to any complex matrix.

$$\begin{bmatrix}2+i & 1+i\\ 3-i & 2-2i\end{bmatrix}\begin{bmatrix}a_1+ib_1\\ a_2+ib_2\end{bmatrix} = \begin{bmatrix}3+6i\\ 7-i\end{bmatrix}$$
$$\left(\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix} + \begin{bmatrix}i & i\\ -i & -2i\end{bmatrix}\right)\left(\begin{bmatrix}a_1\\ a_2\end{bmatrix} + \begin{bmatrix}ib_1\\ ib_2\end{bmatrix}\right) = \left(\begin{bmatrix}3\\ 7\end{bmatrix} + \begin{bmatrix}6i\\ -i\end{bmatrix}\right)$$
$$\underbrace{\left(\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix} + i\begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}\right)}_{A}\underbrace{\left(\begin{bmatrix}a_1\\ a_2\end{bmatrix} + i\begin{bmatrix}b_1\\ b_2\end{bmatrix}\right)}_{x} = \underbrace{\left(\begin{bmatrix}3\\ 7\end{bmatrix} + i\begin{bmatrix}6\\ -1\end{bmatrix}\right)}_{b}$$

- Foil the left side of the above equation[1].

$$\left(\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}\begin{bmatrix}a_1\\ a_2\end{bmatrix} - \begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}b_1\\ b_2\end{bmatrix}\right) + i\left(\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}\begin{bmatrix}b_1\\ b_2\end{bmatrix} + \begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}a_1\\ a_2\end{bmatrix}\right) = \begin{bmatrix}3\\ 7\end{bmatrix} + i\begin{bmatrix}6\\ -1\end{bmatrix}$$

- Split the above system of equations into a real system of equations and a complex system of equations by setting equal to each other the real components of each side and the imaginary components of each side.

$$\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}\begin{bmatrix}a_1\\ a_2\end{bmatrix} - \begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}b_1\\ b_2\end{bmatrix} = \begin{bmatrix}3\\ 7\end{bmatrix}$$
$$\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}\begin{bmatrix}b_1\\ b_2\end{bmatrix} + \begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}a_1\\ a_2\end{bmatrix} = \begin{bmatrix}6\\ -1\end{bmatrix}$$

- Multiply out the matrices above to yield a system of four equations.

$$\begin{aligned}2a_1 + a_2 - b_1 - b_2 &= 3\\ 3a_1 + 2a_2 + b_1 + 2b_2 &= 7\\ a_1 + a_2 + 2b_1 + b_2 &= 6\\ -a_1 - 2a_2 + 3b_1 + 2b_2 &= -1\end{aligned}$$

- Condense the above system of equations into a single matrix system of equations.

$$\begin{bmatrix}2 & 1 & -1 & -1\\ 3 & 2 & 1 & 2\\ 1 & 1 & 2 & 1\\ -1 & -2 & 3 & 2\end{bmatrix}\begin{bmatrix}a_1\\ a_2\\ b_1\\ b_2\end{bmatrix} = \begin{bmatrix}3\\ 7\\ 6\\ -1\end{bmatrix}$$

[1]Note that the minus sign appears in the real component because, when multiplying the two "last" parts, $i^2 = -1$.

- Solve for a_1, a_2, b_1, and b_2 using an augmented matrix and Gauss-Jordan elimination.

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & -1 & 3 \\ 3 & 2 & 1 & 2 & 7 \\ 1 & 1 & 2 & 1 & 6 \\ -1 & -2 & 3 & 2 & -1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}$$

- From these four values, the original solutions $x_1 = a_1 + ib_1$ and $x_2 = a_2 + ib_2$ can be found.

$$x_1 = 1 + 2i$$
$$x_2 = 2 - i$$

Hermitian, Unitary, and Normal Matrices

4/13:

- What necessitates different categorizations of complex vectors and matrices?
- Consider a vector v.

$$v = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

- If you want to find $||v||$, you typically evaluate $\sqrt{v^{\mathrm{T}}v}$. However, this equals to 0 (see the following), which is clearly not the magnitude of v.

$$\begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 - 1 = 0$$

  - Note that $||v||$ must be an element of $\mathbb{R}$ because it measures a distance.

- With complex vectors, it is necessary to evaluate $\sqrt{\bar{v}^{\mathrm{T}}v}$ to find $||v||$.

$$\begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = 1 + 1 = 2$$

$$||v|| = \sqrt{2}$$

  - This makes sense because $\begin{bmatrix} 1 \\ i \end{bmatrix}$ extends one unit into $\mathbb{R}^1$ and one unit into $\mathbb{C}^2$.
  - If $z\bar{z} = |z|^2$ and $\bar{v}^{\mathrm{T}}v = v \cdot \bar{v}$, it stands to reason that $\bar{v}^{\mathrm{T}}v = ||v||^2$. Essentially, the dot product multiplies every element of v by its complex conjugate and sums them.

- Instead of writing $\bar{v}^{\mathrm{T}}$[2] every time, mathematicians shorthand to v^{H}[3].
  - v^{H} works for all vectors, but it is necessary for complex ones.

- **Hermitian** (matrix): A matrix A such that $A = A^{\mathrm{H}}$.
  - Typically defined for $A \in \mathbb{C}^n$, but holds for $A \in \mathbb{R}^n$, too.
  - Parallel to how if $A \in \mathbb{R}^n$ and $A = A^{\mathrm{T}}$, A is symmetrical.
  - Also note that if $A^{\mathrm{H}}A = A^2 = AA^{\mathrm{H}}$, A is Hermitian.

[2] "v conjugate transpose"
[3] "v Hermitian" after French mathematician Charles Hermite.

  - A Hermitian matrix has to have real values on the principal diagonal. When A is transposed and conjugated, the diagonal entries are the only values that don't move. Thus, their conjugates must equal themselves, so they must be real[4].

- **Unitary** (matrix): A matrix A such that $A^{-1} = A^{\mathrm{H}}$.
  - Typically defined for $A \in \mathbb{C}^n$, but holds for $A \in \mathbb{R}^n$, too.
  - Parallel to how if $A \in \mathbb{R}^n$ and $A^{-1} = A^{\mathrm{T}}$, A is orthonormal.
  - Also note that if $A^{\mathrm{H}}A = I = AA^{\mathrm{H}}$, A is unitary.
- **Normal** (matrix): A matrix that is unitarily diagonalizable.
  - Typically defined for $A \in \mathbb{C}^n$, but holds for $A \in \mathbb{R}^n$, too.
  - Parallel to matrices $A \in \mathbb{R}^n$ such that A is orthonormally diagonalizable.
- Note that not every complex matrix has to be one of these three types.
- When $A^{\mathrm{H}}A = AA^{\mathrm{H}}$, $A = U\Lambda U^{\mathrm{H}}$.

$$\begin{aligned}
AA^{\mathrm{H}} &= \left(U\Lambda U^{\mathrm{H}}\right)\left(U\Lambda U^{\mathrm{H}}\right)^{\mathrm{H}} \\
&= U\Lambda U^{\mathrm{H}} U \Lambda^{\mathrm{H}} U^{\mathrm{H}} \\
&= U\Lambda\Lambda^{\mathrm{H}} U^{\mathrm{H}} \\
&= U\Lambda^{\mathrm{H}}\Lambda U^{\mathrm{H}\,[5]} \\
&= U\Lambda^{\mathrm{H}} U^{\mathrm{H}} U\Lambda U^{\mathrm{H}} \\
&= \left(U\Lambda U^{\mathrm{H}}\right)^{\mathrm{H}}\left(U\Lambda U^{\mathrm{H}}\right) \\
&= A^{\mathrm{H}}A
\end{aligned}$$

- When $A = A^{\mathrm{H}}$, all eigenvalues are elements of $\mathbb{R}$ (similar to spectral theorem).

$$v^{\mathrm{H}}Av = \left(v^{\mathrm{H}}Av\right)^{\mathrm{H}} = v^{\mathrm{H}}Av$$

  - The above proves that $v^{\mathrm{H}}Av \in \mathbb{R}$ because it's its own conjugate[4].

$$\begin{aligned}
Av &= \lambda v \\
v^{\mathrm{H}}Av &= \lambda v^{\mathrm{H}}v
\end{aligned}$$

  - $\lambda = \frac{v^{\mathrm{H}}Av}{v^{\mathrm{H}}v} \to \frac{\mathbb{R}}{\mathbb{R}} = \mathbb{R}$[6].
- When $A = A^{\mathrm{H}}$ and $Ax = \lambda x$, all x's can be chosen orthonormally (also similar to spectral theorem).
  - Normality is implied because any eigenvector can be scaled to any version (including a normal version) and still be an eigenvector.

$$x_i = \begin{bmatrix} x_{i_1} \\ x_{i_2} \\ \vdots \\ x_{i_n} \end{bmatrix} \qquad\qquad x_i^{\mathrm{H}} = \begin{bmatrix} \bar{x}_{i_1} & \bar{x}_{i_2} & \cdots & \bar{x}_{i_n} \end{bmatrix}$$

[4] Recall that only real quantities can be their own conjugates because $a + 0i = a - 0i$.

[5] Since $\Lambda = \Lambda^{\mathrm{H}}$.

[6] Note that the denominator is real because it's how one finds $||v||$, and $||v||$ must be real, as discussed above.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \qquad A^{\mathrm{H}} = \begin{bmatrix} a_{11} & \bar{a}_{21} & \cdots & \bar{a}_{n1} \\ \bar{a}_{12} & a_{22} & \cdots & \bar{a}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_{1n} & \bar{a}_{2n} & \cdots & a_{nn} \end{bmatrix}$$

- Define an arbitrary vector x_i and matrix A, along with their conjugate transposes (or Hermitian versions). Note that the diagonal entries of A^{H} aren't shown as conjugated because their conjugates equal themselves.

$$\begin{aligned} Ax_1 &= \lambda_1 x_1 \\ x_2^{\mathrm{H}} A x_1 &= \lambda_1 x_2^{\mathrm{H}} x_1 \end{aligned}$$

$$\begin{aligned} Ax_2 &= \lambda_2 x_2 \\ (Ax_2)^{\mathrm{H}} &= (\lambda_2 x_2)^{\mathrm{H}} \\ x_2^{\mathrm{H}} A^{\mathrm{H}} &= \lambda_2 x_2^{\mathrm{H}} \\ x_2^{\mathrm{H}} A x_1 &= \lambda_2 x_2^{\mathrm{H}} x_1 \end{aligned}$$

- $\lambda_1 x_2^{\mathrm{H}} x_1 = \lambda_2 x_2^{\mathrm{H}} x_1$ implies that, since $\lambda_1 \neq \lambda_2$, $x_2^{\mathrm{H}} x_1$ must equal 0, proving orthogonality.

Complex Diagonalization

4/15: • Diagonalize the following matrix A.

$$A = \begin{bmatrix} 0.9 & -0.4 \\ 0.1 & 0.9 \end{bmatrix}$$

- Find the characteristic polynomial.

$$\begin{aligned} 0 &= \begin{vmatrix} 0.9-\lambda & -0.4 \\ 0.1 & 0.9-\lambda \end{vmatrix} \\ &= (0.9-\lambda)^2 - (-0.4)(0.1) \\ &= 0.81 - 1.8\lambda + \lambda^2 + 0.04 \\ &= \lambda^2 - 1.8\lambda + 0.85 \end{aligned}$$

- Find the eigenvalues[7].

$$\begin{aligned} \lambda &= \frac{-(-1.8) \pm \sqrt{(-1.8)^2 - 4(1)(0.85)}}{2(1)} \\ &= 0.9 \pm \frac{\sqrt{-0.16}}{2} \\ &= 0.9 \pm \frac{\sqrt{-1}\sqrt{0.16}}{2} \\ &= 0.9 \pm \frac{0.4i}{2} \\ &= 0.9 \pm 0.2i \end{aligned}$$

$$\lambda_1 = 0.9 + 0.2i \qquad \lambda_2 = 0.9 - 0.2i$$

[7]It is interesting that the eigenvalues are complex conjugates of each other.

– Find the eigenvectors[8].

$$(A-(0.9+0.2i))x_1 = \begin{bmatrix} 0.9-(0.9+0.2i) & -0.4 \\ 0.1 & 0.9-(0.9+0.2i) \end{bmatrix} \begin{bmatrix} x_{1_1} \\ x_{1_2} \end{bmatrix}$$
$$= \begin{bmatrix} -0.2i & -0.4 \\ 0.1 & -0.2i \end{bmatrix} \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(A-(0.9-0.2i))x_1 = \begin{bmatrix} 0.9-(0.9-0.2i) & -0.4 \\ 0.1 & 0.9-(0.9-0.2i) \end{bmatrix} \begin{bmatrix} x_{1_1} \\ x_{1_2} \end{bmatrix}$$
$$= \begin{bmatrix} 0.2i & -0.4 \\ 0.1 & 0.2i \end{bmatrix} \begin{bmatrix} -2i \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 2i \\ 1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} -2i \\ 1 \end{bmatrix}$$

– Compile the diagonalization.

$$A = \frac{1}{4i} \begin{bmatrix} 2i & -2i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0.9+0.2i & 0 \\ 0 & 0.9-0.2i \end{bmatrix} \begin{bmatrix} 1 & 2i \\ -1 & 2i \end{bmatrix}$$

Real versus Complex

4/16:

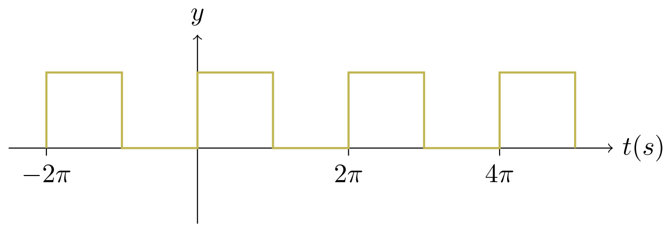
Real	**Complex**
$\mathbb{R}^n$: vectors with n real components	$\mathbb{C}^n$: vectors with n complex components
length: $\|\|x\|\|^2 = x_1^2 + \cdots + x_n^2$	length: $\|\|z\|\|^2 = \|z_1\|^2 + \cdots + \|z_n\|^2$
transpose: $\left(A^{\mathrm{T}}\right)_{ij} = A_{ji}$	conjugate transpose: $\left(A^{\mathrm{H}}\right)_{ij} = \overline{A_{ji}}$
product rule: $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$	product rule: $(AB)^{\mathrm{H}} = B^{\mathrm{H}}A^{\mathrm{H}}$
dot product: $x^{\mathrm{T}}y = x_1y_1 + \cdots + x_ny_n$	inner product: $u^{\mathrm{H}}v = \bar{u}_1u_1 + \cdots + \bar{u}_nv_n$
reason for A^{T}: $(Ax)^{\mathrm{T}}y = x^{\mathrm{T}}(A^{\mathrm{T}}y)$	reason for A^{H}: $(Au)^{\mathrm{H}}v = u^{\mathrm{H}}(A^{\mathrm{H}}v)$
orthogonality: $x^{\mathrm{T}}y = 0$	orthogonality: $u^{\mathrm{H}}v = 0$.
symmetric matrices: $A = A^{\mathrm{T}}$	Hermitian matrices: $A = A^{\mathrm{H}}$
$A = Q\Lambda Q^{-1} = Q\Lambda Q^{\mathrm{T}}$ (real Λ)	$A = U\Lambda U^{-1} = U\Lambda U^{\mathrm{H}}$ (real Λ)
skew-symmetric matrices: $k^{\mathrm{T}} = -K$	skew-Hermitian matrices: $K^{\mathrm{H}} = -K$
orthogonal matrices: $Q^{\mathrm{T}} = Q^{-1}$	unitary matrices: $U^{\mathrm{H}} = U^{-1}$
orthonormal columns: $Q^{\mathrm{T}}Q = I$	orthonormal columns: $U^{\mathrm{H}} = U^{-1}$
$(Qx)^{\mathrm{T}}(Qy) = x^{\mathrm{T}}y$ and $\|\|Qx\|\| = \|\|x\|\|$	$(Ux)^{\mathrm{H}}(Uy) = x^{\mathrm{H}}y$ and $\|\|Uz\|\| = \|\|z\|\|$

- Note that the columns and eigenvectors of Q and U are orthonormal, and all of their eigenvalues λ satisfy $|\lambda| = 1$.

[8] It is interesting that the eigenvectors are *also* complex conjugates of each other.

Fourier Series

4/21:
- Consider the square wave $f(t)$.

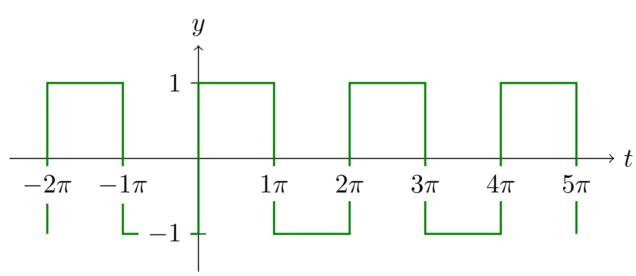


- Its period is $2\pi\frac{\text{sec}}{\text{cycle}}$, and its frequency is $\frac{1}{2\pi}$Hz.
- Can we write $f(t)$ as a sum of sines and cosines?

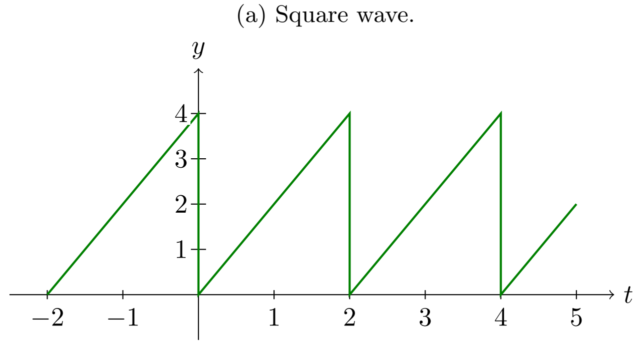
$$f(t) = a_0 + a_1\cos(t) + b_1\sin(t) + a_2\cos(2t) + b_2\sin(2t) + a_3\cos(3t) + b_3\sin(3t) + \cdots$$

  - Be general.
  - Since $T = 2\pi$, it makes sense to use some functions with $T = 2\pi$ to model it.
  - The weighting coefficients account for how much each function contributes to the whole.
- Historically studied by Fourier, who studied differential equations. Differential equations were often easy to solve for sines and cosines, so if a function could be modeled by a sum of sines and cosines, a related differential equation would be easier to solve.
- Fourier series, transforms, and analysis also tell us how much of each frequency a function contains (as measured by the weight coefficients).

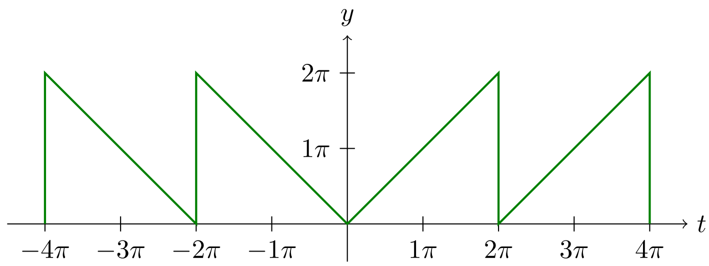
Wave Type Sketches



(a) Square wave.



(b) Sawtooth wave.



(c) Triangular wave.

- Square wave: $f(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & \pi < t < 2\pi \end{cases}$ where periodicity is defined by $f(t + 2\pi) = f(t)$.
- Sawtooth wave: $f(t) = 2t \qquad 0 < t < 2$ where periodicity is defined by $f(t + 2) = f(t)$.
- Triangular wave: $f(t) = |t| \qquad 0 < t < 2\pi$ where periodicity is defined by $f(t + 2\pi) = f(t)$.

1) $\int_{-\pi}^{\pi} \sin(nt)dt = \left[-\frac{1}{n}\cos(nt)\right]_{-\pi}^{\pi} = \frac{1}{n}\left(-\cos(n\pi)+\cos(n\pi)\right) = 0, n \neq 0$

2) $\int_{-\pi}^{\pi} \cos(nt)dt = \left[\frac{1}{n}\sin(nt)\right]_{-\pi}^{\pi} = 0, n \neq 0$

3) $\int_{-\pi}^{\pi} f(t)dt = \frac{1}{2}\int_{-\pi}^{\pi} a_0 dt + \sum_{n=1}^{\infty}\left(\int_{-\pi}^{\pi} a_n \cos(nt)dt + \int_{-\pi}^{\pi} b_n \sin(nt)dt\right) = \frac{1}{2}\left[a_0 t\right]_{-\pi}^{\pi} + \sum_{n=1}^{\infty}(0+0)$

$\therefore \int_{-\pi}^{\pi} f(t)dt = \frac{1}{2}(2a_0\pi) \to a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt \to \frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt$

4) $\int_{-\pi}^{\pi} \sin(nt)\cos(mt)dt = \frac{1}{2}\left(\int_{-\pi}^{\pi} \sin(n+m)tdt + \int_{-\pi}^{\pi} \sin(n-m)tdt\right) = \frac{1}{2}(0+0) = 0$ using the results from problems 1 and 2 since if n is an integer and m is an integer then $n+m$ and $n-m$ are also integers.

5) $\int_{-\pi}^{\pi} \cos(nt)\cos(mt)dt = \frac{1}{2}\left(\int_{-\pi}^{\pi} \cos(n+m)tdt + \int_{-\pi}^{\pi} \cos(n-m)tdt\right) = \frac{1}{2}(0+0) = 0$ when $n \neq m$

6) $\int_{-\pi}^{\pi} \cos^2(nt)dt = \frac{1}{2}\int_{-\pi}^{\pi} \left(1+\cos(2n)t\right)dt = \frac{1}{2}\left[t + \frac{1}{2n}\sin(2n)t\right]_{-\pi}^{\pi} = \pi$ when $n \neq 0$

7) $\pi - (-\pi) = 2\pi$

8) a) $\int_{-\pi}^{\pi} \sin(nt)\sin(mt)dt = 0$

b) $\int_{-\pi}^{\pi} \sin^2(nt)dt = \frac{1}{2}\int_{-\pi}^{\pi} \left(1-\cos(2nt)\right)dt = \pi$

c) $\int_{-\pi}^{\pi} \sin(nt)\sin(mt)dt = 0$

For integers m, n:

$$\int_{-\pi}^{\pi} \sin(nt)\cos(mt)dt = 0$$

$$\int_{-\pi}^{\pi} \cos(nt)\cos(mt)dt = \left\{\begin{matrix} 0, n \neq m \\ \pi, n = m \neq 0 \\ 2\pi, n = m = 0 \end{matrix}\right\}$$

$$\int_{-\pi}^{\pi} \sin(nt)\sin(mt)dt = \left\{\begin{matrix} 0, n \neq m, n = m = 0 \\ \pi, n = m \end{matrix}\right\}$$

FOURIER SERIES

- We have shown that when $v^T w = 0$, then vectors *v* and *w* are orthogonal.
- Fourier series ask us to think of continuous functions as vectors.
- Let $f(t) = \cos(t)$ and let $g(t) = \sin(t)$
- To find $f \bullet g$ of these two function "vectors" would be asking us to take a dot product that has infinitely many terms. All vectors we have used have had a finite number of components. Because these "vectors" would have infinitely many terms, it would be like asking us to take an integral!
- $f \bullet g = \int_0^{2\pi} \cos(t)\sin(t)dt$ (We limit the domain because these are sinusoidal periodic functions and thus they repeat after a period of 2π .)

$$\int_0^{2\pi} \cos(t)\sin(t)dt$$

Let $u = \cos(t)$ and $du = -\sin(t)$

$$-\int_1^1 udu$$

$$\left[-\frac{u^2}{2}\right]_1^1$$

$$0$$

OR

$$\left[-\frac{1}{2}\cos^2(t)\right]_0^{2\pi}$$

$$-\frac{1}{2}(\cos^2 2\pi - \cos^2 0)$$

$-\frac{1}{2}(1-1)$ =

$$0$$

- Therefore $\cos t \bullet \sin t = 0$, therefore these function "vectors" are orthogonal, therefore they serve as basis vectors for the function space of continuous periodic functions in F^2, therefore any continuous function also in this space can be written as a linear combination of $\cos(t)$ and $\sin(t)$!!!

$$f(t) = \frac{a_0}{2} + a_1\cos(t) + b_1\sin(t) + a_2\cos(2t) + b_2\sin(2t) + \cdots$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nt) + b_n\sin(nt)\right)$$

*Why divide the constant term by 2?

CALCULATING THE FOURIER COEFFICENTS

As sine and cosine can serve as an orthogonal basis for periodic functions, consider the Fourier Series for a function $f(t)$ of period 2π to be $f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos(nt)+b_n\sin(nt)\right)$

In problem 3 from your handout you already found a formula for a_0, the constant term.

$$\boxed{\frac{a_0}{2}=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)dt}$$

To obtain the coefficients $a_n, n\in\mathbb{Z}$, multiply both sides of the equation by $\cos(mt)$ where $m\in\mathbb{Z}$ and $m>0$ and integrate both sides from $-\pi$ to π since this is a periodic function of period 2π.

$$\int_{-\pi}^{\pi}f(t)\cos(mt)dt=\frac{a_0}{2}\int_{-\pi}^{\pi}\cos(mt)dt+\sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos(nt)\cos(mt)dt+b_n\int_{-\pi}^{\pi}\sin(nt)\cos(mt)dt\right)$$

Simplify this result using the integrals on the right side of this equation from the handout problems 5-7. The only nonzero integral results from $\int_{-\pi}^{\pi}\cos^2(mt)dt=\pi$ in the case where $n=m$.

$$\therefore\int_{-\pi}^{\pi}\cos(mt)dt=a_m\pi$$

Sine this is the case where $n=m$, replacing m with n and solving for a_n we obtain:

$$\boxed{a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos(nt)dt\text{ , }n\in\mathbb{Z}\text{ and }n>0}$$

To obtain the coefficients $b_{n,}b\in\mathbb{Z},n>0$, multiply both sides of the equation by $\sin(mt)$ where $m\in\mathbb{Z}$ and $m>0$ and integrate both sides from $-\pi$ to π since this is a periodic function of period 2π.

$$\int_{-\pi}^{\pi}f(t)\sin(mt)dt=\frac{a_0}{2}\int_{-\pi}^{\pi}\sin(mt)dt+\sum_{n=1}^{\infty}\left(a_n\int_{-\pi}^{\pi}\cos(nt)\sin(mt)dt+b_n\int_{-\pi}^{\pi}\sin(nt)\sin(mt)dt\right)$$

Simplify this result using the integrals on the right side of this equation using the handout problems you did for 8. The only nonzero integral results from $\int_{-\pi}^{\pi}b_m\sin^2(mt)dt=b_m\pi$ in the case where $n=m$. Relabeling m as n and solving for b_n we obtain:

$$\boxed{b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin(nt)dt\text{ , }n\in\mathbb{Z}\text{ and }n>0}$$

Square Wave

4/29:
- **Hilbert space**: An infinite-dimensional vector space — extends a lot of the ideas of linear algebra to functions in infinite dimensions.
  - Recall that functions behave with linearity ($\alpha f(t) + \beta g(t)$ is still a function).
- Square wave (period 2π): $f(t) = \begin{cases} 0 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases}$ and $f(t+2\pi) = f(t)$.
- Find $\frac{a_0}{2}$ term:

$$\begin{aligned}
\frac{a_0}{2} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\pi}^{0} (0)\,\mathrm{d}t + \frac{1}{2\pi}\int_{0}^{\pi} (1)\,\mathrm{d}t \\
&= 0 + \left[\frac{t}{2\pi}\right]_0^{\pi} \\
&= \frac{\pi}{2\pi}
\end{aligned}$$

$$\boxed{\frac{a_0}{2} = \frac{1}{2}}$$

  - Makes sense because $\frac{a_0}{2}$ is like the sinusoidal axis and $\frac{1}{2}$ is half way between 0 and 1.
- Find a_n terms:

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,\mathrm{d}t \\
&= \frac{1}{\pi}\int_{-\pi}^{0} (0)\cos(nt)\,\mathrm{d}t + \frac{1}{\pi}\int_{0}^{\pi} (1)\cos(nt)\,\mathrm{d}t \\
&= 0 + \left[\frac{1}{n\pi}\sin(nt)\right]_0^{\pi}
\end{aligned}$$

$$\boxed{a_n = 0}$$

- Find b_n terms:

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,\mathrm{d}t \\
&= \frac{1}{\pi}\int_{-\pi}^{0} (0)\sin(nt)\,\mathrm{d}t + \frac{1}{\pi}\int_{0}^{\pi} (1)\sin(nt)\,\mathrm{d}t \\
&= 0 - \frac{1}{n\pi}[\cos(nt)]_0^{\pi} \\
&= -\frac{1}{n\pi}[\cos(n\pi) - \cos(0)] \\
&= -\frac{1}{n\pi}[\cos(n\pi) - 1]^{[9]}
\end{aligned}$$

$$\boxed{b_n = \begin{cases} 0 & n = 2, 4, 6, \dots \\ \frac{2}{n\pi} & n = 1, 3, 5, \dots \end{cases}}$$

[9] $\cos(n\pi)$ equals 1 when n is even and -1 when n is odd.

- Assemble the Fourier series for the square wave function:

$$\begin{aligned} f(t) &= \frac{1}{2} + \frac{2}{\pi}\sin(t) + \frac{2}{3\pi}\sin(3t) + \frac{2}{5\pi}\sin(5t) + \cdots \\ &= \frac{1}{2} + \frac{2}{\pi}\left(\frac{1}{1}\sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \cdots\right) \\ &= \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((2n-1)t) \end{aligned}$$