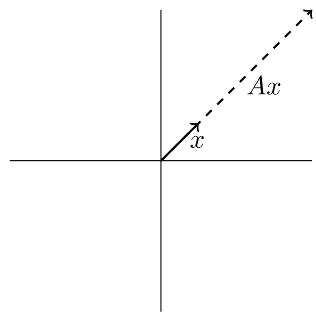
1/28: Introduction to Eigenvalues and Eigenvectors

- $Ax = b = \lambda x$
- $Ax = \lambda x$, $\lambda \in \mathbb{F}$, $x \in \mathbb{R}^n$
- λ is an eigenvalue. λx is an eigenvector.

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

- $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with corresponding eigenvalue of 4.
- $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



$$\begin{aligned} Ax &= \lambda x \\ Ax - \lambda x &= 0 \\ Ax - \lambda I x &= 0 \\ (A - \lambda I)x &= 0 \end{aligned}$$

- $(A - \lambda I)x = 0 \Rightarrow x \in N(A - \lambda I)$[1] $\Rightarrow |A - \lambda I| = 0$
- $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}$
- $\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$$\begin{aligned} 0 &= (3-\lambda)^2 - 1^2 \\ &= 3^2 - 6\lambda + \lambda^2 - 1 \\ &= \lambda^2 - 6\lambda + 8 \\ &= (\lambda - 4)(\lambda - 2) \end{aligned}$$

- $\lambda = 4, 2.$
- $\lambda^2 - 6\lambda + 8$ is the **characteristic polynomial** of A.
- $A - 2I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in N(A - 2I)$.
- $A - 4I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \in N(A - 4I)$.

[1] To have a null space, $A - \lambda I$ has free columns.

- "Eigenspace" is not $\mathbb{R}^2$, but two lines in $\mathbb{R}^2$, specifically $y = \pm x$.
  - $y = \pm x$ comes from $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

1/29:

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\begin{aligned}
P(\lambda) &= |A - \lambda I| \\
&= \begin{vmatrix} 2-\lambda & -2 & 3 \\ 0 & 3-\lambda & -2 \\ 0 & -1 & 2-\lambda \end{vmatrix} \\
&= -1 \begin{vmatrix} 2-\lambda & 3 \\ 0 & -2 \end{vmatrix} (-1)^{3+2} + (2-\lambda) \begin{vmatrix} 2-\lambda & -2 \\ 0 & 3-\lambda \end{vmatrix} (-1)^{3+3} \\
&= ((2-\lambda)(-2)) + (2-\lambda)((2-\lambda)(3-\lambda)) \\
&= -4 + 2\lambda + (2-\lambda)^2(3-\lambda) \\
&= -4 + 2\lambda + (4 - 4\lambda + \lambda^2)(3-\lambda) \\
&= -4 + 2\lambda + 12 - 4\lambda - 12\lambda + 4\lambda^2 + 3\lambda^2 - \lambda^3 \\
&= -\lambda^3 + 7\lambda^2 - 14\lambda + 8 \\
&= -(\lambda - 1)(\lambda - 2)(\lambda - 4)
\end{aligned}$$

$$A - I = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -2 \\ 0 & -1 & 1 \end{bmatrix} \qquad A - 2I = \begin{bmatrix} 0 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & -1 & 0 \end{bmatrix} \qquad A - 4I = \begin{bmatrix} -2 & -2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix}$$

- $P(\lambda)$ is positive when $n \in 2\mathbb{N}$, negative otherwise.
  - Signs flip term to term (think about binomial expansion).
- Coefficients of the $n-1$ degree term is the sum of the diagonal entries.
- Coefficient of the 0^{th} degree term is $|A|$.
  - $P_\lambda(0) = |A - 0 \cdot I| = |A|$.
- Product of the eigenvalues is $|A|$.
  - Think about expanding the factorization.
- Eigenvalues of U are the diagonal values.
  - $\lambda_1 \lambda_2 \cdots \lambda_n = |A|$, which is the product of the diagonal entries.
  - $\lambda_1 + \cdots + \lambda_n = \text{trace}(A)$, which is the sum of the diagonal entries.
- $Ax = \lambda x$
  - $A^2 x = AAx = A\lambda x = \lambda Ax = \lambda\lambda x = \lambda^2 x$

Similarity

1/30:

- $A \sim B$[2] iff $\exists\ S : A = SBS^{-1},\ B = S^{-1}AS$.
  1. If $A \sim B$, then $|A| = |B|$.
  $$\begin{aligned} B &= S^{-1}AS \\ |B| &= |S^{-1}AS| \\ |B| &= |S^{-1}||A||S| \\ |B| &= \frac{1}{|S|}|A||S| \\ |B| &= |A| \end{aligned}$$
  2. If $A \sim B$, then they share the same characteristic polynomial.
  $$\begin{aligned} B &= S^{-1}AS \\ |B - \lambda I| &= |S^{-1}AS - \lambda I| \\ &= |S^{-1}AS - \lambda S^{-1}IS| \\ &= |S^{-1}S(A - \lambda I)| \\ &= |I(A - \lambda I)| \\ |B - \lambda I| &= |A - \lambda I| \end{aligned}$$
     - If they have the same characteristic polynomial, $\therefore$ A and B have the same eigenvalues.
- What is the best possible B if $A \sim B$?
  - Sparse.
  - Diagonal.
  - $A = [\text{ugly}] \quad \rightarrow \quad B = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = \Lambda$
- **Diagonalization**:
$$\begin{aligned} A &= S\Lambda S^{-1} \\ AS &= S\Lambda \\ \Lambda &= S^{-1}AS \end{aligned}$$
- $A = S\Lambda S^{-1}$
  - $A^2 = AA = S\Lambda S^{-1}S\Lambda S^{-1} = S\Lambda\Lambda S^{-1} = S\Lambda^2 S^{-1}$
  - $A^k = S\Lambda^k S^{-1}$
  - $A^k = S \begin{bmatrix} {\lambda_1}^k & & 0 \\ & \ddots & \\ 0 & & {\lambda_n}^k \end{bmatrix} S^{-1}$
- Diagonalize the following matrix A.
$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

[2] A "is similar to" B

- Find the characteristic polynomial.

$$\begin{aligned}|A-\lambda I| &= \begin{vmatrix}-1-\lambda & 0 & 1\\ 3 & 0-\lambda & -3\\ 1 & 0 & -1-\lambda\end{vmatrix}\\ &= (-1-\lambda)((-\lambda)(-1-\lambda))+(-1)(-\lambda)\\ &= -\lambda(-1-\lambda)^2+\lambda\\ &= -\lambda(1+2\lambda+\lambda^2)+\lambda\\ &= -\lambda^3-2\lambda^2\\ &= -\lambda^2(\lambda+2)\end{aligned}$$

- Find the eigenvalues: $\lambda_1=\lambda_2=0$, $\lambda_3=-2$
- **Algebraic multiplicity** of λ_1,λ_2 is 2.
- A.M. of λ_3 is 1.
- $A-0I=\begin{bmatrix}-1 & 0 & 1\\ 3 & 0 & -3\\ 1 & 0 & -1\end{bmatrix}$
- $\text{rank}(A-0I)=1 \Rightarrow \dim(N(A-0I))=2$
- The 2 directly above is the **geometric multiplicity**.
- A is diagonalizable iff A.M. of λ_i = G.M.

1/31: - Eigenvectors are $x_1=\begin{bmatrix}1\\0\\1\end{bmatrix}$ and $x_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$.
- $A+2I=\begin{bmatrix}1 & 0 & 1\\ 3 & 2 & -3\\ 1 & 0 & 1\end{bmatrix}$
- Eigenvector is $x_3=\begin{bmatrix}1\\-3\\-1\end{bmatrix}$
- Use an S matrix of eigenvectors.
- $A=S\Lambda S^{-1}=\frac{1}{2}\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & -3\\ 0 & 1 & -1\end{bmatrix}\begin{bmatrix}0 & & \\ & 0 & \\ & & -2\end{bmatrix}\begin{bmatrix}3 & 2 & -3\\ 1 & 0 & 1\\ 1 & 0 & -1\end{bmatrix}$
- Note that $A^{9752}=\frac{1}{2}\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & -3\\ 0 & 1 & -1\end{bmatrix}\begin{bmatrix}0 & & \\ & 0 & \\ & & (-2)^{9752}\end{bmatrix}\begin{bmatrix}3 & 2 & -3\\ 1 & 0 & 1\\ 1 & 0 & -1\end{bmatrix}$

- **Algebraic multiplicity**: The number of repeated roots to a polynomial. For all of the roots, it adds up to n (n-square matrix). *Also known as* **A.M.**
- **Geometric multiplicity**: The number of eigenvectors produced from each root. For all of the roots, it may not add up to n (n-square matrix). $\dim(N(A-\lambda I))$. *Also known as* **G.M.**
- A nondiagonalizable example:

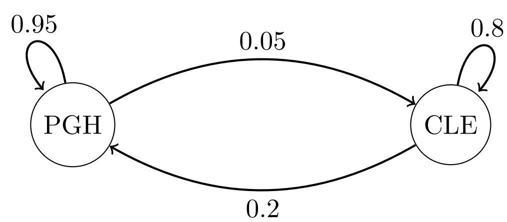
$$A=\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 4\end{bmatrix}$$

  - $\lambda_1=\lambda_2=1$ and $\lambda_3=4$.
  - λ_1 and λ_2 have A.M. = 2.

  - λ_3 has A.M. $= 1$.
  - $A - I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix}$
  - $\text{rank}(A - I) = 2 \Rightarrow \dim(N(A - I)) = 1 \Rightarrow$ G.M. $= 1$.
  - $x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
  - $A - 4I = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
  - $x_2 = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$
  - S would be 3×2 and, thus, not square, so $\nexists S^{-1}$[3].
- **Canonical** (form): An accepted way of expressing something.

Markov Chains

2/3:



- $u_0 = \begin{bmatrix} 500,000 \\ 500,000 \end{bmatrix}$
- $Au_0 = u_1$.
- $Au_1 = u_2$, $A(Au_0) = u_2$, $A^2 u_0 = u_2$, $A^k u_0 = u_k$, $(S\Lambda S^{-1})^k u_0 = u_k$, $S\Lambda^k S^{-1} u_0 = u_k$.

$$A = \begin{bmatrix} 0.95 & 0.20 \\ 0.05 & 0.80 \end{bmatrix} \qquad u_k = \begin{bmatrix} \text{PGH} \\ \text{CLE} \end{bmatrix}$$

- A is a **Markov matrix**, where all columns and rows add to 1.
- $Au_0 = \begin{bmatrix} 0.95 & 0.20 \\ 0.05 & 0.80 \end{bmatrix} \begin{bmatrix} 500,000 \\ 500,000 \end{bmatrix} = u_1$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 0.95 - \lambda & 0.20 \\ 0.05 & 0.80 - \lambda \end{vmatrix} \\ &= (0.95 - \lambda)(0.8 - \lambda) - (0.2)(0.05) \\ &= (\lambda - 1)(\lambda - 0.75) \end{aligned}$$

- $\lambda_1 = 1, \ \lambda_2 = 0.75$.

[3]At a later date, we will look at an analogy of projections to diagonalization that finds the "best possible" diagonalization (which may not be perfectly diagonal).

- $A - I = \begin{bmatrix} -0.05 & 0.2 \\ 0.05 & -0.2 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$
- $A - 0.75I = \begin{bmatrix} 0.2 & 0.2 \\ 0.05 & 0.05 \end{bmatrix} \Rightarrow x = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\begin{aligned} A^k u_0 &= \frac{1}{5} \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.75 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 500,000 \\ 500,000 \end{bmatrix} \\ &= \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.75^k \end{bmatrix} \begin{bmatrix} 200,000 \\ 300,000 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 1 \end{bmatrix} (200,000) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} (0.75)^k (300,000) \end{aligned}$$

- $\begin{bmatrix} 800,000 \\ 200,000 \end{bmatrix}$ is the steady-state vector.
- $\begin{bmatrix} -(0.75)^k(300,000) \\ (0.75)^k(300,000) \end{bmatrix}$ is the dynamically changing vector.
- $\lim_{k \to \infty} A^k u_0 = \begin{bmatrix} 800,000 \\ 200,000 \end{bmatrix} = \begin{bmatrix} \text{PGH} \\ \text{CLE} \end{bmatrix}$

Explicit Formula for the Fibonacci Sequence

2/4:

- 1, 1, 2, 3, 5, 8, ...
- Recursively defined formula: F_n[4]$= F_{n-1} + F_{n-2}$.

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} \\ F_{n-1} &= F_{n-1} \\ \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix} \end{aligned}$$

- $u_n = A^n u_0 = S \Lambda^n S^{-1} u_0$.

$$\begin{aligned} 0 &= |A - \lambda I| \\ &= \begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} \\ &= -\lambda(1-\lambda) - 1 \\ &= \lambda^2 - \lambda - 1 \end{aligned}$$

- $\lambda = \frac{1 \pm \sqrt{5}}{2}$[5].
- $\lambda_1 = \frac{1+\sqrt{5}}{2}$.

$$\begin{aligned} N(A - \lambda_1 I) &= N\left(\begin{bmatrix} 1 - \frac{1+\sqrt{5}}{2} & 1 \\ 1 & -\frac{1+\sqrt{5}}{2} \end{bmatrix} \right) \\ &= N\left(\begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{-1-\sqrt{5}}{2} \end{bmatrix} \right) \end{aligned}$$

[4]The n-th Fibonacci number.
[5]This is the Golden ratio!

- $\begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & \frac{-1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- Let $x_2 = 1$.

$$\begin{aligned} \frac{1-\sqrt{5}}{2}x_1 + 1 &= 0 \\ \frac{1-\sqrt{5}}{2}x_1 &= -\frac{2}{2} \\ x_1 &= \frac{-2}{1-\sqrt{5}} \times \frac{1+\sqrt{5}}{1+\sqrt{5}} \\ &= \frac{-2-2\sqrt{5}}{-4} \\ &= \frac{1+\sqrt{5}}{2} \end{aligned}$$

- $s_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$, $s_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$.
- $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} S^{-1}$
- $S^{-1} = \frac{1}{|S|} C_S^{\mathrm{T}} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}$
- $u_k = A^k u_0 = S\Lambda^k S^{-1} u_0$.

$$\begin{aligned} S^{-1}u_0 &= \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ \frac{-1+\sqrt{5}}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1+\sqrt{5}}{2\sqrt{5}} \\ \frac{-1+\sqrt{5}}{2\sqrt{5}} \end{bmatrix} \\ &= \begin{bmatrix} \frac{5+\sqrt{5}}{10} \\ \frac{5-\sqrt{5}}{10} \end{bmatrix} \end{aligned}$$

- $u_k = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^k \left(\frac{5+\sqrt{5}}{10}\right) + \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix} \left(\frac{1-\sqrt{5}}{2}\right)^k \left(\frac{5-\sqrt{5}}{10}\right)$

Systems of First-Order Ordinary Differential Equations

2/11:

- Let $f(x) = y$ and $a, c, K \in \mathbb{F}$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= ay \\
\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= a \\
\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x &= a\,\mathrm{d}x \\
\frac{1}{y}\,\mathrm{d}y &= a\,\mathrm{d}x \\
\int \frac{1}{y}\,\mathrm{d}y &= \int a\,\mathrm{d}x \\
\ln y &= ax + c \\
y &= \mathrm{e}^{ax+c} \\
&= \mathrm{e}^{ax}\mathrm{e}^{c} \\
&= K\mathrm{e}^{ax}
\end{aligned}$$

- Let $\frac{\mathrm{d}y}{\mathrm{d}x} = y'$.
  - $y_1' = a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n$.
  - $y_2' = a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n$.
  - $\vdots$
  - $y_n' = a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n$.
- This is a **square system** of equations.
- Rewrite as $y' = Ay$.

$$\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

- Solve the following system of differential equations.

$$\begin{aligned}
y_1' &= 3y_1 \\
y_2' &= -2y_2 \\
y_3' &= 5y_3
\end{aligned}$$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

  - General Solution:

$$\begin{aligned}
y_1 &= k_1\mathrm{e}^{3x} \\
y_2 &= k_2\mathrm{e}^{-2x} \\
y_3 &= k_3\mathrm{e}^{5x}
\end{aligned}$$

  - Particular Solution (where $y_1(0) = 2$, $y_2(0) = -1$, and $y_3(0) = 7$ are the initial conditions):

$$\begin{aligned} y_1 &= 2\mathrm{e}^{3x} \\ y_2 &= -\mathrm{e}^{-2x} \\ y_3 &= 7\mathrm{e}^{5x} \end{aligned}$$

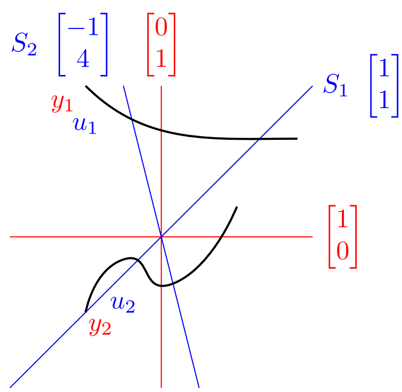
- Consider a different system. Remember throughout that we are solving for y.

$$\begin{aligned} y_1' &= y_1 + y_2 \\ y_2' &= 4y_1 - 2y_2 \end{aligned}$$

  - The previous system was so easy to solve because the matrix was diagonal. This one (as follows) will not be. Therefore, we should diagonalize it.

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

  - Start with $y' = Ay$.
  - Substitute $y = Su$.
    - Note that $y = Su \Rightarrow y' = Su'$[6].
    - If we can find u' in terms of a diagonal matrix and u, we can solve for y.



  - We seek to find a new basis S such that the matrix scaling u will be diagonal.

$$\begin{aligned} Su' &= Ay \\ Su' &= ASu \\ u' &= S^{-1}ASu \\ u' &= \Lambda u \end{aligned}$$

  - The last substitution above is legal because if $A = S\Lambda S^{-1}$, then $\Lambda = S^{-1}AS$.

$$\begin{aligned} 0 &= \begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} \\ &= (1-\lambda)(-2-\lambda) - 4 \\ &= -2 - \lambda + 2\lambda + \lambda^2 - 4 \\ &= \lambda^2 + \lambda - 6 \\ &= (\lambda - 2)(\lambda + 3) \end{aligned}$$

$$\lambda_1 = 2 \qquad\qquad \lambda_2 = -3$$

[6]Think about differentiating both sides: $y \to y'$ is obvious, S will be unchanged because it's just coefficients, and the functions of u will be differentiated.

- $A-2I=\begin{bmatrix}-1 & 1\\ 4 & -4\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$
- $A+3I=\begin{bmatrix}4 & 1\\ 4 & 1\end{bmatrix}\begin{bmatrix}-1\\4\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$$u'=\Lambda u$$
$$\begin{bmatrix}u_1'\\u_2'\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & -3\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}$$

$$u_1=k_1\mathrm{e}^{2x}$$
$$u_2=k_2\mathrm{e}^{-3x}$$

$$y=Su$$
$$\begin{bmatrix}y_1\\y_2\end{bmatrix}=\begin{bmatrix}1 & -1\\ 1 & 4\end{bmatrix}\begin{bmatrix}k_1\mathrm{e}^{2x}\\k_2\mathrm{e}^{-3x}\end{bmatrix}$$

$$y_1=k_1\mathrm{e}^{2x}-k_2\mathrm{e}^{-3x}$$
$$y_2=k_1\mathrm{e}^{2x}+4k_2\mathrm{e}^{-3x}$$

2/12:
- Initial conditions: $y_1(0)=1$ and $y_2(0)=6$.
  - Use augmented matrices to solve a system of equations.

$$\left[\begin{array}{cc|c}1 & -1 & 1\\ 1 & 4 & 6\end{array}\right]\to\left[\begin{array}{cc|c}1 & 0 & 2\\ 0 & 1 & 1\end{array}\right]$$

- Particular solution:

$$y_1=2\mathrm{e}^{2x}-\mathrm{e}^{-3x}$$
$$y_2=2\mathrm{e}^{2x}+4\mathrm{e}^{-3x}$$

Matrix Exponentiation

- e^A is a matrix defined as the infinite sum of a power series.
- $f(t)=\mathrm{e}^t$.

Differential Equations	**Power Series**
$f'(t)=f(t)$	$f(t)=1+t+\frac{t^2}{2}+\frac{t^3}{3!}+\cdots$
$f(0)=1$	$\frac{\mathrm{d}}{\mathrm{d}t}(t)=1,\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^2}{2}\right)=t,\ \ldots$
	$f(t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}$

- $f(t)=\mathrm{e}^{at}$.

Differential Equations	**Power Series**
$f'(t)=af(t)$	$f(t)=\sum_{n=0}^{\infty}\frac{a^nt^n}{n!}$
$f(0)=1$	

- $F(t) = \mathrm{e}^{At}$.
  - A matrix-valued function.
  - Ex. $F(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$
  - $F(\theta)A$ rotates points (arrows) of A by θ.

Differential Equations	**Power Series**
$F'(t) = A\mathrm{e}^{At}$	$F(t) = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$
$F(0) = I$	$F(t) = \sum_{n=0}^{\infty} \frac{A^nt^n}{n!}$

Diagonalization of e^{At}

- Find an alternate form for e^{At} by manipulating its power series definition:

2/13:

$$\begin{aligned}
\mathrm{e}^{At} &= \sum_{n=0}^{\infty} \frac{A^nt^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{S\Lambda^nS^{-1}t^n}{n!} \\
&= \sum_{n=0}^{\infty} S\left(\frac{\Lambda^nt^n}{n!}\right)S^{-1} \\
&= S\left(\sum_{n=0}^{\infty} \frac{t^n}{n!}\Lambda^n\right)S^{-1} \\
&= S\left(\sum_{n=0}^{\infty} \frac{t^n}{n!}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix}^n\right)S^{-1} \\
&= S\left(\sum_{n=0}^{\infty} \frac{t^n}{n!}\begin{bmatrix} \lambda_1^n & & \\ & \ddots & \\ & & \lambda_k^n \end{bmatrix}\right)S^{-1} \\
&= S\left(\sum_{n=0}^{\infty} \begin{bmatrix} \frac{t^n}{n!}\lambda_1^n & & \\ & \ddots & \\ & & \frac{t^n}{n!}\lambda_k^n \end{bmatrix}\right)S^{-1} \\
&= S\left(\sum_{n=0}^{\infty} \begin{bmatrix} \frac{\lambda_1^nt^n}{n!} & & \\ & \ddots & \\ & & \frac{\lambda_k^nt^n}{n!} \end{bmatrix}\right)S^{-1} \\
&= S\begin{bmatrix} \sum_{n=0}^{\infty} \frac{\lambda_1^nt^n}{n!} & & \\ & \ddots & \\ & & \sum_{n=0}^{\infty} \frac{\lambda_k^nt^n}{n!} \end{bmatrix}S^{-1} \\
&= S\begin{bmatrix} \mathrm{e}^{\lambda_1t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_kt} \end{bmatrix}S^{-1} \\
&= S\mathrm{e}^{\Lambda t}S^{-1} \\
&= F(t)
\end{aligned}$$

- Prove, using the above result, that $F'(t)$ can be defined in terms of $F(t)$:

$$\begin{aligned} F(t) &= \mathrm{e}^{At} \\ &= S \begin{bmatrix} \mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_k t} \end{bmatrix} S^{-1} \end{aligned}$$

$$\begin{aligned} F'(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \left(S \begin{bmatrix} \mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_k t} \end{bmatrix} S^{-1} \right) \\ &= S \frac{\mathrm{d}}{\mathrm{d}t} \left(\begin{bmatrix} \mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_k t} \end{bmatrix} \right) S^{-1} \\ &= S \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\lambda_k t} \end{bmatrix} S^{-1} \\ &= S \begin{bmatrix} \lambda_1 \mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \lambda_k \mathrm{e}^{\lambda_k t} \end{bmatrix} S^{-1} \\ &= S \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} \begin{bmatrix} \mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_k t} \end{bmatrix} S^{-1} \\ &= S \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} I_k \begin{bmatrix} \mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_k t} \end{bmatrix} S^{-1} \\ &= {\color{red} S \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{bmatrix} S^{-1}} {\color{blue} S \begin{bmatrix} \mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_k t} \end{bmatrix} S^{-1}} \\ &= {\color{red} A}{\color{blue} F(t)} \\ &= A\mathrm{e}^{At} \end{aligned}$$

- In other words, $y'(t) = Ay(t)$ and $y(0) = y_0$. The solution is $y = \mathrm{e}^{At} y_0$.
- Example:

$$\begin{aligned} y_1' &= 5y_1 + y_2 & y_1(0) &= -3 \\ y_2' &= -2y_1 + 2y_2 & y_2(0) &= 8 \end{aligned}$$

$$\begin{aligned} y(t) &= \mathrm{e}^{At} y(0) \\ &= S\mathrm{e}^{\Lambda t} S^{-1} y(0) \end{aligned}$$

$$\begin{aligned} 0 &= |A - \lambda I| \\ &= \begin{vmatrix} 5-\lambda & 1 \\ -2 & 2-\lambda \end{vmatrix} \\ &= (\lambda - 3)(\lambda - 4) \end{aligned}$$

- $A - 3I = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$
- $N(A - 3I) = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$
- $A - 4I = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}$
- $N(A - 4I) = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

$$\begin{aligned}
y(t) &= S\mathrm{e}^{\Lambda t} S^{-1} y(0) \\
&= \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{e}^{3t} & 0 \\ 0 & \mathrm{e}^{4t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 8 \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{e}^{3t} & -\mathrm{e}^{4t} \\ -2\mathrm{e}^{3t} & \mathrm{e}^{4t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} -3 \\ 8 \end{bmatrix} \\
&= \begin{bmatrix} -\mathrm{e}^{3t} + 2\mathrm{e}^{4t} & -\mathrm{e}^{3t} + \mathrm{e}^{4t} \\ 2\mathrm{e}^{3t} - 2\mathrm{e}^{4t} & 2\mathrm{e}^{3t} - \mathrm{e}^{4t} \end{bmatrix} \begin{bmatrix} -3 \\ 8 \end{bmatrix} \\
&= \begin{bmatrix} 3\mathrm{e}^{3t} - 6\mathrm{e}^{4t} - 8\mathrm{e}^{3t} + 8\mathrm{e}^{4t} \\ -6\mathrm{e}^{3t} + 6\mathrm{e}^{4t} + 16\mathrm{e}^{3t} - 8\mathrm{e}^{4t} \end{bmatrix} \\
&= \begin{bmatrix} -5\mathrm{e}^{3t} + 2\mathrm{e}^{4t} \\ 10\mathrm{e}^{3t} - 2\mathrm{e}^{4t} \end{bmatrix} \\
&= \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}
\end{aligned}$$

Orthonormally Diagonalizable Matrices

2/19:

- $A = Q\Lambda Q^{\mathrm{T}}$.
  - Eigenvectors are orthonormal.
- $A^{\mathrm{T}} = \left(Q\Lambda Q^{\mathrm{T}}\right)^{\mathrm{T}} = Q^{\mathrm{TT}}\Lambda^{\mathrm{T}}Q^{\mathrm{T}} = Q\Lambda Q^{\mathrm{T}} = A$.
- Prove that the symmetric matrices are exactly those that are orthonormally diagonalizable.
  - Let $A = A^{\mathrm{T}}$.

$$Ax_1 = \lambda_1 x_1 \tag{1}$$
$$Ax_2 = \lambda_2 x_2 \tag{2}$$

  - Multiply Equation 1 by x_2^{T} from Equation 2.
    - We have to relate the two equations.
    - Later, we transpose, because we have to specifically target the properties of symmetric matrices.

$$\begin{aligned}
\lambda_1 x_2^{\mathrm{T}} x_1 &= x_2^{\mathrm{T}} A x_1 \\
&= (x_2^{\mathrm{T}} A) x_1 \\
&= \left(A^{\mathrm{T}} x_2\right)^{\mathrm{T}} x_1 \\
&= (Ax_2)^{\mathrm{T}} x_1 \\
\lambda_1 x_2^{\mathrm{T}} x_1 &= \lambda_2 x_2^T x_1 \\
\lambda_1 x_2^{\mathrm{T}} x_1 - \lambda_2 x_2^T x_1 &= 0 \\
x_2^{\mathrm{T}} x_1 (\lambda_1 - \lambda_2) &= 0
\end{aligned}$$

  - The last line above implies that $x_2^{\mathrm{T}} x_1 = 0$ iff $\lambda_1 \neq \lambda_2$.
- The only matrices that we can guarantee will never have complex eigenvalues are symmetric matrices.
- On complex numbers/vectors:
  - $z = a + bi$ and $\bar{z} = a - bi$, where $a, b \in \mathbb{R}$, $i = \sqrt{-1}$.
  - $z\bar{z} = a^2 + b^2$.
  - $x = \begin{bmatrix} a_1 + b_1 i \\ \vdots \\ a_n + b_n i \end{bmatrix}$ and $\bar{x} = \begin{bmatrix} a_1 - b_1 i \\ \vdots \\ a_n - b_n i \end{bmatrix}$.
- Prove that when $A = A^{\mathrm{T}}$, $\lambda_n \in \mathbb{R}$.
  - Let $A = A^{\mathrm{T}}$, $A \in \mathbb{R}^n$.

$$\begin{aligned} Ax &= \lambda x \qquad (3) \\ \bar{A}\bar{x} &= \bar{\lambda}\bar{x} \end{aligned}$$

  - If $A \in \mathbb{R}^n$, then $A = \bar{A}$.

$$\begin{aligned} A\bar{x} &= \bar{\lambda}\bar{x} \\ (A\bar{x})^{\mathrm{T}} &= \left(\bar{\lambda}\bar{x}\right)^{\mathrm{T}} \\ \bar{x}^{\mathrm{T}} A^{\mathrm{T}} &= \bar{\lambda}\bar{x}^{\mathrm{T}} \\ \bar{x}^{\mathrm{T}} A &= \bar{\lambda}\bar{x}^{\mathrm{T}} \qquad (4) \end{aligned}$$

  - Multiply Equation 3 by $\bar{x}^{\mathrm{T}}$ from the left.
    - $\bar{x}^{\mathrm{T}} A x = \lambda \bar{x}^{\mathrm{T}} x$.
  - Multiply Equation 4 by x from the right.
    - $\bar{x}^{\mathrm{T}} A x = \bar{\lambda} \bar{x}^{\mathrm{T}} x$.
  - $\lambda \bar{x}^{\mathrm{T}} x = \bar{\lambda} \bar{x}^{\mathrm{T}} x \Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}$.

Spectral Decomposition

2/20:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

- $\lambda_1 = 4$, $\lambda_2 = \lambda_3 = 1$.
- $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $x_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.
  - $x_1^{\mathrm{T}} x_2 = 0$, $x_1^{\mathrm{T}} x_3 = 0$, $x_2^{\mathrm{T}} x_3 = -1$.
- Orthogonalize by Gram-Schmidt, inspection, put the vectors in a matrix and find the null space (the null vector will be orthogonal by the fundamental theorem).
- $x_3' = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$.
  - x_3' is not scaled along its line by A, but it is scaled in the plane of x_2 and x_3 by A.

  - $x_1^{\mathrm{T}} x_3' = 0$, $x_2^{\mathrm{T}} x_3' = 0$.

- $q_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $q_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$.

$$\begin{aligned} A &= Q\Lambda Q^{\mathrm{T}} \\ &= \begin{bmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} — & q_1^{\mathrm{T}} & — \\ & \vdots & \\ — & q_n^{\mathrm{T}} & — \end{bmatrix} \\ &= \begin{bmatrix} | & & | \\ q_1\lambda_1 & \cdots & q_n\lambda_n \\ | & & | \end{bmatrix} \begin{bmatrix} — & q_1^{\mathrm{T}} & — \\ & \vdots & \\ — & q_n^{\mathrm{T}} & — \end{bmatrix} \\ &= \lambda_1 q_1 q_1^{\mathrm{T}} + \cdots + \lambda_n q_n q_n^{\mathrm{T}} \end{aligned}$$

- $q_1 q_1^{\mathrm{T}} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$

- $q_2 q_2^{\mathrm{T}} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$

- $q_3 q_3^{\mathrm{T}} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{bmatrix}$

- $A = 4 \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{bmatrix}$

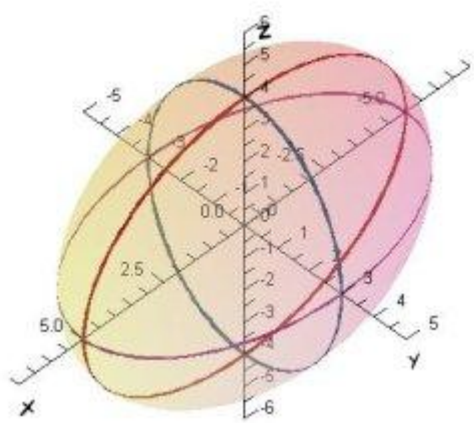
- $A = 4 \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} + \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$ is the spectral decomposition of A.

QUADRIC SURFACES

- A **quadric surface** is the graph of a second-degree equation in three variables: x, y, z
- Its **general form** equation is $Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$ for which $A, B, C, \ldots, J$ are constants
- Through appropriate translation and rotation the general form may be reduced to either $Ax^2 + By^2 + Cz^2 + J = 0$ or $Ax^2 + By^2 + Iz = 0$
- Quadric surfaces are analogous in three dimensions to the conic sections in two dimensions

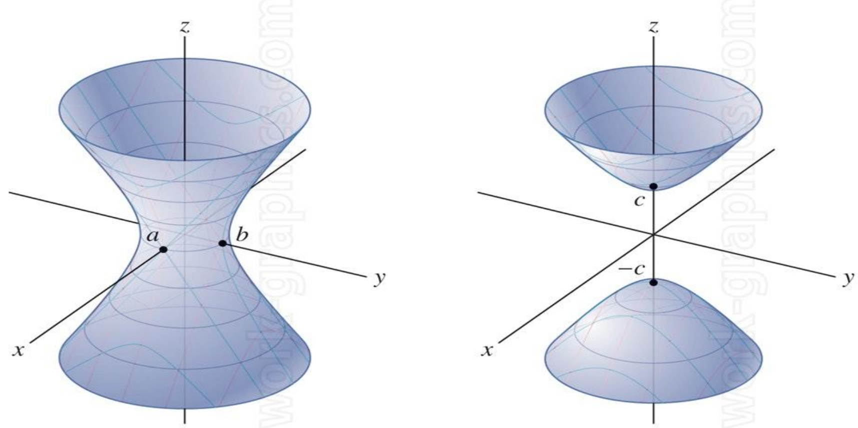
- In order to sketch the graph of a quadric surface (or any surface for that matter) it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes. These are called the **traces**

ELLIPSOIDS



- **Ellipsoids** have elliptical traces
- Equations are in the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ (Equation 1)

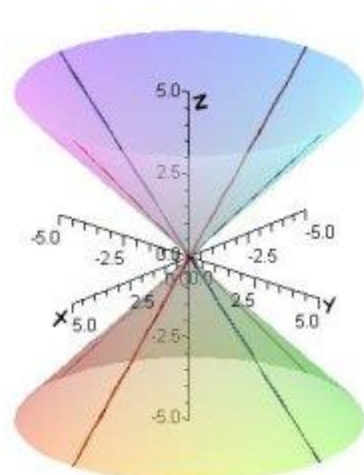
HYPERBOLOIDS



(A) Hyperboloid of one sheet (B) Hyperboloid of two sheets

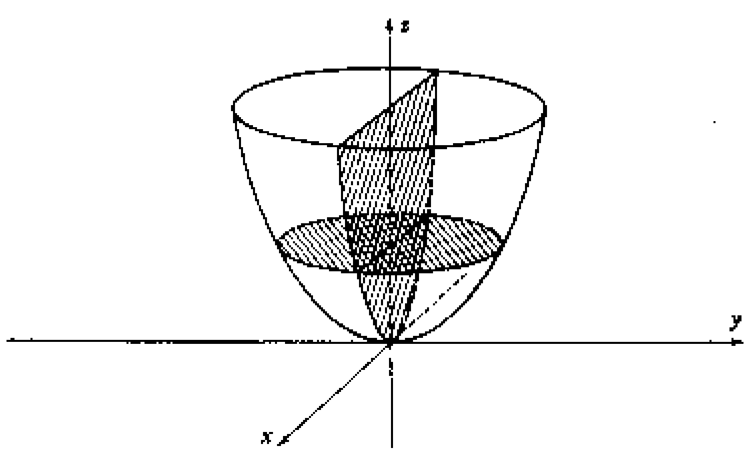
- Equations for (A) are of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ (Equation 2)
- Horizontal traces are ellipses
- Vertical traces are hyperbolas
- The axis of symmetry corresponds to the variables whose coefficient is negatives
- Equations for (B) are of the form $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ (Equation 3)
- Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$
- Vertical traces are hyperbolas
- The two negative signs indicate the two sheets

CONES



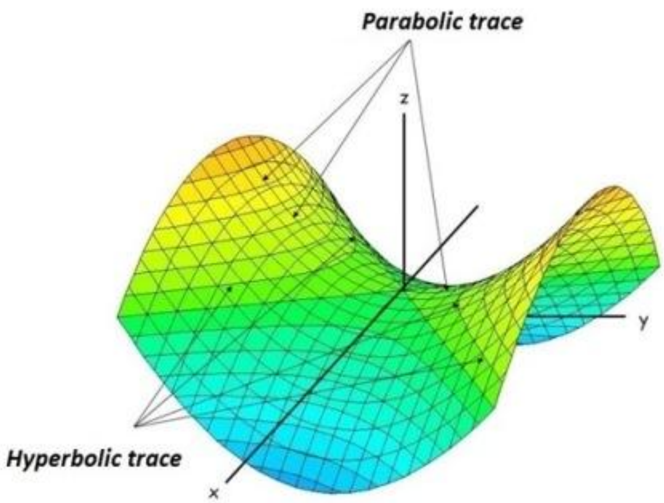
- Equations of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c}$ (Equation 4)
- Horizontal traces are ellipses
- Vertical lines in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$

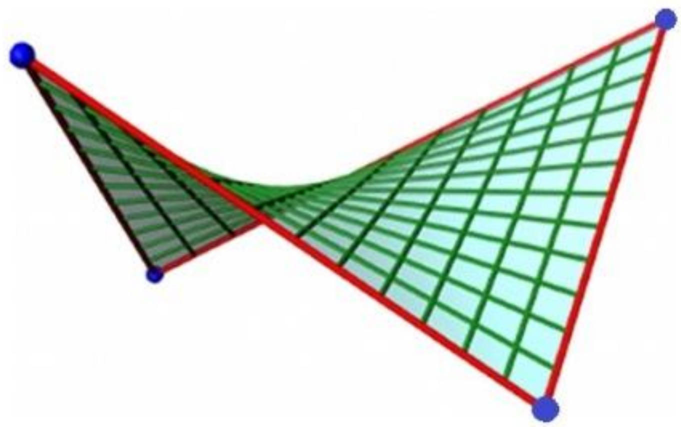
ELLIPTIC PARABOLOIDS



- Equations of the form $\dfrac{z}{c}=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}$ (Equation 5)
- Horizontal traces are ellipses
- Vertical traces are parabolas
- The variables raised to the first power indicates the axis of the paraboloid

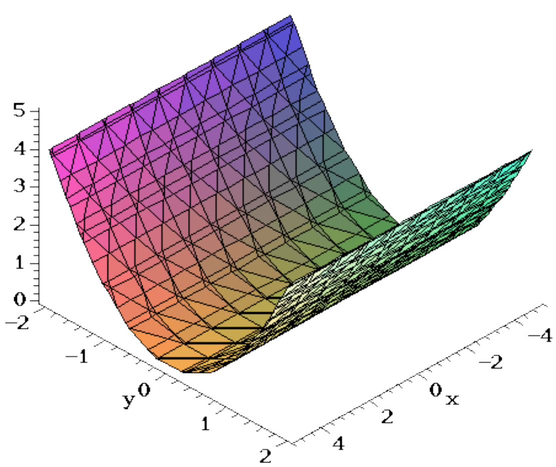
HYPERBOLIC PARABOLOIDS



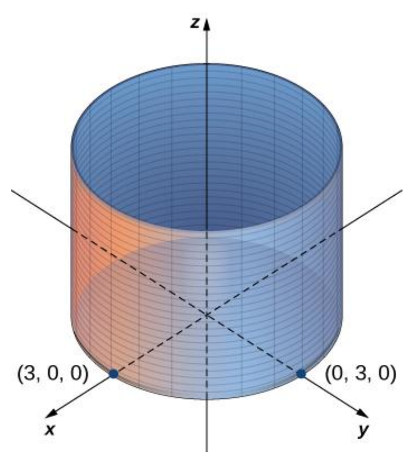


- Equations of the form $\dfrac{z}{c}=\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}$ (Equation 6)
- Horizontal traces are hyperbolas
- Vertical traces are parabolas

QUADRIC CYLINDERS



Parabolic Cylinder



Elliptic Cylinder

- Equations of the form $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ (Equation 7)

Example: Use traces to sketch the quadric surface $x^2+\frac{y^2}{9}+\frac{z^2}{4}=1$

- Let $z=0$ to see that the trace in the $xy-$plane is $x^2+\frac{y^2}{9}=1$, an ellipse.
- Generally speaking, the horizontal trace in plane $z=k$ is $x^2+\frac{y^2}{9}=1-\frac{k^2}{4}$ which is an ellipse provided that $k^2<4$, or $-2<k<2$.
- Vertical traces are also ellipses: $$\begin{aligned}&\frac{y^2}{9}+\frac{z^2}{4}=1-k^2, x=k, -1<k<1\\&x^2+\frac{z^2}{4}=1-\frac{k^2}{9}, y=k, -3<k<3\end{aligned}$$

Example: Use traces to sketch $z=4x^2+y^2$

- Let $x=0$ to see $z=y^2$, indicating that the $xy-$plane intersects the surface in a parabola
- Let $x=k$ to yield $z=y^2+4k^2$, indicating that if we were to slice the graph with any plane parallel to the $yz-$plane we obtain a parabola that opens upward
- Let $y=k$ to yield $z=4x^2+k^2$, indicating another parabola opening upward
- Let $z=k$ to yield horizontal traces $4x^2+y^2=k$, which are ellipse.

Example: Sketch the surface $z=y^2-x^2$

- The traces in vertical planes $x=k$ are parabolas $z=y^2-k^2$,which open upward
- The traces in $y=k$ are parabolas $z=-x^2+k^2$, which open downward
- Horizontal traces are $y^2-x^2=k$, a family of hyperbolas
- Fitting together all these traces, we have a hyperbolic paraoloid

Example: Sketch the surface $\frac{x^2}{4}+y^2-\frac{z^2}{4}=1$

- The trace in any horizontal plane $z=k$ is the ellipse $\frac{x^2}{4}+y^2=1+\frac{k^2}{4}, z=k$
- Traces in $xz-$and $yz-$planes are the hyperbolas $\frac{x^2}{4}-\frac{z^2}{4}=1, y=0$ and $y^2-\frac{z^2}{4}=1, z=0$

Example : Identify and sketch the surface $4x^2 - y^2 + 2z^2 + 4 = 0$.

- First divide the equation by -4 to put it in its standard form: $-x^2 + \frac{y^2}{4} - \frac{z^2}{2} = 1$
- Next match its form with one of the seven equations above. Notice its standard form resembles Equation 3: $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$; this is a hyperboloid of two sheets. Notice the difference is that since the y^2 term of our equation is positive, this hyperboloid will be formed along the y-axis. Equation 3 has a positive z^2 term meaning it would be formed along the z-axis
- This means we will need to sketch the traces of the hyperboloid along the xy-plane and the yz-plane
- For the xy-plane set $z = 0$ and sketch the hyperbola $-x^2 + \frac{y^2}{4} = 1$
- For the yz-plane set $x = 0$ and sketch the hyperbola $\frac{y^2}{4} - \frac{z^2}{2} = 1$
- Since this hyperboloid is formed along the y-axis, there will be no xz-plane traces
- Set $y = k$
- $-x^2 + \frac{k^2}{4} - \frac{z^2}{2} = 1 \rightarrow x^2 + \frac{z^2}{4} = \frac{k^2}{4} - 1$
- Dividing each term in the equation above yields $\frac{x^2}{\frac{k^2}{4} - 1} + \frac{z^2}{2\left(\frac{k^2}{4} - 1\right)} = 1$; This is the trace the plane $y = k$ and

  we can see it is in the form of an ellipse

Try sketching the following surfaces: (a) $x^2 + 2z^2 - 6x - y + 10 = 0$ (Hint: complete the square)

(b) $z^2 = 4x^2 + 9y^2 + 36$ (c) $4x^2 + y^2 + 4z^2 - 4y - 24z + 36 = 0$ (d) $x^2 - y^2 + z^2 - 2x + 2y + 4z + 2 = 0$