THE NORM OF A MATRIX

Let $A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & -1 & -2 \\ -5 & 1 & 3 \end{bmatrix}$

Find $\|A\|_1$ and $\|A\|_\infty$

THE CONDITION NUMBER OF A MATRIX

Show that $A = \begin{bmatrix} 1 & 1 \\ 1 & 1.0005 \end{bmatrix}$ is ill-conditioned.

Let $b = \begin{bmatrix} 3 \\ 3.0010 \end{bmatrix}$. The solution to Ax = b is $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. If A changes to $A' = \begin{bmatrix} 1 & 1 \\ 1 & 1.0010 \end{bmatrix}$, then $x' = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

A relative change of $0.0005 / 1.0005 \approx 0.0005$ or about 0.05% causes a change of (2-1)/1=1 or 100% in x_1 and (1-2)/2=-0.5 or -50% in x_2. A is ill-conditioned.

Let the change from A to A' be an error ΔA that introduces an error Δx in the solution x to Ax =b. Then $A' = A + \Delta A$ and $x' = x + \Delta x$.

$$\Delta A = \begin{bmatrix} 0 & 0 \\ 0 & 0.0005 \end{bmatrix} \text{ and } \Delta x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Since Ax = b and A'x' = b, we have $(A + \Delta A)(x + \Delta x) = b$

$$A(\Delta x) + (\Delta A)x + (\Delta A)(\Delta x) = 0 \quad \text{or} \quad A(\Delta x) = -\Delta A(x + \Delta x)$$

A is invertible if it has a solution, so

$$\Delta x = -A^{-1}(\Delta A)(x + \Delta x) = -A^{-1}(\Delta A)x'$$

Taking norms of all sides

$$\|\Delta x\| = \|-A^{-1}(\Delta A)(x + \Delta x)\| = \|-A^{-1}(\Delta A)x'\|$$

$$\leq \|A^{-1}(\Delta A)\|\|x'\|$$
$$\leq \|A^{-1}\|\|\Delta A\|\|x'\|$$

Therefore $\dfrac{\|\Delta x\|}{\|x'\|} \leq \|A^{-1}\|\|\Delta A\| = \left(\|A^{-1}\|\|A\|\right)\dfrac{\|\Delta A\|}{\|A\|}$

$\|A^{-1}\|\|A\|$ is the CONDITION NUMBER. Cond(A)

If A is not invertible, cond(A) = ∞

$\frac{\|\Delta A\|}{\|A\|}$ is a measure of the relative change in the matrix A, which we assume to be small.

$\frac{\|\Delta x\|}{\|x'\|}$ is a measure of the relative error created in the solution to Ax = b. In this case, the error is measured relative to the new solution x'.

$\frac{\|\Delta x\|}{\|x'\|} \le cond(A)\frac{\|\Delta A\|}{\|A\|}$ gives an upper bound on how large the relative error in the solution can be in terms of the relative error in the coefficient matrix. The larger the condition number, the more ill-conditioned the matrix, since there is more "room" for the error to be large relative to the solution.