THE MATRIX OF A LINEAR TRANSFORMATION

Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ **be defined by** $T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-2y \\ x+y-3z \end{bmatrix}$ **with bases** $B = \ e_1, e_2, e_3$ **for** $\mathbb{R}^3$ **and** $C = \ e_2, e_1$ **for** $\mathbb{R}^2$**. Find the matrix of** T **with respect to** B **and** C**.**

- **Compute** $T(e_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $T(e_2) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $T(e_3) = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$
- **Find their coordinate vectors with respect to** C

  Since $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = e_2 + e_1, \begin{bmatrix} -2 \\ 1 \end{bmatrix} = e_2 - 2e_1, \begin{bmatrix} -3 \\ 0 \end{bmatrix} = -3e_2 + 0e_1$, **we have**

  $$T(e_1)\ _C = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\ T(e_2)\ _C = \begin{bmatrix} 1 \\ -2 \end{bmatrix},\ T(e_3)\ _C = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$

- **Therefore, the matrix of** T **with respect to** B **and** C **is** $\begin{bmatrix} 1 & 1 & -3 \\ 1 & -2 & 0 \end{bmatrix}$.

Let $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$

$$T(v) = T\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ 10 \end{bmatrix}$$

$v\ _B = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ **and** $T(v)\ _C = \begin{bmatrix} -5 \\ 10 \end{bmatrix}_C = \begin{bmatrix} 10 \\ -5 \end{bmatrix}$

$$A\ v\ _B = \begin{bmatrix} 1 & 1 & -3 \\ 1 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \end{bmatrix} = T(v)\ _C$$

Let $D: P_3 \to P_2$ be the differential operator $D(p(x)) = p'(x)$. A basis for P_3 would be $B = \ 1, x, x^2, x^3 \ $ and a basis for P_2 would be $C = \ 1, x, x^2 \ $.

Find the matrix of D with respect to B and C.

The images of B **under** D **are** $D(1) = 0, D(x) = 1, D(x^2) = 2x, D(x^3) = 3x^2$ **and therefore their coordinate vectors with respect to** C **are:**

$$D(1)\ _C = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ D(x)\ _C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \left[D(x^2)\right]_C = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \left[D(x^3)\right]_C = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

Therefore $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Find the matrix A' of D with respect to B' and C where $B' = \ x^3, x^2, x, 1$

The order of the vectors in a basis will affect the matrix of a transformation with respect to the basis. Since basis B' is simply B in reverse order, we see that

$$A' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}$$

Use matrix A to compute $D(5 - x + 2x^3)$

$$A\left[5 - x + 2x^3\right]_B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix} = \left[D(5 - x + 2x^3\right]_C$$

Let $T: P_2 \rightarrow P_2 : T(p(x) = p(2x-1)$

Find the matrix of the linear transformation with respect to $B = \ 1, x, x^2$

$$T(1) = 1, T(x) = 2x-1, T(x^2) = (2x-1)^2 = 1-4x+4x^2$$

$$T(1)\ _B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ T(x)\ _B = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \left[T(x^2)\right]_B = \begin{bmatrix} 1 \\ -4 \\ 4 \end{bmatrix}$$

Therefore $T\ _B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}$

Compute $T(3+2x-x^2)$