Script 5

Connectedness and Boundedness

5.1 Journal

11/19: **Axiom 4.** *A continuum is connected.*

Theorem 5.1. *The only subsets of a continuum C that are both open and closed are $\emptyset$ and C.*

Proof. To prove that the only subsets of C that are both open and closed are $\emptyset$ and C, it will suffice to show that if $A \subset C$ is both open and closed, then $A = \emptyset$ or $A = C$. Let $A \subset C$ be both open and closed. We divide into two cases ($A = \emptyset$ and $A \neq \emptyset$). If $A = \emptyset$, then we are done. On the other hand, if $A \neq \emptyset$, we have a bit more work to do. Basically, we will end up proving that the facts that A is open, A is closed, and $A \neq \emptyset$ imply that $A = C$. Let's begin.

First off, the fact that A is closed implies by Definition 4.8 that $C \setminus A$ is open. Additionally, we have by Script 1 that $A \cap (C \setminus A) = \emptyset$ and $A \cup (C \setminus A) = C$. Now suppose for the sake of contradiction that $A \neq C$. It follows since $A \subset C$ that we must have $C \not\subset A$, i.e., there is some object in C that is not an element of A. This object would clearly be an element of $C \setminus A$ in this case, meaning that $C \setminus A$ is nonempty. Thus, we have that A and $C \setminus A$ are disjoint, open, nonempty sets such that $A \cup (C \setminus A) = C$. Consequently, by consecutive applications of Definition 4.22, we know that C is disconnected, i.e., C is not connected. But this contradicts Axiom 4, which asserts that C is connected. Therefore, we must have that $A = C$, as desired. ☐

Theorem 5.2. *For all $x, y \in C$, if $x < y$, then there exists a point $z \in C$ such that z is between x and y.*

Proof. Suppose for the sake of contradiction that no point $z \in C$ exists such that z is between x and y. To find a contradiction, we will let $A = \{c \in C \mid c < y\}$ and $B = \{c \in C \mid x < c\}$ and prove that $A \cup B = C$, and that A and B are disjoint, nonempty, open sets. This will imply that C is disconnected, contradicting Axiom 4. Let's begin.

Suppose for the sake of contradiction that $C \neq A \cup B$. Then by Theorem 1.7, $C \not\subset A \cup B$ or $A \cup B \not\subset C$. Since $A \subset C$ and $B \subset C$ by their definitions, we have $A \cup B \subset C$, so it must be that $C \not\subset A \cup B$. Thus, by Definition 1.3, there exists a point $p \in C$ such that $p \notin A \cup B$. From the latter condition, we have by Definition 1.5 that $p \notin A$ and $p \notin B$. It follows from the definitions of A and B that $p \notin C$, or $p \not< y$ and $x \not< p$. But we know that $p \in C$, so it must be that $p \not< y$ and $x \not< p$. Equivalently, $p \geq y$ and $x \geq p$. But this implies that $x \geq y$, which contradicts the fact that $x < y$ by hypothesis. Therefore, we must have $C = A \cup B$, as desired.

Suppose for the sake of contradiction that A and B are not disjoint. Then by Definition 1.9, $A \cap B \neq \emptyset$. Thus, Definition 1.8 tells us that there exists some object $p \in A \cap B$. By Definition 1.6, this implies that $p \in A$ and $p \in B$. It follows by the definitions of A and B that $p \in C$, $p < y$, and $x < p$. Since $x < p < y$, Definition 3.6 tells us that p is between x and y, contradicting the supposition that no such point exists. Therefore, A and B are disjoint, as desired.

To prove that A and B are nonempty, Definition 1.8 tells us that it will suffice to show that there exists an object in each set. Since $x \in C$ and $x < y$, $x \in A$. Similarly, since $y \in C$ and $x < y$, $y \in B$. Therefore, A and B are nonempty, as desired.

By Corollary 4.13, A and B are open, as desired.

Since C can be written as $A \cup B$ where A and B are disjoint, nonempty, open sets, we have by Definition 4.22 that C is disconnected. But this contradicts Axiom 4, which asserts that C is connected. Therefore, there must exist a point $z \in C$ such that z is between x and y, as desired. $\square$

Corollary 5.3. *Every region is infinite.*

Proof. Let $\underline{ab}$ be a region, and suppose for the sake of contradiction that $\underline{ab}$ is finite. Then by Definitions 1.30 and 1.33, $\underline{ab} = \emptyset$, or $\underline{ab}$ has cardinality n. We divide into two cases. Suppose first that $\underline{ab} = \emptyset$. Then by Definitions 3.10 and 3.6, no point p exists such that $a < p < b$. Thus, by the contrapositive of Theorem 5.2, $a = b$. But this implies by Definition 3.10 that $\underline{ab}$ is not a region (since $a \not< b$), a contradiction. Now suppose that $\underline{ab}$ has cardinality n. Then by Theorem 3.5, the symbols $a_1, \dots, a_n$ may be assigned to each point of $\underline{ab}$ so that $a_1 < a_2 < \cdots < a_n$. But by Theorem 5.2, there exists a point $z \in C$ such that z is between a and a_1. Since $a < z < a_1 < b$, we clearly have that $z \in \underline{ab}$, yet it was not assigned a symbol a_k, a contradiction. Therefore, $\underline{ab}$ is infinite, as desired. $\square$