MATH 16110 (Honors Calculus I IBL) Notes

Steven Labalme

September 30, 2020

Contents

Introduction to Proofs

9/27:

- Note: These answers address the exercises on the following document.
- We will prove Lemma 4 (i) by contrapositive.

  Lemma 4. *Let x, y be positive integers. Then xy is odd if and only if x and y are both odd.*

  Proof. We wish to prove that if x and y are not both odd, then xy is not odd. In other words, we wish to prove that if at least one of x or y is even, then xy is even. Let's begin. WLOG, let x be even. Then $x = 2k$ for some $k \in \mathbb{N}$. Thus, $xy = 2(ky)$, proving that xy is even since $ky \in \mathbb{N}$. The proof is symmetric for y. □

- We now prove Corollary 5.

  Corollary 5. *Let x, y be positive integers. Then xy is even if and only if at least one of x and y is even.*

  Proof. We wish to prove that xy is even if and only if at least one of x and y is even. Consequently, we must prove the dual implications "if xy is even, then at least one of x and y is even" and "if at least one of x and y is even, then xy is even." Let's begin. For the first statement, let xy be even and suppose for the sake of contradiction that and both x and y are not even, i.e., are odd. But by Lemma 4, it follows from the assumption that x and y are both odd that xy is odd, which contradicts the fact that xy is even. Therefore, at least one of x or y must be even. As to the second statement, suppose that at least one of x or y is even. In this case, x and y are not both odd. Thus, by Lemma 4, xy is not odd, or, equivalently, xy is even. □

- Are there positive integers m, n such that m and n have no common factors (other than 1) and $m^2 = 3n^2$? Either give an example or prove that no example is possible.

  Proof. Let m, n be relatively prime positive integers and suppose for the sake of contradiction that $m^2 = 3n^2$. We divide into two cases (the case where n is even, and the case where n is odd); we seek contradictions in both cases. First off, if n is even, then $n = 2k$ for some $k \in \mathbb{N}$. Thus, $3n^2 = 3(2k)^2 = 12k^2 = 2(6k^2) = m^2$, proving that m^2 is even since $6k^2 \in \mathbb{N}$. By Corollary 5, this implies that m is even. Therefore, since m and n are both even, they have a common factor, a contradiction. On the other hand, if n is odd, then $n = 2k+1$ for some $k \in \mathbb{N}$. Thus, $3n^2 = 3(2k+1)^2 = 12k^2+12k+3 = 2(6k^2+6k+1)+1 = m^2$, proving that m^2 is odd since $6k^2+6k+1 \in \mathbb{N}$. Thus, by Lemma 4, m is odd. Consequently, $m = 2l+1$ for some $l \in \mathbb{N}$, so $m^2 = (2l+1)^2 = 4l^2+4l+1 = 12k^2+12k+3$, the last equality holding because we also have $m^2 = 3n^2 = 12k^2+12k+3$. This implies the following.

  $$\begin{aligned} 4l^2 + 4l + 1 &= 12k^2 + 12k + 3 \\ 4l^2 + 4l &= 12k^2 + 12k + 2 \\ 2l^2 + 2l &= 6k^2 + 6k + 1 \\ 2(l^2 + l) &= 2(3k^2 + 3k) + 1 \end{aligned}$$

  Since $l^2 + l$ and $3k^2 + 3k$ are both natural numbers, the above asserts that an odd number equals an even number, a contradiction. Hence, in both cases, we must have that $m^2 \neq 3n^2$. □

- Are there positive integers m, n such that m and n have no common factors (other than 1) and $m^2 = 6n^2$? Either give an example or prove that no example is possible.

  Proof. Let $m, n \in \mathbb{N}$ have no common factors (other than 1), and suppose for the sake of contradiction that $m^2 = 6n^2$. Since $m^2 = 6n^2 = 2(3n^2)$, m^2 is even. It follows by Corollary 5 that m is even, implying that $m = 2k$ for some $k \in \mathbb{N}$. Thus, $6n^2 = m^2 = (2k)^2 = 4k^2$, so $3n^2 = 2k^2$. Since $k^2 \in \mathbb{N}$, $3n^2$ is even. Consequently, we have that n^2 is even by Corollary 5 (since at least one of 3 or n^2 is even and $3 = 2(1) + 1$ is odd). By Corollary 5 again, n is even. Thus, m and n are both even, contradicting the assumption that they have no common factors other than 1. □

- Are there positive integers m, n such that m and n have no common factors (other than 1) and $m^2 = 4n^2$? Either give an example or prove that no example is possible.

  Proof. Let $m = 2$ and $n = 1$. Then $m^2 = 2^2 = 4 = 4 \cdot 1^2 = 4n^2$. $\square$

Notes on Proofs

Mathematical symbols used in proofs: In the following p and q represent statements that must be either true or false.

1. $p \implies q$ means 'p implies q' or 'if p, then q.'
2. $p \impliedby q$ means 'p is implied by q' or 'p only if q' or 'if q then p.'
3. $p \iff q$ means 'p if and only if q,' i.e. 'if p then q' and 'if q then p.'
4. $\exists$ means 'there exists' or 'there is a.'
5. $\forall$ means 'for all' or 'for every.'
6. $\in$ means 'is a member of'
7. $\subset$ means 'is a subset of' *(See Definition 1.3)*
8. $\supset$ means 'is a superset of.' (A is a superset of $B \iff B$ is a subset of A.)

Remark 1. *These symbols are appropriate for use on the board but should be used sparingly and very carefully in formal write ups. More explanation of this follows in the discussion below.*

Key Points

1. You will rarely be able to write down a completely correct proof, with good mathematical style, on your first attempt. Usually you need to work out how the proof will go and then work out the best way to write it down. Proofs by induction (Sheet 0) are unusual in that there is a basic format that you follow.
2. Proofs must be **written entirely in complete sentences** and should not just consist of calculations (see Example 3). Sentences should not start with symbols.
3. Another student should be able to read your proof and understand it easily. This should be kept in mind especially when writing your journal. You should think of your journal as a textbook that you are writing.
4. You have to be very careful about your use of logical connectives.

   For example, if you want to prove

   $$x^2 + 2xy + y^2 \geq 0, \tag{1}$$

   it is not correct to write

$$\begin{aligned} x^2 + 2xy + y^2 &\geq 0 \\ (x+y)^2 &\geq 0 \\ \text{True} &. \end{aligned}$$

It is clear from this kind of work that you know what you are doing, however it is not a *proof*. You can easily turn it into a proof however. One temptation is to add $\Longleftrightarrow$ symbols where appropriate. ($\Longleftrightarrow$ means 'if and only if,' so can be used to join two equivalent statements.) For example,

$$\begin{aligned} x^2 + 2xy + y^2 &\geq 0 \\ \Longleftrightarrow (x+y)^2 &\geq 0, \end{aligned}$$

which is true.

However, although this is correct, it is not good mathematical style. It is much preferable to write something like:

Since the square of any real number is non-negative, $(x+y)^2 \geq 0$, and so $x^2+2xy+y^2 = (x+y)^2 \geq 0$.

In general, it is much better to use 'so' or 'therefore' than the implication symbol $\Longrightarrow$, which can be ambiguous. Does "so $a = 0 \Longrightarrow b = 0$" mean "so $a = 0$ and therefore $b = 0$," or "so if $a = 0$ then $b = 0$"? It is much better to avoid the ambiguity by using the words rather than the symbol.

5. "If p then q" is not the same is "if q then p," so you must be sure that you are writing your proof in the correct direction.

   For example

$$\begin{aligned} x^2 + 2xy + y^2 &\geq 0 \\ \Longrightarrow (x+y)^2 &\geq 0 \end{aligned}$$

   is true but will not help for the proof of (1) as the implication is in the wrong direction. The fact that (1) implies something true does not mean that (1) must itself be true. It is easy to come up with examples of false statements that imply true ones. For example:

$$\begin{aligned} 0 &= 1 \\ \Longrightarrow 0 \cdot 1 &= 1 \cdot 0 \\ \Longrightarrow 0 &= 0. \end{aligned}$$

6. The equals sign, =, should only be used when 2 quantities are equal and not as a logical connective. For example you should not write

$$\begin{aligned} x^2 + 2xy + y^2 &\geq 0 \\ = (x+y)^2 &\geq 0; \end{aligned}$$

this makes no sense grammatically. (Try reading it aloud.)

7. Make sure that every variable you use has been introduced with a quantifier such as "for every", "there exists", or 'for some". You should generally state what set the variables lie in, unless this is completely obvious from the context. It is usually better to write out quantifiers in words although on the board you may use the symbols above (4 and 5) for brevity.

8. Placement of quantifiers is crucial. Consider the following examples:

   There is a student in this class who is sick every day.

   or

   Every day there is a student in this class who is sick.

   A more mathematical example:

   For every natural number n there is natural number m such that $n < m$.

   or

   There is a natural number m such that $n < m$ for every natural number n.

Do you see the difference?

A couple of examples of proofs

For the sake of the following discussion we will assume that we know the following definition.

Definition 2. a) A positive integer n is even if it can be written as $n = 2k$ for some positive integer k.

b) A positive integer n is odd if it can be written as $n = 2k + 1$ for some positive integer k.

Example 3. If $n \in \mathbb{N}$ and n is odd then n^2 is also odd.

Proof. (first 'proof')
$n = 2k + 1, n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$ □

Proof. (second proof) Since n is odd, we can write n as $n = 2k + 1$ for some $k \in \mathbb{N}$. Then $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$. Since $n^2 = 2m + 1$ for some $m \in \mathbb{N}$, n^2 is odd. □

The first 'proof' is not really a proof. It is simply the calculation that is needed for the proof. In your journals, you need to be writing your proofs more like the second proof. You needn't write as much when presenting at the board, but you do need to write enough that your audience is able to follow your argument. **It usually will not be sufficient to copy down what is written on the board.**

Types of Proof

To *disprove* a statement we need only come up with a *counterexample*, i.e. an example that shows that the statement cannot hold. To *prove* a statement, there are three possible strategies, besides mathematical induction. We suppose that we want to prove a statement of the form **'if** p **then** q**'** (i.e. p is the assumption, q is the conclusion):

1. Direct Proof - we start by assuming that p is true and argue directly, using the assumption, p, and any other statement that has been previously established, until we arrive at statement q.

2. Proof by Contradiction - we start by assuming that the implication is false and argue until we arrive at a false statement, at which point we have a contradiction to the assumption that our implication was false, which proves that the implication is in fact true. For 'if p then q' to be false it must be that p holds but q does not, so a proof by contradiction always starts by assuming that p holds and q does not.

3. Proof by Contraposition - 'if p then q ' is logically equivalent to 'if not q then not p.' A proof by contraposition is a direct proof of the implication 'if not q then not p.'

Some results can be proved by any one of these methods while others may only readily be proven by one of them. Figuring out which proof technique is the best for the problem at hand takes practice. As an example we consider the following lemma.

Lemma 4. *Let x, y be positive integers. Then xy is odd if, and only if, x and y are both odd.*

Proof. Note that since this is an 'if and only if' statement we must prove two implications, namely

(i) If xy is odd, then x and y are both odd, and

(ii) If x and y are both odd, then xy is odd.

(For (i) it is hard to see how to proceed with a direct proof since we need to somehow separate out the x and y, so it is more natural to try one of the other approaches. In this case, the argument is similar whether we use proof by contradiction or contrapositve. We will give one here - you should try the other.)

We will prove (i) by contradiction. So we suppose that xy is odd but (at least) one of x or y is not odd. It follows that (at least) one of x and y must be even. If x is even then

$x = 2k$ for some positive integer k and then $xy = 2(ky)$ which is even also. Similarly, if y is even then xy is even. This contradicts our assumption that xy is odd. Hence x and y must both be odd.

(For (ii) a direct proof is straightforward.)

To prove (ii) we suppose that x and y are both odd. So, by the definition above, there are positive integers k and l with $x = 2k + 1$ and $y = 2l + 1$. Then $xy = (2k + 1)(2l + 1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1$, which is of the form $2m + 1$ for $m = 2kl + k + l$, hence is odd.

$\square$

Corollary 5. *Let x, y be positive integers. Then xy is even if, and only if, at least one of x and y is even.*

Exercise 6. Prove Corollary 5.

Theorem 7. *There are no positive integers m, n such that m and n have no common factors (other than 1) and $m^2 = 2n^2$.*

Proof. We first want to rephrase the theorem as an 'if p then q ' implication. So we want to prove that 'if $m, n \in \mathbb{N}$ have no common factors (other than 1), then $m^2 \neq 2n^2$. *(It seems very hard to know how to approach this either directly or by the contrapositive but proof by contradiction looks more approachable).*

We assume, for the sake of contradiction, that $m, n \in \mathbb{N}$ have no common factors (other than 1) and $m^2 = 2n^2$. Since m^2 is even, we know from Corollary 5 that m must be even. Then $m = 2k$, for some $k \in \mathbb{N}$ and $m^2 = 4k^2 = 2n^2$, so $n^2 = 2k^2$. But then n^2 is even, and from Corollary 5 it follows that n is even. But if m and n are both even then we have a contradiction since we assumed that they have no common factors (other than 1).

$\square$

Exercise 8.

a) Are there positive integers m, n such that m and n have no common factors (other than 1) and $m^2 = 3n^2$? Either give an example or prove that no example is possible.

b) Are then positive integers m, n such that m and n have no common factors (other than 1) and $m^2 = 6n^2$? Either give an example or prove that no example is possible.

c) Are then positive integers m, n such that m and n have no common factors (other than 1) and $m^2 = 4n^2$? Either give an example or prove that no example is possible.

Remark 9. *Note that Theorem 7 tells us that there is no rational whose square is equal to 2, i.e. that $\sqrt{2}$ is irrational. However at this point in the course we have yet to meet rationals and irrationals.*

We finish with some examples of proofs that involve sets. This section should be read *after you have seen Script 1, Theorem 1.7.*

Example 10. Let A and B be two sets. If $A \cup B = A \cap B$ then $A = B$.

Proof. When giving a direct proof that 2 sets are equal Theorem 1.7a) is usually invoked. We suppose that $A \cup B = A \cap B$. We first note that A and B can be interchanged in Theorem 1.7. Suppose that $x \in A$. Then, by Theorem 1.7b) , $x \in A \cup B$ and so, since $A \cup B = A \cap B$, by Theorem 1.7c), $x \in B$. Hence every element in A is also in B and so $A \subset B$. A similar argument shows that $B \subset A$ and so $A = B$ (by Theorem 1.7a)). □

This result could also be proved by the contrapositive or by contradiction. We give an alternative proof by contrapositive.

Proof. We want to show that if $A \neq B$, then $A \cup B \neq A \cap B$. Well, if $A \neq B$, then by Definition 1.2, either there is some $a \in A$ such that $a \notin B$, or there is some $b \in B$ such that $b \notin A$. We suppose that $a \in A, a \notin B$. (The other case is similar.) Then $a \in A \cup B$, by Theorem 1.7b) but $a \notin A \cap B$, by definition of intersection. So $A \cup B \neq A \cap B$.

□

Exercise 11. Use Theorem 1.7 to prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Exercise 12. (See Definition 1.11) Let A, B be subsets of a set X. Show that

a) if $A \subset B$ then $X \setminus B \subset X \setminus A$.

b) $X \setminus (X \setminus A) = A$.

Introduction to IBL

9/29:
- ZZ or zih-HWAY, Dixon Instructor in Department of Mathematics.
- Judson (super reader) is an advanced undergraduate who has taken this class before.
- Honors Calculus uses Spivak — we do not have a textbook, just scripts!
  - Few lectures in the traditional sense.
  - Majority of material is presented and developed by the students.
  - Several scripts will be covered throughout the quarter.
    - In scripts: It is our job to complete the exercises, prove the theorems/lemmas/propositions, etc.
  - Be on the look-out for "no proof required" theorems.
  - 3 chances to learn/review scripts material:
    1. Before class, you prepare your own proof.
    2. During class, we discuss.
    3. After class and before the journal is due, we type up our own record of the proof in LaTeX.
- Before each class, she will tell us which theorems/exercises we need to work through.
- Your proofs do not have to be perfect in the beginning! Judson and ZZ will help us. Expect to present every other week.
  - For the first two scripts, you have the ability to rewrite your journal after Judson reviews it to recover up to half of the lost credit.
  - You only recover credit if your new solution is perfect.
  - Return your changes one week after Judson grades it.
    - Mark what parts/problems you have rewritten, and turn in the original as well.
- Later this afternoon, ZZ will share which Script 0 problems we should do before Thursday. Sign up for problems on a Google Sheet before 7:00 PM on Wednesday.
- She chooses a presenter based on our 0-3 rankings.
  - A 0 means you don't know stuff or don't want to present.
  - A 3 means you really want to present stuff.
  - Other numbers are in between.
- Class participation: When and how often and the quality of our presentations, and also how good are our questions that help presenters fill in the gaps.
- For hard proofs she may designate a backup presenter.
- We can use Overleaf for collaborative LaTeX projects.
- We can check in with ZZ on our progress whenever throughout the quarter.
- She won't assign homework for the first week so that we can familiarize ourselves with LaTeX.
  - First HW assignment is due Thursday next week (10/8/2020)?
- Judson's office hours: We get to talk to him one-on-one with questions.
  - Problem session: we're all working collaboratively to figure something out.
- You have one chance to ask for a 24-hour extension on HW (like if you're sick).

- In the case of a switch to virtual class:
  - We can present by turning our phone into a document camera or using a white board behind us or typing up in LaTeX (in real time?).
- Get good at writing — you cannot type up your solutions during exams!
- We submit HW assignments through Canvas if we type it up in LaTeX, or in class by hand. It's nice if we can type it up.

Problems

Exercise 0.2 (PMI Exercise 2)**.** *Prove that if $x > -1$, then $(1+x)^n \geq 1 + nx$ for any natural number n. (Note that although this script is focused on the natural numbers, your argument should hold for any real number $x > -1$.)*

Proof. We induct on n. For the base case $n = 1$, we have $(1+x)^1 = 1+x \geq 1+(1)x$, where the greater than or equal to relation could be strengthened to equality but will be left as such for the sake of the argument. Now suppose inductively that we have proven the claim for some natural number n, i.e., we know that $(1+x)^n \geq 1+nx$ if $x > -1$. We now seek to prove it for $n+1$. To begin, we have $(1+x)^{n+1} = (1+x)^n(1+x)$ by the laws of exponents. By the inductive hypothesis and the fact that $ac \geq bc$ if and only if a, b, c are positive numbers and $a \geq b$ (note that $x > -1$ implies $1 + x > 0$ along with $(1+x)^n > 0$), we have $(1+x)^n(1+x) \geq (1+nx)(1+x) = 1+nx+x+nx^2$. Since x^2 must be positive or zero and $n \in \mathbb{N}$ is clearly positive, we have that $nx^2 \geq 0$ so that $1 + nx + x + nx^2 \geq 1 + nx + x = 1 + (n+1)x$. To recap,

$$\begin{aligned}
(1+x)^{n+1} &= (1+x)^n(1+x) \\
&\geq (1+nx)(1+x) \\
&= 1 + nx + x + nx^2 \\
&= 1 + (n+1)x + nx^2 \\
&\geq 1 + (n+1)x
\end{aligned}$$

thus closing the induction. ☐

Theorem 1.12. *Let X be a set, and let $A, B \subset X$. Then*

a) $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$

b) $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$

Proof of a. To prove that $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$, Definition 1.2 tells us that it will suffice to prove that $x \in X \setminus (A \cup B)$ if and only if $x \in (X \setminus A) \cap (X \setminus B)$, i.e., that if $x \in X \setminus (A \cup B)$, then $x \in (X \setminus A) \cap (X \setminus B)$ and if $x \in (X \setminus A) \cap (X \setminus B)$, then $x \in X \setminus (A \cup B)$. To begin, let $x \in X \setminus (A \cup B)$. By Definition 1.11, $x \in X$ and $x \notin A \cup B$. By Definition 1.5, it follows that $x \notin A$ and $x \notin B$. Since we know that $x \in X$ and $x \notin A$, Definition 1.11 tells us that $x \in X \setminus A$. Similarly, $x \in X \setminus B$. Since $x \in X \setminus A$ and $x \in X \setminus B$, we have by Definition 1.6 that $x \in (X \setminus A) \cap (X \setminus B)$, as desired. The proof of the other implication is the preceding proof "in reverse." For clarity, let $x \in (X \setminus A) \cap (X \setminus B)$. By Definition 1.6, $x \in X \setminus A$ and $x \in X \setminus B$. By consecutive applications of Definition 1.11, $x \in X$, $x \notin A$, and $x \notin B$. Since $x \notin A$ and $x \notin B$, Definition 1.5 reveals that $x \notin A \cup B$. But as previously established, $x \in X$, so Definition 1.11 tells us that $x \in X \setminus (A \cup B)$. ☐

Proof of b. To prove that $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$, Definition 1.2 tells us that it will suffice to prove that $x \in X \setminus (A \cap B)$ if and only if $x \in (X \setminus A) \cup (X \setminus B)$. To begin, let $x \in X \setminus (A \cap B)$. By Definition 1.11, $x \in X$ and $x \notin A \cap B$. By Definition 1.6, it follows that $x \notin A$ or $x \notin B$. We divide into two cases. If $x \notin A$, then since we know that $x \in X$, Definition 1.11 tells us that $x \in X \setminus A$. It naturally follows that $x \in (X \setminus A) \cup (X \setminus B)$, since x need only be an element of one of the two unionized sets (see Definition 1.5). The proof is symmetric if $x \notin B$. Now let $x \in (X \setminus A) \cup (X \setminus B)$. By Definition 1.5, $x \in X \setminus A$ or $x \in X \setminus B$. Once again, we divide into two cases. If $x \in X \setminus A$, then $x \in X$ and $x \notin A$ by Definition 1.11. Consequently, by Definition 1.6, $x \notin A \cap B$. Therefore, $x \in X \setminus (A \cap B)$ by Defintion 1.11. The proof is symmetric if $x \in X \setminus B$. $\square$

Exercise 1.19. *Must $f(f^{-1}(Y)) = Y$ and $f^{-1}(f(X)) = X$? For each, either prove that it always holds or give a counterexample.*

Proof. We will address each statement in turn.

Consider the set $f(f^{-1}(Y))$. By consecutive applications of Definition 1.18, we have that $f(f^{-1}(Y)) = f(\{a \in A \mid f(a) \in Y\}) = \{f(x) \in B \mid x \in \{a \in A \mid f(a) \in Y\}\}$. Now the rightmost set in the previous equality expresses the fact that $f(f^{-1}(Y))$ is the set of all elements $f(x)$ in B such that "$x \in \{a \in A \mid f(a) \in Y\}$" is a true statement. Equivalently, $f(f^{-1}(Y))$ is the set of all elements $f(x)$ in B such that $x \in A$ and $f(x) \in Y$. But since $Y \subset B$, the constraint that $f(x) \in Y$ implies that $f(x) \in B$ by Definition 1.3, meaning that $f(f^{-1}(Y))$ can be thought of as the set of all elements $f(x)$ in Y such that $x \in A$. Since $x \in A$ for every $f(x) \in Y$, $f(f^{-1}(Y))$ is just the set of all $f(x)$ in Y, i.e., the set of all elements of Y, i.e., Y.

Consider the sets $\{1, 2\}$ and $\{3, 4\}$, and let $f : \{1, 3\} \to \{3, 4\}$ be a function defined by $f(1) = 3$ and $f(2) = 3$. Let $X = \{1\}$ (we clearly have $X \subset \{1, 2\}$ since 1 is the only element of X and $1 \in \{1, 2\}$ [see Definition 1.3]). Then $f(X) = \{f(x) \in \{3, 4\} \mid x \in \{1\}\} = \{f(1)\} = \{3\}$ and $f^{-1}(f(X)) = \{a \in \{1, 2\} \mid f(a) \in \{3\}\} = \{1, 2\} \neq X$ by consecutive applications of Definition 1.18. $\square$

Proposition 1.26. *Let A, B, and C be sets and suppose that $f : A \to B$ and $g : B \to C$. Then $g \circ f : A \to C$ and*

a) if f and g are both injections, so is $g \circ f$.

b) if f and g are both surjections, so is $g \circ f$.

c) if f and g are both bijections, so is $g \circ f$.

Proof of a. Suppose that $(g \circ f)(a) = (g \circ f)(a')$. By Definition 1.25, this implies that $g(f(a)) = g(f(a'))$. Since g is injective, Definition 1.20 tells us that $f(a) = f(a')$. Similarly, the fact that f is injective tells us that $a = a'$. Since we have shown that $(g \circ f)(a) = (g \circ f)(a')$ implies that $a = a'$ under the given conditions, we know by Definition 1.20 that $g \circ f$ is injective. $\square$

Proof of b. Let c be an arbitrary element of C. We wish to prove that there exists some $a \in A$ such that $(g \circ f)(a) = c$ (Definition 1.20). By Definition 1.25, it will suffice to show that there exists some $a \in A$ such that $g(f(a)) = c$. Let's begin. By the surjectivity of g, there exists some $b \in B$ such that $g(b) = c$ (see Definition 1.20). If we now consider this b, we have by the surjectivity of f that there exists some $a \in A$ such that $f(a) = b$ (see Definition 1.20). But this a is an element of A such that $g(f(a)) = g(b) = c$, as desired. $\square$

Proof of c. Suppose that f and g are two bijective functions. By Definition 1.20, this implies that f and g are both injections and are both surjections. Thus, by part (a), $g \circ f$ is an injection, and by part (b), $g \circ f$ is a surjection. Therefore, by Definition 1.20, $g \circ f$ is a bijection. $\square$