Script 2

The Rationals

2.1 Journal

10/15: **Definition 2.1.** Let X be a set. A **relation** R on X is a subset of $X \times X$. The statement $(x, y) \in R$ is read "x is related to y by the relation R" and is often denoted $x \sim y$.

A relation is **reflexive** if $x \sim x$ for all $x \in X$.
A relation is **symmetric** if $y \sim x$ whenever $x \sim y$.
A relation is **transitive** if $x \sim z$ whenever $x \sim y$ and $y \sim z$.
A relation is an **equivalence relation** if it is reflexive, symmetric, and transitive.

Exercise 2.2. Determine which of the following are equivalence relations.

a) Any set X with the relation $=$. So $x \sim y$ if and only if $x = y$.

Proof. To prove that the relation $=$ is reflexive, Definition 2.1 tells us that it will suffice to show that $x \sim x$ for all $x \in X$. Clearly, $x = x$ for all $x \in X$. It follows by the definition of $=$ that $x \sim x$ for all $x \in X$. For symmetry, we must verify that $x \sim y$ implies $y \sim x$ for $x, y \in X$. Let $x \sim y$ for some $x, y \in X$. Consequently, by the definition of $=$, $x = y$. It follows that $y = x$, and thus that $y \sim x$. For transitivity, we must show that $x \sim y$ and $y \sim z$ imply that $x \sim z$ for $x, y, z \in X$. Let $x \sim y$ and $y \sim z$ for some $x, y, z \in X$. By the definition of $=$, $x \sim y$ and $y \sim z$ imply that $x = y$ and $y = z$, respectively. Thus, $x = y = z$, so $x = z$, meaning that $x \sim z$ by the definition of the relation $=$. Since the relation $=$ is reflexive, symmetric, and transitive, it is an equivalence relation. $\square$

b) $\mathbb{Z}$ with the relation $<$.

Proof. Consider $1 \in \mathbb{Z}$, and note that $1 = 1$. Since $1 = 1$, $1 \not< 1$ by the trichotomy. Thus, $1 \nsim 1$ by the relation $<$, proving that $<$ is not reflexive for all $z \in \mathbb{Z}$. Therefore, by Definition 2.1, $<$ is not an equivalence relation. $\square$

c) Any subset X of $\mathbb{Z}$ with the relation $\leq$. So $x \sim y$ if and only if $x \leq y$.

Proof. Let $X = \{1, 2\}$. Clearly, $X \subset \mathbb{Z}$. Now, $1 \leq 2$, so $1 \sim 2$ by the relation $\leq$, but $2 \nleq 1$ so $2 \nsim 1$. Thus, $x \sim y$ for $x, y \in X$ does not necessarily imply that $y \sim x$. It follows by Definition 2.1 that $\leq$ is not an equivalence relation on *any* subset of $\mathbb{Z}$. $\square$

d) $X = \mathbb{Z}$ with $x \sim y$ if and only if $y - x$ is divisible by 5.

Proof. To prove that the described relation is an equivalence relation, Definition 2.1 tells us that we must verify that it is reflexive, symmetric, and transitive. To prove these properties, it will suffice to show that $x \sim x$ for all $x \in X$, $x \sim y$ implies $y \sim x$ for any $x, y \in X$, and $x \sim y$ and $y \sim z$ implies $x \sim z$ for any $x, y, z \in X$, respectively. Let's begin.

To prove that $x \sim x$ for all $x \in X$, the definition of $\sim$ and Additional Exercise 0.8 tell us that it will suffice to show that $x - x = 5a$ for an arbitrary $x \in X$ and some $a \in \mathbb{Z}$. Let x be an arbitrary element of X. It follows that $x - x = 0 = 5(0)$ where $0 = a$ is clearly an element of $\mathbb{Z}$. In sum, $x - x = 5a$ for an $a \in \mathbb{Z}$, as desired.

To prove that $x \sim y$ implies that $y \sim x$, the definition of $\sim$ tells us that it will suffice to show that $x-y$ is divisible by 5 given that $y - x$ is so divisible for $x, y \in X$. Let $y - x$ be divisible by 5. It follows by Additional Exercise 0.8-ii that $-1 \cdot (y-x)$ is divisible by 5 (since $-1 \in \mathbb{Z}$). But since $-(y-x) = x-y$, this means that $x - y$ is divisible by 5, as desired.

To prove that $x \sim y$ and $y \sim z$ imply that $x \sim z$, the definition of $\sim$ tells us that it will suffice to show that $z - x$ is divisible by 5 given that $y - x$ and $z - y$ are also so divisible for $x, y, z \in X$. Let $y - x$ and $z - y$ be divisible by 5. It follows by Additional Exercise 0.8-i that $(z - y) + (y - x)$ is divisible by 5. But since $(z - y) + (y - x) = z - x$, this means that $z - x$ is divisible by 5, as desired. □

e) $X = \{(a, b) \mid a, b \in \mathbb{Z}, b \neq 0\}$ with the relation $\sim$ defined by $(a, b) \sim (c, d) \iff ad = bc$.

Proof. Reflexivity: Let (a, b) be an arbitrary element of X. Since $a, b \in \mathbb{Z}$ and integer multiplication is commutative, it is true that $ab = ba$. Therefore, by the definition of the relation $\sim$, $(a, b) \sim (a, b)$.

Symmetry: Let $(a, b) \sim (c, d)$ for some $(a, b), (c, d) \in X$. By the definition of the relation $\sim$, $ad = bc$. Thus, $cb = da$ by the symmetry of $=$ (see Exercise 2.2a) and the commutativity of integer multiplication. Therefore, by the definition of the relation $\sim$, $(c, d) \sim (a, b)$.

Transitivity: Let $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$ for some $(a, b), (c, d), (e, f) \in X$. By consecutive applications of the definition of $\sim$, $ad = bc$ and $cf = de$. We now divide into two cases ($c \neq 0$ and $c = 0$; the reason for doing so will become clear later). Suppose first that $c \neq 0$. By the multiplicative property of equality, we can multiply $cf = de$ to both sides, creating the equation $ad \cdot cf = bc \cdot de$. By the commutativity of multiplication, we have $afcd = becd$. Since $c \neq 0$ by assumption and $d \neq 0$ by the definition of X, $cd \neq 0$ and the cancellation law for multiplication applies, giving us $af = be$. Therefore, by the definition of the relation $\sim$, $(a, b) \sim (e, f)$. Now suppose that $c = 0$. Consequently, $bc = 0$, implying by the equality $ad = bc$ that $ad = 0$. Thus, $a = 0$ or $d = 0$ (or both) by the zero product property. Since $d \neq 0$ by the definition of X (d is the second element in the ordered pair (c, d)), we must have $a = 0$. A similar analysis can be performed on the equation $cf = de$ to show that $e = 0$. Since $a = 0$ and $e = 0$, $af = 0$ and $be = 0$, implying by transitivity that $af = be$. Therefore, by the definition of the relation $\sim$, $(a, b) \sim (e, f)$.

Since the relation $\sim$ is reflexive, symmetric, and transitive, Definition 2.1 tells us that it is an equivalence relation. □

Remark 2.3. A **partition** of a set is a collection of non-empty disjoint subsets whose union is the original set. Any equivalence relation on a set creates a partition of that set by collecting into subsets all of the elements that are equivalent (related) to each other. When the partition of a set arises from an equivalence relation in this manner, the subsets are referred to as **equivalence classes**.

Remark 2.4. If we think of the set X in Exercise 2.2e as representing the collection of all fractions whose denominators are not zero, then the relation $\sim$ may be thought of as representing the equivalence of two fractions.

Definition 2.5. As a set, the **rational numbers**, denoted $\mathbb{Q}$, are the equivalence classes in the set $X = \{(a, b) \mid a, b \in \mathbb{Z}, b \neq 0\}$ under the equivalence relation $\sim$ as defined in Exercise 2.2e. If $(a, b) \in X$, we denote the equivalence class of this element as $\left[\frac{a}{b}\right]$. So

$$\left[\frac{a}{b}\right] = \{(x_1, x_2) \in X \mid (x_1, x_2) \sim (a, b)\} = \{(x_1, x_2) \in X \mid x_1 b = x_2 a\}$$

Then,

$$\mathbb{Q} = \left\{ \left[\frac{a}{b}\right] \middle| (a,b) \in X \right\}$$

Exercise 2.6. $\left[\frac{a}{b}\right] = \left[\frac{a'}{b'}\right] \iff (a,b) \sim (a',b')$

Proof. Suppose first that $\left[\frac{a}{b}\right] = \left[\frac{a'}{b'}\right]$. Since $\left[\frac{a}{b}\right] \in \mathbb{Q}$, Definition 2.5 implies that $(a,b) \in X$. It follows by Exercise 2.2e that $(a,b) \sim (a,b)$. The last two results imply by Definition 2.5 that $(a,b) \in \left[\frac{a}{b}\right]$. Consequently, set equality implies that $(a,b) \in \left[\frac{a'}{b'}\right]$. But by Definition 2.5, this means that $(a,b) \sim (a',b')$, as desired. Now suppose that $(a,b) \sim (a',b')$. To prove that $\left[\frac{a}{b}\right] = \left[\frac{a'}{b'}\right]$, Definition 1.2 tells us that we must verify that every element of $\left[\frac{a}{b}\right]$ is an element of $\left[\frac{a'}{b'}\right]$ and vice versa. Let (x_1,x_2) be an arbitrary element of $\left[\frac{a}{b}\right]$. It follows by Definition 2.5 that $(x_1,x_2) \in X$ and that $(x_1,x_2) \sim (a,b)$. The latter result combined with the hypothesis that $(a,b) \sim (a',b')$ implies by the transitivity of $\sim$ (see Exercise 2.2e) that $(x_1,x_2) \sim (a',b')$. This new finding coupled with the fact that $(x_1,x_2) \in X$ implies by Definition 2.5 that $(x_1,x_2) \in \left[\frac{a'}{b'}\right]$, as desired. The proof is symmetric if we first let that (x_1,x_2) be an arbitrary element of $\left[\frac{a'}{b'}\right]$. $\square$

Definition 2.7. We define the binary operations addition and multiplication on $\mathbb{Q}$ as follows. If $\left[\frac{a}{b}\right], \left[\frac{c}{d}\right] \in \mathbb{Q}$, then

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{ad+bc}{bd}\right]$$
$$\left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{ac}{bd}\right]$$

We use the notation $+_{\mathbb{Q}}$ and $\cdot_{\mathbb{Q}}$ to represent addition and multiplication in $\mathbb{Q}$ so as to distinguish these operations from the usual addition $(+)$ and multiplication $(\cdot)$ in $\mathbb{Z}$.

Theorem 2.8. *Addition in $\mathbb{Q}$ is well-defined. That is, if $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$, then*

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{a'}{b'}\right] +_{\mathbb{Q}} \left[\frac{c'}{d'}\right]$$

Proof. By consecutive applications of the definition of $\sim$, we have from the hypotheses that

$$ab' = ba' \qquad\qquad cd' = dc'$$

It follows by the multiplicative property of equality that

$$ab'dd' = ba'dd' \qquad\qquad bb'cd' = bb'dc'$$

The above two results can be combined via the additive property of equality, giving the following, which will be algebraically manipulated further.

$$\begin{aligned} ab'dd' + bb'cd' &= ba'dd' + bb'dc' \\ adb'd' + bcb'd' &= bda'd' + bdb'c' \\ (ad+bc)(b'd') &= (bd)(a'd'+b'c') \end{aligned}$$

The last line above implies by the definition of $\sim$ that $(ad+bc, bd) \sim (a'd'+b'c', b'd')$. It follows by Exercise 2.6 that

$$\left[\frac{ad+bc}{bd}\right] = \left[\frac{a'd'+b'c'}{b'd'}\right]$$

Therefore, by two applications of Definition 2.7,

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{a'}{b'}\right] +_{\mathbb{Q}} \left[\frac{c'}{d'}\right]$$

as desired. $\square$

Theorem 2.9. *Multiplication in $\mathbb{Q}$ is well-defined. That is, if $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$, then*

$$\left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{a'}{b'}\right] \cdot_{\mathbb{Q}} \left[\frac{c'}{d'}\right]$$

Proof. By consecutive applications of the definition of $\sim$, we have from the hypotheses that $ab' = ba'$ and $cd' = dc'$. Thus, by the multiplicative property of equality, $ab'cd' = ba'dc'$. This can be algebraically rearranged into $(ac)(b'd') = (bd)(a'c')$. It follows by the definition of $\sim$ that $(ac, bd) \sim (a'c', b'd')$. But this implies by Exercise 2.6 that

$$\left[\frac{ac}{bd}\right] = \left[\frac{a'c'}{b'd'}\right]$$

Consequently, by Definition 2.7,

$$\left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{a'}{b'}\right] \cdot_{\mathbb{Q}} \left[\frac{c'}{d'}\right]$$

as desired. □

Theorem 2.10.

a) *Commutativity of addition*

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{c}{d}\right] +_{\mathbb{Q}} \left[\frac{a}{b}\right] \text{ for all } \left[\frac{a}{b}\right], \left[\frac{c}{d}\right] \in \mathbb{Q}$$

Proof. By Definition 2.7,

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{ad+bc}{bd}\right]$$

With integer algebra, we can rearrange the above expression into

$$= \left[\frac{cb+da}{db}\right]$$

By Definition 2.7 again, the above

$$= \left[\frac{c}{d}\right] +_{\mathbb{Q}} \left[\frac{a}{b}\right]$$

□

b) *Associativity of addition*

$$\left(\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{c}{d}\right]\right) +_{\mathbb{Q}} \left[\frac{e}{f}\right] = \left[\frac{a}{b}\right] +_{\mathbb{Q}} \left(\left[\frac{c}{d}\right] +_{\mathbb{Q}} \left[\frac{e}{f}\right]\right) \text{ for all } \left[\frac{a}{b}\right], \left[\frac{c}{d}\right], \left[\frac{e}{f}\right] \in \mathbb{Q}$$

Proof. By consecutive applications of Definition 2.7,

$$\begin{aligned}\left(\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{c}{d}\right]\right) +_{\mathbb{Q}} \left[\frac{e}{f}\right] &= \left[\frac{ad+bc}{bd}\right] +_{\mathbb{Q}} \left[\frac{e}{f}\right] \\ &= \left[\frac{(ad+bc)(f)+(bd)(e)}{(bd)(f)}\right]\end{aligned}$$

With integer algebra, we can rearrange the above as follows.

$$\begin{aligned}&= \left[\frac{adf+bcf+bde}{bdf}\right] \\ &= \left[\frac{(a)(df)+(b)(cf+de)}{(b)(df)}\right]\end{aligned}$$

Now apply Definition 2.7 twice, again.

$$= \left[\frac{a}{b}\right] +_{\mathbb{Q}} \left(\left[\frac{cf+de}{df}\right]\right)$$
$$= \left[\frac{a}{b}\right] +_{\mathbb{Q}} \left(\left[\frac{c}{d}\right] +_{\mathbb{Q}} \left[\frac{e}{f}\right]\right)$$

□

c) *Existence of an additive identity*

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{0}{1}\right] = \left[\frac{a}{b}\right] \textit{ for all } \left[\frac{a}{b}\right] \in \mathbb{Q}$$

Proof. Via Definition 2.7 and integer algebra, we can show that

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{0}{1}\right] = \left[\frac{a \cdot 1 + b \cdot 0}{b \cdot 1}\right]$$
$$= \left[\frac{a}{b}\right]$$

as desired. □

d) *Existence of additive inverses*

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{-a}{b}\right] = \left[\frac{0}{1}\right] \textit{ for all } \left[\frac{a}{b}\right] \in \mathbb{Q}$$

Proof. Through various application of Definition 2.7 and integer algebra, we can show that

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{-a}{b}\right] = \left[\frac{ab + b \cdot -a}{bb}\right]$$
$$= \left[\frac{ab - ab}{bb}\right]$$
$$= \left[\frac{0}{bb}\right]$$

Since $0 \cdot 1 = 0$ and $bb \cdot 0 = 0$, transitivity implies that $(0)(1) = (bb)(0)$. By the definition of $\sim$, this means that $(0, bb) \sim (0, 1)$. It follows by Exercise 2.6 that the above equals the following, as desired.

$$= \left[\frac{0}{1}\right]$$

□

e) *Commutativity of multiplication*

$$\left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{c}{d}\right] \cdot_{\mathbb{Q}} \left[\frac{a}{b}\right] \textit{ for all } \left[\frac{a}{b}\right], \left[\frac{c}{d}\right] \in \mathbb{Q}$$

Proof. Via Definition 2.7 and integer algebra, we can show that

$$\left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{ac}{bd}\right]$$
$$= \left[\frac{ca}{db}\right]$$
$$= \left[\frac{c}{d}\right] \cdot_{\mathbb{Q}} \left[\frac{a}{b}\right]$$

as desired. □

f) *Associativity of multiplication*

$$\left(\left[\frac{a}{b}\right]\cdot_{\mathbb{Q}}\left[\frac{c}{d}\right]\right)\cdot_{\mathbb{Q}}\left[\frac{e}{f}\right]=\left[\frac{a}{b}\right]\cdot_{\mathbb{Q}}\left(\left[\frac{c}{d}\right]\cdot_{\mathbb{Q}}\left[\frac{e}{f}\right]\right)\text{ for all }\left[\frac{a}{b}\right],\left[\frac{c}{d}\right],\left[\frac{e}{f}\right]\in\mathbb{Q}$$

Proof. Through various application of Definition 2.7 and integer algebra, we can show that

$$\begin{aligned}\left(\left[\frac{a}{b}\right]\cdot_{\mathbb{Q}}\left[\frac{c}{d}\right]\right)\cdot_{\mathbb{Q}}\left[\frac{e}{f}\right]&=\left[\frac{ac}{bd}\right]\cdot_{\mathbb{Q}}\left[\frac{e}{f}\right]\\&=\left[\frac{(ac)(e)}{(bd)(f)}\right]\\&=\left[\frac{(a)(ce)}{(b)(df)}\right]\\&=\left[\frac{a}{b}\right]\cdot_{\mathbb{Q}}\left(\left[\frac{ce}{df}\right]\right)\\&=\left[\frac{a}{b}\right]\cdot_{\mathbb{Q}}\left(\left[\frac{c}{d}\right]\cdot_{\mathbb{Q}}\left[\frac{e}{f}\right]\right)\end{aligned}$$

as desired. □

g) *Existence of a multiplicative identity*

$$\left[\frac{a}{b}\right]\cdot_{\mathbb{Q}}\left[\frac{1}{1}\right]=\left[\frac{a}{b}\right]\text{ for all }\left[\frac{a}{b}\right]\in\mathbb{Q}$$

Proof. Via Definition 2.7 and integer algebra, we can show that

$$\begin{aligned}\left[\frac{a}{b}\right]\cdot_{\mathbb{Q}}\left[\frac{1}{1}\right]&=\left[\frac{a\cdot 1}{b\cdot 1}\right]\\&=\left[\frac{a}{b}\right]\end{aligned}$$

as desired. □

h) *Existence of multiplicative inverses for nonzero elements*

$$\left[\frac{a}{b}\right]\cdot_{\mathbb{Q}}\left[\frac{b}{a}\right]=\left[\frac{1}{1}\right]\text{ for all }\left[\frac{a}{b}\right]\in\mathbb{Q}\text{ such that }\left[\frac{a}{b}\right]\neq\left[\frac{0}{1}\right]$$

Proof. By Definition 2.7,

$$\left[\frac{a}{b}\right]\cdot_{\mathbb{Q}}\left[\frac{b}{a}\right]=\left[\frac{ab}{ba}\right]$$

Since $(ab)(1)=(ba)(1)$, we have by the definition of $\sim$ that $(ab,ba)\sim(1,1)$. It follows by Exercise 2.6 that the above equals the following, as desired.

$$=\left[\frac{1}{1}\right]$$

□

i) *Distributivity*

$$\left[\frac{a}{b}\right]\cdot_{\mathbb{Q}}\left(\left[\frac{c}{d}\right]+_{\mathbb{Q}}\left[\frac{e}{f}\right]\right)=\left(\left[\frac{a}{b}\right]\cdot_{\mathbb{Q}}\left[\frac{c}{d}\right]\right)+_{\mathbb{Q}}\left(\left[\frac{a}{b}\right]\cdot_{\mathbb{Q}}\left[\frac{e}{f}\right]\right)\text{ for all }\left[\frac{a}{b}\right],\left[\frac{c}{d}\right],\left[\frac{e}{f}\right]\in\mathbb{Q}$$

Proof. By Definition 2.7 and integer algebra,

$$\begin{aligned}\left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left(\left[\frac{c}{d}\right] +_{\mathbb{Q}} \left[\frac{e}{f}\right]\right) &= \left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left[\frac{cf+de}{df}\right] \\ &= \left[\frac{a(cf+de)}{bdf}\right] \\ &= \left[\frac{acf+ade}{bdf}\right]\end{aligned}$$

Use Theorem 2.10g.

$$= \left[\frac{acf+ade}{bdf}\right] \cdot_{\mathbb{Q}} \left[\frac{1}{1}\right]$$

Use the lemma from the proof of Theorem 2.10h.

$$= \left[\frac{acf+ade}{bdf}\right] \cdot_{\mathbb{Q}} \left[\frac{b}{b}\right]$$

Use various applications of Definition 2.7 and integer algebra to finish.

$$\begin{aligned} &= \left[\frac{(acf+ade)b}{(bdf)b}\right] \\ &= \left[\frac{acfb+adeb}{bdfb}\right] \\ &= \left[\frac{(ac)(bf)+(bd)(ae)}{(bd)(bf)}\right] \\ &= \left[\frac{ac}{bd}\right] +_{\mathbb{Q}} \left[\frac{ae}{bf}\right] \\ &= \left(\left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left[\frac{c}{d}\right]\right) +_{\mathbb{Q}} \left(\left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left[\frac{e}{f}\right]\right)\end{aligned}$$

as desired. □

10/20: **Theorem 2.11.** $\mathbb{Q}$ *is countable.*

Lemma.

a) *If there exists a surjection $g : B \to A$, then there exists an injection $f : A \to B$.*

b) *The set $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ is countable.*

Proof of a. Let $f : A \to B$ be defined such that for all $a \in A$, $f(a) \in g^{-1}(\{a\})$[1]. To prove that this condition is well-defined (i.e., there is no $a \in A$ such that $f(a)$ *cannot* be an element of $g^{-1}(\{a\})$), we will show that for all $a \in A$, $g^{-1}(\{a\})$ contains at least one element of B. Let a be an arbitrary element of A. Since g is surjective, we know by Definition 1.20 that there is a $b \in B$ such that $g(b) = a$. Let's consider this b more closely. As an element of B satisfying the condition that $g(b) = a$, b is naturally an element of the set $\{b' \in B \mid g(b') = a\}$. Clearly, this set is equivalent to $\{b' \in B \mid g(b') \in \{a\}\}$, so b is also an element of this new set. But by Definition 1.18, this set is equal to $g^{-1}(\{a\})$. Thus, $b \in B$ and $b \in g^{-1}(\{a\})$, as desired.

To prove that f is injective, Definition 1.20 tells us that it will suffice to show that $f(a) = f(a')$ implies that $a = a'$. Let $f(a) = f(a')$. It follows by the condition imposed on f that $f(a) \in g^{-1}(\{a\})$ and $f(a') \in g^{-1}(\{a'\})$. With respect to the latter case, the fact that $f(a) = f(a')$ also implies that $f(a) \in g^{-1}(\{a'\})$. Because $f(a) \in g^{-1}(\{a\})$ and $f(a) \in g^{-1}(\{a'\})$, Definition 1.18 tells us that $g(f(a)) \in \{a\}$ and $g(f(a)) \in \{a'\}$, respectively. Consequently, $g(f(a)) = a$ and $g(f(a)) = a'$, respectively. Since g is a function, Definition 1.16 implies that $g(f(a))$ is a unique, well-defined object, so $a = g(f(a)) = a'$, i.e., $a = a'$, as desired. □

[1] Note that we are not defining f explicitly, but rather providing a rule that means that some matchings will not suffice to define f, namely ones for which $f(a) \notin g^{-1}(\{a\})$ for all $a \in A$.

Proof of b. By Exercise 1.36, $\mathbb{Z}$ is countable, i.e.[2], there exists a bijection $f_1 : \mathbb{Z} \to \mathbb{N}$. Since $\mathbb{Z} \setminus \{0\} \subset \mathbb{Z}$, Exercise 1.37 implies that $\mathbb{Z} \setminus \{0\}$ is countable, i.e., there exists a bijection $f_2 : \mathbb{Z} \setminus \{0\} \to \mathbb{N}$. Now let $f : \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \to \mathbb{N} \times \mathbb{N}$ be defined by $f(a,b) = (f_1(a), f_2(b))$. To prove that f is a function, Definition 1.16 tells us that we must show that for every $(a,b) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$, there is a unique $(c,d) \in \mathbb{N} \times \mathbb{N}$ such that $f(a,b) = (c,d)$. Let (a,b) be an arbitrary element of $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. Then by Definition 1.15, $a \in \mathbb{Z}$ and $b \in \mathbb{Z} \setminus \{0\}$. Thus, by the definitions of f_1 and f_2, $f_1(a)$ and $f_2(b)$ are defined objects and elements of $\mathbb{N}$. Consequently, Definition 1.15 implies that $(f_1(a), f_2(a)) \in \mathbb{N} \times \mathbb{N}$. Since $f(a,b) = (f_1(a), f_2(b))$ by the definition of f, it follows that $(f_1(a), f_2(b))$ is *an* element of $\mathbb{N} \times \mathbb{N}$ to which f maps (a,b). On the uniqueness of this object, suppose that $f(a,b) = (c,d)$ and $f(a,b) = (c',d')$ for some $(a,b) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. By the definition of f, this implies that $(f_1(a), f_2(b)) = (c,d)$ and $(f_1(a), f_2(b)) = (c',d')$. Thus, by multiple applications of Definition 1.15, $f_1(a) = c$, $f_1(a) = c'$, $f_2(b) = d$, and $f_2(b) = d'$. But since f_1 and f_2 are both functions, Definition 1.16 implies that $f_1(a)$ and $f_2(b)$ are both unique, well-defined objects. Consequently, transitivity applies and implies that $c = f_1(a) = c'$ and $d = f_2(b) = d'$. It follows by Definition 1.15 once again that $(c,d) = (c',d')$, as desired.

To prove that f is injective, Definition 1.20 tells us that we must verify that $f(a,b) = f(a',b')$ implies that $(a,b) = (a',b')$. Let $f(a,b) = f(a',b')$. By the definition of f, $(f_1(a), f_2(b)) = (f_1(a'), f_2(b'))$. Thus, by Definition 1.15, $f_1(a) = f_1(a')$ and $f_2(b) = f_2(b')$. Consequently, by the injectivity of f_1 and f_2 (which follows from their respective bijectivity by Definition 1.20), $a = a'$ and $b = b'$. Therefore, by Definition 1.15 once again, $(a,b) = (a',b')$.

By Exercise 1.39, $\mathbb{N} \times \mathbb{N}$ is countable, i.e., there exists a bijection $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Since g is bijective, Definition 1.20 implies that it is injective. Thus, by Proposition 1.26, $g \circ f : \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \to \mathbb{N}$ (which Definition 1.25 guarantees exists) is injective.

Since there exists an injection $g \circ f : \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \to \mathbb{N}$ where $\mathbb{N}$ is clearly countable and $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ is clearly infinite, Exercise 1.38 implies that $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ is countable, as desired. ☐

Proof of Theorem 2.11. Let $g : \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \to \mathbb{Q}$ be defined by $g(a,b) = \left[\frac{a}{b}\right]$. For g to be a function as defined by Definition 1.16, g must map every ordered pair $(a,b) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ to a unique element $\left[\frac{a}{b}\right]$ in $\mathbb{Q}$. Let (a,b) be an arbitrary element of $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. By definition, g clearly maps (a,b) to only one (i.e., a unique) object, namely the equivalence class $\left[\frac{a}{b}\right]$. But we must still show that this $\left[\frac{a}{b}\right]$ is an element of $\mathbb{Q}$ (note that this is not immediately obvious as equivalence classes such as $\left[\frac{0}{0}\right]$ [which is still a well-defined equivalence class, just an empty one] are not elements of $\mathbb{Q}$). For $\left[\frac{a}{b}\right]$ to be an element of $\mathbb{Q}$, Definition 2.5 tells us that it will suffice to show that $(a,b) \in X$. For (a,b) to be an element of X, Exercise 2.2e asserts that it will suffice to show that $a,b \in \mathbb{Z}$ and $b \neq 0$. But since $(a,b) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ by assumption, Definition 1.15 tells us that $a \in \mathbb{Z}$ and $b \in \mathbb{Z} \setminus \{0\}$. Expounding on the latter result, Definition 1.11 tells us that $b \in \mathbb{Z}$ and $b \notin \{0\}$, i.e., $b \in \mathbb{Z}$ and $b \neq 0$. Combining the last three results, we have that $a,b \in \mathbb{Z}$ and $b \neq 0$, as desired.

To prove that g is surjective, Definition 1.20 tells us that we must verify that for all $\left[\frac{a}{b}\right] \in \mathbb{Q}$, there exists an $(a,b) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ such that $g(a,b) = \left[\frac{a}{b}\right]$. Let $\left[\frac{a}{b}\right]$ be an arbitrary element of $\mathbb{Q}$. By Definition 2.5, $(a,b) \in X$. Thus, by Exercise 2.2e, $a,b \in \mathbb{Z}$ and $b \neq 0$. Since $b \in \mathbb{Z}$ and $b \neq 0$, i.e., $b \in \mathbb{Z}$ and $b \notin \{0\}$, Definition 1.11 tells us that $b \in \mathbb{Z} \setminus \{0\}$. To recap, $a \in \mathbb{Z}$ and $b \in \mathbb{Z} \setminus \{0\}$. But by Definition 1.15, this implies that $(a,b) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. With regards to this (a,b), we have by the definition of g that $g(a,b) = \left[\frac{a}{b}\right]$, as desired.

Since there exists a surjection $g : \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \to \mathbb{Q}$, we have by Lemma (a) that there exists an injection $f : \mathbb{Q} \to \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. Thus, we have an injection $f : \mathbb{Q} \to \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ where $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ is countable (by Lemma (b)) and $\mathbb{Q}$ is clearly infinite. By Exercise 1.38, this means that $\mathbb{Q}$ is countable. ☐

[2]Let A be a set (such as $\mathbb{Z}$). Technically, Definition 1.35 must be invoked to move from "A is countable" to "A is in bijective correspondence with $\mathbb{N}$," and Definition 1.28 must be invoked to move from "A is in bijective correspondence with $\mathbb{N}$" to "there exists a bijection $f : A \to \mathbb{N}$." However, as we are no longer in Script 1, such justifications will not be supplied beyond this footnote.