# Script 5

# Connectedness and Boundedness

## 5.1 Journal

11/19: **Axiom 4.** *A continuum is connected.*

**Theorem 5.1.** *The only subsets of a continuum $C$ that are both open and closed are $\emptyset$ and $C$.*

*Proof.* To prove that the only subsets of $C$ that are both open and closed are $\emptyset$ and $C$, it will suffice to show that if $A \subset C$ is both open and closed, then $A = \emptyset$ or $A = C$. Let $A \subset C$ be both open and closed. We divide into two cases ($A = \emptyset$ and $A \neq \emptyset$). If $A = \emptyset$, then we are done. On the other hand, if $A \neq \emptyset$, we have a bit more work to do. Basically, we will end up proving that the facts that $A$ is open, $A$ is closed, and $A \neq \emptyset$ imply that $A = C$. Let's begin.

First off, the fact that $A$ is closed implies by Definition 4.8 that $C \setminus A$ is open. Additionally, we have by Script 1 that $A \cap (C \setminus A) = \emptyset$ and $A \cup (C \setminus A) = C$. Now suppose for the sake of contradiction that $A \neq C$. It follows since $A \subset C$ that we must have $C \not\subset A$, i.e., there is some object in $C$ that is not an element of $A$. This object would clearly be an element of $C \setminus A$ in this case, meaning that $C \setminus A$ is nonempty. Thus, we have that $A$ and $C \setminus A$ are disjoint, open, nonempty sets such that $A \cup (C \setminus A) = C$. Consequently, by consecutive applications of Definition 4.22, we know that $C$ is disconnected, i.e., $C$ is not connected. But this contradicts Axiom 4, which asserts that $C$ is connected. Therefore, we must have that $A = C$, as desired. □

**Theorem 5.2.** *For all $x, y \in C$, if $x < y$, then there exists a point $z \in C$ such that $z$ is between $x$ and $y$.*

*Proof.* Suppose for the sake of contradiction that no point $z \in C$ exists such that $z$ is between $x$ and $y$. To find a contradiction, we will let $A = \{c \in C \mid c < y\}$ and $B = \{c \in C \mid x < c\}$ and prove that $A \cup B = C$, and that $A$ and $B$ are disjoint, nonempty, open sets. This will imply that $C$ is disconnected, contradicting Axiom 4. Let's begin.

Suppose for the sake of contradiction that $C \neq A \cup B$. Then by Theorem 1.7, $C \not\subset A \cup B$ or $A \cup B \not\subset C$. Since $A \subset C$ and $B \subset C$ by their definitions, we have $A \cup B \subset C$, so it must be that $C \not\subset A \cup B$. Thus, by Definition 1.3, there exists a point $p \in C$ such that $p \notin A \cup B$. From the latter condition, we have by Definition 1.5 that $p \notin A$ and $p \notin B$. It follows from the definitions of $A$ and $B$ that $p \notin C$, or $p \not< y$ and $x \not< p$. But we know that $p \in C$, so it must be that $p \not< y$ and $x \not< p$. Equivalently, $p \geq y$ and $x \geq p$. But this implies that $x \geq y$, which contradicts the fact that $x < y$ by hypothesis. Therefore, we must have $C = A \cup B$, as desired.

Suppose for the sake of contradiction that $A$ and $B$ are not disjoint. Then by Definition 1.9, $A \cap B \neq \emptyset$. Thus, Definition 1.8 tells us that there exists some object $p \in A \cap B$. By Definition 1.6, this implies that $p \in A$ and $p \in B$. It follows by the definitions of $A$ and $B$ that $p \in C$, $p < y$, and $x < p$. Since $x < p < y$, Definition 3.6 tells us that $p$ is between $x$ and $y$, contradicting the supposition that no such point exists. Therefore, $A$ and $B$ are disjoint, as desired.

To prove that $A$ and $B$ are nonempty, Definition 1.8 tells us that it will suffice to show that there exists an object in each set. Since $x \in C$ and $x < y$, $x \in A$. Similarly, since $y \in C$ and $x < y$, $y \in B$. Therefore, $A$ and $B$ are nonempty, as desired.

By Corollary 4.13, $A$ and $B$ are open, as desired.

Since $C$ can be written as $A \cup B$ where $A$ and $B$ are disjoint, nonempty, open sets, we have by Definition 4.22 that $C$ is disconnected. But this contradicts Axiom 4, which asserts that $C$ is connected. Therefore, there must exist a point $z \in C$ such that $z$ is between $x$ and $y$, as desired. $\square$

**Corollary 5.3.** *Every region is infinite.*

*Proof.* Let $\underline{ab}$ be a region, and suppose for the sake of contradiction that $\underline{ab}$ is finite. Then by Definitions 1.30 and 1.33, $\underline{ab} = \emptyset$, or $\underline{ab}$ has cardinality $n$. We divide into two cases. Suppose first that $\underline{ab} = \emptyset$. Then by Definitions 3.10 and 3.6, no point $p$ exists such that $a < p < b$. Thus, by the contrapositive of Theorem 5.2, $a = b$. But this implies by Definition 3.10 that $\underline{ab}$ is not a region (since $a \not< b$), a contradiction. Now suppose that $\underline{ab}$ has cardinality $n$. Then by Theorem 3.5, the symbols $a_1, \dots, a_n$ may be assigned to each point of $\underline{ab}$ so that $a_1 < a_2 < \cdots < a_n$. But by Theorem 5.2, there exists a point $z \in C$ such that $z$ is between $a$ and $a_1$. Since $a < z < a_1 < b$, we clearly have that $z \in \underline{ab}$, yet it was not assigned a symbol $a_k$, a contradiction. Therefore, $\underline{ab}$ is infinite, as desired. $\square$

12/1: **Corollary 5.4.** *Every point of $C$ is a limit point of $C$.*

*Proof.* Let $p$ be an arbitrary element of $C$. To prove that $p$ is a limit point of $C$, Definition 3.13 tells us that it will suffice to show that for all regions $R$ with $p \in R$, $R \cap (C \setminus \{p\}) \neq \emptyset$. Let $R$ be an arbitrary region with $p \in R$. By Corollary 5.3, $R$ is infinite, so there exists a point $q \in R$ such that $q \neq p$. Additionally, since $q \in R$, we have $q \in C$. Thus, since $q \in C$ and $q \neq p$ (i.e., $q \notin \{p\}$), we have by Definition 1.11 that $q \in C \setminus \{p\}$. This combined with the fact that $q \in R$ implies by Definition 1.6 that $q \in R \cap (C \setminus \{p\})$, so $R \cap (C \setminus \{p\}) \neq \emptyset$, as desired. $\square$

**Corollary 5.5.** *Every point of the region $\underline{ab}$ is a limit point of $\underline{ab}$.*

*Proof.* Let $p$ be an arbitrary element of the region $\underline{ab}$. Since $p \in C$, by Corollary 5.4, $p \in LP(C)$. Thus, by Script 1, $p \in LP(\underline{ab} \cup (C \setminus \underline{ab}))$. It follows by Theorem 3.20 that $p \in LP(\underline{ab})$ or $p \in LP(C \setminus \underline{ab})$. Now suppose $p \in LP(C \setminus \underline{ab})$. Then by Definition 3.13, for all regions $R$ with $p \in R$, $R \cap ((C \setminus \underline{ab}) \setminus \{p\}) \neq \emptyset$. However, by Script 1, $\underline{ab}$ is a region with $p \in \underline{ab}$ such that $\emptyset = \underline{ab} \cap (C \setminus \underline{ab}) = \underline{ab} \cap ((C \setminus \underline{ab}) \setminus \{p\})$. Therefore, it cannot be that $p \in LP(C \setminus \underline{ab})$, so it must be that $p \in LP(\underline{ab})$. $\square$

**Definition 5.6.** Let $X$ be a subset of $C$. A point $u$ is called an **upper bound** of $X$ if for all $x \in X$, $x \leq u$. A point $l$ is called a **lower bound** of $X$ if for all $x \in X$, $l \leq x$. If there exists an upper bound of $X$, then we say that $X$ is **bounded above**. If there exists a lower bound of $X$, then we say that $X$ is **bounded below**. If $X$ is bounded above and below, then we simply say that $X$ is **bounded**.

**Definition 5.7.** Let $X$ be a subset of $C$. We say that $u$ is a **least upper bound** of $X$ and write $u = \sup X$ if:

1. $u$ is an upper bound of $X$;
2. if $u'$ is an upper bound of $X$, then $u \leq u'$.

We say that $l$ is a **greatest lower bound** and write $l = \inf X$ if:

1. $l$ is a lower bound of $X$;
2. if $l'$ is a lower bound of $X$, then $l' \leq l$.

The notation sup comes from the word **supremum**, which is another name for least upper bound. The notation inf comes from the word **infimum**, which is another name for greatest lower bound.

**Exercise 5.8.** If $\sup X$ exists, then it is unique, and similarly for $\inf X$.

*Proof.* Let $X$ be a subset of a continuum $C$ such that $\sup X$ exists, and suppose that both $u$ and $u'$ are least upper bounds of $X$. It follows from the supposition and Definition 5.7 that $u, u'$ are both upper bounds of $X$. Thus, since $u$ is a least upper bound of $X$ and $u'$ is an upper bound of $X$, we have by Definition 5.7 again that $u \leq u'$. By a symmetric argument, we also have that $u' \leq u$. But since $u \leq u'$ and $u' \leq u$, $u = u'$, proving the uniqueness of $\sup X$.

The proof is symmetric for $\inf X$. ☐

**Exercise 5.9.** If $X$ has a first point $L$, then $\inf X$ exists and equals $L$. Similarly, if $X$ has a last point $U$, then $\sup X$ exists and equals $U$.

*Proof.* Let $L$ be the first point of $X$. Then by Definition 3.3, for all $x \in X$, $L \leq x$. Thus, by Definition 5.6, $L$ is a lower bound of $X$. Now suppose for the sake of contradiction that there exists a lower bound $L'$ of $X$ such that $L' > L$. Since $L'$ is a lower bound, Definition 5.6 implies that for all $x \in X$, $L' \leq x$. But $L$ is an element of $X$ and $L < L'$, a contradiction. Therefore, if $L'$ is a lower bound of $X$, then $L' \leq L$. This result coupled with the fact that $L$ is a lower bound of $X$ implies by Definition 5.7 that $L = \inf X$.

The proof is symmetric in the other case. ☐

**Exercise 5.10.** For this exercise, we assume that $C = \mathbb{R}$. Find $\sup X$ and $\inf X$ for each of the following subsets of $\mathbb{R}$, or state that they do not exist. You need not give proofs.

1. $X = \mathbb{N}$.

   *Answer.* $\sup X$ does not exist because the natural numbers continue on forever to positive infinity. However, $\inf X = 1$ since we know that $1 \leq n$ for all $n \in \mathbb{N}$. ☐

2. $X = \mathbb{Q}$.

   *Answer.* Neither $\sup X$ nor $\inf X$ exists because the rational numbers continue on forever to both positive and negative infinity. ☐

3. $X = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$.

   *Answer.* For $n = 1$, $\frac{1}{n} = 1$. From here, as $n$ increases, $\frac{1}{n}$ decreases asymptotically toward zero but always remains a positive nonzero rational number. Thus, $\sup X = 1$ and $\inf X = 0$. ☐

4. $X = \{x \in \mathbb{R} \mid 0 < x < 1\}$.

   *Answer.* $\sup X = 1$ and $\inf X = 0$. In the case of $\sup X$, any number slightly less than 1 would be included in $X$ and have a number in $X$ between it and 1 by Theorem 5.2, i.e., greater than it. A symmetric argument can treat the other case. ☐

5. $X = \{3\} \cup \{x \in \mathbb{R} \mid -7 \leq x \leq -5\}$.

   *Answer.* $\sup X = 3$ (3 is the greatest element of the set) and $\inf X = -7$ (for a similar reason to part 4, above). ☐

**Lemma 5.11.** *Suppose that $X \subset C$ and $s = \sup X$. If $p < s$, then there exists an $x \in X$ such that $p < x \leq s$. Similarly, suppose that $X \subset C$ and $l = \inf X$. If $l < p$, then there exists an $x \in X$ such that $l \leq x < p$.*

*Proof.* Suppose for the sake of contradiction that for some $p < s$, no $x \in X$ exists such that $p < x \leq s$. Since $s$ is a least upper bound of $X$, Definitions 5.7 and 5.6 imply[1] that for all $x \in X$, $x \leq s$. Consequently, by the supposition, it is true that for all $x \in X$, $x \leq p$ (if there existed an $x > p$, then this point would satisfy $p < x \leq s$, contradicting the supposition). Thus, by Definition 5.6, $p$ is a upper bound of $X$. But since

[1]Technically, Definition 5.7 implies that $s$ is an upper bound of $X$ and Definition 5.6 implies based off of this result that for all $x \in X$, $x \leq s$. However, to avoid having to write this every time, I will shorthand this concept in this fashion.

$p < s$, it is not true that $s \leq s'$ for all upper bounds $s'$ of $X$, meaning by Definition 5.7 that $s$ is not a least upper bound of $X$, a contradiction. Therefore, if $p < s$, then there exists an $x \in X$ such that $p < x \leq s$, as desired.

The proof is symmetric in the other case. □

**Theorem 5.12.** *Let $a < b$. The least upper bound and greatest lower bound of the region $\underline{ab}$ are $\sup \underline{ab} = b$ and $\inf \underline{ab} = a$.*

*Proof.* To prove that $\sup \underline{ab} = b$, Definition 5.7 tells us that it will suffice to show that $b$ is an upper bound of $\underline{ab}$ and that if $u$ is an upper bound of $\underline{ab}$, then $b \leq u$. For the first condition, Definition 5.6 tells us that it will suffice to confirm that for all $x \in \underline{ab}$, $x \leq b$. Let $x$ be an arbitrary element of $\underline{ab}$. Then by Definitions 3.10 and 3.6, we know that $a < x < b$, i.e., $x \leq b$, as desired. For the second condition, suppose for the sake of contradiction that $u$ is an upper bound of $\underline{ab}$ such that $u < b$. Then by Definition 5.6, for all $x \in \underline{ab}$, $x \leq u$. Additionally, since $\underline{ab}$ is infinite by Corollary 5.3, we know that at least one such $x$ exists, which we shall hereafter refer to as $y$. Note that as an element of $\underline{ab}$, $y$ satisfies $a < y < b$ by Definitions 3.10 and 3.6. Furthermore, since $u < b$, Theorem 5.2 implies that there exists a point $z$ such that $z$ is between $u$ and $b$. Thus, by Definition 3.6, $u < z < b$. Combining the last few results, we have $a < y \leq u < z < b$. Consequently, since $a < z < b$, we have by Definitions 3.6 and 3.10 that $z \in \underline{ab}$ and $u < z$, contradicting the statement that for all $x \in \underline{ab}$, $x \leq u$. Therefore, if $u$ is an upper bound of $\underline{ab}$, then $b \leq u$, as desired.

The proof is symmetric in the other case. □

12/3: **Lemma 5.13.** *Let $X$ be a subset of $C$. Suppose that $\sup X$ exists and $\sup X \notin X$. Then $\sup X$ is a limit point of $X$. The same holds for $\inf X$.*

*Proof.* To prove that $\sup X$ is a limit point of $X$, Definition 3.13 tells us that it will suffice to verify that for all regions $\underline{ab}$ with $\sup X \in \underline{ab}$, we have $\underline{ab} \cap (X \setminus \{\sup X\}) \neq \emptyset$. Let $\underline{ab}$ be an arbitrary region with $\sup X \in \underline{ab}$. Then by Definitions 3.10 and 3.6, $a < \sup X < b$. It follows by Lemma 5.11 that there exists an $x \in X$ such that $a < x \leq \sup X$. Additionally, since $x \in X$ and $\sup X \notin X$, we cannot have $\sup X = x$, meaning that $a < x < \sup X$. Combining the last few results, we have $a < x < \sup X < b$. Thus, by Definitions 3.6 and 3.10, $x \in \underline{ab}$. Consequently, since $x \in X$ and $x \neq \sup X$ implies $x \notin \{\sup X\}$, Definition 1.11 asserts that $x \in X \setminus \{\sup X\}$. Therefore, since we also know that $x \in \underline{ab}$, we have by Definition 1.6 that $x \in \underline{ab} \cap (X \setminus \{\sup X\})$, meaning by Definition 1.8 that $\underline{ab} \cap (X \setminus \{\sup X\}) \neq \emptyset$, as desired.

The proof is symmetric in the other case. □

**Corollary 5.14.** *Both $a$ and $b$ are limit points of the region $\underline{ab}$.*

*Proof.* Clearly, $\underline{ab} \subset C$. Additionally, by Theorem 5.12, $\sup \underline{ab}$ and $\inf \underline{ab}$ exist and are equal to $b$ and $a$, respectively. Furthermore, it follows from Definition 3.10 that neither $b$ nor $a$ (i.e., neither $\sup \underline{ab}$ nor $\inf \underline{ab}$) are elements of $\underline{ab}$. Therefore, by Lemma 5.13, $\inf \underline{ab} = a$ and $\sup \underline{ab} = b$ are limit points of the region $\underline{ab}$. □

**Corollary 5.15.** *Let $[a, b]$ denote the closure $\overline{\underline{ab}}$ of the region $\underline{ab}$. Then $[a, b] = \{x \in C \mid a \leq x \leq b\}$.*

*Proof.* To prove that $[a, b] = \{x \in C \mid a \leq x \leq b\}$, Definition 4.4 tells us that it will suffice to show that $\underline{ab} \cup LP(\underline{ab}) = \{x \in C \mid a \leq x \leq b\}$. To show this, we will verify that $\underline{ab} \cup LP(\underline{ab}) = \{a\} \cup \underline{ab} \cup \{b\}$ and that $\{x \in C \mid a \leq x \leq b\} = \{a\} \cup \underline{ab} \cup \{b\}$ and use transitivity. Let's begin.

By Corollaries 5.5 and 5.14, every point in $\underline{ab}$ as well as $a$ and $b$ are limit points of $\underline{ab}$. Thus, $\{a\} \cup \underline{ab} \cup \{b\} \subset LP(\underline{ab}) \subset \underline{ab} \cup LP(\underline{ab})$. On the other hand, by Lemma 3.17 and Definition 3.15, no element of $C \setminus (\{a\} \cup \underline{ab} \cup \{b\})$ is a limit point of $\underline{ab}$. Additionally, it is clear that no element of $C \setminus (\{a\} \cup \underline{ab} \cup \{b\})$ is an element of $\underline{ab}$. Thus, $\underline{ab} \cup LP(\underline{ab}) \subset \{a\} \cup \underline{ab} \cup \{b\}$. Therefore, by Theorem 1.7a, $\underline{ab} \cup LP(\underline{ab}) = \{a\} \cup \underline{ab} \cup \{b\}$, as desired.

By Lemma 3.16 and Definition 3.15, $C \setminus (\{a\} \cup \underline{ab} \cup \{b\}) = \{x \in C \mid x < a \text{ or } x > b\}$. It follows that

$$\begin{aligned} C \setminus (C \setminus (\{a\} \cup \underline{ab} \cup \{b\})) &= C \setminus \{x \in C \mid x < a \text{ or } x > b\} \\ \{a\} \cup \underline{ab} \cup \{b\} &= \{x \in C \mid x \not< a \text{ and } x \not> b\} \\ &= \{x \in C \mid a \leq x \leq b\} \end{aligned}$$

as desired. □

**Lemma 5.16.** *Let $X \subset C$ and define:*

$$\Psi(X) = \{x \in C \mid x \text{ is not an upper bound of } X\} \qquad \Omega(X) = \{x \in C \mid x \text{ is not a lower bound of } X\}$$

*Then both $\Psi(X)$ and $\Omega(X)$ are open.*

*Proof.* We will take this one set at a time.

To prove that $\Psi(X)$ is open, Theorem 4.10 tells us that it will suffice to confirm that for all $y \in \Psi(X)$, there exists a region containing $y$ that is a subset of $\Psi(X)$. Let $y$ be an arbitrary element of $\Psi(X)$. Then by the definition of $\Psi(X)$, $y$ is not an upper bound of $X$. Thus, by Definition 5.6, there exists some $x \in X$ such that $x > y$. Now let $a \in C$ be a point such that $a < y$ (Axiom 3 and Definition 3.3 imply the existence of such a point) and consider the region $\underline{ax}$. We will demonstrate that $\underline{ax}$ is the desired region, i.e., that $y \in \underline{ax}$ and $\underline{ax} \subset \Psi(X)$. For the first condition, since $a < y < x$, it immediately follows from Definitions 3.6 and 3.10 that $y \in \underline{ax}$, as desired. As to the second condition, Definition 1.3 tells us that it will suffice to show that every element $z \in \underline{ax}$ is an element of $\Psi(X)$. Let $z$ be an arbitrary element of $\underline{ax}$. Then by Definitions 3.10 and 3.6, $z < x$. Since $z$ is less than an element of $X$, Definition 5.6 asserts that $z$ is not an upper bound of $X$. Thus, by the definition of $\Psi(X)$, $z \in \Psi(X)$, as desired. Therefore, for all $y \in \Psi(X)$, there exists a region containing $y$ that is a subset of $\Psi(X)$.

The proof is symmetric in the other case. □

**Theorem 5.17.** *Suppose that $X$ is nonempty and bounded above. Then $\sup X$ exists. Similarly, if $X$ is nonempty and bounded below, then $\inf X$ exists.*

*Proof.* We begin by proving that $\Psi(X)$ is not closed. This can be accomplished by using the two constraints on $X$ (that it is nonempty and bounded) to demonstrate that it is is neither equal to $C$ nor $\emptyset$, respectively; it follows from the facts that $\Psi(X)$ is open, $\Psi(X) \neq C$, and $\Psi(X) \neq \emptyset$ that $\Psi(X)$ is not closed. Once we've established that $\Psi(X)$ is not closed, we have that there exists a limit point of $\Psi(X)$ that is not an element of $\Psi(X)$. This limit point will be $\sup X$. Let's begin.

To prove that $\Psi(X) \neq C$, Definition 1.2 tells us that it will suffice to show that there exists some point in $C$ that is not an element of $\Psi(X)$. By definition, $X$ is bounded above. Thus, by Definition 5.6, there exists an upper bound $u$ of $X$, where $u \in C$ by definition. Since $u$ is an upper bound of $X$, we have by the definition of $\Psi(X)$ that $u \notin \Psi(X)$. Since $u \in C$ and $u \notin \Psi(X)$, $\Psi(X) \neq C$, as desired.

To prove that $\Psi(X) \neq \emptyset$, Definition 1.8 tells us that it will suffice to show that there exists some point $y \in \Psi(X)$. By Definition, $X$ is nonempty. Thus, there exists some $x \in X$. It follows by Axiom 3 and Definition 3.3 that there exists some $y \in C$ such that $y < x$. Since there is an element of $X$ that is greater than $y$, we have by Definition 5.6 that $y$ is not an upper bound of $X$. Thus, by the definition or $\Psi(X)$, $y \in \Psi(X)$. Therefore, $\Psi(X) \neq \emptyset$, as desired.

Now the last two main results state that $\Psi(X) \neq C$ and $\Psi(X) \neq \emptyset$, respectively. Thus, by Theorem 5.1, $\Psi(X)$ cannot be both open and closed. Consequently, since $\Psi(X)$ is open by Lemma 5.16, we have that $\Psi(X)$ is not closed.

Since $\Psi(X)$ is not closed, we have by Definition 4.1 that $\Psi(X)$ does not contain all of its limit points, i.e., there exists some $u \in LP(\Psi(X))$ such that $u \notin \Psi(X)$. It follows from the latter condition and the definition of $\Psi(X)$ that $u$ is an upper bound of $X$. Now suppose for the sake of contradiction that there exists an upper bound $u'$ of $X$ such that $u' < u$. First off, we have by Axiom 3 and Definition 3.3 that there exists some point $b \in C$ such that $u < b$. Now consider the region $\underline{u'b}$. Since $u' < u < b$, we have by Definitions 3.6 and 3.10 that $u \in \underline{u'b}$. Additionally, we can verify that $\underline{u'b} \cap \Psi(X) = \emptyset$ (if $z \in \underline{u'b} \cap \Psi(X)$, then $z \in \underline{u'b}$ and $z \in \Psi(X)$; it follows from the first result that $z > u'$, i.e., $z > x$ for all $x \in X$, but it follows from the second that $z$ is not an upper bound of $X$, i.e., $z < x$ for some $x \in X$, a contradiction). Consequently, $\underline{u'b} \cap (\Psi(X) \setminus \{u\}) = \emptyset$, implying by Definition 3.13 that $u \notin LP(\Psi(X))$, a contradiction. Therefore, if $u'$ is an upper bound of $X$, then $u \leq u'$. This combined with the fact that $u$ is an upper bound of $X$ implies by Definition 5.7 that $u = \sup X$, meaning that $\sup X$ exists, as desired.

The proof is symmetric in the other case. □

**Corollary 5.18.** *Every nonempty, closed, and bounded set has a first and a last point.*

*Proof.* Let $X$ be an arbitrary nonempty, closed, and bounded set. We will start by proving that $X$ has a first point, and then we will prove that it has a last point. Let's begin.

Since $X$ is bounded, Definition 5.7 implies that $X$ is bounded below. This result combined with the fact that $X$ is nonempty proves by Theorem 5.17 that $\inf X$ exists. Now suppose for the sake of contradiction that $\inf X \notin X$. Then by Lemma 5.13, $\inf X \in LP(X)$. But since $X$ is closed, this result implies by Definition 4.1 that $\inf X \in X$, a contradiction. Thus, $\inf X \in X$. To summarize, at this point we know that $\inf X$ is a point of $X$ such that (by Definitions 5.7 and 5.6) for every element $x \in X$, $\inf X \leq x$. Therefore, by Definition 3.3, $\inf X$ is the first point of $X$.

The proof is symmetric in the other case. □

**Exercise 5.19.** Is this true for $\mathbb{Q}$?

*Proof.* No. Consider the set $A = \{x \in \mathbb{Q} \mid x^2 < 2\}$. By the proof of Exercise 4.24, $A$ is nonempty. Additionally, by the proof of Exercise 4.24, $\mathbb{Q} \setminus A$ is open, meaning that $A$ is closed. To prove that $A$ is bounded, Definition 5.6 tells us that it will suffice to show that there exists an upper and lower bound on $A$. Now it can be deduced from the properties in Script 0 that $x^2 < y^2$ and $y > 0$ imply that $x < y$. Thus, if we let our upper bound be 3, we have that $3^2 = 9 > 2 > x^2$ for all $x \in A$ and $3 > 0$, so $3 > x$ for all $x \in A$. A similar argument proves that $-3$ is a lower bound.

However, $A$ has no first or last point — for any value in $A$, no matter how large, we can apply the technique of Exercise 4.24 to find a larger value. Similarly, for any value in $A$, no matter how small, we can find a smaller value with the same procedure. □