Script 4

The Topology of a Continuum

4.1 Journal

11/3: **Definition 4.1.** A subset of a continuum is **closed** if it contains all of its limit points.

Theorem 4.2. *The sets $\emptyset$ and C are closed.*

Proof. We will address the two sets individually.

To prove that $\emptyset$ is closed, Definition 4.1 tells us that it will suffice to show that $\emptyset \subset C$ and $\emptyset$ contains all of its limit points. By Exercise 1.10, $\emptyset \subset C$. We now prove that $\emptyset$ has no limit points. Suppose for the sake of contradiction that some point $p \in C$ is a limit point of $\emptyset$. Then by Definition 3.13, for all regions R with $p \in R$, $R \cap (\emptyset \setminus \{p\}) \neq \emptyset$. But clearly, $R \cap (\emptyset \setminus \{p\}) = R \cap \emptyset = \emptyset$, a contradiction. Therefore, since $\emptyset$ has no limit points, the statement "$\emptyset$ contains all of its limit points" is vacuously true.

To prove that C is closed, Definition 4.1 tells us that it will suffice to show that $C \subset C$ and C contains all of its limit points. Since $C = C$, Theorem 1.7 implies that $C \subset C$. Now suppose for the sake of contradiction that C does not contain all of its limit points. Then there exists a point $p \in C$ that is a limit point of C such that $p \notin C$. But we cannot have $p \in C$ and $p \notin C$, so it must be that the initial hypothesis was incorrect, meaning that C does, in fact, contain all of its limit points. ☐

Theorem 4.3. *A subset of C containing a finite number of points is closed.*

Proof. Let A be a finite subset of C. To prove that A is closed, Definition 4.1 tells us that it will suffice to show that A contains all of its limit points. But by Theorem 3.24, A has no limit points, so the statement "A contains all of its limit points" is vacuously true. ☐

Definition 4.4. Let X be a subset of C. The **closure** of X is the subset $\overline{X}$ of C defined by

$$\overline{X} = X \cup LP(X)$$

Theorem 4.5. *$X \subset C$ is closed if and only if $X = \overline{X}$.*

Proof. Suppose first that X is closed. To prove that $X = \overline{X}$, Definition 4.4 tells us that it will suffice to show that $X = X \cup LP(X)$. To show this, Definition 1.2 tells us that we must verify that every element x of X is an element of $X \cup LP(X)$ and vice versa. First, let x be an arbitrary element of X. Then by Definition 1.5, $x \in X \cup LP(X)$, as desired. Now let x be an arbitrary element of $X \cup LP(X)$. Then by Definition 1.5, $x \in X$ or $x \in LP(X)$. We divide into two cases. If $x \in X$, then we are done. If $x \in LP(X)$, then $x \in X$ as desired for the following reason: Since X is closed by hypothesis, Definition 4.1 implies that X contains all of its limit points, i.e., for all $y \in LP(X)$, $y \in X$; this implication notably applies to the x in question.

Now suppose that $X = \overline{X}$. To prove that X is closed, Definition 4.1 tells us that it will suffice to show that X contains all of its limit points. By Theorem 1.7, $LP(X) \subset X \cup LP(X)$. This combined with the fact that $X = X \cup LP(X)$ (by Definition 4.4, since $X = \overline{X}$) implies that $LP(X) \subset X$. It follows by Definition 1.3 that every element of $LP(X)$ is an element of X, i.e., every limit point of X is an element of X, i.e., X contains all of its limit points, as desired. ☐

11/5: **Theorem 4.6.** *Let $X \subset C$. Then $\overline{X} = \overline{\overline{X}}$.*

Lemma. *If p is an element of $LP(LP(X))$, then p is an element of $LP(X)$.*

Proof. We will prove the claim by contrapositive. Because of the complexity of the argument used, a short outline follows. Essentially, we let $p \in C$ such that $p \notin LP(X)$. We show that this implies that there exists a region R of C with $p \in R$ such that for every point $q \in R$, $q \notin LP(X)$. This will imply that $R \cap LP(X) = \emptyset$, i.e., $R \cap (LP(X) \setminus \{p\}) = \emptyset$, which means that p is not an element of $LP(LP(X))$. Let's begin.

Let p be an arbitrary point of C such that $p \notin LP(X)$. Then by Definition 3.13, there exists a region $\underline{ab}$ (where $a, b \in C$) with $p \in \underline{ab}$ such that $\underline{ab} \cap (X \setminus \{p\}) = \emptyset$. Now let q be an arbitrary point of $\underline{ab}$ such that $q \neq p$. To prove that $q \notin LP(X)$, we divide into two cases ($q < p$ and $p < q$; recall that $q = p$ is covered by the definition of p, which directly asserts that $p \notin LP(X)$). Note that if no $q \in \underline{ab}$ exists such that $q \neq p$, then the statement "$q \notin LP(X)$" is vacuously true.

Suppose first that $q < p$. Since $p \in \underline{ab}$, we have by Definitions 3.10 and 3.6 that $a < p$ and $p < b$. It follows from the former result and Definition 3.10 that we the region $\underline{ap}$ is well defined. During our treatment of this case, we will spend a great deal of time examining this region. However, before we begin in earnest, we will prove two preliminary results (that $q \in \underline{ap}$ and that $\underline{ap} \subset \underline{ab}$).

To prove that $q \in \underline{ap}$, Definitions 3.10 and 3.6 tell us that it will suffice to show that $a < q$ and $q < p$. We already know that $q < p$ by hypothesis, and since $q \in \underline{ab}$ by definition, Definitions 3.10 and 3.6 imply that $a < q$ (and $q < b$, but this is unimportant), as desired.

To prove that $\underline{ap} \subset \underline{ab}$, Definition 1.3 tells us that it will suffice to show that every element $x \in \underline{ap}$ is an element of $\underline{ab}$. Let x be an arbitrary point of $\underline{ap}$. It follows by Definitions 3.10 and 3.6 that $a < x$ and $x < p$. By Definitions 3.10 and 3.6 again, the fact that $p \in \underline{ab}$ implies that $a < p$ and $p < b$. Thus, since $x < p$ and $p < b$, Definition 3.1 asserts that $x < b$. Therefore, since $a < x$ and $x < b$, Definitions 3.6 and 3.10 imply that $x \in \underline{ab}$, as desired.

With these two claims proven, we may now begin in earnest. Suppose for the sake of contradiction that there exists an object $x \in \underline{ap} \cap (X \setminus \{q\})$. Then by Definition 1.6, $x \in \underline{ap}$ and $x \in X \setminus \{q\}$. We investigate each result in turn. Using the former result and the fact that $\underline{ap} \subset \underline{ab}$, Definition 1.3 tells us that $x \in \underline{ab}$ (this is an important result; remember it). Additionally, $x \in \underline{ap}$ reveals via Definitions 3.10 and 3.6 that $a < x$ and $x < p$. By Definition 3.1, $x < p$ implies that $x \neq p$, i.e., that $x \notin \{p\}$ (this is another important result). This concludes our investigation of the first result. With respect to the latter result, Definition 1.11 implies that $x \in X$ (this is our final important result) and $x \notin \{q\}$. We now combine the three important results into a whole that leads to a contradiction. First off, since $x \in X$ and $x \notin \{p\}$, Definition 1.11 tells us that $x \in X \setminus \{p\}$. This combined with the fact that $x \in \underline{ab}$ implies by Definition 1.6 that $x \in \underline{ab} \cap (X \setminus \{p\})$. Consequently, since we also know from early on in this proof that $\underline{ab} \cap (X \setminus \{p\}) = \emptyset$, we have by Definition 1.2 that $x \in \emptyset$. But this contradicts Definition 1.8. Therefore, we have that $\underline{ap} \cap (X \setminus \{q\}) = \emptyset$ for all $q < p$.

The proof is symmetric if $p < q$.

Therefore, for all $q \in \underline{ab}$ (the objects equal to, less than, and greater than p), there exists a region S with $q \in S$ (recall that we proved $q \in \underline{ap}$ in the case we treated, and $p \in \underline{ab}$ by definition) such that $S \cap (X \setminus \{q\}) = \emptyset$. It follows by the contrapositive of Definition 3.13 that for all $q \in \underline{ab}$, $q \notin LP(X)$.

We are now very close to being done. To wrap it up, suppose for the sake of contradiction that there exists an object $x \in \underline{ab} \cap LP(X)$. Then by Definition 1.6, $x \in \underline{ab}$ and $x \in LP(X)$. But this contradicts the previously proven implication that if $x \in \underline{ab}$, we must also have $x \notin LP(X)$. Therefore, no object x is an element of $\underline{ab} \cap LP(X)$, so by Definition 1.8, $\underline{ab} \cap LP(X) = \emptyset$. Consequently, $\underline{ab} \cap (LP(X) \setminus \{p\}) = \emptyset$. Therefore, by the contrapositive of Definition 1.13, $p \notin LP(LP(X))$, as desired. ☐

Proof of Theorem 4.6. To prove that $\overline{X} = \overline{\overline{X}}$, repeated applications of Definition 4.4 tell us that it will suffice to show that

$$X \cup LP(X) = (X \cup LP(X)) \cup LP(X \cup LP(X))$$

To show this, Theorem 1.7a tells us that it will suffice to verify the two statements

$$X \cup LP(X) \subset (X \cup LP(X)) \cup LP(X \cup LP(X)) \qquad (X \cup LP(X)) \cup LP(X \cup LP(X)) \subset X \cup LP(X)$$

By Theorem 1.7b, the left statement above is true. Consequently, all that's left at this point is to verify the right statement. To do so, Definition 1.3 tells us that it will suffice to demonstrate that every point $p \in (X \cup LP(X)) \cup LP(X \cup LP(X))$ is an element of $X \cup LP(X)$. Let's begin.

Let p be an arbitrary element of $(X \cup LP(X)) \cup LP(X \cup LP(X))$. Then by Definition 1.5, $p \in X \cup LP(X)$ or $p \in LP(X \cup LP(X))$. We divide into two cases. Suppose first that $p \in X \cup LP(X)$. Since this is actually exactly what we want to prove, we are done. Now suppose that $p \in LP(X \cup LP(X))$. Then we have by Theorem 3.20 that $p \in LP(X)$ or $p \in LP(LP(X))$. We divide into two cases again. If $p \in LP(X)$, then by Definition 1.5, $p \in X \cup LP(X)$, and we are done. On the other hand, if $p \in LP(LP(X))$, then by the lemma, $p \in LP(X)$. Therefore, as before, $p \in X \cup LP(X)$, and we are done. $\square$

Corollary 4.7. *Let $X \subset C$. Then $\overline{X}$ is closed.*

Proof. By Theorem 4.6, $\overline{X} = \overline{\overline{X}}$. Thus, if we let $Y = \overline{X}$, we know that $Y = \overline{Y}$. But by Theorem 4.5, this implies that Y, i.e., $\overline{X}$, is closed, as desired. $\square$

Definition 4.8. A subset G of a continuum C is **open** if its complement $C \setminus G$ is closed.

Theorem 4.9. *The sets $\emptyset$ and C are open.*

Proof. We will address the two sets individually. to prove that $\emptyset$ is open, Definition 4.8 tells us that it will suffice to show that $C \setminus \emptyset$ is closed. But $C \setminus \emptyset = C$, and by Theorem 4.2, C is closed. The proof is symmetric for C. $\square$