Script 5

Connectedness and Boundedness

5.1 Journal

11/19: **Axiom 4.** *A continuum is connected.*

Theorem 5.1. *The only subsets of a continuum C that are both open and closed are $\emptyset$ and C.*

Proof. To prove that the only subsets of C that are both open and closed are $\emptyset$ and C, it will suffice to show that if $A \subset C$ is both open and closed, then $A = \emptyset$ or $A = C$. Let $A \subset C$ be both open and closed. We divide into two cases ($A = \emptyset$ and $A \neq \emptyset$). If $A = \emptyset$, then we are done. On the other hand, if $A \neq \emptyset$, we have a bit more work to do. Basically, we will end up proving that the facts that A is open, A is closed, and $A \neq \emptyset$ imply that $A = C$. Let's begin.

First off, the fact that A is closed implies by Definition 4.8 that $C \setminus A$ is open. Additionally, we have by Script 1 that $A \cap (C \setminus A) = \emptyset$ and $A \cup (C \setminus A) = C$. Now suppose for the sake of contradiction that $A \neq C$. It follows since $A \subset C$ that we must have $C \not\subset A$, i.e., there is some object in C that is not an element of A. This object would clearly be an element of $C \setminus A$ in this case, meaning that $C \setminus A$ is nonempty. Thus, we have that A and $C \setminus A$ are disjoint, open, nonempty sets such that $A \cup (C \setminus A) = C$. Consequently, by consecutive applications of Definition 4.22, we know that C is disconnected, i.e., C is not connected. But this contradicts Axiom 4, which asserts that C is connected. Therefore, we must have that $A = C$, as desired. ☐

Theorem 5.2. *For all $x, y \in C$, if $x < y$, then there exists a point $z \in C$ such that z is between x and y.*

Proof. Suppose for the sake of contradiction that no point $z \in C$ exists such that z is between x and y. To find a contradiction, we will construct two sets A and B and prove that $A \cup B = C$, and that A and B are disjoint, nonempty, open sets. This will imply that C is disconnected, contradicting Axiom 4. Let's begin.

By consecutive applications of Corollary 4.13, we have that $\{c \in C \mid c < y\}$ and $\{c \in C \mid x < c\}$ are open. It follows by consecutive applications of Definition 4.8 that the complements of these two sets ($\{c \in C \mid c \geq y\}$ and $\{c \in C \mid x \geq c\}$, respectively) are closed. Let $A = \{c \in C \mid c \geq y\}$ and $B = \{c \in C \mid x \geq c\}$.

To prove that $A \cup B = C$, we suppose for the sake of contradiction that $A \cup B \neq C$. Since it is clear from their definitions that $A \subset C$ and $B \subset C$, we know that $A \cup B \subset C$. Thus, we must have $C \not\subset A \cup B$ for the supposition to hold. It follows that there exists a point $p \in C$ such that $p \notin A \cup B$. Since $p \notin A \cup B$, we have by Definition 1.5 that $p \notin A$ and $p \notin B$. Consequently, by the definitions of A and B, $p \ngeq y$ and $x \ngeq p$. Equivalently, $p < y$ and $x < p$. Thus, by Definition 3.6, p is between x and y. But this contradicts the assumption that no point of C is between x and y. Therefore, $A \cup B = C$, as desired.

To prove that $A \cap B = \emptyset$, we suppose for the sake of contradiction that there exists an object $p \in A \cap B$. Then by Definition 1.6, we have that $p \in A$ and $p \in B$. It follows by the definitions of A and B that $p \geq y$ and $x \geq p$. Thus, by transitivity, $x \geq y$. But by hypothesis, $x < y$, contradicting the trichotomy established by Definition 3.1. Therefore, $A \cap B = \emptyset$, as desired.

To prove that A and B are nonempty, Definition 1.8 tells us that it will suffice to an element of each set. From their definitions, it is clear that $y \in A$ and $x \in B$, so we are done.

To prove that A and B are open, Definition 4.8 tells us that it will suffice to show that $C \setminus A$ and $C \setminus B$, respectively, are closed. But since $A \cup B = C$ and $A \cap B = \emptyset$, we know that $C \setminus A = B$ and $C \setminus B = A$. Therefore, since B and A are closed as previously established, $C \setminus A$ and $C \setminus B$, respectively, are closed, too, as desired.

Since C can be written as $A \cup B$ where A and B are disjoint, nonempty, open sets, we have by Definition 4.22 that C is disconnected. But this contradicts Axiom 4, which asserts that C is connected. Therefore, there must exist a point $z \in C$ such that z is between x and y, as desired. $\square$

Corollary 5.3. *Every region is infinite.*

Proof. Let $\underline{ab}$ be a region, and suppose for the sake of contradiction that $\underline{ab}$ is finite. Then by Definitions 1.30 and 1.33, $\underline{ab} = \emptyset$, or $\underline{ab}$ has cardinality n. We divide into two cases. Suppose first that $\underline{ab} = \emptyset$. Then by Definitions 3.10 and 3.6, no point p exists such that $a < p < b$. Thus, by the contrapositive of Theorem 5.2, $a = b$. But this implies by Definition 3.10 that $\underline{ab}$ is not a region (since $a \not< b$), a contradiction. Now suppose that $\underline{ab}$ has cardinality n. Then by Theorem 3.5, the symbols $a_1, \dots, a_n$ may be assigned to each point of $\underline{ab}$ so that $a_1 < a_2 < \cdots < a_n$. But by Theorem 5.2, there exists a point $z \in C$ such that z is between a and a_1. Since $a < z < a_1 < b$, we clearly have that $z \in \underline{ab}$, yet it was not assigned a symbol a_k, a contradiction. Therefore, $\underline{ab}$ is infinite, as desired. $\square$

Corollary 5.4. *Every point of C is a limit point of C.*

Proof. Let p be an arbitrary element of C. To prove that p is a limit point of C, Definition 3.13 tells us that it will suffice to show that for all regions R with $p \in R$, $R \cap (C \setminus \{p\}) \neq \emptyset$. Let R be an arbitrary region with $p \in R$. By Corollary 5.3, R is infinite, so there exists a point $q \in R$ such that $q \neq p$. Additionally, since $q \in R$, we have $q \in C$. Thus, since $q \in C$ and $q \neq p$ (i.e., $q \notin \{p\}$), we have by Definition 1.11 that $q \in C \setminus \{p\}$. This combined with the fact that $q \in R$ implies by Definition 1.6 that $q \in R \cap (C \setminus \{p\})$, so $R \cap (C \setminus \{p\}) \neq \emptyset$, as desired. $\square$

Corollary 5.5. *Every point of the region $\underline{ab}$ is a limit point of $\underline{ab}$.*

Proof. Let p be an arbitrary element of $\underline{ab}$. It follows that $p \in C$. Thus, by Corollary 5.4, $p \in LP(C)$. Consequently, since $C = \underline{ab} \cup (C \setminus \underline{ab})$, Theorem 3.20 implies that $p \in LP(\underline{ab})$ or $p \in LP(C \setminus \underline{ab})$. Suppose for the sake of contradiction that $p \in LP(C \setminus \underline{ab})$. Then $p \in LP((C \setminus (\{a\} \cup \underline{ab} \cup \{b\}) \cup \{a\} \cup \{b\}))$. It follows by Definition 3.15 that $p \in LP(\operatorname{ext} \underline{ab} \cup \{a\} \cup \{b\})$. Consequently, by Corollary 3.21, $p \in LP(\operatorname{ext} \underline{ab})$, $p \in LP(\{a\})$, or $p \in LP(\{b\})$. But by Lemma 3.17, the fact that $p \in \underline{ab}$ prohibits p from being a limit point of $\operatorname{ext} \underline{ab}$, and by Corollary 3.23, $p \notin LP(\{a\})$ and $p \notin LP(\{b\})$, either, a contradiction. Therefore, $p \notin LP(C \setminus \underline{ab})$, so we must have $p \in LP(\underline{ab})$, as desired. $\square$