Script 3

Introducing a Continuum

3.1 Journal

10/20: **Axiom 1.** *A continuum is a nonempty set C.*

Definition 3.1. Let X be a set. An **ordering** on the set X is a subset $<$ of $X \times X$ with elements $(x, y) \in <$ written as $x < y$, satisfying the following properties:

a) (*Trichotomy*) For all $x, y \in X$, exactly one of the following holds: $x < y$, $y < x$, or $x = y$.

b) (*Transitivity*) For all $x, y, z \in X$, if $x < y$ and $y < z$, then $x < z$.

Remark 3.2.

a) In mathematics, "or" is understood to be inclusive unless stated otherwise. So in Definition 3.1a above, the word "exactly" is needed.

b) $x < y$ may also be written as $y > x$.

c) By $x \leq y$, we mean $x < y$ or $x = y$; similarly for $x \geq y$.

d) We often refer to elements of a continuum C as **points**.

Axiom 2. *A continuum C has an ordering $<$.*

Definition 3.3. If $A \subset C$, then a point $a \in A$ is a **first** point of A if for every element $x \in A$, either $a < x$ or $a = x$. Similarly, a point $b \in A$ is called a **last** point of A if, for every $x \in A$, either $x < b$ or $x = b$.

Lemma 3.4. *If A is a nonempty, finite subset of a continuum C, then A has a first and last point.*

Lemma. *Let A be a nonempty, finite subset of a continuum C, let a be any element of A, and let $B = A \setminus \{a\}$. Then $|B| = |A| - 1$.*

Proof. We first prove that $|\{a\}| = 1$. By Definition 1.33, to do so, it will suffice to find a bijection $f : \{a\} \to [1]$. Since $[1] = \{1\}$ by Definition 1.29, $f : \{a\} \to [1]$ defined by $f(a) = 1$ is clearly such a bijection.

We now note that $B \cap \{a\} = (A \setminus \{a\}) \cap \{a\} = \emptyset$ and $B \cup \{a\} = (A \setminus \{a\}) \cup \{a\} = A$; these results imply by Theorem 1.34b that $|A| = |B| + |\{a\}|$. But since $|\{a\}| = 1$, it follows that $|A| = |B| + 1$, i.e., $|B| = |A| - 1$. □

Proof of Lemma 3.4. We consider first points herein (the proof is symmetric for last points). If A is a nonempty, finite set, then by Definition 1.30, $|A| = n$ for some $n \in \mathbb{N}$. Thus, if we prove the claim for each $n \in \mathbb{N}$ individually, we will have proven the claim for all $n \in \mathbb{N}$, i.e., for all nonempty, finite sets A. Logically, to prove a property pertaining to any natural number, we induct on n.

For the base case $n = 1$, there is only one element (which we may call a) in A. Since $a = a$, i.e., "for every $x \in A$, either $a < x$ or $a = x$" is a true statement, it follows by Definition 3.3 that A has a first point. Now suppose inductively that we have proven the claim for n, i.e., we know that if A is a nonempty, finite subset of a continuum C with $|A| = n$, then A has a first point. We wish to prove the same claim if $|A| = n + 1$. Let a be an arbitrary element of A, and consider the set $B = A \setminus \{a\}$. By the lemma, $|B| = (n+1) - 1 = n$. Consequently, the induction hypothesis applies and asserts that B has a first point a_0. Clearly, a_0 is also an element of A, but it may or may not be the first point of A (the first point may now be a). Since C has an ordering $<$ (see Axiom 2) and $A \subset C$, Definition 3.1 asserts that either $a < a_0$, $a_0 < a$, or $a = a_0$. We now divide into three cases. If $a < a_0$, then since $a_0 \leq x$ for all $x \in A$ by Definition 3.3, Definition 3.1 implies that $a \leq x$ for all $x \in A$. Thus, by Definition 3.3, a is the first point in A, and we have proven the claim for $|A| = n + 1$ in this case. If $a_0 < a$, then it is still true that $a_0 \leq x$ for all $x \in A$. This means by Definition 3.3 that a_0 is still the first point in A, proving the claim for $|A| = n + 1$ in this case. If $a = a_0$, then $a \in B$ (since $a_0 \in B$), contradicting the fact that $B = A \setminus \{a\}$, so we need not consider this final case (the contradiction proves that it will never arise). This closes the induction. $\square$

Theorem 3.5. *Suppose that A is a set of n distinct points in a continuum C, or in other words, $A \subset C$ has cardinality n. Then the symbols $a_1, \dots, a_n$ may be assigned to each point of A so that $a_1 < a_2 < \cdots < a_n$, i.e., $a_i < a_{i+1}$ for all $1 \leq i \leq n - 1$.*

Proof. We divide into two cases ($|A| = 0$ and $|A| \in \mathbb{N}$).

If $|A| = 0$, then the statements "the symbols $a_1, \dots, a_n$ may be assigned to each point of A" and "$a_i < a_{i+1}$ for all $1 \leq i \leq n - 1 = -1$" are both vacuously true.

If $|A| \in \mathbb{N}$, we induct on $|A| = n$. For the base case $n = 1$, denote the single element of A by a_1. Since $a_i < a_{i+1}$ for all $1 \leq i \leq n - 1 = 0$ is vacuously true, the base case holds. Now suppose inductively that we have proven the claim for n, i.e., for a set $A \subset C$ satisfying $|A| = n$, the symbols $a_1, \dots, a_n$ may be assigned to each point of A so that $a_1 < a_2 < \cdots < a_n$. We now wish to prove the claim with regard to a set $A \subset C$ with $|A| = n + 1$. By Lemma 3.4, there is a last point $a_{n+1} \in A$, which may be denoted as such (we will rigorously confirm this later). Since the set $A \setminus \{a_{n+1}\}$ has cardinality n (see the lemma from Lemma 3.4), we have by the induction hypothesis that its n elements can be named $a_1, \dots, a_n$ and ordered $a_1 < a_2 < \cdots < a_n$. Clearly these n elements are elements of A and all that's left to do is determine where a_{n+1} fits into the established order. But by Definition 3.3, $x \leq a_{n+1}$ for all $x \in A$, i.e., $x < a_{n+1}$ for all $x \in A \setminus \{a_{n+1}\}$. Consequently, as its name would suggest, it is true that $a_1 < a_2 < \cdots < a_n < a_{n+1}$, as desired. $\square$

Definition 3.6. If $x, y, z \in C$ and either (i) both $x < y$ and $y < z$ or (ii) both $z < y$ and $y < x$, then we say that y is **between** x and z.

Corollary 3.7. *Of three distinct points in a continuum, one must be between the other two.*

Proof. Let A be a subset of the described continuum containing the three distinct points. It follows by Theorem 3.5 that the symbols a_1, a_2, a_3 may be assigned to each point of A so that $a_1 < a_2 < a_3$. Thus, $a_1 < a_2$ and $a_2 < a_3$, so a_2 is between a_1 and a_3 by Definition 3.6. $\square$

10/22: **Axiom 3.** *A continuum C has no first or last point.*

Definition 3.8. We define an ordering on $\mathbb{Z}$ by $m < n$ if $n = m + c$ for some $c \in \mathbb{N}$.

Exercise 3.9.

a) Prove that with this ordering $\mathbb{Z}$ satisfies Axioms 1-3.

Proof. Clearly, $\mathbb{Z}$ is a nonempty set, so Axiom 1 is immediately satisfied.

Axiom 2 asserts that $\mathbb{Z}$ must have an ordering $<$. As such, it will suffice to verify that the ordering given by Definition 3.8 satisfies the stipulations of Definition 3.1. To prove that $<$ satisfies the trichotomy, it will suffice to show that for all $x, y \in \mathbb{Z}$, exactly one of the following holds: $x < y$, $y < x$, or $x = y$.

We first show that *no more than one* of the three statements can simultaneously be true. Let x, y be arbitrary elements of $\mathbb{Z}$. We divide into three cases. First, suppose for the sake of contradiction that $x < y$ and $y < x$. By Definition 3.8, this implies that $y = x + c$ and $x = y + c'$ for some $c, c' \in \mathbb{N}$.

Substituting, we have $y = y + c' + c$, or $0 = c' + c$ by the cancellation law of addition. But since $c', c \in \mathbb{N}$, the closure of addition on $\mathbb{N}$ implies that $(c' + c) \in \mathbb{N}$. Therefore, $c' + c \neq 0$, a contradiction. Second, suppose for the sake of contradiction that $x < y$ and $x = y$. By Definition 3.8, this implies that $y = x + c$ for some $c \in \mathbb{N}$. Substituting, we have $y = y + c$, or $0 = c$ by the cancellation law of addition. But since $c \in \mathbb{N}$, $c \neq 0$, a contradiction. The proof of the third case ($y < x$ and $x = y$) is symmetric to that of the second case.

We now show that *at least one* of the three statements is always true. Let x, y be arbitrary elements of $\mathbb{Z}$, and suppose for the sake of contradiction that $x \not< y$, $y \not< x$, and $x \neq y$. Since $x \not< y$, $y \neq x + c$ for any $c \in \mathbb{N}$. Equivalently, $y - x \neq c$ for any $c \in \mathbb{N}$, i.e., $(y - x) \notin \mathbb{N}$. Similarly, since $y \not< x$, $x - y \neq c'$ for any $c' \in \mathbb{N}$. Equivalently, $y - x \neq c'$ for any $c' \in -\mathbb{N}$, i.e., $(y - x) \notin -\mathbb{N}$. Lastly, since $x \neq y$, $y - x \neq 0$, i.e., $(y - x) \notin \{0\}$. Since $(y - x) \notin -\mathbb{N}$, $(y - x) \notin \{0\}$, and $(y - x) \notin \mathbb{N}$, Definition 1.5 implies that $(y - x) \notin (-\mathbb{N}) \cup \{0\} \cup \mathbb{N}$. Consequently, by Script 0, $(y - x) \notin \mathbb{Z}$. But by the closure of integer subtraction, $(y - x) \in \mathbb{Z}$, a contradiction.

To prove that $<$ is transitive, it will suffice to show that for all $x, y, z \in \mathbb{Z}$, if $x < y$ and $y < z$, then $x < z$. Let x, y, z be arbitrary elements of $\mathbb{Z}$ for which it is true that $x < y$ and $y < z$. By Definition 3.8, we have $y = x + c$ and $z = y + c'$ for some $c, c' \in \mathbb{N}$. Substituting, we have $z = x + c + c'$. Since $(c + c') \in \mathbb{N}$ by the closure of addition on $\mathbb{N}$, Definition 3.8 implies that $x < z$.

Axiom 3 asserts that $\mathbb{Z}$ must have no first or last point. Suppose for the sake of contradiction that $\mathbb{Z}$ has some first point a. Then by Definition 3.3, $a \leq x$ for every $x \in \mathbb{Z}$. However, under the closure of subtraction on $\mathbb{Z}$, $(a - 1) \in \mathbb{Z}$. Since $(a - 1) + 1 = a$, Definition 3.8 asserts that $a - 1 < a$. Therefore, we have $a - 1 < a$ and $a \leq a - 1$ (since, again, $(a - 1) \in \mathbb{Z}$), contradicting the previously demonstrated fact that $<$ is an ordering. The proof is symmetric for the last point. ☐

b) Show that for any $p = \left[\frac{a}{b}\right] \in \mathbb{Q}$, there is some $(a_1, b_1) \in p$ with $0 < b_1$.

Proof. Let $\left[\frac{a}{b}\right]$ be an arbitrary element of $\mathbb{Q}$. It follows by Definition 2.5 that $(a, b) \in X$. Since we also have $(a, b) \sim (a, b)$ by Exercise 2.2e, Definition 2.5 implies that $(a, b) \in \left[\frac{a}{b}\right]$. By the trichotomy of $\mathbb{Z}$ (see Exercise 3.9a), we have $0 < b$, $b < 0$, or $0 = b$. We divide into three cases. First, suppose that $0 < b$. Then (a, b) is an element $(a_1, b_1) \in \left[\frac{a}{b}\right]$ for which $0 < b_1$, and we are done. Second, suppose that $b < 0$. Since $(-a)(b) = (-b)(a)$, we have by the definition of $\sim$ that $(-a, -b) \sim (a, b)$. Additionally, we have by the closure of integer multiplication that $-a, -b \in \mathbb{Z}$, and since $b \neq 0$ by Exercise 2.2e and clearly $-1 \neq 0$, $-1 \cdot b = -b \neq 0$ by the contrapositive of the zero-product property. Thus, by Exercise 2.2e, $(-a, -b) \in X$. This coupled with the previously proven fact that $(-a, -b) \sim (a, b)$ implies by Definition 2.5 that $(-a, -b) \in \left[\frac{a}{b}\right]$. Now recall that $b < 0$ by hypothesis, so we may use Definition 3.8 to see that $b + c = 0$ for some $c \in \mathbb{N}$. It follows that $-(b + c) = 0$, i.e., $-b - c = 0$, i.e., $-b = 0 + c$, meaning that $0 < -b$ by Definition 3.8. Thus, $(-a, -b)$ is an element $(a_1, b_1) \in \left[\frac{a}{b}\right]$ for which $0 < b_1$. Third, suppose that $b = 0$. But this contradicts Exercise 2.2e which asserts that $b \neq 0$, so we need not consider this case (as it will never arise). ☐

c) Define an ordering $<_{\mathbb{Q}}$ on $\mathbb{Q}$ as follows. For $p, q \in \mathbb{Q}$, let $(a_1, b_1) \in p$ be such that $0 < b_1$ and let $(a_2, b_2) \in q$ be such that $0 < b_2$. Then we define $p <_{\mathbb{Q}} q$ if $a_1 b_2 < a_2 b_1$. Show that $<_{\mathbb{Q}}$ is a well-defined relation on $\mathbb{Q}$.

Proof. For the relation $<_{\mathbb{Q}}$ to be well-defined, Definition 3.1 tells us that it must satisfy the trichotomy and be transitive.

To prove that $<_{\mathbb{Q}}$ satisfies the trichotomy, it will suffice to show that for all $p, q \in \mathbb{Q}$, exactly one of the following holds: $p <_{\mathbb{Q}} q$, $q <_{\mathbb{Q}} p$, or $p = q$.

We first show that *no more than one* of the three statements can be simultaneously true. Let p, q be arbitrary elements of $\mathbb{Q}$, let $(a, b) \in p$ be such that $0 < b$ (we know that such an element exists by Exercise 3.9b[1]), and let $(c, d) \in q$ be such that $0 < d$. We divide into three cases. First, suppose for

[1] This justification will not be supplied again to make the proof less repetitive.

the sake of contradiction that $p <_{\mathbb{Q}} q$ and $q <_{\mathbb{Q}} p$. Then $ad < bc$ and $cb < da$ (i.e., $bc < ad$) by the definition of $<_{\mathbb{Q}}$. But this violates the trichotomy known to hold (by Exercise 3.9a) for the ordering $<$ on the integers, a contradiction. Second, suppose for the sake of contradiction that $p <_{\mathbb{Q}} q$ and $p = q$. By the definition of $<_{\mathbb{Q}}$, it follows from the first assumption that $ad < bc$. Additionally, by Exercise 2.6, it follows from the second assumption that $(a,b) \sim (c,d)$, implying by Exercise 2.2e that $ad = bc$. But once again, the simultaneous results that $ad < bc$ and $ad = bc$ violate the trichotomy of the integers, a contradiction. The proof of the third case is symmetric to that of the second.

We now show that *at least one* of the three statements is always true. Let p, q be arbitrary elements of $\mathbb{Q}$, let $(a,b) \in p$, and let $(c,d) \in q$. Suppose for the sake of contradiction that $p \not<_{\mathbb{Q}} q$, $q \not<_{\mathbb{Q}} p$, and $p \neq q$. Since $p \not<_{\mathbb{Q}} q$, we have that $ad \not< bc$. Similarly, since $q \not<_{\mathbb{Q}} p$, we have $cb \not< da$ (i.e., $bc \not< ad$). Lastly, since $p \neq q$, Exercise 2.6 implies that $(a,b) \nsim (c,d)$. It follows by Exercise 2.2e that $ad \neq bc$. To recap, for the integers ad and bc, we have $ad \not< bc$, $bc \not< ad$, and $ad \neq bc$. But by Exercise 3.9a, $ad < bc$, $bc < ad$, or $ad = bc$, a contradiction.

To prove that $<_{\mathbb{Q}}$ is transitive, it will suffice to show that for all $p, q, r \in \mathbb{Q}$, if $p <_{\mathbb{Q}} q$ and $q <_{\mathbb{Q}} r$, then $p <_{\mathbb{Q}} r$. Let p, q, r be arbitrary elements of $\mathbb{Q}$ for which it is true that $p <_{\mathbb{Q}} q$ and $q <_{\mathbb{Q}} r$, let $(a,b) \in p$ be such that $0 < b$, let $(c,d) \in q$ be such that $0 < d$, and let $(e,f) \in r$ such that $0 < f$. By the definition of $<_{\mathbb{Q}}$, we have $ad < bc$ and $cf < de$. Since $0 < f$ and $0 < b$, we can multiply both sides of the inequalities by b or f without affecting the truth of the statement (see Script 0). Thus, we may create the inequalities $adf < bcf$ and $bcf < bde$. So $adf < bde$ by Definition 3.1, implying that $af < be$ by the cancellation law (which we may use since $0 < d$). It follows by the definition of $<_{\mathbb{Q}}$ that $p <_{\mathbb{Q}} r$. □

d) Show that $\mathbb{Q}$ with the ordering $<_{\mathbb{Q}}$ satisfies Axioms 1-3.

Proof. Clearly, $\mathbb{Q}$ is a nonempty set, so Axiom 1 is immediately satisfied.

By Exercise 3.9c, $\mathbb{Q}$ has an ordering, so Axiom 2 is satisfied.

Axiom 3 asserts that $\mathbb{Q}$ must have no first or last point. Suppose for the sake of contradiction that $\mathbb{Q}$ has some first point p. Then by Definition 3.3, $p <_{\mathbb{Q}} x$ or $p = x$ for all $x \in \mathbb{Q}$. Let $(a,b) \in p$ be such that $0 < b$ (see Exercise 3.9b). Under the closure of integer subtraction, $(a-1) \in \mathbb{Z}$, so $\left[\frac{a-1}{b}\right] \in \mathbb{Q}$. Since $ba = ba - b + b = b(a-1) + b$ where $b \in \mathbb{N}$ because $b \in \mathbb{Z}$ and $0 < b$, Definition 3.8 implies that $(a-1)b < ba$. It follows by the definition of $<_{\mathbb{Q}}$ that $\left[\frac{a-1}{b}\right] <_{\mathbb{Q}} \left[\frac{a}{b}\right] = p$, a contradiction. The argument is symmetric for the last point. □

Definition 3.10. If $a, b \in C$ and $a < b$, then the set of points between a and b is called a **region** and denoted by $\underline{ab}$.

Remark 3.11. One often sees the notation (a,b) for regions. We will reserve the notation (a,b) for ordered pairs in a product $A \times B$. These are very different things.

Theorem 3.12. *If x is a point of a continuum C, then there exists a region $\underline{ab}$ such that $x \in \underline{ab}$.*

Proof. Let x be an arbitrary point in a continuum C. By Axiom 2, C has an ordering $<$, which we will frequently make use of throughout the remainder of this proof. By Axiom 3, C has no first or last points, so it cannot be true that $x \leq y$ for all $y \in C$, nor can it be true that $x \geq y$ for all $y \in C$. This implies that there exists an $a \in C$ such that $a < x$ and that there exists a $b \in C$ such that $b > x$. Since $a < x$ and $x < b$, Definition 3.6 implies that x is between a and b. Note also that by Definition 3.1 (specifically transitivity), $a < b$. Therefore, since $a, b \in C$, $a < b$, and x is between a and b, Definition 3.10 implies that $x \in \underline{ab}$. □

Definition 3.13. Let A be a subset of a continuum C. A point p of C is called a **limit point** of A if every region R containing p has nonempty intersection with $A \setminus \{p\}$. Explicitly, this means:

$$\text{for every region } R \text{ with } p \in R, \text{ we have } R \cap (A \setminus \{p\}) \neq \emptyset.$$

Notice that we do not require that a limit point p of A be an element of A. We will use the notation $LP(A)$ to denote the set of limit points of A.

Theorem 3.14. *If p is a limit point of A and $A \subset B$, then p is a limit point of B.*

Proof. To prove that a limit point p of $A \subset B$ is a limit point of B, Definition 3.13 tells us that it will suffice to show that for every region R with $p \in R$, we have $R \cap (B \setminus \{p\}) \neq \emptyset$. Let p be a limit point of A, and let R be an arbitrary region with $p \in R$ (we know that such a region exists because of Theorem 3.12). Then by Definition 3.13, we have $R \cap (A \setminus \{p\}) \neq \emptyset$. Thus, by Definition 1.8, there is an element $x \in R \cap (A \setminus \{p\})$. Since $A \setminus \{p\} \subset B \setminus \{p\}$ (because $A \subset B$), it follows (by the fact that for three sets A, B, C such that $B \subset C$, $A \cap B \subset A \cap C$) that $R \cap (A \setminus \{p\}) \subset R \cap (B \setminus \{p\})$. Consequently, by Definition 1.3, the previously referenced object $x \in R \cap (A \setminus \{p\})$ is also an element of $R \cap (B \setminus \{p\})$. Thus, by Definition 1.8, $R \cap (B \setminus \{p\}) \neq \emptyset$, as desired. ☐

10/27: **Definition 3.15.** If $\underline{ab}$ is a region in a continuum C, then $C \setminus (\{a\} \cup \underline{ab} \cup \{b\})$ is called the **exterior** of $\underline{ab}$ and is denoted by $\operatorname{ext} \underline{ab}$.

Lemma 3.16. *If $\underline{ab}$ is a region in a continuum C, then*

$$\operatorname{ext} \underline{ab} = \{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}$$

Proof. To prove that $\operatorname{ext} \underline{ab} = \{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}$, Definition 3.15 tells us that it will suffice to show that $C \setminus (\{a\} \cup \underline{ab} \cup \{b\}) = \{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}$. To do this, Definition 1.2 tells us that we must verify that every element $y \in C \setminus (\{a\} \cup \underline{ab} \cup \{b\})$ is an element of $\{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}$ and vice versa. Let's begin.

First, let y be an arbitrary element of $C \setminus (\{a\} \cup \underline{ab} \cup \{b\})$. By Definition 1.11, this implies that $y \in C$ and $y \notin \{a\} \cup \underline{ab} \cup \{b\}$. The latter result implies by Definition 1.5 that $y \notin \{a\}$, $y \notin \underline{ab}$, and $y \notin \{b\}$. Since $y \notin \{a\}$ and $y \notin \{b\}$, we know that $y \neq a$ and $y \neq b$. Furthermore, since $y \notin \underline{ab}$, Definition 3.10 asserts that y is not between a and b. Thus, by Definition 3.6[2], we have that $y \leq a$ or $y \geq b$. But as previously established, $y \neq a$ and $y \neq b$, so it must be that $y < a$ or $y > b$. We divide into two cases. If $y < a$, then this fact combined with the fact that $y \in C$ implies that $y \in \{x \in C \mid x < a\}$. Therefore, by Definition 1.5, $y \in \{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}$, as desired. The proof is symmetric for the other case.

Now let y be an arbitrary element of $\{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}$. By Definition 1.5, this implies that $y \in \{x \in C \mid x < a\}$ or $y \in \{x \in C \mid b < x\}$. We divide into two cases. Suppose first that $y \in \{x \in C \mid x < a\}$. Then $y \in C$ and $y < a$. Since $y < a$, Definition 3.1 implies that $y \neq a$, i.e., $y \notin \{a\}$. Since $y < a$ and $a < b$, Definition 3.1 implies that $y < b$. Thus, for similar reasons to before, $y \neq b$, i.e., $y \notin \{b\}$. Lastly, since $a < b$, for y to be between a and b, Definition 3.6 implies that we must have $a < y$ and $y < b$. But $y < a$, so it must be that y is not between a and b. Thus, by Definition 3.10, $y \notin \underline{ab}$. Since $y \notin \{a\}$, $y \notin \underline{ab}$, and $y \notin \{b\}$, Definition 1.5 asserts that $y \notin \{a\} \cup \underline{ab} \cup \{b\}$. Therefore, since we also have $y \in C$ as previously established, Definition 1.11 implies that $y \in C \setminus (\{a\} \cup \underline{ab} \cup \{b\})$, as desired. The proof is symmetric if $y \in \{x \in C \mid b < x\}$. ☐

Lemma 3.17. *No point in the exterior of a region is a limit point of that region. No point of a region is a limit point of the exterior of that region.*

Proof. We will take this one claim at a time, starting with the first listed claim.

Let $\underline{ab}$ be an arbitrary region of a continuum C. To prove that no point in the exterior of $\underline{ab}$ is a limit point of $\underline{ab}$, Definition 3.13 tells us that it will suffice to show that for all points $p \in \operatorname{ext} \underline{ab}$, there exists some region R with $p \in R$ such that $R \cap (\underline{ab} \setminus \{p\}) = \emptyset$. Let p be an arbitrary element of $\operatorname{ext} \underline{ab}$. Then by Lemma 3.16 and Definition 1.5, $p \in \{x \in C \mid x < a\}$ or $p \in \{x \in C \mid b < x\}$. We divide into two cases. Suppose first that $p \in \{x \in C \mid x < a\}$. It follows that $p < a$, so let $c \in C$ be a point such that $c < p$ (Axiom 3 and Definition 3.3 imply that such a point exists). Since $c < p$ and $p < a$, Definition 3.6 implies that p is between c and a. Thus, Definition 3.10 implies that $p \in \underline{ca}$. Now suppose for the sake of contradiction that for some object x, $x \in \underline{ca} \cap (\underline{ab} \setminus \{p\})$. By Definition 1.6, this implies that $x \in \underline{ca}$ and $x \in \underline{ab} \setminus \{p\}$. Since $x \in \underline{ca}$, Definitions 3.10 and 3.6 imply that $c < x$ and $x < a$. Additionally, since $x \in \underline{ab} \setminus \{p\}$, Definition 1.11 implies that $x \in \underline{ab}$ and $x \notin \{p\}$, so with respect to the former claim, $a < x$ and $x < b$, as before. But by Definition 3.1, we cannot have $x < a$ and $a < x$, so we have arrived at a contradiction. Therefore,

[2]Technically, to use Definition 3.6, we also need the fact that $a < b$ to know that we are applying case i not case ii. This detail will not be mentioned again to make future proofs less repetitive.

$x \notin \underline{ca} \cap (\underline{ab} \setminus \{p\})$ for any x, proving by Definition 1.8 that $\underline{ca} \cap (\underline{ab} \setminus \{p\}) = \emptyset$, as desired. The proof is symmetric if $p \in \{x \in C \mid b < x\}$.

Let $\underline{ab}$ be an arbitrary region of a continuum C. To prove that no point of $\underline{ab}$ is a limit point of the exterior of $\underline{ab}$, Definition 3.13 tells us that it will suffice to show that for all points $p \in \underline{ab}$, there exists some region R with $p \in R$ such that $R \cap (\operatorname{ext} \underline{ab} \setminus \{p\}) = \emptyset$. Let p be an arbitrary element of $\underline{ab}$. Then $\underline{ab}$ is actually a p-containing region having empty intersection with $\operatorname{ext} \underline{ab} \setminus \{p\}$, as will now be proven. Suppose for the sake of contradiction that for some object x, $x \in \underline{ab} \cap (\operatorname{ext} \underline{ab} \setminus \{p\})$. By Definition 3.15, this implies that $x \in \underline{ab} \cap ((C \setminus (\{a\} \cup \underline{ab} \cup \{b\})) \setminus \{p\})$. Thus, by Definition 1.6, $x \in \underline{ab}$ and $x \in (C \setminus (\{a\} \cup \underline{ab} \cup \{b\})) \setminus \{p\}$. Consequently, by consecutive applications of Definition 1.11, $x \in C$, $x \notin \{a\} \cup \underline{ab} \cup \{b\}$, and $x \notin \{p\}$. With respect to the middle of the three previous results, Definition 1.5 implies that $x \notin \{a\}$, $x \notin \underline{ab}$, and $x \notin \{b\}$. But we have previously demonstrated that $x \in \underline{ab}$, a contradiction. Therefore, $x \notin \underline{ab} \cap (\operatorname{ext} \underline{ab} \setminus \{p\})$ for any x, proving by Definition 1.8 that $\underline{ab} \cap (\operatorname{ext} \underline{ab} \setminus \{p\}) = \emptyset$, as desired. ☐

Theorem 3.18. *If two regions have a point x in common, their intersection is a region containing x.*

Proof. Let $\underline{ab}$ and $\underline{cd}$ be two regions of a continuum C such that for some point $x \in C$, $x \in \underline{ab}$ and $x \in \underline{cd}$. We divide into two cases ($a \leq c$ and $b \leq d$, and $a \leq c$ and $b > d$) WLOG[3].

Suppose first that $a \leq c$ and $b \leq d$. We seek to prove that $\underline{ab} \cap \underline{cd} = \underline{cb}$ where $\underline{cb}$ is clearly a region, and that $x \in \underline{cb}$. To prove that $\underline{ab} \cap \underline{cd} = \underline{cb}$, Definition 1.2 tells us that it will suffice to show that every element $y \in \underline{ab} \cap \underline{cd}$ is an element of $\underline{cb}$ and vice versa. Suppose first that y is an arbitrary element of $\underline{ab} \cap \underline{cd}$. Then by Definition 1.6, $y \in \underline{ab}$ and $y \in \underline{cd}$. Thus, by Definitions 3.10 and 3.6, $a < y$, $y < b$, $c < y$, and $y < d$. Since $c < y$ and $y < b$, it follows by Definitions 3.6 and 3.10 that $y \in \underline{cb}$, as desired. Now suppose that $y \in \underline{cb}$. Then by Definitions 3.10 and 3.6, $c < y$ and $y < b$. Since $a \leq c$ (by assumption) and $c < y$, $a < y$ (if $a = c$, then $c < y$ implies $a < y$ by substitution; if $a < c$ and $c < y$, then Definition 3.1 implies $a < y$[4]). Thus, since $a < y$ and $y < b$, Definitions 3.6 and 3.10 imply that $y \in \underline{ab}$. Similarly, $y < b$ and $b \leq d$ (by assumption) together imply that $y < d$. Thus, since $c < y$ and $y < d$, Definitions 3.6 and 3.10 imply that $y \in \underline{cd}$. Since $y \in \underline{ab}$ and $y \in \underline{cd}$, Definition 1.6 implies that $y \in \underline{ab} \cap \underline{cd}$, as desired. Lastly, since $x \in \underline{ab}$ and $x \in \underline{cd}$ by hypothesis, Definition 1.6 implies that $x \in \underline{ab} \cap \underline{cd}$, which means by Definition 1.2 and the above that $x \in \underline{cb}$, as desired.

Now suppose that $a \leq c$ and $b > d$. We seek to prove that $\underline{ab} \cap \underline{cd} = \underline{cd}$ where $\underline{cd}$ is clearly a region, and that $x \in \underline{cd}$. To prove that $\underline{ab} \cap \underline{cd} = \underline{cd}$, Theorem 1.7a tells us that it will suffice to show that $\underline{ab} \cap \underline{cd} \subset \underline{cd}$ and $\underline{cd} \subset \underline{ab} \cap \underline{cd}$. But by Theorem 1.7c, $\underline{ab} \cap \underline{cd} \subset \underline{cd}$, so all that's left to prove is that $\underline{cd} \subset \underline{ab} \cap \underline{cd}$. Definition 1.3 tells us that we may do this by demonstrating that every element $y \in \underline{cd}$ is an element of $\underline{ab} \cap \underline{cd}$. Let y be an arbitrary element of $\underline{cd}$. Then by Definitions 3.10 and 3.6, $c < y$ and $y < d$. Since $a \leq c$ (by assumption) and $c < y$, $a < y$, and since $y < d$ and $d < b$, $y < b$. Thus, since $a < y$ and $y < b$, Definitions 3.6 and 3.10 imply that $y \in \underline{ab}$. This fact combined with the hypothesis that $y \in \underline{cd}$ implies by Definition 1.6 that $y \in \underline{ab} \cap \underline{cd}$, as desired. Lastly, since $\underline{cd}$ *is* the intersection of $\underline{ab}$ and $\underline{cd}$, and $x \in \underline{cd}$ by hypothesis, x is clearly an element of the intersection of $\underline{ab}$ and $\underline{cd}$. ☐

Corollary 3.19. *If n regions $R_1, \dots, R_n$ have a point x in common, then their intersection $R_1 \cap \cdots \cap R_n$ is a region containing x.*

Proof. We induct on n from the base case $n_0 = 2$ using the form of induction described in Additional Exercise 0.2a. For the base case $n = 2$, let R_1 and R_2 be two regions that have a point x in common. By Theorem 3.18, it follows that their intersection $R_1 \cap R_2$ is a region containing x, proving the base case. Now suppose inductively that we have proven the claim for some n, i.e., we know that if n regions $R_1, \dots, R_n$ have a point x in common, then their intersection $\bigcap_{k=1}^{n} R_k$ is a region containing x. We wish to prove that if $n+1$ regions $R_1, \dots, R_{n+1}$ have a point x in common, then their intersection $\bigcap_{k=1}^{n+1} R_k$ is a region containing x. Let $R_1, \dots, R_{n+1}$ be $n+1$ regions that have a point x in common. By the induction hypothesis, $\bigcap_{k=1}^{n} R_k$ is a region containing x. Since $\bigcap_{k=1}^{n} R_k$ and R_{n+1} are both regions with a point x in common,

[3]The other two cases ($a > c$ and $b \leq d$, and $a > c$ and $b > d$) can be encapsulated in the first two by switching the names of the two regions: $a > c \wedge b \leq d \xrightarrow{\text{rename}} c > a \wedge d \leq b \implies (a \leq c \wedge b > d) \vee (a \leq c \wedge b = d)$, where both of the latter two cases are covered by one of the original cases.

[4]This justification and simple variations thereof, although used again, will not be stated again.

Theorem 3.18 applies and implies that $(\bigcap_{k=1}^{n} R_k) \cap R_{n+1} = \bigcap_{k=1}^{n+1} R_k$ is a region containing x, thus closing the induction. $\square$

Theorem 3.20. *Let A, B be subsets of a continuum C. Then p is a limit point of $A \cup B$ if and only if p is a limit point of at least one of A or B.*

Proof. To prove that p is a limit point of $A \cup B$ if and only if p is a limit point of at least one of A or B, we must prove the dual implications "$p \in LP(A)$ or $p \in LP(B)$ implies $p \in LP(A \cup B)$" and "$p \in LP(A \cup B)$ implies $p \in LP(A)$ or $p \in LP(B)$." The first implication will be proved directly, but the second one will be proved by contrapositive. Let's begin.

Suppose first that p is a limit point of A or B. We divide into two cases. If $p \in LP(A)$, then since $A \subset A \cup B$ by Theorem 1.7, Theorem 3.14 applies and implies that $p \in LP(A \cup B)$. The proof is symmetric if $p \in LP(B)$.

Now suppose that $p \notin LP(A)$ and $p \notin LP(B)$. Then by the contrapositive of Definition 3.13, there exist regions R_1 and R_2 with $p \in R_1$ and $p \in R_2$ such that $R_1 \cap (A \setminus \{p\}) = \emptyset$ and $R_2 \cap (B \setminus \{p\}) = \emptyset$. Since R_1 and R_2 are two regions that have a point (namely p) in common, Theorem 3.18 asserts that $R_1 \cap R_2$ is a region R with $p \in R$. Additionally, since $R_1 \cap R_2 \subset R_1$ and $R_1 \cap R_2 \subset R_2$ by Theorem 1.7, $R \subset R_1$ and $R \subset R_2$. These two results combined with the previously proven facts that $R_1 \cap (A \setminus \{p\}) = \emptyset$ and $R_2 \cap (B \setminus \{p\}) = \emptyset$ imply that $R \cap (A \setminus \{p\}) = \emptyset$ and $R \cap (B \setminus \{p\}) = \emptyset$. Now since $U \cup \emptyset = U$ for any set U, we know that

$$\begin{aligned} \emptyset &= R \cap (A \setminus \{p\}) \\ &= (R \cap (A \setminus \{p\})) \cup \emptyset \end{aligned}$$

Substitute $\emptyset = R \cap (B \setminus \{p\})$.

$$= (R \cap (A \setminus \{p\})) \cup (R \cap (B \setminus \{p\}))$$

Apply the fact that $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$ for any sets X, Y, Z.

$$= R \cap ((A \setminus \{p\}) \cup (B \setminus \{p\}))$$

Apply the fact that $(X \setminus Z) \cup (Y \setminus Z) = (X \cup Y) \setminus Z$ for any sets X, Y, Z.

$$= R \cap ((A \cup B) \setminus \{p\})$$

Since there exists a region R with $p \in R$ such that $R \cap ((A \cup B) \setminus \{p\}) = \emptyset$, the contrapositive of Definition 3.13 implies that $p \notin LP(A \cup B)$. $\square$

Corollary 3.21. *Let $A_1, \dots, A_n$ be n subsets of a continuum C. Then p is a limit point of $A_1 \cup \cdots \cup A_n$ if and only if p is a limit point of at least one of the sets A_k.*

Proof. We induct on n from the base case $n_0 = 2$ using the form of induction described in Additional Exercise 0.2a. For the base case $n = 2$, let A_1 and A_2 be two subsets of a continuum C. By Theorem 3.20, it follows that p is a limit point of $A_1 \cup A_2$ if and only if p is a limit point of at least one of A_1 or A_2, proving the base case. Now suppose inductively that we have proven the claim for some n, i.e., we know that if there exist n subsets $A_1, \dots, A_n$ of a continuum C, then p is a limit point of $\bigcup_{k=1}^{n} A_k$ if and only if p is a limit point of at least one of the sets A_k. We wish to prove that if there exist $n+1$ subsets $A_1, \dots, A_{n+1}$ of a continuum C, then p is a limit point of $\bigcup_{k=1}^{n+1} A_k$ if and only if p is a limit point of at least one of the sets A_k. Let $A_1, \dots, A_{n+1}$ be $n+1$ subsets of a continuum C. By the induction hypothesis, p is a limit point of $\bigcup_{k=1}^{n} A_k$ if and only if p is a limit point of at least one of the sets A_k. Since $\bigcup_{k=1}^{n} A_k$ and A_{n+1} are both subsets of a continuum C, Theorem 3.20 applies and implies that p is a limit point of $(\bigcup_{k=1}^{n} A_k) \cup A_{n+1} = \bigcup_{k=1}^{n+1} A_k$ if and only if p is a limit point of at least one of $\bigcup_{k=1}^{n} A_k$ or A_{n+1}. But the last two statements combined imply that p is a limit point of $\bigcup_{k=1}^{n+1} A_k$ if and only if p is a limit point of at least one of the sets A_k where $1 \leq k \leq n$ or A_{n+1}, i.e., p is a limit point of $\bigcup_{k=1}^{n+1} A_k$ if and only if p is a limit point of at least one of the sets A_k, thus closing the induction. $\square$

Theorem 3.22. *If p and q are distinct points of a continuum C, then there exist disjoint regions R and S containing p and q, respectively.*

Proof. WLOG, let $p < q$. Additionally, let $a < p$ and $b > q$ be points of C (Axiom 3 and Definition 3.3 imply that such points exist). We divide into two cases (no point $x \in C$ exists between p and q, and there exists a point $x \in C$ between p and q). Let's begin.

Suppose first that no point $x \in C$ exists between p and q. Let $R = \underline{aq}$ and let $S = \underline{pb}$. Thus, Definitions 3.6 and 3.10 imply by the facts that $a < p$ by definition and $p < q$ by hypothesis, and $p < q$ by hypothesis and $q < b$ by definition that $p \in R$ and $q \in S$, respectively. To prove that R and S are disjoint, Definition 1.9 tells us that it will suffice to show that $R \cap S = \emptyset$. Suppose for the sake of contradiction that there exists an object $x \in R \cap S$. By Definition 1.6, this implies that $x \in R$ and $x \in S$ (and, hence, that $x \in C$). Thus, by consecutive applications of Definitions 3.10 and 3.6, we have that $a < x$, $x < q$, $p < x$, and $x < b$. Since $p < x$ and $x < q$, Definition 3.6 implies that x is between p and q. But by hypothesis, no point $x \in C$ exists between p and q, a contradiction. Therefore, since $x \notin R \cap S$ for all x, Definition 1.8 implies that $R \cap S = \emptyset$, as desired.

Now suppose that there exists a point $x \in C$ between between p and q. Let $R = \underline{ax}$ and let $S = \underline{xb}$. It follows from the hypothesis by Definition 3.6 that $p < x$ and $x < q$. Thus, Definitions 3.6 and 3.10 imply by the facts that $a < p$ and $p < x$, and $x < q$ and $q < b$ that $p \in R$ and $q \in S$, respectively. To prove that R and S are disjoint, Definition 1.9 tells us tht it will suffice to show that $R \cap S = \emptyset$. Suppose for the sake of contradiction that there exists an object $y \in R \cap S$. By Definition 1.6, this implies that $y \in R$ and $y \in S$ (and, hence, that $y \in C$). Thus, by consecutive applications of Definitions 3.10 and 3.6, we have that $a < y$, $y < x$, $x < y$, and $y < b$. But by Definition 3.1, we cannot have $y < x$ and $x < y$, a contradiction. Therefore, since $x \notin R \cap S$ for all x, Definition 1.8 implies that $R \cap S = \emptyset$, as desired. □

10/29: **Corollary 3.23.** *A subset of a continuum C consisting of one point has no limit points.*

Proof. Let $\{x\} \subset C$. To prove that $\{x\}$ has no limit points, Definition 3.13 tells us that it will suffice to show that for all $p \in C$, there exists a region R with $p \in R$ such that $R \cap (\{x\} \setminus \{p\}) = \emptyset$. Let p be an arbitrary point in C. We divide into two cases ($p = x$ and $p \neq x$; Definition 3.1 guarantees that these two cases account for all $p \in C$). Suppose first that $p = x$. If we let R be any region of C, it follows that

$$\begin{aligned} R \cap (\{x\} \setminus \{p\}) &= R \cap (\{x\} \setminus \{x\}) \\ &= R \cap \emptyset \\ &= \emptyset \end{aligned}$$

as desired. Now suppose that $p \neq x$. Since p and x are distinct points, Theorem 3.22 applies and implies that there exist disjoint regions R and S containing p and x, respectively. Consequently, by Definition 1.9, $x \notin R$. Thus,

$$\begin{aligned} R \cap (\{x\} \setminus \{p\}) &= R \cap \{x\} \\ &= \emptyset \end{aligned}$$

as desired. □

Theorem 3.24. *A finite subset A of a continuum C has no limit points.*

Proof. We divide into two cases ($A = \emptyset$ and $A \neq \emptyset$).

Suppose that $A = \emptyset$. Then for an arbitrary $p \in C$ and any region R of C,

$$\begin{aligned} R \cap (A \setminus \{p\}) &= R \cap (\emptyset \setminus \{p\}) \\ &= R \cap \emptyset \\ &= \emptyset \end{aligned}$$

proving by the contrapositive of Definition 3.13 that p is not a limit point of A. Since p is an *arbitrary* element of C, no point of C is a limit point of A, i.e., A has no limit points, as desired.

Now suppose that $A \neq \emptyset$. Since A is finite, i.e., is a set of n distinct points of a continuum C for some $n \in \mathbb{N}$, Theorem 3.5 applies and implies that the symbols $a_1, \dots, a_n$ can be assigned to each point of A. It follows that we can write $A = \bigcup_{k=1}^{n} \{a_k\}$. Now suppose for the sake of contradiction that for some $p \in C$, $p \in LP(A)$. Then by Corollary 3.21, p is a limit point of at least one of the sets $\{a_k\}$. But this contradicts Corollary 3.23, which asserts that no singleton set has limit points. Therefore, no point p of C is a limit point of A, i.e., A has no limit points, as desired. ☐

Corollary 3.25. *If A is a finite subset of a continuum C and $x \in A$, then there exists a region R containing x, such that $A \cap R = \{x\}$.*

Proof. Since A is a finite subset of a continuum C, Theorem 3.24 implies that A has no limit points. Thus, for an arbitrary $x \in A$, x is not a limit point of A. Consequently, Definition 3.13 implies that there exists a region R of C with $x \in R$ such that $R \cap (A \setminus \{x\}) = \emptyset$. Moreover, since $x \in A$ and $x \in R$, Definition 1.6 implies that $x \in A \cap R$. Now suppose for the sake of contradiction that there exists some object $y \in A \cap R$ such that $y \neq x$. Then by Definition 1.6, $y \in A$ and $y \in R$. The former discovery combined with the fact that $y \neq x$ (i.e., $y \notin \{x\}$) reveals that $y \in A \setminus \{x\}$ by Definition 1.11. Thus, since $y \in R$ and $y \in A \setminus \{x\}$, $y \in R \cap (A \setminus \{x\})$. Consequently, by Definition 1.2 and the fact that $R \cap (A \setminus \{x\}) = \emptyset$, $y \in \emptyset$. But this contradicts Definition 1.8. Therefore, x is the only element of $A \cap R$, so $A \cap R = \{x\}$, as desired. ☐

Theorem 3.26. *If p is a limit point of A and R is a region containing p, then the set $R \cap A$ is infinite.*

Proof. Suppose for the sake of contradiction that $R \cap A$ is finite. Then by Theorem 3.24, $R \cap A$ has no limit points. Notably, this implies that p is not a limit point of $R \cap A$. It follows by Definition 3.13 that there exists some region S with $p \in S$ such that

$$\emptyset = S \cap ((R \cap A) \setminus \{p\})$$

Since $p \in S$ and $p \in R$, Theorem 3.18 implies that $S \cap R$ is a region containing p. But since the above

$$= (S \cap R) \cap (A \setminus \{p\})$$

where $S \cap R$ is a p-containing region, Definition 3.13 implies that p is not a limit point of A. But this contradicts the hypothesis that p is a limit point of A. Thus, $R \cap A$ must be infinite. ☐

11/3: **Exercise 3.27.** Find realizations of a continuum $(C, <)$. That is, find concrete sets C endowed with a relation $<$ satisfying all of the axioms (so far). Are they the same? What does "the same" mean here?

Explanation. $\mathbb{Z}$ and $\mathbb{Q}$ are complete realizations of a continuum $(C, <)$ (see Exercise 3.9). Additionally, $\mathbb{Q} \setminus \underline{01}$ is a continuum (it is a nonempty set C, it obeys the same ordering as $\mathbb{Q}$ since it is a subset of $\mathbb{Q}$, and it has no first or last points as we only removed a region in the "middle" as opposed to a region containing points from some point, on).

As to the other part of the question, it is hard to define what "the same" means with respect to infinite sets. However, in our class discussion on November 3, 2020, we reached the consensus that two continua A, B are "the same" if and only if

1. There exists a bijection $f : A \to B$.
2. A and B are of the same density category (both dense, both semi-dense, or both buoyant), where the three aforementioned density categories are defined as
   - **Dense** (continuum): A continuum C such that for all regions $R \subset C$, $LP(R) \neq \emptyset$.
   - **Semi-dense** (continuum): A continuum C such that for some region $R \subset C$, $LP(R) \neq \emptyset$, and for some region $S \subset C$, $LP(S) = \emptyset$.
   - **Buoyant** (continuum): A continuum C that is neither dense nor semi-dense. Equivalently, for all regions $R \subset C$, $LP(R) = \emptyset$.

   These categories are mutually exclusive.

Going back to the examples from the beginning, it can be proven that $\mathbb{Q}$ is a dense continuum, $\mathbb{Q} \setminus \underline{01}$ is a semi-dense continuum, and $\mathbb{Z}$ is a buoyant continuum.

It is worth noting that there are multiple semi-dense continua that it could be argued are different (for example, $\mathbb{Q} \setminus \underline{01}$ and $\mathbb{Q} \setminus (\underline{01} \cup \underline{23})$). However, since all semi-dense sets have similar properties, it's not that much of a stretch, in my opinion, to treat all semi-dense sets as effectively "the same." Perhaps the argument could be made for *some* subcategories (let semi-dense sets be dense sets with finite regions of buoyancy, and let hemi-dense sets be buoyant sets with finite regions of density), but then we run into the issue of sets with infinite regions of both density and buoyancy. More generally, we run into issues of where to stop subdividing, so for now, we'll lump all semi-dense continua together. $\square$