Script 1

Sets, Functions, and Cardinality

1.1 Journal

10/1: **Definition 1.2.** Two sets A and B are **equal** if they contain precisely the same elements, that is, $x \in A$ if and only if $x \in B$. When A and B are equal, we denote this by $A = B$.

Definition 1.3. A set A is a **subset** of a set B if every element of A is also an element of B, that is, if $x \in A$, then $x \in B$. When A is a subset of B, we denote this by $A \subset B$. If $A \subset B$ but $A \neq B$, we say that A is a **proper subset** of B, and we denote this by $A \subsetneq B$.

10/6: **Exercise 1.4.** Let $A = \{1, \{2\}\}$. Is $1 \in A$? Is $2 \in A$? Is $\{1\} \subset A$? Is $\{2\} \subset A$? Is $1 \subset A$? Is $\{1\} \in A$? Is $\{2\} \in A$? Is $\{\{2\}\} \subset A$? Explain.

Proof. We list affirmative or negative answers and short explanations.

Yes, $1 \in A$.
No, $2 \notin A$, but $\{2\} \in A$.
Yes, $\{1\} \subset A$ since 1 is the only element of $\{1\}$ and $1 \in A$ (as previously established).
No, $\{2\} \not\subset A$ since $2 \in \{2\}$ but $2 \notin A$ (as previously established).
No, $1 \not\subset A$ since 1 is not a set.
No, $\{1\} \notin A$, but $1 \in A$ and $\{1\} \subset A$ as previously established.
Yes, $\{2\} \in A$.
Yes, $\{\{2\}\} \subset A$ since $\{2\}$ is the only element of $\{\{2\}\}$ and $\{2\} \in A$ (as previously established). □

10/1: **Definition 1.5.** Let A and B be two sets. The **union** of A and B is the set

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Definition 1.6. Let A and B be two sets. The **intersection** of A and B is the set

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

Theorem 1.7. *Let A and B be two sets. Then*

a) $A = B$ *if and only if* $A \subset B$ *and* $B \subset A$.

b) $A \subset A \cup B$.

c) $A \cap B \subset A$.

Definition 1.8. The **empty set** is the set with no elements, and it is denoted $\emptyset$. That is, no matter what x is, we have $x \in \emptyset$.

Definition 1.9. Two sets A and B are **disjoint** if $A \cap B = \emptyset$.

10/6: **Exercise 1.10.** Show that if A is any set, then $\emptyset \subset A$.

Proof. Suppose for the sake of contradiction that there exists a set A such that $\emptyset \not\subset A$. Then by Definition 1.3, not every element of $\emptyset$ is also an element of A, i.e., there exists an element $x \in \emptyset$ such that $x \notin A$. But by Definition 1.8, x (like all other objects) cannot be an element of $\emptyset$, a contradiction. Therefore, $\emptyset \subset A$ for all sets A. $\square$

10/1: **Definition 1.11.** Let A and B be two sets. The **difference** of B from A is the set

$$A \setminus B = \{x \in A \mid x \notin B\}$$

The set $A \setminus B$ is also called the **complement** of B relative to A. When the set A is clear from the context, this set is sometimes denoted B^c, but we will try to avoid this imprecise formulation and use it only with warning.

Theorem 1.12. *Let X be a set, and let $A, B \subset X$. Then*

a) $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$

b) $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$

Proof of a. To prove that $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$, Definition 1.2 tells us that it will suffice to prove that $x \in X \setminus (A \cup B)$ if and only if $x \in (X \setminus A) \cap (X \setminus B)$, i.e., that if $x \in X \setminus (A \cup B)$, then $x \in (X \setminus A) \cap (X \setminus B)$ and if $x \in (X \setminus A) \cap (X \setminus B)$, then $x \in X \setminus (A \cup B)$. To begin, let $x \in X \setminus (A \cup B)$. By Definition 1.11, $x \in X$ and $x \notin A \cup B$. By Definition 1.5, it follows that $x \notin A$ and $x \notin B$. Since we know that $x \in X$ and $x \notin A$, Definition 1.11 tells us that $x \in X \setminus A$. Similarly, $x \in X \setminus B$. Since $x \in X \setminus A$ and $x \in X \setminus B$, we have by Definition 1.6 that $x \in (X \setminus A) \cap (X \setminus B)$, as desired. The proof of the other implication is the preceding proof "in reverse." For clarity, let $x \in (X \setminus A) \cap (X \setminus B)$. By Definition 1.6, $x \in X \setminus A$ and $x \in X \setminus B$. By consecutive applications of Definition 1.11, $x \in X$, $x \notin A$, and $x \notin B$. Since $x \notin A$ and $x \notin B$, Definition 1.5 reveals that $x \notin A \cup B$. But as previously established, $x \in X$, so Definition 1.11 tells us that $x \in X \setminus (A \cup B)$. $\square$

Proof of b. To prove that $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$, Definition 1.2 tells us that it will suffice to prove that $x \in X \setminus (A \cap B)$ if and only if $x \in (X \setminus A) \cup (X \setminus B)$. To begin, let $x \in X \setminus (A \cap B)$. By Definition 1.11, $x \in X$ and $x \notin A \cap B$. By Definition 1.6, it follows that $x \notin A$ or $x \notin B$. We divide into two cases. If $x \notin A$, then since we know that $x \in X$, Definition 1.11 tells us that $x \in X \setminus A$. It naturally follows that $x \in (X \setminus A) \cup (X \setminus B)$, since x need only be an element of one of the two unionized sets (see Definition 1.5). The proof is symmetric if $x \notin B$. Now let $x \in (X \setminus A) \cup (X \setminus B)$. By Definition 1.5, $x \in X \setminus A$ or $x \in X \setminus B$. Once again, we divide into two cases. If $x \in X \setminus A$, then $x \in X$ and $x \notin A$ by Definition 1.11. Consequently, by Definition 1.6, $x \notin A \cap B$. Therefore, $x \in X \setminus (A \cap B)$ by Defintion 1.11. The proof is symmetric if $x \in X \setminus B$. $\square$

10/13: **Definition 1.13.** Let $\mathcal{A} = \{A_\lambda \mid \lambda \in I\}$ be a collection of sets indexed by a nonempty set I. Then the intersection and union of $\mathcal{A}$ are the sets

$$\bigcap_{\lambda \in I} A_\lambda = \{x \mid x \in A_\lambda \text{ for all } \lambda \in I\}$$

and

$$\bigcup_{\lambda \in I} A_\lambda = \{x \mid x \in A_\lambda \text{ for some } \lambda \in I\}$$

Theorem 1.14. *Let X be a set and let $\mathcal{A} = \{A_\lambda \mid \lambda \in I\}$ be a collection of subsets of X. Then*

1. $X \setminus \left(\bigcup_{\lambda \in I} A_\lambda\right) = \bigcap_{\lambda \in I} (X \setminus A_\lambda)$.
2. $X \setminus \left(\bigcap_{\lambda \in I} A_\lambda\right) = \bigcup_{\lambda \in I} (X \setminus A_\lambda)$.

Proof of 1. To prove that $X \setminus \left(\bigcup_{\lambda\in I} A_\lambda\right) = \bigcap_{\lambda\in I}(X \setminus A_\lambda)$, Definition 1.2 tells us that it will suffice to show that every element x of $X \setminus \left(\bigcup_{\lambda\in I} A_\lambda\right)$ is an element of $\bigcap_{\lambda\in I}(X \setminus A_\lambda)$ and vice versa. Suppose first that $x \in X \setminus \left(\bigcup_{\lambda\in I} A_\lambda\right)$. Then by Definition 1.11, $x \in X$ and $x \notin \bigcup_{\lambda\in I} A_\lambda$. By Definition 1.13, the latter result implies that $x \notin A_\lambda$ for any $\lambda \in I$. This combined with the fact that $x \in X$ implies by Definition 1.11 that $x \in X \setminus A_\lambda$ for all $\lambda \in I$. Therefore, by Definition 1.13, $x \in \bigcap_{\lambda\in I}(X \setminus A_\lambda)$. Now suppose that $x \in \bigcap_{\lambda\in I}(X \setminus A_\lambda)$. Then by Definition 1.13, $x \in X \setminus A_\lambda$ for all $\lambda \in I$. By Definition 1.11, this implies that $x \in X$ and $x \notin A_\lambda$ for any $\lambda \in I$. Thus, by Definition 1.13, the latter result implies that $x \notin \bigcup_{\lambda\in I} A_\lambda$. Therefore, since we also have $x \in X$, Definition 1.11 implies that $x \in X \setminus \left(\bigcup_{\lambda\in I} A_\lambda\right)$. □

Proof of 2. To prove that $X \setminus \left(\bigcap_{\lambda\in I} A_\lambda\right) = \bigcup_{\lambda\in I}(X \setminus A_\lambda)$, Definition 1.2 tells us that it will suffice to show that every element x of $X \setminus \left(\bigcap_{\lambda\in I} A_\lambda\right)$ is an element of $\bigcup_{\lambda\in I}(X \setminus A_\lambda)$ and vice versa. Suppose first that $x \in X \setminus \left(\bigcap_{\lambda\in I} A_\lambda\right)$. Then by Definition 1.11, $x \in X$ and $x \notin \bigcap_{\lambda\in I} A_\lambda$. By Definition 1.13, the latter result implies that $x \notin A_\lambda$ for some $\lambda \in I$. This combined with the fact that $x \in X$ implies by Definition 1.11 that $x \in X \setminus A_\lambda$ for some $\lambda \in I$. Therefore, by Definition 1.13, $x \in \bigcup_{\lambda\in I}(X \setminus A_\lambda)$. Now suppose that $x \in \bigcup_{\lambda\in I}(X \setminus A_\lambda)$. Then by Definition 1.13, $x \in X \setminus A_\lambda$ for some $\lambda \in I$. By Definition 1.11, this implies that $x \in X$ and $x \notin A_\lambda$ for some $\lambda \in I$. Thus, by Definition 1.13, the latter result implies that $x \notin \bigcap_{\lambda\in I} A_\lambda$. Therefore, since we also have $x \in X$, Definition 1.11 implies that $x \in X \setminus \left(\bigcap_{\lambda\in I} A_\lambda\right)$. □

10/1: **Definition 1.15.** Let A and B be two nonempty sets. The **Cartesian product** of A and B is the set of ordered pairs

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

If $(a, b), (a', b') \in A \times B$, we say that (a, b) and (a', b') are **equal** if and only if $a = a'$ and $b = b'$. In this case, we write $(a, b) = (a', b')$.

Definition 1.16. Let A and B be two nonempty sets. A **function** f from A to B is a subset $f \subset A \times B$ such that for all $a \in A$, there exists a unique $b \in B$ satisfying $(a, b) \in f$. To express the idea that $(a, b) \in f$, we most often write $f(a) = b$. To express that f is a function from A to B in symbols, we write $f : A \to B$.

10/13: **Exercise 1.17.** Let the function $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = 2n$. Describe f as a subset of $\mathbb{Z} \times \mathbb{Z}$.

Proof. By Definition 1.15, $\mathbb{Z} \times \mathbb{Z}$ is the set of all ordered pairs (a, b) where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$. By Definition 1.16, f is some subset of these ordered pairs. More specifically, elements of f are ordered pairs of the form $(n, f(n)) = (n, 2n)$ for some $n \in \mathbb{Z}$. Since f is a function, Definition 1.16 asserts that an ordered pair of such form must be present for all $n \in \mathbb{Z}$, so every object $(n, 2n)$ where $n \in \mathbb{Z}$ is an element of f. There are no other elements of f. □

10/1: **Definition 1.18.** Let $f : A \to B$ be a function. The **domain** of f is A and the **codomain** of f is B. If $X \subset A$, then the **image** of X under f is the set

$$f(X) = \{f(x) \in B \mid x \in X\}$$

If $Y \subset B$, then the **preimage** of Y under f is the set

$$f^{-1}(Y) = \{a \in A \mid f(a) \in Y\}$$

Exercise 1.19. Must $f(f^{-1}(Y)) = Y$ and $f^{-1}(f(X)) = X$? For each, either prove that it always holds or give a counterexample.

Counterexample to $f(f^{-1}(Y)) = Y$. Consider the sets $\{1\}$ and $\{3, 4\}$, and let $f : \{1\} \to \{3, 4\}$ be a function defined by $f(1) = 3$. Let $Y = \{3, 4\}$ (Theorem 1.7a guarantees that $Y \subset \{3, 4\}$, as needed). Then $f^{-1}(Y) = \{a \in \{1\} \mid f(a) \in \{3, 4\}\} = \{1\}$ and $f(f^{-1}(Y)) = \{f(x) \in \{3, 4\} \mid x \in \{1\}\} = \{3\}$ by consecutive applications of Definition 1.18. Therefore, $f(f^{-1}(Y)) \neq Y$ since $4 \in Y$ but $4 \notin f(f^{-1}(Y))$ (see Definition 1.2)[1]. □

[1]Note that the reason $f(f^{-1}(Y)) \neq Y$ in this case is because f is not surjective.

Counterexample to $f^{-1}(f(X)) = X$. Consider the sets $\{1, 2\}$ and $\{3\}$, and let $f : \{1, 2\} \to \{3\}$ be a function defined by $f(1) = 3$ and $f(2) = 3$. Let $X = \{1\}$ (we have $X \subset \{1, 2\}$ since 1 is the only element of X and $1 \in \{1, 2\}$ [see Definition 1.3]). Then $f(X) = \{f(x) \in \{3\} \mid x \in \{1\}\} = \{f(1)\} = \{3\}$ and $f^{-1}(f(X)) = \{a \in \{1, 2\} \mid f(a) \in \{3\}\} = \{1, 2\}$ by consecutive applications of Definition 1.18. Therefore, $f^{-1}(f(X)) \neq X$ since $2 \in f^{-1}(f(X))$ but $2 \notin X$ (see Definition 1.2)[2]. □

Definition 1.20. A function $f : A \to B$ is **surjective** (*also known as* **onto**) if for every $b \in B$, there is some $a \in A$ such that $f(a) = b$. The function f is **injective** (*also known as* **one-to-one**) if for all $a, a' \in A$, if $f(a) = f(a')$, then $a = a'$. The function f is **bijective** (*also known as* a **bijection**, a **one-to-one correspondence**) if it is surjective and injective.

10/6: **Exercise 1.21.** Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n^2$. Is f injective? Is f surjective?

Proof. f is injective: Let $f(n) = f(n')$. Then $n^2 = (n')^2$, implying that $n = n'$ (note that this last step is not permissible in all number systems, but it is within the naturals).

f is not surjective: For example, $2 \in \mathbb{N}$ but there exists no natural number n such that $f(n) = n^2 = 2$ (suppose for the sake of contradiction that there exists a natural number n such that $n^2 = 2$. Since $n^2 = 2 \neq 1$, we know that $n < 2$ [a natural number is less than its square if its square is unequal to 1, as proven in Additional Exercise 0.9f-i]. Thus, $n = 1$ since 1 is the only natural number less than 2. But then $n^2 = 1^2 = 1 \neq 2$, a contradiction). □

Exercise 1.22. Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n + 2$. Is f injective? Is f surjective?

Proof. f is injective: Let $f(n) = f(n')$. Then $n + 2 = n' + 2$, implying by the cancellation law for addition that $n = n'$.

f is not surjective: For example, $1 \in \mathbb{N}$ but there exists no natural number n such that $n+2 = 1$ (suppose for the sake of contradiction that there exists a natural number n such that $n + 2 = 1$. Because $1 = n + 2$, we know that $1 > n$. But we also know that $1 \leq n$ for all $n \in \mathbb{N}$ (as can be proven by induction). Therefore, n is both > 1 and ≤ 1, contradicting the trichotomy). □

Exercise 1.23. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x^2$. Is f injective? Is f surjective?

Proof. f is not injective: For example, $f(2) = 4 = f(-2)$, but $2 \neq -2$.

f is not surjective: For example, $2 \in \mathbb{Z}$ but there exists no integer x such that $f(x) = x^2 = 2$ (suppose for the sake of contradiction that there exists an integer x such that $x^2 = 2$. Since $x^2 = 2$, $|x| < 2$ for similar reasons to those discussed in Exercise 1.21. Thus, $x = -1$, $x = 0$, or $x = 1$. But $(-1)^2 = 1 \neq 2$, $0^2 = 0 \neq 2$, and $1^2 = 1 \neq 2$, so $x \neq -1$, $x \neq 0$, and $x \neq 1$, a contradiction). □

10/13: **Exercise 1.24.** Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x + 2$. Is f injective? Is f surjective?

Proof. f is injective: Let $f(x) = f(x')$. Then $x + 2 = x' + 2$, implying by the cancellation law for addition that $x = x'$.

f is surjective: Let x be an arbitrary element of $\mathbb{Z}$. We divide into five cases ($x \geq 3$, $x = 2$, $x = 1$, $x = 0$, and $x \leq -1$). If $x \geq 3$, there is a unique natural number (hence an integer) equal to $x - 2$. If $x = 2$, then $x - 2 = 0$ (since $0 + 2 = 2$) and $0 \in \mathbb{Z}$. If $x = 1$, then $x - 2 = -1$ (since $-1 + 2 = 1$) and $-1 \in \mathbb{Z}$. If $x = 0$, then $x - 1 = -2$ (since $-2 + 2 = 0$) and $-2 \in \mathbb{Z}$. And if $x \leq -1$, then subtraction on $-\mathbb{N} \subset \mathbb{Z}$ is identical to addition on $\mathbb{N}$, so a unique $x - 2$ exists by an inverse interpretation of the closure of addition for $\mathbb{N}$. □

10/1: **Definition 1.25.** Let $f : A \to B$ and $g : B \to C$. Then the **composition** $g \circ f : A \to C$ is defined by $(g \circ f)(x) = g(f(x))$ for all $x \in A$.

Proposition 1.26. *Let A, B, and C be sets and suppose that $f : A \to B$ and $g : B \to C$. Then $g \circ f : A \to C$ and*

a) if f and g are both injections, so is $g \circ f$.

b) if f and g are both surjections, so is $g \circ f$.

[2] Note that the reason $f^{-1}(f(X)) \neq X$ in this case is because f is not injective.

c) if f and g are both bijections, so is $g \circ f$.

Proof of a. Suppose that $(g \circ f)(a) = (g \circ f)(a')$. By Definition 1.25, this implies that $g(f(a)) = g(f(a'))$. Since g is injective, Definition 1.20 tells us that $f(a) = f(a')$. Similarly, the fact that f is injective tells us that $a = a'$. Since we have shown that $(g \circ f)(a) = (g \circ f)(a')$ implies that $a = a'$ under the given conditions, we know by Definition 1.20 that $g \circ f$ is injective. □

Proof of b. Let c be an arbitrary element of C. We wish to prove that there exists some $a \in A$ such that $(g \circ f)(a) = c$ (Definition 1.20). By Definition 1.25, it will suffice to show that there exists some $a \in A$ such that $g(f(a)) = c$. Let's begin. By the surjectivity of g, there exists some $b \in B$ such that $g(b) = c$ (see Definition 1.20). If we now consider this b, we have by the surjectivity of f that there exists some $a \in A$ such that $f(a) = b$ (see Definition 1.20). But this a is an element of A such that $g(f(a)) = g(b) = c$, as desired. □

Proof of c. Suppose that f and g are two bijective functions. By Definition 1.20, this implies that f and g are both injections and are both surjections. Thus, by Proposition 1.26a, $g \circ f$ is an injection, and by Proposition 1.26b, $g \circ f$ is a surjection. Therefore, by Definition 1.20, $g \circ f$ is a bijection. □

10/6: **Proposition 1.27.** *Suppose that $f : A \to B$ is bijective. Then there exists a bijection $g : B \to A$ that satisfies $(g \circ f)(a) = a$ for all $a \in A$, and $(f \circ g)(b) = b$ for all $b \in B$.*

Proof. Let $g : B \to A$ be defined by the rule, "$g(b) = a$ if and only if $f(a) = b$." For g to be a function as defined, Definition 1.16 tells us that we must show that for every $b \in B$, there exists a unique $a \in A$ such that $g(b) = a$. By the surjectivity of f, we know that for all $b \in B$, there exists *an* $a \in A$ such that $f(a) = b$. On the uniqueness of this a, let $a \neq a'$ and suppose for the sake of contradiction that $g(b) = a$ and $g(b) = a'$. By the definition of g, we have that $f(a) = b$ and $f(a') = b$, so $f(a) = f(a')$. But by the injectivity of f, this means that $a = a'$, a contradiction. Therefore, g indeed maps every $b \in B$ to a unique $a \in A$. To demonstrate that g satisfies the remainder of the necessary constraints, we will work through them one by one.

To prove that g is injective, Definition 1.20 tells us that we must verify that if $g(b) = g(b')$, then $b = b'$. Let $g(b) = g(b')$. Since g is a function, $g(b) = g(b') = a$, where $a \in A$. This implies by the definition of g that $f(a) = b$ and $f(a) = b'$. But since f is a function (i.e., by Definition 1.16, $f(a)$ is a unique object), this means that $b = f(a) = b'$, as desired. To prove that g is surjective, Definition 1.20 tells us that we must verify that for all $a \in A$, there exists a $b \in B$ such that $g(b) = a$. Let a be an arbitrary element of A. By Definition 1.16 and the status of f as a function, there exists an element $b \in B$ such that $f(a) = b$. But by the definition of g, $f(a) = b$ implies that $g(b) = a$, meaning that this b satisfies the desired constraint. On the basis of this and the previous argument, Definition 1.20 allows us to conclude that g is bijective.

We now prove that $(g \circ f)(a) = a$ for all $a \in A$. Let a be an arbitrary element of A. Then by Definition 1.16 and the status of f as a function, $f(a) = b$ where $b \in B$. Thus, by the definition of g, $g(b) = a$. But by Definition 1.25 and some substitutions, this means that $(g \circ f)(a) = g(f(a)) = g(b) = a$, as desired.

A symmetric argument can demonstrate that $(f \circ g)(b) = b$ for all $b \in B$. □

Definition 1.28. We say that two sets A and B are in **bijective correspondence** when there exists a bijection from A to B or, equivalently, from B to A.

Definition 1.29. Let $n \in \mathbb{N}$ be a natural number. We define $[n]$ to be the set $\{1, 2, \dots, n\}$. Additionally, we define $[0] = \emptyset$.

Definition 1.30. A set A is **finite** if $A = \emptyset$ or if there exists a natural number n and a bijective correspondence between A and the set $[n]$. If A is not finite, we say that A is **infinite**.

Theorem 1.31. *Let $n, m \in \mathbb{N}$ with $n < m$. Then there does not exist an injective function $f : [m] \to [n]$.*

Theorem 1.32. *Let A be a finite set. Suppose that A is in bijective correspondence both with $[m]$ and with $[n]$. Then $m = n$.*

Proof. If A is in bijective correspondence with both $[m]$ and with $[n]$, then Definition 1.28 tells us that there exist bijections $f : [m] \to A$ and $g : A \to [n]$. Thus, by Proposition 1.26, $g \circ f : [m] \to [n]$ is bijective. Now suppose for the sake of contradiction that $m \neq n$. Then by the trichotomy, either $m > n$ or $m < n$. We divide into two cases. If $m > n$, then Theorem 1.31 tells us that no injective function $h : [m] \to [n]$ exists. But $f : [m] \to [n]$ is bijective, hence injective by Definition 1.20, a contradiction. On the other hand, if $m < n$, then Theorem 1.31 tells us that no injective function $h : [n] \to [m]$ exists. But by Proposition 1.27, the existence of the bijection $f : [m] \to [n]$ implies the existence of a bijection $f^{-1} : [n] \to [m]$. As before, the bijectivity of f^{-1} implies that it is also injective by Definition 1.20, a contradiction. Therefore, we must have $m = n$. ☐

10/8: **Definition 1.33** (Cardinality of a finite set)**.** If A is a finite set that is in bijective correspondence with $[n]$, then we say that the **cardinality** of A is n, and we write $|A| = n$. (By Theorem 1.32, there is exactly one such natural number n.) We also say that A contains n elements. We define the cardinality of the empty set to be 0.

Exercise 1.34. Let A and B be finite sets.

a) If $A \subset B$, then $|A| \leq |B|$.

Proof. Let $|A| = m$ and $|B| = n$. Using these variables, Definitions 1.33 and 1.28 tell us that there exist bijections $f : [m] \to A$ and $g : B \to [n]$. Now let $h : A \to B$ be defined by $h(a) = a$ for each $a \in A$. By Definition 1.16, to verify that h is a function, we must show that for every $a \in A$, there exists a unique $b \in B$ such that $h(a) = b$. Let a be an arbitrary element of A. Since $A \subset B$, Definition 1.3 implies that $a \in B$. Thus, since $h(a) = a$, $h(a) \in B$. Now suppose for the sake of contradiction that $h(a) = b$ and $h(a) = b'$ for two elements $b, b' \in B$ such that $b \neq b'$. By the definition of h, $h(a) = a$, so $a = b$ and $a = b'$, implying by transitivity that $b = b'$, a contradiction.

We now demonstrate that h is injective. By Definition 1.20, it will suffice to show that $h(a) = h(a')$ implies that $a = a'$ (where $a, a' \in A$). So suppose that $h(a) = h(a')$. By the definition of h, $h(a) = a$ and $h(a') = a'$, so by assumption, $a = h(a) = h(a') = a'$, as desired.

To recap, at this point we have injective functions $f : [m] \to A$, $h : A \to B$, and $g : B \to [n]$, where the injectivity of f and g follows from their bijectivity (see Definition 1.20). It follows by consecutive applications of Proposition 1.26 that $h \circ f$ is injective, and that $g \circ (h \circ f)$ is injective. Thus, there exists an injective function $g \circ (h \circ f) : [m] \to [n]$, so the contrapositive of Theorem 1.31 implies that it is false that $n < m$. Equivalently, it is true that $n \geq m$, or, to return substitutions, that $|A| \leq |B|$. ☐

b) Let $A \cap B = \emptyset$. Then $|A \cup B| = |A| + |B|$.

Proof. Let $|A| = m$ and $|B| = n$. Thus, $|A| + |B| = m + n$, so to prove that $|A \cup B| = |A| + |B|$, Definition 1.33 and 1.28 tell us that that we must find a bijection $h : A \cup B \to [m + n]$. Let's begin.

Since $|A| = m$ and $|B| = n$, by Definition 1.33 and 1.28, there exist bijections $f : A \to [m]$ and $g : B \to [n]$. As such, let $h : A \cup B \to [m + n]$ be defined as follows:

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) + m & x \in B \end{cases}$$

Since the two cases defining h are both functions, the only possible barrier to h itself being a function is if there exists some $x \in A \cup B$ such that $x \in A$ and $x \in B$. To address this, suppose for the sake of contradiction that this is the case. Fortunately, such a hypothesis implies by Definition 1.6 that $x \in A \cap B$, contradicting the fact that $A \cap B = \emptyset$ (see Definition 1.8).

To prove that h is injective, the contrapositive of Definition 1.20 tells us that we must verify that if $x \neq x'$, then $h(x) \neq h(x')$. We divide into three cases ($x, x' \in A$, $x, x' \in B$, and WLOG $x \in A$ and $x' \in B$[3]). First, suppose that $x, x' \in A$. By the injectivity of f (which follows from its bijectivity by

[3]Note that we do not have to treat the case that $x \in B$ and $x' \in A$ since in this case, we just call the object represented by x, "x'," and vice versa — this reversal of names is what is implied by "Without the Loss Of Generality," or WLOG.

Definition 1.20), we have that $f(x) \neq f(x')$. Since $h(x) = f(x)$ and $h(x') = f(x')$ by the definition of h, we have that $h(x) = f(x) \neq f(x') = h(x')$, as desired. Second, suppose that $x, x' \in B$. By the injectivity of g, we have that $g(x) \neq g(x')$, which implies by the additive property of equality that $g(x) + m \neq g(x') + m$. Since $h(x) = g(x) + m$ and $h(x') = g(x') + m$ by the definition of h, we have that $h(x) = g(x) + m \neq g(x') + m = h(x')$, as desired. Third, suppose that $x \in A$ and $x' \in B$. Then $h(x) = f(x) \leq m$ since $f(x) \in [m]$ while $m < h(x') = g(x') + m \leq m + n$ since $0 < g(x') < n$ as $g(x') \in [n]$. Since $h(x) \leq m$ and $h(x') > m$, we have by the trichotomy that $h(x) \neq h(x')$, as desired.

To prove that h is surjective, Definition 1.20 tells us that we must verify that for every $i \in [m+n]$, there exists an $x \in A \cup B$ such that $h(x) = i$. We divide into two cases ($i \leq m$ and $m + n \geq i > m$). If $i \leq m$, then $i \in [m]$. It follows by the surjectivity of f (which follows from its bijectivity by Definition 1.20) that there exists an $x \in A$ such that $f(x) = i$. Now by Definition 1.5, this x is also an element of $A \cup B$, so $h(x) = f(x) = i$ by the definition of h, as desired. On the other hand, if $m + n \geq i > m$, then $i = m + u$ for some $u \in [n]$. It follows by the surjectivity of g that there exists an $x \in B$ such that $g(x) = u$. Thus, $i = m + u = m + g(x) = h(x)$, as desired.

At this point, Definition 1.20 implies that h is bijective, as desired. ☐

c) $|A \cup B| + |A \cap B| = |A| + |B|$.

Lemma. *Let A and B be sets. Then*

a) $A \cup B = (B \setminus A) \cup A$.

b) $(B \setminus A) \cap A = \emptyset$.

c) $(B \setminus A) \cap (A \cap B) = \emptyset$.

d) $B = (B \setminus A) \cup (A \cap B)$.

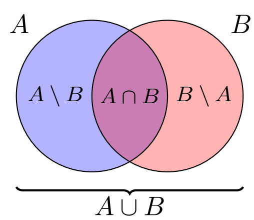


Figure 1.1: Set union Venn diagram.

Proof. All of these claims can be read directly from the above diagram — for the sake of space and because proving these claims is not the main point of this exercise, a rigorous proof of this lemma will be omitted. ☐

Proof of Exercise 1.34c. We want to show that $|A \cup B| + |A \cap B| = |A| + |B|$, which we can do by using the above lemma to justify various manipulations inspired by Exercise 1.34b. To begin, use Lemma (a) as follows.

$$|A \cup B| + |A \cap B| = |(B \setminus A) \cup A| + |A \cap B|$$

Since $(B \setminus A) \cup A$ is a union of two disjoint sets (see Lemma (b)), it follows by Exercise 1.34b that the above

$$= |B \setminus A| + |A| + |A \cap B|$$
$$= |A| + |B \setminus A| + |A \cap B|$$

Since $B \setminus A$ and $A \cap B$ are disjoint (see Lemma (c)), we know that the above

$$= |A| + |(B \setminus A) \cup (A \cap B)|$$

Lastly, apply Lemma (d):

$$= |A| + |B|$$

□

10/13: d) $|A \times B| = |A| \cdot |B|$.

Proof. Let $|A| = n$ and $|B| = m$. Then since A and B are finite, Definition 1.30 and 1.28 tell us that there exist bijections $f : A \to [n]$ and $g : B \to [m]$. By Definition 1.33 and 1.28, to prove the claim, it will suffice to find a bijection $h : A \times B \to [m \cdot n]$.

Let $h : A \times B \to [m \cdot n]$ be defined by

$$h(a, b) = f(a) + n \cdot (g(b) - 1)$$

Clearly, the above rule assigns a unique value to every (a, b), and since f and g map all $a \in A$ and $b \in B$, respectively, the above function is not undefined for any $(a, b) \in A \times B$. Thus, h is a function as defined in Definition 1.16.

We must now prove that h is bijective. By Definition 1.20, it will suffice to prove that h is injective and surjective, which we may do as follows. We will start with injectivity.

Let

$$h(a, b) = h(a', b')$$

Then by the definition of h,

$$\begin{aligned} f(a) + n \cdot (g(b) - 1) &= f(a') + n \cdot (g(b') - 1) \\ f(a) - f(a') &= n \cdot (g(b') - 1) - n \cdot (g(b) - 1) \\ f(a) - f(a') &= n \cdot (g(b') - g(b)) \end{aligned}$$

Since $f(a)$ and $f(a')$ are both elements of $[n]$, we have $|f(a) - f(a')| < n$ (since $\max([n]) - \min([n]) = n - 1 < n$). Substituting, we have that $|n \cdot (g(b') - g(b))| < n$, i.e., $|g(b') - g(b)| < 1$. But since $g(b), g(b') \in \mathbb{N}$, the only way that $|g(b') - g(b)| < 1$ is if $|g(b') - g(b)| = 0$. Consequently, $g(b') - g(b) = 0$, so additionally, $f(a) - f(a') = n \cdot (g(b') - g(b)) = 0$. Having ascertained that $g(b') - g(b) = 0$ and $f(a) - f(a') = 0$, it is a simple matter to find that $g(b) = g(b')$ and $f(a) = f(a')$, meaning by the bijectivity (more specifically, the injectivity) of f and g that $b = b'$ and $a = a'$. But by Definition 1.15, this implies that $(a, b) = (a', b')$, as desired.

As to surjectivity, let c be an arbitrary element of $[n \cdot m]$. As a natural number, c can be written in the form $c = \beta \cdot n + \alpha$ where $1 \leq \alpha \leq n$ and $\beta \in \mathbb{N}$. We know that $\min([n \cdot m]) = 1 = 0 \cdot n + 1$ and $\max([n \cdot m]) = m \cdot n = (m - 1) \cdot n + n$; thus, if we restrict the possible values of β to $0 \leq \beta \leq m - 1$, we still know that $c = \beta \cdot n + \alpha$ for some $1 \leq \alpha \leq n$ and $0 \leq \beta \leq m - 1$. Now by the surjectivity of f, there exists an $a \in A$ such that $f(a) = \alpha$ for any $1 \leq \alpha \leq n$. Similarly, the surjectivity of g implies that there exists a $b \in B$ such that $g(b) = \beta + 1$ for any $1 \leq \beta + 1 \leq m$, i.e., there exists a $b \in B$ such that $g(b) - 1 = \beta$ for any $0 \leq \beta \leq m - 1$. Therefore, c can be written in the form $c = f(a) + n \cdot (g(b) - 1)$ for some $a \in A$ and $b \in B$, which by the definition of h means that $c = h(a, b)$ for some $(a, b) \in A \times B$, as desired. □

10/8: **Definition 1.35.** An infinite set A is said to be **countable** if it is in bijective correspondence with $\mathbb{N}$. An infinite set that is not countable is called **uncountable**.

Exercise 1.36. Prove that $\mathbb{Z}$ is a countable set.

Proof. To prove that $\mathbb{Z}$ is countable, Definition 1.35 and, subsequently, Definition 1.28 tell us that we must find a bijection $f : \mathbb{Z} \to \mathbb{N}$. To do so, we will define a matching and then prove that the guiding rule generates a (1) function that is (2) injective and (3) surjective (demonstrating injectivity and surjectivity verifies bijectivity by Definition 1.20).

Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as follows:

$$f(z) = \begin{cases} -2z+1 & z \in -\mathbb{N} \\ 1 & z \in \{0\} \\ 2z & z \in \mathbb{N} \end{cases}$$

Since $\mathbb{Z} = (-\mathbb{N}) \cup \{0\} \cup (\mathbb{N})$, it is clear that the above mapping sends every element of $\mathbb{Z}$ to *an* element of $\mathbb{N}$. Additionally, since $-\mathbb{N}$, $\{0\}$, and $\mathbb{N}$ are all disjoint from one another by definition, it follows that each element of $\mathbb{Z}$ is only mapped once. Thus, by Definition 1.16, f is a function as defined.

Let $f(z) = f(z')$. Since the outputs of the first case in the definition of f are the odd natural numbers except 1, the output of the second case is 1, and outputs of the third case are the even natural numbers, the outputs form three disjoint sets, so $f(z)$ and $f(z')$ as equal quantities are elements of only one category. We now divide into three cases by category. First, suppose $f(z) = f(z')$ is an odd natural number not equal to 1. Then we have by the definition of f that $-2z+1 = -2z'+1$, implying by the cancellation laws of addition and multiplication, respectively, that $z = z'$. Second, suppose that $f(z) = f(z') = 1$. Then $z = z' = 0$ by the definition of f. Lastly, suppose $f(z) = f(z')$ is an even natural number. Then we have by the definition of f that $2z = 2z'$, implying by the cancellation law of multiplication that $z = z'$. Therefore, in any case, $f(z) = f(z')$ implies that $z = z'$, meaning by Definition 1.20 that f is injective.

Let n be an arbitrary element of $\mathbb{N}$. As noted above, n must be even, 1, or odd but not 1. We now divide into three cases by category, as before. First, suppose that n is even. Then $n = 2z$ for some $z \in \mathbb{N}$ (see "Notes on Proofs"). By the definition of f, this z is the element of $\mathbb{Z}$ that f sends to n. Second, suppose n is odd and not equal to 1. Then $n = 2z+1$ for some $z \in \mathbb{N}$ (see "Notes on Proofs"), or $-2z+1$ for some $z \in -\mathbb{N}$ (the negative signs cancel). By the definition of f, this $z \in -\mathbb{N}$ is the element of $\mathbb{Z}$ that f sends to n. Lastly, suppose that $n = 1$. Then since $f(0) = 1$ by the definition of f, 0 is the element of $\mathbb{Z}$ from which f generates 1. Therefore, for every element $n \in \mathbb{N}$, there exists a $z \in \mathbb{Z}$ satisfying $f(z) = n$, meaning by Definition 1.20 that f is surjective. ☐

10/13: **Exercise 1.37.** Prove that every infinite subset of a countable set is also countable.

Lemma. *Every infinite subset of the natural numbers is countable.*

Proof. Let $A \subset \mathbb{N}$ be infinite. To prove that A is countable, Definition 1.35 tells us that it will suffice to show that there exists a bijection $g : \mathbb{N} \to A$. Let's begin.

We define g recursively with strong induction, as follows (note that $A = A \setminus \{\}$ where $\{\} = \emptyset$). By the well-ordering principle (see Additional Exercise 0.1), there exists a minimum element $\min(A \setminus \{\}) \in A \setminus \{\}$; we define $g(1) = \min(A \setminus \{\})$. Now suppose inductively that we have defined $g(1), g(2), \dots, g(n)$. Then we can define $g(n+1)$ by defining $g(n+1) = \min(A \setminus \{g(1), g(2), \dots, g(n)\})$[4]. By the principle of strong mathematical induction (see Additional Exercise 0.2b), it follows that g is defined for all $n \in \mathbb{N}$, and it is obvious that g is not multiply defined for any $n \in \mathbb{N}$. Thus, g is a function as defined in Definition 1.16.

To prove that g is bijective, Definition 1.20 tells us that it will suffice to show that g is injective and surjective. We will prove each of these qualities in turn. To prove that g is injective, the contrapositive of Definition 1.20 necessitates that we verify that $n \neq n'$ implies $g(n) \neq g(n')$. Suppose that $n \neq n'$. Then by the trichotomy, either $n > n'$ or $n < n'$. If $n > n'$, then $g(n) = \min(A \setminus \{g(1), \dots, g(n'), \dots, g(n-1)\})$, meaning that $g(n)$ cannot equal $g(n')$ since $g(n)$ is an element of a set (namely, $A \setminus \{g(1), \dots, g(n'), \dots, g(n-1)\}$) of which $g(n')$ is explicitly not a member. The proof is symmetric if $n < n'$. To prove that g is surjective, Definition 1.20 tells us that we must verify that for all $a \in A$, there exists an $n \in \mathbb{N}$ such that $g(n) = a$.

[4] Basically, what this definition is doing is mapping 1 to the least element of A, 2 to the second-least element of A, 3 to the third-least element of A, and so on and so forth. Notice how the least element of A is denoted by $g(1)$, and $g(2)$ (for example) is equal to $\min(A \setminus \{g(1)\})$, i.e., the minimum value in A if A's least element did not exist, i.e., the second-least element in A. Additionally, $g(3) = \min(A \setminus \{g(1), g(2)\})$, so we can see how $g(3)$ is the third-least element in A by the same logic used in discussing $g(2)$. Obviously, the pattern continues for all $n \in \mathbb{N}$.

Suppose for the sake of contradiction that there exists some $a \in A$ such that $g(n) \neq a$ for any $n \in \mathbb{N}$. This implies that $a \neq \min(A \setminus \{g(1), \dots, g(n)\})$ for any $n \in \mathbb{N}$, which must mean that $a \notin A$, a contradiction. $\square$

Proof of Exercise 1.37. Let A be a countable set and let $B \subset A$ be an infinite set. By Definitions 1.35 and 1.28, there exists a bijection $f : A \to \mathbb{N}$. Now consider the set $f(B)$. Let $\tilde{f} : B \to f(B)$ be defined by $\tilde{f}(b) = f(b)$. To prove that $\tilde{f}$ is a function, Definition 1.16 tells us that it will suffice to show that for all $b \in B$, there exists a unique $c \in f(B)$ such that $\tilde{f}(b) = c$. Let b be an arbitrary element of B. It follows by Definition 1.18 that $f(b) \in f(B)$, hence $\tilde{f}(b) \in B$ by the definition of $\tilde{f}$. Furthermore, since $f(b)$ is a unique object by Definition 1.16, $\tilde{f}(B)$ is also a unique object.

To prove that $\tilde{f}$ is bijective, Definition 1.20 tells us that it will suffice to show that $\tilde{f}$ is injective and surjective. We will verify these two characteristics in turn. To prove that $\tilde{f}$ is injective, Definition 1.20 tells us that we must demonstrate that $\tilde{f}(b) = \tilde{f}(b')$ implies $b = b'$. Let $\tilde{f}(b) = \tilde{f}(b')$. By the definition of $\tilde{f}$, $\tilde{f}(b) = f(b)$ and $\tilde{f}(b') = f(b')$. Thus, $f(b) = \tilde{f}(b) = \tilde{f}(b') = f(b')$, i.e., $f(b) = f(b')$. As such, by the injectivity of f (which follows from its bijectivity by Definition 1.20), $b = b'$, as desired. To prove that $\tilde{f}$ is surjective, Definition 1.20 tells us that we must demonstrate that for all $c \in f(B)$, there exists a $b \in B$ such that $\tilde{f}(b) = c$. Let c be an arbitrary element of $f(B)$. By Definition 1.18, it follows that $c = f(b)$ for some $b \in B$. But by the definition of $\tilde{f}$, we also have $f(b) = \tilde{f}(b)$, so transitivity implies that $\tilde{f}(b) = c$, as desired.

Since B is infinite, Definitions 1.30 and 1.28 tell us that no bijection $h : B \to [n]$ exists for any $n \in \mathbb{N}$. Consequently, since there exists a bijection $\tilde{f} : B \to f(B)$, no bijection $h : f(B) \to [n]$ exists, implying by Definitions 1.28 and 1.30 that $f(B)$ is similarly infinite. In addition to being infinite, Definition 1.18 asserts that $f(B) \subset \mathbb{N}$. Thus, there exists a bijection $g : f(B) \to \mathbb{N}$ by the lemma, Definition 1.35, and Definition 1.28. It follows by Proposition 1.26 that $g \circ \tilde{f} : B \to \mathbb{N}$ is a bijection, proving that B is countable by Definitions 1.28 and 1.35, as desired. $\square$

Exercise 1.38. Prove that if there is an injection $f : A \to B$ where B is countable and A is infinite, then A is countable.

Proof. Let $\tilde{f} : A \to f(A)$ be defined by $\tilde{f}(a) = f(a)$. To prove that $\tilde{f}$ is a function, Definition 1.16 tells us that it will suffice to show that for all $a \in A$, there exists a unique $b \in f(A)$ such that $\tilde{f}(a) = b$. Let a be an arbitrary element of A. It follows by Definition 1.18 that $f(a) \in f(A)$, hence $\tilde{f}(a) \in A$ by the definition of $\tilde{f}$. Furthermore, since $f(a)$ is a unique object by Definition 1.16, $\tilde{f}(a)$ is also a unique object.

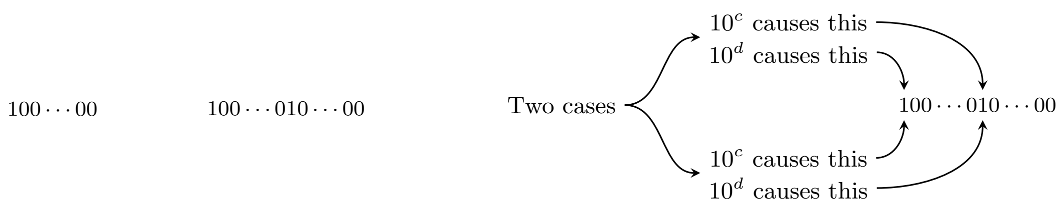
To prove that $\tilde{f}$ is bijective, Definition 1.20 tells us that it will suffice to show that $\tilde{f}$ is injective and surjective. We will verify these two characteristics in turn. To prove that $\tilde{f}$ is injective, Definition 1.20 tells us that we must demonstrate that $\tilde{f}(a) = \tilde{f}(a')$ implies $a = a'$. Let $\tilde{f}(a) = \tilde{f}(a')$. By the definition of $\tilde{f}$, $\tilde{f}(a) = f(a)$ and $\tilde{f}(a') = f(a')$. Thus, $f(a) = \tilde{f}(a) = \tilde{f}(a') = f(a')$, i.e., $f(a) = f(a')$. As such, by the injectivity of f, $a = a'$, as desired. To prove that $\tilde{f}$ is surjective, Definition 1.20 tells us that we must demonstrate that for all $b \in f(A)$, there exists an $a \in A$ such that $\tilde{f}(a) = b$. Let b be an arbitrary element of $f(A)$. By Definition 1.18, it follows that $b = f(a)$ for some $a \in A$. But by the definition of $\tilde{f}$, we also have $f(a) = \tilde{f}(a)$, so transitivity implies that $\tilde{f}(a) = b$, as desired.

Since A is infinite, Definitions 1.30 and 1.28 tell us that no bijection $h : A \to [n]$ exists for any $n \in \mathbb{N}$. Consequently, since there exists a bijection $\tilde{f} : A \to f(A)$, no bijection $h : f(A) \to [n]$ exists, implying by Definitions 1.28 and 1.30 that $f(A)$ is similarly infinite. In addition to being infinite, Definition 1.18 asserts that $f(A) \subset B$. Thus, Exercise 1.37 applies and proves that $f(A)$ is countable. It follows by Definitions 1.35 and 1.28 that there exists a bijection $g : f(A) \to \mathbb{N}$. Since $\tilde{f}$ and g are both bijective, Proposition 1.26 implies that $g \circ \tilde{f} : A \to \mathbb{N}$ is bijective. Therefore, A and $\mathbb{N}$ are in bijective correspondence by Definition 1.28, meaning that A is countable by Definition 1.35. $\square$

Exercise 1.39. Prove that $\mathbb{N} \times \mathbb{N}$ is countable by considering the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by $f(n, m) = (10^n - 1)10^m$.

Lemma (Informal[5]). *If $10^a + 10^b = 10^c + 10^d$ for $a, b, c, d \in \mathbb{N}$, then either $a = c$ and $b = d$, or $a = d$ and $b = c$.*

[5] Dr. Cartee approved this.



(a) 10^a. (b) $10^a + 10^b$. (c) Possible matchings of 10^c and 10^d.

Figure 1.2: Base-10 representations (ignoring the case where $a = b$).

Proof. Refer to Figure 1.2 throughout the following discussion. Think about the base-10 representation of 10^a — it will be a 1 followed by a bunch of 0s. When we add 10^b to 10^a, either one of the 0s becomes a 1, the 1 becomes a 2, or a further string consisting of a 1 (possibly followed by 0s) is concatenated to the beginning of the existing number. In any of these cases, it is clear that for this number to be written in the form $10^c + 10^d$, one of those two terms (10^c or 10^d) must account for one of the 1s, and the other for the other 1 (or both for the 2, in that case). □

Proof of Exercise 1.39. We wish to prove that f is injective, so that Exercise 1.38 applies. By Definition 1.20, proving that f is injective necessitates showing that $f(a,b) = f(c,d)$ implies that $(a,b) = (c,d)$. Suppose that

$$f(a,b) = f(c,d)$$

Substituting the definition of f and algebraically manipulating, we get

$$\begin{aligned}(10^a - 1)(10^b) &= (10^c - 1)(10^d)\\ 10^{a+b} - 10^b &= 10^{c+d} - 10^d\\ 10^{a+b} + 10^d &= 10^{c+d} + 10^b\end{aligned}$$

By the lemma, either $a + b = b$ and $c + d = d$, or $a + b = c + d$ and $b = d$. In the first case, we must have $a = 0$ and $c = 0$ for the equalities to hold. But since $0 \notin \mathbb{N}$, this implies that $a, c \notin \mathbb{N}$, a contradiction. Thus this case does not hold and it must be that the second case is true. In the second case, $b = d$, so by the cancellation law for addition, $a = c$. Since $a = c$ and $b = d$, Definition 1.15 tells us that $(a,b) = (c,d)$, as desired.

Having proven that there exists an injection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ where $\mathbb{N}$ is (clearly) countable and $\mathbb{N} \times \mathbb{N}$ is (clearly) infinite, Exercise 1.38 implies that $\mathbb{N} \times \mathbb{N}$ is countable, as desired. □

Additional Exercises

10/6: 1. In each of the following, write out the elements of the sets.

a) $(\{n \in \mathbb{Z} \mid n \text{ is divisible by } 2\} \cap \mathbb{N}) \cup \{-5\}$

Proof. The elements are -5 as well as 2, 4, 6, and every other even natural number. □

c) $\{[n] \mid n \in \mathbb{N}, 1 \leq n \leq 3\}$

Proof. The elements are the three sets $\{1\}$, $\{1,2\}$, and $\{1,2,3\}$. □

k) $\{\{a\} \cup \{b\} \mid a \in \mathbb{N}, b \in \mathbb{N}, 1 \leq a \leq 4, 3 \leq b \leq 5\}$

Proof. The elements are the 11 sets $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{2,3\}$, $\{2,4\}$, $\{2,5\}$, $\{3\}$, $\{3,4\}$, $\{3,5\}$, $\{4\}$, and $\{4,5\}$. □