MATH 16110 (Honors Calculus I IBL) Notes

Steven Labalme

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Responses

9/27: **Lemma 4.** *Let x, y be positive integers. Then xy is odd if and only if x and y are both odd.*

Proof. We wish to prove that if x and y are not both odd, then xy is not odd. In other words, we wish to prove that if at least one of x or y is even, then xy is even. Let's begin. WLOG, let x be even. Then $x = 2k$ for some $k \in \mathbb{N}$. Thus, $xy = 2(ky)$, proving that xy is even since $ky \in \mathbb{N}$. The proof is symmetric for y. $\square$

Corollary 5. *Let x, y be positive integers. Then xy is even if and only if at least one of x and y is even.*

Proof. We wish to prove that xy is even if and only if at least one of x and y is even. Consequently, we must prove the dual implications "if xy is even, then at least one of x and y is even" and "if at least one of x and y is even, then xy is even." Let's begin. For the first statement, let xy be even and suppose for the sake of contradiction that and both x and y are not even, i.e., are odd. But by Lemma 4, it follows from the assumption that x and y are both odd that xy is odd, which contradicts the fact that xy is even. Therefore, at least one of x or y must be even. As to the second statement, suppose that at least one of x or y is even. In this case, x and y are not both odd. Thus, by Lemma 4, xy is not odd, or, equivalently, xy is even. $\square$

Exercise 8.

a) Are there positive integers m, n such that m and n have no common factors (other than 1) and $m^2 = 3n^2$? Either give an example or prove that no example is possible.

Proof. Let m, n be relatively prime positive integers and suppose for the sake of contradiction that $m^2 = 3n^2$. We divide into two cases (the case where n is even, and the case where n is odd); we seek contradictions in both cases. First off, if n is even, then $n = 2k$ for some $k \in \mathbb{N}$. Thus, $3n^2 = 3(2k)^2 = 12k^2 = 2(6k^2) = m^2$, proving that m^2 is even since $6k^2 \in \mathbb{N}$. By Corollary 5, this implies that m is even. Therefore, since m and n are both even, they have a common factor, a contradiction. On the other hand, if n is odd, then $n = 2k+1$ for some $k \in \mathbb{N}$. Thus, $3n^2 = 3(2k+1)^2 = 12k^2+12k+3 = 2(6k^2+6k+1)+1 = m^2$, proving that m^2 is odd since $6k^2+6k+1 \in \mathbb{N}$. Thus, by Lemma 4, m is odd. Consequently, $m = 2l+1$ for some $l \in \mathbb{N}$, so $m^2 = (2l+1)^2 = 4l^2+4l+1 = 12k^2+12k+3$, the last equality holding because we also have $m^2 = 3n^2 = 12k^2+12k+3$. This implies the following.

$$\begin{aligned} 4l^2 + 4l + 1 &= 12k^2 + 12k + 3 \\ 4l^2 + 4l &= 12k^2 + 12k + 2 \\ 2l^2 + 2l &= 6k^2 + 6k + 1 \\ 2(l^2 + l) &= 2(3k^2 + 3k) + 1 \end{aligned}$$

Since $l^2 + l$ and $3k^2 + 3k$ are both natural numbers, the above asserts that an odd number equals an even number, a contradiction. Hence, in both cases, we must have that $m^2 \neq 3n^2$. $\square$

b) Are there positive integers m, n such that m and n have no common factors (other than 1) and $m^2 = 6n^2$? Either give an example or prove that no example is possible.

Proof. Let $m, n \in \mathbb{N}$ have no common factors (other than 1), and suppose for the sake of contradiction that $m^2 = 6n^2$. Since $m^2 = 6n^2 = 2(3n^2)$, m^2 is even. It follows by Corollary 5 that m is even, implying that $m = 2k$ for some $k \in \mathbb{N}$. Thus, $6n^2 = m^2 = (2k)^2 = 4k^2$, so $3n^2 = 2k^2$. Since $k^2 \in \mathbb{N}$, $3n^2$ is even. Consequently, we have that n^2 is even by Corollary 5 (since at least one of 3 or n^2 is even and $3 = 2(1) + 1$ is odd). By Corollary 5 again, n is even. Thus, m and n are both even, contradicting the assumption that they have no common factors other than 1. ☐

c) Are there positive integers m, n such that m and n have no common factors (other than 1) and $m^2 = 4n^2$? Either give an example or prove that no example is possible.

Proof. Let $m = 2$ and $n = 1$. Then $m^2 = 2^2 = 4 = 4 \cdot 1^2 = 4n^2$. ☐

Discussion

9/29:

- Dr. Cartee.
- Sam Craig (super reader) is an advanced undergraduate who has taken this class before.
- Honors Calculus uses Spivak — we do not have a textbook, just scripts.
  - Few lectures in the traditional sense.
  - Majority of material is presented and developed by the students.
  - Several scripts will be covered throughout the quarter.
    - In scripts: It is our job to complete the exercises, prove the theorems/lemmas/propositions, etc.
  - Be on the look-out for "no proof required" theorems.
  - 3 chances to learn/review scripts material:
    1. Before class, you prepare your own proof.
    2. During class, we discuss.
    3. After class and before the journal is due, we type up our own record of the proof in LaTeX.
- Before each class, he will tell us which theorems/exercises we need to work through.
- Your proofs do not have to be perfect in the beginning! Sam and Dr. Cartee will help us. Expect to present every other week.
  - For the first two scripts, you have the ability to rewrite your journal after Sam reviews it to recover up to half of the lost credit.
  - You only recover credit if your new solution is perfect.
  - Return your changes one week after Sam grades it.
    - Mark what parts/problems you have rewritten, and turn in the original as well.
- Later this afternoon, Dr. Cartee will share which Script 0 problems we should do before Thursday. Sign up for problems on a Google Doc when each script is released. You also get a "buddy," who discusses your proof with you before you present.
- Class participation: When and how often and the quality of our presentations, and also how good are our questions that help presenters fill in the gaps.
- We can use Overleaf for collaborative LaTeX projects.
- We can check in with Dr. Cartee on our progress whenever throughout the quarter.
- Sam's office hours: We get to talk to him one-on-one with questions.

  - 7:00-8:00 PM on Thursdays
- You have one chance to ask for a 24-hour extension on HW (like if you're sick).
- In the case of a switch to virtual class:
  - We can present by turning our phone into a document camera or using a white board behind us or typing up in LaTeX (in real time?).
- Get good at writing — you cannot type up your solutions during exams!
- We submit HW assignments through Canvas if we type it up in LaTeX, or in class by hand. It's nice if we can type it up.

Script 0

The Natural Numbers and Mathematical Induction

0.1 Responses

10/1: **Exercise 0.2** (PMI Exercise 2)**.** Prove that if $x > -1$, then $(1+x)^n \geq 1 + nx$ for any natural number n. (Note that although this script is focused on the natural numbers, your argument should hold for any real number $x > -1$.)

Proof. We induct on n. For the base case $k = 1$, we have $(1+x)^1 = 1 + x \geq 1 + (1)x$, where the greater than or equal to relation could be strengthened to equality but will be left as such for the sake of the argument. Now suppose inductively that we have proven the claim for some natural number k, i.e., we know that $(1+x)^k \geq 1 + kx$ if $x > -1$. We now seek to prove it for $k+1$. To begin, we have

$$(1+x)^{k+1} = (1+x)^k(1+x)$$

by the laws of exponents. By the inductive hypothesis and the fact that $ac \geq bc$ if and only if a, b, c are positive numbers and $a \geq b$ (note that $x > -1$ implies $1 + x > 0$ along with $(1+x)^k > 0$), we have that the above is

$$\geq (1 + kx)(1 + x)$$

Now expand and simplify.

$$\begin{aligned} &= 1 + kx + x + kx^2 \\ &= 1 + (k+1)x + kx^2 \end{aligned}$$

Since x^2 must be positive or zero and $k \in \mathbb{N}$ is clearly positive, we have that $kx^2 \geq 0$ so that the above is

$$\geq 1 + (k+1)x$$

thus closing the induction. □

Additional Exercises

10/13:

1. Prove that if A is a non-empty subset of $\mathbb{N}$, then A has a least element, i.e., there is some $n_0 \in A$ such that for all $n \in A$, we have $n_0 \leq n$.
2. Prove the following variants of the Principle of Mathematical Induction:
   (a) For each $n \in \mathbb{N}$, let $P(n)$ be a proposition and let n_0 be some natural number. Suppose the following two results:

(A) $P(n_0)$ is true.

(B) If $P(k)$ is true, then $P(k+1)$ is also true.

Then $P(n)$ is true for all natural numbers n such that $n \geq n_0$.

(b) For each $n \in \mathbb{N}$, let $P(n)$ be a proposition. Suppose the following two results:

(A) $P(1)$ is true.

(B) If $P(r)$ is true for all r such that $1 \leq r \leq k$, then $P(k+1)$ is true.

Then $P(n)$ is true for all natural numbers n.

10/8: 7. Let n be a natural number and $k \leq n$ also be a natural number. Define

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

where $k! = 1 \times 2 \times \cdots \times k$. Show that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

for all $n \in \mathbb{N}$.

Proof. We begin with a lemma proving some basic properties of combinations.

Lemma. *Let n be a natural number and $k \leq n$ also be a natural number. Define*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

where $k! = 1 \times 2 \times \cdots \times k$. Then

a) $\binom{n}{0} = 1$ *for all* $n \in \mathbb{N}$;

b) $\binom{n}{n} = 1$ *for all* $n \in \mathbb{N}$;

c) $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ *for all natural numbers n and $1 \leq k \leq n$.*

Proof of Lemma 1. We will address each of the three parts of the lemma in turn.

For part (a), we know by the definition of $\binom{n}{k}$ that $\binom{n}{0} = \frac{n!}{0!(n-0)!}$ and by the definition of a factorial that $\frac{n!}{0!(n-0)!} = \frac{1 \cdot n!}{n!} = 1$. Thus, $\binom{n}{0} = 1$, as desired.

For part (b), we proceed in a similar manner to the above: $\binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n! \cdot 1} = 1$.

For part (c), we repeatedly apply the definition of $\binom{n}{k}$ and of a factorial in the following algebra. Note that we proceed from the right side of the equality we seek to prove since it makes the algebra flow more logically (via simplification rather than expansion).

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-(k-1))!} + \frac{n!}{k!(n-k)!} \\
&= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k(k-1)!(n-k)!} \\
&= \frac{n!}{(k-1)!(n-k+1)(n-k)!} + \frac{n!}{k(k-1)!(n-k)!} \\
&= \frac{k \cdot n!}{k(k-1)!(n-k+1)(n-k)!} + \frac{(n-k+1)n!}{k(k-1)!(n-k+1)(n-k)!} \\
&= \frac{k \cdot n! + (n-k+1)n!}{k(k-1)!(n-k+1)(n-k)!}
\end{aligned}$$

$$= \frac{k \cdot n! + (n+1)n! - k \cdot n!}{k!(n-k+1)!}$$
$$= \frac{(n+1)n!}{k!(n+1-k)!}$$
$$= \frac{(n+1)!}{k!((n+1)-k)!}$$
$$= \binom{n+1}{k}$$

$\square$

Now we begin to address the question in earnest by inducting on n. For the base case $n = 1$, begin with the left side of the equality we wish to verify and employ the definition of exponents.

$$(x+y)^1 = x + y$$

Now use a couple of "clever forms of 1," which we can, of course, multiply to the terms in the above equation and still preserve equality.

$$= \frac{1!}{0!(1-0)!}x^1y^0 + \frac{1!}{1!(1-1)!}x^0y^1$$

Now just employ the definition of $\binom{n}{k}$ and use summation notation to simplify the expression.

$$= \binom{1}{0}x^{1-0}y^0 + \binom{1}{1}x^{1-1}y^1$$
$$= \sum_{k=0}^{1}\binom{1}{k}x^{1-k}y^k$$

This proves the base case. Now suppose inductively that we have proven the claim for some natural number n, i.e., we know given the definitions in the question that $(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$. We wish to prove the claim for $n+1$, which can be done as follows. Once again, begin with the left side of the equality we wish to prove and employ a rule of exponents.

$$(x+y)^{n+1} = (x+y)^1(x+y)^n$$

Now substitute using the induction hypothesis.

$$= (x+y)\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$$

Distribute the summation to each term in $x + y$, and then "distribute" x and y into the general term of the summation.

$$= x\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k + y\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$$
$$= \sum_{k=0}^{n}\binom{n}{k}x^{(n+1)-k}y^k + \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^{k+1}$$

Reindex the second summation (instead of iterating from 0 to n, iterate from 1 to $n+1$ [the same number of terms] and subtract 1 from each instance of the index variable k). Note that this does not change the sum at all; it just changes how the sum is written. After the reindexing, algebraically manipulate the exponents into an equivalent form that matches the exponents in the other summation.

$$= \sum_{k=0}^{n}\binom{n}{k}x^{(n+1)-k}y^k + \sum_{k=1}^{n+1}\binom{n}{k-1}x^{n-(k-1)}y^{(k-1)+1}$$
$$= \sum_{k=0}^{n}\binom{n}{k}x^{(n+1)-k}y^k + \sum_{k=1}^{n+1}\binom{n}{k-1}x^{(n+1)-k}y^k$$

Separate the first term of the left summation and the last term of the right summation from the summation notation.

$$= \binom{n}{0} x^{n+1} y^0 + \sum_{k=1}^{n} \binom{n}{k} x^{(n+1)-k} y^k + \sum_{k=1}^{n} \binom{n}{k-1} x^{(n+1)-k} y^k + \binom{n}{n} x^0 y^{n+1}$$

Now that the sums are once again indexed alike, combine them and do some algebraic manipulations to set up a substitution.

$$\begin{aligned} &= \binom{n}{0} x^{n+1} y^0 + \sum_{k=1}^{n} \left(\binom{n}{k} x^{(n+1)-k} y^k + \binom{n}{k-1} x^{(n+1)-k} y^k \right) + \binom{n}{n} x^0 y^{n+1} \\ &= \binom{n}{0} x^{n+1} y^0 + \sum_{k=1}^{n} \left(\binom{n}{k} + \binom{n}{k-1} \right) x^{(n+1)-k} y^k + \binom{n}{n} x^0 y^{n+1} \end{aligned}$$

In the first term, use Lemma 1a to make the substitution $\binom{n}{0} = 1 = \binom{n+1}{0}$. In the last term, use Lemma 1b to make the substitution $\binom{n}{n} = 1 = \binom{n+1}{n+1}$. In the summation, use Lemma 1c to make the substitution $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ (notice how k varies between 1 and n in the summation, just like it is allowed to in the statement of Lemma 1c).

$$= \binom{n+1}{0} x^{n+1} y^0 + \sum_{k=1}^{n} \binom{n+1}{k} x^{(n+1)-k} y^k + \binom{n+1}{n+1} x^0 y^{n+1}$$

Expand the limits of the summation to encompass the first and last terms.

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} x^{(n+1)-k} y^k$$

This closes the induction. □

10/15: 8. Let $a, b \in \mathbb{Z}$. We say that b is **divisible** by a if there is an integer m such that $b = ma$. It is easily shown that

i) if both b_1 and b_2 are divisible by a, then so is their sum;

ii) if b is divisible by a, then so is kb for any integer k.

10/8: 9. From Peano's Postulates (below), prove the following claims.

Axioms (Peano's Postulates)**.** *The natural numbers are defined as a set $\mathbb{N}$ together with a unary "successor" function $S : \mathbb{N} \to \mathbb{N}$ and a special element $1 \in \mathbb{N}$ satisfying the following postulates.*

I. $1 \in \mathbb{N}$.

II. *If $n \in \mathbb{N}$, then $S(n) \in \mathbb{N}$.*

III. *There is no $n \in \mathbb{N}$ such that $S(n) = 1$.*

IV. *If $n, m \in \mathbb{N}$ and $S(n) = S(m)$, then $n = m$.*

V. *If $A \subset \mathbb{N}$ is a subset satisfying the two properties:*

- $1 \in A$;
- *if $n \in A$, then $S(n) \in A$;*

then $A = \mathbb{N}$.

(a) **Bonus exercise**. Show that

$$\mathbb{N} = \{1, S(1), S(S(1)), S(S(S(1))), \cdots\}$$

Proof. We wish to eventually use Axiom V to show that the set on the right side of the above equality (which we shall call A) is equal to $\mathbb{N}$. Thus, we begin by demonstrating that A is a subset of $\mathbb{N}$. To do so, we must verify that every element of A is an element of $\mathbb{N}$. Now A consists of 1 and elements in the codomain of S, so since $1 \in \mathbb{N}$ (Axiom I) and any element of the codomain of S is clearly an element of $\mathbb{N}$ (because the codomain of S *is* $\mathbb{N}$), $A \subset \mathbb{N}$. Moving on, as previously referenced, $1 \in A$, so the first property of Axiom V holds. Additionally, the pattern defining A clearly indicates that for any $a \in A$, $S(a) \in A$, so the second property of Axiom V holds. Therefore, by Axiom V, $A = \mathbb{N}$. □

(b) Prove that the Principle of Mathematical Induction follows from Peano's Postulates.

Proof. We wish to prove, using only Axioms I-V above and set theoretic results, that if $P(n)$ is a proposition pertaining to each natural number n, $P(1)$ is true, and the truth of $P(k)$ implies that $P(S(n))$[1] is also true, then $P(n)$ is true for all natural numbers n. We will do this by defining a set A such that "$P(n)$ is true" is logically equivalent to $n \in A$. Then if we can show that $n \in A$ for all $n \in \mathbb{N}$ (i.e., that $A = \mathbb{N}$), we will have verified that $P(n)$ is true for all $n \in \mathbb{N}$ as desired. Lastly, note that we will show that $A = \mathbb{N}$ by demonstrating that A satisfies the stipulations of Axiom V. Let's begin.

Let $A = \{n \in \mathbb{N} \mid P(n) \text{ is true}\}$. Since every element of A is an element of $\mathbb{N}$ by the definition of A, $A \subset \mathbb{N}$. Additionally, since $P(1)$ is true by hypothesis and $1 \in \mathbb{N}$ by Axiom I, we know by the definition of A that $1 \in A$. Now suppose $n \in A$. It follows that $n \in \mathbb{N}$ and $P(n)$ is true. But by hypothesis, the truth of $P(n)$ implies that $P(S(n))$ is true. This, combined with the fact that $S(n) \in \mathbb{N}$ by Axiom II, shows that $S(n) \in A$. Having now proven that $A \subset \mathbb{N}$, $1 \in A$, and $n \in A$ implies $S(n) \in A$, Axiom V tells us that $A = \mathbb{N}$, as desired. □

(c) Define a special element $0 \notin \mathbb{N}$ and define $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $s : \mathbb{N}_0 \to \mathbb{N}$ be defined by

$$s(0) = 1$$
$$s(n) = S(n), \text{ for } n \in \mathbb{N}$$

where S is the successor function defined in Peano's Postulates.

Definition. We define addition $x + y$ for $x, y \in \mathbb{N}_0$ inductively on y by

$$x + 0 = x$$
$$x + s(y) = s(x + y)$$

Theorem. *The following facts all hold.*

i. If $x, y \in \mathbb{N}_0$, then $x + y \in \mathbb{N}_0$.
ii. $0 + x = x$, for all $x \in \mathbb{N}_0$.
iii. (Commutative Law) $x + y = y + x$ for all $x, y \in \mathbb{N}_0$.
iv. (Associative Law) $x + (y + z) = (x + y) + z$ for all $x, y, z \in \mathbb{N}_0$.
v. Given $x, y, z \in \mathbb{N}_0$, if $x + y = x + z$, then $y = z$.

Prove that $x+1 = s(x)$ for all $x \in \mathbb{N}_0$ and then prove items (i), (ii), and (iv) in the above theorem.

Proof of first claim. Since $1 = s(0)$ by definition, we know by repeated applications of the various parts of the definition of addition that $x + 1 = x + s(0) = s(x + 0) = s(x)$, as desired. □

Proof of i. We keep x fixed and induct on y. For the base case $y = 0$, we have by the definition of addition that $x + 0 = x$. Since $x \in \mathbb{N}_0$ by assumption, it clearly follows that $x + 0 \in \mathbb{N}_0$, thus proving the base case. Now suppose inductively that we have proven that $x + y \in \mathbb{N}_0$ for some $y \in \mathbb{N}_0$; we now seek to prove that $x + (y + 1) \in \mathbb{N}_0$. By the above argument, $y + 1 = s(y)$, so

$$x + (y + 1) = x + s(y)$$

[1] Addition has not yet been defined. Although we do not yet "know" that $n + 1 = S(n)$ we must assume it for the sake of this proof.

It follows by the definition of addition that the above

$$= s(x+y)$$

Since $s : \mathbb{N}_0 \to \mathbb{N}$ and $x+y \in \mathbb{N}_0$ by hypothesis, $s(x+y) \in \mathbb{N}$. Thus, $x+(y+1) \in \mathbb{N}$. Consequently, $x+(y+1) \in \mathbb{N} \cup \{0\}$, implying by the definition of $\mathbb{N}_0$ that $x+(y+1) \in \mathbb{N}_0$. This closes the induction. $\square$

Proof of ii. We induct on x. For the base case $x = 0$, we have by the definition of addition that $0+0 = 0$, thus proving the base case. Now suppose inductively that we have proven that $0+x = x$ for some $x \in \mathbb{N}_0$; we now seek to prove that $0+(x+1) = x+1$. As before, we can write that

$$\begin{aligned} 0+(x+1) &= 0+s(x) \\ &= s(0+x) \end{aligned}$$

But by the inductive hypothesis and the first claim proven herein, it follows that the above

$$\begin{aligned} &= s(x) \\ &= x+1 \end{aligned}$$

This closes the induction. $\square$

Proof of iv. We induct on x (keeping y, z fixed). For the base case $x = 0$, we must consider $0+(y+z)$. By part (i), $y+z \in \mathbb{N}_0$. Thus, part (ii) applies, and implies that

$$0+(y+z) = y+z$$

Since $y \in \mathbb{N}_0$ by assumption, we can apply part (ii) again in reverse to demonstrate that $y = 0+y$. Thus, the above is

$$= (0+y)+z$$

This proves the base case. Now suppose inductively that we have proven that $x+(y+z) = (x+y)+z$ for some $x \in \mathbb{N}_0$; we now seek to prove that $(x+1)+(y+z) = ((x+1)+y)+z$. As before,

$$(x+1)+(y+z) = s(x)+(y+z)$$

By part (iii) (which implies that $s(y)+x = s(y+x)$ is also true), the fact that $y+z \in \mathbb{N}_0$ by part (i), and the definition of addition, we thus have that the above

$$= s(x+(y+z))$$

We now apply the inductive hypothesis.

$$= s((x+y)+z)$$

By the fact that $x+y \in \mathbb{N}_0$ (part i) and consecutive applications of the definition of addition, we find that the above

$$\begin{aligned} &= s(x+y)+z \\ &= (s(x)+y)+z \end{aligned}$$

To finish it off, we once again use the first claim proved herein:

$$= ((x+1)+y)+z$$

This closes the induction. $\square$

(d) **Definition.** We define multiplication $x \cdot y$ for $x, y \in \mathbb{N}_0$ inductively on $\mathbb{N}_0$ by

$$x \cdot 0 = 0$$
$$x \cdot s(y) = x \cdot y + x$$

Prove that $x \cdot 1 = x$ for all $x \in \mathbb{N}_0$.

Proof. Since $s(0) = 1$,

$$x \cdot 1 = x \cdot s(0)$$

By the definition of multiplication, the above is

$$= x \cdot 0 + x$$

From the above, we can use the definition of multiplication to substitute $x \cdot 0 = 0$.

$$= 0 + x$$

Now just apply part (ii) of the Theorem in part (c).

$$= x$$

□

(e) **Definition.** We define $<$ on $\mathbb{N}_0$ by

$$x < y \text{ if and only if } y = x + u \text{ for some } u \in \mathbb{N}.$$

i. Prove that $1 < n$ for all $n \in (\mathbb{N} \setminus \{1\})$.
ii. Prove that if $a, x, y \in \mathbb{N}$ with $x < y$, then $a \cdot x < a \cdot y$.

Proof of i. We induct on n. For the base case $n = 2$, we have $2 = s(1 + 0) = 1 + s(0) = 1 + 1$, so $1 < 2$. Now suppose inductively that $1 < n$ for some $n \in \mathbb{N}$; we wish to prove that $1 < n + 1$. By the induction hypothesis and the definition of $<$, $n = 1 + u$. Thus, $n + 1 = 1 + u + 1$ by the inverse of the cancellation law for addition. Since $u + 1 \in \mathbb{N}$ by part (c) Theorem part (i), we have that $n + 1 = 1 + (u + 1)$, implying that $1 < n + 1$. This closes the induction. □

Proof of ii. We induct on a (keeping x, y fixed). For the base case $a = 1$, we have by part (d) that $x < y$ is equivalent to $1 \cdot x < 1 \cdot y$ since $x = 1 \cdot x$ for all $x \in \mathbb{N}$. Now suppose inductively that we have proven that $a \cdot x < a \cdot y$; we wish to prove that $(a + 1) \cdot x < (a + 1) \cdot y$. Let's start with

$$a \cdot x < a \cdot y$$

By the definition of $<$, we know that this implies

$$a \cdot y = a \cdot x + u$$

By the inverse of the cancellation law for addition, we can add a quantity to both sides, say y.

$$a \cdot y + y = a \cdot x + y + u$$

Since $x < y$ by assumption, $y = x + u'$ for some $u' \in \mathbb{N}$.

$$a \cdot y + y = a \cdot x + x + u' + u$$

Use the definition of multiplication and addition.

$$s(a) \cdot y = s(a) \cdot x + u' + u$$
$$(a + 1) \cdot y = (a + 1) \cdot x + u' + u$$

If we treat $u' + u$ as a single natural number, which we can do because of part (c) Theorem part i, we can employ the definition of $<$ one more time.

$$(a + 1) \cdot x < (a + 1) \cdot y$$

□

10/6: (f) **Definition.** For $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$, we define n^k inductively by

$$n^0 = 1$$
$$n^{k+1} = n \cdot n^k$$

i. Prove that $n < n^2$ for all $n \in \mathbb{N} \setminus \{1\}$.
ii. Prove that $n^k < n^{k+1}$ for all $n \in \mathbb{N} \setminus \{1\}$.

0.2 Discussion

10/8: - How much of the proof of Additional Exercise 0.7 proof could be left out as unnecessary verbage?

Script 1

Sets, Functions, and Cardinality

1.1 Journal

10/1: **Definition 1.2.** Two sets A and B are **equal** if they contain precisely the same elements, that is, $x \in A$ if and only if $x \in B$. When A and B are equal, we denote this by $A = B$.

Definition 1.3. A set A is a **subset** of a set B if every element of A is also an element of B, that is, if $x \in A$, then $x \in B$. When A is a subset of B, we denote this by $A \subset B$. If $A \subset B$ but $A \neq B$, we say that A is a **proper subset** of B, and we denote this by $A \subsetneq B$.

10/6: **Exercise 1.4.** Let $A = \{1, \{2\}\}$. Is $1 \in A$? Is $2 \in A$? Is $\{1\} \subset A$? Is $\{2\} \subset A$? Is $1 \subset A$? Is $\{1\} \in A$? Is $\{2\} \in A$? Is $\{\{2\}\} \subset A$? Explain.

Proof. We list affirmative or negative answers and short explanations.
Yes, $1 \in A$.
No, $2 \notin A$, but $\{2\} \in A$.
Yes, $\{1\} \subset A$ since 1 is the only element of $\{1\}$ and $1 \in A$ (as previously established).
No, $\{2\} \not\subset A$ since $2 \in \{2\}$ but $2 \notin A$ (as previously established).
No, $1 \not\subset A$ since 1 is not a set.
No, $\{1\} \notin A$, but $1 \in A$ and $\{1\} \subset A$ as previously established.
Yes, $\{2\} \in A$.
Yes, $\{\{2\}\} \subset A$ since $\{2\}$ is the only element of $\{\{2\}\}$ and $\{2\} \in A$ (as previously established). □

10/1: **Definition 1.5.** Let A and B be two sets. The **union** of A and B is the set

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Definition 1.6. Let A and B be two sets. The **intersection** of A and B is the set

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

Theorem 1.7. *Let A and B be two sets. Then*

a) $A = B$ *if and only if* $A \subset B$ *and* $B \subset A$.

b) $A \subset A \cup B$.

c) $A \cap B \subset A$.

Definition 1.8. The **empty set** is the set with no elements, and it is denoted $\emptyset$. That is, no matter what x is, we have $x \in \emptyset$.

Definition 1.9. Two sets A and B are **disjoint** if $A \cap B = \emptyset$.

10/6: **Exercise 1.10.** Show that if A is any set, then $\emptyset \subset A$.

Proof. Suppose for the sake of contradiction that there exists a set A such that $\emptyset \not\subset A$. Then by Definition 1.3, not every element of $\emptyset$ is also an element of A, i.e., there exists an element $x \in \emptyset$ such that $x \notin A$. But by Definition 1.8, x (like all other objects) cannot be an element of $\emptyset$, a contradiction. Therefore, $\emptyset \subset A$ for all sets A. ☐

10/1: **Definition 1.11.** Let A and B be two sets. The **difference** of B from A is the set

$$A \setminus B = \{x \in A \mid x \notin B\}$$

The set $A \setminus B$ is also called the **complement** of B relative to A. When the set A is clear from the context, this set is sometimes denoted B^c, but we will try to avoid this imprecise formulation and use it only with warning.

Theorem 1.12. *Let X be a set, and let $A, B \subset X$. Then*

a) $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$

b) $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$

Proof of a. To prove that $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$, Definition 1.2 tells us that it will suffice to prove that $x \in X \setminus (A \cup B)$ if and only if $x \in (X \setminus A) \cap (X \setminus B)$, i.e., that if $x \in X \setminus (A \cup B)$, then $x \in (X \setminus A) \cap (X \setminus B)$ and if $x \in (X \setminus A) \cap (X \setminus B)$, then $x \in X \setminus (A \cup B)$. To begin, let $x \in X \setminus (A \cup B)$. By Definition 1.11, $x \in X$ and $x \notin A \cup B$. By Definition 1.5, it follows that $x \notin A$ and $x \notin B$. Since we know that $x \in X$ and $x \notin A$, Definition 1.11 tells us that $x \in X \setminus A$. Similarly, $x \in X \setminus B$. Since $x \in X \setminus A$ and $x \in X \setminus B$, we have by Definition 1.6 that $x \in (X \setminus A) \cap (X \setminus B)$, as desired. The proof of the other implication is the preceding proof "in reverse." For clarity, let $x \in (X \setminus A) \cap (X \setminus B)$. By Definition 1.6, $x \in X \setminus A$ and $x \in X \setminus B$. By consecutive applications of Definition 1.11, $x \in X$, $x \notin A$, and $x \notin B$. Since $x \notin A$ and $x \notin B$, Definition 1.5 reveals that $x \notin A \cup B$. But as previously established, $x \in X$, so Definition 1.11 tells us that $x \in X \setminus (A \cup B)$. ☐

Proof of b. To prove that $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$, Definition 1.2 tells us that it will suffice to prove that $x \in X \setminus (A \cap B)$ if and only if $x \in (X \setminus A) \cup (X \setminus B)$. To begin, let $x \in X \setminus (A \cap B)$. By Definition 1.11, $x \in X$ and $x \notin A \cap B$. By Definition 1.6, it follows that $x \notin A$ or $x \notin B$. We divide into two cases. If $x \notin A$, then since we know that $x \in X$, Definition 1.11 tells us that $x \in X \setminus A$. It naturally follows that $x \in (X \setminus A) \cup (X \setminus B)$, since x need only be an element of one of the two unionized sets (see Definition 1.5). The proof is symmetric if $x \notin B$. Now let $x \in (X \setminus A) \cup (X \setminus B)$. By Definition 1.5, $x \in X \setminus A$ or $x \in X \setminus B$. Once again, we divide into two cases. If $x \in X \setminus A$, then $x \in X$ and $x \notin A$ by Definition 1.11. Consequently, by Definition 1.6, $x \notin A \cap B$. Therefore, $x \in X \setminus (A \cap B)$ by Defintion 1.11. The proof is symmetric if $x \in X \setminus B$. ☐

10/13: **Definition 1.13.** Let $\mathcal{A} = \{A_\lambda \mid \lambda \in I\}$ be a collection of sets indexed by a nonempty set I. Then the intersection and union of $\mathcal{A}$ are the sets

$$\bigcap_{\lambda \in I} A_\lambda = \{x \mid x \in A_\lambda \text{ for all } \lambda \in I\}$$

and

$$\bigcup_{\lambda \in I} A_\lambda = \{x \mid x \in A_\lambda \text{ for some } \lambda \in I\}$$

Theorem 1.14. *Let X be a set and let $\mathcal{A} = \{A_\lambda \mid \lambda \in I\}$ be a collection of subsets of X. Then*

1. $X \setminus \left(\bigcup_{\lambda \in I} A_\lambda\right) = \bigcap_{\lambda \in I}(X \setminus A_\lambda)$.
2. $X \setminus \left(\bigcap_{\lambda \in I} A_\lambda\right) = \bigcup_{\lambda \in I}(X \setminus A_\lambda)$.

Proof of 1. To prove that $X \setminus \left(\bigcup_{\lambda\in I} A_\lambda\right) = \bigcap_{\lambda\in I}(X \setminus A_\lambda)$, Definition 1.2 tells us that it will suffice to show that every element x of $X \setminus \left(\bigcup_{\lambda\in I} A_\lambda\right)$ is an element of $\bigcap_{\lambda\in I}(X \setminus A_\lambda)$ and vice versa. Suppose first that $x \in X \setminus \left(\bigcup_{\lambda\in I} A_\lambda\right)$. Then by Definition 1.11, $x \in X$ and $x \notin \bigcup_{\lambda\in I} A_\lambda$. By Definition 1.13, the latter result implies that $x \notin A_\lambda$ for any $\lambda \in I$. This combined with the fact that $x \in X$ implies by Definition 1.11 that $x \in X \setminus A_\lambda$ for all $\lambda \in I$. Therefore, by Definition 1.13, $x \in \bigcap_{\lambda\in I}(X \setminus A_\lambda)$. Now suppose that $x \in \bigcap_{\lambda\in I}(X \setminus A_\lambda)$. Then by Definition 1.13, $x \in X \setminus A_\lambda$ for all $\lambda \in I$. By Definition 1.11, this implies that $x \in X$ and $x \notin A_\lambda$ for any $\lambda \in I$. Thus, by Definition 1.13, the latter result implies that $x \notin \bigcup_{\lambda\in I} A_\lambda$. Therefore, since we also have $x \in X$, Definition 1.11 implies that $x \in X \setminus \left(\bigcup_{\lambda\in I} A_\lambda\right)$. ☐

Proof of 2. To prove that $X \setminus \left(\bigcap_{\lambda\in I} A_\lambda\right) = \bigcup_{\lambda\in I}(X \setminus A_\lambda)$, Definition 1.2 tells us that it will suffice to show that every element x of $X \setminus \left(\bigcap_{\lambda\in I} A_\lambda\right)$ is an element of $\bigcup_{\lambda\in I}(X \setminus A_\lambda)$ and vice versa. Suppose first that $x \in X \setminus \left(\bigcap_{\lambda\in I} A_\lambda\right)$. Then by Definition 1.11, $x \in X$ and $x \notin \bigcap_{\lambda\in I} A_\lambda$. By Definition 1.13, the latter result implies that $x \notin A_\lambda$ for some $\lambda \in I$. This combined with the fact that $x \in X$ implies by Definition 1.11 that $x \in X \setminus A_\lambda$ for some $\lambda \in I$. Therefore, by Definition 1.13, $x \in \bigcup_{\lambda\in I}(X \setminus A_\lambda)$. Now suppose that $x \in \bigcup_{\lambda\in I}(X \setminus A_\lambda)$. Then by Definition 1.13, $x \in X \setminus A_\lambda$ for some $\lambda \in I$. By Definition 1.11, this implies that $x \in X$ and $x \notin A_\lambda$ for some $\lambda \in I$. Thus, by Definition 1.13, the latter result implies that $x \notin \bigcap_{\lambda\in I} A_\lambda$. Therefore, since we also have $x \in X$, Definition 1.11 implies that $x \in X \setminus \left(\bigcap_{\lambda\in I} A_\lambda\right)$. ☐

10/1: **Definition 1.15.** Let A and B be two nonempty sets. The **Cartesian product** of A and B is the set of ordered pairs

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$$

If $(a, b), (a', b') \in A \times B$, we say that (a, b) and (a', b') are **equal** if and only if $a = a'$ and $b = b'$. In this case, we write $(a, b) = (a', b')$.

Definition 1.16. Let A and B be two nonempty sets. A **function** f from A to B is a subset $f \subset A \times B$ such that for all $a \in A$, there exists a unique $b \in B$ satisfying $(a, b) \in f$. To express the idea that $(a, b) \in f$, we most often write $f(a) = b$. To express that f is a function from A to B in symbols, we write $f : A \to B$.

10/13: **Exercise 1.17.** Let the function $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = 2n$. Describe f as a subset of $\mathbb{Z} \times \mathbb{Z}$.

Proof. By Definition 1.15, $\mathbb{Z} \times \mathbb{Z}$ is the set of all ordered pairs (a, b) where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$. By Definition 1.16, f is some subset of these ordered pairs. More specifically, elements of f are ordered pairs of the form $(n, f(n)) = (n, 2n)$ for some $n \in \mathbb{Z}$. Since f is a function, Definition 1.16 asserts that an ordered pair of such form must be present for all $n \in \mathbb{Z}$, so every object $(n, 2n)$ where $n \in \mathbb{Z}$ is an element of f. There are no other elements of f. ☐

10/1: **Definition 1.18.** Let $f : A \to B$ be a function. The **domain** of f is A and the **codomain** of f is B. If $X \subset A$, then the **image** of X under f is the set

$$f(X) = \{f(x) \in B \mid x \in X\}$$

If $Y \subset B$, then the **preimage** of Y under f is the set

$$f^{-1}(Y) = \{a \in A \mid f(a) \in Y\}$$

Exercise 1.19. Must $f(f^{-1}(Y)) = Y$ and $f^{-1}(f(X)) = X$? For each, either prove that it always holds or give a counterexample.

Counterexample to $f(f^{-1}(Y)) = Y$. Consider the sets $\{1\}$ and $\{3, 4\}$, and let $f : \{1\} \to \{3, 4\}$ be a function defined by $f(1) = 3$. Let $Y = \{3, 4\}$ (Theorem 1.7a guarantees that $Y \subset \{3, 4\}$, as needed). Then $f^{-1}(Y) = \{a \in \{1\} \mid f(a) \in \{3, 4\}\} = \{1\}$ and $f(f^{-1}(Y)) = \{f(x) \in \{3, 4\} \mid x \in \{1\}\} = \{3\}$ by consecutive applications of Definition 1.18. Therefore, $f(f^{-1}(Y)) \neq Y$ since $4 \in Y$ but $4 \notin f(f^{-1}(Y))$ (see Definition 1.2)[1]. ☐

[1] Note that the reason $f(f^{-1}(Y)) \neq Y$ in this case is because f is not surjective.

Counterexample to $f^{-1}(f(X)) = X$. Consider the sets $\{1,2\}$ and $\{3\}$, and let $f : \{1,2\} \to \{3\}$ be a function defined by $f(1) = 3$ and $f(2) = 3$. Let $X = \{1\}$ (we have $X \subset \{1,2\}$ since 1 is the only element of X and $1 \in \{1,2\}$ [see Definition 1.3]). Then $f(X) = \{f(x) \in \{3\} \mid x \in \{1\}\} = \{f(1)\} = \{3\}$ and $f^{-1}(f(X)) = \{a \in \{1,2\} \mid f(a) \in \{3\}\} = \{1,2\}$ by consecutive applications of Definition 1.18. Therefore, $f^{-1}(f(X)) \neq X$ since $2 \in f^{-1}(f(X))$ but $2 \notin X$ (see Definition 1.2)[2]. $\square$

Definition 1.20. A function $f : A \to B$ is **surjective** (*also known as* **onto**) if for every $b \in B$, there is some $a \in A$ such that $f(a) = b$. The function f is **injective** (*also known as* **one-to-one**) if for all $a, a' \in A$, if $f(a) = f(a')$, then $a = a'$. The function f is **bijective** (*also known as* a **bijection**, a **one-to-one correspondence**) if it is surjective and injective.

10/6: **Exercise 1.21.** Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n^2$. Is f injective? Is f surjective?

Proof. f is injective: Let $f(n) = f(n')$. Then $n^2 = (n')^2$, implying that $n = n'$ (note that this last step is not permissible in all number systems, but it is within the naturals).

f is not surjective: For example, $2 \in \mathbb{N}$ but there exists no natural number n such that $f(n) = n^2 = 2$ (suppose for the sake of contradiction that there exists a natural number n such that $n^2 = 2$. Since $n^2 = 2 \neq 1$, we know that $n < 2$ [a natural number is less than its square if its square is unequal to 1, as proven in Additional Exercise 0.9f-i]. Thus, $n = 1$ since 1 is the only natural number less than 2. But then $n^2 = 1^2 = 1 \neq 2$, a contradiction). $\square$

Exercise 1.22. Let $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = n + 2$. Is f injective? Is f surjective?

Proof. f is injective: Let $f(n) = f(n')$. Then $n + 2 = n' + 2$, implying by the cancellation law for addition that $n = n'$.

f is not surjective: For example, $1 \in \mathbb{N}$ but there exists no natural number n such that $n+2 = 1$ (suppose for the sake of contradiction that there exists a natural number n such that $n + 2 = 1$. Because $1 = n + 2$, we know that $1 > n$. But we also know that $1 \leq n$ for all $n \in \mathbb{N}$ (as can be proven by induction). Therefore, n is both > 1 and ≤ 1, contradicting the trichotomy). $\square$

Exercise 1.23. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x^2$. Is f injective? Is f surjective?

Proof. f is not injective: For example, $f(2) = 4 = f(-2)$, but $2 \neq -2$.

f is not surjective: For example, $2 \in \mathbb{Z}$ but there exists no integer x such that $f(x) = x^2 = 2$ (suppose for the sake of contradiction that there exists an integer x such that $x^2 = 2$. Since $x^2 = 2$, $|x| < 2$ for similar reasons to those discussed in Exercise 1.21. Thus, $x = -1$, $x = 0$, or $x = 1$. But $(-1)^2 = 1 \neq 2$, $0^2 = 0 \neq 2$, and $1^2 = 1 \neq 2$, so $x \neq -1$, $x \neq 0$, and $x \neq 1$, a contradiction). $\square$

10/13: **Exercise 1.24.** Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x + 2$. Is f injective? Is f surjective?

Proof. f is injective: Let $f(x) = f(x')$. Then $x + 2 = x' + 2$, implying by the cancellation law for addition that $x = x'$.

f is surjective: Let x be an arbitrary element of $\mathbb{Z}$. We divide into five cases ($x \geq 3$, $x = 2$, $x = 1$, $x = 0$, and $x \leq -1$). If $x \geq 3$, there is a unique natural number (hence an integer) equal to $x - 2$. If $x = 2$, then $x - 2 = 0$ (since $0 + 2 = 2$) and $0 \in \mathbb{Z}$. If $x = 1$, then $x - 2 = -1$ (since $-1 + 2 = 1$) and $-1 \in \mathbb{Z}$. If $x = 0$, then $x - 1 = -2$ (since $-2 + 2 = 0$) and $-2 \in \mathbb{Z}$. And if $x \leq -1$, then subtraction on $-\mathbb{N} \subset \mathbb{Z}$ is identical to addition on $\mathbb{N}$, so a unique $x - 2$ exists by an inverse interpretation of the closure of addition for $\mathbb{N}$. $\square$

10/1: **Definition 1.25.** Let $f : A \to B$ and $g : B \to C$. Then the **composition** $g \circ f : A \to C$ is defined by $(g \circ f)(x) = g(f(x))$ for all $x \in A$.

Proposition 1.26. *Let A, B, and C be sets and suppose that $f : A \to B$ and $g : B \to C$. Then $g \circ f : A \to C$ and*

a) if f and g are both injections, so is $g \circ f$.

b) if f and g are both surjections, so is $g \circ f$.

[2] Note that the reason $f^{-1}(f(X)) \neq X$ in this case is because f is not injective.

c) *if f and g are both bijections, so is $g \circ f$.*

Proof of a. Suppose that $(g \circ f)(a) = (g \circ f)(a')$. By Definition 1.25, this implies that $g(f(a)) = g(f(a'))$. Since g is injective, Definition 1.20 tells us that $f(a) = f(a')$. Similarly, the fact that f is injective tells us that $a = a'$. Since we have shown that $(g \circ f)(a) = (g \circ f)(a')$ implies that $a = a'$ under the given conditions, we know by Definition 1.20 that $g \circ f$ is injective. □

Proof of b. Let c be an arbitrary element of C. We wish to prove that there exists some $a \in A$ such that $(g \circ f)(a) = c$ (Definition 1.20). By Definition 1.25, it will suffice to show that there exists some $a \in A$ such that $g(f(a)) = c$. Let's begin. By the surjectivity of g, there exists some $b \in B$ such that $g(b) = c$ (see Definition 1.20). If we now consider this b, we have by the surjectivity of f that there exists some $a \in A$ such that $f(a) = b$ (see Definition 1.20). But this a is an element of A such that $g(f(a)) = g(b) = c$, as desired. □

Proof of c. Suppose that f and g are two bijective functions. By Definition 1.20, this implies that f and g are both injections and are both surjections. Thus, by Proposition 1.26a, $g \circ f$ is an injection, and by Proposition 1.26b, $g \circ f$ is a surjection. Therefore, by Definition 1.20, $g \circ f$ is a bijection. □

10/6: **Proposition 1.27.** *Suppose that $f : A \to B$ is bijective. Then there exists a bijection $g : B \to A$ that satisfies $(g \circ f)(a) = a$ for all $a \in A$, and $(f \circ g)(b) = b$ for all $b \in B$.*

Proof. Let $g : B \to A$ be defined by the rule, "$g(b) = a$ if and only if $f(a) = b$." For g to be a function as defined, Definition 1.16 tells us that we must show that for every $b \in B$, there exists a unique $a \in A$ such that $g(b) = a$. By the surjectivity of f, we know that for all $b \in B$, there exists *an* $a \in A$ such that $f(a) = b$. On the uniqueness of this a, let $a \neq a'$ and suppose for the sake of contradiction that $g(b) = a$ and $g(b) = a'$. By the definition of g, we have that $f(a) = b$ and $f(a') = b$, so $f(a) = f(a')$. But by the injectivity of f, this means that $a = a'$, a contradiction. Therefore, g indeed maps every $b \in B$ to a unique $a \in A$. To demonstrate that g satisfies the remainder of the necessary constraints, we will work through them one by one.

To prove that g is injective, Definition 1.20 tells us that we must verify that if $g(b) = g(b')$, then $b = b'$. Let $g(b) = g(b')$. Since g is a function, $g(b) = g(b') = a$, where $a \in A$. This implies by the definition of g that $f(a) = b$ and $f(a) = b'$. But since f is a function (i.e., by Definition 1.16, $f(a)$ is a unique object), this means that $b = f(a) = b'$, as desired. To prove that g is surjective, Definition 1.20 tells us that we must verify that for all $a \in A$, there exists a $b \in B$ such that $g(b) = a$. Let a be an arbitrary element of A. By Definition 1.16 and the status of f as a function, there exists an element $b \in B$ such that $f(a) = b$. But by the definition of g, $f(a) = b$ implies that $g(b) = a$, meaning that this b satisfies the desired constraint. On the basis of this and the previous argument, Definition 1.20 allows us to conclude that g is bijective.

We now prove that $(g \circ f)(a) = a$ for all $a \in A$. Let a be an arbitrary element of A. Then by Definition 1.16 and the status of f as a function, $f(a) = b$ where $b \in B$. Thus, by the definition of g, $g(b) = a$. But by Definition 1.25 and some substitutions, this means that $(g \circ f)(a) = g(f(a)) = g(b) = a$, as desired.

A symmetric argument can demonstrate that $(f \circ g)(b) = b$ for all $b \in B$. □

Definition 1.28. We say that two sets A and B are in **bijective correspondence** when there exists a bijection from A to B or, equivalently, from B to A.

Definition 1.29. Let $n \in \mathbb{N}$ be a natural number. We define $[n]$ to be the set $\{1, 2, \dots, n\}$. Additionally, we define $[0] = \emptyset$.

Definition 1.30. A set A is **finite** if $A = \emptyset$ or if there exists a natural number n and a bijective correspondence between A and the set $[n]$. If A is not finite, we say that A is **infinite**.

Theorem 1.31. *Let $n, m \in \mathbb{N}$ with $n < m$. Then there does not exist an injective function $f : [m] \to [n]$.*

Theorem 1.32. *Let A be a finite set. Suppose that A is in bijective correspondence both with $[m]$ and with $[n]$. Then $m = n$.*

Proof. If A is in bijective correspondence with both $[m]$ and with $[n]$, then Definition 1.28 tells us that there exist bijections $f : [m] \to A$ and $g : A \to [n]$. Thus, by Proposition 1.26, $g \circ f : [m] \to [n]$ is bijective. Now suppose for the sake of contradiction that $m \neq n$. Then by the trichotomy, either $m > n$ or $m < n$. We divide into two cases. If $m > n$, then Theorem 1.31 tells us that no injective function $h : [m] \to [n]$ exists. But $f : [m] \to [n]$ is bijective, hence injective by Definition 1.20, a contradiction. On the other hand, if $m < n$, then Theorem 1.31 tells us that no injective function $h : [n] \to [m]$ exists. But by Proposition 1.27, the existence of the bijection $f : [m] \to [n]$ implies the existence of a bijection $f^{-1} : [n] \to [m]$. As before, the bijectivity of f^{-1} implies that it is also injective by Definition 1.20, a contradiction. Therefore, we must have $m = n$. $\square$

10/8: **Definition 1.33** (Cardinality of a finite set)**.** If A is a finite set that is in bijective correspondence with $[n]$, then we say that the **cardinality** of A is n, and we write $|A| = n$. (By Theorem 1.32, there is exactly one such natural number n.) We also say that A contains n elements. We define the cardinality of the empty set to be 0.

Exercise 1.34. Let A and B be finite sets.

a) If $A \subset B$, then $|A| \leq |B|$.

Proof. Let $|A| = m$ and $|B| = n$. Using these variables, Definitions 1.33 and 1.28 tell us that there exist bijections $f : [m] \to A$ and $g : B \to [n]$. Now let $h : A \to B$ be defined by $h(a) = a$ for each $a \in A$. By Definition 1.16, to verify that h is a function, we must show that for every $a \in A$, there exists a unique $b \in B$ such that $h(a) = b$. Let a be an arbitrary element of A. Since $A \subset B$, Definition 1.3 implies that $a \in B$. Thus, since $h(a) = a$, $h(a) \in B$. Now suppose for the sake of contradiction that $h(a) = b$ and $h(a) = b'$ for two elements $b, b' \in B$ such that $b \neq b'$. By the definition of h, $h(a) = a$, so $a = b$ and $a = b'$, implying by transitivity that $b = b'$, a contradiction.

We now demonstrate that h is injective. By Definition 1.20, it will suffice to show that $h(a) = h(a')$ implies that $a = a'$ (where $a, a' \in A$). So suppose that $h(a) = h(a')$. By the definition of h, $h(a) = a$ and $h(a') = a'$, so by assumption, $a = h(a) = h(a') = a'$, as desired.

To recap, at this point we have injective functions $f : [m] \to A$, $h : A \to B$, and $g : B \to [n]$, where the injectivity of f and g follows from their bijectivity (see Definition 1.20). It follows by consecutive applications of Proposition 1.26 that $h \circ f$ is injective, and that $g \circ (h \circ f)$ is injective. Thus, there exists an injective function $g \circ (h \circ f) : [m] \to [n]$, so the contrapositive of Theorem 1.31 implies that it is false that $n < m$. Equivalently, it is true that $n \geq m$, or, to return substitutions, that $|A| \leq |B|$. $\square$

b) Let $A \cap B = \emptyset$. Then $|A \cup B| = |A| + |B|$.

Proof. Let $|A| = m$ and $|B| = n$. Thus, $|A| + |B| = m + n$, so to prove that $|A \cup B| = |A| + |B|$, Definition 1.33 and 1.28 tell us that that we must find a bijection $h : A \cup B \to [m + n]$. Let's begin.

Since $|A| = m$ and $|B| = n$, by Definition 1.33 and 1.28, there exist bijections $f : A \to [m]$ and $g : B \to [n]$. As such, let $h : A \cup B \to [m + n]$ be defined as follows:

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) + m & x \in B \end{cases}$$

Since the two cases defining h are both functions, the only possible barrier to h itself being a function is if there exists some $x \in A \cup B$ such that $x \in A$ and $x \in B$. To address this, suppose for the sake of contradiction that this is the case. Fortunately, such a hypothesis implies by Definition 1.6 that $x \in A \cap B$, contradicting the fact that $A \cap B = \emptyset$ (see Definition 1.8).

To prove that h is injective, the contrapositive of Definition 1.20 tells us that we must verify that if $x \neq x'$, then $h(x) \neq h(x')$. We divide into three cases ($x, x' \in A$, $x, x' \in B$, and WLOG $x \in A$ and $x' \in B$[3]). First, suppose that $x, x' \in A$. By the injectivity of f (which follows from its bijectivity by

[3] Note that we do not have to treat the case that $x \in B$ and $x' \in A$ since in this case, we just call the object represented by x, "x'," and vice versa — this reversal of names is what is implied by "Without the Loss Of Generality," or WLOG.

Definition 1.20), we have that $f(x) \neq f(x')$. Since $h(x) = f(x)$ and $h(x') = f(x')$ by the definition of h, we have that $h(x) = f(x) \neq f(x') = h(x')$, as desired. Second, suppose that $x, x' \in B$. By the injectivity of g, we have that $g(x) \neq g(x')$, which implies by the additive property of equality that $g(x) + m \neq g(x') + m$. Since $h(x) = g(x) + m$ and $h(x') = g(x') + m$ by the definition of h, we have that $h(x) = g(x) + m \neq g(x') + m = h(x')$, as desired. Third, suppose that $x \in A$ and $x' \in B$. Then $h(x) = f(x) \leq m$ since $f(x) \in [m]$ while $m < h(x') = g(x') + m \leq m + n$ since $0 < g(x') < n$ as $g(x') \in [n]$. Since $h(x) \leq m$ and $h(x') > m$, we have by the trichotomy that $h(x) \neq h(x')$, as desired.

To prove that h is surjective, Definition 1.20 tells us that we must verify that for every $i \in [m+n]$, there exists an $x \in A \cup B$ such that $h(x) = i$. We divide into two cases ($i \leq m$ and $m + n \geq i > m$). If $i \leq m$, then $i \in [m]$. It follows by the surjectivity of f (which follows from its bijectivity by Definition 1.20) that there exists an $x \in A$ such that $f(x) = i$. Now by Definition 1.5, this x is also an element of $A \cup B$, so $h(x) = f(x) = i$ by the definition of h, as desired. On the other hand, if $m + n \geq i > m$, then $i = m + u$ for some $u \in [n]$. It follows by the surjectivity of g that there exists an $x \in B$ such that $g(x) = u$. Thus, $i = m + u = m + g(x) = h(x)$, as desired.

At this point, Definition 1.20 implies that h is bijective, as desired. □

c) $|A \cup B| + |A \cap B| = |A| + |B|$.

Lemma. *Let A and B be sets. Then*

a) $A \cup B = (B \setminus A) \cup A$.
b) $(B \setminus A) \cap A = \emptyset$.
c) $(B \setminus A) \cap (A \cap B) = \emptyset$.
d) $B = (B \setminus A) \cup (A \cap B)$.

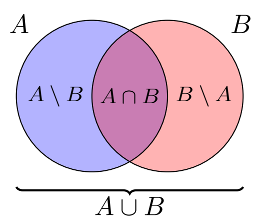


Figure 1.1: Set union Venn diagram.

Proof. All of these claims can be read directly from the above diagram — for the sake of space and because proving these claims is not the main point of this exercise, a rigorous proof of this lemma will be omitted. □

Proof of Theorem 1.34c. We want to show that $|A \cup B| + |A \cap B| = |A| + |B|$, which we can do by using the above lemma to justify various manipulations inspired by Theorem 1.34b. To begin, use Lemma (a) as follows.

$$|A \cup B| + |A \cap B| = |(B \setminus A) \cup A| + |A \cap B|$$

Since $(B \setminus A) \cup A$ is a union of two disjoint sets (see Lemma (b)), it follows by Theorem 1.34b that the above

$$\begin{aligned} &= |B \setminus A| + |A| + |A \cap B| \\ &= |A| + |B \setminus A| + |A \cap B| \end{aligned}$$

Since $B \setminus A$ and $A \cap B$ are disjoint (see Lemma (c)), we know that the above

$$= |A| + |(B \setminus A) \cup (A \cap B)|$$

Lastly, apply Lemma (d):

$$= |A| + |B|$$

$\square$

10/13: d) $|A \times B| = |A| \cdot |B|$.

Proof. Let $|A| = n$ and $|B| = m$. Then since A and B are finite, Definition 1.30 and 1.28 tell us that there exist bijections $f : A \to [n]$ and $g : B \to [m]$. By Definition 1.33 and 1.28, to prove the claim, it will suffice to find a bijection $h : A \times B \to [m \cdot n]$.

Let $h : A \times B \to [m \cdot n]$ be defined by

$$h(a, b) = f(a) + n \cdot (g(b) - 1)$$

Clearly, the above rule assigns a unique value to every (a, b), and since f and g map all $a \in A$ and $b \in B$, respectively, the above function is not undefined for any $(a, b) \in A \times B$. Thus, h is a function as defined in Definition 1.16.

We must now prove that h is bijective. By Definition 1.20, it will suffice to prove that h is injective and surjective, which we may do as follows. We will start with injectivity.

Let

$$h(a, b) = h(a', b')$$

Then by the definition of h,

$$\begin{aligned} f(a) + n \cdot (g(b) - 1) &= f(a') + n \cdot (g(b') - 1) \\ f(a) - f(a') &= n \cdot (g(b') - 1) - n \cdot (g(b) - 1) \\ f(a) - f(a') &= n \cdot (g(b') - g(b)) \end{aligned}$$

Since $f(a)$ and $f(a')$ are both elements of $[n]$, we have $|f(a) - f(a')| < n$ (since $\max([n]) - \min([n]) = n - 1 < n$). Substituting, we have that $|n \cdot (g(b') - g(b))| < n$, i.e., $|g(b') - g(b)| < 1$. But since $g(b), g(b') \in \mathbb{N}$, the only way that $|g(b') - g(b)| < 1$ is if $|g(b') - g(b)| = 0$. Consequently, $g(b') - g(b) = 0$, so additionally, $f(a) - f(a') = n \cdot (g(b') - g(b)) = 0$. Having ascertained that $g(b') - g(b) = 0$ and $f(a) - f(a') = 0$, it is a simple matter to find that $g(b) = g(b')$ and $f(a) = f(a')$, meaning by the bijectivity (more specifically, the injectivity) of f and g that $b = b'$ and $a = a'$. But by Definition 1.15, this implies that $(a, b) = (a', b')$, as desired.

As to surjectivity, let c be an arbitrary element of $[n \cdot m]$. As a natural number, c can be written in the form $c = \beta \cdot n + \alpha$ where $1 \leq \alpha \leq n$ and $\beta \in \mathbb{N}$. We know that $\min([n \cdot m]) = 1 = 0 \cdot n + 1$ and $\max([n \cdot m]) = m \cdot n = (m - 1) \cdot n + n$; thus, if we restrict the possible values of β to $0 \leq \beta \leq m - 1$, we still know that $c = \beta \cdot n + \alpha$ for some $1 \leq \alpha \leq n$ and $0 \leq \beta \leq m - 1$. Now by the surjectivity of f, there exists an $a \in A$ such that $f(a) = \alpha$ for any $1 \leq \alpha \leq n$. Similarly, the surjectivity of g implies that there exists a $b \in B$ such that $g(b) = \beta + 1$ for any $1 \leq \beta + 1 \leq m$, i.e., there exists a $b \in B$ such that $g(b) - 1 = \beta$ for any $0 \leq \beta \leq m - 1$. Therefore, c can be written in the form $c = f(a) + n \cdot (g(b) - 1)$ for some $a \in A$ and $b \in B$, which by the definition of h means that $c = h(a, b)$ for some $(a, b) \in A \times B$, as desired. $\square$

10/8: **Definition 1.35.** An infinite set A is said to be **countable** if it is in bijective correspondence with $\mathbb{N}$. An infinite set that is not countable is called **uncountable**.

Exercise 1.36. Prove that $\mathbb{Z}$ is a countable set.

Proof. To prove that $\mathbb{Z}$ is countable, Definition 1.35 and, subsequently, Definition 1.28 tell us that we must find a bijection $f : \mathbb{Z} \to \mathbb{N}$. To do so, we will define a matching and then prove that the guiding rule generates a (1) function that is (2) injective and (3) surjective (demonstrating injectivity and surjectivity verifies bijectivity by Definition 1.20).

Let $f : \mathbb{Z} \to \mathbb{N}$ be defined as follows:

$$f(z) = \begin{cases} -2z+1 & z \in -\mathbb{N} \\ 1 & z \in \{0\} \\ 2z & z \in \mathbb{N} \end{cases}$$

Since $\mathbb{Z} = (-\mathbb{N}) \cup \{0\} \cup (\mathbb{N})$, it is clear that the above mapping sends every element of $\mathbb{Z}$ to *an* element of $\mathbb{N}$. Additionally, since $-\mathbb{N}$, $\{0\}$, and $\mathbb{N}$ are all disjoint from one another by definition, it follows that each element of $\mathbb{Z}$ is only mapped once. Thus, by Definition 1.16, f is a function as defined.

Let $f(z) = f(z')$. Since the outputs of the first case in the definition of f are the odd natural numbers except 1, the output of the second case is 1, and outputs of the third case are the even natural numbers, the outputs form three disjoint sets, so $f(z)$ and $f(z')$ as equal quantities are elements of only one category. We now divide into three cases by category. First, suppose $f(z) = f(z')$ is an odd natural number not equal to 1. Then we have by the definition of f that $-2z+1 = -2z'+1$, implying by the cancellation laws of addition and multiplication, respectively, that $z = z'$. Second, suppose that $f(z) = f(z') = 1$. Then $z = z' = 0$ by the definition of f. Lastly, suppose $f(z) = f(z')$ is an even natural number. Then we have by the definition of f that $2z = 2z'$, implying by the cancellation law of multiplication that $z = z'$. Therefore, in any case, $f(z) = f(z')$ implies that $z = z'$, meaning by Definition 1.20 that f is injective.

Let n be an arbitrary element of $\mathbb{N}$. As noted above, n must be even, 1, or odd but not 1. We now divide into three cases by category, as before. First, suppose that n is even. Then $n = 2z$ for some $z \in \mathbb{N}$ (see "Notes on Proofs"). By the definition of f, this z is the element of $\mathbb{Z}$ that f sends to n. Second, suppose n is odd and not equal to 1. Then $n = 2z+1$ for some $z \in \mathbb{N}$ (see "Notes on Proofs"), or $-2z+1$ for some $z \in -\mathbb{N}$ (the negative signs cancel). By the definition of f, this $z \in -\mathbb{N}$ is the element of $\mathbb{Z}$ that f sends to n. Lastly, suppose that $n = 1$. Then since $f(0) = 1$ by the definition of f, 0 is the element of $\mathbb{Z}$ from which f generates 1. Therefore, for every element $n \in \mathbb{N}$, there exists a $z \in \mathbb{Z}$ satisfying $f(z) = n$, meaning by Definition 1.20 that f is surjective. ☐

10/13: **Exercise 1.37.** Prove that every infinite subset of a countable set is also countable.

Lemma. *Every infinite subset of the natural numbers is countable.*

Proof. Let $A \subset \mathbb{N}$ be infinite. To prove that A is countable, Definition 1.35 tells us that it will suffice to show that there exists a bijection $g : \mathbb{N} \to A$. Let's begin.

We define g recursively with strong induction, as follows (note that $A = A \setminus \{\}$ where $\{\} = \emptyset$). By the well-ordering principle (see Additional Exercise 0.1), there exists a minimum element $\min(A \setminus \{\}) \in A \setminus \{\}$; we define $g(1) = \min(A \setminus \{\})$. Now suppose inductively that we have defined $g(1), g(2), \dots, g(n)$. Then we can define $g(n+1)$ by defining $g(n+1) = \min(A \setminus \{g(1), g(2), \dots, g(n)\})$[4]. By the principle of strong mathematical induction (see Additional Exercise 0.2b), it follows that g is defined for all $n \in \mathbb{N}$, and it is obvious that g is not multiply defined for any $n \in \mathbb{N}$. Thus, g is a function as defined in Definition 1.16.

To prove that g is bijective, Definition 1.20 tells us that it will suffice to show that g is injective and surjective. We will prove each of these qualities in turn. To prove that g is injective, the contrapositive of Definition 1.20 necessitates that we verify that $n \neq n'$ implies $g(n) \neq g(n')$. Suppose that $n \neq n'$. Then by the trichotomy, either $n > n'$ or $n < n'$. If $n > n'$, then $g(n) = \min(A \setminus \{g(1), \dots, g(n'), \dots, g(n-1)\})$, meaning that $g(n)$ cannot equal $g(n')$ since $g(n)$ is an element of a set (namely, $A \setminus \{g(1), \dots, g(n'), \dots, g(n-1)\}$) of which $g(n')$ is explicitly not a member. The proof is symmetric if $n < n'$. To prove that g is surjective, Definition 1.20 tells us that we must verify that for all $a \in A$, there exists an $n \in \mathbb{N}$ such that $g(n) = a$.

[4]Basically, what this definition is doing is mapping 1 to the least element of A, 2 to the second-least element of A, 3 to the third-least element of A, and so on and so forth. Notice how the least element of A is denoted by $g(1)$, and $g(2)$ (for example) is equal to $\min(A \setminus \{g(1)\})$, i.e., the minimum value in A if A's least element did not exist, i.e., the second-least element in A. Additionally, $g(3) = \min(A \setminus \{g(1), g(2)\})$, so we can see how $g(3)$ is the third-least element in A by the same logic used in discussing $g(2)$. Obviously, the pattern continues for all $n \in \mathbb{N}$.

Suppose for the sake of contradiction that there exists some $a \in A$ such that $g(n) \neq a$ for any $n \in \mathbb{N}$. This implies that $a \neq \min(A \setminus \{g(1), \dots, g(n)\})$ for any $n \in \mathbb{N}$, which must mean that $a \notin A$, a contradiction. $\square$

Proof of Exercise 1.37. Let A be a countable set and let $B \subset A$ be an infinite set. By Definitions 1.35 and 1.28, there exists a bijection $f : A \to \mathbb{N}$. Now consider the set $f(B)$. Let $\tilde{f} : B \to f(B)$ be defined by $\tilde{f}(b) = f(b)$. To prove that $\tilde{f}$ is a function, Definition 1.16 tells us that it will suffice to show that for all $b \in B$, there exists a unique $c \in f(B)$ such that $\tilde{f}(b) = c$. Let b be an arbitrary element of B. It follows by Definition 1.18 that $f(b) \in f(B)$, hence $\tilde{f}(b) \in B$ by the definition of $\tilde{f}$. Furthermore, since $f(b)$ is a unique object by Definition 1.16, $\tilde{f}(B)$ is also a unique object.

To prove that $\tilde{f}$ is bijective, Definition 1.20 tells us that it will suffice to show that $\tilde{f}$ is injective and surjective. We will verify these two characteristics in turn. To prove that $\tilde{f}$ is injective, Definition 1.20 tells us that we must demonstrate that $\tilde{f}(b) = \tilde{f}(b')$ implies $b = b'$. Let $\tilde{f}(b) = \tilde{f}(b')$. By the definition of $\tilde{f}$, $\tilde{f}(b) = f(b)$ and $\tilde{f}(b') = f(b')$. Thus, $f(b) = \tilde{f}(b) = \tilde{f}(b') = f(b')$, i.e., $f(b) = f(b')$. As such, by the injectivity of f (which follows from its bijectivity by Definition 1.20), $b = b'$, as desired. To prove that $\tilde{f}$ is surjective, Definition 1.20 tells us that we must demonstrate that for all $c \in f(B)$, there exists a $b \in B$ such that $\tilde{f}(b) = c$. Let c be an arbitrary element of $f(B)$. By Definition 1.18, it follows that $c = f(b)$ for some $b \in B$. But by the definition of $\tilde{f}$, we also have $f(b) = \tilde{f}(b)$, so transitivity implies that $\tilde{f}(b) = c$, as desired.

Since B is infinite, Definitions 1.30 and 1.28 tell us that no bijection $h : B \to [n]$ exists for any $n \in \mathbb{N}$. Consequently, since there exists a bijection $\tilde{f} : B \to f(B)$, no bijection $h : f(B) \to [n]$ exists, implying by Definitions 1.28 and 1.30 that $f(B)$ is similarly infinite. In addition to being infinite, Definition 1.18 asserts that $f(B) \subset \mathbb{N}$. Thus, there exists a bijection $g : f(B) \to \mathbb{N}$ by the lemma, Definition 1.35, and Definition 1.28. It follows by Proposition 1.26 that $g \circ \tilde{f} : B \to \mathbb{N}$ is a bijection, proving that B is countable by Definitions 1.28 and 1.35, as desired. $\square$

Exercise 1.38. Prove that if there is an injection $f : A \to B$ where B is countable and A is infinite, then A is countable.

Proof. Let $\tilde{f} : A \to f(A)$ be defined by $\tilde{f}(a) = f(a)$. To prove that $\tilde{f}$ is a function, Definition 1.16 tells us that it will suffice to show that for all $a \in A$, there exists a unique $b \in f(A)$ such that $\tilde{f}(a) = b$. Let a be an arbitrary element of A. It follows by Definition 1.18 that $f(a) \in f(A)$, hence $\tilde{f}(a) \in A$ by the definition of $\tilde{f}$. Furthermore, since $f(a)$ is a unique object by Definition 1.16, $\tilde{f}(a)$ is also a unique object.

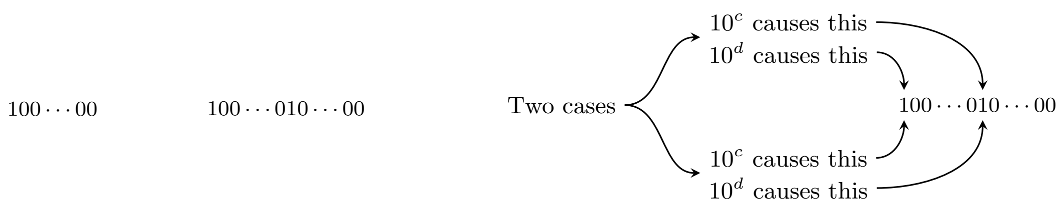
To prove that $\tilde{f}$ is bijective, Definition 1.20 tells us that it will suffice to show that $\tilde{f}$ is injective and surjective. We will verify these two characteristics in turn. To prove that $\tilde{f}$ is injective, Definition 1.20 tells us that we must demonstrate that $\tilde{f}(a) = \tilde{f}(a')$ implies $a = a'$. Let $\tilde{f}(a) = \tilde{f}(a')$. By the definition of $\tilde{f}$, $\tilde{f}(a) = f(a)$ and $\tilde{f}(a') = f(a')$. Thus, $f(a) = \tilde{f}(a) = \tilde{f}(a') = f(a')$, i.e., $f(a) = f(a')$. As such, by the injectivity of f, $a = a'$, as desired. To prove that $\tilde{f}$ is surjective, Definition 1.20 tells us that we must demonstrate that for all $b \in f(A)$, there exists an $a \in A$ such that $\tilde{f}(a) = b$. Let b be an arbitrary element of $f(A)$. By Definition 1.18, it follows that $b = f(a)$ for some $a \in A$. But by the definition of $\tilde{f}$, we also have $f(a) = \tilde{f}(a)$, so transitivity implies that $\tilde{f}(a) = b$, as desired.

Since A is infinite, Definitions 1.30 and 1.28 tell us that no bijection $h : A \to [n]$ exists for any $n \in \mathbb{N}$. Consequently, since there exists a bijection $\tilde{f} : A \to f(A)$, no bijection $h : f(A) \to [n]$ exists, implying by Definitions 1.28 and 1.30 that $f(A)$ is similarly infinite. In addition to being infinite, Definition 1.18 asserts that $f(A) \subset B$. Thus, Exercise 1.37 applies and proves that $f(A)$ is countable. It follows by Definitions 1.35 and 1.28 that there exists a bijection $g : f(A) \to \mathbb{N}$. Since $\tilde{f}$ and g are both bijective, Proposition 1.26 implies that $g \circ \tilde{f} : A \to \mathbb{N}$ is bijective. Therefore, A and $\mathbb{N}$ are in bijective correspondence by Definition 1.28, meaning that A is countable by Definition 1.35. $\square$

Exercise 1.39. Prove that $\mathbb{N} \times \mathbb{N}$ is countable by considering the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by $f(n, m) = (10^n - 1)10^m$.

Lemma (Informal[5])**.** *If $10^a + 10^b = 10^c + 10^d$ for $a, b, c, d \in \mathbb{N}$, then either $a = c$ and $b = d$, or $a = d$ and $b = c$.*

[5] Dr. Cartee approved this.



(a) 10^a. (b) $10^a + 10^b$. (c) Possible matchings of 10^c and 10^d.

Figure 1.2: Base-10 representations (ignoring the case where $a = b$).

Proof. Refer to Figure 1.2 throughout the following discussion. Think about the base-10 representation of 10^a — it will be a 1 followed by a bunch of 0s. When we add 10^b to 10^a, either one of the 0s becomes a 1, the 1 becomes a 2, or a further string consisting of a 1 (possibly followed by 0s) is concatenated to the beginning of the existing number. In any of these cases, it is clear that for this number to be written in the form $10^c + 10^d$, one of those two terms (10^c or 10^d) must account for one of the 1s, and the other for the other 1 (or both for the 2, in that case). □

Proof of Exercise 1.39. We wish to prove that f is injective, so that Exercise 1.38 applies. By Definition 1.20, proving that f is injective necessitates showing that $f(a,b) = f(c,d)$ implies that $(a,b) = (c,d)$. Suppose that

$$f(a,b) = f(c,d)$$

Substituting the definition of f and algebraically manipulating, we get

$$\begin{aligned}(10^a - 1)(10^b) &= (10^c - 1)(10^d)\\ 10^{a+b} - 10^b &= 10^{c+d} - 10^d\\ 10^{a+b} + 10^d &= 10^{c+d} + 10^b\end{aligned}$$

By the lemma, either $a + b = b$ and $c + d = d$, or $a + b = c + d$ and $b = d$. In the first case, we must have $a = 0$ and $c = 0$ for the equalities to hold. But since $0 \notin \mathbb{N}$, this implies that $a, c \notin \mathbb{N}$, a contradiction. Thus this case does not hold and it must be that the second case is true. In the second case, $b = d$, so by the cancellation law for addition, $a = c$. Since $a = c$ and $b = d$, Definition 1.15 tells us that $(a,b) = (c,d)$, as desired.

Having proven that there exists an injection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ where $\mathbb{N}$ is (clearly) countable and $\mathbb{N} \times \mathbb{N}$ is (clearly) infinite, Exercise 1.38 implies that $\mathbb{N} \times \mathbb{N}$ is countable, as desired. □

Additional Exercises

10/6: 1. In each of the following, write out the elements of the sets.

a) $(\{n \in \mathbb{Z} \mid n \text{ is divisible by } 2\} \cap \mathbb{N}) \cup \{-5\}$

Proof. The elements are -5 as well as 2, 4, 6, and every other even natural number. □

c) $\{[n] \mid n \in \mathbb{N}, 1 \leq n \leq 3\}$

Proof. The elements are the three sets $\{1\}$, $\{1,2\}$, and $\{1,2,3\}$. □

k) $\{\{a\} \cup \{b\} \mid a \in \mathbb{N}, b \in \mathbb{N}, 1 \leq a \leq 4, 3 \leq b \leq 5\}$

Proof. The elements are the 11 sets $\{1,3\}$, $\{1,4\}$, $\{1,5\}$, $\{2,3\}$, $\{2,4\}$, $\{2,5\}$, $\{3\}$, $\{3,4\}$, $\{3,5\}$, $\{4\}$, and $\{4,5\}$. □

1.2 Discussion

10/1:
- Always make sure you use all given assumptions.
- We are allowed to assume that $x \in \{A : Q\}$ tells us that $x \in A$ and Q is true? — yes.
- We can let x be an arbitrary element of a set and deduce stuff like in Tao.
- When we're writing proofs (consider Theorem 1.12), do we do not have to show the definition of $A \cap B$? — we can just say "by Definition 1.6, $y \notin A \cap B$ implies that $y \notin a$ or $y \notin b$."

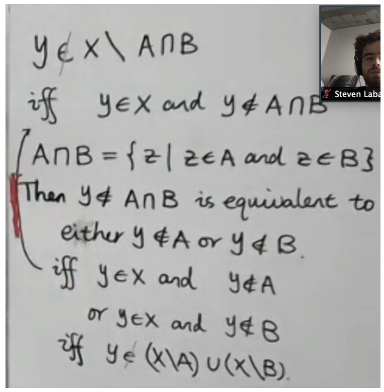


Figure 1.3: Sample exam-ready proof of Theorem 1.12.

- What he wrote for the beginning of the proof of Theorem 1.12 (see Figure 1.3) is acceptable on an exam; in our exams, it will be the same as when presenting to class (we do not need complete sentences).
- Can we say "A similar argument works in reverse?"

10/6:
- Vacuous truths were introduced.
- If you had to prove your answers to Additional Exercise 1.1, you would write out the elements of the first set, and rewrite the elements with each additional constraint.
  - For example, $(\{n \in \mathbb{Z} \mid n \text{ is divisible by } 2\} \cap \mathbb{N}) \cup \{-5\} = (\{\cdots, -4, -2, 0, 2, 4, \cdots\} \cap \{1, 2, 3, \cdots\}) \cup \{-5\} = \{2, 4, 6, \cdots\} \cup \{-5\}$.
- In this class, $0 \notin \mathbb{N}$, but $0 \in \mathbb{N}_0$.
- For Exercise 1.21, we can refer to Theorem 7 in "Notes on proofs" to demonstrate that $\sqrt{2} \notin \mathbb{N}$.
- When presenting, write on the board more like I would in a journal.
- Ask about my contradiction proofs for 1.21-1.23!

10/8:
- ! means "unique."
- $\because$ means "since."

10/13:
- What are your office hours? Mondays 4-6 PM.
- Do I need to submit the LaTeX assignment to you? Email it to him!
- Edit this document to reflect switch to section 22!
- Script 2 sign up sheet is on Canvas (sign up within 24 hours)!

10/15:
- Informal explanation of $10^a + 10^b = 10^c + 10^d$ is ok.

Script 2

The Rationals

2.1 Journal

10/15: **Definition 2.1.** Let X be a set. A **relation** R on X is a subset of $X \times X$. The statement $(x, y) \in R$ is read "x is related to y by the relation R" and is often denoted $x \sim y$.

A relation is **reflexive** if $x \sim x$ for all $x \in X$.
A relation is **symmetric** if $y \sim x$ whenever $x \sim y$.
A relation is **transitive** if $x \sim z$ whenever $x \sim y$ and $y \sim z$.
A relation is an **equivalence relation** if it is reflexive, symmetric, and transitive.

Exercise 2.2. Determine which of the following are equivalence relations.

a) Any set X with the relation $=$. So $x \sim y$ if and only if $x = y$.

Proof. To prove that the relation $=$ is reflexive, Definition 2.1 tells us that it will suffice to show that $x \sim x$ for all $x \in X$. Clearly, $x = x$ for all $x \in X$. It follows by the definition of $=$ that $x \sim x$ for all $x \in X$. For symmetry, we must verify that $x \sim y$ implies $y \sim x$ for $x, y \in X$. Let $x \sim y$ for some $x, y \in X$. Consequently, by the definition of $=$, $x = y$. It follows that $y = x$, and thus that $y \sim x$. For transitivity, we must show that $x \sim y$ and $y \sim z$ imply that $x \sim z$ for $x, y, z \in X$. Let $x \sim y$ and $y \sim z$ for some $x, y, z \in X$. By the definition of $=$, $x \sim y$ and $y \sim z$ imply that $x = y$ and $y = z$, respectively. Thus, $x = y = z$, so $x = z$, meaning that $x \sim z$ by the definition of the relation $=$. Since the relation $=$ is reflexive, symmetric, and transitive, it is an equivalence relation. □

b) $\mathbb{Z}$ with the relation $<$.

Proof. Consider $1 \in \mathbb{Z}$, and note that $1 = 1$. Since $1 = 1$, $1 \not< 1$ by the trichotomy. Thus, $1 \nsim 1$ by the relation $<$, proving that $<$ is not reflexive for all $z \in \mathbb{Z}$. Therefore, by Definition 2.1, $<$ is not an equivalence relation. □

c) Any subset X of $\mathbb{Z}$ with the relation $\leq$. So $x \sim y$ if and only if $x \leq y$.

Proof. Let $X = \{1, 2\}$. Clearly, $X \subset \mathbb{Z}$. Now, $1 \leq 2$, so $1 \sim 2$ by the relation $\leq$, but $2 \not\leq 1$ so $2 \nsim 1$. Thus, $x \sim y$ for $x, y \in X$ does not necessarily imply that $y \sim x$. It follows by Definition 2.1 that $\leq$ is not an equivalence relation on *any* subset of $\mathbb{Z}$. □

d) $X = \mathbb{Z}$ with $x \sim y$ if and only if $y - x$ is divisible by 5.

Proof. To prove that the described relation is an equivalence relation, Definition 2.1 tells us that we must verify that it is reflexive, symmetric, and transitive. To prove these properties, it will suffice to show that $x \sim x$ for all $x \in X$, $x \sim y$ implies $y \sim x$ for any $x, y \in X$, and $x \sim y$ and $y \sim z$ implies $x \sim z$ for any $x, y, z \in X$, respectively. Let's begin.

To prove that $x \sim x$ for all $x \in X$, the definition of $\sim$ and Additional Exercise 0.8 tell us that it will suffice to show that $x - x = 5a$ for an arbitrary $x \in X$ and some $a \in \mathbb{Z}$. Let x be an arbitrary element of X. It follows that $x - x = 0 = 5(0)$ where $0 = a$ is clearly an element of $\mathbb{Z}$. In sum, $x - x = 5a$ for an $a \in \mathbb{Z}$, as desired.

To prove that $x \sim y$ implies that $y \sim x$, the definition of $\sim$ tells us that it will suffice to show that $x-y$ is divisible by 5 given that $y - x$ is so divisible for $x, y \in X$. Let $y - x$ be divisible by 5. It follows by Additional Exercise 0.8-ii that $-1 \cdot (y-x)$ is divisible by 5 (since $-1 \in \mathbb{Z}$). But since $-(y-x) = x-y$, this means that $x - y$ is divisible by 5, as desired.

To prove that $x \sim y$ and $y \sim z$ imply that $x \sim z$, the definition of $\sim$ tells us that it will suffice to show that $z - x$ is divisible by 5 given that $y - x$ and $z - y$ are also so divisible for $x, y, z \in X$. Let $y - x$ and $z - y$ be divisible by 5. It follows by Additional Exercise 0.8-i that $(z - y) + (y - x)$ is divisible by 5. But since $(z - y) + (y - x) = z - x$, this means that $z - x$ is divisible by 5, as desired. □

e) $X = \{(a, b) \mid a, b \in \mathbb{Z}, b \neq 0\}$ with the relation $\sim$ defined by $(a, b) \sim (c, d) \iff ad = bc$.

Proof. Reflexivity: Let (a, b) be an arbitrary element of X. Since $a, b \in \mathbb{Z}$ and integer multiplication is commutative, it is true that $ab = ba$. Therefore, by the definition of the relation $\sim$, $(a, b) \sim (a, b)$.

Symmetry: Let $(a, b) \sim (c, d)$ for some $(a, b), (c, d) \in X$. By the definition of the relation $\sim$, $ad = bc$. Thus, $cb = da$ by the symmetry of $=$ (see Exercise 2.2a) and the commutativity of integer multiplication. Therefore, by the definition of the relation $\sim$, $(c, d) \sim (a, b)$.

Transitivity: Let $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$ for some $(a, b), (c, d), (e, f) \in X$. By consecutive applications of the definition of $\sim$, $ad = bc$ and $cf = de$. We now divide into two cases ($c \neq 0$ and $c = 0$; the reason for doing so will become clear later). Suppose first that $c \neq 0$. By the multiplicative property of equality, we can multiply $cf = de$ to both sides, creating the equation $ad \cdot cf = bc \cdot de$. By the commutativity of multiplication, we have $afcd = becd$. Since $c \neq 0$ by assumption and $d \neq 0$ by the definition of X, $cd \neq 0$ and the cancellation law for multiplication applies, giving us $af = be$. Therefore, by the definition of the relation $\sim$, $(a, b) \sim (e, f)$. Now suppose that $c = 0$. Consequently, $bc = 0$, implying by the equality $ad = bc$ that $ad = 0$. Thus, $a = 0$ or $d = 0$ (or both) by the zero product property. Since $d \neq 0$ by the definition of X (d is the second element in the ordered pair (c, d)), we must have $a = 0$. A similar analysis can be performed on the equation $cf = de$ to show that $e = 0$. Since $a = 0$ and $e = 0$, $af = 0$ and $be = 0$, implying by transitivity that $af = be$. Therefore, by the definition of the relation $\sim$, $(a, b) \sim (e, f)$.

Since the relation $\sim$ is reflexive, symmetric, and transitive, Definition 2.1 tells us that it is an equivalence relation. □

Remark 2.3. A **partition** of a set is a collection of non-empty disjoint subsets whose union is the original set. Any equivalence relation on a set creates a partition of that set by collecting into subsets all of the elements that are equivalent (related) to each other. When the partition of a set arises from an equivalence relation in this manner, the subsets are referred to as **equivalence classes**.

Remark 2.4. If we think of the set X in Exercise 2.2e as representing the collection of all fractions whose denominators are not zero, then the relation $\sim$ may be thought of as representing the equivalence of two fractions.

Definition 2.5. As a set, the **rational numbers**, denoted $\mathbb{Q}$, are the equivalence classes in the set $X = \{(a, b) \mid a, b \in \mathbb{Z}, b \neq 0\}$ under the equivalence relation $\sim$ as defined in Exercise 2.2e. If $(a, b) \in X$, we denote the equivalence class of this element as $\left[\frac{a}{b}\right]$. So

$$\left[\frac{a}{b}\right] = \{(x_1, x_2) \in X \mid (x_1, x_2) \sim (a, b)\} = \{(x_1, x_2) \in X \mid x_1 b = x_2 a\}$$

Then,

$$\mathbb{Q} = \left\{ \left[\frac{a}{b}\right] \middle| (a,b) \in X \right\}$$

Exercise 2.6. $\left[\frac{a}{b}\right] = \left[\frac{a'}{b'}\right] \iff (a,b) \sim (a',b')$

Proof. Suppose first that $\left[\frac{a}{b}\right] = \left[\frac{a'}{b'}\right]$. Since $\left[\frac{a}{b}\right] \in \mathbb{Q}$, Definition 2.5 implies that $(a,b) \in X$. It follows by Exercise 2.2e that $(a,b) \sim (a,b)$. The last two results imply by Definition 2.5 that $(a,b) \in \left[\frac{a}{b}\right]$. Consequently, set equality implies that $(a,b) \in \left[\frac{a'}{b'}\right]$. But by Definition 2.5, this means that $(a,b) \sim (a',b')$, as desired. Now suppose that $(a,b) \sim (a',b')$. To prove that $\left[\frac{a}{b}\right] = \left[\frac{a'}{b'}\right]$, Definition 1.2 tells us that we must verify that every element of $\left[\frac{a}{b}\right]$ is an element of $\left[\frac{a'}{b'}\right]$ and vice versa. Let (x_1,x_2) be an arbitrary element of $\left[\frac{a}{b}\right]$. It follows by Definition 2.5 that $(x_1,x_2) \in X$ and that $(x_1,x_2) \sim (a,b)$. The latter result combined with the hypothesis that $(a,b) \sim (a',b')$ implies by the transitivity of $\sim$ (see Exercise 2.2e) that $(x_1,x_2) \sim (a',b')$. This new finding coupled with the fact that $(x_1,x_2) \in X$ implies by Definition 2.5 that $(x_1,x_2) \in \left[\frac{a'}{b'}\right]$, as desired. The proof is symmetric if we first let that (x_1,x_2) be an arbitrary element of $\left[\frac{a'}{b'}\right]$. ☐

Definition 2.7. We define the binary operations addition and multiplication on $\mathbb{Q}$ as follows. If $\left[\frac{a}{b}\right], \left[\frac{c}{d}\right] \in \mathbb{Q}$, then

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{ad+bc}{bd}\right]$$
$$\left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{ac}{bd}\right]$$

We use the notation $+_{\mathbb{Q}}$ and $\cdot_{\mathbb{Q}}$ to represent addition and multiplication in $\mathbb{Q}$ so as to distinguish these operations from the usual addition $(+)$ and multiplication $(\cdot)$ in $\mathbb{Z}$.

Theorem 2.8. *Addition in $\mathbb{Q}$ is well-defined. That is, if $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$, then*

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{a'}{b'}\right] +_{\mathbb{Q}} \left[\frac{c'}{d'}\right]$$

Proof. By consecutive applications of the definition of $\sim$, we have from the hypotheses that

$$ab' = ba' \qquad\qquad cd' = dc'$$

It follows by the multiplicative property of equality that

$$ab'dd' = ba'dd' \qquad\qquad bb'cd' = bb'dc'$$

The above two results can be combined via the additive property of equality, giving the following, which will be algebraically manipulated further.

$$\begin{aligned} ab'dd' + bb'cd' &= ba'dd' + bb'dc' \\ adb'd' + bcb'd' &= bda'd' + bdb'c' \\ (ad+bc)(b'd') &= (bd)(a'd' + b'c') \end{aligned}$$

The last line above implies by the definition of $\sim$ that $(ad+bc, bd) \sim (a'd'+b'c', b'd')$. It follows by Exercise 2.6 that

$$\left[\frac{ad+bc}{bd}\right] = \left[\frac{a'd'+b'c'}{b'd'}\right]$$

Therefore, by two applications of Definition 2.7,

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{a'}{b'}\right] +_{\mathbb{Q}} \left[\frac{c'}{d'}\right]$$

as desired. ☐

Theorem 2.9. *Multiplication in $\mathbb{Q}$ is well-defined. That is, if $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$, then*

$$\left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{a'}{b'}\right] \cdot_{\mathbb{Q}} \left[\frac{c'}{d'}\right]$$

Proof. By consecutive applications of the definition of $\sim$, we have from the hypotheses that $ab' = ba'$ and $cd' = dc'$. Thus, by the multiplicative property of equality, $ab'cd' = ba'dc'$. This can be algebraically rearranged into $(ac)(b'd') = (bd)(a'c')$. It follows by the definition of $\sim$ that $(ac, bd) \sim (a'c', b'd')$. But this implies by Exercise 2.6 that

$$\left[\frac{ac}{bd}\right] = \left[\frac{a'c'}{b'd'}\right]$$

Consequently, by Definition 2.7,

$$\left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{a'}{b'}\right] \cdot_{\mathbb{Q}} \left[\frac{c'}{d'}\right]$$

as desired. □

Theorem 2.10.

a) *Commutativity of addition*

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{c}{d}\right] +_{\mathbb{Q}} \left[\frac{a}{b}\right] \text{ for all } \left[\frac{a}{b}\right], \left[\frac{c}{d}\right] \in \mathbb{Q}$$

Proof. By Definition 2.7,

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{ad+bc}{bd}\right]$$

With integer algebra, we can rearrange the above expression into

$$= \left[\frac{cb+da}{db}\right]$$

By Definition 2.7 again, the above

$$= \left[\frac{c}{d}\right] +_{\mathbb{Q}} \left[\frac{a}{b}\right]$$

□

b) *Associativity of addition*

$$\left(\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{c}{d}\right]\right) +_{\mathbb{Q}} \left[\frac{e}{f}\right] = \left[\frac{a}{b}\right] +_{\mathbb{Q}} \left(\left[\frac{c}{d}\right] +_{\mathbb{Q}} \left[\frac{e}{f}\right]\right) \text{ for all } \left[\frac{a}{b}\right], \left[\frac{c}{d}\right], \left[\frac{e}{f}\right] \in \mathbb{Q}$$

Proof. By consecutive applications of Definition 2.7,

$$\begin{aligned}\left(\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{c}{d}\right]\right) +_{\mathbb{Q}} \left[\frac{e}{f}\right] &= \left[\frac{ad+bc}{bd}\right] +_{\mathbb{Q}} \left[\frac{e}{f}\right] \\ &= \left[\frac{(ad+bc)(f)+(bd)(e)}{(bd)(f)}\right]\end{aligned}$$

With integer algebra, we can rearrange the above as follows.

$$\begin{aligned}&= \left[\frac{adf+bcf+bde}{bdf}\right] \\ &= \left[\frac{(a)(df)+(b)(cf+de)}{(b)(df)}\right]\end{aligned}$$

Now apply Definition 2.7 twice, again.

$$= \left[\frac{a}{b}\right] +_{\mathbb{Q}} \left(\left[\frac{cf+de}{df}\right]\right)$$
$$= \left[\frac{a}{b}\right] +_{\mathbb{Q}} \left(\left[\frac{c}{d}\right] +_{\mathbb{Q}} \left[\frac{e}{f}\right]\right)$$

□

c) *Existence of an additive identity*

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{0}{1}\right] = \left[\frac{a}{b}\right] \textit{ for all } \left[\frac{a}{b}\right] \in \mathbb{Q}$$

Proof. Via Definition 2.7 and integer algebra, we can show that

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{0}{1}\right] = \left[\frac{a\cdot 1 + b\cdot 0}{b\cdot 1}\right]$$
$$= \left[\frac{a}{b}\right]$$

as desired. □

d) *Existence of additive inverses*

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{-a}{b}\right] = \left[\frac{0}{1}\right] \textit{ for all } \left[\frac{a}{b}\right] \in \mathbb{Q}$$

Proof. Through various application of Definition 2.7 and integer algebra, we can show that

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{-a}{b}\right] = \left[\frac{ab + b\cdot -a}{bb}\right]$$
$$= \left[\frac{ab - ab}{bb}\right]$$
$$= \left[\frac{0}{bb}\right]$$

Since $0 \cdot 1 = 0$ and $bb \cdot 0 = 0$, transitivity implies that $(0)(1) = (bb)(0)$. By the definition of $\sim$, this means that $(0, bb) \sim (0, 1)$. It follows by Exercise 2.6 that the above equals the following, as desired.

$$= \left[\frac{0}{1}\right]$$

□

e) *Commutativity of multiplication*

$$\left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{c}{d}\right] \cdot_{\mathbb{Q}} \left[\frac{a}{b}\right] \textit{ for all } \left[\frac{a}{b}\right], \left[\frac{c}{d}\right] \in \mathbb{Q}$$

Proof. Via Definition 2.7 and integer algebra, we can show that

$$\left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{ac}{bd}\right]$$
$$= \left[\frac{ca}{db}\right]$$
$$= \left[\frac{c}{d}\right] \cdot_{\mathbb{Q}} \left[\frac{a}{b}\right]$$

as desired. □

f) *Associativity of multiplication*

$$\left(\left[\frac{a}{b}\right]\cdot_{\mathbb{Q}}\left[\frac{c}{d}\right]\right)\cdot_{\mathbb{Q}}\left[\frac{e}{f}\right]=\left[\frac{a}{b}\right]\cdot_{\mathbb{Q}}\left(\left[\frac{c}{d}\right]\cdot_{\mathbb{Q}}\left[\frac{e}{f}\right]\right)\text{ for all }\left[\frac{a}{b}\right],\left[\frac{c}{d}\right],\left[\frac{e}{f}\right]\in\mathbb{Q}$$

Proof. Through various application of Definition 2.7 and integer algebra, we can show that

$$\begin{aligned}\left(\left[\frac{a}{b}\right]\cdot_{\mathbb{Q}}\left[\frac{c}{d}\right]\right)\cdot_{\mathbb{Q}}\left[\frac{e}{f}\right]&=\left[\frac{ac}{bd}\right]\cdot_{\mathbb{Q}}\left[\frac{e}{f}\right]\\&=\left[\frac{(ac)(e)}{(bd)(f)}\right]\\&=\left[\frac{(a)(ce)}{(b)(df)}\right]\\&=\left[\frac{a}{b}\right]\cdot_{\mathbb{Q}}\left(\left[\frac{ce}{df}\right]\right)\\&=\left[\frac{a}{b}\right]\cdot_{\mathbb{Q}}\left(\left[\frac{c}{d}\right]\cdot_{\mathbb{Q}}\left[\frac{e}{f}\right]\right)\end{aligned}$$

as desired. □

g) *Existence of a multiplicative identity*

$$\left[\frac{a}{b}\right]\cdot_{\mathbb{Q}}\left[\frac{1}{1}\right]=\left[\frac{a}{b}\right]\text{ for all }\left[\frac{a}{b}\right]\in\mathbb{Q}$$

Proof. Via Definition 2.7 and integer algebra, we can show that

$$\begin{aligned}\left[\frac{a}{b}\right]\cdot_{\mathbb{Q}}\left[\frac{1}{1}\right]&=\left[\frac{a\cdot 1}{b\cdot 1}\right]\\&=\left[\frac{a}{b}\right]\end{aligned}$$

as desired. □

h) *Existence of multiplicative inverses for nonzero elements*

$$\left[\frac{a}{b}\right]\cdot_{\mathbb{Q}}\left[\frac{b}{a}\right]=\left[\frac{1}{1}\right]\text{ for all }\left[\frac{a}{b}\right]\in\mathbb{Q}\text{ such that }\left[\frac{a}{b}\right]\neq\left[\frac{0}{1}\right]$$

Proof. By Definition 2.7,

$$\left[\frac{a}{b}\right]\cdot_{\mathbb{Q}}\left[\frac{b}{a}\right]=\left[\frac{ab}{ba}\right]$$

Since $(ab)(1)=(ba)(1)$, we have by the definition of $\sim$ that $(ab,ba)\sim(1,1)$. It follows by Exercise 2.6 that the above equals the following, as desired.

$$=\left[\frac{1}{1}\right]$$

□

i) *Distributivity*

$$\left[\frac{a}{b}\right]\cdot_{\mathbb{Q}}\left(\left[\frac{c}{d}\right]+_{\mathbb{Q}}\left[\frac{e}{f}\right]\right)=\left(\left[\frac{a}{b}\right]\cdot_{\mathbb{Q}}\left[\frac{c}{d}\right]\right)+_{\mathbb{Q}}\left(\left[\frac{a}{b}\right]\cdot_{\mathbb{Q}}\left[\frac{e}{f}\right]\right)\text{ for all }\left[\frac{a}{b}\right],\left[\frac{c}{d}\right],\left[\frac{e}{f}\right]\in\mathbb{Q}$$

Proof. By Definition 2.7 and integer algebra,

$$\begin{aligned}\left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left(\left[\frac{c}{d}\right] +_{\mathbb{Q}} \left[\frac{e}{f}\right]\right) &= \left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left[\frac{cf+de}{df}\right] \\ &= \left[\frac{a(cf+de)}{bdf}\right] \\ &= \left[\frac{acf+ade}{bdf}\right]\end{aligned}$$

Use Theorem 2.10g.

$$= \left[\frac{acf+ade}{bdf}\right] \cdot_{\mathbb{Q}} \left[\frac{1}{1}\right]$$

Use the lemma from the proof of Theorem 2.10h.

$$= \left[\frac{acf+ade}{bdf}\right] \cdot_{\mathbb{Q}} \left[\frac{b}{b}\right]$$

Use various applications of Definition 2.7 and integer algebra to finish.

$$\begin{aligned}&= \left[\frac{(acf+ade)b}{(bdf)b}\right] \\ &= \left[\frac{acfb+adeb}{bdfb}\right] \\ &= \left[\frac{(ac)(bf)+(bd)(ae)}{(bd)(bf)}\right] \\ &= \left[\frac{ac}{bd}\right] +_{\mathbb{Q}} \left[\frac{ae}{bf}\right] \\ &= \left(\left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left[\frac{c}{d}\right]\right) +_{\mathbb{Q}} \left(\left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left[\frac{e}{f}\right]\right)\end{aligned}$$

as desired. □

10/20: **Theorem 2.11.** $\mathbb{Q}$ *is countable.*

Lemma.

a) *If there exists a surjection $g : B \to A$, then there exists an injection $f : A \to B$.*

b) *The set $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ is countable.*

Proof of a. Let $f : A \to B$ be defined such that for all $a \in A$, $f(a) \in g^{-1}(\{a\})$[1]. To prove that this condition is well-defined (i.e., there is no $a \in A$ such that $f(a)$ *cannot* be an element of $g^{-1}(\{a\})$), we will show that for all $a \in A$, $g^{-1}(\{a\})$ contains at least one element of B. Let a be an arbitrary element of A. Since g is surjective, we know by Definition 1.20 that there is a $b \in B$ such that $g(b) = a$. Let's consider this b more closely. As an element of B satisfying the condition that $g(b) = a$, b is naturally an element of the set $\{b' \in B \mid g(b') = a\}$. Clearly, this set is equivalent to $\{b' \in B \mid g(b') \in \{a\}\}$, so b is also an element of this new set. But by Definition 1.18, this set is equal to $g^{-1}(\{a\})$. Thus, $b \in B$ and $b \in g^{-1}(\{a\})$, as desired.

To prove that f is injective, Definition 1.20 tells us that it will suffice to show that $f(a) = f(a')$ implies that $a = a'$. Let $f(a) = f(a')$. It follows by the condition imposed on f that $f(a) \in g^{-1}(\{a\})$ and $f(a') \in g^{-1}(\{a'\})$. With respect to the latter case, the fact that $f(a) = f(a')$ also implies that $f(a) \in g^{-1}(\{a'\})$. Because $f(a) \in g^{-1}(\{a\})$ and $f(a) \in g^{-1}(\{a'\})$, Definition 1.18 tells us that $g(f(a)) \in \{a\}$ and $g(f(a)) \in \{a'\}$, respectively. Consequently, $g(f(a)) = a$ and $g(f(a)) = a'$, respectively. Since g is a function, Definition 1.16 implies that $g(f(a))$ is a unique, well-defined object, so $a = g(f(a)) = a'$, i.e., $a = a'$, as desired. □

[1] Note that we are not defining f explicitly, but rather providing a rule that means that some matchings will not suffice to define f, namely ones for which $f(a) \notin g^{-1}(\{a\})$ for all $a \in A$.

Proof of b. By Exercise 1.36, $\mathbb{Z}$ is countable, i.e.[2], there exists a bijection $f_1 : \mathbb{Z} \to \mathbb{N}$. Since $\mathbb{Z} \setminus \{0\} \subset \mathbb{Z}$, Exercise 1.37 implies that $\mathbb{Z} \setminus \{0\}$ is countable, i.e., there exists a bijection $f_2 : \mathbb{Z} \setminus \{0\} \to \mathbb{N}$. Now let $f : \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \to \mathbb{N} \times \mathbb{N}$ be defined by $f(a,b) = (f_1(a), f_2(b))$. To prove that f is a function, Definition 1.16 tells us that we must show that for every $(a,b) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$, there is a unique $(c,d) \in \mathbb{N} \times \mathbb{N}$ such that $f(a,b) = (c,d)$. Let (a,b) be an arbitrary element of $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. Then by Definition 1.15, $a \in \mathbb{Z}$ and $b \in \mathbb{Z} \setminus \{0\}$. Thus, by the definitions of f_1 and f_2, $f_1(a)$ and $f_2(b)$ are defined objects and elements of $\mathbb{N}$. Consequently, Definition 1.15 implies that $(f_1(a), f_2(a)) \in \mathbb{N} \times \mathbb{N}$. Since $f(a,b) = (f_1(a), f_2(b))$ by the definition of f, it follows that $(f_1(a), f_2(b))$ is *an* element of $\mathbb{N} \times \mathbb{N}$ to which f maps (a,b). On the uniqueness of this object, suppose that $f(a,b) = (c,d)$ and $f(a,b) = (c',d')$ for some $(a,b) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. By the definition of f, this implies that $(f_1(a), f_2(b)) = (c,d)$ and $(f_1(a), f_2(b)) = (c',d')$. Thus, by multiple applications of Definition 1.15, $f_1(a) = c$, $f_1(a) = c'$, $f_2(b) = d$, and $f_2(b) = d'$. But since f_1 and f_2 are both functions, Definition 1.16 implies that $f_1(a)$ and $f_2(b)$ are both unique, well-defined objects. Consequently, transitivity applies and implies that $c = f_1(a) = c'$ and $d = f_2(b) = d'$. It follows by Definition 1.15 once again that $(c,d) = (c',d')$, as desired.

To prove that f is injective, Definition 1.20 tells us that we must verify that $f(a,b) = f(a',b')$ implies that $(a,b) = (a',b')$. Let $f(a,b) = f(a',b')$. By the definition of f, $(f_1(a), f_2(b)) = (f_1(a'), f_2(b'))$. Thus, by Definition 1.15, $f_1(a) = f_1(a')$ and $f_2(b) = f_2(b')$. Consequently, by the injectivity of f_1 and f_2 (which follows from their respective bijectivity by Definition 1.20), $a = a'$ and $b = b'$. Therefore, by Definition 1.15 once again, $(a,b) = (a',b')$.

By Exercise 1.39, $\mathbb{N} \times \mathbb{N}$ is countable, i.e., there exists a bijection $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Since g is bijective, Definition 1.20 implies that it is injective. Thus, by Proposition 1.26, $g \circ f : \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \to \mathbb{N}$ (which Definition 1.25 guarantees exists) is injective.

Since there exists an injection $g \circ f : \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \to \mathbb{N}$ where $\mathbb{N}$ is clearly countable and $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ is clearly infinite, Exercise 1.38 implies that $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ is countable, as desired. ☐

Proof of Theorem 2.11. Let $g : \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \to \mathbb{Q}$ be defined by $g(a,b) = \left[\frac{a}{b}\right]$. For g to be a function as defined by Definition 1.16, g must map every ordered pair $(a,b) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ to a unique element $\left[\frac{a}{b}\right]$ in $\mathbb{Q}$. Let (a,b) be an arbitrary element of $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. By definition, g clearly maps (a,b) to only one (i.e., a unique) object, namely the equivalence class $\left[\frac{a}{b}\right]$. But we must still show that this $\left[\frac{a}{b}\right]$ is an element of $\mathbb{Q}$ (note that this is not immediately obvious as equivalence classes such as $\left[\frac{0}{0}\right]$ [which is still a well-defined equivalence class, just an empty one] are not elements of $\mathbb{Q}$). For $\left[\frac{a}{b}\right]$ to be an element of $\mathbb{Q}$, Definition 2.5 tells us that it will suffice to show that $(a,b) \in X$. For (a,b) to be an element of X, Exercise 2.2e asserts that it will suffice to show that $a,b \in \mathbb{Z}$ and $b \neq 0$. But since $(a,b) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ by assumption, Definition 1.15 tells us that $a \in \mathbb{Z}$ and $b \in \mathbb{Z} \setminus \{0\}$. Expounding on the latter result, Definition 1.11 tells us that $b \in \mathbb{Z}$ and $b \notin \{0\}$, i.e., $b \in \mathbb{Z}$ and $b \neq 0$. Combining the last three results, we have that $a,b \in \mathbb{Z}$ and $b \neq 0$, as desired.

To prove that g is surjective, Definition 1.20 tells us that we must verify that for all $\left[\frac{a}{b}\right] \in \mathbb{Q}$, there exists an $(a,b) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ such that $g(a,b) = \left[\frac{a}{b}\right]$. Let $\left[\frac{a}{b}\right]$ be an arbitrary element of $\mathbb{Q}$. By Definition 2.5, $(a,b) \in X$. Thus, by Exercise 2.2e, $a,b \in \mathbb{Z}$ and $b \neq 0$. Since $b \in \mathbb{Z}$ and $b \neq 0$, i.e., $b \in \mathbb{Z}$ and $b \notin \{0\}$, Definition 1.11 tells us that $b \in \mathbb{Z} \setminus \{0\}$. To recap, $a \in \mathbb{Z}$ and $b \in \mathbb{Z} \setminus \{0\}$. But by Definition 1.15, this implies that $(a,b) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. With regards to this (a,b), we have by the definition of g that $g(a,b) = \left[\frac{a}{b}\right]$, as desired.

Since there exists a surjection $g : \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \to \mathbb{Q}$, we have by Lemma (a) that there exists an injection $f : \mathbb{Q} \to \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$. Thus, we have an injection $f : \mathbb{Q} \to \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ where $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ is countable (by Lemma (b)) and $\mathbb{Q}$ is clearly infinite. By Exercise 1.38, this means that $\mathbb{Q}$ is countable. ☐

2.2 Discussion

10/15: • Divisibility (see Exercise 2.2d) is defined in terms of multiplication in Script 0. Thus, I should use $y - x = 5a$ for some $a \in X$ instead of $\frac{y-x}{5} \in X$.

[2]Let A be a set (such as $\mathbb{Z}$). Technically, Definition 1.35 must be invoked to move from "A is countable" to "A is in bijective correspondence with $\mathbb{N}$," and Definition 1.28 must be invoked to move from "A is in bijective correspondence with $\mathbb{N}$" to "there exists a bijection $f : A \to \mathbb{N}$." However, as we are no longer in Script 1, such justifications will not be supplied beyond this footnote.

- At the end of the proof of Exercise 2.2e, do casework on c ($c = 0$ and $c \neq 0$) to stymie accusations of dividing by 0.
- Exercise 2.6:
  - Note that we cannot always pick an arbitrary element of a set (e.g., if a set is empty). However, by definition, $\left[\frac{a}{b}\right]$ is nonempty, so we *can* pick an arbitrary element in it.
    - Additionally, note that we are probably fine not getting into this technicality because performing any operation on a hypothetical $x \in \emptyset$ invokes vacuous (i.e., weird but still sound) logic.
  - Let (x_1, x_2) be an arbitrary element of the left set. It is therefore related to (a, b). By set equality, it is an element of the right set. Thus, $(x_1, x_2) \sim (a', b')$. Use symmetry/transitivity to show $(a, b) \sim (a', b')$.
- Rewrite Theorem 2.8 as a direct proof.
- Journal format: Turning in what I have in this document is fine.

10/20:
- Edit Exercise 2.2d to use Additional Exercise 0.8-i!
- Do we have to prove injectivity for h in Theorem 2.11? (Injectivity implies bijectivity by Exercise 1.38.)
  - Yes, we still have to prove injectivity here.
- Technically, you have to prove that $\mathbb{N}$ is countable, but Cartee won't be looking for it — you won't have to prove it for credit.
- You can also use a lemma that a surjection $f : A \to B$ where A is countable and B is infinite implies that B is countable.
- $\gcd(a, b)$ is defined as the largest natural number by which a and b are both divisible.
- We want to pick the pair in the equivalence class that has the lowest possible, positive denominator. Think about it in terms of the well-ordering principle. You get a huge set of pairs, look at the subset with positive denominators, then look at just the denominators. You would have to explain why one denominator corresponds to one ordered pair.
- Use $f(\left[\frac{a}{b}\right]) = (a', b')$ to denote that $a \neq a'$ and $b \neq b'$ in every case although the numbers are related.
- Two things in the same equivalence class go to the same place, whereas two things in different equivalent classes go to different places.
- We're restricting the range with the well-ordering principle, not the domain.

Script 3

Introducing a Continuum

3.1 Journal

10/20: **Axiom 1.** *A continuum is a nonempty set C.*

Definition 3.1. Let X be a set. An **ordering** on the set X is a subset $<$ of $X \times X$ with elements $(x, y) \in <$ written as $x < y$, satisfying the following properties:

a) (*Trichotomy*) For all $x, y \in X$, exactly one of the following holds: $x < y$, $y < x$, or $x = y$.

b) (*Transitivity*) For all $x, y, z \in X$, if $x < y$ and $y < z$, then $x < z$.

Remark 3.2.

a) In mathematics, "or" is understood to be inclusive unless stated otherwise. So in Definition 3.1a above, the word "exactly" is needed.

b) $x < y$ may also be written as $y > x$.

c) By $x \leq y$, we mean $x < y$ or $x = y$; similarly for $x \geq y$.

d) We often refer to elements of a continuum C as **points**.

Axiom 2. *A continuum C has an ordering $<$.*

Definition 3.3. If $A \subset C$, then a point $a \in A$ is a **first** point of A if for every element $x \in A$, either $a < x$ or $a = x$. Similarly, a point $b \in A$ is called a **last** point of A if, for every $x \in A$, either $x < b$ or $x = b$.

Lemma 3.4. *If A is a nonempty, finite subset of a continuum C, then A has a first and last point.*

Lemma. *Let A be a nonempty, finite subset of a continuum C, let a be any element of A, and let $B = A \backslash \{a\}$. Then $|B| = |A| - 1$.*

Proof. We first prove that $|\{a\}| = 1$. By Definition 1.33, to do so, it will suffice to find a bijection $f : \{a\} \to [1]$. Since $[1] = \{1\}$ by Definition 1.29, $f : \{a\} \to [1]$ defined by $f(a) = 1$ is clearly such a bijection.

We now note that $B \cap \{a\} = (A \setminus \{a\}) \cap \{a\} = \emptyset$ and $B \cup \{a\} = (A \setminus \{a\}) \cup \{a\} = A$; these results imply by Theorem 1.34b that $|A| = |B| + |\{a\}|$. But since $|\{a\}| = 1$, it follows that $|A| = |B| + 1$, i.e., $|B| = |A| - 1$. □

Proof of Lemma 3.4. We consider first points herein (the proof is symmetric for last points). If A is a nonempty, finite set, then by Definition 1.30, $|A| = n$ for some $n \in \mathbb{N}$. Thus, if we prove the claim for each $n \in \mathbb{N}$ individually, we will have proven the claim for all $n \in \mathbb{N}$, i.e., for all nonempty, finite sets A. Logically, to prove a property pertaining to any natural number, we induct on n.

For the base case $n = 1$, there is only one element (which we may call a) in A. Since $a = a$, i.e., "for every $x \in A$, either $a < x$ or $a = x$" is a true statement, it follows by Definition 3.3 that A has a first point. Now suppose inductively that we have proven the claim for n, i.e., we know that if A is a nonempty, finite subset of a continuum C with $|A| = n$, then A has a first point. We wish to prove the same claim if $|A| = n + 1$. Let a be an arbitrary element of A, and consider the set $B = A \setminus \{a\}$. By the lemma, $|B| = (n+1) - 1 = n$. Consequently, the induction hypothesis applies and asserts that B has a first point a_0. Clearly, a_0 is also an element of A, but it may or may not be the first point of A (the first point may now be a). Since C has an ordering $<$ (see Axiom 2) and $A \subset C$, Definition 3.1 asserts that either $a < a_0$, $a_0 < a$, or $a = a_0$. We now divide into three cases. If $a < a_0$, then since $a_0 \leq x$ for all $x \in A$ by Definition 3.3, Definition 3.1 implies that $a \leq x$ for all $x \in A$. Thus, by Definition 3.3, a is the first point in A, and we have proven the claim for $|A| = n + 1$ in this case. If $a_0 < a$, then it is still true that $a_0 \leq x$ for all $x \in A$. This means by Definition 3.3 that a_0 is still the first point in A, proving the claim for $|A| = n + 1$ in this case. If $a = a_0$, then $a \in B$ (since $a_0 \in B$), contradicting the fact that $B = A \setminus \{a\}$, so we need not consider this final case (the contradiction proves that it will never arise). This closes the induction. ☐

Theorem 3.5. *Suppose that A is a set of n distinct points in a continuum C, or in other words, $A \subset C$ has cardinality n. Then the symbols $a_1, \dots, a_n$ may be assigned to each point of A so that $a_1 < a_2 < \cdots < a_n$, i.e., $a_i < a_{i+1}$ for all $1 \leq i \leq n - 1$.*

Proof. We divide into two cases ($|A| = 0$ and $|A| \in \mathbb{N}$).

If $|A| = 0$, then the statements "the symbols $a_1, \dots, a_n$ may be assigned to each point of A" and "$a_i < a_{i+1}$ for all $1 \leq i \leq n - 1 = -1$" are both vacuously true.

If $|A| \in \mathbb{N}$, we induct on $|A| = n$. For the base case $n = 1$, denote the single element of A by a_1. Since $a_i < a_{i+1}$ for all $1 \leq i \leq n - 1 = 0$ is vacuously true, the base case holds. Now suppose inductively that we have proven the claim for n, i.e., for a set $A \subset C$ satisfying $|A| = n$, the symbols $a_1, \dots, a_n$ may be assigned to each point of A so that $a_1 < a_2 < \cdots < a_n$. We now wish to prove the claim with regard to a set $A \subset C$ with $|A| = n + 1$. By Lemma 3.4, there is a last point $a_{n+1} \in A$, which may be denoted as such (we will rigorously confirm this later). Since the set $A \setminus \{a_{n+1}\}$ has cardinality n (see the lemma from Lemma 3.4), we have by the induction hypothesis that its n elements can be named $a_1, \dots, a_n$ and ordered $a_1 < a_2 < \cdots < a_n$. Clearly these n elements are elements of A and all that's left to do is determine where a_{n+1} fits into the established order. But by Definition 3.3, $x \leq a_{n+1}$ for all $x \in A$, i.e., $x < a_{n+1}$ for all $x \in A \setminus \{a_{n+1}\}$. Consequently, as its name would suggest, it is true that $a_1 < a_2 < \cdots < a_n < a_{n+1}$, as desired. ☐

Definition 3.6. If $x, y, z \in C$ and either (i) both $x < y$ and $y < z$ or (ii) both $z < y$ and $y < x$, then we say that y is **between** x and z.

Corollary 3.7. *Of three distinct points in a continuum, one must be between the other two.*

Proof. Let A be a subset of the described continuum containing the three distinct points. It follows by Theorem 3.5 that the symbols a_1, a_2, a_3 may be assigned to each point of A so that $a_1 < a_2 < a_3$. Thus, $a_1 < a_2$ and $a_2 < a_3$, so a_2 is between a_1 and a_3 by Definition 3.6. ☐

10/22: **Axiom 3.** *A continuum C has no first or last point.*

Definition 3.8. We define an ordering on $\mathbb{Z}$ by $m < n$ if $n = m + c$ for some $c \in \mathbb{N}$.

Exercise 3.9.

a) Prove that with this ordering $\mathbb{Z}$ satisfies Axioms 1-3.

Proof. Clearly, $\mathbb{Z}$ is a nonempty set, so Axiom 1 is immediately satisfied.

Axiom 2 asserts that $\mathbb{Z}$ must have an ordering $<$. As such, it will suffice to verify that the ordering given by Definition 3.8 satisfies the stipulations of Definition 3.1. To prove that $<$ satisfies the trichotomy, it will suffice to show that for all $x, y \in \mathbb{Z}$, exactly one of the following holds: $x < y$, $y < x$, or $x = y$.

We first show that *no more than one* of the three statements can simultaneously be true. Let x, y be arbitrary elements of $\mathbb{Z}$. We divide into three cases. First, suppose for the sake of contradiction that $x < y$ and $y < x$. By Definition 3.8, this implies that $y = x + c$ and $x = y + c'$ for some $c, c' \in \mathbb{N}$.

Substituting, we have $y = y + c' + c$, or $0 = c' + c$ by the cancellation law of addition. But since $c', c \in \mathbb{N}$, the closure of addition on $\mathbb{N}$ implies that $(c' + c) \in \mathbb{N}$. Therefore, $c' + c \neq 0$, a contradiction. Second, suppose for the sake of contradiction that $x < y$ and $x = y$. By Definition 3.8, this implies that $y = x + c$ for some $c \in \mathbb{N}$. Substituting, we have $y = y + c$, or $0 = c$ by the cancellation law of addition. But since $c \in \mathbb{N}$, $c \neq 0$, a contradiction. The proof of the third case ($y < x$ and $x = y$) is symmetric to that of the second case.

We now show that *at least one* of the three statements is always true. Let x, y be arbitrary elements of $\mathbb{Z}$, and suppose for the sake of contradiction that $x \not< y$, $y \not< x$, and $x \neq y$. Since $x \not< y$, $y \neq x + c$ for any $c \in \mathbb{N}$. Equivalently, $y - x \neq c$ for any $c \in \mathbb{N}$, i.e., $(y - x) \notin \mathbb{N}$. Similarly, since $y \not< x$, $x - y \neq c'$ for any $c' \in \mathbb{N}$. Equivalently, $y - x \neq c'$ for any $c' \in -\mathbb{N}$, i.e., $(y - x) \notin -\mathbb{N}$. Lastly, since $x \neq y$, $y - x \neq 0$, i.e., $(y - x) \notin \{0\}$. Since $(y - x) \notin -\mathbb{N}$, $(y - x) \notin \{0\}$, and $(y - x) \notin \mathbb{N}$, Definition 1.5 implies that $(y - x) \notin (-\mathbb{N}) \cup \{0\} \cup \mathbb{N}$. Consequently, by Script 0, $(y - x) \notin \mathbb{Z}$. But by the closure of integer subtraction, $(y - x) \in \mathbb{Z}$, a contradiction.

To prove that $<$ is transitive, it will suffice to show that for all $x, y, z \in \mathbb{Z}$, if $x < y$ and $y < z$, then $x < z$. Let x, y, z be arbitrary elements of $\mathbb{Z}$ for which it is true that $x < y$ and $y < z$. By Definition 3.8, we have $y = x + c$ and $z = y + c'$ for some $c, c' \in \mathbb{N}$. Substituting, we have $z = x + c + c'$. Since $(c + c') \in \mathbb{N}$ by the closure of addition on $\mathbb{N}$, Definition 3.8 implies that $x < z$.

Axiom 3 asserts that $\mathbb{Z}$ must have no first or last point. Suppose for the sake of contradiction that $\mathbb{Z}$ has some first point a. Then by Definition 3.3, $a \leq x$ for every $x \in \mathbb{Z}$. However, under the closure of subtraction on $\mathbb{Z}$, $(a - 1) \in \mathbb{Z}$. Since $(a - 1) + 1 = a$, Definition 3.8 asserts that $a - 1 < a$. Therefore, we have $a - 1 < a$ and $a \leq a - 1$ (since, again, $(a - 1) \in \mathbb{Z}$), contradicting the previously demonstrated fact that $<$ is an ordering. The proof is symmetric for the last point. $\square$

b) Show that for any $p = \left[\frac{a}{b}\right] \in \mathbb{Q}$, there is some $(a_1, b_1) \in p$ with $0 < b_1$.

Proof. Let $\left[\frac{a}{b}\right]$ be an arbitrary element of $\mathbb{Q}$. It follows by Definition 2.5 that $(a, b) \in X$. Since we also have $(a, b) \sim (a, b)$ by Exercise 2.2e, Definition 2.5 implies that $(a, b) \in \left[\frac{a}{b}\right]$. By the trichotomy of $\mathbb{Z}$ (see Exercise 3.9a), we have $0 < b$, $b < 0$, or $0 = b$. We divide into three cases. First, suppose that $0 < b$. Then (a, b) is an element $(a_1, b_1) \in \left[\frac{a}{b}\right]$ for which $0 < b_1$, and we are done. Second, suppose that $b < 0$. Since $(-a)(b) = (-b)(a)$, we have by the definition of $\sim$ that $(-a, -b) \sim (a, b)$. Additionally, we have by the closure of integer multiplication that $-a, -b \in \mathbb{Z}$, and since $b \neq 0$ by Exercise 2.2e and clearly $-1 \neq 0$, $-1 \cdot b = -b \neq 0$ by the contrapositive of the zero-product property. Thus, by Exercise 2.2e, $(-a, -b) \in X$. This coupled with the previously proven fact that $(-a, -b) \sim (a, b)$ implies by Definition 2.5 that $(-a, -b) \in \left[\frac{a}{b}\right]$. Now recall that $b < 0$ by hypothesis, so we may use Definition 3.8 to see that $b + c = 0$ for some $c \in \mathbb{N}$. It follows that $-(b + c) = 0$, i.e., $-b - c = 0$, i.e., $-b = 0 + c$, meaning that $0 < -b$ by Definition 3.8. Thus, $(-a, -b)$ is an element $(a_1, b_1) \in \left[\frac{a}{b}\right]$ for which $0 < b_1$. Third, suppose that $b = 0$. But this contradicts Exercise 2.2e which asserts that $b \neq 0$, so we need not consider this case (as it will never arise). $\square$

c) Define an ordering $<_{\mathbb{Q}}$ on $\mathbb{Q}$ as follows. For $p, q \in \mathbb{Q}$, let $(a_1, b_1) \in p$ be such that $0 < b_1$ and let $(a_2, b_2) \in q$ be such that $0 < b_2$. Then we define $p <_{\mathbb{Q}} q$ if $a_1 b_2 < a_2 b_1$. Show that $<_{\mathbb{Q}}$ is a well-defined relation on $\mathbb{Q}$.

Proof. For the relation $<_{\mathbb{Q}}$ to be well-defined, Definition 3.1 tells us that it must satisfy the trichotomy and be transitive.

To prove that $<_{\mathbb{Q}}$ satisfies the trichotomy, it will suffice to show that for all $p, q \in \mathbb{Q}$, exactly one of the following holds: $p <_{\mathbb{Q}} q$, $q <_{\mathbb{Q}} p$, or $p = q$.

We first show that *no more than one* of the three statements can be simultaneously true. Let p, q be arbitrary elements of $\mathbb{Q}$, let $(a, b) \in p$ be such that $0 < b$ (we know that such an element exists by Exercise 3.9b[1]), and let $(c, d) \in q$ be such that $0 < d$. We divide into three cases. First, suppose for

[1]This justification will not be supplied again to make the proof less repetitive.

the sake of contradiction that $p <_{\mathbb{Q}} q$ and $q <_{\mathbb{Q}} p$. Then $ad < bc$ and $cb < da$ (i.e., $bc < ad$) by the definition of $<_{\mathbb{Q}}$. But this violates the trichotomy known to hold (by Exercise 3.9a) for the ordering $<$ on the integers, a contradiction. Second, suppose for the sake of contradiction that $p <_{\mathbb{Q}} q$ and $p = q$. By the definition of $<_{\mathbb{Q}}$, it follows from the first assumption that $ad < bc$. Additionally, by Exercise 2.6, it follows from the second assumption that $(a, b) \sim (c, d)$, implying by Exercise 2.2e that $ad = bc$. But once again, the simultaneous results that $ad < bc$ and $ad = bc$ violate the trichotomy of the integers, a contradiction. The proof of the third case is symmetric to that of the second.

We now show that *at least one* of the three statements is always true. Let p, q be arbitrary elements of $\mathbb{Q}$, let $(a, b) \in p$, and let $(c, d) \in q$. Suppose for the sake of contradiction that $p \not<_{\mathbb{Q}} q$, $q \not<_{\mathbb{Q}} p$, and $p \neq q$. Since $p \not<_{\mathbb{Q}} q$, we have that $ad \not< bc$. Similarly, since $q \not<_{\mathbb{Q}} p$, we have $cb \not< da$ (i.e., $bc \not< ad$). Lastly, since $p \neq q$, Exercise 2.6 implies that $(a, b) \nsim (c, d)$. It follows by Exercise 2.2e that $ad \neq bc$. To recap, for the integers ad and bc, we have $ad \not< bc$, $bc \not< ad$, and $ad \neq bc$. But by Exercise 3.9a, $ad < bc$, $bc < ad$, or $ad = bc$, a contradiction.

To prove that $<_{\mathbb{Q}}$ is transitive, it will suffice to show that for all $p, q, r \in \mathbb{Q}$, if $p <_{\mathbb{Q}} q$ and $q <_{\mathbb{Q}} r$, then $p <_{\mathbb{Q}} r$. Let p, q, r be arbitrary elements of $\mathbb{Q}$ for which it is true that $p <_{\mathbb{Q}} q$ and $q <_{\mathbb{Q}} r$, let $(a, b) \in p$ be such that $0 < b$, let $(c, d) \in q$ be such that $0 < d$, and let $(e, f) \in r$ such that $0 < f$. By the definition of $<_{\mathbb{Q}}$, we have $ad < bc$ and $cf < de$. Since $0 < f$ and $0 < b$, we can multiply both sides of the inequalities by b or f without affecting the truth of the statement (see Script 0). Thus, we may create the inequalities $adf < bcf$ and $bcf < bde$. So $adf < bde$ by Definition 3.1, implying that $af < be$ by the cancellation law (which we may use since $0 < d$). It follows by the definition of $<_{\mathbb{Q}}$ that $p <_{\mathbb{Q}} r$. □

d) Show that $\mathbb{Q}$ with the ordering $<_{\mathbb{Q}}$ satisfies Axioms 1-3.

Proof. Clearly, $\mathbb{Q}$ is a nonempty set, so Axiom 1 is immediately satisfied.

By Exercise 3.9c, $\mathbb{Q}$ has an ordering, so Axiom 2 is satisfied.

Axiom 3 asserts that $\mathbb{Q}$ must have no first or last point. Suppose for the sake of contradiction that $\mathbb{Q}$ has some first point p. Then by Definition 3.3, $p <_{\mathbb{Q}} x$ or $p = x$ for all $x \in \mathbb{Q}$. Let $(a, b) \in p$ be such that $0 < b$ (see Exercise 3.9b). Under the closure of integer subtraction, $(a - 1) \in \mathbb{Z}$, so $\left[\frac{a-1}{b}\right] \in \mathbb{Q}$. Since $ba = ba - b + b = b(a - 1) + b$ where $b \in \mathbb{N}$ because $b \in \mathbb{Z}$ and $0 < b$, Definition 3.8 implies that $(a - 1)b < ba$. It follows by the definition of $<_{\mathbb{Q}}$ that $\left[\frac{a-1}{b}\right] <_{\mathbb{Q}} \left[\frac{a}{b}\right] = p$, a contradiction. The argument is symmetric for the last point. □

Definition 3.10. If $a, b \in C$ and $a < b$, then the set of points between a and b is called a **region** and denoted by $\underline{ab}$.

Remark 3.11. One often sees the notation (a, b) for regions. We will reserve the notation (a, b) for ordered pairs in a product $A \times B$. These are very different things.

Theorem 3.12. *If x is a point of a continuum C, then there exists a region $\underline{ab}$ such that $x \in \underline{ab}$.*

Proof. Let x be an arbitrary point in a continuum C. By Axiom 2, C has an ordering $<$, which we will frequently make use of throughout the remainder of this proof. By Axiom 3, C has no first or last points, so it cannot be true that $x \leq y$ for all $y \in C$, nor can it be true that $x \geq y$ for all $y \in C$. This implies that there exists an $a \in C$ such that $a < x$ and that there exists a $b \in C$ such that $b > x$. Since $a < x$ and $x < b$, Definition 3.6 implies that x is between a and b. Note also that by Definition 3.1 (specifically transitivity), $a < b$. Therefore, since $a, b \in C$, $a < b$, and x is between a and b, Definition 3.10 implies that $x \in \underline{ab}$. □

Definition 3.13. Let A be a subset of a continuum C. A point p of C is called a **limit point** of A if every region R containing p has nonempty intersection with $A \setminus \{p\}$. Explicitly, this means:

$$\text{for every region } R \text{ with } p \in R, \text{ we have } R \cap (A \setminus \{p\}) \neq \emptyset.$$

Notice that we do not require that a limit point p of A be an element of A. We will use the notation $LP(A)$ to denote the set of limit points of A.

Theorem 3.14. *If p is a limit point of A and $A \subset B$, then p is a limit point of B.*

Proof. To prove that a limit point p of $A \subset B$ is a limit point of B, Definition 3.13 tells us that it will suffice to show that for every region R with $p \in R$, we have $R \cap (B \setminus \{p\}) \neq \emptyset$. Let p be a limit point of A, and let R be an arbitrary region with $p \in R$ (we know that such a region exists because of Theorem 3.12). Then by Definition 3.13, we have $R \cap (A \setminus \{p\}) \neq \emptyset$. Thus, by Definition 1.8, there is an element $x \in R \cap (A \setminus \{p\})$. Since $A \setminus \{p\} \subset B \setminus \{p\}$ (because $A \subset B$), it follows (by the fact that for three sets A, B, C such that $B \subset C$, $A \cap B \subset A \cap C$) that $R \cap (A \setminus \{p\}) \subset R \cap (B \setminus \{p\})$. Consequently, by Definition 1.3, the previously referenced object $x \in R \cap (A \setminus \{p\})$ is also an element of $R \cap (B \setminus \{p\})$. Thus, by Definition 1.8, $R \cap (B \setminus \{p\}) \neq \emptyset$, as desired. ☐

10/27: **Definition 3.15.** If $\underline{ab}$ is a region in a continuum C, then $C \setminus (\{a\} \cup \underline{ab} \cup \{b\})$ is called the **exterior** of $\underline{ab}$ and is denoted by $\operatorname{ext} \underline{ab}$.

Lemma 3.16. *If $\underline{ab}$ is a region in a continuum C, then*

$$\operatorname{ext} \underline{ab} = \{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}$$

Proof. To prove that $\operatorname{ext} \underline{ab} = \{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}$, Definition 3.15 tells us that it will suffice to show that $C \setminus (\{a\} \cup \underline{ab} \cup \{b\}) = \{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}$. To do this, Definition 1.2 tells us that we must verify that every element $y \in C \setminus (\{a\} \cup \underline{ab} \cup \{b\})$ is an element of $\{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}$ and vice versa. Let's begin.

First, let y be an arbitrary element of $C \setminus (\{a\} \cup \underline{ab} \cup \{b\})$. By Definition 1.11, this implies that $y \in C$ and $y \notin \{a\} \cup \underline{ab} \cup \{b\}$. The latter result implies by Definition 1.5 that $y \notin \{a\}$, $y \notin \underline{ab}$, and $y \notin \{b\}$. Since $y \notin \{a\}$ and $y \notin \{b\}$, we know that $y \neq a$ and $y \neq b$. Furthermore, since $y \notin \underline{ab}$, Definition 3.10 asserts that y is not between a and b. Thus, by Definition 3.6[2], we have that $y \leq a$ or $y \geq b$. But as previously established, $y \neq a$ and $y \neq b$, so it must be that $y < a$ or $y > b$. We divide into two cases. If $y < a$, then this fact combined with the fact that $y \in C$ implies that $y \in \{x \in C \mid x < a\}$. Therefore, by Definition 1.5, $y \in \{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}$, as desired. The proof is symmetric for the other case.

Now let y be an arbitrary element of $\{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}$. By Definition 1.5, this implies that $y \in \{x \in C \mid x < a\}$ or $y \in \{x \in C \mid b < x\}$. We divide into two cases. Suppose first that $y \in \{x \in C \mid x < a\}$. Then $y \in C$ and $y < a$. Since $y < a$, Definition 3.1 implies that $y \neq a$, i.e., $y \notin \{a\}$. Since $y < a$ and $a < b$, Definition 3.1 implies that $y < b$. Thus, for similar reasons to before, $y \neq b$, i.e., $y \notin \{b\}$. Lastly, since $a < b$, for y to be between a and b, Definition 3.6 implies that we must have $a < y$ and $y < b$. But $y < a$, so it must be that y is not between a and b. Thus, by Definition 3.10, $y \notin \underline{ab}$. Since $y \notin \{a\}$, $y \notin \underline{ab}$, and $y \notin \{b\}$, Definition 1.5 asserts that $y \notin \{a\} \cup \underline{ab} \cup \{b\}$. Therefore, since we also have $y \in C$ as previously established, Definition 1.11 implies that $y \in C \setminus (\{a\} \cup \underline{ab} \cup \{b\})$, as desired. The proof is symmetric if $y \in \{x \in C \mid b < x\}$. ☐

Lemma 3.17. *No point in the exterior of a region is a limit point of that region. No point of a region is a limit point of the exterior of that region.*

Proof. We will take this one claim at a time, starting with the first listed claim.

Let $\underline{ab}$ be an arbitrary region of a continuum C. To prove that no point in the exterior of $\underline{ab}$ is a limit point of $\underline{ab}$, Definition 3.13 tells us that it will suffice to show that for all points $p \in \operatorname{ext} \underline{ab}$, there exists some region R with $p \in R$ such that $R \cap (\underline{ab} \setminus \{p\}) = \emptyset$. Let p be an arbitrary element of $\operatorname{ext} \underline{ab}$. Then by Lemma 3.16 and Definition 1.5, $p \in \{x \in C \mid x < a\}$ or $p \in \{x \in C \mid b < x\}$. We divide into two cases. Suppose first that $p \in \{x \in C \mid x < a\}$. It follows that $p < a$, so let $c \in C$ be a point such that $c < p$ (Axiom 3 and Definition 3.3 imply that such a point exists). Since $c < p$ and $p < a$, Definition 3.6 implies that p is between c and a. Thus, Definition 3.10 implies that $p \in \underline{ca}$. Now suppose for the sake of contradiction that for some object x, $x \in \underline{ca} \cap (\underline{ab} \setminus \{p\})$. By Definition 1.6, this implies that $x \in \underline{ca}$ and $x \in \underline{ab} \setminus \{p\}$. Since $x \in \underline{ca}$, Definitions 3.10 and 3.6 imply that $c < x$ and $x < a$. Additionally, since $x \in \underline{ab} \setminus \{p\}$, Definition 1.11 implies that $x \in \underline{ab}$ and $x \notin \{p\}$, so with respect to the former claim, $a < x$ and $x < b$, as before. But by Definition 3.1, we cannot have $x < a$ and $a < x$, so we have arrived at a contradiction. Therefore,

[2]Technically, to use Definition 3.6, we also need the fact that $a < b$ to know that we are applying case i not case ii. This detail will not be mentioned again to make future proofs less repetitive.

$x \notin \underline{ca} \cap (\underline{ab} \setminus \{p\})$ for any x, proving by Definition 1.8 that $\underline{ca} \cap (\underline{ab} \setminus \{p\}) = \emptyset$, as desired. The proof is symmetric if $p \in \{x \in C \mid b < x\}$.

Let $\underline{ab}$ be an arbitrary region of a continuum C. To prove that no point of $\underline{ab}$ is a limit point of the exterior of $\underline{ab}$, Definition 3.13 tells us that it will suffice to show that for all points $p \in \underline{ab}$, there exists some region R with $p \in R$ such that $R \cap (\text{ext}\,\underline{ab} \setminus \{p\}) = \emptyset$. Let p be an arbitrary element of $\underline{ab}$. Then $\underline{ab}$ is actually a p-containing region having empty intersection with $\text{ext}\,\underline{ab} \setminus \{p\}$, as will now be proven. Suppose for the sake of contradiction that for some object x, $x \in \underline{ab} \cap (\text{ext}\,\underline{ab} \setminus \{p\})$. By Definition 3.15, this implies that $x \in \underline{ab} \cap ((C \setminus (\{a\} \cup \underline{ab} \cup \{b\})) \setminus \{p\})$. Thus, by Definition 1.6, $x \in \underline{ab}$ and $x \in (C \setminus (\{a\} \cup \underline{ab} \cup \{b\})) \setminus \{p\}$. Consequently, by consecutive applications of Definition 1.11, $x \in C$, $x \notin \{a\} \cup \underline{ab} \cup \{b\}$, and $x \notin \{p\}$. With respect to the middle of the three previous results, Definition 1.5 implies that $x \notin \{a\}$, $x \notin \underline{ab}$, and $x \notin \{b\}$. But we have previously demonstrated that $x \in \underline{ab}$, a contradiction. Therefore, $x \notin \underline{ab} \cap (\text{ext}\,\underline{ab} \setminus \{p\})$ for any x, proving by Definition 1.8 that $\underline{ab} \cap (\text{ext}\,\underline{ab} \setminus \{p\}) = \emptyset$, as desired. ☐

Theorem 3.18. *If two regions have a point x in common, their intersection is a region containing x.*

Proof. Let $\underline{ab}$ and $\underline{cd}$ be two regions of a continuum C such that for some point $x \in C$, $x \in \underline{ab}$ and $x \in \underline{cd}$. We divide into two cases ($a \leq c$ and $b \leq d$, and $a \leq c$ and $b > d$) WLOG[3].

Suppose first that $a \leq c$ and $b \leq d$. We seek to prove that $\underline{ab} \cap \underline{cd} = \underline{cb}$ where $\underline{cb}$ is clearly a region, and that $x \in \underline{cb}$. To prove that $\underline{ab} \cap \underline{cd} = \underline{cb}$, Definition 1.2 tells us that it will suffice to show that every element $y \in \underline{ab} \cap \underline{cd}$ is an element of $\underline{cb}$ and vice versa. Suppose first that y is an arbitrary element of $\underline{ab} \cap \underline{cd}$. Then by Definition 1.6, $y \in \underline{ab}$ and $y \in \underline{cd}$. Thus, by Definitions 3.10 and 3.6, $a < y$, $y < b$, $c < y$, and $y < d$. Since $c < y$ and $y < b$, it follows by Definitions 3.6 and 3.10 that $y \in \underline{cb}$, as desired. Now suppose that $y \in \underline{cb}$. Then by Definitions 3.10 and 3.6, $c < y$ and $y < b$. Since $a \leq c$ (by assumption) and $c < y$, $a < y$ (if $a = c$, then $c < y$ implies $a < y$ by substitution; if $a < c$ and $c < y$, then Definition 3.1 implies $a < y$[4]). Thus, since $a < y$ and $y < b$, Definitions 3.6 and 3.10 imply that $y \in \underline{ab}$. Similarly, $y < b$ and $b \leq d$ (by assumption) together imply that $y < d$. Thus, since $c < y$ and $y < d$, Definitions 3.6 and 3.10 imply that $y \in \underline{cd}$. Since $y \in \underline{ab}$ and $y \in \underline{cd}$, Definition 1.6 implies that $y \in \underline{ab} \cap \underline{cd}$, as desired. Lastly, since $x \in \underline{ab}$ and $x \in \underline{cd}$ by hypothesis, Definition 1.6 implies that $x \in \underline{ab} \cap \underline{cd}$, which means by Definition 1.2 and the above that $x \in \underline{cb}$, as desired.

Now suppose that $a \leq c$ and $b > d$. We seek to prove that $\underline{ab} \cap \underline{cd} = \underline{cd}$ where $\underline{cd}$ is clearly a region, and that $x \in \underline{cd}$. To prove that $\underline{ab} \cap \underline{cd} = \underline{cd}$, Theorem 1.7a tells us that it will suffice to show that $\underline{ab} \cap \underline{cd} \subset \underline{cd}$ and $\underline{cd} \subset \underline{ab} \cap \underline{cd}$. But by Theorem 1.7c, $\underline{ab} \cap \underline{cd} \subset \underline{cd}$, so all that's left to prove is that $\underline{cd} \subset \underline{ab} \cap \underline{cd}$. Definition 1.3 tells us that we may do this by demonstrating that every element $y \in \underline{cd}$ is an element of $\underline{ab} \cap \underline{cd}$. Let y be an arbitrary element of $\underline{cd}$. Then by Definitions 3.10 and 3.6, $c < y$ and $y < d$. Since $a \leq c$ (by assumption) and $c < y$, $a < y$, and since $y < d$ and $d < b$, $y < b$. Thus, since $a < y$ and $y < b$, Definitions 3.6 and 3.10 imply that $y \in \underline{ab}$. This fact combined with the hypothesis that $y \in \underline{cd}$ implies by Definition 1.6 that $y \in \underline{ab} \cap \underline{cd}$, as desired. Lastly, since $\underline{cd}$ *is* the intersection of $\underline{ab}$ and $\underline{cd}$, and $x \in \underline{cd}$ by hypothesis, x is clearly an element of the intersection of $\underline{ab}$ and $\underline{cd}$. ☐

Corollary 3.19. *If n regions $R_1, \dots, R_n$ have a point x in common, then their intersection $R_1 \cap \cdots \cap R_n$ is a region containing x.*

Proof. We induct on n from the base case $n_0 = 2$ using the form of induction described in Additional Exercise 0.2a. For the base case $n = 2$, let R_1 and R_2 be two regions that have a point x in common. By Theorem 3.18, it follows that their intersection $R_1 \cap R_2$ is a region containing x, proving the base case. Now suppose inductively that we have proven the claim for some n, i.e., we know that if n regions $R_1, \dots, R_n$ have a point x in common, then their intersection $\bigcap_{k=1}^{n} R_k$ is a region containing x. We wish to prove that if $n+1$ regions $R_1, \dots, R_{n+1}$ have a point x in common, then their intersection $\bigcap_{k=1}^{n+1} R_k$ is a region containing x. Let $R_1, \dots, R_{n+1}$ be $n+1$ regions that have a point x in common. By the induction hypothesis, $\bigcap_{k=1}^{n} R_k$ is a region containing x. Since $\bigcap_{k=1}^{n} R_k$ and R_{n+1} are both regions with a point x in common,

[3]The other two cases ($a > c$ and $b \leq d$, and $a > c$ and $b > d$) can be encapsulated in the first two by switching the names of the two regions: $a > c \land b \leq d \xRightarrow{\text{rename}} c > a \land d \leq b \implies (a \leq c \land b > d) \lor (a \leq c \land b = d)$, where both of the latter two cases are covered by one of the original cases.

[4]This justification and simple variations thereof, although used again, will not be stated again.

Theorem 3.18 applies and implies that $(\bigcap_{k=1}^{n} R_k) \cap R_{n+1} = \bigcap_{k=1}^{n+1} R_k$ is a region containing x, thus closing the induction. $\square$

Theorem 3.20. *Let A, B be subsets of a continuum C. Then p is a limit point of $A \cup B$ if and only if p is a limit point of at least one of A or B.*

Proof. To prove that p is a limit point of $A \cup B$ if and only if p is a limit point of at least one of A or B, we must prove the dual implications "$p \in LP(A)$ or $p \in LP(B)$ implies $p \in LP(A \cup B)$" and "$p \in LP(A \cup B)$ implies $p \in LP(A)$ or $p \in LP(B)$." The first implication will be proved directly, but the second one will be proved by contrapositive. Let's begin.

Suppose first that p is a limit point of A or B. We divide into two cases. If $p \in LP(A)$, then since $A \subset A \cup B$ by Theorem 1.7, Theorem 3.14 applies and implies that $p \in LP(A \cup B)$. The proof is symmetric if $p \in LP(B)$.

Now suppose that $p \notin LP(A)$ and $p \notin LP(B)$. Then by the contrapositive of Definition 3.13, there exist regions R_1 and R_2 with $p \in R_1$ and $p \in R_2$ such that $R_1 \cap (A \setminus \{p\}) = \emptyset$ and $R_2 \cap (B \setminus \{p\}) = \emptyset$. Since R_1 and R_2 are two regions that have a point (namely p) in common, Theorem 3.18 asserts that $R_1 \cap R_2$ is a region R with $p \in R$. Additionally, since $R_1 \cap R_2 \subset R_1$ and $R_1 \cap R_2 \subset R_2$ by Theorem 1.7, $R \subset R_1$ and $R \subset R_2$. These two results combined with the previously proven facts that $R_1 \cap (A \setminus \{p\}) = \emptyset$ and $R_2 \cap (B \setminus \{p\}) = \emptyset$ imply that $R \cap (A \setminus \{p\}) = \emptyset$ and $R \cap (B \setminus \{p\}) = \emptyset$. Now since $U \cup \emptyset = U$ for any set U, we know that

$$\begin{aligned} \emptyset &= R \cap (A \setminus \{p\}) \\ &= (R \cap (A \setminus \{p\})) \cup \emptyset \end{aligned}$$

Substitute $\emptyset = R \cap (B \setminus \{p\})$.

$$= (R \cap (A \setminus \{p\})) \cup (R \cap (B \setminus \{p\}))$$

Apply the fact that $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$ for any sets X, Y, Z.

$$= R \cap ((A \setminus \{p\}) \cup (B \setminus \{p\}))$$

Apply the fact that $(X \setminus Z) \cup (Y \setminus Z) = (X \cup Y) \setminus Z$ for any sets X, Y, Z.

$$= R \cap ((A \cup B) \setminus \{p\})$$

Since there exists a region R with $p \in R$ such that $R \cap ((A \cup B) \setminus \{p\}) = \emptyset$, the contrapositive of Definition 3.13 implies that $p \notin LP(A \cup B)$. $\square$

Corollary 3.21. *Let $A_1, \dots, A_n$ be n subsets of a continuum C. Then p is a limit point of $A_1 \cup \cdots \cup A_n$ if and only if p is a limit point of at least one of the sets A_k.*

Proof. We induct on n from the base case $n_0 = 2$ using the form of induction described in Additional Exercise 0.2a. For the base case $n = 2$, let A_1 and A_2 be two subsets of a continuum C. By Theorem 3.20, it follows that p is a limit point of $A_1 \cup A_2$ if and only if p is a limit point of at least one of A_1 or A_2, proving the base case. Now suppose inductively that we have proven the claim for some n, i.e., we know that if there exist n subsets $A_1, \dots, A_n$ of a continuum C, then p is a limit point of $\bigcup_{k=1}^{n} A_k$ if and only if p is a limit point of at least one of the sets A_k. We wish to prove that if there exist $n+1$ subsets $A_1, \dots, A_{n+1}$ of a continuum C, then p is a limit point of $\bigcup_{k=1}^{n+1} A_k$ if and only if p is a limit point of at least one of the sets A_k. Let $A_1, \dots, A_{n+1}$ be $n+1$ subsets of a continuum C. By the induction hypothesis, p is a limit point of $\bigcup_{k=1}^{n} A_k$ if and only if p is a limit point of at least one of the sets A_k. Since $\bigcup_{k=1}^{n} A_k$ and A_{n+1} are both subsets of a continuum C, Theorem 3.20 applies and implies that p is a limit point of $(\bigcup_{k=1}^{n} A_k) \cup A_{n+1} = \bigcup_{k=1}^{n+1} A_k$ if and only if p is a limit point of at least one of $\bigcup_{k=1}^{n} A_k$ or A_{n+1}. But the last two statements combined imply that p is a limit point of $\bigcup_{k=1}^{n+1} A_k$ if and only if p is a limit point of at least one of the sets A_k where $1 \leq k \leq n$ or A_{n+1}, i.e., p is a limit point of $\bigcup_{k=1}^{n+1} A_k$ if and only if p is a limit point of at least one of the sets A_k, thus closing the induction. $\square$

Theorem 3.22. *If p and q are distinct points of a continuum C, then there exist disjoint regions R and S containing p and q, respectively.*

Proof. WLOG, let $p < q$. Additionally, let $a < p$ and $b > q$ be points of C (Axiom 3 and Definition 3.3 imply that such points exist). We divide into two cases (no point $x \in C$ exists between p and q, and there exists a point $x \in C$ between p and q). Let's begin.

Suppose first that no point $x \in C$ exists between p and q. Let $R = \underline{aq}$ and let $S = \underline{pb}$. Thus, Definitions 3.6 and 3.10 imply by the facts that $a < p$ by definition and $p < q$ by hypothesis, and $p < q$ by hypothesis and $q < b$ by definition that $p \in R$ and $q \in S$, respectively. To prove that R and S are disjoint, Definition 1.9 tells us that it will suffice to show that $R \cap S = \emptyset$. Suppose for the sake of contradiction that there exists an object $x \in R \cap S$. By Definition 1.6, this implies that $x \in R$ and $x \in S$ (and, hence, that $x \in C$). Thus, by consecutive applications of Definitions 3.10 and 3.6, we have that $a < x$, $x < q$, $p < x$, and $x < b$. Since $p < x$ and $x < q$, Definition 3.6 implies that x is between p and q. But by hypothesis, no point $x \in C$ exists between p and q, a contradiction. Therefore, since $x \notin R \cap S$ for all x, Definition 1.8 implies that $R \cap S = \emptyset$, as desired.

Now suppose that there exists a point $x \in C$ between between p and q. Let $R = \underline{ax}$ and let $S = \underline{xb}$. It follows from the hypothesis by Definition 3.6 that $p < x$ and $x < q$. Thus, Definitions 3.6 and 3.10 imply by the facts that $a < p$ and $p < x$, and $x < q$ and $q < b$ that $p \in R$ and $q \in S$, respectively. To prove that R and S are disjoint, Definition 1.9 tells us tht it will suffice to show that $R \cap S = \emptyset$. Suppose for the sake of contradiction that there exists an object $y \in R \cap S$. By Definition 1.6, this implies that $y \in R$ and $y \in S$ (and, hence, that $y \in C$). Thus, by consecutive applications of Definitions 3.10 and 3.6, we have that $a < y$, $y < x$, $x < y$, and $y < b$. But by Definition 3.1, we cannot have $y < x$ and $x < y$, a contradiction. Therefore, since $x \notin R \cap S$ for all x, Definition 1.8 implies that $R \cap S = \emptyset$, as desired. □

10/29: **Corollary 3.23.** *A subset of a continuum C consisting of one point has no limit points.*

Proof. Let $\{x\} \subset C$. To prove that $\{x\}$ has no limit points, Definition 3.13 tells us that it will suffice to show that for all $p \in C$, there exists a region R with $p \in R$ such that $R \cap (\{x\} \setminus \{p\}) = \emptyset$. Let p be an arbitrary point in C. We divide into two cases ($p = x$ and $p \neq x$; Definition 3.1 guarantees that these two cases account for all $p \in C$). Suppose first that $p = x$. If we let R be any region of C, it follows that

$$\begin{aligned} R \cap (\{x\} \setminus \{p\}) &= R \cap (\{x\} \setminus \{x\}) \\ &= R \cap \emptyset \\ &= \emptyset \end{aligned}$$

as desired. Now suppose that $p \neq x$. Since p and x are distinct points, Theorem 3.22 applies and implies that there exist disjoint regions R and S containing p and x, respectively. Consequently, by Definition 1.9, $x \notin R$. Thus,

$$\begin{aligned} R \cap (\{x\} \setminus \{p\}) &= R \cap \{x\} \\ &= \emptyset \end{aligned}$$

as desired. □

Theorem 3.24. *A finite subset A of a continuum C has no limit points.*

Proof. We divide into two cases ($A = \emptyset$ and $A \neq \emptyset$).

Suppose that $A = \emptyset$. Then for an arbitrary $p \in C$ and any region R of C,

$$\begin{aligned} R \cap (A \setminus \{p\}) &= R \cap (\emptyset \setminus \{p\}) \\ &= R \cap \emptyset \\ &= \emptyset \end{aligned}$$

proving by the contrapositive of Definition 3.13 that p is not a limit point of A. Since p is an *arbitrary* element of C, no point of C is a limit point of A, i.e., A has no limit points, as desired.

Now suppose that $A \neq \emptyset$. Since A is finite, i.e., is a set of n distinct points of a continuum C for some $n \in \mathbb{N}$, Theorem 3.5 applies and implies that the symbols $a_1, \dots, a_n$ can be assigned to each point of A. It follows that we can write $A = \bigcup_{k=1}^{n} \{a_k\}$. Now suppose for the sake of contradiction that for some $p \in C$, $p \in LP(A)$. Then by Corollary 3.21, p is a limit point of at least one of the sets $\{a_k\}$. But this contradicts Corollary 3.23, which asserts that no singleton set has limit points. Therefore, no point p of C is a limit point of A, i.e., A has no limit points, as desired. ☐

Corollary 3.25. *If A is a finite subset of a continuum C and $x \in A$, then there exists a region R containing x, such that $A \cap R = \{x\}$.*

Proof. Since A is a finite subset of a continuum C, Theorem 3.24 implies that A has no limit points. Thus, for an arbitrary $x \in A$, x is not a limit point of A. Consequently, Definition 3.13 implies that there exists a region R of C with $x \in R$ such that $R \cap (A \setminus \{x\}) = \emptyset$. Moreover, since $x \in A$ and $x \in R$, Definition 1.6 implies that $x \in A \cap R$. Now suppose for the sake of contradiction that there exists some object $y \in A \cap R$ such that $y \neq x$. Then by Definition 1.6, $y \in A$ and $y \in R$. The former discovery combined with the fact that $y \neq x$ (i.e., $y \notin \{x\}$) reveals that $y \in A \setminus \{x\}$ by Definition 1.11. Thus, since $y \in R$ and $y \in A \setminus \{x\}$, $y \in R \cap (A \setminus \{x\})$. Consequently, by Definition 1.2 and the fact that $R \cap (A \setminus \{x\}) = \emptyset$, $y \in \emptyset$. But this contradicts Definition 1.8. Therefore, x is the only element of $A \cap R$, so $A \cap R = \{x\}$, as desired. ☐

Theorem 3.26. *If p is a limit point of A and R is a region containing p, then the set $R \cap A$ is infinite.*

Proof. Suppose for the sake of contradiction that $R \cap A$ is finite. Then by Theorem 3.24, $R \cap A$ has no limit points. Notably, this implies that p is not a limit point of $R \cap A$. It follows by Definition 3.13 that there exists some region S with $p \in S$ such that

$$\emptyset = S \cap ((R \cap A) \setminus \{p\})$$

Since $p \in S$ and $p \in R$, Theorem 3.18 implies that $S \cap R$ is a region containing p. But since the above

$$= (S \cap R) \cap (A \setminus \{p\})$$

where $S \cap R$ is a p-containing region, Definition 3.13 implies that p is not a limit point of A. But this contradicts the hypothesis that p is a limit point of A. Thus, $R \cap A$ must be infinite. ☐

11/3: **Exercise 3.27.** Find realizations of a continuum $(C, <)$. That is, find concrete sets C endowed with a relation $<$ satisfying all of the axioms (so far). Are they the same? What does "the same" mean here?

Explanation. $\mathbb{Z}$ and $\mathbb{Q}$ are complete realizations of a continuum $(C, <)$ (see Exercise 3.9). Additionally, $\mathbb{Q} \setminus \underline{01}$ is a continuum (it is a nonempty set C, it obeys the same ordering as $\mathbb{Q}$ since it is a subset of $\mathbb{Q}$, and it has no first or last points as we only removed a region in the "middle" as opposed to a region containing points from some point, on).

As to the other part of the question, it is hard to define what "the same" means with respect to infinite sets. However, in our class discussion on November 3, 2020, we reached the consensus that two continua A, B are "the same" if and only if

1. There exists a bijection $f : A \to B$.
2. A and B are of the same density category (both dense, both semi-dense, or both buoyant), where the three aforementioned density categories are defined as
   - **Dense** (continuum): A continuum C such that for all regions $R \subset C$, $LP(R) \neq \emptyset$.
   - **Semi-dense** (continuum): A continuum C such that for some region $R \subset C$, $LP(R) \neq \emptyset$, and for some region $S \subset C$, $LP(S) = \emptyset$.
   - **Buoyant** (continuum): A continuum C that is neither dense nor semi-dense. Equivalently, for all regions $R \subset C$, $LP(R) = \emptyset$.

These categories are mutually exclusive.

Going back to the examples from the beginning, it can be proven that $\mathbb{Q}$ is a dense continuum, $\mathbb{Q} \setminus \underline{01}$ is a semi-dense continuum, and $\mathbb{Z}$ is a buoyant continuum.

It is worth noting that there are multiple semi-dense continua that it could be argued are different (for example, $\mathbb{Q} \setminus \underline{01}$ and $\mathbb{Q} \setminus (\underline{01} \cup \underline{23})$). However, since all semi-dense sets have similar properties, it's not that much of a stretch, in my opinion, to treat all semi-dense sets as effectively "the same." Perhaps the argument could be made for *some* subcategories (let semi-dense sets be dense sets with finite regions of buoyancy, and let hemi-dense sets be buoyant sets with finite regions of density), but then we run into the issue of sets with infinite regions of both density and buoyancy. More generally, we run into issues of where to stop subdividing, so for now, we'll lump all semi-dense continua together. $\square$

3.2 Discussion

10/20:

- Show that $|\{x\}| = 1$? Perhaps in a lemma by defining a singleton set. Also make sure that it's clear that $A \cap \{x\} = \emptyset$ to use Theorem 1.34b.
  - Using $B = A \setminus \{x\}$ might help in identifying that x exists, and would help in $B \cap \{x\} = \emptyset$. Thus, $|\{x\}| + |B| = |A|$, it follows by subtraction that $|B| = n$. Potentially as a lemma so I can use it in Theorem 3.5, too?
- Consider case in Theorem 3.5 that A is empty.
- What is a limit point?
  - A point p would be a limit point of a set A if there's a point p to which the elements of A converge. So on an open interval, the limit points would be the two "end points" and every point therein.
  - No limit points on the integers. Yes on the rationals and reals.

10/22:

- For Exercises 3.9a and 3.9c, it is necessary to show that at least one of the three trichotomy statements holds.

10/27:

- Do we have to show the explicit proof of both cases in the second part of Lemma 3.16, or can we just say, "the proof is symmetric?"
- In Theorem 3.18, you can define two functions $\min : C \times C \to C$ and $\max : C \times C \to C$, where C is the continuum in which the two regions exist, by

$$\min(i,j) = \begin{cases} i & i \leq j \\ j & i > j \end{cases}$$
$$\max(i,j) = \begin{cases} i & i \geq j \\ j & i < j \end{cases}$$

  for any $i, j \in C$.
  - This allows you to define $m = \max(a, c)$ and $n = \min(b, d)$. It can then be proven that $m < n$ in every case, that $\underline{ab} \cap \underline{cd} = \underline{mn}$, and that $x \in \underline{mn}$.
- Note that Lemmas 3.16 and 3.17 cannot be used to prove Theorem 3.18 — the ordering in the script is just misleading (these lemmas will only be needed later).
- You don't need to treat all four cases in Theorem 3.18; rather, just two and then the others are covered by WLOG.
  - $a > c \wedge b \leq d \xRightarrow{\text{rename}} c > a \wedge d \leq b \Rightarrow (a \leq c \wedge b > d) \vee (a \leq c \wedge b = d)$, where both of the latter two cases are covered by one of the original cases.
- What is the base case in Corollary 3.19? Although the question does not explicitly state that we can, we should use $n_0 = 2$ and the form of induction in Additional Exercise 0.2a.

- In Corollary 3.19, use summation notation, e.g., $\bigcap_{i=1}^{k} R_i$!
- Do the forward implication of Theorem 3.20 by contrapositive.
- Use Theorem 3.18 to justify $R_1 \cap R_2 = R$ in the proof of Theorem 3.20.
- Do I have to prove the two set theory things I use in my proof of Theorem 3.20?

10/29:

- There are several possible proofs for Theorem 3.24. While induction is correct, the simplest is as follows.
  - Label the elements $a_1, \dots, a_n$ (Theorem 3.5).
  - Partition A into n singleton sets (such a partitioning may be verified via induction).
  - If $p \in LP(A)$, then $p \in \{a_k\}$ (Corollary **??**), but we cannot have $p \in \{a_k\}$ by Corollary 3.23.
- Alternate proof of Corollary 3.25:
  - Finite $A \subset C \xRightarrow{\text{Theorem 3.24}} LP(A) = \emptyset \Longrightarrow p \notin LP(A)\ \forall\ x \in C$.
  - $\xRightarrow{\text{Definition 3.13}} \exists\ R : p \in R \vee R \cap (A \setminus \{p\}) = \emptyset$.
    - $\Rightarrow (R \cap A) \setminus \{p\} = \emptyset$.
    - $\xRightarrow{\text{Definition 1.11}} \nexists\ y \in R \cap A : y \notin \{p\}$, i.e., $\forall\ y \in R \cap A$, $y \in \{p\}$.
    - $\xRightarrow{\text{Definition 1.3}} R \cap A \subset \{p\}$.
  - $p \in A \vee p \in R \xRightarrow{\text{Definition 1.6}} p \in A \cap R$
    - $\xRightarrow{\text{Definition 1.3}} \{p\} \subset A \cap R$
  - $R \cap A \subset \{p\} \vee \{p\} \subset R \cap A \xRightarrow{\text{Theorem 1.7}} R \cap A = \{x\}$.
- The method used in Theorem 3.26, is the best, but it could likely be done by inducting to prove that $|R \cap A|$ cannot equal any natural number (to find contradictions for all $n \in \mathbb{N}$).
  - If there was a finite number of elements in R, they could be ordered as in Theorem 3.5. If $a_i < p < a_{i+1}$, then $\underline{a_i a_{i+1}}$ would be an empty p-containing region.
- The set of all grammatically correct English sentences is a continuum: Define f : alphabet $\to \mathbb{Z}$ by

$$f(\alpha) = \begin{cases} 1 & \alpha = a \\ -1 & \alpha = y \\ 0 & \alpha \neq a \vee \alpha \neq y \end{cases}$$

  Add "which is yellow" to sentences to prove that there is no first point. Add "which is amusing" to sentences to prove that there is no last point. These modifiers can be added on indefinitely, and although you generally wouldn't, it's all grammatically correct.
- $\mathbb{Z}$ and $\mathbb{Q}$ both satisfy all of the axioms, and there is a bijection between them, yet one is dense and the other isn't. So…

11/3:

- First attempt at formalizing "the same": Two continuua are the same iff they are in bijective correspondence.
  - Flaws: A bijection between $\mathbb{Z}$ and $\mathbb{Q}$ is a necessary condition for them being "the same," but it's not a sufficient condition.
  - $\mathbb{Q}$ is "dense" whereas $\mathbb{Z}$ is not.

  - Thus, we must include a proto-definition of density (or somehow assert that the bijection "preserves the ordering").

- Second attempt at formalizing "the same": Two continuua are the same iff there exists a bijection $f : A \to B$ such that $f(a) < f(b)$ iff $a < b$.
  - Alternate formulation for density: Let C be a continnuum. C is dense iff for all regions $R \subset C$, $LP(R) \neq \emptyset$.
  - Flaws: If there are two continua A, B that meet the condition "there exists a bijection $f : A \to B$ such that $f(a) < f(b)$ iff $a < b$," then the latter condition implies that they are either both dense or both not dense. Basically, dense/non-dense is a dichotomy; we need a categorization method that is a *trichotomy* (as per Cartee), perhaps one that includes dense, "non-dense," and some third category.

- Proof that the condition implies that A and B are either both dense or both not dense:
  - Let a, b be arbitrary elements of A such that $a < b$.
  - If A is dense, then there exists an element $c \in A$ such that $a < c < b$. Now if B is not dense, then we cannot guarantee that there exists an element $d \in B$ such that $f(a) < d < f(b)$. Thus, we cannot guarantee that $f(a) < f(c) < f(b)$. So B must be dense.
  - If A is not dense, the proof is similar to the above argument in reverse.

- Final conclusion: Constraints for two continua A, B to be "the same":
  1. There must exist a bijection $f : A \to B$.
  2. A and B must be of the same density category (both dense, both semi-dense, or both buoyant).

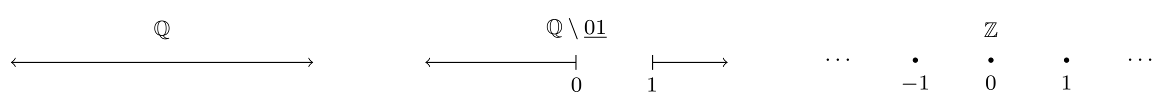
- **Dense** (continuum): A continuum C such that for all regions $R \subset C$, $LP(R) \neq \emptyset$.

- **Semi-dense** (continuum): A continuum C such that for some region $R \subset C$, $LP(R) \neq \emptyset$, and for some region $S \subset C$, $LP(S) = \emptyset$.

- **Buoyant** (continuum): A continuum C that is neither dense nor semi-dense. Equivalently, for all regions $R \subset C$, $LP(R) = \emptyset$.

- Note that there are multiple semi-dense continuua that could be argued are different but are the same under this definition (say if there were only one dense "region" versus two dense "regions"). We might want to split into **hemi-dense** and other categories, but for now, we're going with this.

- We conclude by considering three continua, one of each kind, as the concrete realizations requested by Exercise 3.27: $\mathbb{Z}$ (buoyant), $\mathbb{Q}$ (dense), and $\mathbb{Q} \setminus \underline{01}$ (semi-dense).



(a) Dense set. (b) Semi-dense set. (c) Buoyant set.

Figure 3.1: A dense, semi-dense, and buoyant number line.

  - A pictoral representation is given above in Figure 3.1 for each of these sets.

- Additional problem: If there exists *a* bijection $f : \mathbb{Z} \to \mathbb{Q}$ (which there is), shouldn't there be one that preserves ordering?
  - For one, $f(1)$ cannot be mapped to the smallest, positive, nonzero element of $\mathbb{Q}$, since no such object exists.

- Another thing is that since $\mathbb{Q}$ is dense, there exists a $p \in Q$ such that $f(1) < p < f(2)$. But there does not exist any integer between 1 and 2.
- And yet *an* element of $\mathbb{Z}$ does map to every element of $\mathbb{Q}$, so can't we "flip" which object maps to which other object? So if $f(2) = \frac{1}{2}$ and $f(z) = \frac{1}{4}$, then can't we let $f(2) = \frac{1}{4}$ and $f(z) = \frac{1}{2}$, and keep "flipping" indefinitely? But then, would we ever actually reach a "limit bijection" with the desired quality? It would simultaneously seem that we would have to and that we never could.
- Basically, what the issue boils down to is a potential problem with the ZFC-axioms of set theory and the objects we've defined since. All of these results follow directly from what we assume, but there is the possibility that the base assumptions are wrong. If I can assemble my disparate ideas into a formal paradox (à la Russell's Paradox with respect to naïve set theory), then I would be able to challenge this idea, but until then, I'll let it go.

Script 4

The Topology of a Continuum

4.1 Journal

11/3: **Definition 4.1.** A subset of a continuum is **closed** if it contains all of its limit points.

Theorem 4.2. *The sets $\emptyset$ and C are closed.*

Proof. We will address the two sets individually.

To prove that $\emptyset$ is closed, Definition 4.1 tells us that it will suffice to show that $\emptyset \subset C$ and $\emptyset$ contains all of its limit points. By Exercise 1.10, $\emptyset \subset C$. We now prove that $\emptyset$ has no limit points. Suppose for the sake of contradiction that some point $p \in C$ is a limit point of $\emptyset$. Then by Definition 3.13, for all regions R with $p \in R$, $R \cap (\emptyset \setminus \{p\}) \neq \emptyset$. But clearly, $R \cap (\emptyset \setminus \{p\}) = R \cap \emptyset = \emptyset$, a contradiction. Therefore, since $\emptyset$ has no limit points, the statement "$\emptyset$ contains all of its limit points" is vacuously true.

To prove that C is closed, Definition 4.1 tells us that it will suffice to show that $C \subset C$ and C contains all of its limit points. Since $C = C$, Theorem 1.7 implies that $C \subset C$. Now suppose for the sake of contradiction that C does not contain all of its limit points. Then there exists a point $p \in C$ that is a limit point of C such that $p \notin C$. But we cannot have $p \in C$ and $p \notin C$, so it must be that the initial hypothesis was incorrect, meaning that C does, in fact, contain all of its limit points. □

Theorem 4.3. *A subset of C containing a finite number of points is closed.*

Proof. Let A be a finite subset of C. To prove that A is closed, Definition 4.1 tells us that it will suffice to show that A contains all of its limit points. But by Theorem 3.24, A has no limit points, so the statement "A contains all of its limit points" is vacuously true. □

Definition 4.4. Let X be a subset of C. The **closure** of X is the subset $\overline{X}$ of C defined by

$$\overline{X} = X \cup LP(X)$$

Theorem 4.5. *$X \subset C$ is closed if and only if $X = \overline{X}$.*

Proof. Suppose first that X is closed. To prove that $X = \overline{X}$, Definition 4.4 tells us that it will suffice to show that $X = X \cup LP(X)$. To show this, Definition 1.2 tells us that we must verify that every element x of X is an element of $X \cup LP(X)$ and vice versa. First, let x be an arbitrary element of X. Then by Definition 1.5, $x \in X \cup LP(X)$, as desired. Now let x be an arbitrary element of $X \cup LP(X)$. Then by Definition 1.5, $x \in X$ or $x \in LP(X)$. We divide into two cases. If $x \in X$, then we are done. If $x \in LP(X)$, then $x \in X$ as desired for the following reason: Since X is closed by hypothesis, Definition 4.1 implies that X contains all of its limit points, i.e., for all $y \in LP(X)$, $y \in X$; this implication notably applies to the x in question.

Now suppose that $X = \overline{X}$. To prove that X is closed, Definition 4.1 tells us that it will suffice to show that X contains all of its limit points. By Theorem 1.7, $LP(X) \subset X \cup LP(X)$. This combined with the fact that $X = X \cup LP(X)$ (by Definition 4.4, since $X = \overline{X}$) implies that $LP(X) \subset X$. It follows by Definition 1.3 that every element of $LP(X)$ is an element of X, i.e., every limit point of X is an element of X, i.e., X contains all of its limit points, as desired. □

11/5: **Theorem 4.6.** *Let $X \subset C$. Then $\overline{X} = \overline{\overline{X}}$.*

Lemma. *If p is an element of $LP(LP(X))$, then p is an element of $LP(X)$.*

Proof. Let p be an arbitrary element of $LP(LP(X))$. To prove that $p \in LP(X)$, Definition 3.13 tells us that it will suffice to show that for all regions R containing p, $R \cap (X \setminus \{p\}) \neq \emptyset$. Let R be an arbitrary region with $p \in R$ (we know that such a region exists because of Theorem 3.12[1]). Then since we know that $p \in LP(LP(X))$, we have by Theorem 3.26 that $R \cap LP(X)$ is infinite. Thus, we know that there exists an object $x \in R \cap LP(X)$ such that $x \neq p$. By Definition 1.6, it follows that $x \in R$ and $x \in LP(X)$. Since $x \in LP(X)$, Theorem 3.26 tells us that all regions containing x (including R) have infinite intersection with X, i.e., $R \cap X$ is infinite. Consequently, $R \cap (X \setminus \{p\})$ is still infinite (since $\{p\}$ is finite), so $R \cap (X \setminus \{p\}) \neq \emptyset$, as desired. ☐

Proof of Theorem 4.6. To prove that $\overline{X} = \overline{\overline{X}}$, repeated applications of Definition 4.4 tell us that it will suffice to show that

$$X \cup LP(X) = (X \cup LP(X)) \cup LP(X \cup LP(X))$$

To show this, Theorem 1.7a tells us that it will suffice to verify the two statements

$$X \cup LP(X) \subset (X \cup LP(X)) \cup LP(X \cup LP(X)) \qquad (X \cup LP(X)) \cup LP(X \cup LP(X)) \subset X \cup LP(X)$$

By Theorem 1.7b, the left statement above is true. Consequently, all that's left at this point is to verify the right statement. To do so, Definition 1.3 tells us that it will suffice to demonstrate that every point $p \in (X \cup LP(X)) \cup LP(X \cup LP(X))$ is an element of $X \cup LP(X)$. Let's begin.

Let p be an arbitrary element of $(X \cup LP(X)) \cup LP(X \cup LP(X))$. Then by Definition 1.5, $p \in X \cup LP(X)$ or $p \in LP(X \cup LP(X))$. We divide into two cases. Suppose first that $p \in X \cup LP(X)$. Since this is actually exactly what we want to prove, we are done. Now suppose that $p \in LP(X \cup LP(X))$. Then we have by Theorem 3.20 that $p \in LP(X)$ or $p \in LP(LP(X))$. We divide into two cases again. If $p \in LP(X)$, then by Definition 1.5, $p \in X \cup LP(X)$, and we are done. On the other hand, if $p \in LP(LP(X))$, then by the lemma, $p \in LP(X)$. Therefore, as before, $p \in X \cup LP(X)$, and we are done. ☐

Corollary 4.7. *Let $X \subset C$. Then $\overline{X}$ is closed.*

Proof. By Theorem 4.6, $\overline{X} = \overline{\overline{X}}$. Thus, if we let $Y = \overline{X}$, we know that $Y = \overline{Y}$. But by Theorem 4.5, this implies that Y, i.e., $\overline{X}$, is closed, as desired. ☐

Definition 4.8. A subset G of a continuum C is **open** if its complement $C \setminus G$ is closed.

Theorem 4.9. *The sets $\emptyset$ and C are open.*

Proof. We will address the two sets individually. to prove that $\emptyset$ is open, Definition 4.8 tells us that it will suffice to show that $C \setminus \emptyset$ is closed. But $C \setminus \emptyset = C$, and by Theorem 4.2, C is closed. The proof is symmetric for C. ☐

Theorem 4.10. *Let $G \subset C$. Then G is open if and only if for all $x \in G$, there exists a region R such that $x \in R$ and $R \subset G$.*

Proof. To prove that G is open if and only if for all $x \in G$, there exists a region R such that $x \in R$ and $R \subset G$, we will take a similar approach to the proof of Theorem 3.20. Indeed, to prove the dual implications "if there exists a region R such that $x \in R$ and $R \subset G$ for all $x \in G$, then G is open" and "if G is open, then for all $x \in G$, there exists a region R such that $x \in R$ and $R \subset G$," we will prove the first implication directly and the second one by contrapositive. Let's begin.

Suppose first that for all $x \in G$, there exists a region R such that $x \in R$ and $R \subset G$. To prove that G is open, Definition 4.8 tells us that it will suffice to confirm that $C \setminus G$ is closed. To confirm this, Definition 4.1 tells us that it will suffice to show that $C \setminus G$ contains all of its limit points. Suppose for the sake of contradiction that for some limit point p of $C \setminus G$, $p \notin C \setminus G$. Since $p \notin C \setminus G$, Definition 1.11 tells us that $p \notin C$ or $p \in G$. But we must have $p \in C$, so necessarily $p \in G$. It follows by the hypothesis that

[1]This justification will not be supplied in similar cases beyond this point.

there exists a region R such that $p \in R$ and $R \subset G$. The fact that $R \subset G$ implies that $R \cap (C \setminus G) = \emptyset$. Consequently, $R \cap ((C \setminus G) \setminus \{p\}) = \emptyset$. But this implies by Definition 3.13 that p is not a limit point of $C \setminus G$, a contradiction. Therefore, $C \setminus G$ contains all of its limit points, as desired.

Now suppose that for some $x \in G$, there does not exist a region R such that $x \in R$ and $R \subset G$. To prove that G is not open, Definition 4.8 tells us that it will suffice to show that its complement $C \setminus G$ is not closed. To show this, Definition 4.1 tells us that it will suffice to verify that $C \setminus G$ does not contain all of its limit points, i.e., it will suffice to find some limit point of $C \setminus G$ that is not an element of this set. Consider the $x \in G$ introduced by the hypothesis; we will prove that this x is the desired limit point of $C \setminus G$ that is also not an element of $C \setminus G$. By Definition 1.11, $x \in G$ implies $x \notin C \setminus G$, so all that's left at this point is to prove that $x \in LP(C \setminus G)$. To do this, Definition 3.13 tells us that it will suffice to demonstrate that for all R with $x \in R$, $R \cap ((C \setminus G) \setminus \{x\}) \neq \emptyset$. Let R be an arbitrary region with $x \in R$. By the hypothesis, $R \not\subset G$, so Definition 1.3 implies that there exists some $y \in R$ such that $y \notin G$. Since $y \notin G$, we know two things: First, $y \neq x$ (since $x \in G$ and y cannot be both an element of and not an element of G) and second, $y \in C \setminus G$ (see Definition 1.11). Consequently, we have $y \in R$ and $y \in C \setminus G$, so by Definition 1.6, $y \in R \cap (C \setminus G)$. It follows since $y \neq x$ by Definition 1.11 that $y \in R \cap ((C \setminus G) \setminus \{x\})$. Therefore, by Definition 1.8, $R \cap ((C \setminus G) \setminus \{x\}) \neq \emptyset$, as desired. □

Corollary 4.11. *Every region R is open. Every complement of a region $C \setminus R$ is closed.*

Proof. Let R be an arbitrary region. Clearly, for all $x \in R$, there exists a region (namely R) such that $x \in R$ and $R \subset R$. Thus, by Theorem 4.10, R is open, as desired. It follows by Definition 4.8 that $C \setminus R$ is closed, as desired. □

Corollary 4.12. *Let $G \subset C$. Then G is open if and only if for all $x \in G$, there exists a subset $V \subset G$ such that $x \in V$ and V is open.*

Proof. Suppose first that G is open. Then by Theorem 4.10, for all $x \in G$, there exists a region R such that $x \in R$ and $R \subset G$. Additionally, by Corollary 4.11, each of these regions R is open. Thus, R is the desired open subset $V \subset G$ with $x \in V$.

Now suppose that for all $x \in G$, there exists a subset $V \subset G$ such that $x \in V$ and V is open. To prove that G is open, Theorem 4.10 tells us that it will suffice to show that for all $x \in G$, there exists a region R such that $x \in R$ and $R \subset G$. Let x be an arbitrary element of G. By the hypothesis, we know that $x \in V$ where V is an open subset of G. It follows by Theorem 4.10 that there exists a region R such that $x \in R$ and $R \subset V$. But by subset transitivity, for any R, $R \subset G$. Thus, there exists a region R such that $x \in R$ and $R \subset G$, as desired. □

Corollary 4.13. *Let $a \in C$. Then the sets $\{x \in C \mid x < a\}$ and $\{x \in C \mid a < x\}$ are open.*

Proof. We divide into two cases.

First, consider the set $\{x \in C \mid x < a\}$, which we will henceforth call G for ease of use. To prove that G is open, Theorem 4.10 tells us that it will suffice to show that for all $y \in G$, there exists a region R such that $y \in R$ and $R \subset G$. Let y be an arbitrary element of G. By Axiom 3 and Definition 3.3, we can choose a $z < y$. Since we also have $y < a$ (by the definition of G and the fact that $y \in G$), Definitions 3.6 and 3.10 assert that $y \in \underline{za}$. Now we must prove that $\underline{za} \subset G$, which Definition 1.3 tells us we can do by showing that every element of $\underline{za}$ is an element of G. But since every element of $\underline{za}$ is less than a (and greater than z, but this is not relevant) by Definitions 3.10 and 3.6, they *are* all elements of G by the definition of G, as desired.

The proof is symmetric in the other case. □

Theorem 4.14. *Let G be a nonempty open set. Then G is the union of a collection of regions.*

Proof. Since G is an open set, Theorem 4.10 implies that for all $x \in G$, there exists a region R_x such that $x \in R_x$ and $R_x \subset G$. Thus, we can create the set $\mathcal{R} = \{R_x \mid x \in G\}$ and define its union $\bigcup_{x \in G} R_x$ by Definition 1.13. We now prove that $\bigcup_{x \in G} R_x = G$, which will suffice to show that G can be described as a collection of regions.

To prove that $\bigcup_{x \in G} R_x = G$, Definition 1.2 tells us that it will suffice to show that every element $y \in \bigcup_{x \in G} R_x$ is an element of G and vice versa. Suppose first that y is an arbitrary element of $\bigcup_{x \in G} R_x$. Then by Definition 1.13, $y \in R_x$ for some $x \in G$. Thus, since $R_x \subset G$ by definition, Definition 1.3 implies that $y \in G$. Now suppose that y is an arbitrary element of G. Then $y \in R_y$, so Definition 1.13 implies that $y \in \bigcup_{x \in G} R_x$, as desired. $\square$

4.2 Discussion

11/3:
- Emily Wilson's more lyrical translation of *The Odyssey*.

11/5:
- In Theorem 4.6, is there a simpler way to prove the lemma?
  - Direct approach: $R \cap (LP(X) \setminus \{p\}) \neq \emptyset$ implies R contains at least one other limit point of $LP(X)$ besides p. Let m be this point. Then $R \cap (X \setminus \{m\}) \neq \emptyset$. Therefore, $R \cap X$ is infinite. So $R \cap (X \setminus \{p\}) \neq \emptyset$ (it will just be "slightly less infinite"). Therefore, $p \in LP(X)$.
- Alternate proof (Theorem 4.10 part 2): Now suppose that G is open. Then by Definition 4.8, $C \setminus G$ is closed. Let x be an arbitrary element of G. We wish to prove that there exists a region R with $x \in R$ such that $R \cap (C \setminus G) = \emptyset$, i.e., $R \cap ((C \setminus G) \setminus \{x\}) = \emptyset$, i.e., $x \notin LP(C \setminus G)$. Since $C \setminus G$ is closed, $C \setminus G = (C \setminus G) \cup LP(C \setminus G)$ by Definition 4.1. Thus, for all $a \in G$, $a \notin C \setminus G$, i.e., $a \notin LP(C \setminus G)$ by the closure of $C \setminus G$. Additionally, we necessarily have that $a \in R$ where R is some region. Thus, $R \cap ((C \setminus G) \setminus \{a\}) = \emptyset$, i.e., for all $r \in R$, $r \notin C \setminus G$. Thus, $R \subset C \setminus (C \setminus G) = G$.
- Alternate proof (Corollary 4.11): First, address "every $C \setminus R$ is closed." We must show that $LP(C \setminus R) \subset C \setminus R$. If $p \in LP(C \setminus R)$, then by Definition 3.13, $S \cap ((C \setminus R) \setminus \{p\}) \neq \emptyset$ for all $S \subset C$ such that $p \in S$. Now suppose for the sake of contradiction that $p \notin C \setminus R$. Then $S \cap (C \setminus R) \neq \emptyset$ for all $S \subset C$ with $p \in S$. But since $p \in R$, if we let $S = R$, then $R \cap (C \setminus R) \neq \emptyset$, a contradiction. Thus, $p \in C \setminus R$, so $C \setminus R$ is closed for all regions $R \subset C$.

  We have by the first part that $C \setminus R$ is closed for all $R \subset C$. Thus, R is open.
- Shiv approached Theorem 4.14 inductively and then considered the infinite case. This is far overcomplicated and likely incorrect, though.
- Definitions can be treated as if and only if. More accurately, a definition assigns a term to a certain characteristic in a way similar to if and only if but not identical.