Notes on Proofs

Responses

9/27: **Lemma 4.** *Let x, y be positive integers. Then xy is odd if and only if x and y are both odd.*

Proof. We wish to prove that if x and y are not both odd, then xy is not odd. In other words, we wish to prove that if at least one of x or y is even, then xy is even. Let's begin. WLOG, let x be even. Then $x = 2k$ for some $k \in \mathbb{N}$. Thus, $xy = 2(ky)$, proving that xy is even since $ky \in \mathbb{N}$. The proof is symmetric for y. $\square$

Corollary 5. *Let x, y be positive integers. Then xy is even if and only if at least one of x and y is even.*

Proof. We wish to prove that xy is even if and only if at least one of x and y is even. Consequently, we must prove the dual implications "if xy is even, then at least one of x and y is even" and "if at least one of x and y is even, then xy is even." Let's begin. For the first statement, let xy be even and suppose for the sake of contradiction that and both x and y are not even, i.e., are odd. But by Lemma 4, it follows from the assumption that x and y are both odd that xy is odd, which contradicts the fact that xy is even. Therefore, at least one of x or y must be even. As to the second statement, suppose that at least one of x or y is even. In this case, x and y are not both odd. Thus, by Lemma 4, xy is not odd, or, equivalently, xy is even. $\square$

Exercise 8.

a) Are there positive integers m, n such that m and n have no common factors (other than 1) and $m^2 = 3n^2$? Either give an example or prove that no example is possible.

Proof. Let m, n be relatively prime positive integers and suppose for the sake of contradiction that $m^2 = 3n^2$. We divide into two cases (the case where n is even, and the case where n is odd); we seek contradictions in both cases. First off, if n is even, then $n = 2k$ for some $k \in \mathbb{N}$. Thus, $3n^2 = 3(2k)^2 = 12k^2 = 2(6k^2) = m^2$, proving that m^2 is even since $6k^2 \in \mathbb{N}$. By Corollary 5, this implies that m is even. Therefore, since m and n are both even, they have a common factor, a contradiction. On the other hand, if n is odd, then $n = 2k+1$ for some $k \in \mathbb{N}$. Thus, $3n^2 = 3(2k+1)^2 = 12k^2+12k+3 = 2(6k^2+6k+1)+1 = m^2$, proving that m^2 is odd since $6k^2+6k+1 \in \mathbb{N}$. Thus, by Lemma 4, m is odd. Consequently, $m = 2l+1$ for some $l \in \mathbb{N}$, so $m^2 = (2l+1)^2 = 4l^2+4l+1 = 12k^2+12k+3$, the last equality holding because we also have $m^2 = 3n^2 = 12k^2+12k+3$. This implies the following.

$$\begin{aligned}
4l^2 + 4l + 1 &= 12k^2 + 12k + 3 \\
4l^2 + 4l &= 12k^2 + 12k + 2 \\
2l^2 + 2l &= 6k^2 + 6k + 1 \\
2(l^2 + l) &= 2(3k^2 + 3k) + 1
\end{aligned}$$

Since $l^2 + l$ and $3k^2 + 3k$ are both natural numbers, the above asserts that an odd number equals an even number, a contradiction. Hence, in both cases, we must have that $m^2 \neq 3n^2$. $\square$

b) Are there positive integers m, n such that m and n have no common factors (other than 1) and $m^2 = 6n^2$? Either give an example or prove that no example is possible.

Proof. Let $m, n \in \mathbb{N}$ have no common factors (other than 1), and suppose for the sake of contradiction that $m^2 = 6n^2$. Since $m^2 = 6n^2 = 2(3n^2)$, m^2 is even. It follows by Corollary 5 that m is even, implying that $m = 2k$ for some $k \in \mathbb{N}$. Thus, $6n^2 = m^2 = (2k)^2 = 4k^2$, so $3n^2 = 2k^2$. Since $k^2 \in \mathbb{N}$, $3n^2$ is even. Consequently, we have that n^2 is even by Corollary 5 (since at least one of 3 or n^2 is even and $3 = 2(1) + 1$ is odd). By Corollary 5 again, n is even. Thus, m and n are both even, contradicting the assumption that they have no common factors other than 1. ☐

c) Are there positive integers m, n such that m and n have no common factors (other than 1) and $m^2 = 4n^2$? Either give an example or prove that no example is possible.

Proof. Let $m = 2$ and $n = 1$. Then $m^2 = 2^2 = 4 = 4 \cdot 1^2 = 4n^2$. ☐

Discussion

9/29:

- Dr. Cartee.
- Sam Craig (super reader) is an advanced undergraduate who has taken this class before.
- Honors Calculus uses Spivak — we do not have a textbook, just scripts.
  - Few lectures in the traditional sense.
  - Majority of material is presented and developed by the students.
  - Several scripts will be covered throughout the quarter.
    - In scripts: It is our job to complete the exercises, prove the theorems/lemmas/propositions, etc.
  - Be on the look-out for "no proof required" theorems.
  - 3 chances to learn/review scripts material:
    1. Before class, you prepare your own proof.
    2. During class, we discuss.
    3. After class and before the journal is due, we type up our own record of the proof in LaTeX.
- Before each class, he will tell us which theorems/exercises we need to work through.
- Your proofs do not have to be perfect in the beginning! Sam and Dr. Cartee will help us. Expect to present every other week.
  - For the first two scripts, you have the ability to rewrite your journal after Sam reviews it to recover up to half of the lost credit.
  - You only recover credit if your new solution is perfect.
  - Return your changes one week after Sam grades it.
    - Mark what parts/problems you have rewritten, and turn in the original as well.
- Later this afternoon, Dr. Cartee will share which Script 0 problems we should do before Thursday. Sign up for problems on a Google Doc when each script is released. You also get a "buddy," who discusses your proof with you before you present.
- Class participation: When and how often and the quality of our presentations, and also how good are our questions that help presenters fill in the gaps.
- We can use Overleaf for collaborative LaTeX projects.
- We can check in with Dr. Cartee on our progress whenever throughout the quarter.
- Sam's office hours: We get to talk to him one-on-one with questions.

  - 7:00-8:00 PM on Thursdays

- You have one chance to ask for a 24-hour extension on HW (like if you're sick).
- In the case of a switch to virtual class:
  - We can present by turning our phone into a document camera or using a white board behind us or typing up in LaTeX (in real time?).
- Get good at writing — you cannot type up your solutions during exams!
- We submit HW assignments through Canvas if we type it up in LaTeX, or in class by hand. It's nice if we can type it up.