Script 3

Introducing a Continuum

3.1 Journal

10/20: **Axiom 1.** *A continuum is a nonempty set C.*

Definition 3.1. Let X be a set. An **ordering** on the set X is a subset $<$ of $X \times X$ with elements $(x, y) \in <$ written as $x < y$, satisfying the following properties:

a) (*Trichotomy*) For all $x, y \in X$, exactly one of the following holds: $x < y$, $y < x$, or $x = y$.

b) (*Transitivity*) For all $x, y, z \in X$, if $x < y$ and $y < z$, then $x < z$.

Remark 3.2.

a) In mathematics, "or" is understood to be inclusive unless stated otherwise. So in Definition 3.1a above, the word "exactly" is needed.

b) $x < y$ may also be written as $y > x$.

c) By $x \leq y$, we mean $x < y$ or $x = y$; similarly for $x \geq y$.

d) We often refer to elements of a continuum C as **points**.

Axiom 2. *A continuum C has an ordering $<$.*

Definition 3.3. If $A \subset C$, then a point $a \in A$ is a **first** point of A if for every element $x \in A$, either $a < x$ or $a = x$. Similarly, a point $b \in A$ is called a **last** point of A if, for every $x \in A$, either $x < b$ or $x = b$.

Lemma 3.4. *If A is a nonempty, finite subset of a continuum C, then A has a first and last point.*

Lemma. *Let A be a nonempty, finite set (i.e., $|A| = n$ for some $n \in \mathbb{N}$), let a be any element of A, and let the set $B = A \setminus \{a\}$. Then $|B| = n - 1$.*

Proof. We first prove that $|\{a\}| = 1$. By Definition 1.33, to do so, it will suffice to find a bijection $f : \{a\} \to [1]$. Since $[1] = \{1\}$ by Definition 1.29, $f : \{a\} \to \{1\}$ defined by $f(a) = 1$ is clearly such a bijection. We now demonstrate that $B \cap \{a\} = (A \setminus \{a\}) \cap \{a\} = \emptyset$. The previous two results combined with the fact that $B \cup \{a\} = (A \setminus \{a\}) \cup \{a\} = A$ imply by Theorem 1.34b that $|A| = |B| + |\{a\}|$. It follows that $n = |B| + 1$, so $|B| = n - 1$. $\square$

Proof of Lemma 3.4. We consider first points herein (the proof is symmetric for last points). If A is a finite set, then by Definition 1.30, $|A| = n$ for some $n \in \mathbb{N}$. Thus, if we prove the claim for each $n \in \mathbb{N}$ individually, we will have proven the claim. To prove a property pertaining to any natural number, we induct on n.

For the base case $n = 1$, there is only one element (which we may call a) in A. Since $a = a$, i.e., "for every $x \in A$, either $a < x$ or $a = x$" is a true statement, it follows by Definition 3.3 that A has a first point. Now suppose inductively that we have proven the claim for n, i.e., we know that if A is a nonempty, finite subset of a continuum C with $|A| = n$, then A has a first point. We wish to prove the same claim if $|A| = n+1$. Let a be an arbitrary element of A, and consider the set $B = A \setminus \{a\}$. By the lemma, $|B| = n$. Consequently, the induction hypothesis applies and asserts that B has a first point a_0. Clearly, a_0 is also an element of A, but it may or may not be the first point of A (the first point may now be a). Since C has an ordering $<$ (see Axiom 2), Definition 3.1 asserts that either $a < a_0$, $a_0 < a$, or $a = a_0$. We now divide into three cases. If $a < a_0$, then since $a_0 \leq x$ for all $x \in A$ by Definition 3.3, Definition 3.1 implies that $a \leq x$ for all $x \in A$. Thus, by Definition 3.3, a is the first point in A, and we have proven the claim for $|A| = n+1$ in this case. If $a_0 < a$, then it is still true that $a_0 \leq x$ for all $x \in A$. This means by Definition 3.3 that a_0 is still the first point in A, proving the claim for $|A| = n+1$ in this case. If $a = a_0$, then $a \in B$, contradicting the fact that $B = A \setminus \{a\}$, so we need not consider this final case. This closes the induction. $\square$

Theorem 3.5. *Suppose that A is a set of n distinct points in a continuum C, or in other words, $A \subset C$ has cardinality n. Then the symbols $a_1, \dots, a_n$ may be assigned to each point of A so that $a_1 < a_2 < \cdots < a_n$, i.e., $a_i < a_{i+1}$ for all $1 \leq i \leq n-1$.*

Proof. We divide into two cases ($|A| = 0$ and $|A| \in \mathbb{N}$).

If $|A| = 0$, then the statements "the symbols $a_1, \dots, a_n$ may be assigned to each point of A" and "$a_i < a_{i+1}$ for all $1 \leq i \leq n-1 = -1$" are both vacuously true.

If $|A| \in \mathbb{N}$, we induct on $|A| = n$. For the base case $n = 1$, denote the single element of A by a_1. Since $a_i < a_{i+1}$ for all $1 \leq i \leq n-1 = 0$ is vacuously true, the base case holds. Now suppose inductively that we have proven the claim for n, i.e., for a set $A \subset C$ satisfying $|A| = n$, the symbols $a_1, \dots, a_n$ may be assigned to each point of A so that $a_1 < a_2 < \cdots < a_n$. We now wish to prove the claim with regard to a set $A \subset C$ with $|A| = n+1$. By Lemma 3.4, there is a last point $a_{n+1} \in A$, which may be denoted as such (we will rigorously confirm this later). Since the set $A \setminus \{a_{n+1}\}$ has cardinality n (see the lemma from Lemma 3.4), we have by the induction hypothesis that its n elements can be named $a_1, \dots, a_n$ and ordered $a_1 < a_2 < \cdots < a_n$. Clearly these n elements are elements of A and all that's left to do is determine where a_{n+1} fits into the established order. But by Definition 3.3, $x \leq a_{n+1}$ for all $x \in A$, i.e., $x < a_{n+1}$ for all $x \in A \setminus \{a_{n+1}\}$. Consequently, as its name would suggest, it is true that $a_1 < a_2 < \cdots < a_n < a_{n+1}$, as desired. $\square$

Definition 3.6. If $x, y, z \in C$ and either (i) both $x < y$ and $y < z$ or (ii) both $z < y$ and $y < x$, then we say that y is **between** x and z.

Corollary 3.7. *Of three distinct points in a continuum, one must be between the other two.*

Proof. Let A be a subset of the described continuum containing the three distinct points. It follows by Theorem 3.5 that the symbols a_1, a_2, a_3 may be assigned to each point of A so that $a_1 < a_2 < a_3$. Thus, $a_1 < a_2$ and $a_2 < a_3$, so a_2 is between a_1 and a_3 by Definition 3.6. $\square$

10/22: **Axiom 3.** *A continuum C has no first or last point.*

Definition 3.8. We define an ordering on $\mathbb{Z}$ by $m < n$ if $n = m + c$ for some $c \in \mathbb{N}$.

Exercise 3.9.

a) Prove that with this ordering $\mathbb{Z}$ satisfies Axioms 1-3.

Proof. Clearly, $\mathbb{Z}$ is a nonempty set, so Axiom 1 is immediately satisfied.

Axiom 2 asserts that $\mathbb{Z}$ must have an ordering $<$. As such, it will suffice to verify that the ordering given by Definition 3.8 satisfies the stipulations of Definition 3.1. To prove that $<$ satisfies the trichotomy, it will suffice to show that for all $x, y \in \mathbb{Z}$, exactly one of the following holds: $x < y$, $y < x$, or $x = y$.

We first show that *no more than one* of the three statements can simultaneously be true. Let x, y be arbitrary elements of $\mathbb{Z}$. We divide into three cases. First, suppose for the sake of contradiction that $x < y$ and $y < x$. By Definition 3.8, this implies that $y = x + c$ and $x = y + c'$ for some $c, c' \in \mathbb{N}$. Substituting, we have $y = y + c' + c$, or $0 = c' + c$ by the cancellation law of addition. But since $c', c \in \mathbb{N}$, the closure of addition on $\mathbb{N}$ implies that $(c' + c) \in \mathbb{N}$. Therefore, $c' + c \neq 0$, a contradiction.

Second, suppose for the sake of contradiction that $x < y$ and $x = y$. By Definition 3.8, this implies that $y = x + c$ for some $c \in \mathbb{N}$. Substituting, we have $y = y + c$, or $0 = c$ by the cancellation law of addition. But since $c \in \mathbb{N}$, $c \neq 0$, a contradiction. The proof of the third case ($y < x$ and $x = y$) is symmetric to that of the second case.

We now show that *at least one* of the three statements is always true. Let x, y be arbitrary elements of $\mathbb{Z}$, and suppose for the sake of contradiction that $x \not< y$, $y \not< x$, and $x \neq y$. Since $x \not< y$, $y \neq x + c$ for any $c \in \mathbb{N}$. Equivalently, $y - x \neq c$ for any $c \in \mathbb{N}$, i.e., $(y - x) \notin \mathbb{N}$. Similarly, since $y \not< x$, $x - y \neq c'$ for any $c' \in \mathbb{N}$. Equivalently, $y - x \neq c'$ for any $c' \in -\mathbb{N}$, i.e., $(y - x) \notin -\mathbb{N}$. Lastly, since $x \neq y$, $y - x \neq 0$, i.e., $(y - x) \notin \{0\}$. Since $(y - x) \notin -\mathbb{N}$, $(y - x) \notin \{0\}$, and $(y - x) \notin \mathbb{N}$, Definition 1.5 implies that $(y - x) \notin (-\mathbb{N}) \cup \{0\} \cup \mathbb{N}$. Consequently, by Script 0, $(y - x) \notin \mathbb{Z}$. But by the closure of integer subtraction, $(y - x) \in \mathbb{Z}$, a contradiction.

To prove that $<$ is transitive, it will suffice to show that for all $x, y, z \in \mathbb{Z}$, if $x < y$ and $y < z$, then $x < z$. Let x, y, z be arbitrary elements of $\mathbb{Z}$ for which it is true that $x < y$ and $y < z$. By Definition 3.8, we have $y = x + c$ and $z = y + c'$ for some $c, c' \in \mathbb{N}$. Substituting, we have $z = x + c + c'$. Since $(c + c') \in \mathbb{N}$ by the closure of addition on $\mathbb{N}$, Definition 3.8 implies that $x < z$.

Axiom 3 asserts that $\mathbb{Z}$ must have no first or last point. Suppose for the sake of contradiction that $\mathbb{Z}$ has some first point a. Then by Definition 3.3, $a \leq x$ for every $x \in \mathbb{Z}$. However, under the closure of subtraction on $\mathbb{Z}$, $(a - 1) \in \mathbb{Z}$. Since $(a - 1) + 1 = a$, Definition 3.8 asserts that $a - 1 < a$, a contradiction. The proof is symmetric for the last point. $\square$

b) Show that for any $p = \left[\frac{a}{b}\right] \in \mathbb{Q}$, there is some $(a_1, b_1) \in p$ with $0 < b_1$.

Proof. Let $\left[\frac{a}{b}\right]$ be an arbitrary element of $\mathbb{Q}$. It follows by Definition 2.5 that $(a, b) \in X$. Since we also have $(a, b) \sim (a, b)$ by Exercise 2.2e, Definition 2.5 implies that $(a, b) \in \left[\frac{a}{b}\right]$. By the trichotomy on $\mathbb{Z}$ (see Exercise 3.9a), we have $0 < b$, $b < 0$, or $0 = b$. We divide into three cases. First, suppose that $0 < b$. Then (a, b) is an element $(a_1, b_1) \in \left[\frac{a}{b}\right]$ for which $0 < b_1$, and we are done. Second, suppose that $b < 0$. Since $(-a)(b) = (-b)(a)$, we have by the definition of $\sim$ that $(-a, -b) \sim (a, b)$. Additionally, we have by the closure of integer multiplication that $-a, -b \in \mathbb{Z}$, and since $b \neq 0$ by Exercise 2.2e and clearly $-1 \neq 0$, $-b \neq 0$ by the contrapositive of the zero-product property. Thus, by Exercise 2.2e, $(-a, -b) \in X$. This coupled with the previously proven fact that $(-a, -b) \sim (a, b)$ implies by Definition 2.5 that $(-a, -b) \in \left[\frac{a}{b}\right]$. Now recall that $b < 0$ by hypothesis, so we may use Definition 3.8 to see that $b + c = 0$ for some $c \in \mathbb{N}$. It follows that $-(b + c) = 0$, i.e., $-b - c = 0$, i.e., $-b = 0 + c$, meaning that $0 < -b$ by Definition 3.8. Thus, $(-a, -b)$ is an element $(a_1, b_1) \in \left[\frac{a}{b}\right]$ for which $0 < b_1$. Third, suppose that $b = 0$. But this contradicts Exercise 2.2e which asserts that $b \neq 0$, so we need not consider this case. $\square$

c) Define an ordering $<_{\mathbb{Q}}$ on $\mathbb{Q}$ as follows. For $p, q \in \mathbb{Q}$, let $(a_1, b_1) \in p$ be such that $0 < b_1$ and let $(a_2, b_2) \in q$ be such that $0 < b_2$. Then we define $p <_{\mathbb{Q}} q$ if $a_1 b_2 < a_2 b_1$. Show that $<_{\mathbb{Q}}$ is a well-defined relation on $\mathbb{Q}$.

Proof. For the relation $<_{\mathbb{Q}}$ to be well-defined, Definition 3.1 tells us that it must satisfy the trichotomy and be transitive.

To prove that $<_{\mathbb{Q}}$ satisfies the trichotomy, it will suffice to show that for all $p, q \in \mathbb{Q}$, exactly one of the following holds: $p <_{\mathbb{Q}} q$, $q <_{\mathbb{Q}} p$, or $p = q$.

We first show that *no more than one* of the three statements can be simultaneously true. Let p, q be arbitrary elements of $\mathbb{Q}$, let $(a, b) \in p$ be such that $0 < b$ (we know that such an element exists by Exercise 3.9b[1]), and let $(c, d) \in q$ be such that $0 < d$. We divide into three cases. First, suppose for the sake of contradiction that $p <_{\mathbb{Q}} q$ and $q <_{\mathbb{Q}} p$. Then $ad < bc$ and $cb < da$ by the definition of $<_{\mathbb{Q}}$. But this violates the trichotomy known to hold for the ordering $<$ on the integers by Exercise 3.9a, a

[1]This justification will not be supplied every subsequent time we choose such an element to make the proof less repetitive.

contradiction. Second, suppose for the sake of contradiction that $p <_{\mathbb{Q}} q$ and $p = q$. By the definition of $<_{\mathbb{Q}}$, it follows from the first assumption that $ad < bc$. Additionally, by Exercise 2.6, it follows from the second assumption that $(a, b) \sim (c, d)$, implying by Exercise 2.2e that $ad = bc$. But once again, the simultaneous results that $ad < bc$ and $ad = bc$ violate the trichotomy of the integers, a contradiction. The proof of the third case is symmetric to that of the second.

We now show that *at least one* of the three statements is always true. Let p, q be arbitrary elements of $\mathbb{Q}$, let $(a, b) \in p$, and let $(c, d) \in q$. Suppose for the sake of contradiction that $p \not<_{\mathbb{Q}} q$, $q \not<_{\mathbb{Q}} p$, and $p \neq q$. Since $p \not<_{\mathbb{Q}} q$, we have that $ad \not< bc$. Similarly, since $q \not<_{\mathbb{Q}} p$, we have $cb \not< da$. Equivalently, $bc \not< ad$. Lastly, since $p \neq q$, Exercise 2.6 implies that $(a, b) \nsim (c, d)$. It follows by Exercise 2.2e that $ad \neq bc$. To recap, for the integers ad and bc, we have $ad \not< bc$, $bc \not< ad$, and $ad \neq bc$. But by Exercise 3.9a, $ad < bc$, $bc < ad$, or $ad = bc$, a contradiction.

To prove that $<_{\mathbb{Q}}$ is transitive, it will suffice to show that for all $p, q, r \in \mathbb{Q}$, if $p <_{\mathbb{Q}} q$ and $q <_{\mathbb{Q}} r$, then $p <_{\mathbb{Q}} r$. Let p, q, r be arbitrary elements of $\mathbb{Q}$ for which it is true that $p <_{\mathbb{Q}} q$ and $q <_{\mathbb{Q}} r$, let $(a, b) \in p$ be such that $0 < b$, let $(c, d) \in q$ be such that $0 < d$, and let $(e, f) \in r$ such that $0 < f$. By the definition of $<_{\mathbb{Q}}$, we have $ad < bc$ and $cf < de$. Since $0 < f$ and $0 < b$, we can multiply both sides of the inequalities by b or f without affecting the truth of the statement (see Script 0). Thus, we may create the inequalities $adf < bcf$ and $bcf < bde$. So $adf < bde$ by Definition 3.1, implying that $af < be$ by the cancellation law (which we may use since $0 < d$). It follows by the definition of $<_{\mathbb{Q}}$ that $p <_{\mathbb{Q}} r$. □

d) Show that $\mathbb{Q}$ with the ordering $<_{\mathbb{Q}}$ satisfies Axioms 1-3.

Proof. Clearly, $\mathbb{Q}$ is a nonempty set, so Axiom 1 is immediately satisfied.

By Exercise 3.9c, $\mathbb{Q}$ has an ordering, so Axiom 2 is satisfied.

Axiom 3 asserts that $\mathbb{Q}$ must have no first or last point. Suppose for the sake of contradiction that $\mathbb{Q}$ has some first point p. Then by Definition 3.3, $p <_{\mathbb{Q}} x$ or $p = x$ for all $x \in \mathbb{Q}$. Let $(a, b) \in p$ be such that $0 < b$ (see Exercise 3.9b). Under the closure of integer subtraction, $(a-1) \in \mathbb{Z}$, so $\left[\frac{a-1}{b}\right] \in \mathbb{Q}$. Since $ba = ba - b + b = b(a-1) + b$ where $b \in \mathbb{N}$ since $b \in \mathbb{Z}$ and $0 < b$, Definition 3.8 implies that $(a-1)b < ba$. It follows by the definition of $<_{\mathbb{Q}}$ that $\left[\frac{a-1}{b}\right] <_{\mathbb{Q}} \left[\frac{a}{b}\right] = p$, a contradiction. The argument is symmetric for the last point. □

Definition 3.10. If $a, b \in C$ and $a < b$, then the set of points between a and b is called a **region** and denoted by $\underline{ab}$.

Remark 3.11. One often sees the notation (a, b) for regions. We will reserve the notation (a, b) for ordered pairs in a product $A \times B$. These are very different things.

Theorem 3.12. *If x is a point of a continuum C, then there exists a region $\underline{ab}$ such that $x \in \underline{ab}$.*

Proof. Let x be an arbitrary point in a continuum C. By Axiom 2, C has an ordering $<$, which we will frequently make use of throughout the remainder of this proof. By Axiom 3, C has no first or last points, so it cannot be true that $x \leq y$ for all $y \in C$, nor can it be true that $x \geq y$ for all $y \in C$. This implies that there exists an $a \in C$ such that $a < x$ and that there exists a $b \in C$ such that $b > x$. Since $a < x$ and $x < b$, Definition 3.6 implies that x is between a and b. Note also that by Definition 3.1 (specifically transitivity), $a < b$. Therefore, since $a, b \in C$, $a < b$, and x is between a and b, Definition 3.10 implies that $x \in \underline{ab}$. □

Definition 3.13. Let A be a subset of a continuum C. A point p of C is called a **limit point** of A if every region R containing p has nonempty intersection with $A \setminus \{p\}$. Explicitly, this means:

$$\text{for every region } R \text{ with } p \in R, \text{ we have } R \cap (A \setminus \{p\}) \neq \emptyset.$$

Notice that we do not require that a limit point p of A be an element of A. We will use the notation $LP(A)$ to denote the set of limit points of A.

Theorem 3.14. *If p is a limit point of A and $A \subset B$, then p is a limit point of B.*

Lemma. *Let A, B, C be sets such that $A \subset B$. Then $A \cap C \subset B \cap C$.*

Proof. Let x be an arbitrary element of $A \cap C$. By Definition 1.6, this implies that $x \in A$ and $x \in C$. Since $x \in A$ and $A \subset B$, Definition 1.3 implies that $x \in B$. Thus, $x \in B$ and $x \in C$, so $x \in B \cap C$ by Definition 1.6. ☐

Proof. To prove that a limit point p of $A \subset B$ is a limit point of B, Definition 3.13 tells us that it will suffice to show that for every region R with $p \in R$, we have $R \cap (B \setminus \{p\}) \neq \emptyset$. Let p be a limit point of A, and let R be an arbitrary region with $p \in R$. Then by Definition 3.13, we have $R \cap (A \setminus \{p\}) \neq \emptyset$. Thus, by Definition 1.8, there is an element $x \in R \cap (A \setminus \{p\})$. Since $A \setminus \{p\} \subset B \setminus \{p\}$ (because $A \subset B$ and $\{p\} = \{p\}$), it follows by the lemma that $R \cap (A \setminus \{p\}) \subset R \cap (B \setminus \{p\})$. Consequently, by Definition 1.3, the previously referenced object $x \in R \cap (A \setminus \{p\})$ is also an element of $R \cap (B \setminus \{p\})$. Thus, by Definition 1.8, $R \cap (B \setminus \{p\}) \neq \emptyset$, as desired. ☐

10/27: **Definition 3.15.** If $\underline{ab}$ is a region in a continuum C, then $C \setminus (\{a\} \cup \underline{ab} \cup \{b\})$ is called the **exterior** of $\underline{ab}$ and is denoted by $\operatorname{ext} \underline{ab}$.

Lemma 3.16. *If $\underline{ab}$ is a region in a continuum C, then*

$$\operatorname{ext} \underline{ab} = \{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}$$

Proof. To prove that $\operatorname{ext} \underline{ab} = \{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}$, Definition 3.15 tells us that it will suffice to show that $C \setminus (\{a\} \cup \underline{ab} \cup \{b\}) = \{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}$. To do this, Definition 1.2 tells us that we must verify that every element $y \in C \setminus (\{a\} \cup \underline{ab} \cup \{b\})$ is an element of $\{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}$ and vice versa. Let's begin.

First, let y be an arbitrary element of $C \setminus (\{a\} \cup \underline{ab} \cup \{b\})$. By Definition 1.11, this implies that $y \in C$ and $y \notin \{a\} \cup \underline{ab} \cup \{b\}$. The latter result implies by Definition 1.5 that $y \notin \{a\}$, $y \notin \underline{ab}$, and $y \notin \{b\}$. Since $y \notin \{a\}$ and $y \notin \{b\}$, we know that $y \neq a$ and $y \neq b$. Furthermore, since $y \notin \underline{ab}$, Definition 3.10 asserts that y is not between a and b. Thus, by Definition 3.6 and the fact that $a < b$ (i.e., case ii of Definition 3.6 does not apply), we have that $y \leq a$ or $y \geq b$. But as previously established, $y \neq a$ and $y \neq b$, so it must be that $y < a$ or $y > b$. We divide into two cases. If $y < a$, then this fact combined with the fact that $y \in C$ implies that $y \in \{x \in C \mid x < a\}$. Therefore, by Definition 1.5, $y \in \{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}$, as desired. Similarly, if $y > b$, we have $y \in \{x \in C \mid b < x\}$, meaning that $y \in \{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}$, as desired.

Now let y be an arbitrary element of $\{x \in C \mid x < a\} \cup \{x \in C \mid b < x\}$. By Definition 1.5, this implies that $y \in \{x \in C \mid x < a\}$ or $y \in \{x \in C \mid b < x\}$. We divide into two cases. Suppose first that $y \in \{x \in C \mid x < a\}$. Then $y \in C$ and $y < a$. Since $y < a$, Definition 3.1 implies that $y \neq a$, i.e., $y \notin \{a\}$. Since $y < a$ and $a < b$, Definition 3.1 implies that $y < b$. Thus, for similar reasons to before, $y \neq b$, i.e., $y \notin \{b\}$. Lastly, since $a < b$, for y to be between a and b, Definition 3.6 implies that we must have $a < y$ and $y < b$. But $y < a$, so it must be that y is not between a and b. Thus, by Definition 3.10, $y \notin \underline{ab}$. Since $y \notin \{a\}$, $y \notin \underline{ab}$, and $y \notin \{b\}$, Definition 1.5 asserts that $y \notin \{a\} \cup \underline{ab} \cup \{b\}$. Therefore, since we also have $y \in C$ as previously established, Definition 1.11 implies that $y \in C \setminus (\{a\} \cup \underline{ab} \cup \{b\})$, as desired. The proof is symmetric if $y \in \{x \in C \mid b < x\}$. ☐

Lemma 3.17. *No point in the exterior of a region is a limit point of that region. No point of a region is a limit point of the exterior of that region.*

Proof. We will take this one claim at a time, starting with the first listed claim.

Let $\underline{ab}$ be an arbitrary region of a continuum C. To prove that no point in the exterior of $\underline{ab}$ is a limit point of $\underline{ab}$, Definition 3.13 tells us that it will suffice to show that for all points $p \in \operatorname{ext} \underline{ab}$, there exists some region R with $p \in R$ such that $R \cap (\underline{ab} \setminus \{p\}) = \emptyset$. Let p be an arbitrary element of $\operatorname{ext} \underline{ab}$. Then by Lemma 3.16 and Definition 1.5, $p \in \{x \in C \mid x < a\}$ or $p \in \{x \in C \mid b < x\}$. We divide into two cases. Suppose first that $p \in \{x \in C \mid x < a\}$. It follows that $p < a$, so let $c \in C$ be a point such that $c < p$ (Axiom 3 and Definition 3.3 imply that such a point exists). Since $c < p$ and $p < a$, Definition 3.6 implies that p is between c and a. Thus, Definition 3.10 implies that $p \in \underline{ca}$. Now suppose for the sake of contradiction that for some object x, $x \in \underline{ca} \cap (\underline{ab} \setminus \{p\})$. By Definition 1.6, this implies that $x \in \underline{ca}$ and $x \in \underline{ab} \setminus \{p\}$. Since

$x \in \underline{ca}$, Definitions 3.10 and 3.6 imply that $c < x$ and $x < a$. Additionally, since $x \in \underline{ab} \setminus \{p\}$, Definition 1.11 implies that $x \in \underline{ab}$ and $x \notin \{p\}$, so with respect to the former claim, $a < x$ and $x < b$, as before. But by Definition 3.1, we cannot have $x < a$ and $a < x$, so we have arrived at a contradiction. Therefore, $x \notin \underline{ca} \cap (\underline{ab} \setminus \{p\})$ for any x, proving by Definition 1.8 that $\underline{ca} \cap (\underline{ab} \setminus \{p\}) = \emptyset$, as desired. The proof is symmetric if $p \in \{x \in C \mid b < x\}$.

Let $\underline{ab}$ be an arbitrary region of a continuum C. To prove that no point of $\underline{ab}$ is a limit point of the exterior of $\underline{ab}$, Definition 3.13 tells us that it will suffice to show that for all points $p \in \underline{ab}$, there exists some region R with $p \in R$ such that $R \cap (\operatorname{ext} \underline{ab} \setminus \{p\}) = \emptyset$. Let p be an arbitrary element of $\underline{ab}$. Then $\underline{ab}$ is actually a p-containing region having empty intersection with $\operatorname{ext} \underline{ab} \setminus \{p\}$, as will now be proven. Suppose for the sake of contradiction that for some object x, $x \in \underline{ab} \cap (\operatorname{ext} \underline{ab} \setminus \{p\})$. By Definition 3.15, this implies that $x \in \underline{ab} \cap ((C \setminus (\{a\} \cup \underline{ab} \cup \{b\})) \setminus \{p\})$. Thus, by Definition 1.6, $x \in \underline{ab}$ and $x \in (C \setminus (\{a\} \cup \underline{ab} \cup \{b\})) \setminus \{p\}$. Consequently, by consecutive applications of Definition 1.11, $x \in C$, $x \notin \{a\} \cup \underline{ab} \cup \{b\}$, and $x \notin \{p\}$. With respect to the middle of the three previous results, Definition 1.5 implies that $x \notin \{a\}$, $x \notin \underline{ab}$, and $x \notin \{b\}$. But we have previously demonstrated that $x \in \underline{ab}$, a contradiction. Therefore, $x \notin \underline{ab} \cap (\operatorname{ext} \underline{ab} \setminus \{p\})$ for any x, proving by Definition 1.8 that $\underline{ab} \cap (\operatorname{ext} \underline{ab} \setminus \{p\}) = \emptyset$, as desired. □

Theorem 3.18. *If two regions have a point x in common, their intersection is a region containing x.*

Proof. Let $\underline{ab}$ and $\underline{cd}$ be two regions of a continuum C such that for some point $x \in C$, $x \in \underline{ab}$ and $x \in \underline{cd}$. We divide into two cases ($a \leq c$ and $b \leq d$, and $a \leq c$ and $b > d$) WLOG[2].

Suppose first that $a \leq c$ and $b \leq d$. We seek to prove that $\underline{ab} \cap \underline{cd} = \underline{cb}$ where $\underline{cb}$ is clearly a region, and that $x \in \underline{cb}$. To prove that $\underline{ab} \cap \underline{cd} = \underline{cb}$, Definition 1.2 tells us that it will suffice to show that every element $y \in \underline{ab} \cap \underline{cd}$ is an element of $\underline{cb}$ and vice versa. Suppose first that y is an arbitrary element of $\underline{ab} \cap \underline{cd}$. Then by Definition 1.6, $y \in \underline{ab}$ and $y \in \underline{cd}$. Thus, by Definitions 3.10 and 3.6, $a < y$, $y < b$, $c < y$, and $y < d$. Since $c < y$ and $y < b$, it follows by Definitions 3.6 and 3.10 that $y \in \underline{cb}$, as desired. Now suppose that $y \in \underline{cb}$. Then by Definitions 3.10 and 3.6, $c < y$ and $y < b$. Since $a \leq c$ (by assumption) and $c < y$, $a < y$ (if $a = c$, then $c < y$ implies $a < y$ by substitution; if $a < c$ and $c < y$, then Definition 3.1 implies $a < y$[3]). Thus, since $a < y$ and $y < b$, Definitions 3.6 and 3.10 imply that $y \in \underline{ab}$. Similarly, $y < b$ and $b \leq d$ (by assumption) together imply that $y < d$. Thus, since $c < y$ and $y < d$, Definitions 3.6 and 3.10 imply that $y \in \underline{cd}$. Since $y \in \underline{ab}$ and $y \in \underline{cd}$, Definition 1.6 implies that $y \in \underline{ab} \cap \underline{cd}$, as desired. Lastly, since $x \in \underline{ab}$ and $x \in \underline{cd}$ by hypothesis, Definition 1.6 implies that $x \in \underline{ab} \cap \underline{cd}$, which means by Definition 1.2 and the above that $x \in \underline{cb}$, as desired.

Now suppose that $a \leq c$ and $b > d$. We seek to prove that $\underline{ab} \cap \underline{cd} = \underline{cd}$ where $\underline{cd}$ is clearly a region, and that $x \in \underline{cd}$. To prove that $\underline{ab} \cap \underline{cd} = \underline{cd}$, Theorem 1.7a tells us that it will suffice to show that $\underline{ab} \cap \underline{cd} \subset \underline{cd}$ and $\underline{cd} \subset \underline{ab} \cap \underline{cd}$. But by Theorem 1.7c, $\underline{ab} \cap \underline{cd} \subset \underline{cd}$, so all that's left to prove is that $\underline{cd} \subset \underline{ab} \cap \underline{cd}$. Definition 1.3 tells us that we may do this by demonstrating that every element $y \in \underline{cd}$ is an element of $\underline{ab} \cap \underline{cd}$. Let y be an arbitrary element of $\underline{cd}$. Then by Definitions 3.10 and 3.6, $c < y$ and $y < d$. Since $a \leq c$ (by assumption) and $c < y$, $a < y$, and since $y < d$ and $d < b$, $y < b$. Thus, since $a < y$ and $y < b$, Definitions 3.6 and 3.10 imply that $y \in \underline{ab}$. This fact combined with the hypothesis that $y \in \underline{cd}$ implies by Definition 1.6 that $y \in \underline{ab} \cap \underline{cd}$, as desired. Lastly, since $\underline{cd}$ *is* the intersection of $\underline{ab}$ and $\underline{cd}$, and $x \in \underline{cd}$ by hypothesis, x is clearly an element of the intersection of $\underline{ab}$ and $\underline{cd}$. □

Corollary 3.19. *If n regions $R_1, \dots, R_n$ have a point x in common, then their intersection $R_1 \cap \cdots \cap R_n$ is a region containing x.*

Proof. We induct on n from the base case $n_0 = 2$ using the form of induction described in Additional Exercise 0.2a. For the base case $n = 2$, let R_1 and R_2 be two regions that have a point x in common. By Theorem 3.18, it follows that their intersection $R_1 \cap R_2$ is a region containing x, proving the base case. Now suppose inductively that we have proven the claim for some n, i.e., we know that if n regions $R_1, \dots, R_n$ have a point x in common, then their intersection $\bigcap_{k=1}^{n} R_k$ is a region containing x. We wish to prove that if $n+1$ regions $R_1, \dots, R_{n+1}$ have a point x in common, then their intersection $\bigcap_{k=1}^{n+1} R_k$ is a region containing x. Let $R_1, \dots, R_{n+1}$ be $n+1$ regions that have a point x in common. By the induction hypothesis,

[2] The other two cases ($a > c$ and $b \leq d$, and $a > c$ and $b > d$) can be encapsulated in the first two by switching the names of the two regions.

[3] This justification and simple variations thereof, although used again, will not be stated again.

$\bigcap_{k=1}^{n} R_k$ is a region containing x. Since $\bigcap_{k=1}^{n} R_k$ and R_{n+1} are both regions with a point x in common, Theorem 3.18 applies and implies that $(\bigcap_{k=1}^{n} R_k) \cap R_{n+1} = \bigcap_{k=1}^{n+1} R_k$ is a region containing x, thus closing the induction. □

Theorem 3.20. *Let A, B be subsets of a continuum C. Then p is a limit point of $A \cup B$ if and only if p is a limit point of at least one of A or B.*

Proof. To prove that p is a limit point of $A \cup B$ if and only if p is a limit point of at least one of A or B, we must prove the dual implications "$p \in LP(A)$ or $p \in LP(B)$ implies $p \in LP(A \cup B)$" and "$p \in LP(A \cup B)$ implies $p \in LP(A)$ or $p \in LP(B)$." The first implication will be proved directly, but the second one will be proved by contrapositive. Let's begin.

Suppose first that p is a limit point of A or B. We divide into two cases. If $p \in LP(A)$, then since $A \subset A \cup B$ by Theorem 1.7, Theorem 3.14 applies and implies that $p \in LP(A \cup B)$. The proof is symmetric if $p \in LP(B)$.

Now suppose that $p \notin LP(A)$ and $p \notin LP(B)$. Then by the contrapositive of Definition 3.13, there exist regions R_1 and R_2 with $p \in R_1$ and $p \in R_2$ such that $R_1 \cap (A \setminus \{p\}) = \emptyset$ and $R_2 \cap (B \setminus \{p\}) = \emptyset$. Since R_1 and R_2 are two regions that have a point (namely p) in common, Theorem 3.18 asserts that $R_1 \cap R_2$ is a region R with $p \in R$. Additionally, since $R_1 \cap R_2 \subset R_1$ and $R_1 \cap R_2 \subset R_2$ by Theorem 1.7, $R \subset R_1$ and $R \subset R_2$. Thus, $R \cap (A \setminus \{p\}) = \emptyset$ and $R \cap (B \setminus \{p\}) = \emptyset$. Now since $U \cup \emptyset = U$ for any set U, we know that

$$\begin{aligned}\emptyset &= R \cap (A \setminus \{p\}) \\ &= (R \cap (A \setminus \{p\})) \cup \emptyset\end{aligned}$$

Substitute $\emptyset = R \cap (B \setminus \{p\})$.

$$= (R \cap (A \setminus \{p\})) \cup (R \cap (B \setminus \{p\}))$$

Apply the fact that $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$ for any sets X, Y, Z.

$$= R \cap ((A \setminus \{p\}) \cup (B \setminus \{p\}))$$

Apply the fact that $(X \setminus Z) \cup (Y \setminus Z) = (X \cup Y) \setminus Z$ for any sets X, Y, Z.

$$= R \cap ((A \cup B) \setminus \{p\})$$

Since there exists a region R with $p \in R$ such that $R \cap ((A \cup B) \setminus \{p\}) = \emptyset$, the contrapositive of Definition 3.13 implies that $p \notin LP(A \cup B)$. □

Corollary 3.21. *Let $A_1, \dots, A_n$ be n subsets of a continuum C. Then p is a limit point of $A_1 \cup \cdots \cup A_n$ if and only if p is a limit point of at least one of the sets A_k.*

Proof. We induct on n from the base case $n_0 = 2$ using the form of induction described in Additional Exercise 0.2a. For the base case $n = 2$, let A_1 and A_2 be two subsets of a continuum C. By Theorem 3.20, it follows that p is a limit point of $A_1 \cup A_2$ if and only if p is a limit point of at least one of A_1 or A_2, proving the base case. Now suppose inductively that we have proven the claim for some n, i.e., we know that if there exist n subsets $A_1, \dots, A_n$ of a continuum C, then p is a limit point of $\bigcup_{k=1}^{n} A_k$ if and only if p is a limit point of at least one of the sets A_k. We wish to prove that if there exist $n+1$ subsets $A_1, \dots, A_{n+1}$ of a continuum C, then p is a limit point of $\bigcup_{k=1}^{n+1} A_k$ if and only if p is a limit point of at least one of the sets A_k. Let $A_1, \dots, A_n$ be $n+1$ subsets of a continuum C. By the induction hypothesis, p is a limit point of $\bigcup_{k=1}^{n} A_k$ if and only if p is a limit point of at least one of the sets A_k. Since $\bigcup_{k=1}^{n} A_k$ and A_{n+1} are both subsets of a continuum C, Theorem 3.20 applies and implies that p is a limit point of $(\bigcup_{k=1}^{n} A_k) \cup A_{n+1} = \bigcup_{k=1}^{n+1} A_k$ if and only if p is a limit point of at least one of $\bigcup_{k=1}^{n} A_k$ or A_{n+1}. But the last two statements combined imply that p is a limit point of $\bigcup_{k=1}^{n+1} A_k$ if and only if p is a limit point of at least one of the sets A_k where $1 \leq k \leq n$ or A_{n+1}, i.e., p is a limit point of $\bigcup_{k=1}^{n+1} A_k$ if and only if p is a limit point of at least one of the sets A_k, thus closing the induction. □

Theorem 3.22. *If p and q are distinct points of a continuum C, then there exist disjoint regions R and S containing p and q, respectively.*

Proof. WLOG, let $p < q$. Additionally, let $a < p$ and $b > q$ be points of C (Axiom 3 and Definition 3.3 imply that such points exist). We divide into two cases (no point $x \in C$ exists between p and q, and there exists a point $x \in C$ between p and q). Let's begin.

Suppose first that no point $x \in C$ exists between p and q. Let $R = \underline{aq}$ and let $S = \underline{pb}$. Thus, Definitions 3.6 and 3.10 imply by the facts that $a < p$ by definition and $p < q$ by hypothesis, and $p < q$ by hypothesis and $q < b$ by definition that $p \in R$ and $q \in S$, respectively. To prove that R and S are disjoint, Definition 1.9 tells us that it will suffice to show that $R \cap S = \emptyset$. Suppose for the sake of contradiction that there exists an object $x \in R \cap S$. By Definition 1.6, this implies that $x \in R$ and $x \in S$ (and, hence, that $x \in C$). Thus, by consecutive applications of Definitions 3.10 and 3.6, we have that $a < x$, $x < q$, $p < x$, and $x < b$. Since $p < x$ and $x < q$, Definition 3.6 implies that x is between p and q. But by hypothesis, no point $x \in C$ exists between p and q, a contradiction. Therefore, since $x \notin R \cap S$ for all x, Definition 1.8 implies that $R \cap S = \emptyset$, as desired.

Now suppose that there exists a point $x \in C$ between between p and q. Let $R = \underline{ax}$ and let $S = \underline{xb}$. It follows from the hypothesis by Definition 3.6 that $p < x$ and $x < q$. Thus, Definitions 3.6 and 3.10 imply by the facts that $a < p$ and $p < x$, and $x < q$ and $q < b$ that $p \in R$ and $q \in S$, respectively. To prove that R and S are disjoint, Definition 1.9 tells us tht it will suffice to show that $R \cap S = \emptyset$. Suppose for the sake of contradiction that there exists an object $y \in R \cap S$. By Definition 1.6, this implies that $y \in R$ and $y \in S$ (and, hence, that $y \in C$). Thus, by consecutive applications of Definitions 3.10 and 3.6, we have that $a < y$, $y < x$, $x < y$, and $y < b$. But by Definition 3.1, we cannot have $y < x$ and $x < y$, a contradiction. Therefore, since $x \notin R \cap S$ for all x, Definition 1.8 implies that $R \cap S = \emptyset$, as desired. ☐

10/29: **Corollary 3.23.** *A subset of a continuum C consisting of one point has no limit points.*

Proof. Let $\{x\} \subset C$. To prove that $\{x\}$ has no limit points, Definition 3.13 tells us that it will suffice to show that for all $p \in C$, there exists a region R with $p \in R$ such that $R \cap (\{x\} \setminus \{p\}) = \emptyset$. Let p be an arbitrary point in C. We divide into two cases ($p = x$ and $p \neq x$). Suppose first that $p = x$. If we let R be any region of C, it follows that

$$\begin{aligned} R \cap (\{x\} \setminus \{p\}) &= R \cap (\{x\} \setminus \{x\}) \\ &= R \cap \emptyset \\ &= \emptyset \end{aligned}$$

as desired. Now suppose that $p \neq x$. Since p and x are distinct points, Theorem 3.22 applies and implies that there exist disjoint regions R and S containing p and x, respectively. Consequently, by Definition 1.9, $x \notin R$. Thus,

$$\begin{aligned} R \cap (\{x\} \setminus \{p\}) &= R \cap \{x\} \\ &= \emptyset \end{aligned}$$

as desired. ☐

Theorem 3.24. *A finite subset A of a continuum C has no limit points.*

Proof. We divide into two cases ($|A| = 0$ and $|A| \in \mathbb{N}$).

If $|A| = 0$, then $A = \emptyset$. Clearly it follows that for an arbitrary $p \in C$ and any region R of C,

$$\begin{aligned} R \cap (A \setminus \{p\}) &= R \cap (\emptyset \setminus \{p\}) \\ &= R \cap \emptyset \\ &= \emptyset \end{aligned}$$

proving by the contrapositive of Definition 3.13 that p is not a limit point of A, as desired.

If $|A| \in \mathbb{N}$, we induct on $|A| = n$. For the base case $n = 1$, A is a subset of C consisting of one point. Thus, Corollary 3.23 applies and implies that A has no limit points, proving the base case. Now suppose inductively that we have proven the claim for n, i.e., we know that a finite subset A of a continuum C satisfying $|A| = n$ has no limit points. We now wish to prove the claim with regard to a subset A of a continuum C with $|A| = n+1$. Let a be any element of A, and let $B = A \setminus \{a\}$. Since $(n+1) \in \mathbb{N}$ (because n and 1 are), all of the tenets of the lemma of Lemma 3.4 are satisfied, implying that $|B| = n$. This revelation combined with the fact that $B \subset C$ means that the induction hypothesis applies; consequently, B has no limit points. Since B has no limit points and $\{a\}$ has no limit points (see Corollary 3.23), Theorem 3.20 applies and implies that $B \cup \{a\} = (A \setminus \{a\}) \cup \{a\} = A$ has no limit points as well, as desired. □

Corollary 3.25. *If A is a finite subset of a continuum C and $x \in A$, then there exists a region R containing x, such that $A \cap R = \{x\}$.*

Proof. Since A is a finite subset of a continuum C, Theorem 3.24 implies that A has no limit points. Thus, for an arbitrary $x \in A$, x is not a limit point of A. Consequently, Definition 3.13 implies that there exists a region R of C with $x \in R$ such that $R \cap (A \setminus \{x\}) = \emptyset$. Moreover, since $x \in A$ and $x \in R$, Definition 1.6 implies that $x \in A \cap R$. Now suppose for the sake of contradiction that there exists some object $y \in A \cap R$ such that $y \neq x$. Then by Definition 1.6, $y \in A$ and $y \in R$. The former discovery combined with the fact that $y \neq x$, i.e., $y \notin \{x\}$ reveals that $y \in A \setminus \{x\}$ by Definition 1.11. Thus, since $y \in R$ and $y \in A \setminus \{x\}$, $y \in R \cap (A \setminus \{x\})$. It follows since $R \cap (A \setminus \{x\}) = \emptyset$ that $y \in \emptyset$, but this contradicts Definition 1.8. Therefore, x is the only element of $A \cap R$, so $A \cap R = \{x\}$, as desired. □