2 Script

2.1 Journal

10/15: **Definition 2.1.** Let X be a set. A **relation** R on X is a subset of $X \times X$. The statement $(x, y) \in R$ is read "x is related to y by the relation R" and is often denoted $x \sim y$.

A relation is **reflexive** if $x \sim x$ for all $x \in X$.
A relation is **symmetric** if $y \sim x$ whenever $x \sim y$.
A relation is **transitive** if $x \sim z$ whenever $x \sim y$ and $y \sim z$.
A relation is an **equivalence relation** if it is reflexive, symmetric, and transitive.

Exercise 2.2. Determine which of the following are equivalence relations.

a) Any set X with the relation $=$. So $x \sim y$ if and only if $x = y$.

Proof. To prove that the relation $=$ is reflexive, Definition 2.1 tells us that it will suffice to show that $x \sim x$ for all $x \in X$. Clearly, $x = x$ for all $x \in X$. It follows by the definition of $=$ that $x \sim x$ for all $x \in X$. For symmetry, we must verify that $x \sim y$ implies $y \sim x$ for any $x, y \in X$. Let $x \sim y$ for some $x, y \in X$. Consequently, by the definition of $=$, $x = y$. It follows that $y = x$, and thus that $y \sim x$. For transitivity, we must show that $x \sim y$ and $y \sim z$ imply that $x \sim z$ for any $x, y, z \in X$. Let $x \sim y$ and $y \sim z$ for some $x, y, z \in X$. By the definition of $=$, $x \sim y$ and $y \sim z$ imply that $x = y$ and $y = z$, respectively. Thus, $x = y = z$, so $x = z$, meaning that $x \sim z$ by the definition of the relation $=$. Since the relation $=$ is reflexive, symmetric, and transitive, it is an equivalence relation. $\square$

b) $\mathbb{Z}$ with the relation $<$.

Proof. The relation $<$ is neither reflexive nor symmetric, although it is transitive. Since demonstrating that $<$ does not satisfy any one of the three properties proves that $<$ is not an equivalence relation, we shall arbitrarily choose to prove that $<$ is not reflexive. Consider $1 \in \mathbb{Z}$, and note that $1 = 1$. Since $1 = 1$, $1 \not< 1$ by the trichotomy. Thus, $1 \nsim 1$ by the relation $<$, proving that $<$ is not reflexive for all $z \in Z$, i.e., $<$ is not an equivalence relation. $\square$

c) Any subset X of $\mathbb{Z}$ with the relation $\leq$. So $x \sim y$ if and only if $x \leq y$.

Proof. Here, we demonstrate a failure of symmetry. Let $X = \{1, 2\}$. Clearly, $X \subset \mathbb{Z}$. Now, $1 \leq 2$, so $1 \sim 2$ by the relation $\leq$, but $2 \nleq 1$ so $2 \nsim 1$. Thus, $x \sim x'$ for $x, x' \in X$ does not necessarily imply that $x' \sim x$. It follows that $\leq$ is not an equivalence relation on *any* subset of $\mathbb{Z}$. $\square$

d) $X = \mathbb{Z}$ with $x \sim y$ if and only if $y - x$ is divisible by 5.

Proof. To prove that the described relation is an equivalence relation, Definition 2.1 tells us that we must verify that it is reflexive, symmetric, and transitive. To prove these properties, it will suffice to show that $x \sim x$ for all $x \in X$, $x \sim y$ implies $y \sim x$ for any $x, y \in X$, and $x \sim y$ and $y \sim z$ implies $x \sim z$ for any $x, y, z \in X$, respectively. Let's begin.

To prove that $x \sim x$ for all $x \in X$, it will suffice to show that $\frac{x-x}{5} \in X$ for an arbitrary element $x \in X$. Let x be such an object. It follows that $\frac{x-x}{5} = \frac{0}{5} = 0$. Since $0 \in X$, $\frac{x-x}{5} \in X$, as desired.

To prove that $x \sim y$ implies that $y \sim x$ for any $x, y \in X$, it will suffice to show that $\frac{x-y}{5} \in X$ given that $\frac{y-x}{5} \in X$. Since $\frac{y-x}{5} \in X$, it follows by the set theoretic definition of $\mathbb{Z}$ that $-\frac{y-x}{5} \in X$. But $-\frac{y-x}{5} = \frac{x-y}{5}$, so $\frac{x-y}{5} \in X$, as desired.

To prove that $x \sim y$ and $y \sim z$ implies that $x \sim y$ for any $x, y, z \in X$, it will suffice to show that $\frac{z-x}{5} \in X$ given that $\frac{y-x}{5} \in X$ and $\frac{z-y}{5} \in X$. Since $\frac{y-x}{5} \in X$ and $\frac{z-y}{5} \in X$, it follows by the closure of addition for integers that $\left(\frac{z-y}{5} + \frac{y-x}{5}\right) \in X$. But $\frac{z-y}{5} + \frac{y-x}{5} = \frac{z-y+y-x}{5} = \frac{z-x}{5}$, so $\frac{z-x}{5} \in X$, as desired. $\square$

e) $X = \{(a,b) \mid a,b \in \mathbb{Z}, b \neq 0\}$ with the relation $\sim$ defined by $(a,b) \sim (c,d) \iff ad = bc$.

Proof. Reflexivity: Let (a,b) be an arbitrary element of X. Since $a,b \in \mathbb{Z}$ and integer multiplication is commutative, $ab = ba$. Therefore, by the definition of the relation $\sim$, $(a,b) \sim (a,b)$.

Symmetry: Let $(a,b) \sim (c,d)$ for some $(a,b),(c,d) \in X$. By the definition of the relation $\sim$, $ad = bc$. Thus, $cb = da$ by the symmetry of $=$ (see part (a)) and the commutativity of integer multiplication. Consequently, by the definition of the relation $\sim$, $(c,d) \sim (a,b)$.

Transitivity: Let $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$ for some $(a,b),(c,d),(e,f) \in X$. By consecutive applications of the definition of $\sim$, $ad = bc$ and $cf = de$. Now if we consider $ad = bc$, we can multiply an equal quantity to each side and still preserve the equality. As such, we choose to multiply $cf = de$ to both sides, creating the equation $ad \cdot cf = bc \cdot de$. By the commutativity of multiplication, we have $afcd = becd$. By the cancellation law for multiplication, we have $af = be$ (we cancel out cd from both sides). Therefore, by the definition of the relation $\sim$, $(a,b) \sim (e,f)$. $\square$

Remark 2.3. A **partition** of a set is a collection of non-empty disjoint subsets whose union is the original set. Any equivalence relation on a set creates a partition of that set by collecting into subsets all of the elements that are equivalent (related) to each other. When the partition of a set arises from an equivalence relation in this manner, the subsets are referred to as **equivalence classes**.

Remark 2.4. If we think of the set X in Exercise 2.2e as representing the collection of all fractions whose denominators are not zero, then the relation $\sim$ may be thought of as representing the equivalence of two fractions.

Definition 2.5. As a set, the **rational numbers**, denoted $\mathbb{Q}$, are the equivalence classes in the set $X = \{(a,b) \mid a,b \in \mathbb{Z}, b \neq 0\}$ under the equivalence relation $\sim$ as defined in Exercise 2.2e. If $(a,b) \in X$, we denote the equivalence class of this element as $\left[\frac{a}{b}\right]$. So

$$\left[\frac{a}{b}\right] = \{(x_1,x_2) \in X \mid (x_1,x_2) \sim (a,bb)\} = \{(x_1,x_2) \in X \mid x_1 b = x_2 a\}$$

Then,

$$\mathbb{Q} = \left\{ \left[\frac{a}{b}\right] \middle| (a,b) \in X \right\}$$

Exercise 2.6. $\left[\frac{a}{b}\right] = \left[\frac{a'}{b'}\right] \iff (a,b) \sim (a',b')$

Proof. To prove this claim, we must prove the two implications

$$\left[\frac{a}{b}\right] = \left[\frac{a'}{b'}\right] \implies (a,b) \sim (a',b') \qquad\qquad (a,b) \sim (a',b') \implies \left[\frac{a}{b}\right] = \left[\frac{a'}{b'}\right]$$

Suppose first that $\left[\frac{a}{b}\right] = \left[\frac{a'}{b'}\right]$. Then by Definition 2.5,

$$\{(x_1,x_2) \in X \mid (x_1,x_2) \sim (a,b)\} = \{(x_1,x_2) \in X \mid (x_1,x_2) \sim (a',b')\}$$

Since $(a,b) \sim (a,b)$ by Exercise 2.2e and clearly $(a,b) \in X$, it follows that $(a,b) \in \{(x_1,x_2) \in X \mid (x_1,x_2) \sim (a,b)\}$. Consequently, set equality implies that $(a,b) \in \{(x_1,x_2) \in X \mid (x_1,x_2) \sim (a',b')\}$. Thus, $(a,b) \sim (a',b')$, as desired.

Now suppose that $(a,b) \sim (a',b')$. To prove that $\left[\frac{a}{b}\right] = \left[\frac{a'}{b'}\right]$, Definition 2.5 tells us that it will suffice to show that

$$\{(x_1,x_2) \in X \mid (x_1,x_2) \sim (a,b)\} = \{(x_1,x_2) \in X \mid (x_1,x_2) \sim (a',b')\}$$

Let (x_1,x_2) be an arbitrary element of $\{(x_1,x_2) \in X \mid (x_1,x_2) \sim (a,b)\}$. It follows that $(x_1,x_2) \sim (a,b)$. Thus, since $(a,b) \sim (a',b')$, the transitivity of $\sim$ (see Exercise 2.2e) implies that $(x_1,x_2) \sim (a',b')$. This coupled with the fact that $(x_1,x_2) \in X$ means that $(x_1,x_2) \in \{(x_1,x_2) \in X \mid (x_1,x_2) \sim (a',b')\}$. The proof is symmetric if we first let that (x_1,x_2) be an arbitrary element of $\{(x_1,x_2) \in X \mid (x_1,x_2) \sim (a',b')\}$. $\square$

Definition 2.7. We define the binary operations addition and multiplication on $\mathbb{Q}$ as follows. If $\left[\frac{a}{b}\right], \left[\frac{c}{d}\right] \in \mathbb{Q}$, then

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{ad+bc}{bd}\right]$$
$$\left[\frac{a}{b}\right] \cdot_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{ac}{bd}\right]$$

We use the notation $+_{\mathbb{Q}}$ and $\cdot_{\mathbb{Q}}$ to represent addition and multiplication in $\mathbb{Q}$ so as to distinguish these operations from the usual addition $(+)$ and multiplication $(\cdot)$ in $\mathbb{Z}$.

Theorem 2.8. *Addition in $\mathbb{Q}$ is well-defined. That is, if $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$, then*

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{a'}{b'}\right] +_{\mathbb{Q}} \left[\frac{c'}{d'}\right]$$

Lemma. *If $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$, then $(ad+bc, bd) \sim (a'd'+b'c', b'd')$.*

Proof. By consecutive applications of the definition of $\sim$, we have that

$$ab' = ba' \qquad\qquad cd' = dc'$$

It follows by the multiplicative property of equality that

$$ab'dd' = ba'dd' \qquad\qquad bb'cd' = bb'dc'$$

The above two results can be combined via the additive property of equality, giving the following, which will further be algebraically manipulated.

$$\begin{aligned} ab'dd' + bb'cd' &= ba'dd' + bb'dc' \\ adb'd' + bcb'd' &= bda'd' + bdb'c' \\ (ad+bc)(b'd') &= (bd)(a'd'+b'c') \end{aligned}$$

The last line above implies by the definition of $\sim$ that $(ad+bc, bd) \sim (a'd'+b'c', b'd')$, as desired. □

Proof. Suppose for the sake of contradiction that

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{c}{d}\right] \neq \left[\frac{a'}{b'}\right] +_{\mathbb{Q}} \left[\frac{c'}{d'}\right] \text{ for some } \left[\frac{a}{b}\right], \left[\frac{c}{d}\right], \left[\frac{a'}{b'}\right], \left[\frac{c'}{d'}\right] \in \mathbb{Q}$$

Since

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{ad+bc}{bd}\right] \qquad\qquad \left[\frac{a'}{b'}\right] +_{\mathbb{Q}} \left[\frac{c'}{d'}\right] = \left[\frac{a'd'+b'c'}{b'd'}\right]$$

by Definition 2.7, the supposition implies that

$$\left[\frac{ad+bc}{bd}\right] \neq \left[\frac{a'd'+b'c'}{b'd'}\right]$$

By Exercise 2.6, this means that $(ad+bc, bd) \nsim (a'd'+b'c', b'd')$. But since $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$ by hypothesis, the lemma tells us that $(ad+bc, bd) \sim (a'd'+b'c', b'd')$, a contradiction. Therefore,

$$\left[\frac{a}{b}\right] +_{\mathbb{Q}} \left[\frac{c}{d}\right] = \left[\frac{a'}{b'}\right] +_{\mathbb{Q}} \left[\frac{c'}{d'}\right]$$

under the given conditions, as desired. □