Script 4

The Topology of a Continuum

4.1 Journal

11/3: **Definition 4.1.** A subset of a continuum is **closed** if it contains all of its limit points.

Theorem 4.2. *The sets $\emptyset$ and C are closed.*

Proof. Let C be a continuum. We will address the two sets individually.

To prove that $\emptyset$ is closed, Definition 4.1 tells us that it will suffice to show that $\emptyset \subset C$ and $\emptyset$ contains all of its limit points. By Exercise 1.10, $\emptyset \subset C$. We now prove that $\emptyset$ has no limit points. Suppose for the sake of contradiction that some point $p \in C$ is a limit point of $\emptyset$. Then by Definition 3.13, for all regions R with $p \in R$, $R \cap (\emptyset \setminus \{p\}) \neq \emptyset$. But clearly, $R \cap (\emptyset \setminus \{p\}) = R \cap \emptyset = \emptyset$, a contradiction. Therefore, since $\emptyset$ has no limit points, the statement "$\emptyset$ contains all of its limit points" is vacuously true.

To prove that C is closed, Definition 4.1 tells us that it will suffice to show that $C \subset C$ and C contains all of its limit points. Since $C = C$, Theorem 1.7 implies that $C \subset C$. Now suppose for the sake of contradiction that C does not contain all of its limit points. Then there exists a point $p \in C$ that is a limit point of C such that $p \notin C$. But we cannot have $p \in C$ and $p \notin C$, so it must be that the initial hypothesis was incorrect, meaning that C does, in fact, contain all of its limit points. $\square$

Theorem 4.3. *A subset of C containing a finite number of points is closed.*

Proof. Let A be a finite subset of C. To prove that A is closed, Definition 4.1 tells us that it will suffice to show that A contains all of its limit points. But by Theorem 3.24, A has no limit points, so the statement "A contains all of its limit points" is vacuously true. $\square$

Definition 4.4. Let X be a subset of C. The **closure** of X is the subset $\overline{X}$ of C defined by

$$\overline{X} = X \cup LP(X)$$

Theorem 4.5. *$X \subset C$ is closed if and only if $X = \overline{X}$.*

Proof. Suppose first that X is closed. To prove that $X = \overline{X}$, Definition 4.4 tells us that it will suffice to show that $X = X \cup LP(X)$. To show this, Definition 1.2 tells us that we must verify that every element x of X is an element of $X \cup LP(X)$ and vice versa. First, let x be an arbitrary element of X. Then by Definition 1.5, $x \in X \cup LP(X)$, as desired. Now let x be an arbitrary element of $X \cup LP(X)$. Then by Definition 1.5, $x \in X$ or $x \in LP(X)$. We divide into two cases. If $x \in X$, then we are done. If $x \in LP(X)$, then $x \in X$ as desired for the following reason: Since X is closed by hypothesis, Definition 4.1 implies that X contains all of its limit points, i.e., for all $y \in LP(X)$, $y \in X$; this implication notably applies to the x in question.

Now suppose that $X = \overline{X}$. To prove that X is closed, Definition 4.1 tells us that it will suffice to show that X contains all of its limit points. By Theorem 1.7, $LP(X) \subset X \cup LP(X)$. This combined with the fact that $X = X \cup LP(X)$ (by Definition 4.4, since $X = \overline{X}$) implies that $LP(X) \subset X$. It follows by Definition 1.3 that every element of $LP(X)$ is an element of X, i.e., every limit point of X is an element of X, i.e., X contains all of its limit points, as desired. $\square$

Theorem 4.6. *Let $X \subset C$. Then $\overline{X} = \overline{\overline{X}}$.*

Proof. By Theorem 1.7, $LP(X) \subset X \cup LP(X)$. Thus, $(X \cup LP(X)) \cup LP(X) = X \cup LP(X)$. But this implies that $X \cup LP(X) = X \cup LP(X) \cup LP(X)$, meaning by several applications of Definition 4.4 that $\overline{X} = \overline{\overline{X}}$, as desired. $\square$

Corollary 4.7. *Let $X \subset C$. Then $\overline{X}$ is closed.*

Proof. By Theorem 4.6, $\overline{X} = \overline{\overline{X}}$. Thus, if we let $Y = \overline{X}$, we know that $Y = \overline{Y}$. But by Theorem 4.5, this implies that Y, i.e., $\overline{X}$, is closed, as desired. $\square$