Script 5

Connectedness and Boundedness

5.1 Journal

11/19: **Axiom 4.** *A continuum is connected.*

Theorem 5.1. *The only subsets of a continuum C that are both open and closed are $\emptyset$ and C.*

Proof. To prove that the only subsets of C that are both open and closed are $\emptyset$ and C, it will suffice to show that if $A \subset C$ is both open and closed, then $A = \emptyset$ or $A = C$. Let $A \subset C$ be both open and closed. We divide into two cases ($A = \emptyset$ and $A \neq \emptyset$). If $A = \emptyset$, then we are done. On the other hand, if $A \neq \emptyset$, we have a bit more work to do. Basically, we will end up proving that the facts that A is open, A is closed, and $A \neq \emptyset$ imply that $A = C$. Let's begin.

First off, the fact that A is closed implies by Definition 4.8 that $C \setminus A$ is open. Additionally, we have by Script 1 that $A \cap (C \setminus A) = \emptyset$ and $A \cup (C \setminus A) = C$. Now suppose for the sake of contradiction that $A \neq C$. It follows since $A \subset C$ that we must have $C \not\subset A$, i.e., there is some object in C that is not an element of A. This object would clearly be an element of $C \setminus A$ in this case, meaning that $C \setminus A$ is nonempty. Thus, we have that A and $C \setminus A$ are disjoint, open, nonempty sets such that $A \cup (C \setminus A) = C$. Consequently, by consecutive applications of Definition 4.22, we know that C is disconnected, i.e., C is not connected. But this contradicts Axiom 4, which asserts that C is connected. Therefore, we must have that $A = C$, as desired. $\square$

Theorem 5.2. *For all $x, y \in C$, if $x < y$, then there exists a point $z \in C$ such that z is between x and y.*

Proof. Suppose for the sake of contradiction that no point $z \in C$ exists such that z is between x and y. To find a contradiction, we will let $A = \{c \in C \mid c < y\}$ and $B = \{c \in C \mid x < c\}$ and prove that $A \cup B = C$, and that A and B are disjoint, nonempty, open sets. This will imply that C is disconnected, contradicting Axiom 4. Let's begin.

Suppose for the sake of contradiction that $C \neq A \cup B$. Then by Theorem 1.7, $C \not\subset A \cup B$ or $A \cup B \not\subset C$. Since $A \subset C$ and $B \subset C$ by their definitions, we have $A \cup B \subset C$, so it must be that $C \not\subset A \cup B$. Thus, by Definition 1.3, there exists a point $p \in C$ such that $p \notin A \cup B$. From the latter condition, we have by Definition 1.5 that $p \notin A$ and $p \notin B$. It follows from the definitions of A and B that $p \notin C$, or $p \not< y$ and $x \not< p$. But we know that $p \in C$, so it must be that $p \not< y$ and $x \not< p$. Equivalently, $p \geq y$ and $x \geq p$. But this implies that $x \geq y$, which contradicts the fact that $x < y$ by hypothesis. Therefore, we must have $C = A \cup B$, as desired.

Suppose for the sake of contradiction that A and B are not disjoint. Then by Definition 1.9, $A \cap B \neq \emptyset$. Thus, Definition 1.8 tells us that there exists some object $p \in A \cap B$. By Definition 1.6, this implies that $p \in A$ and $p \in B$. It follows by the definitions of A and B that $p \in C$, $p < y$, and $x < p$. Since $x < p < y$, Definition 3.6 tells us that p is between x and y, contradicting the supposition that no such point exists. Therefore, A and B are disjoint, as desired.

To prove that A and B are nonempty, Definition 1.8 tells us that it will suffice to show that there exists an object in each set. Since $x \in C$ and $x < y$, $x \in A$. Similarly, since $y \in C$ and $x < y$, $y \in B$. Therefore, A and B are nonempty, as desired.

By Corollary 4.13, A and B are open, as desired.

Since C can be written as $A \cup B$ where A and B are disjoint, nonempty, open sets, we have by Definition 4.22 that C is disconnected. But this contradicts Axiom 4, which asserts that C is connected. Therefore, there must exist a point $z \in C$ such that z is between x and y, as desired. $\square$

Corollary 5.3. *Every region is infinite.*

Proof. Let $\underline{ab}$ be a region, and suppose for the sake of contradiction that $\underline{ab}$ is finite. Then by Definitions 1.30 and 1.33, $\underline{ab} = \emptyset$, or $\underline{ab}$ has cardinality n. We divide into two cases. Suppose first that $\underline{ab} = \emptyset$. Then by Definitions 3.10 and 3.6, no point p exists such that $a < p < b$. Thus, by the contrapositive of Theorem 5.2, $a = b$. But this implies by Definition 3.10 that $\underline{ab}$ is not a region (since $a \not< b$), a contradiction. Now suppose that $\underline{ab}$ has cardinality n. Then by Theorem 3.5, the symbols $a_1, \dots, a_n$ may be assigned to each point of $\underline{ab}$ so that $a_1 < a_2 < \cdots < a_n$. But by Theorem 5.2, there exists a point $z \in C$ such that z is between a and a_1. Since $a < z < a_1 < b$, we clearly have that $z \in \underline{ab}$, yet it was not assigned a symbol a_k, a contradiction. Therefore, $\underline{ab}$ is infinite, as desired. $\square$

12/1: **Corollary 5.4.** *Every point of C is a limit point of C.*

Proof. Let p be an arbitrary element of C. To prove that p is a limit point of C, Definition 3.13 tells us that it will suffice to show that for all regions R with $p \in R$, $R \cap (C \setminus \{p\}) \neq \emptyset$. Let R be an arbitrary region with $p \in R$. By Corollary 5.3, R is infinite, so there exists a point $q \in R$ such that $q \neq p$. Additionally, since $q \in R$, we have $q \in C$. Thus, since $q \in C$ and $q \neq p$ (i.e., $q \notin \{p\}$), we have by Definition 1.11 that $q \in C \setminus \{p\}$. This combined with the fact that $q \in R$ implies by Definition 1.6 that $q \in R \cap (C \setminus \{p\})$, so $R \cap (C \setminus \{p\}) \neq \emptyset$, as desired. $\square$

Corollary 5.5. *Every point of the region $\underline{ab}$ is a limit point of $\underline{ab}$.*

Proof. Let p be an arbitrary element of $\underline{ab}$. It follows that $p \in C$. Thus, by Corollary 5.4, $p \in LP(C)$. Consequently, since $C = \underline{ab} \cup (C \setminus \underline{ab})$, Theorem 3.20 implies that $p \in LP(\underline{ab})$ or $p \in LP(C \setminus \underline{ab})$. Suppose for the sake of contradiction that $p \in LP(C \setminus \underline{ab})$. Then $p \in LP((C \setminus (\{a\} \cup \underline{ab} \cup \{b\}) \cup \{a\} \cup \{b\}))$. It follows by Definition 3.15 that $p \in LP(\operatorname{ext} \underline{ab} \cup \{a\} \cup \{b\})$. Consequently, by Corollary 3.21, $p \in LP(\operatorname{ext} \underline{ab})$, $p \in LP(\{a\})$, or $p \in LP(\{b\})$. But by Lemma 3.17, the fact that $p \in \underline{ab}$ prohibits p from being a limit point of $\operatorname{ext} \underline{ab}$, and by Corollary 3.23, $p \notin LP(\{a\})$ and $p \notin LP(\{b\})$, either, a contradiction. Therefore, $p \notin LP(C \setminus \underline{ab})$, so we must have $p \in LP(\underline{ab})$, as desired. $\square$

Definition 5.6. Let X be a subset of C. A point u is called an **upper bound** of X if for all $x \in X$, $x \leq u$. A point l is called a **lower bound** of X if for all $x \in X$, $l \leq x$. If there exists an upper bound of X, then we say that X is **bounded above**. If there exists a lower bound of X, then we say that X is **bounded below**. If X is bounded above and below, then we simply say that X is **bounded**.

Definition 5.7. Let X be a subset of C. We say that u is a **least upper bound** of X and write $u = \sup X$ if:

1. u is an upper bound of X;
2. if u' is an upper bound of X, then $u \leq u'$.

We say that l is a **greatest lower bound** and write $l = \inf X$ if:

1. l is a lower bound of X;
2. if l' is a lower bound of X, then $l' \leq l$.

The notation sup comes from the word **supremum**, which is another name for least upper bound. The notation inf comes from the word **infimum**, which is another name for greatest lower bound.

Exercise 5.8. If $\sup X$ exists, then it is unique, and similarly for $\inf X$.

Proof. Let X be a subset of a continuum C such that $\sup X$ exists, and suppose that both u and u' are least upper bounds of X. It follows from the supposition and Definition 5.7 that u, u' are both upper bounds of X. Thus, since u is a least upper bound of X and u' is an upper bound of X, we have by Definition 5.7 again that $u \leq u'$. By a symmetric argument, we also have that $u' \leq u$. But since $u \leq u'$ and $u' \leq u$, $u = u'$, proving the uniqueness of $\sup X$.

The proof is symmetric for $\inf X$. □

Exercise 5.9. If X has a first point L, then $\inf X$ exists and equals L. Similarly, if X has a last point U, then $\sup X$ exists and equals U.

Proof. Let L be the first point of X. Then by Definition 3.3, for all $x \in X$, $L \leq x$. Thus, by Definition 5.6, L is a lower bound of X. Now suppose for the sake of contradiction that there exists a lower bound L' of X such that $L' > L$. Since L' is a lower bound, Definition 5.6 implies that for all $x \in X$, $L' \leq x$. But L is an element of X and $L < L'$, a contradiction. Therefore, if L' is a lower bound of X, then $L' \leq L$. This result coupled with the fact that L is a lower bound of X implies by Definition 5.7 that $L = \inf X$.

The proof is symmetric in the other case. □

Exercise 5.10. For this exercise, we assume that $C = \mathbb{R}$. Find $\sup X$ and $\inf X$ for each of the following subsets of $\mathbb{R}$, or state that they do not exist. You need not give proofs.

1. $X = \mathbb{N}$.

   Answer. $\sup X$ does not exist, but $\inf X = 1$. □

2. $X = \mathbb{Q}$.

   Answer. Neither $\sup X$ nor $\inf X$ exists. □

3. $X = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$.

   Answer. $\sup X = 1$ and $\inf X = 0$. □

4. $X = \{x \in \mathbb{R} \mid 0 < x < 1\}$.

   Answer. $\sup X = 1$ and $\inf X = 0$. □

5. $X = \{3\} \cup \{x \in \mathbb{R} \mid -7 \leq x \leq -5\}$.

   Answer. $\sup X = 3$ and $\inf X = -7$. □

Lemma 5.11. *Suppose that $X \subset C$ and $s = \sup X$. If $p < s$, then there exists an $x \in X$ such that $p < x \leq s$. Similarly, suppose that $X \subset C$ and $l = \inf X$. If $l < p$, then there exists an $x \in X$ such that $l \leq x < p$.*

Proof. We divide into two cases ($s \in X$ and $s \notin X$).

Suppose first that $s \in X$. Let $x = s$. Then if $p < s$, we know that $p < x = s$, i.e., $p < x \leq s$ where $x \in X$ by definition, as desired.

Now suppose that $s \notin X$. Suppose for the sake of contradiction that for some $p < s$, no $x \in X$ exists such that $p < x \leq s$. Since s is a least upper bound of X, Definitions 5.7 and 5.6 imply[1] that for all $x \in X$, $x \leq s$. Consequently, by the supposition, it is true that for all $x \in X$, $x \leq p$ (if there existed an $x > p$, then this point would satisfy $p < x \leq s$, contradicting the supposition). Thus, by Definition 5.6, p is a upper bound of X. But since $p < s$, it is not true that $s \leq s'$ for all upper bounds s' of X, meaning by Definition 5.7 that s is not a least upper bound of X, a contradiction. Therefore, if $p < s$, then there exists an $x \in X$ such that $p < x \leq s$, as desired.

The proof is symmetric in the other case. □

[1] Technically, Definition 5.7 implies that s is an upper bound of X and Definition 5.6 implies based off of this result that for all $x \in X$, $x \leq s$. However, to avoid having to write this every time, I will shorthand this concept in this fashion.

Theorem 5.12. *Let $a < b$. The least upper bound and greatest lower bound of the region $\underline{ab}$ are* $\sup \underline{ab} = b$ *and* $\inf \underline{ab} = a$.

Proof. To prove that $\sup \underline{ab} = b$, Definition 5.7 tells us that it will suffice to show that b is an upper bound of $\underline{ab}$ and that if u is an upper bound of $\underline{ab}$, then $b \leq u$. For the first condition, Definition 5.6 tells us that it will suffice to confirm that for all $x \in \underline{ab}$, $x \leq b$. Let x be an arbitrary element of $\underline{ab}$. Then by Definitions 3.10 and 3.6, we know that $a < x < b$, i.e., $x \leq b$, as desired. For the second condition, suppose for the sake of contradiction that u is an upper bound of $\underline{ab}$ such that $u < b$. Then by Definition 5.6, for all $x \in \underline{ab}$, $x \leq u$. Additionally, since $\underline{ab}$ is infinite by Corollary 5.3, we know that at least one such x exists, which we shall hereafter refer to as y. Note that as an element of $\underline{ab}$, y satisfies $a < y < b$ by Definitions 3.10 and 3.6. Furthermore, since $u < b$, Theorem 5.2 implies that there exists a point z such that z is between u and b. Thus, by Definition 3.6, $u < z < b$. Combining the last few results, we have $a < y \leq u < z < b$. Consequently, since $a < z < b$, we have by Definitions 3.6 and 3.10 that $z \in \underline{ab}$ and $u < z$, contradicting the statement that for all $x \in \underline{ab}$, $x \leq u$. Therefore, if u is an upper bound of $\underline{ab}$, then $b \leq u$, as desired.

The proof is symmetric in the other case. □