Script 3

Introducing a Continuum

3.1 Journal

10/20: **Axiom 1.** *A continuum is a nonempty set C.*

Definition 3.1. Let X be a set. An **ordering** on the set X is a subset $<$ of $X \times X$ with elements $(x, y) \in <$ written as $x < y$, satisfying the following properties:

a) (*Trichotomy*) For all $x, y \in X$, exactly one of the following holds: $x < y$, $y < x$, or $x = y$.

b) (*Transitivity*) For all $x, y, z \in X$, if $x < y$ and $y < z$, then $x < z$.

Remark 3.2.

a) In mathematics, "or" is understood to be inclusive unless stated otherwise. So in Definition 3.1a above, the word "exactly" is needed.

b) $x < y$ may also be written as $y > x$.

c) By $x \leq y$, we mean $x < y$ or $x = y$; similarly for $x \geq y$.

d) We often refer to elements of a continuum C as **points**.

Axiom 2. *A continuum C has an ordering $<$.*

Definition 3.3. If $A \subset C$, then a point $a \in A$ is a **first** point of A if for every element $x \in A$, either $a < x$ or $a = x$. Similarly, a point $b \in A$ is called a **last** point of A if, for every $x \in A$, either $x < b$ or $x = b$.

Lemma 3.4. *If A is a nonempty, finite subset of a continuum C, then A has a first and last point.*

Lemma. *Let A be a nonempty, finite set (i.e., $|A| = n$ for some $n \in \mathbb{N}$), let a be any element of A, and let the set $B = A \setminus \{a\}$. Then $|B| = n - 1$.*

Proof. We first prove that $|\{a\}| = 1$. By Definition 1.33, to do so, it will suffice to find a bijection $f : \{a\} \to [1]$. Since $[1] = \{1\}$ by Definition 1.29, $f : \{a\} \to \{1\}$ defined by $f(a) = 1$ is clearly such a bijection. We now demonstrate that $B \cap \{a\} = (A \setminus \{a\}) \cap \{a\} = \emptyset$. The previous two results combined with the fact that $B \cup \{a\} = (A \setminus \{a\}) \cup \{a\} = A$ imply by Theorem 1.34b that $|A| = |B| + |\{a\}|$. It follows that $n = |B| + 1$, so $|B| = n - 1$. $\square$

Proof of Lemma 3.4. We consider first points herein (the proof is symmetric for last points). If A is a finite set, then by Definition 1.30, $|A| = n$ for some $n \in \mathbb{N}$. Thus, if we prove the claim for each $n \in \mathbb{N}$ individually, we will have proven the claim. To prove a property pertaining to any natural number, we induct on n.

For the base case $n = 1$, there is only one element (which we may call a) in A. Since $a = a$, i.e., "for every $x \in A$, either $a < x$ or $a = x$" is a true statement, it follows by Definition 3.3 that A has a first point. Now suppose inductively that we have proven the claim for n, i.e., we know that if A is a nonempty, finite subset of a continuum C with $|A| = n$, then A has a first point. We wish to prove the same claim if $|A| = n + 1$. Let a be an arbitrary element of A, and consider the set $B = A \setminus \{a\}$. By the lemma, $|B| = n$. Consequently, the induction hypothesis applies and asserts that B has a first point a_0. Clearly, a_0 is also an element of A, but it may or may not be the first point of A (the first point may now be a). Since C has an ordering $<$ (see Axiom 2), Definition 3.1 asserts that either $a < a_0$, $a_0 < a$, or $a = a_0$. We now divide into three cases. If $a < a_0$, then since $a_0 \leq x$ for all $x \in A$ by Definition 3.3, Definition 3.1 implies that $a \leq x$ for all $x \in A$. Thus, by Definition 3.3, a is the first point in A, and we have proven the claim for $|A| = n + 1$ in this case. If $a_0 < a$, then it is still true that $a_0 \leq x$ for all $x \in A$. This means by Definition 3.3 that a_0 is still the first point in A, proving the claim for $|A| = n + 1$ in this case. If $a = a_0$, then $a \in B$, contradicting the fact that $B = A \setminus \{a\}$, so we need not consider this final case. This closes the induction. □

Theorem 3.5. *Suppose that A is a set of n distinct points in a continuum C, or in other words, $A \subset C$ has cardinality n. Then the symbols $a_1, \dots, a_n$ may be assigned to each point of A so that $a_1 < a_2 < \cdots < a_n$, i.e., $a_i < a_{i+1}$ for all $1 \leq i \leq n - 1$.*

Proof. We divide into two cases ($|A| = 0$ and $|A| \in \mathbb{N}$).

If $|A| = 0$, then the statements "the symbols $a_1, \dots, a_n$ may be assigned to each point of A" and "$a_i < a_{i+1}$ for all $1 \leq i \leq n - 1 = -1$" are both vacuously true.

If $|A| \in \mathbb{N}$, we induct on $|A| = n$. For the base case $n = 1$, denote the single element of A by a_1. Since $a_i < a_{i+1}$ for all $1 \leq i \leq n - 1 = 0$ is vacuously true, the base case holds. Now suppose inductively that we have proven the claim for n, i.e., for a set $A \subset C$ satisfying $|A| = n$, the symbols $a_1, \dots, a_n$ may be assigned to each point of A so that $a_1 < a_2 < \cdots < a_n$. We now wish to prove the claim with regards to a set $A \subset C$ with $|A| = n + 1$. By Lemma 3.4, there is a last point $a_{n+1} \in A$, which may be denoted as such (we will rigorously confirm this later). Since the set $A \setminus \{a_{n+1}\}$ has cardinality n (see the lemma from Lemma 3.4), we have by the induction hypothesis that its n elements can be named $a_1, \dots, a_n$ and ordered $a_1 < a_2 < \cdots < a_n$. Clearly these n elements are elements of A and all that's left to do is determine where a_{n+1} fits into the established order. But by Definition 3.3, $x \leq a_{n+1}$ for all $x \in A$, i.e., $x < a_{n+1}$ for all $x \in A \setminus \{a_{n+1}\}$. Consequently, as its name would suggest, it is true that $a_1 < a_2 < \cdots < a_n < a_{n+1}$, as desired. □

Definition 3.6. If $x, y, z \in C$ and either (i) both $x < y$ and $y < z$ or (ii) both $z < y$ and $y < x$, then we say that y is **between** x and z.

Corollary 3.7. *Of three distinct points in a continuum, one must be between the other two.*

Proof. Let A be a subset of the described continuum containing the three distinct points. It follows by Theorem 3.5 that the symbols a_1, a_2, a_3 may be assigned to each point of A so that $a_1 < a_2 < a_3$. Thus, $a_1 < a_2$ and $a_2 < a_3$, so a_2 is between a_1 and a_3 by Definition 3.6. □