Script 3

Introducing a Continuum

3.1 Journal

10/20: **Axiom 1.** *A continuum is a nonempty set C.*

Definition 3.1. Let X be a set. An **ordering** on the set X is a subset $<$ of $X \times X$ with elements $(x, y) \in <$ written as $x < y$, satisfying the following properties:

a) (*Trichotomy*) For all $x, y \in X$, exactly one of the following holds: $x < y$, $y < x$, or $x = y$.

b) (*Transitivity*) For all $x, y, z \in X$, if $x < y$ and $y < z$, then $x < z$.

Remark 3.2.

a) In mathematics, "or" is understood to be inclusive unless stated otherwise. So in Definition 3.1a above, the word "exactly" is needed.

b) $x < y$ may also be written as $y > x$.

c) By $x \leq y$, we mean $x < y$ or $x = y$; similarly for $x \geq y$.

d) We often refer to elements of a continuum C as **points**.

Axiom 2. *A continuum C has an ordering $<$.*

Definition 3.3. If $A \subset C$, then a point $a \in A$ is a **first** point of A if for every element $x \in A$, either $a < x$ or $a = x$. Similarly, a point $b \in A$ is called a **last** point of A if, for every $x \in A$, either $x < b$ or $x = b$.

Lemma 3.4. *If A is a nonempty, finite subset of a continuum C, then A has a first and last point.*

Lemma. *Let A be a nonempty, finite set (i.e., $|A| = n$ for some $n \in \mathbb{N}$), let a be any element of A, and let the set $B = A \setminus \{a\}$. Then $|B| = n - 1$.*

Proof. We first prove that $|\{a\}| = 1$. By Definition 1.33, to do so, it will suffice to find a bijection $f : \{a\} \to [1]$. Since $[1] = \{1\}$ by Definition 1.29, $f : \{a\} \to \{1\}$ defined by $f(a) = 1$ is clearly such a bijection. We now demonstrate that $B \cap \{a\} = (A \setminus \{a\}) \cap \{a\} = \emptyset$. The previous two results combined with the fact that $B \cup \{a\} = (A \setminus \{a\}) \cup \{a\} = A$ imply by Theorem 1.34b that $|A| = |B| + |\{a\}|$. It follows that $n = |B| + 1$, so $|B| = n - 1$. $\square$

Proof of Lemma 3.4. We consider first points herein (the proof is symmetric for last points). If A is a finite set, then by Definition 1.30, $|A| = n$ for some $n \in \mathbb{N}$. Thus, if we prove the claim for each $n \in \mathbb{N}$ individually, we will have proven the claim. To prove a property pertaining to any natural number, we induct on n.

For the base case $n = 1$, there is only one element (which we may call a) in A. Since $a = a$, i.e., "for every $x \in A$, either $a < x$ or $a = x$" is a true statement, it follows by Definition 3.3 that A has a first point. Now suppose inductively that we have proven the claim for n, i.e., we know that if A is a nonempty, finite subset of a continuum C with $|A| = n$, then A has a first point. We wish to prove the same claim if $|A| = n+1$. Let a be an arbitrary element of A, and consider the set $B = A \setminus \{a\}$. By the lemma, $|B| = n$. Consequently, the induction hypothesis applies and asserts that B has a first point a_0. Clearly, a_0 is also an element of A, but it may or may not be the first point of A (the first point may now be a). Since C has an ordering $<$ (see Axiom 2), Definition 3.1 asserts that either $a < a_0$, $a_0 < a$, or $a = a_0$. We now divide into three cases. If $a < a_0$, then since $a_0 \leq x$ for all $x \in A$ by Definition 3.3, Definition 3.1 implies that $a \leq x$ for all $x \in A$. Thus, by Definition 3.3, a is the first point in A, and we have proven the claim for $|A| = n+1$ in this case. If $a_0 < a$, then it is still true that $a_0 \leq x$ for all $x \in A$. This means by Definition 3.3 that a_0 is still the first point in A, proving the claim for $|A| = n+1$ in this case. If $a = a_0$, then $a \in B$, contradicting the fact that $B = A \setminus \{a\}$, so we need not consider this final case. This closes the induction. $\square$

Theorem 3.5. *Suppose that A is a set of n distinct points in a continuum C, or in other words, $A \subset C$ has cardinality n. Then the symbols $a_1, \dots, a_n$ may be assigned to each point of A so that $a_1 < a_2 < \cdots < a_n$, i.e., $a_i < a_{i+1}$ for all $1 \leq i \leq n-1$.*

Proof. We divide into two cases ($|A| = 0$ and $|A| \in \mathbb{N}$).

If $|A| = 0$, then the statements "the symbols $a_1, \dots, a_n$ may be assigned to each point of A" and "$a_i < a_{i+1}$ for all $1 \leq i \leq n-1 = -1$" are both vacuously true.

If $|A| \in \mathbb{N}$, we induct on $|A| = n$. For the base case $n = 1$, denote the single element of A by a_1. Since $a_i < a_{i+1}$ for all $1 \leq i \leq n-1 = 0$ is vacuously true, the base case holds. Now suppose inductively that we have proven the claim for n, i.e., for a set $A \subset C$ satisfying $|A| = n$, the symbols $a_1, \dots, a_n$ may be assigned to each point of A so that $a_1 < a_2 < \cdots < a_n$. We now wish to prove the claim with regards to a set $A \subset C$ with $|A| = n+1$. By Lemma 3.4, there is a last point $a_{n+1} \in A$, which may be denoted as such (we will rigorously confirm this later). Since the set $A \setminus \{a_{n+1}\}$ has cardinality n (see the lemma from Lemma 3.4), we have by the induction hypothesis that its n elements can be named $a_1, \dots, a_n$ and ordered $a_1 < a_2 < \cdots < a_n$. Clearly these n elements are elements of A and all that's left to do is determine where a_{n+1} fits into the established order. But by Definition 3.3, $x \leq a_{n+1}$ for all $x \in A$, i.e., $x < a_{n+1}$ for all $x \in A \setminus \{a_{n+1}\}$. Consequently, as its name would suggest, it is true that $a_1 < a_2 < \cdots < a_n < a_{n+1}$, as desired. $\square$

Definition 3.6. If $x, y, z \in C$ and either (i) both $x < y$ and $y < z$ or (ii) both $z < y$ and $y < x$, then we say that y is **between** x and z.

Corollary 3.7. *Of three distinct points in a continuum, one must be between the other two.*

Proof. Let A be a subset of the described continuum containing the three distinct points. It follows by Theorem 3.5 that the symbols a_1, a_2, a_3 may be assigned to each point of A so that $a_1 < a_2 < a_3$. Thus, $a_1 < a_2$ and $a_2 < a_3$, so a_2 is between a_1 and a_3 by Definition 3.6. $\square$

10/22: **Axiom 3.** *A continuum C has no first or last point.*

Definition 3.8. We define an ordering on $\mathbb{Z}$ by $m < n$ if $n = m + c$ for some $c \in \mathbb{N}$.

Exercise 3.9.

a) Prove that with this ordering $\mathbb{Z}$ satisfies Axioms 1-3.

Proof. Clearly, $\mathbb{Z}$ is a nonempty set, so Axiom 1 is immediately satisfied.

Axiom 2 asserts that $\mathbb{Z}$ must have an ordering $<$. As such, it will suffice to verify that the ordering given by Definition 3.8 satisfies the stipulations of Definition 3.1. To prove that $<$ satisfies the trichotomy, it will suffice to show that for all $x, y \in \mathbb{Z}$, exactly one of the following holds: $x < y$, $y < x$, or $x = y$.

We first show that *no more than one* of the three statements can simultaneously be true. Let x, y be arbitrary elements of $\mathbb{Z}$. We divide into three cases. First, suppose for the sake of contradiction that $x < y$ and $y < x$. By Definition 3.8, this implies that $y = x + c$ and $x = y + c'$ for some $c, c' \in \mathbb{N}$. Substituting, we have $y = y + c' + c$, or $0 = c' + c$ by the cancellation law of addition. But since $c', c \in \mathbb{N}$, the closure of addition on $\mathbb{N}$ implies that $(c' + c) \in \mathbb{N}$. Therefore, $c' + c \neq 0$, a contradiction.

Second, suppose for the sake of contradiction that $x < y$ and $x = y$. By Definition 3.8, this implies that $y = x + c$ for some $c \in \mathbb{N}$. Substituting, we have $y = y + c$, or $0 = c$ by the cancellation law of addition. But since $c \in \mathbb{N}$, $c \neq 0$, a contradiction. The proof of the third case ($y < x$ and $x = y$) is symmetric to that of the second case.

We now show that *at least one* of the three statements is always true. Let x, y be arbitrary elements of $\mathbb{Z}$, and suppose for the sake of contradiction that $x \not< y$, $y \not< x$, and $x \neq y$. Since $x \not< y$, $y \neq x + c$ for any $c \in \mathbb{N}$. Equivalently, $y - x \neq c$ for any $c \in \mathbb{N}$, i.e., $(y - x) \notin \mathbb{N}$. Similarly, since $y \not< x$, $x - y \neq c'$ for any $c' \in \mathbb{N}$. Equivalently, $y - x \neq c'$ for any $c' \in -\mathbb{N}$, i.e., $(y - x) \notin -\mathbb{N}$. Lastly, since $x \neq y$, $y - x \neq 0$, i.e., $(y - x) \notin \{0\}$. Since $(y - x) \notin -\mathbb{N}$, $(y - x) \notin \{0\}$, and $(y - x) \notin \mathbb{N}$, Definition 1.5 implies that $(y - x) \notin (-\mathbb{N}) \cup \{0\} \cup \mathbb{N}$. Consequently, by Script 0, $(y - x) \notin \mathbb{Z}$. But by the closure of integer subtraction, $(y - x) \in \mathbb{Z}$, a contradiction.

To prove that $<$ is transitive, it will suffice to show that for all $x, y, z \in \mathbb{Z}$, if $x < y$ and $y < z$, then $x < z$. Let x, y, z be arbitrary elements of $\mathbb{Z}$ for which it is true that $x < y$ and $y < z$. By Definition 3.8, we have $y = x + c$ and $z = y + c'$ for some $c, c' \in \mathbb{N}$. Substituting, we have $z = x + c + c'$. Since $(c + c') \in \mathbb{N}$ by the closure of addition on $\mathbb{N}$, Definition 3.8 implies that $x < z$.

Axiom 3 asserts that $\mathbb{Z}$ must have no first or last point. Suppose for the sake of contradiction that $\mathbb{Z}$ has some first point a. Then by Definition 3.3, $a \leq x$ for every $x \in \mathbb{Z}$. However, under the closure of subtraction on $\mathbb{Z}$, $(a - 1) \in \mathbb{Z}$. Since $(a - 1) + 1 = a$, Definition 3.8 asserts that $a - 1 < a$, a contradiction. The proof is symmetric for the last point. □

b) Show that for any $p = \left[\frac{a}{b}\right] \in \mathbb{Q}$, there is some $(a_1, b_1) \in p$ with $0 < b_1$.

Proof. Let $\left[\frac{a}{b}\right]$ be an arbitrary element of $\mathbb{Q}$. It follows by Definition 2.5 that $(a, b) \in X$. Since we also have $(a, b) \sim (a, b)$ by Exercise 2.2e, Definition 2.5 implies that $(a, b) \in \left[\frac{a}{b}\right]$. By the trichotomy on $\mathbb{Z}$ (see Exercise 3.9a), we have $0 < b$, $b < 0$, or $0 = b$. We divide into three cases. First, suppose that $0 < b$. Then (a, b) is an element $(a_1, b_1) \in \left[\frac{a}{b}\right]$ for which $0 < b_1$, and we are done. Second, suppose that $b < 0$. Since $(-a)(b) = (-b)(a)$, we have by the definition of $\sim$ that $(-a, -b) \sim (a, b)$. Additionally, we have by the closure of integer multiplication that $-a, -b \in \mathbb{Z}$, and since $b \neq 0$ by Exercise 2.2e and clearly $-1 \neq 0$, $-b \neq 0$ by the contrapositive of the zero-product property. Thus, by Exercise 2.2e, $(-a, -b) \in X$. This coupled with the previously proven fact that $(-a, -b) \sim (a, b)$ implies by Definition 2.5 that $(-a, -b) \in \left[\frac{a}{b}\right]$. Now recall that $b < 0$ by hypothesis, so we may use Definition 3.8 to see that $b + c = 0$ for some $c \in \mathbb{N}$. It follows that $-(b + c) = 0$, i.e., $-b - c = 0$, i.e., $-b = 0 + c$, meaning that $0 < -b$ by Definition 3.8. Thus, $(-a, -b)$ is an element $(a_1, b_1) \in \left[\frac{a}{b}\right]$ for which $0 < b_1$. Third, suppose that $b = 0$. But this contradicts Exercise 2.2e which asserts that $b \neq 0$, so we need not consider this case. □

c) Define an ordering $<_{\mathbb{Q}}$ on $\mathbb{Q}$ as follows. For $p, q \in \mathbb{Q}$, let $(a_1, b_1) \in p$ be such that $0 < b_1$ and let $(a_2, b_2) \in q$ be such that $0 < b_2$. Then we define $p <_{\mathbb{Q}} q$ if $a_1 b_2 < a_2 b_1$. Show that $<_{\mathbb{Q}}$ is a well-defined relation on $\mathbb{Q}$.

Proof. For the relation $<_{\mathbb{Q}}$ to be well-defined, Definition 3.1 tells us that it must satisfy the trichotomy and be transitive.

To prove that $<_{\mathbb{Q}}$ satisfies the trichotomy, it will suffice to show that for all $p, q \in \mathbb{Q}$, exactly one of the following holds: $p <_{\mathbb{Q}} q$, $q <_{\mathbb{Q}} p$, or $p = q$.

We first show that *no more than one* of the three statements can be simultaneously true. Let p, q be arbitrary elements of $\mathbb{Q}$, let $(a, b) \in p$ be such that $0 < b$ (we know that such an element exists by Exercise 3.9b[1]), and let $(c, d) \in q$ be such that $0 < d$. We divide into three cases. First, suppose for the sake of contradiction that $p <_{\mathbb{Q}} q$ and $q <_{\mathbb{Q}} p$. Then $ad < bc$ and $cb < da$ by the definition of $<_{\mathbb{Q}}$. But this violates the trichotomy known to hold for the ordering $<$ on the integers by Exercise 3.9a, a

[1]This justification will not be supplied every subsequent time we choose such an element to make the proof less repetitive.

contradiction. Second, suppose for the sake of contradiction that $p <_{\mathbb{Q}} q$ and $p = q$. By the definition of $<_{\mathbb{Q}}$, it follows from the first assumption that $ad < bc$. Additionally, by Exercise 2.6, it follows from the second assumption that $(a, b) \sim (c, d)$, implying by Exercise 2.2e that $ad = bc$. But once again, the simultaneous results that $ad < bc$ and $ad = bc$ violate the trichotomy of the integers, a contradiction. The proof of the third case is symmetric to that of the second.

We now show that *at least one* of the three statements is always true. Let p, q be arbitrary elements of $\mathbb{Q}$, let $(a, b) \in p$, and let $(c, d) \in q$. Suppose for the sake of contradiction that $p \not<_{\mathbb{Q}} q$, $q \not<_{\mathbb{Q}} p$, and $p \neq q$. Since $p \not<_{\mathbb{Q}} q$, we have that $ad \not< bc$. Similarly, since $q \not<_{\mathbb{Q}} p$, we have $cb \not< da$. Equivalently, $bc \not< ad$. Lastly, since $p \neq q$, Exercise 2.6 implies that $(a, b) \nsim (c, d)$. It follows by Exercise 2.2e that $ad \neq bc$. To recap, for the integers ad and bc, we have $ad \not< bc$, $bc \not< ad$, and $ad \neq bc$. But by Exercise 3.9a, $ad < bc$, $bc < ad$, or $ad = bc$, a contradiction.

To prove that $<_{\mathbb{Q}}$ is transitive, it will suffice to show that for all $p, q, r \in \mathbb{Q}$, if $p <_{\mathbb{Q}} q$ and $q <_{\mathbb{Q}} r$, then $p <_{\mathbb{Q}} r$. Let p, q, r be arbitrary elements of $\mathbb{Q}$ for which it is true that $p <_{\mathbb{Q}} q$ and $q <_{\mathbb{Q}} r$, let $(a, b) \in p$ be such that $0 < b$, let $(c, d) \in q$ be such that $0 < d$, and let $(e, f) \in r$ such that $0 < f$. By the definition of $<_{\mathbb{Q}}$, we have $ad < bc$ and $cf < de$. Since $0 < f$ and $0 < b$, we can multiply both sides of the inequalities by b or f without affecting the truth of the statement (see Script 0). Thus, we may create the inequalities $adf < bcf$ and $bcf < bde$. So $adf < bde$ by Definition 3.1, implying that $af < be$ by the cancellation law (which we may use since $0 < d$). It follows by the definition of $<_{\mathbb{Q}}$ that $p <_{\mathbb{Q}} r$. $\square$

d) Show that $\mathbb{Q}$ with the ordering $<_{\mathbb{Q}}$ satisfies Axioms 1-3.

Proof. Clearly, $\mathbb{Q}$ is a nonempty set, so Axiom 1 is immediately satisfied.

By Exercise 3.9c, $\mathbb{Q}$ has an ordering, so Axiom 2 is satisfied.

Axiom 3 asserts that $\mathbb{Q}$ must have no first or last point. Suppose for the sake of contradiction that $\mathbb{Q}$ has some first point p. Then by Definition 3.3, $p <_{\mathbb{Q}} x$ or $p = x$ for all $x \in \mathbb{Q}$. Let $(a, b) \in p$ be such that $0 < b$ (see Exercise 3.9b). Under the closure of integer subtraction, $(a - 1) \in \mathbb{Z}$, so $\left[\frac{a-1}{b}\right] \in \mathbb{Q}$. Since $ba = ba - b + b = b(a - 1) + b$ where $b \in \mathbb{N}$ since $b \in \mathbb{Z}$ and $0 < b$, Definition 3.8 implies that $(a-1)b < ba$. It follows by the definition of $<_{\mathbb{Q}}$ that $\left[\frac{a-1}{b}\right] <_{\mathbb{Q}} \left[\frac{a}{b}\right] = p$, a contradiction. The argument is symmetric for the last point. $\square$

Definition 3.10. If $a, b \in C$ and $a < b$, then the set of points between a and b is called a **region** and denoted by $\underline{ab}$.

Remark 3.11. One often sees the notation (a, b) for regions. We will reserve the notation (a, b) for ordered pairs in a product $A \times B$. These are very different things.

Theorem 3.12. *If x is a point of a continuum C, then there exists a region $\underline{ab}$ such that $x \in \underline{ab}$.*

Proof. Let x be an arbitrary point in a continuum C. By Axiom 2, C has an ordering $<$, which we will frequently make use of throughout the remainder of this proof. By Axiom 3, C has no first or last points, so it cannot be true that $x \leq y$ for all $y \in C$, nor can it be true that $x \geq y$ for all $y \in C$. This implies that there exists an $a \in C$ such that $a < x$ and that there exists a $b \in C$ such that $b > x$. Since $a < x$ and $x < b$, Definition 3.6 implies that x is between a and b. Note also that by Definition 3.1 (specifically transitivity), $a < b$. Therefore, since $a, b \in C$, $a < b$, and x is between a and b, Definition 3.10 implies that $x \in \underline{ab}$. $\square$

Definition 3.13. Let A be a subset of a continuum C. A point p of C is called a **limit point** of A if every region R containing p has nonempty intersection with $A \setminus \{p\}$. Explicitly, this means:

$$\text{for every region } R \text{ with } p \in R, \text{ we have } R \cap (A \setminus \{p\}) \neq \emptyset.$$

Notice that we do not require that a limit point p of A be an element of A. We will use the notation $LP(A)$ to denote the set of limit points of A.

Theorem 3.14. *If p is a limit point of A and $A \subset B$, then p is a limit point of B.*

Lemma. *Let A, B, C be sets such that $A \subset B$. Then $A \cap C \subset B \cap C$.*

Proof. Let x be an arbitrary element of $A \cap C$. By Definition 1.6, this implies that $x \in A$ and $x \in C$. Since $x \in A$ and $A \subset B$, Definition 1.3 implies that $x \in B$. Thus, $x \in B$ and $x \in C$, so $x \in B \cap C$ by Definition 1.6. $\square$

Proof. To prove that a limit point p of $A \subset B$ is a limit point of B, Definition 3.13 tells us that it will suffice to show that for every region R with $p \in R$, we have $R \cap (B \setminus \{p\}) \neq \emptyset$. Let p be a limit point of A, and let R be an arbitrary region with $p \in R$. Then by Definition 3.13, we have $R \cap (A \setminus \{p\}) \neq \emptyset$. Thus, by Definition 1.8, there is an element $x \in R \cap (A \setminus \{p\})$. Since $A \setminus \{p\} \subset B \setminus \{p\}$ (because $A \subset B$ and $\{p\} = \{p\}$), it follows by the lemma that $R \cap (A \setminus \{p\}) \subset R \cap (B \setminus \{p\})$. Consequently, by Definition 1.3, the previously referenced object $x \in R \cap (A \setminus \{p\})$ is also an element of $R \cap (B \setminus \{p\})$. Thus, by Definition 1.8, $R \cap (B \setminus \{p\}) \neq \emptyset$, as desired. $\square$