Final-Specific Questions

1. Define an ordering on $\mathbb{Z} \times \mathbb{Z}$ by $(a,b) < (c,d)$ if and only if either $a < c$, or $a = c$ and $b < d$.

   a) Prove that $<$ is indeed an ordering.

   b) Prove that with this ordering, $\mathbb{Z} \times \mathbb{Z}$ satisfies Axioms 1, 2, and 3 (as stated in Script 5).

Proof of a. To prove that $<$ is an ordering, Definition 3.1 tells us that it will suffice to show that $<$ satisfies the trichotomy and transitivity. We will address each stipulation in turn.

To show that $<$ satisfies the trichotomy, Definition 3.1 tells us that it will suffice to verify that for all $(a,b),(c,d) \in \mathbb{Z} \times \mathbb{Z}$, exactly one of the following holds: $(a,b) < (c,d)$, $(c,d) < (a,b)$, or $(a,b) = (c,d)$. We first show that *no more than one* of the three statements can simultaneously be true, and then show that *at least one* of the three statements is always true. Let's begin.

Let $(a,b),(c,d)$ be arbitrary elements of $\mathbb{Z} \times \mathbb{Z}$. We divide into three cases. First, suppose for the sake of contradiction that $(a,b) < (c,d)$ and $(c,d) < (a,b)$. By the definition of $<$, the former statement implies that either $a < c$, or $a = c$ and $b < d$, and the latter statement implies that either $c < a$, or $c = a$ and $d < b$. We divide into cases once again, this time into two (one for each of the possibilities implied by the former statement). If we have $a < c$, then Exercise 3.9 asserts that $c \not< a$, so we must have $c = a$ and $d < b$. But by Exercise 3.9, we cannot have both $a < c$ and $a = c$, a contradiction. On the other hand, if we have $a = c$ and $b < d$, then Exercise 3.9 asserts that $c \not< a$, so we must have $c = a$ and $d < b$. But by Exercise 3.9, we cannot have both $b < d$ and $d < b$, a contradiction. Therefore, we have a contradiction in every case, so we cannot have both $(a,b) < (c,d)$ and $(c,d) < (a,b)$. Second, suppose for the sake of contradiction that $(a,b) < (c,d)$ and $(a,b) = (c,d)$. By the definition of $<$, either $a < c$, or $a = c$ and $b < d$. We divide into two cases as before. If $a < c$, then we have a contradiction with the fact that $a = c$, as implied by $(a,b) = (c,d)$ and Definition 1.15. On the other hand, if $a = c$ and $b < d$, we arrive at the same contradiction as before except with regard to b and d. Therefore, we have a contradiction in every case, so we cannot have both $(a,b) < (c,d)$ and $(a,b) = (c,d)$. The proof of the third case ($(c,d) < (a,b)$ and $(a,b) = (c,d)$) is symmetric to that of the second case.

Let $(a,b),(c,d)$ be arbitrary elements of $\mathbb{Z} \times \mathbb{Z}$, and suppose for the sake of contradiction that $(a,b) \not< (c,d)$, $(c,d) \not< (a,b)$, and $(a,b) \neq (c,d)$. By consecutive applications of the definition of $<$, we have from the first statement that $a \not< c$, and $a \neq c$ or $b \not< d$, and from the second statement that $c \not< a$, and $c \neq a$ or $d \not< b$. Since $a \not< c$ and $c \not< a$, Definition 3.1 asserts that $a = c$. But by Definition 1.15, we have from the third statement that $a \neq c$, a contradiction.

To show that $<$ is transitive, Definition 3.1 tells us that it will suffice to verify that for all $(a,b),(c,d),(e,f) \in \mathbb{Z} \times \mathbb{Z}$, if $(a,b) < (c,d)$ and $(c,d) < (e,f)$, then $(a,b) < (e,f)$. Let $(a,b),(c,d),(e,f)$ be elements of $\mathbb{Z} \times \mathbb{Z}$ for which it is true that $(a,b) < (c,d)$ and $(c,d) < (e,f)$. Then by consecutive applications of the definition of $<$, it follows from the first statement that either $a < c$, or $a = c$ and $b < d$, and it follows from the second statement that either $c < e$, or $c = e$ and $d < f$. We divide into four cases ($a < c$ and $c < e$; $a < c$, $c = e$, and $d < f$; $a = c$, $b < d$, and $c < e$; and $a = c$, $b < d$, $c = e$, and $d < f$). In the first case, it follows from the facts that $a < c$ and $c < e$ by Exercise 3.9 that $a < e$, implying that $(a,b) < (e,f)$ by the definition of $<$. In the second case, it follows from the facts that $a < c$ and $c = e$ by substitution that $a < e$, implying that $(a,b) < (e,f)$ by the definition of $<$. In the third case, it follows from the facts that $a = c$ and $c < e$ by substitution that $a < e$, implying that $(a,b) < (e,f)$

by the definition of $<$. In the fourth case, it follows from the facts that $a = c$ and $c = e$ by transitivity that $a = e$, and it follows from the facts that $b < d$ and $d < f$ by Exercise 3.9 that $b < f$; these two results imply that $(a, b) < (e, f)$ by the definition of $<$. $\square$

Proof of b. To prove that $\mathbb{Z} \times \mathbb{Z}$ satisfies Axiom 1, we must show that it is nonempty. To show this, Definition 1.8 tells us that it will suffice to find an element of $\mathbb{Z} \times \mathbb{Z}$. Since $0 \in \mathbb{Z}$ by definition, $(0, 0) \in \mathbb{Z} \times \mathbb{Z}$ by Definition 1.15, as desired.

By part (a), $\mathbb{Z} \times \mathbb{Z}$ has an ordering $<$; thus, Axiom 2 is satisfied.

To prove that $\mathbb{Z} \times \mathbb{Z}$ satisfies Axiom 3, we must show that it has no first or last point. Suppose for the sake of contradiction that $\mathbb{Z} \times \mathbb{Z}$ has some first point (a, b). Then by Definition 3.3, $(a, b) \leq (x, y)$ for every $(x, y) \in \mathbb{Z} \times \mathbb{Z}$. However, under the closure of subtraction on $\mathbb{Z}$, $(a-1) \in \mathbb{Z}$. Thus, $(a-1, b) \in \mathbb{Z} \times \mathbb{Z}$. But since $a - 1 < a$ by Exercise 3.9, we have by the definition of $<$ that $(a - 1, b) < (a, b)$. Therefore, we have $(a - 1, b) < (a, b)$ and $(a, b) \leq (a - 1, b)$ (since, again, $(a - 1, b) \in \mathbb{Z} \times \mathbb{Z}$), contradicting the previously demonstrated fact that $<$ is an ordering. The proof is symmetric for the last point. $\square$

2. Let C be a continuum satisfying Axioms 1-4, and let $S \subset C$.

   a) Show that $\overline{S} = \{x \in C \mid \text{for all } R \text{ containing } x,\ R \cap S \neq \emptyset\}$.

   b) Let A and B be subsets of C. Show that if $A \subset B$, then $\overline{A} \subset \overline{B}$.

Proof of a. To show that $\overline{S} = \{x \in C \mid \text{for all } R \text{ containing } x,\ R \cap S \neq \emptyset\}$, Definition 1.2 tells us that it will suffice to prove that every element $y \in \overline{S}$ is an element of $\{x \in C \mid \text{for all } R \text{ containing } x,\ R \cap S \neq \emptyset\}$ and vice versa. First off, let $y \in \overline{S}$. Then by Definition 4.4, $y \in S \cup LP(S)$. Thus, by Definition 1.5, $y \in S$ or $y \in LP(S)$. We now divide into two cases. Suppose first that $y \in S$. Clearly, this implies that $y \in C$. As to the other stipulation, let R be any region containing y. Since $y \in S$ and $y \in R$, Definition 1.6 asserts that $y \in R \cap S$, which implies by Definition 1.8 that $R \cap S \neq \emptyset$. Thus, for all R containing y, $R \cap S \neq \emptyset$. It follows from this result and the previous finding that $y \in C$ that $y \in \{x \in C \mid \text{for all } R \text{ containing } x,\ R \cap S \neq \emptyset\}$. Now suppose that $y \in LP(S)$. As before, this implies that $y \in C$. Additionally, by Definition 3.13, for all R containing y, $R \cap (S \setminus \{y\}) \neq \emptyset$. Consequently, we have that for all R containing y, $R \cap S \neq \emptyset$. It follows from this result and the previous finding that $y \in C$ that $y \in \{x \in C \mid \text{for all } R \text{ containing } x,\ R \cap S \neq \emptyset\}$. Now let $y \in \{x \in C \mid \text{for all } R \text{ containing } x,\ R \cap S \neq \emptyset\}$. Then $y \in C$ and for all R containing y, $R \cap S \neq \emptyset$. We divide into two cases ($y \in S$ and $y \notin S$). If $y \in S$, then by Definitions 1.5 and 4.4, $y \in \overline{S}$. On the other hand, if $y \notin S$, then for all R containing y, $R \cap (S \setminus \{y\}) \neq \emptyset$. It follows by Definition 3.13 that $y \in LP(S)$. Thus, by Definitions 1.5 and 4.4, $y \in \overline{S}$. $\square$

Proof of b. To prove that $\overline{A} \subset \overline{B}$, Definition 1.3 tells us that it will suffice to show that every element $x \in \overline{A}$ is an element of $\overline{B}$. Let x be an arbitrary element of $\overline{A}$. Then by Definitions 4.4 and 1.5, $x \in A$ or $x \in LP(A)$. We now divide into two cases. Suppose first that $x \in A$. Then since $A \subset B$, we have by Definition 1.3 that $x \in B$. Consequently, by Definitions 1.5 and 4.4, $x \in \overline{B}$. On the other hand, suppose that $x \in LP(A)$. Then since $A \subset B$, by Theorem 3.14, $x \in LP(B)$. Consequently, by Definitions 1.5 and 4.4, $x \in \overline{B}$. $\square$

3. Let $A \subset C$ where C is a continuum. We say that $x \in A$ is an **interior** point of A if there is a region R such that $x \in R$ and $R \subset A$. We let $\text{int}(A) = \{a \in A \mid a \text{ is an interior point of } A\}$.

   a) Show that $\text{int}(A)$ is open.

   b) Show that A is open if and only if $A = \text{int}(A)$.

Proof of a. To prove that $\text{int}(A)$ is open, Theorem 4.10 tells us that it will suffice to show that for all $x \in \text{int}(A)$, there exists a region R such that $x \in R$ and $R \subset \text{int}(A)$. Let x be an arbitrary element of $\text{int}(A)$. Then by the definition of $\text{int}(A)$, $x \in A$ and x is an interior point of A. It follows from the latter result that there exists a region R such that $x \in R$ and $R \subset A$. We now demonstrate that this R

is a subset of $\text{int}(A)$, too. Let y be an arbitrary element of R. Then since $R \subset A$, $y \in A$. Additionally, R is a region such that $y \in R$ and $R \subset A$, meaning that y is an interior point of A. These last two results imply that $y \in \text{int}(A)$. It follows by Definition 1.3 that $R \subset \text{int}(A)$. Therefore, there exists a region R such that $x \in R$ and $R \subset \text{int}(A)$, as desired. □

Proof of b. Suppose first that A is open, and suppose for the sake of contradiction that $A \neq \text{int}(A)$. It follows from the supposition by Definition 1.2 that there exists a point $x \in A$ such that $x \notin \text{int}(A)$ (since all elements of $\text{int}(A)$ are elements of A by definition). Since $x \notin \text{int}(A)$ but $x \in A$, we have by the definition of $\text{int}(A)$ that x is not an interior point of A. Thus, there does not exist a region R such that $x \in R$ and $R \subset A$. But since $x \in A$, this implies by Theorem 4.10 that A is not open, a contradiction. Therefore, $A = \text{int}(A)$, as desired.

Now suppose that $A = \text{int}(A)$. Then since $\text{int}(A)$ is open by part (a), clearly A is open, too. □

4. Let A and B be disjoint, countable sets. Show that $A \cup B$ is also countable.

Proof. By Exercise 1.36, $\mathbb{Z}$ is countable. Since $\mathbb{Z} \setminus \{0\}$ is an infinite subset of $\mathbb{Z}$, Exercise 1.37 implies that $\mathbb{Z} \setminus \{0\}$ is countable. To prove that $A \cup B$ is countable, Exercise 1.38 tells us that it will suffice to find an injection $h : A \cup B \to \mathbb{Z} \setminus \{0\}$ (note that $A \cup B$ is clearly infinite). First off, by consecutive applications of Definitions 1.35 and 1.28, the fact that A and B are both countable implies that there exist bijections $f : A \to \mathbb{N}$ and $g : B \to \mathbb{N}$. Now let $h : A \cup B \to \mathbb{Z} \setminus \{0\}$ be defined as follows:

$$h(x) = \begin{cases} f(x) & x \in A \\ -g(x) & x \in B \end{cases}$$

To prove that h is a function, Definition 1.16 tells us that it will suffice to show that for all $x \in A \cup B$, there exists a unique $y \in \mathbb{Z} \setminus \{0\}$ such that $h(x) = y$. Let x be an arbitrary element of $A \cup B$. Then since A and B are disjoint, either $x \in A$ or $x \in B$ (but not both). We now divide into two cases. If $x \in A$, then $h(x) = f(x)$, which is a well-defined element of $\mathbb{N}$, i.e., of $\mathbb{Z} \setminus \{0\}$, since f is a function. If $x \in B$, then $h(x) = -g(x)$, which is a well-defined element of $-\mathbb{N}$, i.e., of $\mathbb{Z} \setminus \{0\}$, since g is a function.

To prove that h is injective, Definition 1.20 tells us that it will suffice to show that $h(a) = h(b)$ implies $a = b$. We divide into two cases ($h(a), h(b) \in \mathbb{N}$ and $h(a), h(b) \in -\mathbb{N}$). If $h(a) = h(b)$ is an element of $\mathbb{N}$, then $f(a) = h(a) = h(b) = f(b)$, so by the injectivity of f (which follows from its bijectivity by Definition 1.20), $a = b$. If $h(a) = h(b)$ is an element of $-\mathbb{N}$, then $-g(a) = h(a) = h(b) = -g(b)$, i.e., $g(a) = g(b)$, so by the injectivity of g (which follows from its bijectivity as with f), $a = b$. □

5. If $A \subset C$, where C is a continuum satisfying Axioms 1-4, we define the **boundary** of A by the equation

$$\text{Bd}(A) = \overline{A} \cap \overline{(C \setminus A)}$$

a) Show that if A is a closed set, then $\text{Bd}(A) \subset A$.
b) Show that if A is an open set, then $A \cap \text{Bd}(A) = \emptyset$.

Proof of a. To prove that $\text{Bd}(A) \subset A$, Definition 1.3 tells us that it will suffice to show that every element $x \in \text{Bd}(A)$ is an element of A. Let x be an arbitrary element of $\text{Bd}(A)$. Then by the definition of the boundary of A, $x \in \overline{A} \cap \overline{(C \setminus A)}$. Thus, by Definition 1.6, $x \in \overline{A}$ and $x \in \overline{C \setminus A}$. Additionally, since A is closed, Theorem 4.5 asserts that $\overline{A} = A$. Therefore, since $x \in \overline{A}$ and $\overline{A} = A$, Definition 1.2 implies that $x \in A$, as desired. □

Proof of b. Suppose for the sake of contradiction that $A \cap \text{Bd}(A) \neq \emptyset$. By Definition 1.8, this implies that there exists an object $x \in A \cap \text{Bd}(A)$. Consequently, by Definition 1.6, $x \in A$ and $x \in \text{Bd}(A)$. It follows from the latter result by the definition of the boundary of A that $x \in \overline{A} \cap \overline{(C \setminus A)}$. Thus, by Definition 1.6, $x \in \overline{A}$ and $x \in \overline{C \setminus A}$. By consecutive applications of Definitions 4.4 and 1.5, we have from the former result that $x \in A$ or $x \in LP(A)$, and from the latter result that $x \in C \setminus A$ or

$x \in LP(C \setminus A)$. Since $x \in A$, as previously established, Definition 1.11 implies that $x \notin C \setminus A$, meaning that $x \in LP(C \setminus A)$. But since A is open, Definition 4.4 asserts that $C \setminus A$ is closed, and this implies by Definition 4.1 that $x \in C \setminus A$ (since $x \in LP(C \setminus A)$), a contradiction. Therefore, $A \cap \text{Bd}(A) = \emptyset$, as desired. $\square$