Script 5

Connectedness and Boundedness

5.1 Journal

11/19: **Axiom 4.** *A continuum is connected.*

Theorem 5.1. *The only subsets of a continuum C that are both open and closed are $\emptyset$ and C.*

Proof. To prove that the only subsets of C that are both open and closed are $\emptyset$ and C, it will suffice to show that if $A \subset C$ is both open and closed, then $A = \emptyset$ or $A = C$. Let $A \subset C$ be both open and closed. We divide into two cases ($A = \emptyset$ and $A \neq \emptyset$). If $A = \emptyset$, then we are done. On the other hand, if $A \neq \emptyset$, we have a bit more work to do. Basically, we will end up proving that the facts that A is open, A is closed, and $A \neq \emptyset$ imply that $A = C$. Let's begin.

First off, the fact that A is closed implies by Definition 4.8 that $C \setminus A$ is open. Additionally, we have by Script 1 that $A \cap (C \setminus A) = \emptyset$ and $A \cup (C \setminus A) = C$. Now suppose for the sake of contradiction that $A \neq C$. It follows since $A \subset C$ that we must have $C \not\subset A$, i.e., there is some object in C that is not an element of A. This object would clearly be an element of $C \setminus A$ in this case, meaning that $C \setminus A$ is nonempty. Thus, we have that A and $C \setminus A$ are disjoint, open, nonempty sets such that $A \cup (C \setminus A) = C$. Consequently, by consecutive applications of Definition 4.22, we know that C is disconnected, i.e., C is not connected. But this contradicts Axiom 4, which asserts that C is connected. Therefore, we must have that $A = C$, as desired. $\square$

Theorem 5.2. *For all $x, y \in C$, if $x < y$, then there exists a point $z \in C$ such that z is between x and y.*

Proof. Suppose for the sake of contradiction that no point $z \in C$ exists such that z is between x and y. To find a contradiction, we will let $A = \{c \in C \mid c < y\}$ and $B = \{c \in C \mid x < c\}$ and prove that $A \cup B = C$, and that A and B are disjoint, nonempty, open sets. This will imply that C is disconnected, contradicting Axiom 4. Let's begin.

Suppose for the sake of contradiction that $C \neq A \cup B$. Then by Theorem 1.7, $C \not\subset A \cup B$ or $A \cup B \not\subset C$. Since $A \subset C$ and $B \subset C$ by their definitions, we have $A \cup B \subset C$, so it must be that $C \not\subset A \cup B$. Thus, by Definition 1.3, there exists a point $p \in C$ such that $p \notin A \cup B$. From the latter condition, we have by Definition 1.5 that $p \notin A$ and $p \notin B$. It follows from the definitions of A and B that $p \notin C$, or $p \not< y$ and $x \not< p$. But we know that $p \in C$, so it must be that $p \not< y$ and $x \not< p$. Equivalently, $p \geq y$ and $x \geq p$. But this implies that $x \geq y$, which contradicts the fact that $x < y$ by hypothesis. Therefore, we must have $C = A \cup B$, as desired.

Suppose for the sake of contradiction that A and B are not disjoint. Then by Definition 1.9, $A \cap B \neq \emptyset$. Thus, Definition 1.8 tells us that there exists some object $p \in A \cap B$. By Definition 1.6, this implies that $p \in A$ and $p \in B$. It follows by the definitions of A and B that $p \in C$, $p < y$, and $x < p$. Since $x < p < y$, Definition 3.6 tells us that p is between x and y, contradicting the supposition that no such point exists. Therefore, A and B are disjoint, as desired.

To prove that A and B are nonempty, Definition 1.8 tells us that it will suffice to show that there exists an object in each set. Since $x \in C$ and $x < y$, $x \in A$. Similarly, since $y \in C$ and $x < y$, $y \in B$. Therefore, A and B are nonempty, as desired.

By Corollary 4.13, A and B are open, as desired.

Since C can be written as $A \cup B$ where A and B are disjoint, nonempty, open sets, we have by Definition 4.22 that C is disconnected. But this contradicts Axiom 4, which asserts that C is connected. Therefore, there must exist a point $z \in C$ such that z is between x and y, as desired. $\square$

Corollary 5.3. *Every region is infinite.*

Proof. Let $\underline{ab}$ be a region, and suppose for the sake of contradiction that $\underline{ab}$ is finite. Then by Definitions 1.30 and 1.33, $\underline{ab} = \emptyset$, or $\underline{ab}$ has cardinality n. We divide into two cases. Suppose first that $\underline{ab} = \emptyset$. Then by Definitions 3.10 and 3.6, no point p exists such that $a < p < b$. Thus, by the contrapositive of Theorem 5.2, $a = b$. But this implies by Definition 3.10 that $\underline{ab}$ is not a region (since $a \not< b$), a contradiction. Now suppose that $\underline{ab}$ has cardinality n. Then by Theorem 3.5, the symbols $a_1, \dots, a_n$ may be assigned to each point of $\underline{ab}$ so that $a_1 < a_2 < \cdots < a_n$. But by Theorem 5.2, there exists a point $z \in C$ such that z is between a and a_1. Since $a < z < a_1 < b$, we clearly have that $z \in \underline{ab}$, yet it was not assigned a symbol a_k, a contradiction. Therefore, $\underline{ab}$ is infinite, as desired. $\square$

12/1: **Corollary 5.4.** *Every point of C is a limit point of C.*

Proof. Let p be an arbitrary element of C. To prove that p is a limit point of C, Definition 3.13 tells us that it will suffice to show that for all regions R with $p \in R$, $R \cap (C \setminus \{p\}) \neq \emptyset$. Let R be an arbitrary region with $p \in R$. By Corollary 5.3, R is infinite, so there exists a point $q \in R$ such that $q \neq p$. Additionally, since $q \in R$, we have $q \in C$. Thus, since $q \in C$ and $q \neq p$ (i.e., $q \notin \{p\}$), we have by Definition 1.11 that $q \in C \setminus \{p\}$. This combined with the fact that $q \in R$ implies by Definition 1.6 that $q \in R \cap (C \setminus \{p\})$, so $R \cap (C \setminus \{p\}) \neq \emptyset$, as desired. $\square$

Corollary 5.5. *Every point of the region $\underline{ab}$ is a limit point of $\underline{ab}$.*

Proof. Suppose for the sake of contradiction that there exists a point $p \in \underline{ab}$ such that $p \notin LP(\underline{ab})$. Then since $p \notin LP(\underline{ab})$, we have by Definition 3.13 that there exists a region R with $p \in R$ such that $R \cap (\underline{ab} \setminus \{p\}) = \emptyset$. It follows that from the facts that $p \in R$, $p \in \underline{ab}$, and $R \cap (\underline{ab} \setminus \{p\}) = \emptyset$ that $R \cap \underline{ab} = \{p\}$. Additionally, since R and $\underline{ab}$ are two regions with a point in common (namely p), Theorem 3.18 asserts that $R \cap \underline{ab}$ is a region. Consequently, by Corollary 5.3, $R \cap \underline{ab}$ is infinite. But this contradicts the result that $R \cap \underline{ab} = \{p\}$, a notably finite set. Therefore, it must be that every point of the region $\underline{ab}$ is a limit point of $\underline{ab}$. $\square$

Definition 5.6. Let X be a subset of C. A point u is called an **upper bound** of X if for all $x \in X$, $x \leq u$. A point l is called a **lower bound** of X if for all $x \in X$, $l \leq x$. If there exists an upper bound of X, then we say that X is **bounded above**. If there exists a lower bound of X, then we say that X is **bounded below**. If X is bounded above and below, then we simply say that X is **bounded**.

Definition 5.7. Let X be a subset of C. We say that u is a **least upper bound** of X and write $u = \sup X$ if:

1. u is an upper bound of X;
2. if u' is an upper bound of X, then $u \leq u'$.

We say that l is a **greatest lower bound** and write $l = \inf X$ if:

1. l is a lower bound of X;
2. if l' is a lower bound of X, then $l' \leq l$.

The notation sup comes from the word **supremum**, which is another name for least upper bound. The notation inf comes from the word **infimum**, which is another name for greatest lower bound.

Exercise 5.8. If $\sup X$ exists, then it is unique, and similarly for $\inf X$.

Proof. Let X be a subset of a continuum C such that $\sup X$ exists, and suppose that both u and u' are least upper bounds of X. It follows from the supposition and Definition 5.7 that u, u' are both upper bounds of X. Thus, since u is a least upper bound of X and u' is an upper bound of X, we have by Definition 5.7 again that $u \leq u'$. By a symmetric argument, we also have that $u' \leq u$. But since $u \leq u'$ and $u' \leq u$, $u = u'$, proving the uniqueness of $\sup X$.

The proof is symmetric for $\inf X$. $\square$

Exercise 5.9. If X has a first point L, then $\inf X$ exists and equals L. Similarly, if X has a last point U, then $\sup X$ exists and equals U.

Proof. Let L be the first point of X. Then by Definition 3.3, for all $x \in X$, $L \leq x$. Thus, by Definition 5.6, L is a lower bound of X. Now suppose for the sake of contradiction that there exists a lower bound L' of X such that $L' > L$. Since L' is a lower bound, Definition 5.6 implies that for all $x \in X$, $L' \leq x$. But L is an element of X and $L < L'$, a contradiction. Therefore, if L' is a lower bound of X, then $L' \leq L$. This result coupled with the fact that L is a lower bound of X implies by Definition 5.7 that $L = \inf X$.

The proof is symmetric in the other case. $\square$

Exercise 5.10. For this exercise, we assume that $C = \mathbb{R}$. Find $\sup X$ and $\inf X$ for each of the following subsets of $\mathbb{R}$, or state that they do not exist. You need not give proofs.

1. $X = \mathbb{N}$.

   Answer. $\sup X$ does not exist because the natural numbers continue on forever to positive infinity. However, $\inf X = 1$ since we know that $1 \leq n$ for all $n \in \mathbb{N}$. $\square$

2. $X = \mathbb{Q}$.

   Answer. Neither $\sup X$ nor $\inf X$ exists because the rational numbers continue on forever to both positive and negative infinity. $\square$

3. $X = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$.

   Answer. For $n = 1$, $\frac{1}{n} = 1$. From here, as n increases, $\frac{1}{n}$ decreases asymptotically toward zero but always remains a positive nonzero rational number. Thus, $\sup X = 1$ and $\inf X = 0$. $\square$

4. $X = \{x \in \mathbb{R} \mid 0 < x < 1\}$.

   Answer. $\sup X = 1$ and $\inf X = 0$. In the case of $\sup X$, any number slightly less than 1 would be included in X and have a number in X between it and 1 by Theorem 5.2, i.e., greater than it. A symmetric argument can treat the other case. $\square$

5. $X = \{3\} \cup \{x \in \mathbb{R} \mid -7 \leq x \leq -5\}$.

   Answer. $\sup X = 3$ (3 is the greatest element of the set) and $\inf X = -7$ (for a similar reason to part 4, above). $\square$

Lemma 5.11. *Suppose that $X \subset C$ and $s = \sup X$. If $p < s$, then there exists an $x \in X$ such that $p < x \leq s$. Similarly, suppose that $X \subset C$ and $l = \inf X$. If $l < p$, then there exists an $x \in X$ such that $l \leq x < p$.*

Proof. Suppose for the sake of contradiction that for some $p < s$, no $x \in X$ exists such that $p < x \leq s$. Since s is a least upper bound of X, Definitions 5.7 and 5.6 imply[1] that for all $x \in X$, $x \leq s$. Consequently, by the supposition, it is true that for all $x \in X$, $x \leq p$ (if there existed an $x > p$, then this point would satisfy $p < x \leq s$, contradicting the supposition). Thus, by Definition 5.6, p is a upper bound of X. But since

[1]Technically, Definition 5.7 implies that s is an upper bound of X and Definition 5.6 implies based off of this result that for all $x \in X$, $x \leq s$. However, to avoid having to write this every time, I will shorthand this concept in this fashion.

$p < s$, it is not true that $s \leq s'$ for all upper bounds s' of X, meaning by Definition 5.7 that s is not a least upper bound of X, a contradiction. Therefore, if $p < s$, then there exists an $x \in X$ such that $p < x \leq s$, as desired.

The proof is symmetric in the other case. $\square$

Theorem 5.12. *Let $a < b$. The least upper bound and greatest lower bound of the region $\underline{ab}$ are $\sup \underline{ab} = b$ and $\inf \underline{ab} = a$.*

Proof. To prove that $\sup \underline{ab} = b$, Definition 5.7 tells us that it will suffice to show that b is an upper bound of $\underline{ab}$ and that if u is an upper bound of $\underline{ab}$, then $b \leq u$. For the first condition, Definition 5.6 tells us that it will suffice to confirm that for all $x \in \underline{ab}$, $x \leq b$. Let x be an arbitrary element of $\underline{ab}$. Then by Definitions 3.10 and 3.6, we know that $a < x < b$, i.e., $x \leq b$, as desired. For the second condition, suppose for the sake of contradiction that u is an upper bound of $\underline{ab}$ such that $u < b$. Then by Definition 5.6, for all $x \in \underline{ab}$, $x \leq u$. Additionally, since $\underline{ab}$ is infinite by Corollary 5.3, we know that at least one such x exists, which we shall hereafter refer to as y. Note that as an element of $\underline{ab}$, y satisfies $a < y < b$ by Definitions 3.10 and 3.6. Furthermore, since $u < b$, Theorem 5.2 implies that there exists a point z such that z is between u and b. Thus, by Definition 3.6, $u < z < b$. Combining the last few results, we have $a < y \leq u < z < b$. Consequently, since $a < z < b$, we have by Definitions 3.6 and 3.10 that $z \in \underline{ab}$ and $u < z$, contradicting the statement that for all $x \in \underline{ab}$, $x \leq u$. Therefore, if u is an upper bound of $\underline{ab}$, then $b \leq u$, as desired.

The proof is symmetric in the other case. $\square$

12/3: **Lemma 5.13.** *Let X be a subset of C. Suppose that $\sup X$ exists and $\sup X \notin X$. Then $\sup X$ is a limit point of X. The same holds for $\inf X$.*

Proof. To prove that $\sup X$ is a limit point of X, Definition 3.13 tells us that it will suffice to verify that for all regions $\underline{ab}$ with $\sup X \in \underline{ab}$, we have $\underline{ab} \cap (X \setminus \{\sup X\}) \neq \emptyset$. Let $\underline{ab}$ be an arbitrary region with $\sup X \in \underline{ab}$. Then by Definitions 3.10 and 3.6, $a < \sup X < b$. It follows by Lemma 5.11 that there exists an $x \in X$ such that $a < x \leq \sup X$. Additionally, since $x \in X$ and $\sup X \notin X$, we cannot have $\sup X = x$, meaning that $a < x < \sup X$. Combining the last few results, we have $a < x < \sup X < b$. Thus, by Definitions 3.6 and 3.10, $x \in \underline{ab}$. Consequently, since $x \in X$ and $x \neq \sup X$ implies $x \notin \{\sup X\}$, Definition 1.11 asserts that $x \in X \setminus \{\sup X\}$. Therefore, since we also know that $x \in \underline{ab}$, we have by Definition 1.6 that $x \in \underline{ab} \cap (X \setminus \{\sup X\})$, meaning by Definition 1.8 that $\underline{ab} \cap (X \setminus \{\sup X\}) \neq \emptyset$, as desired.

The proof is symmetric in the other case. $\square$

Corollary 5.14. *Both a and b are limit points of the region $\underline{ab}$.*

Proof. Clearly, $\underline{ab} \subset C$. Additionally, by Theorem 5.12, $\sup \underline{ab}$ and $\inf \underline{ab}$ exist and are equal to b and a, respectively. Furthermore, it follows from Definition 3.10 that neither b nor a (i.e., neither $\sup \underline{ab}$ nor $\inf \underline{ab}$) are elements of $\underline{ab}$. Therefore, by Lemma 5.13, $\inf \underline{ab} = a$ and $\sup \underline{ab} = b$ are limit points of the region $\underline{ab}$. $\square$

Corollary 5.15. *Let $[a, b]$ denote the closure $\overline{\underline{ab}}$ of the region $\underline{ab}$. Then $[a, b] = \{x \in C \mid a \leq x \leq b\}$.*

Proof. To prove that $[a, b] = \{x \in C \mid a \leq x \leq b\}$, Definition 1.2 tells us that it will suffice to show that every element $y \in [a, b]$ is an element of $\{x \in C \mid a \leq x \leq b\}$ and vice versa.

First, let $y \in [a, b]$. Then by Definition 4.4, $y \in \underline{ab} \cup LP(\underline{ab})$. It follows by Definition 1.5 that $y \in \underline{ab}$ or $y \in LP(\underline{ab})$. We divide into two cases. Suppose first that $y \in \underline{ab}$. Then by Definitions 3.10 and 3.6, $y \in C$ and $a < y < b$. The latter condition implies that $a \leq y \leq b$. Therefore, since $y \in C$ and $a \leq y \leq b$, we have that $y \in \{x \in C \mid a \leq x \leq b\}$ in this case. Now suppose that $y \in LP(\underline{ab})$. Then by Lemma 3.17, $y \notin \operatorname{ext} \underline{ab}$. By Definition 3.15, this implies that $y \notin C \setminus (\{a\} \cup \underline{ab} \cup \{b\})$. It follows by Definition 1.11 that $y \notin C$ or $y \in \{a\} \cup \underline{ab} \cup \{b\}$. Now as a limit point of $\underline{ab}$, Definition 3.13 asserts that $y \in C$, so it must be that $y \in \{a\} \cup \underline{ab} \cup \{b\}$. Consequently, by two applications of Definition 1.5 as well as Definitions 3.10 and 3.6, $y = a$, $a < y < b$, or $y = b$. In other words, $a \leq y \leq b$. Therefore, since $y \in C$ and $a \leq y \leq b$, we have that $y \in \{x \in C \mid a \leq x \leq b\}$ in this case, too, as desired.

Now let $y \in \{x \in C \mid a \leq x \leq b\}$. Then $y \in C$ and $a \leq y \leq b$. We divide into three cases ($y = a$, $y = b$, and $a < y < b$). Suppose first that $y = a$. Then by Corollary 5.14, $y \in LP(\underline{ab})$. Thus, by Definition 1.5, $y \in \underline{ab} \cup LP(\underline{ab})$. Consequently, by Definition 4.4, $y \in [a, b]$ in this case. The proof of the second case is symmetric to that of the first. Lastly, suppose that $a < y < b$. Then by Definitions 3.6 and 3.10, $y \in \underline{ab}$.

Thus, by Definition 1.5, $y \in \underline{ab} \cup LP(\underline{ab})$. Consequently, by Definition 4.4, $y \in [a, b]$ in this case, too, as desired. □

Lemma 5.16. *Let $X \subset C$ and define:*

$$\Psi(X) = \{x \in C \mid x \textit{ is not an upper bound of } X\} \qquad \Omega(X) = \{x \in C \mid x \textit{ is not a lower bound of } X\}$$

Then both $\Psi(X)$ and $\Omega(X)$ are open.

Proof. We will take this one set at a time.

To prove that $\Psi(X)$ is open, Theorem 4.10 tells us that it will suffice to confirm that for all $y \in \Psi(X)$, there exists a region containing y that is a subset of $\Psi(X)$. Let y be an arbitrary element of $\Psi(X)$. Then by the definition of $\Psi(X)$, y is not an upper bound of X. Thus, by Definition 5.6, there exists some $x \in X$ such that $x > y$. Now let $a \in C$ be a point such that $a < y$ (Axiom 3 and Definition 3.3 imply the existence of such a point) and consider the region $\underline{ax}$. We will demonstrate that $\underline{ax}$ is the desired region, i.e., that $y \in \underline{ax}$ and $\underline{ax} \subset \Psi(X)$. For the first condition, since $a < y < x$, it immediately follows from Definitions 3.6 and 3.10 that $y \in \underline{ax}$, as desired. As to the second condition, Definition 1.3 tells us that it will suffice to show that every element $z \in \underline{ax}$ is an element of $\Psi(X)$. Let z be an arbitrary element of $\underline{ax}$. Then by Definitions 3.10 and 3.6, $z < x$. Since z is less than an element of X, Definition 5.6 asserts that z is not an upper bound of X. Thus, by the definition of $\Psi(X)$, $z \in \Psi(X)$, as desired. Therefore, for all $y \in \Psi(X)$, there exists a region containing y that is a subset of $\Psi(X)$.

The proof is symmetric in the other case. □