MATH 16210 (Honors Calculus II IBL) Notes

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Contents

Script 6

Construction of the Real Numbers

6.1 Journal

1/12: **Definition 6.1.** A subset A of $\mathbb{Q}$ is said to be a **cut** (or **Dedekind cut**) if it satisfies the following:

(a) $A \neq \emptyset$ and $A \neq \mathbb{Q}$.

(b) If $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$.

(c) A does not have a last point; i.e., if $r \in A$, then there is some $s \in A$ with $s > r$.

We denote the collection of all cuts by $\mathbb{R}$.

Lemma 6.2. *Let A be a Dedekind cut and $x \in \mathbb{Q}$. Then $x \notin A$ if and only if x is an upper bound for A.*

Proof. Suppose first that x is an element of $\mathbb{Q}$ such that $x \notin A$. To prove that x is an upper bound for A, Definition 5.6 tells us that it will suffice to show that for all $r \in A$, $r \leq x$. Let r be an arbitrary element of A. Then since $r \in A$, $x \in \mathbb{Q}$, and $x \notin A$, the contrapositive of Definition 6.1b asserts that $x \nless r$. Therefore, $r \leq x$, as desired.

Now suppose that x is an upper bound for A. By Definition 5.6, this implies that for all $r \in A$, $r \leq x$. Therefore, since there is no $r \in A$ with $r > x$, by the contrapositive of Definition 6.1c, $x \notin A$, as desired. $\square$

Exercise 6.3.

(a) Prove that for any $q \in \mathbb{Q}$, $\{x \in \mathbb{Q} \mid x < q\}$ is a Dedekind cut. We then define $\mathbf{0} = \{x \in \mathbb{Q} \mid x < 0\}$.

(b) Prove that $\{x \in \mathbb{Q} \mid x \leq 0\}$ is not a Dedekind cut.

(c) Prove that $\{x \in \mathbb{Q} \mid x < 0\} \cup \{x \in \mathbb{Q} \mid x^2 < 2\}$ is a Dedekind cut.

Proof of a. Let q be an arbitrary element of $\mathbb{Q}$. To prove that $A = \{x \in \mathbb{Q} \mid x < q\}$ is a Dedekind cut, Definition 6.1 tells us that it will suffice to show that $A \neq \emptyset$; $A \neq \mathbb{Q}$; if $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$; and if $r \in A$, then there is some $s \in A$ with $s > r$. We will take this one claim at a time.

To show that $A \neq \emptyset$, Definition 1.8 tells us that it will suffice to find an element of A. By Exercise 3.9d, q is not the first point of $\mathbb{Q}$. Thus, by Definition 3.3, there exists an object $x \in \mathbb{Q}$ such that $x < q$. By the definition of A, this implies that $x \in A$, as desired.

To show that $A \neq \mathbb{Q}$, Definition 1.2 tells us that it will suffice to find an element of $\mathbb{Q}$ that is not an element of A. By hypothesis, $q \in \mathbb{Q}$. By Exercise 3.9d, $q \nless q$. Therefore, $q \in \mathbb{Q}$ but $q \notin A$, as desired.

To show that if $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$, we let $r \in A$ and $s \in \mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s < r$ and seek to verify that $s \in A$. Since $r \in A$, $r < q$. This combined with the fact that $s < r$ implies by transitivity that $s < q$. Therefore, since $s \in \mathbb{Q}$ and $s < q$, $s \in A$, as desired.

To show that if $r \in A$, then there is some $s \in A$ with $s > r$, we let $r \in A$ and seek to find such an s. By the definition of A, $r < q$. Thus, by Additional Exercise 3.1, there exists a point $s \in \mathbb{Q}$ such that $r < s < q$. Since $s \in \mathbb{Q}$ and $s < q$, $s \in A$. It follows that s is the desired element of A which satisfies $s > r$. $\square$

Proof of b. To prove that $A = \{x \in \mathbb{Q} \mid x \leq 0\}$ is not a Dedekind cut, Definition 6.1 tells us that it will suffice to show that A *does* have a last point. To show this, we will demonstrate that 0 is the last point of A. To demonstrate this, Definition 3.1 tells us that it will suffice to confirm that $0 \in A$ and for all $x \in A$, $x \leq 0$. Since $0 \leq 0$ and $0 \in \mathbb{Q}$, $0 \in A$. Additionally, by the definition of A, it is true that for all $x \in A$, $x \leq 0$. ☐

Proof of c. Let $B = \{x \in \mathbb{Q} \mid x < 0\}$ and let $C = \{x \in \mathbb{Q} \mid x^2 < 2\}$. To prove that $A = B \cup C$ is a Dedekind cut, Definition 6.1 tells us that it will suffice to show that $A \neq \emptyset$; $A \neq \mathbb{Q}$; if $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$; and if $r \in A$, then there is some $s \in A$ with $s > r$. We will take this one claim at a time.

To show that $A \neq \emptyset$, Definition 1.8 tells us that it will suffice to find an element of A. Since $-1 \in \mathbb{Q}$ and $-1 < 0$, $-1 \in B$. Therefore, by Definition 1.5, $-1 \in A$, as desired.

To show that $A \neq \mathbb{Q}$, Definition 1.2 tells us that it will suffice to find an element of $\mathbb{Q}$ that is not an element of A. Since $2 \geq 0$, $2 \notin B$. Additionally, since $2^2 \geq 2$, $2 \notin C$. Therefore, by Definition 1.5, $2 \notin A$, as desired.

To show that if $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$, we let $r \in A$ and $s \in \mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s < r$ and seek to verify that $s \in A$. We divide into two cases ($s < 0$ and $s \geq 0$). Suppose first that $s < 0$. Then $s \in B$, meaning that $s \in A$. Now suppose that $s \geq 0$. Then by Script 0, we have $0 \leq s^2 < r^2 < 2$. Thus, by the definition of C, $s \in C$, implying that $s \in A$.

To show that A does not have a last point, suppose for the sake of contradiction that A has a last point p. We now divide into two cases ($p \leq 0$ and $p > 0$). Suppose first that $p \leq 0$. Since p is the last point of A, Definition 3.3 tells us that $x \leq p$ for all $x \in A$. But $1 \in A$ (since $1 \in \mathbb{Q}$ and $1^2 = 1 < 2$ implies $1 \in B$, implies $1 \in A$) and $1 > 0 \geq p$, a contradiction. Now suppose that $p > 0$. Definition 3.3 tells us that $p \in A$, but the condition that $p > 0$ means $p \notin B$, so we must have $p \in C$. However, by the proof of Exercise 4.24, $\frac{2(p+1)}{p+2}$ will be an element of B (and therefore A) that is greater than p no matter how large p is, a contradiction. ☐

Definition 6.4. If $A, B \in \mathbb{R}$, we say that $A < B$ if A is a proper subset of B.

Exercise 6.5. Show that $\mathbb{R}$ satisfies Axioms 1, 2, and 3.

Proof. By Exercise 6.3a, $\{x \in \mathbb{Q} \mid x < 0\} \in \mathbb{R}$ since $0 \in \mathbb{Q}$. Therefore, Axiom 1 is immediately satisfied.

Axiom 2 asserts that $\mathbb{R}$ must have an ordering $<$. As such, it will suffice to verify that the ordering given by Definition 6.4 satisfies the stipulations of Definition 3.1. To prove that $<$ satisfies the trichotomy, it will suffice to show that for all $A, B \in \mathbb{R}$, exactly one of the following holds: $A < B$, $B < A$, or $A = B$.

We first show that *no more than one* of the three statements can simultaneously be true. Let A, B be arbitrary elements of $\mathbb{R}$. We divide into three cases. First, suppose for the sake of contradiction that $A < B$ and $B < A$. By Definition 6.4, this implies that $A \subsetneq B$ and $B \subsetneq A$. Thus, by Definition 1.3, $A \subset B$, $B \subset A$, and $A \neq B$. But by Theorem 1.7, $A \subset B$ and $B \subset A$ implies that $A = B$, a contradiction. Second, suppose for the sake of contradiction that $A < B$ and $A = B$. By substitution, we have that $A < A$. But by Definitions 6.4 and 1.3, it follows that $A \neq A$. The proof of the third case ($B < A$ and $A = B$) is symmetric to that of the second case.

We now show that *at least one* of the three statements is always true. Let A, B be arbitrary elements of $\mathbb{R}$, and suppose for the sake of contradiction that $A \not< B$, $B \not< A$, and $A \neq B$. Since $A \not< B$ and $B \not< A$, we have by Definition 6.4 that $A \not\subsetneq B$ and $B \not\subsetneq A$. Thus, by Definition 1.3, $A \not\subset B$ or $A = B$, and $B \not\subset A$ or $A = B$. But $A \neq B$ by hypothesis, so it must be that $A \not\subset B$ and $B \not\subset A$. It follows from the first statement by Definition 1.3 that there exists an object $x \in A$ such that $x \notin B$, and there exists an object $y \in B$ such that $y \notin A$. Since $x \notin B$, Lemma 6.2 implies that x is an upper bound of B. Consequently, by Definition 5.6, $p \leq x$ for all $p \in B$, including y. Similarly, $p \leq y$ for all $p \in A$, including x. Thus, we have $y \leq x$ and $x \leq y$, implying that $x = y$. But since $y \in B$, this implies that $x \in B$, a contradiction.

To prove that $<$ is transitive, it will suffice to show that for all $A, B, C \in \mathbb{R}$, if $A < B$ and $B < C$, then $A < C$. Let A, B, C be arbitrary elements of $\mathbb{R}$ for which it is true that $A < B$ and $B < C$. By Definition 6.4, we have $A \subsetneq B$ and $B \subsetneq C$. Thus, by Script 1, $A \subsetneq C$. Therefore, by Definition 6.4, $A < C$.

Axiom 3 asserts that $\mathbb{R}$ must have no first or last point. We will take this one argument at a time

Suppose for the sake of contradiction that $\mathbb{R}$ has some first point A. Then by Definition 3.3, $A \leq X$ for every $X \in \mathbb{R}$. Now since A is a Dedekind cut, Definition 6.1 tells us that $A \neq \emptyset$. Thus, by Definition 1.8, there exists some $q \in A$. Additionally, $A \subset \mathbb{Q}$ by Definition 6.1, so $q \in A$ implies that $q \in \mathbb{Q}$. It follows by Exercise 6.3a that $B = \{x \in \mathbb{Q} \mid x < q\}$ is a Dedekind cut. We now seek to prove that $B \subsetneq A$. To do this, Definition 1.3 tells us that it will suffice to show that $B \neq A$ and $B \subset A$. To show that $B \neq A$, Definition 1.2 tells us that it will suffice to find an element of A that is not an element of B. Conveniently, q is clearly such an object. To show that $B \subset A$, Definition 1.3 tells us that we must confirm that every element of B is an element of A. Let p be an arbitrary element of B. Then by the definition of B, $p \in \mathbb{Q}$ and $p < q$. It follows by Definition 6.1b (which clearly applies to A) that $p \in A$, as desired. Having proven that $B \subsetneq A$, Definition 6.4 tells us that $B < A$. But this contradicts the previously demonstrated fact that $A \leq X$ for every $X \in \mathbb{R}$, including B.

Suppose for the sake of contradiction that $\mathbb{R}$ has some last point A. Then by Definition 3.3, $X \leq A$ for every $X \in \mathbb{R}$. Now since A is a Dedekind cut, Definition 6.1 tells us that $A \neq \mathbb{Q}$. Thus, by Definition 1.2, there exists some $q \in \mathbb{Q}$ such that $q \notin A$. It follows by Lemma 6.2 that q is an upper bound of A. Consequently, by Definition 5.6, $x \leq q$ for all $x \in A$. Additionally, by Exercise 6.3a, $B = \{x \in \mathbb{Q} \mid x < q + 1\}$[1] is a Dedekind cut. We now seek to prove that $A \subsetneq B$. As before, this means we must show that $A \neq B$ and $A \subset B$. To show that $A \neq B$, Definition 1.2 tells us that it will suffice to find an element of B that is not an element of A. Since $x \leq q$ for all $x \in A$ and $q < q + 0.5 < q + 1$, $q + 0.5 \notin A$ and $q + 0.5 \in B$ is one such desired object. To show that $A \subset B$, Definition 1.3 tells us that we must confirm that every element of A is an element of B. Let p be an arbitrary element of A. As an element of A, we know that $p \leq q$. Thus, $p < q + 1$, so $p \in B$, as desired. Having proven that $A \subsetneq B$, Definition 6.4 tells us that $A < B$. But this contradicts the previously demonstrated fact that $X \leq A$ for every $X \in \mathbb{R}$, including B. ☐

1/14: **Lemma 6.6.** *A nonempty subset of* $\mathbb{R}$ *that is bounded above has a supremum.*

Proof. Let X be an arbitrary nonempty subset of $\mathbb{R}$ that is bounded above. To prove that $\sup X$ exists, we will show that $\sup X = U = \bigcup\{Y \mid Y \in X\}$. To show this, Definition 5.7 tells us that it will suffice to demonstrate that $U \in \mathbb{R}$, U is an upper bound of X, and if U' is an upper bound of X, then $U \leq U'$. Let's begin.

To demonstrate that $U \in \mathbb{R}$, Definition 6.1 tells us that it will suffice to confirm that $U \neq \emptyset$; $U \neq \mathbb{Q}$; if $r \in U$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in U$; and if $r \in U$, then there is some $s \in U$ with $s > r$.

As the union of a nonempty set of nonempty sets, Script 1 implies that $U \neq \emptyset$.

To demonstrate that $U \neq \mathbb{Q}$, Definition 1.2 tells us that it will suffice to find a point $p \in \mathbb{Q}$ such that $p \notin U$. Since X is bounded above, we have by Definition 5.6 that there exists a Dedekind cut $V \in \mathbb{R}$ such that $A \leq V$ for all $A \in X$. It follows by Definition 6.4 that $A \subset V$ for all $A \in X$. Thus, by Script 1, $U \subset V$. Now since V is a Dedekind cut, we know by Definition 6.1 that $V \subset \mathbb{Q}$ and $V \neq \mathbb{Q}$, meaning that there exists a point $p \in \mathbb{Q}$ such that $p \notin V$. Consequently, since $U \subset V$, $p \notin U$, as desired.

To demonstrate that if $r \in U$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in U$, we let $r \in U$ and $s \in \mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s < r$ and seek to verify that $s \in U$. Since $r \in U$, Definition 1.13 tells us that $r \in A$ for some $A \in X$. Thus, since A is a Dedekind cut, $s \in \mathbb{Q}$ and $s < r$ implies that $s \in A$. Therefore, $s \in U$.

To demonstrate that if $r \in U$, then there is some $s \in U$ with $s > r$, we let $r \in U$ and seek to find such an s. Since $r \in U$, Definition 1.13 tells us that $r \in A$ for some $A \in X$. Thus, since A is a Dedekind cut, there exists a point $s \in A$ with $s > r$. Therefore, $s \in U$.

To demonstrate that U is an upper bound of X, Definition 5.6 tells us that it will suffice to confirm that $A \leq U$ for all $A \in X$. To confirm this, Definition 6.4 tells us that it will suffice to verify that $A \subset U$ for all $A \in X$. But by an extension of Theorem 1.7b, this is true.

Now suppose for the sake of contradiction that there exists an upper bound U' of X such that $U' < U$. It follows by Definitions 6.4 and 1.3 that there exists a point $p \in U$ such that $p \notin U'$. Thus, by the former statement and Definition 1.13, $p \in A$ for some $A \in X$. Additionally, since U' is an upper bound of X, we have by Definitions 5.6 and 6.4 that $A \subset U'$ for all $A \in X$. But this implies by Definition 1.3 that $p \in U'$, a contradiction. ☐

[1] Note that we add 1 to q to treat the case that $q = \sup A$, a case in which we would have $B = A$ if B were defined as $\{x \in \mathbb{Q} \mid x < q\}$.

1/19: **Exercise 6.7.** Show that $\mathbb{R}$ satisfies Axiom 4.

Proof. Suppose for the sake of contradiction that $\mathbb{R}$ does not satisfy Axiom 4. It follows that $\mathbb{R}$ is not connected, implying by Definition 4.22 that $\mathbb{R} = A \cup B$ where A, B are disjoint, nonempty, open sets. Since A, B are disjoint and nonempty, we know that there exist distinct objects $a \in A$ and $b \in B$. WLOG, let $a < b$.

We now seek to prove that the set $A \cap \underline{ab}$ is nonempty and bounded above. To prove that $A \cap \underline{ab}$ is nonempty, Definition 1.8 tells us that it will suffice to find an element of $A \cap \underline{ab}$. Since $a \in A$ and A is open, we have by Theorem 4.10 that there exists a region $\underline{cd}$ such that $a \in \underline{cd}$ and $\underline{cd} \subset A$. It follows by Definitions 3.10 and 3.6 that $a < d$, implying by Lemma 6.10[2] that there exists some point $x \in \mathbb{R}$ such that $c < a < x < d < b$ (note that $d < b$ since if $b < d$, then $b \in \underline{cd}$ would contradict the fact that $\underline{cd} \subset A$). Consequently, $x \in \underline{cd}$, meaning that $x \in A$, and $x \in \underline{ab}$. Therefore, $x \in A \cap \underline{ab}$, as desired. To prove that $A \cap \underline{ab}$ is bounded above, Definition 5.6 tells us that it will suffice to show that b is an upper bound of $A \cap \underline{ab}$. To show this, Definition 5.6 tells us that it will suffice to confirm that $y \leq b$ for all $y \in A \cap \underline{ab}$. Let y be an arbitrary element of $A \cap \underline{ab}$. Then by Definition 1.6, $y \in A$ and $y \in \underline{ab}$. It follows from the latter statement by Definitions 3.10 and 3.6 that $y < b$, i.e., $y \leq b$, as desired.

Having established that $A \cap \underline{ab} \subset \mathbb{R}$ is nonempty and bounded above, we can invoke Lemma 6.6 to learn that $A \cap \underline{ab}$ has a supremum $\sup(A \cap \underline{ab})$. We now divide into two cases ($\sup(A \cap \underline{ab}) \in A$ and $\sup(A \cap \underline{ab}) \in B$; it follows from the definitions of A and B that exactly one of these cases is true). Suppose first that $\sup(A \cap \underline{ab}) \in A$. Then since A is open, we have by Theorem 4.10 that there exists a region $\underline{ef}$ such that $\sup(A \cap \underline{ab}) \in \underline{ef}$ and $\underline{ef} \subset A$. It follows from the former condition that $\sup(A \cap \underline{ab}) < f$. Thus, by Lemma 6.10, there exists an object $z \in \mathbb{R}$ such that $e < \sup(A \cap \underline{ab}) < z < f < b$ (note that $f < b$ for the same reason that $d < b$). Consequently, $z \in \underline{ef}$, implying that $z \in A$, and $z \in \underline{ab}$. Thus, we have found an element of $A \cap \underline{ab}$ that is greater than $\sup(A \cap \underline{ab})$, contradicting Definitions 5.7 and 5.6. The proof is symmetric in the other case (except that we find an element of B less than $\sup(A \cap \underline{ab})$). $\square$

1/14: **Definition 6.8.** Let C be a continuum satisfying Axioms 1-4. Consider a subset $X \subset C$. We say that X is **dense** in C if every $p \in C$ is a limit point of X.

Lemma 6.9. *A subset $X \subset C$ is dense in C if and only if $\overline{X} = C$.*

Proof. Suppose first that $X \subset C$ is dense in C. To prove that $\overline{X} = C$, Definition 1.2 tells us that it will suffice to show that every point $p \in \overline{X}$ is an element of C and vice versa. Clearly, every element of $\overline{X}$ is an element of C. On the other hand, let p be an arbitrary element of C. Since X is dense in C, Definition 6.8 tells us that $p \in LP(X)$. Therefore, by Definitions 1.5 and 4.4, $p \in \overline{X}$.

Now suppose that $\overline{X} = C$. To prove that X is dense in C, Definition 6.8 tells us that it will suffice to show that every $p \in C$ is a limit point of X. Let p be an arbitrary element of C. By Corollary 5.4, this implies that $p \in LP(C)$. It follows that $p \in LP(\overline{X})$. Thus, by Definition 4.4, $p \in LP(X \cup LP(X))$. Consequently, by Theorem 3.20, $p \in LP(X)$ or $p \in LP(LP(X))$. We now divide into two cases. If $p \in LP(X)$, then we are done. On the other hand, if $p \in LP(LP(X))$, the lemma from Theorem 4.6 asserts that $p \in LP(X)$, and we are done again. $\square$

Our next goal is to prove that $\mathbb{Q}$ is dense in $\mathbb{R}$. Just to make sense of that statement, we need to decide how to think of $\mathbb{Q}$ as a subset of $\mathbb{R}$. For every rational number $q \in \mathbb{Q}$, define the corresponding real number as the Dedekind cut

$$i(q) = \{x \in \mathbb{Q} \mid x < q\}$$

For example, $\mathbf{0} = i(0)$. It can be verified that this gives a well-defined injective function $i : \mathbb{Q} \to \mathbb{R}$. We identify $\mathbb{Q}$ with its image $i(\mathbb{Q}) \subset \mathbb{R}$ so that the rational numbers $\mathbb{Q}$ are a subset of the real numbers $\mathbb{R}$. (Similarly, $\mathbb{N}$ and $\mathbb{Z}$ can be understood as subsets of $\mathbb{R}$.)

[2]We may use this lemma since it does not depend on this result, Definition 6.8, or Lemma 6.9.

Lemma 6.10. *Given $A, B \in \mathbb{R}$ with $A < B$, there exists $p \in \mathbb{Q}$ such that $A < i(p) < B$.*

Proof. Since $A < B$, Definition 6.4 tells us that $A \subsetneq B$. Thus, by Definition 1.3, there exists a point q such that $q \in B$ and $q \notin A$. Since $q \in B$ where B is a Dedekind cut, we have by Definition 6.1 that there exists a point $p \in B$ with $p > q$. Additionally, since $q \notin A$ implies that q is an upper bound of A by Lemma 6.2, we know by Definition 5.6 that $x \leq q$ for all $x \in A$. It follows since $q < p$ that $x \leq p$ for all $x \in A$, meaning by Definition 5.6 and Lemma 6.2 that $p \notin A$. Having established that $p, q \in B$, $p, q \notin A$, and $q < p$, we are now ready to prove that $A < i(p) < B$. Definition 6.4 tells us that we may do so by showing that $A \subsetneq i(p)$ and $i(p) \subsetneq B$. We will take this one argument at a time.

To show that $A \subsetneq i(p)$, Definition 1.3 tells us that it will suffice to verify that every element of A is an element of $i(p)$ and that there exists an element of $i(p)$ that is not an element of A. We treat the former statement first. As previously mentioned, $x \leq p$ for all $x \in A$. This combined with the fact that $p \notin A$ implies that $x < p$ for all $x \in A$. Thus, by the definition of $i(p)$, $x \in i(p)$ for all $x \in A$, as desired. As to the latter statement, since $q < p$, we have by the definition of $i(p)$ that $q \in i(p)$. However, we also know that $q \notin A$, as desired.

To show that $i(p) \subsetneq B$, we must verify symmetric arguments to before. For the former statement, let r be an arbitrary element of $i(p)$. Then by the definition of $i(p)$, $r < p$. Since $p \in B$ and $r \in \mathbb{Q}$ satisfy $r < p$, we have by Definition 6.1 that $r \in B$, as desired. As to the latter statement, p is clearly an element of B that is not an element of $i(p)$, as desired. □

1/19: **Theorem 6.11.** *$i(\mathbb{Q})$ is dense in $\mathbb{R}$.*

Proof. To prove that $i(\mathbb{Q})$ is dense in $\mathbb{R}$, Definition 6.8 tells us that it will suffice to show the every point $X \in \mathbb{R}$ is a limit point of $i(\mathbb{Q})$. Let X be an arbitrary element of $\mathbb{R}$. To show that $X \in LP(i(\mathbb{Q}))$, Definition 3.13 tells us that it will suffice to verify that for every region $\underline{AB}$ with $X \in \underline{AB}$, we have $\underline{AB} \cap (i(\mathbb{Q}) \setminus \{X\}) \neq \emptyset$. Let $\underline{AB}$ be an arbitrary region with $X \in \underline{AB}$. It follows by Definitions 3.10 and 3.6 that $A < X < B$. Thus, by Lemma 6.10, there exists $p \in \mathbb{Q}$ such that $A < i(p) < X < B$. By Definitions 3.6 and 3.10, $i(p) \in \underline{AB}$. By Definition 1.18, $i(p) \in i(\mathbb{Q})$. By Exercise 6.5, $i(p) < X$ implies that $i(p) \neq X$. Combining the last three results with Definitions 1.11 and 1.6, we have that $i(p) \in \underline{AB} \cap (i(\mathbb{Q}) \setminus \{X\})$, as desired. □

Corollary 6.12 (The Archimedean Property)**.** *Let $A \in \mathbb{R}$ be a positive real number. Then there exist nonzero natural numbers $n, m \in \mathbb{N}$ such that $i(\frac{1}{n}) < A < i(m)$.*

Proof. We will first prove that there exists a nonzero natural number n such that $i(\frac{1}{n}) < A$. We will then prove that there exists a nonzero natural number m such that $A < i(m)$. Let's begin.

Since $A \in \mathbb{R}$ is positive, we know that $0 < A$. Thus, by Lemma 6.10, there exists $\frac{p}{n} \in \mathbb{Q}$ such that $0 < i(\frac{p}{n}) < A$. As permitted by Exercise 3.9b, we choose $\frac{p}{n} \in \left[\frac{p}{n}\right]$ to be an object such that $0 < n$ (this also means that $n \in \mathbb{N}$). Consequently, by Scripts 2 and 3, we know that $0 < \frac{1}{n} \leq \frac{p}{n}$. It follows that $i(\frac{1}{n}) \leq i(\frac{p}{n})$ since $x \in i(\frac{1}{n})$ implies $x < \frac{1}{n} \leq \frac{p}{n}$ implies $x \in i(\frac{p}{n})$, implies $i(\frac{1}{n}) \subset i(\frac{p}{n})$. Therefore, $i(\frac{1}{n}) \leq i(\frac{p}{n}) < A$, as desired.

By Exercise 6.5, Axiom 3, and Definition 3.3, there exists a point $B \in \mathbb{R}$ such that $A < B$. It follows by Lemma 6.10 that there exists $\frac{m}{q} \in \mathbb{Q}$ such that $A < i(\frac{m}{q}) < B$. As before, let $\frac{m}{q}$ be an object such that $0 < q$. Consequently, by Scripts 2 and 3, we know that $0 < \frac{m}{q} \leq m$. Once again, for the same reasons as before, $i(\frac{m}{q}) \leq i(m)$. Therefore, $A < i(\frac{m}{q}) \leq i(m)$, as desired. □

Corollary 6.13. *$i(\mathbb{N})$ is an unbounded subset of $\mathbb{R}$.*

Proof. Suppose for the sake of contradiction that $i(\mathbb{N})$ is bounded above. Then by Definition 5.6, there exists a point $A \in \mathbb{R}$ such that $i(n) \leq A$ for all $n \in \mathbb{N}$. Note that A is a positive real number since $i(0) < i(0) \leq A$. But by Corollary 6.12, $A < i(n)$ for some $n \in \mathbb{N}$, a contradiction. □

1/21: **Corollary 6.14.** *If $A \in \mathbb{R}$ is a real number, then there is an integer n such that $i(n-1) \leq A < i(n)$.*

Proof. Let X be be the set of all integers z such that $i(z) \leq A$. Symbolically,

$$X = \{z \mid z \in \mathbb{Z} \text{ and } i(z) \leq A\}$$

Since $A \neq \emptyset$ by Definition 6.1, there exists a point $\frac{p}{q} \in \mathbb{Q}$ such that $\frac{p}{q} \in A$. As in Corollary 6.12, we let $q > 0$. It follows by Scripts 2 and 3 that if $p \geq 0$, then $0 \leq \frac{p}{q}$, i.e.[3], $i(0) \leq A$ and if $p < 0$, then $p \leq \frac{p}{q}$, i.e., $i(p) \leq A$. Thus, in either case, X is nonempty.

Now there exists a nonzero natural number m such that $A < i(m)$ (if $A \leq i(0)$, then $A < i(1)$; if $A > 0$, then apply Corollary 6.12). Let $f : X \to \mathbb{N}$ be defined by the rule

$$f(x) = m - x$$

By Script 1, f is an injective function, $f(X) \subset \mathbb{N}$, and $f(X)$ is nonempty (since X is nonempty). Thus, by the well-ordering principle (Additional Exercise 0.1), there is a least element, which we shall call y, in $f(X)$. Since f is injective, there exists exactly one object $n - 1 \in X$ such that $f(n-1) = y$.

By the definition of X, $i(n-1) \leq A$. To prove that $A < i(n)$, suppose for the sake of contradiction that $i(n) \leq A$. This coupled with the fact that $n \in \mathbb{Z}$ implies that $n \in X$. Thus, $f(n) \in f(X)$. But $f(n) = m - n < m - n + 1 = m - (n-1) = f(n-1)$, contradicting the fact that $f(n-1)$ is the least element of $f(X)$. ☐

1/26: **Axiom 1.** *The continuum contains a countable dense subset.*

Definition 6.15. Let X and Y be sets with orderings $<_X$ an $<_Y$, respectively. A function $f : X \to Y$ is **order-preserving** if for all $r, s \in X$,

$$r <_X s \Longrightarrow f(r) <_Y f(s)$$

Note that the function $i : \mathbb{Q} \to \mathbb{R}$ discussed above is order-preserving.

Exercise 6.16. Let C satisfy Axioms 1-5. Let $K \subset C$ be a countable dense subset of C. Construct an order-preserving bijection $f : \mathbb{Q} \to K$.

Lemma.

a) *K satisfies Axiom 3.*

b) (Density Lemma) *For all $x, y \in K$, if $x < y$, then there exists a point $z \in K$ such that z is between x and y.*

Proof of a. To prove that K satisfies Axiom 3, we must verify that K has neither a first nor a last point. We will address the first point question first. Suppose for the sake of contradiction that K has a first point x. Then by Definition 3.3, $x \leq y$ for all $y \in K$. However, since C satisfies Axiom 3, there exists an object $a \in C$ such that $a < x$. Now consider the region $\underline{ax}$. We have by Corollary 5.3 that there exists a point $p \in \underline{ax}$. Additionally, we have by Script 3 that $\underline{ax} \cap K = \emptyset$. Thus, $\underline{ax} \cap (K \setminus \{p\}) = \emptyset$, implying by Definition 3.13 that $p \notin LP(K)$. But since $p \in C$ and $p \notin LP(K)$, we have by Definition 6.8 that K is not dense in C, a contradiction.

The proof is symmetric for last points. ☐

Proof of b. Suppose for the sake of contradiction that that there exist $x, y \in K$ with $x < y$ such that no point $z \in K$ is between x and y. By Theorem 5.2, there exists $p \in C$ such that p is between x and y. Consequently, by Definition 3.10, $p \in \underline{xy}$. Additionally, we have by Script 3 that $\underline{xy} \cap K = \emptyset$. It follows that $\underline{xy} \cap (K \setminus \{p\}) = \emptyset$, implying by Definition 3.13 that $p \notin LP(K)$. But since $p \in C$ and $p \notin LP(K)$, we have by Definition 6.8 that K is not dense in C, a contradiction. ☐

[3]For the same reasons as in Corollary 6.12.

Proof of Exercise 6.16. By Theorem 2.11, $\mathbb{Q}$ is countable, implying by Definition 1.35 that there exists a bijection $g : \mathbb{N} \to \mathbb{Q}$. The existence of this bijection means that we can refer to an arbitrary element q of $\mathbb{Q}$ by the number n for which $g(n) = q$; in another notation, we can refer to q as q_n. Thus, since every element of $\mathbb{Q}$ can be written as q_n for some $n \in \mathbb{N}$, we can write $\mathbb{Q} = \{q_1, q_2, \dots\}$. Similarly, we can express K as $K = \{k_1, k_2, \dots\}$. We will use this method of referring to the elements of $\mathbb{Q}$ to construct f.

We define f recursively with strong induction. For the base case q_1, we define $f(q_1) = k_1$. Now suppose inductively that we have defined $f(q_1), f(q_2), \dots, f(q_n)$; we now seek to define $f(q_{n+1})$. By Theorem 3.5, the symbols $a_1, \dots, a_{n+1}$ can be assigned to $q_1, \dots, q_{n+1}$ so that $a_1 <_{\mathbb{Q}} a_2 <_{\mathbb{Q}} \cdots <_{\mathbb{Q}} a_{n+1}$. We divide into three cases ($q_{n+1} = a_1$, $q_{n+1} = a_{n+1}$, and $q_{n+1} = a_i$ where $1 < i < n+1$). First, suppose that $q_{n+1} = a_1$. By the inductive hypothesis, $f(a_2), f(a_3), \dots, f(a_{n+1})$ are defined elements of K. At this point, define the set $X = \{k \in K \mid k <_K f(a_2)\}$. It follows by Lemma (a) that this set is nonempty. Thus, by the well-ordering principle, there exists a $k_i \in X$ such that $i \leq j$ for all $k_j \in X$. We let $f(q_{n+1}) = k_i$. The second case is symmetric to the first. Third, suppose that $q_{n+1} = a_i$ where $1 < i < n+1$. By the inductive hypothesis, $f(a_1), \dots, f(a_{i-1}), f(a_{i+1}), \dots, f(a_{n+1})$ are defined elements of K. At this point, define the set $X = \{k \in K \mid f(a_{i-1}) <_K k <_K f(a_{i+1})\}$. It follows by Lemma (b) that this set is nonempty. Thus, by the well-ordering principle, there exists a $k_i \in X$ such that $i \leq j$ for all $k_j \in X$. We let $f(q_{n+1}) = k_i$.

To prove that f is a function, Definition 1.16 tells us that it will suffice to show that for all $q \in \mathbb{Q}$, there exists a unique $k \in K$ such that $f(q) = k$. First, we will prove that for all $q \in \mathbb{Q}$, there exists *some* $k \in K$ such that $f(q) = k$. Let q_i be an arbitrary element of $\mathbb{Q}$. Then $i \in \mathbb{N}$, and by the principle of strong mathematical induction (Additional Exercise 0.2b), $f(q_i)$ is assigned to an element of k. As to proving the uniqueness of the k to which q_i is defined, each q is assigned once, in one of three mutually exclusive cases, to an unambiguously defined (as guaranteed by the well-ordering principle) element of K.

To prove that f is order-preserving, we will first verify the following claim (which we will refer to as Lemma (c) for future reference): Consider the set $\{q_1, \dots, q_n\} \subset \mathbb{Q}$; if the symbols $a_1, \dots, a_n$ are assigned to $q_1, \dots, q_n$ such that $a_1 <_{\mathbb{Q}} a_2 <_{\mathbb{Q}} \cdots <_{\mathbb{Q}} a_n$, then $f(a_1) <_K f(a_2) <_K \cdots <_K f(a_n)$. We will then use this result to prove that f is order-preserving for any two arbitrary elements $q_i, q_j \in \mathbb{Q}$. Let's begin.

To verify the above claim, we induct on n. The base case $n = 1$ is vacuously true. Now suppose inductively that we have proven the claim for n; we now seek to prove it for $n+1$. By Theorem 3.5, the symbols $a_1, \dots, a_{n+1}$ can be assigned to $q_1, \dots, q_{n+1}$ so that $a_1 <_{\mathbb{Q}} a_2 <_{\mathbb{Q}} \cdots <_{\mathbb{Q}} a_{n+1}$. We divide into three cases ($q_{n+1} = a_1$, $q_{n+1} = a_{n+1}$, and $q_{n+1} = a_i$ where $1 < i < n+1$). First, suppose that $q_{n+1} = a_1$. By the definition of f, $f(q_{n+1}) \in \{k \in K \mid k <_K f(a_2)\}$, meaning that $f(q_{n+1}) = f(a_1) <_K f(a_2)$. Additionally, by the inductive hypothesis, we know that $f(a_2) <_K f(a_3) <_K \cdots <_K f(a_{n+1})$ (since $a_2, \dots, a_{n+1}$ correspond to $q_1, \dots, q_n$). Together, these two results imply that $f(a_1) <_K f(a_2) <_K \cdots <_K f(a_{n+1})$. The proof of the second case is symmetric to that of the first. Third, suppose that $q_{n+1} = a_i$ where $1 < i < n+1$. By the definition of f, $f(q_{n+1}) \in \{k \in K \mid f(a_{i-1}) <_K k <_K f(a_{i+1})\}$, meaning that $f(a_{i-1}) <_K f(q_{n+1}) = f(a_i) <_K f(a_{i+1})$. Additionally, by the inductive hypothesis, we know that $f(a_1) <_K \cdots <_K f(a_{i-1}) <_K f(a_{i+1}) <_K \cdots <_K f(a_{n+1})$ (for an analogous reason to before). These two results imply that $f(a_1) <_K f(a_2) <_K \cdots <_K f(a_{n+1})$.

We are now ready to actually prove that f is order-preserving. To do so, Definition 6.15 tells us that it will suffice to show that for all $q_i, q_j \in \mathbb{Q}$, $q_i <_{\mathbb{Q}} q_j$ implies $f(q_i) <_K f(q_j)$. Let q_i, q_j be arbitrary elements of $\mathbb{Q}$ such that $q_i <_{\mathbb{Q}} q_j$. Since $q_i <_{\mathbb{Q}} q_j$, $q_i \neq q_j$, implying that $i \neq j$. We divide into two cases ($i < j$ and $i > j$). Suppose first that $i < j$. By Theorem 3.5, the symbols $a_1, \dots, a_j$ can be assigned to $q_1, \dots, q_j$ so that $a_1 <_{\mathbb{Q}} a_2 <_{\mathbb{Q}} \cdots <_{\mathbb{Q}} a_j$. Let $q_j = a_l$. Since $q_i <_{\mathbb{Q}} q_j$, we know that $q_i = a_m$ where $m < l$. Additionally, by Lemma (c), we know that $f(a_1) <_K f(a_2) <_K \cdots <_K f(a_j)$. It follows that $f(a_m) <_K f(a_l)$, implying that $f(q_i) <_K f(q_j)$, as desired. The proof is symmetric in the other case.

To prove that f is bijective, Definition 1.20 tells us that it will suffice to show that f is injective and surjective.

To show that f is injective, Definition 1.20 tells us that it will suffice to demonstrate that $q_i \neq q_j$ implies $f(q_i) \neq f(q_j)$. WLOG let $q_i <_{\mathbb{Q}} q_j$. Then since f is order-preserving, Definition 6.15 implies that $f(q_i) <_K f(q_j)$. It follows that $f(q_i) \neq f(q_j)$, as desired.

We are now ready to actually show that f is surjective. To do so, Definition 1.20 tells us that it will suffice to demonstrate that for all $k_n \in K$, there exists a $q_i \in \mathbb{Q}$ such that $f(q_i) = k_n$. To do this, we induct on n. For the base case $n = 1$, it follows from the definition of f that $f(q_1) = k_1$. Now suppose inductively that for each $k_1, \dots, k_n$, there exists a $q_i \in \mathbb{Q}$ such that $f(q_i) = k_n$; we now seek to prove the claim for $n+1$.

By Theorem 3.5, the symbols $b_1, \dots, b_{n+1}$ can be assigned to $k_1, \dots, k_{n+1}$ so that $b_1 <_K b_2 <_K \cdots <_K b_{n+1}$. We divide into three cases ($k_{n+1} = b_1$, $k_{n+1} = b_{n+1}$, and $k_{n+1} = b_i$ where $1 < i < n+1$). First, suppose that $k_{n+1} = b_1$. By the inductive hypothesis, $b_2 = f(q_i) <_K b_3 = f(q_j) <_K \cdots <_K b_{n+1} = f(q_l)$. It follows by Definition 6.15 that $q_i <_{\mathbb{Q}} q_j <_{\mathbb{Q}} \cdots <_{\mathbb{Q}} q_l$. At this point, define the set $X = \{q \in \mathbb{Q} \mid q <_{\mathbb{Q}} q_i\}$. It follows from Exercise 3.9d that this set is nonempty. Thus, by the well-ordering principle, there exists a $q_m \in X$ such that $m \leq m'$ for all $q_{m'} \in X$. By the definition of f, $f(q_m) = k_{n+1}$. The proof of the second case is symmetric to that of the first. Third, suppose that $k_{n+1} = b_i$ where $1 < i < n+1$. By the inductive hypothesis, $b_2 = f(q_j) <_K \cdots <_K b_{i-1} = f(q_{j'}) <_K b_{i+1} = f(q_l) <_K \cdots <_K b_{n+1} = f(q_{l'})$. It follows by Definition 6.15 that $q_j <_{\mathbb{Q}} \cdots <_{\mathbb{Q}} q_{j'} <_{\mathbb{Q}} q_l <_{\mathbb{Q}} \cdots <_{\mathbb{Q}} q_{l'}$. At this point, define the set $X = \{q \in \mathbb{Q} \mid q_{j'} <_{\mathbb{Q}} q <_{\mathbb{Q}} q_l\}$. It follows from Additional Exercise 3.1 that this set is nonempty. Thus, by the well-ordering principle, there exists a $q_m \in X$ such that $m \leq m'$ for all $q_{m'} \in X$. By the definition of f, $f(q_m) = k_{n+1}$. $\square$

Exercise 6.17. Let $f : \mathbb{Q} \to K$ be an order-preserving bijection, as found in Exercise 6.16. Let $A \in \mathbb{R}$. Then $A \subset \mathbb{Q}$ and so $f(A) \subset K \subset C$. Define $F : \mathbb{R} \to C$ by

$$F(A) = \sup f(A)$$

1. Show $\sup f(A)$ exists, so F is well-defined.
2. Show F is injective and order-preserving.

Proof of 1. To prove that $\sup f(A)$ exists, Theorem 5.17 tells us that it will suffice to show that $f(A)$ is nonempty and bounded above. To show that $f(A)$ is nonempty, Definition 1.8 tells us that it will suffice to find an element of $f(A)$. By Definition 6.1, $A \neq \emptyset$. Thus, by Definition 1.8, there exists an object $x \in A$. It follows by Definition 1.18 that $f(x) \in f(A)$, as desired. To show that $f(A)$ is bounded above, Definition 5.6 tells us that it will suffice to find an element of K such that $f(x) \geq_C f(a)$ for all $f(a) \in f(A)$. By Definition 6.1, $A \neq \mathbb{Q}$ and $A \subset \mathbb{Q}$. Thus, by Definition 1.2, there exists an object $x \in \mathbb{Q}$ such that $x \notin A$. It follows from the latter condition by Lemma 6.2 that x is an upper bound for A. Thus, by Definition 5.6, $x \geq a$ for all $a \in A$. Consequently, by Definition 6.15, $f(x)$ is an element of K such that $f(x) \geq_C f(a)$ for all $f(a) \in f(A)$, as desired. $\square$

Proof of 2. To prove that F is order-preserving, Definition 6.15 tells us that it will suffice to show that for all $A, B \in \mathbb{R}$, $A <_{\mathbb{R}} B$ implies $F(A) <_C F(B)$. Let A, B be two arbitrary elements of $\mathbb{R}$ satisfying $A <_{\mathbb{R}} B$. Then by Definitions 6.4 and 1.3, there exists a point $x \in B$ such that $x \notin A$. It follows from the latter condition by Lemma 6.2 and Definition 5.6 that $x \geq a$ for all $a \in A$. Thus, by Definition 6.15, $f(x) \geq_C f(a)$ for all $f(a) \in f(A)$. Consequently, by Definition 5.7, $\sup f(A) \leq_C f(x)$. Additionally, by Definition 6.1, there exists a point $y \in B$ such that $y > x$. Thus, by Definition 6.15, we have that $f(y) >_C f(x)$. It follows by Definitions 5.6 and 5.7 that $f(y) \leq_C \sup f(B)$. Combining two results, we therefore have that $\sup f(A) \leq_C f(x) <_C f(y) \leq_C \sup f(B)$, meaning that $F(A) = \sup f(A) <_C \sup f(B) = F(B)$, as desired.

To prove that F is injective, Definition 1.20 tells us that it will suffice to show that if $A \neq B$, then $F(A) \neq F(B)$. Let A, B be two distinct real numbers. Then by Exercise 6.5, $A < B$ or $B < A$. We now divide into two cases. Suppose first that $A < B$. Then $F(A) < F(B)$ by Definition 6.15 (which we have just proven applies to F). This implies by Definition 3.1 that $F(A) \neq F(B)$, as desired. The proof is symmetric in the other case. $\square$

Theorem 6.18. *Suppose that C is a continuum satisfying Axioms 1-5. Then C is isomorphic to the real numbers $\mathbb{R}$; i.e., there is an order-preserving bijection $F : \mathbb{R} \to C$.*

Lemma. *Let K be a dense subset of C. For all $x, y \in C$, if $x < y$, then there exists a point $z \in K$ such that z is between x and y.*

Proof. Suppose for the sake of contradiction that there exist two points $x, y \in C$ with $x < y$ such that no point $z \in K$ is between x and y. By Corollary 5.3, the region $\underline{xy}$ is infinite. Thus, we can pick a point $p \in \underline{xy}$. Additionally, by Definition 1.6, we have that $\underline{xy} \cap K = \emptyset$. Thus, $\underline{xy} \cap (K \setminus \{p\}) = \emptyset$, implying by Definition 3.13 that $p \notin LP(K)$. But since $p \in C$ and $p \notin LP(K)$, we have by Definition 6.8 that K is not dense in C, a contradiction. $\square$

Proof of Theorem 6.18. By Axiom 1, C contains a countable dense subset K. By Exercise 6.16, there exists an order-preserving bijection $f : \mathbb{Q} \to K$. By Exercise 6.17, there exists an order-preserving injection $F : \mathbb{R} \to C$. To prove that there is an order-preserving bijection $F : \mathbb{R} \to C$, all that is left to do is to demonstrate that F (as defined in Exercise 6.17) is surjective.

To do this, Definition 1.20 tells us that it will suffice to show that for all $X \in C$, there exists an object $A \in \mathbb{R}$ such that $F(A) = X$. Put more simply, we must find a Dedekind cut A such that $\sup f(A) = X$ for every $X \in C$. To do this, we will begin by constructing the set $S = \{k \in K \mid k < X\}$. We will then verify that the preimage $f^{-1}(S)$ is a Dedekind cut. Lastly, we will verify that $\sup f(f^{-1}(S)) = X$. Let's begin.

Let X be an arbitrary element of C. Define S as above. To verify that $f^{-1}(S)$ is a Dedekind cut, Definition 6.1 tells us that it will suffice to confirm that $f^{-1}(S) \neq \emptyset$; $f^{-1}(S) \neq \mathbb{Q}$; if $r \in f^{-1}(S)$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in f^{-1}(S)$; and if $r \in f^{-1}(S)$, then there is some $s \in f^{-1}(S)$ with $s > r$. We will take this one claim at a time.

To confirm that $f^{-1}(S) \neq \emptyset$, Definition 1.8 tells us that it will suffice to find an element of $f^{-1}(S)$. By Axiom 3 and Definition 3.3, there exists some point $Y \in C$ such that $Y < X$. Consequently, by the lemma and Definition 3.6, there exists a point $f(p) \in K$[4] such that $Y < f(p) < X$. It follows by the definition of S that $f(p) \in S$. Therefore, by Definition 1.18, $p \in f^{-1}(S)$, as desired.

To confirm that $f^{-1}(S) \neq \mathbb{Q}$, Definition 1.2 tells us that it will suffice to find an element of $\mathbb{Q}$ that is not an element of $f^{-1}(S)$. By Axiom 3 and Definition 3.3, there exists some point $Y \in C$ such that $X < Y$. Consequently, by the lemma and Definition 3.6, there exists a point $f(p) \in K$ such that $X < f(p) < Y$. It follows by the definition of S that $f(p) \notin S$. Therefore, by Definition 6.18, $p \in \mathbb{Q}$ but $p \notin f^{-1}(S)$, as desired.

To confirm that if $r \in f^{-1}(S)$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in f^{-1}(S)$, we let $r \in f^{-1}(S)$ and $s \in \mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s < r$ and seek to verify that $s \in f^{-1}(S)$. By Definition 1.18, the fact that $r \in f^{-1}(S)$ implies that $f(r) \in S$. Thus, by the definition of S, $f(r) < X$. Additionally, by the definition of f and Definition 6.15, $f(s) \in K$ and $f(s) < f(r)$, respectively. Since $f(s) < f(r)$ and $f(r) < X$, transitivity implies that $f(s) < X$. This combined with the previously established fact that $f(s) \in K$ implies that $f(s) \in S$. Therefore, by Definition 1.18, $s \in f^{-1}(S)$, as desired.

To confirm that if $r \in f^{-1}(S)$, then there is some $s \in f^{-1}(S)$ with $s > r$, we let $r \in f^{-1}(S)$ and seek to find such an s. As before, $r \in f^{-1}(S)$ implies that $f(r) \in S$. Thus, by the definition of S, $f(r) < X$. It follows by the lemma and Definition 3.6 that there exists a point $f(s) \in K$ such that $f(r) < f(s) < X$. Consequently, by the definition of S, we have that $f(s) \in S$. Therefore, by Definitions 1.18 and 6.15, $s \in f^{-1}(S)$ and $r < s$, respectively, as desired.

Since f is bijective, Script 1 asserts that $f(f^{-1}(S)) = S$. Thus, $\sup f(f^{-1}(S)) = \sup S$. To verify that $\sup S = X$, Definition 5.7 tells us that it will suffice to confirm that X is an upper bound of S and if U is an upper bound of S, $X \leq U$. To confirm the former statement, Definition 5.6 tells us that it will suffice to show that $k \leq X$ for all $k \in S$. But by the definition of S, this is true. To confirm the latter statement, suppose for the sake of contradiction that there exists an upper bound U of S such that $U < X$. Since $U < X$, the lemma and Definition 3.6 imply that there exists a point $Z \in K$ such that $U < Z < X$. It follows by the definition of S that $Z \in S$. Since there exists an element of S greater than U, Definition 5.6 asserts that U is not an upper bound of S, a contradiction. ☐

6.2 Discussion

1/12:

- Upper limit at signing up for 4-5 across the script.
- Lemma 6.2 is probably more straightforward using a contradiction argument.
- Briefly restate the algebra of Exercise 4.24 in Exercise 6.3c.

1/14:

- Turning in Script 5 journals is optional — it will boost your grade a bit if you do.
  - Your journal grade will be whichever is higher: the average of all your journal grades with and without Script 5.

[4]Note that we know that the element of K (the existence of which is implied by the lemma) can be written in the form $f(p)$ because f is bijective.

  - Script 5 will probably be due Wednesday, 1/20.
- In Lemma 6.6, do we need to prove that the union of arbitrarily many Dedekind cuts is, itself, a Dedekind cut? Yes.

1/18:

- Is there a way to prove something else besides A is not open in Exercise 6.7?
  - This is probably it as far as proving that continuua are connected.
  - It may not be possible to prove that *any* of the statements are wrong, but he's not sure.
- Is Lemma 6.9 used in the proofs of any subsequent results, or is it just a less important result (hence the lemma designation)?
  - We can think of it as an alternate definition for density — we could prove Definition 6.8 from it.
- Is my handwavey use of Scripts 2 and 3 ok in Corollary 6.12?
  - I'm fine.
- Is there a simpler way to prove Corollaries 6.12 and 6.14?
  - hi
- Is the math REU still running this summer?
  - He's not sure; UChicago's may not be NSF approved, hence why its not on the website rn.
- What other summer opportunities would you recommend for a student at my level?
  - He did an REU at UWisconsin when he was an undergrad.
  - Sounds like its pretty much just REUs for undergrads.
  - I could ask around to see if anyone is a Knot Theorist/willing to sponsor me.

1/19:

- Easier Corollary 6.12:
  - Let $B > A$. Then $A < i(\frac{m}{q}) < B$. Then $A < i(m)$.
- Several proofs were given for Corollary 6.14. One other correct one constructed the nonempty, bounded above set of all $i(n)$ less than or equal to A and considered its supremum.

1/21:

- Now graded a bit more critically on presentations.
  - Write big, talk loudly, don't talk to the blackboard.
- My original proof of Corollary 6.14 is incorrect because I can't split into cases the way I did (*longer expo*).
  - Instead, use Seb's approach.

1/26:

- Stray thoughts on Exercise 6.16:
  - Any property we can prove for $\mathbb{Q}$ (e.g., betweenness, Axioms 1-3, etc.) we should be able to prove for K.
    * Many of these follow from $\mathbb{Q}$'s density! This is how we can make use of this condition.
  - We think of 0 as being somehow the "midpoint" of $\mathbb{Q}$. But since $\mathbb{Q}$ diverges in both directions, it doesn't really have a midpoint; we just assert this rather arbitrary structure on a more foundational algebraic construct.
    * The same would hold for K. Thus, we can choose an arbitrary point $x \in K$ and let it be the "midpoint," i.e., let $f(0) = x$.
  - Can we induct on the elements of $\mathbb{Q}$? Since there exists a bijection $\mathbb{Q} \to \mathbb{N}$.

  - We can construct an order preserving bijection between any finite subsets of $\mathbb{Q}$ and K with equal cardinality.
  - $f : \mathbb{Q} \to K$, $g : \mathbb{N} \to \mathbb{Q}$, $h : \mathbb{N} \to K$. If $g(n) < g(n')$, then $h(n) < h(n')$.
  - Let $h(n) < h(n')$. WLOG let $n < n'$, too. Now consider $N = \{n \in \mathbb{N} \mid n \leq n'\}$. This is a finite set. Now create a new set $g(N)$. There will be an order-preserving bijection $\tilde{f} : h(N) \to g(N)$.
  - Let $g : \mathbb{N} \to \mathbb{Q}$ be a bijection (we know one exists by countability). We presently seek to define $h : \mathbb{N} \to K$ recursively. Let x_1 be an arbitrary element of K (Axiom 1). We define $h(1) = x_1$. Now suppose inductively that we have defined $h(n)$. We now seek to define $h(n+1)$. Consider the set $A = \{g(m) \mid m \leq n+1\}$. By Theorem 3.5, we can assign the symbols $a_1, \dots, a_{n+1}$ to each point of A so that $a_1 < a_2 < \cdots < a_{n+1}$. We know that $g(n+1) = a_i$ for some $i \in [n+1]$. We divide into three cases ($g(n+1) = b_1$, $g(n+1) = b_{n+1}$, and $g(n+1) = b_i$ where $1 < i < n+1$). First, suppose that $g(n+1) = b_1$. By the inductive hypothesis, $h(g^{-1}(b_2)) \in K$. By Axiom 3, $h(g^{-1}(b_2))$ is not the first point of K. Thus, there exists an $x \in K$ such that $x < h(g^{-1}(b_2))$. Consequently, let $h(n+1) = x$. The proof of the second case is symmetric to that of the first. Third, suppose that $g(n+1) = b_i$ where $1 < i < n+1$. By the inductive hypothesis, $h(g^{-1}(b_{i-1})), h(g^{-1}(b_{i+1})) \in K$. Thus, there exists an $x \in K$ such that $h(b_{i-1}) < x < h(b_{i+1})$. Consequently, let $h(n+1) = x$.
  - We define $f : \mathbb{Q} \to K$ by $f(p) = h(g^{-1}(p))$.
  - Function diagram: The characteristic of an order preserving bijection is no intersections between lines connecting elements of different sets.

- Do we need to have subscripts on our orderings? Yes.

- The canonical way of doing Exercise 6.16 is with the **back and forth method**.
  - Because both are countable, $\mathbb{Q} = \{q_1, q_2, \dots\}$. Likewise, $K = \{k_1, k_2, \dots\}$.
  - To create the bijection, we have two repeating steps.
    1. Let i be the smallest index such that q_i has not been paired. Let j be an index such that k_j hasn't been paired, and assigning $f(q_i) = k_j$ preserves ordering (we have to prove that such a j exists). To prove this, we know that we can order the elements of $\mathbb{Q}$ that have already been paired. We can also order the elements of K that have already been paired. Case 1: q_i is between some preexisting q's. Then there exists some k_j between. Case 2: $q_i < \cdots < q_n$ implies there exists some k_j less than all other k so far. Case 3: q_i is a last element; symmetric to Case 2.
    2. Smallest j, smallest i such that order is preserved. Then we let $f(q_i) = k_j$.
    3. Repeat.
  - Injectivity: Suppose $f(q_i) = f(q_j)$. Each q_k is assigned to a unique k_k, so if they're equal, they must have been assigned at the same time. Therefore, $q_i = q_j$.
  - Surjectivity: Let $k_j \in K$. By jth step at most, k_j will be paired.

- Do summer research things every happen with graduate students, or is it just with professors? It pretty much only happens with professors, but DRP could be a good way to get your foot in the door.

Script 7

The Field Axioms

7.1 Journal

1/28: **Definition 7.1.** A **binary operation** on a set X is a function

$$f : X \times X \to X$$

We say that f is **associative** if

$$f(f(x,y),z) = f(x,f(y,z)) \quad \text{for all } x,y,z \in X$$

We say that f is **commutative** if

$$f(x,y) = f(y,x) \quad \text{for all } x,y \in X$$

An **identity element** of a binary operation f is an element $e \in X$ such that

$$f(x,e) = f(e,x) = x \quad \text{for all } x \in X$$

Remark 7.2. Frequently, we denote a binary operation differently. If $* : X \times X \to X$ is the binary operation, we often write $a * b$ in place of $*(a,b)$. We sometimes indicate this same operation by writing $(a,b) \mapsto a * b$.

Exercise 7.3. Rewrite Definition 7.1 using the notation of Remark 7.2.

Answer. A **binary operation** on a set X is a function

$$* : X \times X \to X$$

We say that $*$ is **associative** if

$$(x * y) * z = x * (y * z) \quad \text{for all } x,y,z \in X$$

We say that $*$ is **commutative** if

$$x * y = y * x \quad \text{for all } x,y \in X$$

An **identity element** of a binary operation $*$ is an element $e \in X$ such that

$$x * e = e * x = x \quad \text{for all } x \in X$$

□

Examples 7.4.

1. *The function* $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ *which sends a pair of integers* (m,n) *to* $+(m,n) = m + n$ *is a binary operation on the integers, called addition. Addition is associative, commutative, and has identity element 0.*

2. *The maximum of m and n, denoted* $\max(m,n)$*, is an associative and commutative binary operation on* $\mathbb{Z}$*. Is there an identity element for* $\max$*?*

   Proof. Suppose for the sake of contradiction that there exists an identity element e for max. But $\max(e-1,e) = e \neq e-1$, a contradiction. Therefore, no identity element exists for max. $\square$

3. *Let $\wp(Y)$ be the power set of a set Y. Recall that the power set consists of all subsets of Y. Then the intersection of sets, $(A,B) \mapsto A \cap B$, defines an associative and commutative binary operation on $\wp(Y)$. Is there an identity element for $\cap$?*

   Proof. Clearly, $Y \in \wp(Y)$. By Script 1, $Y \cap A = A \cap Y = A$ where $A \subset Y$. Therefore, Y is an identity element for $\cap$. $\square$

Exercise 7.5. Find a binary operation on a set that is not commutative. Find a binary operation on a set that is not associative.

Proof. We will prove that the subtraction operation on the integers ($- : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$) is neither commutative nor associative. To prove that it's not commutative, Definition 7.1 tells us that it will suffice to show that $x - y \neq y - x$ for some $x, y \in \mathbb{Z}$. Since $2 - 1 = 1$ but $1 - 2 = -1$, we can see that $1, 2 \in \mathbb{Z}$ clearly meet this requirement. To prove that it's not associative, Definition 7.1 tells us that it will suffice to show that $(x-y)-z \neq x-(y-z)$ for some $x,y,z \in \mathbb{Z}$. Since $(3-2)-1 = 0$ but $3-(2-1) = 2$, we can see that $1, 2, 3 \in \mathbb{Z}$ clearly meet this requirement. $\square$

Exercise 7.6. Let X be a finite set, and let $Y = \{f : X \to X \mid f \text{ is bijective}\}$. Consider the binary operation of composition of functions, denoted $\circ : Y \times Y \to Y$ and defined by $(f \circ g)(x) = f(g(x))$ as seen in Definition 1.25. Decide whether or not composition is commutative and/or associative and whether or not it has an identity.

Proof. To prove that composition is not commutative, Definition 7.1 tells us that it will suffice to find a finite set X paired with two bijections in Y that do not commute. Let $X = \{1, 2, 3\}$ and consider the bijections $f : X \to X$ (defined by $f(1) = 2$, $f(2) = 3$, $f(3) = 1$) and $g : X \to X$ (defined by $g(1) = 1$, $g(2) = 3$, $g(3) = 2$). In this case, $f \circ g$ would be defined by $f(g(1)) = 2$, $f(g(2)) = 1$, and $f(g(3)) = 3$, but $g \circ f$ would be defined by $g(f(1)) = 3$, $g(f(2)) = 2$, and $g(f(3)) = 1$.

To prove that composition is associative, Definition 7.1 tells us that it will suffice to show that $((f \circ g) \circ h)(x) = (f \circ (g \circ h))(x)$. We may do this with the following algebra.

$$\begin{aligned}((f \circ g) \circ h)(x) &= (f \circ g)(h(x)) \\ &= f(g(h(x))) \\ &= f((g \circ h)(x)) \\ &= (f \circ (g \circ h))(x)\end{aligned}$$

With respect to any finite set X, there will always be a bijection $i : X \to X$ defined by $i(x) = x$. To prove that i is an identity element, Definition 7.1 tells us that it will suffice to show that for all $f \in Y$, $f \circ i = i \circ f = f$. We may do this with the following algebra.

$$\begin{aligned}(f \circ i)(x) &= f(i(x)) \\ &= f(x) \\ &= i(f(x)) \\ &= (i \circ f)(x)\end{aligned}$$

$\square$

Theorem 7.7. *Identity elements are unique. That is, suppose that f is a binary operation on a set X that has two identity elements e and e'. Then $e = e'$.*

Proof. Let $f : X \times X \to X$ be a binary operation on a set X with two identity elements e, e'. By Definition 7.1, we know that $f(e, e') = e$ and $f(e, e') = e'$. Since f is a well-defined function by definition, it must be that $e = f(e, e') = e'$. $\square$

Definition 7.8. A **field** is a set F with two binary operations on F called addition, denoted $+$, and multiplication, denoted $\cdot$, satisfying the following **field axioms**:

FA1 (Commutativity of Addition) For all $x, y \in F$, $x + y = y + x$.

FA2 (Associativity of Addition) For all $x, y, z \in F$, $(x + y) + z = x + (y + z)$.

FA3 (Additive Identity) There exists an element $0 \in F$ such that $x + 0 = 0 + x = x$ for all $x \in F$.

FA4 (Additive Inverses) For any $x \in F$, there exists $y \in F$ such that $x + y = y + x = 0$, called an additive inverse of x.

FA5 (Commutativity of Multiplication) For all $x, y \in F$, $x \cdot y = y \cdot x$.

FA6 (Associativity of Multiplication) For all $x, y, z \in F$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

FA7 (Multiplicative Identity) There exists an element $1 \in F$ such that $x \cdot 1 = 1 \cdot x = x$ for all $x \in F$.

FA8 (Multiplicative Inverses) For any $x \in F$ such that $x \neq 0$, there exists $y \in F$ such that $x \cdot y = y \cdot x = 1$, called a multiplicative inverse of x.

FA9 (Distributivity of Multiplication over Addition) For all $x, y, z \in F$, $x \cdot (y + z) = x \cdot y + x \cdot z$.

FA10 (Distinct Additive and Multiplicative Identities) $1 \neq 0$.

Exercise 7.9. Consider the set $\mathbb{F}_2 = \{0, 1\}$, and define binary operations $+$ and $\cdot$ on $\mathbb{F}_2$ by

$$\begin{array}{cccc} 0+0=0 & 0+1=1 & 1+0=1 & 1+1=0 \\ 0\cdot 0=0 & 0\cdot 1=0 & 1\cdot 0=0 & 1\cdot 1=1 \end{array}$$

Show that $\mathbb{F}_2$ is a field.

Proof. To prove that $\mathbb{F}_2$ obeys FA1 from Definition 7.8, it will suffice to show that $0+0 = 0+0$, $0+1 = 1+0$, and $1+1 = 1+1$. The first and third of these are evidently true. For the second, we have $0+1 = 1 = 1+0$, so it is good, too.

To prove that $\mathbb{F}_2$ obeys FA2 from Definition 7.8, the following casework will suffice.

$$\begin{array}{ll} (0+0)+0 = 0 = 0+(0+0) & (0+0)+1 = 1 = 0+(0+1) \\ (0+1)+0 = 1 = 0+(1+0) & (1+0)+0 = 1 = 1+(0+0) \\ (0+1)+1 = 0 = 0+(1+1) & (1+1)+0 = 0 = 1+(1+0) \\ (1+0)+1 = 0 = 1+(0+1) & (1+1)+1 = 1 = 1+(1+1) \end{array}$$

To prove that $\mathbb{F}_2$ obeys FA3 from Definition 7.8, it will suffice to find an element $0 \in \mathbb{F}_2$ such that $x + 0 = 0 + x = x$. Since $0 + 0 = 0$, $1 + 0 = 0$, and by commutativity, it is clear that 0 is an additive identity in $\mathbb{F}_2$.

To prove that $\mathbb{F}_2$ obeys FA4 from Definition 7.8, it will suffice to show that for all $x \in \mathbb{F}_2$, there exists a $y \in \mathbb{F}_2$ such that $x + y = y + x = 0$. For 0, this object is 0 (since $0 + 0 = 0 + 0 = 0$), and for 1, this object is 1 (since $1 + 1 = 1 + 1 = 0$).

To prove that $\mathbb{F}_2$ obeys FA5 from Definition 7.8, it will suffice to show that $0 \cdot 0 = 0 \cdot 0$, $0 \cdot 1 = 1 \cdot 0$, and $1 \cdot 1 = 1 \cdot 1$. The first and third of these are evidently true. For the second, we have $0 \cdot 1 = 0 = 1 \cdot 0$, so it is good, too.

To prove that $\mathbb{F}_2$ obeys FA6 from Definition 7.8, the following casework will suffice.

$$\begin{aligned}
(0\cdot 0)\cdot 0 &= 0 = 0\cdot(0\cdot 0) & (0\cdot 0)\cdot 1 &= 0 = 0\cdot(0\cdot 1)\\
(0\cdot 1)\cdot 0 &= 0 = 0\cdot(1\cdot 0) & (1\cdot 0)\cdot 0 &= 0 = 1\cdot(0\cdot 0)\\
(0\cdot 1)\cdot 1 &= 0 = 0\cdot(1\cdot 1) & (1\cdot 1)\cdot 0 &= 0 = 1\cdot(1\cdot 0)\\
(1\cdot 0)\cdot 1 &= 0 = 1\cdot(0\cdot 1) & (1\cdot 1)\cdot 1 &= 1 = 1\cdot(1\cdot 1)
\end{aligned}$$

To prove that $\mathbb{F}_2$ obeys FA7 from Definition 7.8, it will suffice to find an element $1 \in \mathbb{F}_2$ such that $x\cdot 1 = 1\cdot x = x$. Since $0\cdot 1 = 0$, $1\cdot 1 = 1$, and by commutativity, it is clear that 1 is a multiplicative identity in $\mathbb{F}_2$.

To prove that $\mathbb{F}_2$ obeys FA8 from Definition 7.8, it will suffice to show that for all $x \in \mathbb{F}_2$ such that $x \neq 0$, there exists a $y \in \mathbb{F}_2$ such that $x\cdot y = y\cdot x = 1$. For 1, this object is 1 (since $1\cdot 1 = 1\cdot 1 = 1$).

To prove that $\mathbb{F}_2$ obeys FA9 from Definition 7.8, the following casework will suffice.

$$\begin{aligned}
0\cdot(0+0) &= 0 = 0\cdot 0 + 0\cdot 0 & 0\cdot(0+1) &= 0 = 0\cdot 0 + 0\cdot 1\\
0\cdot(1+0) &= 0 = 0\cdot 1 + 0\cdot 0 & 1\cdot(0+0) &= 0 = 1\cdot 0 + 1\cdot 0\\
0\cdot(1+1) &= 0 = 0\cdot 1 + 0\cdot 1 & 1\cdot(1+0) &= 1 = 1\cdot 1 + 1\cdot 0\\
1\cdot(0+1) &= 1 = 1\cdot 0 + 1\cdot 1 & 1\cdot(1+1) &= 0 = 1\cdot 1 + 1\cdot 1
\end{aligned}$$

To prove that $\mathbb{F}_2$ obeys FA10 from Definition 7.8, it will suffice to show that $0 \neq 1$. Clearly this is true. $\square$

Theorem 7.10. *Suppose that F is a field. Then additive inverses are unique. This means: Let $x \in F$. If $y, y' \in F$ satisfy $x + y = 0$ and $x + y' = 0$, then $y = y'$.*

Proof. Let $x, y, y' \in F$ be such that $x + y = 0$ and $x + y' = 0$. From Definition 7.8, we have

$$\begin{aligned}
y' + (x + y) &= (y' + x) + y && \text{FA2}\\
y' + 0 &= 0 + y && \text{FA4}\\
y' &= y && \text{FA3}
\end{aligned}$$

$\square$

We usually write $-x$ for the additive inverse of x.

Corollary 7.11. *If $x \in F$, then $-(-x) = x$.*

Proof. Let $x \in F$. Then by consecutive applications of FA4 from Definition 7.8, $-x + (-(-x)) = 0$ and $-x + x = 0$. Therefore, by Theorem 7.10, we have that $-(-x) = x$. $\square$

Theorem 7.12. *Let F be a field, and let $a, b, c \in F$. If $a + b = a + c$, then $b = c$.*

Proof. Let $a, b, c \in F$ be such that $a + b = a + c$. By FA4 from Definition 7.8, there exists $-a \in F$ such that $-a + a = a + (-a) = 0$. Having established that $-a$ exists, we can prove from Definition 7.8 that

$$\begin{aligned}
-a + (a + b) &= -a + (a + c) &&\\
(-a + a) + b &= (-a + a) + c && \text{FA2}\\
0 + b &= 0 + c && \text{FA4}\\
b &= c && \text{FA3}
\end{aligned}$$

$\square$

Theorem 7.13. *Let F be a field. If $a \in F$, then $a \cdot 0 = 0$.*

Proof. Let $a \in F$. From Definition 7.8, we have

$$\begin{aligned} a &= a \cdot 1 && \text{FA7} \\ &= a \cdot (1 + 0) && \text{FA3} \\ &= a \cdot 1 + a \cdot 0 && \text{FA9} \\ &= a + a \cdot 0 && \text{FA7} \\ 0 &= a \cdot 0 && \text{Theorem 7.12} \end{aligned}$$

□

2/2: **Theorem 7.14.** *Suppose that F is a field. Then multiplicative inverses are unique. This means: Let $x \in F$. If $y, y' \in F$ satisfy $x \cdot y = 1$ and $x \cdot y' = 1$, then $y = y'$.*

Proof. Let $x, y, y' \in F$ be such that $x \cdot y = 1$ and $x \cdot y' = 1$. From Definition 7.8, we have

$$\begin{aligned} (y \cdot x) \cdot y' &= y \cdot (x \cdot y') && \text{FA6} \\ 1 \cdot y' &= y \cdot 1 && \text{FA8} \\ y' &= y && \text{FA7} \end{aligned}$$

□

We usually write x^{-1} or $\frac{1}{x}$ for the multiplicative inverse of x.

Corollary 7.15. *If $x \in F$ and $x \neq 0$, then $(x^{-1})^{-1} = x$.*

Proof. Let $x \in F \setminus \{0\}$. Then by FA8 from Definition 7.8, there exists $x^{-1} \in F$ such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$. It follows from Theorem 7.13 that $x^{-1} \neq 0$ (if $x^{-1} = 0$, then Theorem 7.13 would imply that $1 = x \cdot x^{-1} = 0$, contradicting FA10). Thus, by FA8 from Definition 7.8 again, there exists $(x^{-1})^{-1} \in F$ such that $x^{-1} \cdot (x^{-1})^{-1} = (x^{-1})^{-1} \cdot x^{-1} = 1$. Having established that $(x^{-1})^{-1}$ exists, $x^{-1} \cdot (x^{-1})^{-1} = 1$, and $x^{-1} \cdot x = 1$, we have by Theorem 7.14 that $(x^{-1})^{-1} = x$. □

Theorem 7.16. *Let F be a field, and let $a, b, c \in F$. If $a \cdot b = a \cdot c$ and $a \neq 0$, then $b = c$.*

Proof. Let $a, b, c \in F$ be such that $a \cdot b = a \cdot c$ and $a \neq 0$. By FA8 from Definition 7.8, there exists $a^{-1} \in F$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$. Having established that a^{-1} exists, we can prove from Definition 7.8 that

$$\begin{aligned} a^{-1} \cdot (a \cdot b) &= a^{-1} \cdot (a \cdot c) && \\ (a^{-1} \cdot a) \cdot b &= (a^{-1} \cdot a) \cdot c && \text{FA6} \\ 1 \cdot b &= 1 \cdot c && \text{FA8} \\ b &= c && \text{FA7} \end{aligned}$$

□

Theorem 7.17. *Let F be a field, and let $a, b \in F$. If $a \cdot b = 0$, then $a = 0$ or $b = 0$.*

Proof. Let $a, b \in F$ be such that $a \cdot b = 0$, and suppose for the sake of contradiction that $a \neq 0$ and $b \neq 0$. It follows from the supposition by consecutive applications of FA8 from Definition 7.8 that a^{-1} and b^{-1} exist. Thus, from Definition 7.8, we have

$$\begin{aligned} 1 &= 1 \cdot 1 && \text{FA7} \\ &= (a \cdot a^{-1}) \cdot (b \cdot b^{-1}) && \text{FA8} \\ &= (a \cdot b) \cdot (a^{-1} \cdot b^{-1}) && \text{FA6 and FA7} \\ &= 0 \cdot (a^{-1} \cdot b^{-1}) && \text{Substitution} \\ &= 0 && \text{Theorem 7.13} \end{aligned}$$

But this contradicts FA10 from Definition 7.8. □

Lemma 7.18. *Let F be a field. If $a \in F$, then $-a = (-1)a$.*

Proof. Let $a \in F$. From Definition 7.8, we have

$$
\begin{aligned}
0 &= a \cdot 0 && \text{Theorem 7.13}\\
a + (-a) &= a \cdot (1 + (-1)) && \text{FA4}\\
a + (-a) &= a \cdot 1 + a \cdot (-1) && \text{FA9}\\
a + (-a) &= a + a \cdot (-1) && \text{FA7}\\
a + (-a) &= a + (-1)a && \text{FA5}\\
-a &= (-1)a && \text{Theorem 7.12}
\end{aligned}
$$

□

Lemma 7.19. *Let F be a field. If $a, b \in F$, then $a \cdot (-b) = -(a \cdot b) = (-a) \cdot b$.*

Proof. Let $a, b \in F$. From Definition 7.8, we have

$$
\begin{aligned}
a \cdot (-b) &= a \cdot ((-1) \cdot b) && \text{Lemma 7.18}\\
&= a \cdot (b \cdot (-1)) && \text{FA5}\\
&= (a \cdot b) \cdot (-1) && \text{FA6}\\
&= (-1) \cdot (a \cdot b) && \text{FA5}\\
&\boxed{= -(a \cdot b)} && \text{Lemma 7.18}\\
&= (-1) \cdot (a \cdot b) && \text{Lemma 7.18}\\
&= ((-1) \cdot a) \cdot b && \text{FA6}\\
&\boxed{= (-a) \cdot b} && \text{Lemma 7.18}
\end{aligned}
$$

□

Lemma 7.20. *Let F be a field. If $a, b \in F$, then $a \cdot b = (-a) \cdot (-b)$.*

Proof. Let $a, b \in F$. Thus, we have

$$
\begin{aligned}
(-a) \cdot (-b) &= -(-a) \cdot b && \text{Lemma 7.19}\\
&= a \cdot b && \text{Corollary 7.11}
\end{aligned}
$$

□

Definition 7.21. An **ordered field** is a field F equipped with an ordering $<$ (satisfying Definition 3.1) such that also:

(a) Addition respects the ordering: if $x < y$, then $x + z < y + z$ for all $z \in F$.

(b) Multiplication respects the ordering: if $0 < x$ and $0 < y$, then $0 < x \cdot y$.

Definition 7.22. Suppose F is an ordered field and $x \in F$. If $0 < x$, we say that x is **positive**. If $x < 0$, we say that x is **negative**.

Lemma 7.23. *Let F be an ordered field, and let $x \in F$. If $0 < x$, then $-x < 0$. Similarly, if $x < 0$, then $0 < -x$.*

Proof. Let $x \in F$ be such that $0 < x$. Then by Definition 7.21a, $0 + (-x) < x + (-x)$. Consequently, from Definition 7.8, we have

$$
\begin{aligned}
-x &< x + (-x) && \text{FA3}\\
-x &< 0 && \text{FA4}
\end{aligned}
$$

The proof is symmetric if $x < 0$. □

Lemma 7.24. *Let F be an ordered field, and let $x, y, z \in F$.*

(*a*) *If $x > 0$ and $y < z$, then $x \cdot y < x \cdot z$.*

(*b*) *If $x < 0$ and $y < z$, then $x \cdot z < x \cdot y$.*

Proof of a. Let $x, y, z \in F$ be such that $x > 0$ and $y < z$. It follows from the latter condition by Definition 7.21a that $y + (-y) < z + (-y)$. Thus, by FA4 from Definition 7.8, we have $0 < z + (-y)$. This combined with the fact that $0 < x$ implies by Definition 7.21b that $0 < x \cdot (z + (-y))$. Consequently, from Definition 7.8, we have

$$
\begin{aligned}
0 &< x \cdot z + x \cdot (-y) && \text{FA9} \\
0 &< x \cdot z + (-(x \cdot y)) && \text{Lemma 7.19} \\
0 + x \cdot y &< (x \cdot z + (-(x \cdot y))) + x \cdot y && \text{Definition 7.21a} \\
0 + x \cdot y &< x \cdot z + (-(x \cdot y) + x \cdot y) && \text{FA2} \\
0 + x \cdot y &< x \cdot z + 0 && \text{FA4} \\
x \cdot y &< x \cdot z && \text{FA3}
\end{aligned}
$$

□

Proof of b. Let $x, y, z \in F$ be such that $x < 0$ and $y < z$. It follows from the former condition by Lemma 7.23 that $0 < -x$. Thus, by Lemma 7.24a, $(-x) \cdot y < (-x) \cdot z$. Consequently, from Definition 7.8, we have

$$
\begin{aligned}
-(x \cdot y) &< -(x \cdot z) && \text{Lemma 7.19} \\
-(x \cdot y) + (x \cdot y + x \cdot z) &< -(x \cdot z) + (x \cdot y + x \cdot z) && \text{Definition 7.21a} \\
-(x \cdot y) + (x \cdot y + x \cdot z) &< -(x \cdot z) + (x \cdot z + x \cdot y) && \text{FA1} \\
(-(x \cdot y) + x \cdot y) + x \cdot z &< (-(x \cdot z) + x \cdot z) + x \cdot y && \text{FA2} \\
0 + x \cdot z &< 0 + x \cdot y && \text{FA4} \\
x \cdot z &< x \cdot y && \text{FA3}
\end{aligned}
$$

□

Remark 7.25. An immediate consequence of this lemma is the fact that if x and y are both positive or both negative, their product is positive.

Lemma 7.26. *Let F be an ordered field, and let $x \in F$. Then $0 \leq x^2$. Moreover, if $x \neq 0$, then $0 < x^2$.*

Proof. We divide into two cases ($x = 0$ and $x \neq 0$). Suppose first that $x = 0$. Then by Theorem 7.13, $0 \leq 0 = 0 \cdot 0 = 0^2 = x^2$, as desired. Now suppose that $x \neq 0$. We divide into two cases again ($x > 0$ and $x < 0$). If $x > 0$, then by Lemma 7.24a, $x > 0$ and $0 < x$ imply that $x \cdot 0 < x \cdot x$, from which it follows by Theorem 7.13 that $0 < x^2$, as desired. On the other hand, if $x < 0$, then by Lemma 7.24b, $x < 0$ and $x < 0$ imply that $x \cdot 0 < x \cdot x$, from which it follows for the same reason as before that $0 < x^2$, as desired. Both of the original two cases together prove the first statement, while the second original case alone proves the second statement. □

Corollary 7.27. *Let F be an ordered field. Then $0 < 1$.*

Proof. By FA10 from Definition 7.8, $1 \neq 0$. Thus, by Lemma 7.26, $0 < 1^2 = 1$, as desired. □

Theorem 7.28. *If F is an ordered field, then F has no first or last point.*

Proof. Suppose for the sake of contradiction that F has a first point a. By Corollary 7.27, we have that $0 < 1$, which implies by Lemma 7.23 that $-1 < 0$. It follows by Definition 7.21a that $-1 + a < 0 + a$. Thus, by FA3 from Definition 7.8, $-1 + a < a$. Since there exists an object in F (namely $-1 + a$) that is less than a, Definition 3.3 tells us that a is not the first point of F, a contradiction.

The proof is symmetric in the other case. □

Theorem 7.29. *The rational numbers $\mathbb{Q}$ form an ordered field.*

Proof. To prove that $\mathbb{Q}$ forms an ordered field, Definition 7.21 tells us that it will suffice to show that $\mathbb{Q}$ forms a field; has an ordering $<$; satisfies $x+z<y+z$ if $x<y$ for all $z\in\mathbb{Q}$; and satisfies $0<x\cdot y$ if $0<x$ and $0<y$. We will take this one constraint at a time.

To show that $\mathbb{Q}$ forms a field, Definition 7.8 tells us that it will suffice to verify that $\mathbb{Q}$ has two binary operations ($+$ and $\cdot$), and satisfies field axioms 1-10. Define $+$ and $\cdot$ as in Definition 2.7. Under these definitions, parts a-i of Theorem 2.10 guarantee that $\mathbb{Q}$ satisfies FA1-FA9, respectively. As to FA10, to verify that $\left[\frac{1}{1}\right]\neq\left[\frac{0}{1}\right]$, Exercise 2.6 tells us that it will suffice to confirm that $(1,1)\nsim(1,0)$. But since $1\cdot 0=0\neq 1=1\cdot 1$, Exercise 2.2e confirms that $(1,1)\nsim(1,0)$, as desired.

$\mathbb{Q}$ has an ordering by Exercise 3.9d, as desired.

To show that $x+z<y+z$ if $x<y$ for all $z\in\mathbb{Q}$, let $\left[\frac{a}{b}\right],\left[\frac{c}{d}\right],\left[\frac{x}{z}\right]$ be arbitrary elements of $\mathbb{Q}$ with positive denominators (we can choose these WLOG by Exercise 3.9b) and the first two satisfying $\left[\frac{a}{b}\right]<\left[\frac{c}{d}\right]$; we seek to verify that $\left[\frac{a}{b}\right]+\left[\frac{x}{z}\right]<\left[\frac{c}{d}\right]+\left[\frac{x}{z}\right]$. Since $\left[\frac{a}{b}\right]<\left[\frac{c}{d}\right]$, we have by Exercise 3.9c that $ad<bc$. It follows by Script 0 that

$$\begin{gathered}
ad<bc\\
adzz<bczz\\
adzz+bdxz<bczz+bdxz\\
azdz+bxdz<bzcz+bzdx\\
(az+bx)(dz)<(bz)(cz+dx)
\end{gathered}$$

Thus, by Exercise 3.9c, $\left[\frac{az+bx}{bz}\right]<\left[\frac{cz+dx}{dz}\right]$. Therefore, by Definition 2.7, $\left[\frac{a}{b}\right]+\left[\frac{x}{z}\right]<\left[\frac{c}{d}\right]+\left[\frac{x}{z}\right]$, as desired.

To show that $0<x\cdot y$ if $0<x$ and $0<y$, let $\left[\frac{a}{b}\right],\left[\frac{c}{d}\right]$ be arbitrary elements of $\mathbb{Q}$ with positive denominators (which we can choose for the same reason as before) and such that $\left[\frac{0}{1}\right]<\left[\frac{a}{b}\right]$ and $\left[\frac{0}{1}\right]<\left[\frac{c}{d}\right]$; we seek to verify that $\left[\frac{0}{1}\right]<\left[\frac{a}{b}\right]\cdot\left[\frac{c}{d}\right]$. Since $\left[\frac{0}{1}\right]<\left[\frac{a}{b}\right]$ and $\left[\frac{0}{1}\right]<\left[\frac{c}{d}\right]$, we have by Exercise 3.9c that $0\cdot b<1\cdot a$ and $0\cdot d<1\cdot c$. It follows by Script 0 that $0\cdot bd<1\cdot ac$. Thus, by Exercise 3.9c, $\left[\frac{0}{1}\right]<\left[\frac{ac}{bd}\right]$. Therefore, by Definition 2.7, $\left[\frac{0}{1}\right]<\left[\frac{a}{b}\right]\cdot\left[\frac{c}{d}\right]$, as desired. □

2/4: **Definition 7.31.** We define $\oplus$ on $\mathbb{R}$ as follows. Let $A,B\in\mathbb{R}$ be Dedekind cuts. Define

$$A\oplus B=\{a+b\mid a\in A\text{ and }b\in B\}$$

Exercise 7.32.

(a) Prove that $A\oplus B$ is a Dedekind cut.

(b) Prove that $\oplus$ is commutative and associative.

(c) Prove that if $A\in\mathbb{R}$, then $A=\mathbf{0}\oplus A$.

Proof of a. To prove that $A\oplus B$ is a Dedekind cut, Definition 6.1 tells us that it will suffice to show that $A\oplus B\neq\emptyset$; $A\oplus B\neq\mathbb{Q}$; if $r\in A\oplus B$ and $s\in\mathbb{Q}$ satisfy $s<r$, then $s\in A\oplus B$; and if $r\in A\oplus B$, then there is some $s\in A\oplus B$ with $s>r$. We will take this one claim at a time.

To show that $A\oplus B\neq\emptyset$, Definition 1.8 tells us that it will suffice to find an element of $A\oplus B$. Since A,B are Dedekind cuts, Definition 6.1 asserts that they are nonempty. Thus, there exist rational numbers $x\in A$ and $y\in B$. Therefore, by the definition of $A\oplus B$, the sum $x+y\in A\oplus B$, as desired.

To show that $A\oplus B\neq\mathbb{Q}$, Definition 1.2 tells us that it will suffice to find an element of $\mathbb{Q}$ that is not an element of $A\oplus B$. For an analogous reason to before, we can choose $x,y\in\mathbb{Q}$ such that $x\notin A$ and $y\notin B$. It follows by Lemma 6.2 and Definition 5.6 that $x\geq a$ for all $a\in A$ and $y\geq b$ for all $b\in B$. Additionally, since $x\notin A$, we have that $x\neq a$ for any $a\in A$; thus, $x>a$ for all $a\in A$. Similarly, $y>b$ for all $b\in B$. Consequently, by Script 2, $x+y>a+b$ for all $a+b\in A\oplus B$. Therefore, $x+y\notin A\oplus B$, as desired.

To show that if $r\in A\oplus B$ and $s\in\mathbb{Q}$ satisfy $s<r$, then $s\in A\oplus B$, we let $r\in A\oplus B$ and $s\in\mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s<r$ and seek to verify that $s\in A\oplus B$. Since $r\in A\oplus B$, $r=x+y$ for some $x\in A$ and $y\in B$. Additionally, it follows from the fact that $s<r$ that

$s = r - q = x + y - q$ for some $q \in \mathbb{Q}^+$. Since $y \in B$ and $y - q \in \mathbb{Q}$ satisfy $y - q < y$, we have by Definition 6.1b that $y - q \in B$. Therefore, $s = (x) + (y - q)$ is an element of $A \oplus B$, as desired.

To show that if $r \in A \oplus B$, then there is some $s \in A \oplus B$ with $s > r$, we let $r \in A \oplus B$ and seek to find such an s. Since $r \in A \oplus B$, $r = x + y$ for some $x \in A$ and $y \in B$. It follows from the fact that $x \in A$ by Definition 6.1c that there exists a $z \in A$ with $z > x$. Consequently, by Script 0, $z + y > x + y$ is the desired element of $A \oplus B$. ☐

Proof of b. To prove that $\oplus$ is commutative, Definition 7.1 tells us that it will suffice to show that for all $A, B \in \mathbb{R}$, we have $A \oplus B = B \oplus A$. Let A, B be arbitrary elements of $\mathbb{R}$. Then by Definition 7.31, we clearly have

$$\begin{aligned} A \oplus B &= \{a + b \mid a \in A \text{ and } b \in B\} \\ &= \{b + a \mid b \in B \text{ and } a \in A\} \\ &= B \oplus A \end{aligned}$$

To prove that $\oplus$ is associative, Definition 7.1 tells us that it will suffice to show that for all $A, B, C \in \mathbb{R}$, we have $(A \oplus B) \oplus C = A \oplus (B \oplus C)$. Let A, B, C be arbitrary elements of $\mathbb{R}$. Then by Definition 7.31, we clearly have

$$\begin{aligned} (A \oplus B) \oplus C &= \{a + b \mid a \in A \text{ and } b \in B\} \oplus C \\ &= \{d + c \mid d \in \{a + b \mid a \in A \text{ and } b \in B\} \text{ and } c \in C\} \\ &= \{d + c \mid d = a + b \text{ for some } a \in A \text{ and } b \in B, \text{ and } c \in C\} \\ &= \{a + b + c \mid a \in A \text{ and } b \in B \text{ and } c \in C\} \\ &= \{a + e \mid a \in A, \text{ and } e = b + c \text{ for some } b \in B \text{ and } c \in C\} \\ &= \{a + e \mid c \in C \text{ and } e \in \{b + c \mid b \in B \text{ and } c \in C\}\} \\ &= A \oplus \{b + c \mid b \in B \text{ and } c \in C\} \\ &= A \oplus (B \oplus C) \end{aligned}$$

Note that we also make use of Exercise 7.32a to guarantee $A \oplus B \in \mathbb{R}$, so that we can apply $\oplus$ to $A \oplus B$ and C. We similarly invoke Exercise 7.32a to take the sum of A and $B \oplus C$. ☐

Proof of c. To prove that for all $A \in \mathbb{R}$, $A = \mathbf{0} \oplus A$, we will show for an arbitrary $A \in \mathbb{R}$ that every element of A is an element of $\mathbf{0} \oplus A$ and vice versa. Let A be an arbitrary element of $\mathbb{R}$. Suppose first that $x \in A$. Then by Definition 6.1c, there exists $y \in A$ such that $y > x$. Let $z = x - y$. Clearly, $z \in \mathbb{Q}$ and $z < 0$, so we know that $z \in \mathbf{0}$. Additionally, since $x - z = y$, we know that $x - z \in A$. Therefore, since $x = (z) + (x - z)$, we have by Definition 7.31 that $x \in \mathbf{0} \oplus A$. Now suppose that $z \in \mathbf{0} \oplus A$. Then by Definition 7.31, $z = x + y$ for some $x \in \mathbf{0}$ and $y \in A$. Since $x \in \mathbf{0}$, we know that $x < 0$, which means that $y > z$. This combined with the fact that $y \in A$ and $z \in \mathbb{Q}$ implies by Definition 6.1b that $z \in A$. ☐

2/9: **Definition 7.39.** For $A, B \in \mathbb{R}$, $\mathbf{0} < A$, $\mathbf{0} < B$, we define

$$A \otimes B = \{r \in \mathbb{Q} \mid r \leq 0\} \cup \{ab \mid a \in A, b \in B, a > 0, b > 0\}$$

If $A = \mathbf{0}$ or $B = \mathbf{0}$, we define $A \otimes B = \mathbf{0}$. If $A < \mathbf{0}$ but $\mathbf{0} < B$, we replace A with $-A$ and use the definition of multiplication of positive elements. Hence, in this case,

$$A \otimes B = -[(-A) \otimes B]$$

Similarly, if $\mathbf{0} < A$ but $B < \mathbf{0}$, then

$$A \otimes B = -[A \otimes (-B)]$$

and if $A < \mathbf{0}$, $B < \mathbf{0}$, then

$$A \otimes B = (-A) \otimes (-B)$$

Exercise 7.40. [1]

(a) Show that if $A, B \in \mathbb{R}$, then $A \otimes B \in \mathbb{R}$.

(b) Show that $\otimes$ is commutative and associative.

(c) Show that if $A, B \in \mathbb{R}$, $\mathbf{0} < A$, and $\mathbf{0} < B$, then $\mathbf{0} < A \otimes B$.

(d) Let $\mathbf{1} = \{x \in \mathbb{Q} \mid x < 1\}$. Show that if $A \in \mathbb{R}$, then $\mathbf{1} \otimes A = A$.

Proof of a. To prove that $A \otimes B$ where $\mathbf{0} < A, \mathbf{0} < B$ are Dedekind cuts, Definition 6.1 tells us that it will suffice to show that $A \otimes B \neq \emptyset$; $A \otimes B \neq \mathbb{Q}$; if $r \in A \otimes B$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A \otimes B$; and if $r \in A \otimes B$, then there is some $s \in A \otimes B$ with $s > r$. We will take this one claim at a time.

To show that $A \otimes B \neq \emptyset$, Definition 1.8 tells us that it will suffice to find an element of $A \otimes B$. Since $0 \in \mathbb{Q}$ and $0 \leq 0$, $0 \in \{r \in \mathbb{Q} \mid r \leq 0\}$. It follows by Definition 1.5 that $0 \in \{r \in \mathbb{Q} \mid r \leq 0\} \cup \{ab \mid a \in A, b \in B, a > 0, b > 0\}$. Therefore, by Definition 7.39, $0 \in A \otimes B$, as desired.

To show that $A \otimes B \neq \mathbb{Q}$, Definition 1.2 tells us that it will suffice to find an element of $\mathbb{Q}$ that is not an element of $A \otimes B$. Since $\mathbf{0} < A$ and $\mathbf{0} < B$, Definitions 6.4 and 1.3 assert that there exist points $a \in A$ and $b \in B$ such that $a, b \notin \mathbf{0}$, i.e., $a, b \geq 0$. Furthermore, since a, b are not the last points of A, B, respectively, by Definition 6.1c, there exist points $c \in A$ and $d \in B$ such that $c > 0$ and $d > 0$. Now, for an analogous reason to before, we can choose $x, y \in \mathbb{Q}$ such that $x \notin A$ and $y \notin B$. It follows by Lemma 6.2 and Definition 5.6 that $x \geq e$ for all $e \in A$ and $y \geq f$ for all $f \in B$, meaning (when combined with the last result) that $x > 0$ and $y > 0$. Thus, $xy > 0$, so $xy \notin \{r \in \mathbb{Q} \mid r \leq 0\}$. Additionally, we have by Script 2 that $xy > ef$ for all ef formed from the product of positive elements of A and B. Thus, $xy \notin \{ab \mid a \in A, b \in B, a > 0, b > 0\}$. Therefore, by Definitions 1.5 and 7.39, $xy \notin A \otimes B$, as desired.

To show that if $r \in A \otimes B$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A \otimes B$, we let $r \in A \otimes B$ and $s \in \mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s < r$ and seek to verify that $s \in A \otimes B$. We divide into two cases ($s \leq 0$ and $s > 0$). Suppose first that $s \leq 0$. Then $s \in \{r \in \mathbb{Q} \mid r \leq 0\}$. It follows by Definition 1.5 that $0 \in \{r \in \mathbb{Q} \mid r \leq 0\} \cup \{ab \mid a \in A, b \in B, a > 0, b > 0\}$. Therefore, by Definition 7.39, $0 \in A \otimes B$. Now suppose that $s > 0$. Then $r > 0$. Since $r \in A \otimes B$ and $r > 0$, $r = xy$ where $x \in A, y \in B, x > 0, y > 0$. Additionally, it follows from the fact that $s < r$ that $s = r - q = xy - q = (x - \frac{q}{y})y$ for some $q \in \mathbb{Q}^+$. Since $x - \frac{q}{y} \in \mathbb{Q}$ and $x - \frac{q}{y} < x$, we have by Definition 6.1b that $x - \frac{q}{y} \in A$. Therefore, $s = (x - \frac{q}{y})(y)$ is an element of $\{ab \mid a \in A, b \in B, a > 0, b > 0\}$, and hence by Definition 7.39, $A \otimes B$.

To show that if $r \in A \otimes B$, then there is some $s \in A \otimes B$ with $s > r$, we let $r \in A \otimes B$ and seek to find such an s. We divide into two cases ($r \leq 0$ and $r > 0$). Suppose first that $r \leq 0$. Then for the same reasons outlined in the proof of the second condition, there exist positive elements of $A \otimes B$ that are greater than r. Now suppose that $r > 0$. This implies that $r = xy$ for some $x \in A, y \in B, x > 0, y > 0$. It follows from the fact that $x \in A$ by Definition 6.1c that there exists a $z \in A$ with $z > x$. Consequently, by Lemma 7.24[2], $zy > xy$ is the desired element of $A \otimes B$. ☐

Proof of b. To prove that $\otimes$ is commutative for $\mathbf{0} < A, \mathbf{0} < B$, Definition 7.1 tells us that it will suffice to show that for all such $A, B \in \mathbb{R}$, we have $A \otimes B = B \otimes A$. Let A, B be arbitrary elements of $\mathbb{R}$ where $\mathbf{0} < A, \mathbf{0} < B$. Then by Definition 7.39, we clearly have

$$\begin{aligned}
A \otimes B &= \{r \in \mathbb{Q} \mid r \leq 0\} \cup \{ab \mid a \in A, b \in B, a > 0, b > 0\} \\
&= \{r \in \mathbb{Q} \mid r \leq 0\} \cup \{ba \mid b \in B, a \in A, b > 0, a > 0\} \\
&= B \oplus A
\end{aligned}$$

To prove that $\otimes$ is associative for $\mathbf{0} < A, \mathbf{0} < B, \mathbf{0} < C$, Definition 7.1 tells us that it will suffice to show that for all such $A, B, C \in \mathbb{R}$, we have $(A \otimes B) \otimes C = A \otimes (B \otimes C)$. Let A, B, C be arbitrary elements of $\mathbb{R}$ where $\mathbf{0} < A, \mathbf{0} < B, \mathbf{0} < C$. To show that $(A \otimes B) \otimes C = A \otimes (B \otimes C)$, Definition 1.2 tells us that it will suffice to verify that every element of $(A \otimes B) \otimes C$ is an element of $A \otimes (B \otimes C)$ and vice versa. Suppose first that $x \in (A \otimes B) \otimes C$. Then by Definition 7.39, $x \leq 0$ or $x = dc$ where $d \in A \otimes B, c \in C, d > 0, c > 0$. If $x \leq 0$, then by Definition 7.39, $x \in A \otimes (B \otimes C)$ since it's an element of $\{r \in \mathbb{Q} \mid r \leq 0\}$, as desired. On the

[1] Note that the proofs given here only address the case where $\mathbf{0} < A$ and $\mathbf{0} < B$.

[2] And, technically, Theorem 7.29.

other hand, if $x = dc$ where $d \in A \otimes B, c \in C, d > 0, c > 0$, we continue. Now $d \in A \otimes B$ implies that $d \leq 0$ or $d = ab$ where $a \in A, b \in B, a > 0, b > 0$. However, the prior constraint that $d > 0$ guarantees that $d \not\leq 0$, so we know that $d = ab$ where a, b satisfy the above conditions. Combining the last two results, we have $x = (ab)(c)$ where $a \in A, b \in B, c \in C, a > 0, b > 0, c > 0$. It follows that we also have $x = (a)(bc)$ under the same conditions. If we let $e = bc$ where $b \in B, c \in C, b > 0, c > 0$, then $e \in \{bc \mid b \in B, c \in C, b > 0, c > 0\}$. Consequently, by Definition 7.31, $e \in B \otimes C$. Additionally, $b > 0, c > 0$ imply by Definition 7.21 that $e > 0$. To recap, at this point we have $x = ae$ where $a \in A, e \in B \otimes C, a > 0, e > 0$. It follows by a similar process to before that $x \in A \otimes (B \otimes C)$. The proof is symmetric in the other direction. ☐

Proof of c. To prove that $\mathbf{0} < A \otimes B$, Definitions 6.4 and 1.3 tell us that it will suffice to show that every $x \in \mathbf{0}$ is an element of $A \otimes B$ and find an $x \in A \otimes B$ such that $x \notin \mathbf{0}$. Let x be an arbitrary element of $\mathbf{0}$. Then $x \in \mathbb{Q}$ and $x < 0$. But it follows that $x \in \{r \in \mathbb{Q} \mid r \leq 0\}$, which implies by Definition 7.39 that $x \in A \otimes B$. As to the other stipulation, clearly $0 \in \{r \in \mathbb{Q} \mid r \leq 0\}$ but since $0 \not< 0$, $0 \notin \mathbf{0}$. Therefore, by Definition 7.39, $0 \in A \otimes B$, but $0 \notin \mathbf{0}$, as desired. ☐

Proof of d. To prove that $\mathbf{1} \otimes A = A$, Definition 1.2 tells us that it will suffice to show that every $x \in \mathbf{1} \otimes A$ is an element of A and vice versa.

Let x be an arbitrary element of $\mathbf{1} \otimes A$. Then by Definition 7.39, $x \leq 0$ or $x = da$ where $d \in \mathbf{1}, a \in A, d > 0, a > 0$. We now divide into two cases. Suppose first that $x \leq 0$. We divide into two cases again ($x < 0$ and $x = 0$). If $x < 0$, then $x \in \mathbf{0}$, which implies by Definitions 6.4, 1.3, and the fact that $\mathbf{0} < A$ that $x \in A$, as desired. On the other hand, if $x = 0$, suppose for the sake of contradiction that $x \notin A$. Then by Lemma 6.2 and Definition 5.6, $a \leq x$ for all $a \in A$. This combined with the fact that $x \notin A$ implies that $a < x$ for all $a \in A$. Consequently, since $\mathbf{0} = \{q \in \mathbb{Q} \mid q < 0\}$, it follows that $A \subset \mathbf{0}$. But by Definition 6.4, this implies $A \leq \mathbf{0}$, contradicting the fact that $\mathbf{0} < A$, as desired. Now suppose that $x = da$ where $d \in \mathbf{1}, a \in A, d > 0, a > 0$. Then by Script 2, $d < 1$ implies that $x = da < a$. Therefore, by Definition 6.1b, $x \in A$, as desired.

Let x be an arbitrary element of A. We divide into two cases ($x \leq 0$ and $x > 0$). If $x \leq 0$, then $x \in \{r \in \mathbb{Q} \mid r \leq 0\}$, which implies by Definition 7.39 that $x \in \underline{1} \otimes A$. On the other hand, suppose $x > 0$. Then by Definition 6.1c, there is some $y \in A$ with $y > x$. It follows by Script 2 that $1 > \frac{x}{y} > 0$, so we have that $\frac{x}{y} \in \mathbf{1}$. Thus, since $x = \frac{x}{y} \cdot y$, we know that x is the product of a positive element of $\mathbf{1}$ and a positive element of A (since $y > x > 0$). Therefore, $x \in \{da \mid d \in \mathbf{1}, a \in A, d > 0, a > 0\}$, which implies by Definition 7.39 that $x \in \mathbf{1} \otimes A$. ☐

7.2 Discussion

1/28:

- Script 6 journals due Wednesday.
- We'll also have to prove a density lemma:
  - Let X be a dense subset of a continuum C. Show that for all $x, y \in X$, if $x < y$, then there exists a $z \in X$ such that $x < z < y$.
  - Mark in Exercise 6.16 as "Density Lemma."
- Explicitly cite Field Axioms as you go.

2/2:

- For Theorem 7.30 in class, he wants a simple explanation of what the injective map looks like and why, but not a full-on rigorous proof.
  - Nothing in the journal for Theorem 7.30, though.
- He also wants to see Exercises 7.32 and 7.40 in the journal.
- For Corollary 7.15, we can write that $x^{-1} \cdot x = 1$ and $x^{-1} \cdot (x^{-1})^{-1} = 1$, and know by the uniqueness of multiplicative inverses (Theorem 7.14) that $x = (x^{-1})^{-1}$. For Corollary 7.11, we have an analogous proof.

- Alternate Theorem 7.17:

$$\begin{aligned}
1 &= 1 \cdot 1 \\
&= (a \cdot a^{-1})(b \cdot b^{-1}) \\
&= (ab)(a^{-1}b^{-1}) \\
&= 0
\end{aligned}$$

- Alternate Lemma 7.18: $a + (-a) = 0$. $a + (-1)a = a(1 + (-1)) = a \cdot 0 = 0$. Thus, by Theorem 7.10, $-a = (-1)a$.

- Alternate Lemma 7.19: We can use the uniqueness of additive inverses (Theorem 7.10).

- We can also cite Remark 7.25 in Lemma 7.26.

2/4:
- Thoughts on Theorem 7.30:

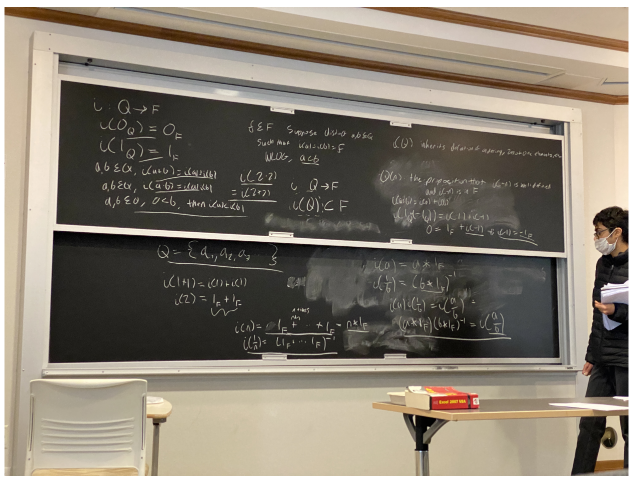

Figure 7.1: Theorem 7.30 discussion.

Script 8

Intervals

8.1 Journal

2/9: Now that we have constructed $\mathbb{R}$ and proved the fundamental facts about it, we will work with the real numbers $\mathbb{R}$ instead of an arbitrary continuum C. We will leave behind Dedekind cuts and think of elements of $\mathbb{R}$ as numbers. Accordingly, from now on, we will use lower-case letters like x for real numbers and will write $+$ and $\cdot$ for $\oplus$ and $\otimes$. We will also now use the standard notation (a, b) for the region $\underline{ab} = \{x \in \mathbb{R} \mid a < x < b\}$. Even though the notation is the same, this is *not* the same object as the ordered pair (a, b).

More generally, we adopt the following standard notation:

$$\begin{aligned}
(a,b) &= \{x \in \mathbb{R} \mid a < x < b\} \\
[a,b) &= \{x \in \mathbb{R} \mid a \leq x < b\} \\
(a,b] &= \{x \in \mathbb{R} \mid a < x \leq b\} \\
[a,b] &= \{x \in \mathbb{R} \mid a \leq x \leq b\} \\
(a,\infty) &= \{x \in \mathbb{R} \mid a < x\} \\
[a,\infty) &= \{x \in \mathbb{R} \mid a \leq x\} \\
(-\infty,b) &= \{x \in \mathbb{R} \mid x < b\} \\
(-\infty,b] &= \{x \in \mathbb{R} \mid x \leq b\}
\end{aligned} \tag{8.1}$$

Exercise 8.1. Identify the sets in Equations 8.1 that are open/closed/neither.

Proof. Note that by Theorem 5.1, any of these sets proven to be just one of open or closed will not be the other, i.e., a set proven to be open will not be closed and vice versa.

By Corollary 4.11, (a, b) is open.

By an adaptation of Corollary 5.14, $b \in LP([a, b))$ but $b \notin [a, b)$. Since $[a, b)$ doesn't contain all of its limit points, Definition 4.1 tells us that it is not closed. Additionally, since $a \in LP(C \setminus [a, b))$ but $a \notin C \setminus [a, b)$, Definition 4.8 tells us that it is not open. Therefore, it is neither.

The proof that $(a, b]$ is neither is symmetric to the previous case.

By Corollaries 5.15 and 4.7, $[a, b]$ is closed.

By Corollary 4.13, (a, ∞) is open.

By Corollary 4.13 and Definition 4.8, $[a, \infty) = C \setminus (-\infty, a)$ is closed.

The proofs that $(-\infty, b)$ and $(-\infty, b]$ are open and closed, respectively, are symmetric to the previous two cases, respectively. $\square$

Definition 8.2. A set $I \subset \mathbb{R}$ is an **interval** if for all $x, y \in I$ with $x < y$, $[x, y] \subset I$.

Lemma 8.3. *A proper subset $I \subsetneq \mathbb{R}$ is an interval if and only if it takes one of the eight forms in Equations 8.1.*

Proof. Suppose first that $I \subsetneq \mathbb{R}$ is an interval. If $I = \emptyset$, then $I = (a, a)$ for any $a \in \mathbb{R}$, and we are done. Thus, we will assume for the remainder of the proof of the forward direction that I is nonempty. To address this case, we will first prove that the facts that $I \subsetneq \mathbb{R}$, $I \neq \emptyset$, and I is an interval imply that I is bounded above, bounded below, or both. Then in each of these three cases, we will look at whether $\sup I$ and $\inf I$ (if they exist) are elements of the set or not to differentiate between the eight forms in Equations 8.1. Let's begin.

Suppose for the sake of contradiction that there exists a nonempty interval $I \subsetneq \mathbb{R}$ that is neither bounded above nor bounded below. Since $I \subsetneq \mathbb{R}$, we have by Definition 1.3 that there exists a point $p \in \mathbb{R}$ such that $p \notin I$. Additionally, since I is neither bounded above nor below, Definition 5.6 implies that p is neither an upper nor a lower bound of I. Thus, there exist $x, y \in I$ such that $x < p$ and $y > p$. Now by Definition 8.2, $[x, y] \subset I$. But it follows by Definition 1.3 that every point in $[x, y]$, including p, is an element of I, a contradiction.

We now divide into three cases (I is exclusively bounded below, I is exclusively bounded above, and I is bounded both below and above).

First, suppose that I is only bounded below. Since I is a nonempty subset of $\mathbb{R}$ that is bounded below, we have by Theorem 5.17 that $\inf I$ exists. We divide into two cases again ($\inf I \in I$ and $\inf I \notin I$).

If $\inf I \in I$, then we can demonstrate that $I = [\inf I, \infty)$. To do this, Definition 1.2 tells us that it will suffice to verify that every $p \in I$ is an element of $[\inf I, \infty)$ and vice versa. Let p be an arbitrary element of I. Then $p \in \mathbb{R}$ and by Definitions 5.7 and 5.6, $\inf I \leq p$. Therefore, $p \in [\inf I, \infty)$, as desired. Now let p be an arbitrary element of $[\inf I, \infty)$. Then $\inf I \leq p$. Additionally, since I is not bounded above, we have by Definition 5.6 that there exists $y \in I$ such that $y > p$. Since $\inf I \in I$, $y \in I$, and $\inf I < y$ (by transitivity), $[\inf I, y] \subset I$ by Definition 8.2. This combined with the fact that $p \in [\inf I, y]$ (we know that $\inf I \leq p < y$, so $\inf I \leq p \leq y$) implies that $p \in I$, as desired.

If $\inf I \notin I$, then we can demonstrate that $I = (\inf I, \infty)$. As before, to do so, it will suffice to verify that every $p \in I$ is an element of $(\inf I, \infty)$ and vice versa. Let p be an arbitrary element of I. Then $p \in \mathbb{R}$ and by Definitions 5.7 and 5.6, $\inf I \leq p$. The additional constraint that $\inf I \notin I$ implies that $\inf I < p$. Therefore, $p \in (\inf I, \infty)$, as desired. Now let p be an arbitrary element of $(\inf I, \infty)$. Then $\inf I < p$. It follows by Lemma 5.11 that there exists a $z \in I$ such that $\inf I \leq z < p$. Additionally, since I is not bounded above, we have by Definition 5.6 that there exists $y \in I$ such that $y > p$. Since $z \in I$, $y \in I$, and $z < y$ (by transitivity), $[z, y] \subset I$ by Definition 8.2. This combined with the fact that $p \in [z, y]$ (we know that $z < p < y$, so $z \leq p \leq y$) implies that $p \in I$, as desired.

Second, suppose that I is only bounded above. The proof of this case is symmetric to that of the first.

Third, suppose that I is bounded below and above. Since I is a nonempty subset of $\mathbb{R}$ that is bounded above and below, we have by consecutive applications of Theorem 5.17 that both $\sup I$ and $\inf I$ exist. We divide into four cases ($\inf I \in I$ and $\sup I \in I$, $\inf I \in I$ and $\sup I \notin I$, $\inf I \notin I$ and $\sup I \in I$, and $\inf I \notin I$ and $\sup I \notin I$).

If $\inf I \in I$ and $\sup I \in I$, then we can demonstrate that $I = [\inf I, \sup I]$. We divide into two cases again ($\inf I = \sup I$ and $\inf I \neq \sup I$). If $\inf I = \sup I \in I$, then $I = \{\inf I\} = \{\sup I\} = [\inf I, \sup I]$, as desired. On the other hand, if $\inf I \neq \sup I$, we continue. To demonstrate that $I = [\inf I, \sup I]$, Theorem 1.7 tells us that it will suffice to verify that $I \subset [\inf I, \sup I]$ and $[\inf I, \sup I] \subset I$. To verify that the former claim, Definition 1.3 tells us that it will suffice to confirm that every $p \in I$ is an element of $[\inf I, \sup I]$. Let p be an arbitrary element of I. Then $p \in \mathbb{R}$ and by consecutive applications of Definitions 5.7 and 5.6, $\inf I \leq p$ and $p \leq \sup I$. Therefore, $p \in [\inf I, \sup I]$, as desired. On the other hand, since $\inf I \in I$, $\sup I \in I$, and $\inf I < \sup I$ (as follows from Definition 5.7 and the fact that they are unequal), $[\inf I, \sup I] \subset I$ by Definition 8.2, as desired.

If $\inf I \in I$ and $\sup I \notin I$, then we can demonstrate that $I = [\inf I, \sup I)$. To do so, it will suffice to verify that every $p \in I$ is an element of $[\inf I, \sup I)$ and vice versa. Let p be an arbitrary element of I. Then $p \in \mathbb{R}$ and by Definitions 5.7 and 5.6, $\inf I \leq p$ and $p \leq \sup I$. The additional constraint that $\sup I \notin I$ implies that $p < \sup I$. Therefore, $p \in [\inf I, \sup I)$, as desired. Now let p be an arbitrary element of $[\inf I, \sup I)$. Then $\inf I \leq p < \sup I$. It follows by Lemma 5.11 that there exists a $y \in I$ such that $p < y \leq \sup I$. Since

$\inf I \in I$, $y \in I$, and $\inf I < y$ (by transitivity), $[\inf I, y] \subset I$ by Definition 8.2. This combined with the fact that $p \in [\inf I, y]$ (we know that $\inf I \leq p < y$, so $\inf I \leq p \leq y$) implies that $p \in I$, as desired.

If $\inf I \notin I$ and $\sup I \in I$, the proof is symmetric to that of the previous case.

If $\inf I \notin I$ and $\sup I \notin I$, then we can demonstrate that $I = (\inf I, \sup I)$. To do so, it will suffice to verify that every $p \in I$ is an element of $(\inf I, \sup I)$ and vice versa. Let p be an arbitrary element of I. Then $p \in \mathbb{R}$ and by Definitions 5.7 and 5.6, $\inf I \leq p$ and $p \leq \sup I$. The additional constraints that $\inf I \notin I$ and $\sup I \notin I$ imply that $\inf I < p$ and $p < \sup I$, respectively. Therefore, $p \in (\inf I, \sup I)$, as desired. Now let p be an arbitrary element of $(\inf I, \sup I)$. Then $\inf I < p < \sup I$. It follows by consecutive applications of Lemma 5.11 that there exist $x, y \in I$ such that $\inf I \leq x < p$ and $p < y \leq \sup I$. Since $x \in I$, $y \in I$, and $x < y$ (by transitivity), $[x, y] \subset I$ by Definition 8.2. This combined with the fact that $p \in [x, y]$ (we know that $x < p < y$, so $x \leq p \leq y$) implies that $p \in I$, as desired.

Now suppose that $I \subsetneq \mathbb{R}$ takes one of the eight forms in Equations 8.1. To prove that I is an interval, Definition 8.2 tells us that it will suffice to show that for all $x, y \in I$ with $x < y$, $[x, y] \subset I$. Let x, y be arbitrary elements of I with $x < y$. We divide into eight cases (one for each equation in Equations 8.1).

First, suppose that $I = (a, b)$. To demonstrate that $[x, y] \subset I$, Definition 1.3 tells us that it will suffice to confirm that every $z \in [x, y]$ is an element of I. Let z be an arbitrary element of $[x, y]$. Then by Corollary 5.15, $x \leq z \leq y$. But since $a < x < y < b$ by Equations 8.1, the fact that $a < x \leq z \leq y < b$ implies by Equations 8.1 that $z \in (a, b)$, as desired.

The proofs of the second, third, and fourth equations are symmetric to that of the first.

Fifth, suppose that $I = (a, \infty)$. To demonstrate that $[x, y] \subset I$, Definition 1.3 tells us that it will suffice to confirm that every $z \in [x, y]$ is an element of I. Let z be an arbitrary element of $[x, y]$. Then by Corollary 5.15, $x \leq z \leq y$. But since $a < x$ by Equations 8.1, the fact that $a < x \leq z$ implies by Equations 8.1 that $z \in (a, \infty)$, as desired.

The proofs of the sixth, seventh, and eighth equations are symmetric to that of the first. $\square$

Definition 8.4. The **absolute value** of a real number x is the non-negative number $|x|$ defined by

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Exercise 8.5. Show that $|x| = |-x|$ for all $x \in \mathbb{R}$. (Note that this also means that $|x - y| = |y - x|$ for any $x, y \in \mathbb{R}$.)

Proof. Let x be an arbitrary element of $\mathbb{R}$. We divide into three cases ($x = 0$, $x > 0$, and $x < 0$). First, suppose that $x = 0$. Then since $0 = -0$, clearly $|0| = |-0|$, as desired. Second, suppose that $x > 0$. Then by Lemma 7.23[1] $-x < 0$. Thus, by consecutive applications of Definition 8.4, $|x| = x$ and $|-x| = -(-x)$. Therefore, since $-(-x) = x$ by Corollary 7.11, $|x| = x = |-x|$, as desired. Third, suppose that $x < 0$. Then by Lemma 7.23, $-x > 0$. Thus, by consecutive applications of Definition 8.4, $|x| = -x$ and $|-x| = -x$. Therefore, $|x| = -x = |-x|$, as desired. $\square$

Definition 8.6. The **distance** between $x \in \mathbb{R}$ and $y \in \mathbb{R}$ is defined to be $|x - y|$.

Remark 8.7. It follows from Definition 8.6 that $|x|$ is the distance between x and 0.

Lemma 8.8. *For any real numbers x, y, z, we have*

(*a*) $|x + y| \leq |x| + |y|$.

(*b*) $|x - z| \leq |x - y| + |y - z|$.

(*c*) $||x| - |y|| \leq |x - y|$.

[1] And, technically, Theorem 7.47.

Proof of a. We divide into four cases ($x \geq 0$ and $y \geq 0$, $x \geq 0$ and $y < 0$, $x < 0$ and $y \geq 0$, and $x < 0$ and $y < 0$).

First, suppose that $x \geq 0$ and $y \geq 0$. Then by Definition 7.21, $x+y \geq 0$. Thus, by consecutive applications of Definition 8.4, $|x+y| = x+y$, $|x| = x$, and $|y| = y$. Therefore, $|x+y| = x+y \leq x+y = |x|+|y|$, as desired.

Second, suppose that $x \geq 0$ and $y < 0$. By Definition 8.4, $|x| = x$ and $|y| = -y$. We now divide into two cases ($x+y \geq 0$ and $x+y < 0$). If $x+y \geq 0$, then $|x+y| = x+y$. Additionally, since $y < 0$, Lemma 7.23 implies that $0 < -y$. Consequently, by transitivity, $y < -y = |y|$. It follows by Definition 7.21 that $x+y < x+|y|$. Therefore, $|x+y| = x+y < x+|y| = |x|+|y|$, so $|x+y| \leq |x|+|y|$, as desired. On the other hand, if $x+y < 0$, then $|x+y| = -(x+y) = -x+(-y) = -x+|y|$. Additionally, by Lemma 7.23, $x \geq 0$ implies that $-x \leq 0$. It follows by Definition 7.21 since $-x \leq x$ that $-x+|y| \leq x+|y|$. Therefore, $|x+y| = -x+|y| \leq x+|y| = |x|+|y|$, as desired.

The proof of the third case is symmetric to that of the second.

The proof of the fourth case is symmetric to that of the first. □

Proof of b. By part (a), $|x-z| = |x-y+y-z| \leq |x-y|+|y-z|$, as desired. □

Proof of c. To prove that $||x|-|y|| \leq |x-y|$, Definition 8.4 tells us that it will suffice to show that $|x|-|y| \leq |x-y|$ and $-(|x|-|y|) \leq |x-y$. By part (a), $|x| = |x-y+y| \leq |x-y|+|y|$, so $|x|-|y| \leq |x-y|$. Similarly, $|y|-|x| \leq |x-y|$, so $-(|x|-|y|) \leq |x-y|$, as desired. □

Exercise 8.9. Let $a, \delta \in \mathbb{R}$ with $\delta > 0$. Prove that

$$(a-\delta, a+\delta) = \{x \in \mathbb{R} \mid |x-a| < \delta\}$$

Lemma. *For any $a, b \in \mathbb{R}$ such that $0 < b$, $|a| < b$ if and only if $-b < a < b$.*

Proof. Suppose first that $|a| < b$. We divide into two cases ($a \geq 0$ and $a < 0$). If $a \geq 0$, then by Definition 8.4, $0 \leq a = |a| < b$. Additionally, by Lemma 7.23, $-b < 0$. Therefore, $-b < 0 \leq a < b$, as desired. If $a < 0$, then by Definition 8.4, $-a = |a| < b$. It follows by Definition 7.21 (by adding $a-b$ to both sides) that $-b < a$. Additionally, by Lemma 7.23, $a < 0$ implies $0 < -a$, so we know that $a < -a$. Therefore, $-b < a < -a < b$, as desired.

Now suppose that $-b < a < b$. We divide into two cases ($a \geq 0$ and $a > 0$). If $a \geq 0$, then by Definition 8.4, $|a| = a < b$, as desired. If $a < 0$, then by Definition 8.4, $|a| = -a$. Since $-b < a$, Definition 7.21 implies (by adding $b-a$ to both sides) that $-a < b$. Therefore, $|a| = -a < b$, as desired. □

Proof of Exercise 8.9. To prove that $(a-\delta, a+\delta) = \{x \in \mathbb{R} \mid |x-a| < \delta\}$, Definition 1.2 tells us that it will suffice to show that every $p \in (a-\delta, a+\delta)$ is an element of $\{x \in \mathbb{R} \mid |x-a| < \delta\}$ and vice versa.

Suppose first that $p \in (a-\delta, a+\delta)$. Then by Equations 8.1, $a-\delta < p$ and $p < a+\delta$. It follows by consecutive applications of Definition 7.21 from the former condition that $-\delta < p-a$, and from the latter condition that $p-a < \delta$. Since $-\delta < p-a < \delta$, the lemma asserts that $|p-a| < \delta$. Therefore, $p \in \{x \in \mathbb{R} \mid |x-a| < \delta\}$.

Now suppose that $p \in \{x \in \mathbb{R} \mid |x-a| < \delta\}$. Then $|p-a| < \delta$. Thus, by the lemma, $-\delta < p-a$ and $p-a < \delta$. It follows by consecutive applications of Definition 7.21 from the former condition that $a-\delta < p$, and from the latter condition that $p < a+\delta$. Therefore, since $a-\delta < p < a+\delta$, we have that $p \in (a-\delta, a+\delta)$. □

2/11: **Lemma 8.10.** *Let I be an open interval containing the point $p \in \mathbb{R}$. Then*

a) *There exists a number $\delta > 0$ such that $(p-\delta, p+\delta) \subset I$.*

b) *There exists a natural number N such that for all natural numbers $k \geq N$ we have $(p-\frac{1}{k}, p+\frac{1}{k}) \subset I$.*

Proof of a. Since I is open, we have by Theorem 4.10 that there exists a region (a,b) such that $p \in (a,b) \subset I$. Let $\delta = \min(p-a, b-p)$. To show that $(p-\delta, p+\delta) \subset I$, we will demonstrate that $(p-\delta, p+\delta) \subset (a,b) \subset I$. To do this, Definition 1.3 tells us that it will suffice to verify that every element $x \in (p-\delta, p+\delta)$ is an element of (a,b). Let x be an arbitrary element of $(p-\delta, p+\delta)$. Then by Equations 8.1, $p-\delta < x < p+\delta$. We divide into two cases ($\delta = p-a$ and $\delta = b-p$). Suppose first that $\delta = p-a$. Then $p-(p-a) < x < p+(p-a)$, i.e., $a < x < p + (p - a)$. Additionally, the fact that $p - a = \min(p - a, b - p)$ implies that $p - a \leq b - p$. Combining these last two results gives us $a < x < p + (p - a) \leq p + (b - p) = b$. Since $a < x < b$, Equations 8.1 imply that $x \in (a, b)$, as desired. The proof is symmetric if $\delta = b - p$. $\square$

Proof of b. By Lemma 8.10a, there exists a number $\delta > 0$ such that $(p - \delta, p + \delta) \subset I$. Since δ is a positive real number, Corollary 6.12 implies that there exists a nonzero natural number N such that $\frac{1}{N} < \delta$. To prove that for all numbers $k \geq N$, we have $(p - \frac{1}{k}, p + \frac{1}{k}) \subset I$, we will show that $(p - \frac{1}{k}, p + \frac{1}{k}) \subset (p - \delta, p + \delta) \subset I$. To do this, Definition 1.3 tells us that it will suffice to show that every $x \in (p - \frac{1}{k}, p + \frac{1}{k})$ is an element of $(p - \delta, p + \delta)$. Let k be an arbitrary natural number such that $k \geq N$, and let x be an arbitrary element of $(p - \frac{1}{k}, p + \frac{1}{k})$. It follows from the latter condition by Equations 8.1 that $p - \frac{1}{k} < x < p + \frac{1}{k}$. Since $\frac{1}{k} \leq \frac{1}{N}$ by Scripts 2 and 3, we have that $p - \frac{1}{N} < x < p + \frac{1}{N}$. Since $\frac{1}{N} < \delta$ by definition, $p - \delta < x < p + \delta$. Therefore, by Equations 8.1, $x \in (p - \delta, p + \delta)$, as desired. $\square$

Definition 8.11. Let $A \subset X \subset \mathbb{R}$. We say that A is **open** (in X) if it is the intersection of X with an open set, and **closed** (in X) if it is the intersection of X with a closed set. (This is called the subspace topology on X.)

Remark 8.12. $A \subset \mathbb{R}$ open, as defined in Script 3, is equivalent to A open in $\mathbb{R}$.

Exercise 8.13. Let $A \subset X \subset \mathbb{R}$. Show that $X \setminus A$ is closed in X if and only if A is open in X.

Proof. Suppose first that $X \setminus A$ is closed in X. Then by Definition 8.11, $X \setminus A = X \cap B$ where B is a closed set. It follows by Script 1 that

$$\begin{aligned}
X \setminus A &= X \cap B \\
\mathbb{R} \setminus (X \setminus A) &= \mathbb{R} \setminus (X \cap B) \\
(\mathbb{R} \setminus X) \cup A &= (\mathbb{R} \setminus X) \cup (\mathbb{R} \setminus B) \\
X \cap ((\mathbb{R} \setminus X) \cup A) &= X \cap ((\mathbb{R} \setminus X) \cup (\mathbb{R} \setminus B)) \\
(X \cap (\mathbb{R} \setminus X)) \cup (X \cap A) &= (X \cap (\mathbb{R} \setminus X)) \cup (X \cap (\mathbb{R} \setminus B)) \\
\emptyset \cup (X \cap A) &= \emptyset \cup (X \cap (\mathbb{R} \setminus B)) \\
A &= X \cap (\mathbb{R} \setminus B)
\end{aligned}$$

Since $\mathbb{R} \setminus B$ is open by Definition 4.4, we have by Definition 8.11 that A is open in X.

Now suppose that A is open in X. Then by Definition 8.11, $A = X \cap B$ where B is an open set. It follows by Script 1 that

$$\begin{aligned}
A &= X \cap B \\
\mathbb{R} \setminus A &= \mathbb{R} \setminus (X \cap B) \\
&= (\mathbb{R} \setminus X) \cup (\mathbb{R} \setminus B) \\
X \cap (\mathbb{R} \setminus A) &= X \cap ((\mathbb{R} \setminus X) \cup (\mathbb{R} \setminus B)) \\
X \setminus A &= (X \cap (\mathbb{R} \setminus X)) \cup (X \cap (\mathbb{R} \setminus B)) \\
&= X \cap (\mathbb{R} \setminus B)
\end{aligned}$$

Since $\mathbb{R} \setminus B$ is closed by Definition 4.4, we have by Definition 8.11 that $X \setminus A$ is closed in X. $\square$

Exercise 8.14.

a) Let $[a, b] \subset \mathbb{R}$. Give an example of a set $A \subset [a, b]$ such that A is open in $[a, b]$ but not in $\mathbb{R}$.

b) Give an example of sets $A \subset X \subset \mathbb{R}$ such that A is closed in X but not in $\mathbb{R}$.

Proof of a. We first briefly consider the case where $a = b$. In this case, let $c < a < d$; then $\{a\} = [a, a] \cap (c, d)$ is a subset of $[a, b]$ that is open in $[a, b]$ (by Definition 8.11; (c, d) is open by Exercise 8.1) but closed in $\mathbb{R}$ (by Corollary 3.23, Definition 4.1, and Theorem 5.1).

We now direct our attention to the case where $a \neq b$. Let $c \in [a, b]$ be a point such that $a < c < b$ (we know at least one such point exists by Theorem 5.2). If we define the set $(c, b] = [a, b] \cap (c, \infty)$, we have by

Definition 8.11 that $(c, b]$ is open in $[a, b]$ (since (c, ∞) is open as per Exercise 8.1). However, we know that $(c, b]$ is not open in $\mathbb{R}$ by Theorem 4.10 (b is an element of $(c, b]$ such that any region containing b necessarily contains an element that is not in $(c, b]$; this element will be greater than b but less than the right bound of the region, and its existence is guaranteed by Theorem 5.2). $\square$

Proof of b. Let $X = (a, b) \subset \mathbb{R}$. Then $(a, b) = X \cap [a, b]$, so $(a, b) = X \cap [a, b]$ is closed in (a, b) by Definition 8.11. However, by Corollary 5.14, a, b are limit points of (a, b) that are not contained within (a, b). It follows by Definition 4.1 that (a, b) is not closed in $\mathbb{R}$. $\square$

Theorem 8.15. *Let $X \subset \mathbb{R}$. Then X is connected if and only if X is an interval.*

Proof. Suppose first that X is connected. To prove that X is an interval, Definition 8.2 tells us that it will suffice to show that for all $x, y \in X$ with $x < y$, $[x, y] \subset X$. Let x, y be arbitrary elements of X satisfying $x < y$, and suppose for the sake of contradiction that $[x, y] \not\subset X$. Then there exists $z \in [x, y]$ such that $z \notin X$. Let $A = \{a \in X \mid a < z\}$ and $B = \{b \in X \mid z < b\}$. It follows from Script 1 that $X = A \cup B$ and $A \cap B = \emptyset$. To verify that A is nonempty, Definition 1.8 tells us that it will suffice to find an element in it. Since $z \notin X$ but $x \in X$, we know that $z \neq x$. This combined with the fact that $x \leq z$ by Equations 8.1 implies that $x < z$. Thus, since $x \in X$ and $x < z$, $x \in A$. Similarly, $y \in B$. To verify that A is open in X, Definition 8.11 tells us that it will suffice to demonstrate that A is the intersection of X with an open set. Since we clearly have $A = X \cap (-\infty, z)$ where $(-\infty, z)$ is open by Exercise 8.1, we are done. We can do something similar for B. But the existence of two disjoint, nonempty, open (in X) sets A, B whose union equals X demonstrates by Definition 4.22 that X is disconnected, a contradiction.

Now suppose that X is an interval, and suppose for the sake of contradiction that X is disconnected. Then by Definition 4.22, $X = A \cup B$ where A, B are disjoint, nonempty sets that are open in X. Since A, B are disjoint and nonempty, we know that there exist distinct objects $a \in A$ and $b \in B$. WLOG, let $a < b$.

To prove that $\sup(A \cap [a, b])$ exists, Theorem 5.17 tells us that it will suffice to show that $A \cap [a, b]$ is nonempty and bounded above. To show that $A \cap [a, b]$ is nonempty, Definition 1.8 tells us that it will suffice to find an element of $A \cap [a, b]$. By Equations 8.1, $a \in [a, b]$. By Definition, $a \in A$. Thus, by Definition 1.6, $a \in A \cap [a, b]$, as desired. To show that $A \cap [a, b]$ is bounded above, consecutive applications of Definition 5.6 tell us that it will suffice to verify that $x \leq b$ for all $x \in A \cap [a, b]$. Let x be an arbitrary element of $A \cap [a, b]$. It follows by Definition 1.6 that $x \in [a, b]$. Thus, by Equations 8.1, $x \leq b$, as desired.

Let $s = \sup(A \cap [a, b])$. To prove that $\inf(B \cap [s, b])$ exists, it will suffice to utilize a symmetric argument to the above.

Let $i = \inf(B \cap [s, b])$. We divide into three cases ($s > i$, $s = i$, and $s < i$).

First, suppose that $s > i$. To show that s is a lower bound of $B \cap [s, b]$, Definition 5.6 tells us that it will suffice to verify that $s \leq x$ for all $x \in B \cap [s, b]$. Let x be an arbitrary element of $B \cap [s, b]$. By Definition 1.6, $x \in [s, b]$. Thus, by Equations 8.1, $s \leq x$, as desired. Since s is a lower bound of $B \cap [s, b]$, Definition 5.7 asserts that $i \geq s$, contradicting the hypothesis that $s > i$.

Second, suppose that $s = i$. We divide into three cases ($s \in A$, $s \in B$, and $s \notin A$ and $s \notin B$).

If $s \in A$, then since A is open in X, Definition 8.11 implies that $A = X \cap G$ where G is open. It follows by the hypothesis that $s \in A$ along with Definitions 1.2 and 1.6 that $s \in G$. Consequently, by Theorem 4.10, there exists a region (c, d) such that $s \in (c, d)$ and $(c, d) \subset G$. From the former condition, we have by Equations 8.1 that $c < s < d$. Thus, by Lemma 5.11, there exists a point $x \in B \cap [s, b]$ such that $s = i \leq x < d$. Since $c < s \leq x < d$, Equations 8.1 imply that $x \in (c, d)$. This combined with the fact that $(c, d) \subset G$ implies by Definition 1.3 that $x \in G$. Additionally, we know that $x \in B$ (since $x \in B \cap [s, b]$ by Definition 1.6). It follows from this and the fact that $X = A \cup B$ by Definitions 1.5 and 1.2 that $x \in X$. Thus, since $x \in X$ and $x \in G$, Definition 1.6 asserts that $x \in X \cap G$, meaning that $x \in A$. But if $x \in A$ and $x \in B$, then Definition 1.6 implies that $x \in A \cap B$, contradicting the supposition that A and B are disjoint.

If $s \in B$, then the proof is symmetric to the previous case.

If $s \notin A$ and $s \notin B$, then by Definition 1.5, $s \notin A \cup B$, implying that $s \notin X$. Additionally, the facts that $a \in A$, $b \in B$, and $X = A \cup B$ imply that $a, b \in X$. It follows since $a < b$ by Definition 8.2 that $[a, b] \subset X$. We now show that $s \in [a, b]$ via Equations 8.1, which tell us that it will suffice to verify that $a \leq s \leq b$. As previously shown, b is an upper bound of $A \cap [a, b]$. Thus, by Definition 5.7, we have that $s \leq b$, and we are half done. As to the other half, we have also previously shown that $a \in A \cap [a,b]$. Additionally, by Definitions 5.7 and 5.6, $s \geq x$ for all $x \in A \cap [a,b]$, including a. Thus, $s \geq a$. Having shown that $s \in [a,b]$ and $[a,b] \subset X$, we may invoke Definition 1.3 to learn that $s \in X$, contradicting the previously proven statement that $s \notin X$.

Third, suppose that $s < i$. Then by Theorem 5.2 and Definition 3.6, there exists a $z \in \mathbb{R}$ such that $s < z < i$. We now show that $i \in [a,b]$ via Equations 8.1, which tell us that it wil suffice to verify that $a \leq i \leq b$. As previously shown, s is a lower bound of $B \cap [s,b]$. Thus, by Definition 5.7, we have that $i \geq s$. We have also previously shown that $s \geq a$, so by transitivity, $i \geq a$, and we are half done. As to the other half, we now confirm that $b \in B \cap [s,b]$. By Equations 8.1, $b \in [s,b]$. By definition, $b \in B$. Thus, by Definition 1.6, $b \in B \cap [s,b]$, as desired. Additionally, by Definitions 5.7 and 5.6, $i \leq x$ for all $x \in B \cap [s,b]$, including b. Thus, $i \leq b$, concluding our argument that $i \in [a,b]$. Moving on, the fact that $s < z$ implies by Definition 5.6 that $z \notin A \cap [a,b]$. Additionally, we know from the facts that $s, i \in [a,b]$ that $a \leq s < z < i \leq b$, meaning that $z \in [a,b]$. Combining the previous two results with Definition 1.6, we have that $z \notin A$. By a symmetric argument, we can show that $z \notin B$. Since $z \notin A$ and $z \notin B$, Definition 1.5 asserts that $z \notin A \cup B$, i.e., $z \notin X$. But as before, $[a,b] \subset X$, so the fact that $z \in [a,b]$ combined with Definition 1.3 implies that $z \in X$, a contradiction. ☐

2/16: **Definition 8.16.** Let I be an interval and let $f : I \to \mathbb{R}$.

a) We say that f is **increasing** on I if, whenever $x, y \in I$ with $x < y$, $f(x) \leq f(y)$.

b) We say that f is **decreasing** on I if, whenever $x, y \in I$ with $x < y$, $f(x) \geq f(y)$.

c) We say that f is **strictly increasing** on I if, whenever $x, y \in I$ with $x < y$, $f(x) < f(y)$.

d) We say that f is **strictly decreasing** on I if, whenever $x, y \in I$ with $x < y$, $f(x) > f(y)$.

Lemma 8.17. *If f is strictly increasing or strictly decreasing on an interval I then f is injective on I.*

Proof. We divide into two cases (f is strictly increasing, and f is strictly decreasing). Suppose first that f is strictly increasing. To prove that f is injective on I, Definition 1.20 tells us that it will suffice to show that for all $a, b \in I$, $a \neq b$ implies that $f(a) \neq f(b)$. Let a, b be arbitrary elements of I such that $a \neq b$. WLOG, let $a < b$. Then by Definition 8.16, $f(a) < f(b)$. Therefore, $f(a) \neq f(b)$, as desired. The proof is symmetric for the other case. ☐

8.2 Discussion

2/9:

- Due date: Feb. 19; if there's anything I can provide that would facilitate the process, lmk; Know anything about mixed-integer nonlinear programming?
- Lemma 8.3 more efficiently by proving that every $x \in (a,b)$ is an element of I and then just working with the boundary conditions?
  - Make first four cases of second direction symmetric.
- Rewrite Exercise 8.5 with three cases: $x = 0$, $x > 0$, $x < 0$ with the last two symmetric.
- We don't have to cite every algebraic manipulations from Script 7.

2/11:

- Use a bidirectional inclusion proof instead of set algebra for Exercise 8.13.
- What is the problem with Exercise 8.14b?

Script 9

Continuous Functions

9.1 Journal

2/16: **Lemma 9.1.** *Let $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$. If $A, B \subset \mathbb{R}$, then*

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$
$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$
$$f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$$
$$f^{-1}(\mathbb{R}) = X$$

Proof. To prove that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, Definition 1.2 tells us that it will suffice to show that every $x \in f^{-1}(A \cup B)$ is an element of $f^{-1}(A) \cup f^{-1}(B)$ and vice versa. Suppose first that x is an arbitrary element of $f^{-1}(A \cup B)$. Then by Definition 1.18, $f(x) \in A \cup B$. Thus, by Definition 1.5, $f(x) \in A$ or $f(x) \in B$. We now divide into two cases. If $f(x) \in A$, then by Definition 1.18, $x \in f^{-1}(A)$. It follows by Definition 1.5 that $x \in f^{-1}(A) \cup f^{-1}(B)$, as desired. The argument is symmetric in the other case. Now suppose that $x \in f^{-1}(A) \cup f^{-1}(B)$. Then by Definition 1.5, $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. We now divide into two cases. If $x \in f^{-1}(A)$, then by Definition 1.18, $f(x) \in A$. It follows by Definition 1.5 that $f(x) \in A \cup B$. Therefore, by Definition 1.18, $x \in f^{-1}(A \cup B)$. The argument is symmetric in the other case, as desired.

To prove that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$, Definition 1.2 tells us that it will suffice to show that every $x \in f^{-1}(A \cap B)$ is an element of $f^{-1}(A) \cap f^{-1}(B)$ and vice versa. Suppose first that x is an arbitrary element of $f^{-1}(A \cap B)$. Then by Definition 1.18, $f(x) \in A \cap B$. Thus, by Definition 1.6, $f(x) \in A$ and $f(x) \in B$. It follows by consecutive applications of Definition 1.18 that $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$. Therefore, by Definition 1.6, $x \in f^{-1}(A) \cap f^{-1}(B)$, as desired. Now suppose that $x \in f^{-1}(A) \cap f^{-1}(B)$. Then by Definition 1.6, $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$. It follows by consecutive applications of Definition 1.18 that $f(x) \in A$ and $f(x) \in B$. Thus, by Definition 1.6, $f(x) \in A \cap B$. Therefore, by Definition 1.18, $x \in f^{-1}(A \cap B)$, as desired.

To prove that $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$, Definition 1.2 tells us that it will suffice to show that every $x \in f^{-1}(A \setminus B)$ is an element of $f^{-1}(A) \setminus f^{-1}(B)$ and vice versa. Suppose first that x is an arbitrary element of $f^{-1}(A \setminus B)$. Then by Definition 1.18, $f(x) \in A \setminus B$. Thus, by Definition 1.11, $f(x) \in A$ and $f(x) \notin B$. It follows by consecutive applications of Definition 1.18 that $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$. Therefore, by Definition 1.11, $x \in f^{-1}(A) \setminus f^{-1}(B)$, as desired. Now suppose that $x \in f^{-1}(A) \setminus f^{-1}(B)$. Then by Definition 1.11, $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$. It follows by consecutive applications of Definition 1.18 that $f(x) \in A$ and $f(x) \notin B$. Thus, by Definition 1.11, $f(x) \in A \setminus B$. Therefore, by Definition 1.18, $x \in f^{-1}(A \setminus B)$, as desired.

To prove that $f^{-1}(\mathbb{R}) = X$, Definition 1.2 tells us that it will suffice to show that every $x \in f^{-1}(\mathbb{R})$ is an element of X and vice versa. Suppose first that x is an arbitrary element of $f^{-1}(\mathbb{R})$. Then by Definition 1.18, $x \in X$, as desired. Now suppose that $x \in X$. Then by Definition 1.16, $f(x) \in \mathbb{R}$. It follows by Definition 1.18 that $x \in f^{-1}(\mathbb{R})$, as desired. $\square$

Exercise 9.2. Let $f : X \to \mathbb{R}$. Let $A \subset X$ and $B \subset \mathbb{R}$. Show that

$$f(f^{-1}(B)) \subset B$$
$$A \subset f^{-1}(f(A))$$

Give examples to show that the inclusions can be proper.

Proof. To prove that $f(f^{-1}(B)) \subset B$, Definition 1.3 tells us that it will suffice to show that every $y \in f(f^{-1}(B))$ is an element of B. Let y be an arbitrary element of $f(f^{-1}(B))$. Then by Definition 1.18, $y = f(x)$ for some $x \in f^{-1}(B)$. By Definition 1.18 again, $f(x) \in B$. Therefore, since $y = f(x)$, it follows that $y \in B$, as desired.

To prove that $A \subset f^{-1}(f(A))$, Definition 1.3 tells us that it will suffice to show that every $x \in A$ is an element of $f^{-1}(f(A))$. Let x be an arbitrary element of A. Then by Definition 1.18, $f(x) \in f(A)$. Therefore, by Definition 1.18, we have $x \in f^{-1}(f(A))$, as desired.

Let $X = \{1, 2\}$ and let $f : X \to \mathbb{R}$ be defined by $f(1) = 3$ and $f(2) = 3$. If we let $B = \{3, 4\}$, then $f(f^{-1}(B)) = \{3\} \subsetneq \{3, 4\}$. Additionally, if we let $A = \{1\}$, then $A \subsetneq f^{-1}(f(A)) = \{1, 2\}$. $\square$

Exercise 9.3. Let $f : X \to \mathbb{R}$. Let $A \subset X$ and $B \subset \mathbb{R}$. Then $f(A) \subset B \iff A \subset f^{-1}(B)$.

Proof. Suppose first that $f(A) \subset B$. To prove that $A \subset f^{-1}(B)$, Definition 1.3 tells us that it will suffice to show that every $x \in A$ is an element of $f^{-1}(B)$. Let x be an arbitrary element of A. Then by Definition 1.18, $f(x) \in f(A)$. It follows by the hypothesis and Definition 1.3 that $f(x) \in B$. Therefore, by Definition 1.18 again, $x \in f^{-1}(B)$.

Now suppose that $A \subset f^{-1}(B)$. To prove that $f(A) \subset B$, Definition 1.3 tells us that it will suffice to show that every $y \in f(A)$ is an element of B. Let y be an arbitrary element of $f(A)$. Then by Definition 1.18, $y = f(x)$ for some $x \in A$. It follows by the hypothesis and Definition 1.3 that $x \in f^{-1}(B)$. Therefore, by Definition 1.18 again, $y = f(x) \in B$. $\square$

Definition 9.4. Let $X \subset \mathbb{R}$. A function $f : X \to \mathbb{R}$ is **continuous** if for every open set $U \subset \mathbb{R}$, the preimage $f^{-1}(U)$ is open in X.

Proposition 9.5. *Let $X \subset \mathbb{R}$. A function $f : X \to \mathbb{R}$ is continuous if and only if for every closed set $F \subset \mathbb{R}$, the preimage $f^{-1}(F)$ is closed in X.*

Proof. Suppose first that f is continuous. We seek to prove that for every closed set $F \subset \mathbb{R}$, the preimage $f^{-1}(F)$ is closed in X. Let F be an arbitrary closed subset of $\mathbb{R}$. Then by Definition 4.8, $F = \mathbb{R} \setminus U$ for some open set $U \subset \mathbb{R}$. It follows by Definition 9.4 since f is continuous that $f^{-1}(U)$ is open in X. Additionally, by consecutive applications of Lemma 9.1, $f^{-1}(F) = f^{-1}(\mathbb{R} \setminus U) = f^{-1}(\mathbb{R}) \setminus f^{-1}(U) = X \setminus f^{-1}(U)$. Therefore, since $f^{-1}(U)$ is open in X, Exercise 8.13 implies that $X \setminus f^{-1}(U) = f^{-1}(F)$ is closed in X.

The proof is symmetric in the other direction. $\square$

Definition 9.6. Let $X \subset Y \subset \mathbb{R}$ and let $f : Y \to \mathbb{R}$. Then the **restriction** (of f to X), written $f|_X$ is the function $f|_X : X \to \mathbb{R}$ defined by

$$f|_X(x) = f(x)$$

for all $x \in X$.

Proposition 9.7. *Let $X \subset Y \subset \mathbb{R}$. If $f : Y \to \mathbb{R}$ is continuous, then the restriction of f to X is continuous.*

Proof. To prove that $f|_X$ is continuous, Definition 9.4 tells us that it will suffice to show that for every open set $U \subset \mathbb{R}$, the preimage $f|_X^{-1}(U)$ is open in X. Let U be an open subset of $\mathbb{R}$. Then

$$\begin{aligned}
f|_X^{-1}(U) &= \{x \in X \mid f|_X(x) \in U\} && \text{Definition 1.18} \\
&= \{x \in X \mid f(x) \in U\} && \text{Definition 9.6} \\
&= \{x \in Y \mid f(x) \in U\} \cap X && \text{Script 1} \\
&= f^{-1}(U) \cap X && \text{Definition 1.18} \\
&= (Y \cap G) \cap X && \text{Definitions 9.4 and 8.11}
\end{aligned}$$

$$= X \cap G \qquad \text{Script 1}$$

Since $f|_X^{-1}(U) = X \cap G$ where G is an open set, Definition 8.11 asserts that $f|_X^{-1}(U)$ is open in X. □

Exercise 9.8. Show that for any $X \subsetneq \mathbb{R}$ that is not open and any continuous function $f : X \to \mathbb{R}$, there is an open set U for which $f^{-1}(U)$ is open in X but is not open in $\mathbb{R}$.

Proof. We will prove that $\mathbb{R}$ is an open set for which $f^{-1}(\mathbb{R})$ is open in X but not in $\mathbb{R}$. First, by Theorem 5.1, $\mathbb{R}$ is open. Next, by Lemma 9.1, $f^{-1}(\mathbb{R}) = X$. It follows since $f^{-1}(\mathbb{R}) = X = X \cap \mathbb{R}$ (where $\mathbb{R}$ is an open set) by Definition 8.11 that $f^{-1}(\mathbb{R})$ is open in X. Last, since X is not open (in $\mathbb{R}$) by definition, $f^{-1}(\mathbb{R}) = X$ is not open in $\mathbb{R}$. □

Definition 9.9. The function $f : X \to \mathbb{R}$ is **continuous** (at $x \in X$) if for every region R containing $f(x)$, there exists an open set S containing x such that $S \cap X \subset f^{-1}(R)$.

Theorem 9.10. *The function $f : X \to \mathbb{R}$ is continuous if and only if it is continuous at every $x \in X$.*

Proof. Suppose first that f is continuous, and suppose for the sake of contradiction that f is not continuous at every $x \in X$. Then by Definition 9.9, there exists some $x \in X$ such that f is not continuous at x. Thus, there exists a region R with $f(x) \in R$ such that for all open sets S containing x, $S \cap X \not\subset f^{-1}(R)$. Since f is continuous by hypothesis and R is open by Corollary 4.11, $f^{-1}(R)$ is open in X. It follows by Definition 8.11 that $f^{-1}(R) = X \cap S$ for some open set S. But this implies that $f^{-1}(R) \not\subset f^{-1}(R)$, a contradiction.

Now suppose that f is continuous at every $x \in X$. To prove that f is continuous, Definition 9.4 tells us that it will suffice to show that for every open set $U \subset \mathbb{R}$, the preimage $f^{-1}(U)$ is open in X. We divide into two cases ($f^{-1}(U) = \emptyset$ and $f^{-1}(U) \neq \emptyset$). If $f^{-1}(U) = \emptyset$, then since $\emptyset \cap X = \emptyset$ by Script 1 where $\emptyset$ is open by Theorem 5.1, Definition 8.11 tells us that $\emptyset = f^{-1}(U)$ is open in X, as desired. On the other hand, if $f^{-1}(U) \neq \emptyset$, Definition 8.11 tells us that it will suffice to show that $f^{-1}(U) = S \cap X$ where S is an open set. We first seek to show that for every $x \in f^{-1}(U)$, there exists an open set S_x containing x such that $S_x \cap X \subset f^{-1}(U)$. Let x be an arbitrary element of $f^{-1}(U)$. It follows by Definition 1.18 that $f(x) \in U$. Thus, since U is open, we have by Theorem 4.10 that there exists a region R such that $f(x) \in R$ and $R \subset U$. Consequently, since R is open by Corollary 4.11, we have by Definition 9.9 that there exists an open set S_x containing x such that $S_x \cap X \subset f^{-1}(R)$. Additionally, Script 1 tells us based off of the fact that $R \subset U$ that $f^{-1}(R) \subset f^{-1}(U)$. Thus, by subset transitivity, $S_x \cap X \subset f^{-1}(U)$. At this point, let $S = \bigcup_{x \in f^{-1}(U)} S_x$. It follows immediately from Corollary 4.18 that S is open. Additionally, since the intersection of each set in the union with X is a subset of $f^{-1}(U)$, it follows by Script 1 that $S \cap X \subset f^{-1}(U)$. Furthermore, for all $x \in f^{-1}(U)$, Definition 1.18 asserts that $x \in X$. In addition, we have defined an S_x such that $x \in S_x$. These last two results combined demonstrate by Definition 1.6 that $x \in S \cap X$. Thus, by Definition 1.3, $f^{-1}(U) \subset S \cap X$. Consequently, by Theorem 1.7, $f^{-1}(U) = S \cap X$. Since $f^{-1}(U)$ is the intersection of X with an open set, Definition 8.11 asserts that it is open in X, as desired. □

2/18: **Theorem 9.11.** *Suppose that $f : X \to \mathbb{R}$ is continuous. If X is connected, then $f(X)$ is connected.*

Proof. This will be a proof by contrapositive; as such, suppose that $f(X)$ is disconnected. Then by Definition 4.22, $f(X) = A \cup B$ where A, B are nonempty, disjoint sets that are open in $f(X)$. It follows from the last condition by Definition 8.11 that $A = G \cap f(X)$ and $B = H \cap f(X)$, where G, H are open sets. Since for all $x \in X$, $f(x) \in A$ or $f(x) \in B$, Definitions 1.2 and 1.6 imply that for all $x \in X$, $f(x) \in G$ and $f(x) \in H$. Thus, by Script 1, $X \subset f^{-1}(G) \cup f^{-1}(H)$. Additionally, we know by Definition 1.18 that for all $x \in f^{-1}(G) \cup f^{-1}(H)$, $x \in X$. Thus, by Definition 1.3, $f^{-1}(G) \cup f^{-1}(H) \subset X$. Consequently, by Theorem 1.7, we have that $X = f^{-1}(G) \cup f^{-1}(H)$.

To show that $f^{-1}(G), f^{-1}(H)$ are disjoint, Definition 1.9 tells us that it will suffice to verify that $f^{-1}(G) \cap f^{-1}(H) = \emptyset$. As such, suppose for the sake of contradiction that $x \in f^{-1}(G) \cap f^{-1}(H)$. Then by consecutive applications of Definition 1.6, $x \in f^{-1}(G)$ and $x \in f^{-1}(H)$. Thus, by multiple applications of Definition 1.18, $x \in X$, $f(x) \in G$, and $f(x) \in H$. It follows from the first condition by Definition 1.18 that $f(x) \in f(X)$. The facts that $f(x) \in f(X)$ and $f(x) \in G$ imply by Definitions 1.6 and 1.2 that $f(x) \in A$. Similarly, $f(x) \in B$. But these last two statements imply by Definition 1.6 that $f(x) \in A \cap B$, a contradiction.

To show that $f^{-1}(G), f^{-1}(H)$ are nonempty, Definition 1.8 tells us that it will suffice to find an element of each set. As previously mentioned, A, B are nonempty. Thus, by consecutive applications of Definition 1.8, there exist $f(x) \in A$ and $f(y) \in B$. Consequently, by Definitions 1.2 and 1.6, $f(x) \in G$ and $f(y) \in H$. Therefore, by consecutive applications of Definition 1.18, $x \in f^{-1}(G)$ and $y \in f^{-1}(H)$, as desired.

To show that $f^{-1}(G)$ and $f^{-1}(H)$ are open, Definition 9.4 tells us that it will suffice to verify (since f is continuous by hypothesis) that G, H are open subsets of $\mathbb{R}$. But by definition, they are exactly that. ☐

Exercise 9.12. Use Theorem 9.11 to prove that if $f : [a, b] \to \mathbb{R}$ is continuous, then for every point p between $f(a)$ and $f(b)$, there exists c such that $a < c < b$ and $f(c) = p$.

Proof. Suppose that $a < b$. Then by Lemma 8.3, $[a, b]$ is an interval. Thus, by Theorem 8.15, $[a, b]$ is connected. It follows by Theorem 9.11 that $f([a, b])$ is connected. Consequently, by Theorem 8.15, $f([a, b])$ is an interval. We divide into three cases ($f(a) < f(b)$, $f(a) > f(b)$, and $f(a) = f(b)$).

First, suppose that $f(a) < f(b)$, and let p be an arbitrary point between $f(a)$ and $f(b)$ (we know that at least one such point exists by Theorem 5.2). Then by Definition 3.6, $f(a) < p < f(b)$. Now $a, b \in [a, b]$ by Equations 8.1, so by Definition 1.18, $f(a), f(b) \in f([a, b])$. It follows by Definition 8.2 since $f([a, b])$ is an interval that $[f(a), f(b)] \subset f([a, b])$. Thus, since $f(a) < p < f(b)$ implies $p \in [f(a), f(b)]$ by Equations 8.1, Definition 1.3 asserts that $p \in f([a, b])$. Consequently, by Definition 1.18, $p = f(c)$ for some $c \in [a, b]$. Additionally, since $f(a) < p < f(b)$, we know that $p \neq f(a)$ and $p \neq f(b)$. It follows that $p = f(c)$ for some $c \in (a, b)$, as desired.

The proof of the second case is symmetric to that of the first.

Third, suppose that $f(a) = f(b)$. This implies that there are no points p between $f(a)$ and $f(b)$ by Definition 3.6, so the statement is vacuously true in this case. ☐

Lemma 9.13. *If $f : (a, b) \to \mathbb{R}$ is continuous and injective, then f is either strictly increasing or strictly decreasing on (a, b).*

Proof. Suppose that $a < b$. Then by Corollary 5.3, we can pick two distinct points $c, d \in (a, b)$. Since f is injective, we have from the fact that $x \neq y$ by Definition 1.20 that $f(c) \neq f(d)$. We divide into two cases ($f(c) < f(d)$ and $f(c) > f(d)$).

Suppose first that $f(c) < f(d)$. To prove that f is strictly increasing, Definition 8.16 tells us that it will suffice to show that for all $x, y \in (a, b)$ with $x < y$, $f(x) < f(y)$. Let x, y be arbitrary elements of (a, b) that satisfy $x < y$, and suppose for the sake of contradiction that $f(x) \geq f(y)$. If $f(x) = f(y)$, then by Definition 1.20, $x = y$, a contradiction. If $f(x) > f(y)$, then we divide into five cases ($x < y < c < d$, $x < c < y < d$, $x < c < d < y$, $c < x < d < y$, and $c < d < x < y$).

Let $x < y < c < d$. If $f(x) > f(c)$, let p_1 be a point between $f(x)$ and $f(c)$ and let p_2 be a point between $f(d)$ and $f(c)$. Let $p = \min(p_1, p_2)$. Since $f(c) < p < f(x)$, Exercise 9.12 implies that there exists $e \in (x, c)$ such that $f(e) = p$. Similarly, there exists $e' \in (c, d)$ such that $f(e') = p$. Since (x, c) and (c, d) are clearly disjoint, $e \neq e'$. But by Definition 1.20, $f(e) = p = f(e')$ implies that $e = e'$, a contradiction. If $f(x) < f(c)$, then we can arrive at a similar contradiction by considering values in the regions (x, y) and (y, c).

The proof is symmetric in the second case if we consider values in the regions (c, y) and (y, d) when $f(y)$ is not between $f(c)$ and $f(d)$, and values in the regions (x, c) and (c, y) when $f(y)$ is.

The proof is symmetric in the third case if we consider values in the regions (x, c) and (c, d) when $f(x) > f(c)$, and values in the regions (x, d) and (d, y) when $f(x) < f(c)$.

The proof is symmetric in the fourth case if we consider values in the regions (c, x) and (x, d) when $f(x) > f(d)$, and values in the regions (x, d) and (d, y) when $f(x) < f(d)$.

The proof is symmetric in the fifth case if we consider values in the regions (c, d) and (d, x) when $f(d) > f(x)$, and values in the regions (d, x) and (x, y) when $f(d) < f(x)$. ☐

9.2 Discussion

2/16: • For Proposition 9.5, do an iff proof — use $\iff$ steps instead of back-and-forth work.

2/18: • To what extent do we need casework in Theorem **??**?