MATH 16210 (Honors Calculus II IBL) Notes

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Script 6

Construction of the Real Numbers

6.1 Journal

1/12: **Definition 6.1.** A subset A of $\mathbb{Q}$ is said to be a **cut** (or **Dedekind cut**) if it satisfies the following:

(a) $A \neq \emptyset$ and $A \neq \mathbb{Q}$.

(b) If $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$.

(c) A does not have a last point; i.e., if $r \in A$, then there is some $s \in A$ with $s > r$.

We denote the collection of all cuts by $\mathbb{R}$.

Lemma 6.2. *Let A be a Dedekind cut and $x \in \mathbb{Q}$. Then $x \notin A$ if and only if x is an upper bound for A.*

Proof. Suppose first that x is an element of $\mathbb{Q}$ such that $x \notin A$. To prove that x is an upper bound for A, Definition 5.6 tells us that it will suffice to show that for all $r \in A$, $r \leq x$. Let r be an arbitrary element of A. Then since $r \in A$, $x \in \mathbb{Q}$, and $x \notin A$, the contrapositive of Definition 6.1b asserts that $x \not< r$. Therefore, $r \leq x$, as desired.

Now suppose that x is an upper bound for A. By Definition 5.6, this implies that for all $r \in A$, $r \leq x$. Therefore, since there is no $r \in A$ with $r > x$, by the contrapositive of Definition 6.1c, $x \notin A$, as desired. ☐

Exercise 6.3.

(a) Prove that for any $q \in \mathbb{Q}$, $\{x \in \mathbb{Q} \mid x < q\}$ is a Dedekind cut. We then define $\mathbf{0} = \{x \in \mathbb{Q} \mid x < 0\}$.

(b) Prove that $\{x \in \mathbb{Q} \mid x \leq 0\}$ is not a Dedekind cut.

(c) Prove that $\{x \in \mathbb{Q} \mid x < 0\} \cup \{x \in \mathbb{Q} \mid x^2 < 2\}$ is a Dedekind cut.

Proof of a. Let q be an arbitrary element of $\mathbb{Q}$. To prove that $A = \{x \in \mathbb{Q} \mid x < q\}$ is a Dedekind cut, Definition 6.1 tells us that it will suffice to show that $A \neq \emptyset$; $A \neq \mathbb{Q}$; if $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$; and if $r \in A$, then there is some $s \in A$ with $s > r$. We will take this one claim at a time.

To show that $A \neq \emptyset$, Definition 1.8 tells us that it will suffice to find an element of A. By Exercise 3.9d, q is not the first point of $\mathbb{Q}$. Thus, by Definition 3.3, there exists an object $x \in \mathbb{Q}$ such that $x < q$. By the definition of A, this implies that $x \in A$, as desired.

To show that $A \neq \mathbb{Q}$, Definition 1.2 tells us that it will suffice to find an element of $\mathbb{Q}$ that is not an element of A. By hypothesis, $q \in \mathbb{Q}$. By Exercise 3.9d, $q \not< q$. Therefore, $q \in \mathbb{Q}$ but $q \notin A$, as desired.

To show that if $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$, we let $r \in A$ and $s \in \mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s < r$ and seek to verify that $s \in A$. Since $r \in A$, $r < q$. This combined with the fact that $s < r$ implies by transitivity that $s < q$. Therefore, since $s \in \mathbb{Q}$ and $s < q$, $s \in A$, as desired.

To show that if $r \in A$, then there is some $s \in A$ with $s > r$, we let $r \in A$ and seek to find such an s. By the definition of A, $r < q$. Thus, by Additional Exercise 3.1, there exists a point $s \in \mathbb{Q}$ such that $r < s < q$. Since $s \in \mathbb{Q}$ and $s < q$, $s \in A$. It follows that s is the desired element of A which satisfies $s > r$. ☐

Proof of b. To prove that $A = \{x \in \mathbb{Q} \mid x \leq 0\}$ is not a Dedekind cut, Definition 6.1 tells us that it will suffice to show that A *does* have a last point. To show this, we will demonstrate that 0 is the last point of A. To demonstrate this, Definition 3.1 tells us that it will suffice to confirm that $0 \in A$ and for all $x \in A$, $x \leq 0$. Since $0 \leq 0$ and $0 \in \mathbb{Q}$, $0 \in A$. Additionally, by the definition of A, it is true that for all $x \in A$, $x \leq 0$. □

Proof of c. Let $B = \{x \in \mathbb{Q} \mid x < 0\}$ and let $C = \{x \in \mathbb{Q} \mid x^2 < 2\}$. To prove that $A = B \cup C$ is a Dedekind cut, Definition 6.1 tells us that it will suffice to show that $A \neq \emptyset$; $A \neq \mathbb{Q}$; if $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$; and if $r \in A$, then there is some $s \in A$ with $s > r$. We will take this one claim at a time.

To show that $A \neq \emptyset$, Definition 1.8 tells us that it will suffice to find an element of A. Since $-1 \in \mathbb{Q}$ and $-1 < 0$, $-1 \in B$. Therefore, by Definition 1.5, $-1 \in A$, as desired.

To show that $A \neq \mathbb{Q}$, Definition 1.2 tells us that it will suffice to find an element of $\mathbb{Q}$ that is not an element of A. Since $2 \geq 0$, $2 \notin B$. Additionally, since $2^2 \geq 2$, $2 \notin C$. Therefore, by Definition 1.5, $2 \notin A$, as desired.

To show that if $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$, we let $r \in A$ and $s \in \mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s < r$ and seek to verify that $s \in A$. We divide into two cases ($s < 0$ and $s \geq 0$). Suppose first that $s < 0$. Then $s \in B$, meaning that $s \in A$. Now suppose that $s \geq 0$. Then by Script 0, we have $0 \leq s^2 < r^2 < 2$. Thus, by the definition of C, $s \in C$, implying that $s \in A$.

To show that A does not have a last point, suppose for the sake of contradiction that A has a last point p. We now divide into two cases ($p \leq 0$ and $p > 0$). Suppose first that $p \leq 0$. Since p is the last point of A, Definition 3.3 tells us that $x \leq p$ for all $x \in A$. But $1 \in A$ (since $1 \in \mathbb{Q}$ and $1^2 = 1 < 2$ implies $1 \in B$, implies $1 \in A$) and $1 > 0 \geq p$, a contradiction. Now suppose that $p > 0$. Definition 3.3 tells us that $p \in A$, but the condition that $p > 0$ means $p \notin B$, so we must have $p \in C$. However, by the proof of Exercise 4.24, $\frac{2(p+1)}{p+2}$ will be an element of B (and therefore A) that is greater than p no matter how large p is, a contradiction. □

Definition 6.4. If $A, B \in \mathbb{R}$, we say that $A < B$ if A is a proper subset of B.

Exercise 6.5. Show that $\mathbb{R}$ satisfies Axioms 1, 2, and 3.

Proof. By Exercise 6.3a, $\{x \in \mathbb{Q} \mid x < 0\} \in \mathbb{R}$ since $0 \in \mathbb{Q}$. Therefore, Axiom 1 is immediately satisfied.

Axiom 2 asserts that $\mathbb{R}$ must have an ordering $<$. As such, it will suffice to verify that the ordering given by Definition 6.4 satisfies the stipulations of Definition 3.1. To prove that $<$ satisfies the trichotomy, it will suffice to show that for all $A, B \in \mathbb{R}$, exactly one of the following holds: $A < B$, $B < A$, or $A = B$.

We first show that *no more than one* of the three statements can simultaneously be true. Let A, B be arbitrary elements of $\mathbb{R}$. We divide into three cases. First, suppose for the sake of contradiction that $A < B$ and $B < A$. By Definition 6.4, this implies that $A \subsetneq B$ and $B \subsetneq A$. Thus, by Definition 1.3, $A \subset B$, $B \subset A$, and $A \neq B$. But by Theorem 1.7, $A \subset B$ and $B \subset A$ implies that $A = B$, a contradiction. Second, suppose for the sake of contradiction that $A < B$ and $A = B$. By substitution, we have that $A < A$. But by Definitions 6.4 and 1.3, it follows that $A \neq A$. The proof of the third case ($B < A$ and $A = B$) is symmetric to that of the second case.

We now show that *at least one* of the three statements is always true. Let A, B be arbitrary elements of $\mathbb{R}$, and suppose for the sake of contradiction that $A \not< B$, $B \not< A$, and $A \neq B$. Since $A \not< B$ and $B \not< A$, we have by Definition 6.4 that $A \not\subsetneq B$ and $B \not\subsetneq A$. Thus, by Definition 1.3, $A \not\subset B$ or $A = B$, and $B \not\subset A$ or $A = B$. But $A \neq B$ by hypothesis, so it must be that $A \not\subset B$ and $B \not\subset A$. It follows from the first statement by Definition 1.3 that there exists an object $x \in A$ such that $x \notin B$, and there exists an object $y \in B$ such that $y \notin A$. Since $x \notin B$, Lemma 6.2 implies that x is an upper bound of B. Consequently, by Definition 5.6, $p \leq x$ for all $p \in B$, including y. Similarly, $p \leq y$ for all $p \in A$, including x. Thus, we have $y \leq x$ and $x \leq y$, implying that $x = y$. But since $y \in B$, this implies that $x \in B$, a contradiction.

To prove that $<$ is transitive, it will suffice to show that for all $A, B, C \in \mathbb{R}$, if $A < B$ and $B < C$, then $A < C$. Let A, B, C be arbitrary elements of $\mathbb{R}$ for which it is true that $A < B$ and $B < C$. By Definition 6.4, we have $A \subsetneq B$ and $B \subsetneq C$. Thus, by Script 1, $A \subsetneq C$. Therefore, by Definition 6.4, $A < C$.

Axiom 3 asserts that $\mathbb{R}$ must have no first or last point. We will take this one argument at a time

Suppose for the sake of contradiction that $\mathbb{R}$ has some first point A. Then by Definition 3.3, $A \leq X$ for every $X \in \mathbb{R}$. Now since A is a Dedekind cut, Definition 6.1 tells us that $A \neq \emptyset$. Thus, by Definition 1.8, there exists some $q \in A$. Additionally, $A \subset \mathbb{Q}$ by Definition 6.1, so $q \in A$ implies that $q \in \mathbb{Q}$. It follows by Exercise 6.3a that $B = \{x \in \mathbb{Q} \mid x < q\}$ is a Dedekind cut. We now seek to prove that $B \subsetneq A$. To do this, Definition 1.3 tells us that it will suffice to show that $B \neq A$ and $B \subset A$. To show that $B \neq A$, Definition 1.2 tells us that it will suffice to find an element of A that is not an element of B. Conveniently, q is clearly such an object. To show that $B \subset A$, Definition 1.3 tells us that we must confirm that every element of B is an element of A. Let p be an arbitrary element of B. Then by the definition of B, $p \in \mathbb{Q}$ and $p < q$. It follows by Definition 6.1b (which clearly applies to A) that $p \in A$, as desired. Having proven that $B \subsetneq A$, Definition 6.4 tells us that $B < A$. But this contradicts the previously demonstrated fact that $A \leq X$ for every $X \in \mathbb{R}$, including B.

Suppose for the sake of contradiction that $\mathbb{R}$ has some last point A. Then by Definition 3.3, $X \leq A$ for every $X \in \mathbb{R}$. Now since A is a Dedekind cut, Definition 6.1 tells us that $A \neq \mathbb{Q}$. Thus, by Definition 1.2, there exists some $q \in \mathbb{Q}$ such that $q \notin A$. It follows by Lemma 6.2 that q is an upper bound of A. Consequently, by Definition 5.6, $x \leq q$ for all $x \in A$. Additionally, by Exercise 6.3a, $B = \{x \in \mathbb{Q} \mid x < q + 1\}$[1] is a Dedekind cut. We now seek to prove that $A \subsetneq B$. As before, this means we must show that $A \neq B$ and $A \subset B$. To show that $A \neq B$, Definition 1.2 tells us that it will suffice to find an element of B that is not an element of A. Since $x \leq q$ for all $x \in A$ and $q < q + 0.5 < q + 1$, $q + 0.5 \notin A$ and $q + 0.5 \in B$ is one such desired object. To show that $A \subset B$, Definition 1.3 tells us that we must confirm that every element of A is an element of B. Let p be an arbitrary element of A. As an element of A, we know that $p \leq q$. Thus, $p < q + 1$, so $p \in B$, as desired. Having proven that $A \subsetneq B$, Definition 6.4 tells us that $A < B$. But this contradicts the previously demonstrated fact that $X \leq A$ for every $X \in \mathbb{R}$, including B. ☐

1/14: **Lemma 6.6.** *A nonempty subset of* $\mathbb{R}$ *that is bounded above has a supremum.*

Proof. Let X be an arbitrary nonempty subset of $\mathbb{R}$ that is bounded above. To prove that $\sup X$ exists, we will show that $\sup X = U = \bigcup\{Y \mid Y \in X\}$. To show this, Definition 5.7 tells us that it will suffice to demonstrate that $U \in \mathbb{R}$, U is an upper bound of X, and if U' is an upper bound of X, then $U \leq U'$. Let's begin.

To demonstrate that $U \in \mathbb{R}$, Definition 6.1 tells us that it will suffice to confirm that $U \neq \emptyset$; $U \neq \mathbb{Q}$; if $r \in U$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in U$; and if $r \in U$, then there is some $s \in U$ with $s > r$.

As the union of a nonempty set of nonempty sets, Script 1 implies that $U \neq \emptyset$.

To demonstrate that $U \neq \mathbb{Q}$, Definition 1.2 tells us that it will suffice to find a point $p \in \mathbb{Q}$ such that $p \notin U$. Since X is bounded above, we have by Definition 5.6 that there exists a Dedekind cut $V \in \mathbb{R}$ such that $A \leq V$ for all $A \in X$. It follows by Definition 6.4 that $A \subset V$ for all $A \in X$. Thus, by Script 1, $U \subset V$. Now since V is a Dedekind cut, we know by Definition 6.1 that $V \subset \mathbb{Q}$ and $V \neq \mathbb{Q}$, meaning that there exists a point $p \in \mathbb{Q}$ such that $p \notin V$. Consequently, since $U \subset V$, $p \notin U$, as desired.

To demonstrate that if $r \in U$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in U$, we let $r \in U$ and $s \in \mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s < r$ and seek to verify that $s \in U$. Since $r \in U$, Definition 1.13 tells us that $r \in A$ for some $A \in X$. Thus, since A is a Dedekind cut, $s \in \mathbb{Q}$ and $s < r$ implies that $s \in A$. Therefore, $s \in U$.

To demonstrate that if $r \in U$, then there is some $s \in U$ with $s > r$, we let $r \in U$ and seek to find such an s. Since $r \in U$, Definition 1.13 tells us that $r \in A$ for some $A \in X$. Thus, since A is a Dedekind cut, there exists a point $s \in A$ with $s > r$. Therefore, $s \in U$.

To demonstrate that U is an upper bound of X, Definition 5.6 tells us that it will suffice to confirm that $A \leq U$ for all $A \in X$. To confirm this, Definition 6.4 tells us that it will suffice to verify that $A \subset U$ for all $A \in X$. But by an extension of Theorem 1.7b, this is true.

Now suppose for the sake of contradiction that there exists an upper bound U' of X such that $U' < U$. It follows by Definitions 6.4 and 1.3 that there exists a point $p \in U$ such that $p \notin U'$. Thus, by the former statement and Definition 1.13, $p \in A$ for some $A \in X$. Additionally, since U' is an upper bound of X, we have by Definitions 5.6 and 6.4 that $A \subset U'$ for all $A \in X$. But this implies by Definition 1.3 that $p \in U'$, a contradiction. ☐

[1]Note that we add 1 to q to treat the case that $q = \sup A$, a case in which we would have $B = A$ if B were defined as $\{x \in \mathbb{Q} \mid x < q\}$.

1/19: **Exercise 6.7.** Show that $\mathbb{R}$ satisfies Axiom 4.

Proof. Suppose for the sake of contradiction that $\mathbb{R}$ does not satisfy Axiom 4. It follows that $\mathbb{R}$ is not connected, implying by Definition 4.22 that $\mathbb{R} = A \cup B$ where A, B are disjoint, nonempty, open sets. Since A, B are disjoint and nonempty, we know that there exist distinct objects $a \in A$ and $b \in B$. WLOG, let $a < b$.

We now seek to prove that the set $A \cap \underline{ab}$ is nonempty and bounded above. To prove that $A \cap \underline{ab}$ is nonempty, Definition 1.8 tells us that it will suffice to find an element of $A \cap \underline{ab}$. Since $a \in A$ and A is open, we have by Theorem 4.10 that there exists a region $\underline{cd}$ such that $a \in \underline{cd}$ and $\underline{cd} \subset A$. It follows by Definitions 3.10 and 3.6 that $a < d$, implying by Lemma 6.10[2] that there exists some point $x \in \mathbb{R}$ such that $c < a < x < d < b$ (note that $d < b$ since if $b < d$, then $b \in \underline{cd}$ would contradict the fact that $\underline{cd} \subset A$). Consequently, $x \in \underline{cd}$, meaning that $x \in A$, and $x \in \underline{ab}$. Therefore, $x \in A \cap \underline{ab}$, as desired. To prove that $A \cap \underline{ab}$ is bounded above, Definition 5.6 tells us that it will suffice to show that b is an upper bound of $A \cap \underline{ab}$. To show this, Definition 5.6 tells us that it will suffice to confirm that $y \leq b$ for all $y \in A \cap \underline{ab}$. Let y be an arbitrary element of $A \cap \underline{ab}$. Then by Definition 1.6, $y \in A$ and $y \in \underline{ab}$. It follows from the latter statement by Definitions 3.10 and 3.6 that $y < b$, i.e., $y \leq b$, as desired.

Having established that $A \cap \underline{ab} \subset \mathbb{R}$ is nonempty and bounded above, we can invoke Lemma 6.6 to learn that $A \cap \underline{ab}$ has a supremum $\sup(A \cap \underline{ab})$. We now divide into two cases ($\sup(A \cap \underline{ab}) \in A$ and $\sup(A \cap \underline{ab}) \in B$; it follows from the definitions of A and B that exactly one of these cases is true). Suppose first that $\sup(A \cap \underline{ab}) \in A$. Then since A is open, we have by Theorem 4.10 that there exists a region $\underline{ef}$ such that $\sup(A \cap \underline{ab}) \in \underline{ef}$ and $\underline{ef} \subset A$. It follows from the former condition that $\sup(A \cap \underline{ab}) < f$. Thus, by Lemma 6.10, there exists an object $z \in \mathbb{R}$ such that $e < \sup(A \cap \underline{ab}) < z < f < b$ (note that $f < b$ for the same reason that $d < b$). Consequently, $z \in \underline{ef}$, implying that $z \in A$, and $z \in \underline{ab}$. Thus, we have found an element of $A \cap \underline{ab}$ that is greater than $\sup(A \cap \underline{ab})$, contradicting Definitions 5.7 and 5.6. The proof is symmetric in the other case (except that we find an element of B less than $\sup(A \cap \underline{ab})$). $\square$

1/14: **Definition 6.8.** Let C be a continuum satisfying Axioms 1-4. Consider a subset $X \subset C$. We say that X is **dense** in C if every $p \in C$ is a limit point of X.

Lemma 6.9. *A subset $X \subset C$ is dense in C if and only if $\overline{X} = C$.*

Proof. Suppose first that $X \subset C$ is dense in C. To prove that $\overline{X} = C$, Definition 1.2 tells us that it will suffice to show that every point $p \in \overline{X}$ is an element of C and vice versa. Clearly, every element of $\overline{X}$ is an element of C. On the other hand, let p be an arbitrary element of C. Since X is dense in C, Definition 6.8 tells us that $p \in LP(X)$. Therefore, by Definitions 1.5 and 4.4, $p \in \overline{X}$.

Now suppose that $\overline{X} = C$. To prove that X is dense in C, Definition 6.8 tells us that it will suffice to show that every $p \in C$ is a limit point of X. Let p be an arbitrary element of C. By Corollary 5.4, this implies that $p \in LP(C)$. It follows that $p \in LP(\overline{X})$. Thus, by Definition 4.4, $p \in LP(X \cup LP(X))$. Consequently, by Theorem 3.20, $p \in LP(X)$ or $p \in LP(LP(X))$. We now divide into two cases. If $p \in LP(X)$, then we are done. On the other hand, if $p \in LP(LP(X))$, the lemma from Theorem 4.6 asserts that $p \in LP(X)$, and we are done again. $\square$

Our next goal is to prove that $\mathbb{Q}$ is dense in $\mathbb{R}$. Just to make sense of that statement, we need to decide how to think of $\mathbb{Q}$ as a subset of $\mathbb{R}$. For every rational number $q \in \mathbb{Q}$, define the corresponding real number as the Dedekind cut

$$i(q) = \{x \in \mathbb{Q} \mid x < q\}$$

For example, $\mathbf{0} = i(0)$. It can be verified that this gives a well-defined injective function $i : \mathbb{Q} \to \mathbb{R}$. We identify $\mathbb{Q}$ with its image $i(\mathbb{Q}) \subset \mathbb{R}$ so that the rational numbers $\mathbb{Q}$ are a subset of the real numbers $\mathbb{R}$. (Similarly, $\mathbb{N}$ and $\mathbb{Z}$ can be understood as subsets of $\mathbb{R}$.)

[2] We may use this lemma since it does not depend on this result, Definition 6.8, or Lemma 6.9.

Lemma 6.10. *Given $A, B \in \mathbb{R}$ with $A < B$, there exists $p \in \mathbb{Q}$ such that $A < i(p) < B$.*

Proof. Since $A < B$, Definition 6.4 tells us that $A \subsetneq B$. Thus, by Definition 1.3, there exists a point q such that $q \in B$ and $q \notin A$. Since $q \in B$ where B is a Dedekind cut, we have by Definition 6.1 that there exists a point $p \in B$ with $p > q$. Additionally, since $q \notin A$ implies that q is an upper bound of A by Lemma 6.2, we know by Definition 5.6 that $x \leq q$ for all $x \in A$. It follows since $q < p$ that $x \leq p$ for all $x \in A$, meaning by Definition 5.6 and Lemma 6.2 that $p \notin A$. Having established that $p, q \in B$, $p, q \notin A$, and $q < p$, we are now ready to prove that $A < i(p) < B$. Definition 6.4 tells us that we may do so by showing that $A \subsetneq i(p)$ and $i(p) \subsetneq B$. We will take this one argument at a time.

To show that $A \subsetneq i(p)$, Definition 1.3 tells us that it will suffice to verify that every element of A is an element of $i(p)$ and that there exists an element of $i(p)$ that is not an element of A. We treat the former statement first. As previously mentioned, $x \leq p$ for all $x \in A$. This combined with the fact that $p \notin A$ implies that $x < p$ for all $x \in A$. Thus, by the definition of $i(p)$, $x \in i(p)$ for all $x \in A$, as desired. As to the latter statement, since $q < p$, we have by the definition of $i(p)$ that $q \in i(p)$. However, we also know that $q \notin A$, as desired.

To show that $i(p) \subsetneq B$, we must verify symmetric arguments to before. For the former statement, let r be an arbitrary element of $i(p)$. Then by the definition of $i(p)$, $r < p$. Since $p \in B$ and $r \in \mathbb{Q}$ satisfy $r < p$, we have by Definition 6.1 that $r \in B$, as desired. As to the latter statement, p is clearly an element of B that is not an element of $i(p)$, as desired. ☐

1/19: **Theorem 6.11.** *$i(\mathbb{Q})$ is dense in $\mathbb{R}$.*

Proof. To prove that $i(\mathbb{Q})$ is dense in $\mathbb{R}$, Definition 6.8 tells us that it will suffice to show the every point $X \in \mathbb{R}$ is a limit point of $i(\mathbb{Q})$. Let X be an arbitrary element of $\mathbb{R}$. To show that $X \in LP(i(\mathbb{Q}))$, Definition 3.13 tells us that it will suffice to verify that for every region $\underline{AB}$ with $X \in \underline{AB}$, we have $\underline{AB} \cap (i(\mathbb{Q}) \setminus \{X\}) \neq \emptyset$. Let $\underline{AB}$ be an arbitrary region with $X \in \underline{AB}$. It follows by Definitions 3.10 and 3.6 that $A < X < B$. Thus, by Lemma 6.10, there exists $p \in \mathbb{Q}$ such that $A < i(p) < X < B$. By Definitions 3.6 and 3.10, $i(p) \in \underline{AB}$. By Definition 1.18, $i(p) \in i(\mathbb{Q})$. By Exercise 6.5, $i(p) < X$ implies that $i(p) \neq X$. Combining the last three results with Definitions 1.11 and 1.6, we have that $i(p) \in \underline{AB} \cap (i(\mathbb{Q}) \setminus \{X\})$, as desired. ☐

Corollary 6.12 (The Archimedean Property)**.** *Let $A \in \mathbb{R}$ be a positive real number. Then there exist nonzero natural numbers $n, m \in \mathbb{N}$ such that $i(\frac{1}{n}) < A < i(m)$.*

Proof. We will first prove that there exists a nonzero natural number n such that $i(\frac{1}{n}) < A$. We will then prove that there exists a nonzero natural number m such that $A < i(m)$. Let's begin.

Since $A \in \mathbb{R}$ is positive, we know that $0 < A$. Thus, by Lemma 6.10, there exists $\frac{p}{n} \in \mathbb{Q}$ such that $0 < i(\frac{p}{n}) < A$. As permitted by Exercise 3.9b, we choose $\frac{p}{n} \in \left[\frac{p}{n}\right]$ to be an object such that $0 < n$ (this also means that $n \in \mathbb{N}$). Consequently, by Scripts 2 and 3, we know that $0 < \frac{1}{n} \leq \frac{p}{n}$. It follows that $i(\frac{1}{n}) \leq i(\frac{p}{n})$ since $x \in i(\frac{1}{n})$ implies $x < \frac{1}{n} \leq \frac{p}{n}$ implies $x \in i(\frac{p}{n})$, implies $i(\frac{1}{n}) \subset i(\frac{p}{n})$. Therefore, $i(\frac{1}{n}) \leq i(\frac{p}{n}) < A$, as desired.

By Exercise 6.5, Axiom 3, and Definition 3.3, there exists a point $B \in \mathbb{R}$ such that $A < B$. It follows by Lemma 6.10 that there exists $\frac{m}{q} \in \mathbb{Q}$ such that $A < i(\frac{m}{q}) < B$. As before, let $\frac{m}{q}$ be an object such that $0 < q$. Consequently, by Scripts 2 and 3, we know that $0 < \frac{m}{q} \leq m$. Once again, for the same reasons as before, $i(\frac{m}{q}) \leq i(m)$. Therefore, $A < i(\frac{m}{q}) \leq i(m)$, as desired. ☐

Corollary 6.13. *$i(\mathbb{N})$ is an unbounded subset of $\mathbb{R}$.*

Proof. Suppose for the sake of contradiction that $i(\mathbb{N})$ is bounded above. Then by Definition 5.6, there exists a point $A \in \mathbb{R}$ such that $i(n) \leq A$ for all $n \in \mathbb{N}$. Note that A is a positive real number since $i(0) < i(0) \leq A$. But by Corollary 6.12, $A < i(n)$ for some $n \in \mathbb{N}$, a contradiction. ☐

1/21: **Corollary 6.14.** *If $A \in \mathbb{R}$ is a real number, then there is an integer n such that $i(n-1) \leq A < i(n)$.*

Proof. Let X be be the set of all integers z such that $i(z) \leq A$. Symbolically,

$$X = \{z \mid z \in \mathbb{Z} \text{ and } i(z) \leq A\}$$

Since $A \neq \emptyset$ by Definition 6.1, there exists a point $\frac{p}{q} \in \mathbb{Q}$ such that $\frac{p}{q} \in A$. As in Corollary 6.12, we let $q > 0$. It follows by Scripts 2 and 3 that if $p \geq 0$, then $0 \leq \frac{p}{q}$, i.e.[3], $i(0) \leq A$ and if $p < 0$, then $p \leq \frac{p}{q}$, i.e., $i(p) \leq A$. Thus, in either case, X is nonempty.

Now there exists a nonzero natural number m such that $A < i(m)$ (if $A \leq i(0)$, then $A < i(1)$; if $A > 0$, then apply Corollary 6.12). Let $f : X \to \mathbb{N}$ be defined by the rule

$$f(x) = m - x$$

By Script 1, f is an injective function, $f(X) \subset \mathbb{N}$, and $f(X)$ is nonempty (since X is nonempty). Thus, by the well-ordering principle (Additional Exercise 0.1), there is a least element, which we shall call y, in $f(X)$. Since f is injective, there exists exactly one object $n - 1 \in X$ such that $f(n-1) = y$.

By the definition of X, $i(n-1) \leq A$. To prove that $A < i(n)$, suppose for the sake of contradiction that $i(n) \leq A$. This coupled with the fact that $n \in \mathbb{Z}$ implies that $n \in X$. Thus, $f(n) \in f(X)$. But $f(n) = m - n < m - n + 1 = m - (n-1) = f(n-1)$, contradicting the fact that $f(n-1)$ is the least element of $f(X)$. $\square$

1/26: **Axiom 1.** *The continuum contains a countable dense subset.*

Definition 6.15. Let X and Y be sets with orderings $<_X$ an $<_Y$, respectively. A function $f : X \to Y$ is **order-preserving** if for all $r, s \in X$,

$$r <_X s \Longrightarrow f(r) <_Y f(s)$$

Note that the function $i : \mathbb{Q} \to \mathbb{R}$ discussed above is order-preserving.

Exercise 6.16. Let C satisfy Axioms 1-5. Let $K \subset C$ be a countable dense subset of C. Construct an order-preserving bijection $f : \mathbb{Q} \to K$.

Lemma.

a) *K satisfies Axiom 3.*

b) (Density Lemma) *For all $x, y \in K$, if $x < y$, then there exists a point $z \in K$ such that z is between x and y.*

Proof of a. To prove that K satisfies Axiom 3, we must verify that K has neither a first nor a last point. We will address the first point question first. Suppose for the sake of contradiction that K has a first point x. Then by Definition 3.3, $x \leq y$ for all $y \in K$. However, since C satisfies Axiom 3, there exists an object $a \in C$ such that $a < x$. Now consider the region $\underline{ax}$. We have by Corollary 5.3 that there exists a point $p \in \underline{ax}$. Additionally, we have by Script 3 that $\underline{ax} \cap K = \emptyset$. Thus, $\underline{ax} \cap (K \setminus \{p\}) = \emptyset$, implying by Definition 3.13 that $p \notin LP(K)$. But since $p \in C$ and $p \notin LP(K)$, we have by Definition 6.8 that K is not dense in C, a contradiction.

The proof is symmetric for last points. $\square$

Proof of b. Suppose for the sake of contradiction that that there exist $x, y \in K$ with $x < y$ such that no point $z \in K$ is between x and y. By Theorem 5.2, there exists $p \in C$ such that p is between x and y. Consequently, by Definition 3.10, $p \in \underline{xy}$. Additionally, we have by Script 3 that $\underline{xy} \cap K = \emptyset$. It follows that $\underline{xy} \cap (K \setminus \{p\}) = \emptyset$, implying by Definition 3.13 that $p \notin LP(K)$. But since $p \in C$ and $p \notin LP(K)$, we have by Definition 6.8 that K is not dense in C, a contradiction. $\square$

[3]For the same reasons as in Corollary 6.12.

Proof of Exercise 6.16. By Theorem 2.11, $\mathbb{Q}$ is countable, implying by Definition 1.35 that there exists a bijection $g : \mathbb{N} \to \mathbb{Q}$. The existence of this bijection means that we can refer to an arbitrary element q of $\mathbb{Q}$ by the number n for which $g(n) = q$; in another notation, we can refer to q as q_n. Thus, since every element of $\mathbb{Q}$ can be written as q_n for some $n \in \mathbb{N}$, we can write $\mathbb{Q} = \{q_1, q_2, \dots\}$. Similarly, we can express K as $K = \{k_1, k_2, \dots\}$. We will use this method of referring to the elements of $\mathbb{Q}$ to construct f.

We define f recursively with strong induction. For the base case q_1, we define $f(q_1) = k_1$. Now suppose inductively that we have defined $f(q_1), f(q_2), \dots, f(q_n)$; we now seek to define $f(q_{n+1})$. By Theorem 3.5, the symbols $a_1, \dots, a_{n+1}$ can be assigned to $q_1, \dots, q_{n+1}$ so that $a_1 <_{\mathbb{Q}} a_2 <_{\mathbb{Q}} \cdots <_{\mathbb{Q}} a_{n+1}$. We divide into three cases ($q_{n+1} = a_1$, $q_{n+1} = a_{n+1}$, and $q_{n+1} = a_i$ where $1 < i < n+1$). First, suppose that $q_{n+1} = a_1$. By the inductive hypothesis, $f(a_2), f(a_3), \dots, f(a_{n+1})$ are defined elements of K. At this point, define the set $X = \{k \in K \mid k <_K f(a_2)\}$. It follows by Lemma (a) that this set is nonempty. Thus, by the well-ordering principle, there exists a $k_i \in X$ such that $i \leq j$ for all $k_j \in X$. We let $f(q_{n+1}) = k_i$. The second case is symmetric to the first. Third, suppose that $q_{n+1} = a_i$ where $1 < i < n+1$. By the inductive hypothesis, $f(a_1), \dots, f(a_{i-1}), f(a_{i+1}), \dots, f(a_{n+1})$ are defined elements of K. At this point, define the set $X = \{k \in K \mid f(a_{i-1}) <_K k <_K f(a_{i+1})\}$. It follows by Lemma (b) that this set is nonempty. Thus, by the well-ordering principle, there exists a $k_i \in X$ such that $i \leq j$ for all $k_j \in X$. We let $f(q_{n+1}) = k_i$.

To prove that f is a function, Definition 1.16 tells us that it will suffice to show that for all $q \in \mathbb{Q}$, there exists a unique $k \in K$ such that $f(q) = k$. First, we will prove that for all $q \in \mathbb{Q}$, there exists *some* $k \in K$ such that $f(q) = k$. Let q_i be an arbitrary element of $\mathbb{Q}$. Then $i \in \mathbb{N}$, and by the principle of strong mathematical induction (Additional Exercise 0.2b), $f(q_i)$ is assigned to an element of k. As to proving the uniqueness of the k to which q_i is defined, each q is assigned once, in one of three mutually exclusive cases, to an unambiguously defined (as guaranteed by the well-ordering principle) element of K.

To prove that f is order-preserving, we will first verify the following claim (which we will refer to as Lemma (c) for future reference): Consider the set $\{q_1, \dots, q_n\} \subset \mathbb{Q}$; if the symbols $a_1, \dots, a_n$ are assigned to $q_1, \dots, q_n$ such that $a_1 <_{\mathbb{Q}} a_2 <_{\mathbb{Q}} \cdots <_{\mathbb{Q}} a_n$, then $f(a_1) <_K f(a_2) <_K \cdots <_K f(a_n)$. We will then use this result to prove that f is order-preserving for any two arbitrary elements $q_i, q_j \in \mathbb{Q}$. Let's begin.

To verify the above claim, we induct on n. The base case $n = 1$ is vacuously true. Now suppose inductively that we have proven the claim for n; we now seek to prove it for $n+1$. By Theorem 3.5, the symbols $a_1, \dots, a_{n+1}$ can be assigned to $q_1, \dots, q_{n+1}$ so that $a_1 <_{\mathbb{Q}} a_2 <_{\mathbb{Q}} \cdots <_{\mathbb{Q}} a_{n+1}$. We divide into three cases ($q_{n+1} = a_1$, $q_{n+1} = a_{n+1}$, and $q_{n+1} = a_i$ where $1 < i < n+1$). First, suppose that $q_{n+1} = a_1$. By the definition of f, $f(q_{n+1}) \in \{k \in K \mid k <_K f(a_2)\}$, meaning that $f(q_{n+1}) = f(a_1) <_K f(a_2)$. Additionally, by the inductive hypothesis, we know that $f(a_2) <_K f(a_3) <_K \cdots <_K f(a_{n+1})$ (since $a_2, \dots, a_{n+1}$ correspond to $q_1, \dots, q_n$). Together, these two results imply that $f(a_1) <_K f(a_2) <_K \cdots <_K f(a_{n+1})$. The proof of the second case is symmetric to that of the first. Third, suppose that $q_{n+1} = a_i$ where $1 < i < n+1$. By the definition of f, $f(q_{n+1}) \in \{k \in K \mid f(a_{i-1}) <_K k <_K f(a_{i+1})\}$, meaning that $f(a_{i-1}) <_K f(q_{n+1}) = f(a_i) <_K f(a_{i+1})$. Additionally, by the inductive hypothesis, we know that $f(a_1) <_K \cdots <_K f(a_{i-1}) <_K f(a_{i+1}) <_K \cdots <_K f(a_{n+1})$ (for an analogous reason to before). These two results imply that $f(a_1) <_K f(a_2) <_K \cdots <_K f(a_{n+1})$.

We are now ready to actually prove that f is order-preserving. To do so, Definition 6.15 tells us that it will suffice to show that for all $q_i, q_j \in \mathbb{Q}$, $q_i <_{\mathbb{Q}} q_j$ implies $f(q_i) <_K f(q_j)$. Let q_i, q_j be arbitrary elements of $\mathbb{Q}$ such that $q_i <_{\mathbb{Q}} q_j$. Since $q_i <_{\mathbb{Q}} q_j$, $q_i \neq q_j$, implying that $i \neq j$. We divide into two cases ($i < j$ and $i > j$). Suppose first that $i < j$. By Theorem 3.5, the symbols $a_1, \dots, a_j$ can be assigned to $q_1, \dots, q_j$ so that $a_1 <_{\mathbb{Q}} a_2 <_{\mathbb{Q}} \cdots <_{\mathbb{Q}} a_j$. Let $q_j = a_l$. Since $q_i <_{\mathbb{Q}} q_j$, we know that $q_i = a_m$ where $m < l$. Additionally, by Lemma (c), we know that $f(a_1) <_K f(a_2) <_K \cdots <_K f(a_j)$. It follows that $f(a_m) <_K f(a_l)$, implying that $f(q_i) <_K f(q_j)$, as desired. The proof is symmetric in the other case.

To prove that f is bijective, Definition 1.20 tells us that it will suffice to show that f is injective and surjective.

To show that f is injective, Definition 1.20 tells us that it will suffice to demonstrate that $q_i \neq q_j$ implies $f(q_i) \neq f(q_j)$. WLOG let $q_i <_{\mathbb{Q}} q_j$. Then since f is order-preserving, Definition 6.15 implies that $f(q_i) <_K f(q_j)$. It follows that $f(q_i) \neq f(q_j)$, as desired.

We are now ready to actually show that f is surjective. To do so, Definition 1.20 tells us that it will suffice to demonstrate that for all $k_n \in K$, there exists a $q_i \in \mathbb{Q}$ such that $f(q_i) = k_n$. To do this, we induct on n. For the base case $n = 1$, it follows from the definition of f that $f(q_1) = k_1$. Now suppose inductively that for each $k_1, \dots, k_n$, there exists a $q_i \in \mathbb{Q}$ such that $f(q_i) = k_n$; we now seek to prove the claim for $n+1$.

By Theorem 3.5, the symbols $b_1, \dots, b_{n+1}$ can be assigned to $k_1, \dots, k_{n+1}$ so that $b_1 <_K b_2 <_K \cdots <_K b_{n+1}$. We divide into three cases ($k_{n+1} = b_1$, $k_{n+1} = b_{n+1}$, and $k_{n+1} = b_i$ where $1 < i < n+1$). First, suppose that $k_{n+1} = b_1$. By the inductive hypothesis, $b_2 = f(q_i) <_K b_3 = f(q_j) <_K \cdots <_K b_{n+1} = f(q_l)$. It follows by Definition 6.15 that $q_i <_{\mathbb{Q}} q_j <_{\mathbb{Q}} \cdots <_{\mathbb{Q}} q_l$. At this point, define the set $X = \{q \in \mathbb{Q} \mid q <_{\mathbb{Q}} q_i\}$. It follows from Exercise 3.9d that this set is nonempty. Thus, by the well-ordering principle, there exists a $q_m \in X$ such that $m \leq m'$ for all $q_{m'} \in X$. By the definition of f, $f(q_m) = k_{n+1}$. The proof of the second case is symmetric to that of the first. Third, suppose that $k_{n+1} = b_i$ where $1 < i < n+1$. By the inductive hypothesis, $b_2 = f(q_j) <_K \cdots <_K b_{i-1} = f(q_{j'}) <_K b_{i+1} = f(q_l) <_K \cdots <_K b_{n+1} = f(q_{l'})$. It follows by Definition 6.15 that $q_j <_{\mathbb{Q}} \cdots <_{\mathbb{Q}} q_{j'} <_{\mathbb{Q}} q_l <_{\mathbb{Q}} \cdots <_{\mathbb{Q}} q_{l'}$. At this point, define the set $X = \{q \in \mathbb{Q} \mid q_{j'} <_{\mathbb{Q}} q <_{\mathbb{Q}} q_l\}$. It follows from Additional Exercise 3.1 that this set is nonempty. Thus, by the well-ordering principle, there exists a $q_m \in X$ such that $m \leq m'$ for all $q_{m'} \in X$. By the definition of f, $f(q_m) = k_{n+1}$. ☐

Exercise 6.17. Let $f : \mathbb{Q} \to K$ be an order-preserving bijection, as found in Exercise 6.16. Let $A \in \mathbb{R}$. Then $A \subset \mathbb{Q}$ and so $f(A) \subset K \subset C$. Define $F : \mathbb{R} \to C$ by

$$F(A) = \sup f(A)$$

1. Show $\sup f(A)$ exists, so F is well-defined.
2. Show F is injective and order-preserving.

Proof of 1. To prove that $\sup f(A)$ exists, Theorem 5.17 tells us that it will suffice to show that $f(A)$ is nonempty and bounded above. To show that $f(A)$ is nonempty, Definition 1.8 tells us that it will suffice to find an element of $f(A)$. By Definition 6.1, $A \neq \emptyset$. Thus, by Definition 1.8, there exists an object $x \in A$. It follows by Definition 1.18 that $f(x) \in f(A)$, as desired. To show that $f(A)$ is bounded above, Definition 5.6 tells us that it will suffice to find an element of K such that $f(x) \geq_C f(a)$ for all $f(a) \in f(A)$. By Definition 6.1, $A \neq \mathbb{Q}$ and $A \subset \mathbb{Q}$. Thus, by Definition 1.2, there exists an object $x \in \mathbb{Q}$ such that $x \notin A$. It follows from the latter condition by Lemma 6.2 that x is an upper bound for A. Thus, by Definition 5.6, $x \geq a$ for all $a \in A$. Consequently, by Definition 6.15, $f(x)$ is an element of K such that $f(x) \geq_C f(a)$ for all $f(a) \in f(A)$, as desired. ☐

Proof of 2. To prove that F is order-preserving, Definition 6.15 tells us that it will suffice to show that for all $A, B \in \mathbb{R}$, $A <_{\mathbb{R}} B$ implies $F(A) <_C F(B)$. Let A, B be two arbitrary elements of $\mathbb{R}$ satisfying $A <_{\mathbb{R}} B$. Then by Definitions 6.4 and 1.3, there exists a point $x \in B$ such that $x \notin A$. It follows from the latter condition by Lemma 6.2 and Definition 5.6 that $x \geq a$ for all $a \in A$. Thus, by Definition 6.15, $f(x) \geq_C f(a)$ for all $f(a) \in f(A)$. Consequently, by Definition 5.7, $\sup f(A) \leq_C f(x)$. Additionally, by Definition 6.1, there exists a point $y \in B$ such that $y > x$. Thus, by Definition 6.15, we have that $f(y) >_C f(x)$. It follows by Definitions 5.6 and 5.7 that $f(y) \leq_C \sup f(B)$. Combining two results, we therefore have that $\sup f(A) \leq_C f(x) <_C f(y) \leq_C \sup f(B)$, meaning that $F(A) = \sup f(A) <_C \sup f(B) = F(B)$, as desired.

To prove that F is injective, Definition 1.20 tells us that it will suffice to show that if $A \neq B$, then $F(A) \neq F(B)$. Let A, B be two distinct real numbers. Then by Exercise 6.5, $A < B$ or $B < A$. We now divide into two cases. Suppose first that $A < B$. Then $F(A) < F(B)$ by Definition 6.15 (which we have just proven applies to F). This implies by Definition 3.1 that $F(A) \neq F(B)$, as desired. The proof is symmetric in the other case. ☐

Theorem 6.18. *Suppose that C is a continuum satisfying Axioms 1-5. Then C is isomorphic to the real numbers $\mathbb{R}$; i.e., there is an order-preserving bijection $F : \mathbb{R} \to C$.*

Lemma. *Let K be a dense subset of C. For all $x, y \in C$, if $x < y$, then there exists a point $z \in K$ such that z is between x and y.*

Proof. Suppose for the sake of contradiction that there exist two points $x, y \in C$ with $x < y$ such that no point $z \in K$ is between x and y. By Corollary 5.3, the region $\underline{xy}$ is infinite. Thus, we can pick a point $p \in \underline{xy}$. Additionally, by Definition 1.6, we have that $\underline{xy} \cap K = \emptyset$. Thus, $\underline{xy} \cap (K \setminus \{p\}) = \emptyset$, implying by Definition 3.13 that $p \notin LP(K)$. But since $p \in C$ and $p \notin LP(K)$, we have by Definition 6.8 that K is not dense in C, a contradiction. ☐

Proof of Theorem 6.18. By Axiom 1, C contains a countable dense subset K. By Exercise 6.16, there exists an order-preserving bijection $f : \mathbb{Q} \to K$. By Exercise 6.17, there exists an order-preserving injection $F : \mathbb{R} \to C$. To prove that there is an order-preserving bijection $F : \mathbb{R} \to C$, all that is left to do is to demonstrate that F (as defined in Exercise 6.17) is surjective.

To do this, Definition 1.20 tells us that it will suffice to show that for all $X \in C$, there exists an object $A \in \mathbb{R}$ such that $F(A) = X$. Put more simply, we must find a Dedekind cut A such that $\sup f(A) = X$ for every $X \in C$. To do this, we will begin by constructing the set $S = \{k \in K \mid k < X\}$. We will then verify that the preimage $f^{-1}(S)$ is a Dedekind cut. Lastly, we will verify that $\sup f(f^{-1}(S)) = X$. Let's begin.

Let X be an arbitrary element of C. Define S as above. To verify that $f^{-1}(S)$ is a Dedekind cut, Definition 6.1 tells us that it will suffice to confirm that $f^{-1}(S) \neq \emptyset$; $f^{-1}(S) \neq \mathbb{Q}$; if $r \in f^{-1}(S)$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in f^{-1}(S)$; and if $r \in f^{-1}(S)$, then there is some $s \in f^{-1}(S)$ with $s > r$. We will take this one claim at a time.

To confirm that $f^{-1}(S) \neq \emptyset$, Definition 1.8 tells us that it will suffice to find an element of $f^{-1}(S)$. By Axiom 3 and Definition 3.3, there exists some point $Y \in C$ such that $Y < X$. Consequently, by the lemma and Definition 3.6, there exists a point $f(p) \in K$[4] such that $Y < f(p) < X$. It follows by the definition of S that $f(p) \in S$. Therefore, by Definition 1.18, $p \in f^{-1}(S)$, as desired.

To confirm that $f^{-1}(S) \neq \mathbb{Q}$, Definition 1.2 tells us that it will suffice to find an element of $\mathbb{Q}$ that is not an element of $f^{-1}(S)$. By Axiom 3 and Definition 3.3, there exists some point $Y \in C$ such that $X < Y$. Consequently, by the lemma and Definition 3.6, there exists a point $f(p) \in K$ such that $X < f(p) < Y$. It follows by the definition of S that $f(p) \notin S$. Therefore, by Definition 6.18, $p \in \mathbb{Q}$ but $p \notin f^{-1}(S)$, as desired.

To confirm that if $r \in f^{-1}(S)$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in f^{-1}(S)$, we let $r \in f^{-1}(S)$ and $s \in \mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s < r$ and seek to verify that $s \in f^{-1}(S)$. By Definition 1.18, the fact that $r \in f^{-1}(S)$ implies that $f(r) \in S$. Thus, by the definition of S, $f(r) < X$. Additionally, by the definition of f and Definition 6.15, $f(s) \in K$ and $f(s) < f(r)$, respectively. Since $f(s) < f(r)$ and $f(r) < X$, transitivity implies that $f(s) < X$. This combined with the previously established fact that $f(s) \in K$ implies that $f(s) \in S$. Therefore, by Definition 1.18, $s \in f^{-1}(S)$, as desired.

To confirm that if $r \in f^{-1}(S)$, then there is some $s \in f^{-1}(S)$ with $s > r$, we let $r \in f^{-1}(S)$ and seek to find such an s. As before, $r \in f^{-1}(S)$ implies that $f(r) \in S$. Thus, by the definition of S, $f(r) < X$. It follows by the lemma and Definition 3.6 that there exists a point $f(s) \in K$ such that $f(r) < f(s) < X$. Consequently, by the definition of S, we have that $f(s) \in S$. Therefore, by Definitions 1.18 and 6.15, $s \in f^{-1}(S)$ and $r < s$, respectively, as desired.

Since f is bijective, Script 1 asserts that $f(f^{-1}(S)) = S$. Thus, $\sup f(f^{-1}(S)) = \sup S$. To verify that $\sup S = X$, Definition 5.7 tells us that it will suffice to confirm that X is an upper bound of S and if U is an upper bound of S, $X \leq U$. To confirm the former statement, Definition 5.6 tells us that it will suffice to show that $k \leq X$ for all $k \in S$. But by the definition of S, this is true. To confirm the latter statement, suppose for the sake of contradiction that there exists an upper bound U of S such that $U < X$. Since $U < X$, the lemma and Definition 3.6 imply that there exists a point $Z \in K$ such that $U < Z < X$. It follows by the definition of S that $Z \in S$. Since there exists an element of S greater than U, Definition 5.6 asserts that U is not an upper bound of S, a contradiction. $\square$

6.2 Discussion

1/12:

- Upper limit at signing up for 4-5 across the script.
- Lemma 6.2 is probably more straightforward using a contradiction argument.
- Briefly restate the algebra of Exercise 4.24 in Exercise 6.3c.

1/14:

- Turning in Script 5 journals is optional — it will boost your grade a bit if you do.
  - Your journal grade will be whichever is higher: the average of all your journal grades with and without Script 5.

[4]Note that we know that the element of K (the existence of which is implied by the lemma) can be written in the form $f(p)$ because f is bijective.

  - Script 5 will probably be due Wednesday, 1/20.
- In Lemma 6.6, do we need to prove that the union of arbitrarily many Dedekind cuts is, itself, a Dedekind cut? Yes.

1/18:
- Is there a way to prove something else besides A is not open in Exercise 6.7?
  - This is probably it as far as proving that continuua are connected.
  - It may not be possible to prove that *any* of the statements are wrong, but he's not sure.
- Is Lemma 6.9 used in the proofs of any subsequent results, or is it just a less important result (hence the lemma designation)?
  - We can think of it as an alternate definition for density — we could prove Definition 6.8 from it.
- Is my handwavey use of Scripts 2 and 3 ok in Corollary 6.12?
  - I'm fine.
- Is there a simpler way to prove Corollaries 6.12 and 6.14?
  - hi
- Is the math REU still running this summer?
  - He's not sure; UChicago's may not be NSF approved, hence why its not on the website rn.
- What other summer opportunities would you recommend for a student at my level?
  - He did an REU at UWisconsin when he was an undergrad.
  - Sounds like its pretty much just REUs for undergrads.
  - I could ask around to see if anyone is a Knot Theorist/willing to sponsor me.

1/19:
- Easier Corollary 6.12:
  - Let $B > A$. Then $A < i(\frac{m}{q}) < B$. Then $A < i(m)$.
- Several proofs were given for Corollary 6.14. One other correct one constructed the nonempty, bounded above set of all $i(n)$ less than or equal to A and considered its supremum.

1/21:
- Now graded a bit more critically on presentations.
  - Write big, talk loudly, don't talk to the blackboard.
- My original proof of Corollary 6.14 is incorrect because I can't split into cases the way I did (*longer expo*).
  - Instead, use Seb's approach.

1/26:
- Stray thoughts on Exercise 6.16:
  - Any property we can prove for $\mathbb{Q}$ (e.g., betweenness, Axioms 1-3, etc.) we should be able to prove for K.
    * Many of these follow from $\mathbb{Q}$'s density! This is how we can make use of this condition.
  - We think of 0 as being somehow the "midpoint" of $\mathbb{Q}$. But since $\mathbb{Q}$ diverges in both directions, it doesn't really have a midpoint; we just assert this rather arbitrary structure on a more foundational algebraic construct.
    * The same would hold for K. Thus, we can choose an arbitrary point $x \in K$ and let it be the "midpoint," i.e., let $f(0) = x$.
  - Can we induct on the elements of $\mathbb{Q}$? Since there exists a bijection $\mathbb{Q} \to \mathbb{N}$.

- We can construct an order preserving bijection between any finite subsets of $\mathbb{Q}$ and K with equal cardinality.
- $f : \mathbb{Q} \to K$, $g : \mathbb{N} \to \mathbb{Q}$, $h : \mathbb{N} \to K$. If $g(n) < g(n')$, then $h(n) < h(n')$.
- Let $h(n) < h(n')$. WLOG let $n < n'$, too. Now consider $N = \{n \in \mathbb{N} \mid n \leq n'\}$. This is a finite set. Now create a new set $g(N)$. There will be an order-preserving bijection $\tilde{f} : h(N) \to g(N)$.
- Let $g : \mathbb{N} \to \mathbb{Q}$ be a bijection (we know one exists by countability). We presently seek to define $h : \mathbb{N} \to K$ recursively. Let x_1 be an arbitrary element of K (Axiom 1). We define $h(1) = x_1$. Now suppose inductively that we have defined $h(n)$. We now seek to define $h(n+1)$. Consider the set $A = \{g(m) \mid m \leq n+1\}$. By Theorem 3.5, we can assign the symbols $a_1, \dots, a_{n+1}$ to each point of A so that $a_1 < a_2 < \cdots < a_{n+1}$. We know that $g(n+1) = a_i$ for some $i \in [n+1]$. We divide into three cases ($g(n+1) = b_1$, $g(n+1) = b_{n+1}$, and $g(n+1) = b_i$ where $1 < i < n+1$). First, suppose that $g(n+1) = b_1$. By the inductive hypothesis, $h(g^{-1}(b_2)) \in K$. By Axiom 3, $h(g^{-1}(b_2))$ is not the first point of K. Thus, there exists an $x \in K$ such that $x < h(g^{-1}(b_2))$. Consequently, let $h(n+1) = x$. The proof of the second case is symmetric to that of the first. Third, suppose that $g(n+1) = b_i$ where $1 < i < n+1$. By the inductive hypothesis, $h(g^{-1}(b_{i-1})), h(g^{-1}(b_{i+1})) \in K$. Thus, there exists an $x \in K$ such that $h(b_{i-1}) < x < h(b_{i+1})$. Consequently, let $h(n+1) = x$.
- We define $f : \mathbb{Q} \to K$ by $f(p) = h(g^{-1}(p))$.
- Function diagram: The characteristic of an order preserving bijection is no intersections between lines connecting elements of different sets.

- Do we need to have subscripts on our orderings? Yes.
- The canonical way of doing Exercise 6.16 is with the **back and forth method**.
  - Because both are countable, $\mathbb{Q} = \{q_1, q_2, \dots\}$. Likewise, $K = \{k_1, k_2, \dots\}$.
  - To create the bijection, we have two repeating steps.
    1. Let i be the smallest index such that q_i has not been paired. Let j be an index such that k_j hasn't been paired, and assigning $f(q_i) = k_j$ preserves ordering (we have to prove that such a j exists). To prove this, we know that we can order the elements of $\mathbb{Q}$ that have already been paired. We can also order the elements of K that have already been paired. Case 1: q_i is between some preexisting q's. Then there exists some k_j between. Case 2: $q_i < \cdots < q_n$ implies there exists some k_j less than all other k so far. Case 3: q_i is a last element; symmetric to Case 2.
    2. Smallest j, smallest i such that order is preserved. Then we let $f(q_i) = k_j$.
    3. Repeat.
  - Injectivity: Suppose $f(q_i) = f(q_j)$. Each q_k is assigned to a unique k_k, so if they're equal, they must have been assigned at the same time. Therefore, $q_i = q_j$.
  - Surjectivity: Let $k_j \in K$. By jth step at most, k_j will be paired.
- Do summer research things every happen with graduate students, or is it just with professors? It pretty much only happens with professors, but DRP could be a good way to get your foot in the door.

Script 7

The Field Axioms

7.1 Journal

1/28: **Definition 7.1.** A **binary operation** on a set X is a function

$$f : X \times X \to X$$

We say that f is **associative** if

$$f(f(x,y),z) = f(x, f(y,z)) \quad \text{for all } x,y,z \in X$$

We say that f is **commutative** if

$$f(x,y) = f(y,x) \quad \text{for all } x,y \in X$$

An **identity element** of a binary operation f is an element $e \in X$ such that

$$f(x,e) = f(e,x) = x \quad \text{for all } x \in X$$

Remark 7.2. Frequently, we denote a binary operation differently. If $* : X \times X \to X$ is the binary operation, we often write $a * b$ in place of $*(a,b)$. We sometimes indicate this same operation by writing $(a,b) \mapsto a * b$.

Exercise 7.3. Rewrite Definition 7.1 using the notation of Remark 7.2.

Answer. A **binary operation** on a set X is a function

$$* : X \times X \to X$$

We say that $*$ is **associative** if

$$(x * y) * z = x * (y * z) \quad \text{for all } x,y,z \in X$$

We say that $*$ is **commutative** if

$$x * y = y * x \quad \text{for all } x,y \in X$$

An **identity element** of a binary operation $*$ is an element $e \in X$ such that

$$x * e = e * x = x \quad \text{for all } x \in X$$

□

Examples 7.4.

1. *The function $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ which sends a pair of integers (m,n) to $+(m,n) = m + n$ is a binary operation on the integers, called addition. Addition is associative, commutative, and has identity element 0.*

2. *The maximum of m and n, denoted* $\max(m,n)$*, is an associative and commutative binary operation on* $\mathbb{Z}$*. Is there an identity element for* $\max$*?*

   Proof. Suppose for the sake of contradiction that there exists an identity element e for max. But $\max(e-1,e) = e \neq e-1$, a contradiction. Therefore, no identity element exists for max. $\square$

3. *Let $\wp(Y)$ be the power set of a set Y. Recall that the power set consists of all subsets of Y. Then the intersection of sets, $(A,B) \mapsto A \cap B$, defines an associative and commutative binary operation on $\wp(Y)$. Is there an identity element for $\cap$?*

   Proof. Clearly, $Y \in \wp(Y)$. By Script 1, $Y \cap A = A \cap Y = A$ where $A \subset Y$. Therefore, Y is an identity element for $\cap$. $\square$

Exercise 7.5. Find a binary operation on a set that is not commutative. Find a binary operation on a set that is not associative.

Proof. We will prove that the subtraction operation on the integers ($- : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$) is neither commutative nor associative. To prove that it's not commutative, Definition 7.1 tells us that it will suffice to show that $x - y \neq y - x$ for some $x, y \in \mathbb{Z}$. Since $2-1 = 1$ but $1-2 = -1$, we can see that $1, 2 \in \mathbb{Z}$ clearly meet this requirement. To prove that it's not associative, Definition 7.1 tells us that it will suffice to show that $(x-y)-z \neq x-(y-z)$ for some $x,y,z \in \mathbb{Z}$. Since $(3-2)-1 = 0$ but $3-(2-1) = 2$, we can see that $1, 2, 3 \in \mathbb{Z}$ clearly meet this requirement. $\square$

Exercise 7.6. Let X be a finite set, and let $Y = \{f : X \to X \mid f \text{ is bijective}\}$. Consider the binary operation of composition of functions, denoted $\circ : Y \times Y \to Y$ and defined by $(f \circ g)(x) = f(g(x))$ as seen in Definition 1.25. Decide whether or not composition is commutative and/or associative and whether or not it has an identity.

Proof. To prove that composition is not commutative, Definition 7.1 tells us that it will suffice to find a finite set X paired with two bijections in Y that do not commute. Let $X = \{1,2,3\}$ and consider the bijections $f : X \to X$ (defined by $f(1) = 2$, $f(2) = 3$, $f(3) = 1$) and $g : X \to X$ (defined by $g(1) = 1$, $g(2) = 3$, $g(3) = 2$). In this case, $f \circ g$ would be defined by $f(g(1)) = 2$, $f(g(2)) = 1$, and $f(g(3)) = 3$, but $g \circ f$ would be defined by $g(f(1)) = 3$, $g(f(2)) = 2$, and $g(f(3)) = 1$.

To prove that composition is associative, Definition 7.1 tells us that it will suffice to show that $((f \circ g) \circ h)(x) = (f \circ (g \circ h))(x)$. We may do this with the following algebra.

$$\begin{aligned} ((f \circ g) \circ h)(x) &= (f \circ g)(h(x)) \\ &= f(g(h(x))) \\ &= f((g \circ h)(x)) \\ &= (f \circ (g \circ h))(x) \end{aligned}$$

With respect to any finite set X, there will always be a bijection $i : X \to X$ defined by $i(x) = x$. To prove that i is an identity element, Definition 7.1 tells us that it will suffice to show that for all $f \in Y$, $f \circ i = i \circ f = f$. We may do this with the following algebra.

$$\begin{aligned} (f \circ i)(x) &= f(i(x)) \\ &= f(x) \\ &= i(f(x)) \\ &= (i \circ f)(x) \end{aligned}$$

$\square$

Theorem 7.7. *Identity elements are unique. That is, suppose that f is a binary operation on a set X that has two identity elements e and e'. Then $e = e'$.*

Proof. Let $f : X \times X \to X$ be a binary operation on a set X with two identity elements e, e'. By Definition 7.1, we know that $f(e, e') = e$ and $f(e, e') = e'$. Since f is a well-defined function by definition, it must be that $e = f(e, e') = e'$. ☐

Definition 7.8. A **field** is a set F with two binary operations on F called addition, denoted $+$, and multiplication, denoted $\cdot$, satisfying the following **field axioms**:

FA1 (Commutativity of Addition) For all $x, y \in F$, $x + y = y + x$.

FA2 (Associativity of Addition) For all $x, y, z \in F$, $(x + y) + z = x + (y + z)$.

FA3 (Additive Identity) There exists an element $0 \in F$ such that $x + 0 = 0 + x = x$ for all $x \in F$.

FA4 (Additive Inverses) For any $x \in F$, there exists $y \in F$ such that $x + y = y + x = 0$, called an additive inverse of x.

FA5 (Commutativity of Multiplication) For all $x, y \in F$, $x \cdot y = y \cdot x$.

FA6 (Associativity of Multiplication) For all $x, y, z \in F$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

FA7 (Multiplicative Identity) There exists an element $1 \in F$ such that $x \cdot 1 = 1 \cdot x = x$ for all $x \in F$.

FA8 (Multiplicative Inverses) For any $x \in F$ such that $x \neq 0$, there exists $y \in F$ such that $x \cdot y = y \cdot x = 1$, called a multiplicative inverse of x.

FA9 (Distributivity of Multiplication over Addition) For all $x, y, z \in F$, $x \cdot (y + z) = x \cdot y + x \cdot z$.

FA10 (Distinct Additive and Multiplicative Identities) $1 \neq 0$.

Exercise 7.9. Consider the set $\mathbb{F}_2 = \{0, 1\}$, and define binary operations $+$ and $\cdot$ on $\mathbb{F}_2$ by

$$\begin{array}{cccc} 0+0=0 & 0+1=1 & 1+0=1 & 1+1=0 \\ 0\cdot 0=0 & 0\cdot 1=0 & 1\cdot 0=0 & 1\cdot 1=1 \end{array}$$

Show that $\mathbb{F}_2$ is a field.

Proof. To prove that $\mathbb{F}_2$ obeys FA1 from Definition 7.8, it will suffice to show that $0+0 = 0+0$, $0+1 = 1+0$, and $1+1 = 1+1$. The first and third of these are evidently true. For the second, we have $0+1 = 1 = 1+0$, so it is good, too.

To prove that $\mathbb{F}_2$ obeys FA2 from Definition 7.8, the following casework will suffice.

$$\begin{array}{ll} (0+0)+0 = 0 = 0+(0+0) & (0+0)+1 = 1 = 0+(0+1) \\ (0+1)+0 = 1 = 0+(1+0) & (1+0)+0 = 1 = 1+(0+0) \\ (0+1)+1 = 0 = 0+(1+1) & (1+1)+0 = 0 = 1+(1+0) \\ (1+0)+1 = 0 = 1+(0+1) & (1+1)+1 = 1 = 1+(1+1) \end{array}$$

To prove that $\mathbb{F}_2$ obeys FA3 from Definition 7.8, it will suffice to find an element $0 \in \mathbb{F}_2$ such that $x + 0 = 0 + x = x$. Since $0 + 0 = 0$, $1 + 0 = 0$, and by commutativity, it is clear that 0 is an additive identity in $\mathbb{F}_2$.

To prove that $\mathbb{F}_2$ obeys FA4 from Definition 7.8, it will suffice to show that for all $x \in \mathbb{F}_2$, there exists a $y \in \mathbb{F}_2$ such that $x + y = y + x = 0$. For 0, this object is 0 (since $0 + 0 = 0 + 0 = 0$), and for 1, this object is 1 (since $1 + 1 = 1 + 1 = 0$).

To prove that $\mathbb{F}_2$ obeys FA5 from Definition 7.8, it will suffice to show that $0 \cdot 0 = 0 \cdot 0$, $0 \cdot 1 = 1 \cdot 0$, and $1 \cdot 1 = 1 \cdot 1$. The first and third of these are evidently true. For the second, we have $0 \cdot 1 = 0 = 1 \cdot 0$, so it is good, too.

To prove that $\mathbb{F}_2$ obeys FA6 from Definition 7.8, the following casework will suffice.

$$\begin{aligned}
(0\cdot 0)\cdot 0 &= 0 = 0\cdot(0\cdot 0) & (0\cdot 0)\cdot 1 &= 0 = 0\cdot(0\cdot 1)\\
(0\cdot 1)\cdot 0 &= 0 = 0\cdot(1\cdot 0) & (1\cdot 0)\cdot 0 &= 0 = 1\cdot(0\cdot 0)\\
(0\cdot 1)\cdot 1 &= 0 = 0\cdot(1\cdot 1) & (1\cdot 1)\cdot 0 &= 0 = 1\cdot(1\cdot 0)\\
(1\cdot 0)\cdot 1 &= 0 = 1\cdot(0\cdot 1) & (1\cdot 1)\cdot 1 &= 1 = 1\cdot(1\cdot 1)
\end{aligned}$$

To prove that $\mathbb{F}_2$ obeys FA7 from Definition 7.8, it will suffice to find an element $1 \in \mathbb{F}_2$ such that $x \cdot 1 = 1 \cdot x = x$. Since $0 \cdot 1 = 0$, $1 \cdot 1 = 1$, and by commutativity, it is clear that 1 is a multiplicative identity in $\mathbb{F}_2$.

To prove that $\mathbb{F}_2$ obeys FA8 from Definition 7.8, it will suffice to show that for all $x \in \mathbb{F}_2$ such that $x \neq 0$, there exists a $y \in \mathbb{F}_2$ such that $x \cdot y = y \cdot x = 1$. For 1, this object is 1 (since $1 \cdot 1 = 1 \cdot 1 = 1$).

To prove that $\mathbb{F}_2$ obeys FA9 from Definition 7.8, the following casework will suffice.

$$\begin{aligned}
0\cdot(0+0) &= 0 = 0\cdot 0+0\cdot 0 & 0\cdot(0+1) &= 0 = 0\cdot 0+0\cdot 1\\
0\cdot(1+0) &= 0 = 0\cdot 1+0\cdot 0 & 1\cdot(0+0) &= 0 = 1\cdot 0+1\cdot 0\\
0\cdot(1+1) &= 0 = 0\cdot 1+0\cdot 1 & 1\cdot(1+0) &= 1 = 1\cdot 1+1\cdot 0\\
1\cdot(0+1) &= 1 = 1\cdot 0+1\cdot 1 & 1\cdot(1+1) &= 0 = 1\cdot 1+1\cdot 1
\end{aligned}$$

To prove that $\mathbb{F}_2$ obeys FA10 from Definition 7.8, it will suffice to show that $0 \neq 1$. Clearly this is true. □

Theorem 7.10. *Suppose that F is a field. Then additive inverses are unique. This means: Let $x \in F$. If $y, y' \in F$ satisfy $x + y = 0$ and $x + y' = 0$, then $y = y'$.*

Proof. Let $x, y, y' \in F$ be such that $x + y = 0$ and $x + y' = 0$. From Definition 7.8, we have

$$\begin{aligned}
y' + (x + y) &= (y' + x) + y && \text{FA2}\\
y' + 0 &= 0 + y && \text{FA4}\\
y' &= y && \text{FA3}
\end{aligned}$$

□

We usually write $-x$ for the additive inverse of x.

Corollary 7.11. *If $x \in F$, then $-(-x) = x$.*

Proof. Let $x \in F$. Then by consecutive applications of FA4 from Definition 7.8, $-x + (-(-x)) = 0$ and $-x + x = 0$. Therefore, by Theorem 7.10, we have that $-(-x) = x$. □

Theorem 7.12. *Let F be a field, and let $a, b, c \in F$. If $a + b = a + c$, then $b = c$.*

Proof. Let $a, b, c \in F$ be such that $a + b = a + c$. By FA4 from Definition 7.8, there exists $-a \in F$ such that $-a + a = a + (-a) = 0$. Having established that $-a$ exists, we can prove from Definition 7.8 that

$$\begin{aligned}
-a + (a + b) &= -a + (a + c)\\
(-a + a) + b &= (-a + a) + c && \text{FA2}\\
0 + b &= 0 + c && \text{FA4}\\
b &= c && \text{FA3}
\end{aligned}$$

□

Theorem 7.13. *Let F be a field. If $a \in F$, then $a \cdot 0 = 0$.*

Proof. Let $a \in F$. From Definition 7.8, we have

$$\begin{aligned} a &= a \cdot 1 && \text{FA7} \\ &= a \cdot (1+0) && \text{FA3} \\ &= a \cdot 1 + a \cdot 0 && \text{FA9} \\ &= a + a \cdot 0 && \text{FA7} \\ 0 &= a \cdot 0 && \text{Theorem 7.12} \end{aligned}$$

□

2/2: **Theorem 7.14.** *Suppose that F is a field. Then multiplicative inverses are unique. This means: Let $x \in F$. If $y, y' \in F$ satisfy $x \cdot y = 1$ and $x \cdot y' = 1$, then $y = y'$.*

Proof. Let $x, y, y' \in F$ be such that $x \cdot y = 1$ and $x \cdot y' = 1$. From Definition 7.8, we have

$$\begin{aligned} (y \cdot x) \cdot y' &= y \cdot (x \cdot y') && \text{FA6} \\ 1 \cdot y' &= y \cdot 1 && \text{FA8} \\ y' &= y && \text{FA7} \end{aligned}$$

□

We usually write x^{-1} or $\frac{1}{x}$ for the multiplicative inverse of x.

Corollary 7.15. *If $x \in F$ and $x \neq 0$, then $(x^{-1})^{-1} = x$.*

Proof. Let $x \in F \setminus \{0\}$. Then by FA8 from Definition 7.8, there exists $x^{-1} \in F$ such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$. It follows from Theorem 7.13 that $x^{-1} \neq 0$ (if $x^{-1} = 0$, then Theorem 7.13 would imply that $1 = x \cdot x^{-1} = 0$, contradicting FA10). Thus, by FA8 from Definition 7.8 again, there exists $(x^{-1})^{-1} \in F$ such that $x^{-1} \cdot (x^{-1})^{-1} = (x^{-1})^{-1} \cdot x^{-1} = 1$. Having established that $(x^{-1})^{-1}$ exists, $x^{-1} \cdot (x^{-1})^{-1} = 1$, and $x^{-1} \cdot x = 1$, we have by Theorem 7.14 that $(x^{-1})^{-1} = x$. □

Theorem 7.16. *Let F be a field, and let $a, b, c \in F$. If $a \cdot b = a \cdot c$ and $a \neq 0$, then $b = c$.*

Proof. Let $a, b, c \in F$ be such that $a \cdot b = a \cdot c$ and $a \neq 0$. By FA8 from Definition 7.8, there exists $a^{-1} \in F$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$. Having established that a^{-1} exists, we can prove from Definition 7.8 that

$$\begin{aligned} a^{-1} \cdot (a \cdot b) &= a^{-1} \cdot (a \cdot c) && \\ (a^{-1} \cdot a) \cdot b &= (a^{-1} \cdot a) \cdot c && \text{FA6} \\ 1 \cdot b &= 1 \cdot c && \text{FA8} \\ b &= c && \text{FA7} \end{aligned}$$

□

Theorem 7.17. *Let F be a field, and let $a, b \in F$. If $a \cdot b = 0$, then $a = 0$ or $b = 0$.*

Proof. Let $a, b \in F$ be such that $a \cdot b = 0$, and suppose for the sake of contradiction that $a \neq 0$ and $b \neq 0$. It follows from the supposition by consecutive applications of FA8 from Definition 7.8 that a^{-1} and b^{-1} exist. Thus, from Definition 7.8, we have

$$\begin{aligned} 1 &= 1 \cdot 1 && \text{FA7} \\ &= (a \cdot a^{-1}) \cdot (b \cdot b^{-1}) && \text{FA8} \\ &= (a \cdot b) \cdot (a^{-1} \cdot b^{-1}) && \text{FA6 and FA7} \\ &= 0 \cdot (a^{-1} \cdot b^{-1}) && \text{Substitution} \\ &= 0 && \text{Theorem 7.13} \end{aligned}$$

But this contradicts FA10 from Definition 7.8. □

Lemma 7.18. *Let F be a field. If $a \in F$, then $-a = (-1)a$.*

Proof. Let $a \in F$. From Definition 7.8, we have

$$\begin{aligned} 0 &= a \cdot 0 && \text{Theorem 7.13} \\ a + (-a) &= a \cdot (1 + (-1)) && \text{FA4} \\ a + (-a) &= a \cdot 1 + a \cdot (-1) && \text{FA9} \\ a + (-a) &= a + a \cdot (-1) && \text{FA7} \\ a + (-a) &= a + (-1)a && \text{FA5} \\ -a &= (-1)a && \text{Theorem 7.12} \end{aligned}$$

□

Lemma 7.19. *Let F be a field. If $a, b \in F$, then $a \cdot (-b) = -(a \cdot b) = (-a) \cdot b$.*

Proof. Let $a, b \in F$. From Definition 7.8, we have

$$\begin{aligned} a \cdot (-b) &= a \cdot ((-1) \cdot b) && \text{Lemma 7.18} \\ &= a \cdot (b \cdot (-1)) && \text{FA5} \\ &= (a \cdot b) \cdot (-1) && \text{FA6} \\ &= (-1) \cdot (a \cdot b) && \text{FA5} \\ &\boxed{= -(a \cdot b)} && \text{Lemma 7.18} \\ &= (-1) \cdot (a \cdot b) && \text{Lemma 7.18} \\ &= ((-1) \cdot a) \cdot b && \text{FA6} \\ &\boxed{= (-a) \cdot b} && \text{Lemma 7.18} \end{aligned}$$

□

Lemma 7.20. *Let F be a field. If $a, b \in F$, then $a \cdot b = (-a) \cdot (-b)$.*

Proof. Let $a, b \in F$. Thus, we have

$$\begin{aligned} (-a) \cdot (-b) &= -(-a) \cdot b && \text{Lemma 7.19} \\ &= a \cdot b && \text{Corollary 7.11} \end{aligned}$$

□

Definition 7.21. An **ordered field** is a field F equipped with an ordering $<$ (satisfying Definition 3.1) such that also:

(a) Addition respects the ordering: if $x < y$, then $x + z < y + z$ for all $z \in F$.

(b) Multiplication respects the ordering: if $0 < x$ and $0 < y$, then $0 < x \cdot y$.

Definition 7.22. Suppose F is an ordered field and $x \in F$. If $0 < x$, we say that x is **positive**. If $x < 0$, we say that x is **negative**.

Lemma 7.23. *Let F be an ordered field, and let $x \in F$. If $0 < x$, then $-x < 0$. Similarly, if $x < 0$, then $0 < -x$.*

Proof. Let $x \in F$ be such that $0 < x$. Then by Definition 7.21a, $0 + (-x) < x + (-x)$. Consequently, from Definition 7.8, we have

$$\begin{aligned} -x &< x + (-x) && \text{FA3} \\ -x &< 0 && \text{FA4} \end{aligned}$$

The proof is symmetric if $x < 0$. □

Lemma 7.24. *Let F be an ordered field, and let $x, y, z \in F$.*

(a) *If $x > 0$ and $y < z$, then $x \cdot y < x \cdot z$.*

(b) *If $x < 0$ and $y < z$, then $x \cdot z < x \cdot y$.*

Proof of a. Let $x, y, z \in F$ be such that $x > 0$ and $y < z$. It follows from the latter condition by Definition 7.21a that $y + (-y) < z + (-y)$. Thus, by FA4 from Definition 7.8, we have $0 < z + (-y)$. This combined with the fact that $0 < x$ implies by Definition 7.21b that $0 < x \cdot (z + (-y))$. Consequently, from Definition 7.8, we have

$$
\begin{aligned}
0 &< x \cdot z + x \cdot (-y) && \text{FA9} \\
0 &< x \cdot z + (-(x \cdot y)) && \text{Lemma 7.19} \\
0 + x \cdot y &< (x \cdot z + (-(x \cdot y))) + x \cdot y && \text{Definition 7.21a} \\
0 + x \cdot y &< x \cdot z + (-(x \cdot y) + x \cdot y) && \text{FA2} \\
0 + x \cdot y &< x \cdot z + 0 && \text{FA4} \\
x \cdot y &< x \cdot z && \text{FA3}
\end{aligned}
$$

□

Proof of b. Let $x, y, z \in F$ be such that $x < 0$ and $y < z$. It follows from the former condition by Lemma 7.23 that $0 < -x$. Thus, by Lemma 7.24a, $(-x) \cdot y < (-x) \cdot z$. Consequently, from Definition 7.8, we have

$$
\begin{aligned}
-(x \cdot y) &< -(x \cdot z) && \text{Lemma 7.19} \\
-(x \cdot y) + (x \cdot y + x \cdot z) &< -(x \cdot z) + (x \cdot y + x \cdot z) && \text{Definition 7.21a} \\
-(x \cdot y) + (x \cdot y + x \cdot z) &< -(x \cdot z) + (x \cdot z + x \cdot y) && \text{FA1} \\
(-(x \cdot y) + x \cdot y) + x \cdot z &< (-(x \cdot z) + x \cdot z) + x \cdot y && \text{FA2} \\
0 + x \cdot z &< 0 + x \cdot y && \text{FA4} \\
x \cdot z &< x \cdot y && \text{FA3}
\end{aligned}
$$

□

Remark 7.25. An immediate consequence of this lemma is the fact that if x and y are both positive or both negative, their product is positive.

Lemma 7.26. *Let F be an ordered field, and let $x \in F$. Then $0 \leq x^2$. Moreover, if $x \neq 0$, then $0 < x^2$.*

Proof. We divide into two cases ($x = 0$ and $x \neq 0$). Suppose first that $x = 0$. Then by Theorem 7.13, $0 \leq 0 = 0 \cdot 0 = 0^2 = x^2$, as desired. Now suppose that $x \neq 0$. We divide into two cases again ($x > 0$ and $x < 0$). If $x > 0$, then by Lemma 7.24a, $x > 0$ and $0 < x$ imply that $x \cdot 0 < x \cdot x$, from which it follows by Theorem 7.13 that $0 < x^2$, as desired. On the other hand, if $x < 0$, then by Lemma 7.24b, $x < 0$ and $x < 0$ imply that $x \cdot 0 < x \cdot x$, from which it follows for the same reason as before that $0 < x^2$, as desired. Both of the original two cases together prove the first statement, while the second original case alone proves the second statement. □

Corollary 7.27. *Let F be an ordered field. Then $0 < 1$.*

Proof. By FA10 from Definition 7.8, $1 \neq 0$. Thus, by Lemma 7.26, $0 < 1^2 = 1$, as desired. □

Theorem 7.28. *If F is an ordered field, then F has no first or last point.*

Proof. Suppose for the sake of contradiction that F has a first point a. By Corollary 7.27, we have that $0 < 1$, which implies by Lemma 7.23 that $-1 < 0$. It follows by Definition 7.21a that $-1 + a < 0 + a$. Thus, by FA3 from Definition 7.8, $-1 + a < a$. Since there exists an object in F (namely $-1 + a$) that is less than a, Definition 3.3 tells us that a is not the first point of F, a contradiction.

The proof is symmetric in the other case. □

Theorem 7.29. *The rational numbers $\mathbb{Q}$ form an ordered field.*

Proof. To prove that $\mathbb{Q}$ forms an ordered field, Definition 7.21 tells us that it will suffice to show that $\mathbb{Q}$ forms a field; has an ordering $<$; satisfies $x+z<y+z$ if $x<y$ for all $z \in \mathbb{Q}$; and satisfies $0<x \cdot y$ if $0<x$ and $0<y$. We will take this one constraint at a time.

To show that $\mathbb{Q}$ forms a field, Definition 7.8 tells us that it will suffice to verify that $\mathbb{Q}$ has two binary operations ($+$ and $\cdot$), and satisfies field axioms 1-10. Define $+$ and $\cdot$ as in Definition 2.7. Under these definitions, parts a-i of Theorem 2.10 guarantee that $\mathbb{Q}$ satisfies FA1-FA9, respectively. As to FA10, to verify that $\left[\frac{1}{1}\right] \neq \left[\frac{0}{1}\right]$, Exercise 2.6 tells us that it will suffice to confirm that $(1,1) \not\sim (1,0)$. But since $1 \cdot 0 = 0 \neq 1 = 1 \cdot 1$, Exercise 2.2e confirms that $(1,1) \not\sim (1,0)$, as desired.

$\mathbb{Q}$ has an ordering by Exercise 3.9d, as desired.

To show that $x+z<y+z$ if $x<y$ for all $z \in \mathbb{Q}$, let $\left[\frac{a}{b}\right], \left[\frac{c}{d}\right], \left[\frac{x}{z}\right]$ be arbitrary elements of $\mathbb{Q}$ with positive denominators (we can choose these WLOG by Exercise 3.9b) and the first two satisfying $\left[\frac{a}{b}\right] < \left[\frac{c}{d}\right]$; we seek to verify that $\left[\frac{a}{b}\right] + \left[\frac{x}{z}\right] < \left[\frac{c}{d}\right] + \left[\frac{x}{z}\right]$. Since $\left[\frac{a}{b}\right] < \left[\frac{c}{d}\right]$, we have by Exercise 3.9c that $ad<bc$. It follows by Script 0 that

$$\begin{aligned} ad &< bc \\ adzz &< bczz \\ adzz + bdxz &< bczz + bdxz \\ azdz + bxdz &< bzcz + bzdx \\ (az+bx)(dz) &< (bz)(cz+dx) \end{aligned}$$

Thus, by Exercise 3.9c, $\left[\frac{az+bx}{bz}\right] < \left[\frac{cz+dx}{dz}\right]$. Therefore, by Definition 2.7, $\left[\frac{a}{b}\right] + \left[\frac{x}{z}\right] < \left[\frac{c}{d}\right] + \left[\frac{x}{z}\right]$, as desired.

To show that $0<x \cdot y$ if $0<x$ and $0<y$, let $\left[\frac{a}{b}\right], \left[\frac{c}{d}\right]$ be arbitrary elements of $\mathbb{Q}$ with positive denominators (which we can choose for the same reason as before) and such that $\left[\frac{0}{1}\right] < \left[\frac{a}{b}\right]$ and $\left[\frac{0}{1}\right] < \left[\frac{c}{d}\right]$; we seek to verify that $\left[\frac{0}{1}\right] < \left[\frac{a}{b}\right] \cdot \left[\frac{c}{d}\right]$. Since $\left[\frac{0}{1}\right] < \left[\frac{a}{b}\right]$ and $\left[\frac{0}{1}\right] < \left[\frac{c}{d}\right]$, we have by Exercise 3.9c that $0 \cdot b < 1 \cdot a$ and $0 \cdot d < 1 \cdot c$. It follows by Script 0 that $0 \cdot bd < 1 \cdot ac$. Thus, by Exercise 3.9c, $\left[\frac{0}{1}\right] < \left[\frac{ac}{bd}\right]$. Therefore, by Definition 2.7, $\left[\frac{0}{1}\right] < \left[\frac{a}{b}\right] \cdot \left[\frac{c}{d}\right]$, as desired. □

2/4: **Definition 7.31.** We define $\oplus$ on $\mathbb{R}$ as follows. Let $A, B \in \mathbb{R}$ be Dedekind cuts. Define

$$A \oplus B = \{a+b \mid a \in A \text{ and } b \in B\}$$

Exercise 7.32.

(a) Prove that $A \oplus B$ is a Dedekind cut.

(b) Prove that $\oplus$ is commutative and associative.

(c) Prove that if $A \in \mathbb{R}$, then $A = \mathbf{0} \oplus A$.

Proof of a. To prove that $A \oplus B$ is a Dedekind cut, Definition 6.1 tells us that it will suffice to show that $A \oplus B \neq \emptyset$; $A \oplus B \neq \mathbb{Q}$; if $r \in A \oplus B$ and $s \in \mathbb{Q}$ satisfy $s<r$, then $s \in A \oplus B$; and if $r \in A \oplus B$, then there is some $s \in A \oplus B$ with $s>r$. We will take this one claim at a time.

To show that $A \oplus B \neq \emptyset$, Definition 1.8 tells us that it will suffice to find an element of $A \oplus B$. Since A, B are Dedekind cuts, Definition 6.1 asserts that they are nonempty. Thus, there exist rational numbers $x \in A$ and $y \in B$. Therefore, by the definition of $A \oplus B$, the sum $x+y \in A \oplus B$, as desired.

To show that $A \oplus B \neq \mathbb{Q}$, Definition 1.2 tells us that it will suffice to find an element of $\mathbb{Q}$ that is not an element of $A \oplus B$. For an analogous reason to before, we can choose $x, y \in \mathbb{Q}$ such that $x \notin A$ and $y \notin B$. It follows by Lemma 6.2 and Definition 5.6 that $x \geq a$ for all $a \in A$ and $y \geq b$ for all $b \in B$. Additionally, since $x \notin A$, we have that $x \neq a$ for any $a \in A$; thus, $x>a$ for all $a \in A$. Similarly, $y>b$ for all $b \in B$. Consequently, by Script 2, $x+y>a+b$ for all $a+b \in A \oplus B$. Therefore, $x+y \notin A \oplus B$, as desired.

To show that if $r \in A \oplus B$ and $s \in \mathbb{Q}$ satisfy $s<r$, then $s \in A \oplus B$, we let $r \in A \oplus B$ and $s \in \mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s<r$ and seek to verify that $s \in A \oplus B$. Since $r \in A \oplus B$, $r=x+y$ for some $x \in A$ and $y \in B$. Additionally, it follows from the fact that $s<r$ that

$s = r - q = x + y - q$ for some $q \in \mathbb{Q}^+$. Since $y \in B$ and $y - q \in \mathbb{Q}$ satisfy $y - q < y$, we have by Definition 6.1b that $y - q \in B$. Therefore, $s = (x) + (y - q)$ is an element of $A \oplus B$, as desired.

To show that if $r \in A \oplus B$, then there is some $s \in A \oplus B$ with $s > r$, we let $r \in A \oplus B$ and seek to find such an s. Since $r \in A \oplus B$, $r = x + y$ for some $x \in A$ and $y \in B$. It follows from the fact that $x \in A$ by Definition 6.1c that there exists a $z \in A$ with $z > x$. Consequently, by Script 0, $z + y > x + y$ is the desired element of $A \oplus B$. ☐

Proof of b. To prove that $\oplus$ is commutative, Definition 7.1 tells us that it will suffice to show that for all $A, B \in \mathbb{R}$, we have $A \oplus B = B \oplus A$. Let A, B be arbitrary elements of $\mathbb{R}$. Then by Definition 7.31, we clearly have

$$\begin{aligned} A \oplus B &= \{a + b \mid a \in A \text{ and } b \in B\} \\ &= \{b + a \mid b \in B \text{ and } a \in A\} \\ &= B \oplus A \end{aligned}$$

To prove that $\oplus$ is associative, Definition 7.1 tells us that it will suffice to show that for all $A, B, C \in \mathbb{R}$, we have $(A \oplus B) \oplus C = A \oplus (B \oplus C)$. Let A, B, C be arbitrary elements of $\mathbb{R}$. Then by Definition 7.31, we clearly have

$$\begin{aligned} (A \oplus B) \oplus C &= \{a + b \mid a \in A \text{ and } b \in B\} \oplus C \\ &= \{d + c \mid d \in \{a + b \mid a \in A \text{ and } b \in B\} \text{ and } c \in C\} \\ &= \{d + c \mid d = a + b \text{ for some } a \in A \text{ and } b \in B, \text{ and } c \in C\} \\ &= \{a + b + c \mid a \in A \text{ and } b \in B \text{ and } c \in C\} \\ &= \{a + e \mid a \in A, \text{ and } e = b + c \text{ for some } b \in B \text{ and } c \in C\} \\ &= \{a + e \mid c \in C \text{ and } e \in \{b + c \mid b \in B \text{ and } c \in C\}\} \\ &= A \oplus \{b + c \mid b \in B \text{ and } c \in C\} \\ &= A \oplus (B \oplus C) \end{aligned}$$

Note that we also make use of Exercise 7.32a to guarantee $A \oplus B \in \mathbb{R}$, so that we can apply $\oplus$ to $A \oplus B$ and C. We similarly invoke Exercise 7.32a to take the sum of A and $B \oplus C$. ☐

Proof of c. To prove that for all $A \in \mathbb{R}$, $A = \mathbf{0} \oplus A$, we will show for an arbitrary $A \in \mathbb{R}$ that every element of A is an element of $\mathbf{0} \oplus A$ and vice versa. Let A be an arbitrary element of $\mathbb{R}$. Suppose first that $x \in A$. Then by Definition 6.1c, there exists $y \in A$ such that $y > x$. Let $z = x - y$. Clearly, $z \in \mathbb{Q}$ and $z < 0$, so we know that $z \in \mathbf{0}$. Additionally, since $x - z = y$, we know that $x - z \in A$. Therefore, since $x = (z) + (x - z)$, we have by Definition 7.31 that $x \in \mathbf{0} \oplus A$. Now suppose that $z \in \mathbf{0} \oplus A$. Then by Definition 7.31, $z = x + y$ for some $x \in \mathbf{0}$ and $y \in A$. Since $x \in \mathbf{0}$, we know that $x < 0$, which means that $y > z$. This combined with the fact that $y \in A$ and $z \in \mathbb{Q}$ implies by Definition 6.1b that $z \in A$. ☐

2/9: **Definition 7.39.** For $A, B \in \mathbb{R}$, $\mathbf{0} < A$, $\mathbf{0} < B$, we define

$$A \otimes B = \{r \in \mathbb{Q} \mid r \leq 0\} \cup \{ab \mid a \in A, b \in B, a > 0, b > 0\}$$

If $A = \mathbf{0}$ or $B = \mathbf{0}$, we define $A \otimes B = \mathbf{0}$. If $A < \mathbf{0}$ but $\mathbf{0} < B$, we replace A with $-A$ and use the definition of multiplication of positive elements. Hence, in this case,

$$A \otimes B = -[(-A) \otimes B]$$

Similarly, if $\mathbf{0} < A$ but $B < \mathbf{0}$, then

$$A \otimes B = -[A \otimes (-B)]$$

and if $A < \mathbf{0}$, $B < \mathbf{0}$, then

$$A \otimes B = (-A) \otimes (-B)$$

Exercise 7.40. [1]

(a) Show that if $A, B \in \mathbb{R}$, then $A \otimes B \in \mathbb{R}$.

(b) Show that $\otimes$ is commutative and associative.

(c) Show that if $A, B \in \mathbb{R}$, $\mathbf{0} < A$, and $\mathbf{0} < B$, then $\mathbf{0} < A \otimes B$.

(d) Let $\mathbf{1} = \{x \in \mathbb{Q} \mid x < 1\}$. Show that if $A \in \mathbb{R}$, then $\mathbf{1} \otimes A = A$.

Proof of a. To prove that $A \otimes B$ where $\mathbf{0} < A, \mathbf{0} < B$ are Dedekind cuts, Definition 6.1 tells us that it will suffice to show that $A \otimes B \neq \emptyset$; $A \otimes B \neq \mathbb{Q}$; if $r \in A \otimes B$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A \otimes B$; and if $r \in A \otimes B$, then there is some $s \in A \otimes B$ with $s > r$. We will take this one claim at a time.

To show that $A \otimes B \neq \emptyset$, Definition 1.8 tells us that it will suffice to find an element of $A \otimes B$. Since $0 \in \mathbb{Q}$ and $0 \leq 0$, $0 \in \{r \in \mathbb{Q} \mid r \leq 0\}$. It follows by Definition 1.5 that $0 \in \{r \in \mathbb{Q} \mid r \leq 0\} \cup \{ab \mid a \in A, b \in B, a > 0, b > 0\}$. Therefore, by Definition 7.39, $0 \in A \otimes B$, as desired.

To show that $A \otimes B \neq \mathbb{Q}$, Definition 1.2 tells us that it will suffice to find an element of $\mathbb{Q}$ that is not an element of $A \otimes B$. Since $\mathbf{0} < A$ and $\mathbf{0} < B$, Definitions 6.4 and 1.3 assert that there exist points $a \in A$ and $b \in B$ such that $a, b \notin \mathbf{0}$, i.e., $a, b \geq 0$. Furthermore, since a, b are not the last points of A, B, respectively, by Definition 6.1c, there exist points $c \in A$ and $d \in B$ such that $c > 0$ and $d > 0$. Now, for an analogous reason to before, we can choose $x, y \in \mathbb{Q}$ such that $x \notin A$ and $y \notin B$. It follows by Lemma 6.2 and Definition 5.6 that $x \geq e$ for all $e \in A$ and $y \geq f$ for all $f \in B$, meaning (when combined with the last result) that $x > 0$ and $y > 0$. Thus, $xy > 0$, so $xy \notin \{r \in \mathbb{Q} \mid r \leq 0\}$. Additionally, we have by Script 2 that $xy > ef$ for all ef formed from the product of positive elements of A and B. Thus, $xy \notin \{ab \mid a \in A, b \in B, a > 0, b > 0\}$. Therefore, by Definitions 1.5 and 7.39, $xy \notin A \otimes B$, as desired.

To show that if $r \in A \otimes B$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A \otimes B$, we let $r \in A \otimes B$ and $s \in \mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s < r$ and seek to verify that $s \in A \otimes B$. We divide into two cases ($s \leq 0$ and $s > 0$). Suppose first that $s \leq 0$. Then $s \in \{r \in \mathbb{Q} \mid r \leq 0\}$. It follows by Definition 1.5 that $0 \in \{r \in \mathbb{Q} \mid r \leq 0\} \cup \{ab \mid a \in A, b \in B, a > 0, b > 0\}$. Therefore, by Definition 7.39, $0 \in A \otimes B$. Now suppose that $s > 0$. Then $r > 0$. Since $r \in A \otimes B$ and $r > 0$, $r = xy$ where $x \in A, y \in B, x > 0, y > 0$. Additionally, it follows from the fact that $s < r$ that $s = r - q = xy - q = (x - \frac{q}{y})y$ for some $q \in \mathbb{Q}^+$. Since $x - \frac{q}{y} \in \mathbb{Q}$ and $x - \frac{q}{y} < x$, we have by Definition 6.1b that $x - \frac{q}{y} \in A$. Therefore, $s = (x - \frac{q}{y})(y)$ is an element of $\{ab \mid a \in A, b \in B, a > 0, b > 0\}$, and hence by Definition 7.39, $A \otimes B$.

To show that if $r \in A \otimes B$, then there is some $s \in A \otimes B$ with $s > r$, we let $r \in A \otimes B$ and seek to find such an s. We divide into two cases ($r \leq 0$ and $r > 0$). Suppose first that $r \leq 0$. Then for the same reasons outlined in the proof of the second condition, there exist positive elements of $A \otimes B$ that are greater than r. Now suppose that $r > 0$. This implies that $r = xy$ for some $x \in A, y \in B, x > 0, y > 0$. It follows from the fact that $x \in A$ by Definition 6.1c that there exists a $z \in A$ with $z > x$. Consequently, by Lemma 7.24[2], $zy > xy$ is the desired element of $A \otimes B$. $\square$

Proof of b. To prove that $\otimes$ is commutative for $\mathbf{0} < A, \mathbf{0} < B$, Definition 7.1 tells us that it will suffice to show that for all such $A, B \in \mathbb{R}$, we have $A \otimes B = B \otimes A$. Let A, B be arbitrary elements of $\mathbb{R}$ where $\mathbf{0} < A, \mathbf{0} < B$. Then by Definition 7.39, we clearly have

$$\begin{aligned} A \otimes B &= \{r \in \mathbb{Q} \mid r \leq 0\} \cup \{ab \mid a \in A, b \in B, a > 0, b > 0\} \\ &= \{r \in \mathbb{Q} \mid r \leq 0\} \cup \{ba \mid b \in B, a \in A, b > 0, a > 0\} \\ &= B \oplus A \end{aligned}$$

To prove that $\otimes$ is associative for $\mathbf{0} < A, \mathbf{0} < B, \mathbf{0} < C$, Definition 7.1 tells us that it will suffice to show that for all such $A, B, C \in \mathbb{R}$, we have $(A \otimes B) \otimes C = A \otimes (B \otimes C)$. Let A, B, C be arbitrary elements of $\mathbb{R}$ where $\mathbf{0} < A, \mathbf{0} < B, \mathbf{0} < C$. To show that $(A \otimes B) \otimes C = A \otimes (B \otimes C)$, Definition 1.2 tells us that it will suffice to verify that every element of $(A \otimes B) \otimes C$ is an element of $A \otimes (B \otimes C)$ and vice versa. Suppose first that $x \in (A \otimes B) \otimes C$. Then by Definition 7.39, $x \leq 0$ or $x = dc$ where $d \in A \otimes B, c \in C, d > 0, c > 0$. If $x \leq 0$, then by Definition 7.39, $x \in A \otimes (B \otimes C)$ since it's an element of $\{r \in \mathbb{Q} \mid r \leq 0\}$, as desired. On the

[1] Note that the proofs given here only address the case where $\mathbf{0} < A$ and $\mathbf{0} < B$.

[2] And, technically, Theorem 7.29.

other hand, if $x = dc$ where $d \in A \otimes B, c \in C, d > 0, c > 0$, we continue. Now $d \in A \otimes B$ implies that $d \leq 0$ or $d = ab$ where $a \in A, b \in B, a > 0, b > 0$. However, the prior constraint that $d > 0$ guarantees that $d \nleq 0$, so we know that $d = ab$ where a, b satisfy the above conditions. Combining the last two results, we have $x = (ab)(c)$ where $a \in A, b \in B, c \in C, a > 0, b > 0, c > 0$. It follows that we also have $x = (a)(bc)$ under the same conditions. If we let $e = bc$ where $b \in B, c \in C, b > 0, c > 0$, then $e \in \{bc \mid b \in B, c \in C, b > 0, c > 0\}$. Consequently, by Definition 7.31, $e \in B \otimes C$. Additionally, $b > 0, c > 0$ imply by Definition 7.21 that $e > 0$. To recap, at this point we have $x = ae$ where $a \in A, e \in B \otimes C, a > 0, e > 0$. It follows by a similar process to before that $x \in A \otimes (B \otimes C)$. The proof is symmetric in the other direction. $\square$

Proof of c. To prove that $\mathbf{0} < A \otimes B$, Definitions 6.4 and 1.3 tell us that it will suffice to show that every $x \in \mathbf{0}$ is an element of $A \otimes B$ and find an $x \in A \otimes B$ such that $x \notin \mathbf{0}$. Let x be an arbitrary element of $\mathbf{0}$. Then $x \in \mathbb{Q}$ and $x < 0$. But it follows that $x \in \{r \in \mathbb{Q} \mid r \leq 0\}$, which implies by Definition 7.39 that $x \in A \otimes B$. As to the other stipulation, clearly $0 \in \{r \in \mathbb{Q} \mid r \leq 0\}$ but since $0 \not< 0$, $0 \notin \mathbf{0}$. Therefore, by Definition 7.39, $0 \in A \otimes B$, but $0 \notin \mathbf{0}$, as desired. $\square$

Proof of d. To prove that $\mathbf{1} \otimes A = A$, Definition 1.2 tells us that it will suffice to show that every $x \in \mathbf{1} \otimes A$ is an element of A and vice versa.

Let x be an arbitrary element of $\mathbf{1} \otimes A$. Then by Definition 7.39, $x \leq 0$ or $x = da$ where $d \in \mathbf{1}, a \in A, d > 0, a > 0$. We now divide into two cases. Suppose first that $x \leq 0$. We divide into two cases again ($x < 0$ and $x = 0$). If $x < 0$, then $x \in \mathbf{0}$, which implies by Definitions 6.4, 1.3, and the fact that $\mathbf{0} < A$ that $x \in A$, as desired. On the other hand, if $x = 0$, suppose for the sake of contradiction that $x \notin A$. Then by Lemma 6.2 and Definition 5.6, $a \leq x$ for all $a \in A$. This combined with the fact that $x \notin A$ implies that $a < x$ for all $a \in A$. Consequently, since $\mathbf{0} = \{q \in \mathbb{Q} \mid q < 0\}$, it follows that $A \subset \mathbf{0}$. But by Definition 6.4, this implies $A \leq \mathbf{0}$, contradicting the fact that $\mathbf{0} < A$, as desired. Now suppose that $x = da$ where $d \in \mathbf{1}, a \in A, d > 0, a > 0$. Then by Script 2, $d < 1$ implies that $x = da < a$. Therefore, by Definition 6.1b, $x \in A$, as desired.

Let x be an arbitrary element of A. We divide into two cases ($x \leq 0$ and $x > 0$). If $x \leq 0$, then $x \in \{r \in \mathbb{Q} \mid r \leq 0\}$, which implies by Definition 7.39 that $x \in \underline{1} \otimes A$. On the other hand, suppose $x > 0$. Then by Definition 6.1c, there is some $y \in A$ with $y > x$. It follows by Script 2 that $1 > \frac{x}{y} > 0$, so we have that $\frac{x}{y} \in \mathbf{1}$. Thus, since $x = \frac{x}{y} \cdot y$, we know that x is the product of a positive element of $\mathbf{1}$ and a positive element of A (since $y > x > 0$). Therefore, $x \in \{da \mid d \in \mathbf{1}, a \in A, d > 0, a > 0\}$, which implies by Definition 7.39 that $x \in \mathbf{1} \otimes A$. $\square$

7.2 Discussion

1/28:
- Script 6 journals due Wednesday.
- We'll also have to prove a density lemma:
  - Let X be a dense subset of a continuum C. Show that for all $x, y \in X$, if $x < y$, then there exists a $z \in X$ such that $x < z < y$.
  - Mark in Exercise 6.16 as "Density Lemma."
- Explicitly cite Field Axioms as you go.

2/2:
- For Theorem 7.30 in class, he wants a simple explanation of what the injective map looks like and why, but not a full-on rigorous proof.
  - Nothing in the journal for Theorem 7.30, though.
- He also wants to see Exercises 7.32 and 7.40 in the journal.
- For Corollary 7.15, we can write that $x^{-1} \cdot x = 1$ and $x^{-1} \cdot (x^{-1})^{-1} = 1$, and know by the uniqueness of multiplicative inverses (Theorem 7.14) that $x = (x^{-1})^{-1}$. For Corollary 7.11, we have an analogous proof.

- Alternate Theorem 7.17:

$$\begin{aligned} 1 &= 1 \cdot 1 \\ &= (a \cdot a^{-1})(b \cdot b^{-1}) \\ &= (ab)(a^{-1}b^{-1}) \\ &= 0 \end{aligned}$$

- Alternate Lemma 7.18: $a + (-a) = 0$. $a + (-1)a = a(1 + (-1)) = a \cdot 0 = 0$. Thus, by Theorem 7.10, $-a = (-1)a$.
- Alternate Lemma 7.19: We can use the uniqueness of additive inverses (Theorem 7.10).
- We can also cite Remark 7.25 in Lemma 7.26.

2/4:

- Thoughts on Theorem 7.30:

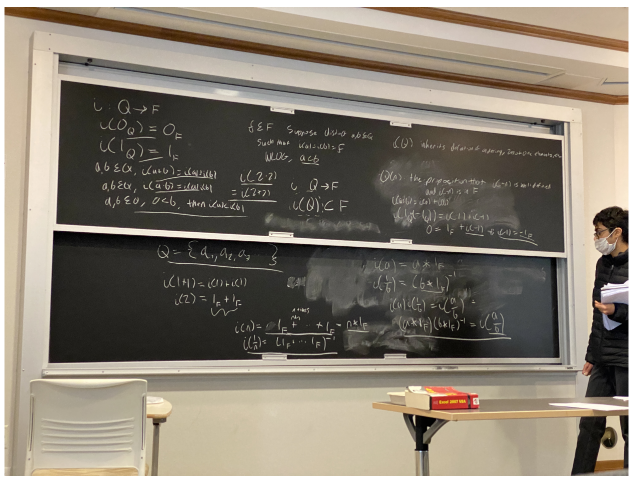


Figure 7.1: Theorem 7.30 discussion.

Script 8

Intervals

8.1 Journal

2/9: Now that we have constructed $\mathbb{R}$ and proven the fundamental facts about it, we will work with the real numbers $\mathbb{R}$ instead of an arbitrary continuum C. We will leave behind Dedekind cuts and think of elements of $\mathbb{R}$ as numbers. Accordingly, from now on, we will use lower-case letters like x for real numbers and will write $+$ and $\cdot$ for $\oplus$ and $\otimes$. We will also now use the standard notation (a,b) for the region $\underline{ab} = \{x \in \mathbb{R} \mid a < x < b\}$. Even though the notation is the same, this is *not* the same object as the ordered pair (a,b).

More generally, we adopt the following standard notation:

$$\begin{aligned}
(a,b) &= \{x \in \mathbb{R} \mid a < x < b\} \\
[a,b) &= \{x \in \mathbb{R} \mid a \leq x < b\} \\
(a,b] &= \{x \in \mathbb{R} \mid a < x \leq b\} \\
[a,b] &= \{x \in \mathbb{R} \mid a \leq x \leq b\} \\
(a,\infty) &= \{x \in \mathbb{R} \mid a < x\} \\
[a,\infty) &= \{x \in \mathbb{R} \mid a \leq x\} \\
(-\infty,b) &= \{x \in \mathbb{R} \mid x < b\} \\
(-\infty,b] &= \{x \in \mathbb{R} \mid x \leq b\}
\end{aligned} \tag{8.1}$$

Exercise 8.1. Identify the sets in Equations 8.1 that are open/closed/neither.

Proof. For this proof, we assume when applicable that $a < b$ (cases where $a = b$ are trivially simple and uncommon). Additionally, note that by Theorem 5.1, any of these sets proven to be just one of open or closed will not be the other, i.e., a set proven to be open will not be closed and vice versa.

By Corollary 4.11, (a,b) is open.

By an adaptation of Corollary 5.14, $b \in LP([a,b))$ but $b \notin [a,b)$. Since $[a,b)$ doesn't contain all of its limit points, Definition 4.1 tells us that it is not closed. Additionally, since $a \in LP(C \setminus [a,b))$ but $a \notin C \setminus [a,b)$, Definition 4.8 tells us that it is not open. Therefore, it is neither.

The proof that $(a,b]$ is neither is symmetric to the previous case.

By Corollaries 5.15 and 4.7, $[a,b]$ is closed.

By Corollary 4.13, (a,∞) is open.

By Corollary 4.13 and Definition 4.8, $[a,\infty) = C \setminus (-\infty,a)$ is closed.

The proofs that $(-\infty,b)$ and $(-\infty,b]$ are open and closed, respectively, are symmetric to the previous two cases, respectively. ☐

Definition 8.2. A set $I \subset \mathbb{R}$ is an **interval** if for all $x, y \in I$ with $x < y$, $[x,y] \subset I$.

Lemma 8.3. *A proper subset $I \subsetneq \mathbb{R}$ is an interval if and only if it takes one of the eight forms in Equations 8.1.*

Proof. Suppose first that $I \subsetneq \mathbb{R}$ is an interval. If $I = \emptyset$, then $I = (a, a)$ for any $a \in \mathbb{R}$, and we are done. Thus, we will assume for the remainder of the proof of the forward direction that I is nonempty. To address this case, we will first prove that the facts that $I \subsetneq \mathbb{R}$, $I \neq \emptyset$, and I is an interval imply that I is bounded above, bounded below, or both. Then in each of these three cases, we will look at whether $\sup I$ and $\inf I$ (if they exist) are elements of the set or not to differentiate between the eight forms in Equations 8.1. Let's begin.

Suppose for the sake of contradiction that there exists a nonempty interval $I \subsetneq \mathbb{R}$ that is neither bounded above nor bounded below. Since $I \subsetneq \mathbb{R}$, we have by Definition 1.3 that there exists a point $p \in \mathbb{R}$ such that $p \notin I$. Additionally, since I is neither bounded above nor below, Definition 5.6 implies that p is neither an upper nor a lower bound of I. Thus, there exist $x, y \in I$ such that $x < p$ and $y > p$. Now by Definition 8.2, $[x, y] \subset I$. But it follows by Definition 1.3 that every point in $[x, y]$, including p, is an element of I, a contradiction.

We now divide into three cases (I is exclusively bounded below, I is exclusively bounded above, and I is bounded both below and above).

First, suppose that I is only bounded below. Since I is a nonempty subset of $\mathbb{R}$ that is bounded below, we have by Theorem 5.17 that $\inf I$ exists. We divide into two cases again ($\inf I \in I$ and $\inf I \notin I$).

If $\inf I \in I$, then we can demonstrate that $I = [\inf I, \infty)$. To do this, Definition 1.2 tells us that it will suffice to verify that every $p \in I$ is an element of $[\inf I, \infty)$ and vice versa. Let p be an arbitrary element of I. Then $p \in \mathbb{R}$ and by Definitions 5.7 and 5.6, $\inf I \leq p$. Therefore, $p \in [\inf I, \infty)$, as desired. Now let p be an arbitrary element of $[\inf I, \infty)$. Then $\inf I \leq p$. Additionally, since I is not bounded above, we have by Definition 5.6 that there exists $y \in I$ such that $y > p$. Since $\inf I \in I$, $y \in I$, and $\inf I < y$ (by transitivity), $[\inf I, y] \subset I$ by Definition 8.2. This combined with the fact that $p \in [\inf I, y]$ (we know that $\inf I \leq p < y$, so $\inf I \leq p \leq y$) implies that $p \in I$, as desired.

If $\inf I \notin I$, then we can demonstrate that $I = (\inf I, \infty)$. As before, to do so, it will suffice to verify that every $p \in I$ is an element of $(\inf I, \infty)$ and vice versa. Let p be an arbitrary element of I. Then $p \in \mathbb{R}$ and by Definitions 5.7 and 5.6, $\inf I \leq p$. The additional constraint that $\inf I \notin I$ combined with the fact that $p \in I$ implies that $p \neq \inf I$, so $\inf I < p$. Therefore, $p \in (\inf I, \infty)$, as desired. Now let p be an arbitrary element of $(\inf I, \infty)$. Then $\inf I < p$. It follows by Lemma 5.11 that there exists a $z \in I$ such that $\inf I \leq z < p$. Additionally, since I is not bounded above, we have by Definition 5.6 that there exists $y \in I$ such that $y > p$. Since $z \in I$, $y \in I$, and $z < y$ (by transitivity), $[z, y] \subset I$ by Definition 8.2. This combined with the fact that $p \in [z, y]$ (we know that $z < p < y$, so $z \leq p \leq y$) implies that $p \in I$, as desired.

Second, suppose that I is only bounded above. The proof of this case is symmetric to that of the first.

Third, suppose that I is bounded below and above. Since I is a nonempty subset of $\mathbb{R}$ that is bounded above and below, we have by consecutive applications of Theorem 5.17 that both $\sup I$ and $\inf I$ exist. We divide into four cases ($\inf I \in I$ and $\sup I \in I$, $\inf I \in I$ and $\sup I \notin I$, $\inf I \notin I$ and $\sup I \in I$, and $\inf I \notin I$ and $\sup I \notin I$).

If $\inf I \in I$ and $\sup I \in I$, then we can demonstrate that $I = [\inf I, \sup I]$. We divide into two cases again ($\inf I = \sup I$ and $\inf I \neq \sup I$). If $\inf I = \sup I \in I$, then $I = \{\inf I\} = \{\sup I\} = [\inf I, \sup I]$, as desired. On the other hand, if $\inf I \neq \sup I$, we continue. To demonstrate that $I = [\inf I, \sup I]$, Theorem 1.7 tells us that it will suffice to verify that $I \subset [\inf I, \sup I]$ and $[\inf I, \sup I] \subset I$. To verify that the former claim, Definition 1.3 tells us that it will suffice to confirm that every $p \in I$ is an element of $[\inf I, \sup I]$. Let p be an arbitrary element of I. Then $p \in \mathbb{R}$ and by consecutive applications of Definitions 5.7 and 5.6, $\inf I \leq p$ and $p \leq \sup I$. Therefore, $p \in [\inf I, \sup I]$, as desired. On the other hand, since $\inf I \in I$, $\sup I \in I$, and $\inf I < \sup I$ (as follows from Definition 5.7 and the fact that they are unequal), $[\inf I, \sup I] \subset I$ by Definition 8.2, as desired.

If $\inf I \in I$ and $\sup I \notin I$, then we can demonstrate that $I = [\inf I, \sup I)$. To do so, it will suffice to verify that every $p \in I$ is an element of $[\inf I, \sup I)$ and vice versa. Let p be an arbitrary element of I. Then $p \in \mathbb{R}$ and by Definitions 5.7 and 5.6, $\inf I \leq p$ and $p \leq \sup I$. The additional constraint that $\sup I \notin I$ combined with the fact that $p \in I$ implies that $p \neq \sup I$, so $p < \sup I$. Therefore, $p \in [\inf I, \sup I)$, as desired. Now let p be an arbitrary element of $[\inf I, \sup I)$. Then $\inf I \leq p < \sup I$. It follows by Lemma

5.11 that there exists a $y \in I$ such that $p < y \leq \sup I$. Since $\inf I \in I$, $y \in I$, and $\inf I < y$ (by transitivity), $[\inf I, y] \subset I$ by Definition 8.2. This combined with the fact that $p \in [\inf I, y]$ (we know that $\inf I \leq p < y$, so $\inf I \leq p \leq y$) implies that $p \in I$, as desired.

If $\inf I \notin I$ and $\sup I \in I$, the proof is symmetric to that of the previous case.

If $\inf I \notin I$ and $\sup I \notin I$, then we can demonstrate that $I = (\inf I, \sup I)$. To do so, it will suffice to verify that every $p \in I$ is an element of $(\inf I, \sup I)$ and vice versa. Let p be an arbitrary element of I. Then $p \in \mathbb{R}$ and by Definitions 5.7 and 5.6, $\inf I \leq p$ and $p \leq \sup I$. The additional constraints that $\inf I \notin I$ and $\sup I \notin I$ combined with the fact that $p \in I$ imply that $p \neq \inf I$ and $p \neq \sup I$, respectively, so $\inf I < p$ and $p < \sup I$, respectively. Therefore, $p \in (\inf I, \sup I)$, as desired. Now let p be an arbitrary element of $(\inf I, \sup I)$. Then $\inf I < p < \sup I$. It follows by consecutive applications of Lemma 5.11 that there exist $x, y \in I$ such that $\inf I \leq x < p$ and $p < y \leq \sup I$. Since $x \in I$, $y \in I$, and $x < y$ (by transitivity), $[x, y] \subset I$ by Definition 8.2. This combined with the fact that $p \in [x, y]$ (we know that $x < p < y$, so $x \leq p \leq y$) implies that $p \in I$, as desired.

Now suppose that $I \subsetneq \mathbb{R}$ takes one of the eight forms in Equations 8.1. To prove that I is an interval, Definition 8.2 tells us that it will suffice to show that for all $x, y \in I$ with $x < y$, $[x, y] \subset I$. Let x, y be arbitrary elements of I with $x < y$. We divide into eight cases (one for each equation in Equations 8.1).

First, suppose that $I = (a, b)$. To demonstrate that $[x, y] \subset I$, Definition 1.3 tells us that it will suffice to confirm that every $z \in [x, y]$ is an element of I. Let z be an arbitrary element of $[x, y]$. Then by Equations 8.1, $x \leq z \leq y$. But since $a < x < y < b$ by Equations 8.1, the fact that $a < x \leq z \leq y < b$ implies by Equations 8.1 that $z \in (a, b)$, as desired.

The proofs of the second, third, and fourth equations are symmetric to that of the first.

Fifth, suppose that $I = (a, \infty)$. To demonstrate that $[x, y] \subset I$, Definition 1.3 tells us that it will suffice to confirm that every $z \in [x, y]$ is an element of I. Let z be an arbitrary element of $[x, y]$. Then by Equations 8.1, $x \leq z \leq y$. But since $a < x$ by Equations 8.1, the fact that $a < x \leq z$ implies by Equations 8.1 that $z \in (a, \infty)$, as desired.

The proofs of the sixth, seventh, and eighth equations are symmetric to that of the first. □

Definition 8.4. The **absolute value** of a real number x is the non-negative number $|x|$ defined by

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Exercise 8.5. Show that $|x| = |-x|$ for all $x \in \mathbb{R}$. (Note that this also means that $|x - y| = |y - x|$ for any $x, y \in \mathbb{R}$.)

Proof. Let x be an arbitrary element of $\mathbb{R}$. We divide into three cases ($x = 0$, $x > 0$, and $x < 0$). First, suppose that $x = 0$. Then since $0 = -0$, clearly $|0| = |-0|$, as desired. Second, suppose that $x > 0$. Then by Lemma 7.23[1] $-x < 0$. Thus, by consecutive applications of Definition 8.4, $|x| = x$ and $|-x| = -(-x)$. Therefore, since $-(-x) = x$ by Corollary 7.11, $|x| = x = |-x|$, as desired. Third, suppose that $x < 0$. Then by Lemma 7.23, $-x > 0$. Thus, by consecutive applications of Definition 8.4, $|x| = -x$ and $|-x| = -x$. Therefore, $|x| = -x = |-x|$, as desired. □

Definition 8.6. The **distance** between $x \in \mathbb{R}$ and $y \in \mathbb{R}$ is defined to be $|x - y|$.

Remark 8.7. It follows from Definition 8.6 that $|x|$ is the distance between x and 0.

Lemma 8.8. *For any real numbers x, y, z, we have*

(a) $|x + y| \leq |x| + |y|$.

(b) $|x - z| \leq |x - y| + |y - z|$.

(c) $||x| - |y|| \leq |x - y|$.

[1] And, technically, Theorem 7.47.

Proof of a. We divide into four cases ($x \geq 0$ and $y \geq 0$, $x \geq 0$ and $y < 0$, $x < 0$ and $y \geq 0$, and $x < 0$ and $y < 0$).

First, suppose that $x \geq 0$ and $y \geq 0$. Then by Definition 7.21, $x+y \geq 0$. Thus, by consecutive applications of Definition 8.4, $|x+y| = x+y$, $|x| = x$, and $|y| = y$. Therefore, $|x+y| = x+y \leq x+y = |x|+|y|$, as desired.

Second, suppose that $x \geq 0$ and $y < 0$. By Definition 8.4, $|x| = x$ and $|y| = -y$. We now divide into two cases ($x+y \geq 0$ and $x+y < 0$). If $x+y \geq 0$, then $|x+y| = x+y$. Additionally, since $y < 0$, Lemma 7.23 implies that $0 < -y$. Consequently, by transitivity, $y < -y = |y|$. It follows by Definition 7.21 that $x+y < x+|y|$. Therefore, $|x+y| = x+y < x+|y| = |x|+|y|$, so $|x+y| \leq |x|+|y|$, as desired. On the other hand, if $x+y < 0$, then $|x+y| = -(x+y) = -x+(-y) = -x+|y|$. Additionally, by Lemma 7.23, $x \geq 0$ implies that $-x \leq 0$. It follows by Definition 7.21 since $-x \leq x$ that $-x+|y| \leq x+|y|$. Therefore, $|x+y| = -x+|y| \leq x+|y| = |x|+|y|$, as desired.

The proof of the third case is symmetric to that of the second.

The proof of the fourth case is symmetric to that of the first. ☐

Proof of b. By part (a), $|x-z| = |x-y+y-z| \leq |x-y|+|y-z|$, as desired. ☐

Proof of c. To prove that $||x|-|y|| \leq |x-y|$, Definition 8.4 tells us that it will suffice to show that $|x|-|y| \leq |x-y|$ and $-(|x|-|y|) \leq |x-y|$. By part (a), $|x| = |x-y+y| \leq |x-y|+|y|$, so $|x|-|y| \leq |x-y|$. Similarly, $|y|-|x| \leq |x-y|$, so $-(|x|-|y|) \leq |x-y|$, as desired. ☐

Exercise 8.9. Let $a, \delta \in \mathbb{R}$ with $\delta > 0$. Prove that

$$(a-\delta, a+\delta) = \{x \in \mathbb{R} \mid |x-a| < \delta\}$$

Lemma. *For any $a, b \in \mathbb{R}$ such that $0 < b$, $|a| < b$ if and only if $-b < a < b$.*

Proof. Suppose first that $|a| < b$. We divide into two cases ($a \geq 0$ and $a < 0$). If $a \geq 0$, then by Definition 8.4, $0 \leq a = |a| < b$. Additionally, by Lemma 7.23, $-b < 0$. Therefore, $-b < 0 \leq a < b$, as desired. If $a < 0$, then by Definition 8.4, $-a = |a| < b$. It follows by Definition 7.21 (by adding $a-b$ to both sides) that $-b < a$. Additionally, by Lemma 7.23, $a < 0$ implies $0 < -a$, so we know that $a < -a$. Therefore, $-b < a < -a < b$, as desired.

Now suppose that $-b < a < b$. We divide into two cases ($a \geq 0$ and $a > 0$). If $a \geq 0$, then by Definition 8.4, $|a| = a < b$, as desired. If $a < 0$, then by Definition 8.4, $|a| = -a$. Since $-b < a$, Definition 7.21 implies (by adding $b-a$ to both sides) that $-a < b$. Therefore, $|a| = -a < b$, as desired. ☐

Proof of Exercise 8.9. To prove that $(a-\delta, a+\delta) = \{x \in \mathbb{R} \mid |x-a| < \delta\}$, Definition 1.2 tells us that it will suffice to show that every $p \in (a-\delta, a+\delta)$ is an element of $\{x \in \mathbb{R} \mid |x-a| < \delta\}$ and vice versa.

Suppose first that $p \in (a-\delta, a+\delta)$. Then by Equations 8.1, $a-\delta < p < a+\delta$. It follows by consecutive applications of Definition 7.21 that $-\delta < p-a < \delta$. Thus, the lemma asserts that $|p-a| < \delta$. Therefore, $p \in \{x \in \mathbb{R} \mid |x-a| < \delta\}$.

Now suppose that $p \in \{x \in \mathbb{R} \mid |x-a| < \delta\}$. Then $|p-a| < \delta$. Thus, by the lemma, $-\delta < p-a < \delta$. It follows by consecutive applications of Definition 7.21 that $a-\delta < p < a+\delta$. Therefore, since $a-\delta < p < a+\delta$, we have that $p \in (a-\delta, a+\delta)$. ☐

2/11: **Lemma 8.10.** *Let I be an open interval containing the point $p \in \mathbb{R}$. Then*

a) There exists a number $\delta > 0$ such that $(p-\delta, p+\delta) \subset I$.

b) There exists a natural number N such that for all natural numbers $k \geq N$ we have $(p-\frac{1}{k}, p+\frac{1}{k}) \subset I$.

Proof of a. Since I is open, we have by Theorem 4.10 that there exists a region (a,b) such that $p \in (a,b)$ and $(a,b) \subset I$. Let $\delta = \min(p-a, b-p)$. To show that $(p-\delta, p+\delta) \subset I$, we will demonstrate that $(p-\delta, p+\delta) \subset (a,b) \subset I$. To do this, Definition 1.3 tells us that it will suffice to verify that every element $x \in (p-\delta, p+\delta)$ is an element of (a,b). Let x be an arbitrary element of $(p-\delta, p+\delta)$. Then by Equations 8.1, $p-\delta < x < p+\delta$. We divide into two cases ($\delta = p-a$ and $\delta = b-p$). Suppose first that $\delta = p-a$. Then $p-(p-a) < x < p+(p-a)$, i.e., $a < x < p+(p-a)$. Additionally, the fact that $p-a = \min(p-a, b-p)$ implies that $p-a \leq b-p$. Combining these last two results gives us $a < x < p+(p-a) \leq p+(b-p) = b$. Since $a < x < b$, Equations 8.1 imply that $x \in (a,b)$, as desired. The proof is symmetric if $\delta = b-p$. ☐

Proof of b. By Lemma 8.10a, there exists a number $\delta > 0$ such that $(p-\delta, p+\delta) \subset I$. Since δ is a positive real number, Corollary 6.12 implies that there exists a nonzero natural number N such that $\frac{1}{N} < \delta$. To prove that for all numbers $k \geq N$, we have $(p - \frac{1}{k}, p + \frac{1}{k}) \subset I$, we will show that $(p - \frac{1}{k}, p + \frac{1}{k}) \subset (p - \delta, p + \delta) \subset I$. To do this, Definition 1.3 tells us that it will suffice to show that every $x \in (p - \frac{1}{k}, p + \frac{1}{k})$ is an element of $(p - \delta, p + \delta)$. Let k be an arbitrary natural number such that $k \geq N$, and let x be an arbitrary element of $(p - \frac{1}{k}, p + \frac{1}{k})$. It follows from the latter condition by Equations 8.1 that $p - \frac{1}{k} < x < p + \frac{1}{k}$. Since $\frac{1}{k} \leq \frac{1}{N}$ by Scripts 2 and 3, we have that $p - \frac{1}{N} < x < p + \frac{1}{N}$. Since $\frac{1}{N} < \delta$ by definition, $p - \delta < x < p + \delta$. Therefore, by Equations 8.1, $x \in (p - \delta, p + \delta)$, as desired. $\square$

Definition 8.11. Let $A \subset X \subset \mathbb{R}$. We say that A is **open** (in X) if it is the intersection of X with an open set, and **closed** (in X) if it is the intersection of X with a closed set. (This is called the subspace topology on X.)

Remark 8.12. $A \subset \mathbb{R}$ open, as defined in Script 3, is equivalent to A open in $\mathbb{R}$.

Exercise 8.13. Let $A \subset X \subset \mathbb{R}$. Show that $X \setminus A$ is closed in X if and only if A is open in X.

Proof. Suppose first that $X \setminus A$ is closed in X. Then by Definition 8.11, $X \setminus A = X \cap B$ where B is a closed set. It follows by Script 1 that

$$\begin{aligned} X \setminus A &= X \cap B \\ \mathbb{R} \setminus (X \setminus A) &= \mathbb{R} \setminus (X \cap B) \\ (\mathbb{R} \setminus X) \cup A &= (\mathbb{R} \setminus X) \cup (\mathbb{R} \setminus B) \\ X \cap ((\mathbb{R} \setminus X) \cup A) &= X \cap ((\mathbb{R} \setminus X) \cup (\mathbb{R} \setminus B)) \\ (X \cap (\mathbb{R} \setminus X)) \cup (X \cap A) &= (X \cap (\mathbb{R} \setminus X)) \cup (X \cap (\mathbb{R} \setminus B)) \\ \emptyset \cup (X \cap A) &= \emptyset \cup (X \cap (\mathbb{R} \setminus B)) \\ A &= X \cap (\mathbb{R} \setminus B) \end{aligned}$$

Since $\mathbb{R} \setminus B$ is open by Definition 4.4, we have by Definition 8.11 that A is open in X.

Now suppose that A is open in X. Then by Definition 8.11, $A = X \cap B$ where B is an open set. It follows by Script 1 that

$$\begin{aligned} A &= X \cap B \\ \mathbb{R} \setminus A &= \mathbb{R} \setminus (X \cap B) \\ &= (\mathbb{R} \setminus X) \cup (\mathbb{R} \setminus B) \\ X \cap (\mathbb{R} \setminus A) &= X \cap ((\mathbb{R} \setminus X) \cup (\mathbb{R} \setminus B)) \\ X \setminus A &= (X \cap (\mathbb{R} \setminus X)) \cup (X \cap (\mathbb{R} \setminus B)) \\ &= X \cap (\mathbb{R} \setminus B) \end{aligned}$$

Since $\mathbb{R} \setminus B$ is closed by Definition 4.4, we have by Definition 8.11 that $X \setminus A$ is closed in X. $\square$

Exercise 8.14.

a) Let $[a, b] \subset \mathbb{R}$. Give an example of a set $A \subset [a, b]$ such that A is open in $[a, b]$ but not in $\mathbb{R}$.

b) Give an example of sets $A \subset X \subset \mathbb{R}$ such that A is closed in X but not in $\mathbb{R}$.

Proof of a. We first briefly consider the case where $a = b$. In this case, let $c < a < d$; then $\{a\} = [a, a] \cap (c, d)$ is a subset of $[a, b]$ that is open in $[a, b]$ (by Definition 8.11; (c, d) is open by Exercise 8.1) but closed in $\mathbb{R}$ (by Corollary 3.23, Definition 4.1, and Theorem 5.1).

We now direct our attention to the case where $a \neq b$. Let $c \in [a, b]$ be a point such that $a < c < b$ (we know at least one such point exists by Theorem 5.2). If we define the set $(c, b] = [a, b] \cap (c, \infty)$, we have by Definition 8.11 that $(c, b]$ is open in $[a, b]$ (since (c, ∞) is open per Exercise 8.1). However, we know that $(c, b]$ is not open in $\mathbb{R}$ by Theorem 4.10 (b is an element of $(c, b]$ such that any region containing b necessarily contains an element that is not in $(c, b]$; this element will be greater than b but less than the right bound of the region, and its existence is guaranteed by Theorem 5.2). $\square$

Proof of b. Let $X = (a,b) \subset \mathbb{R}$. Then $(a,b) = X \cap [a,b]$, so (a,b) is closed in (a,b) by Definition 8.11. However, by Corollary 5.14, a,b are limit points of (a,b) that are not contained within (a,b). It follows by Definition 4.1 that (a,b) is not closed in $\mathbb{R}$. □

Theorem 8.15. *Let $X \subset \mathbb{R}$. Then X is connected if and only if X is an interval.*

Proof. Suppose first that X is connected. To prove that X is an interval, Definition 8.2 tells us that it will suffice to show that for all $x,y \in X$ with $x < y$, $[x,y] \subset X$. Let x,y be arbitrary elements of X satisfying $x < y$, and suppose for the sake of contradiction that $[x,y] \not\subset X$. Then there exists $z \in [x,y]$ such that $z \notin X$. Let $A = \{a \in X \mid a < z\}$ and $B = \{b \in X \mid z < b\}$. It follows from Script 1 that $X = A \cup B$ and $A \cap B = \emptyset$. To verify that A is nonempty, Definition 1.8 tells us that it will suffice to find an element in it. Since $z \notin X$ but $x \in X$, we know that $z \neq x$. This combined with the fact that $x \leq z$ by Equations 8.1 implies that $x < z$. Thus, since $x \in X$ and $x < z$, $x \in A$. Similarly, $y \in B$. To verify that A is open in X, Definition 8.11 tells us that it will suffice to demonstrate that A is the intersection of X with an open set. Since we clearly have $A = X \cap (-\infty, z)$ where $(-\infty, z)$ is open by Exercise 8.1, we are done. We can do something similar for B. But the existence of two disjoint, nonempty, open (in X) sets A, B whose union equals X demonstrates by Definition 4.22 that X is disconnected, a contradiction.

Now suppose that X is an interval, and suppose for the sake of contradiction that X is disconnected. Then by Definition 4.22, $X = A \cup B$ where A, B are disjoint, nonempty sets that are open in X. Since A, B are disjoint and nonempty, we know that there exist distinct objects $a \in A$ and $b \in B$. WLOG, let $a < b$.

To prove that $\sup(A \cap [a,b])$ exists, Theorem 5.17 tells us that it will suffice to show that $A \cap [a,b]$ is nonempty and bounded above. To show that $A \cap [a,b]$ is nonempty, Definition 1.8 tells us that it will suffice to find an element of $A \cap [a,b]$. By Equations 8.1, $a \in [a,b]$. By Definition, $a \in A$. Thus, by Definition 1.6, $a \in A \cap [a,b]$, as desired. To show that $A \cap [a,b]$ is bounded above, consecutive applications of Definition 5.6 tell us that it will suffice to verify that $x \leq b$ for all $x \in A \cap [a,b]$. Let x be an arbitrary element of $A \cap [a,b]$. It follows by Definition 1.6 that $x \in [a,b]$. Thus, by Equations 8.1, $x \leq b$, as desired.

Let $s = \sup(A \cap [a,b])$. To prove that $\inf(B \cap [s,b])$ exists, it will suffice to utilize a symmetric argument to the above.

Let $i = \inf(B \cap [s,b])$. We divide into three cases ($s > i$, $s = i$, and $s < i$).

First, suppose that $s > i$. To show that s is a lower bound of $B \cap [s,b]$, Definition 5.6 tells us that it will suffice to verify that $s \leq x$ for all $x \in B \cap [s,b]$. Let x be an arbitrary element of $B \cap [s,b]$. By Definition 1.6, $x \in [s,b]$. Thus, by Equations 8.1, $s \leq x$, as desired. Since s is a lower bound of $B \cap [s,b]$, Definition 5.6 asserts that $i \geq s$, contradicting the hypothesis that $s > i$.

Second, suppose that $s = i$. We divide into three cases ($s \in A$, $s \in B$, and $s \notin A$ and $s \notin B$).

If $s \in A$, then since A is open in X, Definition 8.11 implies that $A = X \cap G$ where G is open. It follows by the hypothesis that $s \in A$ along with Definitions 1.2 and 1.6 that $s \in G$. Consequently, by Theorem 4.10, there exists a region (c,d) such that $s \in (c,d)$ and $(c,d) \subset G$. From the former condition, we have by Equations 8.1 that $c < s < d$. Thus, by Lemma 5.11, there exists a point $x \in B \cap [s,b]$ such that $s = i \leq x < d$. Since $c < s \leq x < d$, Equations 8.1 imply that $x \in (c,d)$. This combined with the fact that $(c,d) \subset G$ implies by Definition 1.3 that $x \in G$. Additionally, we know that $x \in B$ (since $x \in B \cap [s,b]$ by Definition 1.6). It follows from this and the fact that $X = A \cup B$ by Definitions 1.5 and 1.2 that $x \in X$. Thus, since $x \in X$ and $x \in G$, Definition 1.6 asserts that $x \in X \cap G$, meaning that $x \in A$. But if $x \in A$ and $x \in B$, then Definition 1.6 implies that $x \in A \cap B$, contradicting the supposition that A and B are disjoint.

If $s \in B$, then the proof is symmetric to the previous case.

If $s \notin A$ and $s \notin B$, then by Definition 1.5, $s \notin A \cup B$, implying that $s \notin X$. Additionally, the facts that $a \in A$, $b \in B$, and $X = A \cup B$ imply that $a, b \in X$. It follows since $a < b$ by Definition 8.2 that $[a,b] \subset X$. We now show that $s \in [a,b]$ via Equations 8.1, which tell us that it will suffice to verify that $a \leq s \leq b$. As previously shown, b is an upper bound of $A \cap [a,b]$. Thus, by Definition 5.7, we have that $s \leq b$, and we are half done. As to the other half, we have also previously shown that $a \in A \cap [a,b]$. Additionally, by Definitions 5.7 and 5.6, $s \geq x$ for all $x \in A \cap [a,b]$, including a. Thus, $s \geq a$. Having shown that $s \in [a,b]$ and $[a,b] \subset X$, we may invoke Definition 1.3 to learn that $s \in X$, contradicting the previously proven statement that $s \notin X$.

Third, suppose that $s < i$. Then by Theorem 5.2 and Definition 3.6, there exists a $z \in \mathbb{R}$ such that $s < z < i$. We now show that $i \in [a, b]$ via Equations 8.1, which tell us that it wil suffice to verify that $a \leq i \leq b$. As previously shown, s is a lower bound of $B \cap [s, b]$. Thus, by Definition 5.7, we have that $i \geq s$. We have also previously shown that $s \geq a$, so by transitivity, $i \geq a$, and we are half done. As to the other half, we now confirm that $b \in B \cap [s, b]$. By Equations 8.1, $b \in [s, b]$. By definition, $b \in B$. Thus, by Definition 1.6, $b \in B \cap [s, b]$, as desired. Additionally, by Definitions 5.7 and 5.6, $i \leq x$ for all $x \in B \cap [s, b]$, including b. Thus, $i \leq b$, concluding our argument that $i \in [a, b]$. Moving on, the fact that $s < z$ implies by Definition 5.6 that $z \notin A \cap [a, b]$. Additionally, we know from the facts that $s, i \in [a, b]$ that $a \leq s < z < i \leq b$, meaning that $z \in [a, b]$. Combining the previous two results with Definition 1.6, we have that $z \notin A$. By a symmetric argument, we can show that $z \notin B$. Since $z \notin A$ and $z \notin B$, Definition 1.5 asserts that $z \notin A \cup B$, i.e., $z \notin X$. But as before, $[a, b] \subset X$, so the fact that $z \in [a, b]$ combined with Definition 1.3 implies that $z \in X$, a contradiction. ☐

2/16: **Definition 8.16.** Let I be an interval and let $f : I \to \mathbb{R}$.

a) We say that f is **increasing** on I if, whenever $x, y \in I$ with $x < y$, $f(x) \leq f(y)$.

b) We say that f is **decreasing** on I if, whenever $x, y \in I$ with $x < y$, $f(x) \geq f(y)$.

c) We say that f is **strictly increasing** on I if, whenever $x, y \in I$ with $x < y$, $f(x) < f(y)$.

d) We say that f is **strictly decreasing** on I if, whenever $x, y \in I$ with $x < y$, $f(x) > f(y)$.

Lemma 8.17. *If f is strictly increasing or strictly decreasing on an interval I then f is injective on I.*

Proof. We divide into two cases (f is strictly increasing, and f is strictly decreasing). Suppose first that f is strictly increasing. To prove that f is injective on I, Definition 1.20 tells us that it will suffice to show that for all $a, b \in I$, $a \neq b$ implies that $f(a) \neq f(b)$. Let a, b be arbitrary elements of I such that $a \neq b$. WLOG, let $a < b$. Then by Definition 8.16, $f(a) < f(b)$. Therefore, $f(a) \neq f(b)$, as desired. The proof is symmetric for the other case. ☐

8.2 Discussion

2/9:

- Due date: Feb. 19; if there's anything I can provide that would facilitate the process, lmk; Know anything about mixed-integer nonlinear programming?
- Lemma 8.3 more efficiently by proving that every $x \in (a, b)$ is an element of I and then just working with the boundary conditions?
  - Make first four cases of second direction symmetric.
- Rewrite Exercise 8.5 with three cases: $x = 0$, $x > 0$, $x < 0$ with the last two symmetric.
- We don't have to cite every algebraic manipulations from Script 7.

2/11:

- Use a bidirectional inclusion proof instead of set algebra for Exercise 8.13.
- What is the problem with Exercise 8.14b?

Script 9

Continuous Functions

9.1 Journal

2/16: **Lemma 9.1.** *Let $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$. If $A, B \subset \mathbb{R}$, then*

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$
$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$
$$f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$$
$$f^{-1}(\mathbb{R}) = X$$

Proof. To prove that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, Definition 1.2 tells us that it will suffice to show that every $x \in f^{-1}(A \cup B)$ is an element of $f^{-1}(A) \cup f^{-1}(B)$ and vice versa. Suppose first that x is an arbitrary element of $f^{-1}(A \cup B)$. Then by Definition 1.18, $f(x) \in A \cup B$. Thus, by Definition 1.5, $f(x) \in A$ or $f(x) \in B$. We now divide into two cases. If $f(x) \in A$, then by Definition 1.18, $x \in f^{-1}(A)$. It follows by Definition 1.5 that $x \in f^{-1}(A) \cup f^{-1}(B)$, as desired. The argument is symmetric in the other case. Now suppose that $x \in f^{-1}(A) \cup f^{-1}(B)$. Then by Definition 1.5, $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. We now divide into two cases. If $x \in f^{-1}(A)$, then by Definition 1.18, $f(x) \in A$. It follows by Definition 1.5 that $f(x) \in A \cup B$. Therefore, by Definition 1.18, $x \in f^{-1}(A \cup B)$. The argument is symmetric in the other case, as desired.

To prove that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$, Definition 1.2 tells us that it will suffice to show that every $x \in f^{-1}(A \cap B)$ is an element of $f^{-1}(A) \cap f^{-1}(B)$ and vice versa. Suppose first that x is an arbitrary element of $f^{-1}(A \cap B)$. Then by Definition 1.18, $f(x) \in A \cap B$. Thus, by Definition 1.6, $f(x) \in A$ and $f(x) \in B$. It follows by consecutive applications of Definition 1.18 that $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$. Therefore, by Definition 1.6, $x \in f^{-1}(A) \cap f^{-1}(B)$, as desired. Now suppose that $x \in f^{-1}(A) \cap f^{-1}(B)$. Then by Definition 1.6, $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$. It follows by consecutive applications of Definition 1.18 that $f(x) \in A$ and $f(x) \in B$. Thus, by Definition 1.6, $f(x) \in A \cap B$. Therefore, by Definition 1.18, $x \in f^{-1}(A \cap B)$, as desired.

To prove that $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$, Definition 1.2 tells us that it will suffice to show that every $x \in f^{-1}(A \setminus B)$ is an element of $f^{-1}(A) \setminus f^{-1}(B)$ and vice versa. Suppose first that x is an arbitrary element of $f^{-1}(A \setminus B)$. Then by Definition 1.18, $f(x) \in A \setminus B$. Thus, by Definition 1.11, $f(x) \in A$ and $f(x) \notin B$. It follows by consecutive applications of Definition 1.18 that $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$. Therefore, by Definition 1.11, $x \in f^{-1}(A) \setminus f^{-1}(B)$, as desired. Now suppose that $x \in f^{-1}(A) \setminus f^{-1}(B)$. Then by Definition 1.11, $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$. It follows by consecutive applications of Definition 1.18 that $f(x) \in A$ and $f(x) \notin B$. Thus, by Definition 1.11, $f(x) \in A \setminus B$. Therefore, by Definition 1.18, $x \in f^{-1}(A \setminus B)$, as desired.

To prove that $f^{-1}(\mathbb{R}) = X$, Definition 1.2 tells us that it will suffice to show that every $x \in f^{-1}(\mathbb{R})$ is an element of X and vice versa. Suppose first that x is an arbitrary element of $f^{-1}(\mathbb{R})$. Then by Definition 1.18, $x \in X$, as desired. Now suppose that $x \in X$. Then by Definition 1.16, $f(x) \in \mathbb{R}$. It follows by Definition 1.18 that $x \in f^{-1}(\mathbb{R})$, as desired. □

Exercise 9.2. Let $f : X \to \mathbb{R}$. Let $A \subset X$ and $B \subset \mathbb{R}$. Show that

$$f(f^{-1}(B)) \subset B$$
$$A \subset f^{-1}(f(A))$$

Give examples to show that the inclusions can be proper.

Proof. To prove that $f(f^{-1}(B)) \subset B$, Definition 1.3 tells us that it will suffice to show that every $y \in f(f^{-1}(B))$ is an element of B. Let y be an arbitrary element of $f(f^{-1}(B))$. Then by Definition 1.18, $y = f(x)$ for some $x \in f^{-1}(B)$. By Definition 1.18 again, $f(x) \in B$. Therefore, since $y = f(x)$, it follows that $y \in B$, as desired.

To prove that $A \subset f^{-1}(f(A))$, Definition 1.3 tells us that it will suffice to show that every $x \in A$ is an element of $f^{-1}(f(A))$. Let x be an arbitrary element of A. Then by Definition 1.18, $f(x) \in f(A)$. Therefore, by Definition 1.18, we have $x \in f^{-1}(f(A))$, as desired.

Let $X = \{1, 2\}$ and let $f : X \to \mathbb{R}$ be defined by $f(1) = 3$ and $f(2) = 3$. If we let $B = \{3, 4\}$, then $f(f^{-1}(B)) = \{3\} \subsetneq \{3, 4\}$. Additionally, if we let $A = \{1\}$, then $A \subsetneq f^{-1}(f(A)) = \{1, 2\}$. $\square$

Exercise 9.3. Let $f : X \to \mathbb{R}$. Let $A \subset X$ and $B \subset \mathbb{R}$. Then $f(A) \subset B \iff A \subset f^{-1}(B)$.

Proof. Suppose first that $f(A) \subset B$. To prove that $A \subset f^{-1}(B)$, Definition 1.3 tells us that it will suffice to show that every $x \in A$ is an element of $f^{-1}(B)$. Let x be an arbitrary element of A. Then by Definition 1.18, $f(x) \in f(A)$. It follows by the hypothesis and Definition 1.3 that $f(x) \in B$. Therefore, by Definition 1.18 again, $x \in f^{-1}(B)$.

Now suppose that $A \subset f^{-1}(B)$. To prove that $f(A) \subset B$, Definition 1.3 tells us that it will suffice to show that every $y \in f(A)$ is an element of B. Let y be an arbitrary element of $f(A)$. Then by Definition 1.18, $y = f(x)$ for some $x \in A$. It follows by the hypothesis and Definition 1.3 that $x \in f^{-1}(B)$. Consequently, by Definition 1.18 again, $f(x) \in B$. Therefore, since $y = f(x)$, $y \in B$. $\square$

Definition 9.4. Let $X \subset \mathbb{R}$. A function $f : X \to \mathbb{R}$ is **continuous** if for every open set $U \subset \mathbb{R}$, the preimage $f^{-1}(U)$ is open in X.

Proposition 9.5. *Let $X \subset \mathbb{R}$. A function $f : X \to \mathbb{R}$ is continuous if and only if for every closed set $F \subset \mathbb{R}$, the preimage $f^{-1}(F)$ is closed in X.*

Proof. Suppose first that f is continuous. We seek to prove that for every closed set $F \subset \mathbb{R}$, the preimage $f^{-1}(F)$ is closed in X. Let F be an arbitrary closed subset of $\mathbb{R}$. Then by Definition 4.8, $F = \mathbb{R} \setminus U$ for some open set $U \subset \mathbb{R}$. It follows by Definition 9.4 since f is continuous that $f^{-1}(U)$ is open in X. Additionally, by consecutive applications of Lemma 9.1, $f^{-1}(F) = f^{-1}(\mathbb{R} \setminus U) = f^{-1}(\mathbb{R}) \setminus f^{-1}(U) = X \setminus f^{-1}(U)$. Therefore, since $f^{-1}(U)$ is open in X, Exercise 8.13 implies that $X \setminus f^{-1}(U) = f^{-1}(F)$ is closed in X.

The proof is symmetric in the other direction. $\square$

Definition 9.6. Let $X \subset Y \subset \mathbb{R}$ and let $f : Y \to \mathbb{R}$. Then the **restriction** (of f to X), written $f|_X$ is the function $f|_X : X \to \mathbb{R}$ defined by

$$f|_X(x) = f(x)$$

for all $x \in X$.

Proposition 9.7. *Let $X \subset Y \subset \mathbb{R}$. If $f : Y \to \mathbb{R}$ is continuous, then the restriction of f to X is continuous.*

Proof. To prove that $f|_X$ is continuous, Definition 9.4 tells us that it will suffice to show that for every open set $U \subset \mathbb{R}$, the preimage $f|_X^{-1}(U)$ is open in X. Let U be an open subset of $\mathbb{R}$. Then

$$\begin{aligned}
f|_X^{-1}(U) &= \{x \in X \mid f|_X(x) \in U\} && \text{Definition 1.18} \\
&= \{x \in X \mid f(x) \in U\} && \text{Definition 9.6} \\
&= \{x \in Y \mid f(x) \in U\} \cap X && \text{Script 1} \\
&= f^{-1}(U) \cap X && \text{Definition 1.18} \\
&= (Y \cap G) \cap X && \text{Definitions 9.4 and 8.11}
\end{aligned}$$

$$= X \cap G \qquad \text{Script 1}$$

Since $f|_X^{-1}(U) = X \cap G$ where G is an open set, Definition 8.11 asserts that $f|_X^{-1}(U)$ is open in X. $\square$

Exercise 9.8. Show that for any $X \subsetneq \mathbb{R}$ that is not open and any continuous function $f : X \to \mathbb{R}$, there is an open set U for which $f^{-1}(U)$ is open in X but is not open in $\mathbb{R}$.

Proof. We will prove that $\mathbb{R}$ is an open set for which $f^{-1}(\mathbb{R})$ is open in X but not in $\mathbb{R}$. First, by Theorem 5.1, $\mathbb{R}$ is open. Next, by Lemma 9.1, $f^{-1}(\mathbb{R}) = X$. It follows since $f^{-1}(\mathbb{R}) = X = X \cap \mathbb{R}$ (where $\mathbb{R}$ is an open set) by Definition 8.11 that $f^{-1}(\mathbb{R})$ is open in X. Last, since X is not open (in $\mathbb{R}$) by definition, $f^{-1}(\mathbb{R}) = X$ is not open in $\mathbb{R}$. $\square$

Definition 9.9. The function $f : X \to \mathbb{R}$ is **continuous** (at $x \in X$) if for every region R containing $f(x)$, there exists an open set S containing x such that $S \cap X \subset f^{-1}(R)$.

Theorem 9.10. *The function $f : X \to \mathbb{R}$ is continuous if and only if it is continuous at every $x \in X$.*

Proof. Suppose first that f is continuous, and suppose for the sake of contradiction that f is not continuous at every $x \in X$. Then by Definition 9.9, there exists some $x \in X$ such that f is not continuous at x. Thus, there exists a region R with $f(x) \in R$ such that for all open sets S containing x, $S \cap X \not\subset f^{-1}(R)$. Since f is continuous by hypothesis and R is open by Corollary 4.11, $f^{-1}(R)$ is open in X. It follows by Definition 8.11 that $f^{-1}(R) = X \cap S$ for some open set S. But this implies that $f^{-1}(R) \not\subset f^{-1}(R)$, a contradiction.

Now suppose that f is continuous at every $x \in X$. To prove that f is continuous, Definition 9.4 tells us that it will suffice to show that for every open set $U \subset \mathbb{R}$, the preimage $f^{-1}(U)$ is open in X. We divide into two cases ($f^{-1}(U) = \emptyset$ and $f^{-1}(U) \neq \emptyset$). If $f^{-1}(U) = \emptyset$, then since $\emptyset \cap X = \emptyset$ by Script 1 where $\emptyset$ is open by Theorem 5.1, Definition 8.11 tells us that $\emptyset = f^{-1}(U)$ is open in X, as desired. On the other hand, if $f^{-1}(U) \neq \emptyset$, Definition 8.11 tells us that it will suffice to show that $f^{-1}(U) = S \cap X$ where S is an open set. We first seek to show that for every $x \in f^{-1}(U)$, there exists an open set S_x containing x such that $S_x \cap X \subset f^{-1}(U)$. Let x be an arbitrary element of $f^{-1}(U)$. It follows by Definition 1.18 that $f(x) \in U$. Thus, since U is open, we have by Theorem 4.10 that there exists a region R such that $f(x) \in R$ and $R \subset U$. Consequently, since R is open by Corollary 4.11, we have by Definition 9.9 that there exists an open set S_x containing x such that $S_x \cap X \subset f^{-1}(R)$. Additionally, Script 1 tells us based off of the fact that $R \subset U$ that $f^{-1}(R) \subset f^{-1}(U)$. Thus, by subset transitivity, $S_x \cap X \subset f^{-1}(U)$. At this point, let $S = \bigcup_{x \in f^{-1}(U)} S_x$. It follows immediately from Corollary 4.18 that S is open. Additionally, since the intersection of each set in the union with X is a subset of $f^{-1}(U)$, it follows by Script 1 that $S \cap X \subset f^{-1}(U)$. Furthermore, for all $x \in f^{-1}(U)$, Definition 1.18 asserts that $x \in X$. In addition, we have defined an S_x such that $x \in S_x$. These last two results combined demonstrate by Definition 1.6 that $x \in S \cap X$. Thus, by Definition 1.3, $f^{-1}(U) \subset S \cap X$. Consequently, by Theorem 1.7, $f^{-1}(U) = S \cap X$. Since $f^{-1}(U)$ is the intersection of X with an open set, Definition 8.11 asserts that it is open in X, as desired. $\square$

2/18: **Theorem 9.11.** *Suppose that $f : X \to \mathbb{R}$ is continuous. If X is connected, then $f(X)$ is connected.*

Proof. This will be a proof by contrapositive; as such, suppose that $f(X)$ is disconnected. Then by Definition 4.22, $f(X) = A \cup B$ where A, B are nonempty, disjoint sets that are open in $f(X)$. It follows from the last condition by Definition 8.11 that $A = G \cap f(X)$ and $B = H \cap f(X)$, where G, H are open sets. Since for all $x \in X$, $f(x) \in A$ or $f(x) \in B$, Definitions 1.2 and 1.6 imply that for all $x \in X$, $f(x) \in G$ or $f(x) \in H$. Thus, by Script 1, $X \subset f^{-1}(G) \cup f^{-1}(H)$. Additionally, we know by Definition 1.18 that for all $x \in f^{-1}(G) \cup f^{-1}(H)$, $x \in X$. Thus, by Definition 1.3, $f^{-1}(G) \cup f^{-1}(H) \subset X$. Consequently, by Theorem 1.7, we have that $X = f^{-1}(G) \cup f^{-1}(H)$.

To show that $f^{-1}(G)$ and $f^{-1}(H)$ are disjoint, Definition 1.9 tells us that it will suffice to verify that $f^{-1}(G) \cap f^{-1}(H) = \emptyset$. As such, suppose for the sake of contradiction that $x \in f^{-1}(G) \cap f^{-1}(H)$. Then by Definition 1.6, $x \in f^{-1}(G)$ and $x \in f^{-1}(H)$. Thus, by multiple applications of Definition 1.18, $x \in X$, $f(x) \in G$, and $f(x) \in H$. It follows from the first condition by Definition 1.18 that $f(x) \in f(X)$. The facts that $f(x) \in f(X)$ and $f(x) \in G$ imply by Definitions 1.6 and 1.2 that $f(x) \in A$. Similarly, $f(x) \in B$. But these last two statements imply by Definition 1.6 that $f(x) \in A \cap B$, a contradiction.

To show that $f^{-1}(G)$ and $f^{-1}(H)$ are nonempty, Definition 1.8 tells us that it will suffice to find an element of each set. As previously mentioned, A, B are nonempty. Thus, by consecutive applications of Definition 1.8, there exist $f(x) \in A$ and $f(y) \in B$. Consequently, by Definitions 1.2 and 1.6, $f(x) \in G$ and $f(y) \in H$. Therefore, by consecutive applications of Definition 1.18, $x \in f^{-1}(G)$ and $y \in f^{-1}(H)$, as desired.

To show that $f^{-1}(G)$ and $f^{-1}(H)$ are open in X, Definition 9.4 tells us that it will suffice to verify (since f is continuous by hypothesis) that G, H are open subsets of $\mathbb{R}$. But by definition, they are exactly that. $\square$

Exercise 9.12. Use Theorem 9.11 to prove that if $f : [a, b] \to \mathbb{R}$ is continuous, then for every point p between $f(a)$ and $f(b)$, there exists c such that $a < c < b$ and $f(c) = p$.

Proof. Suppose that $a < b$. Then by Lemma 8.3, $[a, b]$ is an interval. Thus, by Theorem 8.15, $[a, b]$ is connected. It follows by Theorem 9.11 that $f([a, b])$ is connected. Consequently, by Theorem 8.15, $f([a, b])$ is an interval. We divide into three cases ($f(a) < f(b)$, $f(a) > f(b)$, and $f(a) = f(b)$).

First, suppose that $f(a) < f(b)$, and let p be an arbitrary point between $f(a)$ and $f(b)$ (we know that at least one such point exists by Theorem 5.2). Then by Definition 3.6, $f(a) < p < f(b)$. Now $a, b \in [a, b]$ by Equations 8.1, so by Definition 1.18, $f(a), f(b) \in f([a, b])$. It follows by Definition 8.2 since $f([a, b])$ is an interval that $[f(a), f(b)] \subset f([a, b])$. Thus, since $f(a) < p < f(b)$ implies $p \in [f(a), f(b)]$ by Equations 8.1, Definition 1.3 asserts that $p \in f([a, b])$. Consequently, by Definition 1.18, $p = f(c)$ for some $c \in [a, b]$. Additionally, since $f(a) < p < f(b)$, we know that $p \neq f(a)$ and $p \neq f(b)$. It follows that $p = f(c)$ for some $c \in (a, b)$, as desired.

The proof of the second case is symmetric to that of the first.

Third, suppose that $f(a) = f(b)$. This implies that there are no points p between $f(a)$ and $f(b)$ by Definition 3.6, so the statement is vacuously true in this case. $\square$

Lemma 9.13. *If $f : (a, b) \to \mathbb{R}$ is continuous and injective, then f is either strictly increasing or strictly decreasing on (a, b).*

Proof. Suppose for the sake of contradiction that f is neither strictly increasing nor strictly decreasing. Then by Definition 8.16 there exist $x, y, z \in (a, b)$ with $x < y < z$ such that $f(x) \leq f(y)$ and $f(z) \leq f(y)$. Additionally, since f is injective and x, y, z are distinct, we have by Definition 1.20 that $f(x) < f(y)$ and $f(z) < f(y)$.

We divide into two cases ($f(x) < f(z)$ and $f(x) > f(z)$). Suppose first that $f(x) < f(z)$. Since $x, y \in (a, b)$, we have that $[x, y] \subset (a, b)$. It follows by Proposition 9.7 that $f|_{[x,y]} : [x, y] \to \mathbb{R}$ is continuous. Thus, we have by the supposition and Definition 9.6 that $f|_{[x,y]}(x) < f|_{[x,y]}(z) < f|_{[x,y]}(y)$. It follows by Exercise 9.12 that there exists c with $x < c < y$ such that $f|_{[x,y]}(c) = f|_{[x,y]}(z)$. Consequently, by Definition 9.6, $f(c) = f(z)$. Thus, by Definition 1.20, $c = z$. But this implies that $x < z < y$, contradicting the fact that $x < y < z$. The proof is symmetric in the other case. $\square$

2/23: **Theorem 9.14.** *If $f : (a, b) \to \mathbb{R}$ is continuous and injective, then the inverse function $g : f((a, b)) \to (a, b)$ is continuous.*

Lemma. *Let $f : (a, b) \to \mathbb{R}$ be continuous and injective, and let $(x, y) \subset (a, b)$ be a region. Then $f((x, y))$ is also a region.*

Proof. Since $f : (a, b) \to \mathbb{R}$ is continuous and injective, Lemma 9.13 implies that f is either strictly increasing or strictly decreasing on (a, b). We now divide into two cases.

Suppose first that f is strictly increasing. To prove that $f((x, y))$ is a region, Definition 3.10 tells us that it will suffice to show that $f((x, y)) = (f(x), f(y))$. To show this, Definition 1.2 tells us that it will suffice to verify that every $p \in f((x, y))$ is an element of $(f(x), f(y))$ and vice versa. Let p be an arbitrary element of $f((x, y))$. Then by Definition 1.18, $p = f(z)$ for some $z \in (x, y)$. Since $z \in (x, y)$, we have by Equations 8.1 that $x < z < y$. Since f is strictly increasing on (a, b), by Definition 8.16, $x < z < y$ implies that $f(x) < f(z) < f(y)$. But this implies by Equations 8.1 that $f(z) = p$ is an element of $(f(x), f(y))$, as desired. Now let p be an arbitrary element of $(f(x), f(y))$. Then by Equations 8.1, $f(x) < p < f(y)$. We seek to prove that $[x, y] \subset (a, b)$. Let q be an arbitrary element of $[x, y]$. Then by Equations 8.1, $x \leq q \leq y$. Additionally, since $x, y \in (a, b)$, Equations 8.1 imply that $a < x < b$ and $a < y < b$. Thus, $a < x \leq q \leq y < b$, meaning by Equations 8.1 that $q \in (a, b)$. Consequently, by Definition 1.3, $[x, y] \subset (a, b)$. If we now consider the restriction $f|_{[x,y]}$, we have by Proposition 9.7 that $f|_{[x,y]}$ is continuous. Thus, since $f|_{[x,y]} : [x,y] \to \mathbb{R}$ is continuous and $f|_{[x,y]}(x) = f(x) < p < f(y) = f|_{[x,y]}(y)$ (by Definition 9.6), Exercise 9.12 implies that there exists $c \in (x,y)$ such that $f|_{[x,y]}(c) = f(c) = p$. But by Definition 1.18, this implies that $p \in f((x,y))$.

The proof is symmetric in the other case. $\square$

Proof of Theorem 9.14. We first show that g exists, and then show that it is continuous.

To prove that g is a function, Definition 1.16 tells us that it will suffice to show that for all $y \in f((a,b))$, there exists a unique $x \in (a,b)$ such that $g(y) = x$. We will first show that for each y, such an element exists, and then show that it is unique. Let y be an arbitrary element of $f((a,b))$. Then by Definition 1.18, $y = f(x)$ for some $x \in (a,b)$. Thus, since we require that $g(f(x')) = x'$ and $f(g(y')) = y'$ for g to be an inverse function, we assign $g(y) = x$. Now suppose that $g(y) = x_1$ and $g(y) = x_2$. Then by the definition of g, $f(x_1) = y$ and $f(x_2) = y$. It follows that $f(x_1) = f(x_2)$, implying since f is injective by Definition 1.20 that $x_1 = x_2$, as desired.

To prove that g is continuous, Definition 9.15 tells us that it will suffice to show that for every $U \subset (a,b)$ that is open in (a,b), the preimage $g^{-1}(U)$ is open in $f((a,b))$. Let U be an arbitrary subset of (a,b) that is open in (a,b). To show that $g^{-1}(U)$ is open in $f((a,b))$, Definition 8.11 tells us that it will suffice to confirm that $g^{-1}(U) = f((a,b)) \cap G$, where G is an open set.

To begin, we have

$$\begin{aligned} g^{-1}(U) &= \{y \in f((a,b)) \mid g(y) \in U\} && \text{Definition 1.18} \\ &= \{f(x) \in f((a,b)) \mid g(f(x)) \in U\} && \text{Definition 1.18} \end{aligned}$$

By the definition of g, we have $g(f(x)) = x$.

$$\begin{aligned} &= \{f(x) \in f((a,b)) \mid x \in U\} \\ &= \{f(x) \in \{f(x') \in \mathbb{R} \mid x' \in (a,b)\} \mid x \in U\} && \text{Definition 1.18} \end{aligned}$$

This next transition is mostly notational in nature. $f(x)$ being an element of the set of all $f(x') \in \mathbb{R}$ that meet a certain condition means that $f(x) \in \mathbb{R}$. Additionally, since that condition is $x' \in (a,b)$, we know that $x \in (a,b)$. But if $x \in (a,b)$ and (from the condition in the original set) $x \in U$, we have by Definition 1.6 that $x \in U \cap (a,b)$.

$$\begin{aligned} &= \{f(x) \in \mathbb{R} \mid x \in U \cap (a,b)\} \\ &= f(U \cap (a,b)) && \text{Definition 1.18} \end{aligned}$$

By definition, U is open in (a,b). Consequently, by Definition 8.11, $U = (a,b) \cap V$ where V is open.

$$\begin{aligned} &= f(((a,b) \cap V) \cap (a,b)) \\ &= f((a,b) \cap V) && \text{Definition 1.6} \\ &= f((a,b)) \cap f(V) && \text{Additional Exercise 9.2b} \end{aligned}$$

All that's left at this point is to prove that $f(V)$ is open. By Theorem 4.14, $V = \bigcup_{\lambda \in I}\{R_\lambda\}$ is a collection of regions. It follows by an extension of Additional Exercise 9.2a that $f(V) = \bigcup_{\lambda \in I}\{f(R_\lambda)\}$. Additionally, by the lemma, each $f(R_\lambda)$ is a region; hence, by Corollary 4.11, each $f(R_\lambda)$ is open. Thus, $f(V)$ is the union of a collection of open subsets of $\mathbb{R}$, so by Corollary 4.18, $f(V)$ is open. $\square$

We denote the inverse function g by f^{-1}. In this result, g has codomain (a,b) but our definition of continuity (Definition 9.4) only applies to functions with codomain $\mathbb{R}$. Our definitions/results are easily adapted. The definitions are as given below and we give a sample theorem. Other results can be adjusted in a similar fashion.

Definition 9.15. Let $X, Y \subset \mathbb{R}$. A function $f : X \to Y$ is **continuous** if for every U that is open in Y, the preimage $f^{-1}(U)$ is open in X.

Definition 9.16. The function $f : X \to Y$ is **continuous** (at $x \in X$) if for every region R containing $f(x)$, there exists an open set S containing x such that $S \cap X \subset f^{-1}(R \cap Y)$.

Theorem 9.17. *The function $f : X \to Y$ is continuous if and only if it is continuous at every $x \in X$.*

Additional Exercises

2. Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$. Let $A, B \subset \mathbb{R}$. Either prove or give a counterexample to each of the following:

   a) $f(A \cup B) = f(A) \cup f(B)$.

   b) $f(A \cap B) = f(A) \cap f(B)$.

   c) $f(A \setminus B) = f(A) \setminus f(B)$.

9.2 Discussion

2/16: • For Proposition 9.5, do an iff proof — use $\iff$ steps instead of back-and-forth work.

2/18: • To what extent do we need casework in Theorem 9.14?

Script 10

Compactness

10.1 Journal

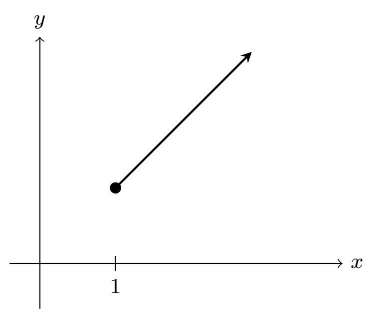
2/23: **Definition 10.1.** We say that a function $f : A \to \mathbb{R}$ is **bounded** if $f(A)$ is a bounded subset of $\mathbb{R}$. We say that f is **bounded above** if $f(A)$ is bounded above and that f is **bounded below** if $f(A)$ is bounded below.

If $f : A \to \mathbb{R}$ is bounded above, we say that f **attains** (its least upper bound) if there is some $a \in A$ such that $f(a) = \sup f(A)$. Similarly, if $f : A \to \mathbb{R}$ is bounded below, we say that f **attains** (its greatest lower bound) if there is some $a \in A$ such that $f(a) = \inf f(A)$.

Exercise 10.2. If possible, find examples of each of the following: a picture suffices.

a) A continuous function on $[1, \infty)$ that is not bounded above.

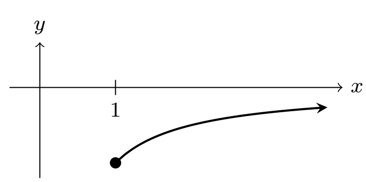
Example. Let $f : [1, \infty) \to \mathbb{R}$ be defined by $f(x) = x$.



□

b) A continuous function on $[1, \infty)$ that is bounded above but does not attain its least upper bound.

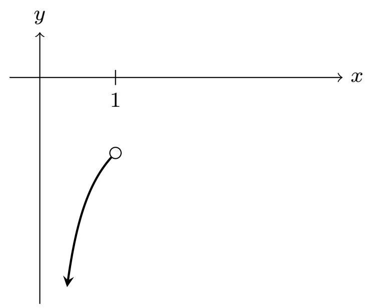
Example. Let $f : [1, \infty) \to \mathbb{R}$ be defined by $f(x) = -\frac{1}{x}$.



□

c) A continuous function on $(0, 1)$ that is not bounded below.

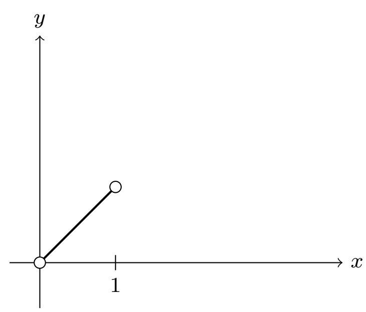
Example. Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) = -\frac{1}{x}$.



$\square$

d) A continuous function on $(0, 1)$ that is bounded below but does not attain its greatest lower bound.

Example. Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) = x$.



$\square$

Definition 10.3. Let X be a subset of $\mathbb{R}$ and let $\mathcal{G} = \{G_\lambda\}_{\lambda\in\Lambda}$ be a collection of subsets of $\mathbb{R}$. We say that $\mathcal{G}$ is a **cover** of X if every point of X is in some G_λ, or in other words:

$$X \subset \bigcup_{\lambda\in\Lambda} G_\lambda$$

We say that the collection $\mathcal{G}$ is an **open cover** if each G_λ is open.

Definition 10.4. Let X be a subset of $\mathbb{R}$. X is **compact** if for every open cover $\mathcal{G}$ of X, there exists a finite subset $\mathcal{G}' \subset \mathcal{G}$ that is also an open cover.

A good summary of the definition of compactness is "every open cover contains a finite subcover."

Exercise 10.5. Show that all finite subsets of $\mathbb{R}$ are compact.

Proof. Let X be an arbitrary finite subset of $\mathbb{R}$. To prove that X is compact, Definition 10.4 tells us that it will suffice to show that for every open cover $\mathcal{G}$ of X, there exists a finite subset $\mathcal{G}' \subset \mathcal{G}$ that is also an open cover. Let $\mathcal{G}$ be an arbitrary open cover of X. By Definition 10.3, every point $x \in X$ is an element of G_λ for some $G_\lambda \in \mathcal{G}$. Thus, for each $x \in X$, let $G_x \in \mathcal{G}$ be a set that contains x. Since X is finite, we do not need the axiom of choice to make these selections. Additionally, since there are finitely many X, we know that there are finitely many distinct G_x[1]. Thus, $\mathcal{G}' = \{G_x\}_{x\in X}$ is finite. Additionally, it is a subset of $\mathcal{G}$ by definition (each G_x is defined to be an element of $\mathcal{G}$). Furthermore, each G_x is open (again, each G_x is an element of $\mathcal{G}$, which is a collection of open sets by definition). Lastly, every point $x \in X$ is an element of $G_x \in \mathcal{G}'$, so $\mathcal{G}'$ is a cover. Therefore, by Definition 10.3, $\mathcal{G}' \subset \mathcal{G}$ is a finite open cover of X. $\square$

[1] In fact, the number of G_x is less than or equal to the cardinality of X since we may choose the same G_x for multiple x but may not choose multiple G_x for the same x.

2/25: **Lemma 10.6.** *No finite collection of regions covers* $\mathbb{R}$.

Lemma. *If* X *is nonempty, then* $\emptyset$ *does not cover* X.

Proof. Suppose for the sake of contradiction that $\emptyset$ covers X. By Definition 1.8, there exists $x \in X$. It follows by Definition 10.3 that $x \in \bigcup \emptyset$. But since $\bigcup \emptyset = \emptyset$, we have by Definition 1.2 that $x \in \emptyset$, contradicting Definition 1.8. ☐

Proof of Lemma 10.6. Suppose for the sake of contradiction that $\mathcal{G}$ is a finite collection of regions that covers $\mathbb{R}$. Since $\mathbb{R}$ is nonempty (by Axiom 1), the lemma asserts that $\mathcal{G} \neq \emptyset$. It follows that $\mathcal{G} = \{(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)\}$. Considering the set $\{a_1, \dots, a_n\}$ of lower bounds of all regions in $\mathcal{G}$, we can determine that it is nonempty and finite since $\mathcal{G}$ itself is nonempty and finite. Thus, by Lemma 3.4, $\{a_1, \dots, a_n\}$ has a first point a_i. It follows by Axiom 3 and Definition 3.3 that there exists a point $x \in \mathbb{R}$ such that $x < a_i$. Since $\mathcal{G}$ is an open cover of $\mathbb{R}$, we know by Definition 10.3 that $x \in (a_j, b_j)$ for some $(a_j, b_j) \in \mathcal{G}$. But this implies by Equations 8.1 that $a_j < x$ for some $a_j \in \{a_1, \dots, a_n\}$, contradicting the fact that $x < a_i \leq a_j$ for all $a_j \in \{a_1, \dots, a_n\}$ (the latter inequality being true by Definition 3.3). ☐

Theorem 10.7. $\mathbb{R}$ *is not compact.*

Proof. To prove that $\mathbb{R}$ is not compact, Definition 10.4 tells us that it will suffice to find an open cover $\mathcal{G}$ of $\mathbb{R}$ such that no finite subset $\mathcal{G}' \subset \mathcal{G}$ exists that is also an open cover. Let $\mathcal{G}$ be the collection of all regions in $\mathbb{R}$.

To confirm that $\mathcal{G}$ is an open cover, Definition 10.3 tells us that it will suffice to demonstrate that every $x \in \mathbb{R}$ is an element of R_λ for some region $R_\lambda \in \mathcal{G}$, and that every R_λ is open. For the first condition, let x be an arbitrary element of $\mathbb{R}$. Clearly, we have that $x \in (x-1, x+1)$ where $(x-1, x+1)$ is a region. Thus, x is an element of a set in $\mathcal{G}$, as desired. As to the other condition, we have by Corollary 4.11 that every region (i.e., every set in $\mathcal{G}$) is open, as desired.

To confirm that no finite subset $\mathcal{G}' \subset \mathcal{G}$ exists that is also an open cover, we invoke Lemma 10.6, which asserts as much[2]. ☐

Exercise 10.8. Show that regions are not compact.

Proof. Let (a, b) be an arbitrary region. To prove that (a, b) is not compact, Definition 10.4 tells us that it will suffice to find an open cover $\mathcal{G}$ of $\mathbb{R}$ such that no finite subset $\mathcal{G}' \subset \mathcal{G}$ exists that is also an open cover. Let $\mathcal{G}$ be the collection of all regions (a, c) where $c \in (a, b)$.

To confirm that $\mathcal{G}$ is an open cover, Definition 10.3 tells us that it will suffice to demonstrate that every $x \in (a, b)$ is an element of (a, c) for some $(a, c) \in \mathcal{G}$, and that every (a, c) is open. For the first condition, let x be an arbitrary element of (a, b). Then by Equations 8.1, $a < x < b$. It follows by Theorem 5.2 that there exists some c such that $x < c < b$. Since $a < x < c < b$, we have by consecutive applications of Equations 8.1 that $x \in (a, c)$ and $c \in (a, b)$. The latter result shows that $(a, c) \in \mathcal{G}$, as desired. As to the other condition, we have by Corollary 4.11 that every region (notably including all those in $\mathcal{G}$) is open, as desired.

To confirm that no finite subset $\mathcal{G}' \subset \mathcal{G}$ exists that is also an open cover, let $\mathcal{G}'$ be an arbitrary finite subset of $\mathcal{G}$. Since (a, b) is nonempty, the lemma from Lemma 10.6 asserts that $\mathcal{G}' \neq \emptyset$. It follows that $\mathcal{G}' = \{(a, c_1), (a, c_2), \dots, (a, c_n)\}$. Considering the set $\{c_1, \dots, c_n\}$ of upper bounds of all regions in $\mathcal{G}'$, we can determine that it is nonempty and finite since $\mathcal{G}'$ itself is nonempty and finite. Thus, by Lemma 3.4, $\{c_1, \dots, c_n\}$ has a last point c_i. Since $c_i \in (a, b)$, Equations 8.1 assert that $a < c_i < b$. Consequently, by Theorem 5.2, there exists a point $x \in \mathbb{R}$ such that $c_i < x < b$. Since $a < c_i < x < b$, we have by Equations 8.1 that $x \in (a, b)$. Since $\mathcal{G}'$ is an open cover of (a, b), we know by Definition 10.3 that $x \in (a, c_j)$ for some $(a, c_j) \in \mathcal{G}'$. But this implies by Equations 8.1 that $x < c_j$ for some $c_j \in \{c_1, \dots, c_n\}$, contradicting the fact that $c_j \leq c_i < x$ (the latter inequality being true by Definition 3.3). ☐

[2]Technically, Lemma 10.6 forbids $\mathcal{G}'$ from being a cover, but a set that is not a cover cannot be an open cover by Definition 10.3.

Theorem 10.9. *If X is compact, then X is bounded.*

Proof. We divide into two cases ($X = \emptyset$ and $X \neq \emptyset$). Suppose first that $X = \emptyset$. Then if we let a, b be arbitrary elements of $\mathbb{R}$, it is vacuously true that $a \leq x$ for all $x \in X$ and $x \leq b$ for all $x \in X$. Therefore, by consecutive applications of Definition 5.6, a and b are lower and upper bounds of X, respectively, and thus X is bounded, as desired.

Now suppose that $X \neq \emptyset$. Let $\mathcal{G} = \{(x-1, x+1) \mid x \in X\}$. To confirm that $\mathcal{G}$ is an open cover of X, Definition 10.3 tells us that it will suffice to demonstrate that every $x \in X$ is an element of some set in $\mathcal{G}$, and that every set in $\mathcal{G}$ is open. For the first condition, let x be an arbitrary element of X. Clearly, we have that $x \in (x-1, x+1)$. Additionally, it follows from the fact that $x \in X$ that $(x-1, x+1) \in \mathcal{G}$. Thus, x is an element of a set in $\mathcal{G}$, as desired. As to the other condition, we have by Corollary 4.11 that every region (notably including all those in $\mathcal{G}$) is open, as desired.

Since $\mathcal{G}$ is an open cover of X and X is compact, Definition 10.4 asserts that there exists a finite subset $\mathcal{G}'$ of $\mathcal{G}$ that is also an open cover of X. Since X is nonempty and $\mathcal{G}'$ is a cover of X, we have by the lemma from Lemma 10.6 that $\mathcal{G}' \neq \emptyset$. It follows since $\mathcal{G}$ is a collection of regions that $\mathcal{G}' = \{(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)\}$. Considering the sets $\{a_1, \dots, a_n\}$ of lower bounds of all regions in $\mathcal{G}'$ and $\{b_1, \dots, b_n\}$ of upper bounds of all regions in $\mathcal{G}'$, we can determine that both are nonempty and finite since $\mathcal{G}'$ itself is nonempty and finite. Thus, by consecutive applications of Lemma 3.4, $\{a_1, \dots, a_n\}$ has a first point a_i and $\{b_1, \dots, b_n\}$ has a last point b_j. To confirm that a_i is a lower bound of X, Definition 5.6 tells us that it will suffice to demonstrate that $a_i \leq x$ for all $x \in X$. Let x be an arbitrary element of X. Then by Definition 10.3, $x \in (a_k, b_k)$ for some $(a_k, b_k) \in \mathcal{G}$. Thus, by Equations 8.1, $a_k < x < b_k$. Additionally, since a_i is the first point of $\{a_1, \dots, a_n\}$, Definition 3.3 asserts that $a_i \leq a_k$. Combining the last two results, we have by transitivity that $a_i < x$, which we may weaken to $a_i \leq x$, as desired. The proof that b_j is an upper bound is symmetric. Therefore, since X has a lower and an upper bound, Definition 5.6 implies that X is bounded. □

Lemma 10.10. *Let $X \subset \mathbb{R}$ and $p \in \mathbb{R} \setminus X$. Then $\mathcal{G} = \{\text{ext}(a, b) \mid p \in (a, b)\}$ is an open cover of X.*

Proof. To prove that $\mathcal{G}$ is an open cover of X, Definition 10.3 tells us that it will suffice to show that every $x \in X$ is an element of some set in $\mathcal{G}$, and that every set in $\mathcal{G}$ is open. For the first condition, let x, p be arbitrary elements of $X, \mathbb{R} \setminus X$, respectively. Since x, p are elements of disjoint sets by Script 1, we know that $x \neq p$. Thus, we can apply Theorem 3.22 to learn that there exist disjoint regions (c, d) and (a, b) containing x and p, respectively. We now seek to verify that $x \in \text{ext}(a, b)$. To do so, Definition 3.15 tells that it will suffice to verify that $x \notin (a, b)$, $x \neq a$, and $x \neq b$. First, suppose for the sake of contradiction that $x \in (a, b)$. Then since $x \in (c, d)$, too, $x \in (a, b) \cap (c, d)$, contradicting Definition 1.9 and the fact that $(a, b), (c, d)$ are disjoint. Second, suppose for the sake of contradiction that $x = a$. Since $x \in (c, d)$, we have by Equations 8.1 that $c < x = a < d$. We divide into two cases ($d \leq b$ and $b < d$). If $d \leq b$, then by Theorem 5.2, we can choose z such that $c < x = a < z < d \leq b$. It follows by the same logic as in the first case that $z \in (a, b) \cap (c, d)$, and we arrive at the same contradiction. If $b < d$, then we similarly choose $c < x = a < z < b < d$, and arrive at the same contradiction again. The proof of the third claim is symmetric to that of the second. Therefore, $x \in \text{ext}(a, b)$, so we have by the definition of $\mathcal{G}$ that x is an element of a set in $\mathcal{G}$, as desired. As to the other condition, we have by Corollary 4.21 that every exterior of a region (notably including all those in $\mathcal{G}$) is open, as desired. □

Theorem 10.11. *If X is compact, then X is closed.*

Proof. We divide into two cases ($X = \emptyset$ and $X \neq \emptyset$). Suppose first that $X = \emptyset$. Then by Theorem 4.2, X is closed, as desired.

Now suppose that $X \neq \emptyset$, and suppose for the sake of contradiction that X is not closed. Then by Definition 4.1, there exists a limit point p of X such that $p \notin X$. Since $p \in \mathbb{R}$ and $p \notin X$, Definition 1.11 implies that $p \in \mathbb{R} \setminus X$. Thus, by Lemma 10.10, $\mathcal{G} = \{\text{ext}(a, b) \mid p \in (a, b)\}$ is an open cover of X. Additionally, since X is compact by hypothesis, we have by Definition 10.4 that there exists a finite subset $\mathcal{G}' \subset \mathcal{G}$ that is also an open cover of X. Since X is nonempty and $\mathcal{G}'$ is a cover of X, we have by the lemma from Lemma 10.6 that $\mathcal{G}' \neq \emptyset$. It follows since $\mathcal{G}$ is a collection of exteriors of regions that $\mathcal{G}' = \{\text{ext}(a_1, b_1), \text{ext}(a_2, b_2), \dots, \text{ext}(a_n, b_n)\}$. Considering the sets $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$, we can determine that both are nonempty and finite since $\mathcal{G}'$ itself is nonempty and finite. Thus, by consecutive applications of Lemma 3.4, $\{a_1, \dots, a_n\}$ has a last point a_i and $\{b_1, \dots, b_n\}$ has a first point b_j. We now seek to verify that $p \in (a_i, b_j)$. By consecutive applications of the definition of $\mathcal{G}$, $p \in (a_i, b_i)$ and $p \in (a_j, b_j)$. Consequently, by consecutive applications of Equations 8.1, $a_i < p < b_i$ and $a_j < p < b_j$. Since $a_i < p < b_j$, we have by Equations 8.1 that $p \in (a_i, b_j)$, as desired. Thus, since $p \in (a_i, b_j)$ and $p \in LP(X)$, Definition 3.13 asserts that $(a_i, b_j) \cap (X \setminus \{p\}) \neq \emptyset$. Consequently, by Definition 1.8, there exists some $x \in (a_i, b_j) \cap (X \setminus \{p\})$. Thus, by Definitions 1.6 and 1.11, $x \in (a_i, b_j)$ and $x \in X$. Since $\mathcal{G}'$ is an open cover of X, it follows from the latter condition by Definition 10.3 that $x \in \text{ext}(a_k, b_k)$ for some $\text{ext}(a_k, b_k) \in \mathcal{G}'$. Consequently, by Lemma 3.16, $x < a_k$ or $b_k < x$. We now divide into two cases. If $x < a_k$, this contradicts the fact that $a_k \leq a_i < x$ (the former inequality being true by Definition 3.3 since a_i is the last point of $\{a_1, \dots, a_n\}$). If $b_k < x$, we arrive at a symmetric contradiction. □

It will turn out that the two properties of compactness in Theorems 10.9 and 10.11 characterize compact sets completely, meaning that every bounded closed set is compact. We will see this in Theorem 10.16. First, however, we need some preliminary results.

For the next three statements, fix points $a, b \in \mathbb{R}$ and suppose $\mathcal{G}$ is an open cover of $[a, b]$.

Lemma 10.12. *For all $s \in [a, b]$, there exists $G \in \mathcal{G}$ and $p, q \in \mathbb{R}$ such that $p < s < q$ and $[p, q] \subset G$.*

Proof. Let s be an arbitrary element of $[a, b]$. Since $\mathcal{G}$ is an open cover of $[a, b]$, Definition 10.3 implies that there exists a $G \in \mathcal{G}$ such that $s \in G$ and G is open. It follows from the latter condition by Theorem 4.10 that there exists a region (x, y) such that $s \in (x, y)$ and $(x, y) \subset G$. Since $s \in (x, y)$, we have by Equations 8.1 that $x < s < y$. Thus, by consecutive applications of Theorem 5.2, we can pick $p, q \in \mathbb{R}$ such that $x < p < s < q < y$. Clearly, $p < s < q$. To verify that $[p, q] \subset G$, Definition 1.3 tells us that it will suffice to confirm that every $z \in [p, q]$ is an element of G. Let z be an arbitrary element of $[p, q]$. Then by Equations 8.1, $p \leq z \leq q$. It follows since $x < p < q < y$ that $x < p \leq z \leq q < y$, meaning by Equations 8.1 that $z \in (x, y)$. Since $(x, y) \subset G$ by definition, we have by Definition 1.3 that $z \in G$, as desired. □

3/2: **Lemma 10.13.** *Let X be the set of all $x \in \mathbb{R}$ that are **reachable** from a, by which we mean the following: there exist $n \in \mathbb{N} \cup \{0\}$, $x_0, ..., x_n \in \mathbb{R}$, and $G_1, \dots, G_n \in \mathcal{G}$ such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = x$ and $[x_{i-1}, x_i] \subset G_i$ for $i = 1, \dots, n$. Note in particular that $a \in X$, by choosing $n = 0$. Then the point b is not an upper bound for X.*

Proof. Suppose for the sake of contradiction that b is an upper bound of X. We know by hypothesis that $a \in X$. Thus, X is a nonempty set that is bounded above, so by Theorem 5.17, $\sup X$ exists. Let $s = \sup X$. Since b is an upper bound of X, we have by Definition 5.7 that $s \leq b$. Additionally, since $a \in X$, we have by Definitions 5.7 and 5.6 that $a \leq s$. Combining these last two results, we have by Equations 8.1 that $s \in [a, b]$. Thus, by Lemma 10.12, there exists $G \in \mathcal{G}$ and $p, q \in \mathbb{R}$ such that $p < s < q$ and $[p, q] \subset G$. Additionally, we have by Lemma 5.11 that there exists an $x \in X$ such that $p < x \leq s$. Since $p < x \leq s < q$, we have by Equations 8.1 that $x \in [p, q]$. It follows by Definition 1.3 since $[p, q] \subset G$ that $x \in G$. Furthermore, by Theorem 5.2, there exists $s' \in \mathbb{R}$ such that $s < s' < q$. We will now demonstrate that $s' \in X$, which will contradict the previously proven statement that s is an upper bound on X.

To demonstrate that $s' \in X$, the definition of X tells us that it will suffice to confirm that there exist $n + 1 \in \mathbb{N} \cup \{0\}$, $x_0, \dots, x_{n+1} \in \mathbb{R}$, and $G_1, \dots, G_{n+1} \in \mathcal{G}$ such that $a = x_0 < \cdots < x_{n+1} = s'$ and $[x_{i-1}, x_i] \subset G_i$ for $i = 1, \dots, n + 1$. To begin, since $x \in X$, we have that there exist $n \in \mathbb{N} \cup \{0\}$, $x_0, \dots, x_n \in \mathbb{R}$, and $G_1, \dots, G_n \in \mathcal{G}$ such that $a = x_0 < \cdots < x_n = x$ and $[x_{i-1}, x_i] \subset G_i$ for $i = 1, \dots, n$. Let $x_{n+1} = s'$ and $G_{n+1} = G$. Then since we can carry over all of the variable assignments from the definition of x and add in $x_n = x \leq s < s' = x_{n+1}$ as well as $[x_n, x_{n+1}] = [x, s'] \subset [p, q] \subset G = G_{n+1}$, we have that $s' \in X$, as desired. □

Theorem 10.14. *The set $[a, b]$ is compact.*

Proof. To prove that $[a, b]$ is compact, Definition 10.4 tells us that it will suffice to show that for every open cover $\mathcal{G}$ of $[a, b]$, there exists a finite subset $\mathcal{G}' \subset \mathcal{G}$ that is also an open cover. Let $\mathcal{G}$ be an arbitrary open cover of $[a, b]$. By Lemma 10.13, b is not an upper bound of X where X is the set of all $x \in \mathbb{R}$ that are reachable from a. Thus, by Definition 5.6, there exists an $x \in X$ such that $x > b$. Since $x \in X$, we have by Lemma 10.13 that there exist $n \in \mathbb{N} \cup \{0\}$, $x_0, \dots, x_n \in \mathbb{R}$, and $G_1, \dots, G_n \in \mathcal{G}$ such that $a = x_0 < \cdots < x_n = x$ and $[x_{i-1}, x_i] \subset G_i$ for $i = 1, \dots, n$. Let $\mathcal{G}' = \{G_1, \dots, G_n\}$. We will now show that $\mathcal{G}'$ is finite, a subset of $\mathcal{G}$, and an open cover of $[a, b]$.

Clearly, $\mathcal{G}'$ is finite.

Since every element of $\mathcal{G}'$ is an element of $\mathcal{G}$ by definition, we have by Definition 1.3 that $\mathcal{G}' \subset \mathcal{G}$.

To show that $\mathcal{G}'$ is an open cover of $[a,b]$, Definition 10.3 tells us that it will suffice to verify that every $y \in [a,b]$ is an element of G_i for some $G_i \in \mathcal{G}'$, and that every G_i is open. For the first condition, suppose for the sake of contradiction that there exists a $y \in [a,b]$ such that $y \notin G_i$ for any $G_i \in \mathcal{G}'$. Then since $[x_{i-1}, x_i] \subset G_i$ for all $i = 1, \dots, n$, $y \notin [x_{i-1}, x_i]$ for any $i = 1, \dots, n$. It follows from Equations 8.1 that $y < x_{i-1}$ or $y > x_i$ for all $i = 1, \dots, n$. In particular, we have that $y < x_0 = a$ or $y > x_n = x$. We now divide into two cases. If $y < a$, then by Equations 8.1, $y \notin [a,b]$, a contradiction. If $y > x$, then since $x > b$, we have that $y > b$, which implies by Equations 8.1 that $y \notin [a,b]$, a contradiction. As to the other condition, we have by the definition of $\mathcal{G}$ that every G_i is open, as desired. ☐

Theorem 10.15. *A closed subset Y of a compact set $X \subset \mathbb{R}$ is compact.*

Lemma. *Y is closed in X if and only if Y is closed.*

Proof. Suppose first that Y is closed in X. Then by Definition 8.11, $Y = X \cap B$ where B is closed. Additionally, since X is compact, Theorem 10.11 asserts that X is closed. Therefore, since Y is the intersection of two closed sets, we have by Theorem 4.16 that Y is closed, as desired.

Now suppose that Y is closed. Since $Y = X \cap Y$ by Script 1, we have by Definition 8.11 that Y is closed in X, as desired. ☐

Proof of Theorem 10.15. To prove that Y is compact, Definition 10.4 tells us that it will suffice to show that for every open cover $\mathcal{G}$ of Y, there exists a finite subset $\mathcal{G}' \subset \mathcal{G}$ that is also an open cover. To do this, we will first demonstrate that $\mathcal{H} = \mathcal{G} \cup \{\mathbb{R} \setminus Y\}$ is an open cover of X. It will follow that there exists a finite subset $\mathcal{H}' \subset \mathcal{H}$ that is an open cover of X. Lastly, we will demonstrate that $\mathcal{G}' = \mathcal{H}' \setminus \{\mathbb{R} \setminus Y\}$ is the desired finite open cover subset of $\mathcal{G}$.

Let $\mathcal{G}$ be an arbitrary open cover of Y, and let $\mathcal{H} = \mathcal{G} \cup \{\mathbb{R} \setminus Y\}$. To demonstrate that $\mathcal{H}$ is an open cover of X, Definition 10.3 tells us that it will suffice to confirm that every $x \in X$ is an element of H for some $H \in \mathcal{H}$, and that every H is open. For the first condition, let x be an arbitrary element of X. We divide into two cases ($x \in Y$ and $x \notin Y$). If $x \in Y$, then since $\mathcal{G}$ is an open cover of Y, Definition 10.3 implies that $x \in G$ for some $G \in \mathcal{G}$. But since $\mathcal{G} \subset \mathcal{H}$, $x \in G$ for some $G \in \mathcal{H}$, as desired. On the other hand, if $x \notin Y$, then this combined with the fact that $x \in \mathbb{R}$ implies by Definition 1.11 that $x \in \mathbb{R} \setminus Y \in \mathcal{H}$, as desired. As to the other condition, let H be an arbitrary element of $\mathcal{H}$. We divide into two cases ($H \in \mathcal{G}$ and $H \notin \mathcal{G}$). If $H \in \mathcal{G}$, then by Definition 10.3, H is open, as desired. On the other hand, if $H \notin \mathcal{G}$, then $H = \mathbb{R} \setminus Y$ by Script 1. It follows since Y is closed by Definition 4.8 that $\mathbb{R} \setminus Y$ is open, so H is open, as desired.

Since $\mathcal{H}$ is an open cover of X and X is compact, we have by Definition 10.4 that there exists a finite subset $\mathcal{H}' \subset \mathcal{H}$ that is also an open cover of X. Let $\mathcal{G}' = \mathcal{H}' \setminus \{\mathbb{R} \setminus Y\}$. By Script 1, $\mathcal{G}' \subset \mathcal{G}$. Additionally, since $\mathcal{H}'$ is finite, Script 1 also implies that $\mathcal{G}'$ is finite. However, to demonstrate that $\mathcal{G}'$ is an open cover of Y, Definition 10.3 tells us that we must confirm that every $y \in Y$ is an element of G for some $G \in \mathcal{G}'$, and that every G is open. For the first condition, let y be an arbitrary element of Y. Then since $Y \subset X$, Definition 10.3 asserts that $y \in H$ for some $H \in \mathcal{H}'$. Additionally, since $y \in Y$, Definition 1.11 implies that $y \notin \mathbb{R} \setminus Y$. Thus, $H \neq \mathbb{R} \setminus Y$, which implies by Definition 1.11 that $H \in \mathcal{H}' \setminus \{\mathbb{R} \setminus Y\} = \mathcal{G}'$. Therefore, $y \in H$ for some $H \in \mathcal{G}'$, as desired. As to the other condition, since every $G \in \mathcal{G}'$ is an element of $\mathcal{G}$ (i.e., open by Definition 10.3), every $G \in \mathcal{G}'$ is open, as desired. ☐

Theorem 10.16. *Let $X \subset \mathbb{R}$. X is compact if and only if X is closed and bounded.*

Proof. Suppose first that X is compact. Then by Theorems 10.11 and 10.9, respectively, X is closed and bounded.

Now suppose that X is closed and bounded. It follows from the latter condition by Definition 5.6 that X has a lower bound a and an upper bound b. Constructing the region $[a,b]$, we have from Theorem 10.14 that $[a,b]$ is compact. Additionally, we know that $X \subset [a,b]$ since $x \in X$ implies by consecutive applications of Definition 5.6 that $a \leq x \leq b$, from which it follows by Equations 8.1 that $x \in [a,b]$. Thus, X is a closed (by hypothesis) subset of a compact set, so by Theorem 10.15, X is compact. ☐

Lemma 10.17. *A compact set $X \subset \mathbb{R}$ with no limit points must be finite.*

Proof. Suppose for the sake of contradiction that X is infinite, and let x be an arbitrary element of X. Since X has no limit points, we know that $x \notin LP(X)$. Thus, by Definition 3.13, there exists a region R_x with $x \in R_x$ such that $R_x \cap (X \setminus \{x\}) = \emptyset$. Let $\mathcal{G} = \bigcup_{x \in X} R_x$, where R_x is similarly defined for each $x \in X$. Then since every $x \in R_x \in \mathcal{G}$ and every R_x is open by Corollary 4.11, Definition 10.3 implies that $\mathcal{G}$ is an open cover of X. It follows since X is compact that there exists a finite subset of $\mathcal{G}' \subset \mathcal{G}$ that is also an open cover of X. Additionally, since $\mathcal{G}'$ is finite whereas $\mathcal{G}$ is infinite (infinitely many x imply infinitely many R_x since each R_x contains only one x), we know that $\mathcal{G}' \neq \mathcal{G}$. Thus, there exists an $R_x \in \mathcal{G}$ such that $R_x \notin \mathcal{G}'$. However, since $\mathcal{G}'$ is an open cover of X, Definition 10.3 implies that $x \in R_y$ for some $R_y \in \mathcal{G}'$ where $y \neq x$. But since $x \in R_y$, $x \in X$, and $x \neq y$, it follows by Script 1 that $R_y \cap (X \setminus \{y\}) \neq \emptyset$, a contradiction. □

Theorem 10.18. *Every bounded infinite subset of $\mathbb{R}$ has at least one limit point.*

Proof. Suppose for the sake of contradiction that X is a bounded, infinite subset of $\mathbb{R}$ with no limit points. Since X has no limit points, it is vacuously true that it contains all of its limit points. Thus, by Definition 4.1, X is closed. This combined with the fact that X is bounded implies by Theorem 10.16 that X is compact. But this combined with the fact that X has no limit points implies by Lemma 10.17 that X is finite, a contradiction. □

3/4: **Theorem 10.19.** *Suppose that $f : X \to \mathbb{R}$ is continuous. If X is compact, the $f(X)$ is compact.*

Proof. To prove that $f(X)$ is compact, Definition 10.4 tells us that it will suffice to show that for every open cover $\mathcal{G}$ of $f(X)$, there exists a finite subset $\mathcal{G}' \subset \mathcal{G}$ that is also an open cover. Let $\mathcal{G}$ be an arbitrary open cover of $f(X)$. Then Definition 10.3 implies that for every point $f(x) \in f(X)$, we have that $f(x) \in G_{f(x)}$ where $G_{f(x)}$ is an element of $\mathcal{G}$. It follows since every $G_{f(x)} \in \mathcal{G}$ by Definition 10.3 that every $G_{f(x)}$ is open. Thus, by Definition 9.4, every $f^{-1}(G_{f(x)})$ is open in X. Consequently, by Definition 8.11, every $f^{-1}(G_{f(x)}) = X \cap A_x$ for some open set A_x.

We will now show that $\mathcal{H} = \{A_x\}_{x \in X}$ is an open cover of X. Since every $f(x) \in G_{f(x)}$, Definition 1.18 implies that every $x \in f^{-1}(G_{f(x)})$. It follows by Definitions 1.2 and 1.6 that every $x \in A_x$. But since every $x \in X$ is an element of $A_x \in \mathcal{H}$ and each A_x is open by definition, Definition 10.3 implies that $\mathcal{H}$ is an open cover of X.

It follows since X is compact by Definition 10.4 that there exists a finite subset $\mathcal{H}' \subset \mathcal{H}$ that is also an open cover of X. Since $\mathcal{H}'$ is finite, we know that $\mathcal{H}' = \{A_{x_1}, \dots, A_{x_n}\}$ where $x_1, \dots, x_n$ are elements of X. Let $\mathcal{G}' = \{G_{f(x_1)}, \dots, G_{f(x_n)}\}$. By Script 1, $\mathcal{G}' \subset \mathcal{G}$. Script 1 also implies that $\mathcal{G}'$ is finite. However, to demonstrate that $\mathcal{G}'$ is an open cover of $f(X)$, Definition 10.3 tells us that we must confirm that every $f(x) \in f(X)$ is an element of G for some $G \in \mathcal{G}'$, and that every G is open. For the first condition, let $f(x)$ be an arbitrary element of $f(X)$. Then since $x \in X$, $x \in A_{x_i}$ for some $A_{x_i} \in \mathcal{H}'$. It follows by the definition of A_{x_i} that $x \in f^{-1}(G_{f(x_i)})$. Thus, by Definition 1.18, $f(x) \in G_{f(x_i)}$. Therefore, $f(x)$ is an element of an element of $\mathcal{G}'$, as desired. As to the other condition, since every $G \in \mathcal{G}'$ is an element of $\mathcal{G}$ (i.e., open by Definition 10.3), every $G \in \mathcal{G}'$ is open, as desired. □

Corollary 10.20. *If $X \subset \mathbb{R}$ is nonempty, closed, and bounded and $f : X \to \mathbb{R}$ is continuous, then $f(X)$ has a first and last point.*

Proof. Since X is closed and bounded, we have by Theorem 10.16 that X is compact. This result combined with the fact that f is continuous implies by Theorem 10.19 that $f(X)$ is compact. Thus, by Theorem 10.16 again, $f(X)$ is closed and bounded. Additionally, since X is nonempty, there exists $x \in X$. Consequently, $f(x) \in f(X)$, so $f(X)$ is nonempty. Therefore, since $f(X)$ is nonempty, closed, and bounded, we have by Corollary 5.18 that $f(X)$ has a first and last point. □

Exercise 10.21. Use Corollary 10.20 to prove that if $f : [a, b] \to \mathbb{R}$ is continuous, then there exists a point $c \in [a, b]$ such that $f(c) \geq f(x)$ for all $x \in [a, b]$. Similarly, there exists a point $d \in [a, b]$ such that $f(d) \leq f(x)$ for all $x \in [a, b]$.

Proof. By Equations 8.1, $a \in [a,b]$, so $[a,b]$ is nonempty. By Corollaries 5.15 and 4.7, $[a,b]$ is closed. Since a,b are lower and upper bounds on $[a,b]$, respectively ($x \in [a,b]$ implies $a \leq x$ and $x \leq b$), Definition 5.6 asserts that $[a,b]$ is bounded. These three results combined with the fact that f is continuous imply by Corollary 10.20 that $f([a,b])$ has a first and last point. With respect to the first point, by Definition 3.3, there exists some $f(c) \in f([a,b])$ such that $f(c) \geq f(x)$ for all $f(x) \in f([a,b])$. By consecutive applications of Definition 1.18, $f(c) \in f([a,b])$ implies that $c \in [a,b]$, and $x \in [a,b]$ implies that $f(x) \in f([a,b])$, i.e., $f(c) \geq f(x)$. Therefore, there exists a point $c \in [a,b]$ such that $f(c) \geq f(x)$ for all $x \in [a,b]$. The argument is symmetric with respect to the last point. □

10.2 Discussion

2/25:

- Script 9 journals due next Wednesday.

3/2:

- (a,b) is not closed, compact, so I have to change my answer to no for the question associated with Theorem 10.15.
  - Also show this as a lemma.
- For Lemma 10.13, s' is a better variable name than y.
- For Theorem 10.14, I don't need casework — if we negate $x_{i-1} < y \leq x_i$, we have $y < x_0$ or $y > x_n$, contradictions in either way, so straight-up suppose it.
- Dramatically simplify Theorem 10.15 with $\mathcal{G} \cup (\mathbb{R} \setminus Y)$ as a cover of X where $\mathcal{G}$ is an arbitrary open cover of Y.
- FSC as an abbreviation for <u>F</u>or the <u>S</u>ake of <u>C</u>ontradiction.
- Simplify Theorem 10.18 by supposing X has no limit points, noting that this implies it's closed. It follows that it's compact and that it's finite, a contradiction.
- Is it true that a set is compact iff it is the union of finitely many closed intervals?

Script 11

Limits and Continuity

11.1 Journal

3/4: Throughout this sheet, we let $f, g : A \to \mathbb{R}$ be real-valued functions with domain $A \subset \mathbb{R}$, unless otherwise specified.

Definition 11.1. Let $a \in LP(A) \subset \mathbb{R}$. A **limit** of f at a is a number $L \in \mathbb{R}$ satisfying the following condition: for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Lemma 11.2. *Limits are unique: if L and L' are both limits of f at a point a, then $L = L'$.*

Proof. Let the limit of f at a be L, and suppose for the sake of contradiction that the limit of f at a is also equal to L' where $L \neq L'$. If we let $\epsilon = \frac{|L-L'|}{2}$, then $\epsilon > 0$ by Script 8. Thus, by consecutive applications of Definition 11.1, we have that there exists a $\delta_1 > 0$ such that if $x \in A$ and $0 < |x-a| < \delta_1$, then $|f(x)-L| < \epsilon$; and that there exists a $\delta_2 > 0$ such that if $x \in A$ and $0 < |x - a| < \delta_2$, then $|f(x) - L'| < \epsilon$. Now choose $\delta = \min(\delta_1, \delta_2)$. This makes it so that for any $x \in A$ such that $0 < |x - a| < \delta$, we have $|f(x) - L| < \epsilon$ and $|f(x) - L'| < \epsilon$, so

$$\begin{aligned}
|L - L'| &= |L - f(x) + f(x) - L'| \\
&\leq |L - f(x)| + |f(x) - L'| && \text{Lemma 8.8} \\
&= |f(x) - L| + |f(x) - L'| && \text{Exercise 8.5} \\
&< 2\epsilon \\
&= |L - L'|
\end{aligned}$$

But this implies that $|L - L'| < |L - L'|$, a contradiction. ☐

Definition 11.3. If L is the limit of f at a, we write

$$\lim_{x \to a} f(x) = L$$

Exercise 11.4. Give an example of a set $A \subset \mathbb{R}$, a function $f : A \to \mathbb{R}$, and a point $a \in LP(A)$ such that $\lim_{x\to a} f(x)$ does not exist.

Proof. Let $A = \mathbb{R}$, let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

and consider $0 \in LP(\mathbb{R})$ (by Corollary 5.4). Now suppose for the sake of contradiction that $\lim_{x\to a} f(x) = L$. Then by Definitions 11.3[1] and 11.1, for all $\epsilon > 0$, there exists some $\delta > 0$ such that if $x \in \mathbb{R}$ and

[1] I will not cite this definition again for the sake of concision.

$0 < |x - 0| = |x| < \delta$, then $|f(x) - L| < \epsilon$. If we let $\epsilon = 0.5$, then $\epsilon > 0$. Choosing a corresponding δ, we have by an extension of Exercise 8.9 that all $x \in (-\delta, 0) \cup (0, \delta)$ satisfy $|f(x) - L| < \epsilon$. This would include objects $y \in (0, \delta)$ and $z \in (-\delta, 0)$. We have by the definition of f that $f(y) = 1$ and $f(z) = 0$; thus, we have

$$\begin{aligned} 1 &= |f(y) - f(z)| \\ &= |f(y) - L + L - f(z)| \\ &\leq |f(y) - L| + |f(z) - L| \\ &< 0.5 + 0.5 \\ &= 1 \end{aligned}$$

But this implies that $1 < 1$, a contradiction. □

Theorem 11.5. *Let $x \in A$. Then the following are equivalent:*

(*a*) *f is continuous at x.*

(*b*) *For every $\epsilon > 0$, there exists $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$.*

(*c*) *Either $x \notin LP(A)$ or $\lim_{y \to x} f(y) = f(x)$.*

Proof. To illustrate that statements a-c are equivalent, it will suffice to verify that $a \Rightarrow b$, $b \Rightarrow c$, and $c \Rightarrow a$. Note that this foregoes the need for explicit proofs of "backwards implications" such as $b \Rightarrow a$ since that implication, for example, follows from $b \Rightarrow c \Rightarrow a$. Let's begin.

To prove that $a \Rightarrow b$, let $\epsilon > 0$ be arbitrary and look to find a $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then $|f(y) - f(x)|$.

We first locate δ. To do so, begin by defining the region $R = (f(x) - \epsilon, f(x) + \epsilon)$ (clearly R contains $f(x)$). Since R is open by Corollary 4.11 and f is continuous at x, we have by Definition 9.9 that there exists an open set S with $x \in S$ such that $S \cap A \subset f^{-1}(R)$. It follows by Theorem 4.10 that there exists a region (a, b) such that $x \in (a, b)$ and $(a, b) \subset S$. Thus, since (a, b) is an open interval by Corollary 4.11 and Lemma 8.3, we have by Lemma 8.10 that there exists a number $\delta > 0$ such that $(x - \delta, x + \delta) \subset (a, b)$.

As we will now show, this δ satisfies the desired property. Let y be an arbitrary element of A such that $|y - x| < \delta$. Then by Exercise 8.9, $y \in (x - \delta, x + \delta)$. It follows by consecutive applications of Definition 1.3 that $y \in (a, b)$, hence $y \in S$. This result combined with the fact that $y \in A$ by definition implies by Definition 1.6 that $y \in S \cap A$. Thus, by Definition 1.3 again, $y \in f^{-1}(R)$. Consequently, by Definition 1.18, $f(y) \in R$. Therefore, by Exercise 8.9 one more time, $|f(y) - f(x)| < \epsilon$.

To prove that $b \Rightarrow c$, let x be an arbitrary element of $\mathbb{R}$. We divide into two cases ($x \notin LP(A)$ and $x \in LP(A)$). If $x \notin LP(A)$, then we are done. If $x \in LP(A)$, then by the hypothesis, we know that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in A$ and $0 < |y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. It follows by Definition 11.1 that $f(x)$ is the limit of f at x, meaning that $\lim_{y \to x} f(y) = f(x)$, and we are done.

To prove that $c \Rightarrow a$, we divide into two cases ($x \notin LP(A)$ and $\lim_{y \to x} f(y) = f(x)$).

Suppose first that $x \notin LP(A)$. To demonstrate that f is continuous at x, Definition 9.9 tells us that it will suffice to confirm that for every region R containing $f(x)$, there exists an open set S containing x such that $S \cap A \subset f^{-1}(R)$. Let R be an arbitrary region with $f(x) \in R$. Since $x \notin LP(A)$, Definition 3.13 asserts that there exists a region (hence an open set by Corollary 4.11) S such that $x \in S$ and $S \cap (A \setminus \{x\}) = \emptyset$. It follows by Script 1 that $S \cap A = \{x\}$. But since $f(x) \in R$ implies by Definition 1.18 that $x \in f^{-1}(R)$, we have by Definition 1.3 that $S \cap A \subset f^{-1}(R)$. Therefore, S is an open set containing x such that $S \cap A \subset f^{-1}(R)$.

Now suppose that $\lim_{y \to x} f(y) = f(x)$. To demonstrate that f is continuous at x, Definition 9.9 tells us that it will suffice to confirm that for every region (a, b) containing $f(x)$, there exists an open set S containing x such that $S \cap A \subset f^{-1}((a, b))$. Let (a, b) be an arbitrary region with $f(x) \in (a, b)$. Then since (a, b) is an open interval by Lemma 8.3, Lemma 8.10 asserts that there exists $\epsilon > 0$ such that $(f(x) - \epsilon, f(x) + \epsilon) \subset (a, b)$. With regard to this ϵ, since $\lim_{y \to x} f(y) = f(x)$ by hypothesis, we have by Definition 11.1 that there exists a $\delta > 0$ such that if $y \in A$ and $0 < |y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $S = (x - \delta, x + \delta)$. Clearly, S contains x. Additionally, we can confirm that $S \cap A \subset f^{-1}((a, b))$: if we let y be an arbitrary element of $S \cap A$, then Definition 1.6 asserts that $y \in S$ and $y \in A$. It follows from the former condition by Exercise 8.9 that $|y - x| < \delta$. This combined with the fact that $y \in A$ implies that $|f(y) - f(x)| < \epsilon$. Thus, by Exercise

8.9 again, $f(y) \in (f(x) - \epsilon, f(x) + \epsilon)$. Consequently, by Definition 1.3, $f(y) \in (a, b)$. As such, we have by Definition 1.18 that $y \in f^{-1}((a, b))$, as desired. $\square$

Exercise 11.6.

(a) Let $a, b \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = ax + b$. Show that f is continuous at every $x \in \mathbb{R}$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$. Show that f is not continuous at 0.

Proof of a. To prove that f is continuous at every $x \in \mathbb{R}$, let x be an arbitrary element of $\mathbb{R}$; then by Theorem 11.5, it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. We divide into two cases ($a = 0$ and $a \neq 0$). If $a = 0$, then choose $\delta = 1$[2]. This makes it so that for any $y \in A$ such that $|y - x| < \delta = 1$, we have $|f(y) - f(x)| = |b - b| = 0 < \epsilon$, as desired. If $a \neq 0$, then choose $\delta = \frac{\epsilon}{|a|}$. This makes it so that for any $y \in A$ such that $|y - x| < \delta = \frac{\epsilon}{|a|}$, we have

$$\begin{aligned} |a|\,|y - x| &< \epsilon \\ |ay - ax| &< \epsilon \\ |ay + b - (ax + b)| &< \epsilon \\ |f(y) - f(x)| &< \epsilon \end{aligned}$$

as desired. $\square$

Proof of b. To prove that f is not continuous at 0, Theorem 11.5 tells us that it will suffice to show that for some $\epsilon > 0$, no $\delta > 0$ exists such that if $x \in \mathbb{R}$ and $|x - 0| = |x| < \delta$, then $|f(x) - 1| < \epsilon$. Let $\epsilon = 1$, and suppose for the sake of contradiction that $\delta > 0$ is a number such that if $x \in \mathbb{R}$ and $|x| < \delta$, then $|f(x) - 1| < \epsilon$. Clearly, $0 \in \mathbb{R}$ and by the definition of δ and Definition 8.4, $|0| < \delta$. However, $|f(x) - 1| = |0 - 1| = 1 \not< 1 = \epsilon$, a contradiction. $\square$

3/9: **Exercise 11.7.** Show that the absolute value function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ is continuous.

Proof. To prove that the absolute value function is continuous, Theorem 9.10 tells us that it will suffice to show that it is continuous at every $x \in \mathbb{R}$. To do this, let x be an arbitrary element of $\mathbb{R}$; then by Theorem 11.5, it will suffice to verify that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in \mathbb{R}$ and $|y - x| < \delta$, then $||y| - |x|| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Then choose $\delta = \epsilon$. This makes it so that for any $y \in A$ such that $|y - x| < \delta$, we have $||y| - |x|| \leq |y - x| < \delta = \epsilon$ (with the first inequality coming from Lemma 8.8), as desired. $\square$

Given real-valued functions f and g, we define new functions $f + g$, fg, and $\frac{1}{f}$ by

$$(f + g)(x) = f(x) + g(x) \qquad (fg)(x) = f(x) \cdot g(x) \qquad \frac{1}{f}(x) = \frac{1}{f(x)}$$

where $f(x) \neq 0$ in the definition of $\frac{1}{f}$. We wish to understand the limits of $f + g$, fg, and $\frac{1}{f}$ in terms of the limits of f and g.

Lemma 11.8. *If $\lim_{x \to a} f(x) = L > 0$, then there exists a region R with $a \in R$ such that $f(x) > 0$ for all $x \in R \cap A$ such that $x \neq a$. Moreover, if f is continuous at a, then $f(x) > 0$ for all $x \in R \cap A$. The analogous statement is true if $\lim_{x \to a} f(x) = L < 0$.*

Proof. We divide into two cases ($\lim_{x \to a} f(x) = L > 0$ and $\lim_{x \to a} f(x) = L < 0$).

Suppose first that $\lim_{x \to a} f(x) = L > 0$. Choose $\epsilon = L$. Then we have by Definition 11.1 that there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|f(x) - L| < L$. Let $R = (a - \delta, a + \delta)$. Clearly, $a \in R$. Now let x be an arbitrary element of $R \cap A$ such that $x \neq a$. It follows from the first condition by

[2] This choice is arbitrary; it can be any nonzero value, as we will soon see.

Definition 1.6 that $x \in R$ and $x \in A$. Thus, we have by Exercise 8.9 that $|x-a| < \delta$. Additionally, since $x \neq a$, Definition 3.1 asserts that $x > a$ or $x < a$, i.e., $x - a > 0$ or $x - a < 0$; either way, Script 8 implies that $0 < |x-a|$. To recap, we know that $x \in A$ and $0 < |x-a| < \delta$, so we have by the initial implication that $|f(x) - L| < L = \epsilon$. Therefore, by the lemma from Exercise 8.9, we have $-L < f(x) - L < L$, i.e., $0 < f(x)$ (which we obtain by adding L to both sides of the inequality as permitted by Definition 7.21).

Moreover, if f is continuous at a, then by Theorem 11.5, $a \notin LP(A)$ or $\lim_{x\to a} f(x) = f(a)$. But by Definition 11.1, $a \in LP(A)$, so we have $\lim_{x\to a} f(x) = f(a)$. The first part of this proof guarantees the existence of a region R such that $f(x) > 0$ for all $x \in R \cap A$ such that $x \neq a$. The fact that $f(a) = L > 0$ takes care of the case where $x = a$.

The proof is symmetric in the other case. ☐

3/11: **Theorem 11.9.** *Suppose that* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$. *Then*

(*a*) $\lim_{x\to a}(f+g)(x) = L + M$.

(*b*) $\lim_{x\to a}(fg)(x) = L \cdot M$.

(*c*) *Suppose that* $\lim_{x\to a} f(x) = L \neq 0$. *Then* $\lim_{x\to a} \frac{1}{f}(x) = \frac{1}{L}$.

Proof of a. To prove that $\lim_{x\to a}(f+g)(x) = L+M$, Definition 11.1 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x-a| < \delta$, then $|(f+g)(x) - (L+M)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Consider $\frac{\epsilon}{2}$. Since $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, we have by consecutive applications of Definition 11.1 that there exists a $\delta_1 > 0$ such that if $x \in A$ and $0 < |x-a| < \delta_1$, then $|f(x) - L| < \frac{\epsilon}{2}$; and there exists a $\delta_2 > 0$ such that if $x \in A$ and $0 < |x-a| < \delta_2$, then $|g(x) - M| < \frac{\epsilon}{2}$. Now choose $\delta = \min(\delta_1, \delta_2)$. This makes it so that for any $x \in A$ such that $0 < |x-a| < \delta$, we have $|f(x) - L| < \frac{\epsilon}{2}$ and $|g(x) - M| < \frac{\epsilon}{2}$, so

$$\begin{aligned}
|(f+g)(x) - (L+M)| &= |f(x) - L + g(x) - M| \\
&\leq |f(x) - L| + |g(x) - M| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

as desired. ☐

Proof of b. To prove that $\lim_{x\to a}(fg)(x) = LM$, Definition 11.1 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x-a| < \delta$, then $|(fg)(x) - LM| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ (and $\min(\frac{\epsilon}{2(|M|+1)}, 1)$ and $\frac{\epsilon}{2(|L|+1)}$ are both greater than zero), we have by consecutive applications of Definition 11.1 that there exists a $\delta_1 > 0$ such that if $x \in A$ and $0 < |x-a| < \delta_1$, then $|f(x) - L| < \min(\frac{\epsilon}{2(|M|+1)}, 1)$[3]; and there exists a $\delta_2 > 0$ such that if $x \in A$ and $0 < |x-a| < \delta_2$, then $|g(x) - M| < \frac{\epsilon}{2(|L|+1)}$. Now choose $\delta = \min(\delta_1, \delta_2)$. This makes it so that for any $x \in A$ such that $0 < |x-a| < \delta$, we have $|f(x) - L| < \min(\frac{\epsilon}{2(|M|+1)}, 1)$ and $|g(x) - M| < \frac{\epsilon}{2(|L|+1)}$.

Before we get into the body of the proof, we need a couple of preliminary results. By Script 8, we have $|a| = |a-b+b| \leq |a-b| + |b|$, so $|a| - |b| \leq |a-b|$. Thus, since $|f(x) - L| < 1$, we have $|f(x)| - |L| \leq |f(x) - L| < 1$, which means that $|f(x)| < 1 + |L|$. Additionally, we have

$$\begin{aligned}
f(x)g(x) - LM &= f(x)g(x) - f(x)M + f(x)M - LM \\
&= f(x)(g(x) - M) + M(f(x) - L)
\end{aligned}$$

[3]When we choose a compound ϵ (i.e., one that makes use of a min expression), it means that $|f(x) - L| < \frac{\epsilon}{2(|M|+1)}$ and $|f(x) - L| < 1$. Basically, whichever quantity is smaller is what min evaluates to, and whichever is bigger is still greater than $|f(x) - L|$ by transitivity.

With these results, we are ready to introduce the main inequality:

$$\begin{aligned}
|(fg)(x) - LM| &= |f(x)(g(x) - M) + M(f(x) - L)| \\
&\leq |f(x)| \cdot |g(x) - M| + |M| \cdot |f(x) - L| \\
&< (1 + |L|) \cdot \frac{\epsilon}{2(|L| + 1)} + |M| \cdot \frac{\epsilon}{2(|M| + 1)} \\
&= \frac{\epsilon}{2} \cdot \frac{1 + |L|}{1 + |L|} + \frac{\epsilon}{2} \cdot \frac{|M|}{|M| + 1} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

□

Proof of c. To prove that $\lim_{x\to a} \frac{1}{f}(x) = \frac{1}{L}$, Definition 11.1 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x-a| < \delta$, then $|\frac{1}{f}(x) - \frac{1}{L}| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{x\to a} f(x) = L$ and $\min(\frac{|L|}{2}, \frac{\epsilon|L|^2}{2}) > 0$, we have by Definition 11.1 that there exists a $\delta_1 > 0$ such that if $x \in A$ and $0 < |x-a| < \delta_1$, then $|f(x) - L| < \min(\frac{|L|}{2}, \frac{\epsilon|L|^2}{2})$. Additionally, since $L \neq 0$, Theorem 11.8 asserts that there exists a region R containing a such that for all $x \in (R \cap A) \setminus \{a\}$, $f(x) \neq 0$ (since $L \neq 0$ implies $L > 0$ or $L < 0$). It follows by Lemma 8.10 that there exists a δ_2 such that $(a - \delta_2, a + \delta_2) \subset R$. Let $\delta = \min(\delta_1, \delta_2)$.

At this point, we know by this definition of δ that if $x \in A$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \min(\frac{|L|}{2}, \frac{\epsilon|L|^2}{2})$ and $f(x) \neq 0$. Before we get into the body of the proof, we need one more preliminary result. It follows from the fact that $|f(x) - L| < \frac{|L|}{2}$ that

$$\begin{aligned}
|L| &= 2|L| - |L| \\
&= 2(|L| - |f(x)| + |f(x)|) - |L| \\
&\leq 2(|L - f(x)| + |f(x)|) - |L| \\
&= 2(|f(x) - L| + |f(x)|) - |L| \\
&< 2\left(\frac{|L|}{2} + |f(x)|\right) - |L| \\
&= |L| + 2|f(x)| - |L| \\
&= 2|f(x)|
\end{aligned}$$

With these results, we are ready to introduce the main inequality:

$$\begin{aligned}
\left|\frac{1}{f}(x) - \frac{1}{L}\right| &= \left|\frac{1}{f(x)} - \frac{1}{L}\right| \\
&= \left|\frac{L - f(x)}{f(x) \cdot L}\right| \\
&= \frac{|f(x) - L|}{|f(x)| \cdot |L|} \\
&< \frac{\epsilon|L|^2}{2|f(x)| \cdot |L|} \\
&= \frac{\epsilon|L|}{2|f(x)|} \\
&< \frac{\epsilon|L|}{|L|} \\
&= \epsilon
\end{aligned}$$

□

Corollary 11.10. *If f and g are continuous at a, then $f+g$ and fg are continuous at a. Also, $\frac{1}{f}$ and $\frac{g}{f}$ are continuous at a, provided that $f(a) \neq 0$.*

Proof. Since f is continuous at a, we have by Theorem 11.5 that either $a \notin LP(A)$ or $\lim_{x\to a} f(x) = f(a)$. Similarly, we have that either $a \notin LP(A)$ or $\lim_{x\to a} g(x) = g(a)$. We divide into four cases ($a \notin LP(A)$ and $a \notin LP(A)$, $a \notin LP(A)$ and $\lim_{x\to a} g(x) = g(a)$, $\lim_{x\to a} f(x) = f(a)$ and $a \notin LP(A)$, and $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$). If $a \notin LP(A)$ (this takes care of the first three cases), then by Theorem 11.5, $f+g$ is continuous at a. If $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$, then by Theorem 11.9, $\lim_{x\to a}(f+g)(x) = f(a)+g(a) = (f+g)(a)$. Therefore, by Theorem 11.5, $f+g$ is continuous at a.

The proofs of the second and third cases are symmetric to that of the first. The fourth case can be handled by letting $\frac{g}{f} = g \cdot \frac{1}{f}$ and applying the third and second cases. $\square$

Definition 11.11. A **polynomial** (in one variable with real coefficients) is a function f of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

for some $n \in \mathbb{N}\cup\{0\}$, where $a_i \in \mathbb{R}$ for $0 \leq i \leq n$. A **rational function** (in one variable with real coefficients) is a function of the form $h(x) = \frac{f(x)}{g(x)}$ where f and g are polynomials in one variable with real coefficients.

Corollary 11.12. *Polynomials in one variable with real coefficients are continuous. A rational function in one variable with real coefficients $h(x) = \frac{f(x)}{g(x)}$ is continuous at all $a \in \mathbb{R}$ where $g(a) \neq 0$.*

Proof. We induct on the degree n of the polynomial. For the base case $n = 0$, we have by Definition 11.11 that f is a function of the form $f(x) = a_0$ for some $a_0 \in \mathbb{R}$. Thus, by Exercise 11.6, f is continuous at every $x \in \mathbb{R}$ (since $ax+b$ is continuous for $a = 0$ and $b = a_0$). Therefore, by Theorem 9.10, f is continuous. Now suppose inductively that all functions of the form $f(x) = a_n x^n + \cdots + a_0$ are continuous; we seek to prove that all functions of the form $g(x) = a_{n+1}x^{n+1} + \cdots + a_0$ are continuous. By the inductive hypothesis, we know that x^n is continuous. Thus, by Theorem 9.10, x^n is continuous at every $x \in \mathbb{R}$. Additionally, by Exercise 11.6, $a_{n+1}x$ (where a_{n+1} is an arbitrary element of $\mathbb{R}$) is continuous at every $x \in \mathbb{R}$. The last two results combined with Corollary 11.10 tell us that $a_{n+1}x \cdot x^n = a_{n+1}x^{n+1}$ is continuous at every $x \in \mathbb{R}$. In addition, since f is continuous, Theorem 9.10 asserts that f is continuous at every $x \in \mathbb{R}$. Thus, by Corollary 11.10 again, $g(x) = a_{n+1}x^{n+1} + f(x) = a_{n+1}x^{n+1} + \cdots + a_0$ is continuous at every $x \in \mathbb{R}$. By one more application of Theorem 9.10, $g(x)$ is continuous.

Now we move on to proving that $h(x) = \frac{f(x)}{g(x)}$ is continuous at all $a \in \mathbb{R}$ such that $g(a) \neq 0$. By the above, f and g (as polynomials) are continuous. Thus, by Theorem 9.10, f and g are continuous at every $a \in \mathbb{R}$. Therefore, by Corollary 11.10, $h = \frac{f}{g}$ is continuous at all $a \in \mathbb{R}$, provided that $g(a) \neq 0$, as desired. $\square$

Now we want to look at limits of the composition of functions. We assume here (for 11.13-11.15) that $a \in A$, $g : A \to \mathbb{R}$, and $f : I \to \mathbb{R}$, where I is an open interval containing $\overline{g(A)}$. It is not quite true in general that if $\lim_{x\to a} g(x) = M$ and $\lim_{y\to M} f(y) = L$, then $\lim_{x\to a} f(g(x)) = L$, but it is true in some cases.

Theorem 11.13. *If $\lim_{x\to a} g(x) = M$ and f is continuous at M, then $\lim_{x\to a} f(g(x)) = f(M)$.*

Proof. To prove that $\lim_{x\to a} f(g(x)) = f(M)$, Definition 11.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x-a| < \delta$, then $|f(g(x)) - f(M)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Then since f is continuous at M, Theorem 11.5 implies that there exists a $\delta' > 0$ such that if $y \in I$ and $|y - M| < \delta'$, then $|f(y) - f(M)| < \epsilon$. It follows by Definition 11.1 that there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x-a| < \delta$, then $|g(x) - M| < \delta'$.

We will now prove that this δ is the desired δ. Let x be an arbitrary element of A that satisfies $0 < |x-a| < \delta$. Then we know that $|g(x)-M| < \delta'$. Additionally, it follows from the fact that $x \in A$ by Definition 1.18 that $g(x) \in g(A)$. Thus, by Definition 4.4, $g(x) \in \overline{g(A)}$. Consequently, by Definition 1.3, $g(x) \in I$. Indeed, we now know that $g(x) \in I$ and $|g(x) - M| < \delta'$, so we can determine that $|f(g(x)) - f(M)| < \epsilon$, as desired. $\square$

3/17: **Remark 11.14.** This theorem can also be rewritten as

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

which can be remembered as "limits pass through continuous functions."

Corollary 11.15. *If g is continuous at a and f is continuous at $g(a)$, then $f\circ g$ is continuous at a.*

Proof. Since g is continuous at a, we have by Theorem 11.5 that either $a\notin LP(A)$ or $\lim_{x\to a} g(x) = g(a)$. We now divide into two cases. If $a\notin LP(A)$, then by Theorem 11.5, $f\circ g$ is continuous at a. If $\lim_{x\to a} g(x) = g(a)$, then this combined with the fact that f is continuous at $g(a)$ implies by Theorem 11.13 that $\lim_{x\to a} f(g(x)) = f(g(a))$. It follows by Definition 1.25 that $\lim_{x\to a}(f\circ g)(x) = (f\circ g)(a)$. Therefore, by Theorem 11.5, $f\circ g$ is continuous at a. □

11.2 Discussion

3/9:
- Final exams will be in person from Tuesday to Friday during the reading week.
  - We'll turn in all of the written questions this time, and then orally present a short proof from this class.
  - Open note again, too.
- The original statement of Lemma 11.8 is wrong! Doesn't work at endpoints of closed intervals or in subsets of $\mathbb{R}$ such as the rationals (where we'll never find a region that's a subset of $\mathbb{Q}$ with the desired property).

3/11:
- For Corollary 11.10, do we need $a\in LP(A)$ for the other case?
  - Perhaps also show limits for the last three cases?
- For Corollary 11.12, present the 0 case and then induct from 1?
- 9.11?
- 11.7: $x\in R\cap A$ the second time around?

Final-Specific Questions

1. Suppose that X and Y are compact subsets of $\mathbb{R}$. For this problem, use only results up to and including Theorem 10.11, and not any of the subsequent results in Script 10.

   (a) Show that $X \cup Y$ is compact.

   (b) Show that $X \cap Y$ is compact.

   (c) Suppose $X_1, X_2, \dots$ are compact sets. Are the following compact? Either prove that the set is always compact or provide a counterexample that is not compact.

$$\bigcap_{n\in\mathbb{N}} X_n \qquad\qquad \bigcup_{n\in\mathbb{N}} X_n$$

Proof of a. To prove that $X \cup Y$ is compact, Definition 10.4 tells us that it will suffice to show that for every open cover $\mathcal{G}$ of $X \cup Y$, there exists a finite subset $\mathcal{G}' \subset \mathcal{G}$ that is also an open cover. Let $\mathcal{G}$ be an arbitrary open cover of $X \cup Y$. We now seek to demonstrate that $\mathcal{G}$ is an open cover of X and Y, starting with X. To do so, Definition 10.3 tells us that it will suffice to confirm that every $x \in X$ is an element of G for some $G \in \mathcal{G}$, and that every G is open. For the first condition, let x be an arbitrary element of X. Then by Definition 1.5, $x \in X \cup Y$. It follows by Definition 10.3 that $x \in G$ for some $G \in \mathcal{G}$. As to the other condition, every $G \in \mathcal{G}$ is open by Definition 10.3, as desired. The argument is symmetric for Y.

We now invoke Definition 10.4 to find finite subcovers $\mathcal{G}_X \subset \mathcal{G}$ and $\mathcal{G}_Y \subset \mathcal{G}$ of X and Y, respectively. Let $\mathcal{G}' = \mathcal{G}_X \cup \mathcal{G}_Y$. By Script 1, $\mathcal{G}' \subset \mathcal{G}$. Additionally, since $\mathcal{G}_X$ and $\mathcal{G}_Y$ are both finite, Script 1 also implies that $\mathcal{G}'$ is finite. However, to demonstrate that $\mathcal{G}'$ is an open cover of $X \cup Y$, Definition 10.3 tells us that we must confirm that every $z \in X \cup Y$ is an element of G for some $G \in \mathcal{G}'$, and that every G is open. For the first condition, let z be an arbitrary element of $X \cup Y$. Thus, by Definition 1.5, $z \in X$ or $z \in Y$. We now divide into two cases. If $z \in X$, then by Definition 10.3, $z \in G$ for some $G \in \mathcal{G}_X$. But since $\mathcal{G}_X \subset \mathcal{G}'$ by Script 1, $z \in G$ implies that z is an element of a set that is an element of $\mathcal{G}'$, as desired. The argument is symmetric in the other case. As to the other condition, every $G \in \mathcal{G}$ is open by Definition 10.3, as desired. □

Proof of b. To prove that $X \cap Y$ is compact, Definition 10.4 tells us that it will suffice to show that for every open cover $\mathcal{G}$ of $X \cap Y$, there exists a finite subset $\mathcal{G}' \subset \mathcal{G}$ that is also an open cover. To do this, we will first demonstrate that $\mathcal{H} = \mathcal{G} \cup \{\mathbb{R} \setminus (X \cap Y)\}$ is an open cover of X. It will follow that there exists a finite subset $\mathcal{H}' \subset \mathcal{H}$ that is an open cover of X. Lastly, we will demonstrate that $\mathcal{G}' = \mathcal{H}' \setminus \{\mathbb{R} \setminus (X \cap Y)\}$ is the desired finite open cover subset of $\mathcal{G}$.

Let $\mathcal{G}$ be an arbitrary open cover of $X \cap Y$, and let $\mathcal{H} = \mathcal{G} \cup \{\mathbb{R} \setminus (X \cap Y)\}$. To demonstrate that $\mathcal{H}$ is an open cover of X, Definition 10.3 tells us that it will suffice to confirm that every $x \in X$ is an element of H for some $H \in \mathcal{H}$, and that every H is open. For the first condition, let x be an arbitrary element of X. We divide into two cases ($x \in X \cap Y$ and $x \notin X \cap Y$). If $x \in X \cap Y$, then since $\mathcal{G}$ is an open cover of $X \cap Y$, Definition 10.3 implies that $x \in G$ for some $G \in \mathcal{G}$. But since $\mathcal{G} \subset \mathcal{H}$, $x \in G$ for some $G \in \mathcal{H}$, as desired. On the other hand, if $x \notin X \cap Y$, then this combined with the fact that $x \in \mathbb{R}$ implies by Definition 1.11 that $x \in \mathbb{R} \setminus (X \cap Y) \in \mathcal{H}$, as desired. As to the other condition, let H be an arbitrary element of $\mathcal{H}$. We divide into two cases ($H \in \mathcal{G}$ and $H \notin \mathcal{G}$). If $H \in \mathcal{G}$, then by

Definition 10.3, H is open, as desired. On the other hand, if $H \notin \mathcal{G}$, then $H = \mathbb{R} \setminus (X \cap Y)$ by Script 1. X and Y are closed by Theorem 10.11, so $X \cap Y$ is closed by Theorem 4.16. It follows by Definition 4.8 that $\mathbb{R} \setminus (X \cap Y)$ is open, so H is open, as desired.

Since $\mathcal{H}$ is an open cover of X and X is compact, we have by Definition 10.4 that there exists a finite subset $\mathcal{H}' \subset \mathcal{H}$ that is also an open cover of X. Let $\mathcal{G}' = \mathcal{H}' \setminus \{\mathbb{R} \setminus (X \cap Y)\}$. By Script 1, $\mathcal{G}' \subset \mathcal{G}$. Additionally, since $\mathcal{H}'$ is finite, Script 1 also implies that $\mathcal{G}'$ is finite. However, to demonstrate that $\mathcal{G}'$ is an open cover of $X \cap Y$, Definition 10.3 tells us that we must confirm that every $z \in X \cap Y$ is an element of G for some $G \in \mathcal{G}'$, and that every G is open. For the first condition, let z be an arbitrary element of $X \cap Y$. Then since $X \cap Y \subset X$ by Theorem 1.7, Definition 10.3 asserts that $z \in H$ for some $H \in \mathcal{H}'$. Additionally, since $z \in X \cap Y$, Definition 1.11 implies that $z \notin \mathbb{R} \setminus (X \cap Y)$. Thus, $H \neq \mathbb{R} \setminus (X \cap Y)$, which implies by Definition 1.11 that $H \in \mathcal{H}' \setminus \{\mathbb{R} \setminus (X \cap Y)\} = \mathcal{G}'$. Therefore, $z \in H$ for some $H \in \mathcal{G}'$, as desired. As to the other condition, since every $G \in \mathcal{G}'$ is an element of $\mathcal{G}$ (i.e., open by Definition 10.3), every $G \in \mathcal{G}'$ is open, as desired. □

Proof of c. To prove that $\bigcap_{n\in\mathbb{N}} X_n$ is compact, Definition 10.4 tells us that it will suffice to show that for every open cover $\mathcal{G}$ of $\bigcap_{n\in\mathbb{N}} X_n$, there exists a finite subset $\mathcal{G}' \subset \mathcal{G}$ that is also an open cover. To do this, we will use an analogous process to part (b).

Let $\mathcal{G}$ be an arbitrary open cover of $\bigcap_{n\in\mathbb{N}} X_n$, and let $\mathcal{H} = \mathcal{G} \cup \{\mathbb{R} \setminus (\bigcap_{n\in\mathbb{N}} X_n)\}$. To demonstrate that $\mathcal{H}$ is an open cover of X_1, Definition 10.3 tells us that it will suffice to confirm that every $x \in X_1$ is an element of H for some $H \in \mathcal{H}$, and that every H is open. For the first condition, let x be an arbitrary element of X_1. We divide into two cases ($x \in \bigcap_{n\in\mathbb{N}} X_n$ and $x \notin \bigcap_{n\in\mathbb{N}} X_n$). If $x \in \bigcap_{n\in\mathbb{N}} X_n$, then since $\mathcal{G}$ is an open cover of $\bigcap_{n\in\mathbb{N}} X_n$, Definition 10.3 implies that $x \in G$ for some $G \in \mathcal{G}$. But since $\mathcal{G} \subset \mathcal{H}$, $x \in G$ for some $G \in \mathcal{H}$, as desired. On the other hand, if $x \notin \bigcap_{n\in\mathbb{N}} X_n$, then this combined with the fact that $x \in \mathbb{R}$ implies by Definition 1.11 that $x \in \mathbb{R} \setminus (\bigcap_{n\in\mathbb{N}} X_n) \in \mathcal{H}$, as desired. As to the other condition, let H be an arbitrary element of $\mathcal{H}$. We divide into two cases ($H \in \mathcal{G}$ and $H \notin \mathcal{G}$). If $H \in \mathcal{G}$, then by Definition 10.3, H is open, as desired. On the other hand, if $H \notin \mathcal{G}$, then $H = \mathbb{R} \setminus (\bigcap_{n\in\mathbb{N}} X_n)$ by Script 1. Each X_n is closed by Theorem 10.11, so $\bigcap_{n\in\mathbb{N}} X_n$ is closed by Theorem 4.16. It follows by Definition 4.8 that $\mathbb{R} \setminus (\bigcap_{n\in\mathbb{N}} X_n)$ is open, so H is open, as desired.

Since $\mathcal{H}$ is an open cover of X_1 and X_1 is compact, we have by Definition 10.4 that there exists a finite subset $\mathcal{H}' \subset \mathcal{H}$ that is also an open cover of X_1. Let $\mathcal{G}' = \mathcal{H}' \setminus \{\mathbb{R} \setminus (\bigcap_{n\in\mathbb{N}} X_n)\}$. By Script 1, $\mathcal{G}' \subset \mathcal{G}$. Additionally, since $\mathcal{H}'$ is finite, Script 1 also implies that $\mathcal{G}'$ is finite. However, to demonstrate that $\mathcal{G}'$ is an open cover of $\bigcap_{n\in\mathbb{N}} X_n$, Definition 10.3 tells us that we must confirm that every $y \in \bigcap_{n\in\mathbb{N}} X_n$ is an element of G for some $G \in \mathcal{G}'$, and that every G is open. For the first condition, let y be an arbitrary element of $\bigcap_{n\in\mathbb{N}} X_n$. Then since $\bigcap_{n\in\mathbb{N}} X_n \subset X_1$, Definition 10.3 asserts that $y \in H$ for some $H \in \mathcal{H}'$. Additionally, since $y \in \bigcap_{n\in\mathbb{N}} X_n$, Definition 1.11 implies that $y \notin \mathbb{R} \setminus (\bigcap_{n\in\mathbb{N}} X_n)$. Thus, $H \neq \mathbb{R} \setminus (\bigcap_{n\in\mathbb{N}} X_n)$, which implies by Definition 1.11 that $H \in \mathcal{H}' \setminus \{\mathbb{R} \setminus (\bigcap_{n\in\mathbb{N}} X_n)\} = \mathcal{G}'$. Therefore, $y \in H$ for some $H \in \mathcal{G}'$, as desired. As to the other condition, since every $G \in \mathcal{G}'$ is an element of $\mathcal{G}$ (i.e., open by Definition 10.3), every $G \in \mathcal{G}'$ is open, as desired.

Let $X_n = \{n\}$ for all $n \in \mathbb{N}$. By Exercise 10.5, each X_n is compact, as desired. However, $\bigcup_{n\in\mathbb{N}} X_n = \mathbb{N}$ is not compact: if we let $\mathcal{G} = \{(n-1, n+1) : n \in \mathbb{N}\}$, then we have an open cover (clearly) that is infinite (clearly) and yet from which no term can be removed without revoking its status as an open cover. □

2. Let $f, g : A \to \mathbb{R}$. In each of the following, justify your answer fully:
   (a) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ do not exist, can $\lim_{x\to a}[f(x) + g(x)]$ exist?
   (b) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a}[f(x) + g(x)]$ exists, must $\lim_{x\to a} g(x)$ exist?

Justification of a. Let $f, g : A \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad g(x) = \begin{cases} 0 & x \geq 0 \\ 1 & x < 0 \end{cases}$$

By Exercise 11.4, $\lim_{x\to 0} f(x)$ does not exist. Similarly, $\lim_{x\to 0} g(x)$ does not exist. However, by Exercise 11.6, $\lim_{x\to 0}[f(x)+g(x)] = 1$. ☐

Justification of b. Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a}[f(x)+g(x)] = M$. If we let $h(x) = -1$, then we have by Theorem 11.9 that $\lim_{x\to a} -f(x) = -L$. Applying Theorem 11.9 again, we have $\lim_{x\to a}[f(x)+g(x)-f(x)] = \lim_{x\to a} g(x) = M - L$. ☐

3. For the problem you may assume $f : \mathbb{R} \to \mathbb{R}$.
   (a) Show that for all $x \in \mathbb{R}$ there exists a unique $y \in \mathbb{R}$ such that $y^3 = x$.
   (b) Using part (a), we can define the cube root function $g(x) = x^{1/3}$ in the usual way. Show that g is continuous and strictly increasing.
   (c) Suppose $\lim_{x\to 0} f(x) = L$. Show that $\lim_{x\to 0} f(x^3)$ exists and equals L.
   (d) Suppose $\lim_{x\to 0} f(x^3) = L$. Show that $\lim_{x\to 0} f(x)$ exists and equals L.

Proof of a. We first show that there always *exists* a y such that $y^3 = x$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(y) = y^3$, and let x be an arbitrary element of $\mathbb{R}$. By Definition 11.11, f is a polynomial, meaning by Corollary 11.12 that f is continuous. We divide into three cases ($x = 0$, $x > 0$, and $x < 0$). If $x = 0$, then $x = 0 = 0^3 = y^3$. If $x > 0$, consider the closed interval $[0, x+1]$. By Proposition 9.7, $f|_{[0,x+1]} : [0, x+1] \to \mathbb{R}$ is continuous. Additionally, $f(0) = 0 < x < x^3 + 3x^2 + 3x + 1 = (x+1)^3 = f(x+1)$. Thus, we have by Exercise 9.12 that there exists $y \in (0, x+1)$ such that $f(y) = y^3 = x$. The argument is symmetric if $x < 0$.

We now show that this y is unique, by proving that f is injective. To do so, Definition 1.20 tells us that it will suffice to show that if $f(x) = f(x')$, then $x = x'$. Let $x^3 = x'^3$. Then $0 = x^3 - x'^3 = (x - x')(x^2 + xx' + x'^2)$. It follows by Script 0 that $x = x'$ or $x = x' = 0$ (the latter result is trivial, but obtained by setting $x^2 + xx' + x'^2 = 0$), as desired. ☐

Proof of b. Suppose for the sake of contradiction that g is not continuous. Then by Theorem 9.10, there exists some $x \in \mathbb{R}$ at which g is not continuous. It follows by Theorem 11.5 that $\lim_{y\to x} g(y) \neq g(x)$. Additionally, by part (a), there exists some $z \in \mathbb{R}$ such that $f(z) = x$. Also, since f is continuous by part (a), Theorem 9.10 asserts that it is continuous at at x. Thus, by Theorem 11.5, $\lim_{y\to z} f(y) = f(z)$ (we know that $z \in LP(\mathbb{R})$ by Corollary 5.4). Consequently, $\lim_{y\to z} g(f(y))$ does not exist. But this contradicts Exercise 11.6, which asserts that $g(f(x)) = x$ is continuous, i.e., $\lim_{y\to z} g(f(y))$ should exist.

By the proof of part (a), f is injective. Additionally, by part (a), f is surjective. Thus, by Definition 1.20, f is bijective. It follows by Proposition 1.27 that g is bijective. Thus, by Definition 1.20 again, g is injective. Consequently, by Proposition 9.7, $g|_{(a,b)} : (a,b) \to \mathbb{R}$ is continuous for any open interval (a,b), and by Script 1, $g|_{(a,b)}$ is also injective. Thus, by Lemma 9.13, g is strictly increasing on any open interval (a,b). It follows that g is strictly increasing overall. ☐

Proof of c. To prove that $\lim_{x\to 0} f(x^3) = L$, Definition 11.1 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in \mathbb{R}$ and $0 < |x| < \delta$, $|f(x^3) - L| < \epsilon$. Let $\epsilon > 0$ be arbitrary. It follows from the hypothesis that $\lim_{x\to 0} f(x) = L$ by Definition 11.1 that there exists $\delta_1 > 0$ such that if $x \in \mathbb{R}$ and $0 < |x| < \delta_1$, then $|f(x) - L| < \epsilon$. Additionally, by Theorem 11.5, the facts that the x^3 function is continuous and $0 \in LP(\mathbb{R})$ imply that $\lim_{x\to 0} x^3 = 0^3 = 0$. Thus, by Definition 11.1, there exists a $\delta_2 > 0$ such that if $x \in \mathbb{R}$ and $0 < |x| < \delta_2$, then $|x^3 - 0| = |x^3| < \delta_1$. Choose $\delta = \delta_2$. Then if x is an arbitrary element of $\mathbb{R}$ such that $0 < |x| < \delta$, we have $|x^3| < \delta_1$. Additionally, since $x \neq 0$, $x^3 \neq 0$, so $0 < |x^3| < \delta_1$. It follows that $|f(x^3) - L| < \epsilon$, as desired. ☐

Proof of d. To prove that $\lim_{x\to 0} f(x) = L$, Definition 11.1 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in \mathbb{R}$ and $0 < |x| < \delta$, $|f(x) - L| < \epsilon$. Let $\epsilon > 0$ be arbitrary. It follows from the hypothesis that $\lim_{x\to 0} f(x^3) = L$ by Definition 11.1 that there exists $\delta_1 > 0$ such that if $x \in \mathbb{R}$ and $0 < |x| < \delta_1$, then $|f(x^3) - L| < \epsilon$. Additionally, by Theorem 11.5, the facts that the $x^{1/3}$ function is continuous (part (b)) and $0 \in LP(\mathbb{R})$ imply that $\lim_{x\to 0} x^{1/3} = 0^{1/3} = 0$. Thus, by Definition 11.1, there exists a $\delta_2 > 0$ such that if $x \in \mathbb{R}$ and $0 < |x| < \delta_2$, then $|x^{1/3} - 0| = |x^{1/3}| < \delta_1$. Choose $\delta = \delta_2$. Then if x is an arbitrary element of $\mathbb{R}$ such that $0 < |x| < \delta$, we have $|x^{1/3}| < \delta_1$. Additionally, since $x \neq 0$, $x^{1/3} \neq 0$, so $0 < |x^{1/3}| < \delta_1$. It follows that $|f((x^{1/3})^3) - L| = |f(x) - L| < \epsilon$, as desired. $\square$

4. For this problem, suppose $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, and that A is a dense subset of $\mathbb{R}$.

   (a) Prove that if f is continuous and $f(x) = 0$ for all $x \in A$, then $f(x) = 0$ for all $x \in \mathbb{R}$.

   (b) Prove that if f and g are continuous and $f(x) = g(x)$ for all $x \in A$, then $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Proof of a. Suppose for the sake of contradiction that $f(x) \neq 0$ for some $x \in \mathbb{R}$. Since f is continuous, Theorem 9.10 asserts that f is continuous at x. It follows by Theorem 11.5 that $\lim_{y\to x} f(y) = f(x)$ (since $x \in LP(\mathbb{R})$ by Corollary 5.4). Choose $\epsilon = |f(x)|$. Consequently, by Definition 11.1, there exists a $\delta > 0$ such that if $y \in \mathbb{R}$ and $|y - x| < \delta$, then $|f(y) - f(x)| < |f(x)|$.

Switching gears for a moment, consider the fact that A is dense in $\mathbb{R}$. It follows by Definition 6.8 that $x \in LP(A)$. Thus, by Definition 3.13, for every region R with $x \in R$, $R \cap (A \setminus \{x\}) \neq \emptyset$.

We merge the above two ideas by considering the region $R = (x - \delta, x + \delta)$, which is clearly an x-containing region. By the above, $R \cap (A \setminus \{x\}) \neq \emptyset$, i.e., there exists a point y such that $y \in R$, $y \in A$, and $y \neq x$. It follows from the former claim by Exercise 8.9 that $|y - x| < \delta$, from the middle claim by hypothesis that $f(y) = 0$, and from the latter claim that $y - x \neq 0$, i.e., $0 < |y - x| < \delta$. Therefore, $|f(y) - f(x)| < |f(x)|$. But this implies that $|0 - f(x)| = |f(x)| < |f(x)|$, a contradiction. $\square$

Proof of b. Let $h(x) = f(x) - g(x)$. We will prove that h is continuous and that $h(x) = 0$ for all $x \in A$. It will then follow from part (a) that $h(x) = 0$ for all $x \in \mathbb{R}$, implying that $f(x) = g(x)$ for all $x \in \mathbb{R}$. Let's begin.

Since f, g are continuous by Theorem 9.10, f, g are continuous at every $x \in \mathbb{R}$. Thus, by consecutive applications of Corollary 11.10, $-g$ is continuous at every $x \in \mathbb{R}$, so $f - g$ is continuous at every $x \in \mathbb{R}$. Consequently, by Theorem 9.10 again, $h = f - g$ is continuous.

Since $f(x) = g(x)$ for all $x \in A$, it naturally follows that $h(x) = f(x) - g(x) = 0$ for all $x \in A$.

Since h is continuous and $h(x) = 0$ for all $x \in A$, part (a) asserts that $h(x) = 0$ for all $x \in \mathbb{R}$. Thus, $f(x) - g(x) = 0$ for all $x \in \mathbb{R}$, meaning that $f(x) = g(x)$ for all $x \in \mathbb{R}$, as desired. $\square$