Script 6

Construction of the Real Numbers

6.1 Journal

1/12: **Definition 6.1.** A subset A of $\mathbb{Q}$ is said to be a **cut** (or **Dedekind cut**) if it satisfies the following:

(a) $A \neq \emptyset$ and $A \neq \mathbb{Q}$.

(b) If $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$.

(c) A does not have a last point; i.e., if $r \in A$, then there is some $s \in A$ with $s > r$.

We denote the collection of all cuts by $\mathbb{R}$.

Lemma 6.2. *Let A be a Dedekind cut and $x \in \mathbb{Q}$. Then $x \notin A$ if and only if x is an upper bound for A.*

Proof. Suppose first that $x \notin A$. To prove that x is an upper bound for A, Definition 5.6 tells us that it will suffice to show that for all $r \in A$, $r \leq x$. Let r be an arbitrary element of A. Then by the contrapositive of Definition 6.1b and the hypothesis that $x \notin A$, we know that $r \notin A$, $x \notin \mathbb{Q}$, or $x \not< r$. But since $r \in A$ and $x \in \mathbb{Q}$, it must be that $x \not< r$. Therefore, $r \leq x$, as desired.

Now suppose that x is an upper bound for A. By Definition 5.6, this implies that for all $r \in A$, $r \leq x$. Therefore, since there is no $r \in A$ with $r > x$, by the contrapositive of Definition 6.1c, $x \notin A$, as desired. $\square$

Exercise 6.3.

(a) Prove that for any $q \in \mathbb{Q}$, $\{x \in \mathbb{Q} \mid x < q\}$ is a Dedekind cut. We then define $\mathbf{0} = \{x \in \mathbb{Q} \mid x < 0\}$.

(b) Prove that $\{x \in \mathbb{Q} \mid x \leq 0\}$ is not a Dedekind cut.

(c) Prove that $\{x \in \mathbb{Q} \mid x < 0\} \cup \{x \in \mathbb{Q} \mid x^2 < 2\}$ is a Dedekind cut.

Proof of a. Let q be an arbitrary element of $\mathbb{Q}$. To prove that $A = \{x \in \mathbb{Q} \mid x < q\}$ is a Dedekind cut, Definition 6.1 tells us that it will suffice to show that $A \neq \emptyset$; $A \neq \mathbb{Q}$; if $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$; and if $r \in A$, then there is some $s \in A$ with $s > r$. We will take this one claim at a time.

To show that $A \neq \emptyset$, Definition 1.8 tells us that it will suffice to find an element of A. By Exercise 3.9d, q is not the first point of $\mathbb{Q}$. Thus, by Definition 3.3, there exists an object $x \in \mathbb{Q}$ such that $x < q$. By the definition of A, this implies that $x \in A$, as desired.

To show that $A \neq \mathbb{Q}$, Definition 1.2 tells us that it will suffice to find an element of $\mathbb{Q}$ that is not an element of A. By hypothesis, $q \in \mathbb{Q}$. By Exercise 3.9d, $q \not< q$. Therefore, $q \in \mathbb{Q}$ but $q \notin A$, as desired.

To show that if $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$, we let $r \in A$ and $s \in \mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s < r$ and seek to verify that $s \in A$. Since $r \in A$, $r < q$. This combined with the fact that $s < r$ implies by transitivity that $s < q$. Therefore, since $s \in \mathbb{Q}$ and $s < q$, $s \in A$, as desired.

To show that if $r \in A$, then there is some $s \in A$ with $s > r$, we let $r \in A$ and seek to find such an s. By the definition of A, $r < q$. Thus, by Additional Exercise 3.1, there exists a point $s \in \mathbb{Q}$ such that $r < s < q$. Since $s \in \mathbb{Q}$ and $s < q$, $s \in A$. It follows that s is the desired element of A satisfying $s > r$. $\square$

Proof of b. To prove that $A = \{x \in \mathbb{Q} \mid x \leq 0\}$ is not a Dedekind cut, Definition 6.1 tells us that it will suffice to show that A does have a last point. To show this, we will demonstrate that 0 is the last point of A. To demonstrate this, Definition 3.1 tells us that it will suffice to confirm that $0 \in A$ and for all $x \in A$, $x \leq 0$. Since $0 \leq 0$ and $0 \in \mathbb{Q}$, $0 \in A$. Additionally, by the definition of A, it is true that for all $x \in A$, $x \leq 0$. □

Proof of c. Let $B = \{x \in \mathbb{Q} \mid x < 0\}$ and let $C = \{x \in \mathbb{Q} \mid x^2 < 2\}$. To prove that $A = B \cup C$ is a Dedekind cut, Definition 6.1 tells us that it will suffice to show that $A \neq \emptyset$; $A \neq \mathbb{Q}$; if $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$; and if $r \in A$, then there is some $s \in A$ with $s > r$. We will take this one claim at a time.

To show that $A \neq \emptyset$, Definition 1.8 tells us that it will suffice to find an element of A. Since $-1 \in \mathbb{Q}$ and $-1 < 0$, $-1 \in B$. Therefore, by Definition 1.5, $-1 \in A$, as desired.

To show that $A \neq \mathbb{Q}$, Definition 1.2 tells us that it will suffice to find an element of $\mathbb{Q}$ that is not an element of A. Since $2 \in \mathbb{Q}$ and $2 \geq 0$, $2 \notin B$. Additionally, since $2^2 \geq 2$, $2 \notin C$. Therefore, by Definition 1.5, $2 \notin A$, as desired.

To show that if $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$, we let $r \in A$ and $s \in \mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s < r$ and seek to verify that $s \in A$. Since $r \in A$, Definition 1.5 tells us that $r \in B$ or $r \in C$. We now divide into two cases. Suppose first that $r \in B$. Then $s < r < 0$, which implies that $s \in B$, meaning that $s \in A$. Now suppose that $r \in C$. We divide into two cases again ($r \leq 0$ and $r > 0$). If $r \leq 0$, then $s < r \leq 0$ implies that $s < 0$. Thus, by the definition of B, $s \in B$, implying that $s \in A$. On the other hand, if $r > 0$, then $0 < s^2 < r^2 < 2$. Thus, by the definition of C, $s \in C$, implying that $s \in A$.

To show that A does not have a last point, suppose for the sake of contradiction that A has a last point p. We now divide into two cases ($p \leq 0$ and $p > 0$). Suppose first that $p \leq 0$. Since p is the last point of A, Definition 3.3 tells us that $x \leq p$ for all $x \in A$. But $1 \in A$ (since $1 \in \mathbb{Q}$ and $1^2 = 1 < 2$ implies $1 \in B$, implies $1 \in A$) and $1 > 0 \geq p$, a contradiction. Now suppose that $p > 0$. Definition 3.3 tells us that $p \in A$, but the condition that $p > 0$ means $p \notin B$, so we must have $p \in C$. However, by the proof of Exercise 4.24, $\frac{2(p+1)}{p+2}$ will be an element of B (and therefore A) that is greater than p no matter how large p is, a contradiction. □

Definition 6.4. If $A, B \in \mathbb{R}$, we say that $A < B$ if A is a proper subset of B.

Exercise 6.5. Show that $\mathbb{R}$ satisfies Axioms 1, 2, and 3.

Proof. By Exercise 6.3a, $\{x \in \mathbb{Q} \mid x < 0\} \in \mathbb{R}$ since $0 \in \mathbb{Q}$. Therefore, Axiom 1 is immediately satisfied.

Axiom 2 asserts that $\mathbb{R}$ must have an ordering $<$. As such, it will suffice to verify that the ordering given by Definition 6.4 satisfies the stipulations of Definition 3.1. To prove that $<$ satisfies the trichotomy, it will suffice to show that for all $A, B \in \mathbb{R}$, exactly one of the following holds: $A < B$, $B < A$, or $A = B$.

We first show that *no more than one* of the three statements can simultaneously be true. Let A, B be arbitrary elements of $\mathbb{R}$. We divide into three cases. First, suppose for the sake of contradiction that $A < B$ and $B < A$. By Definition 6.4, this implies that $A \subsetneq B$ and $B \subsetneq A$. Thus, by Definition 1.3, $A \subset B$, $B \subset A$, and $A \neq B$. But by Theorem 1.7, $A \subset B$ and $B \subset A$ implies that $A = B$, a contradiction. Second, suppose for the sake of contradiction that $A < B$ and $A = B$. By Definition 6.4, the former statement implies that $A \subsetneq B$. Thus, by Definition 1.3, $A \neq B$, a contradiction. The proof of the third case ($B < A$ and $A = B$) is symmetric to that of the second case.

We now show that *at least one* of the three statements is always true. Let A, B be arbitrary elements of $\mathbb{R}$, and suppose for the sake of contradiction that $A \not< B$, $B \not< A$, and $A \neq B$. Since $A \not< B$ and $B \not< A$, we have by Definition 6.4 that $A \not\subsetneq B$ and $B \not\subsetneq A$. Thus, by Definition 1.3, $A \not\subset B$ or $A = B$, and $B \not\subset A$ or $A = B$. But $A \neq B$ by hypothesis, so it must be that $A \not\subset B$ and $B \not\subset A$. It follows from the first statement by Definition 1.3 that there exists an object $x \in A$ such that $x \notin B$, and there exists an object $y \in B$ such that $y \notin A$. Since $x \notin B$, Lemma 6.2 implies that x is an upper bound of B. Consequently, by Definition 5.6, $p \leq x$ for all $p \in B$, including y. Similarly, $p \leq y$ for all $p \in A$, including x. Thus, we have $y \leq x$ and $x \leq y$, implying that $x = y$. But since $y \in B$, this implies that $x \in B$, a contradiction.

To prove that $<$ is transitive, it will suffice to show that for all $A, B, C \in \mathbb{R}$, if $A < B$ and $B < C$, then $A < C$. Let A, B, C be arbitrary elements of $\mathbb{R}$ for which it is true that $A < B$ and $B < C$. By Definition 6.4, we have $A \subsetneq B$ and $B \subsetneq C$. Thus, by Script 1, $A \subsetneq C$. Therefore, by Definition 6.4, $A < C$.

Axiom 3 asserts that $\mathbb{R}$ must have no first or last point. We will take this one argument at a time

Suppose for the sake of contradiction that $\mathbb{R}$ has some first point A. Then by Definition 3.3, $A \leq X$ for every $X \in \mathbb{R}$. Now since A is a Dedekind cut, Definition 6.1 tells us that $A \neq \emptyset$. Thus, by Definition 1.8, there exists some $q \in A$. Additionally, $A \subset \mathbb{Q}$ by Definition 6.1, so $q \in A$ implies that $q \in \mathbb{Q}$. It follows by Exercise 6.3a that $B = \{x \in \mathbb{Q} \mid x < q\}$ is a Dedekind cut. We now seek to prove that $B \subsetneq A$. To do this, Definition 1.3 tells us that it will suffice to show that $B \neq A$ and $B \subset A$. To show that $B \neq A$, Definition 1.2 tells us that it will suffice to find an element of A that is not an element of B. Conveniently, q is clearly such an object. To show that $B \subset A$, Definition 1.3 tells us that we must confirm that every element of B is an element of A. Let p be an arbitrary element of B. Then by the definition of B, $p \in \mathbb{Q}$ and $p < q$. It follows by Definition 6.1b (which clearly applies to A) that $p \in A$, as desired. Having proven that $B \subsetneq A$, Definition 6.4 tells us that $B < A$. But this contradicts the previously demonstrated fact that $A \leq X$ for every $X \in \mathbb{R}$, including B.

Suppose for the sake of contradiction that $\mathbb{R}$ has some last point A. Then by Definition 3.3, $X \leq A$ for every $X \in \mathbb{R}$. Now since A is a Dedekind cut, Definition 6.1 tells us that $A \neq \mathbb{Q}$. Thus, by Definition 1.2, there exists some $q \in \mathbb{Q}$ such that $q \notin A$. It follows by Lemma 6.2 that q is an upper bound of A. Consequently, by Definition 5.6, $x \leq q$ for all $x \in A$. Additionally, by Exercise 6.3a, $B = \{x \in \mathbb{Q} \mid x < q+1\}$[1] is a Dedekind cut. We now seek to prove that $A \subsetneq B$. As before, this means we must show that $A \neq B$ and $A \subset B$. To show that $A \neq B$, Definition 1.2 tells us that it will suffice to find an element of B that is not an element of A. Since $x \leq q$ for all $x \in A$ and $q < q + 0.5 < q + 1$, $q + 0.5 \notin A$ and $q + 0.5 \in B$ is the desired object. To show that $A \subset B$, Definition 1.3 tells us that we must confirm that every element of A is an element of B. Let p be an arbitrary element of A. As an element of A, we know that $p \leq q$. Thus, $p < q + 1$, so $p \in B$, as desired. Having proven that $A \subsetneq B$, Definition 6.4 tells us that $A < B$. But this contradicts the previously demonstrated fact that $X \leq A$ for every $X \in \mathbb{R}$, including B. ☐

1/14: **Lemma 6.6.** *A nonempty subset of* $\mathbb{R}$ *that is bounded above has a supremum.*

Proof. Let X be an arbitrary nonempty subset of $\mathbb{R}$ that is bounded above. To prove that $\sup X$ exists, we will show that $\sup X = U = \bigcup\{Y \mid Y \in X\}$. To show this, Definition 5.7 tells us that it will suffice to demonstrate that $U \in \mathbb{R}$, U is an upper bound of X, and if U' is an upper bound of X, then $U \leq U'$. Let's begin.

To demonstrate that $U \in \mathbb{R}$, Definition 6.1 tells us that it will suffice to confirm that $U \neq \emptyset$; $U \neq \mathbb{Q}$; if $r \in U$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in U$; and if $r \in U$, then there is some $s \in U$ with $s > r$.

As the union of a nonempty subset of nonempty sets, Script 1 implies that $U \neq \emptyset$.

To demonstrate that $U \neq \mathbb{Q}$, Definition 1.2 tells us that it will suffice to find a point $p \in \mathbb{Q}$ such that $p \notin U$. Since X is bounded above, we have by Definition 5.6 that there exists a Dedekind cut $V \in \mathbb{R}$ such that $Y \leq V$ for all $Y \in X$. It follows by Definition 6.4 that $Y \subset V$ for all $Y \in X$. Thus, by Script 1, $U \subset V$. Now since V is a Dedekind cut, we know by Definition 6.1 that $V \subset \mathbb{Q}$ and $V \neq \mathbb{Q}$, meaning that there exists a point $p \in \mathbb{Q}$ such that $p \notin V$. Consequently, since $U \subset V$, $p \notin U$, as desired.

To demonstrate that if $r \in U$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in U$, we let $r \in U$ and $s \in \mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s < r$ and seek to verify that $s \in U$. Since $r \in U$, Definition 1.13 tells us that $r \in Y$ for some $Y \in X$. Thus, since Y is a Dedekind cut, $s \in \mathbb{Q}$ and $s < r$ implies that $s \in Y$. Therefore, $s \in U$.

To demonstrate that if $r \in U$, then there is some $s \in U$ with $s > r$, we let $r \in U$ and seek to find such an s. Since $r \in U$, Definition 1.13 tells us that $r \in Y$ for some $Y \in X$. Thus, since Y is a Dedekind cut, there exists a point $s \in Y$ with $s > r$. Therefore, $s \in U$.

To demonstrate that U is an upper bound of X, Definition 5.6 tells us that it will suffice to confirm that $Y \leq U$ for all $Y \in X$. To confirm this, Definition 6.4 tells us that it will suffice to verify that $Y \subset U$ for all $Y \in X$. But by an extension of Theorem 1.7b, this is true.

Now suppose for the sake of contradiction that there exists an upper bound U' of X such that $U' < U$. It follows by Definitions 6.4 and 1.3 that there exists a point $p \in U$ such that $p \notin U'$. Thus, by the former statement and Definition 1.13, $p \in Y$ for some $Y \in X$. Additionally, since U' is an upper bound of X, we

[1]Note that we add 1 to q to treat the case that $q = \sup A$, a case in which we would have $B = A$ if B were defined as $\{x \in \mathbb{Q} \mid x < q\}$.

have by Definitions 5.6 and 6.4 that $Y \subset U'$ for all $Y \in X$. But this implies by Definition 1.3 that $p \in U'$, a contradiction. $\square$

Exercise 6.7. Show that $\mathbb{R}$ satisfies Axiom 4.

Proof. $\square$

Definition 6.8. Let C be a continuum satisfying Axioms 1-4. Consider a subset $X \subset C$. We say that X is **dense** in C if every $p \in C$ is a limit point of X.

Lemma 6.9. *A subset $X \subset C$ is dense in C if and only if $\overline{X} = C$.*

Proof. Suppose first that $X \subset C$ is dense in C. To prove that $\overline{X} = C$, Definition 1.2 tells us that it will suffice to show that every point $p \in \overline{X}$ is an element of C and vice versa. Clearly, every element of $\overline{X}$ is an element of C. On the other hand, let p be an arbitrary element of C. Since X is dense in C, Definition 6.8 tells us that $p \in LP(X)$. Therefore, by Definitions 1.5 and 4.4, $p \in \overline{X}$.

Now suppose that $\overline{X} = C$. To prove that X is dense in C, Definition 6.8 tells us that it will suffice to show that every $p \in C$ is a limit point of X. Let p be an arbitrary element of C. By Corollary 5.4, this implies that $p \in LP(C)$. It follows that $p \in LP(\overline{X})$. Thus, by Definition 4.4, $p \in LP(X \cup LP(X))$. Consequently, by Theorem 3.20, $p \in LP(X)$ or $p \in LP(LP(X))$. We now divide into two cases. If $p \in LP(X)$, then we are done. On the other hand, if $p \in LP(LP(X))$, the lemma from Theorem 4.6 asserts that $p \in LP(X)$, and we are done again. $\square$

Our next goal is to prove that $\mathbb{Q}$ is dense in $\mathbb{R}$. Just to make sense of that statement, we need to decide how to think of $\mathbb{Q}$ as a subset of $\mathbb{R}$. For every rational number $q \in \mathbb{Q}$, define the corresponding real number as the Dedekind cut

$$i(q) = \{x \in \mathbb{Q} \mid x < q\}$$

For example, $\mathbf{0} = i(0)$. It can be verified that this gives a well-defined injective function $i : \mathbb{Q} \to \mathbb{R}$. We identify $\mathbb{Q}$ with its image $i(\mathbb{Q}) \subset \mathbb{R}$ so that the rational numbers $\mathbb{Q}$ are a subset of the real numbers $\mathbb{R}$. (Similarly, $\mathbb{N}$ and $\mathbb{Z}$ can be understood as subsets of $\mathbb{R}$.)

Lemma 6.10. *Given $A, B \in \mathbb{R}$ with $A < B$, there exists $p \in \mathbb{Q}$ such that $A < i(p) < B$.*

Proof. Since $A < B$, Definition 6.4 tells us that $A \subsetneq B$. Thus, by Definition 1.3, there exists a point q such that $q \in B$ and $q \notin A$. Since $q \in B$ where B is a Dedekind cut, we have by Definition 6.1 that there exists a point $p \in B$ with $p > q$. Additionally, since $q \notin A$ implies that q is an upper bound of A by Lemma 6.2, we know by Definition 5.6 that $x \leq q$ for all $x \in A$. It follows since $q < p$ that $x \leq p$ for all $x \in A$, meaning by Definition 5.6 and Lemma 6.2 that $p \notin A$. Having established that $p, q \in B$, $p, q \notin A$, and $q < p$, we are now ready to prove that $A < i(p) < B$. Definition 6.4 tells us that we may do so by showing that $A \subsetneq i(p)$ and $i(p) \subsetneq B$. We will take this one argument at a time.

To show that $A \subsetneq i(p)$, Definition 1.3 tells us that it will suffice to verify that every element of A is an element of $i(p)$ and that there exists an element of $i(p)$ that is not an element of A. We treat the former statement first. As previously mentioned, $x \leq p$ for all $x \in A$. This combined with the fact that $p \notin A$ implies that $x < p$ for all $x \in A$. Thus, by the definition of $i(p)$, $x \in i(p)$ for all $x \in A$, as desired. As to the latter statement, since $q < p$, we have by the definition of $i(p)$ that $q \in i(p)$. However, we also know that $q \notin A$, as desired.

To show that $i(p) \subsetneq B$, we must verify symmetric arguments to before. For the former statement, let r be an arbitrary element of $i(p)$. Then by the definition of $i(p)$, $r < p$. Since $p \in B$ and $r \in \mathbb{Q}$ satisfy $r < p$, we have by Definition 6.1 that $r \in B$, as desired. As to the latter statement, p is clearly an element of B that is not an element of $i(p)$, as desired. $\square$