

MATH 16210 (Honors Calculus II IBL) Notes

Steven Labalme

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Contents

6	Construction of the Real Numbers	1
6.1	Journal	1
6.2	Discussion	6

Script 6

Construction of the Real Numbers

6.1 Journal

1/12: **Definition 6.1.** A subset A of \mathbb{Q} is said to be a **cut** (or **Dedekind cut**) if it satisfies the following:

- (a) $A \neq \emptyset$ and $A \neq \mathbb{Q}$.
- (b) If $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$.
- (c) A does not have a last point; i.e., if $r \in A$, then there is some $s \in A$ with $s > r$.

We denote the collection of all cuts by \mathbb{R} .

Lemma 6.2. Let A be a Dedekind cut and $x \in \mathbb{Q}$. Then $x \notin A$ if and only if x is an upper bound for A .

Proof. Suppose first that $x \notin A$. To prove that x is an upper bound for A , Definition 5.6 tells us that it will suffice to show that for all $r \in A$, $r \leq x$. Let r be an arbitrary element of A . Then by the contrapositive of Definition 6.1b and the hypothesis that $x \notin A$, we know that $r \notin A$, $x \notin \mathbb{Q}$, or $x \not\leq r$. But since $r \in A$ and $x \in \mathbb{Q}$, it must be that $x \not\leq r$. Therefore, $r \leq x$, as desired.

Now suppose that x is an upper bound for A . By Definition 5.6, this implies that for all $r \in A$, $r \leq x$. Therefore, since there is no $r \in A$ with $r > x$, by the contrapositive of Definition 6.1c, $x \notin A$, as desired. \square

Exercise 6.3.

- (a) Prove that for any $q \in \mathbb{Q}$, $\{x \in \mathbb{Q} \mid x < q\}$ is a Dedekind cut. We then define $\mathbf{0} = \{x \in \mathbb{Q} \mid x < 0\}$.
- (b) Prove that $\{x \in \mathbb{Q} \mid x \leq 0\}$ is not a Dedekind cut.
- (c) Prove that $\{x \in \mathbb{Q} \mid x < 0\} \cup \{x \in \mathbb{Q} \mid x^2 < 2\}$ is a Dedekind cut.

Proof of a. Let q be an arbitrary element of \mathbb{Q} . To prove that $A = \{x \in \mathbb{Q} \mid x < q\}$ is a Dedekind cut, Definition 6.1 tells us that it will suffice to show that $A \neq \emptyset$; $A \neq \mathbb{Q}$; if $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$; and if $r \in A$, then there is some $s \in A$ with $s > r$. We will take this one claim at a time.

To show that $A \neq \emptyset$, Definition 1.8 tells us that it will suffice to find an element of A . By Exercise 3.9d, q is not the first point of \mathbb{Q} . Thus, by Definition 3.3, there exists an object $x \in \mathbb{Q}$ such that $x < q$. By the definition of A , this implies that $x \in A$, as desired.

To show that $A \neq \mathbb{Q}$, Definition 1.2 tells us that it will suffice to find an element of \mathbb{Q} that is not an element of A . By hypothesis, $q \in \mathbb{Q}$. By Exercise 3.9d, $q \not< q$. Therefore, $q \in \mathbb{Q}$ but $q \notin A$, as desired.

To show that if $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$, we let $r \in A$ and $s \in \mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s < r$ and seek to verify that $s \in A$. Since $r \in A$, $r < q$. This combined with the fact that $s < r$ implies by transitivity that $s < q$. Therefore, since $s \in \mathbb{Q}$ and $s < q$, $s \in A$, as desired.

To show that if $r \in A$, then there is some $s \in A$ with $s > r$, we let $r \in A$ and seek to find such an s . By the definition of A , $r < q$. Thus, by Additional Exercise 3.1, there exists a point $s \in \mathbb{Q}$ such that $r < s < q$. Since $s \in \mathbb{Q}$ and $s < q$, $s \in A$. It follows that s is the desired element of A satisfying $s > r$. \square

Proof of b. To prove that $A = \{x \in \mathbb{Q} \mid x \leq 0\}$ is not a Dedekind cut, Definition 6.1 tells us that it will suffice to show that A does have a last point. To show this, we will demonstrate that 0 is the last point of A . To demonstrate this, Definition 3.1 tells us that it will suffice to confirm that $0 \in A$ and for all $x \in A$, $x \leq 0$. Since $0 \leq 0$ and $0 \in \mathbb{Q}$, $0 \in A$. Additionally, by the definition of A , it is true that for all $x \in A$, $x \leq 0$. \square

Proof of c. Let $B = \{x \in \mathbb{Q} \mid x < 0\}$ and let $C = \{x \in \mathbb{Q} \mid x^2 < 2\}$. To prove that $A = B \cup C$ is a Dedekind cut, Definition 6.1 tells us that it will suffice to show that $A \neq \emptyset$; $A \neq \mathbb{Q}$; if $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$; and if $r \in A$, then there is some $s \in A$ with $s > r$. We will take this one claim at a time.

To show that $A \neq \emptyset$, Definition 1.8 tells us that it will suffice to find an element of A . Since $-1 \in \mathbb{Q}$ and $-1 < 0$, $-1 \in B$. Therefore, by Definition 1.5, $-1 \in A$, as desired.

To show that $A \neq \mathbb{Q}$, Definition 1.2 tells us that it will suffice to find an element of \mathbb{Q} that is not an element of A . Since $2 \in \mathbb{Q}$ and $2 \geq 0$, $2 \notin B$. Additionally, since $2^2 \geq 2$, $2 \notin C$. Therefore, by Definition 1.5, $2 \notin A$, as desired.

To show that if $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$, we let $r \in A$ and $s \in \mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s < r$ and seek to verify that $s \in A$. Since $r \in A$, Definition 1.5 tells us that $r \in B$ or $r \in C$. We now divide into two cases. Suppose first that $r \in B$. Then $s < r < 0$, which implies that $s \in B$, meaning that $s \in A$. Now suppose that $r \in C$. We divide into two cases again ($r \leq 0$ and $r > 0$). If $r \leq 0$, then $s < r \leq 0$ implies that $s < 0$. Thus, by the definition of B , $s \in B$, implying that $s \in A$. On the other hand, if $r > 0$, then $0 < s^2 < r^2 < 2$. Thus, by the definition of C , $s \in C$, implying that $s \in A$.

To show that A does not have a last point, suppose for the sake of contradiction that A has a last point p . We now divide into two cases ($p \leq 0$ and $p > 0$). Suppose first that $p \leq 0$. Since p is the last point of A , Definition 3.3 tells us that $x \leq p$ for all $x \in A$. But $1 \in A$ (since $1 \in \mathbb{Q}$ and $1^2 = 1 < 2$ implies $1 \in B$, implies $1 \in A$) and $1 > 0 \geq p$, a contradiction. Now suppose that $p > 0$. Definition 3.3 tells us that $p \in A$, but the condition that $p > 0$ means $p \notin B$, so we must have $p \in C$. However, by the proof of Exercise 4.24, $\frac{2(p+1)}{p+2}$ will be an element of B (and therefore A) that is greater than p no matter how large p is, a contradiction. \square

Definition 6.4. If $A, B \in \mathbb{R}$, we say that $A < B$ if A is a proper subset of B .

Exercise 6.5. Show that \mathbb{R} satisfies Axioms 1, 2, and 3.

Proof. By Exercise 6.3a, $\{x \in \mathbb{Q} \mid x < 0\} \in \mathbb{R}$ since $0 \in \mathbb{Q}$. Therefore, Axiom 1 is immediately satisfied.

Axiom 2 asserts that \mathbb{R} must have an ordering $<$. As such, it will suffice to verify that the ordering given by Definition 6.4 satisfies the stipulations of Definition 3.1. To prove that $<$ satisfies the trichotomy, it will suffice to show that for all $A, B \in \mathbb{R}$, exactly one of the following holds: $A < B$, $B < A$, or $A = B$.

We first show that *no more than one* of the three statements can simultaneously be true. Let A, B be arbitrary elements of \mathbb{R} . We divide into three cases. First, suppose for the sake of contradiction that $A < B$ and $B < A$. By Definition 6.4, this implies that $A \subsetneq B$ and $B \subsetneq A$. Thus, by Definition 1.3, $A \subset B$, $B \subset A$, and $A \neq B$. But by Theorem 1.7, $A \subset B$ and $B \subset A$ implies that $A = B$, a contradiction. Second, suppose for the sake of contradiction that $A < B$ and $A = B$. By Definition 6.4, the former statement implies that $A \subsetneq B$. Thus, by Definition 1.3, $A \neq B$, a contradiction. The proof of the third case ($B < A$ and $A = B$) is symmetric to that of the second case.

We now show that *at least one* of the three statements is always true. Let A, B be arbitrary elements of \mathbb{R} , and suppose for the sake of contradiction that $A \not< B$, $B \not< A$, and $A \neq B$. Since $A \not< B$ and $B \not< A$, we have by Definition 6.4 that $A \not\subsetneq B$ and $B \not\subsetneq A$. Thus, by Definition 1.3, $A \not\subset B$ or $A = B$, and $B \not\subset A$ or $A = B$. But $A \neq B$ by hypothesis, so it must be that $A \not\subset B$ and $B \not\subset A$. It follows from the first statement by Definition 1.3 that there exists an object $x \in A$ such that $x \notin B$, and there exists an object $y \in B$ such that $y \notin A$. Since $x \notin B$, Lemma 6.2 implies that x is an upper bound of B . Consequently, by Definition 5.6, $p \leq x$ for all $p \in B$, including y . Similarly, $p \leq y$ for all $p \in A$, including x . Thus, we have $y \leq x$ and $x \leq y$, implying that $x = y$. But since $y \in B$, this implies that $x \in B$, a contradiction.

To prove that $<$ is transitive, it will suffice to show that for all $A, B, C \in \mathbb{R}$, if $A < B$ and $B < C$, then $A < C$. Let A, B, C be arbitrary elements of \mathbb{R} for which it is true that $A < B$ and $B < C$. By Definition 6.4, we have $A \subsetneq B$ and $B \subsetneq C$. Thus, by Script 1, $A \subsetneq C$. Therefore, by Definition 6.4, $A < C$.

Axiom 3 asserts that \mathbb{R} must have no first or last point. We will take this one argument at a time

Suppose for the sake of contradiction that \mathbb{R} has some first point A . Then by Definition 3.3, $A \leq X$ for every $X \in \mathbb{R}$. Now since A is a Dedekind cut, Definition 6.1 tells us that $A \neq \emptyset$. Thus, by Definition 1.8, there exists some $q \in A$. Additionally, $A \subset \mathbb{Q}$ by Definition 6.1, so $q \in A$ implies that $q \in \mathbb{Q}$. It follows by Exercise 6.3a that $B = \{x \in \mathbb{Q} \mid x < q\}$ is a Dedekind cut. We now seek to prove that $B \subsetneq A$. To do this, Definition 1.3 tells us that it will suffice to show that $B \neq A$ and $B \subset A$. To show that $B \neq A$, Definition 1.2 tells us that it will suffice to find an element of A that is not an element of B . Conveniently, q is clearly such an object. To show that $B \subset A$, Definition 1.3 tells us that we must confirm that every element of B is an element of A . Let p be an arbitrary element of B . Then by the definition of B , $p \in \mathbb{Q}$ and $p < q$. It follows by Definition 6.1b (which clearly applies to A) that $p \in A$, as desired. Having proven that $B \subsetneq A$, Definition 6.4 tells us that $B < A$. But this contradicts the previously demonstrated fact that $A \leq X$ for every $X \in \mathbb{R}$, including B .

Suppose for the sake of contradiction that \mathbb{R} has some last point A . Then by Definition 3.3, $X \leq A$ for every $X \in \mathbb{R}$. Now since A is a Dedekind cut, Definition 6.1 tells us that $A \neq \mathbb{Q}$. Thus, by Definition 1.2, there exists some $q \in \mathbb{Q}$ such that $q \notin A$. It follows by Lemma 6.2 that q is an upper bound of A . Consequently, by Definition 5.6, $x \leq q$ for all $x \in A$. Additionally, by Exercise 6.3a, $B = \{x \in \mathbb{Q} \mid x < q + 1\}$ ^[1] is a Dedekind cut. We now seek to prove that $A \subsetneq B$. As before, this means we must show that $A \neq B$ and $A \subset B$. To show that $A \neq B$, Definition 1.2 tells us that it will suffice to find an element of B that is not an element of A . Since $x \leq q$ for all $x \in A$ and $q < q + 0.5 < q + 1$, $q + 0.5 \notin A$ and $q + 0.5 \in B$ is the desired object. To show that $A \subset B$, Definition 1.3 tells us that we must confirm that every element of A is an element of B . Let p be an arbitrary element of A . As an element of A , we know that $p \leq q$. Thus, $p < q + 1$, so $p \in B$, as desired. Having proven that $A \subsetneq B$, Definition 6.4 tells us that $A < B$. But this contradicts the previously demonstrated fact that $X \leq A$ for every $X \in \mathbb{R}$, including B . \square

1/14: **Lemma 6.6.** *A nonempty subset of \mathbb{R} that is bounded above has a supremum.*

Proof. Let X be an arbitrary nonempty subset of \mathbb{R} that is bounded above. To prove that $\sup X$ exists, we will show that $\sup X = U = \bigcup\{Y \mid Y \in X\}$. To show this, Definition 5.7 tells us that it will suffice to demonstrate that $U \in \mathbb{R}$, U is an upper bound of X , and if U' is an upper bound of X , then $U \leq U'$. Let's begin.

To demonstrate that $U \in \mathbb{R}$, Definition 6.1 tells us that it will suffice to confirm that $U \neq \emptyset$; $U \neq \mathbb{Q}$; if $r \in U$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in U$; and if $r \in U$, then there is some $s \in U$ with $s > r$.

As the union of a nonempty subset of nonempty sets, Script 1 implies that $U \neq \emptyset$.

To demonstrate that $U \neq \mathbb{Q}$, Definition 1.2 tells us that it will suffice to find a point $p \in \mathbb{Q}$ such that $p \notin U$. Since X is bounded above, we have by Definition 5.6 that there exists a Dedekind cut $V \in \mathbb{R}$ such that $Y \leq V$ for all $Y \in X$. It follows by Definition 6.4 that $Y \subset V$ for all $Y \in X$. Thus, by Script 1, $U \subset V$. Now since V is a Dedekind cut, we know by Definition 6.1 that $V \subset \mathbb{Q}$ and $V \neq \mathbb{Q}$, meaning that there exists a point $p \in \mathbb{Q}$ such that $p \notin V$. Consequently, since $U \subset V$, $p \notin U$, as desired.

To demonstrate that if $r \in U$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in U$, we let $r \in U$ and $s \in \mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s < r$ and seek to verify that $s \in U$. Since $r \in U$, Definition 1.13 tells us that $r \in Y$ for some $Y \in X$. Thus, since Y is a Dedekind cut, $s \in \mathbb{Q}$ and $s < r$ implies that $s \in Y$. Therefore, $s \in U$.

To demonstrate that if $r \in U$, then there is some $s \in U$ with $s > r$, we let $r \in U$ and seek to find such an s . Since $r \in U$, Definition 1.13 tells us that $r \in Y$ for some $Y \in X$. Thus, since Y is a Dedekind cut, there exists a point $s \in Y$ with $s > r$. Therefore, $s \in U$.

To demonstrate that U is an upper bound of X , Definition 5.6 tells us that it will suffice to confirm that $Y \leq U$ for all $Y \in X$. To confirm this, Definition 6.4 tells us that it will suffice to verify that $Y \subset U$ for all $Y \in X$. But by an extension of Theorem 1.7b, this is true.

Now suppose for the sake of contradiction that there exists an upper bound U' of X such that $U' < U$. It follows by Definitions 6.4 and 1.3 that there exists a point $p \in U$ such that $p \notin U'$. Thus, by the former statement and Definition 1.13, $p \in Y$ for some $Y \in X$. Additionally, since U' is an upper bound of X , we

¹Note that we add 1 to q to treat the case that $q = \sup A$, a case in which we would have $B = A$ if B were defined as $\{x \in \mathbb{Q} \mid x < q\}$.

have by Definitions 5.6 and 6.4 that $Y \subset U'$ for all $Y \in X$. But this implies by Definition 1.3 that $p \in U'$, a contradiction. \square

1/19: **Exercise 6.7.** Show that \mathbb{R} satisfies Axiom 4.

Proof. Suppose for the sake of contradiction that \mathbb{R} does not satisfy Axiom 4. It follows that \mathbb{R} is not connected, implying by Definition 4.22 that $\mathbb{R} = A \cup B$ where A, B are disjoint, nonempty, open sets. Since A, B are disjoint and nonempty, we know that there exist distinct objects $a \in A$ and $b \in B$. WLOG, let $a < b$.

We now seek to prove that the set $A \cap \underline{ab}$ is nonempty and bounded above. To prove that $A \cap \underline{ab}$ is nonempty, Definition 1.8 tells us that it will suffice to find an element of $A \cap \underline{ab}$. Since $a \in A$ and A is open, we have by Theorem 4.10 that there exists a region \underline{cd} such that $a \in \underline{cd}$ and $\underline{cd} \subset A$. It follows by Definitions 3.10 and 3.6 that $a < d$, implying by Lemma 6.10^[2] that there exists some point $x \in \mathbb{R}$ such that $c < a < x < d < b$ (note that $d < b$ since if $b < d$, then $b \in \underline{cd}$ would contradict the fact that $\underline{cd} \subset A$). Consequently, $x \in \underline{cd}$, meaning that $x \in A$, and $x \in \underline{ab}$. Therefore, $x \in A \cap \underline{ab}$, as desired. To prove that $A \cap \underline{ab}$ is bounded above, Definition 5.6 tells us that it will suffice to show that b is an upper bound of $A \cap \underline{ab}$. To show this, Definition 5.6 tells us that it will suffice to confirm that $y \leq b$ for all $y \in A \cap \underline{ab}$. Let y be an arbitrary element of $A \cap \underline{ab}$. Then by Definition 1.6, $y \in A$ and $y \in \underline{ab}$. It follows from the latter statement by Definitions 3.10 and 3.6 that $y < b$, i.e., $y \leq b$, as desired.

Having established that $A \cap \underline{ab} \subset \mathbb{R}$ is nonempty and bounded above, we can invoke Lemma 6.6 to learn that $A \cap \underline{ab}$ has a supremum $\sup(A \cap \underline{ab})$. We now divide into two cases ($\sup(A \cap \underline{ab}) \in A$ and $\sup(A \cap \underline{ab}) \in B$; it follows from the definitions of A and B that exactly one of these cases is true). Suppose first that $\sup(A \cap \underline{ab}) \in A$. Then since A is open, we have by Theorem 4.10 that there exists a region \underline{ef} such that $\sup(A \cap \underline{ab}) \in \underline{ef}$ and $\underline{ef} \subset A$. It follows from the former condition that $\sup(A \cap \underline{ab}) < f$. Thus, by Lemma 6.10, there exists an object $z \in \mathbb{R}$ such that $e < \sup(A \cap \underline{ab}) < z < f < b$ (note that $f < b$ for the same reason that $d < b$). Consequently, $z \in \underline{ef}$, implying that $z \in A$, and $z \in \underline{ab}$. Thus, we have found an element of $A \cap \underline{ab}$ that is greater than $\sup(A \cap \underline{ab})$, contradicting Definitions 5.7 and 5.6. The proof is symmetric in the other case. \square

1/14: **Definition 6.8.** Let C be a continuum satisfying Axioms 1-4. Consider a subset $X \subset C$. We say that X is **dense** in C if every $p \in C$ is a limit point of X .

Lemma 6.9. A subset $X \subset C$ is dense in C if and only if $\overline{X} = C$.

Proof. Suppose first that $X \subset C$ is dense in C . To prove that $\overline{X} = C$, Definition 1.2 tells us that it will suffice to show that every point $p \in \overline{X}$ is an element of C and vice versa. Clearly, every element of \overline{X} is an element of C . On the other hand, let p be an arbitrary element of C . Since X is dense in C , Definition 6.8 tells us that $p \in LP(X)$. Therefore, by Definitions 1.5 and 4.4, $p \in \overline{X}$.

Now suppose that $\overline{X} = C$. To prove that X is dense in C , Definition 6.8 tells us that it will suffice to show that every $p \in C$ is a limit point of X . Let p be an arbitrary element of C . By Corollary 5.4, this implies that $p \in LP(C)$. It follows that $p \in LP(\overline{X})$. Thus, by Definition 4.4, $p \in LP(X \cup LP(X))$. Consequently, by Theorem 3.20, $p \in LP(X)$ or $p \in LP(LP(X))$. We now divide into two cases. If $p \in LP(X)$, then we are done. On the other hand, if $p \in LP(LP(X))$, the lemma from Theorem 4.6 asserts that $p \in LP(X)$, and we are done again. \square

Our next goal is to prove that \mathbb{Q} is dense in \mathbb{R} . Just to make sense of that statement, we need to decide how to think of \mathbb{Q} as a subset of \mathbb{R} . For every rational number $q \in \mathbb{Q}$, define the corresponding real number as the Dedekind cut

$$i(q) = \{x \in \mathbb{Q} \mid x < q\}$$

For example, $\mathbf{0} = i(0)$. It can be verified that this gives a well-defined injective function $i : \mathbb{Q} \rightarrow \mathbb{R}$. We identify \mathbb{Q} with its image $i(\mathbb{Q}) \subset \mathbb{R}$ so that the rational numbers \mathbb{Q} are a subset of the real numbers \mathbb{R} . (Similarly, \mathbb{N} and \mathbb{Z} can be understood as subsets of \mathbb{R} .)

Lemma 6.10. Given $A, B \in \mathbb{R}$ with $A < B$, there exists $p \in \mathbb{Q}$ such that $A < i(p) < B$.

²We may use this lemma since it does not depend on this result, Definition 6.8, or Lemma 6.9.

Proof. Since $A < B$, Definition 6.4 tells us that $A \subsetneq B$. Thus, by Definition 1.3, there exists a point q such that $q \in B$ and $q \notin A$. Since $q \in B$ where B is a Dedekind cut, we have by Definition 6.1 that there exists a point $p \in B$ with $p > q$. Additionally, since $q \notin A$ implies that q is an upper bound of A by Lemma 6.2, we know by Definition 5.6 that $x \leq q$ for all $x \in A$. It follows since $q < p$ that $x \leq p$ for all $x \in A$, meaning by Definition 5.6 and Lemma 6.2 that $p \notin A$. Having established that $p, q \in B$, $p, q \notin A$, and $q < p$, we are now ready to prove that $A < i(p) < B$. Definition 6.4 tells us that we may do so by showing that $A \subsetneq i(p)$ and $i(p) \subsetneq B$. We will take this one argument at a time.

To show that $A \subsetneq i(p)$, Definition 1.3 tells us that it will suffice to verify that every element of A is an element of $i(p)$ and that there exists an element of $i(p)$ that is not an element of A . We treat the former statement first. As previously mentioned, $x \leq p$ for all $x \in A$. This combined with the fact that $p \notin A$ implies that $x < p$ for all $x \in A$. Thus, by the definition of $i(p)$, $x \in i(p)$ for all $x \in A$, as desired. As to the latter statement, since $q < p$, we have by the definition of $i(p)$ that $q \in i(p)$. However, we also know that $q \notin A$, as desired.

To show that $i(p) \subsetneq B$, we must verify symmetric arguments to before. For the former statement, let r be an arbitrary element of $i(p)$. Then by the definition of $i(p)$, $r < p$. Since $p \in B$ and $r \in \mathbb{Q}$ satisfy $r < p$, we have by Definition 6.1 that $r \in B$, as desired. As to the latter statement, p is clearly an element of B that is not an element of $i(p)$, as desired. \square

1/19: **Theorem 6.11.** $i(\mathbb{Q})$ is dense in \mathbb{R} .

Proof. To prove that $i(\mathbb{Q})$ is dense in \mathbb{R} , Definition 6.8 tells us that it will suffice to show the every point $X \in \mathbb{R}$ is a limit point of $i(\mathbb{Q})$. Let X be an arbitrary element of \mathbb{R} . To show that $X \in LP(i(\mathbb{Q}))$, Definition 3.13 tells us that it will suffice to verify that for every region \underline{AB} with $X \in \underline{AB}$, we have $\underline{AB} \cap (i(\mathbb{Q}) \setminus \{X\}) \neq \emptyset$. Let \underline{AB} be an arbitrary region with $X \in \underline{AB}$. It follows by Definitions 3.10 and 3.6 that $A < X < B$. Thus, by Lemma 6.10, there exists $p \in \mathbb{Q}$ such that $A < i(p) < X < B$. By Definitions 3.6 and 3.10, $i(p) \in \underline{AB}$. By Definition 1.18, $i(p) \in i(\mathbb{Q})$. By Exercise 6.5, $i(p) < X$ implies that $i(p) \neq X$. Combining the last three results with Definitions 1.11 and 1.6, we have that $i(p) \in \underline{AB} \cap (i(\mathbb{Q}) \setminus \{X\})$, as desired. \square

Corollary 6.12 (The Archimedean Property). *Let $A \in \mathbb{R}$ be a positive real number. Then there exist nonzero natural numbers $n, m \in \mathbb{N}$ such that $i(\frac{1}{n}) < A < i(m)$.*

Proof. We will first prove that there exists a nonzero natural number n such that $i(\frac{1}{n}) < A$. We will then prove that there exists a nonzero natural number m such that $A < i(m)$. Let's begin.

Since $A \in \mathbb{R}$ is positive, we know that $0 < A$. Thus, by Lemma 6.10, there exists $\frac{p}{n} \in \mathbb{Q}$ such that $0 < i(\frac{p}{n}) < A$. As permitted by Exercise 3.9b, we choose $\frac{p}{n} \in [\frac{p}{n}]$ to be an object such that $0 < n$ (this means that $n \in \mathbb{N}$). Consequently, by Scripts 2 and 3, we know that $0 < \frac{1}{n} \leq \frac{p}{n}$. It follows that $i(\frac{1}{n}) \leq i(\frac{p}{n})$ since $x \in i(\frac{1}{n})$ implies $x < \frac{1}{n} \leq \frac{p}{n}$ implies $x \in i(\frac{p}{n})$, implies $i(\frac{1}{n}) \subset i(\frac{p}{n})$. Therefore, $i(\frac{1}{n}) \leq i(\frac{p}{n}) < A$, as desired.

By Exercise 6.5, Axiom 3, and Definition 3.3, there exists a point $B \in \mathbb{R}$ such that $A < B$. It follows by Lemma 6.10 that there exists $\frac{m}{q} \in \mathbb{Q}$ such that $A < i(\frac{m}{q}) < B$. As before, let $\frac{m}{q}$ be an object such that $0 < q$. Consequently, by Scripts 2 and 3, we know that $0 < \frac{m}{q} \leq m$. Once again, for the same reasons as before, $i(\frac{m}{q}) \leq i(m)$. Therefore, $A < i(\frac{m}{q}) \leq i(m)$, as desired. \square

Corollary 6.13. $i(\mathbb{N})$ is an unbounded subset of \mathbb{R} .

Proof. Suppose for the sake of contradiction that $i(\mathbb{N})$ is bounded above. Then by Definition 5.6, there exists a point $A \in \mathbb{R}$ such that $i(n) \leq A$ for all $n \in \mathbb{N}$. Note that A is a positive real number since $0 = i(0) \leq A$. But by Corollary 6.12, $A < i(n)$ for some $n \in \mathbb{N}$, a contradiction. \square

1/21: **Corollary 6.14.** *If $A \in \mathbb{R}$ is a real number, then there is an integer n such that $i(n-1) \leq A < i(n)$.*

Proof. Let X be the set of all integers z such that $i(z) \leq A$. Symbolically,

$$X = \{z \mid z \in \mathbb{Z} \text{ and } i(z) \leq A\}$$

Since $A \neq \emptyset$ by Definition 6.1, there exists a point $\frac{p}{q} \in \mathbb{Q}$ such that $\frac{p}{q} \in A$. As in Corollary 6.12, we let $q > 0$. It follows by Scripts 2 and 3 that if $p \geq 0$, then $0 \leq \frac{p}{q}$, i.e.^[3], $i(0) \leq A$ and if $p < 0$, then $p \leq \frac{p}{q}$, i.e.,

³For the same reasons as in Corollary 6.12.

$i(p) \leq A$. Thus, in either case, X is nonempty.

Now by Corollary 6.12, there exists a nonzero natural number m such that $A < i(m)$. Let $f : X \rightarrow \mathbb{N}$ be defined by the rule

$$f(x) = m - x$$

By Script 1, f is an injective function, $f(X) \subset \mathbb{N}$, and $f(X)$ is nonempty (since X is nonempty). Thus, by Additional Exercise 0.1, there is a least element, which we shall call y , in $f(X)$. Since f is injective, there exists exactly one object $n - 1 \in X$ such that $f(n - 1) = y$.

By the definition of X , $i(n - 1) \leq A$. To prove that $A < i(n)$, suppose for the sake of contradiction that $i(n) \leq A$. This coupled with the fact that $n \in \mathbb{Z}$ implies that $n \in X$. Thus, $f(n) \in f(X)$. But $f(n) = m - n < m - n + 1 = m - (n - 1) = f(n - 1)$, contradicting the fact that $f(n - 1)$ is the least element of $f(X)$. \square

6.2 Discussion

- 1/12:
- Upper limit at signing up for 4-5 across the script.
 - Lemma 6.2 is probably more straightforward using a contradiction argument.
 - Briefly restate the algebra of Exercise 4.24 in Exercise 6.3c.
- 1/14:
- Turning in Script 5 journals is optional — it will boost your grade a bit if you do.
 - Your journal grade will be whichever is higher: the average of all your journal grades with and without Script 5.
 - Script 5 will probably be due Wednesday, 1/20.
 - In Lemma 6.6, do we need to prove that the union of arbitrarily many Dedekind cuts is, itself, a Dedekind cut? Yes.
- 1/18:
- Is there a way to prove something else besides A is not open in Exercise 6.7?
 - This is probably it as far as proving that continua are connected.
 - It may not be possible to prove that *any* of the statements are wrong, but he's not sure.
 - Is Lemma 6.9 used in the proofs of any subsequent results, or is it just a less important result (hence the lemma designation)?
 - We can think of it as an alternate definition for density — we could prove Definition 6.8 from it.
 - Is my handwavey use of Scripts 2 and 3 ok in Corollary 6.12?
 - I'm fine.
 - Is there a simpler way to prove Corollaries 6.12 and 6.14?
 - hi
 - Is the math REU still running this summer?
 - He's not sure; UChicago's may not be NSF approved, hence why its not on the website rn.
 - What other summer opportunities would you recommend for a student at my level?
 - He did an REU at UWisconsin when he was an undergrad.
 - Sounds like its pretty much just REUs for undergrads.
 - I could ask around to see if anyone is a Knot Theorist/willing to sponsor me.
- 1/19:
- Easier Corollary 6.12:

– Let $B > A$. Then $A < i(\frac{m}{q}) < B$. Then $A < i(m)$.

- Several proofs were given for Corollary 6.14. One other correct one constructed the nonempty, bounded above set of all $i(n)$ less than or equal to A and considered its supremum.

1/21:

- Now graded a bit more critically on presentations.

– Write big, talk loudly, don't talk to the blackboard.

- My original proof of Corollary 6.14 is incorrect because I can't split into cases the way I did (*longer expo*).

– Instead, use Seb's approach.