

# Script 8

## Intervals

### 8.1 Journal

2/9: Now that we have constructed  $\mathbb{R}$  and proved the fundamental facts about it, we will work with the real numbers  $\mathbb{R}$  instead of an arbitrary continuum  $C$ . We will leave behind Dedekind cuts and think of elements of  $\mathbb{R}$  as numbers. Accordingly, from now on, we will use lower-case letters like  $x$  for real numbers and will write  $+$  and  $\cdot$  for  $\oplus$  and  $\otimes$ . We will also now use the standard notation  $(a, b)$  for the region  $ab = \{x \in \mathbb{R} \mid a < x < b\}$ . Even though the notation is the same, this is *not* the same object as the ordered pair  $(a, b)$ .

More generally, we adopt the following standard notation:

$$\begin{aligned}(a, b) &= \{x \in \mathbb{R} \mid a < x < b\} \\ [a, b) &= \{x \in \mathbb{R} \mid a \leq x < b\} \\ (a, b] &= \{x \in \mathbb{R} \mid a < x \leq b\} \\ [a, b] &= \{x \in \mathbb{R} \mid a \leq x \leq b\} \\ (a, \infty) &= \{x \in \mathbb{R} \mid a < x\} \\ [a, \infty) &= \{x \in \mathbb{R} \mid a \leq x\} \\ (-\infty, b) &= \{x \in \mathbb{R} \mid x < b\} \\ (-\infty, b] &= \{x \in \mathbb{R} \mid x \leq b\}\end{aligned}\tag{8.1}$$

**Exercise 8.1.** Identify the sets in Equations 8.1 that are open/closed/neither.

*Proof.* Note that by Theorem 5.1, any of these sets proven to be just one of open or closed will not be the other, i.e., a set proven to be open will not be closed and vice versa.

By Corollary 4.11,  $(a, b)$  is open.

By an adaptation of Corollary 5.14,  $b \in LP([a, b))$  but  $b \notin [a, b)$ . Since  $[a, b)$  doesn't contain all of its limit points, Definition 4.1 tells us that it is not closed. Additionally, since  $a \in LP(C \setminus [a, b))$  but  $a \notin C \setminus [a, b)$ , Definition 4.8 tells us that it is not open. Therefore, it is neither.

The proof that  $(a, b]$  is neither is symmetric to the previous case.

By Corollaries 5.15 and 4.7,  $[a, b]$  is closed.

By Corollary 4.13,  $(a, \infty)$  is open.

By Corollary 4.13 and Definition 4.8,  $[a, \infty) = C \setminus (-\infty, a)$  is closed.

The proofs that  $(-\infty, b)$  and  $(-\infty, b]$  are open and closed, respectively, are symmetric to the previous two cases, respectively.  $\square$

**Definition 8.2.** A set  $I \subset \mathbb{R}$  is an **interval** if for all  $x, y \in I$  with  $x < y$ ,  $[x, y] \subset I$ .

**Lemma 8.3.** *A proper subset  $I \subsetneq \mathbb{R}$  is an interval if and only if it takes one of the eight forms in Equations 8.1.*

*Proof.* Suppose first that  $I \subsetneq \mathbb{R}$  is an interval. If  $I = \emptyset$ , then  $I = (a, a)$  for any  $a \in \mathbb{R}$ , and we are done. Thus, we will assume for the remainder of the proof of the forward direction that  $I$  is nonempty. To address this case, we will first prove that the facts that  $I \subsetneq \mathbb{R}$ ,  $I \neq \emptyset$ , and  $I$  is an interval imply that  $I$  is bounded above, bounded below, or both. Then in each of these three cases, we will look at whether  $\sup I$  and  $\inf I$  (if they exist) are elements of the set or not to differentiate between the eight forms in Equations 8.1. Let's begin.

Suppose for the sake of contradiction that there exists a nonempty interval  $I \subsetneq \mathbb{R}$  that is neither bounded above nor bounded below. Since  $I \subsetneq \mathbb{R}$ , we have by Definition 1.3 that there exists a point  $p \in \mathbb{R}$  such that  $p \notin I$ . Additionally, since  $I$  is neither bounded above nor below, Definition 5.6 implies that  $p$  is neither an upper nor a lower bound of  $I$ . Thus, there exist  $x, y \in I$  such that  $x < p$  and  $y > p$ . Now by Definition 8.2,  $[x, y] \subset I$ . But it follows by Definition 1.3 that every point in  $[x, y]$ , including  $p$ , is an element of  $I$ , a contradiction.

We now divide into three cases ( $I$  is exclusively bounded below,  $I$  is exclusively bounded above, and  $I$  is bounded both below and above).

First, suppose that  $I$  is only bounded below. Since  $I$  is a nonempty subset of  $\mathbb{R}$  that is bounded below, we have by Theorem 5.17 that  $\inf I$  exists. We divide into two cases again ( $\inf I \in I$  and  $\inf I \notin I$ ).

If  $\inf I \in I$ , then we can demonstrate that  $I = [\inf I, \infty)$ . To do this, Definition 1.2 tells us that it will suffice to verify that every  $p \in I$  is an element of  $[\inf I, \infty)$  and vice versa. Let  $p$  be an arbitrary element of  $I$ . Then  $p \in \mathbb{R}$  and by Definitions 5.7 and 5.6,  $\inf I \leq p$ . Therefore,  $p \in [\inf I, \infty)$ , as desired. Now let  $p$  be an arbitrary element of  $[\inf I, \infty)$ . Then  $\inf I \leq p$ . Additionally, since  $I$  is not bounded above, we have by Definition 5.6 that there exists  $y \in I$  such that  $y > p$ . Since  $\inf I \in I$ ,  $y \in I$ , and  $\inf I < y$  (by transitivity),  $[\inf I, y] \subset I$  by Definition 8.2. This combined with the fact that  $p \in [\inf I, y]$  (we know that  $\inf I \leq p < y$ , so  $\inf I \leq p \leq y$ ) implies that  $p \in I$ , as desired.

If  $\inf I \notin I$ , then we can demonstrate that  $I = (\inf I, \infty)$ . As before, to do so, it will suffice to verify that every  $p \in I$  is an element of  $(\inf I, \infty)$  and vice versa. Let  $p$  be an arbitrary element of  $I$ . Then  $p \in \mathbb{R}$  and by Definitions 5.7 and 5.6,  $\inf I \leq p$ . The additional constraint that  $\inf I \notin I$  implies that  $\inf I < p$ . Therefore,  $p \in (\inf I, \infty)$ , as desired. Now let  $p$  be an arbitrary element of  $(\inf I, \infty)$ . Then  $\inf I < p$ . It follows by Lemma 5.11 that there exists a  $z \in I$  such that  $\inf I \leq z < p$ . Additionally, since  $I$  is not bounded above, we have by Definition 5.6 that there exists  $y \in I$  such that  $y > p$ . Since  $z \in I$ ,  $y \in I$ , and  $z < y$  (by transitivity),  $[z, y] \subset I$  by Definition 8.2. This combined with the fact that  $p \in [z, y]$  (we know that  $z < p < y$ , so  $z \leq p \leq y$ ) implies that  $p \in I$ , as desired.

Second, suppose that  $I$  is only bounded above. The proof of this case is symmetric to that of the first.

Third, suppose that  $I$  is bounded below and above. Since  $I$  is a nonempty subset of  $\mathbb{R}$  that is bounded above and below, we have by consecutive applications of Theorem 5.17 that both  $\sup I$  and  $\inf I$  exist. We divide into four cases ( $\inf I \in I$  and  $\sup I \in I$ ,  $\inf I \in I$  and  $\sup I \notin I$ ,  $\inf I \notin I$  and  $\sup I \in I$ , and  $\inf I \notin I$  and  $\sup I \notin I$ ).

If  $\inf I \in I$  and  $\sup I \in I$ , then we can demonstrate that  $I = [\inf I, \sup I]$ . We divide into two cases again ( $\inf I = \sup I$  and  $\inf I \neq \sup I$ ). If  $\inf I = \sup I \in I$ , then  $I = \{\inf I\} = \{\sup I\} = [\inf I, \sup I]$ , as desired. On the other hand, if  $\inf I \neq \sup I$ , we continue. To demonstrate that  $I = [\inf I, \sup I]$ , Theorem 1.7 tells us that it will suffice to verify that  $I \subset [\inf I, \sup I]$  and  $[\inf I, \sup I] \subset I$ . To verify that the former claim, Definition 1.3 tells us that it will suffice to confirm that every  $p \in I$  is an element of  $[\inf I, \sup I]$ . Let  $p$  be an arbitrary element of  $I$ . Then  $p \in \mathbb{R}$  and by consecutive applications of Definitions 5.7 and 5.6,  $\inf I \leq p$  and  $p \leq \sup I$ . Therefore,  $p \in [\inf I, \sup I]$ , as desired. On the other hand, since  $\inf I \in I$ ,  $\sup I \in I$ , and  $\inf I < \sup I$  (as follows from Definition 5.7 and the fact that they are unequal),  $[\inf I, \sup I] \subset I$  by Definition 8.2, as desired.

If  $\inf I \in I$  and  $\sup I \notin I$ , then we can demonstrate that  $I = [\inf I, \sup I)$ . To do so, it will suffice to verify that every  $p \in I$  is an element of  $[\inf I, \sup I)$  and vice versa. Let  $p$  be an arbitrary element of  $I$ . Then  $p \in \mathbb{R}$  and by Definitions 5.7 and 5.6,  $\inf I \leq p$  and  $p \leq \sup I$ . The additional constraint that  $\sup I \notin I$  implies that  $p < \sup I$ . Therefore,  $p \in [\inf I, \sup I)$ , as desired. Now let  $p$  be an arbitrary element of  $[\inf I, \sup I)$ . Then  $\inf I \leq p < \sup I$ . It follows by Lemma 5.11 that there exists a  $y \in I$  such that  $p < y \leq \sup I$ . Since

$\inf I \in I$ ,  $y \in I$ , and  $\inf I < y$  (by transitivity),  $[\inf I, y] \subset I$  by Definition 8.2. This combined with the fact that  $p \in [\inf I, y]$  (we know that  $\inf I \leq p < y$ , so  $\inf I \leq p \leq y$ ) implies that  $p \in I$ , as desired.

If  $\inf I \notin I$  and  $\sup I \in I$ , the proof is symmetric to that of the previous case.

If  $\inf I \notin I$  and  $\sup I \notin I$ , then we can demonstrate that  $I = (\inf I, \sup I)$ . To do so, it will suffice to verify that every  $p \in I$  is an element of  $(\inf I, \sup I)$  and vice versa. Let  $p$  be an arbitrary element of  $I$ . Then  $p \in \mathbb{R}$  and by Definitions 5.7 and 5.6,  $\inf I \leq p$  and  $p \leq \sup I$ . The additional constraints that  $\inf I \notin I$  and  $\sup I \notin I$  imply that  $\inf I < p$  and  $p < \sup I$ , respectively. Therefore,  $p \in (\inf I, \sup I)$ , as desired. Now let  $p$  be an arbitrary element of  $(\inf I, \sup I)$ . Then  $\inf I < p < \sup I$ . It follows by consecutive applications of Lemma 5.11 that there exist  $x, y \in I$  such that  $\inf I \leq x < p$  and  $p < y \leq \sup I$ . Since  $x \in I$ ,  $y \in I$ , and  $x < y$  (by transitivity),  $[x, y] \subset I$  by Definition 8.2. This combined with the fact that  $p \in [x, y]$  (we know that  $x < p < y$ , so  $x \leq p \leq y$ ) implies that  $p \in I$ , as desired.

Now suppose that  $I \subsetneq \mathbb{R}$  takes one of the eight forms in Equations 8.1. To prove that  $I$  is an interval, Definition 8.2 tells us that it will suffice to show that for all  $x, y \in I$  with  $x < y$ ,  $[x, y] \subset I$ . Let  $x, y$  be arbitrary elements of  $I$  with  $x < y$ . We divide into eight cases (one for each equation in Equations 8.1).

First, suppose that  $I = (a, b)$ . To demonstrate that  $[x, y] \subset I$ , Definition 1.3 tells us that it will suffice to confirm that every  $z \in [x, y]$  is an element of  $I$ . Let  $z$  be an arbitrary element of  $[x, y]$ . Then by Corollary 5.15,  $x \leq z \leq y$ . But since  $a < x < y < b$  by Equations 8.1, the fact that  $a < x \leq z \leq y < b$  implies by Equations 8.1 that  $z \in (a, b)$ , as desired.

The proofs of the second, third, and fourth equations are symmetric to that of the first.

Fifth, suppose that  $I = (a, \infty)$ . To demonstrate that  $[x, y] \subset I$ , Definition 1.3 tells us that it will suffice to confirm that every  $z \in [x, y]$  is an element of  $I$ . Let  $z$  be an arbitrary element of  $[x, y]$ . Then by Corollary 5.15,  $x \leq z \leq y$ . But since  $a < x$  by Equations 8.1, the fact that  $a < x \leq z$  implies by Equations 8.1 that  $z \in (a, \infty)$ , as desired.

The proofs of the sixth, seventh, and eighth equations are symmetric to that of the first.  $\square$

**Definition 8.4.** The **absolute value** of a real number  $x$  is the non-negative number  $|x|$  defined by

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

**Exercise 8.5.** Show that  $|x| = |-x|$  for all  $x \in \mathbb{R}$ . (Note that this also means that  $|x - y| = |y - x|$  for any  $x, y \in \mathbb{R}$ .)

*Proof.* Let  $x$  be an arbitrary element of  $\mathbb{R}$ . We divide into three cases ( $x = 0$ ,  $x > 0$ , and  $x < 0$ ). First, suppose that  $x = 0$ . Then since  $0 = -0$ , clearly  $|0| = |-0|$ , as desired. Second, suppose that  $x > 0$ . Then by Lemma 7.23<sup>[1]</sup>  $-x < 0$ . Thus, by consecutive applications of Definition 8.4,  $|x| = x$  and  $|-x| = -(-x)$ . Therefore, since  $-(-x) = x$  by Corollary 7.11,  $|x| = x = |-x|$ , as desired. Third, suppose that  $x < 0$ . Then by Lemma 7.23,  $-x > 0$ . Thus, by consecutive applications of Definition 8.4,  $|x| = -x$  and  $|-x| = -x$ . Therefore,  $|x| = -x = |-x|$ , as desired.  $\square$

**Definition 8.6.** The **distance** between  $x \in \mathbb{R}$  and  $y \in \mathbb{R}$  is defined to be  $|x - y|$ .

**Remark 8.7.** It follows from Definition 8.6 that  $|x|$  is the distance between  $x$  and 0.

**Lemma 8.8.** For any real numbers  $x, y, z$ , we have

- (a)  $|x + y| \leq |x| + |y|$ .
- (b)  $|x - z| \leq |x - y| + |y - z|$ .
- (c)  $||x| - |y|| \leq |x - y|$ .

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<sup>1</sup>And, technically, Theorem 7.47.

*Proof of a.* We divide into four cases ( $x \geq 0$  and  $y \geq 0$ ,  $x \geq 0$  and  $y < 0$ ,  $x < 0$  and  $y \geq 0$ , and  $x < 0$  and  $y < 0$ ).

First, suppose that  $x \geq 0$  and  $y \geq 0$ . Then by Definition 7.21,  $x + y \geq 0$ . Thus, by consecutive applications of Definition 8.4,  $|x + y| = x + y$ ,  $|x| = x$ , and  $|y| = y$ . Therefore,  $|x + y| = x + y \leq x + y = |x| + |y|$ , as desired.

Second, suppose that  $x \geq 0$  and  $y < 0$ . By Definition 8.4,  $|x| = x$  and  $|y| = -y$ . We now divide into two cases ( $x + y \geq 0$  and  $x + y < 0$ ). If  $x + y \geq 0$ , then  $|x + y| = x + y$ . Additionally, since  $y < 0$ , Lemma 7.23 implies that  $0 < -y$ . Consequently, by transitivity,  $y < -y = |y|$ . It follows by Definition 7.21 that  $x + y < x + |y|$ . Therefore,  $|x + y| = x + y < x + |y| = |x| + |y|$ , so  $|x + y| \leq |x| + |y|$ , as desired. On the other hand, if  $x + y < 0$ , then  $|x + y| = -(x + y) = -x + (-y) = -x + |y|$ . Additionally, by Lemma 7.23,  $x \geq 0$  implies that  $-x \leq 0$ . It follows by Definition 7.21 since  $-x \leq x$  that  $-x + |y| \leq x + |y|$ . Therefore,  $|x + y| = -x + |y| \leq x + |y| = |x| + |y|$ , as desired.

The proof of the third case is symmetric to that of the second.

The proof of the fourth case is symmetric to that of the first. □

*Proof of b.* By part (a),  $|x - z| = |x - y + y - z| \leq |x - y| + |y - z|$ , as desired. □

*Proof of c.* To prove that  $||x| - |y|| \leq |x - y|$ , Definition 8.4 tells us that it will suffice to show that  $|x| - |y| \leq |x - y|$  and  $-(|x| - |y|) \leq |x - y|$ . By part (a),  $|x| = |x - y + y| \leq |x - y| + |y|$ , so  $|x| - |y| \leq |x - y|$ . Similarly,  $|y| - |x| \leq |x - y|$ , so  $-(|x| - |y|) \leq |x - y|$ , as desired. □

**Exercise 8.9.** Let  $a, \delta \in \mathbb{R}$  with  $\delta > 0$ . Prove that

$$(a - \delta, a + \delta) = \{x \in \mathbb{R} \mid |x - a| < \delta\}$$

**Lemma.** For any  $a, b \in \mathbb{R}$  such that  $0 < b$ ,  $|a| < b$  if and only if  $-b < a < b$ .

*Proof.* Suppose first that  $|a| < b$ . We divide into two cases ( $a \geq 0$  and  $a < 0$ ). If  $a \geq 0$ , then by Definition 8.4,  $0 \leq a = |a| < b$ . Additionally, by Lemma 7.23,  $-b < 0$ . Therefore,  $-b < 0 \leq a < b$ , as desired. If  $a < 0$ , then by Definition 8.4,  $-a = |a| < b$ . It follows by Definition 7.21 (by adding  $a - b$  to both sides) that  $-b < a$ . Additionally, by Lemma 7.23,  $a < 0$  implies  $0 < -a$ , so we know that  $a < -a$ . Therefore,  $-b < a < -a < b$ , as desired.

Now suppose that  $-b < a < b$ . We divide into two cases ( $a \geq 0$  and  $a < 0$ ). If  $a \geq 0$ , then by Definition 8.4,  $|a| = a < b$ , as desired. If  $a < 0$ , then by Definition 8.4,  $|a| = -a$ . Since  $-b < a$ , Definition 7.21 implies (by adding  $b - a$  to both sides) that  $-a < b$ . Therefore,  $|a| = -a < b$ , as desired. □

*Proof of Exercise 8.9.* To prove that  $(a - \delta, a + \delta) = \{x \in \mathbb{R} \mid |x - a| < \delta\}$ , Definition 1.2 tells us that it will suffice to show that every  $p \in (a - \delta, a + \delta)$  is an element of  $\{x \in \mathbb{R} \mid |x - a| < \delta\}$  and vice versa.

Suppose first that  $p \in (a - \delta, a + \delta)$ . Then by Equations 8.1,  $a - \delta < p$  and  $p < a + \delta$ . It follows by consecutive applications of Definition 7.21 from the former condition that  $-\delta < p - a$ , and from the latter condition that  $p - a < \delta$ . Since  $-\delta < p - a < \delta$ , the lemma asserts that  $|p - a| < \delta$ . Therefore,  $p \in \{x \in \mathbb{R} \mid |x - a| < \delta\}$ .

Now suppose that  $p \in \{x \in \mathbb{R} \mid |x - a| < \delta\}$ . Then  $|p - a| < \delta$ . Thus, by the lemma,  $-\delta < p - a$  and  $p - a < \delta$ . It follows by consecutive applications of Definition 7.21 from the former condition that  $a - \delta < p$ , and from the latter condition that  $p < a + \delta$ . Therefore, since  $a - \delta < p < a + \delta$ , we have that  $p \in (a - \delta, a + \delta)$ . □

2/11: **Lemma 8.10.** Let  $I$  be an open interval containing the point  $p \in \mathbb{R}$ . Then

a) There exists a number  $\delta > 0$  such that  $(p - \delta, p + \delta) \subset I$ .

b) There exists a natural number  $N$  such that for all natural numbers  $k \geq N$  we have  $(p - \frac{1}{k}, p + \frac{1}{k}) \subset I$ .

*Proof of a.* Since  $I$  is open, we have by Theorem 4.10 that there exists a region  $(a, b)$  such that  $p \in (a, b) \subset I$ . Let  $\delta = \min(p - a, b - p)$ . To show that  $(p - \delta, p + \delta) \subset I$ , we will demonstrate that  $(p - \delta, p + \delta) \subset (a, b) \subset I$ . To do this, Definition 1.3 tells us that it will suffice to verify that every element  $x \in (p - \delta, p + \delta)$  is an element of  $(a, b)$ . Let  $x$  be an arbitrary element of  $(p - \delta, p + \delta)$ . Then by Equations 8.1,  $p - \delta < x < p + \delta$ . We divide into two cases ( $\delta = p - a$  and  $\delta = b - p$ ). Suppose first that  $\delta = p - a$ . Then  $p - (p - a) < x < p + (p - a)$ ,

i.e.,  $a < x < p + (p - a)$ . Additionally, the fact that  $p - a = \min(p - a, b - p)$  implies that  $p - a \leq b - p$ . Combining these last two results gives us  $a < x < p + (p - a) \leq p + (b - p) = b$ . Since  $a < x < b$ , Equations 8.1 imply that  $x \in (a, b)$ , as desired. The proof is symmetric if  $\delta = b - p$ .  $\square$

*Proof of b.* By Lemma 8.10a, there exists a number  $\delta > 0$  such that  $(p - \delta, p + \delta) \subset I$ . Since  $\delta$  is a positive real number, Corollary 6.12 implies that there exists a nonzero natural number  $N$  such that  $\frac{1}{N} < \delta$ . To prove that for all numbers  $k \geq N$ , we have  $(p - \frac{1}{k}, p + \frac{1}{k}) \subset I$ , we will show that  $(p - \frac{1}{k}, p + \frac{1}{k}) \subset (p - \delta, p + \delta) \subset I$ . To do this, Definition 1.3 tells us that it will suffice to show that every  $x \in (p - \frac{1}{k}, p + \frac{1}{k})$  is an element of  $(p - \delta, p + \delta)$ . Let  $k$  be an arbitrary natural number such that  $k \geq N$ , and let  $x$  be an arbitrary element of  $(p - \frac{1}{k}, p + \frac{1}{k})$ . It follows from the latter condition by Equations 8.1 that  $p - \frac{1}{k} < x < p + \frac{1}{k}$ . Since  $\frac{1}{k} \leq \frac{1}{N}$  by Scripts 2 and 3, we have that  $p - \frac{1}{N} < x < p + \frac{1}{N}$ . Since  $\frac{1}{N} < \delta$  by definition,  $p - \delta < x < p + \delta$ . Therefore, by Equations 8.1,  $x \in (p - \delta, p + \delta)$ , as desired.  $\square$

**Definition 8.11.** Let  $A \subset X \subset \mathbb{R}$ . We say that  $A$  is **open** (in  $X$ ) if it is the intersection of  $X$  with an open set, and **closed** (in  $X$ ) if it is the intersection of  $X$  with a closed set. (This is called the subspace topology on  $X$ .)

**Remark 8.12.**  $A \subset \mathbb{R}$  open, as defined in Script 3, is equivalent to  $A$  open in  $\mathbb{R}$ .

**Exercise 8.13.** Let  $A \subset X \subset \mathbb{R}$ . Show that  $X \setminus A$  is closed in  $X$  if and only if  $A$  is open in  $X$ .

*Proof.* Suppose first that  $X \setminus A$  is closed in  $X$ . Then by Definition 8.11,  $X \setminus A = X \cap B$  where  $B$  is a closed set. It follows by Script 1 that

$$\begin{aligned} X \setminus A &= X \cap B \\ \mathbb{R} \setminus (X \setminus A) &= \mathbb{R} \setminus (X \cap B) \\ (\mathbb{R} \setminus X) \cup A &= (\mathbb{R} \setminus X) \cup (\mathbb{R} \setminus B) \\ X \cap ((\mathbb{R} \setminus X) \cup A) &= X \cap ((\mathbb{R} \setminus X) \cup (\mathbb{R} \setminus B)) \\ (X \cap (\mathbb{R} \setminus X)) \cup (X \cap A) &= (X \cap (\mathbb{R} \setminus X)) \cup (X \cap (\mathbb{R} \setminus B)) \\ \emptyset \cup (X \cap A) &= \emptyset \cup (X \cap (\mathbb{R} \setminus B)) \\ A &= X \cap (\mathbb{R} \setminus B) \end{aligned}$$

Since  $\mathbb{R} \setminus B$  is open by Definition 4.4, we have by Definition 8.11 that  $A$  is open in  $X$ .

Now suppose that  $A$  is open in  $X$ . Then by Definition 8.11,  $A = X \cap B$  where  $B$  is an open set. It follows by Script 1 that

$$\begin{aligned} A &= X \cap B \\ \mathbb{R} \setminus A &= \mathbb{R} \setminus (X \cap B) \\ &= (\mathbb{R} \setminus X) \cup (\mathbb{R} \setminus B) \\ X \cap (\mathbb{R} \setminus A) &= X \cap ((\mathbb{R} \setminus X) \cup (\mathbb{R} \setminus B)) \\ X \setminus A &= (X \cap (\mathbb{R} \setminus X)) \cup (X \cap (\mathbb{R} \setminus B)) \\ &= X \cap (\mathbb{R} \setminus B) \end{aligned}$$

Since  $\mathbb{R} \setminus B$  is closed by Definition 4.4, we have by Definition 8.11 that  $X \setminus A$  is closed in  $X$ .  $\square$

**Exercise 8.14.**

- Let  $[a, b] \subset \mathbb{R}$ . Give an example of a set  $A \subset [a, b]$  such that  $A$  is open in  $[a, b]$  but not in  $\mathbb{R}$ .
- Give an example of sets  $A \subset X \subset \mathbb{R}$  such that  $A$  is closed in  $X$  but not in  $\mathbb{R}$ .

*Proof of a.* We first briefly consider the case where  $a = b$ . In this case, let  $c < a < d$ ; then  $\{a\} = [a, a] \cap (c, d)$  is a subset of  $[a, b]$  that is open in  $[a, b]$  (by Definition 8.11;  $(c, d)$  is open by Exercise 8.1) but closed in  $\mathbb{R}$  (by Corollary 3.23, Definition 4.1, and Theorem 5.1).

We now direct our attention to the case where  $a \neq b$ . Let  $c \in [a, b]$  be a point such that  $a < c < b$  (we know at least one such point exists by Theorem 5.2). If we define the set  $(c, b] = [a, b] \cap (c, \infty)$ , we have by

Definition 8.11 that  $(c, b]$  is open in  $[a, b]$  (since  $(c, \infty)$  is open as per Exercise 8.1). However, we know that  $(c, b]$  is not open in  $\mathbb{R}$  by Theorem 4.10 ( $b$  is an element of  $(c, b]$  such that any region containing  $b$  necessarily contains an element that is not in  $(c, b]$ ; this element will be greater than  $b$  but less than the right bound of the region, and its existence is guaranteed by Theorem 5.2).  $\square$

*Proof of b.* Let  $X = (a, b) \subset \mathbb{R}$ . Then  $(a, b) = X \cap [a, b]$ , so  $(a, b) = X \cap [a, b]$  is closed in  $(a, b)$  by Definition 8.11. However, by Corollary 5.14,  $a, b$  are limit points of  $(a, b)$  that are not contained within  $(a, b)$ . It follows by Definition 4.1 that  $(a, b)$  is not closed in  $\mathbb{R}$ .  $\square$

**Theorem 8.15.** *Let  $X \subset \mathbb{R}$ . Then  $X$  is connected if and only if  $X$  is an interval.*

*Proof.* Suppose first that  $X$  is connected. To prove that  $X$  is an interval, Definition 8.2 tells us that it will suffice to show that for all  $x, y \in X$  with  $x < y$ ,  $[x, y] \subset X$ . Let  $x, y$  be arbitrary elements of  $X$  satisfying  $x < y$ , and suppose for the sake of contradiction that  $[x, y] \not\subset X$ . Then there exists  $z \in [x, y]$  such that  $z \notin X$ . Let  $A = \{a \in X \mid a < z\}$  and  $B = \{b \in X \mid z < b\}$ . It follows from Script 1 that  $X = A \cup B$  and  $A \cap B = \emptyset$ . To verify that  $A$  is nonempty, Definition 1.8 tells us that it will suffice to find an element in it. Since  $z \notin X$  but  $x \in X$ , we know that  $z \neq x$ . This combined with the fact that  $x \leq z$  by Equations 8.1 implies that  $x < z$ . Thus, since  $x \in X$  and  $x < z$ ,  $x \in A$ . Similarly,  $y \in B$ . To verify that  $A$  is open in  $X$ , Definition 8.11 tells us that it will suffice to demonstrate that  $A$  is the intersection of  $X$  with an open set. Since we clearly have  $A = X \cap (-\infty, z)$  where  $(-\infty, z)$  is open by Exercise 8.1, we are done. We can do something similar for  $B$ . But the existence of two disjoint, nonempty, open (in  $X$ ) sets  $A, B$  whose union equals  $X$  demonstrates by Definition 4.22 that  $X$  is disconnected, a contradiction.

Now suppose that  $X$  is an interval, and suppose for the sake of contradiction that  $X$  is disconnected. Then by Definition 4.22,  $X = A \cup B$  where  $A, B$  are disjoint, nonempty sets that are open in  $X$ . Since  $A, B$  are disjoint and nonempty, we know that there exist distinct objects  $a \in A$  and  $b \in B$ . WLOG, let  $a < b$ .

To prove that  $\sup(A \cap [a, b])$  exists, Theorem 5.17 tells us that it will suffice to show that  $A \cap [a, b]$  is nonempty and bounded above. To show that  $A \cap [a, b]$  is nonempty, Definition 1.8 tells us that it will suffice to find an element of  $A \cap [a, b]$ . By Equations 8.1,  $a \in [a, b]$ . By Definition,  $a \in A$ . Thus, by Definition 1.6,  $a \in A \cap [a, b]$ , as desired. To show that  $A \cap [a, b]$  is bounded above, consecutive applications of Definition 5.6 tell us that it will suffice to verify that  $x \leq b$  for all  $x \in A \cap [a, b]$ . Let  $x$  be an arbitrary element of  $A \cap [a, b]$ . It follows by Definition 1.6 that  $x \in [a, b]$ . Thus, by Equations 8.1,  $x \leq b$ , as desired.

Let  $s = \sup(A \cap [a, b])$ . To prove that  $\inf(B \cap [s, b])$  exists, it will suffice to utilize a symmetric argument to the above.

Let  $i = \inf(B \cap [s, b])$ . We divide into three cases ( $s > i$ ,  $s = i$ , and  $s < i$ ).

First, suppose that  $s > i$ . To show that  $s$  is a lower bound of  $B \cap [s, b]$ , Definition 5.6 tells us that it will suffice to verify that  $s \leq x$  for all  $x \in B \cap [s, b]$ . Let  $x$  be an arbitrary element of  $B \cap [s, b]$ . By Definition 1.6,  $x \in [s, b]$ . Thus, by Equations 8.1,  $s \leq x$ , as desired. Since  $s$  is a lower bound of  $B \cap [s, b]$ , Definition 5.7 asserts that  $i \geq s$ , contradicting the hypothesis that  $s > i$ .

Second, suppose that  $s = i$ . We divide into three cases ( $s \in A$ ,  $s \in B$ , and  $s \notin A$  and  $s \notin B$ ).

If  $s \in A$ , then since  $A$  is open in  $X$ , Definition 8.11 implies that  $A = X \cap G$  where  $G$  is open. It follows by the hypothesis that  $s \in A$  along with Definitions 1.2 and 1.6 that  $s \in G$ . Consequently, by Theorem 4.10, there exists a region  $(c, d)$  such that  $s \in (c, d)$  and  $(c, d) \subset G$ . From the former condition, we have by Equations 8.1 that  $c < s < d$ . Thus, by Lemma 5.11, there exists a point  $x \in B \cap [s, b]$  such that  $s = i \leq x < d$ . Since  $c < s \leq x < d$ , Equations 8.1 imply that  $x \in (c, d)$ . This combined with the fact that  $(c, d) \subset G$  implies by Definition 1.3 that  $x \in G$ . Additionally, we know that  $x \in B$  (since  $x \in B \cap [s, b]$  by Definition 1.6). It follows from this and the fact that  $X = A \cup B$  by Definitions 1.5 and 1.2 that  $x \in X$ . Thus, since  $x \in X$  and  $x \in G$ , Definition 1.6 asserts that  $x \in X \cap G$ , meaning that  $x \in A$ . But if  $x \in A$  and  $x \in B$ , then Definition 1.6 implies that  $x \in A \cap B$ , contradicting the supposition that  $A$  and  $B$  are disjoint.

If  $s \in B$ , then the proof is symmetric to the previous case.

If  $s \notin A$  and  $s \notin B$ , then by Definition 1.5,  $s \notin A \cup B$ , implying that  $s \notin X$ . Additionally, the facts that  $a \in A$ ,  $b \in B$ , and  $X = A \cup B$  imply that  $a, b \in X$ . It follows since  $a < b$  by Definition 8.2 that  $[a, b] \subset X$ . We now show that  $s \in [a, b]$  via Equations 8.1, which tell us that it will suffice to verify that  $a \leq s \leq b$ . As previously shown,  $b$  is an upper bound of  $A \cap [a, b]$ . Thus, by Definition 5.7, we have that  $s \leq b$ , and

we are half done. As to the other half, we have also previously shown that  $a \in A \cap [a, b]$ . Additionally, by Definitions 5.7 and 5.6,  $s \geq x$  for all  $x \in A \cap [a, b]$ , including  $a$ . Thus,  $s \geq a$ . Having shown that  $s \in [a, b]$  and  $[a, b] \subset X$ , we may invoke Definition 1.3 to learn that  $s \in X$ , contradicting the previously proven statement that  $s \notin X$ .

Third, suppose that  $s < i$ . Then by Theorem 5.2 and Definition 3.6, there exists a  $z \in \mathbb{R}$  such that  $s < z < i$ . We now show that  $i \in [a, b]$  via Equations 8.1, which tell us that it will suffice to verify that  $a \leq i \leq b$ . As previously shown,  $s$  is a lower bound of  $B \cap [s, b]$ . Thus, by Definition 5.7, we have that  $i \geq s$ . We have also previously shown that  $s \geq a$ , so by transitivity,  $i \geq a$ , and we are half done. As to the other half, we now confirm that  $b \in B \cap [s, b]$ . By Equations 8.1,  $b \in [s, b]$ . By definition,  $b \in B$ . Thus, by Definition 1.6,  $b \in B \cap [s, b]$ , as desired. Additionally, by Definitions 5.7 and 5.6,  $i \leq x$  for all  $x \in B \cap [s, b]$ , including  $b$ . Thus,  $i \leq b$ , concluding our argument that  $i \in [a, b]$ . Moving on, the fact that  $s < z$  implies by Definition 5.6 that  $z \notin A \cap [a, b]$ . Additionally, we know from the facts that  $s, i \in [a, b]$  that  $a \leq s < z < i \leq b$ , meaning that  $z \in [a, b]$ . Combining the previous two results with Definition 1.6, we have that  $z \notin A$ . By a symmetric argument, we can show that  $z \notin B$ . Since  $z \notin A$  and  $z \notin B$ , Definition 1.5 asserts that  $z \notin A \cup B$ , i.e.,  $z \notin X$ . But as before,  $[a, b] \subset X$ , so the fact that  $z \in [a, b]$  combined with Definition 1.3 implies that  $z \in X$ , a contradiction.  $\square$

2/16: **Definition 8.16.** Let  $I$  be an interval and let  $f : I \rightarrow \mathbb{R}$ .

- a) We say that  $f$  is **increasing** on  $I$  if, whenever  $x, y \in I$  with  $x < y$ ,  $f(x) \leq f(y)$ .
- b) We say that  $f$  is **decreasing** on  $I$  if, whenever  $x, y \in I$  with  $x < y$ ,  $f(x) \geq f(y)$ .
- c) We say that  $f$  is **strictly increasing** on  $I$  if, whenever  $x, y \in I$  with  $x < y$ ,  $f(x) < f(y)$ .
- d) We say that  $f$  is **strictly decreasing** on  $I$  if, whenever  $x, y \in I$  with  $x < y$ ,  $f(x) > f(y)$ .

**Lemma 8.17.** *If  $f$  is strictly increasing or strictly decreasing on an interval  $I$  then  $f$  is injective on  $I$ .*

*Proof.* We divide into two cases ( $f$  is strictly increasing, and  $f$  is strictly decreasing). Suppose first that  $f$  is strictly increasing. To prove that  $f$  is injective on  $I$ , Definition 1.20 tells us that it will suffice to show that for all  $a, b \in I$ ,  $a \neq b$  implies that  $f(a) \neq f(b)$ . Let  $a, b$  be arbitrary elements of  $I$  such that  $a \neq b$ . WLOG, let  $a < b$ . Then by Definition 8.16,  $f(a) < f(b)$ . Therefore,  $f(a) \neq f(b)$ , as desired. The proof is symmetric for the other case.  $\square$