Script 10

Compactness

10.1 Journal

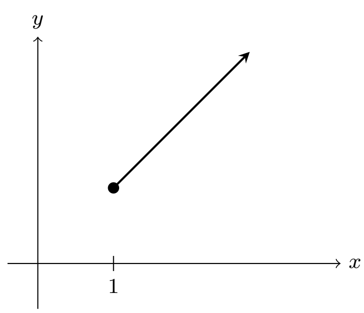
2/23: **Definition 10.1.** We say that a function $f : A \to \mathbb{R}$ is **bounded** if $f(A)$ is a bounded subset of $\mathbb{R}$. We say that f is **bounded above** if $f(A)$ is bounded above and that f is **bounded below** if $f(A)$ is bounded below.

If $f : A \to \mathbb{R}$ is bounded above, we say that f **attains** (its least upper bound) if there is some $a \in A$ such that $f(a) = \sup f(A)$. Similarly, if $f : A \to \mathbb{R}$ is bounded below, we say that f **attains** (its greatest lower bound) if there is some $a \in A$ such that $f(a) = \inf f(A)$.

Exercise 10.2. If possible, find examples of each of the following: a picture suffices.

a) A continuous function on $[1, \infty)$ that is not bounded above.

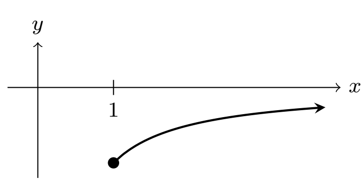
Example. Let $f : [1, \infty) \to \mathbb{R}$ be defined by $f(x) = x$.



□

b) A continuous function on $[1, \infty)$ that is bounded above but does not attain its least upper bound.

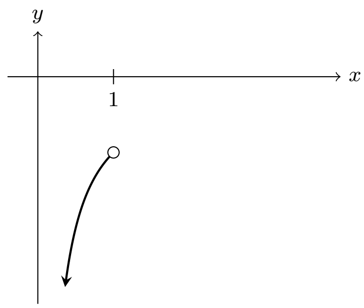
Example. Let $f : [1, \infty) \to \mathbb{R}$ be defined by $f(x) = -\frac{1}{x}$.



□

c) A continuous function on $(0, 1)$ that is not bounded below.

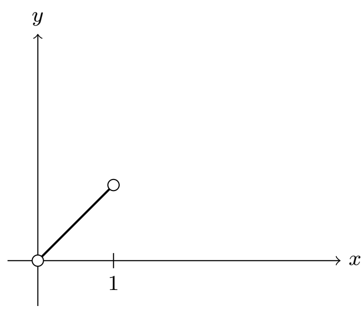
Example. Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) = -\frac{1}{x}$.



$\square$

d) A continuous function on $(0, 1)$ that is bounded below but does not attain its greatest lower bound.

Example. Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) = x$.



$\square$

Definition 10.3. Let X be a subset of $\mathbb{R}$ and let $\mathcal{G} = \{G_\lambda\}_{\lambda \in \Lambda}$ be a collection of subsets of $\mathbb{R}$. We say that $\mathcal{G}$ is a **cover** of X if every point of X is in some G_λ, or in other words:

$$X \subset \bigcup_{\lambda \in \Lambda} G_\lambda$$

We say that the collection $\mathcal{G}$ is an **open cover** if each G_λ is open.

Definition 10.4. Let X be a subset of $\mathbb{R}$. X is **compact** if for every open cover $\mathcal{G}$ of X, there exists a finite subset $\mathcal{G}' \subset \mathcal{G}$ that is also an open cover.

A good summary of the definition of compactness is "every open cover contains a finite subcover."

Exercise 10.5. Show that all finite subsets of $\mathbb{R}$ are compact.

Proof. Let X be an arbitrary finite subset of $\mathbb{R}$. To prove that X is compact, Definition 10.4 tells us that it will suffice to show that for every open cover $\mathcal{G}$ of X, there exists a finite subset $\mathcal{G}' \subset \mathcal{G}$ that is also an open cover. Let $\mathcal{G}$ be an arbitrary open cover of X. By Definition 10.3, every point $x \in X$ is an element of G_λ for some $G_\lambda \in \mathcal{G}$. Thus, for each $x \in X$, let $G_x \in \mathcal{G}$ be a set that contains x. Since X is finite, we do not need the axiom of choice to make these selections. Additionally, since there are finitely many X, we know that there are finitely many distinct G_x[1]. Thus, $\mathcal{G}' = \{G_x\}_{x \in X}$ is finite. Additionally, it is a subset of $\mathcal{G}$ by definition (each G_x is defined to be an element of $\mathcal{G}$). Furthermore, each G_x is open (again, each G_x is an element of $\mathcal{G}$, which is a collection of open sets by definition). Lastly, every point $x \in X$ is an element of $G_x \in \mathcal{G}'$, so $\mathcal{G}'$ is a cover. Therefore, by Definition 10.3, $\mathcal{G}' \subset \mathcal{G}$ is a finite open cover of X. $\square$

[1]In fact, the number of G_x is less than or equal to the cardinality of X since we may choose the same G_x for multiple x but may not choose multiple G_x for the same x.