

# Script 8

## Intervals

### 8.1 Journal

2/9: Now that we have constructed  $\mathbb{R}$  and proved the fundamental facts about it, we will work with the real numbers  $\mathbb{R}$  instead of an arbitrary continuum  $C$ . We will leave behind Dedekind cuts and think of elements of  $\mathbb{R}$  as numbers. Accordingly, from now on, we will use lower-case letters like  $x$  for real numbers and will write  $+$  and  $\cdot$  for  $\oplus$  and  $\otimes$ . We will also now use the standard notation  $(a, b)$  for the region  $ab = \{x \in \mathbb{R} \mid a < x < b\}$ . Even though the notation is the same, this is *not* the same object as the ordered pair  $(a, b)$ .

More generally, we adopt the following standard notation:

$$\begin{aligned}(a, b) &= \{x \in \mathbb{R} \mid a < x < b\} \\ [a, b) &= \{x \in \mathbb{R} \mid a \leq x < b\} \\ (a, b] &= \{x \in \mathbb{R} \mid a < x \leq b\} \\ [a, b] &= \{x \in \mathbb{R} \mid a \leq x \leq b\} \\ (a, \infty) &= \{x \in \mathbb{R} \mid a < x\} \\ [a, \infty) &= \{x \in \mathbb{R} \mid a \leq x\} \\ (-\infty, b) &= \{x \in \mathbb{R} \mid x < b\} \\ (-\infty, b] &= \{x \in \mathbb{R} \mid x \leq b\}\end{aligned}\tag{8.1}$$

**Exercise 8.1.** Identify the sets in Equations 8.1 that are open/closed/neither.

*Proof.* Note that by Theorem 5.1, any of these sets proven to be just one of open or closed will not be the other, i.e., a set proven to be open will not be closed and vice versa.

By Corollary 4.11,  $(a, b)$  is open.

By an adaptation of Corollary 5.14,  $b \in LP([a, b))$  but  $b \notin [a, b)$ . Since  $[a, b)$  doesn't contain all of its limit points, Definition 4.1 tells us that it is not closed. Additionally, since  $a \in LP(C \setminus [a, b))$  but  $a \notin C \setminus [a, b)$ , Definition 4.8 tells us that it is not open. Therefore, it is neither.

The proof that  $(a, b]$  is neither is symmetric to the previous case.

By Corollaries 5.15 and 4.7,  $[a, b]$  is closed.

By Corollary 4.13,  $(a, \infty)$  is open.

By Corollary 4.13 and Definition 4.8,  $[a, \infty) = C \setminus (-\infty, a)$  is closed.

The proofs that  $(-\infty, b)$  and  $(-\infty, b]$  are open and closed, respectively, are symmetric to the previous two cases, respectively.  $\square$

**Definition 8.2.** A set  $I \subset \mathbb{R}$  is an **interval** if for all  $x, y \in I$  with  $x < y$ ,  $[x, y] \subset I$ .

**Lemma 8.3.** *A proper subset  $I \subsetneq \mathbb{R}$  is an interval if and only if it takes one of the eight forms in Equations 8.1.*

*Proof.* Suppose first that  $I \subsetneq \mathbb{R}$  is an interval. If  $I = \emptyset$ , then  $I = (a, a)$  for any  $a \in \mathbb{R}$ , and we are done. Thus, we will assume for the remainder of the proof of the forward direction that  $I$  is nonempty. To address this case, we will first prove that the facts that  $I \subsetneq \mathbb{R}$ ,  $I \neq \emptyset$ , and  $I$  is an interval imply that  $I$  is bounded above, bounded below, or both. Then in each of these three cases, we will look at whether  $\sup I$  and  $\inf I$  (if they exist) are elements of the set or not to differentiate between the eight forms in Equations 8.1. Let's begin.

Suppose for the sake of contradiction that there exists a nonempty interval  $I \subsetneq \mathbb{R}$  that is neither bounded above nor bounded below. Since  $I \subsetneq \mathbb{R}$ , we have by Definition 1.3 that there exists a point  $p \in \mathbb{R}$  such that  $p \notin I$ . Additionally, since  $I$  is neither bounded above nor below, Definition 5.6 implies that  $p$  is neither an upper nor a lower bound on  $I$ . Thus, there exist  $x, y \in I$  such that  $x < p$  and  $y > p$ . Now by Definition 8.2,  $[x, y] \subset I$ . But it follows by Definition 1.3 that every point in  $[x, y]$ , including  $p$ , is an element of  $I$ , a contradiction.

We now divide into three cases ( $I$  is exclusively bounded below,  $I$  is exclusively bounded above, and  $I$  is bounded both below and above).

First, suppose that  $I$  is only bounded below. Since  $I$  is a nonempty subset of  $\mathbb{R}$  that is bounded below, we have by Theorem 5.17 that  $\inf I$  exists. We divide into two cases again ( $\inf I \in I$  and  $\inf I \notin I$ ).

If  $\inf I \in I$ , then we can demonstrate that  $I = [\inf I, \infty)$ . To do this, Definition 1.2 tells us that it will suffice to verify that every  $p \in I$  is an element of  $[\inf I, \infty)$  and vice versa. Let  $p$  be an arbitrary element of  $I$ . Then  $p \in \mathbb{R}$  and by Definitions 5.7 and 5.6,  $\inf I \leq p$ . Therefore,  $p \in [\inf I, \infty)$ , as desired. Now let  $p$  be an arbitrary element of  $[\inf I, \infty)$ . Then  $\inf I \leq p$ . Additionally, since  $I$  is not bounded above, we have by Definition 5.6 that there exists  $y \in I$  such that  $y > p$ . Since  $\inf I \in I$ ,  $y \in I$ , and  $\inf I < y$  (by transitivity),  $[\inf I, y] \subset I$  by Definition 8.2. This combined with the fact that  $p \in [\inf I, y]$  (we know that  $\inf I \leq p < y$ , so  $\inf I \leq p \leq y$ ) implies that  $p \in I$ , as desired.

If  $\inf I \notin I$ , then we can demonstrate that  $I = (\inf I, \infty)$ . As before, to do so, it will suffice to verify that every  $p \in I$  is an element of  $(\inf I, \infty)$  and vice versa. Let  $p$  be an arbitrary element of  $I$ . Then  $p \in \mathbb{R}$  and by Definitions 5.7 and 5.6,  $\inf I \leq p$ . The additional constraint that  $\inf I \notin I$  implies that  $\inf I < p$ . Therefore,  $p \in (\inf I, \infty)$ , as desired. Now let  $p$  be an arbitrary element of  $(\inf I, \infty)$ . Then  $\inf I < p$ . It follows by Lemma 5.11 that there exists a  $z \in I$  such that  $\inf I \leq z < p$ . Additionally, since  $I$  is not bounded above, we have by Definition 5.6 that there exists  $y \in I$  such that  $y > p$ . Since  $z \in I$ ,  $y \in I$ , and  $z < y$  (by transitivity),  $[z, y] \subset I$  by Definition 8.2. This combined with the fact that  $p \in [z, y]$  (we know that  $z < p < y$ , so  $z \leq p \leq y$ ) implies that  $p \in I$ , as desired.

Second, suppose that  $I$  is only bounded above. The proof of this case is symmetric to that of the first.

Third, suppose that  $I$  is bounded below and above. Since  $I$  is a nonempty subset of  $\mathbb{R}$  that is bounded above and below, we have by consecutive applications of Theorem 5.17 that both  $\sup I$  and  $\inf I$  exist. We divide into four cases ( $\inf I \in I$  and  $\sup I \in I$ ,  $\inf I \in I$  and  $\sup I \notin I$ ,  $\inf I \notin I$  and  $\sup I \in I$ , and  $\inf I \notin I$  and  $\sup I \notin I$ ).

If  $\inf I \in I$  and  $\sup I \in I$ , then we can demonstrate that  $I = [\inf I, \sup I]$ . We divide into two cases again ( $\inf I = \sup I$  and  $\inf I \neq \sup I$ ). If  $\inf I = \sup I \in I$ , then  $I = \{\inf I\} = \{\sup I\} = [\inf I, \sup I]$ , as desired. On the other hand, if  $\inf I \neq \sup I$ , we continue. To demonstrate that  $I = [\inf I, \sup I]$ , Theorem 1.7 tells us that it will suffice to verify that  $I \subset [\inf I, \sup I]$  and  $[\inf I, \sup I] \subset I$ . To verify that the former claim, Definition 1.3 tells us that it will suffice to confirm that every  $p \in I$  is an element of  $[\inf I, \sup I]$ . Let  $p$  be an arbitrary element of  $I$ . Then  $p \in \mathbb{R}$  and by consecutive applications of Definitions 5.7 and 5.6,  $\inf I \leq p$  and  $p \leq \sup I$ . Therefore,  $p \in [\inf I, \sup I]$ , as desired. On the other hand, since  $\inf I \in I$ ,  $\sup I \in I$ , and  $\inf I < \sup I$  (as follows from Definition 5.7 and the fact that they are unequal),  $[\inf I, \sup I] \subset I$  by Definition 8.2, as desired.

If  $\inf I \in I$  and  $\sup I \notin I$ , then we can demonstrate that  $I = [\inf I, \sup I)$ . To do so, it will suffice to verify that every  $p \in I$  is an element of  $[\inf I, \sup I)$  and vice versa. Let  $p$  be an arbitrary element of  $I$ . Then  $p \in \mathbb{R}$  and by Definitions 5.7 and 5.6,  $\inf I \leq p$  and  $p \leq \sup I$ . The additional constraint that  $\sup I \notin I$  implies that  $p < \sup I$ . Therefore,  $p \in [\inf I, \sup I)$ , as desired. Now let  $p$  be an arbitrary element of  $[\inf I, \sup I)$ . Then  $\inf I \leq p < \sup I$ . It follows by Lemma 5.11 that there exists a  $y \in I$  such that  $p < y \leq \sup I$ . Since

$\inf I \in I$ ,  $y \in I$ , and  $\inf I < y$  (by transitivity),  $[\inf I, y] \subset I$  by Definition 8.2. This combined with the fact that  $p \in [\inf I, y]$  (we know that  $\inf I \leq p < y$ , so  $\inf I \leq p \leq y$ ) implies that  $p \in I$ , as desired.

If  $\inf I \notin I$  and  $\sup I \in I$ , the proof is symmetric to that of the previous case.

If  $\inf I \notin I$  and  $\sup I \notin I$ , then we can demonstrate that  $I = (\inf I, \sup I)$ . To do so, it will suffice to verify that every  $p \in I$  is an element of  $(\inf I, \sup I)$  and vice versa. Let  $p$  be an arbitrary element of  $I$ . Then  $p \in \mathbb{R}$  and by Definitions 5.7 and 5.6,  $\inf I \leq p$  and  $p \leq \sup I$ . The additional constraints that  $\inf I \notin I$  and  $\sup I \notin I$  imply that  $\inf I < p$  and  $p < \sup I$ , respectively. Therefore,  $p \in (\inf I, \sup I)$ , as desired. Now let  $p$  be an arbitrary element of  $(\inf I, \sup I)$ . Then  $\inf I < p < \sup I$ . It follows by consecutive applications of Lemma 5.11 that there exist  $x, y \in I$  such that  $\inf I \leq x < p$  and  $p < y \leq \sup I$ . Since  $x \in I$ ,  $y \in I$ , and  $x < y$  (by transitivity),  $[x, y] \subset I$  by Definition 8.2. This combined with the fact that  $p \in [x, y]$  (we know that  $x < p < y$ , so  $x \leq p \leq y$ ) implies that  $p \in I$ , as desired.

Now suppose that  $I \subsetneq \mathbb{R}$  takes one of the eight forms in Equations 8.1. To prove that  $I$  is an interval, Definition 8.2 tells us that it will suffice to show that for all  $x, y \in I$  with  $x < y$ ,  $[x, y] \subset I$ . Let  $x, y$  be arbitrary elements of  $I$  with  $x < y$ . We divide into eight cases (one for each equation in Equations 8.1).

First, suppose that  $I = (a, b)$ . To demonstrate that  $[x, y] \subset I$ , Definition 1.3 tells us that it will suffice to confirm that every  $z \in [x, y]$  is an element of  $I$ . Let  $z$  be an arbitrary element of  $[x, y]$ . Then by Corollary 5.15,  $x \leq z \leq y$ . But since  $a < x < y < b$  by Equations 8.1, the fact that  $a < x \leq z \leq y < b$  implies by Equations 8.1 that  $z \in (a, b)$ , as desired.

The proofs of the second, third, and fourth equations are symmetric to that of the first.

Fifth, suppose that  $I = (a, \infty)$ . To demonstrate that  $[x, y] \subset I$ , Definition 1.3 tells us that it will suffice to confirm that every  $z \in [x, y]$  is an element of  $I$ . Let  $z$  be an arbitrary element of  $[x, y]$ . Then by Corollary 5.15,  $x \leq z \leq y$ . But since  $a < x$  by Equations 8.1, the fact that  $a < x \leq z$  implies by Equations 8.1 that  $z \in (a, \infty)$ , as desired.

The proofs of the sixth, seventh, and eighth equations are symmetric to that of the first.  $\square$

**Definition 8.4.** The **absolute value** of a real number  $x$  is the non-negative number  $|x|$  defined by

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

**Exercise 8.5.** Show that  $|x| = |-x|$  for all  $x \in \mathbb{R}$ . (Note that this also means that  $|x - y| = |y - x|$  for any  $x, y \in \mathbb{R}$ .)

*Proof.* Let  $x$  be an arbitrary element of  $\mathbb{R}$ . We divide into three cases ( $x = 0$ ,  $x > 0$ , and  $x < 0$ ). First, suppose that  $x = 0$ . Then since  $0 = -0$ , clearly  $|0| = |-0|$ , as desired. Second, suppose that  $x > 0$ . Then by Lemma 7.23<sup>[1]</sup>  $-x < 0$ . Thus, by consecutive applications of Definition 8.4,  $|x| = x$  and  $|-x| = -(-x)$ . Therefore, since  $-(-x) = x$  by Corollary 7.11,  $|x| = x = |-x|$ , as desired. Third, suppose that  $x < 0$ . Then by Lemma 7.23,  $-x > 0$ . Thus, by consecutive applications of Definition 8.4,  $|x| = -x$  and  $|-x| = -x$ . Therefore,  $|x| = -x = |-x|$ , as desired.  $\square$

**Definition 8.6.** The **distance** between  $x \in \mathbb{R}$  and  $y \in \mathbb{R}$  is defined to be  $|x - y|$ .

**Remark 8.7.** It follows from Definition 8.6 that  $|x|$  is the distance between  $x$  and 0.

**Lemma 8.8.** For any real numbers  $x, y, z$ , we have

- (a)  $|x + y| \leq |x| + |y|$ .
- (b)  $|x - z| \leq |x - y| + |y - z|$ .
- (c)  $||x| - |y|| \leq |x - y|$ .

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<sup>1</sup>And, technically, Theorem 7.47.

*Proof of a.* We divide into four cases ( $x \geq 0$  and  $y \geq 0$ ,  $x \geq 0$  and  $y < 0$ ,  $x < 0$  and  $y \geq 0$ , and  $x < 0$  and  $y < 0$ ).

First, suppose that  $x \geq 0$  and  $y \geq 0$ . Then by Definition 7.21,  $x+y \geq 0$ . Thus, by consecutive applications of Definition 8.4,  $|x+y| = x+y$ ,  $|x| = x$ , and  $|y| = y$ . Therefore,  $|x+y| = x+y \leq x+y = |x|+|y|$ , as desired.

Second, suppose that  $x \geq 0$  and  $y < 0$ . By Definition 8.4,  $|x| = x$  and  $|y| = -y$ . We now divide into two cases ( $x+y \geq 0$  and  $x+y < 0$ ). If  $x+y \geq 0$ , then  $|x+y| = x+y$ . Additionally, since  $y < 0$ , Lemma 7.23 implies that  $0 < -y$ . Consequently, by transitivity,  $y < -y = |y|$ . It follows by Definition 7.21 that  $x+y < x+|y|$ . Therefore,  $|x+y| = x+y < x+|y| = |x|+|y|$ , so  $|x+y| \leq |x|+|y|$ , as desired. On the other hand, if  $x+y < 0$ , then  $|x+y| = -(x+y) = -x+(-y) = -x+|y|$ . Additionally, by Lemma 7.23,  $x \geq 0$  implies that  $-x \leq 0$ . It follows by Definition 7.21 since  $-x \leq x$  that  $-x+|y| \leq x+|y|$ . Therefore,  $|x+y| = -x+|y| \leq x+|y| = |x|+|y|$ , as desired.

The proof of the third case is symmetric to that of the second.

The proof of the fourth case is symmetric to that of the first. □

*Proof of b.* By part (a),  $|x-z| = |x-y+y-z| \leq |x-y|+|y-z|$ , as desired. □

*Proof of c.* To prove that  $||x|-|y|| \leq |x-y|$ , Definition 8.4 tells us that it will suffice to show that  $|x|-|y| \leq |x-y|$  and  $-(|x|-|y|) \leq |x-y|$ . By part (a),  $|x| = |x-y+y| \leq |x-y|+|y|$ , so  $|x|-|y| \leq |x-y|$ . Similarly,  $|y|-|x| \leq |x-y|$ , so  $-(|x|-|y|) \leq |x-y|$ , as desired. □

**Exercise 8.9.** Let  $a, \delta \in \mathbb{R}$  with  $\delta > 0$ . Prove that

$$(a - \delta, a + \delta) = \{x \in \mathbb{R} \mid |x - a| < \delta\}$$

**Lemma.** For any  $a, b \in \mathbb{R}$  such that  $0 < b$ ,  $|a| < b$  if and only if  $-b < a < b$ .

*Proof.* Suppose first that  $|a| < b$ . We divide into two cases ( $a \geq 0$  and  $a < 0$ ). If  $a \geq 0$ , then by Definition 8.4,  $0 \leq a = |a| < b$ . Additionally, by Lemma 7.23,  $-b < 0$ . Therefore,  $-b < 0 \leq a < b$ , as desired. If  $a < 0$ , then by Definition 8.4,  $-a = |a| < b$ . It follows by Definition 7.21 (by adding  $a-b$  to both sides) that  $-b < a$ . Additionally, by Lemma 7.23,  $a < 0$  implies  $0 < -a$ , so we know that  $a < -a$ . Therefore,  $-b < a < -a < b$ , as desired.

Now suppose that  $-b < a < b$ . We divide into two cases ( $a \geq 0$  and  $a < 0$ ). If  $a \geq 0$ , then by Definition 8.4,  $|a| = a < b$ , as desired. If  $a < 0$ , then by Definition 8.4,  $|a| = -a$ . Since  $-b < a$ , Definition 7.21 implies (by adding  $b-a$  to both sides) that  $-a < b$ . Therefore,  $|a| = -a < b$ , as desired. □

*Proof of Exercise 8.9.* To prove that  $(a - \delta, a + \delta) = \{x \in \mathbb{R} \mid |x - a| < \delta\}$ , Definition 1.2 tells us that it will suffice to show that every  $p \in (a - \delta, a + \delta)$  is an element of  $\{x \in \mathbb{R} \mid |x - a| < \delta\}$  and vice versa.

Suppose first that  $p \in (a - \delta, a + \delta)$ . Then by Equations 8.1,  $a - \delta < p$  and  $p < a + \delta$ . It follows by consecutive applications of Definition 7.21 from the former condition that  $-\delta < p - a$ , and from the latter condition that  $p - a < \delta$ . Since  $-\delta < p - a < \delta$ , the lemma asserts that  $|p - a| < \delta$ . Therefore,  $p \in \{x \in \mathbb{R} \mid |x - a| < \delta\}$ .

Now suppose that  $p \in \{x \in \mathbb{R} \mid |x - a| < \delta\}$ . Then  $|p - a| < \delta$ . Thus, by the lemma,  $-\delta < p - a$  and  $p - a < \delta$ . It follows by consecutive applications of Definition 7.21 from the former condition that  $a - \delta < p$ , and from the latter condition that  $p < a + \delta$ . Therefore, since  $a - \delta < p < a + \delta$ , we have that  $p \in (a - \delta, a + \delta)$ . □