Script 9

Continuous Functions

9.1 Journal

2/16: **Lemma 9.1.** *Let $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$. If $A, B \subset \mathbb{R}$, then*

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$
$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$
$$f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$$
$$f^{-1}(\mathbb{R}) = X$$

Proof. To prove that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, Definition 1.2 tells us that it will suffice to show that every $x \in f^{-1}(A \cup B)$ is an element of $f^{-1}(A) \cup f^{-1}(B)$ and vice versa. Suppose first that x is an arbitrary element of $f^{-1}(A \cup B)$. Then by Definition 1.18, $f(x) \in A \cup B$. Thus, by Definition 1.5, $f(x) \in A$ or $f(x) \in B$. We now divide into two cases. If $f(x) \in A$, then by Definition 1.18, $x \in f^{-1}(A)$. It follows by Definition 1.5 that $x \in f^{-1}(A) \cup f^{-1}(B)$, as desired. The argument is symmetric in the other case. Now suppose that $x \in f^{-1}(A) \cup f^{-1}(B)$. Then by Definition 1.5, $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. We now divide into two cases. If $x \in f^{-1}(A)$, then by Definition 1.18, $f(x) \in A$. It follows by Definition 1.5 that $f(x) \in A \cup B$. Therefore, by Definition 1.18, $x \in f^{-1}(A \cup B)$. The argument is symmetric in the other case, as desired.

To prove that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$, Definition 1.2 tells us that it will suffice to show that every $x \in f^{-1}(A \cap B)$ is an element of $f^{-1}(A) \cap f^{-1}(B)$ and vice versa. Suppose first that x is an arbitrary element of $f^{-1}(A \cap B)$. Then by Definition 1.18, $f(x) \in A \cap B$. Thus, by Definition 1.6, $f(x) \in A$ and $f(x) \in B$. It follows by consecutive applications of Definition 1.18 that $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$. Therefore, by Definition 1.6, $x \in f^{-1}(A) \cap f^{-1}(B)$, as desired. Now suppose that $x \in f^{-1}(A) \cap f^{-1}(B)$. Then by Definition 1.6, $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$. It follows by consecutive applications of Definition 1.18 that $f(x) \in A$ and $f(x) \in B$. Thus, by Definition 1.6, $f(x) \in A \cap B$. Therefore, by Definition 1.18, $x \in f^{-1}(A \cap B)$, as desired.

To prove that $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$, Definition 1.2 tells us that it will suffice to show that every $x \in f^{-1}(A \setminus B)$ is an element of $f^{-1}(A) \setminus f^{-1}(B)$ and vice versa. Suppose first that x is an arbitrary element of $f^{-1}(A \setminus B)$. Then by Definition 1.18, $f(x) \in A \setminus B$. Thus, by Definition 1.11, $f(x) \in A$ and $f(x) \notin B$. It follows by consecutive applications of Definition 1.18 that $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$. Therefore, by Definition 1.11, $x \in f^{-1}(A) \setminus f^{-1}(B)$, as desired. Now suppose that $x \in f^{-1}(A) \setminus f^{-1}(B)$. Then by Definition 1.11, $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$. It follows by consecutive applications of Definition 1.18 that $f(x) \in A$ and $f(x) \notin B$. Thus, by Definition 1.11, $f(x) \in A \setminus B$. Therefore, by Definition 1.18, $x \in f^{-1}(A \setminus B)$, as desired.

To prove that $f^{-1}(\mathbb{R}) = X$, Definition 1.2 tells us that it will suffice to show that every $x \in f^{-1}(\mathbb{R})$ is an element of X and vice versa. Suppose first that x is an arbitrary element of $f^{-1}(\mathbb{R})$. Then by Definition 1.18, $x \in X$, as desired. Now suppose that $x \in X$. Then by Definition 1.16, $f(x) \in \mathbb{R}$. It follows by Definition 1.18 that $x \in f^{-1}(\mathbb{R})$, as desired. $\square$

Exercise 9.2. Let $f : X \to \mathbb{R}$. Let $A \subset X$ and $B \subset \mathbb{R}$. Show that

$$f(f^{-1}(B)) \subset B$$
$$A \subset f^{-1}(f(A))$$

Give examples to show that the inclusions can be proper.

Proof. To prove that $f(f^{-1}(B)) \subset B$, Definition 1.3 tells us that it will suffice to show that every $y \in f(f^{-1}(B))$ is an element of B. Let y be an arbitrary element of $f(f^{-1}(B))$. Then by Definition 1.18, $y = f(x)$ for some $x \in f^{-1}(B)$. By Definition 1.18 again, $f(x) \in B$. Therefore, since $y = f(x)$, it follows that $y \in B$, as desired.

To prove that $A \subset f^{-1}(f(A))$, Definition 1.3 tells us that it will suffice to show that every $x \in A$ is an element of $f^{-1}(f(A))$. Let x be an arbitrary element of A. Then by Definition 1.18, $f(x) \in f(A)$. Therefore, by Definition 1.18, we have $x \in f^{-1}(f(A))$, as desired.

Let $X = \{1, 2\}$ and let $f : X \to \mathbb{R}$ be defined by $f(1) = 3$ and $f(2) = 3$. If we let $B = \{3, 4\}$, then $f(f^{-1}(B)) = \{3\} \subsetneq \{3, 4\}$. Additionally, if we let $A = \{1\}$, then $A \subsetneq f^{-1}(f(A)) = \{1, 2\}$. $\square$

Exercise 9.3. Let $f : X \to \mathbb{R}$. Let $A \subset X$ and $B \subset \mathbb{R}$. Then $f(A) \subset B \iff A \subset f^{-1}(B)$.

Proof. Suppose first that $f(A) \subset B$. To prove that $A \subset f^{-1}(B)$, Definition 1.3 tells us that it will suffice to show that every $x \in A$ is an element of $f^{-1}(B)$. Let x be an arbitrary element of A. Then by Definition 1.18, $f(x) \in f(A)$. It follows by the hypothesis and Definition 1.3 that $f(x) \in B$. Therefore, by Definition 1.18 again, $x \in f^{-1}(B)$.

Now suppose that $A \subset f^{-1}(B)$. To prove that $f(A) \subset B$, Definition 1.3 tells us that it will suffice to show that every $y \in f(A)$ is an element of B. Let y be an arbitrary element of $f(A)$. Then by Definition 1.18, $y = f(x)$ for some $x \in A$. It follows by the hypothesis and Definition 1.3 that $x \in f^{-1}(B)$. Therefore, by Definition 1.18 again, $y = f(x) \in B$. $\square$

Definition 9.4. Let $X \subset \mathbb{R}$. A function $f : X \to \mathbb{R}$ is **continuous** if for every open set $U \subset \mathbb{R}$, the preimage $f^{-1}(U)$ is open in X.

Proposition 9.5. *Let $X \subset \mathbb{R}$. A function $f : X \to \mathbb{R}$ is continuous if and only if for every closed set $F \subset \mathbb{R}$, the preimage $f^{-1}(F)$ is closed in X.*

Proof. Suppose first that f is continuous. We seek to prove that for every closed set $F \subset \mathbb{R}$, the preimage $f^{-1}(F)$ is closed in X. Let F be an arbitrary closed subset of $\mathbb{R}$. Then by Definition 4.8, $F = \mathbb{R} \setminus U$ for some open set $U \subset \mathbb{R}$. It follows by Definition 9.4 since f is continuous that $f^{-1}(U)$ is open in X. Additionally, by consecutive applications of Lemma 9.1, $f^{-1}(F) = f^{-1}(\mathbb{R} \setminus U) = f^{-1}(\mathbb{R}) \setminus f^{-1}(U) = X \setminus f^{-1}(U)$. Therefore, since $f^{-1}(U)$ is open in X, Exercise 8.13 implies that $X \setminus f^{-1}(U) = f^{-1}(F)$ is closed in X.

The proof is symmetric in the other direction. $\square$

Definition 9.6. Let $X \subset Y \subset \mathbb{R}$ and let $f : Y \to \mathbb{R}$. Then the **restriction** (of f to X), written $f|_X$ is the function $f|_X : X \to \mathbb{R}$ defined by

$$f|_X(x) = f(x)$$

for all $x \in X$.

Proposition 9.7. *Let $X \subset Y \subset \mathbb{R}$. If $f : Y \to \mathbb{R}$ is continuous, then the restriction of f to X is continuous.*

Proof. To prove that $f|_X$ is continuous, Definition 9.4 tells us that it will suffice to show that for every open set $U \subset \mathbb{R}$, the preimage $f|_X^{-1}(U)$ is open in X. Let U be an open subset of $\mathbb{R}$. Then

$$\begin{aligned}
f|_X^{-1}(U) &= \{x \in X \mid f|_X(x) \in U\} && \text{Definition 1.18} \\
&= \{x \in X \mid f(x) \in U\} && \text{Definition 9.6} \\
&= \{x \in Y \mid f(x) \in U\} \cap X && \text{Script 1} \\
&= f^{-1}(U) \cap X && \text{Definition 1.18} \\
&= (Y \cap G) \cap X && \text{Definitions 9.4 and 8.11}
\end{aligned}$$

$$= X \cap G \qquad \text{Script 1}$$

Since $f|_X^{-1}(U) = X \cap G$ where G is an open set, Definition 8.11 asserts that $f|_X^{-1}(U)$ is open in X. $\square$

Exercise 9.8. Show that for any $X \subsetneq \mathbb{R}$ that is not open and any continuous function $f : X \to \mathbb{R}$, there is an open set U for which $f^{-1}(U)$ is open in X but is not open in $\mathbb{R}$.

Proof. We will prove that $\mathbb{R}$ is an open set for which $f^{-1}(\mathbb{R})$ is open in X but not in $\mathbb{R}$. First, by Theorem 5.1, $\mathbb{R}$ is open. Next, by Lemma 9.1, $f^{-1}(\mathbb{R}) = X$. It follows since $f^{-1}(\mathbb{R}) = X = X \cap \mathbb{R}$ (where $\mathbb{R}$ is an open set) by Definition 8.11 that $f^{-1}(\mathbb{R})$ is open in X. Last, since X is not open (in $\mathbb{R}$) by definition, $f^{-1}(\mathbb{R}) = X$ is not open in $\mathbb{R}$. $\square$

Definition 9.9. The function $f : X \to \mathbb{R}$ is **continuous** (at $x \in X$) if for every region R containing $f(x)$, there exists an open set S containing x such that $S \cap X \subset f^{-1}(R)$.

Theorem 9.10. *The function $f : X \to \mathbb{R}$ is continuous if and only if it is continuous at every $x \in X$.*

Proof. Suppose first that f is continuous, and suppose for the sake of contradiction that f is not continuous at every $x \in X$. Then by Definition 9.9, there exists some $x \in X$ such that f is not continuous at x. Thus, there exists a region R with $f(x) \in R$ such that for all open sets S containing x, $S \cap X \not\subset f^{-1}(R)$. Since f is continuous by hypothesis and R is open by Corollary 4.11, $f^{-1}(R)$ is open in X. It follows by Definition 8.11 that $f^{-1}(R) = X \cap S$ for some open set S. But this implies that $f^{-1}(R) \not\subset f^{-1}(R)$, a contradiction.

Now suppose that f is continuous at every $x \in X$. To prove that f is continuous, Definition 9.4 tells us that it will suffice to show that for every open set $U \subset \mathbb{R}$, the preimage $f^{-1}(U)$ is open in X. We divide into two cases ($f^{-1}(U) = \emptyset$ and $f^{-1}(U) \neq \emptyset$). If $f^{-1}(U) = \emptyset$, then since $\emptyset \cap X = \emptyset$ by Script 1 where $\emptyset$ is open by Theorem 5.1, Definition 8.11 tells us that $\emptyset = f^{-1}(U)$ is open in X, as desired. On the other hand, if $f^{-1}(U) \neq \emptyset$, Definition 8.11 tells us that it will suffice to show that $f^{-1}(U) = S \cap X$ where S is an open set. We first seek to show that for every $x \in f^{-1}(U)$, there exists an open set S_x containing x such that $S_x \cap X \subset f^{-1}(U)$. Let x be an arbitrary element of $f^{-1}(U)$. It follows by Definition 1.18 that $f(x) \in U$. Thus, since U is open, we have by Theorem 4.10 that there exists a region R such that $f(x) \in R$ and $R \subset U$. Consequently, since R is open by Corollary 4.11, we have by Definition 9.9 that there exists an open set S_x containing x such that $S_x \cap X \subset f^{-1}(R)$. Additionally, Script 1 tells us based off of the fact that $R \subset U$ that $f^{-1}(R) \subset f^{-1}(U)$. Thus, by subset transitivity, $S_x \cap X \subset f^{-1}(U)$. At this point, let $S = \bigcup_{x \in f^{-1}(U)} S_x$. It follows immediately from Corollary 4.18 that S is open. Additionally, since the intersection of each set in the union with X is a subset of $f^{-1}(U)$, it follows by Script 1 that $S \cap X \subset f^{-1}(U)$. Furthermore, for all $x \in f^{-1}(U)$, Definition 1.18 asserts that $x \in X$. In addition, we have defined an S_x such that $x \in S_x$. These last two results combined demonstrate by Definition 1.6 that $x \in S \cap X$. Thus, by Definition 1.3, $f^{-1}(U) \subset S \cap X$. Consequently, by Theorem 1.7, $f^{-1}(U) = S \cap X$. Since $f^{-1}(U)$ is the intersection of X with an open set, Definition 8.11 asserts that it is open in X, as desired. $\square$

2/18: **Theorem 9.11.** *Suppose that $f : X \to \mathbb{R}$ is continuous. If X is connected, then $f(X)$ is connected.*

Proof. This will be a proof by contrapositive; as such, suppose that $f(X)$ is disconnected. Then by Definition 4.22, $f(X) = A \cup B$ where A, B are nonempty, disjoint sets that are open in $f(X)$. It follows from the last condition by Definition 8.11 that $A = G \cap f(X)$ and $B = H \cap f(X)$, where G, H are open sets. Since for all $x \in X$, $f(x) \in A$ or $f(x) \in B$, Definitions 1.2 and 1.6 imply that for all $x \in X$, $f(x) \in G$ and $f(x) \in H$. Thus, by Script 1, $X \subset f^{-1}(G) \cup f^{-1}(H)$. Additionally, we know by Definition 1.18 that for all $x \in f^{-1}(G) \cup f^{-1}(H)$, $x \in X$. Thus, by Definition 1.3, $f^{-1}(G) \cup f^{-1}(H) \subset X$. Consequently, by Theorem 1.7, we have that $X = f^{-1}(G) \cup f^{-1}(H)$.

To show that $f^{-1}(G), f^{-1}(H)$ are disjoint, Definition 1.9 tells us that it will suffice to verify that $f^{-1}(G) \cap f^{-1}(H) = \emptyset$. As such, suppose for the sake of contradiction that $x \in f^{-1}(G) \cap f^{-1}(H)$. Then by consecutive applications of Definition 1.6, $x \in f^{-1}(G)$ and $x \in f^{-1}(H)$. Thus, by multiple applications of Definition 1.18, $x \in X$, $f(x) \in G$, and $f(x) \in H$. It follows from the first condition by Definition 1.18 that $f(x) \in f(X)$. The facts that $f(x) \in f(X)$ and $f(x) \in G$ imply by Definitions 1.6 and 1.2 that $f(x) \in A$. Similarly, $f(x) \in B$. But these last two statements imply by Definition 1.6 that $f(x) \in A \cap B$, a contradiction.

To show that $f^{-1}(G), f^{-1}(H)$ are nonempty, Definition 1.8 tells us that it will suffice to find an element of each set. As previously mentioned, A, B are nonempty. Thus, by consecutive applications of Definition 1.8, there exist $f(x) \in A$ and $f(y) \in B$. Consequently, by Definitions 1.2 and 1.6, $f(x) \in G$ and $f(y) \in H$. Therefore, by consecutive applications of Definition 1.18, $x \in f^{-1}(G)$ and $y \in f^{-1}(H)$, as desired.

To show that $f^{-1}(G)$ and $f^{-1}(H)$ are open, Definition 9.4 tells us that it will suffice to verify (since f is continuous by hypothesis) that G, H are open subsets of $\mathbb{R}$. But by definition, they are exactly that. $\square$

Exercise 9.12. Use Theorem 9.11 to prove that if $f : [a, b] \to \mathbb{R}$ is continuous, then for every point p between $f(a)$ and $f(b)$, there exists c such that $a < c < b$ and $f(c) = p$.

Proof. Suppose that $a < b$. Then by Lemma 8.3, $[a, b]$ is an interval. Thus, by Theorem 8.15, $[a, b]$ is connected. It follows by Theorem 9.11 that $f([a, b])$ is connected. Consequently, by Theorem 8.15, $f([a, b])$ is an interval. We divide into three cases ($f(a) < f(b)$, $f(a) > f(b)$, and $f(a) = f(b)$).

First, suppose that $f(a) < f(b)$, and let p be an arbitrary point between $f(a)$ and $f(b)$ (we know that at least one such point exists by Theorem 5.2). Then by Definition 3.6, $f(a) < p < f(b)$. Now $a, b \in [a, b]$ by Equations 8.1, so by Definition 1.18, $f(a), f(b) \in f([a, b])$. It follows by Definition 8.2 since $f([a, b])$ is an interval that $[f(a), f(b)] \subset f([a, b])$. Thus, since $f(a) < p < f(b)$ implies $p \in [f(a), f(b)]$ by Equations 8.1, Definition 1.3 asserts that $p \in f([a, b])$. Consequently, by Definition 1.18, $p = f(c)$ for some $c \in [a, b]$. Additionally, since $f(a) < p < f(b)$, we know that $p \neq f(a)$ and $p \neq f(b)$. It follows that $p = f(c)$ for some $c \in (a, b)$, as desired.

The proof of the second case is symmetric to that of the first.

Third, suppose that $f(a) = f(b)$. This implies that there are no points p between $f(a)$ and $f(b)$ by Definition 3.6, so the statement is vacuously true in this case. $\square$

Lemma 9.13. *If $f : (a, b) \to \mathbb{R}$ is continuous and injective, then f is either strictly increasing or strictly decreasing on (a, b).*

Proof. Suppose that $a < b$. Then by Corollary 5.3, we can pick two distinct points $c, d \in (a, b)$. Since f is injective, we have from the fact that $x \neq y$ by Definition 1.20 that $f(c) \neq f(d)$. We divide into two cases ($f(c) < f(d)$ and $f(c) > f(d)$).

Suppose first that $f(c) < f(d)$. To prove that f is strictly increasing, Definition 8.16 tells us that it will suffice to show that for all $x, y \in (a, b)$ with $x < y$, $f(x) < f(y)$. Let x, y be arbitrary elements of (a, b) that satisfy $x < y$, and suppose for the sake of contradiction that $f(x) \geq f(y)$. If $f(x) = f(y)$, then by Definition 1.20, $x = y$, a contradiction. If $f(x) > f(y)$, then we divide into five cases ($x < y < c < d$, $x < c < y < d$, $x < c < d < y$, $c < x < d < y$, and $c < d < x < y$).

Let $x < y < c < d$. If $f(x) > f(c)$, let p_1 be a point between $f(x)$ and $f(c)$ and let p_2 be a point between $f(d)$ and $f(c)$. Let $p = \min(p_1, p_2)$. Since $f(c) < p < f(x)$, Exercise 9.12 implies that there exists $e \in (x, c)$ such that $f(e) = p$. Similarly, there exists $e' \in (c, d)$ such that $f(e') = p$. Since (x, c) and (c, d) are clearly disjoint, $e \neq e'$. But by Definition 1.20, $f(e) = p = f(e')$ implies that $e = e'$, a contradiction. If $f(x) < f(c)$, then we can arrive at a similar contradiction by considering values in the regions (x, y) and (y, c).

The proof is symmetric in the second case if we consider values in the regions (c, y) and (y, d) when $f(y)$ is not between $f(c)$ and $f(d)$, and values in the regions (x, c) and (c, y) when $f(y)$ is.

The proof is symmetric in the third case if we consider values in the regions (x, c) and (c, d) when $f(x) > f(c)$, and values in the regions (x, d) and (d, y) when $f(x) < f(c)$.

The proof is symmetric in the fourth case if we consider values in the regions (c, x) and (x, d) when $f(x) > f(d)$, and values in the regions (x, d) and (d, y) when $f(x) < f(d)$.

The proof is symmetric in the fifth case if we consider values in the regions (c, d) and (d, x) when $f(d) > f(x)$, and values in the regions (d, x) and (x, y) when $f(d) < f(x)$. $\square$

2/23: **Theorem 9.14.** *If $f : (a, b) \to \mathbb{R}$ is continuous and injective, then the inverse function $g : f((a, b)) \to (a, b)$ is continuous.*

Lemma. *Let $f : (a, b) \to \mathbb{R}$ be continuous and injective, and let $(x, y) \subset (a, b)$ be a region. Then $f((x, y))$ is also a region.*

Proof. Since $f : (a, b) \to \mathbb{R}$ is continuous and injective, Lemma 9.13 implies that f is either strictly increasing or strictly decreasing on (a, b). We now divide into two cases.

Suppose first that f is strictly increasing. To prove that $f((x, y))$ is a region, Definition 3.10 tells us that it will suffice to show that $f((x, y)) = (f(x), f(y))$. To show this, Definition 1.2 tells us that it will suffice to verify that every $p \in f((x, y))$ is an element of $(f(x), f(y))$ and vice versa. Let p be an arbitrary element of $f((x, y))$. Then by Definition 1.18, $p = f(z)$ for some $z \in (x, y)$. Since $z \in (x, y)$, we have by Equations 8.1 that $x < z < y$. Since f is strictly increasing on (a, b), by Definition 8.16, $x < z < y$ implies that $f(x) < f(z) < f(y)$. But this implies by Equations 8.1 that $f(z) = p$ is an element of $(f(x), f(y))$, as desired. Now let p be an arbitrary element of $(f(x), f(y))$. Then by Equations 8.1, $f(x) < p < f(y)$. We seek to prove that $[x, y] \subset (a, b)$. Let q be an arbitrary element of $[x, y]$. Then by Equations 8.1, $x \leq q \leq y$. Additionally, since $x, y \in (a, b)$, Equations 8.1 imply that $a < x < b$ and $a < y < b$. Thus, $a < x \leq q \leq y < b$, meaning by Equations 8.1 that $q \in (a, b)$. Consequently, by Definition 1.3, $[x, y] \subset (a, b)$. If we now consider the restriction $f|_{[x,y]}$, we have by Proposition 9.7 that $f|_{[x,y]}$ is continuous. Thus, since $f|_{[x,y]} : [x, y] \to \mathbb{R}$ is continuous and $f|_{[x,y]}(x) = f(x) < p < f(y) = f|_{[x,y]}(y)$ (by Definition 9.6), Exercise 9.12 implies that there exists $c \in (x, y)$ such that $f|_{[x,y]}(c) = f(c) = p$. But by Definition 1.18, this implies that $p \in f((x, y))$.

The proof is symmetric in the other case. □

Proof of Theorem 9.14. We first show that g exists, and then show that it is continuous.

To prove that g is a function, Definition 1.16 tells us that it will suffice to show that for all $y \in f((a, b))$, there exists a unique $x \in (a, b)$ such that $g(y) = x$. We will first show that for each y, such an element exists, and then show that it is unique. Let y be an arbitrary element of $f((a, b))$. Then by Definition 1.18, $y = f(x)$ for some $x \in (a, b)$. Thus, since we require that $g(f(x')) = x'$ and $f(g(y')) = y'$ for g to be an inverse function, we assign $g(y) = x$. Now suppose that $g(y) = x_1$ and $g(y) = x_2$. Then by the definition of g, $f(x_1) = y$ and $f(x_2) = y$. It follows that $f(x_1) = f(x_2)$, implying since f is injective by Definition 1.20 that $x_1 = x_2$, as desired.

To prove that g is continuous, Definition 9.15 tells us that it will suffice to show that for every $U \subset (a, b)$ that is open in (a, b), the preimage $g^{-1}(U)$ is open in $f((a, b))$. Let U be an arbitrary subset of (a, b) that is open in (a, b). To show that $g^{-1}(U)$ is open in $f((a, b))$, Definition 8.11 tells us that it will suffice to confirm that $g^{-1}(U) = f((a, b)) \cap G$, where G is an open set.

To begin, we have

$$
\begin{aligned}
g^{-1}(U) &= \{y \in f((a, b)) \mid g(y) \in U\} && \text{Definition 1.18} \\
&= \{f(x) \in f((a, b)) \mid g(f(x)) \in U\} && \text{Definition 1.18}
\end{aligned}
$$

By the definition of g, we have $g(f(x)) = x$.

$$
\begin{aligned}
&= \{f(x) \in f((a, b)) \mid x \in U\} \\
&= \{f(x) \in \{f(x') \in \mathbb{R} \mid x' \in (a, b)\} \mid x \in U\} && \text{Definition 1.18}
\end{aligned}
$$

This next transition is mostly notational in nature. $f(x)$ being an element of the set of all $f(x') \in \mathbb{R}$ that meet a certain condition means that $f(x) \in \mathbb{R}$. Additionally, since that condition is $x' \in (a, b)$, we know that $x \in (a, b)$. But if $x \in (a, b)$ and (from the condition in the original set) $x \in U$, we have by Definition 1.6 that $x \in U \cap (a, b)$.

$$
\begin{aligned}
&= \{f(x) \in \mathbb{R} \mid x \in U \cap (a, b)\} \\
&= f(U \cap (a, b)) && \text{Definition 1.18}
\end{aligned}
$$

By definition, U is open in (a, b). Consequently, by Definition 8.11, $U = (a, b) \cap V$ where V is open.

$$
\begin{aligned}
&= f(((a, b) \cap V) \cap (a, b)) \\
&= f((a, b) \cap V) && \text{Definition 1.6} \\
&= f((a, b)) \cap f(V) && \text{Additional Exercise 9.2b}
\end{aligned}
$$

All that's left at this point is to prove that $f(V)$ is open. By Theorem 4.14, $V = \bigcup_{\lambda \in I}\{R_\lambda\}$ is a collection of regions. It follows by an extension of Additional Exercise 9.2a that $f(V) = \bigcup_{\lambda \in I}\{f(R_\lambda)\}$. Additionally, by the lemma, each $f(R_\lambda)$ is a region; hence, by Corollary 4.11, each $f(R_\lambda)$ is open. Thus, $f(V)$ is the union of a collection of open subsets of $\mathbb{R}$, so by Corollary 4.18, $f(V)$ is open. □

We denote the inverse function g by f^{-1}. In this result, g has codomain (a, b) but our definition of continuity (Definition 9.4) only applies to functions with codomain $\mathbb{R}$. Our definitions/results are easily adapted. The definitions are as given below and we give a sample theorem. Other results can be adjusted in a similar fashion.

Definition 9.15. Let $X, Y \subset \mathbb{R}$. A function $f : X \to Y$ is **continuous** if for every U that is open in Y, the preimage $f^{-1}(U)$ is open in X.

Definition 9.16. The function $f : X \to Y$ is **continuous** (at $x \in X$) if for every region R containing $f(x)$, there exists an open set S containing x such that $S \cap X \subset f^{-1}(R \cap Y)$.

Theorem 9.17. *The function $f : X \to Y$ is continuous if and only if it is continuous at every $x \in X$.*

Additional Exercises

2. Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$. Let $A, B \subset \mathbb{R}$. Either prove or give a counterexample to each of the following:

   a) $f(A \cup B) = f(A) \cup f(B)$.

   b) $f(A \cap B) = f(A) \cap f(B)$.

   c) $f(A \setminus B) = f(A) \setminus f(B)$.