Final-Specific Questions

1. Suppose that X and Y are compact subsets of $\mathbb{R}$. For this problem, use only results up to and including Theorem 10.11, and not any of the subsequent results in Script 10.
   (a) Show that $X \cup Y$ is compact.
   (b) Show that $X \cap Y$ is compact.
   (c) Suppose $X_1, X_2, \dots$ are compact sets. Are the following compact? Either prove that the set is always compact or provide a counterexample that is not compact.

$$\bigcap_{n \in \mathbb{N}} X_n \qquad\qquad \bigcup_{n \in \mathbb{N}} X_n$$

Proof of a. To prove that $X \cup Y$ is compact, Definition 10.4 tells us that it will suffice to show that for every open cover $\mathcal{G}$ of $X \cup Y$, there exists a finite subset $\mathcal{G}' \subset \mathcal{G}$ that is also an open cover. Let $\mathcal{G}$ be an arbitrary open cover of $X \cup Y$. We now seek to demonstrate that $\mathcal{G}$ is an open cover of X and Y, starting with X. To do so, Definition 10.3 tells us that it will suffice to confirm that every $x \in X$ is an element of G for some $G \in \mathcal{G}$, and that every G is open. For the first condition, let x be an arbitrary element of X. Then by Definition 1.5, $x \in X \cup Y$. It follows by Definition 10.3 that $x \in G$ for some $G \in \mathcal{G}$. As to the other condition, every $G \in \mathcal{G}$ is open by Definition 10.3, as desired. The argument is symmetric for Y.

We now invoke Definition 10.4 to find finite subcovers $\mathcal{G}_X \subset \mathcal{G}$ and $\mathcal{G}_Y \subset \mathcal{G}$ of X and Y, respectively. Let $\mathcal{G}' = \mathcal{G}_X \cup \mathcal{G}_Y$. By Script 1, $\mathcal{G}' \subset \mathcal{G}$. Additionally, since $\mathcal{G}_X$ and $\mathcal{G}_Y$ are both finite, Script 1 also implies that $\mathcal{G}'$ is finite. However, to demonstrate that $\mathcal{G}'$ is an open cover of $X \cup Y$, Definition 10.3 tells us that we must confirm that every $z \in X \cup Y$ is an element of G for some $G \in \mathcal{G}'$, and that every G is open. For the first condition, let z be an arbitrary element of $X \cup Y$. Thus, by Definition 1.5, $z \in X$ or $z \in Y$. We now divide into two cases. If $z \in X$, then by Definition 10.3, $z \in G$ for some $G \in \mathcal{G}_X$. But since $\mathcal{G}_X \subset \mathcal{G}'$ by Script 1, $z \in G$ implies that z is an element of a set that is an element of $\mathcal{G}'$, as desired. The argument is symmetric in the other case. As to the other condition, every $G \in \mathcal{G}$ is open by Definition 10.3, as desired. $\square$

Proof of b. To prove that $X \cap Y$ is compact, Definition 10.4 tells us that it will suffice to show that for every open cover $\mathcal{G}$ of $X \cap Y$, there exists a finite subset $\mathcal{G}' \subset \mathcal{G}$ that is also an open cover. To do this, we will first demonstrate that $\mathcal{H} = \mathcal{G} \cup \{\mathbb{R} \setminus (X \cap Y)\}$ is an open cover of X. It will follow that there exists a finite subset $\mathcal{H}' \subset \mathcal{H}$ that is an open cover of X. Lastly, we will demonstrate that $\mathcal{G}' = \mathcal{H}' \setminus \{\mathbb{R} \setminus (X \cap Y)\}$ is the desired finite open cover subset of $\mathcal{G}$.

Let $\mathcal{G}$ be an arbitrary open cover of $X \cap Y$, and let $\mathcal{H} = \mathcal{G} \cup \{\mathbb{R} \setminus (X \cap Y)\}$. To demonstrate that $\mathcal{H}$ is an open cover of X, Definition 10.3 tells us that it will suffice to confirm that every $x \in X$ is an element of H for some $H \in \mathcal{H}$, and that every H is open. For the first condition, let x be an arbitrary element of X. We divide into two cases ($x \in X \cap Y$ and $x \notin X \cap Y$). If $x \in X \cap Y$, then since $\mathcal{G}$ is an open cover of $X \cap Y$, Definition 10.3 implies that $x \in G$ for some $G \in \mathcal{G}$. But since $\mathcal{G} \subset \mathcal{H}$, $x \in G$ for some $G \in \mathcal{H}$, as desired. On the other hand, if $x \notin X \cap Y$, then this combined with the fact that $x \in \mathbb{R}$ implies by Definition 1.11 that $x \in \mathbb{R} \setminus (X \cap Y) \in \mathcal{H}$, as desired. As to the other condition, let H be an arbitrary element of $\mathcal{H}$. We divide into two cases ($H \in \mathcal{G}$ and $H \notin \mathcal{G}$). If $H \in \mathcal{G}$, then by

Definition 10.3, H is open, as desired. On the other hand, if $H \notin \mathcal{G}$, then $H = \mathbb{R} \setminus (X \cap Y)$ by Script 1. X and Y are closed by Theorem 10.11, so $X \cap Y$ is closed by Theorem 4.16. It follows by Definition 4.8 that $\mathbb{R} \setminus (X \cap Y)$ is open, so H is open, as desired.

Since $\mathcal{H}$ is an open cover of X and X is compact, we have by Definition 10.4 that there exists a finite subset $\mathcal{H}' \subset \mathcal{H}$ that is also an open cover of X. Let $\mathcal{G}' = \mathcal{H}' \setminus \{\mathbb{R} \setminus (X \cap Y)\}$. By Script 1, $\mathcal{G}' \subset \mathcal{G}$. Additionally, since $\mathcal{H}'$ is finite, Script 1 also implies that $\mathcal{G}'$ is finite. However, to demonstrate that $\mathcal{G}'$ is an open cover of $X \cap Y$, Definition 10.3 tells us that we must confirm that every $z \in X \cap Y$ is an element of G for some $G \in \mathcal{G}'$, and that every G is open. For the first condition, let z be an arbitrary element of $X \cap Y$. Then since $X \cap Y \subset X$ by Theorem 1.7, Definition 10.3 asserts that $z \in H$ for some $H \in \mathcal{H}'$. Additionally, since $z \in X \cap Y$, Definition 1.11 implies that $z \notin \mathbb{R} \setminus (X \cap Y)$. Thus, $H \neq \mathbb{R} \setminus (X \cap Y)$, which implies by Definition 1.11 that $H \in \mathcal{H}' \setminus \{\mathbb{R} \setminus (X \cap Y)\} = \mathcal{G}'$. Therefore, $z \in H$ for some $H \in \mathcal{G}'$, as desired. As to the other condition, since every $G \in \mathcal{G}'$ is an element of $\mathcal{G}$ (i.e., open by Definition 10.3), every $G \in \mathcal{G}'$ is open, as desired. □

Proof of c. To prove that $\bigcap_{n\in\mathbb{N}} X_n$ is compact, Definition 10.4 tells us that it will suffice to show that for every open cover $\mathcal{G}$ of $\bigcap_{n\in\mathbb{N}} X_n$, there exists a finite subset $\mathcal{G}' \subset \mathcal{G}$ that is also an open cover. To do this, we will use an analogous process to part (b).

Let $\mathcal{G}$ be an arbitrary open cover of $\bigcap_{n\in\mathbb{N}} X_n$, and let $\mathcal{H} = \mathcal{G} \cup \{\mathbb{R} \setminus (\bigcap_{n\in\mathbb{N}} X_n)\}$. To demonstrate that $\mathcal{H}$ is an open cover of X_1, Definition 10.3 tells us that it will suffice to confirm that every $x \in X_1$ is an element of H for some $H \in \mathcal{H}$, and that every H is open. For the first condition, let x be an arbitrary element of X_1. We divide into two cases ($x \in \bigcap_{n\in\mathbb{N}} X_n$ and $x \notin \bigcap_{n\in\mathbb{N}} X_n$). If $x \in \bigcap_{n\in\mathbb{N}} X_n$, then since $\mathcal{G}$ is an open cover of $\bigcap_{n\in\mathbb{N}} X_n$, Definition 10.3 implies that $x \in G$ for some $G \in \mathcal{G}$. But since $\mathcal{G} \subset \mathcal{H}$, $x \in G$ for some $G \in \mathcal{H}$, as desired. On the other hand, if $x \notin \bigcap_{n\in\mathbb{N}} X_n$, then this combined with the fact that $x \in \mathbb{R}$ implies by Definition 1.11 that $x \in \mathbb{R} \setminus (\bigcap_{n\in\mathbb{N}} X_n) \in \mathcal{H}$, as desired. As to the other condition, let H be an arbitrary element of $\mathcal{H}$. We divide into two cases ($H \in \mathcal{G}$ and $H \notin \mathcal{G}$). If $H \in \mathcal{G}$, then by Definition 10.3, H is open, as desired. On the other hand, if $H \notin \mathcal{G}$, then $H = \mathbb{R} \setminus (\bigcap_{n\in\mathbb{N}} X_n)$ by Script 1. Each X_n is closed by Theorem 10.11, so $\bigcap_{n\in\mathbb{N}} X_n$ is closed by Theorem 4.16. It follows by Definition 4.8 that $\mathbb{R} \setminus (\bigcap_{n\in\mathbb{N}} X_n)$ is open, so H is open, as desired.

Since $\mathcal{H}$ is an open cover of X_1 and X_1 is compact, we have by Definition 10.4 that there exists a finite subset $\mathcal{H}' \subset \mathcal{H}$ that is also an open cover of X_1. Let $\mathcal{G}' = \mathcal{H}' \setminus \{\mathbb{R} \setminus (\bigcap_{n\in\mathbb{N}} X_n)\}$. By Script 1, $\mathcal{G}' \subset \mathcal{G}$. Additionally, since $\mathcal{H}'$ is finite, Script 1 also implies that $\mathcal{G}'$ is finite. However, to demonstrate that $\mathcal{G}'$ is an open cover of $\bigcap_{n\in\mathbb{N}} X_n$, Definition 10.3 tells us that we must confirm that every $y \in \bigcap_{n\in\mathbb{N}} X_n$ is an element of G for some $G \in \mathcal{G}'$, and that every G is open. For the first condition, let y be an arbitrary element of $\bigcap_{n\in\mathbb{N}} X_n$. Then since $\bigcap_{n\in\mathbb{N}} X_n \subset X_1$, Definition 10.3 asserts that $y \in H$ for some $H \in \mathcal{H}'$. Additionally, since $y \in \bigcap_{n\in\mathbb{N}} X_n$, Definition 1.11 implies that $y \notin \mathbb{R} \setminus (\bigcap_{n\in\mathbb{N}} X_n)$. Thus, $H \neq \mathbb{R} \setminus (\bigcap_{n\in\mathbb{N}} X_n)$, which implies by Definition 1.11 that $H \in \mathcal{H}' \setminus \{\mathbb{R} \setminus (\bigcap_{n\in\mathbb{N}} X_n)\} = \mathcal{G}'$. Therefore, $y \in H$ for some $H \in \mathcal{G}'$, as desired. As to the other condition, since every $G \in \mathcal{G}'$ is an element of $\mathcal{G}$ (i.e., open by Definition 10.3), every $G \in \mathcal{G}'$ is open, as desired.

Let $X_n = \{n\}$ for all $n \in \mathbb{N}$. By Exercise 10.5, each X_n is compact, as desired. However, $\bigcup_{n\in\mathbb{N}} X_n = \mathbb{N}$ is not compact: if we let $\mathcal{G} = \{(n-1, n+1) : n \in \mathbb{N}\}$, then we have an open cover (clearly) that is infinite (clearly) and yet from which no term can be removed without revoking its status as an open cover. □

2. Let $f, g : A \to \mathbb{R}$. In each of the following, justify your answer fully:
   (a) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ do not exist, can $\lim_{x\to a}[f(x) + g(x)]$ exist?
   (b) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a}[f(x) + g(x)]$ exists, must $\lim_{x\to a} g(x)$ exist?

Justification of a. Let $f, g : A \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \qquad g(x) = \begin{cases} 0 & x \geq 0 \\ 1 & x < 0 \end{cases}$$

By Exercise 11.4, $\lim_{x\to 0} f(x)$ does not exist. Similarly, $\lim_{x\to 0} g(x)$ does not exist. However, by Exercise 11.6, $\lim_{x\to 0}[f(x) + g(x)] = 1$. $\square$

Justification of b. Let $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a}[f(x) + g(x)] = M$. If we let $h(x) = -1$, then we have by Theorem 11.9 that $\lim_{x\to a} -f(x) = -L$. Applying Theorem 11.9 again, we have $\lim_{x\to a}[f(x) + g(x) - f(x)] = \lim_{x\to a} g(x) = M - L$. $\square$

3. For the problem you may assume $f : \mathbb{R} \to \mathbb{R}$.
   (a) Show that for all $x \in \mathbb{R}$ there exists a unique $y \in \mathbb{R}$ such that $y^3 = x$.
   (b) Using part (a), we can define the cube root function $g(x) = x^{1/3}$ in the usual way. Show that g is continuous and strictly increasing.
   (c) Suppose $\lim_{x\to 0} f(x) = L$. Show that $\lim_{x\to 0} f(x^3)$ exists and equals L.
   (d) Suppose $\lim_{x\to 0} f(x^3) = L$. Show that $\lim_{x\to 0} f(x)$ exists and equals L.

Proof of a. We first show that there always *exists* a y such that $y^3 = x$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(y) = y^3$, and let x be an arbitrary element of $\mathbb{R}$. By Definition 11.11, f is a polynomial, meaning by Corollary 11.12 that f is continuous. We divide into three cases ($x = 0$, $x > 0$, and $x < 0$). If $x = 0$, then $x = 0 = 0^3 = y^3$. If $x > 0$, consider the closed interval $[0, x + 1]$. By Proposition 9.7, $f|_{[0,x+1]} : [0, x + 1] \to \mathbb{R}$ is continuous. Additionally, $f(0) = 0 < x < x^3 + 3x^2 + 3x + 1 = (x + 1)^3 = f(x + 1)$. Thus, we have by Exercise 9.12 that there exists $y \in (0, x+1)$ such that $f(y) = y^3 = x$. The argument is symmetric if $x < 0$.

We now show that this y is unique, by proving that f is injective. To do so, Definition 1.20 tells us that it will suffice to show that if $f(x) = f(x')$, then $x = x'$. Let $x^3 = x'^3$. Then $0 = x^3 - x'^3 = (x - x')(x^2 + xx' + x'^2)$. It follows by Script 0 that $x = x'$ or $x = x' = 0$ (the latter result is trivial, but obtained by setting $x^2 + xx' + x'^2 = 0$), as desired. $\square$

Proof of b. Suppose for the sake of contradiction that g is not continuous. Then by Theorem 9.10, there exists some $x \in \mathbb{R}$ at which g is not continuous. It follows by Theorem 11.5 that $\lim_{y\to x} g(y) \neq g(x)$. Additionally, by part (a), there exists some $z \in \mathbb{R}$ such that $f(z) = x$. Also, since f is continuous by part (a), Theorem 9.10 asserts that it is continuous at at x. Thus, by Theorem 11.5, $\lim_{y\to z} f(y) = f(z)$ (we know that $z \in LP(\mathbb{R})$ by Corollary 5.4). Consequently, $\lim_{y\to z} g(f(y))$ does not exist. But this contradicts Exercise 11.6, which asserts that $g(f(x)) = x$ is continuous, i.e., $\lim_{y\to z} g(f(y))$ should exist.

By the proof of part (a), f is injective. Additionally, by part (a), f is surjective. Thus, by Definition 1.20, f is bijective. It follows by Proposition 1.27 that g is bijective. Thus, by Definition 1.20 again, g is injective. Consequently, by Proposition 9.7, $g|_{(a,b)} : (a, b) \to \mathbb{R}$ is continuous for any open interval (a, b), and by Script 1, $g|_{(a,b)}$ is also injective. Thus, by Lemma 9.13, g is strictly increasing on any open interval (a, b). It follows that g is strictly increasing overall. $\square$

Proof of c. To prove that $\lim_{x\to 0} f(x^3) = L$, Definition 11.1 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in \mathbb{R}$ and $0 < |x| < \delta$, $|f(x^3) - L| < \epsilon$. Let $\epsilon > 0$ be arbitrary. It follows from the hypothesis that $\lim_{x\to 0} f(x) = L$ by Definition 11.1 that there exists $\delta_1 > 0$ such that if $x \in \mathbb{R}$ and $0 < |x| < \delta_1$, then $|f(x) - L| < \epsilon$. Additionally, by Theorem 11.5, the facts that the x^3 function is continuous and $0 \in LP(\mathbb{R})$ imply that $\lim_{x\to 0} x^3 = 0^3 = 0$. Thus, by Definition 11.1, there exists a $\delta_2 > 0$ such that if $x \in \mathbb{R}$ and $0 < |x| < \delta_2$, then $|x^3 - 0| = |x^3| < \delta_1$. Choose $\delta = \delta_2$. Then if x is an arbitrary element of $\mathbb{R}$ such that $0 < |x| < \delta$, we have $|x^3| < \delta_1$. Additionally, since $x \neq 0$, $x^3 \neq 0$, so $0 < |x^3| < \delta_1$. It follows that $|f(x^3) - L| < \epsilon$, as desired. $\square$

Proof of d. To prove that $\lim_{x\to 0} f(x) = L$, Definition 11.1 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in \mathbb{R}$ and $0 < |x| < \delta$, $|f(x) - L| < \epsilon$. Let $\epsilon > 0$ be arbitrary. It follows from the hypothesis that $\lim_{x\to 0} f(x^3) = L$ by Definition 11.1 that there exists $\delta_1 > 0$ such that if $x \in \mathbb{R}$ and $0 < |x| < \delta_1$, then $|f(x^3) - L| < \epsilon$. Additionally, by Theorem 11.5, the facts that the $x^{1/3}$ function is continuous (part (b)) and $0 \in LP(\mathbb{R})$ imply that $\lim_{x\to 0} x^{1/3} = 0^{1/3} = 0$. Thus, by Definition 11.1, there exists a $\delta_2 > 0$ such that if $x \in \mathbb{R}$ and $0 < |x| < \delta_2$, then $|x^{1/3} - 0| = |x^{1/3}| < \delta_1$. Choose $\delta = \delta_2$. Then if x is an arbitrary element of $\mathbb{R}$ such that $0 < |x| < \delta$, we have $|x^{1/3}| < \delta_1$. Additionally, since $x \neq 0$, $x^{1/3} \neq 0$, so $0 < |x^{1/3}| < \delta_1$. It follows that $|f((x^{1/3})^3) - L| = |f(x) - L| < \epsilon$, as desired. □

4. For this problem, suppose $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, and that A is a dense subset of $\mathbb{R}$.

   (a) Prove that if f is continuous and $f(x) = 0$ for all $x \in A$, then $f(x) = 0$ for all $x \in \mathbb{R}$.

   (b) Prove that if f and g are continuous and $f(x) = g(x)$ for all $x \in A$, then $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Proof of a. Suppose for the sake of contradiction that $f(x) \neq 0$ for some $x \in \mathbb{R}$. Since f is continuous, Theorem 9.10 asserts that f is continuous at x. It follows by Theorem 11.5 that $\lim_{y\to x} f(y) = f(x)$ (since $x \in LP(\mathbb{R})$ by Corollary 5.4). Choose $\epsilon = |f(x)|$. Consequently, by Definition 11.1, there exists a $\delta > 0$ such that if $y \in \mathbb{R}$ and $|y - x| < \delta$, then $|f(y) - f(x)| < |f(x)|$.

Switching gears for a moment, consider the fact that A is dense in $\mathbb{R}$. It follows by Definition 6.8 that $x \in LP(A)$. Thus, by Definition 3.13, for every region R with $x \in R$, $R \cap (A \setminus \{x\}) \neq \emptyset$.

We merge the above two ideas by considering the region $R = (x - \delta, x + \delta)$, which is clearly an x-containing region. By the above, $R \cap (A \setminus \{x\}) \neq \emptyset$, i.e., there exists a point y such that $y \in R$, $y \in A$, and $y \neq x$. It follows from the former claim by Exercise 8.9 that $|y - x| < \delta$, from the middle claim by hypothesis that $f(y) = 0$, and from the latter claim that $y - x \neq 0$, i.e., $0 < |y - x| < \delta$. Therefore, $|f(y) - f(x)| < |f(x)|$. But this implies that $|0 - f(x)| = |f(x)| < |f(x)|$, a contradiction. □

Proof of b. Let $h(x) = f(x) - g(x)$. We will prove that h is continuous and that $h(x) = 0$ for all $x \in A$. It will then follow from part (a) that $h(x) = 0$ for all $x \in \mathbb{R}$, implying that $f(x) = g(x)$ for all $x \in \mathbb{R}$. Let's begin.

Since f, g are continuous by Theorem 9.10, f, g are continuous at every $x \in \mathbb{R}$. Thus, by consecutive applications of Corollary 11.10, $-g$ is continuous at every $x \in \mathbb{R}$, so $f - g$ is continuous at every $x \in \mathbb{R}$. Consequently, by Theorem 9.10 again, $h = f - g$ is continuous.

Since $f(x) = g(x)$ for all $x \in A$, it naturally follows that $h(x) = f(x) - g(x) = 0$ for all $x \in A$.

Since h is continuous and $h(x) = 0$ for all $x \in A$, part (a) asserts that $h(x) = 0$ for all $x \in \mathbb{R}$. Thus, $f(x) - g(x) = 0$ for all $x \in \mathbb{R}$, meaning that $f(x) = g(x)$ for all $x \in \mathbb{R}$, as desired. □