Script 10

Compactness

10.1 Journal

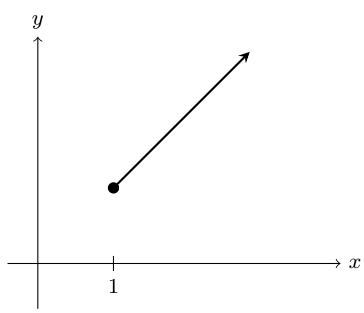
2/23: **Definition 10.1.** We say that a function $f : A \to \mathbb{R}$ is **bounded** if $f(A)$ is a bounded subset of $\mathbb{R}$. We say that f is **bounded above** if $f(A)$ is bounded above and that f is **bounded below** if $f(A)$ is bounded below.

If $f : A \to \mathbb{R}$ is bounded above, we say that f **attains** (its least upper bound) if there is some $a \in A$ such that $f(a) = \sup f(A)$. Similarly, if $f : A \to \mathbb{R}$ is bounded below, we say that f **attains** (its greatest lower bound) if there is some $a \in A$ such that $f(a) = \inf f(A)$.

Exercise 10.2. If possible, find examples of each of the following: a picture suffices.

a) A continuous function on $[1, \infty)$ that is not bounded above.

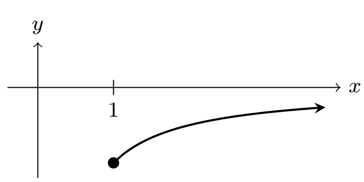
Example. Let $f : [1, \infty) \to \mathbb{R}$ be defined by $f(x) = x$.



□

b) A continuous function on $[1, \infty)$ that is bounded above but does not attain its least upper bound.

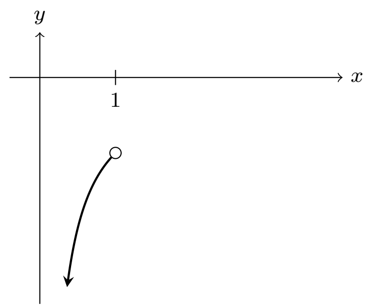
Example. Let $f : [1, \infty) \to \mathbb{R}$ be defined by $f(x) = -\frac{1}{x}$.



□

c) A continuous function on $(0, 1)$ that is not bounded below.

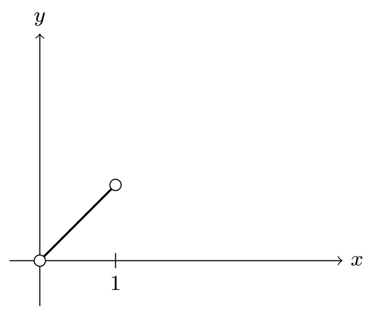
Example. Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) = -\frac{1}{x}$.



$\square$

d) A continuous function on $(0, 1)$ that is bounded below but does not attain its greatest lower bound.

Example. Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) = x$.



$\square$

Definition 10.3. Let X be a subset of $\mathbb{R}$ and let $\mathcal{G} = \{G_\lambda\}_{\lambda \in \Lambda}$ be a collection of subsets of $\mathbb{R}$. We say that $\mathcal{G}$ is a **cover** of X if every point of X is in some G_λ, or in other words:

$$X \subset \bigcup_{\lambda \in \Lambda} G_\lambda$$

We say that the collection $\mathcal{G}$ is an **open cover** if each G_λ is open.

Definition 10.4. Let X be a subset of $\mathbb{R}$. X is **compact** if for every open cover $\mathcal{G}$ of X, there exists a finite subset $\mathcal{G}' \subset \mathcal{G}$ that is also an open cover.

A good summary of the definition of compactness is "every open cover contains a finite subcover."

Exercise 10.5. Show that all finite subsets of $\mathbb{R}$ are compact.

Proof. Let X be an arbitrary finite subset of $\mathbb{R}$. To prove that X is compact, Definition 10.4 tells us that it will suffice to show that for every open cover $\mathcal{G}$ of X, there exists a finite subset $\mathcal{G}' \subset \mathcal{G}$ that is also an open cover. Let $\mathcal{G}$ be an arbitrary open cover of X. By Definition 10.3, every point $x \in X$ is an element of G_λ for some $G_\lambda \in \mathcal{G}$. Thus, for each $x \in X$, let $G_x \in \mathcal{G}$ be a set that contains x. Since X is finite, we do not need the axiom of choice to make these selections. Additionally, since there are finitely many X, we know that there are finitely many distinct G_x[1]. Thus, $\mathcal{G}' = \{G_x\}_{x \in X}$ is finite. Additionally, it is a subset of $\mathcal{G}$ by definition (each G_x is defined to be an element of $\mathcal{G}$). Furthermore, each G_x is open (again, each G_x is an element of $\mathcal{G}$, which is a collection of open sets by definition). Lastly, every point $x \in X$ is an element of $G_x \in \mathcal{G}'$, so $\mathcal{G}'$ is a cover. Therefore, by Definition 10.3, $\mathcal{G}' \subset \mathcal{G}$ is a finite open cover of X. $\square$

[1] In fact, the number of G_x is less than or equal to the cardinality of X since we may choose the same G_x for multiple x but may not choose multiple G_x for the same x.

2/25: **Lemma 10.6.** *No finite collection of regions covers* $\mathbb{R}$.

Lemma. *If* X *is nonempty, then* $\emptyset$ *does not cover* X.

Proof. Suppose for the sake of contradiction that $\emptyset$ covers X. By Definition 1.8, there exists $x \in X$. It follows by Definition 10.3 that $x \in \bigcup \emptyset$. But since $\bigcup \emptyset = \emptyset$, we have by Definition 1.2 that $x \in \emptyset$, contradicting Definition 1.8. ☐

Proof of Lemma 10.6. Suppose for the sake of contradiction that $\mathcal{G}$ is a finite collection of regions that covers $\mathbb{R}$. Since $\mathbb{R}$ is nonempty (by Axiom 1), the lemma asserts that $\mathcal{G} \neq \emptyset$. It follows that $\mathcal{G} = \{(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)\}$. Considering the set $\{a_1, \dots, a_n\}$ of lower bounds of all regions in $\mathcal{G}$, we can determine that it is nonempty and finite since $\mathcal{G}$ itself is nonempty and finite. Thus, by Lemma 3.4, $\{a_1, \dots, a_n\}$ has a first point a_i. It follows by Axiom 3 and Definition 3.3 that there exists a point $x \in \mathbb{R}$ such that $x < a_i$. Since $\mathcal{G}$ is an open cover of $\mathbb{R}$, we know by Definition 10.3 that $x \in (a_j, b_j)$ for some $(a_j, b_j) \in \mathcal{G}$. But this implies by Equations 8.1 that $a_j < x$ for some $a_j \in \{a_1, \dots, a_n\}$, contradicting the fact that $x < a_i \leq a_j$ for all $a_j \in \{a_1, \dots, a_n\}$ (the latter inequality being true by Definition 3.3). ☐

Theorem 10.7. $\mathbb{R}$ *is not compact.*

Proof. To prove that $\mathbb{R}$ is not compact, Definition 10.4 tells us that it will suffice to find an open cover $\mathcal{G}$ of $\mathbb{R}$ such that no finite subset $\mathcal{G}' \subset \mathcal{G}$ exists that is also an open cover. Let $\mathcal{G}$ be the collection of all regions in $\mathbb{R}$.

To confirm that $\mathcal{G}$ is an open cover, Definition 10.3 tells us that it will suffice to demonstrate that every $x \in \mathbb{R}$ is an element of R_λ for some region $R_\lambda \in \mathcal{G}$, and that every R_λ is open. For the first condition, let x be an arbitrary element of $\mathbb{R}$. Clearly, we have that $x \in (x-1, x+1)$ where $(x-1, x+1)$ is a region. Thus, x is an element of a set in $\mathcal{G}$, as desired. As to the other condition, we have by Corollary 4.11 that every region (i.e., every set in $\mathcal{G}$) is open, as desired.

To confirm that no finite subset $\mathcal{G}' \subset \mathcal{G}$ exists that is also an open cover, we invoke Lemma 10.6, which asserts as much[2]. ☐

Exercise 10.8. Show that regions are not compact.

Proof. Let (a, b) be an arbitrary region. To prove that (a, b) is not compact, Definition 10.4 tells us that it will suffice to find an open cover $\mathcal{G}$ of $\mathbb{R}$ such that no finite subset $\mathcal{G}' \subset \mathcal{G}$ exists that is also an open cover. Let $\mathcal{G}$ be the collection of all regions (a, c) where $c \in (a, b)$.

To confirm that $\mathcal{G}$ is an open cover, Definition 10.3 tells us that it will suffice to demonstrate that every $x \in (a, b)$ is an element of (a, c) for some $(a, c) \in \mathcal{G}$, and that every (a, c) is open. For the first condition, let x be an arbitrary element of (a, b). Then by Equations 8.1, $a < x < b$. It follows by Theorem 5.2 that there exists some c such that $x < c < b$. Since $a < x < c < b$, we have by consecutive applications of Equations 8.1 that $x \in (a, c)$ and $c \in (a, b)$. The latter result shows that $(a, c) \in \mathcal{G}$, as desired. As to the other condition, we have by Corollary 4.11 that every region (notably including all those in $\mathcal{G}$) is open, as desired.

To confirm that no finite subset $\mathcal{G}' \subset \mathcal{G}$ exists that is also an open cover, let $\mathcal{G}'$ be an arbitrary finite subset of $\mathcal{G}$. Since (a, b) is nonempty, the lemma from Lemma 10.6 asserts that $\mathcal{G}' \neq \emptyset$. It follows that $\mathcal{G}' = \{(a, c_1), (a, c_2), \dots, (a, c_n)\}$. Considering the set $\{c_1, \dots, c_n\}$ of upper bounds of all regions in $\mathcal{G}'$, we can determine that it is nonempty and finite since $\mathcal{G}'$ itself is nonempty and finite. Thus, by Lemma 3.4, $\{c_1, \dots, c_n\}$ has a last point c_i. Since $c_i \in (a, b)$, Equations 8.1 assert that $a < c_i < b$. Consequently, by Theorem 5.2, there exists a point $x \in \mathbb{R}$ such that $c_i < x < b$. Since $a < c_i < x < b$, we have by Equations 8.1 that $x \in (a, b)$. Since $\mathcal{G}'$ is an open cover of (a, b), we know by Definition 10.3 that $x \in (a, c_j)$ for some $(a, c_j) \in \mathcal{G}'$. But this implies by Equations 8.1 that $x < c_j$ for some $c_j \in \{c_1, \dots, c_n\}$, contradicting the fact that $c_j \leq c_i < x$ (the latter inequality being true by Definition 3.3). ☐

[2] Technically, Lemma 10.6 forbids $\mathcal{G}'$ from being a cover, but a set that is not a cover cannot be an open cover by Definition 10.3.

Theorem 10.9. *If X is compact, then X is bounded.*

Proof. We divide into two cases ($X = \emptyset$ and $X \neq \emptyset$). Suppose first that $X = \emptyset$. Then if we let a, b be arbitrary elements of $\mathbb{R}$, it is vacuously true that $a \leq x$ for all $x \in X$ and $x \leq b$ for all $x \in X$. Therefore, by consecutive applications of Definition 5.6, a and b are lower and upper bounds of X, respectively, and thus X is bounded, as desired.

Now suppose that $X \neq \emptyset$. Let $\mathcal{G} = \{(x-1, x+1) \mid x \in X\}$. To confirm that $\mathcal{G}$ is an open cover of X, Definition 10.3 tells us that it will suffice to demonstrate that every $x \in X$ is an element of some set in $\mathcal{G}$, and that every set in $\mathcal{G}$ is open. For the first condition, let x be an arbitrary element of X. Clearly, we have that $x \in (x-1, x+1)$. Additionally, it follows from the fact that $x \in X$ that $(x-1, x+1) \in \mathcal{G}$. Thus, x is an element of a set in $\mathcal{G}$, as desired. As to the other condition, we have by Corollary 4.11 that every region (notably including all those in $\mathcal{G}$) is open, as desired.

Since $\mathcal{G}$ is an open cover of X and X is compact, Definition 10.4 asserts that there exists a finite subset $\mathcal{G}'$ of $\mathcal{G}$ that is also an open cover of X. Since X is nonempty and $\mathcal{G}'$ is a cover of X, we have by the lemma from Lemma 10.6 that $\mathcal{G}' \neq \emptyset$. It follows since $\mathcal{G}$ is a collection of regions that $\mathcal{G}' = \{(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)\}$. Considering the sets $\{a_1, \dots, a_n\}$ of lower bounds of all regions in $\mathcal{G}'$ and $\{b_1, \dots, b_n\}$ of upper bounds of all regions in $\mathcal{G}'$, we can determine that both are nonempty and finite since $\mathcal{G}'$ itself is nonempty and finite. Thus, by consecutive applications of Lemma 3.4, $\{a_1, \dots, a_n\}$ has a first point a_i and $\{b_1, \dots, b_n\}$ has a last point b_j. To confirm that a_i is a lower bound of X, Definition 5.6 tells us that it will suffice to demonstrate that $a_i \leq x$ for all $x \in X$. Let x be an arbitrary element of X. Then by Definition 10.3, $x \in (a_k, b_k)$ for some $(a_k, b_k) \in \mathcal{G}$. Thus, by Equations 8.1, $a_k < x < b_k$. Additionally, since a_i is the first point of $\{a_1, \dots, a_n\}$, Definition 3.3 asserts that $a_i \leq a_k$. Combining the last two results, we have by transitivity that $a_i < x$, which we may weaken to $a_i \leq x$, as desired. The proof that b_j is an upper bound is symmetric. Therefore, since X has a lower and an upper bound, Definition 5.6 implies that X is bounded. $\square$

Lemma 10.10. *Let $X \subset \mathbb{R}$ and $p \in \mathbb{R} \setminus X$. Then $\mathcal{G} = \{\text{ext}(a, b) \mid p \in (a, b)\}$ is an open cover of X.*

Proof. To prove that $\mathcal{G}$ is an open cover of X, Definition 10.3 tells us that it will suffice to show that every $x \in X$ is an element of some set in $\mathcal{G}$, and that every set in $\mathcal{G}$ is open. For the first condition, let x, p be arbitrary elements of $X, \mathbb{R} \setminus X$, respectively. Since x, p are elements of disjoint sets by Script 1, we know that $x \neq p$. Thus, we can apply Theorem 3.22 to learn that there exist disjoint regions (c, d) and (a, b) containing x and p, respectively. We now seek to verify that $x \in \text{ext}(a, b)$. To do so, Definition 3.15 tells that it will suffice to verify that $x \notin (a, b)$, $x \neq a$, and $x \neq b$. First, suppose for the sake of contradiction that $x \in (a, b)$. Then since $x \in (c, d)$, too, $x \in (a, b) \cap (c, d)$, contradicting Definition 1.9 and the fact that $(a, b), (c, d)$ are disjoint. Second, suppose for the sake of contradiction that $x = a$. Since $x \in (c, d)$, we have by Equations 8.1 that $c < x = a < d$. We divide into two cases ($d \leq b$ and $b < d$). If $d \leq b$, then by Theorem 5.2, we can choose z such that $c < x = a < z < d \leq b$. It follows by the same logic as in the first case that $z \in (a, b) \cap (c, d)$, and we arrive at the same contradiction. If $b < d$, then we similarly choose $c < x = a < z < b < d$, and arrive at the same contradiction again. The proof of the third claim is symmetric to that of the second. Therefore, $x \in \text{ext}(a, b)$, so we have by the definition of $\mathcal{G}$ that x is an element of a set in $\mathcal{G}$, as desired. As to the other condition, we have by Corollary 4.21 that every exterior of a region (notably including all those in $\mathcal{G}$) is open, as desired. $\square$

Theorem 10.11. *If X is compact, then X is closed.*

Proof. We divide into two cases ($X = \emptyset$ and $X \neq \emptyset$). Suppose first that $X = \emptyset$. Then by Theorem 4.2, X is closed, as desired.

Now suppose that $X \neq \emptyset$, and suppose for the sake of contradiction that X is not closed. Then by Definition 4.1, there exists a limit point p of X such that $p \notin X$. Since $p \in \mathbb{R}$ and $p \notin X$, Definition 1.11 implies that $p \in \mathbb{R} \setminus X$. Thus, by Lemma 10.10, $\mathcal{G} = \{\text{ext}(a, b) \mid p \in (a, b)\}$ is an open cover of X. Additionally, since X is compact by hypothesis, we have by Definition 10.4 that there exists a finite subset $\mathcal{G}' \subset \mathcal{G}$ that is also an open cover of X. Since X is nonempty and $\mathcal{G}'$ is a cover of X, we have by the lemma from Lemma 10.6 that $\mathcal{G}' \neq \emptyset$. It follows since $\mathcal{G}$ is a collection of exteriors of regions that $\mathcal{G}' = \{\text{ext}(a_1, b_1), \text{ext}(a_2, b_2), \dots, \text{ext}(a_n, b_n)\}$. Considering the sets $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$, we can determine that both are nonempty and finite since $\mathcal{G}'$ itself is nonempty and finite. Thus, by consecutive applications of Lemma 3.4, $\{a_1, \dots, a_n\}$ has a last point a_i and $\{b_1, \dots, b_n\}$ has a first point b_j. We now seek to verify that $p \in (a_i, b_j)$. By consecutive applications of the definition of $\mathcal{G}$, $p \in (a_i, b_i)$ and $p \in (a_j, b_j)$. Consequently, by consecutive applications of Equations 8.1, $a_i < p < b_i$ and $a_j < p < b_j$. Since $a_i < p < b_j$, we have by Equations 8.1 that $p \in (a_i, b_j)$, as desired. Thus, since $p \in (a_i, b_j)$ and $p \in LP(X)$, Definition 3.13 asserts that $(a_i, b_j) \cap (X \setminus \{p\}) \neq \emptyset$. Consequently, by Definition 1.8, there exists some $x \in (a_i, b_j) \cap (X \setminus \{p\})$. Thus, by Definitions 1.6 and 1.11, $x \in (a_i, b_j)$ and $x \in X$. Since $\mathcal{G}'$ is an open cover of X, it follows from the latter condition by Definition 10.3 that $x \in \text{ext}(a_k, b_k)$ for some $\text{ext}(a_k, b_k) \in \mathcal{G}'$. Consequently, by Lemma 3.16, $x < a_k$ or $b_k < x$. We now divide into two cases. If $x < a_k$, this contradicts the fact that $a_k \leq a_i < x$ (the former inequality being true by Definition 3.3 since a_i is the last point of $\{a_1, \dots, a_n\}$). If $b_k < x$, we arrive at a symmetric contradiction. □

It will turn out that the two properties of compactness in Theorems 10.9 and 10.11 characterize compact sets completely, meaning that every bounded closed set is compact. We will see this in Theorem 10.16. First, however, we need some preliminary results.

For the next three statements, fix points $a, b \in \mathbb{R}$ and suppose $\mathcal{G}$ is an open cover of $[a, b]$.

Lemma 10.12. *For all $s \in [a, b]$, there exists $G \in \mathcal{G}$ and $p, q \in \mathbb{R}$ such that $p < s < q$ and $[p, q] \subset G$.*

Proof. Let s be an arbitrary element of $[a, b]$. Since $\mathcal{G}$ is an open cover of $[a, b]$, Definition 10.3 implies that there exists a $G \in \mathcal{G}$ such that $s \in G$ and G is open. It follows from the latter condition by Theorem 4.10 that there exists a region (x, y) such that $s \in (x, y)$ and $(x, y) \subset G$. Since $s \in (x, y)$, we have by Equations 8.1 that $x < s < y$. Thus, by consecutive applications of Theorem 5.2, we can pick $p, q \in \mathbb{R}$ such that $x < p < s < q < y$. Clearly, $p < s < q$. To verify that $[p, q] \subset G$, Definition 1.3 tells us that it will suffice to confirm that every $z \in [p, q]$ is an element of G. Let z be an arbitrary element of $[p, q]$. Then by Equations 8.1, $p \leq z \leq q$. It follows since $x < p < q < y$ that $x < p \leq z \leq q < y$, meaning by Equations 8.1 that $z \in (x, y)$. Since $(x, y) \subset G$ by definition, we have by Definition 1.3 that $z \in G$, as desired. □

3/2: **Lemma 10.13.** *Let X be the set of all $x \in \mathbb{R}$ that are* ***reachable*** *from a, by which we mean the following: there exist $n \in \mathbb{N} \cup \{0\}$, $x_0, ..., x_n \in \mathbb{R}$, and $G_1, \dots, G_n \in \mathcal{G}$ such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = x$ and $[x_{i-1}, x_i] \subset G_i$ for $i = 1, \dots, n$. Note in particular that $a \in X$, by choosing $n = 0$. Then the point b is not an upper bound for X.*

Proof. Suppose for the sake of contradiction that b is an upper bound of X. We know by hypothesis that $a \in X$. Thus, X is a nonempty set that is bounded above, so by Theorem 5.17, $\sup X$ exists. Let $s = \sup X$. Since b is an upper bound of X, we have by Definition 5.7 that $s \leq b$. Additionally, since $a \in X$, we have by Definitions 5.7 and 5.6 that $a \leq s$. Combining these last two results, we have by Equations 8.1 that $s \in [a, b]$. Thus, by Lemma 10.12, there exists $G \in \mathcal{G}$ and $p, q \in \mathbb{R}$ such that $p < s < q$ and $[p, q] \subset G$. Additionally, we have by Lemma 5.11 that there exists an $x \in X$ such that $p < x \leq s$. Since $p < x \leq s < q$, we have by Equations 8.1 that $x \in [p, q]$. It follows by Definition 1.3 since $[p, q] \subset G$ that $x \in G$. Furthermore, by Theorem 5.2, there exists $s' \in \mathbb{R}$ such that $s < s' < q$. We will now demonstrate that $s' \in X$, which will contradict the previously proven statement that s is an upper bound on X.

To demonstrate that $s' \in X$, the definition of X tells us that it will suffice to confirm that there exist $n + 1 \in \mathbb{N} \cup \{0\}$, $x_0, \dots, x_{n+1} \in \mathbb{R}$, and $G_1, \dots, G_{n+1} \in \mathcal{G}$ such that $a = x_0 < \cdots < x_{n+1} = s'$ and $[x_{i-1}, x_i] \subset G_i$ for $i = 1, \dots, n + 1$. To begin, since $x \in X$, we have that there exist $n \in \mathbb{N} \cup \{0\}$, $x_0, \dots, x_n \in \mathbb{R}$, and $G_1, \dots, G_n \in \mathcal{G}$ such that $a = x_0 < \cdots < x_n = x$ and $[x_{i-1}, x_i] \subset G_i$ for $i = 1, \dots, n$. Let $x_{n+1} = s'$ and $G_{n+1} = G$. Then since we can carry over all of the variable assignments from the definition of x and add in $x_n = x \leq s < s' = x_{n+1}$ as well as $[x_n, x_{n+1}] = [x, s'] \subset [p, q] \subset G = G_{n+1}$, we have that $s' \in X$, as desired. □

Theorem 10.14. *The set $[a, b]$ is compact.*

Proof. To prove that $[a, b]$ is compact, Definition 10.4 tells us that it will suffice to show that for every open cover $\mathcal{G}$ of $[a, b]$, there exists a finite subset $\mathcal{G}' \subset \mathcal{G}$ that is also an open cover. Let $\mathcal{G}$ be an arbitrary open cover of $[a, b]$. By Lemma 10.13, b is not an upper bound of X where X is the set of all $x \in \mathbb{R}$ that are reachable from a. Thus, by Definition 5.6, there exists an $x \in X$ such that $x > b$. Since $x \in X$, we have by Lemma 10.13 that there exist $n \in \mathbb{N} \cup \{0\}$, $x_0, \dots, x_n \in \mathbb{R}$, and $G_1, \dots, G_n \in \mathcal{G}$ such that $a = x_0 < \cdots < x_n = x$ and $[x_{i-1}, x_i] \subset G_i$ for $i = 1, \dots, n$. Let $\mathcal{G}' = \{G_1, \dots, G_n\}$. We will now show that $\mathcal{G}'$ is finite, a subset of $\mathcal{G}$, and an open cover of $[a, b]$.

Clearly, $\mathcal{G}'$ is finite.

Since every element of $\mathcal{G}'$ is an element of $\mathcal{G}$ by definition, we have by Definition 1.3 that $\mathcal{G}' \subset \mathcal{G}$.

To show that $\mathcal{G}'$ is an open cover of $[a, b]$, Definition 10.3 tells us that it will suffice to verify that every $y \in [a, b]$ is an element of G_i for some $G_i \in \mathcal{G}'$, and that every G_i is open. For the first condition, suppose for the sake of contradiction that there exists a $y \in [a, b]$ such that $y \notin G_i$ for any $G_i \in \mathcal{G}'$. Then since $[x_{i-1}, x_i] \subset G_i$ for all $i = 1, \ldots, n$, $y \notin [x_{i-1}, x_i]$ for any $i = 1, \ldots, n$. It follows from Equations 8.1 that $y < x_{i-1}$ or $y > x_i$ for all $i = 1, \ldots, n$. In particular, we have that $y < x_0 = a$ or $y > x_n = x$. We now divide into two cases. If $y < a$, then by Equations 8.1, $y \notin [a, b]$, a contradiction. If $y > x$, then since $x > b$, we have that $y > b$, which implies by Equations 8.1 that $y \notin [a, b]$, a contradiction. As to the other condition, we have by the definition of $\mathcal{G}$ that every G_i is open, as desired. □

Theorem 10.15. *A closed subset Y of a compact set $X \subset \mathbb{R}$ is compact.*

Lemma. *Y is closed in X if and only if Y is closed.*

Proof. Suppose first that Y is closed in X. Then by Definition 8.11, $Y = X \cap B$ where B is closed. Additionally, since X is compact, Theorem 10.11 asserts that X is closed. Therefore, since Y is the intersection of two closed sets, we have by Theorem 4.16 that Y is closed, as desired.

Now suppose that Y is closed. Since $Y = X \cap Y$ by Script 1, we have by Definition 8.11 that Y is closed in X, as desired. □

Proof of Theorem 10.15. To prove that Y is compact, Definition 10.4 tells us that it will suffice to show that for every open cover $\mathcal{G}$ of Y, there exists a finite subset $\mathcal{G}' \subset \mathcal{G}$ that is also an open cover. To do this, we will first demonstrate that $\mathcal{H} = \mathcal{G} \cup \{\mathbb{R} \setminus Y\}$ is an open cover of X. It will follow that there exists a finite subset $\mathcal{H}' \subset \mathcal{H}$ that is an open cover of X. Lastly, we will demonstrate that $\mathcal{G}' = \mathcal{H}' \setminus \{\mathbb{R} \setminus Y\}$ is the desired finite open cover subset of $\mathcal{G}$.

Let $\mathcal{G}$ be an arbitrary open cover of Y, and let $\mathcal{H} = \mathcal{G} \cup \{\mathbb{R} \setminus Y\}$. To demonstrate that $\mathcal{H}$ is an open cover of X, Definition 10.3 tells us that it will suffice to confirm that every $x \in X$ is an element of H for some $H \in \mathcal{H}$, and that every H is open. For the first condition, let x be an arbitrary element of X. We divide into two cases ($x \in Y$ and $x \notin Y$). If $x \in Y$, then since $\mathcal{G}$ is an open cover of Y, Definition 10.3 implies that $x \in G$ for some $G \in \mathcal{G}$. But since $\mathcal{G} \subset \mathcal{H}$, $x \in G$ for some $G \in \mathcal{H}$, as desired. On the other hand, if $x \notin Y$, then this combined with the fact that $x \in \mathbb{R}$ implies by Definition 1.11 that $x \in \mathbb{R} \setminus Y \in \mathcal{H}$, as desired. As to the other condition, let H be an arbitrary element of $\mathcal{H}$. We divide into two cases ($H \in \mathcal{G}$ and $H \notin \mathcal{G}$). If $H \in \mathcal{G}$, then by Definition 10.3, H is open, as desired. On the other hand, if $H \notin \mathcal{G}$, then $H = \mathbb{R} \setminus Y$ by Script 1. It follows since Y is closed by Definition 4.8 that $\mathbb{R} \setminus Y$ is open, so H is open, as desired.

Since $\mathcal{H}$ is an open cover of X and X is compact, we have by Definition 10.4 that there exists a finite subset $\mathcal{H}' \subset \mathcal{H}$ that is also an open cover of X. Let $\mathcal{G}' = \mathcal{H}' \setminus \{\mathbb{R} \setminus Y\}$. By Script 1, $\mathcal{G}' \subset \mathcal{G}$. Additionally, since $\mathcal{H}'$ is finite, Script 1 also implies that $\mathcal{G}'$ is finite. However, to demonstrate that $\mathcal{G}'$ is an open cover of Y, Definition 10.3 tells us that we must confirm that every $y \in Y$ is an element of G for some $G \in \mathcal{G}'$, and that every G is open. For the first condition, let y be an arbitrary element of Y. Then since $Y \subset X$, Definition 10.3 asserts that $y \in H$ for some $H \in \mathcal{H}'$. Additionally, since $y \in Y$, Definition 1.11 implies that $y \notin \mathbb{R} \setminus Y$. Thus, $H \neq \mathbb{R} \setminus Y$, which implies by Definition 1.11 that $H \in \mathcal{H}' \setminus \{\mathbb{R} \setminus Y\} = \mathcal{G}'$. Therefore, $y \in H$ for some $H \in \mathcal{G}'$, as desired. As to the other condition, since every $G \in \mathcal{G}'$ is an element of $\mathcal{G}$ (i.e., open by Definition 10.3), every $G \in \mathcal{G}'$ is open, as desired. □

Theorem 10.16. *Let $X \subset \mathbb{R}$. X is compact if and only if X is closed and bounded.*

Proof. Suppose first that X is compact. Then by Theorems 10.11 and 10.9, respectively, X is closed and bounded.

Now suppose that X is closed and bounded. It follows from the latter condition by Definition 5.6 that X has a lower bound a and an upper bound b. Constructing the region $[a, b]$, we have from Theorem 10.14 that $[a, b]$ is compact. Additionally, we know that $X \subset [a, b]$ since $x \in X$ implies by consecutive applications of Definition 5.6 that $a \leq x \leq b$, from which it follows by Equations 8.1 that $x \in [a, b]$. Thus, X is a closed (by hypothesis) subset of a compact set, so by Theorem 10.15, X is compact. □

Lemma 10.17. *A compact set $X \subset \mathbb{R}$ with no limit points must be finite.*

Proof. Suppose for the sake of contradiction that X is infinite, and let x be an arbitrary element of X. Since X has no limit points, we know that $x \notin LP(X)$. Thus, by Definition 3.13, there exists a region R_x with $x \in R_x$ such that $R_x \cap (X \setminus \{x\}) = \emptyset$. Let $\mathcal{G} = \bigcup_{x \in X} R_x$, where R_x is similarly defined for each $x \in X$. Then since every $x \in R_x \in \mathcal{G}$ and every R_x is open by Corollary 4.11, Definition 10.3 implies that $\mathcal{G}$ is an open cover of X. It follows since X is compact that there exists a finite subset of $\mathcal{G}' \subset \mathcal{G}$ that is also an open cover of X. Additionally, since $\mathcal{G}'$ is finite whereas $\mathcal{G}$ is infinite (infinitely many x imply infinitely many R_x since each R_x contains only one x), we know that $\mathcal{G}' \neq \mathcal{G}$. Thus, there exists an $R_x \in \mathcal{G}$ such that $R_x \notin \mathcal{G}'$. However, since $\mathcal{G}'$ is an open cover of X, Definition 10.3 implies that $x \in R_y$ for some $R_y \in \mathcal{G}'$ where $y \neq x$. But since $x \in R_y$, $x \in X$, and $x \neq y$, it follows by Script 1 that $R_y \cap (X \setminus \{y\}) \neq \emptyset$, a contradiction. $\square$

Theorem 10.18. *Every bounded infinite subset of $\mathbb{R}$ has at least one limit point.*

Proof. Suppose for the sake of contradiction that X is a bounded, infinite subset of $\mathbb{R}$ with no limit points. Since X has no limit points, it is vacuously true that it contains all of its limit points. Thus, by Definition 4.1, X is closed. This combined with the fact that X is bounded implies by Theorem 10.16 that X is compact. But this combined with the fact that X has no limit points implies by Lemma 10.17 that X is finite, a contradiction. $\square$

3/4: **Theorem 10.19.** *Suppose that $f : X \to \mathbb{R}$ is continuous. If X is compact, the $f(X)$ is compact.*

Proof. To prove that $f(X)$ is compact, Definition 10.4 tells us that it will suffice to show that for every open cover $\mathcal{G}$ of $f(X)$, there exists a finite subset $\mathcal{G}' \subset \mathcal{G}$ that is also an open cover. Let $\mathcal{G}$ be an arbitrary open cover of $f(X)$. Then Definition 10.3 implies that for every point $f(x) \in f(X)$, we have that $f(x) \in G_{f(x)}$ where $G_{f(x)}$ is an element of $\mathcal{G}$. It follows since every $G_{f(x)} \in \mathcal{G}$ by Definition 10.3 that every $G_{f(x)}$ is open. Thus, by Definition 9.4, every $f^{-1}(G_{f(x)})$ is open in X. Consequently, by Definition 8.11, every $f^{-1}(G_{f(x)}) = X \cap A_x$ for some open set A_x.

We will now show that $\mathcal{H} = \{A_x\}_{x \in X}$ is an open cover of X. Since every $f(x) \in G_{f(x)}$, Definition 1.18 implies that every $x \in f^{-1}(G_{f(x)})$. It follows by Definitions 1.2 and 1.6 that every $x \in A_x$. But since every $x \in X$ is an element of $A_x \in \mathcal{H}$ and each A_x is open by definition, Definition 10.3 implies that $\mathcal{H}$ is an open cover of X.

It follows since X is compact by Definition 10.4 that there exists a finite subset $\mathcal{H}' \subset \mathcal{H}$ that is also an open cover of X. Since $\mathcal{H}'$ is finite, we know that $\mathcal{H}' = \{A_{x_1}, \dots, A_{x_n}\}$ where $x_1, \dots, x_n$ are elements of X. Let $\mathcal{G}' = \{G_{f(x_1)}, \dots, G_{f(x_n)}\}$. By Script 1, $\mathcal{G}' \subset \mathcal{G}$. Script 1 also implies that $\mathcal{G}'$ is finite. However, to demonstrate that $\mathcal{G}'$ is an open cover of $f(X)$, Definition 10.3 tells us that we must confirm that every $f(x) \in f(X)$ is an element of G for some $G \in \mathcal{G}'$, and that every G is open. For the first condition, let $f(x)$ be an arbitrary element of $f(X)$. Then since $x \in X$, $x \in A_{x_i}$ for some $A_{x_i} \in \mathcal{H}'$. It follows by the definition of A_{x_i} that $x \in f^{-1}(G_{f(x_i)})$. Thus, by Definition 1.18, $f(x) \in G_{f(x_i)}$. Therefore, $f(x)$ is an element of an element of $\mathcal{G}'$, as desired. As to the other condition, since every $G \in \mathcal{G}'$ is an element of $\mathcal{G}$ (i.e., open by Definition 10.3), every $G \in \mathcal{G}'$ is open, as desired. $\square$

Corollary 10.20. *If $X \subset \mathbb{R}$ is nonempty, closed, and bounded and $f : X \to \mathbb{R}$ is continuous, then $f(X)$ has a first and last point.*

Proof. Since X is closed and bounded, we have by Theorem 10.16 that X is compact. This result combined with the fact that f is continuous implies by Theorem 10.19 that $f(X)$ is compact. Thus, by Theorem 10.16 again, $f(X)$ is closed and bounded. Additionally, since X is nonempty, there exists $x \in X$. Consequently, $f(x) \in f(X)$, so $f(X)$ is nonempty. Therefore, since $f(X)$ is nonempty, closed, and bounded, we have by Corollary 5.18 that $f(X)$ has a first and last point. $\square$

Exercise 10.21. Use Corollary 10.20 to prove that if $f : [a, b] \to \mathbb{R}$ is continuous, then there exists a point $c \in [a, b]$ such that $f(c) \geq f(x)$ for all $x \in [a, b]$. Similarly, there exists a point $d \in [a, b]$ such that $f(d) \leq f(x)$ for all $x \in [a, b]$.

Proof. By Equations 8.1, $a \in [a, b]$, so $[a, b]$ is nonempty. By Corollaries 5.15 and 4.7, $[a, b]$ is closed. Since a, b are lower and upper bounds on $[a, b]$, respectively ($x \in [a, b]$ implies $a \leq x$ and $x \leq b$), Definition 5.6 asserts that $[a, b]$ is bounded. These three results combined with the fact that f is continuous imply by Corollary 10.20 that $f([a, b])$ has a first and last point. With respect to the first point, by Definition 3.3, there exists some $f(c) \in f([a, b])$ such that $f(c) \geq f(x)$ for all $f(x) \in f([a, b])$. By consecutive applications of Definition 1.18, $f(c) \in f([a, b])$ implies that $c \in [a, b]$, and $x \in [a, b]$ implies that $f(x) \in f([a, b])$, i.e., $f(c) \geq f(x)$. Therefore, there exists a point $c \in [a, b]$ such that $f(c) \geq f(x)$ for all $x \in [a, b]$. The argument is symmetric with respect to the last point. $\square$