

# Script 6

## Construction of the Real Numbers

### 6.1 Journal

1/12: **Definition 6.1.** A subset  $A$  of  $\mathbb{Q}$  is said to be a **cut** (or **Dedekind cut**) if it satisfies the following:

- (a)  $A \neq \emptyset$  and  $A \neq \mathbb{Q}$ .
- (b) If  $r \in A$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in A$ .
- (c)  $A$  does not have a last point; i.e., if  $r \in A$ , then there is some  $s \in A$  with  $s > r$ .

We denote the collection of all cuts by  $\mathbb{R}$ .

**Lemma 6.2.** Let  $A$  be a Dedekind cut and  $x \in \mathbb{Q}$ . Then  $x \notin A$  if and only if  $x$  is an upper bound for  $A$ .

*Proof.* Suppose first that  $x \notin A$ . To prove that  $x$  is an upper bound for  $A$ , Definition 5.6 tells us that it will suffice to show that for all  $r \in A$ ,  $r \leq x$ . Let  $r$  be an arbitrary element of  $A$ . Then by the contrapositive of Definition 6.1b and the hypothesis that  $x \notin A$ , we know that  $r \notin A$ ,  $x \notin \mathbb{Q}$ , or  $x \not\leq r$ . But since  $r \in A$  and  $x \in \mathbb{Q}$ , it must be that  $x \not\leq r$ . Therefore,  $r \leq x$ , as desired.

Now suppose that  $x$  is an upper bound for  $A$ . By Definition 5.6, this implies that for all  $r \in A$ ,  $r \leq x$ . Therefore, since there is no  $r \in A$  with  $r > x$ , by the contrapositive of Definition 6.1c,  $x \notin A$ , as desired.  $\square$

#### Exercise 6.3.

- (a) Prove that for any  $q \in \mathbb{Q}$ ,  $\{x \in \mathbb{Q} \mid x < q\}$  is a Dedekind cut. We then define  $\mathbf{0} = \{x \in \mathbb{Q} \mid x < 0\}$ .
- (b) Prove that  $\{x \in \mathbb{Q} \mid x \leq 0\}$  is not a Dedekind cut.
- (c) Prove that  $\{x \in \mathbb{Q} \mid x < 0\} \cup \{x \in \mathbb{Q} \mid x^2 < 2\}$  is a Dedekind cut.

*Proof of a.* Let  $q$  be an arbitrary element of  $\mathbb{Q}$ . To prove that  $A = \{x \in \mathbb{Q} \mid x < q\}$  is a Dedekind cut, Definition 6.1 tells us that it will suffice to show that  $A \neq \emptyset$ ;  $A \neq \mathbb{Q}$ ; if  $r \in A$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in A$ ; and if  $r \in A$ , then there is some  $s \in A$  with  $s > r$ . We will take this one claim at a time.

To show that  $A \neq \emptyset$ , Definition 1.8 tells us that it will suffice to find an element of  $A$ . By Exercise 3.9d,  $q$  is not the first point of  $\mathbb{Q}$ . Thus, by Definition 3.3, there exists an object  $x \in \mathbb{Q}$  such that  $x < q$ . By the definition of  $A$ , this implies that  $x \in A$ , as desired.

To show that  $A \neq \mathbb{Q}$ , Definition 1.2 tells us that it will suffice to find an element of  $\mathbb{Q}$  that is not an element of  $A$ . By hypothesis,  $q \in \mathbb{Q}$ . By Exercise 3.9d,  $q \not< q$ . Therefore,  $q \in \mathbb{Q}$  but  $q \notin A$ , as desired.

To show that if  $r \in A$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in A$ , we let  $r \in A$  and  $s \in \mathbb{Q}$  be arbitrary elements of their respective sets that satisfy  $s < r$  and seek to verify that  $s \in A$ . Since  $r \in A$ ,  $r < q$ . This combined with the fact that  $s < r$  implies by transitivity that  $s < q$ . Therefore, since  $s \in \mathbb{Q}$  and  $s < q$ ,  $s \in A$ , as desired.

To show that if  $r \in A$ , then there is some  $s \in A$  with  $s > r$ , we let  $r \in A$  and seek to find such an  $s$ . By the definition of  $A$ ,  $r < q$ . Thus, by Additional Exercise 3.1, there exists a point  $s \in \mathbb{Q}$  such that  $r < s < q$ . Since  $s \in \mathbb{Q}$  and  $s < q$ ,  $s \in A$ . It follows that  $s$  is the desired element of  $A$  satisfying  $s > r$ .  $\square$

*Proof of b.* To prove that  $A = \{x \in \mathbb{Q} \mid x \leq 0\}$  is not a Dedekind cut, Definition 6.1 tells us that it will suffice to show that  $A$  does have a last point. To show this, we will demonstrate that 0 is the last point of  $A$ . To demonstrate this, Definition 3.1 tells us that it will suffice to confirm that  $0 \in A$  and for all  $x \in A$ ,  $x \leq 0$ . Since  $0 \leq 0$  and  $0 \in \mathbb{Q}$ ,  $0 \in A$ . Additionally, by the definition of  $A$ , it is true that for all  $x \in A$ ,  $x \leq 0$ .  $\square$

*Proof of c.* Let  $B = \{x \in \mathbb{Q} \mid x < 0\}$  and let  $C = \{x \in \mathbb{Q} \mid x^2 < 2\}$ . To prove that  $A = B \cup C$  is a Dedekind cut, Definition 6.1 tells us that it will suffice to show that  $A \neq \emptyset$ ;  $A \neq \mathbb{Q}$ ; if  $r \in A$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in A$ ; and if  $r \in A$ , then there is some  $s \in A$  with  $s > r$ . We will take this one claim at a time.

To show that  $A \neq \emptyset$ , Definition 1.8 tells us that it will suffice to find an element of  $A$ . Since  $-1 \in \mathbb{Q}$  and  $-1 < 0$ ,  $-1 \in B$ . Therefore, by Definition 1.5,  $-1 \in A$ , as desired.

To show that  $A \neq \mathbb{Q}$ , Definition 1.2 tells us that it will suffice to find an element of  $\mathbb{Q}$  that is not an element of  $A$ . Since  $2 \in \mathbb{Q}$  and  $2 \geq 0$ ,  $2 \notin B$ . Additionally, since  $2^2 \geq 2$ ,  $2 \notin C$ . Therefore, by Definition 1.5,  $2 \notin A$ , as desired.

To show that if  $r \in A$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in A$ , we let  $r \in A$  and  $s \in \mathbb{Q}$  be arbitrary elements of their respective sets that satisfy  $s < r$  and seek to verify that  $s \in A$ . Since  $r \in A$ , Definition 1.5 tells us that  $r \in B$  or  $r \in C$ . We now divide into two cases. Suppose first that  $r \in B$ . Then  $s < r < 0$ , which implies that  $s \in B$ , meaning that  $s \in A$ . Now suppose that  $r \in C$ . We divide into two cases again ( $r \leq 0$  and  $r > 0$ ). If  $r \leq 0$ , then  $s < r \leq 0$  implies that  $s < 0$ . Thus, by the definition of  $B$ ,  $s \in B$ , implying that  $s \in A$ . On the other hand, if  $r > 0$ , then  $0 < s^2 < r^2 < 2$ . Thus, by the definition of  $C$ ,  $s \in C$ , implying that  $s \in A$ .

To show that  $A$  does not have a last point, suppose for the sake of contradiction that  $A$  has a last point  $p$ . We now divide into two cases ( $p \leq 0$  and  $p > 0$ ). Suppose first that  $p \leq 0$ . Since  $p$  is the last point of  $A$ , Definition 3.3 tells us that  $x \leq p$  for all  $x \in A$ . But  $1 \in A$  (since  $1 \in \mathbb{Q}$  and  $1^2 = 1 < 2$  implies  $1 \in B$ , implies  $1 \in A$ ) and  $1 > 0 \geq p$ , a contradiction. Now suppose that  $p > 0$ . Definition 3.3 tells us that  $p \in A$ , but the condition that  $p > 0$  means  $p \notin B$ , so we must have  $p \in C$ . However, by the proof of Exercise 4.24,  $\frac{2(p+1)}{p+2}$  will be an element of  $B$  (and therefore  $A$ ) that is greater than  $p$  no matter how large  $p$  is, a contradiction.  $\square$

**Definition 6.4.** If  $A, B \in \mathbb{R}$ , we say that  $A < B$  if  $A$  is a proper subset of  $B$ .

**Exercise 6.5.** Show that  $\mathbb{R}$  satisfies Axioms 1, 2, and 3.

*Proof.* By Exercise 6.3a,  $\{x \in \mathbb{Q} \mid x < 0\} \in \mathbb{R}$  since  $0 \in \mathbb{Q}$ . Therefore, Axiom 1 is immediately satisfied.

Axiom 2 asserts that  $\mathbb{R}$  must have an ordering  $<$ . As such, it will suffice to verify that the ordering given by Definition 6.4 satisfies the stipulations of Definition 3.1. To prove that  $<$  satisfies the trichotomy, it will suffice to show that for all  $A, B \in \mathbb{R}$ , exactly one of the following holds:  $A < B$ ,  $B < A$ , or  $A = B$ .

We first show that *no more than one* of the three statements can simultaneously be true. Let  $A, B$  be arbitrary elements of  $\mathbb{R}$ . We divide into three cases. First, suppose for the sake of contradiction that  $A < B$  and  $B < A$ . By Definition 6.4, this implies that  $A \subsetneq B$  and  $B \subsetneq A$ . Thus, by Definition 1.3,  $A \subset B$ ,  $B \subset A$ , and  $A \neq B$ . But by Theorem 1.7,  $A \subset B$  and  $B \subset A$  implies that  $A = B$ , a contradiction. Second, suppose for the sake of contradiction that  $A < B$  and  $A = B$ . By Definition 6.4, the former statement implies that  $A \subsetneq B$ . Thus, by Definition 1.3,  $A \neq B$ , a contradiction. The proof of the third case ( $B < A$  and  $A = B$ ) is symmetric to that of the second case.

We now show that *at least one* of the three statements is always true. Let  $A, B$  be arbitrary elements of  $\mathbb{R}$ , and suppose for the sake of contradiction that  $A \not< B$ ,  $B \not< A$ , and  $A \neq B$ . Since  $A \not< B$  and  $B \not< A$ , we have by Definition 6.4 that  $A \not\subsetneq B$  and  $B \not\subsetneq A$ . Thus, by Definition 1.3,  $A \not\subset B$  or  $A = B$ , and  $B \not\subset A$  or  $A = B$ . But  $A \neq B$  by hypothesis, so it must be that  $A \not\subset B$  and  $B \not\subset A$ . It follows from the first statement by Definition 1.3 that there exists an object  $x \in A$  such that  $x \notin B$ , and there exists an object  $y \in B$  such that  $y \notin A$ . Since  $x \notin B$ , Lemma 6.2 implies that  $x$  is an upper bound of  $B$ . Consequently, by Definition 5.6,  $p \leq x$  for all  $p \in B$ , including  $y$ . Similarly,  $p \leq y$  for all  $p \in A$ , including  $x$ . Thus, we have  $y \leq x$  and  $x \leq y$ , implying that  $x = y$ . But since  $y \in B$ , this implies that  $x \in B$ , a contradiction.

To prove that  $<$  is transitive, it will suffice to show that for all  $A, B, C \in \mathbb{R}$ , if  $A < B$  and  $B < C$ , then  $A < C$ . Let  $A, B, C$  be arbitrary elements of  $\mathbb{R}$  for which it is true that  $A < B$  and  $B < C$ . By Definition 6.4, we have  $A \subsetneq B$  and  $B \subsetneq C$ . Thus, by Script 1,  $A \subsetneq C$ . Therefore, by Definition 6.4,  $A < C$ .

Axiom 3 asserts that  $\mathbb{R}$  must have no first or last point. We will take this one argument at a time

Suppose for the sake of contradiction that  $\mathbb{R}$  has some first point  $A$ . Then by Definition 3.3,  $A \leq X$  for every  $X \in \mathbb{R}$ . Now since  $A$  is a Dedekind cut, Definition 6.1 tells us that  $A \neq \emptyset$ . Thus, by Definition 1.8, there exists some  $q \in A$ . Additionally,  $A \subset \mathbb{Q}$  by Definition 6.1, so  $q \in A$  implies that  $q \in \mathbb{Q}$ . It follows by Exercise 6.3a that  $B = \{x \in \mathbb{Q} \mid x < q\}$  is a Dedekind cut. We now seek to prove that  $B \subsetneq A$ . To do this, Definition 1.3 tells us that it will suffice to show that  $B \neq A$  and  $B \subset A$ . To show that  $B \neq A$ , Definition 1.2 tells us that it will suffice to find an element of  $A$  that is not an element of  $B$ . Conveniently,  $q$  is clearly such an object. To show that  $B \subset A$ , Definition 1.3 tells us that we must confirm that every element of  $B$  is an element of  $A$ . Let  $p$  be an arbitrary element of  $B$ . Then by the definition of  $B$ ,  $p \in \mathbb{Q}$  and  $p < q$ . It follows by Definition 6.1b (which clearly applies to  $A$ ) that  $p \in A$ , as desired. Having proven that  $B \subsetneq A$ , Definition 6.4 tells us that  $B < A$ . But this contradicts the previously demonstrated fact that  $A \leq X$  for every  $X \in \mathbb{R}$ , including  $B$ .

Suppose for the sake of contradiction that  $\mathbb{R}$  has some last point  $A$ . Then by Definition 3.3,  $X \leq A$  for every  $X \in \mathbb{R}$ . Now since  $A$  is a Dedekind cut, Definition 6.1 tells us that  $A \neq \mathbb{Q}$ . Thus, by Definition 1.2, there exists some  $q \in \mathbb{Q}$  such that  $q \notin A$ . It follows by Lemma 6.2 that  $q$  is an upper bound of  $A$ . Consequently, by Definition 5.6,  $x \leq q$  for all  $x \in A$ . Additionally, by Exercise 6.3a,  $B = \{x \in \mathbb{Q} \mid x < q + 1\}$ <sup>[1]</sup> is a Dedekind cut. We now seek to prove that  $A \subsetneq B$ . As before, this means we must show that  $A \neq B$  and  $A \subset B$ . To show that  $A \neq B$ , Definition 1.2 tells us that it will suffice to find an element of  $B$  that is not an element of  $A$ . Since  $x \leq q$  for all  $x \in A$  and  $q < q + 0.5 < q + 1$ ,  $q + 0.5 \notin A$  and  $q + 0.5 \in B$  is the desired object. To show that  $A \subset B$ , Definition 1.3 tells us that we must confirm that every element of  $A$  is an element of  $B$ . Let  $p$  be an arbitrary element of  $A$ . As an element of  $A$ , we know that  $p \leq q$ . Thus,  $p < q + 1$ , so  $p \in B$ , as desired. Having proven that  $A \subsetneq B$ , Definition 6.4 tells us that  $A < B$ . But this contradicts the previously demonstrated fact that  $X \leq A$  for every  $X \in \mathbb{R}$ , including  $B$ .  $\square$

1/14: **Lemma 6.6.** *A nonempty subset of  $\mathbb{R}$  that is bounded above has a supremum.*

*Proof.* Let  $X$  be an arbitrary nonempty subset of  $\mathbb{R}$  that is bounded above. To prove that  $\sup X$  exists, we will show that  $\sup X = U = \bigcup\{Y \mid Y \in X\}$ . To show this, Definition 5.7 tells us that it will suffice to demonstrate that  $U \in \mathbb{R}$ ,  $U$  is an upper bound of  $X$ , and if  $U'$  is an upper bound of  $X$ , then  $U \leq U'$ . Let's begin.

To demonstrate that  $U \in \mathbb{R}$ , Definition 6.1 tells us that it will suffice to confirm that  $U \neq \emptyset$ ;  $U \neq \mathbb{Q}$ ; if  $r \in U$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in U$ ; and if  $r \in U$ , then there is some  $s \in U$  with  $s > r$ .

As the union of a nonempty subset of nonempty sets, Script 1 implies that  $U \neq \emptyset$ .

To demonstrate that  $U \neq \mathbb{Q}$ , Definition 1.2 tells us that it will suffice to find a point  $p \in \mathbb{Q}$  such that  $p \notin U$ . Since  $X$  is bounded above, we have by Definition 5.6 that there exists a Dedekind cut  $V \in \mathbb{R}$  such that  $Y \leq V$  for all  $Y \in X$ . It follows by Definition 6.4 that  $Y \subset V$  for all  $Y \in X$ . Thus, by Script 1,  $U \subset V$ . Now since  $V$  is a Dedekind cut, we know by Definition 6.1 that  $V \subset \mathbb{Q}$  and  $V \neq \mathbb{Q}$ , meaning that there exists a point  $p \in \mathbb{Q}$  such that  $p \notin V$ . Consequently, since  $U \subset V$ ,  $p \notin U$ , as desired.

To demonstrate that if  $r \in U$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in U$ , we let  $r \in U$  and  $s \in \mathbb{Q}$  be arbitrary elements of their respective sets that satisfy  $s < r$  and seek to verify that  $s \in U$ . Since  $r \in U$ , Definition 1.13 tells us that  $r \in Y$  for some  $Y \in X$ . Thus, since  $Y$  is a Dedekind cut,  $s \in \mathbb{Q}$  and  $s < r$  implies that  $s \in Y$ . Therefore,  $s \in U$ .

To demonstrate that if  $r \in U$ , then there is some  $s \in U$  with  $s > r$ , we let  $r \in U$  and seek to find such an  $s$ . Since  $r \in U$ , Definition 1.13 tells us that  $r \in Y$  for some  $Y \in X$ . Thus, since  $Y$  is a Dedekind cut, there exists a point  $s \in Y$  with  $s > r$ . Therefore,  $s \in U$ .

To demonstrate that  $U$  is an upper bound of  $X$ , Definition 5.6 tells us that it will suffice to confirm that  $Y \leq U$  for all  $Y \in X$ . To confirm this, Definition 6.4 tells us that it will suffice to verify that  $Y \subset U$  for all  $Y \in X$ . But by an extension of Theorem 1.7b, this is true.

Now suppose for the sake of contradiction that there exists an upper bound  $U'$  of  $X$  such that  $U' < U$ . It follows by Definitions 6.4 and 1.3 that there exists a point  $p \in U$  such that  $p \notin U'$ . Thus, by the former statement and Definition 1.13,  $p \in Y$  for some  $Y \in X$ . Additionally, since  $U'$  is an upper bound of  $X$ , we

<sup>1</sup>Note that we add 1 to  $q$  to treat the case that  $q = \sup A$ , a case in which we would have  $B = A$  if  $B$  were defined as  $\{x \in \mathbb{Q} \mid x < q\}$ .

have by Definitions 5.6 and 6.4 that  $Y \subset U'$  for all  $Y \in X$ . But this implies by Definition 1.3 that  $p \in U'$ , a contradiction.  $\square$

1/19: **Exercise 6.7.** Show that  $\mathbb{R}$  satisfies Axiom 4.

*Proof.* Suppose for the sake of contradiction that  $\mathbb{R}$  does not satisfy Axiom 4. It follows that  $\mathbb{R}$  is not connected, implying by Definition 4.22 that  $\mathbb{R} = A \cup B$  where  $A, B$  are disjoint, nonempty, open sets. Since  $A, B$  are disjoint and nonempty, we know that there exist distinct objects  $a \in A$  and  $b \in B$ . WLOG, let  $a < b$ .

We now seek to prove that the set  $A \cap \underline{ab}$  is nonempty and bounded above. To prove that  $A \cap \underline{ab}$  is nonempty, Definition 1.8 tells us that it will suffice to find an element of  $A \cap \underline{ab}$ . Since  $a \in A$  and  $A$  is open, we have by Theorem 4.10 that there exists a region  $\underline{cd}$  such that  $a \in \underline{cd}$  and  $\underline{cd} \subset A$ . It follows by Definitions 3.10 and 3.6 that  $a < d$ , implying by Lemma 6.10<sup>[2]</sup> that there exists some point  $x \in \mathbb{R}$  such that  $c < a < x < d < b$  (note that  $d < b$  since if  $b < d$ , then  $b \in \underline{cd}$  would contradict the fact that  $\underline{cd} \subset A$ ). Consequently,  $x \in \underline{cd}$ , meaning that  $x \in A$ , and  $x \in \underline{ab}$ . Therefore,  $x \in A \cap \underline{ab}$ , as desired. To prove that  $A \cap \underline{ab}$  is bounded above, Definition 5.6 tells us that it will suffice to show that  $b$  is an upper bound of  $A \cap \underline{ab}$ . To show this, Definition 5.6 tells us that it will suffice to confirm that  $y \leq b$  for all  $y \in A \cap \underline{ab}$ . Let  $y$  be an arbitrary element of  $A \cap \underline{ab}$ . Then by Definition 1.6,  $y \in A$  and  $y \in \underline{ab}$ . It follows from the latter statement by Definitions 3.10 and 3.6 that  $y < b$ , i.e.,  $y \leq b$ , as desired.

Having established that  $A \cap \underline{ab} \subset \mathbb{R}$  is nonempty and bounded above, we can invoke Lemma 6.6 to learn that  $A \cap \underline{ab}$  has a supremum  $\sup(A \cap \underline{ab})$ . We now divide into two cases ( $\sup(A \cap \underline{ab}) \in A$  and  $\sup(A \cap \underline{ab}) \in B$ ; it follows from the definitions of  $A$  and  $B$  that exactly one of these cases is true). Suppose first that  $\sup(A \cap \underline{ab}) \in A$ . Then since  $A$  is open, we have by Theorem 4.10 that there exists a region  $\underline{ef}$  such that  $\sup(A \cap \underline{ab}) \in \underline{ef}$  and  $\underline{ef} \subset A$ . It follows from the former condition that  $\sup(A \cap \underline{ab}) < f$ . Thus, by Lemma 6.10, there exists an object  $z \in \mathbb{R}$  such that  $e < \sup(A \cap \underline{ab}) < z < f < b$  (note that  $f < b$  for the same reason that  $d < b$ ). Consequently,  $z \in \underline{ef}$ , implying that  $z \in A$ , and  $z \in \underline{ab}$ . Thus, we have found an element of  $A \cap \underline{ab}$  that is greater than  $\sup(A \cap \underline{ab})$ , contradicting Definitions 5.7 and 5.6. The proof is symmetric in the other case.  $\square$

1/14: **Definition 6.8.** Let  $C$  be a continuum satisfying Axioms 1-4. Consider a subset  $X \subset C$ . We say that  $X$  is **dense** in  $C$  if every  $p \in C$  is a limit point of  $X$ .

**Lemma 6.9.** A subset  $X \subset C$  is dense in  $C$  if and only if  $\overline{X} = C$ .

*Proof.* Suppose first that  $X \subset C$  is dense in  $C$ . To prove that  $\overline{X} = C$ , Definition 1.2 tells us that it will suffice to show that every point  $p \in \overline{X}$  is an element of  $C$  and vice versa. Clearly, every element of  $\overline{X}$  is an element of  $C$ . On the other hand, let  $p$  be an arbitrary element of  $C$ . Since  $X$  is dense in  $C$ , Definition 6.8 tells us that  $p \in LP(X)$ . Therefore, by Definitions 1.5 and 4.4,  $p \in \overline{X}$ .

Now suppose that  $\overline{X} = C$ . To prove that  $X$  is dense in  $C$ , Definition 6.8 tells us that it will suffice to show that every  $p \in C$  is a limit point of  $X$ . Let  $p$  be an arbitrary element of  $C$ . By Corollary 5.4, this implies that  $p \in LP(C)$ . It follows that  $p \in LP(\overline{X})$ . Thus, by Definition 4.4,  $p \in LP(X \cup LP(X))$ . Consequently, by Theorem 3.20,  $p \in LP(X)$  or  $p \in LP(LP(X))$ . We now divide into two cases. If  $p \in LP(X)$ , then we are done. On the other hand, if  $p \in LP(LP(X))$ , the lemma from Theorem 4.6 asserts that  $p \in LP(X)$ , and we are done again.  $\square$

Our next goal is to prove that  $\mathbb{Q}$  is dense in  $\mathbb{R}$ . Just to make sense of that statement, we need to decide how to think of  $\mathbb{Q}$  as a subset of  $\mathbb{R}$ . For every rational number  $q \in \mathbb{Q}$ , define the corresponding real number as the Dedekind cut

$$i(q) = \{x \in \mathbb{Q} \mid x < q\}$$

For example,  $\mathbf{0} = i(0)$ . It can be verified that this gives a well-defined injective function  $i : \mathbb{Q} \rightarrow \mathbb{R}$ . We identify  $\mathbb{Q}$  with its image  $i(\mathbb{Q}) \subset \mathbb{R}$  so that the rational numbers  $\mathbb{Q}$  are a subset of the real numbers  $\mathbb{R}$ . (Similarly,  $\mathbb{N}$  and  $\mathbb{Z}$  can be understood as subsets of  $\mathbb{R}$ .)

**Lemma 6.10.** Given  $A, B \in \mathbb{R}$  with  $A < B$ , there exists  $p \in \mathbb{Q}$  such that  $A < i(p) < B$ .

<sup>2</sup>We may use this lemma since it does not depend on this result, Definition 6.8, or Lemma 6.9.

*Proof.* Since  $A < B$ , Definition 6.4 tells us that  $A \subsetneq B$ . Thus, by Definition 1.3, there exists a point  $q$  such that  $q \in B$  and  $q \notin A$ . Since  $q \in B$  where  $B$  is a Dedekind cut, we have by Definition 6.1 that there exists a point  $p \in B$  with  $p > q$ . Additionally, since  $q \notin A$  implies that  $q$  is an upper bound of  $A$  by Lemma 6.2, we know by Definition 5.6 that  $x \leq q$  for all  $x \in A$ . It follows since  $q < p$  that  $x \leq p$  for all  $x \in A$ , meaning by Definition 5.6 and Lemma 6.2 that  $p \notin A$ . Having established that  $p, q \in B$ ,  $p, q \notin A$ , and  $q < p$ , we are now ready to prove that  $A < i(p) < B$ . Definition 6.4 tells us that we may do so by showing that  $A \subsetneq i(p)$  and  $i(p) \subsetneq B$ . We will take this one argument at a time.

To show that  $A \subsetneq i(p)$ , Definition 1.3 tells us that it will suffice to verify that every element of  $A$  is an element of  $i(p)$  and that there exists an element of  $i(p)$  that is not an element of  $A$ . We treat the former statement first. As previously mentioned,  $x \leq p$  for all  $x \in A$ . This combined with the fact that  $p \notin A$  implies that  $x < p$  for all  $x \in A$ . Thus, by the definition of  $i(p)$ ,  $x \in i(p)$  for all  $x \in A$ , as desired. As to the latter statement, since  $q < p$ , we have by the definition of  $i(p)$  that  $q \in i(p)$ . However, we also know that  $q \notin A$ , as desired.

To show that  $i(p) \subsetneq B$ , we must verify symmetric arguments to before. For the former statement, let  $r$  be an arbitrary element of  $i(p)$ . Then by the definition of  $i(p)$ ,  $r < p$ . Since  $p \in B$  and  $r \in \mathbb{Q}$  satisfy  $r < p$ , we have by Definition 6.1 that  $r \in B$ , as desired. As to the latter statement,  $p$  is clearly an element of  $B$  that is not an element of  $i(p)$ , as desired.  $\square$

1/19: **Theorem 6.11.**  $i(\mathbb{Q})$  is dense in  $\mathbb{R}$ .

*Proof.* To prove that  $i(\mathbb{Q})$  is dense in  $\mathbb{R}$ , Definition 6.8 tells us that it will suffice to show the every point  $X \in \mathbb{R}$  is a limit point of  $i(\mathbb{Q})$ . Let  $X$  be an arbitrary element of  $\mathbb{R}$ . To show that  $X \in LP(i(\mathbb{Q}))$ , Definition 3.13 tells us that it will suffice to verify that for every region  $\underline{AB}$  with  $X \in \underline{AB}$ , we have  $\underline{AB} \cap (i(\mathbb{Q}) \setminus \{X\}) \neq \emptyset$ . Let  $\underline{AB}$  be an arbitrary region with  $X \in \underline{AB}$ . It follows by Definitions 3.10 and 3.6 that  $A < X < B$ . Thus, by Lemma 6.10, there exists  $p \in \mathbb{Q}$  such that  $A < i(p) < X < B$ . By Definitions 3.6 and 3.10,  $i(p) \in \underline{AB}$ . By Definition 1.18,  $i(p) \in i(\mathbb{Q})$ . By Exercise 6.5,  $i(p) < X$  implies that  $i(p) \neq X$ . Combining the last three results with Definitions 1.11 and 1.6, we have that  $i(p) \in \underline{AB} \cap (i(\mathbb{Q}) \setminus \{X\})$ , as desired.  $\square$

**Corollary 6.12** (The Archimedean Property). *Let  $A \in \mathbb{R}$  be a positive real number. Then there exist nonzero natural numbers  $n, m \in \mathbb{N}$  such that  $i(\frac{1}{n}) < A < i(m)$ .*

*Proof.* We will first prove that there exists a nonzero natural number  $n$  such that  $i(\frac{1}{n}) < A$ . We will then prove that there exists a nonzero natural number  $m$  such that  $A < i(m)$ . Let's begin.

Since  $A \in \mathbb{R}$  is positive, we know that  $0 < A$ . Thus, by Lemma 6.10, there exists  $\frac{p}{n} \in \mathbb{Q}$  such that  $0 < i(\frac{p}{n}) < A$ . As permitted by Exercise 3.9b, we choose  $\frac{p}{n} \in [\frac{p}{n}]$  to be an object such that  $0 < n$  (this means that  $n \in \mathbb{N}$ ). Consequently, by Scripts 2 and 3, we know that  $0 < \frac{1}{n} \leq \frac{p}{n}$ . It follows that  $i(\frac{1}{n}) \leq i(\frac{p}{n})$  since  $x \in i(\frac{1}{n})$  implies  $x < \frac{1}{n} \leq \frac{p}{n}$  implies  $x \in i(\frac{p}{n})$ , implies  $i(\frac{1}{n}) \subset i(\frac{p}{n})$ . Therefore,  $i(\frac{1}{n}) \leq i(\frac{p}{n}) < A$ , as desired.

By Exercise 6.5, Axiom 3, and Definition 3.3, there exists a point  $B \in \mathbb{R}$  such that  $A < B$ . It follows by Lemma 6.10 that there exists  $\frac{m}{q} \in \mathbb{Q}$  such that  $A < i(\frac{m}{q}) < B$ . As before, let  $\frac{m}{q}$  be an object such that  $0 < q$ . Consequently, by Scripts 2 and 3, we know that  $0 < \frac{m}{q} \leq m$ . Once again, for the same reasons as before,  $i(\frac{m}{q}) \leq i(m)$ . Therefore,  $A < i(\frac{m}{q}) \leq i(m)$ , as desired.  $\square$

**Corollary 6.13.**  $i(\mathbb{N})$  is an unbounded subset of  $\mathbb{R}$ .

*Proof.* Suppose for the sake of contradiction that  $i(\mathbb{N})$  is bounded above. Then by Definition 5.6, there exists a point  $A \in \mathbb{R}$  such that  $i(n) \leq A$  for all  $n \in \mathbb{N}$ . Note that  $A$  is a positive real number since  $0 = i(0) \leq A$ . But by Corollary 6.12,  $A < i(n)$  for some  $n \in \mathbb{N}$ , a contradiction.  $\square$

1/21: **Corollary 6.14.** *If  $A \in \mathbb{R}$  is a real number, then there is an integer  $n$  such that  $i(n-1) \leq A < i(n)$ .*

*Proof.* Let  $X$  be the set of all integers  $z$  such that  $i(z) \leq A$ . Symbolically,

$$X = \{z \mid z \in \mathbb{Z} \text{ and } i(z) \leq A\}$$

Since  $A \neq \emptyset$  by Definition 6.1, there exists a point  $\frac{p}{q} \in \mathbb{Q}$  such that  $\frac{p}{q} \in A$ . As in Corollary 6.12, we let  $q > 0$ . It follows by Scripts 2 and 3 that if  $p \geq 0$ , then  $0 \leq \frac{p}{q}$ , i.e.<sup>[3]</sup>,  $i(0) \leq A$  and if  $p < 0$ , then  $p \leq \frac{p}{q}$ , i.e.,

<sup>3</sup>For the same reasons as in Corollary 6.12.

$i(p) \leq A$ . Thus, in either case,  $X$  is nonempty.

Now by Corollary 6.12, there exists a nonzero natural number  $m$  such that  $A < i(m)$ . Let  $f : X \rightarrow \mathbb{N}$  be defined by the rule

$$f(x) = m - x$$

By Script 1,  $f$  is an injective function,  $f(X) \subset \mathbb{N}$ , and  $f(X)$  is nonempty (since  $X$  is nonempty). Thus, by Additional Exercise 0.1, there is a least element, which we shall call  $y$ , in  $f(X)$ . Since  $f$  is injective, there exists exactly one object  $n - 1 \in X$  such that  $f(n - 1) = y$ .

By the definition of  $X$ ,  $i(n - 1) \leq A$ . To prove that  $A < i(n)$ , suppose for the sake of contradiction that  $i(n) \leq A$ . This coupled with the fact that  $n \in \mathbb{Z}$  implies that  $n \in X$ . Thus,  $f(n) \in f(X)$ . But  $f(n) = m - n < m - (n - 1) = f(n - 1)$ , contradicting the fact that  $f(n - 1)$  is the least element of  $f(X)$ .  $\square$

1/26: **Axiom 5.** *The continuum contains a countable dense subset.*

**Definition 6.15.** Let  $X$  and  $Y$  be sets with orderings  $<_X$  and  $<_Y$ , respectively. A function  $f : X \rightarrow Y$  is **order-preserving** if for all  $r, s \in X$ ,

$$r <_X s \implies f(r) <_Y f(s)$$

Note that the function  $i : \mathbb{Q} \rightarrow \mathbb{R}$  discussed above is order preserving.

**Exercise 6.16.** Let  $C$  satisfy Axioms 1-5. Let  $K \subset C$  be a countable dense subset of  $C$ . Construct an order-preserving bijection  $f : \mathbb{Q} \rightarrow K$ .

**Lemma.**

- a)  $K$  satisfies Axiom 3.
- b) (Density Lemma) For all  $x, y \in K$ , if  $x < y$ , then there exists a point  $z \in K$  such that  $z$  is between  $x$  and  $y$ .

*Proof of a.* To prove that  $K$  satisfies Axiom 3, we must verify that  $K$  has neither a first nor a last point. We will address the first point question first. Suppose for the sake of contradiction that  $K$  has a first point  $x$ . Then by Definition 3.3,  $x \leq y$  for all  $y \in K$ . However, since  $C$  satisfies Axiom 3, there exists an object  $a \in C$  such that  $a < x$ . Now consider the region  $\underline{ax}$ . We have by Corollary 5.3 that there exists a point  $p \in \underline{ax}$ . Additionally, we have by Script 3 that  $\underline{ax} \cap K = \emptyset$ . Thus,  $\underline{ax} \cap (K \setminus \{p\}) = \emptyset$ , implying by Definition 3.13 that  $p \notin LP(K)$ . But since  $p \in C$  and  $p \notin LP(K)$ , we have by Definition 6.8 that  $K$  is not dense in  $C$ , a contradiction.

The proof is symmetric for last points.  $\square$

*Proof of b.* Suppose for the sake of contradiction that there exist  $x, y \in K$  with  $x < y$  such that no point  $z \in K$  is between  $x$  and  $y$ . By Theorem 5.2, there exists  $p \in C$  such that  $p$  is between  $x$  and  $y$ . Consequently, by Definition 3.10,  $p \in \overline{xy}$ . Additionally, we have by Script 3 that  $\overline{xy} \cap K = \emptyset$ . It follows that  $\overline{xy} \cap (K \setminus \{p\}) = \emptyset$ , implying by Definition 3.13 that  $p \notin LP(K)$ . But since  $p \in C$  and  $p \notin LP(K)$ , we have by Definition 6.8 that  $K$  is not dense in  $C$ , a contradiction.  $\square$

*Proof of Exercise 6.16.* By Theorem 2.11,  $\mathbb{Q}$  is countable, implying by Definition 1.35 that there exists a bijection  $g : \mathbb{N} \rightarrow \mathbb{Q}$ . The existence of this bijection means that we can refer to an arbitrary element  $q$  of  $\mathbb{Q}$  by the number  $n$  for which  $g(n) = q$ ; in another notation, we can refer to  $q$  as  $q_n$ . Thus, since every element of  $\mathbb{Q}$  can be written as  $q_n$  for some  $n \in \mathbb{N}$ , we can write  $\mathbb{Q} = \{q_1, q_2, \dots\}$ . Similarly, we can express  $K$  as  $K = \{k_1, k_2, \dots\}$ . We will use this method of referring to the elements of  $\mathbb{Q}$  to construct  $f$ .

We define  $f$  recursively with strong induction. For the base case  $q_1$ , we define  $f(q_1) = k_1$ . Now suppose inductively that we have defined  $f(q_1), f(q_2), \dots, f(q_n)$ ; we now seek to define  $f(q_{n+1})$ . By Theorem 3.5, the symbols  $a_1, \dots, a_{n+1}$  can be assigned to  $q_1, \dots, q_{n+1}$  so that  $a_1 <_{\mathbb{Q}} a_2 <_{\mathbb{Q}} \dots <_{\mathbb{Q}} a_{n+1}$ . We divide into three cases ( $q_{n+1} = a_1$ ,  $q_{n+1} = a_{n+1}$ , and  $q_{n+1} = a_i$  where  $1 < i < n + 1$ ). First, suppose that  $q_{n+1} = a_1$ . By the inductive hypothesis,  $f(a_2), f(a_3), \dots, f(a_{n+1})$  are defined elements of  $K$ . At this point, define the set  $X = \{k \in K \mid k <_K f(a_2)\}$ . It follows by Lemma (a) that this set is nonempty. Thus, by

the well-ordering principle, there exists a  $k_i \in X$  such that  $i \leq j$  for all  $k_j \in X$ . We let  $f(q_{n+1}) = k_i$ . The second case is symmetric to the first. Third, suppose that  $q_{n+1} = a_i$  where  $1 < i < n + 1$ . By the inductive hypothesis,  $f(a_1), \dots, f(a_{i-1}), f(a_{i+1}), \dots, f(a_{n+1})$  are defined elements of  $K$ . At this point, define the set  $X = \{k \in K \mid f(a_{i-1}) <_K k <_K f(a_{i+1})\}$ . It follows by Lemma (b) that this set is nonempty. Thus, by the well-ordering principle, there exists a  $k_i \in X$  such that  $i \leq j$  for all  $k_j \in X$ . We let  $f(q_{n+1}) = k_i$ .

To prove that  $f$  is order-preserving, we will first verify the following claim (which we will refer to as Lemma (c) for future reference): Consider the set  $\{q_1, \dots, q_n\} \subset \mathbb{Q}$ ; if the symbols  $a_1, \dots, a_n$  are assigned to  $q_1, \dots, q_n$  such that  $a_1 <_{\mathbb{Q}} a_2 <_{\mathbb{Q}} \dots <_{\mathbb{Q}} a_n$ , then  $f(a_1) <_K f(a_2) <_K \dots <_K f(a_n)$ . We will then use this result to prove that  $f$  is order-preserving for any two arbitrary elements  $q_i, q_j \in \mathbb{Q}$ . Let's begin.

To verify the above claim, we induct on  $n$ . The base case  $n = 1$  is vacuously true. Now suppose inductively that we have proven the claim for  $n$ ; we now seek to prove it for  $n + 1$ . By Theorem 3.5, the symbols  $a_1, \dots, a_{n+1}$  can be assigned to  $q_1, \dots, q_{n+1}$  so that  $a_1 <_{\mathbb{Q}} a_2 <_{\mathbb{Q}} \dots <_{\mathbb{Q}} a_{n+1}$ . We divide into three cases ( $q_{n+1} = a_1$ ,  $q_{n+1} = a_{n+1}$ , and  $q_{n+1} = a_i$  where  $1 < i < n + 1$ ). First, suppose that  $q_{n+1} = a_1$ . By the definition of  $f$ ,  $f(q_{n+1}) \in \{k \in K \mid k <_K f(a_2)\}$ , meaning that  $f(q_{n+1}) = f(a_1) <_K f(a_2)$ . Additionally, by the inductive hypothesis, we know that  $f(a_2) <_K f(a_3) <_K \dots <_K f(a_{n+1})$  (since  $a_2, \dots, a_{n+1}$  correspond to  $q_1, \dots, q_n$ ). These two results imply that  $f(a_1) <_K f(a_2) <_K \dots <_K f(a_{n+1})$ . The proof of the second case is symmetric to that of the first. Third, suppose that  $q_{n+1} = a_i$  where  $1 < i < n + 1$ . By the definition of  $f$ ,  $f(q_{n+1}) \in \{k \in K \mid f(a_{i-1}) <_K k <_K f(a_{i+1})\}$ , meaning that  $f(a_{i-1}) <_K f(q_{n+1}) = f(a_i) <_K f(a_{i+1})$ . Additionally, by the inductive hypothesis, we know that  $f(a_1) <_K \dots <_K f(a_{i-1}) <_K f(a_{i+1}) <_K \dots <_K f(a_{n+1})$  (for an analogous reason to before). These two results imply that  $f(a_1) <_K f(a_2) <_K \dots <_K f(a_{n+1})$ .

We are now ready to actually prove that  $f$  is order-preserving. To do so, Definition 6.15 tells us that it will suffice to show that for all  $q_i, q_j \in \mathbb{Q}$ ,  $q_i <_{\mathbb{Q}} q_j$  implies  $f(q_i) <_K f(q_j)$ . Let  $q_i, q_j$  be arbitrary elements of  $\mathbb{Q}$  such that  $q_i <_{\mathbb{Q}} q_j$ . Since  $q_i <_{\mathbb{Q}} q_j$ ,  $q_i \neq q_j$ , implying that  $i \neq j$ . We divide into two cases ( $i < j$  and  $i > j$ ). Suppose first that  $i < j$ . By Theorem 3.5, the symbols  $a_1, \dots, a_j$  can be assigned to  $q_1, \dots, q_j$  so that  $a_1 <_{\mathbb{Q}} a_2 <_{\mathbb{Q}} \dots <_{\mathbb{Q}} a_j$ . Let  $q_j = a_l$ . Since  $q_i <_{\mathbb{Q}} q_j$ , we know that  $q_i = a_m$  where  $m < l$ . Additionally, by Lemma (c), we know that  $f(a_1) <_K f(a_2) <_K \dots <_K f(a_j)$ . It follows that  $f(a_m) <_K f(a_l)$ , implying that  $f(q_i) <_K f(q_j)$ , as desired. The proof is symmetric in the other case.

To prove that  $f$  is bijective, Definition 1.20 tells us that it will suffice to show that  $f$  is injective and surjective.

To show that  $f$  is injective, Definition 1.20 tells us that it will suffice to demonstrate that  $q_i \neq q_j$  implies  $f(q_i) \neq f(q_j)$ . WLOG let  $q_i <_{\mathbb{Q}} q_j$ . Then since  $f$  is order-preserving, Definition 6.15 implies that  $f(q_i) <_K f(q_j)$ . It follows that  $f(q_i) \neq f(q_j)$ , as desired.

To show that  $f$  is surjective, Definition 1.20 tells us that it will suffice to demonstrate that for all  $k_n \in K$ , there exists a  $q_i \in \mathbb{Q}$  such that  $f(q_i) = k_n$ . To do this, we induct on  $n$ . For the base case  $n = 1$ , it follows from the definition of  $f$  that  $f(q_1) = k_1$ . Now suppose inductively that for each  $k_1, \dots, k_n$ , there exists a  $q_i \in \mathbb{Q}$  such that  $f(q_i) = k_n$ ; we now seek to prove the claim for  $n + 1$ . By Theorem 3.5, the symbols  $b_1, \dots, b_{n+1}$  can be assigned to  $k_1, \dots, k_{n+1}$  so that  $b_1 <_K b_2 <_K \dots <_K b_{n+1}$ . We divide into three cases ( $k_{n+1} = b_1$ ,  $k_{n+1} = b_{n+1}$ , and  $k_{n+1} = b_i$  where  $1 < i < n + 1$ ). First, suppose that  $k_{n+1} = b_1$ . By the inductive hypothesis,  $b_2 = f(q_i) <_K b_3 = f(q_j) <_K \dots <_K b_{n+1} = f(q_l)$ . It follows by Definition 6.15 that  $q_i <_{\mathbb{Q}} q_j <_{\mathbb{Q}} \dots <_{\mathbb{Q}} q_l$ . At this point, define the set  $X = \{q \in \mathbb{Q} \mid q <_{\mathbb{Q}} q_i\}$ . It follows from Exercise 3.9d that this set is nonempty. Thus, by the well-ordering principle, there exists a  $q_m \in X$  such that  $m \leq m'$  for all  $k_{m'} \in X$ . By the definition of  $f$ ,  $f(q_m) = k_{n+1}$ . The proof of the second case is symmetric to that of the first. Third, suppose that  $k_{n+1} = b_i$  where  $1 < i < n + 1$ . By the inductive hypothesis,  $b_2 = f(q_j) <_K \dots <_K b_{i-1} = f(q_{j'}) <_K b_{i+1} = f(q_l) <_K \dots <_K b_{n+1} = f(q_{l'})$ . It follows by Definition 6.15 that  $q_j <_{\mathbb{Q}} \dots <_{\mathbb{Q}} q_{j'} <_{\mathbb{Q}} q_l <_{\mathbb{Q}} \dots <_{\mathbb{Q}} q_{l'}$ . At this point, define the set  $X = \{q \in \mathbb{Q} \mid q_{j'} <_{\mathbb{Q}} q <_{\mathbb{Q}} q_l\}$ . It follows from Additional Exercise 3.1 that this set is nonempty. Thus, by the well-ordering principle, there exists a  $q_m \in X$  such that  $m \leq m'$  for all  $k_{m'} \in X$ . By the definition of  $f$ ,  $f(q_m) = k_{n+1}$ .  $\square$

**Exercise 6.17.** Let  $f : \mathbb{Q} \rightarrow K$  be an order-preserving bijection, as found in Exercise 6.16. Let  $A \in \mathbb{R}$ . Then  $A \subset \mathbb{Q}$  and so  $f(A) \subset K \subset C$ . Define  $F : \mathbb{R} \rightarrow C$  by

$$F(A) = \sup f(A)$$

1. Show  $\sup f(A)$  exists, so  $F$  is well-defined.

2. Show  $F$  is injective and order-preserving.

*Proof of 1.* To prove that  $\sup f(A)$  exists, Theorem 5.17 tells us that it will suffice to show that  $f(A)$  is nonempty and bounded above. To show that  $f(A)$  is nonempty, Definition 1.8 tells us that it will suffice to find an element of  $f(A)$ . By Definition 6.1,  $A \neq \emptyset$ . Thus, by Definition 1.8, there exists an object  $x \in A$ . It follows by Definition 1.18 that  $f(x) \in f(A)$ , as desired. To show that  $f(A)$  is bounded above, Definition 5.6 tells us that it will suffice to find an object  $f(x) \in f(A)$  such that  $f(x) \geq_C f(a)$  for all  $f(a) \in f(A)$ . By Definition 6.1,  $A \neq \mathbb{Q}$  and  $A \subset \mathbb{Q}$ . Thus, by Definition 1.2, there exists an object  $x \in \mathbb{Q}$  such that  $x \notin A$ . It follows from the latter condition by Lemma 6.2 that  $x$  is an upper bound for  $A$ . Thus, by Definition 5.6,  $x \geq a$  for all  $a \in A$ . Consequently, by Definition 6.15,  $f(x)$  is an object such that  $f(x) \geq_C f(a)$  for all  $f(a) \in f(A)$ , as desired.  $\square$

*Proof of 2.* To prove that  $F$  is order-preserving, Definition 6.15 tells us that it will suffice to show that for all  $A, B \in \mathbb{R}$ ,  $A <_{\mathbb{R}} B$  implies  $F(A) <_C F(B)$ . Let  $A, B$  be two arbitrary elements of  $\mathbb{R}$  satisfying  $A <_{\mathbb{R}} B$ . Then by Definitions 6.4 and 1.3, there exists a point  $x \in B$  such that  $x \notin A$ . It follows from the latter condition by Lemma 6.2 and Definition 5.6 that  $x \geq a$  for all  $a \in A$ . Thus, by Definition 6.15,  $f(x) \geq_C f(a)$  for all  $f(a) \in f(A)$ . Consequently, by Definition 5.7,  $\sup f(A) \leq_C f(x)$ . Additionally, by Definition 6.1, there exists a point  $y \in B$  such that  $y > x$ . Thus, by Definition 6.15, we have that  $f(y) >_C f(x)$ . It follows by Definitions 5.6 and 5.7 that  $f(y) \leq_C \sup f(B)$ . Combining two results, we therefore have that  $\sup f(A) \leq_C f(x) <_C f(y) \leq_C \sup f(B)$ , meaning that  $F(A) = \sup f(A) <_C \sup f(B) = F(B)$ , as desired.

To prove that  $F$  is injective, Definition 1.20 tells us that it will suffice to show that if  $A \neq B$ , then  $F(A) \neq F(B)$ . Let  $A, B$  be two distinct real numbers. Then by Exercise 6.5,  $A < B$  or  $B < A$ . We now divide into two cases. Suppose first that  $A < B$ . Then  $F(A) < F(B)$  by Definition 6.15 (which we have just proven applies to  $F$ ). This implies by Definition 3.1 that  $F(A) \neq F(B)$ , as desired. The proof is symmetric in the other case.  $\square$

**Theorem 6.18.** Suppose that  $C$  is a continuum satisfying Axioms 1-5. Then  $C$  is isomorphic to the real numbers  $\mathbb{R}$ ; i.e., there is an order-preserving bijection  $F : \mathbb{R} \rightarrow C$ .

**Lemma.** Let  $K$  be a dense subset of  $C$ . For all  $x, y \in C$ , if  $x < y$ , then there exists a point  $z \in K$  such that  $z$  is between  $x$  and  $y$ .

*Proof.* Suppose for the sake of contradiction that there exist two points  $x, y \in C$  with  $x < y$  such that no point  $z \in K$  is between  $x$  and  $y$ . By Corollary 5.3, the region  $xy$  is infinite. Thus, we can pick a point  $p \in xy$ . Additionally, by Definition 1.6, we have that  $xy \cap K = \emptyset$ . Thus,  $xy \cap (K \setminus \{p\}) = \emptyset$ , implying by Definition 3.13 that  $p \notin LP(K)$ . But since  $p \in C$  and  $p \notin LP(K)$ , we have by Definition 6.8 that  $K$  is not dense in  $C$ , a contradiction.  $\square$

*Proof of Theorem 6.18.* By Axiom 5,  $C$  contains a countable dense subset  $K$ . By Exercise 6.16, there exists an order-preserving bijection  $f : \mathbb{Q} \rightarrow K$ . By Exercise 6.17, there exists an order-preserving injection  $F : \mathbb{R} \rightarrow C$ . To prove that there is an order-preserving bijection  $F : \mathbb{R} \rightarrow C$ , all that is left to do is to demonstrate that  $F$  (as defined in Exercise 6.17) is surjective.

To do this, Definition 1.20 tells us that it will suffice to show that for all  $X \in C$ , there exists an object  $A \in \mathbb{R}$  such that  $F(A) = X$ . Put more simply, we must find a Dedekind cut  $A$  such that  $\sup f(A) = X$  for every  $X \in C$ . To do this, we will begin by constructing the set  $S = \{k \in K \mid k < X\}$ . We will then verify that the preimage  $f^{-1}(S)$  is a Dedekind cut. Lastly, we will verify that  $\sup f(f^{-1}(S)) = X$ . Let's begin.

Let  $X$  be an arbitrary element of  $C$ . Define  $S$  as above. To verify that  $f^{-1}(S)$  is a Dedekind cut, Definition 6.1 tells us that it will suffice to confirm that  $f^{-1}(S) \neq \emptyset$ ;  $f^{-1}(S) \neq \mathbb{Q}$ ; if  $r \in f^{-1}(S)$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in f^{-1}(S)$ ; and if  $r \in f^{-1}(S)$ , then there is some  $s \in f^{-1}(S)$  with  $s > r$ . We will take this one claim at a time.

To confirm that  $f^{-1}(S) \neq \emptyset$ , Definition 1.8 tells us that it will suffice to find an element of  $f^{-1}(S)$ . By Axiom 3 and Definition 3.3, there exists some point  $Y \in C$  such that  $Y < X$ . Consequently, by the lemma and Definition 3.6, there exists a point  $f(p) \in K^{[4]}$  such that  $Y < f(p) < X$ . It follows by the definition of  $S$  that  $f(p) \in S$ . Therefore, by Definition 1.18,  $p \in f^{-1}(S)$ , as desired.

<sup>4</sup>Note that we know that the element of  $K$  (the existence of which is implied by the lemma) can be written in the form  $f(p)$  because  $f$  is bijective.



To confirm that  $f^{-1}(S) \neq \mathbb{Q}$ , Definition 1.2 tells us that it will suffice to find an element of  $\mathbb{Q}$  that is not an element of  $f^{-1}(S)$ . By Axiom 3 and Definition 3.3, there exists some point  $Y \in C$  such that  $X < Y$ . Consequently, by the lemma and Definition 3.6, there exists a point  $f(p) \in K$  such that  $X < f(p) < Y$ . It follows by the definition of  $S$  that  $f(p) \notin S$ . Therefore, by Definition 6.18,  $p \in \mathbb{Q}$  but  $p \notin f^{-1}(S)$ , as desired.

To confirm that if  $r \in f^{-1}(S)$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in f^{-1}(S)$ , we let  $r \in f^{-1}(S)$  and  $s \in \mathbb{Q}$  be arbitrary elements of their respective sets that satisfy  $s < r$  and seek to verify that  $s \in f^{-1}(S)$ . By Definition 1.18, the fact that  $r \in f^{-1}(S)$  implies that  $f(r) \in S$ . Thus, by the definition of  $S$ ,  $f(r) < X$ . Additionally, by the definition of  $f$  and Definition 6.15,  $f(s) \in K$  and  $f(s) < f(r)$ , respectively. Since  $f(s) < f(r)$  and  $f(r) < X$ , transitivity implies that  $f(s) < X$ . This combined with the previously established fact that  $f(s) \in K$  implies that  $f(s) \in S$ . Therefore, by Definition 1.18,  $s \in f^{-1}(S)$ , as desired.

To confirm that if  $r \in f^{-1}(S)$ , then there is some  $s \in f^{-1}(S)$  with  $s > r$ , we let  $r \in f^{-1}(S)$  and seek to find such an  $s$ . As before,  $r \in f^{-1}(S)$  implies that  $f(r) \in S$ . Thus, by the definition of  $S$ ,  $f(r) < X$ . It follows by the lemma and Definition 3.6 that there exists a point  $f(s) \in K$  such that  $f(r) < f(s) < X$ . Consequently, by the definition of  $S$ , we have that  $f(s) \in S$ . Therefore, by Definitions 1.18 and 6.15,  $s \in f^{-1}(S)$  and  $r < s$ , respectively, as desired.

Since  $f$  is bijective, Script 1 asserts that  $f(f^{-1}(S)) = S$ . Thus,  $\sup f(f^{-1}(S)) = \sup S$ . To verify that  $\sup S = X$ , Definition 5.7 tells us that it will suffice to confirm that  $X$  is an upper bound of  $S$  and if  $U$  is an upper bound of  $S$ ,  $X \leq U$ . To confirm the former statement, Definition 5.6 tells us that it will suffice to show that  $k \leq X$  for all  $k \in S$ . But by the definition of  $S$ , this is true. To confirm the latter statement, suppose for the sake of contradiction that there exists an upper bound  $U$  of  $S$  such that  $U < X$ . Since  $U < X$ , the lemma and Definition 3.6 imply that there exists a point  $Z \in K$  such that  $U < Z < X$ . It follows by the definition of  $S$  that  $Z \in S$ . Since there exists an element of  $S$  greater than  $U$ , Definition 5.6 asserts that  $U$  is not an upper bound of  $S$ , a contradiction.  $\square$