MATH 16210 (Honors Calculus II IBL) Notes

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Contents

Script 6

Construction of the Real Numbers

6.1 Journal

1/12: **Definition 6.1.** A subset A of $\mathbb{Q}$ is said to be a **cut** (or **Dedekind cut**) if it satisfies the following:

(a) $A \neq \emptyset$ and $A \neq \mathbb{Q}$.

(b) If $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$.

(c) A does not have a last point; i.e., if $r \in A$, then there is some $s \in A$ with $s > r$.

We denote the collection of all cuts by $\mathbb{R}$.

Lemma 6.2. *Let A be a Dedekind cut and $x \in \mathbb{Q}$. Then $x \notin A$ if and only if x is an upper bound for A.*

Proof. Suppose first that $x \notin A$. To prove that x is an upper bound for A, Definition 5.6 tells us that it will suffice to show that for all $r \in A$, $r \leq x$. Let r be an arbitrary element of A. Then by the contrapositive of Definition 6.1b and the hypothesis that $x \notin A$, we know that $r \notin A$, $x \notin \mathbb{Q}$, or $x \not< r$. But since $r \in A$ and $x \in \mathbb{Q}$, it must be that $x \not< r$. Therefore, $r \leq x$, as desired.

Now suppose that x is an upper bound for A. By Definition 5.6, this implies that for all $r \in A$, $r \leq x$. Therefore, since there is no $r \in A$ with $r > x$, by the contrapositive of Definition 6.1c, $x \notin A$, as desired. $\square$

Exercise 6.3.

(a) Prove that for any $q \in \mathbb{Q}$, $\{x \in \mathbb{Q} \mid x < q\}$ is a Dedekind cut. We then define $\mathbf{0} = \{x \in \mathbb{Q} \mid x < 0\}$.

(b) Prove that $\{x \in \mathbb{Q} \mid x \leq 0\}$ is not a Dedekind cut.

(c) Prove that $\{x \in \mathbb{Q} \mid x < 0\} \cup \{x \in \mathbb{Q} \mid x^2 < 2\}$ is a Dedekind cut.

Proof of a. Let q be an arbitrary element of $\mathbb{Q}$. To prove that $A = \{x \in \mathbb{Q} \mid x < q\}$ is a Dedekind cut, Definition 6.1 tells us that it will suffice to show that $A \neq \emptyset$; $A \neq \mathbb{Q}$; if $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$; and if $r \in A$, then there is some $s \in A$ with $s > r$. We will take this one claim at a time.

To show that $A \neq \emptyset$, Definition 1.8 tells us that it will suffice to find an element of A. By Exercise 3.9d, q is not the first point of $\mathbb{Q}$. Thus, by Definition 3.3, there exists an object $x \in \mathbb{Q}$ such that $x < q$. By the definition of A, this implies that $x \in A$, as desired.

To show that $A \neq \mathbb{Q}$, Definition 1.2 tells us that it will suffice to find an element of $\mathbb{Q}$ that is not an element of A. By hypothesis, $q \in \mathbb{Q}$. By Exercise 3.9d, $q \not< q$. Therefore, $q \in \mathbb{Q}$ but $q \notin A$, as desired.

To show that if $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$, we let $r \in A$ and $s \in \mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s < r$ and seek to verify that $s \in A$. Since $r \in A$, $r < q$. This combined with the fact that $s < r$ implies by transitivity that $s < q$. Therefore, since $s \in \mathbb{Q}$ and $s < q$, $s \in A$, as desired.

To show that if $r \in A$, then there is some $s \in A$ with $s > r$, we let $r \in A$ and seek to find such an s. By the definition of A, $r < q$. Thus, by Additional Exercise 3.1, there exists a point $s \in \mathbb{Q}$ such that $r < s < q$. Since $s \in \mathbb{Q}$ and $s < q$, $s \in A$. It follows that s is the desired element of A satisfying $s > r$. $\square$

Proof of b. To prove that $A = \{x \in \mathbb{Q} \mid x \leq 0\}$ is not a Dedekind cut, Definition 6.1 tells us that it will suffice to show that A does have a last point. To show this, we will demonstrate that 0 is the last point of A. To demonstrate this, Definition 3.1 tells us that it will suffice to confirm that $0 \in A$ and for all $x \in A$, $x \leq 0$. Since $0 \leq 0$ and $0 \in \mathbb{Q}$, $0 \in A$. Additionally, by the definition of A, it is true that for all $x \in A$, $x \leq 0$. □

Proof of c. Let $B = \{x \in \mathbb{Q} \mid x < 0\}$ and let $C = \{x \in \mathbb{Q} \mid x^2 < 2\}$. To prove that $A = B \cup C$ is a Dedekind cut, Definition 6.1 tells us that it will suffice to show that $A \neq \emptyset$; $A \neq \mathbb{Q}$; if $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$; and if $r \in A$, then there is some $s \in A$ with $s > r$. We will take this one claim at a time.

To show that $A \neq \emptyset$, Definition 1.8 tells us that it will suffice to find an element of A. Since $-1 \in \mathbb{Q}$ and $-1 < 0$, $-1 \in B$. Therefore, by Definition 1.5, $-1 \in A$, as desired.

To show that $A \neq \mathbb{Q}$, Definition 1.2 tells us that it will suffice to find an element of $\mathbb{Q}$ that is not an element of A. Since $2 \in \mathbb{Q}$ and $2 \geq 0$, $2 \notin B$. Additionally, since $2^2 \geq 2$, $2 \notin C$. Therefore, by Definition 1.5, $2 \notin A$, as desired.

To show that if $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$, we let $r \in A$ and $s \in \mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s < r$ and seek to verify that $s \in A$. Since $r \in A$, Definition 1.5 tells us that $r \in B$ or $r \in C$. We now divide into two cases. Suppose first that $r \in B$. Then $s < r < 0$, which implies that $s \in B$, meaning that $s \in A$. Now suppose that $r \in C$. We divide into two cases again ($r \leq 0$ and $r > 0$). If $r \leq 0$, then $s < r \leq 0$ implies that $s < 0$. Thus, by the definition of B, $s \in B$, implying that $s \in A$. On the other hand, if $r > 0$, then $0 < s^2 < r^2 < 2$. Thus, by the definition of C, $s \in C$, implying that $s \in A$.

To show that A does not have a last point, suppose for the sake of contradiction that A has a last point p. We now divide into two cases ($p \leq 0$ and $p > 0$). Suppose first that $p \leq 0$. Since p is the last point of A, Definition 3.3 tells us that $x \leq p$ for all $x \in A$. But $1 \in A$ (since $1 \in \mathbb{Q}$ and $1^2 = 1 < 2$ implies $1 \in B$, implies $1 \in A$) and $1 > 0 \geq p$, a contradiction. Now suppose that $p > 0$. Definition 3.3 tells us that $p \in A$, but the condition that $p > 0$ means $p \notin B$, so we must have $p \in C$. However, by the proof of Exercise 4.24, $\frac{2(p+1)}{p+2}$ will be an element of B (and therefore A) that is greater than p no matter how large p is, a contradiction. □

6.2 Discussion

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