# Script 11

# Limits and Continuity

## 11.1 Journal

3/4: Throughout this sheet, we let $f, g : A \to \mathbb{R}$ be real-valued functions with domain $A \subset \mathbb{R}$, unless otherwise specified.

**Definition 11.1.** Let $a \in LP(A) \subset \mathbb{R}$. A **limit** of $f$ at $a$ is a number $L \in \mathbb{R}$ satisfying the following condition: for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

**Lemma 11.2.** *Limits are unique: if $L$ and $L'$ are both limits of $f$ at a point $a$, then $L = L'$.*

*Proof.* Let the limit of $f$ at $a$ be $L$, and suppose for the sake of contradiction that the limit of $f$ at $a$ is also equal to $L'$ where $L \neq L'$. If we let $\epsilon = \frac{|L-L'|}{2}$, then $\epsilon > 0$ by Script 8. Thus, by consecutive applications of Definition 11.1, we have that there exists a $\delta_1 > 0$ such that if $x \in A$ and $0 < |x-a| < \delta_1$, then $|f(x)-L| < \epsilon$; and that there exists a $\delta_2 > 0$ such that if $x \in A$ and $0 < |x - a| < \delta_2$, then $|f(x) - L'| < \epsilon$. Now choose $\delta = \min(\delta_1, \delta_2)$. This makes it so that for any $x \in A$ such that $0 < |x - a| < \delta$, we have $|f(x) - L| < \epsilon$ and $|f(x) - L'| < \epsilon$, so

$$\begin{aligned}
|L - L'| &= |L - f(x) + f(x) - L'| \\
&\leq |L - f(x)| + |f(x) - L'| && \text{Lemma 8.8} \\
&= |f(x) - L| + |f(x) - L'| && \text{Exercise 8.5} \\
&< 2\epsilon \\
&= |L - L'|
\end{aligned}$$

But this implies that $|L - L'| < |L - L'|$, a contradiction. □

**Definition 11.3.** If $L$ is the limit of $f$ at $a$, we write

$$\lim_{x \to a} f(x) = L$$

**Exercise 11.4.** Give an example of a set $A \subset \mathbb{R}$, a function $f : A \to \mathbb{R}$, and a point $a \in LP(A)$ such that $\lim_{x\to a} f(x)$ does not exist.

*Proof.* Let $A = \mathbb{R}$, let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

and consider $0 \in LP(\mathbb{R})$ (by Corollary 5.4). Now suppose for the sake of contradiction that $\lim_{x\to a} f(x) = L$. Then by Definitions 11.3[1] and 11.1, for all $\epsilon > 0$, there exists some $\delta > 0$ such that if $x \in \mathbb{R}$ and

[1] I will not cite this definition again for the sake of concision.

$0 < |x - 0| = |x| < \delta$, then $|f(x) - L| < \epsilon$. If we let $\epsilon = 0.5$, then $\epsilon > 0$. Choosing a corresponding $\delta$, we have by an extension of Exercise 8.9 that all $x \in (-\delta, 0) \cup (0, \delta)$ satisfy $|f(x) - L| < \epsilon$. This would include objects $y \in (0, \delta)$ and $z \in (-\delta, 0)$. We have by the definition of $f$ that $f(y) = 1$ and $f(z) = 0$; thus, we have

$$\begin{aligned} 1 &= |f(y) - f(z)| \\ &= |f(y) - L + L - f(z)| \\ &\leq |f(y) - L| + |f(z) - L| \\ &< 0.5 + 0.5 \\ &= 1 \end{aligned}$$

But this implies that $1 < 1$, a contradiction. □

**Theorem 11.5.** *Let $x \in A$. Then the following are equivalent:*

(*a*) *$f$ is continuous at $x$.*

(*b*) *For every $\epsilon > 0$, there exists $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$.*

(*c*) *Either $x \notin LP(A)$ or $\lim_{y \to x} f(y) = f(x)$.*

*Proof.* To illustrate that statements $a$-$c$ are equivalent, it will suffice to verify that $a \Rightarrow b$, $b \Rightarrow c$, and $c \Rightarrow a$. Note that this foregoes the need for explicit proofs of "backwards implications" such as $b \Rightarrow a$ since that implication, for example, follows from $b \Rightarrow c \Rightarrow a$. Let's begin.

To prove that $a \Rightarrow b$, let $\epsilon > 0$ be arbitrary and look to find a $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then $|f(y) - f(x)|$.

We first locate $\delta$. To do so, begin by defining the region $R = (f(x) - \epsilon, f(x) + \epsilon)$ (clearly $R$ contains $f(x)$). Since $R$ is open by Corollary 4.11 and $f$ is continuous at $x$, we have by Definition 9.9 that there exists an open set $S$ with $x \in S$ such that $S \cap A \subset f^{-1}(R)$. It follows by Theorem 4.10 that there exists a region $(a, b)$ such that $x \in (a, b)$ and $(a, b) \subset S$. Thus, since $(a, b)$ is an open interval by Corollary 4.11 and Lemma 8.3, we have by Lemma 8.10 that there exists a number $\delta > 0$ such that $(x - \delta, x + \delta) \subset (a, b)$.

As we will now show, this $\delta$ satisfies the desired property. Let $y$ be an arbitrary element of $A$ such that $|y - x| < \delta$. Then by Exercise 8.9, $y \in (x - \delta, x + \delta)$. It follows by consecutive applications of Definition 1.3 that $y \in (a, b)$, hence $y \in S$. This result combined with the fact that $y \in A$ by definition implies by Definition 1.6 that $y \in S \cap A$. Thus, by Definition 1.3 again, $y \in f^{-1}(R)$. Consequently, by Definition 1.18, $f(y) \in R$. Therefore, by Exercise 8.9 one more time, $|f(y) - f(x)| < \epsilon$.

To prove that $b \Rightarrow c$, let $x$ be an arbitrary element of $\mathbb{R}$. We divide into two cases ($x \notin LP(A)$ and $x \in LP(A)$). If $x \notin LP(A)$, then we are done. If $x \in LP(A)$, then by the hypothesis, we know that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in A$ and $0 < |y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. It follows by Definition 11.1 that $f(x)$ is the limit of $f$ at $x$, meaning that $\lim_{y \to x} f(y) = f(x)$, and we are done.

To prove that $c \Rightarrow a$, we divide into two cases ($x \notin LP(A)$ and $\lim_{y \to x} f(y) = f(x)$).

Suppose first that $x \notin LP(A)$. To demonstrate that $f$ is continuous at $x$, Definition 9.9 tells us that it will suffice to confirm that for every region $R$ containing $f(x)$, there exists an open set $S$ containing $x$ such that $S \cap A \subset f^{-1}(R)$. Let $R$ be an arbitrary region with $f(x) \in R$. Since $x \notin LP(A)$, Definition 3.13 asserts that there exists a region (hence an open set by Corollary 4.11) $S$ such that $x \in S$ and $S \cap (A \setminus \{x\}) = \emptyset$. It follows by Script 1 that $S \cap A = \{x\}$. But since $f(x) \in R$ implies by Definition 1.18 that $x \in f^{-1}(R)$, we have by Definition 1.3 that $S \cap A \subset f^{-1}(R)$. Therefore, $S$ is an open set containing $x$ such that $S \cap A \subset f^{-1}(R)$.

Now suppose that $\lim_{y \to x} f(y) = f(x)$. To demonstrate that $f$ is continuous at $x$, Definition 9.9 tells us that it will suffice to confirm that for every region $(a, b)$ containing $f(x)$, there exists an open set $S$ containing $x$ such that $S \cap A \subset f^{-1}((a, b))$. Let $(a, b)$ be an arbitrary region with $f(x) \in (a, b)$. Then since $(a, b)$ is an open interval by Lemma 8.3, Lemma 8.10 asserts that there exists $\epsilon > 0$ such that $(f(x) - \epsilon, f(x) + \epsilon) \subset (a, b)$. With regard to this $\epsilon$, since $\lim_{y \to x} f(y) = f(x)$ by hypothesis, we have by Definition 11.1 that there exists a $\delta > 0$ such that if $y \in A$ and $0 < |y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $S = (x - \delta, x + \delta)$. Clearly, $S$ contains $x$. Additionally, we can confirm that $S \cap A \subset f^{-1}((a, b))$: if we let $y$ be an arbitrary element of $S \cap A$, then Definition 1.6 asserts that $y \in S$ and $y \in A$. It follows from the former condition by Exercise 8.9 that $|y - x| < \delta$. This combined with the fact that $y \in A$ implies that $|f(y) - f(x)| < \epsilon$. Thus, by Exercise

8.9 again, $f(y) \in (f(x) - \epsilon, f(x) + \epsilon)$. Consequently, by Definition 1.3, $f(y) \in (a, b)$. As such, we have by Definition 1.18 that $y \in f^{-1}((a, b))$, as desired. ☐

**Exercise 11.6.**

(a) Let $a, b \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = ax + b$. Show that $f$ is continuous at every $x \in \mathbb{R}$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$. Show that $f$ is not continuous at 0.

*Proof of a.* To prove that $f$ is continuous at every $x \in \mathbb{R}$, let $x$ be an arbitrary element of $\mathbb{R}$; then by Theorem 11.5, it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. We divide into two cases ($a = 0$ and $a \neq 0$). If $a = 0$, then choose $\delta = 1$[2]. This makes it so that for any $y \in A$ such that $|y - x| < \delta = 1$, we have $|f(y) - f(x)| = |b - b| = 0 < \epsilon$, as desired. If $a \neq 0$, then choose $\delta = \frac{\epsilon}{|a|}$. This makes it so that for any $y \in A$ such that $|y - x| < \delta = \frac{\epsilon}{|a|}$, we have

$$\begin{aligned} |a|\,|y - x| &< \epsilon \\ |ay - ax| &< \epsilon \\ |ay + b - (ax + b)| &< \epsilon \\ |f(y) - f(x)| &< \epsilon \end{aligned}$$

as desired. ☐

*Proof of b.* To prove that $f$ is not continuous at 0, Theorem 11.5 tells us that it will suffice to show that for some $\epsilon > 0$, no $\delta > 0$ exists such that if $x \in \mathbb{R}$ and $|x - 0| = |x| < \delta$, then $|f(x) - 1| < \epsilon$. Let $\epsilon = 1$, and suppose for the sake of contradiction that $\delta > 0$ is a number such that if $x \in \mathbb{R}$ and $|x| < \delta$, then $|f(x) - 1| < \epsilon$. Clearly, $0 \in \mathbb{R}$ and by the definition of $\delta$ and Definition 8.4, $|0| < \delta$. However, $|f(x) - 1| = |0 - 1| = 1 \nless 1 = \epsilon$, a contradiction. ☐

3/9: **Exercise 11.7.** Show that the absolute value function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ is continuous.

*Proof.* To prove that the absolute value function is continuous, Theorem 9.10 tells us that it will suffice to show that it is continuous at every $x \in \mathbb{R}$. To do this, let $x$ be an arbitrary element of $\mathbb{R}$; then by Theorem 11.5, it will suffice to verify that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in \mathbb{R}$ and $|y - x| < \delta$, then $||y| - |x|| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Then choose $\delta = \epsilon$. This makes it so that for any $y \in A$ such that $|y - x| < \delta$, we have $||y| - |x|| \leq |y - x| < \delta = \epsilon$ (with the first inequality coming from Lemma 8.8), as desired. ☐

Given real-valued functions $f$ and $g$, we define new functions $f + g$, $fg$, and $\frac{1}{f}$ by

$$(f + g)(x) = f(x) + g(x) \qquad (fg)(x) = f(x) \cdot g(x) \qquad \frac{1}{f}(x) = \frac{1}{f(x)}$$

where $f(x) \neq 0$ in the definition of $\frac{1}{f}$. We wish to understand the limits of $f + g$, $fg$, and $\frac{1}{f}$ in terms of the limits of $f$ and $g$.

**Lemma 11.8.** *If $\lim_{x \to a} f(x) = L > 0$, then there exists a region $R$ with $a \in R$ such that $f(x) > 0$ for all $x \in R \cap A$ such that $x \neq a$. Moreover, if $f$ is continuous at $a$, then $f(x) > 0$ for all $x \in R \cap A$. The analogous statement is true if $\lim_{x \to a} f(x) = L < 0$.*

*Proof.* We divide into two cases ($\lim_{x \to a} f(x) = L > 0$ and $\lim_{x \to a} f(x) = L < 0$).

Suppose first that $\lim_{x \to a} f(x) = L > 0$. Choose $\epsilon = L$. Then we have by Definition 11.1 that there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|f(x) - L| < L$. Let $R = (a - \delta, a + \delta)$. Clearly, $a \in R$. Now let $x$ be an arbitrary element of $R \cap A$ such that $x \neq a$. It follows from the first condition by

[2] This choice is arbitrary; it can be any nonzero value, as we will soon see.

Definition 1.6 that $x \in R$ and $x \in A$. Thus, we have by Exercise 8.9 that $|x - a| < \delta$. Additionally, since $x \neq a$, Definition 3.1 asserts that $x > a$ or $x < a$, i.e., $x - a > 0$ or $x - a < 0$; either way, Script 8 implies that $0 < |x - a|$. To recap, we know that $x \in A$ and $0 < |x - a| < \delta$, so we have by the initial implication that $|f(x) - L| < L = \epsilon$. Therefore, by the lemma from Exercise 8.9, we have $-L < f(x) - L < L$, i.e., $0 < f(x)$ (which we obtain by adding $L$ to both sides of the inequality as permitted by Definition 7.21).

Moreover, if $f$ is continuous at $a$, then by Theorem 11.5, $a \notin LP(A)$ or $\lim_{x\to a} f(x) = f(a)$. But by Definition 11.1, $a \in LP(A)$, so we have $\lim_{x\to a} f(x) = f(a)$. The first part of this proof guarantees the existence of a region $R$ such that $f(x) > 0$ for all $x \in R \cap A$ such that $x \neq a$. The fact that $f(a) = L > 0$ takes care of the case where $x = a$.

The proof is symmetric in the other case. □

3/11: **Theorem 11.9.** *Suppose that* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$. *Then*

(*a*) $\lim_{x\to a}(f + g)(x) = L + M$.

(*b*) $\lim_{x\to a}(fg)(x) = L \cdot M$.

(*c*) *Suppose that* $\lim_{x\to a} f(x) = L \neq 0$. *Then* $\lim_{x\to a} \frac{1}{f}(x) = \frac{1}{L}$.

*Proof of a.* To prove that $\lim_{x\to a}(f + g)(x) = L + M$, Definition 11.1 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|(f + g)(x) - (L + M)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Consider $\frac{\epsilon}{2}$. Since $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, we have by consecutive applications of Definition 11.1 that there exists a $\delta_1 > 0$ such that if $x \in A$ and $0 < |x - a| < \delta_1$, then $|f(x) - L| < \frac{\epsilon}{2}$; and there exists a $\delta_2 > 0$ such that if $x \in A$ and $0 < |x - a| < \delta_2$, then $|g(x) - M| < \frac{\epsilon}{2}$. Now choose $\delta = \min(\delta_1, \delta_2)$. This makes it so that for any $x \in A$ such that $0 < |x - a| < \delta$, we have $|f(x) - L| < \frac{\epsilon}{2}$ and $|g(x) - M| < \frac{\epsilon}{2}$, so

$$\begin{aligned}
|(f + g)(x) - (L + M)| &= |f(x) - L + g(x) - M| \\
&\leq |f(x) - L| + |g(x) - M| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

as desired. □

*Proof of b.* To prove that $\lim_{x\to a}(fg)(x) = LM$, Definition 11.1 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|(fg)(x) - LM| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ (and $\min(\frac{\epsilon}{2(|M|+1)}, 1)$ and $\frac{\epsilon}{2(|L|+1)}$ are both greater than zero), we have by consecutive applications of Definition 11.1 that there exists a $\delta_1 > 0$ such that if $x \in A$ and $0 < |x - a| < \delta_1$, then $|f(x) - L| < \min(\frac{\epsilon}{2(|M|+1)}, 1)$[3]; and there exists a $\delta_2 > 0$ such that if $x \in A$ and $0 < |x - a| < \delta_2$, then $|g(x) - M| < \frac{\epsilon}{2(|L|+1)}$. Now choose $\delta = \min(\delta_1, \delta_2)$. This makes it so that for any $x \in A$ such that $0 < |x - a| < \delta$, we have $|f(x) - L| < \min(\frac{\epsilon}{2(|M|+1)}, 1)$ and $|g(x) - M| < \frac{\epsilon}{2(|L|+1)}$.

Before we get into the body of the proof, we need a couple of preliminary results. By Script 8, we have $|a| = |a-b+b| \leq |a-b|+|b|$, so $|a|-|b| \leq |a-b|$. Thus, since $|f(x)-L| < 1$, we have $|f(x)|-|L| \leq |f(x)-L| < 1$, which means that $|f(x)| < 1 + |L|$. Additionally, we have

$$\begin{aligned}
f(x)g(x) - LM &= f(x)g(x) - f(x)M + f(x)M - LM \\
&= f(x)(g(x) - M) + M(f(x) - L)
\end{aligned}$$

[3]When we choose a compound $\epsilon$ (i.e., one that makes use of a min expression), it means that $|f(x) - L| < \frac{\epsilon}{2(|M|+1)}$ and $|f(x) - L| < 1$. Basically, whichever quantity is smaller is what min evaluates to, and whichever is bigger is still greater than $|f(x) - L|$ by transitivity.

With these results, we are ready to introduce the main inequality:

$$\begin{aligned}
|(fg)(x) - LM| &= |f(x)(g(x) - M) + M(f(x) - L)| \\
&\leq |f(x)| \cdot |g(x) - M| + |M| \cdot |f(x) - L| \\
&< (1 + |L|) \cdot \frac{\epsilon}{2(|L| + 1)} + |M| \cdot \frac{\epsilon}{2(|M| + 1)} \\
&= \frac{\epsilon}{2} \cdot \frac{1 + |L|}{1 + |L|} + \frac{\epsilon}{2} \cdot \frac{|M|}{|M| + 1} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

$\square$

*Proof of c.* To prove that $\lim_{x \to a} \frac{1}{f}(x) = \frac{1}{L}$, Definition 11.1 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|\frac{1}{f}(x) - \frac{1}{L}| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{x \to a} f(x) = L$ and $\min(\frac{|L|}{2}, \frac{\epsilon|L|^2}{2}) > 0$, we have by Definition 11.1 that there exists a $\delta_1 > 0$ such that if $x \in A$ and $0 < |x - a| < \delta_1$, then $|f(x) - L| < \min(\frac{|L|}{2}, \frac{\epsilon|L|^2}{2})$. Additionally, since $L \neq 0$, Theorem 11.8 asserts that there exists a region $R$ containing $a$ such that for all $x \in (R \cap A) \setminus \{a\}$, $f(x) \neq 0$ (since $L \neq 0$ implies $L > 0$ or $L < 0$). It follows by Lemma 8.10 that there exists a $\delta_2$ such that $(a - \delta_2, a + \delta_2) \subset R$. Let $\delta = \min(\delta_1, \delta_2)$.

At this point, we know by this definition of $\delta$ that if $x \in A$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \min(\frac{|L|}{2}, \frac{\epsilon|L|^2}{2})$ and $f(x) \neq 0$. Before we get into the body of the proof, we need one more preliminary result. It follows from the fact that $|f(x) - L| < \frac{|L|}{2}$ that

$$\begin{aligned}
|L| &= 2|L| - |L| \\
&= 2(|L| - |f(x)| + |f(x)|) - |L| \\
&\leq 2(|L - f(x)| + |f(x)|) - |L| \\
&= 2(|f(x) - L| + |f(x)|) - |L| \\
&< 2\left(\frac{|L|}{2} + |f(x)|\right) - |L| \\
&= |L| + 2|f(x)| - |L| \\
&= 2|f(x)|
\end{aligned}$$

With these results, we are ready to introduce the main inequality:

$$\begin{aligned}
\left|\frac{1}{f}(x) - \frac{1}{L}\right| &= \left|\frac{1}{f(x)} - \frac{1}{L}\right| \\
&= \left|\frac{L - f(x)}{f(x) \cdot L}\right| \\
&= \frac{|f(x) - L|}{|f(x)| \cdot |L|} \\
&< \frac{\epsilon|L|^2}{2|f(x)| \cdot |L|} \\
&= \frac{\epsilon|L|}{2|f(x)|} \\
&< \frac{\epsilon|L|}{|L|} \\
&= \epsilon
\end{aligned}$$

$\square$

**Corollary 11.10.** *If $f$ and $g$ are continuous at $a$, then $f+g$ and $fg$ are continuous at $a$. Also, $\frac{1}{f}$ and $\frac{g}{f}$ are continuous at $a$, provided that $f(a) \neq 0$.*

*Proof.* Since $f$ is continuous at $a$, we have by Theorem 11.5 that either $a \notin LP(A)$ or $\lim_{x\to a} f(x) = f(a)$. Similarly, we have that either $a \notin LP(A)$ or $\lim_{x\to a} g(x) = g(a)$. We divide into four cases ($a \notin LP(A)$ and $a \notin LP(A)$, $a \notin LP(A)$ and $\lim_{x\to a} g(x) = g(a)$, $\lim_{x\to a} f(x) = f(a)$ and $a \notin LP(A)$, and $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$). If $a \notin LP(A)$ (this takes care of the first three cases), then by Theorem 11.5, $f+g$ is continuous at $a$. If $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$, then by Theorem 11.9, $\lim_{x\to a}(f+g)(x) = f(a)+g(a) = (f+g)(a)$. Therefore, by Theorem 11.5, $f+g$ is continuous at $a$.

The proofs of the second and third cases are symmetric to that of the first. The fourth case can be handled by letting $\frac{g}{f} = g \cdot \frac{1}{f}$ and applying the third and second cases. □

**Definition 11.11.** A **polynomial** (in one variable with real coefficients) is a function $f$ of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

for some $n \in \mathbb{N}\cup\{0\}$, where $a_i \in \mathbb{R}$ for $0 \leq i \leq n$. A **rational function** (in one variable with real coefficients) is a function of the form $h(x) = \frac{f(x)}{g(x)}$ where $f$ and $g$ are polynomials in one variable with real coefficients.

**Corollary 11.12.** *Polynomials in one variable with real coefficients are continuous. A rational function in one variable with real coefficients $h(x) = \frac{f(x)}{g(x)}$ is continuous at all $a \in \mathbb{R}$ where $g(a) \neq 0$.*

*Proof.* We induct on the degree $n$ of the polynomial. For the base case $n = 0$, we have by Definition 11.11 that $f$ is a function of the form $f(x) = a_0$ for some $a_0 \in \mathbb{R}$. Thus, by Exercise 11.6, $f$ is continuous at every $x \in \mathbb{R}$ (since $ax+b$ is continuous for $a = 0$ and $b = a_0$). Therefore, by Theorem 9.10, $f$ is continuous. Now suppose inductively that all functions of the form $f(x) = a_n x^n + \cdots + a_0$ are continuous; we seek to prove that all functions of the form $g(x) = a_{n+1}x^{n+1} + \cdots + a_0$ are continuous. By the inductive hypothesis, we know that $x^n$ is continuous. Thus, by Theorem 9.10, $x^n$ is continuous at every $x \in \mathbb{R}$. Additionally, by Exercise 11.6, $a_{n+1}x$ (where $a_{n+1}$ is an arbitrary element of $\mathbb{R}$) is continuous at every $x \in \mathbb{R}$. The last two results combined with Corollary 11.10 tell us that $a_{n+1}x \cdot x^n = a_{n+1}x^{n+1}$ is continuous at every $x \in \mathbb{R}$. In addition, since $f$ is continuous, Theorem 9.10 asserts that $f$ is continuous at every $x \in \mathbb{R}$. Thus, by Corollary 11.10 again, $g(x) = a_{n+1}x^{n+1} + f(x) = a_{n+1}x^{n+1} + \cdots + a_0$ is continuous at every $x \in \mathbb{R}$. By one more application of Theorem 9.10, $g(x)$ is continuous.

Now we move on to proving that $h(x) = \frac{f(x)}{g(x)}$ is continuous at all $a \in \mathbb{R}$ such that $g(a) \neq 0$. By the above, $f$ and $g$ (as polynomials) are continuous. Thus, by Theorem 9.10, $f$ and $g$ are continuous at every $a \in \mathbb{R}$. Therefore, by Corollary 11.10, $h = \frac{f}{g}$ is continuous at all $a \in \mathbb{R}$, provided that $g(a) \neq 0$, as desired. □

Now we want to look at limits of the composition of functions. We assume here (for 11.13-11.15) that $a \in A$, $g : A \to \mathbb{R}$, and $f : I \to \mathbb{R}$, where $I$ is an open interval containing $\overline{g(A)}$. It is not quite true in general that if $\lim_{x\to a} g(x) = M$ and $\lim_{y\to M} f(y) = L$, then $\lim_{x\to a} f(g(x)) = L$, but it is true in some cases.

**Theorem 11.13.** *If $\lim_{x\to a} g(x) = M$ and $f$ is continuous at $M$, then $\lim_{x\to a} f(g(x)) = f(M)$.*

*Proof.* To prove that $\lim_{x\to a} f(g(x)) = f(M)$, Definition 11.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x-a| < \delta$, then $|f(g(x)) - f(M)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Then since $f$ is continuous at $M$, Theorem 11.5 implies that there exists a $\delta' > 0$ such that if $y \in I$ and $|y - M| < \delta'$, then $|f(y) - f(M)| < \epsilon$. It follows by Definition 11.1 that there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x-a| < \delta$, then $|g(x) - M| < \delta'$.

We will now prove that this $\delta$ is the desired $\delta$. Let $x$ be an arbitrary element of $A$ that satisfies $0 < |x-a| < \delta$. Then we know that $|g(x)-M| < \delta'$. Additionally, it follows from the fact that $x \in A$ by Definition 1.18 that $g(x) \in g(A)$. Thus, by Definition 4.4, $g(x) \in \overline{g(A)}$. Consequently, by Definition 1.3, $g(x) \in I$. Indeed, we now know that $g(x) \in I$ and $|g(x) - M| < \delta'$, so we can determine that $|f(g(x)) - f(M)| < \epsilon$, as desired. □

3/17: **Remark 11.14.** This theorem can also be rewritten as

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

which can be remembered as "limits pass through continuous functions."

**Corollary 11.15.** *If $g$ is continuous at $a$ and $f$ is continuous at $g(a)$, then $f \circ g$ is continuous at $a$.*

*Proof.* Since $g$ is continuous at $a$, we have by Theorem 11.5 that either $a \notin LP(A)$ or $\lim_{x\to a} g(x) = g(a)$. We now divide into two cases. If $a \notin LP(A)$, then by Theorem 11.5, $f \circ g$ is continuous at $a$. If $\lim_{x\to a} g(x) = g(a)$, then this combined with the fact that $f$ is continuous at $g(a)$ implies by Theorem 11.13 that $\lim_{x\to a} f(g(x)) = f(g(a))$. It follows by Definition 1.25 that $\lim_{x\to a}(f \circ g)(x) = (f \circ g)(a)$. Therefore, by Theorem 11.5, $f \circ g$ is continuous at $a$. $\square$