Script 6

Construction of the Real Numbers

6.1 Journal

1/12: **Definition 6.1.** A subset A of $\mathbb{Q}$ is said to be a **cut** (or **Dedekind cut**) if it satisfies the following:

(a) $A \neq \emptyset$ and $A \neq \mathbb{Q}$.

(b) If $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$.

(c) A does not have a last point; i.e., if $r \in A$, then there is some $s \in A$ with $s > r$.

We denote the collection of all cuts by $\mathbb{R}$.

Lemma 6.2. *Let A be a Dedekind cut and $x \in \mathbb{Q}$. Then $x \notin A$ if and only if x is an upper bound for A.*

Proof. Suppose first that x is an element of $\mathbb{Q}$ such that $x \notin A$. To prove that x is an upper bound for A, Definition 5.6 tells us that it will suffice to show that for all $r \in A$, $r \leq x$. Let r be an arbitrary element of A. Then since $r \in A$, $x \in \mathbb{Q}$, and $x \notin A$, the contrapositive of Definition 6.1b asserts that $x \nless r$. Therefore, $r \leq x$, as desired.

Now suppose that x is an upper bound for A. By Definition 5.6, this implies that for all $r \in A$, $r \leq x$. Therefore, since there is no $r \in A$ with $r > x$, by the contrapositive of Definition 6.1c, $x \notin A$, as desired. ☐

Exercise 6.3.

(a) Prove that for any $q \in \mathbb{Q}$, $\{x \in \mathbb{Q} \mid x < q\}$ is a Dedekind cut. We then define $\mathbf{0} = \{x \in \mathbb{Q} \mid x < 0\}$.

(b) Prove that $\{x \in \mathbb{Q} \mid x \leq 0\}$ is not a Dedekind cut.

(c) Prove that $\{x \in \mathbb{Q} \mid x < 0\} \cup \{x \in \mathbb{Q} \mid x^2 < 2\}$ is a Dedekind cut.

Proof of a. Let q be an arbitrary element of $\mathbb{Q}$. To prove that $A = \{x \in \mathbb{Q} \mid x < q\}$ is a Dedekind cut, Definition 6.1 tells us that it will suffice to show that $A \neq \emptyset$; $A \neq \mathbb{Q}$; if $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$; and if $r \in A$, then there is some $s \in A$ with $s > r$. We will take this one claim at a time.

To show that $A \neq \emptyset$, Definition 1.8 tells us that it will suffice to find an element of A. By Exercise 3.9d, q is not the first point of $\mathbb{Q}$. Thus, by Definition 3.3, there exists an object $x \in \mathbb{Q}$ such that $x < q$. By the definition of A, this implies that $x \in A$, as desired.

To show that $A \neq \mathbb{Q}$, Definition 1.2 tells us that it will suffice to find an element of $\mathbb{Q}$ that is not an element of A. By hypothesis, $q \in \mathbb{Q}$. By Exercise 3.9d, $q \nless q$. Therefore, $q \in \mathbb{Q}$ but $q \notin A$, as desired.

To show that if $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$, we let $r \in A$ and $s \in \mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s < r$ and seek to verify that $s \in A$. Since $r \in A$, $r < q$. This combined with the fact that $s < r$ implies by transitivity that $s < q$. Therefore, since $s \in \mathbb{Q}$ and $s < q$, $s \in A$, as desired.

To show that if $r \in A$, then there is some $s \in A$ with $s > r$, we let $r \in A$ and seek to find such an s. By the definition of A, $r < q$. Thus, by Additional Exercise 3.1, there exists a point $s \in \mathbb{Q}$ such that $r < s < q$. Since $s \in \mathbb{Q}$ and $s < q$, $s \in A$. It follows that s is the desired element of A which satisfies $s > r$. ☐

Proof of b. To prove that $A = \{x \in \mathbb{Q} \mid x \leq 0\}$ is not a Dedekind cut, Definition 6.1 tells us that it will suffice to show that A *does* have a last point. To show this, we will demonstrate that 0 is the last point of A. To demonstrate this, Definition 3.1 tells us that it will suffice to confirm that $0 \in A$ and for all $x \in A$, $x \leq 0$. Since $0 \leq 0$ and $0 \in \mathbb{Q}$, $0 \in A$. Additionally, by the definition of A, it is true that for all $x \in A$, $x \leq 0$. □

Proof of c. Let $B = \{x \in \mathbb{Q} \mid x < 0\}$ and let $C = \{x \in \mathbb{Q} \mid x^2 < 2\}$. To prove that $A = B \cup C$ is a Dedekind cut, Definition 6.1 tells us that it will suffice to show that $A \neq \emptyset$; $A \neq \mathbb{Q}$; if $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$; and if $r \in A$, then there is some $s \in A$ with $s > r$. We will take this one claim at a time.

To show that $A \neq \emptyset$, Definition 1.8 tells us that it will suffice to find an element of A. Since $-1 \in \mathbb{Q}$ and $-1 < 0$, $-1 \in B$. Therefore, by Definition 1.5, $-1 \in A$, as desired.

To show that $A \neq \mathbb{Q}$, Definition 1.2 tells us that it will suffice to find an element of $\mathbb{Q}$ that is not an element of A. Since $2 \geq 0$, $2 \notin B$. Additionally, since $2^2 \geq 2$, $2 \notin C$. Therefore, by Definition 1.5, $2 \notin A$, as desired.

To show that if $r \in A$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A$, we let $r \in A$ and $s \in \mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s < r$ and seek to verify that $s \in A$. We divide into two cases ($s < 0$ and $s \geq 0$). Suppose first that $s < 0$. Then $s \in B$, meaning that $s \in A$. Now suppose that $s \geq 0$. Then by Script 0, we have $0 \leq s^2 < r^2 < 2$. Thus, by the definition of C, $s \in C$, implying that $s \in A$.

To show that A does not have a last point, suppose for the sake of contradiction that A has a last point p. We now divide into two cases ($p \leq 0$ and $p > 0$). Suppose first that $p \leq 0$. Since p is the last point of A, Definition 3.3 tells us that $x \leq p$ for all $x \in A$. But $1 \in A$ (since $1 \in \mathbb{Q}$ and $1^2 = 1 < 2$ implies $1 \in B$, implies $1 \in A$) and $1 > 0 \geq p$, a contradiction. Now suppose that $p > 0$. Definition 3.3 tells us that $p \in A$, but the condition that $p > 0$ means $p \notin B$, so we must have $p \in C$. However, by the proof of Exercise 4.24, $\frac{2(p+1)}{p+2}$ will be an element of B (and therefore A) that is greater than p no matter how large p is, a contradiction. □

Definition 6.4. If $A, B \in \mathbb{R}$, we say that $A < B$ if A is a proper subset of B.

Exercise 6.5. Show that $\mathbb{R}$ satisfies Axioms 1, 2, and 3.

Proof. By Exercise 6.3a, $\{x \in \mathbb{Q} \mid x < 0\} \in \mathbb{R}$ since $0 \in \mathbb{Q}$. Therefore, Axiom 1 is immediately satisfied.

Axiom 2 asserts that $\mathbb{R}$ must have an ordering $<$. As such, it will suffice to verify that the ordering given by Definition 6.4 satisfies the stipulations of Definition 3.1. To prove that $<$ satisfies the trichotomy, it will suffice to show that for all $A, B \in \mathbb{R}$, exactly one of the following holds: $A < B$, $B < A$, or $A = B$.

We first show that *no more than one* of the three statements can simultaneously be true. Let A, B be arbitrary elements of $\mathbb{R}$. We divide into three cases. First, suppose for the sake of contradiction that $A < B$ and $B < A$. By Definition 6.4, this implies that $A \subsetneq B$ and $B \subsetneq A$. Thus, by Definition 1.3, $A \subset B$, $B \subset A$, and $A \neq B$. But by Theorem 1.7, $A \subset B$ and $B \subset A$ implies that $A = B$, a contradiction. Second, suppose for the sake of contradiction that $A < B$ and $A = B$. By substitution, we have that $A < A$. But by Definitions 6.4 and 1.3, it follows that $A \neq A$. The proof of the third case ($B < A$ and $A = B$) is symmetric to that of the second case.

We now show that *at least one* of the three statements is always true. Let A, B be arbitrary elements of $\mathbb{R}$, and suppose for the sake of contradiction that $A \not< B$, $B \not< A$, and $A \neq B$. Since $A \not< B$ and $B \not< A$, we have by Definition 6.4 that $A \not\subsetneq B$ and $B \not\subsetneq A$. Thus, by Definition 1.3, $A \not\subset B$ or $A = B$, and $B \not\subset A$ or $A = B$. But $A \neq B$ by hypothesis, so it must be that $A \not\subset B$ and $B \not\subset A$. It follows from the first statement by Definition 1.3 that there exists an object $x \in A$ such that $x \notin B$, and there exists an object $y \in B$ such that $y \notin A$. Since $x \notin B$, Lemma 6.2 implies that x is an upper bound of B. Consequently, by Definition 5.6, $p \leq x$ for all $p \in B$, including y. Similarly, $p \leq y$ for all $p \in A$, including x. Thus, we have $y \leq x$ and $x \leq y$, implying that $x = y$. But since $y \in B$, this implies that $x \in B$, a contradiction.

To prove that $<$ is transitive, it will suffice to show that for all $A, B, C \in \mathbb{R}$, if $A < B$ and $B < C$, then $A < C$. Let A, B, C be arbitrary elements of $\mathbb{R}$ for which it is true that $A < B$ and $B < C$. By Definition 6.4, we have $A \subsetneq B$ and $B \subsetneq C$. Thus, by Script 1, $A \subsetneq C$. Therefore, by Definition 6.4, $A < C$.

Axiom 3 asserts that $\mathbb{R}$ must have no first or last point. We will take this one argument at a time

Suppose for the sake of contradiction that $\mathbb{R}$ has some first point A. Then by Definition 3.3, $A \leq X$ for every $X \in \mathbb{R}$. Now since A is a Dedekind cut, Definition 6.1 tells us that $A \neq \emptyset$. Thus, by Definition 1.8, there exists some $q \in A$. Additionally, $A \subset \mathbb{Q}$ by Definition 6.1, so $q \in A$ implies that $q \in \mathbb{Q}$. It follows by Exercise 6.3a that $B = \{x \in \mathbb{Q} \mid x < q\}$ is a Dedekind cut. We now seek to prove that $B \subsetneq A$. To do this, Definition 1.3 tells us that it will suffice to show that $B \neq A$ and $B \subset A$. To show that $B \neq A$, Definition 1.2 tells us that it will suffice to find an element of A that is not an element of B. Conveniently, q is clearly such an object. To show that $B \subset A$, Definition 1.3 tells us that we must confirm that every element of B is an element of A. Let p be an arbitrary element of B. Then by the definition of B, $p \in \mathbb{Q}$ and $p < q$. It follows by Definition 6.1b (which clearly applies to A) that $p \in A$, as desired. Having proven that $B \subsetneq A$, Definition 6.4 tells us that $B < A$. But this contradicts the previously demonstrated fact that $A \leq X$ for every $X \in \mathbb{R}$, including B.

Suppose for the sake of contradiction that $\mathbb{R}$ has some last point A. Then by Definition 3.3, $X \leq A$ for every $X \in \mathbb{R}$. Now since A is a Dedekind cut, Definition 6.1 tells us that $A \neq \mathbb{Q}$. Thus, by Definition 1.2, there exists some $q \in \mathbb{Q}$ such that $q \notin A$. It follows by Lemma 6.2 that q is an upper bound of A. Consequently, by Definition 5.6, $x \leq q$ for all $x \in A$. Additionally, by Exercise 6.3a, $B = \{x \in \mathbb{Q} \mid x < q + 1\}$[1] is a Dedekind cut. We now seek to prove that $A \subsetneq B$. As before, this means we must show that $A \neq B$ and $A \subset B$. To show that $A \neq B$, Definition 1.2 tells us that it will suffice to find an element of B that is not an element of A. Since $x \leq q$ for all $x \in A$ and $q < q + 0.5 < q + 1$, $q + 0.5 \notin A$ and $q + 0.5 \in B$ is one such desired object. To show that $A \subset B$, Definition 1.3 tells us that we must confirm that every element of A is an element of B. Let p be an arbitrary element of A. As an element of A, we know that $p \leq q$. Thus, $p < q + 1$, so $p \in B$, as desired. Having proven that $A \subsetneq B$, Definition 6.4 tells us that $A < B$. But this contradicts the previously demonstrated fact that $X \leq A$ for every $X \in \mathbb{R}$, including B. $\square$

1/14: **Lemma 6.6.** *A nonempty subset of $\mathbb{R}$ that is bounded above has a supremum.*

Proof. Let X be an arbitrary nonempty subset of $\mathbb{R}$ that is bounded above. To prove that $\sup X$ exists, we will show that $\sup X = U = \bigcup\{Y \mid Y \in X\}$. To show this, Definition 5.7 tells us that it will suffice to demonstrate that $U \in \mathbb{R}$, U is an upper bound of X, and if U' is an upper bound of X, then $U \leq U'$. Let's begin.

To demonstrate that $U \in \mathbb{R}$, Definition 6.1 tells us that it will suffice to confirm that $U \neq \emptyset$; $U \neq \mathbb{Q}$; if $r \in U$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in U$; and if $r \in U$, then there is some $s \in U$ with $s > r$.

As the union of a nonempty set of nonempty sets, Script 1 implies that $U \neq \emptyset$.

To demonstrate that $U \neq \mathbb{Q}$, Definition 1.2 tells us that it will suffice to find a point $p \in \mathbb{Q}$ such that $p \notin U$. Since X is bounded above, we have by Definition 5.6 that there exists a Dedekind cut $V \in \mathbb{R}$ such that $A \leq V$ for all $A \in X$. It follows by Definition 6.4 that $A \subset V$ for all $A \in X$. Thus, by Script 1, $U \subset V$. Now since V is a Dedekind cut, we know by Definition 6.1 that $V \subset \mathbb{Q}$ and $V \neq \mathbb{Q}$, meaning that there exists a point $p \in \mathbb{Q}$ such that $p \notin V$. Consequently, since $U \subset V$, $p \notin U$, as desired.

To demonstrate that if $r \in U$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in U$, we let $r \in U$ and $s \in \mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s < r$ and seek to verify that $s \in U$. Since $r \in U$, Definition 1.13 tells us that $r \in A$ for some $A \in X$. Thus, since A is a Dedekind cut, $s \in \mathbb{Q}$ and $s < r$ implies that $s \in A$. Therefore, $s \in U$.

To demonstrate that if $r \in U$, then there is some $s \in U$ with $s > r$, we let $r \in U$ and seek to find such an s. Since $r \in U$, Definition 1.13 tells us that $r \in A$ for some $A \in X$. Thus, since A is a Dedekind cut, there exists a point $s \in A$ with $s > r$. Therefore, $s \in U$.

To demonstrate that U is an upper bound of X, Definition 5.6 tells us that it will suffice to confirm that $A \leq U$ for all $A \in X$. To confirm this, Definition 6.4 tells us that it will suffice to verify that $A \subset U$ for all $A \in X$. But by an extension of Theorem 1.7b, this is true.

Now suppose for the sake of contradiction that there exists an upper bound U' of X such that $U' < U$. It follows by Definitions 6.4 and 1.3 that there exists a point $p \in U$ such that $p \notin U'$. Thus, by the former statement and Definition 1.13, $p \in A$ for some $A \in X$. Additionally, since U' is an upper bound of X, we have by Definitions 5.6 and 6.4 that $A \subset U'$ for all $A \in X$. But this implies by Definition 1.3 that $p \in U'$, a contradiction. $\square$

[1]Note that we add 1 to q to treat the case that $q = \sup A$, a case in which we would have $B = A$ if B were defined as $\{x \in \mathbb{Q} \mid x < q\}$.

1/19: **Exercise 6.7.** Show that $\mathbb{R}$ satisfies Axiom 4.

Proof. Suppose for the sake of contradiction that $\mathbb{R}$ does not satisfy Axiom 4. It follows that $\mathbb{R}$ is not connected, implying by Definition 4.22 that $\mathbb{R} = A \cup B$ where A, B are disjoint, nonempty, open sets. Since A, B are disjoint and nonempty, we know that there exist distinct objects $a \in A$ and $b \in B$. WLOG, let $a < b$.

We now seek to prove that the set $A \cap \underline{ab}$ is nonempty and bounded above. To prove that $A \cap \underline{ab}$ is nonempty, Definition 1.8 tells us that it will suffice to find an element of $A \cap \underline{ab}$. Since $a \in A$ and A is open, we have by Theorem 4.10 that there exists a region $\underline{cd}$ such that $a \in \underline{cd}$ and $\underline{cd} \subset A$. It follows by Definitions 3.10 and 3.6 that $a < d$, implying by Lemma 6.10[2] that there exists some point $x \in \mathbb{R}$ such that $c < a < x < d < b$ (note that $d < b$ since if $b < d$, then $b \in \underline{cd}$ would contradict the fact that $\underline{cd} \subset A$). Consequently, $x \in \underline{cd}$, meaning that $x \in A$, and $x \in \underline{ab}$. Therefore, $x \in A \cap \underline{ab}$, as desired. To prove that $A \cap \underline{ab}$ is bounded above, Definition 5.6 tells us that it will suffice to show that b is an upper bound of $A \cap \underline{ab}$. To show this, Definition 5.6 tells us that it will suffice to confirm that $y \leq b$ for all $y \in A \cap \underline{ab}$. Let y be an arbitrary element of $A \cap \underline{ab}$. Then by Definition 1.6, $y \in A$ and $y \in \underline{ab}$. It follows from the latter statement by Definitions 3.10 and 3.6 that $y < b$, i.e., $y \leq b$, as desired.

Having established that $A \cap \underline{ab} \subset \mathbb{R}$ is nonempty and bounded above, we can invoke Lemma 6.6 to learn that $A \cap \underline{ab}$ has a supremum $\sup(A \cap \underline{ab})$. We now divide into two cases ($\sup(A \cap \underline{ab}) \in A$ and $\sup(A \cap \underline{ab}) \in B$; it follows from the definitions of A and B that exactly one of these cases is true). Suppose first that $\sup(A \cap \underline{ab}) \in A$. Then since A is open, we have by Theorem 4.10 that there exists a region $\underline{ef}$ such that $\sup(A \cap \underline{ab}) \in \underline{ef}$ and $\underline{ef} \subset A$. It follows from the former condition that $\sup(A \cap \underline{ab}) < f$. Thus, by Lemma 6.10, there exists an object $z \in \mathbb{R}$ such that $e < \sup(A \cap \underline{ab}) < z < f < b$ (note that $f < b$ for the same reason that $d < b$). Consequently, $z \in \underline{ef}$, implying that $z \in A$, and $z \in \underline{ab}$. Thus, we have found an element of $A \cap \underline{ab}$ that is greater than $\sup(A \cap \underline{ab})$, contradicting Definitions 5.7 and 5.6. The proof is symmetric in the other case (except that we find an element of B less than $\sup(A \cap \underline{ab})$). $\square$

1/14: **Definition 6.8.** Let C be a continuum satisfying Axioms 1-4. Consider a subset $X \subset C$. We say that X is **dense** in C if every $p \in C$ is a limit point of X.

Lemma 6.9. *A subset $X \subset C$ is dense in C if and only if $\overline{X} = C$.*

Proof. Suppose first that $X \subset C$ is dense in C. To prove that $\overline{X} = C$, Definition 1.2 tells us that it will suffice to show that every point $p \in \overline{X}$ is an element of C and vice versa. Clearly, every element of $\overline{X}$ is an element of C. On the other hand, let p be an arbitrary element of C. Since X is dense in C, Definition 6.8 tells us that $p \in LP(X)$. Therefore, by Definitions 1.5 and 4.4, $p \in \overline{X}$.

Now suppose that $\overline{X} = C$. To prove that X is dense in C, Definition 6.8 tells us that it will suffice to show that every $p \in C$ is a limit point of X. Let p be an arbitrary element of C. By Corollary 5.4, this implies that $p \in LP(C)$. It follows that $p \in LP(\overline{X})$. Thus, by Definition 4.4, $p \in LP(X \cup LP(X))$. Consequently, by Theorem 3.20, $p \in LP(X)$ or $p \in LP(LP(X))$. We now divide into two cases. If $p \in LP(X)$, then we are done. On the other hand, if $p \in LP(LP(X))$, the lemma from Theorem 4.6 asserts that $p \in LP(X)$, and we are done again. $\square$

Our next goal is to prove that $\mathbb{Q}$ is dense in $\mathbb{R}$. Just to make sense of that statement, we need to decide how to think of $\mathbb{Q}$ as a subset of $\mathbb{R}$. For every rational number $q \in \mathbb{Q}$, define the corresponding real number as the Dedekind cut

$$i(q) = \{x \in \mathbb{Q} \mid x < q\}$$

For example, $\mathbf{0} = i(0)$. It can be verified that this gives a well-defined injective function $i : \mathbb{Q} \to \mathbb{R}$. We identify $\mathbb{Q}$ with its image $i(\mathbb{Q}) \subset \mathbb{R}$ so that the rational numbers $\mathbb{Q}$ are a subset of the real numbers $\mathbb{R}$. (Similarly, $\mathbb{N}$ and $\mathbb{Z}$ can be understood as subsets of $\mathbb{R}$.)

[2] We may use this lemma since it does not depend on this result, Definition 6.8, or Lemma 6.9.

Lemma 6.10. *Given $A, B \in \mathbb{R}$ with $A < B$, there exists $p \in \mathbb{Q}$ such that $A < i(p) < B$.*

Proof. Since $A < B$, Definition 6.4 tells us that $A \subsetneq B$. Thus, by Definition 1.3, there exists a point q such that $q \in B$ and $q \notin A$. Since $q \in B$ where B is a Dedekind cut, we have by Definition 6.1 that there exists a point $p \in B$ with $p > q$. Additionally, since $q \notin A$ implies that q is an upper bound of A by Lemma 6.2, we know by Definition 5.6 that $x \leq q$ for all $x \in A$. It follows since $q < p$ that $x \leq p$ for all $x \in A$, meaning by Definition 5.6 and Lemma 6.2 that $p \notin A$. Having established that $p, q \in B$, $p, q \notin A$, and $q < p$, we are now ready to prove that $A < i(p) < B$. Definition 6.4 tells us that we may do so by showing that $A \subsetneq i(p)$ and $i(p) \subsetneq B$. We will take this one argument at a time.

To show that $A \subsetneq i(p)$, Definition 1.3 tells us that it will suffice to verify that every element of A is an element of $i(p)$ and that there exists an element of $i(p)$ that is not an element of A. We treat the former statement first. As previously mentioned, $x \leq p$ for all $x \in A$. This combined with the fact that $p \notin A$ implies that $x < p$ for all $x \in A$. Thus, by the definition of $i(p)$, $x \in i(p)$ for all $x \in A$, as desired. As to the latter statement, since $q < p$, we have by the definition of $i(p)$ that $q \in i(p)$. However, we also know that $q \notin A$, as desired.

To show that $i(p) \subsetneq B$, we must verify symmetric arguments to before. For the former statement, let r be an arbitrary element of $i(p)$. Then by the definition of $i(p)$, $r < p$. Since $p \in B$ and $r \in \mathbb{Q}$ satisfy $r < p$, we have by Definition 6.1 that $r \in B$, as desired. As to the latter statement, p is clearly an element of B that is not an element of $i(p)$, as desired. $\square$

1/19: **Theorem 6.11.** *$i(\mathbb{Q})$ is dense in $\mathbb{R}$.*

Proof. To prove that $i(\mathbb{Q})$ is dense in $\mathbb{R}$, Definition 6.8 tells us that it will suffice to show the every point $X \in \mathbb{R}$ is a limit point of $i(\mathbb{Q})$. Let X be an arbitrary element of $\mathbb{R}$. To show that $X \in LP(i(\mathbb{Q}))$, Definition 3.13 tells us that it will suffice to verify that for every region $\underline{AB}$ with $X \in \underline{AB}$, we have $\underline{AB} \cap (i(\mathbb{Q}) \setminus \{X\}) \neq \emptyset$. Let $\underline{AB}$ be an arbitrary region with $X \in \underline{AB}$. It follows by Definitions 3.10 and 3.6 that $A < X < B$. Thus, by Lemma 6.10, there exists $p \in \mathbb{Q}$ such that $A < i(p) < X < B$. By Definitions 3.6 and 3.10, $i(p) \in \underline{AB}$. By Definition 1.18, $i(p) \in i(\mathbb{Q})$. By Exercise 6.5, $i(p) < X$ implies that $i(p) \neq X$. Combining the last three results with Definitions 1.11 and 1.6, we have that $i(p) \in \underline{AB} \cap (i(\mathbb{Q}) \setminus \{X\})$, as desired. $\square$

Corollary 6.12 (The Archimedean Property)**.** *Let $A \in \mathbb{R}$ be a positive real number. Then there exist nonzero natural numbers $n, m \in \mathbb{N}$ such that $i(\frac{1}{n}) < A < i(m)$.*

Proof. We will first prove that there exists a nonzero natural number n such that $i(\frac{1}{n}) < A$. We will then prove that there exists a nonzero natural number m such that $A < i(m)$. Let's begin.

Since $A \in \mathbb{R}$ is positive, we know that $0 < A$. Thus, by Lemma 6.10, there exists $\frac{p}{n} \in \mathbb{Q}$ such that $0 < i(\frac{p}{n}) < A$. As permitted by Exercise 3.9b, we choose $\frac{p}{n} \in \left[\frac{p}{n}\right]$ to be an object such that $0 < n$ (this also means that $n \in \mathbb{N}$). Consequently, by Scripts 2 and 3, we know that $0 < \frac{1}{n} \leq \frac{p}{n}$. It follows that $i(\frac{1}{n}) \leq i(\frac{p}{n})$ since $x \in i(\frac{1}{n})$ implies $x < \frac{1}{n} \leq \frac{p}{n}$ implies $x \in i(\frac{p}{n})$, implies $i(\frac{1}{n}) \subset i(\frac{p}{n})$. Therefore, $i(\frac{1}{n}) \leq i(\frac{p}{n}) < A$, as desired.

By Exercise 6.5, Axiom 3, and Definition 3.3, there exists a point $B \in \mathbb{R}$ such that $A < B$. It follows by Lemma 6.10 that there exists $\frac{m}{q} \in \mathbb{Q}$ such that $A < i(\frac{m}{q}) < B$. As before, let $\frac{m}{q}$ be an object such that $0 < q$. Consequently, by Scripts 2 and 3, we know that $0 < \frac{m}{q} \leq m$. Once again, for the same reasons as before, $i(\frac{m}{q}) \leq i(m)$. Therefore, $A < i(\frac{m}{q}) \leq i(m)$, as desired. $\square$

Corollary 6.13. *$i(\mathbb{N})$ is an unbounded subset of $\mathbb{R}$.*

Proof. Suppose for the sake of contradiction that $i(\mathbb{N})$ is bounded above. Then by Definition 5.6, there exists a point $A \in \mathbb{R}$ such that $i(n) \leq A$ for all $n \in \mathbb{N}$. Note that A is a positive real number since $i(0) < i(0) \leq A$. But by Corollary 6.12, $A < i(n)$ for some $n \in \mathbb{N}$, a contradiction. $\square$

1/21: **Corollary 6.14.** *If $A \in \mathbb{R}$ is a real number, then there is an integer n such that $i(n-1) \leq A < i(n)$.*

Proof. Let X be be the set of all integers z such that $i(z) \leq A$. Symbolically,

$$X = \{z \mid z \in \mathbb{Z} \text{ and } i(z) \leq A\}$$

Since $A \neq \emptyset$ by Definition 6.1, there exists a point $\frac{p}{q} \in \mathbb{Q}$ such that $\frac{p}{q} \in A$. As in Corollary 6.12, we let $q > 0$. It follows by Scripts 2 and 3 that if $p \geq 0$, then $0 \leq \frac{p}{q}$, i.e.[3], $i(0) \leq A$ and if $p < 0$, then $p \leq \frac{p}{q}$, i.e., $i(p) \leq A$. Thus, in either case, X is nonempty.

Now there exists a nonzero natural number m such that $A < i(m)$ (if $A \leq i(0)$, then $A < i(1)$; if $A > 0$, then apply Corollary 6.12). Let $f : X \to \mathbb{N}$ be defined by the rule

$$f(x) = m - x$$

By Script 1, f is an injective function, $f(X) \subset \mathbb{N}$, and $f(X)$ is nonempty (since X is nonempty). Thus, by the well-ordering principle (Additional Exercise 0.1), there is a least element, which we shall call y, in $f(X)$. Since f is injective, there exists exactly one object $n-1 \in X$ such that $f(n-1) = y$.

By the definition of X, $i(n-1) \leq A$. To prove that $A < i(n)$, suppose for the sake of contradiction that $i(n) \leq A$. This coupled with the fact that $n \in \mathbb{Z}$ implies that $n \in X$. Thus, $f(n) \in f(X)$. But $f(n) = m - n < m - n + 1 = m - (n-1) = f(n-1)$, contradicting the fact that $f(n-1)$ is the least element of $f(X)$. □

1/26: **Axiom 5.** *The continuum contains a countable dense subset.*

Definition 6.15. Let X and Y be sets with orderings $<_X$ an $<_Y$, respectively. A function $f : X \to Y$ is **order-preserving** if for all $r, s \in X$,

$$r <_X s \Longrightarrow f(r) <_Y f(s)$$

Note that the function $i : \mathbb{Q} \to \mathbb{R}$ discussed above is order-preserving.

Exercise 6.16. Let C satisfy Axioms 1-5. Let $K \subset C$ be a countable dense subset of C. Construct an order-preserving bijection $f : \mathbb{Q} \to K$.

Lemma.

a) *K satisfies Axiom 3.*

b) (Density Lemma) *For all $x, y \in K$, if $x < y$, then there exists a point $z \in K$ such that z is between x and y.*

Proof of a. To prove that K satisfies Axiom 3, we must verify that K has neither a first nor a last point. We will address the first point question first. Suppose for the sake of contradiction that K has a first point x. Then by Definition 3.3, $x \leq y$ for all $y \in K$. However, since C satisfies Axiom 3, there exists an object $a \in C$ such that $a < x$. Now consider the region $\underline{ax}$. We have by Corollary 5.3 that there exists a point $p \in \underline{ax}$. Additionally, we have by Script 3 that $\underline{ax} \cap K = \emptyset$. Thus, $\underline{ax} \cap (K \setminus \{p\}) = \emptyset$, implying by Definition 3.13 that $p \notin LP(K)$. But since $p \in C$ and $p \notin LP(K)$, we have by Definition 6.8 that K is not dense in C, a contradiction.

The proof is symmetric for last points. □

Proof of b. Suppose for the sake of contradiction that that there exist $x, y \in K$ with $x < y$ such that no point $z \in K$ is between x and y. By Theorem 5.2, there exists $p \in C$ such that p is between x and y. Consequently, by Definition 3.10, $p \in \underline{xy}$. Additionally, we have by Script 3 that $\underline{xy} \cap K = \emptyset$. It follows that $\underline{xy} \cap (K \setminus \{p\}) = \emptyset$, implying by Definition 3.13 that $p \notin LP(K)$. But since $p \in C$ and $p \notin LP(K)$, we have by Definition 6.8 that K is not dense in C, a contradiction. □

[3]For the same reasons as in Corollary 6.12.

Proof of Exercise 6.16. By Theorem 2.11, $\mathbb{Q}$ is countable, implying by Definition 1.35 that there exists a bijection $g : \mathbb{N} \to \mathbb{Q}$. The existence of this bijection means that we can refer to an arbitrary element q of $\mathbb{Q}$ by the number n for which $g(n) = q$; in another notation, we can refer to q as q_n. Thus, since every element of $\mathbb{Q}$ can be written as q_n for some $n \in \mathbb{N}$, we can write $\mathbb{Q} = \{q_1, q_2, \dots\}$. Similarly, we can express K as $K = \{k_1, k_2, \dots\}$. We will use this method of referring to the elements of $\mathbb{Q}$ to construct f.

We define f recursively with strong induction. For the base case q_1, we define $f(q_1) = k_1$. Now suppose inductively that we have defined $f(q_1), f(q_2), \dots, f(q_n)$; we now seek to define $f(q_{n+1})$. By Theorem 3.5, the symbols $a_1, \dots, a_{n+1}$ can be assigned to $q_1, \dots, q_{n+1}$ so that $a_1 <_{\mathbb{Q}} a_2 <_{\mathbb{Q}} \cdots <_{\mathbb{Q}} a_{n+1}$. We divide into three cases ($q_{n+1} = a_1$, $q_{n+1} = a_{n+1}$, and $q_{n+1} = a_i$ where $1 < i < n+1$). First, suppose that $q_{n+1} = a_1$. By the inductive hypothesis, $f(a_2), f(a_3), \dots, f(a_{n+1})$ are defined elements of K. At this point, define the set $X = \{k \in K \mid k <_K f(a_2)\}$. It follows by Lemma (a) that this set is nonempty. Thus, by the well-ordering principle, there exists a $k_i \in X$ such that $i \leq j$ for all $k_j \in X$. We let $f(q_{n+1}) = k_i$. The second case is symmetric to the first. Third, suppose that $q_{n+1} = a_i$ where $1 < i < n+1$. By the inductive hypothesis, $f(a_1), \dots, f(a_{i-1}), f(a_{i+1}), \dots, f(a_{n+1})$ are defined elements of K. At this point, define the set $X = \{k \in K \mid f(a_{i-1}) <_K k <_K f(a_{i+1})\}$. It follows by Lemma (b) that this set is nonempty. Thus, by the well-ordering principle, there exists a $k_i \in X$ such that $i \leq j$ for all $k_j \in X$. We let $f(q_{n+1}) = k_i$.

To prove that f is a function, Definition 1.16 tells us that it will suffice to show that for all $q \in \mathbb{Q}$, there exists a unique $k \in K$ such that $f(q) = k$. First, we will prove that for all $q \in \mathbb{Q}$, there exists *some* $k \in K$ such that $f(q) = k$. Let q_i be an arbitrary element of $\mathbb{Q}$. Then $i \in \mathbb{N}$, and by the principle of strong mathematical induction (Additional Exercise 0.2b), $f(q_i)$ is assigned to an element of k. As to proving the uniqueness of the k to which q_i is defined, each q is assigned once, in one of three mutually exclusive cases, to an unambiguously defined (as guaranteed by the well-ordering principle) element of K.

To prove that f is order-preserving, we will first verify the following claim (which we will refer to as Lemma (c) for future reference): Consider the set $\{q_1, \dots, q_n\} \subset \mathbb{Q}$; if the symbols $a_1, \dots, a_n$ are assigned to $q_1, \dots, q_n$ such that $a_1 <_{\mathbb{Q}} a_2 <_{\mathbb{Q}} \cdots <_{\mathbb{Q}} a_n$, then $f(a_1) <_K f(a_2) <_K \cdots <_K f(a_n)$. We will then use this result to prove that f is order-preserving for any two arbitrary elements $q_i, q_j \in \mathbb{Q}$. Let's begin.

To verify the above claim, we induct on n. The base case $n = 1$ is vacuously true. Now suppose inductively that we have proven the claim for n; we now seek to prove it for $n+1$. By Theorem 3.5, the symbols $a_1, \dots, a_{n+1}$ can be assigned to $q_1, \dots, q_{n+1}$ so that $a_1 <_{\mathbb{Q}} a_2 <_{\mathbb{Q}} \cdots <_{\mathbb{Q}} a_{n+1}$. We divide into three cases ($q_{n+1} = a_1$, $q_{n+1} = a_{n+1}$, and $q_{n+1} = a_i$ where $1 < i < n+1$). First, suppose that $q_{n+1} = a_1$. By the definition of f, $f(q_{n+1}) \in \{k \in K \mid k <_K f(a_2)\}$, meaning that $f(q_{n+1}) = f(a_1) <_K f(a_2)$. Additionally, by the inductive hypothesis, we know that $f(a_2) <_K f(a_3) <_K \cdots <_K f(a_{n+1})$ (since $a_2, \dots, a_{n+1}$ correspond to $q_1, \dots, q_n$). Together, these two results imply that $f(a_1) <_K f(a_2) <_K \cdots <_K f(a_{n+1})$. The proof of the second case is symmetric to that of the first. Third, suppose that $q_{n+1} = a_i$ where $1 < i < n+1$. By the definition of f, $f(q_{n+1}) \in \{k \in K \mid f(a_{i-1}) <_K k <_K f(a_{i+1})\}$, meaning that $f(a_{i-1}) <_K f(q_{n+1}) = f(a_i) <_K f(a_{i+1})$. Additionally, by the inductive hypothesis, we know that $f(a_1) <_K \cdots <_K f(a_{i-1}) <_K f(a_{i+1}) <_K \cdots <_K f(a_{n+1})$ (for an analogous reason to before). These two results imply that $f(a_1) <_K f(a_2) <_K \cdots <_K f(a_{n+1})$.

We are now ready to actually prove that f is order-preserving. To do so, Definition 6.15 tells us that it will suffice to show that for all $q_i, q_j \in \mathbb{Q}$, $q_i <_{\mathbb{Q}} q_j$ implies $f(q_i) <_K f(q_j)$. Let q_i, q_j be arbitrary elements of $\mathbb{Q}$ such that $q_i <_{\mathbb{Q}} q_j$. Since $q_i <_{\mathbb{Q}} q_j$, $q_i \neq q_j$, implying that $i \neq j$. We divide into two cases ($i < j$ and $i > j$). Suppose first that $i < j$. By Theorem 3.5, the symbols $a_1, \dots, a_j$ can be assigned to $q_1, \dots, q_j$ so that $a_1 <_{\mathbb{Q}} a_2 <_{\mathbb{Q}} \cdots <_{\mathbb{Q}} a_j$. Let $q_j = a_l$. Since $q_i <_{\mathbb{Q}} q_j$, we know that $q_i = a_m$ where $m < l$. Additionally, by Lemma (c), we know that $f(a_1) <_K f(a_2) <_K \cdots <_K f(a_j)$. It follows that $f(a_m) <_K f(a_l)$, implying that $f(q_i) <_K f(q_j)$, as desired. The proof is symmetric in the other case.

To prove that f is bijective, Definition 1.20 tells us that it will suffice to show that f is injective and surjective.

To show that f is injective, Definition 1.20 tells us that it will suffice to demonstrate that $q_i \neq q_j$ implies $f(q_i) \neq f(q_j)$. WLOG let $q_i <_{\mathbb{Q}} q_j$. Then since f is order-preserving, Definition 6.15 implies that $f(q_i) <_K f(q_j)$. It follows that $f(q_i) \neq f(q_j)$, as desired.

We are now ready to actually show that f is surjective. To do so, Definition 1.20 tells us that it will suffice to demonstrate that for all $k_n \in K$, there exists a $q_i \in \mathbb{Q}$ such that $f(q_i) = k_n$. To do this, we induct on n. For the base case $n = 1$, it follows from the definition of f that $f(q_1) = k_1$. Now suppose inductively that for each $k_1, \dots, k_n$, there exists a $q_i \in \mathbb{Q}$ such that $f(q_i) = k_n$; we now seek to prove the claim for $n+1$.

By Theorem 3.5, the symbols $b_1, \dots, b_{n+1}$ can be assigned to $k_1, \dots, k_{n+1}$ so that $b_1 <_K b_2 <_K \cdots <_K b_{n+1}$. We divide into three cases ($k_{n+1} = b_1$, $k_{n+1} = b_{n+1}$, and $k_{n+1} = b_i$ where $1 < i < n+1$). First, suppose that $k_{n+1} = b_1$. By the inductive hypothesis, $b_2 = f(q_i) <_K b_3 = f(q_j) <_K \cdots <_K b_{n+1} = f(q_l)$. It follows by Definition 6.15 that $q_i <_{\mathbb{Q}} q_j <_{\mathbb{Q}} \cdots <_{\mathbb{Q}} q_l$. At this point, define the set $X = \{q \in \mathbb{Q} \mid q <_{\mathbb{Q}} q_i\}$. It follows from Exercise 3.9d that this set is nonempty. Thus, by the well-ordering principle, there exists a $q_m \in X$ such that $m \leq m'$ for all $q_{m'} \in X$. By the definition of f, $f(q_m) = k_{n+1}$. The proof of the second case is symmetric to that of the first. Third, suppose that $k_{n+1} = b_i$ where $1 < i < n+1$. By the inductive hypothesis, $b_2 = f(q_j) <_K \cdots <_K b_{i-1} = f(q_{j'}) <_K b_{i+1} = f(q_l) <_K \cdots <_K b_{n+1} = f(q_{l'})$. It follows by Definition 6.15 that $q_j <_{\mathbb{Q}} \cdots <_{\mathbb{Q}} q_{j'} <_{\mathbb{Q}} q_l <_{\mathbb{Q}} \cdots <_{\mathbb{Q}} q_{l'}$. At this point, define the set $X = \{q \in \mathbb{Q} \mid q_{j'} <_{\mathbb{Q}} q <_{\mathbb{Q}} q_l\}$. It follows from Additional Exercise 3.1 that this set is nonempty. Thus, by the well-ordering principle, there exists a $q_m \in X$ such that $m \leq m'$ for all $q_{m'} \in X$. By the definition of f, $f(q_m) = k_{n+1}$. ☐

Exercise 6.17. Let $f : \mathbb{Q} \to K$ be an order-preserving bijection, as found in Exercise 6.16. Let $A \in \mathbb{R}$. Then $A \subset \mathbb{Q}$ and so $f(A) \subset K \subset C$. Define $F : \mathbb{R} \to C$ by

$$F(A) = \sup f(A)$$

1. Show $\sup f(A)$ exists, so F is well-defined.
2. Show F is injective and order-preserving.

Proof of 1. To prove that $\sup f(A)$ exists, Theorem 5.17 tells us that it will suffice to show that $f(A)$ is nonempty and bounded above. To show that $f(A)$ is nonempty, Definition 1.8 tells us that it will suffice to find an element of $f(A)$. By Definition 6.1, $A \neq \emptyset$. Thus, by Definition 1.8, there exists an object $x \in A$. It follows by Definition 1.18 that $f(x) \in f(A)$, as desired. To show that $f(A)$ is bounded above, Definition 5.6 tells us that it will suffice to find an element of K such that $f(x) \geq_C f(a)$ for all $f(a) \in f(A)$. By Definition 6.1, $A \neq \mathbb{Q}$ and $A \subset \mathbb{Q}$. Thus, by Definition 1.2, there exists an object $x \in \mathbb{Q}$ such that $x \notin A$. It follows from the latter condition by Lemma 6.2 that x is an upper bound for A. Thus, by Definition 5.6, $x \geq a$ for all $a \in A$. Consequently, by Definition 6.15, $f(x)$ is an element of K such that $f(x) \geq_C f(a)$ for all $f(a) \in f(A)$, as desired. ☐

Proof of 2. To prove that F is order-preserving, Definition 6.15 tells us that it will suffice to show that for all $A, B \in \mathbb{R}$, $A <_{\mathbb{R}} B$ implies $F(A) <_C F(B)$. Let A, B be two arbitrary elements of $\mathbb{R}$ satisfying $A <_{\mathbb{R}} B$. Then by Definitions 6.4 and 1.3, there exists a point $x \in B$ such that $x \notin A$. It follows from the latter condition by Lemma 6.2 and Definition 5.6 that $x \geq a$ for all $a \in A$. Thus, by Definition 6.15, $f(x) \geq_C f(a)$ for all $f(a) \in f(A)$. Consequently, by Definition 5.7, $\sup f(A) \leq_C f(x)$. Additionally, by Definition 6.1, there exists a point $y \in B$ such that $y > x$. Thus, by Definition 6.15, we have that $f(y) >_C f(x)$. It follows by Definitions 5.6 and 5.7 that $f(y) \leq_C \sup f(B)$. Combining two results, we therefore have that $\sup f(A) \leq_C f(x) <_C f(y) \leq_C \sup f(B)$, meaning that $F(A) = \sup f(A) <_C \sup f(B) = F(B)$, as desired.

To prove that F is injective, Definition 1.20 tells us that it will suffice to show that if $A \neq B$, then $F(A) \neq F(B)$. Let A, B be two distinct real numbers. Then by Exercise 6.5, $A < B$ or $B < A$. We now divide into two cases. Suppose first that $A < B$. Then $F(A) < F(B)$ by Definition 6.15 (which we have just proven applies to F). This implies by Definition 3.1 that $F(A) \neq F(B)$, as desired. The proof is symmetric in the other case. ☐

Theorem 6.18. *Suppose that C is a continuum satisfying Axioms 1-5. Then C is isomorphic to the real numbers $\mathbb{R}$; i.e., there is an order-preserving bijection $F : \mathbb{R} \to C$.*

Lemma. *Let K be a dense subset of C. For all $x, y \in C$, if $x < y$, then there exists a point $z \in K$ such that z is between x and y.*

Proof. Suppose for the sake of contradiction that there exist two points $x, y \in C$ with $x < y$ such that no point $z \in K$ is between x and y. By Corollary 5.3, the region $\underline{xy}$ is infinite. Thus, we can pick a point $p \in \underline{xy}$. Additionally, by Definition 1.6, we have that $\underline{xy} \cap K = \emptyset$. Thus, $\underline{xy} \cap (K \setminus \{p\}) = \emptyset$, implying by Definition 3.13 that $p \notin LP(K)$. But since $p \in C$ and $p \notin LP(K)$, we have by Definition 6.8 that K is not dense in C, a contradiction. ☐

Proof of Theorem 6.18. By Axiom 5, C contains a countable dense subset K. By Exercise 6.16, there exists an order-preserving bijection $f : \mathbb{Q} \to K$. By Exercise 6.17, there exists an order-preserving injection $F : \mathbb{R} \to C$. To prove that there is an order-preserving bijection $F : \mathbb{R} \to C$, all that is left to do is to demonstrate that F (as defined in Exercise 6.17) is surjective.

To do this, Definition 1.20 tells us that it will suffice to show that for all $X \in C$, there exists an object $A \in \mathbb{R}$ such that $F(A) = X$. Put more simply, we must find a Dedekind cut A such that $\sup f(A) = X$ for every $X \in C$. To do this, we will begin by constructing the set $S = \{k \in K \mid k < X\}$. We will then verify that the preimage $f^{-1}(S)$ is a Dedekind cut. Lastly, we will verify that $\sup f(f^{-1}(S)) = X$. Let's begin.

Let X be an arbitrary element of C. Define S as above. To verify that $f^{-1}(S)$ is a Dedekind cut, Definition 6.1 tells us that it will suffice to confirm that $f^{-1}(S) \neq \emptyset$; $f^{-1}(S) \neq \mathbb{Q}$; if $r \in f^{-1}(S)$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in f^{-1}(S)$; and if $r \in f^{-1}(S)$, then there is some $s \in f^{-1}(S)$ with $s > r$. We will take this one claim at a time.

To confirm that $f^{-1}(S) \neq \emptyset$, Definition 1.8 tells us that it will suffice to find an element of $f^{-1}(S)$. By Axiom 3 and Definition 3.3, there exists some point $Y \in C$ such that $Y < X$. Consequently, by the lemma and Definition 3.6, there exists a point $f(p) \in K$[4] such that $Y < f(p) < X$. It follows by the definition of S that $f(p) \in S$. Therefore, by Definition 1.18, $p \in f^{-1}(S)$, as desired.

To confirm that $f^{-1}(S) \neq \mathbb{Q}$, Definition 1.2 tells us that it will suffice to find an element of $\mathbb{Q}$ that is not an element of $f^{-1}(S)$. By Axiom 3 and Definition 3.3, there exists some point $Y \in C$ such that $X < Y$. Consequently, by the lemma and Definition 3.6, there exists a point $f(p) \in K$ such that $X < f(p) < Y$. It follows by the definition of S that $f(p) \notin S$. Therefore, by Definition 6.18, $p \in \mathbb{Q}$ but $p \notin f^{-1}(S)$, as desired.

To confirm that if $r \in f^{-1}(S)$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in f^{-1}(S)$, we let $r \in f^{-1}(S)$ and $s \in \mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s < r$ and seek to verify that $s \in f^{-1}(S)$. By Definition 1.18, the fact that $r \in f^{-1}(S)$ implies that $f(r) \in S$. Thus, by the definition of S, $f(r) < X$. Additionally, by the definition of f and Definition 6.15, $f(s) \in K$ and $f(s) < f(r)$, respectively. Since $f(s) < f(r)$ and $f(r) < X$, transitivity implies that $f(s) < X$. This combined with the previously established fact that $f(s) \in K$ implies that $f(s) \in S$. Therefore, by Definition 1.18, $s \in f^{-1}(S)$, as desired.

To confirm that if $r \in f^{-1}(S)$, then there is some $s \in f^{-1}(S)$ with $s > r$, we let $r \in f^{-1}(S)$ and seek to find such an s. As before, $r \in f^{-1}(S)$ implies that $f(r) \in S$. Thus, by the definition of S, $f(r) < X$. It follows by the lemma and Definition 3.6 that there exists a point $f(s) \in K$ such that $f(r) < f(s) < X$. Consequently, by the definition of S, we have that $f(s) \in S$. Therefore, by Definitions 1.18 and 6.15, $s \in f^{-1}(S)$ and $r < s$, respectively, as desired.

Since f is bijective, Script 1 asserts that $f(f^{-1}(S)) = S$. Thus, $\sup f(f^{-1}(S)) = \sup S$. To verify that $\sup S = X$, Definition 5.7 tells us that it will suffice to confirm that X is an upper bound of S and if U is an upper bound of S, $X \leq U$. To confirm the former statement, Definition 5.6 tells us that it will suffice to show that $k \leq X$ for all $k \in S$. But by the definition of S, this is true. To confirm the latter statement, suppose for the sake of contradiction that there exists an upper bound U of S such that $U < X$. Since $U < X$, the lemma and Definition 3.6 imply that there exists a point $Z \in K$ such that $U < Z < X$. It follows by the definition of S that $Z \in S$. Since there exists an element of S greater than U, Definition 5.6 asserts that U is not an upper bound of S, a contradiction. $\square$

[4]Note that we know that the element of K (the existence of which is implied by the lemma) can be written in the form $f(p)$ because f is bijective.