Script 8

Intervals

8.1 Journal

2/9: Now that we have constructed $\mathbb{R}$ and proved the fundamental facts about it, we will work with the real numbers $\mathbb{R}$ instead of an arbitrary continuum C. We will leave behind Dedekind cuts and think of elements of $\mathbb{R}$ as numbers. Accordingly, from now on, we will use lower-case letters like x for real numbers and will write $+$ and $\cdot$ for $\oplus$ and $\otimes$. We will also now use the standard notation (a, b) for the region $\underline{ab} = \{x \in \mathbb{R} \mid a < x < b\}$. Even though the notation is the same, this is *not* the same object as the ordered pair (a, b).

More generally, we adopt the following standard notation:

$$\begin{aligned}
(a, b) &= \{x \in \mathbb{R} \mid a < x < b\} \\
[a, b) &= \{x \in \mathbb{R} \mid a \leq x < b\} \\
(a, b] &= \{x \in \mathbb{R} \mid a < x \leq b\} \\
[a, b] &= \{x \in \mathbb{R} \mid a \leq x \leq b\} \\
(a, \infty) &= \{x \in \mathbb{R} \mid a < x\} \\
[a, \infty) &= \{x \in \mathbb{R} \mid a \leq x\} \\
(-\infty, b) &= \{x \in \mathbb{R} \mid x < b\} \\
(-\infty, b] &= \{x \in \mathbb{R} \mid x \leq b\}
\end{aligned} \tag{8.1}$$

Exercise 8.1. Identify the sets in Equations 8.1 that are open/closed/neither.

Proof. Note that by Theorem 5.1, any of these sets proven to be just one of open or closed will not be the other, i.e., a set proven to be open will not be closed and vice versa.

By Corollary 4.11, (a, b) is open.

By an adaptation of Corollary 5.14, $b \in LP([a, b))$ but $b \notin [a, b)$. Since $[a, b)$ doesn't contain all of its limit points, Definition 4.1 tells us that it is not closed. Additionally, since $a \in LP(C \setminus [a, b))$ but $a \notin C \setminus [a, b)$, Definition 4.8 tells us that it is not open. Therefore, it is neither.

The proof that $(a, b]$ is neither is symmetric to the previous case.

By Corollaries 5.15 and 4.7, $[a, b]$ is closed.

By Corollary 4.13, (a, ∞) is open.

By Corollary 4.13 and Definition 4.8, $[a, \infty) = C \setminus (-\infty, a)$ is closed.

The proofs that $(-\infty, b)$ and $(-\infty, b]$ are open and closed, respectively, are symmetric to the previous two cases, respectively. $\square$

Definition 8.2. A set $I \subset \mathbb{R}$ is an **interval** if for all $x, y \in I$ with $x < y$, $[x, y] \subset I$.

Lemma 8.3. *A proper subset $I \subsetneq \mathbb{R}$ is an interval if and only if it takes one of the eight forms in Equations 8.1.*

Proof. Suppose first that $I \subsetneq \mathbb{R}$ is an interval. If $I = \emptyset$, then $I = (a, a)$ for any $a \in \mathbb{R}$, and we are done. Thus, we will assume for the remainder of the proof of the forward direction that I is nonempty. To address this case, we will first prove that the facts that $I \subsetneq \mathbb{R}$, $I \neq \emptyset$, and I is an interval imply that I is bounded above, bounded below, or both. Then in each of these three cases, we will look at whether $\sup I$ and $\inf I$ (if they exist) are elements of the set or not to differentiate between the eight forms in Equations 8.1. Let's begin.

Suppose for the sake of contradiction that there exists a nonempty interval $I \subsetneq \mathbb{R}$ that is neither bounded above nor bounded below. Since $I \subsetneq \mathbb{R}$, we have by Definition 1.3 that there exists a point $p \in \mathbb{R}$ such that $p \notin I$. Additionally, since I is neither bounded above nor below, Definition 5.6 implies that p is neither an upper nor a lower bound of I. Thus, there exist $x, y \in I$ such that $x < p$ and $y > p$. Now by Definition 8.2, $[x, y] \subset I$. But it follows by Definition 1.3 that every point in $[x, y]$, including p, is an element of I, a contradiction.

We now divide into three cases (I is exclusively bounded below, I is exclusively bounded above, and I is bounded both below and above).

First, suppose that I is only bounded below. Since I is a nonempty subset of $\mathbb{R}$ that is bounded below, we have by Theorem 5.17 that $\inf I$ exists. We divide into two cases again ($\inf I \in I$ and $\inf I \notin I$).

If $\inf I \in I$, then we can demonstrate that $I = [\inf I, \infty)$. To do this, Definition 1.2 tells us that it will suffice to verify that every $p \in I$ is an element of $[\inf I, \infty)$ and vice versa. Let p be an arbitrary element of I. Then $p \in \mathbb{R}$ and by Definitions 5.7 and 5.6, $\inf I \leq p$. Therefore, $p \in [\inf I, \infty)$, as desired. Now let p be an arbitrary element of $[\inf I, \infty)$. Then $\inf I \leq p$. Additionally, since I is not bounded above, we have by Definition 5.6 that there exists $y \in I$ such that $y > p$. Since $\inf I \in I$, $y \in I$, and $\inf I < y$ (by transitivity), $[\inf I, y] \subset I$ by Definition 8.2. This combined with the fact that $p \in [\inf I, y]$ (we know that $\inf I \leq p < y$, so $\inf I \leq p \leq y$) implies that $p \in I$, as desired.

If $\inf I \notin I$, then we can demonstrate that $I = (\inf I, \infty)$. As before, to do so, it will suffice to verify that every $p \in I$ is an element of $(\inf I, \infty)$ and vice versa. Let p be an arbitrary element of I. Then $p \in \mathbb{R}$ and by Definitions 5.7 and 5.6, $\inf I \leq p$. The additional constraint that $\inf I \notin I$ implies that $\inf I < p$. Therefore, $p \in (\inf I, \infty)$, as desired. Now let p be an arbitrary element of $(\inf I, \infty)$. Then $\inf I < p$. It follows by Lemma 5.11 that there exists a $z \in I$ such that $\inf I \leq z < p$. Additionally, since I is not bounded above, we have by Definition 5.6 that there exists $y \in I$ such that $y > p$. Since $z \in I$, $y \in I$, and $z < y$ (by transitivity), $[z, y] \subset I$ by Definition 8.2. This combined with the fact that $p \in [z, y]$ (we know that $z < p < y$, so $z \leq p \leq y$) implies that $p \in I$, as desired.

Second, suppose that I is only bounded above. The proof of this case is symmetric to that of the first.

Third, suppose that I is bounded below and above. Since I is a nonempty subset of $\mathbb{R}$ that is bounded above and below, we have by consecutive applications of Theorem 5.17 that both $\sup I$ and $\inf I$ exist. We divide into four cases ($\inf I \in I$ and $\sup I \in I$, $\inf I \in I$ and $\sup I \notin I$, $\inf I \notin I$ and $\sup I \in I$, and $\inf I \notin I$ and $\sup I \notin I$).

If $\inf I \in I$ and $\sup I \in I$, then we can demonstrate that $I = [\inf I, \sup I]$. We divide into two cases again ($\inf I = \sup I$ and $\inf I \neq \sup I$). If $\inf I = \sup I \in I$, then $I = \{\inf I\} = \{\sup I\} = [\inf I, \sup I]$, as desired. On the other hand, if $\inf I \neq \sup I$, we continue. To demonstrate that $I = [\inf I, \sup I]$, Theorem 1.7 tells us that it will suffice to verify that $I \subset [\inf I, \sup I]$ and $[\inf I, \sup I] \subset I$. To verify that the former claim, Definition 1.3 tells us that it will suffice to confirm that every $p \in I$ is an element of $[\inf I, \sup I]$. Let p be an arbitrary element of I. Then $p \in \mathbb{R}$ and by consecutive applications of Definitions 5.7 and 5.6, $\inf I \leq p$ and $p \leq \sup I$. Therefore, $p \in [\inf I, \sup I]$, as desired. On the other hand, since $\inf I \in I$, $\sup I \in I$, and $\inf I < \sup I$ (as follows from Definition 5.7 and the fact that they are unequal), $[\inf I, \sup I] \subset I$ by Definition 8.2, as desired.

If $\inf I \in I$ and $\sup I \notin I$, then we can demonstrate that $I = [\inf I, \sup I)$. To do so, it will suffice to verify that every $p \in I$ is an element of $[\inf I, \sup I)$ and vice versa. Let p be an arbitrary element of I. Then $p \in \mathbb{R}$ and by Definitions 5.7 and 5.6, $\inf I \leq p$ and $p \leq \sup I$. The additional constraint that $\sup I \notin I$ implies that $p < \sup I$. Therefore, $p \in [\inf I, \sup I)$, as desired. Now let p be an arbitrary element of $[\inf I, \sup I)$. Then $\inf I \leq p < \sup I$. It follows by Lemma 5.11 that there exists a $y \in I$ such that $p < y \leq \sup I$. Since

$\inf I \in I$, $y \in I$, and $\inf I < y$ (by transitivity), $[\inf I, y] \subset I$ by Definition 8.2. This combined with the fact that $p \in [\inf I, y]$ (we know that $\inf I \leq p < y$, so $\inf I \leq p \leq y$) implies that $p \in I$, as desired.

If $\inf I \notin I$ and $\sup I \in I$, the proof is symmetric to that of the previous case.

If $\inf I \notin I$ and $\sup I \notin I$, then we can demonstrate that $I = (\inf I, \sup I)$. To do so, it will suffice to verify that every $p \in I$ is an element of $(\inf I, \sup I)$ and vice versa. Let p be an arbitrary element of I. Then $p \in \mathbb{R}$ and by Definitions 5.7 and 5.6, $\inf I \leq p$ and $p \leq \sup I$. The additional constraints that $\inf I \notin I$ and $\sup I \notin I$ imply that $\inf I < p$ and $p < \sup I$, respectively. Therefore, $p \in (\inf I, \sup I)$, as desired. Now let p be an arbitrary element of $(\inf I, \sup I)$. Then $\inf I < p < \sup I$. It follows by consecutive applications of Lemma 5.11 that there exist $x, y \in I$ such that $\inf I \leq x < p$ and $p < y \leq \sup I$. Since $x \in I$, $y \in I$, and $x < y$ (by transitivity), $[x, y] \subset I$ by Definition 8.2. This combined with the fact that $p \in [x, y]$ (we know that $x < p < y$, so $x \leq p \leq y$) implies that $p \in I$, as desired.

Now suppose that $I \subsetneq \mathbb{R}$ takes one of the eight forms in Equations 8.1. To prove that I is an interval, Definition 8.2 tells us that it will suffice to show that for all $x, y \in I$ with $x < y$, $[x, y] \subset I$. Let x, y be arbitrary elements of I with $x < y$. We divide into eight cases (one for each equation in Equations 8.1).

First, suppose that $I = (a, b)$. To demonstrate that $[x, y] \subset I$, Definition 1.3 tells us that it will suffice to confirm that every $z \in [x, y]$ is an element of I. Let z be an arbitrary element of $[x, y]$. Then by Corollary 5.15, $x \leq z \leq y$. But since $a < x < y < b$ by Equations 8.1, the fact that $a < x \leq z \leq y < b$ implies by Equations 8.1 that $z \in (a, b)$, as desired.

The proofs of the second, third, and fourth equations are symmetric to that of the first.

Fifth, suppose that $I = (a, \infty)$. To demonstrate that $[x, y] \subset I$, Definition 1.3 tells us that it will suffice to confirm that every $z \in [x, y]$ is an element of I. Let z be an arbitrary element of $[x, y]$. Then by Corollary 5.15, $x \leq z \leq y$. But since $a < x$ by Equations 8.1, the fact that $a < x \leq z$ implies by Equations 8.1 that $z \in (a, \infty)$, as desired.

The proofs of the sixth, seventh, and eighth equations are symmetric to that of the first. □

Definition 8.4. The **absolute value** of a real number x is the non-negative number $|x|$ defined by

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Exercise 8.5. Show that $|x| = |-x|$ for all $x \in \mathbb{R}$. (Note that this also means that $|x - y| = |y - x|$ for any $x, y \in \mathbb{R}$.)

Proof. Let x be an arbitrary element of $\mathbb{R}$. We divide into three cases ($x = 0$, $x > 0$, and $x < 0$). First, suppose that $x = 0$. Then since $0 = -0$, clearly $|0| = |-0|$, as desired. Second, suppose that $x > 0$. Then by Lemma 7.23[1] $-x < 0$. Thus, by consecutive applications of Definition 8.4, $|x| = x$ and $|-x| = -(-x)$. Therefore, since $-(-x) = x$ by Corollary 7.11, $|x| = x = |-x|$, as desired. Third, suppose that $x < 0$. Then by Lemma 7.23, $-x > 0$. Thus, by consecutive applications of Definition 8.4, $|x| = -x$ and $|-x| = -x$. Therefore, $|x| = -x = |-x|$, as desired. □

Definition 8.6. The **distance** between $x \in \mathbb{R}$ and $y \in \mathbb{R}$ is defined to be $|x - y|$.

Remark 8.7. It follows from Definition 8.6 that $|x|$ is the distance between x and 0.

Lemma 8.8. *For any real numbers x, y, z, we have*

(*a*) $|x + y| \leq |x| + |y|$.

(*b*) $|x - z| \leq |x - y| + |y - z|$.

(*c*) $||x| - |y|| \leq |x - y|$.

[1] And, technically, Theorem 7.47.

Proof of a. We divide into four cases ($x \geq 0$ and $y \geq 0$, $x \geq 0$ and $y < 0$, $x < 0$ and $y \geq 0$, and $x < 0$ and $y < 0$).

First, suppose that $x \geq 0$ and $y \geq 0$. Then by Definition 7.21, $x+y \geq 0$. Thus, by consecutive applications of Definition 8.4, $|x+y| = x+y$, $|x| = x$, and $|y| = y$. Therefore, $|x+y| = x+y \leq x+y = |x|+|y|$, as desired.

Second, suppose that $x \geq 0$ and $y < 0$. By Definition 8.4, $|x| = x$ and $|y| = -y$. We now divide into two cases ($x+y \geq 0$ and $x+y < 0$). If $x+y \geq 0$, then $|x+y| = x+y$. Additionally, since $y < 0$, Lemma 7.23 implies that $0 < -y$. Consequently, by transitivity, $y < -y = |y|$. It follows by Definition 7.21 that $x+y < x+|y|$. Therefore, $|x+y| = x+y < x+|y| = |x|+|y|$, so $|x+y| \leq |x|+|y|$, as desired. On the other hand, if $x+y < 0$, then $|x+y| = -(x+y) = -x+(-y) = -x+|y|$. Additionally, by Lemma 7.23, $x \geq 0$ implies that $-x \leq 0$. It follows by Definition 7.21 since $-x \leq x$ that $-x+|y| \leq x+|y|$. Therefore, $|x+y| = -x+|y| \leq x+|y| = |x|+|y|$, as desired.

The proof of the third case is symmetric to that of the second.

The proof of the fourth case is symmetric to that of the first. ☐

Proof of b. By part (a), $|x-z| = |x-y+y-z| \leq |x-y|+|y-z|$, as desired. ☐

Proof of c. To prove that $||x|-|y|| \leq |x-y|$, Definition 8.4 tells us that it will suffice to show that $|x|-|y| \leq |x-y|$ and $-(|x|-|y|) \leq |x-y$. By part (a), $|x| = |x-y+y| \leq |x-y|+|y|$, so $|x|-|y| \leq |x-y|$. Similarly, $|y|-|x| \leq |x-y|$, so $-(|x|-|y|) \leq |x-y|$, as desired. ☐

Exercise 8.9. Let $a, \delta \in \mathbb{R}$ with $\delta > 0$. Prove that

$$(a-\delta, a+\delta) = \{x \in \mathbb{R} \mid |x-a| < \delta\}$$

Lemma. *For any $a, b \in \mathbb{R}$ such that $0 < b$, $|a| < b$ if and only if $-b < a < b$.*

Proof. Suppose first that $|a| < b$. We divide into two cases ($a \geq 0$ and $a < 0$). If $a \geq 0$, then by Definition 8.4, $0 \leq a = |a| < b$. Additionally, by Lemma 7.23, $-b < 0$. Therefore, $-b < 0 \leq a < b$, as desired. If $a < 0$, then by Definition 8.4, $-a = |a| < b$. It follows by Definition 7.21 (by adding $a-b$ to both sides) that $-b < a$. Additionally, by Lemma 7.23, $a < 0$ implies $0 < -a$, so we know that $a < -a$. Therefore, $-b < a < -a < b$, as desired.

Now suppose that $-b < a < b$. We divide into two cases ($a \geq 0$ and $a > 0$). If $a \geq 0$, then by Definition 8.4, $|a| = a < b$, as desired. If $a < 0$, then by Definition 8.4, $|a| = -a$. Since $-b < a$, Definition 7.21 implies (by adding $b-a$ to both sides) that $-a < b$. Therefore, $|a| = -a < b$, as desired. ☐

Proof of Exercise 8.9. To prove that $(a-\delta, a+\delta) = \{x \in \mathbb{R} \mid |x-a| < \delta\}$, Definition 1.2 tells us that it will suffice to show that every $p \in (a-\delta, a+\delta)$ is an element of $\{x \in \mathbb{R} \mid |x-a| < \delta\}$ and vice versa.

Suppose first that $p \in (a-\delta, a+\delta)$. Then by Equations 8.1, $a-\delta < p$ and $p < a+\delta$. It follows by consecutive applications of Definition 7.21 from the former condition that $-\delta < p-a$, and from the latter condition that $p-a < \delta$. Since $-\delta < p-a < \delta$, the lemma asserts that $|p-a| < \delta$. Therefore, $p \in \{x \in \mathbb{R} \mid |x-a| < \delta\}$.

Now suppose that $p \in \{x \in \mathbb{R} \mid |x-a| < \delta\}$. Then $|p-a| < \delta$. Thus, by the lemma, $-\delta < p-a$ and $p-a < \delta$. It follows by consecutive applications of Definition 7.21 from the former condition that $a-\delta < p$, and from the latter condition that $p < a+\delta$. Therefore, since $a-\delta < p < a+\delta$, we have that $p \in (a-\delta, a+\delta)$. ☐

2/11: **Lemma 8.10.** *Let I be an open interval containing the point $p \in \mathbb{R}$. Then*

a) *There exists a number $\delta > 0$ such that $(p-\delta, p+\delta) \subset I$.*

b) *There exists a natural number N such that for all natural numbers $k \geq N$ we have $(p-\frac{1}{k}, p+\frac{1}{k}) \subset I$.*

Proof of a. Since I is open, we have by Theorem 4.10 that there exists a region (a,b) such that $p \in (a,b) \subset I$. Let $\delta = \min(p-a, b-p)$. To show that $(p-\delta, p+\delta) \subset I$, we will demonstrate that $(p-\delta, p+\delta) \subset (a,b) \subset I$. To do this, Definition 1.3 tells us that it will suffice to verify that every element $x \in (p-\delta, p+\delta)$ is an element of (a,b). Let x be an arbitrary element of $(p-\delta, p+\delta)$. Then by Equations 8.1, $p-\delta < x < p+\delta$. We divide into two cases ($\delta = p-a$ and $\delta = b-p$). Suppose first that $\delta = p-a$. Then $p-(p-a) < x < p+(p-a)$, i.e., $a < x < p + (p - a)$. Additionally, the fact that $p - a = \min(p - a, b - p)$ implies that $p - a \le b - p$. Combining these last two results gives us $a < x < p + (p - a) \le p + (b - p) = b$. Since $a < x < b$, Equations 8.1 imply that $x \in (a, b)$, as desired. The proof is symmetric if $\delta = b - p$. □

Proof of b. By Lemma 8.10a, there exists a number $\delta > 0$ such that $(p - \delta, p + \delta) \subset I$. Since δ is a positive real number, Corollary 6.12 implies that there exists a nonzero natural number N such that $\frac{1}{N} < \delta$. To prove that for all numbers $k \ge N$, we have $(p - \frac{1}{k}, p + \frac{1}{k}) \subset I$, we will show that $(p - \frac{1}{k}, p + \frac{1}{k}) \subset (p - \delta, p + \delta) \subset I$. To do this, Definition 1.3 tells us that it will suffice to show that every $x \in (p - \frac{1}{k}, p + \frac{1}{k})$ is an element of $(p - \delta, p + \delta)$. Let k be an arbitrary natural number such that $k \ge N$, and let x be an arbitrary element of $(p - \frac{1}{k}, p + \frac{1}{k})$. It follows from the latter condition by Equations 8.1 that $p - \frac{1}{k} < x < p + \frac{1}{k}$. Since $\frac{1}{k} \le \frac{1}{N}$ by Scripts 2 and 3, we have that $p - \frac{1}{N} < x < p + \frac{1}{N}$. Since $\frac{1}{N} < \delta$ by definition, $p - \delta < x < p + \delta$. Therefore, by Equations 8.1, $x \in (p - \delta, p + \delta)$, as desired. □

Definition 8.11. Let $A \subset X \subset \mathbb{R}$. We say that A is **open** (in X) if it is the intersection of X with an open set, and **closed** (in X) if it is the intersection of X with a closed set. (This is called the subspace topology on X.)

Remark 8.12. $A \subset \mathbb{R}$ open, as defined in Script 3, is equivalent to A open in $\mathbb{R}$.

Exercise 8.13. Let $A \subset X \subset \mathbb{R}$. Show that $X \setminus A$ is closed in X if and only if A is open in X.

Proof. Suppose first that $X \setminus A$ is closed in X. Then by Definition 8.11, $X \setminus A = X \cap B$ where B is a closed set. It follows by Script 1 that

$$
\begin{aligned}
X \setminus A &= X \cap B \\
\mathbb{R} \setminus (X \setminus A) &= \mathbb{R} \setminus (X \cap B) \\
(\mathbb{R} \setminus X) \cup A &= (\mathbb{R} \setminus X) \cup (\mathbb{R} \setminus B) \\
X \cap ((\mathbb{R} \setminus X) \cup A) &= X \cap ((\mathbb{R} \setminus X) \cup (\mathbb{R} \setminus B)) \\
(X \cap (\mathbb{R} \setminus X)) \cup (X \cap A) &= (X \cap (\mathbb{R} \setminus X)) \cup (X \cap (\mathbb{R} \setminus B)) \\
\emptyset \cup (X \cap A) &= \emptyset \cup (X \cap (\mathbb{R} \setminus B)) \\
A &= X \cap (\mathbb{R} \setminus B)
\end{aligned}
$$

Since $\mathbb{R} \setminus B$ is open by Definition 4.4, we have by Definition 8.11 that A is open in X.

Now suppose that A is open in X. Then by Definition 8.11, $A = X \cap B$ where B is an open set. It follows by Script 1 that

$$
\begin{aligned}
A &= X \cap B \\
\mathbb{R} \setminus A &= \mathbb{R} \setminus (X \cap B) \\
&= (\mathbb{R} \setminus X) \cup (\mathbb{R} \setminus B) \\
X \cap (\mathbb{R} \setminus A) &= X \cap ((\mathbb{R} \setminus X) \cup (\mathbb{R} \setminus B)) \\
X \setminus A &= (X \cap (\mathbb{R} \setminus X)) \cup (X \cap (\mathbb{R} \setminus B)) \\
&= X \cap (\mathbb{R} \setminus B)
\end{aligned}
$$

Since $\mathbb{R} \setminus B$ is closed by Definition 4.4, we have by Definition 8.11 that $X \setminus A$ is closed in X. □

Exercise 8.14.

a) Let $[a, b] \subset \mathbb{R}$. Give an example of a set $A \subset [a, b]$ such that A is open in $[a, b]$ but not in $\mathbb{R}$.

b) Give an example of sets $A \subset X \subset \mathbb{R}$ such that A is closed in X but not in $\mathbb{R}$.

Proof of a. We first briefly consider the case where $a = b$. In this case, let $c < a < d$; then $\{a\} = [a, a] \cap (c, d)$ is a subset of $[a, b]$ that is open in $[a, b]$ (by Definition 8.11; (c, d) is open by Exercise 8.1) but closed in $\mathbb{R}$ (by Corollary 3.23, Definition 4.1, and Theorem 5.1).

We now direct our attention to the case where $a \ne b$. Let $c \in [a, b]$ be a point such that $a < c < b$ (we know at least one such point exists by Theorem 5.2). If we define the set $(c, b] = [a, b] \cap (c, \infty)$, we have by

Definition 8.11 that $(c, b]$ is open in $[a, b]$ (since (c, ∞) is open as per Exercise 8.1). However, we know that $(c, b]$ is not open in $\mathbb{R}$ by Theorem 4.10 (b is an element of $(c, b]$ such that any region containing b necessarily contains an element that is not in $(c, b]$; this element will be greater than b but less than the right bound of the region, and its existence is guaranteed by Theorem 5.2). $\square$

Proof of b. Let $X = (a, b) \subset \mathbb{R}$. Then $(a, b) = X \cap [a, b]$, so $(a, b) = X \cap [a, b]$ is closed in (a, b) by Definition 8.11. However, by Corollary 5.14, a, b are limit points of (a, b) that are not contained within (a, b). It follows by Definition 4.1 that (a, b) is not closed in $\mathbb{R}$. $\square$

Theorem 8.15. *Let $X \subset \mathbb{R}$. Then X is connected if and only if X is an interval.*

Proof. Suppose first that X is connected. To prove that X is an interval, Definition 8.2 tells us that it will suffice to show that for all $x, y \in X$ with $x < y$, $[x, y] \subset X$. Let x, y be arbitrary elements of X satisfying $x < y$, and suppose for the sake of contradiction that $[x, y] \not\subset X$. Then there exists $z \in [x, y]$ such that $z \notin X$. Let $A = \{a \in X \mid a < z\}$ and $B = \{b \in X \mid z < b\}$. It follows from Script 1 that $X = A \cup B$ and $A \cap B = \emptyset$. To verify that A is nonempty, Definition 1.8 tells us that it will suffice to find an element in it. Since $z \notin X$ but $x \in X$, we know that $z \neq x$. This combined with the fact that $x \leq z$ by Equations 8.1 implies that $x < z$. Thus, since $x \in X$ and $x < z$, $x \in A$. Similarly, $y \in B$. To verify that A is open in X, Definition 8.11 tells us that it will suffice to demonstrate that A is the intersection of X with an open set. Since we clearly have $A = X \cap (-\infty, z)$ where $(-\infty, z)$ is open by Exercise 8.1, we are done. We can do something similar for B. But the existence of two disjoint, nonempty, open (in X) sets A, B whose union equals X demonstrates by Definition 4.22 that X is disconnected, a contradiction.

Now suppose that X is an interval, and suppose for the sake of contradiction that X is disconnected. Then by Definition 4.22, $X = A \cup B$ where A, B are disjoint, nonempty sets that are open in X. Since A, B are disjoint and nonempty, we know that there exist distinct objects $a \in A$ and $b \in B$. WLOG, let $a < b$.

To prove that $\sup(A \cap [a, b])$ exists, Theorem 5.17 tells us that it will suffice to show that $A \cap [a, b]$ is nonempty and bounded above. To show that $A \cap [a, b]$ is nonempty, Definition 1.8 tells us that it will suffice to find an element of $A \cap [a, b]$. By Equations 8.1, $a \in [a, b]$. By Definition, $a \in A$. Thus, by Definition 1.6, $a \in A \cap [a, b]$, as desired. To show that $A \cap [a, b]$ is bounded above, consecutive applications of Definition 5.6 tell us that it will suffice to verify that $x \leq b$ for all $x \in A \cap [a, b]$. Let x be an arbitrary element of $A \cap [a, b]$. It follows by Definition 1.6 that $x \in [a, b]$. Thus, by Equations 8.1, $x \leq b$, as desired.

Let $s = \sup(A \cap [a, b])$. To prove that $\inf(B \cap [s, b])$ exists, it will suffice to utilize a symmetric argument to the above.

Let $i = \inf(B \cap [s, b])$. We divide into three cases ($s > i$, $s = i$, and $s < i$).

First, suppose that $s > i$. To show that s is a lower bound of $B \cap [s, b]$, Definition 5.6 tells us that it will suffice to verify that $s \leq x$ for all $x \in B \cap [s, b]$. Let x be an arbitrary element of $B \cap [s, b]$. By Definition 1.6, $x \in [s, b]$. Thus, by Equations 8.1, $s \leq x$, as desired. Since s is a lower bound of $B \cap [s, b]$, Definition 5.7 asserts that $i \geq s$, contradicting the hypothesis that $s > i$.

Second, suppose that $s = i$. We divide into three cases ($s \in A$, $s \in B$, and $s \notin A$ and $s \notin B$).

If $s \in A$, then since A is open in X, Definition 8.11 implies that $A = X \cap G$ where G is open. It follows by the hypothesis that $s \in A$ along with Definitions 1.2 and 1.6 that $s \in G$. Consequently, by Theorem 4.10, there exists a region (c, d) such that $s \in (c, d)$ and $(c, d) \subset G$. From the former condition, we have by Equations 8.1 that $c < s < d$. Thus, by Lemma 5.11, there exists a point $x \in B \cap [s, b]$ such that $s = i \leq x < d$. Since $c < s \leq x < d$, Equations 8.1 imply that $x \in (c, d)$. This combined with the fact that $(c, d) \subset G$ implies by Definition 1.3 that $x \in G$. Additionally, we know that $x \in B$ (since $x \in B \cap [s, b]$ by Definition 1.6). It follows from this and the fact that $X = A \cup B$ by Definitions 1.5 and 1.2 that $x \in X$. Thus, since $x \in X$ and $x \in G$, Definition 1.6 asserts that $x \in X \cap G$, meaning that $x \in A$. But if $x \in A$ and $x \in B$, then Definition 1.6 implies that $x \in A \cap B$, contradicting the supposition that A and B are disjoint.

If $s \in B$, then the proof is symmetric to the previous case.

If $s \notin A$ and $s \notin B$, then by Definition 1.5, $s \notin A \cup B$, implying that $s \notin X$. Additionally, the facts that $a \in A$, $b \in B$, and $X = A \cup B$ imply that $a, b \in X$. It follows since $a < b$ by Definition 8.2 that $[a, b] \subset X$. We now show that $s \in [a, b]$ via Equations 8.1, which tell us that it will suffice to verify that $a \leq s \leq b$. As previously shown, b is an upper bound of $A \cap [a, b]$. Thus, by Definition 5.7, we have that $s \leq b$, and we are half done. As to the other half, we have also previously shown that $a \in A \cap [a, b]$. Additionally, by Definitions 5.7 and 5.6, $s \geq x$ for all $x \in A \cap [a, b]$, including a. Thus, $s \geq a$. Having shown that $s \in [a, b]$ and $[a, b] \subset X$, we may invoke Definition 1.3 to learn that $s \in X$, contradicting the previously proven statement that $s \notin X$.

Third, suppose that $s < i$. Then by Theorem 5.2 and Definition 3.6, there exists a $z \in \mathbb{R}$ such that $s < z < i$. We now show that $i \in [a, b]$ via Equations 8.1, which tell us that it wil suffice to verify that $a \leq i \leq b$. As previously shown, s is a lower bound of $B \cap [s, b]$. Thus, by Definition 5.7, we have that $i \geq s$. We have also previously shown that $s \geq a$, so by transitivity, $i \geq a$, and we are half done. As to the other half, we now confirm that $b \in B \cap [s, b]$. By Equations 8.1, $b \in [s, b]$. By definition, $b \in B$. Thus, by Definition 1.6, $b \in B \cap [s, b]$, as desired. Additionally, by Definitions 5.7 and 5.6, $i \leq x$ for all $x \in B \cap [s, b]$, including b. Thus, $i \leq b$, concluding our argument that $i \in [a, b]$. Moving on, the fact that $s < z$ implies by Definition 5.6 that $z \notin A \cap [a, b]$. Additionally, we know from the facts that $s, i \in [a, b]$ that $a \leq s < z < i \leq b$, meaning that $z \in [a, b]$. Combining the previous two results with Definition 1.6, we have that $z \notin A$. By a symmetric argument, we can show that $z \notin B$. Since $z \notin A$ and $z \notin B$, Definition 1.5 asserts that $z \notin A \cup B$, i.e., $z \notin X$. But as before, $[a, b] \subset X$, so the fact that $z \in [a, b]$ combined with Definition 1.3 implies that $z \in X$, a contradiction. $\square$