

# MATH 16210 (Honors Calculus II IBL) Notes

Steven Labalme

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## Script 6

# Construction of the Real Numbers

### 6.1 Journal

1/12: **Definition 6.1.** A subset  $A$  of  $\mathbb{Q}$  is said to be a **cut** (or **Dedekind cut**) if it satisfies the following:

- (a)  $A \neq \emptyset$  and  $A \neq \mathbb{Q}$ .
- (b) If  $r \in A$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in A$ .
- (c)  $A$  does not have a last point; i.e., if  $r \in A$ , then there is some  $s \in A$  with  $s > r$ .

We denote the collection of all cuts by  $\mathbb{R}$ .

**Lemma 6.2.** Let  $A$  be a Dedekind cut and  $x \in \mathbb{Q}$ . Then  $x \notin A$  if and only if  $x$  is an upper bound for  $A$ .

*Proof.* Suppose first that  $x$  is an element of  $\mathbb{Q}$  such that  $x \notin A$ . To prove that  $x$  is an upper bound for  $A$ , Definition 5.6 tells us that it will suffice to show that for all  $r \in A$ ,  $r \leq x$ . Let  $r$  be an arbitrary element of  $A$ . Then since  $r \in A$ ,  $x \in \mathbb{Q}$ , and  $x \notin A$ , the contrapositive of Definition 6.1b asserts that  $x \not< r$ . Therefore,  $r \leq x$ , as desired.

Now suppose that  $x$  is an upper bound for  $A$ . By Definition 5.6, this implies that for all  $r \in A$ ,  $r \leq x$ . Therefore, since there is no  $r \in A$  with  $r > x$ , by the contrapositive of Definition 6.1c,  $x \notin A$ , as desired.  $\square$

#### Exercise 6.3.

- (a) Prove that for any  $q \in \mathbb{Q}$ ,  $\{x \in \mathbb{Q} \mid x < q\}$  is a Dedekind cut. We then define  $\mathbf{0} = \{x \in \mathbb{Q} \mid x < 0\}$ .
- (b) Prove that  $\{x \in \mathbb{Q} \mid x \leq 0\}$  is not a Dedekind cut.
- (c) Prove that  $\{x \in \mathbb{Q} \mid x < 0\} \cup \{x \in \mathbb{Q} \mid x^2 < 2\}$  is a Dedekind cut.

*Proof of a.* Let  $q$  be an arbitrary element of  $\mathbb{Q}$ . To prove that  $A = \{x \in \mathbb{Q} \mid x < q\}$  is a Dedekind cut, Definition 6.1 tells us that it will suffice to show that  $A \neq \emptyset$ ;  $A \neq \mathbb{Q}$ ; if  $r \in A$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in A$ ; and if  $r \in A$ , then there is some  $s \in A$  with  $s > r$ . We will take this one claim at a time.

To show that  $A \neq \emptyset$ , Definition 1.8 tells us that it will suffice to find an element of  $A$ . By Exercise 3.9d,  $q$  is not the first point of  $\mathbb{Q}$ . Thus, by Definition 3.3, there exists an object  $x \in \mathbb{Q}$  such that  $x < q$ . By the definition of  $A$ , this implies that  $x \in A$ , as desired.

To show that  $A \neq \mathbb{Q}$ , Definition 1.2 tells us that it will suffice to find an element of  $\mathbb{Q}$  that is not an element of  $A$ . By hypothesis,  $q \in \mathbb{Q}$ . By Exercise 3.9d,  $q \not< q$ . Therefore,  $q \in \mathbb{Q}$  but  $q \notin A$ , as desired.

To show that if  $r \in A$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in A$ , we let  $r \in A$  and  $s \in \mathbb{Q}$  be arbitrary elements of their respective sets that satisfy  $s < r$  and seek to verify that  $s \in A$ . Since  $r \in A$ ,  $r < q$ . This combined with the fact that  $s < r$  implies by transitivity that  $s < q$ . Therefore, since  $s \in \mathbb{Q}$  and  $s < q$ ,  $s \in A$ , as desired.

To show that if  $r \in A$ , then there is some  $s \in A$  with  $s > r$ , we let  $r \in A$  and seek to find such an  $s$ . By the definition of  $A$ ,  $r < q$ . Thus, by Additional Exercise 3.1, there exists a point  $s \in \mathbb{Q}$  such that  $r < s < q$ . Since  $s \in \mathbb{Q}$  and  $s < q$ ,  $s \in A$ . It follows that  $s$  is the desired element of  $A$  which satisfies  $s > r$ .  $\square$

*Proof of b.* To prove that  $A = \{x \in \mathbb{Q} \mid x \leq 0\}$  is not a Dedekind cut, Definition 6.1 tells us that it will suffice to show that  $A$  *does* have a last point. To show this, we will demonstrate that 0 is the last point of  $A$ . To demonstrate this, Definition 3.1 tells us that it will suffice to confirm that  $0 \in A$  and for all  $x \in A$ ,  $x \leq 0$ . Since  $0 \leq 0$  and  $0 \in \mathbb{Q}$ ,  $0 \in A$ . Additionally, by the definition of  $A$ , it is true that for all  $x \in A$ ,  $x \leq 0$ .  $\square$

*Proof of c.* Let  $B = \{x \in \mathbb{Q} \mid x < 0\}$  and let  $C = \{x \in \mathbb{Q} \mid x^2 < 2\}$ . To prove that  $A = B \cup C$  is a Dedekind cut, Definition 6.1 tells us that it will suffice to show that  $A \neq \emptyset$ ;  $A \neq \mathbb{Q}$ ; if  $r \in A$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in A$ ; and if  $r \in A$ , then there is some  $s \in A$  with  $s > r$ . We will take this one claim at a time.

To show that  $A \neq \emptyset$ , Definition 1.8 tells us that it will suffice to find an element of  $A$ . Since  $-1 \in \mathbb{Q}$  and  $-1 < 0$ ,  $-1 \in B$ . Therefore, by Definition 1.5,  $-1 \in A$ , as desired.

To show that  $A \neq \mathbb{Q}$ , Definition 1.2 tells us that it will suffice to find an element of  $\mathbb{Q}$  that is not an element of  $A$ . Since  $2 \geq 0$ ,  $2 \notin B$ . Additionally, since  $2^2 \geq 2$ ,  $2 \notin C$ . Therefore, by Definition 1.5,  $2 \notin A$ , as desired.

To show that if  $r \in A$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in A$ , we let  $r \in A$  and  $s \in \mathbb{Q}$  be arbitrary elements of their respective sets that satisfy  $s < r$  and seek to verify that  $s \in A$ . We divide into two cases ( $s < 0$  and  $s \geq 0$ ). Suppose first that  $s < 0$ . Then  $s \in B$ , meaning that  $s \in A$ . Now suppose that  $s \geq 0$ . Then by Script 0, we have  $0 \leq s^2 < r^2 < 2$ . Thus, by the definition of  $C$ ,  $s \in C$ , implying that  $s \in A$ .

To show that  $A$  does not have a last point, suppose for the sake of contradiction that  $A$  has a last point  $p$ . We now divide into two cases ( $p \leq 0$  and  $p > 0$ ). Suppose first that  $p \leq 0$ . Since  $p$  is the last point of  $A$ , Definition 3.3 tells us that  $x \leq p$  for all  $x \in A$ . But  $1 \in A$  (since  $1 \in \mathbb{Q}$  and  $1^2 = 1 < 2$  implies  $1 \in B$ , implies  $1 \in A$ ) and  $1 > 0 \geq p$ , a contradiction. Now suppose that  $p > 0$ . Definition 3.3 tells us that  $p \in A$ , but the condition that  $p > 0$  means  $p \notin B$ , so we must have  $p \in C$ . However, by the proof of Exercise 4.24,  $\frac{2(p+1)}{p+2}$  will be an element of  $B$  (and therefore  $A$ ) that is greater than  $p$  no matter how large  $p$  is, a contradiction.  $\square$

**Definition 6.4.** If  $A, B \in \mathbb{R}$ , we say that  $A < B$  if  $A$  is a proper subset of  $B$ .

**Exercise 6.5.** Show that  $\mathbb{R}$  satisfies Axioms 1, 2, and 3.

*Proof.* By Exercise 6.3a,  $\{x \in \mathbb{Q} \mid x < 0\} \in \mathbb{R}$  since  $0 \in \mathbb{Q}$ . Therefore, Axiom 1 is immediately satisfied.

Axiom 2 asserts that  $\mathbb{R}$  must have an ordering  $<$ . As such, it will suffice to verify that the ordering given by Definition 6.4 satisfies the stipulations of Definition 3.1. To prove that  $<$  satisfies the trichotomy, it will suffice to show that for all  $A, B \in \mathbb{R}$ , exactly one of the following holds:  $A < B$ ,  $B < A$ , or  $A = B$ .

We first show that *no more than one* of the three statements can simultaneously be true. Let  $A, B$  be arbitrary elements of  $\mathbb{R}$ . We divide into three cases. First, suppose for the sake of contradiction that  $A < B$  and  $B < A$ . By Definition 6.4, this implies that  $A \subsetneq B$  and  $B \subsetneq A$ . Thus, by Definition 1.3,  $A \subset B$ ,  $B \subset A$ , and  $A \neq B$ . But by Theorem 1.7,  $A \subset B$  and  $B \subset A$  implies that  $A = B$ , a contradiction. Second, suppose for the sake of contradiction that  $A < B$  and  $A = B$ . By substitution, we have that  $A < A$ . But by Definitions 6.4 and 1.3, it follows that  $A \neq A$ . The proof of the third case ( $B < A$  and  $A = B$ ) is symmetric to that of the second case.

We now show that *at least one* of the three statements is always true. Let  $A, B$  be arbitrary elements of  $\mathbb{R}$ , and suppose for the sake of contradiction that  $A \not< B$ ,  $B \not< A$ , and  $A \neq B$ . Since  $A \not< B$  and  $B \not< A$ , we have by Definition 6.4 that  $A \not\subsetneq B$  and  $B \not\subsetneq A$ . Thus, by Definition 1.3,  $A \not\subset B$  or  $A = B$ , and  $B \not\subset A$  or  $A = B$ . But  $A \neq B$  by hypothesis, so it must be that  $A \not\subset B$  and  $B \not\subset A$ . It follows from the first statement by Definition 1.3 that there exists an object  $x \in A$  such that  $x \notin B$ , and there exists an object  $y \in B$  such that  $y \notin A$ . Since  $x \notin B$ , Lemma 6.2 implies that  $x$  is an upper bound of  $B$ . Consequently, by Definition 5.6,  $p \leq x$  for all  $p \in B$ , including  $y$ . Similarly,  $p \leq y$  for all  $p \in A$ , including  $x$ . Thus, we have  $y \leq x$  and  $x \leq y$ , implying that  $x = y$ . But since  $y \in B$ , this implies that  $x \in B$ , a contradiction.

To prove that  $<$  is transitive, it will suffice to show that for all  $A, B, C \in \mathbb{R}$ , if  $A < B$  and  $B < C$ , then  $A < C$ . Let  $A, B, C$  be arbitrary elements of  $\mathbb{R}$  for which it is true that  $A < B$  and  $B < C$ . By Definition 6.4, we have  $A \subsetneq B$  and  $B \subsetneq C$ . Thus, by Script 1,  $A \subsetneq C$ . Therefore, by Definition 6.4,  $A < C$ .

Axiom 3 asserts that  $\mathbb{R}$  must have no first or last point. We will take this one argument at a time

Suppose for the sake of contradiction that  $\mathbb{R}$  has some first point  $A$ . Then by Definition 3.3,  $A \leq X$  for every  $X \in \mathbb{R}$ . Now since  $A$  is a Dedekind cut, Definition 6.1 tells us that  $A \neq \emptyset$ . Thus, by Definition 1.8, there exists some  $q \in A$ . Additionally,  $A \subset \mathbb{Q}$  by Definition 6.1, so  $q \in A$  implies that  $q \in \mathbb{Q}$ . It follows by Exercise 6.3a that  $B = \{x \in \mathbb{Q} \mid x < q\}$  is a Dedekind cut. We now seek to prove that  $B \subsetneq A$ . To do this, Definition 1.3 tells us that it will suffice to show that  $B \neq A$  and  $B \subset A$ . To show that  $B \neq A$ , Definition 1.2 tells us that it will suffice to find an element of  $A$  that is not an element of  $B$ . Conveniently,  $q$  is clearly such an object. To show that  $B \subset A$ , Definition 1.3 tells us that we must confirm that every element of  $B$  is an element of  $A$ . Let  $p$  be an arbitrary element of  $B$ . Then by the definition of  $B$ ,  $p \in \mathbb{Q}$  and  $p < q$ . It follows by Definition 6.1b (which clearly applies to  $A$ ) that  $p \in A$ , as desired. Having proven that  $B \subsetneq A$ , Definition 6.4 tells us that  $B < A$ . But this contradicts the previously demonstrated fact that  $A \leq X$  for every  $X \in \mathbb{R}$ , including  $B$ .

Suppose for the sake of contradiction that  $\mathbb{R}$  has some last point  $A$ . Then by Definition 3.3,  $X \leq A$  for every  $X \in \mathbb{R}$ . Now since  $A$  is a Dedekind cut, Definition 6.1 tells us that  $A \neq \mathbb{Q}$ . Thus, by Definition 1.2, there exists some  $q \in \mathbb{Q}$  such that  $q \notin A$ . It follows by Lemma 6.2 that  $q$  is an upper bound of  $A$ . Consequently, by Definition 5.6,  $x \leq q$  for all  $x \in A$ . Additionally, by Exercise 6.3a,  $B = \{x \in \mathbb{Q} \mid x < q + 1\}$ <sup>[1]</sup> is a Dedekind cut. We now seek to prove that  $A \subsetneq B$ . As before, this means we must show that  $A \neq B$  and  $A \subset B$ . To show that  $A \neq B$ , Definition 1.2 tells us that it will suffice to find an element of  $B$  that is not an element of  $A$ . Since  $x \leq q$  for all  $x \in A$  and  $q < q + 0.5 < q + 1$ ,  $q + 0.5 \notin A$  and  $q + 0.5 \in B$  is one such desired object. To show that  $A \subset B$ , Definition 1.3 tells us that we must confirm that every element of  $A$  is an element of  $B$ . Let  $p$  be an arbitrary element of  $A$ . As an element of  $A$ , we know that  $p \leq q$ . Thus,  $p < q + 1$ , so  $p \in B$ , as desired. Having proven that  $A \subsetneq B$ , Definition 6.4 tells us that  $A < B$ . But this contradicts the previously demonstrated fact that  $X \leq A$  for every  $X \in \mathbb{R}$ , including  $B$ .  $\square$

1/14: **Lemma 6.6.** *A nonempty subset of  $\mathbb{R}$  that is bounded above has a supremum.*

*Proof.* Let  $X$  be an arbitrary nonempty subset of  $\mathbb{R}$  that is bounded above. To prove that  $\sup X$  exists, we will show that  $\sup X = U = \bigcup\{Y \mid Y \in X\}$ . To show this, Definition 5.7 tells us that it will suffice to demonstrate that  $U \in \mathbb{R}$ ,  $U$  is an upper bound of  $X$ , and if  $U'$  is an upper bound of  $X$ , then  $U \leq U'$ . Let's begin.

To demonstrate that  $U \in \mathbb{R}$ , Definition 6.1 tells us that it will suffice to confirm that  $U \neq \emptyset$ ;  $U \neq \mathbb{Q}$ ; if  $r \in U$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in U$ ; and if  $r \in U$ , then there is some  $s \in U$  with  $s > r$ .

As the union of a nonempty set of nonempty sets, Script 1 implies that  $U \neq \emptyset$ .

To demonstrate that  $U \neq \mathbb{Q}$ , Definition 1.2 tells us that it will suffice to find a point  $p \in \mathbb{Q}$  such that  $p \notin U$ . Since  $X$  is bounded above, we have by Definition 5.6 that there exists a Dedekind cut  $V \in \mathbb{R}$  such that  $A \leq V$  for all  $A \in X$ . It follows by Definition 6.4 that  $A \subset V$  for all  $A \in X$ . Thus, by Script 1,  $U \subset V$ . Now since  $V$  is a Dedekind cut, we know by Definition 6.1 that  $V \subset \mathbb{Q}$  and  $V \neq \mathbb{Q}$ , meaning that there exists a point  $p \in \mathbb{Q}$  such that  $p \notin V$ . Consequently, since  $U \subset V$ ,  $p \notin U$ , as desired.

To demonstrate that if  $r \in U$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in U$ , we let  $r \in U$  and  $s \in \mathbb{Q}$  be arbitrary elements of their respective sets that satisfy  $s < r$  and seek to verify that  $s \in U$ . Since  $r \in U$ , Definition 1.13 tells us that  $r \in A$  for some  $A \in X$ . Thus, since  $A$  is a Dedekind cut,  $s \in \mathbb{Q}$  and  $s < r$  implies that  $s \in A$ . Therefore,  $s \in U$ .

To demonstrate that if  $r \in U$ , then there is some  $s \in U$  with  $s > r$ , we let  $r \in U$  and seek to find such an  $s$ . Since  $r \in U$ , Definition 1.13 tells us that  $r \in A$  for some  $A \in X$ . Thus, since  $A$  is a Dedekind cut, there exists a point  $s \in A$  with  $s > r$ . Therefore,  $s \in U$ .

To demonstrate that  $U$  is an upper bound of  $X$ , Definition 5.6 tells us that it will suffice to confirm that  $A \leq U$  for all  $A \in X$ . To confirm this, Definition 6.4 tells us that it will suffice to verify that  $A \subset U$  for all  $A \in X$ . But by an extension of Theorem 1.7b, this is true.

Now suppose for the sake of contradiction that there exists an upper bound  $U'$  of  $X$  such that  $U' < U$ . It follows by Definitions 6.4 and 1.3 that there exists a point  $p \in U$  such that  $p \notin U'$ . Thus, by the former statement and Definition 1.13,  $p \in A$  for some  $A \in X$ . Additionally, since  $U'$  is an upper bound of  $X$ , we have by Definitions 5.6 and 6.4 that  $A \subset U'$  for all  $A \in X$ . But this implies by Definition 1.3 that  $p \in U'$ , a contradiction.  $\square$

<sup>1</sup>Note that we add 1 to  $q$  to treat the case that  $q = \sup A$ , a case in which we would have  $B = A$  if  $B$  were defined as  $\{x \in \mathbb{Q} \mid x < q\}$ .

1/19: **Exercise 6.7.** Show that  $\mathbb{R}$  satisfies Axiom 4.

*Proof.* Suppose for the sake of contradiction that  $\mathbb{R}$  does not satisfy Axiom 4. It follows that  $\mathbb{R}$  is not connected, implying by Definition 4.22 that  $\mathbb{R} = A \cup B$  where  $A, B$  are disjoint, nonempty, open sets. Since  $A, B$  are disjoint and nonempty, we know that there exist distinct objects  $a \in A$  and  $b \in B$ . WLOG, let  $a < b$ .

We now seek to prove that the set  $A \cap \underline{ab}$  is nonempty and bounded above. To prove that  $A \cap \underline{ab}$  is nonempty, Definition 1.8 tells us that it will suffice to find an element of  $A \cap \underline{ab}$ . Since  $a \in A$  and  $A$  is open, we have by Theorem 4.10 that there exists a region  $\underline{cd}$  such that  $a \in \underline{cd}$  and  $\underline{cd} \subset A$ . It follows by Definitions 3.10 and 3.6 that  $a < d$ , implying by Lemma 6.10<sup>[2]</sup> that there exists some point  $x \in \mathbb{R}$  such that  $c < a < x < d < b$  (note that  $d < b$  since if  $b < d$ , then  $b \in \underline{cd}$  would contradict the fact that  $\underline{cd} \subset A$ ). Consequently,  $x \in \underline{cd}$ , meaning that  $x \in A$ , and  $x \in \underline{ab}$ . Therefore,  $x \in A \cap \underline{ab}$ , as desired. To prove that  $A \cap \underline{ab}$  is bounded above, Definition 5.6 tells us that it will suffice to show that  $b$  is an upper bound of  $A \cap \underline{ab}$ . To show this, Definition 5.6 tells us that it will suffice to confirm that  $y \leq b$  for all  $y \in A \cap \underline{ab}$ . Let  $y$  be an arbitrary element of  $A \cap \underline{ab}$ . Then by Definition 1.6,  $y \in A$  and  $y \in \underline{ab}$ . It follows from the latter statement by Definitions 3.10 and 3.6 that  $y < b$ , i.e.,  $y \leq b$ , as desired.

Having established that  $A \cap \underline{ab} \subset \mathbb{R}$  is nonempty and bounded above, we can invoke Lemma 6.6 to learn that  $A \cap \underline{ab}$  has a supremum  $\sup(A \cap \underline{ab})$ . We now divide into two cases ( $\sup(A \cap \underline{ab}) \in A$  and  $\sup(A \cap \underline{ab}) \in B$ ; it follows from the definitions of  $A$  and  $B$  that exactly one of these cases is true). Suppose first that  $\sup(A \cap \underline{ab}) \in A$ . Then since  $A$  is open, we have by Theorem 4.10 that there exists a region  $\underline{ef}$  such that  $\sup(A \cap \underline{ab}) \in \underline{ef}$  and  $\underline{ef} \subset A$ . It follows from the former condition that  $\sup(A \cap \underline{ab}) < f$ . Thus, by Lemma 6.10, there exists an object  $z \in \mathbb{R}$  such that  $e < \sup(A \cap \underline{ab}) < z < f < b$  (note that  $f < b$  for the same reason that  $d < b$ ). Consequently,  $z \in \underline{ef}$ , implying that  $z \in A$ , and  $z \in \underline{ab}$ . Thus, we have found an element of  $A \cap \underline{ab}$  that is greater than  $\sup(A \cap \underline{ab})$ , contradicting Definitions 5.7 and 5.6. The proof is symmetric in the other case (except that we find an element of  $B$  less than  $\sup(A \cap \underline{ab})$ ).  $\square$

1/14: **Definition 6.8.** Let  $C$  be a continuum satisfying Axioms 1-4. Consider a subset  $X \subset C$ . We say that  $X$  is **dense** in  $C$  if every  $p \in C$  is a limit point of  $X$ .

**Lemma 6.9.** A subset  $X \subset C$  is dense in  $C$  if and only if  $\overline{X} = C$ .

*Proof.* Suppose first that  $X \subset C$  is dense in  $C$ . To prove that  $\overline{X} = C$ , Definition 1.2 tells us that it will suffice to show that every point  $p \in \overline{X}$  is an element of  $C$  and vice versa. Clearly, every element of  $\overline{X}$  is an element of  $C$ . On the other hand, let  $p$  be an arbitrary element of  $C$ . Since  $X$  is dense in  $C$ , Definition 6.8 tells us that  $p \in LP(X)$ . Therefore, by Definitions 1.5 and 4.4,  $p \in \overline{X}$ .

Now suppose that  $\overline{X} = C$ . To prove that  $X$  is dense in  $C$ , Definition 6.8 tells us that it will suffice to show that every  $p \in C$  is a limit point of  $X$ . Let  $p$  be an arbitrary element of  $C$ . By Corollary 5.4, this implies that  $p \in LP(C)$ . It follows that  $p \in LP(\overline{X})$ . Thus, by Definition 4.4,  $p \in LP(X \cup LP(X))$ . Consequently, by Theorem 3.20,  $p \in LP(X)$  or  $p \in LP(LP(X))$ . We now divide into two cases. If  $p \in LP(X)$ , then we are done. On the other hand, if  $p \in LP(LP(X))$ , the lemma from Theorem 4.6 asserts that  $p \in LP(X)$ , and we are done again.  $\square$

Our next goal is to prove that  $\mathbb{Q}$  is dense in  $\mathbb{R}$ . Just to make sense of that statement, we need to decide how to think of  $\mathbb{Q}$  as a subset of  $\mathbb{R}$ . For every rational number  $q \in \mathbb{Q}$ , define the corresponding real number as the Dedekind cut

$$i(q) = \{x \in \mathbb{Q} \mid x < q\}$$

For example,  $\mathbf{0} = i(0)$ . It can be verified that this gives a well-defined injective function  $i : \mathbb{Q} \rightarrow \mathbb{R}$ . We identify  $\mathbb{Q}$  with its image  $i(\mathbb{Q}) \subset \mathbb{R}$  so that the rational numbers  $\mathbb{Q}$  are a subset of the real numbers  $\mathbb{R}$ . (Similarly,  $\mathbb{N}$  and  $\mathbb{Z}$  can be understood as subsets of  $\mathbb{R}$ .)

<sup>2</sup>We may use this lemma since it does not depend on this result, Definition 6.8, or Lemma 6.9.

**Lemma 6.10.** *Given  $A, B \in \mathbb{R}$  with  $A < B$ , there exists  $p \in \mathbb{Q}$  such that  $A < i(p) < B$ .*

*Proof.* Since  $A < B$ , Definition 6.4 tells us that  $A \subsetneq B$ . Thus, by Definition 1.3, there exists a point  $q$  such that  $q \in B$  and  $q \notin A$ . Since  $q \in B$  where  $B$  is a Dedekind cut, we have by Definition 6.1 that there exists a point  $p \in B$  with  $p > q$ . Additionally, since  $q \notin A$  implies that  $q$  is an upper bound of  $A$  by Lemma 6.2, we know by Definition 5.6 that  $x \leq q$  for all  $x \in A$ . It follows since  $q < p$  that  $x \leq p$  for all  $x \in A$ , meaning by Definition 5.6 and Lemma 6.2 that  $p \notin A$ . Having established that  $p, q \in B$ ,  $p, q \notin A$ , and  $q < p$ , we are now ready to prove that  $A < i(p) < B$ . Definition 6.4 tells us that we may do so by showing that  $A \subsetneq i(p)$  and  $i(p) \subsetneq B$ . We will take this one argument at a time.

To show that  $A \subsetneq i(p)$ , Definition 1.3 tells us that it will suffice to verify that every element of  $A$  is an element of  $i(p)$  and that there exists an element of  $i(p)$  that is not an element of  $A$ . We treat the former statement first. As previously mentioned,  $x \leq p$  for all  $x \in A$ . This combined with the fact that  $p \notin A$  implies that  $x < p$  for all  $x \in A$ . Thus, by the definition of  $i(p)$ ,  $x \in i(p)$  for all  $x \in A$ , as desired. As to the latter statement, since  $q < p$ , we have by the definition of  $i(p)$  that  $q \in i(p)$ . However, we also know that  $q \notin A$ , as desired.

To show that  $i(p) \subsetneq B$ , we must verify symmetric arguments to before. For the former statement, let  $r$  be an arbitrary element of  $i(p)$ . Then by the definition of  $i(p)$ ,  $r < p$ . Since  $p \in B$  and  $r \in \mathbb{Q}$  satisfy  $r < p$ , we have by Definition 6.1 that  $r \in B$ , as desired. As to the latter statement,  $p$  is clearly an element of  $B$  that is not an element of  $i(p)$ , as desired.  $\square$

1/19: **Theorem 6.11.**  *$i(\mathbb{Q})$  is dense in  $\mathbb{R}$ .*

*Proof.* To prove that  $i(\mathbb{Q})$  is dense in  $\mathbb{R}$ , Definition 6.8 tells us that it will suffice to show the every point  $X \in \mathbb{R}$  is a limit point of  $i(\mathbb{Q})$ . Let  $X$  be an arbitrary element of  $\mathbb{R}$ . To show that  $X \in LP(i(\mathbb{Q}))$ , Definition 3.13 tells us that it will suffice to verify that for every region  $\underline{AB}$  with  $X \in \underline{AB}$ , we have  $\underline{AB} \cap (i(\mathbb{Q}) \setminus \{X\}) \neq \emptyset$ . Let  $\underline{AB}$  be an arbitrary region with  $X \in \underline{AB}$ . It follows by Definitions 3.10 and 3.6 that  $A < X < B$ . Thus, by Lemma 6.10, there exists  $p \in \mathbb{Q}$  such that  $A < i(p) < X < B$ . By Definitions 3.6 and 3.10,  $i(p) \in \underline{AB}$ . By Definition 1.18,  $i(p) \in i(\mathbb{Q})$ . By Exercise 6.5,  $i(p) < X$  implies that  $i(p) \neq X$ . Combining the last three results with Definitions 1.11 and 1.6, we have that  $i(p) \in \underline{AB} \cap (i(\mathbb{Q}) \setminus \{X\})$ , as desired.  $\square$

**Corollary 6.12** (The Archimedean Property). *Let  $A \in \mathbb{R}$  be a positive real number. Then there exist nonzero natural numbers  $n, m \in \mathbb{N}$  such that  $i(\frac{1}{n}) < A < i(m)$ .*

*Proof.* We will first prove that there exists a nonzero natural number  $n$  such that  $i(\frac{1}{n}) < A$ . We will then prove that there exists a nonzero natural number  $m$  such that  $A < i(m)$ . Let's begin.

Since  $A \in \mathbb{R}$  is positive, we know that  $0 < A$ . Thus, by Lemma 6.10, there exists  $\frac{p}{n} \in \mathbb{Q}$  such that  $0 < i(\frac{p}{n}) < A$ . As permitted by Exercise 3.9b, we choose  $\frac{p}{n} \in [\frac{p}{n}]$  to be an object such that  $0 < n$  (this also means that  $n \in \mathbb{N}$ ). Consequently, by Scripts 2 and 3, we know that  $0 < \frac{1}{n} \leq \frac{p}{n}$ . It follows that  $i(\frac{1}{n}) \leq i(\frac{p}{n})$  since  $x \in i(\frac{1}{n})$  implies  $x < \frac{1}{n} \leq \frac{p}{n}$  implies  $x \in i(\frac{p}{n})$ , implies  $i(\frac{1}{n}) \subset i(\frac{p}{n})$ . Therefore,  $i(\frac{1}{n}) \leq i(\frac{p}{n}) < A$ , as desired.

By Exercise 6.5, Axiom 3, and Definition 3.3, there exists a point  $B \in \mathbb{R}$  such that  $A < B$ . It follows by Lemma 6.10 that there exists  $\frac{m}{q} \in \mathbb{Q}$  such that  $A < i(\frac{m}{q}) < B$ . As before, let  $\frac{m}{q}$  be an object such that  $0 < q$ . Consequently, by Scripts 2 and 3, we know that  $0 < \frac{m}{q} \leq m$ . Once again, for the same reasons as before,  $i(\frac{m}{q}) \leq i(m)$ . Therefore,  $A < i(\frac{m}{q}) \leq i(m)$ , as desired.  $\square$

**Corollary 6.13.**  *$i(\mathbb{N})$  is an unbounded subset of  $\mathbb{R}$ .*

*Proof.* Suppose for the sake of contradiction that  $i(\mathbb{N})$  is bounded above. Then by Definition 5.6, there exists a point  $A \in \mathbb{R}$  such that  $i(n) \leq A$  for all  $n \in \mathbb{N}$ . Note that  $A$  is a positive real number since  $i(0) < i(0) \leq A$ . But by Corollary 6.12,  $A < i(n)$  for some  $n \in \mathbb{N}$ , a contradiction.  $\square$

1/21: **Corollary 6.14.** *If  $A \in \mathbb{R}$  is a real number, then there is an integer  $n$  such that  $i(n-1) \leq A < i(n)$ .*

*Proof.* Let  $X$  be the set of all integers  $z$  such that  $i(z) \leq A$ . Symbolically,

$$X = \{z \mid z \in \mathbb{Z} \text{ and } i(z) \leq A\}$$

Since  $A \neq \emptyset$  by Definition 6.1, there exists a point  $\frac{p}{q} \in \mathbb{Q}$  such that  $\frac{p}{q} \in A$ . As in Corollary 6.12, we let  $q > 0$ . It follows by Scripts 2 and 3 that if  $p \geq 0$ , then  $0 \leq \frac{p}{q}$ , i.e.<sup>[3]</sup>,  $i(0) \leq A$  and if  $p < 0$ , then  $p \leq \frac{p}{q}$ , i.e.,  $i(p) \leq A$ . Thus, in either case,  $X$  is nonempty.

Now there exists a nonzero natural number  $m$  such that  $A < i(m)$  (if  $A \leq i(0)$ , then  $A < i(1)$ ; if  $A > 0$ , then apply Corollary 6.12). Let  $f : X \rightarrow \mathbb{N}$  be defined by the rule

$$f(x) = m - x$$

By Script 1,  $f$  is an injective function,  $f(X) \subset \mathbb{N}$ , and  $f(X)$  is nonempty (since  $X$  is nonempty). Thus, by the well-ordering principle (Additional Exercise 0.1), there is a least element, which we shall call  $y$ , in  $f(X)$ . Since  $f$  is injective, there exists exactly one object  $n-1 \in X$  such that  $f(n-1) = y$ .

By the definition of  $X$ ,  $i(n-1) \leq A$ . To prove that  $A < i(n)$ , suppose for the sake of contradiction that  $i(n) \leq A$ . This coupled with the fact that  $n \in \mathbb{Z}$  implies that  $n \in X$ . Thus,  $f(n) \in f(X)$ . But  $f(n) = m - n < m - n + 1 = m - (n-1) = f(n-1)$ , contradicting the fact that  $f(n-1)$  is the least element of  $f(X)$ .  $\square$

1/26: **Axiom 1.** *The continuum contains a countable dense subset.*

**Definition 6.15.** Let  $X$  and  $Y$  be sets with orderings  $<_X$  and  $<_Y$ , respectively. A function  $f : X \rightarrow Y$  is **order-preserving** if for all  $r, s \in X$ ,

$$r <_X s \implies f(r) <_Y f(s)$$

Note that the function  $i : \mathbb{Q} \rightarrow \mathbb{R}$  discussed above is order-preserving.

**Exercise 6.16.** Let  $C$  satisfy Axioms 1-5. Let  $K \subset C$  be a countable dense subset of  $C$ . Construct an order-preserving bijection  $f : \mathbb{Q} \rightarrow K$ .

**Lemma.**

a)  $K$  satisfies Axiom 3.

b) (Density Lemma) *For all  $x, y \in K$ , if  $x < y$ , then there exists a point  $z \in K$  such that  $z$  is between  $x$  and  $y$ .*

*Proof of a.* To prove that  $K$  satisfies Axiom 3, we must verify that  $K$  has neither a first nor a last point. We will address the first point question first. Suppose for the sake of contradiction that  $K$  has a first point  $x$ . Then by Definition 3.3,  $x \leq y$  for all  $y \in K$ . However, since  $C$  satisfies Axiom 3, there exists an object  $a \in C$  such that  $a < x$ . Now consider the region  $\underline{ax}$ . We have by Corollary 5.3 that there exists a point  $p \in \underline{ax}$ . Additionally, we have by Script 3 that  $\underline{ax} \cap K = \emptyset$ . Thus,  $\underline{ax} \cap (K \setminus \{p\}) = \emptyset$ , implying by Definition 3.13 that  $p \notin LP(K)$ . But since  $p \in C$  and  $p \notin LP(K)$ , we have by Definition 6.8 that  $K$  is not dense in  $C$ , a contradiction.

The proof is symmetric for last points.  $\square$

*Proof of b.* Suppose for the sake of contradiction that that there exist  $x, y \in K$  with  $x < y$  such that no point  $z \in K$  is between  $x$  and  $y$ . By Theorem 5.2, there exists  $p \in C$  such that  $p$  is between  $x$  and  $y$ . Consequently, by Definition 3.10,  $p \in \overline{xy}$ . Additionally, we have by Script 3 that  $\overline{xy} \cap K = \emptyset$ . It follows that  $\overline{xy} \cap (K \setminus \{p\}) = \emptyset$ , implying by Definition 3.13 that  $p \notin LP(K)$ . But since  $p \in \overline{C}$  and  $p \notin LP(K)$ , we have by Definition 6.8 that  $K$  is not dense in  $C$ , a contradiction.  $\square$

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<sup>3</sup>For the same reasons as in Corollary 6.12.



*Proof of Exercise 6.16.* By Theorem 2.11,  $\mathbb{Q}$  is countable, implying by Definition 1.35 that there exists a bijection  $g : \mathbb{N} \rightarrow \mathbb{Q}$ . The existence of this bijection means that we can refer to an arbitrary element  $q$  of  $\mathbb{Q}$  by the number  $n$  for which  $g(n) = q$ ; in another notation, we can refer to  $q$  as  $q_n$ . Thus, since every element of  $\mathbb{Q}$  can be written as  $q_n$  for some  $n \in \mathbb{N}$ , we can write  $\mathbb{Q} = \{q_1, q_2, \dots\}$ . Similarly, we can express  $K$  as  $K = \{k_1, k_2, \dots\}$ . We will use this method of referring to the elements of  $\mathbb{Q}$  to construct  $f$ .

We define  $f$  recursively with strong induction. For the base case  $q_1$ , we define  $f(q_1) = k_1$ . Now suppose inductively that we have defined  $f(q_1), f(q_2), \dots, f(q_n)$ ; we now seek to define  $f(q_{n+1})$ . By Theorem 3.5, the symbols  $a_1, \dots, a_{n+1}$  can be assigned to  $q_1, \dots, q_{n+1}$  so that  $a_1 <_{\mathbb{Q}} a_2 <_{\mathbb{Q}} \dots <_{\mathbb{Q}} a_{n+1}$ . We divide into three cases ( $q_{n+1} = a_1$ ,  $q_{n+1} = a_{n+1}$ , and  $q_{n+1} = a_i$  where  $1 < i < n+1$ ). First, suppose that  $q_{n+1} = a_1$ . By the inductive hypothesis,  $f(a_2), f(a_3), \dots, f(a_{n+1})$  are defined elements of  $K$ . At this point, define the set  $X = \{k \in K \mid k <_K f(a_2)\}$ . It follows by Lemma (a) that this set is nonempty. Thus, by the well-ordering principle, there exists a  $k_i \in X$  such that  $i \leq j$  for all  $k_j \in X$ . We let  $f(q_{n+1}) = k_i$ . The second case is symmetric to the first. Third, suppose that  $q_{n+1} = a_i$  where  $1 < i < n+1$ . By the inductive hypothesis,  $f(a_1), \dots, f(a_{i-1}), f(a_{i+1}), \dots, f(a_{n+1})$  are defined elements of  $K$ . At this point, define the set  $X = \{k \in K \mid f(a_{i-1}) <_K k <_K f(a_{i+1})\}$ . It follows by Lemma (b) that this set is nonempty. Thus, by the well-ordering principle, there exists a  $k_i \in X$  such that  $i \leq j$  for all  $k_j \in X$ . We let  $f(q_{n+1}) = k_i$ .

To prove that  $f$  is a function, Definition 1.16 tells us that it will suffice to show that for all  $q \in \mathbb{Q}$ , there exists a unique  $k \in K$  such that  $f(q) = k$ . First, we will prove that for all  $q \in \mathbb{Q}$ , there exists *some*  $k \in K$  such that  $f(q) = k$ . Let  $q_i$  be an arbitrary element of  $\mathbb{Q}$ . Then  $i \in \mathbb{N}$ , and by the principle of strong mathematical induction (Additional Exercise 0.2b),  $f(q_i)$  is assigned to an element of  $k$ . As to proving the uniqueness of the  $k$  to which  $q_i$  is defined, each  $q$  is assigned once, in one of three mutually exclusive cases, to an unambiguously defined (as guaranteed by the well-ordering principle) element of  $K$ .

To prove that  $f$  is order-preserving, we will first verify the following claim (which we will refer to as Lemma (c) for future reference): Consider the set  $\{q_1, \dots, q_n\} \subset \mathbb{Q}$ ; if the symbols  $a_1, \dots, a_n$  are assigned to  $q_1, \dots, q_n$  such that  $a_1 <_{\mathbb{Q}} a_2 <_{\mathbb{Q}} \dots <_{\mathbb{Q}} a_n$ , then  $f(a_1) <_K f(a_2) <_K \dots <_K f(a_n)$ . We will then use this result to prove that  $f$  is order-preserving for any two arbitrary elements  $q_i, q_j \in \mathbb{Q}$ . Let's begin.

To verify the above claim, we induct on  $n$ . The base case  $n = 1$  is vacuously true. Now suppose inductively that we have proven the claim for  $n$ ; we now seek to prove it for  $n+1$ . By Theorem 3.5, the symbols  $a_1, \dots, a_{n+1}$  can be assigned to  $q_1, \dots, q_{n+1}$  so that  $a_1 <_{\mathbb{Q}} a_2 <_{\mathbb{Q}} \dots <_{\mathbb{Q}} a_{n+1}$ . We divide into three cases ( $q_{n+1} = a_1$ ,  $q_{n+1} = a_{n+1}$ , and  $q_{n+1} = a_i$  where  $1 < i < n+1$ ). First, suppose that  $q_{n+1} = a_1$ . By the definition of  $f$ ,  $f(q_{n+1}) \in \{k \in K \mid k <_K f(a_2)\}$ , meaning that  $f(q_{n+1}) = f(a_1) <_K f(a_2)$ . Additionally, by the inductive hypothesis, we know that  $f(a_2) <_K f(a_3) <_K \dots <_K f(a_{n+1})$  (since  $a_2, \dots, a_{n+1}$  correspond to  $q_1, \dots, q_n$ ). Together, these two results imply that  $f(a_1) <_K f(a_2) <_K \dots <_K f(a_{n+1})$ . The proof of the second case is symmetric to that of the first. Third, suppose that  $q_{n+1} = a_i$  where  $1 < i < n+1$ . By the definition of  $f$ ,  $f(q_{n+1}) \in \{k \in K \mid f(a_{i-1}) <_K k <_K f(a_{i+1})\}$ , meaning that  $f(a_{i-1}) <_K f(q_{n+1}) = f(a_i) <_K f(a_{i+1})$ . Additionally, by the inductive hypothesis, we know that  $f(a_1) <_K \dots <_K f(a_{i-1}) <_K f(a_{i+1}) <_K \dots <_K f(a_{n+1})$  (for an analogous reason to before). These two results imply that  $f(a_1) <_K f(a_2) <_K \dots <_K f(a_{n+1})$ .

We are now ready to actually prove that  $f$  is order-preserving. To do so, Definition 6.15 tells us that it will suffice to show that for all  $q_i, q_j \in \mathbb{Q}$ ,  $q_i <_{\mathbb{Q}} q_j$  implies  $f(q_i) <_K f(q_j)$ . Let  $q_i, q_j$  be arbitrary elements of  $\mathbb{Q}$  such that  $q_i <_{\mathbb{Q}} q_j$ . Since  $q_i <_{\mathbb{Q}} q_j$ ,  $q_i \neq q_j$ , implying that  $i \neq j$ . We divide into two cases ( $i < j$  and  $i > j$ ). Suppose first that  $i < j$ . By Theorem 3.5, the symbols  $a_1, \dots, a_j$  can be assigned to  $q_1, \dots, q_j$  so that  $a_1 <_{\mathbb{Q}} a_2 <_{\mathbb{Q}} \dots <_{\mathbb{Q}} a_j$ . Let  $q_j = a_l$ . Since  $q_i <_{\mathbb{Q}} q_j$ , we know that  $q_i = a_m$  where  $m < l$ . Additionally, by Lemma (c), we know that  $f(a_1) <_K f(a_2) <_K \dots <_K f(a_j)$ . It follows that  $f(a_m) <_K f(a_l)$ , implying that  $f(q_i) <_K f(q_j)$ , as desired. The proof is symmetric in the other case.

To prove that  $f$  is bijective, Definition 1.20 tells us that it will suffice to show that  $f$  is injective and surjective.

To show that  $f$  is injective, Definition 1.20 tells us that it will suffice to demonstrate that  $q_i \neq q_j$  implies  $f(q_i) \neq f(q_j)$ . WLOG let  $q_i <_{\mathbb{Q}} q_j$ . Then since  $f$  is order-preserving, Definition 6.15 implies that  $f(q_i) <_K f(q_j)$ . It follows that  $f(q_i) \neq f(q_j)$ , as desired.

We are now ready to actually show that  $f$  is surjective. To do so, Definition 1.20 tells us that it will suffice to demonstrate that for all  $k_n \in K$ , there exists a  $q_i \in \mathbb{Q}$  such that  $f(q_i) = k_n$ . To do this, we induct on  $n$ . For the base case  $n = 1$ , it follows from the definition of  $f$  that  $f(q_1) = k_1$ . Now suppose inductively that for each  $k_1, \dots, k_n$ , there exists a  $q_i \in \mathbb{Q}$  such that  $f(q_i) = k_n$ ; we now seek to prove the claim for  $n+1$ .

By Theorem 3.5, the symbols  $b_1, \dots, b_{n+1}$  can be assigned to  $k_1, \dots, k_{n+1}$  so that  $b_1 <_K b_2 <_K \dots <_K b_{n+1}$ . We divide into three cases ( $k_{n+1} = b_1$ ,  $k_{n+1} = b_{n+1}$ , and  $k_{n+1} = b_i$  where  $1 < i < n+1$ ). First, suppose that  $k_{n+1} = b_1$ . By the inductive hypothesis,  $b_2 = f(q_i) <_K b_3 = f(q_j) <_K \dots <_K b_{n+1} = f(q_l)$ . It follows by Definition 6.15 that  $q_i <_{\mathbb{Q}} q_j <_{\mathbb{Q}} \dots <_{\mathbb{Q}} q_l$ . At this point, define the set  $X = \{q \in \mathbb{Q} \mid q <_{\mathbb{Q}} q_i\}$ . It follows from Exercise 3.9d that this set is nonempty. Thus, by the well-ordering principle, there exists a  $q_m \in X$  such that  $m \leq m'$  for all  $q_{m'} \in X$ . By the definition of  $f$ ,  $f(q_m) = k_{n+1}$ . The proof of the second case is symmetric to that of the first. Third, suppose that  $k_{n+1} = b_i$  where  $1 < i < n+1$ . By the inductive hypothesis,  $b_2 = f(q_j) <_K \dots <_K b_{i-1} = f(q_{j'}) <_K b_{i+1} = f(q_l) <_K \dots <_K b_{n+1} = f(q_{l'})$ . It follows by Definition 6.15 that  $q_j <_{\mathbb{Q}} \dots <_{\mathbb{Q}} q_{j'} <_{\mathbb{Q}} q_l <_{\mathbb{Q}} \dots <_{\mathbb{Q}} q_{l'}$ . At this point, define the set  $X = \{q \in \mathbb{Q} \mid q_{j'} <_{\mathbb{Q}} q <_{\mathbb{Q}} q_l\}$ . It follows from Additional Exercise 3.1 that this set is nonempty. Thus, by the well-ordering principle, there exists a  $q_m \in X$  such that  $m \leq m'$  for all  $q_{m'} \in X$ . By the definition of  $f$ ,  $f(q_m) = k_{n+1}$ .  $\square$

**Exercise 6.17.** Let  $f : \mathbb{Q} \rightarrow K$  be an order-preserving bijection, as found in Exercise 6.16. Let  $A \in \mathbb{R}$ . Then  $A \subset \mathbb{Q}$  and so  $f(A) \subset K \subset C$ . Define  $F : \mathbb{R} \rightarrow C$  by

$$F(A) = \sup f(A)$$

1. Show  $\sup f(A)$  exists, so  $F$  is well-defined.
2. Show  $F$  is injective and order-preserving.

*Proof of 1.* To prove that  $\sup f(A)$  exists, Theorem 5.17 tells us that it will suffice to show that  $f(A)$  is nonempty and bounded above. To show that  $f(A)$  is nonempty, Definition 1.8 tells us that it will suffice to find an element of  $f(A)$ . By Definition 6.1,  $A \neq \emptyset$ . Thus, by Definition 1.8, there exists an object  $x \in A$ . It follows by Definition 1.18 that  $f(x) \in f(A)$ , as desired. To show that  $f(A)$  is bounded above, Definition 5.6 tells us that it will suffice to find an element of  $K$  such that  $f(x) \geq_C f(a)$  for all  $f(a) \in f(A)$ . By Definition 6.1,  $A \neq \mathbb{Q}$  and  $A \subset \mathbb{Q}$ . Thus, by Definition 1.2, there exists an object  $x \in \mathbb{Q}$  such that  $x \notin A$ . It follows from the latter condition by Lemma 6.2 that  $x$  is an upper bound for  $A$ . Thus, by Definition 5.6,  $x \geq a$  for all  $a \in A$ . Consequently, by Definition 6.15,  $f(x)$  is an element of  $K$  such that  $f(x) \geq_C f(a)$  for all  $f(a) \in f(A)$ , as desired.  $\square$

*Proof of 2.* To prove that  $F$  is order-preserving, Definition 6.15 tells us that it will suffice to show that for all  $A, B \in \mathbb{R}$ ,  $A <_{\mathbb{R}} B$  implies  $F(A) <_C F(B)$ . Let  $A, B$  be two arbitrary elements of  $\mathbb{R}$  satisfying  $A <_{\mathbb{R}} B$ . Then by Definitions 6.4 and 1.3, there exists a point  $x \in B$  such that  $x \notin A$ . It follows from the latter condition by Lemma 6.2 and Definition 5.6 that  $x \geq a$  for all  $a \in A$ . Thus, by Definition 6.15,  $f(x) \geq_C f(a)$  for all  $f(a) \in f(A)$ . Consequently, by Definition 5.7,  $\sup f(A) \leq_C f(x)$ . Additionally, by Definition 6.1, there exists a point  $y \in B$  such that  $y > x$ . Thus, by Definition 6.15, we have that  $f(y) >_C f(x)$ . It follows by Definitions 5.6 and 5.7 that  $f(y) \leq_C \sup f(B)$ . Combining two results, we therefore have that  $\sup f(A) \leq_C f(x) <_C f(y) \leq_C \sup f(B)$ , meaning that  $F(A) = \sup f(A) <_C \sup f(B) = F(B)$ , as desired.

To prove that  $F$  is injective, Definition 1.20 tells us that it will suffice to show that if  $A \neq B$ , then  $F(A) \neq F(B)$ . Let  $A, B$  be two distinct real numbers. Then by Exercise 6.5,  $A < B$  or  $B < A$ . We now divide into two cases. Suppose first that  $A < B$ . Then  $F(A) < F(B)$  by Definition 6.15 (which we have just proven applies to  $F$ ). This implies by Definition 3.1 that  $F(A) \neq F(B)$ , as desired. The proof is symmetric in the other case.  $\square$

**Theorem 6.18.** Suppose that  $C$  is a continuum satisfying Axioms 1-5. Then  $C$  is isomorphic to the real numbers  $\mathbb{R}$ ; i.e., there is an order-preserving bijection  $F : \mathbb{R} \rightarrow C$ .

**Lemma.** Let  $K$  be a dense subset of  $C$ . For all  $x, y \in C$ , if  $x < y$ , then there exists a point  $z \in K$  such that  $z$  is between  $x$  and  $y$ .

*Proof.* Suppose for the sake of contradiction that there exist two points  $x, y \in C$  with  $x < y$  such that no point  $z \in K$  is between  $x$  and  $y$ . By Corollary 5.3, the region  $xy$  is infinite. Thus, we can pick a point  $p \in xy$ . Additionally, by Definition 1.6, we have that  $xy \cap K = \emptyset$ . Thus,  $xy \cap (K \setminus \{p\}) = \emptyset$ , implying by Definition 3.13 that  $p \notin LP(K)$ . But since  $p \in C$  and  $p \notin LP(K)$ , we have by Definition 6.8 that  $K$  is not dense in  $C$ , a contradiction.  $\square$

*Proof of Theorem 6.18.* By Axiom 1,  $C$  contains a countable dense subset  $K$ . By Exercise 6.16, there exists an order-preserving bijection  $f : \mathbb{Q} \rightarrow K$ . By Exercise 6.17, there exists an order-preserving injection  $F : \mathbb{R} \rightarrow C$ . To prove that there is an order-preserving bijection  $F : \mathbb{R} \rightarrow C$ , all that is left to do is to demonstrate that  $F$  (as defined in Exercise 6.17) is surjective.

To do this, Definition 1.20 tells us that it will suffice to show that for all  $X \in C$ , there exists an object  $A \in \mathbb{R}$  such that  $F(A) = X$ . Put more simply, we must find a Dedekind cut  $A$  such that  $\sup f(A) = X$  for every  $X \in C$ . To do this, we will begin by constructing the set  $S = \{k \in K \mid k < X\}$ . We will then verify that the preimage  $f^{-1}(S)$  is a Dedekind cut. Lastly, we will verify that  $\sup f(f^{-1}(S)) = X$ . Let's begin.

Let  $X$  be an arbitrary element of  $C$ . Define  $S$  as above. To verify that  $f^{-1}(S)$  is a Dedekind cut, Definition 6.1 tells us that it will suffice to confirm that  $f^{-1}(S) \neq \emptyset$ ;  $f^{-1}(S) \neq \mathbb{Q}$ ; if  $r \in f^{-1}(S)$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in f^{-1}(S)$ ; and if  $r \in f^{-1}(S)$ , then there is some  $s \in f^{-1}(S)$  with  $s > r$ . We will take this one claim at a time.

To confirm that  $f^{-1}(S) \neq \emptyset$ , Definition 1.8 tells us that it will suffice to find an element of  $f^{-1}(S)$ . By Axiom 3 and Definition 3.3, there exists some point  $Y \in C$  such that  $Y < X$ . Consequently, by the lemma and Definition 3.6, there exists a point  $f(p) \in K^{[4]}$  such that  $Y < f(p) < X$ . It follows by the definition of  $S$  that  $f(p) \in S$ . Therefore, by Definition 1.18,  $p \in f^{-1}(S)$ , as desired.

To confirm that  $f^{-1}(S) \neq \mathbb{Q}$ , Definition 1.2 tells us that it will suffice to find an element of  $\mathbb{Q}$  that is not an element of  $f^{-1}(S)$ . By Axiom 3 and Definition 3.3, there exists some point  $Y \in C$  such that  $X < Y$ . Consequently, by the lemma and Definition 3.6, there exists a point  $f(p) \in K$  such that  $X < f(p) < Y$ . It follows by the definition of  $S$  that  $f(p) \notin S$ . Therefore, by Definition 6.18,  $p \in \mathbb{Q}$  but  $p \notin f^{-1}(S)$ , as desired.

To confirm that if  $r \in f^{-1}(S)$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in f^{-1}(S)$ , we let  $r \in f^{-1}(S)$  and  $s \in \mathbb{Q}$  be arbitrary elements of their respective sets that satisfy  $s < r$  and seek to verify that  $s \in f^{-1}(S)$ . By Definition 1.18, the fact that  $r \in f^{-1}(S)$  implies that  $f(r) \in S$ . Thus, by the definition of  $S$ ,  $f(r) < X$ . Additionally, by the definition of  $f$  and Definition 6.15,  $f(s) \in K$  and  $f(s) < f(r)$ , respectively. Since  $f(s) < f(r)$  and  $f(r) < X$ , transitivity implies that  $f(s) < X$ . This combined with the previously established fact that  $f(s) \in K$  implies that  $f(s) \in S$ . Therefore, by Definition 1.18,  $s \in f^{-1}(S)$ , as desired.

To confirm that if  $r \in f^{-1}(S)$ , then there is some  $s \in f^{-1}(S)$  with  $s > r$ , we let  $r \in f^{-1}(S)$  and seek to find such an  $s$ . As before,  $r \in f^{-1}(S)$  implies that  $f(r) \in S$ . Thus, by the definition of  $S$ ,  $f(r) < X$ . It follows by the lemma and Definition 3.6 that there exists a point  $f(s) \in K$  such that  $f(r) < f(s) < X$ . Consequently, by the definition of  $S$ , we have that  $f(s) \in S$ . Therefore, by Definitions 1.18 and 6.15,  $s \in f^{-1}(S)$  and  $r < s$ , respectively, as desired.

Since  $f$  is bijective, Script 1 asserts that  $f(f^{-1}(S)) = S$ . Thus,  $\sup f(f^{-1}(S)) = \sup S$ . To verify that  $\sup S = X$ , Definition 5.7 tells us that it will suffice to confirm that  $X$  is an upper bound of  $S$  and if  $U$  is an upper bound of  $S$ ,  $X \leq U$ . To confirm the former statement, Definition 5.6 tells us that it will suffice to show that  $k \leq X$  for all  $k \in S$ . But by the definition of  $S$ , this is true. To confirm the latter statement, suppose for the sake of contradiction that there exists an upper bound  $U$  of  $S$  such that  $U < X$ . Since  $U < X$ , the lemma and Definition 3.6 imply that there exists a point  $Z \in K$  such that  $U < Z < X$ . It follows by the definition of  $S$  that  $Z \in S$ . Since there exists an element of  $S$  greater than  $U$ , Definition 5.6 asserts that  $U$  is not an upper bound of  $S$ , a contradiction.  $\square$

## 6.2 Discussion

- 1/12:     • Upper limit at signing up for 4-5 across the script.
- Lemma 6.2 is probably more straightforward using a contradiction argument.
- Briefly restate the algebra of Exercise 4.24 in Exercise 6.3c.
- 1/14:     • Turning in Script 5 journals is optional — it will boost your grade a bit if you do.
- Your journal grade will be whichever is higher: the average of all your journal grades with and without Script 5.

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<sup>4</sup>Note that we know that the element of  $K$  (the existence of which is implied by the lemma) can be written in the form  $f(p)$  because  $f$  is bijective.

- Script 5 will probably be due Wednesday, 1/20.
  - In Lemma 6.6, do we need to prove that the union of arbitrarily many Dedekind cuts is, itself, a Dedekind cut? Yes.
- 1/18:
- Is there a way to prove something else besides  $A$  is not open in Exercise 6.7?
    - This is probably it as far as proving that continua are connected.
    - It may not be possible to prove that *any* of the statements are wrong, but he's not sure.
  - Is Lemma 6.9 used in the proofs of any subsequent results, or is it just a less important result (hence the lemma designation)?
    - We can think of it as an alternate definition for density — we could prove Definition 6.8 from it.
  - Is my handwavey use of Scripts 2 and 3 ok in Corollary 6.12?
    - I'm fine.
  - Is there a simpler way to prove Corollaries 6.12 and 6.14?
    - hi
  - Is the math REU still running this summer?
    - He's not sure; UChicago's may not be NSF approved, hence why its not on the website rn.
  - What other summer opportunities would you recommend for a student at my level?
    - He did an REU at UWisconsin when he was an undergrad.
    - Sounds like its pretty much just REUs for undergrads.
    - I could ask around to see if anyone is a Knot Theorist/willing to sponsor me.
- 1/19:
- Easier Corollary 6.12:
    - Let  $B > A$ . Then  $A < i(\frac{m}{q}) < B$ . Then  $A < i(m)$ .
  - Several proofs were given for Corollary 6.14. One other correct one constructed the nonempty, bounded above set of all  $i(n)$  less than or equal to  $A$  and considered its supremum.
- 1/21:
- Now graded a bit more critically on presentations.
    - Write big, talk loudly, don't talk to the blackboard.
  - My original proof of Corollary 6.14 is incorrect because I can't split into cases the way I did (*longer expo*).
    - Instead, use Seb's approach.
- 1/26:
- Stray thoughts on Exercise 6.16:
    - Any property we can prove for  $\mathbb{Q}$  (e.g., betweenness, Axioms 1-3, etc.) we should be able to prove for  $K$ .
      - \* Many of these follow from  $\mathbb{Q}$ 's density! This is how we can make use of this condition.
    - We think of 0 as being somehow the "midpoint" of  $\mathbb{Q}$ . But since  $\mathbb{Q}$  diverges in both directions, it doesn't really have a midpoint; we just assert this rather arbitrary structure on a more foundational algebraic construct.
      - \* The same would hold for  $K$ . Thus, we can choose an arbitrary point  $x \in K$  and let it be the "midpoint," i.e., let  $f(0) = x$ .
    - Can we induct on the elements of  $\mathbb{Q}$ ? Since there exists a bijection  $\mathbb{Q} \rightarrow \mathbb{N}$ .

- We can construct an order preserving bijection between any finite subsets of  $\mathbb{Q}$  and  $K$  with equal cardinality.
  - $f : \mathbb{Q} \rightarrow K$ ,  $g : \mathbb{N} \rightarrow \mathbb{Q}$ ,  $h : \mathbb{N} \rightarrow K$ . If  $g(n) < g(n')$ , then  $h(n) < h(n')$ .
  - Let  $h(n) < h(n')$ . WLOG let  $n < n'$ , too. Now consider  $N = \{n \in \mathbb{N} \mid n \leq n'\}$ . This is a finite set. Now create a new set  $g(N)$ . There will be an order-preserving bijection  $\tilde{f} : h(N) \rightarrow g(N)$ .
  - Let  $g : \mathbb{N} \rightarrow \mathbb{Q}$  be a bijection (we know one exists by countability). We presently seek to define  $h : \mathbb{N} \rightarrow K$  recursively. Let  $x_1$  be an arbitrary element of  $K$  (Axiom 1). We define  $h(1) = x_1$ . Now suppose inductively that we have defined  $h(n)$ . We now seek to define  $h(n+1)$ . Consider the set  $A = \{g(m) \mid m \leq n+1\}$ . By Theorem 3.5, we can assign the symbols  $a_1, \dots, a_{n+1}$  to each point of  $A$  so that  $a_1 < a_2 < \dots < a_{n+1}$ . We know that  $g(n+1) = a_i$  for some  $i \in [n+1]$ . We divide into three cases ( $g(n+1) = b_1$ ,  $g(n+1) = b_{n+1}$ , and  $g(n+1) = b_i$  where  $1 < i < n+1$ ). First, suppose that  $g(n+1) = b_1$ . By the inductive hypothesis,  $h(g^{-1}(b_2)) \in K$ . By Axiom 3,  $h(g^{-1}(b_2))$  is not the first point of  $K$ . Thus, there exists an  $x \in K$  such that  $x < h(g^{-1}(b_2))$ . Consequently, let  $h(n+1) = x$ . The proof of the second case is symmetric to that of the first. Third, suppose that  $g(n+1) = b_i$  where  $1 < i < n+1$ . By the inductive hypothesis,  $h(g^{-1}(b_{i-1})), h(g^{-1}(b_{i+1})) \in K$ . Thus, there exists an  $x \in K$  such that  $h(b_{i-1}) < x < h(b_{i+1})$ . Consequently, let  $h(n+1) = x$ .
  - We define  $f : \mathbb{Q} \rightarrow K$  by  $f(p) = h(g^{-1}(p))$ .
  - Function diagram: The characteristic of an order preserving bijection is no intersections between lines connecting elements of different sets.
- Do we need to have subscripts on our orderings? Yes.
  - The canonical way of doing Exercise 6.16 is with the **back and forth method**.
    - Because both are countable,  $\mathbb{Q} = \{q_1, q_2, \dots\}$ . Likewise,  $K = \{k_1, k_2, \dots\}$ .
    - To create the bijection, we have two repeating steps.
      1. Let  $i$  be the smallest index such that  $q_i$  has not been paired. Let  $j$  be an index such that  $k_j$  hasn't been paired, and assigning  $f(q_i) = k_j$  preserves ordering (we have to prove that such a  $j$  exists). To prove this, we know that we can order the elements of  $\mathbb{Q}$  that have already been paired. We can also order the elements of  $K$  that have already been paired. Case 1:  $q_i$  is between some preexisting  $q$ 's. Then there exists some  $k_j$  between. Case 2:  $q_i < \dots < q_n$  implies there exists some  $k_j$  less than all other  $k$  so far. Case 3:  $q_i$  is a last element; symmetric to Case 2.
      2. Smallest  $j$ , smallest  $i$  such that order is preserved. Then we let  $f(q_i) = k_j$ .
      3. Repeat.
    - Injectivity: Suppose  $f(q_i) = f(q_j)$ . Each  $q_k$  is assigned to a unique  $k_k$ , so if they're equal, they must have been assigned at the same time. Therefore,  $q_i = q_j$ .
    - Surjectivity: Let  $k_j \in K$ . By  $j$ th step at most,  $k_j$  will be paired.
  - Do summer research things every happen with graduate students, or is it just with professors? It pretty much only happens with professors, but DRP could be a good way to get your foot in the door.

# Script 7

## The Field Axioms

### 7.1 Journal

1/28: **Definition 7.1.** A **binary operation** on a set  $X$  is a function

$$f : X \times X \rightarrow X$$

We say that  $f$  is **associative** if

$$f(f(x, y), z) = f(x, f(y, z)) \quad \text{for all } x, y, z \in X$$

We say that  $f$  is **commutative** if

$$f(x, y) = f(y, x) \quad \text{for all } x, y \in X$$

An **identity element** of a binary operation  $f$  is an element  $e \in X$  such that

$$f(x, e) = f(e, x) = x \quad \text{for all } x \in X$$

**Remark 7.2.** Frequently, we denote a binary operation differently. If  $*$  :  $X \times X \rightarrow X$  is the binary operation, we often write  $a * b$  in place of  $*(a, b)$ . We sometimes indicate this same operation by writing  $(a, b) \mapsto a * b$ .

**Exercise 7.3.** Rewrite Definition 7.1 using the notation of Remark 7.2.

*Answer.* A **binary operation** on a set  $X$  is a function

$$* : X \times X \rightarrow X$$

We say that  $*$  is **associative** if

$$(x * y) * z = x * (y * z) \quad \text{for all } x, y, z \in X$$

We say that  $*$  is **commutative** if

$$x * y = y * x \quad \text{for all } x, y \in X$$

An **identity element** of a binary operation  $*$  is an element  $e \in X$  such that

$$x * e = e * x = x \quad \text{for all } x \in X$$

□

**Examples 7.4.**

1. The function  $+: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$  which sends a pair of integers  $(m, n)$  to  $+(m, n) = m + n$  is a binary operation on the integers, called addition. Addition is associative, commutative, and has identity element 0.

2. The maximum of  $m$  and  $n$ , denoted  $\max(m, n)$ , is an associative and commutative binary operation on  $\mathbb{Z}$ . Is there an identity element for  $\max$ ?

*Proof.* Suppose for the sake of contradiction that there exists an identity element  $e$  for  $\max$ . But  $\max(e - 1, e) = e \neq e - 1$ , a contradiction. Therefore, no identity element exists for  $\max$ .  $\square$

3. Let  $\wp(Y)$  be the power set of a set  $Y$ . Recall that the power set consists of all subsets of  $Y$ . Then the intersection of sets,  $(A, B) \mapsto A \cap B$ , defines an associative and commutative binary operation on  $\wp(Y)$ . Is there an identity element for  $\cap$ ?

*Proof.* Clearly,  $Y \in \wp(Y)$ . By Script 1,  $Y \cap A = A \cap Y = A$  where  $A \subset Y$ . Therefore,  $Y$  is an identity element for  $\cap$ .  $\square$

**Exercise 7.5.** Find a binary operation on a set that is not commutative. Find a binary operation on a set that is not associative.

*Proof.* We will prove that the subtraction operation on the integers ( $- : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ ) is neither commutative nor associative. To prove that it's not commutative, Definition 7.1 tells us that it will suffice to show that  $x - y \neq y - x$  for some  $x, y \in \mathbb{Z}$ . Since  $2 - 1 = 1$  but  $1 - 2 = -1$ , we can see that  $1, 2 \in \mathbb{Z}$  clearly meet this requirement. To prove that it's not associative, Definition 7.1 tells us that it will suffice to show that  $(x - y) - z \neq x - (y - z)$  for some  $x, y, z \in \mathbb{Z}$ . Since  $(3 - 2) - 1 = 0$  but  $3 - (2 - 1) = 2$ , we can see that  $1, 2, 3 \in \mathbb{Z}$  clearly meet this requirement.  $\square$

**Exercise 7.6.** Let  $X$  be a finite set, and let  $Y = \{f : X \rightarrow X \mid f \text{ is bijective}\}$ . Consider the binary operation of composition of functions, denoted  $\circ : Y \times Y \rightarrow Y$  and defined by  $(f \circ g)(x) = f(g(x))$  as seen in Definition 1.25. Decide whether or not composition is commutative and/or associative and whether or not it has an identity.

*Proof.* To prove that composition is not commutative, Definition 7.1 tells us that it will suffice to find a finite set  $X$  paired with two bijections in  $Y$  that do not commute. Let  $X = \{1, 2, 3\}$  and consider the bijections  $f : X \rightarrow X$  (defined by  $f(1) = 2, f(2) = 3, f(3) = 1$ ) and  $g : X \rightarrow X$  (defined by  $g(1) = 1, g(2) = 3, g(3) = 2$ ). In this case,  $f \circ g$  would be defined by  $f(g(1)) = 2, f(g(2)) = 1$ , and  $f(g(3)) = 3$ , but  $g \circ f$  would be defined by  $g(f(1)) = 3, g(f(2)) = 2$ , and  $g(f(3)) = 1$ .

To prove that composition is associative, Definition 7.1 tells us that it will suffice to show that  $((f \circ g) \circ h)(x) = (f \circ (g \circ h))(x)$ . We may do this with the following algebra.

$$\begin{aligned} ((f \circ g) \circ h)(x) &= (f \circ g)(h(x)) \\ &= f(g(h(x))) \\ &= f((g \circ h)(x)) \\ &= (f \circ (g \circ h))(x) \end{aligned}$$

With respect to any finite set  $X$ , there will always be a bijection  $i : X \rightarrow X$  defined by  $i(x) = x$ . To prove that  $i$  is an identity element, Definition 7.1 tells us that it will suffice to show that for all  $f \in Y$ ,  $f \circ i = i \circ f = f$ . We may do this with the following algebra.

$$\begin{aligned} (f \circ i)(x) &= f(i(x)) \\ &= f(x) \\ &= i(f(x)) \\ &= (i \circ f)(x) \end{aligned}$$

$\square$

**Theorem 7.7.** *Identity elements are unique. That is, suppose that  $f$  is a binary operation on a set  $X$  that has two identity elements  $e$  and  $e'$ . Then  $e = e'$ .*

*Proof.* Let  $f : X \times X \rightarrow X$  be a binary operation on a set  $X$  with two identity elements  $e, e'$ . By Definition 7.1, we know that  $f(e, e') = e$  and  $f(e, e') = e'$ . Since  $f$  is a well-defined function by definition, it must be that  $e = f(e, e') = e'$ .  $\square$

**Definition 7.8.** A **field** is a set  $F$  with two binary operations on  $F$  called addition, denoted  $+$ , and multiplication, denoted  $\cdot$ , satisfying the following **field axioms**:

FA1 (Commutativity of Addition) For all  $x, y \in F$ ,  $x + y = y + x$ .

FA2 (Associativity of Addition) For all  $x, y, z \in F$ ,  $(x + y) + z = x + (y + z)$ .

FA3 (Additive Identity) There exists an element  $0 \in F$  such that  $x + 0 = 0 + x = x$  for all  $x \in F$ .

FA4 (Additive Inverses) For any  $x \in F$ , there exists  $y \in F$  such that  $x + y = y + x = 0$ , called an additive inverse of  $x$ .

FA5 (Commutativity of Multiplication) For all  $x, y \in F$ ,  $x \cdot y = y \cdot x$ .

FA6 (Associativity of Multiplication) For all  $x, y, z \in F$ ,  $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ .

FA7 (Multiplicative Identity) There exists an element  $1 \in F$  such that  $x \cdot 1 = 1 \cdot x = x$  for all  $x \in F$ .

FA8 (Multiplicative Inverses) For any  $x \in F$  such that  $x \neq 0$ , there exists  $y \in F$  such that  $x \cdot y = y \cdot x = 1$ , called a multiplicative inverse of  $x$ .

FA9 (Distributivity of Multiplication over Addition) For all  $x, y, z \in F$ ,  $x \cdot (y + z) = x \cdot y + x \cdot z$ .

FA10 (Distinct Additive and Multiplicative Identities)  $1 \neq 0$ .

**Exercise 7.9.** Consider the set  $\mathbb{F}_2 = \{0, 1\}$ , and define binary operations  $+$  and  $\cdot$  on  $\mathbb{F}_2$  by

$$\begin{array}{llll} 0 + 0 = 0 & 0 + 1 = 1 & 1 + 0 = 1 & 1 + 1 = 0 \\ 0 \cdot 0 = 0 & 0 \cdot 1 = 0 & 1 \cdot 0 = 0 & 1 \cdot 1 = 1 \end{array}$$

Show that  $\mathbb{F}_2$  is a field.

*Proof.* To prove that  $\mathbb{F}_2$  obeys FA1 from Definition 7.8, it will suffice to show that  $0 + 0 = 0 + 0$ ,  $0 + 1 = 1 + 0$ , and  $1 + 1 = 1 + 1$ . The first and third of these are evidently true. For the second, we have  $0 + 1 = 1 = 1 + 0$ , so it is good, too.

To prove that  $\mathbb{F}_2$  obeys FA2 from Definition 7.8, the following casework will suffice.

$$\begin{array}{ll} (0 + 0) + 0 = 0 = 0 + (0 + 0) & (0 + 0) + 1 = 1 = 0 + (0 + 1) \\ (0 + 1) + 0 = 1 = 0 + (1 + 0) & (1 + 0) + 0 = 1 = 1 + (0 + 0) \\ (0 + 1) + 1 = 0 = 0 + (1 + 1) & (1 + 1) + 0 = 0 = 1 + (1 + 0) \\ (1 + 0) + 1 = 0 = 1 + (0 + 1) & (1 + 1) + 1 = 1 = 1 + (1 + 1) \end{array}$$

To prove that  $\mathbb{F}_2$  obeys FA3 from Definition 7.8, it will suffice to find an element  $0 \in \mathbb{F}_2$  such that  $x + 0 = 0 + x = x$ . Since  $0 + 0 = 0$ ,  $1 + 0 = 0$ , and by commutativity, it is clear that  $0$  is an additive identity in  $\mathbb{F}_2$ .

To prove that  $\mathbb{F}_2$  obeys FA4 from Definition 7.8, it will suffice to show that for all  $x \in \mathbb{F}_2$ , there exists a  $y \in \mathbb{F}_2$  such that  $x + y = y + x = 0$ . For  $0$ , this object is  $0$  (since  $0 + 0 = 0 + 0 = 0$ ), and for  $1$ , this object is  $1$  (since  $1 + 1 = 1 + 1 = 0$ ).

To prove that  $\mathbb{F}_2$  obeys FA5 from Definition 7.8, it will suffice to show that  $0 \cdot 0 = 0 \cdot 0$ ,  $0 \cdot 1 = 1 \cdot 0$ , and  $1 \cdot 1 = 1 \cdot 1$ . The first and third of these are evidently true. For the second, we have  $0 \cdot 1 = 0 = 1 \cdot 0$ , so it is good, too.



To prove that  $\mathbb{F}_2$  obeys FA6 from Definition 7.8, the following casework will suffice.

$$\begin{array}{ll}
 (0 \cdot 0) \cdot 0 = 0 = 0 \cdot (0 \cdot 0) & (0 \cdot 0) \cdot 1 = 0 = 0 \cdot (0 \cdot 1) \\
 (0 \cdot 1) \cdot 0 = 0 = 0 \cdot (1 \cdot 0) & (1 \cdot 0) \cdot 0 = 0 = 1 \cdot (0 \cdot 0) \\
 (0 \cdot 1) \cdot 1 = 0 = 0 \cdot (1 \cdot 1) & (1 \cdot 1) \cdot 0 = 0 = 1 \cdot (1 \cdot 0) \\
 (1 \cdot 0) \cdot 1 = 0 = 1 \cdot (0 \cdot 1) & (1 \cdot 1) \cdot 1 = 1 = 1 \cdot (1 \cdot 1)
 \end{array}$$

To prove that  $\mathbb{F}_2$  obeys FA7 from Definition 7.8, it will suffice to find an element  $1 \in \mathbb{F}_2$  such that  $x \cdot 1 = 1 \cdot x = x$ . Since  $0 \cdot 1 = 0$ ,  $1 \cdot 1 = 1$ , and by commutativity, it is clear that 1 is a multiplicative identity in  $\mathbb{F}_2$ .

To prove that  $\mathbb{F}_2$  obeys FA8 from Definition 7.8, it will suffice to show that for all  $x \in \mathbb{F}_2$  such that  $x \neq 0$ , there exists a  $y \in \mathbb{F}_2$  such that  $x \cdot y = y \cdot x = 1$ . For 1, this object is 1 (since  $1 \cdot 1 = 1 \cdot 1 = 1$ ).

To prove that  $\mathbb{F}_2$  obeys FA9 from Definition 7.8, the following casework will suffice.

$$\begin{array}{ll}
 0 \cdot (0 + 0) = 0 = 0 \cdot 0 + 0 \cdot 0 & 0 \cdot (0 + 1) = 0 = 0 \cdot 0 + 0 \cdot 1 \\
 0 \cdot (1 + 0) = 0 = 0 \cdot 1 + 0 \cdot 0 & 1 \cdot (0 + 0) = 0 = 1 \cdot 0 + 1 \cdot 0 \\
 0 \cdot (1 + 1) = 0 = 0 \cdot 1 + 0 \cdot 1 & 1 \cdot (1 + 0) = 1 = 1 \cdot 1 + 1 \cdot 0 \\
 1 \cdot (0 + 1) = 1 = 1 \cdot 0 + 1 \cdot 1 & 1 \cdot (1 + 1) = 0 = 1 \cdot 1 + 1 \cdot 1
 \end{array}$$

To prove that  $\mathbb{F}_2$  obeys FA10 from Definition 7.8, it will suffice to show that  $0 \neq 1$ . Clearly this is true.  $\square$

**Theorem 7.10.** *Suppose that  $F$  is a field. Then additive inverses are unique. This means: Let  $x \in F$ . If  $y, y' \in F$  satisfy  $x + y = 0$  and  $x + y' = 0$ , then  $y = y'$ .*

*Proof.* Let  $x, y, y' \in F$  be such that  $x + y = 0$  and  $x + y' = 0$ . From Definition 7.8, we have

$$\begin{array}{ll}
 y' + (x + y) = (y' + x) + y & \text{FA2} \\
 y' + 0 = 0 + y & \text{FA4} \\
 y' = y & \text{FA3}
 \end{array}$$

$\square$

We usually write  $-x$  for the additive inverse of  $x$ .

**Corollary 7.11.** *If  $x \in F$ , then  $-(-x) = x$ .*

*Proof.* Let  $x \in F$ . Then by consecutive applications of FA4 from Definition 7.8,  $-x + (-(-x)) = 0$  and  $-x + x = 0$ . Therefore, by Theorem 7.10, we have that  $-(-x) = x$ .  $\square$

**Theorem 7.12.** *Let  $F$  be a field, and let  $a, b, c \in F$ . If  $a + b = a + c$ , then  $b = c$ .*

*Proof.* Let  $a, b, c \in F$  be such that  $a + b = a + c$ . By FA4 from Definition 7.8, there exists  $-a \in F$  such that  $-a + a = a + (-a) = 0$ . Having established that  $-a$  exists, we can prove from Definition 7.8 that

$$\begin{array}{ll}
 -a + (a + b) = -a + (a + c) & \\
 (-a + a) + b = (-a + a) + c & \text{FA2} \\
 0 + b = 0 + c & \text{FA4} \\
 b = c & \text{FA3}
 \end{array}$$

$\square$

**Theorem 7.13.** *Let  $F$  be a field. If  $a \in F$ , then  $a \cdot 0 = 0$ .*

*Proof.* Let  $a \in F$ . From Definition 7.8, we have

$$\begin{aligned}
 a &= a \cdot 1 && \text{FA7} \\
 &= a \cdot (1 + 0) && \text{FA3} \\
 &= a \cdot 1 + a \cdot 0 && \text{FA9} \\
 &= a + a \cdot 0 && \text{FA7} \\
 0 &= a \cdot 0 && \text{Theorem 7.12}
 \end{aligned}$$

□

2/2: **Theorem 7.14.** *Suppose that  $F$  is a field. Then multiplicative inverses are unique. This means: Let  $x \in F$ . If  $y, y' \in F$  satisfy  $x \cdot y = 1$  and  $x \cdot y' = 1$ , then  $y = y'$ .*

*Proof.* Let  $x, y, y' \in F$  be such that  $x \cdot y = 1$  and  $x \cdot y' = 1$ . From Definition 7.8, we have

$$\begin{aligned}
 (y \cdot x) \cdot y' &= y \cdot (x \cdot y') && \text{FA6} \\
 1 \cdot y' &= y \cdot 1 && \text{FA8} \\
 y' &= y && \text{FA7}
 \end{aligned}$$

□

We usually write  $x^{-1}$  or  $\frac{1}{x}$  for the multiplicative inverse of  $x$ .

**Corollary 7.15.** *If  $x \in F$  and  $x \neq 0$ , then  $(x^{-1})^{-1} = x$ .*

*Proof.* Let  $x \in F \setminus \{0\}$ . Then by FA8 from Definition 7.8, there exists  $x^{-1} \in F$  such that  $x \cdot x^{-1} = x^{-1} \cdot x = 1$ . It follows from Theorem 7.13 that  $x^{-1} \neq 0$  (if  $x^{-1} = 0$ , then Theorem 7.13 would imply that  $1 = x \cdot x^{-1} = 0$ , contradicting FA10). Thus, by FA8 from Definition 7.8 again, there exists  $(x^{-1})^{-1} \in F$  such that  $x^{-1} \cdot (x^{-1})^{-1} = (x^{-1})^{-1} \cdot x^{-1} = 1$ . Having established that  $(x^{-1})^{-1}$  exists,  $x^{-1} \cdot (x^{-1})^{-1} = 1$ , and  $x^{-1} \cdot x = 1$ , we have by Theorem 7.14 that  $(x^{-1})^{-1} = x$ . □

**Theorem 7.16.** *Let  $F$  be a field, and let  $a, b, c \in F$ . If  $a \cdot b = a \cdot c$  and  $a \neq 0$ , then  $b = c$ .*

*Proof.* Let  $a, b, c \in F$  be such that  $a \cdot b = a \cdot c$  and  $a \neq 0$ . By FA8 from Definition 7.8, there exists  $a^{-1} \in F$  such that  $a \cdot a^{-1} = a^{-1} \cdot a = 1$ . Having established that  $a^{-1}$  exists, we can prove from Definition 7.8 that

$$\begin{aligned}
 a^{-1} \cdot (a \cdot b) &= a^{-1} \cdot (a \cdot c) \\
 (a^{-1} \cdot a) \cdot b &= (a^{-1} \cdot a) \cdot c && \text{FA6} \\
 1 \cdot b &= 1 \cdot c && \text{FA8} \\
 b &= c && \text{FA7}
 \end{aligned}$$

□

**Theorem 7.17.** *Let  $F$  be a field, and let  $a, b \in F$ . If  $a \cdot b = 0$ , then  $a = 0$  or  $b = 0$ .*

*Proof.* Let  $a, b \in F$  be such that  $a \cdot b = 0$ , and suppose for the sake of contradiction that  $a \neq 0$  and  $b \neq 0$ . It follows from the supposition by consecutive applications of FA8 from Definition 7.8 that  $a^{-1}$  and  $b^{-1}$  exist. Thus, from Definition 7.8, we have

$$\begin{aligned}
 1 &= 1 \cdot 1 && \text{FA7} \\
 &= (a \cdot a^{-1}) \cdot (b \cdot b^{-1}) && \text{FA8} \\
 &= (a \cdot b) \cdot (a^{-1} \cdot b^{-1}) && \text{FA6 and FA7} \\
 &= 0 \cdot (a^{-1} \cdot b^{-1}) && \text{Substitution} \\
 &= 0 && \text{Theorem 7.13}
 \end{aligned}$$

But this contradicts FA10 from Definition 7.8. □

**Lemma 7.18.** *Let  $F$  be a field. If  $a \in F$ , then  $-a = (-1)a$ .*

*Proof.* Let  $a \in F$ . From Definition 7.8, we have

$$\begin{aligned}
 0 &= a \cdot 0 && \text{Theorem 7.13} \\
 a + (-a) &= a \cdot (1 + (-1)) && \text{FA4} \\
 a + (-a) &= a \cdot 1 + a \cdot (-1) && \text{FA9} \\
 a + (-a) &= a + a \cdot (-1) && \text{FA7} \\
 a + (-a) &= a + (-1)a && \text{FA5} \\
 -a &= (-1)a && \text{Theorem 7.12}
 \end{aligned}$$

□

**Lemma 7.19.** *Let  $F$  be a field. If  $a, b \in F$ , then  $a \cdot (-b) = -(a \cdot b) = (-a) \cdot b$ .*

*Proof.* Let  $a, b \in F$ . From Definition 7.8, we have

$$\begin{aligned}
 a \cdot (-b) &= a \cdot ((-1) \cdot b) && \text{Lemma 7.18} \\
 &= a \cdot (b \cdot (-1)) && \text{FA5} \\
 &= (a \cdot b) \cdot (-1) && \text{FA6} \\
 &= (-1) \cdot (a \cdot b) && \text{FA5} \\
 &= \boxed{-(a \cdot b)} && \text{Lemma 7.18} \\
 &= (-1) \cdot (a \cdot b) && \text{Lemma 7.18} \\
 &= ((-1) \cdot a) \cdot b && \text{FA6} \\
 &= \boxed{(-a) \cdot b} && \text{Lemma 7.18}
 \end{aligned}$$

□

**Lemma 7.20.** *Let  $F$  be a field. If  $a, b \in F$ , then  $a \cdot b = (-a) \cdot (-b)$ .*

*Proof.* Let  $a, b \in F$ . Thus, we have

$$\begin{aligned}
 (-a) \cdot (-b) &= -(-a) \cdot b && \text{Lemma 7.19} \\
 &= a \cdot b && \text{Corollary 7.11}
 \end{aligned}$$

□

**Definition 7.21.** An **ordered field** is a field  $F$  equipped with an ordering  $<$  (satisfying Definition 3.1) such that also:

- (a) Addition respects the ordering: if  $x < y$ , then  $x + z < y + z$  for all  $z \in F$ .
- (b) Multiplication respects the ordering: if  $0 < x$  and  $0 < y$ , then  $0 < x \cdot y$ .

**Definition 7.22.** Suppose  $F$  is an ordered field and  $x \in F$ . If  $0 < x$ , we say that  $x$  is **positive**. If  $x < 0$ , we say that  $x$  is **negative**.

**Lemma 7.23.** *Let  $F$  be an ordered field, and let  $x \in F$ . If  $0 < x$ , then  $-x < 0$ . Similarly, if  $x < 0$ , then  $0 < -x$ .*

*Proof.* Let  $x \in F$  be such that  $0 < x$ . Then by Definition 7.21a,  $0 + (-x) < x + (-x)$ . Consequently, from Definition 7.8, we have

$$\begin{aligned}
 -x &< x + (-x) && \text{FA3} \\
 -x &< 0 && \text{FA4}
 \end{aligned}$$

The proof is symmetric if  $x < 0$ .

□

**Lemma 7.24.** *Let  $F$  be an ordered field, and let  $x, y, z \in F$ .*

(a) *If  $x > 0$  and  $y < z$ , then  $x \cdot y < x \cdot z$ .*

(b) *If  $x < 0$  and  $y < z$ , then  $x \cdot z < x \cdot y$ .*

*Proof of a.* Let  $x, y, z \in F$  be such that  $x > 0$  and  $y < z$ . It follows from the latter condition by Definition 7.21a that  $y + (-y) < z + (-y)$ . Thus, by FA4 from Definition 7.8, we have  $0 < z + (-y)$ . This combined with the fact that  $0 < x$  implies by Definition 7.21b that  $0 < x \cdot (z + (-y))$ . Consequently, from Definition 7.8, we have

$$\begin{aligned}
 0 &< x \cdot z + x \cdot (-y) && \text{FA9} \\
 0 &< x \cdot z + (-(x \cdot y)) && \text{Lemma 7.19} \\
 0 + x \cdot y &< (x \cdot z + (-(x \cdot y))) + x \cdot y && \text{Definition 7.21a} \\
 0 + x \cdot y &< x \cdot z + (-(x \cdot y) + x \cdot y) && \text{FA2} \\
 0 + x \cdot y &< x \cdot z + 0 && \text{FA4} \\
 x \cdot y &< x \cdot z && \text{FA3}
 \end{aligned}$$

□

*Proof of b.* Let  $x, y, z \in F$  be such that  $x < 0$  and  $y < z$ . It follows from the former condition by Lemma 7.23 that  $0 < -x$ . Thus, by Lemma 7.24a,  $(-x) \cdot y < (-x) \cdot z$ . Consequently, from Definition 7.8, we have

$$\begin{aligned}
 -(x \cdot y) &< -(x \cdot z) && \text{Lemma 7.19} \\
 -(x \cdot y) + (x \cdot y + x \cdot z) &< -(x \cdot z) + (x \cdot y + x \cdot z) && \text{Definition 7.21a} \\
 -(x \cdot y) + (x \cdot y + x \cdot z) &< -(x \cdot z) + (x \cdot z + x \cdot y) && \text{FA1} \\
 (-(x \cdot y) + x \cdot y) + x \cdot z &< (-(x \cdot z) + x \cdot z) + x \cdot y && \text{FA2} \\
 0 + x \cdot z &< 0 + x \cdot y && \text{FA4} \\
 x \cdot z &< x \cdot y && \text{FA3}
 \end{aligned}$$

□

**Remark 7.25.** An immediate consequence of this lemma is the fact that if  $x$  and  $y$  are both positive or both negative, their product is positive.

**Lemma 7.26.** *Let  $F$  be an ordered field, and let  $x \in F$ . Then  $0 \leq x^2$ . Moreover, if  $x \neq 0$ , then  $0 < x^2$ .*

*Proof.* We divide into two cases ( $x = 0$  and  $x \neq 0$ ). Suppose first that  $x = 0$ . Then by Theorem 7.13,  $0 \leq 0 = 0 \cdot 0 = 0^2 = x^2$ , as desired. Now suppose that  $x \neq 0$ . We divide into two cases again ( $x > 0$  and  $x < 0$ ). If  $x > 0$ , then by Lemma 7.24a,  $x > 0$  and  $0 < x$  imply that  $x \cdot 0 < x \cdot x$ , from which it follows by Theorem 7.13 that  $0 < x^2$ , as desired. On the other hand, if  $x < 0$ , then by Lemma 7.24b,  $x < 0$  and  $x < 0$  imply that  $x \cdot 0 < x \cdot x$ , from which it follows for the same reason as before that  $0 < x^2$ , as desired. Both of the original two cases together prove the first statement, while the second original case alone proves the second statement. □

**Corollary 7.27.** *Let  $F$  be an ordered field. Then  $0 < 1$ .*

*Proof.* By FA10 from Definition 7.8,  $1 \neq 0$ . Thus, by Lemma 7.26,  $0 < 1^2 = 1$ , as desired. □

**Theorem 7.28.** *If  $F$  is an ordered field, then  $F$  has no first or last point.*

*Proof.* Suppose for the sake of contradiction that  $F$  has a first point  $a$ . By Corollary 7.27, we have that  $0 < 1$ , which implies by Lemma 7.23 that  $-1 < 0$ . It follows by Definition 7.21a that  $-1 + a < 0 + a$ . Thus, by FA3 from Definition 7.8,  $-1 + a < a$ . Since there exists an object in  $F$  (namely  $-1 + a$ ) that is less than  $a$ , Definition 3.3 tells us that  $a$  is not the first point of  $F$ , a contradiction.

The proof is symmetric in the other case. □

**Theorem 7.29.** *The rational numbers  $\mathbb{Q}$  form an ordered field.*

*Proof.* To prove that  $\mathbb{Q}$  forms an ordered field, Definition 7.21 tells us that it will suffice to show that  $\mathbb{Q}$  forms a field; has an ordering  $<$ ; satisfies  $x + z < y + z$  if  $x < y$  for all  $z \in \mathbb{Q}$ ; and satisfies  $0 < x \cdot y$  if  $0 < x$  and  $0 < y$ . We will take this one constraint at a time.

To show that  $\mathbb{Q}$  forms a field, Definition 7.8 tells us that it will suffice to verify that  $\mathbb{Q}$  has two binary operations ( $+$  and  $\cdot$ ), and satisfies field axioms 1-10. Define  $+$  and  $\cdot$  as in Definition 2.7. Under these definitions, parts a-i of Theorem 2.10 guarantee that  $\mathbb{Q}$  satisfies FA1-FA9, respectively. As to FA10, to verify that  $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ , Exercise 2.6 tells us that it will suffice to confirm that  $(1, 1) \approx (1, 0)$ . But since  $1 \cdot 0 = 0 \neq 1 = 1 \cdot 1$ , Exercise 2.2e confirms that  $(1, 1) \approx (1, 0)$ , as desired.

$\mathbb{Q}$  has an ordering by Exercise 3.9d, as desired.

To show that  $x + z < y + z$  if  $x < y$  for all  $z \in \mathbb{Q}$ , let  $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}, \begin{bmatrix} x \\ z \end{bmatrix}$  be arbitrary elements of  $\mathbb{Q}$  with positive denominators (we can choose these WLOG by Exercise 3.9b) and the first two satisfying  $\begin{bmatrix} a \\ b \end{bmatrix} < \begin{bmatrix} c \\ d \end{bmatrix}$ ; we seek to verify that  $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} x \\ z \end{bmatrix} < \begin{bmatrix} c \\ d \end{bmatrix} + \begin{bmatrix} x \\ z \end{bmatrix}$ . Since  $\begin{bmatrix} a \\ b \end{bmatrix} < \begin{bmatrix} c \\ d \end{bmatrix}$ , we have by Exercise 3.9c that  $ad < bc$ . It follows by Script 0 that

$$\begin{aligned} ad &< bc \\ adzz &< bczz \\ adzz + bdxz &< bczz + bdxz \\ azdz + bxdz &< bczx + bzdz \\ (az + bx)(dz) &< (bz)(cz + dx) \end{aligned}$$

Thus, by Exercise 3.9c,  $\begin{bmatrix} az+bx \\ bz \end{bmatrix} < \begin{bmatrix} cz+dx \\ dz \end{bmatrix}$ . Therefore, by Definition 2.7,  $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} x \\ z \end{bmatrix} < \begin{bmatrix} c \\ d \end{bmatrix} + \begin{bmatrix} x \\ z \end{bmatrix}$ , as desired.

To show that  $0 < x \cdot y$  if  $0 < x$  and  $0 < y$ , let  $\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}$  be arbitrary elements of  $\mathbb{Q}$  with positive denominators (which we can choose for the same reason as before) and such that  $\begin{bmatrix} 0 \\ 1 \end{bmatrix} < \begin{bmatrix} a \\ b \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix} < \begin{bmatrix} c \\ d \end{bmatrix}$ ; we seek to verify that  $\begin{bmatrix} 0 \\ 1 \end{bmatrix} < \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix}$ . Since  $\begin{bmatrix} 0 \\ 1 \end{bmatrix} < \begin{bmatrix} a \\ b \end{bmatrix}$  and  $\begin{bmatrix} 0 \\ 1 \end{bmatrix} < \begin{bmatrix} c \\ d \end{bmatrix}$ , we have by Exercise 3.9c that  $0 \cdot b < 1 \cdot a$  and  $0 \cdot d < 1 \cdot c$ . It follows by Script 0 that  $0 \cdot bd < 1 \cdot ac$ . Thus, by Exercise 3.9c,  $\begin{bmatrix} 0 \\ 1 \end{bmatrix} < \begin{bmatrix} ac \\ bd \end{bmatrix}$ . Therefore, by Definition 2.7,  $\begin{bmatrix} 0 \\ 1 \end{bmatrix} < \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix}$ , as desired.  $\square$

2/4: **Definition 7.31.** We define  $\oplus$  on  $\mathbb{R}$  as follows. Let  $A, B \in \mathbb{R}$  be Dedekind cuts. Define

$$A \oplus B = \{a + b \mid a \in A \text{ and } b \in B\}$$

**Exercise 7.32.**

- (a) Prove that  $A \oplus B$  is a Dedekind cut.
- (b) Prove that  $\oplus$  is commutative and associative.
- (c) Prove that if  $A \in \mathbb{R}$ , then  $A = \mathbf{0} \oplus A$ .

*Proof of a.* To prove that  $A \oplus B$  is a Dedekind cut, Definition 6.1 tells us that it will suffice to show that  $A \oplus B \neq \emptyset$ ;  $A \oplus B \neq \mathbb{Q}$ ; if  $r \in A \oplus B$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in A \oplus B$ ; and if  $r \in A \oplus B$ , then there is some  $s \in A \oplus B$  with  $s > r$ . We will take this one claim at a time.

To show that  $A \oplus B \neq \emptyset$ , Definition 1.8 tells us that it will suffice to find an element of  $A \oplus B$ . Since  $A, B$  are Dedekind cuts, Definition 6.1 asserts that they are nonempty. Thus, there exist rational numbers  $x \in A$  and  $y \in B$ . Therefore, by the definition of  $A \oplus B$ , the sum  $x + y \in A \oplus B$ , as desired.

To show that  $A \oplus B \neq \mathbb{Q}$ , Definition 1.2 tells us that it will suffice to find an element of  $\mathbb{Q}$  that is not an element of  $A \oplus B$ . For an analogous reason to before, we can choose  $x, y \in \mathbb{Q}$  such that  $x \notin A$  and  $y \notin B$ . It follows by Lemma 6.2 and Definition 5.6 that  $x \geq a$  for all  $a \in A$  and  $y \geq b$  for all  $b \in B$ . Additionally, since  $x \notin A$ , we have that  $x \neq a$  for any  $a \in A$ ; thus,  $x > a$  for all  $a \in A$ . Similarly,  $y > b$  for all  $b \in B$ . Consequently, by Script 2,  $x + y > a + b$  for all  $a + b \in A \oplus B$ . Therefore,  $x + y \notin A \oplus B$ , as desired.

To show that if  $r \in A \oplus B$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in A \oplus B$ , we let  $r \in A \oplus B$  and  $s \in \mathbb{Q}$  be arbitrary elements of their respective sets that satisfy  $s < r$  and seek to verify that  $s \in A \oplus B$ . Since  $r \in A \oplus B$ ,  $r = x + y$  for some  $x \in A$  and  $y \in B$ . Additionally, it follows from the fact that  $s < r$  that

$s = r - q = x + y - q$  for some  $q \in \mathbb{Q}^+$ . Since  $y \in B$  and  $y - q \in \mathbb{Q}$  satisfy  $y - q < y$ , we have by Definition 6.1b that  $y - q \in B$ . Therefore,  $s = (x) + (y - q)$  is an element of  $A \oplus B$ , as desired.

To show that if  $r \in A \oplus B$ , then there is some  $s \in A \oplus B$  with  $s > r$ , we let  $r \in A \oplus B$  and seek to find such an  $s$ . Since  $r \in A \oplus B$ ,  $r = x + y$  for some  $x \in A$  and  $y \in B$ . It follows from the fact that  $x \in A$  by Definition 6.1c that there exists a  $z \in A$  with  $z > x$ . Consequently, by Script 0,  $z + y > x + y$  is the desired element of  $A \oplus B$ .  $\square$

*Proof of b.* To prove that  $\oplus$  is commutative, Definition 7.1 tells us that it will suffice to show that for all  $A, B \in \mathbb{R}$ , we have  $A \oplus B = B \oplus A$ . Let  $A, B$  be arbitrary elements of  $\mathbb{R}$ . Then by Definition 7.31, we clearly have

$$\begin{aligned} A \oplus B &= \{a + b \mid a \in A \text{ and } b \in B\} \\ &= \{b + a \mid b \in B \text{ and } a \in A\} \\ &= B \oplus A \end{aligned}$$

To prove that  $\oplus$  is associative, Definition 7.1 tells us that it will suffice to show that for all  $A, B, C \in \mathbb{R}$ , we have  $(A \oplus B) \oplus C = A \oplus (B \oplus C)$ . Let  $A, B, C$  be arbitrary elements of  $\mathbb{R}$ . Then by Definition 7.31, we clearly have

$$\begin{aligned} (A \oplus B) \oplus C &= \{a + b \mid a \in A \text{ and } b \in B\} \oplus C \\ &= \{d + c \mid d \in \{a + b \mid a \in A \text{ and } b \in B\} \text{ and } c \in C\} \\ &= \{d + c \mid d = a + b \text{ for some } a \in A \text{ and } b \in B, \text{ and } c \in C\} \\ &= \{a + b + c \mid a \in A \text{ and } b \in B \text{ and } c \in C\} \\ &= \{a + e \mid a \in A, \text{ and } e = b + c \text{ for some } b \in B \text{ and } c \in C\} \\ &= \{a + e \mid c \in C \text{ and } e \in \{b + c \mid b \in B \text{ and } c \in C\}\} \\ &= A \oplus \{b + c \mid b \in B \text{ and } c \in C\} \\ &= A \oplus (B \oplus C) \end{aligned}$$

Note that we also make use of Exercise 7.32a to guarantee  $A \oplus B \in \mathbb{R}$ , so that we can apply  $\oplus$  to  $A \oplus B$  and  $C$ . We similarly invoke Exercise 7.32a to take the sum of  $A$  and  $B \oplus C$ .  $\square$

*Proof of c.* To prove that for all  $A \in \mathbb{R}$ ,  $A = \mathbf{0} \oplus A$ , we will show for an arbitrary  $A \in \mathbb{R}$  that every element of  $A$  is an element of  $\mathbf{0} \oplus A$  and vice versa. Let  $A$  be an arbitrary element of  $\mathbb{R}$ . Suppose first that  $x \in A$ . Then by Definition 6.1c, there exists  $y \in A$  such that  $y > x$ . Let  $z = x - y$ . Clearly,  $z \in \mathbb{Q}$  and  $z < 0$ , so we know that  $z \in \mathbf{0}$ . Additionally, since  $x - z = y$ , we know that  $x - z \in A$ . Therefore, since  $x = (z) + (x - z)$ , we have by Definition 7.31 that  $x \in \mathbf{0} \oplus A$ . Now suppose that  $z \in \mathbf{0} \oplus A$ . Then by Definition 7.31,  $z = x + y$  for some  $x \in \mathbf{0}$  and  $y \in A$ . Since  $x \in \mathbf{0}$ , we know that  $x < 0$ , which means that  $y > z$ . This combined with the fact that  $y \in A$  and  $z \in \mathbb{Q}$  implies by Definition 6.1b that  $z \in A$ .  $\square$

2/9: **Definition 7.39.** For  $A, B \in \mathbb{R}$ ,  $\mathbf{0} < A$ ,  $\mathbf{0} < B$ , we define

$$A \otimes B = \{r \in \mathbb{Q} \mid r \leq 0\} \cup \{ab \mid a \in A, b \in B, a > 0, b > 0\}$$

If  $A = \mathbf{0}$  or  $B = \mathbf{0}$ , we define  $A \otimes B = \mathbf{0}$ . If  $A < \mathbf{0}$  but  $\mathbf{0} < B$ , we replace  $A$  with  $-A$  and use the definition of multiplication of positive elements. Hence, in this case,

$$A \otimes B = -[(-A) \otimes B]$$

Similarly, if  $\mathbf{0} < A$  but  $B < \mathbf{0}$ , then

$$A \otimes B = -[A \otimes (-B)]$$

and if  $A < \mathbf{0}$ ,  $B < \mathbf{0}$ , then

$$A \otimes B = (-A) \otimes (-B)$$

**Exercise 7.40.** <sup>[1]</sup>

- (a) Show that if  $A, B \in \mathbb{R}$ , then  $A \otimes B \in \mathbb{R}$ .
- (b) Show that  $\otimes$  is commutative and associative.
- (c) Show that if  $A, B \in \mathbb{R}$ ,  $\mathbf{0} < A$ , and  $\mathbf{0} < B$ , then  $\mathbf{0} < A \otimes B$ .
- (d) Let  $\mathbf{1} = \{x \in \mathbb{Q} \mid x < 1\}$ . Show that if  $A \in \mathbb{R}$ , then  $\mathbf{1} \otimes A = A$ .

*Proof of a.* To prove that  $A \otimes B$  where  $\mathbf{0} < A, \mathbf{0} < B$  are Dedekind cuts, Definition 6.1 tells us that it will suffice to show that  $A \otimes B \neq \emptyset$ ;  $A \otimes B \neq \mathbb{Q}$ ; if  $r \in A \otimes B$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in A \otimes B$ ; and if  $r \in A \otimes B$ , then there is some  $s \in A \otimes B$  with  $s > r$ . We will take this one claim at a time.

To show that  $A \otimes B \neq \emptyset$ , Definition 1.8 tells us that it will suffice to find an element of  $A \otimes B$ . Since  $0 \in \mathbb{Q}$  and  $0 \leq 0$ ,  $0 \in \{r \in \mathbb{Q} \mid r \leq 0\}$ . It follows by Definition 1.5 that  $0 \in \{r \in \mathbb{Q} \mid r \leq 0\} \cup \{ab \mid a \in A, b \in B, a > 0, b > 0\}$ . Therefore, by Definition 7.39,  $0 \in A \otimes B$ , as desired.

To show that  $A \otimes B \neq \mathbb{Q}$ , Definition 1.2 tells us that it will suffice to find an element of  $\mathbb{Q}$  that is not an element of  $A \otimes B$ . Since  $\mathbf{0} < A$  and  $\mathbf{0} < B$ , Definitions 6.4 and 1.3 assert that there exist points  $a \in A$  and  $b \in B$  such that  $a, b \notin \mathbf{0}$ , i.e.,  $a, b \geq 0$ . Furthermore, since  $a, b$  are not the last points of  $A, B$ , respectively, by Definition 6.1c, there exist points  $c \in A$  and  $d \in B$  such that  $c > 0$  and  $d > 0$ . Now, for an analogous reason to before, we can choose  $x, y \in \mathbb{Q}$  such that  $x \notin A$  and  $y \notin B$ . It follows by Lemma 6.2 and Definition 5.6 that  $x \geq e$  for all  $e \in A$  and  $y \geq f$  for all  $f \in B$ , meaning (when combined with the last result) that  $x > 0$  and  $y > 0$ . Thus,  $xy > 0$ , so  $xy \notin \{r \in \mathbb{Q} \mid r \leq 0\}$ . Additionally, we have by Script 2 that  $xy > ef$  for all  $ef$  formed from the product of positive elements of  $A$  and  $B$ . Thus,  $xy \notin \{ab \mid a \in A, b \in B, a > 0, b > 0\}$ . Therefore, by Definitions 1.5 and 7.39,  $xy \notin A \otimes B$ , as desired.

To show that if  $r \in A \otimes B$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in A \otimes B$ , we let  $r \in A \otimes B$  and  $s \in \mathbb{Q}$  be arbitrary elements of their respective sets that satisfy  $s < r$  and seek to verify that  $s \in A \otimes B$ . We divide into two cases ( $s \leq 0$  and  $s > 0$ ). Suppose first that  $s \leq 0$ . Then  $s \in \{r \in \mathbb{Q} \mid r \leq 0\}$ . It follows by Definition 1.5 that  $0 \in \{r \in \mathbb{Q} \mid r \leq 0\} \cup \{ab \mid a \in A, b \in B, a > 0, b > 0\}$ . Therefore, by Definition 7.39,  $0 \in A \otimes B$ . Now suppose that  $s > 0$ . Then  $r > 0$ . Since  $r \in A \otimes B$  and  $r > 0$ ,  $r = xy$  where  $x \in A, y \in B, x > 0, y > 0$ . Additionally, it follows from the fact that  $s < r$  that  $s = r - q = xy - q = (x - \frac{q}{y})y$  for some  $q \in \mathbb{Q}^+$ . Since  $x - \frac{q}{y} \in \mathbb{Q}$  and  $x - \frac{q}{y} < x$ , we have by Definition 6.1b that  $x - \frac{q}{y} \in A$ . Therefore,  $s = (x - \frac{q}{y})(y)$  is an element of  $\{ab \mid a \in A, b \in B, a > 0, b > 0\}$ , and hence by Definition 7.39,  $s \in A \otimes B$ .

To show that if  $r \in A \otimes B$ , then there is some  $s \in A \otimes B$  with  $s > r$ , we let  $r \in A \otimes B$  and seek to find such an  $s$ . We divide into two cases ( $r \leq 0$  and  $r > 0$ ). Suppose first that  $r \leq 0$ . Then for the same reasons outlined in the proof of the second condition, there exist positive elements of  $A \otimes B$  that are greater than  $r$ . Now suppose that  $r > 0$ . This implies that  $r = xy$  for some  $x \in A, y \in B, x > 0, y > 0$ . It follows from the fact that  $x \in A$  by Definition 6.1c that there exists a  $z \in A$  with  $z > x$ . Consequently, by Lemma 7.24<sup>[2]</sup>,  $zy > xy$  is the desired element of  $A \otimes B$ .  $\square$

*Proof of b.* To prove that  $\otimes$  is commutative for  $\mathbf{0} < A, \mathbf{0} < B$ , Definition 7.1 tells us that it will suffice to show that for all such  $A, B \in \mathbb{R}$ , we have  $A \otimes B = B \otimes A$ . Let  $A, B$  be arbitrary elements of  $\mathbb{R}$  where  $\mathbf{0} < A, \mathbf{0} < B$ . Then by Definition 7.39, we clearly have

$$\begin{aligned} A \otimes B &= \{r \in \mathbb{Q} \mid r \leq 0\} \cup \{ab \mid a \in A, b \in B, a > 0, b > 0\} \\ &= \{r \in \mathbb{Q} \mid r \leq 0\} \cup \{ba \mid b \in B, a \in A, b > 0, a > 0\} \\ &= B \otimes A \end{aligned}$$

To prove that  $\otimes$  is associative for  $\mathbf{0} < A, \mathbf{0} < B, \mathbf{0} < C$ , Definition 7.1 tells us that it will suffice to show that for all such  $A, B, C \in \mathbb{R}$ , we have  $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ . Let  $A, B, C$  be arbitrary elements of  $\mathbb{R}$  where  $\mathbf{0} < A, \mathbf{0} < B, \mathbf{0} < C$ . To show that  $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ , Definition 1.2 tells us that it will suffice to verify that every element of  $(A \otimes B) \otimes C$  is an element of  $A \otimes (B \otimes C)$  and vice versa. Suppose first that  $x \in (A \otimes B) \otimes C$ . Then by Definition 7.39,  $x \leq 0$  or  $x = dc$  where  $d \in A \otimes B, c \in C, d > 0, c > 0$ . If  $x \leq 0$ , then by Definition 7.39,  $x \in A \otimes (B \otimes C)$  since it's an element of  $\{r \in \mathbb{Q} \mid r \leq 0\}$ , as desired. On the

<sup>1</sup>Note that the proofs given here only address the case where  $\mathbf{0} < A$  and  $\mathbf{0} < B$ .

<sup>2</sup>And, technically, Theorem 7.29.

other hand, if  $x = dc$  where  $d \in A \otimes B, c \in C, d > 0, c > 0$ , we continue. Now  $d \in A \otimes B$  implies that  $d \leq 0$  or  $d = ab$  where  $a \in A, b \in B, a > 0, b > 0$ . However, the prior constraint that  $d > 0$  guarantees that  $d \not\leq 0$ , so we know that  $d = ab$  where  $a, b$  satisfy the above conditions. Combining the last two results, we have  $x = (ab)(c)$  where  $a \in A, b \in B, c \in C, a > 0, b > 0, c > 0$ . It follows that we also have  $x = (a)(bc)$  under the same conditions. If we let  $e = bc$  where  $b \in B, c \in C, b > 0, c > 0$ , then  $e \in \{bc \mid b \in B, c \in C, b > 0, c > 0\}$ . Consequently, by Definition 7.31,  $e \in B \otimes C$ . Additionally,  $b > 0, c > 0$  imply by Definition 7.21 that  $e > 0$ . To recap, at this point we have  $x = ae$  where  $a \in A, e \in B \otimes C, a > 0, e > 0$ . It follows by a similar process to before that  $x \in A \otimes (B \otimes C)$ . The proof is symmetric in the other direction.  $\square$

*Proof of c.* To prove that  $\mathbf{0} < A \otimes B$ , Definitions 6.4 and 1.3 tell us that it will suffice to show that every  $x \in \mathbf{0}$  is an element of  $A \otimes B$  and find an  $x \in A \otimes B$  such that  $x \notin \mathbf{0}$ . Let  $x$  be an arbitrary element of  $\mathbf{0}$ . Then  $x \in \mathbb{Q}$  and  $x < 0$ . But it follows that  $x \in \{r \in \mathbb{Q} \mid r \leq 0\}$ , which implies by Definition 7.39 that  $x \in A \otimes B$ . As to the other stipulation, clearly  $0 \in \{r \in \mathbb{Q} \mid r \leq 0\}$  but since  $0 \not< 0$ ,  $0 \notin \mathbf{0}$ . Therefore, by Definition 7.39,  $0 \in A \otimes B$ , but  $0 \notin \mathbf{0}$ , as desired.  $\square$

*Proof of d.* To prove that  $\mathbf{1} \otimes A = A$ , Definition 1.2 tells us that it will suffice to show that every  $x \in \mathbf{1} \otimes A$  is an element of  $A$  and vice versa.

Let  $x$  be an arbitrary element of  $\mathbf{1} \otimes A$ . Then by Definition 7.39,  $x \leq 0$  or  $x = da$  where  $d \in \mathbf{1}, a \in A, d > 0, a > 0$ . We now divide into two cases. Suppose first that  $x \leq 0$ . We divide into two cases again ( $x < 0$  and  $x = 0$ ). If  $x < 0$ , then  $x \in \mathbf{0}$ , which implies by Definitions 6.4, 1.3, and the fact that  $\mathbf{0} < A$  that  $x \in A$ , as desired. On the other hand, if  $x = 0$ , suppose for the sake of contradiction that  $x \notin A$ . Then by Lemma 6.2 and Definition 5.6,  $a \leq x$  for all  $a \in A$ . This combined with the fact that  $x \notin A$  implies that  $a < x$  for all  $a \in A$ . Consequently, since  $\mathbf{0} = \{q \in \mathbb{Q} \mid q < 0\}$ , it follows that  $A \subset \mathbf{0}$ . But by Definition 6.4, this implies  $A \leq \mathbf{0}$ , contradicting the fact that  $\mathbf{0} < A$ , as desired. Now suppose that  $x = da$  where  $d \in \mathbf{1}, a \in A, d > 0, a > 0$ . Then by Script 2,  $d < 1$  implies that  $x = da < a$ . Therefore, by Definition 6.1b,  $x \in A$ , as desired.

Let  $x$  be an arbitrary element of  $A$ . We divide into two cases ( $x \leq 0$  and  $x > 0$ ). If  $x \leq 0$ , then  $x \in \{r \in \mathbb{Q} \mid r \leq 0\}$ , which implies by Definition 7.39 that  $x \in \mathbf{1} \otimes A$ . On the other hand, suppose  $x > 0$ . Then by Definition 6.1c, there is some  $y \in A$  with  $y > x$ . It follows by Script 2 that  $1 > \frac{x}{y} > 0$ , so we have that  $\frac{x}{y} \in \mathbf{1}$ . Thus, since  $x = \frac{x}{y} \cdot y$ , we know that  $x$  is the product of a positive element of  $\mathbf{1}$  and a positive element of  $A$  (since  $y > x > 0$ ). Therefore,  $x \in \{da \mid d \in \mathbf{1}, a \in A, d > 0, a > 0\}$ , which implies by Definition 7.39 that  $x \in \mathbf{1} \otimes A$ .  $\square$

## 7.2 Discussion

- 1/28:
- Script 6 journals due Wednesday.
  - We'll also have to prove a density lemma:
    - Let  $X$  be a dense subset of a continuum  $C$ . Show that for all  $x, y \in X$ , if  $x < y$ , then there exists a  $z \in X$  such that  $x < z < y$ .
    - Mark in Exercise 6.16 as "Density Lemma."
  - Explicitly cite Field Axioms as you go.
- 2/2:
- For Theorem 7.30 in class, he wants a simple explanation of what the injective map looks like and why, but not a full-on rigorous proof.
    - Nothing in the journal for Theorem 7.30, though.
  - He also wants to see Exercises 7.32 and 7.40 in the journal.
  - For Corollary 7.15, we can write that  $x^{-1} \cdot x = 1$  and  $x^{-1} \cdot (x^{-1})^{-1} = 1$ , and know by the uniqueness of multiplicative inverses (Theorem 7.14) that  $x = (x^{-1})^{-1}$ . For Corollary 7.11, we have an analogous proof.



- Alternate Theorem 7.17:

$$\begin{aligned}
 1 &= 1 \cdot 1 \\
 &= (a \cdot a^{-1})(b \cdot b^{-1}) \\
 &= (ab)(a^{-1}b^{-1}) \\
 &= 0
 \end{aligned}$$

- Alternate Lemma 7.18:  $a + (-a) = 0$ .  $a + (-1)a = a(1 + (-1)) = a \cdot 0 = 0$ . Thus, by Theorem 7.10,  $-a = (-1)a$ .
- Alternate Lemma 7.19: We can use the uniqueness of additive inverses (Theorem 7.10).
- We can also cite Remark 7.25 in Lemma 7.26.

2/4:

- Thoughts on Theorem 7.30:



Figure 7.1: Theorem 7.30 discussion.

# Script 8

## Intervals

### 8.1 Journal

2/9: Now that we have constructed  $\mathbb{R}$  and proven the fundamental facts about it, we will work with the real numbers  $\mathbb{R}$  instead of an arbitrary continuum  $C$ . We will leave behind Dedekind cuts and think of elements of  $\mathbb{R}$  as numbers. Accordingly, from now on, we will use lower-case letters like  $x$  for real numbers and will write  $+$  and  $\cdot$  for  $\oplus$  and  $\otimes$ . We will also now use the standard notation  $(a, b)$  for the region  $\underline{ab} = \{x \in \mathbb{R} \mid a < x < b\}$ . Even though the notation is the same, this is *not* the same object as the ordered pair  $(a, b)$ .

More generally, we adopt the following standard notation:

$$\begin{aligned}(a, b) &= \{x \in \mathbb{R} \mid a < x < b\} \\ [a, b) &= \{x \in \mathbb{R} \mid a \leq x < b\} \\ (a, b] &= \{x \in \mathbb{R} \mid a < x \leq b\} \\ [a, b] &= \{x \in \mathbb{R} \mid a \leq x \leq b\} \\ (a, \infty) &= \{x \in \mathbb{R} \mid a < x\} \\ [a, \infty) &= \{x \in \mathbb{R} \mid a \leq x\} \\ (-\infty, b) &= \{x \in \mathbb{R} \mid x < b\} \\ (-\infty, b] &= \{x \in \mathbb{R} \mid x \leq b\}\end{aligned}\tag{8.1}$$

**Exercise 8.1.** Identify the sets in Equations 8.1 that are open/closed/neither.

*Proof.* For this proof, we assume when applicable that  $a < b$  (cases where  $a = b$  are trivially simple and uncommon). Additionally, note that by Theorem 5.1, any of these sets proven to be just one of open or closed will not be the other, i.e., a set proven to be open will not be closed and vice versa.

By Corollary 4.11,  $(a, b)$  is open.

By an adaptation of Corollary 5.14,  $b \in LP([a, b))$  but  $b \notin [a, b)$ . Since  $[a, b)$  doesn't contain all of its limit points, Definition 4.1 tells us that it is not closed. Additionally, since  $a \in LP(C \setminus [a, b))$  but  $a \notin C \setminus [a, b)$ , Definition 4.8 tells us that it is not open. Therefore, it is neither.

The proof that  $(a, b]$  is neither is symmetric to the previous case.

By Corollaries 5.15 and 4.7,  $[a, b]$  is closed.

By Corollary 4.13,  $(a, \infty)$  is open.

By Corollary 4.13 and Definition 4.8,  $[a, \infty) = C \setminus (-\infty, a)$  is closed.

The proofs that  $(-\infty, b)$  and  $(-\infty, b]$  are open and closed, respectively, are symmetric to the previous two cases, respectively.  $\square$

**Definition 8.2.** A set  $I \subset \mathbb{R}$  is an **interval** if for all  $x, y \in I$  with  $x < y$ ,  $[x, y] \subset I$ .

**Lemma 8.3.** *A proper subset  $I \subsetneq \mathbb{R}$  is an interval if and only if it takes one of the eight forms in Equations 8.1.*

*Proof.* Suppose first that  $I \subsetneq \mathbb{R}$  is an interval. If  $I = \emptyset$ , then  $I = (a, a)$  for any  $a \in \mathbb{R}$ , and we are done. Thus, we will assume for the remainder of the proof of the forward direction that  $I$  is nonempty. To address this case, we will first prove that the facts that  $I \subsetneq \mathbb{R}$ ,  $I \neq \emptyset$ , and  $I$  is an interval imply that  $I$  is bounded above, bounded below, or both. Then in each of these three cases, we will look at whether  $\sup I$  and  $\inf I$  (if they exist) are elements of the set or not to differentiate between the eight forms in Equations 8.1. Let's begin.

Suppose for the sake of contradiction that there exists a nonempty interval  $I \subsetneq \mathbb{R}$  that is neither bounded above nor bounded below. Since  $I \subsetneq \mathbb{R}$ , we have by Definition 1.3 that there exists a point  $p \in \mathbb{R}$  such that  $p \notin I$ . Additionally, since  $I$  is neither bounded above nor below, Definition 5.6 implies that  $p$  is neither an upper nor a lower bound of  $I$ . Thus, there exist  $x, y \in I$  such that  $x < p$  and  $y > p$ . Now by Definition 8.2,  $[x, y] \subset I$ . But it follows by Definition 1.3 that every point in  $[x, y]$ , including  $p$ , is an element of  $I$ , a contradiction.

We now divide into three cases ( $I$  is exclusively bounded below,  $I$  is exclusively bounded above, and  $I$  is bounded both below and above).

First, suppose that  $I$  is only bounded below. Since  $I$  is a nonempty subset of  $\mathbb{R}$  that is bounded below, we have by Theorem 5.17 that  $\inf I$  exists. We divide into two cases again ( $\inf I \in I$  and  $\inf I \notin I$ ).

If  $\inf I \in I$ , then we can demonstrate that  $I = [\inf I, \infty)$ . To do this, Definition 1.2 tells us that it will suffice to verify that every  $p \in I$  is an element of  $[\inf I, \infty)$  and vice versa. Let  $p$  be an arbitrary element of  $I$ . Then  $p \in \mathbb{R}$  and by Definitions 5.7 and 5.6,  $\inf I \leq p$ . Therefore,  $p \in [\inf I, \infty)$ , as desired. Now let  $p$  be an arbitrary element of  $[\inf I, \infty)$ . Then  $\inf I \leq p$ . Additionally, since  $I$  is not bounded above, we have by Definition 5.6 that there exists  $y \in I$  such that  $y > p$ . Since  $\inf I \in I$ ,  $y \in I$ , and  $\inf I < y$  (by transitivity),  $[\inf I, y] \subset I$  by Definition 8.2. This combined with the fact that  $p \in [\inf I, y]$  (we know that  $\inf I \leq p < y$ , so  $\inf I \leq p \leq y$ ) implies that  $p \in I$ , as desired.

If  $\inf I \notin I$ , then we can demonstrate that  $I = (\inf I, \infty)$ . As before, to do so, it will suffice to verify that every  $p \in I$  is an element of  $(\inf I, \infty)$  and vice versa. Let  $p$  be an arbitrary element of  $I$ . Then  $p \in \mathbb{R}$  and by Definitions 5.7 and 5.6,  $\inf I \leq p$ . The additional constraint that  $\inf I \notin I$  combined with the fact that  $p \in I$  implies that  $p \neq \inf I$ , so  $\inf I < p$ . Therefore,  $p \in (\inf I, \infty)$ , as desired. Now let  $p$  be an arbitrary element of  $(\inf I, \infty)$ . Then  $\inf I < p$ . It follows by Lemma 5.11 that there exists a  $z \in I$  such that  $\inf I \leq z < p$ . Additionally, since  $I$  is not bounded above, we have by Definition 5.6 that there exists  $y \in I$  such that  $y > p$ . Since  $z \in I$ ,  $y \in I$ , and  $z < y$  (by transitivity),  $[z, y] \subset I$  by Definition 8.2. This combined with the fact that  $p \in [z, y]$  (we know that  $z < p < y$ , so  $z \leq p \leq y$ ) implies that  $p \in I$ , as desired.

Second, suppose that  $I$  is only bounded above. The proof of this case is symmetric to that of the first.

Third, suppose that  $I$  is bounded below and above. Since  $I$  is a nonempty subset of  $\mathbb{R}$  that is bounded above and below, we have by consecutive applications of Theorem 5.17 that both  $\sup I$  and  $\inf I$  exist. We divide into four cases ( $\inf I \in I$  and  $\sup I \in I$ ,  $\inf I \in I$  and  $\sup I \notin I$ ,  $\inf I \notin I$  and  $\sup I \in I$ , and  $\inf I \notin I$  and  $\sup I \notin I$ ).

If  $\inf I \in I$  and  $\sup I \in I$ , then we can demonstrate that  $I = [\inf I, \sup I]$ . We divide into two cases again ( $\inf I = \sup I$  and  $\inf I \neq \sup I$ ). If  $\inf I = \sup I \in I$ , then  $I = \{\inf I\} = \{\sup I\} = [\inf I, \sup I]$ , as desired. On the other hand, if  $\inf I \neq \sup I$ , we continue. To demonstrate that  $I = [\inf I, \sup I]$ , Theorem 1.7 tells us that it will suffice to verify that  $I \subset [\inf I, \sup I]$  and  $[\inf I, \sup I] \subset I$ . To verify that the former claim, Definition 1.3 tells us that it will suffice to confirm that every  $p \in I$  is an element of  $[\inf I, \sup I]$ . Let  $p$  be an arbitrary element of  $I$ . Then  $p \in \mathbb{R}$  and by consecutive applications of Definitions 5.7 and 5.6,  $\inf I \leq p$  and  $p \leq \sup I$ . Therefore,  $p \in [\inf I, \sup I]$ , as desired. On the other hand, since  $\inf I \in I$ ,  $\sup I \in I$ , and  $\inf I < \sup I$  (as follows from Definition 5.7 and the fact that they are unequal),  $[\inf I, \sup I] \subset I$  by Definition 8.2, as desired.

If  $\inf I \in I$  and  $\sup I \notin I$ , then we can demonstrate that  $I = [\inf I, \sup I)$ . To do so, it will suffice to verify that every  $p \in I$  is an element of  $[\inf I, \sup I)$  and vice versa. Let  $p$  be an arbitrary element of  $I$ . Then  $p \in \mathbb{R}$  and by Definitions 5.7 and 5.6,  $\inf I \leq p$  and  $p \leq \sup I$ . The additional constraint that  $\sup I \notin I$  combined with the fact that  $p \in I$  implies that  $p \neq \sup I$ , so  $p < \sup I$ . Therefore,  $p \in [\inf I, \sup I)$ , as desired. Now let  $p$  be an arbitrary element of  $[\inf I, \sup I)$ . Then  $\inf I \leq p < \sup I$ . It follows by Lemma

5.11 that there exists a  $y \in I$  such that  $p < y \leq \sup I$ . Since  $\inf I \in I$ ,  $y \in I$ , and  $\inf I < y$  (by transitivity),  $[\inf I, y] \subset I$  by Definition 8.2. This combined with the fact that  $p \in [\inf I, y]$  (we know that  $\inf I \leq p < y$ , so  $\inf I \leq p \leq y$ ) implies that  $p \in I$ , as desired.

If  $\inf I \notin I$  and  $\sup I \in I$ , the proof is symmetric to that of the previous case.

If  $\inf I \notin I$  and  $\sup I \notin I$ , then we can demonstrate that  $I = (\inf I, \sup I)$ . To do so, it will suffice to verify that every  $p \in I$  is an element of  $(\inf I, \sup I)$  and vice versa. Let  $p$  be an arbitrary element of  $I$ . Then  $p \in \mathbb{R}$  and by Definitions 5.7 and 5.6,  $\inf I \leq p$  and  $p \leq \sup I$ . The additional constraints that  $\inf I \notin I$  and  $\sup I \notin I$  combined with the fact that  $p \in I$  imply that  $p \neq \inf I$  and  $p \neq \sup I$ , respectively, so  $\inf I < p$  and  $p < \sup I$ , respectively. Therefore,  $p \in (\inf I, \sup I)$ , as desired. Now let  $p$  be an arbitrary element of  $(\inf I, \sup I)$ . Then  $\inf I < p < \sup I$ . It follows by consecutive applications of Lemma 5.11 that there exist  $x, y \in I$  such that  $\inf I \leq x < p$  and  $p < y \leq \sup I$ . Since  $x \in I$ ,  $y \in I$ , and  $x < y$  (by transitivity),  $[x, y] \subset I$  by Definition 8.2. This combined with the fact that  $p \in [x, y]$  (we know that  $x < p < y$ , so  $x \leq p \leq y$ ) implies that  $p \in I$ , as desired.

Now suppose that  $I \subsetneq \mathbb{R}$  takes one of the eight forms in Equations 8.1. To prove that  $I$  is an interval, Definition 8.2 tells us that it will suffice to show that for all  $x, y \in I$  with  $x < y$ ,  $[x, y] \subset I$ . Let  $x, y$  be arbitrary elements of  $I$  with  $x < y$ . We divide into eight cases (one for each equation in Equations 8.1).

First, suppose that  $I = (a, b)$ . To demonstrate that  $[x, y] \subset I$ , Definition 1.3 tells us that it will suffice to confirm that every  $z \in [x, y]$  is an element of  $I$ . Let  $z$  be an arbitrary element of  $[x, y]$ . Then by Equations 8.1,  $x \leq z \leq y$ . But since  $a < x < y < b$  by Equations 8.1, the fact that  $a < x \leq z \leq y < b$  implies by Equations 8.1 that  $z \in (a, b)$ , as desired.

The proofs of the second, third, and fourth equations are symmetric to that of the first.

Fifth, suppose that  $I = (a, \infty)$ . To demonstrate that  $[x, y] \subset I$ , Definition 1.3 tells us that it will suffice to confirm that every  $z \in [x, y]$  is an element of  $I$ . Let  $z$  be an arbitrary element of  $[x, y]$ . Then by Equations 8.1,  $x \leq z \leq y$ . But since  $a < x$  by Equations 8.1, the fact that  $a < x \leq z$  implies by Equations 8.1 that  $z \in (a, \infty)$ , as desired.

The proofs of the sixth, seventh, and eighth equations are symmetric to that of the first.  $\square$

**Definition 8.4.** The **absolute value** of a real number  $x$  is the non-negative number  $|x|$  defined by

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

**Exercise 8.5.** Show that  $|x| = |-x|$  for all  $x \in \mathbb{R}$ . (Note that this also means that  $|x - y| = |y - x|$  for any  $x, y \in \mathbb{R}$ .)

*Proof.* Let  $x$  be an arbitrary element of  $\mathbb{R}$ . We divide into three cases ( $x = 0$ ,  $x > 0$ , and  $x < 0$ ). First, suppose that  $x = 0$ . Then since  $0 = -0$ , clearly  $|0| = |-0|$ , as desired. Second, suppose that  $x > 0$ . Then by Lemma 7.23<sup>[1]</sup>  $-x < 0$ . Thus, by consecutive applications of Definition 8.4,  $|x| = x$  and  $|-x| = -(-x)$ . Therefore, since  $-(-x) = x$  by Corollary 7.11,  $|x| = x = |-x|$ , as desired. Third, suppose that  $x < 0$ . Then by Lemma 7.23,  $-x > 0$ . Thus, by consecutive applications of Definition 8.4,  $|x| = -x$  and  $|-x| = -x$ . Therefore,  $|x| = -x = |-x|$ , as desired.  $\square$

**Definition 8.6.** The **distance** between  $x \in \mathbb{R}$  and  $y \in \mathbb{R}$  is defined to be  $|x - y|$ .

**Remark 8.7.** It follows from Definition 8.6 that  $|x|$  is the distance between  $x$  and 0.

**Lemma 8.8.** For any real numbers  $x, y, z$ , we have

- (a)  $|x + y| \leq |x| + |y|$ .
- (b)  $|x - z| \leq |x - y| + |y - z|$ .
- (c)  $||x| - |y|| \leq |x - y|$ .

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<sup>1</sup>And, technically, Theorem 7.47.

*Proof of a.* We divide into four cases ( $x \geq 0$  and  $y \geq 0$ ,  $x \geq 0$  and  $y < 0$ ,  $x < 0$  and  $y \geq 0$ , and  $x < 0$  and  $y < 0$ ).

First, suppose that  $x \geq 0$  and  $y \geq 0$ . Then by Definition 7.21,  $x + y \geq 0$ . Thus, by consecutive applications of Definition 8.4,  $|x + y| = x + y$ ,  $|x| = x$ , and  $|y| = y$ . Therefore,  $|x + y| = x + y \leq x + y = |x| + |y|$ , as desired.

Second, suppose that  $x \geq 0$  and  $y < 0$ . By Definition 8.4,  $|x| = x$  and  $|y| = -y$ . We now divide into two cases ( $x + y \geq 0$  and  $x + y < 0$ ). If  $x + y \geq 0$ , then  $|x + y| = x + y$ . Additionally, since  $y < 0$ , Lemma 7.23 implies that  $0 < -y$ . Consequently, by transitivity,  $y < -y = |y|$ . It follows by Definition 7.21 that  $x + y < x + |y|$ . Therefore,  $|x + y| = x + y < x + |y| = |x| + |y|$ , so  $|x + y| \leq |x| + |y|$ , as desired. On the other hand, if  $x + y < 0$ , then  $|x + y| = -(x + y) = -x + (-y) = -x + |y|$ . Additionally, by Lemma 7.23,  $x \geq 0$  implies that  $-x \leq 0$ . It follows by Definition 7.21 since  $-x \leq x$  that  $-x + |y| \leq x + |y|$ . Therefore,  $|x + y| = -x + |y| \leq x + |y| = |x| + |y|$ , as desired.

The proof of the third case is symmetric to that of the second.

The proof of the fourth case is symmetric to that of the first. □

*Proof of b.* By part (a),  $|x - z| = |x - y + y - z| \leq |x - y| + |y - z|$ , as desired. □

*Proof of c.* To prove that  $||x| - |y|| \leq |x - y|$ , Definition 8.4 tells us that it will suffice to show that  $|x| - |y| \leq |x - y|$  and  $-(|x| - |y|) \leq |x - y|$ . By part (a),  $|x| = |x - y + y| \leq |x - y| + |y|$ , so  $|x| - |y| \leq |x - y|$ . Similarly,  $|y| - |x| \leq |x - y|$ , so  $-(|x| - |y|) \leq |x - y|$ , as desired. □

**Exercise 8.9.** Let  $a, \delta \in \mathbb{R}$  with  $\delta > 0$ . Prove that

$$(a - \delta, a + \delta) = \{x \in \mathbb{R} \mid |x - a| < \delta\}$$

**Lemma.** For any  $a, b \in \mathbb{R}$  such that  $0 < b$ ,  $|a| < b$  if and only if  $-b < a < b$ .

*Proof.* Suppose first that  $|a| < b$ . We divide into two cases ( $a \geq 0$  and  $a < 0$ ). If  $a \geq 0$ , then by Definition 8.4,  $0 \leq a = |a| < b$ . Additionally, by Lemma 7.23,  $-b < 0$ . Therefore,  $-b < 0 \leq a < b$ , as desired. If  $a < 0$ , then by Definition 8.4,  $-a = |a| < b$ . It follows by Definition 7.21 (by adding  $a - b$  to both sides) that  $-b < a$ . Additionally, by Lemma 7.23,  $a < 0$  implies  $0 < -a$ , so we know that  $a < -a$ . Therefore,  $-b < a < -a < b$ , as desired.

Now suppose that  $-b < a < b$ . We divide into two cases ( $a \geq 0$  and  $a < 0$ ). If  $a \geq 0$ , then by Definition 8.4,  $|a| = a < b$ , as desired. If  $a < 0$ , then by Definition 8.4,  $|a| = -a$ . Since  $-b < a$ , Definition 7.21 implies (by adding  $b - a$  to both sides) that  $-a < b$ . Therefore,  $|a| = -a < b$ , as desired. □

*Proof of Exercise 8.9.* To prove that  $(a - \delta, a + \delta) = \{x \in \mathbb{R} \mid |x - a| < \delta\}$ , Definition 1.2 tells us that it will suffice to show that every  $p \in (a - \delta, a + \delta)$  is an element of  $\{x \in \mathbb{R} \mid |x - a| < \delta\}$  and vice versa.

Suppose first that  $p \in (a - \delta, a + \delta)$ . Then by Equations 8.1,  $a - \delta < p < a + \delta$ . It follows by consecutive applications of Definition 7.21 that  $-\delta < p - a < \delta$ . Thus, the lemma asserts that  $|p - a| < \delta$ . Therefore,  $p \in \{x \in \mathbb{R} \mid |x - a| < \delta\}$ .

Now suppose that  $p \in \{x \in \mathbb{R} \mid |x - a| < \delta\}$ . Then  $|p - a| < \delta$ . Thus, by the lemma,  $-\delta < p - a < \delta$ . It follows by consecutive applications of Definition 7.21 that  $a - \delta < p < a + \delta$ . Therefore, since  $a - \delta < p < a + \delta$ , we have that  $p \in (a - \delta, a + \delta)$ . □

2/11: **Lemma 8.10.** Let  $I$  be an open interval containing the point  $p \in \mathbb{R}$ . Then

a) There exists a number  $\delta > 0$  such that  $(p - \delta, p + \delta) \subset I$ .

b) There exists a natural number  $N$  such that for all natural numbers  $k \geq N$  we have  $(p - \frac{1}{k}, p + \frac{1}{k}) \subset I$ .

*Proof of a.* Since  $I$  is open, we have by Theorem 4.10 that there exists a region  $(a, b)$  such that  $p \in (a, b)$  and  $(a, b) \subset I$ . Let  $\delta = \min(p - a, b - p)$ . To show that  $(p - \delta, p + \delta) \subset I$ , we will demonstrate that  $(p - \delta, p + \delta) \subset (a, b) \subset I$ . To do this, Definition 1.3 tells us that it will suffice to verify that every element  $x \in (p - \delta, p + \delta)$  is an element of  $(a, b)$ . Let  $x$  be an arbitrary element of  $(p - \delta, p + \delta)$ . Then by Equations 8.1,  $p - \delta < x < p + \delta$ . We divide into two cases ( $\delta = p - a$  and  $\delta = b - p$ ). Suppose first that  $\delta = p - a$ . Then  $p - (p - a) < x < p + (p - a)$ , i.e.,  $a < x < p + (p - a)$ . Additionally, the fact that  $p - a = \min(p - a, b - p)$  implies that  $p - a \leq b - p$ . Combining these last two results gives us  $a < x < p + (p - a) \leq p + (b - p) = b$ . Since  $a < x < b$ , Equations 8.1 imply that  $x \in (a, b)$ , as desired. The proof is symmetric if  $\delta = b - p$ . □

*Proof of b.* By Lemma 8.10a, there exists a number  $\delta > 0$  such that  $(p - \delta, p + \delta) \subset I$ . Since  $\delta$  is a positive real number, Corollary 6.12 implies that there exists a nonzero natural number  $N$  such that  $\frac{1}{N} < \delta$ . To prove that for all numbers  $k \geq N$ , we have  $(p - \frac{1}{k}, p + \frac{1}{k}) \subset I$ , we will show that  $(p - \frac{1}{k}, p + \frac{1}{k}) \subset (p - \delta, p + \delta) \subset I$ . To do this, Definition 1.3 tells us that it will suffice to show that every  $x \in (p - \frac{1}{k}, p + \frac{1}{k})$  is an element of  $(p - \delta, p + \delta)$ . Let  $k$  be an arbitrary natural number such that  $k \geq N$ , and let  $x$  be an arbitrary element of  $(p - \frac{1}{k}, p + \frac{1}{k})$ . It follows from the latter condition by Equations 8.1 that  $p - \frac{1}{k} < x < p + \frac{1}{k}$ . Since  $\frac{1}{k} \leq \frac{1}{N}$  by Scripts 2 and 3, we have that  $p - \frac{1}{N} < x < p + \frac{1}{N}$ . Since  $\frac{1}{N} < \delta$  by definition,  $p - \delta < x < p + \delta$ . Therefore, by Equations 8.1,  $x \in (p - \delta, p + \delta)$ , as desired.  $\square$

**Definition 8.11.** Let  $A \subset X \subset \mathbb{R}$ . We say that  $A$  is **open** (in  $X$ ) if it is the intersection of  $X$  with an open set, and **closed** (in  $X$ ) if it is the intersection of  $X$  with a closed set. (This is called the subspace topology on  $X$ .)

**Remark 8.12.**  $A \subset \mathbb{R}$  open, as defined in Script 3, is equivalent to  $A$  open in  $\mathbb{R}$ .

**Exercise 8.13.** Let  $A \subset X \subset \mathbb{R}$ . Show that  $X \setminus A$  is closed in  $X$  if and only if  $A$  is open in  $X$ .

*Proof.* Suppose first that  $X \setminus A$  is closed in  $X$ . Then by Definition 8.11,  $X \setminus A = X \cap B$  where  $B$  is a closed set. It follows by Script 1 that

$$\begin{aligned} X \setminus A &= X \cap B \\ \mathbb{R} \setminus (X \setminus A) &= \mathbb{R} \setminus (X \cap B) \\ (\mathbb{R} \setminus X) \cup A &= (\mathbb{R} \setminus X) \cup (\mathbb{R} \setminus B) \\ X \cap ((\mathbb{R} \setminus X) \cup A) &= X \cap ((\mathbb{R} \setminus X) \cup (\mathbb{R} \setminus B)) \\ (X \cap (\mathbb{R} \setminus X)) \cup (X \cap A) &= (X \cap (\mathbb{R} \setminus X)) \cup (X \cap (\mathbb{R} \setminus B)) \\ \emptyset \cup (X \cap A) &= \emptyset \cup (X \cap (\mathbb{R} \setminus B)) \\ A &= X \cap (\mathbb{R} \setminus B) \end{aligned}$$

Since  $\mathbb{R} \setminus B$  is open by Definition 4.4, we have by Definition 8.11 that  $A$  is open in  $X$ .

Now suppose that  $A$  is open in  $X$ . Then by Definition 8.11,  $A = X \cap B$  where  $B$  is an open set. It follows by Script 1 that

$$\begin{aligned} A &= X \cap B \\ \mathbb{R} \setminus A &= \mathbb{R} \setminus (X \cap B) \\ &= (\mathbb{R} \setminus X) \cup (\mathbb{R} \setminus B) \\ X \cap (\mathbb{R} \setminus A) &= X \cap ((\mathbb{R} \setminus X) \cup (\mathbb{R} \setminus B)) \\ X \setminus A &= (X \cap (\mathbb{R} \setminus X)) \cup (X \cap (\mathbb{R} \setminus B)) \\ &= X \cap (\mathbb{R} \setminus B) \end{aligned}$$

Since  $\mathbb{R} \setminus B$  is closed by Definition 4.4, we have by Definition 8.11 that  $X \setminus A$  is closed in  $X$ .  $\square$

**Exercise 8.14.**

- Let  $[a, b] \subset \mathbb{R}$ . Give an example of a set  $A \subset [a, b]$  such that  $A$  is open in  $[a, b]$  but not in  $\mathbb{R}$ .
- Give an example of sets  $A \subset X \subset \mathbb{R}$  such that  $A$  is closed in  $X$  but not in  $\mathbb{R}$ .

*Proof of a.* We first briefly consider the case where  $a = b$ . In this case, let  $c < a < d$ ; then  $\{a\} = [a, a] \cap (c, d)$  is a subset of  $[a, b]$  that is open in  $[a, b]$  (by Definition 8.11;  $(c, d)$  is open by Exercise 8.1) but closed in  $\mathbb{R}$  (by Corollary 3.23, Definition 4.1, and Theorem 5.1).

We now direct our attention to the case where  $a \neq b$ . Let  $c \in [a, b]$  be a point such that  $a < c < b$  (we know at least one such point exists by Theorem 5.2). If we define the set  $(c, b] = [a, b] \cap (c, \infty)$ , we have by Definition 8.11 that  $(c, b]$  is open in  $[a, b]$  (since  $(c, \infty)$  is open per Exercise 8.1). However, we know that  $(c, b]$  is not open in  $\mathbb{R}$  by Theorem 4.10 ( $b$  is an element of  $(c, b]$  such that any region containing  $b$  necessarily contains an element that is not in  $(c, b]$ ; this element will be greater than  $b$  but less than the right bound of the region, and its existence is guaranteed by Theorem 5.2).  $\square$

*Proof of b.* Let  $X = (a, b) \subset \mathbb{R}$ . Then  $(a, b) = X \cap [a, b]$ , so  $(a, b)$  is closed in  $(a, b)$  by Definition 8.11. However, by Corollary 5.14,  $a, b$  are limit points of  $(a, b)$  that are not contained within  $(a, b)$ . It follows by Definition 4.1 that  $(a, b)$  is not closed in  $\mathbb{R}$ .  $\square$

**Theorem 8.15.** *Let  $X \subset \mathbb{R}$ . Then  $X$  is connected if and only if  $X$  is an interval.*

*Proof.* Suppose first that  $X$  is connected. To prove that  $X$  is an interval, Definition 8.2 tells us that it will suffice to show that for all  $x, y \in X$  with  $x < y$ ,  $[x, y] \subset X$ . Let  $x, y$  be arbitrary elements of  $X$  satisfying  $x < y$ , and suppose for the sake of contradiction that  $[x, y] \not\subset X$ . Then there exists  $z \in [x, y]$  such that  $z \notin X$ . Let  $A = \{a \in X \mid a < z\}$  and  $B = \{b \in X \mid z < b\}$ . It follows from Script 1 that  $X = A \cup B$  and  $A \cap B = \emptyset$ . To verify that  $A$  is nonempty, Definition 1.8 tells us that it will suffice to find an element in it. Since  $z \notin X$  but  $x \in X$ , we know that  $z \neq x$ . This combined with the fact that  $x \leq z$  by Equations 8.1 implies that  $x < z$ . Thus, since  $x \in X$  and  $x < z$ ,  $x \in A$ . Similarly,  $y \in B$ . To verify that  $A$  is open in  $X$ , Definition 8.11 tells us that it will suffice to demonstrate that  $A$  is the intersection of  $X$  with an open set. Since we clearly have  $A = X \cap (-\infty, z)$  where  $(-\infty, z)$  is open by Exercise 8.1, we are done. We can do something similar for  $B$ . But the existence of two disjoint, nonempty, open (in  $X$ ) sets  $A, B$  whose union equals  $X$  demonstrates by Definition 4.22 that  $X$  is disconnected, a contradiction.

Now suppose that  $X$  is an interval, and suppose for the sake of contradiction that  $X$  is disconnected. Then by Definition 4.22,  $X = A \cup B$  where  $A, B$  are disjoint, nonempty sets that are open in  $X$ . Since  $A, B$  are disjoint and nonempty, we know that there exist distinct objects  $a \in A$  and  $b \in B$ . WLOG, let  $a < b$ .

To prove that  $\sup(A \cap [a, b])$  exists, Theorem 5.17 tells us that it will suffice to show that  $A \cap [a, b]$  is nonempty and bounded above. To show that  $A \cap [a, b]$  is nonempty, Definition 1.8 tells us that it will suffice to find an element of  $A \cap [a, b]$ . By Equations 8.1,  $a \in [a, b]$ . By Definition,  $a \in A$ . Thus, by Definition 1.6,  $a \in A \cap [a, b]$ , as desired. To show that  $A \cap [a, b]$  is bounded above, consecutive applications of Definition 5.6 tell us that it will suffice to verify that  $x \leq b$  for all  $x \in A \cap [a, b]$ . Let  $x$  be an arbitrary element of  $A \cap [a, b]$ . It follows by Definition 1.6 that  $x \in [a, b]$ . Thus, by Equations 8.1,  $x \leq b$ , as desired.

Let  $s = \sup(A \cap [a, b])$ . To prove that  $\inf(B \cap [s, b])$  exists, it will suffice to utilize a symmetric argument to the above.

Let  $i = \inf(B \cap [s, b])$ . We divide into three cases ( $s > i$ ,  $s = i$ , and  $s < i$ ).

First, suppose that  $s > i$ . To show that  $s$  is a lower bound of  $B \cap [s, b]$ , Definition 5.6 tells us that it will suffice to verify that  $s \leq x$  for all  $x \in B \cap [s, b]$ . Let  $x$  be an arbitrary element of  $B \cap [s, b]$ . By Definition 1.6,  $x \in [s, b]$ . Thus, by Equations 8.1,  $s \leq x$ , as desired. Since  $s$  is a lower bound of  $B \cap [s, b]$ , Definition 5.6 asserts that  $i \geq s$ , contradicting the hypothesis that  $s > i$ .

Second, suppose that  $s = i$ . We divide into three cases ( $s \in A$ ,  $s \in B$ , and  $s \notin A$  and  $s \notin B$ ).

If  $s \in A$ , then since  $A$  is open in  $X$ , Definition 8.11 implies that  $A = X \cap G$  where  $G$  is open. It follows by the hypothesis that  $s \in A$  along with Definitions 1.2 and 1.6 that  $s \in G$ . Consequently, by Theorem 4.10, there exists a region  $(c, d)$  such that  $s \in (c, d)$  and  $(c, d) \subset G$ . From the former condition, we have by Equations 8.1 that  $c < s < d$ . Thus, by Lemma 5.11, there exists a point  $x \in B \cap [s, b]$  such that  $s = i \leq x < d$ . Since  $c < s \leq x < d$ , Equations 8.1 imply that  $x \in (c, d)$ . This combined with the fact that  $(c, d) \subset G$  implies by Definition 1.3 that  $x \in G$ . Additionally, we know that  $x \in B$  (since  $x \in B \cap [s, b]$  by Definition 1.6). It follows from this and the fact that  $X = A \cup B$  by Definitions 1.5 and 1.2 that  $x \in X$ . Thus, since  $x \in X$  and  $x \in G$ , Definition 1.6 asserts that  $x \in X \cap G$ , meaning that  $x \in A$ . But if  $x \in A$  and  $x \in B$ , then Definition 1.6 implies that  $x \in A \cap B$ , contradicting the supposition that  $A$  and  $B$  are disjoint.

If  $s \in B$ , then the proof is symmetric to the previous case.

If  $s \notin A$  and  $s \notin B$ , then by Definition 1.5,  $s \notin A \cup B$ , implying that  $s \notin X$ . Additionally, the facts that  $a \in A$ ,  $b \in B$ , and  $X = A \cup B$  imply that  $a, b \in X$ . It follows since  $a < b$  by Definition 8.2 that  $[a, b] \subset X$ . We now show that  $s \in [a, b]$  via Equations 8.1, which tell us that it will suffice to verify that  $a \leq s \leq b$ . As previously shown,  $b$  is an upper bound of  $A \cap [a, b]$ . Thus, by Definition 5.7, we have that  $s \leq b$ , and we are half done. As to the other half, we have also previously shown that  $a \in A \cap [a, b]$ . Additionally, by Definitions 5.7 and 5.6,  $s \geq x$  for all  $x \in A \cap [a, b]$ , including  $a$ . Thus,  $s \geq a$ . Having shown that  $s \in [a, b]$  and  $[a, b] \subset X$ , we may invoke Definition 1.3 to learn that  $s \in X$ , contradicting the previously proven statement that  $s \notin X$ .

Third, suppose that  $s < i$ . Then by Theorem 5.2 and Definition 3.6, there exists a  $z \in \mathbb{R}$  such that  $s < z < i$ . We now show that  $i \in [a, b]$  via Equations 8.1, which tell us that it will suffice to verify that  $a \leq i \leq b$ . As previously shown,  $s$  is a lower bound of  $B \cap [s, b]$ . Thus, by Definition 5.7, we have that  $i \geq s$ . We have also previously shown that  $s \geq a$ , so by transitivity,  $i \geq a$ , and we are half done. As to the other half, we now confirm that  $b \in B \cap [s, b]$ . By Equations 8.1,  $b \in [s, b]$ . By definition,  $b \in B$ . Thus, by Definition 1.6,  $b \in B \cap [s, b]$ , as desired. Additionally, by Definitions 5.7 and 5.6,  $i \leq x$  for all  $x \in B \cap [s, b]$ , including  $b$ . Thus,  $i \leq b$ , concluding our argument that  $i \in [a, b]$ . Moving on, the fact that  $s < z$  implies by Definition 5.6 that  $z \notin A \cap [a, b]$ . Additionally, we know from the facts that  $s, i \in [a, b]$  that  $a \leq s < z < i \leq b$ , meaning that  $z \in [a, b]$ . Combining the previous two results with Definition 1.6, we have that  $z \notin A$ . By a symmetric argument, we can show that  $z \notin B$ . Since  $z \notin A$  and  $z \notin B$ , Definition 1.5 asserts that  $z \notin A \cup B$ , i.e.,  $z \notin X$ . But as before,  $[a, b] \subset X$ , so the fact that  $z \in [a, b]$  combined with Definition 1.3 implies that  $z \in X$ , a contradiction.  $\square$

2/16: **Definition 8.16.** Let  $I$  be an interval and let  $f : I \rightarrow \mathbb{R}$ .

- a) We say that  $f$  is **increasing** on  $I$  if, whenever  $x, y \in I$  with  $x < y$ ,  $f(x) \leq f(y)$ .
- b) We say that  $f$  is **decreasing** on  $I$  if, whenever  $x, y \in I$  with  $x < y$ ,  $f(x) \geq f(y)$ .
- c) We say that  $f$  is **strictly increasing** on  $I$  if, whenever  $x, y \in I$  with  $x < y$ ,  $f(x) < f(y)$ .
- d) We say that  $f$  is **strictly decreasing** on  $I$  if, whenever  $x, y \in I$  with  $x < y$ ,  $f(x) > f(y)$ .

**Lemma 8.17.** If  $f$  is strictly increasing or strictly decreasing on an interval  $I$  then  $f$  is injective on  $I$ .

*Proof.* We divide into two cases ( $f$  is strictly increasing, and  $f$  is strictly decreasing). Suppose first that  $f$  is strictly increasing. To prove that  $f$  is injective on  $I$ , Definition 1.20 tells us that it will suffice to show that for all  $a, b \in I$ ,  $a \neq b$  implies that  $f(a) \neq f(b)$ . Let  $a, b$  be arbitrary elements of  $I$  such that  $a \neq b$ . WLOG, let  $a < b$ . Then by Definition 8.16,  $f(a) < f(b)$ . Therefore,  $f(a) \neq f(b)$ , as desired. The proof is symmetric for the other case.  $\square$

## 8.2 Discussion

- 2/9:
  - Due date: Feb. 19; if there's anything I can provide that would facilitate the process, lmk; Know anything about mixed-integer nonlinear programming?
  - Lemma 8.3 more efficiently by proving that every  $x \in (a, b)$  is an element of  $I$  and then just working with the boundary conditions?
    - Make first four cases of second direction symmetric.
  - Rewrite Exercise 8.5 with three cases:  $x = 0$ ,  $x > 0$ ,  $x < 0$  with the last two symmetric.
  - We don't have to cite every algebraic manipulations from Script 7.
- 2/11:
  - Use a bidirectional inclusion proof instead of set algebra for Exercise 8.13.
  - What is the problem with Exercise 8.14b?



## Script 9

# Continuous Functions

### 9.1 Journal

2/16: **Lemma 9.1.** *Let  $X \subset \mathbb{R}$  and  $f : X \rightarrow \mathbb{R}$ . If  $A, B \subset \mathbb{R}$ , then*

$$\begin{aligned}f^{-1}(A \cup B) &= f^{-1}(A) \cup f^{-1}(B) \\f^{-1}(A \cap B) &= f^{-1}(A) \cap f^{-1}(B) \\f^{-1}(A \setminus B) &= f^{-1}(A) \setminus f^{-1}(B) \\f^{-1}(\mathbb{R}) &= X\end{aligned}$$

*Proof.* To prove that  $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ , Definition 1.2 tells us that it will suffice to show that every  $x \in f^{-1}(A \cup B)$  is an element of  $f^{-1}(A) \cup f^{-1}(B)$  and vice versa. Suppose first that  $x$  is an arbitrary element of  $f^{-1}(A \cup B)$ . Then by Definition 1.18,  $f(x) \in A \cup B$ . Thus, by Definition 1.5,  $f(x) \in A$  or  $f(x) \in B$ . We now divide into two cases. If  $f(x) \in A$ , then by Definition 1.18,  $x \in f^{-1}(A)$ . It follows by Definition 1.5 that  $x \in f^{-1}(A) \cup f^{-1}(B)$ , as desired. The argument is symmetric in the other case. Now suppose that  $x \in f^{-1}(A) \cup f^{-1}(B)$ . Then by Definition 1.5,  $x \in f^{-1}(A)$  or  $x \in f^{-1}(B)$ . We now divide into two cases. If  $x \in f^{-1}(A)$ , then by Definition 1.18,  $f(x) \in A$ . It follows by Definition 1.5 that  $f(x) \in A \cup B$ . Therefore, by Definition 1.18,  $x \in f^{-1}(A \cup B)$ . The argument is symmetric in the other case, as desired.

To prove that  $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ , Definition 1.2 tells us that it will suffice to show that every  $x \in f^{-1}(A \cap B)$  is an element of  $f^{-1}(A) \cap f^{-1}(B)$  and vice versa. Suppose first that  $x$  is an arbitrary element of  $f^{-1}(A \cap B)$ . Then by Definition 1.18,  $f(x) \in A \cap B$ . Thus, by Definition 1.6,  $f(x) \in A$  and  $f(x) \in B$ . It follows by consecutive applications of Definition 1.18 that  $x \in f^{-1}(A)$  and  $x \in f^{-1}(B)$ . Therefore, by Definition 1.6,  $x \in f^{-1}(A) \cap f^{-1}(B)$ , as desired. Now suppose that  $x \in f^{-1}(A) \cap f^{-1}(B)$ . Then by Definition 1.6,  $x \in f^{-1}(A)$  and  $x \in f^{-1}(B)$ . It follows by consecutive applications of Definition 1.18 that  $f(x) \in A$  and  $f(x) \in B$ . Thus, by Definition 1.6,  $f(x) \in A \cap B$ . Therefore, by Definition 1.18,  $x \in f^{-1}(A \cap B)$ , as desired.

To prove that  $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$ , Definition 1.2 tells us that it will suffice to show that every  $x \in f^{-1}(A \setminus B)$  is an element of  $f^{-1}(A) \setminus f^{-1}(B)$  and vice versa. Suppose first that  $x$  is an arbitrary element of  $f^{-1}(A \setminus B)$ . Then by Definition 1.18,  $f(x) \in A \setminus B$ . Thus, by Definition 1.11,  $f(x) \in A$  and  $f(x) \notin B$ . It follows by consecutive applications of Definition 1.18 that  $x \in f^{-1}(A)$  and  $x \notin f^{-1}(B)$ . Therefore, by Definition 1.11,  $x \in f^{-1}(A) \setminus f^{-1}(B)$ , as desired. Now suppose that  $x \in f^{-1}(A) \setminus f^{-1}(B)$ . Then by Definition 1.11,  $x \in f^{-1}(A)$  and  $x \notin f^{-1}(B)$ . It follows by consecutive applications of Definition 1.18 that  $f(x) \in A$  and  $f(x) \notin B$ . Thus, by Definition 1.11,  $f(x) \in A \setminus B$ . Therefore, by Definition 1.18,  $x \in f^{-1}(A \setminus B)$ , as desired.

To prove that  $f^{-1}(\mathbb{R}) = X$ , Definition 1.2 tells us that it will suffice to show that every  $x \in f^{-1}(\mathbb{R})$  is an element of  $X$  and vice versa. Suppose first that  $x$  is an arbitrary element of  $f^{-1}(\mathbb{R})$ . Then by Definition 1.18,  $x \in X$ , as desired. Now suppose that  $x \in X$ . Then by Definition 1.16,  $f(x) \in \mathbb{R}$ . It follows by Definition 1.18 that  $x \in f^{-1}(\mathbb{R})$ , as desired.  $\square$

**Exercise 9.2.** Let  $f : X \rightarrow \mathbb{R}$ . Let  $A \subset X$  and  $B \subset \mathbb{R}$ . Show that

$$\begin{aligned} f(f^{-1}(B)) &\subset B \\ A &\subset f^{-1}(f(A)) \end{aligned}$$

Give examples to show that the inclusions can be proper.

*Proof.* To prove that  $f(f^{-1}(B)) \subset B$ , Definition 1.3 tells us that it will suffice to show that every  $y \in f(f^{-1}(B))$  is an element of  $B$ . Let  $y$  be an arbitrary element of  $f(f^{-1}(B))$ . Then by Definition 1.18,  $y = f(x)$  for some  $x \in f^{-1}(B)$ . By Definition 1.18 again,  $f(x) \in B$ . Therefore, since  $y = f(x)$ , it follows that  $y \in B$ , as desired.

To prove that  $A \subset f^{-1}(f(A))$ , Definition 1.3 tells us that it will suffice to show that every  $x \in A$  is an element of  $f^{-1}(f(A))$ . Let  $x$  be an arbitrary element of  $A$ . Then by Definition 1.18,  $f(x) \in f(A)$ . Therefore, by Definition 1.18, we have  $x \in f^{-1}(f(A))$ , as desired.

Let  $X = \{1, 2\}$  and let  $f : X \rightarrow \mathbb{R}$  be defined by  $f(1) = 3$  and  $f(2) = 3$ . If we let  $B = \{3, 4\}$ , then  $f(f^{-1}(B)) = \{3\} \subsetneq \{3, 4\}$ . Additionally, if we let  $A = \{1\}$ , then  $A \subsetneq f^{-1}(f(A)) = \{1, 2\}$ .  $\square$

**Exercise 9.3.** Let  $f : X \rightarrow \mathbb{R}$ . Let  $A \subset X$  and  $B \subset \mathbb{R}$ . Then  $f(A) \subset B \iff A \subset f^{-1}(B)$ .

*Proof.* Suppose first that  $f(A) \subset B$ . To prove that  $A \subset f^{-1}(B)$ , Definition 1.3 tells us that it will suffice to show that every  $x \in A$  is an element of  $f^{-1}(B)$ . Let  $x$  be an arbitrary element of  $A$ . Then by Definition 1.18,  $f(x) \in f(A)$ . It follows by the hypothesis and Definition 1.3 that  $f(x) \in B$ . Therefore, by Definition 1.18 again,  $x \in f^{-1}(B)$ .

Now suppose that  $A \subset f^{-1}(B)$ . To prove that  $f(A) \subset B$ , Definition 1.3 tells us that it will suffice to show that every  $y \in f(A)$  is an element of  $B$ . Let  $y$  be an arbitrary element of  $f(A)$ . Then by Definition 1.18,  $y = f(x)$  for some  $x \in A$ . It follows by the hypothesis and Definition 1.3 that  $x \in f^{-1}(B)$ . Therefore, by Definition 1.18 again,  $y = f(x) \in B$ .  $\square$

**Definition 9.4.** Let  $X \subset \mathbb{R}$ . A function  $f : X \rightarrow \mathbb{R}$  is **continuous** if for every open set  $U \subset \mathbb{R}$ , the preimage  $f^{-1}(U)$  is open in  $X$ .

**Proposition 9.5.** Let  $X \subset \mathbb{R}$ . A function  $f : X \rightarrow \mathbb{R}$  is continuous if and only if for every closed set  $F \subset \mathbb{R}$ , the preimage  $f^{-1}(F)$  is closed in  $X$ .

*Proof.* Suppose first that  $f$  is continuous. We seek to prove that for every closed set  $F \subset \mathbb{R}$ , the preimage  $f^{-1}(F)$  is closed in  $X$ . Let  $F$  be an arbitrary closed subset of  $\mathbb{R}$ . Then by Definition 4.8,  $F = \mathbb{R} \setminus U$  for some open set  $U \subset \mathbb{R}$ . It follows by Definition 9.4 since  $f$  is continuous that  $f^{-1}(U)$  is open in  $X$ . Additionally, by consecutive applications of Lemma 9.1,  $f^{-1}(F) = f^{-1}(\mathbb{R} \setminus U) = f^{-1}(\mathbb{R}) \setminus f^{-1}(U) = X \setminus f^{-1}(U)$ . Therefore, since  $f^{-1}(U)$  is open in  $X$ , Exercise 8.13 implies that  $X \setminus f^{-1}(U) = f^{-1}(F)$  is closed in  $X$ .

The proof is symmetric in the other direction.  $\square$

**Definition 9.6.** Let  $X \subset Y \subset \mathbb{R}$  and let  $f : Y \rightarrow \mathbb{R}$ . Then the **restriction** (of  $f$  to  $X$ ), written  $f|_X$  is the function  $f|_X : X \rightarrow \mathbb{R}$  defined by

$$f|_X(x) = f(x)$$

for all  $x \in X$ .

**Proposition 9.7.** Let  $X \subset Y \subset \mathbb{R}$ . If  $f : Y \rightarrow \mathbb{R}$  is continuous, then the restriction of  $f$  to  $X$  is continuous.

*Proof.* To prove that  $f|_X$  is continuous, Definition 9.4 tells us that it will suffice to show that for every open set  $U \subset \mathbb{R}$ , the preimage  $f|_X^{-1}(U)$  is open in  $X$ . Let  $U$  be an open subset of  $\mathbb{R}$ . Then

$$\begin{aligned} f|_X^{-1}(U) &= \{x \in X \mid f|_X(x) \in U\} && \text{Definition 1.18} \\ &= \{x \in X \mid f(x) \in U\} && \text{Definition 9.6} \\ &= \{x \in Y \mid f(x) \in U\} \cap X && \text{Script 1} \\ &= f^{-1}(U) \cap X && \text{Definition 1.18} \\ &= (Y \cap G) \cap X && \text{Definitions 9.4 and 8.11} \end{aligned}$$

$$= X \cap G$$

Script 1

Since  $f|_X^{-1}(U) = X \cap G$  where  $G$  is an open set, Definition 8.11 asserts that  $f|_X^{-1}(U)$  is open in  $X$ .  $\square$

**Exercise 9.8.** Show that for any  $X \subsetneq \mathbb{R}$  that is not open and any continuous function  $f : X \rightarrow \mathbb{R}$ , there is an open set  $U$  for which  $f^{-1}(U)$  is open in  $X$  but is not open in  $\mathbb{R}$ .

*Proof.* We will prove that  $\mathbb{R}$  is an open set for which  $f^{-1}(\mathbb{R})$  is open in  $X$  but not in  $\mathbb{R}$ . First, by Theorem 5.1,  $\mathbb{R}$  is open. Next, by Lemma 9.1,  $f^{-1}(\mathbb{R}) = X$ . It follows since  $f^{-1}(\mathbb{R}) = X = X \cap \mathbb{R}$  (where  $\mathbb{R}$  is an open set) by Definition 8.11 that  $f^{-1}(\mathbb{R})$  is open in  $X$ . Last, since  $X$  is not open (in  $\mathbb{R}$ ) by definition,  $f^{-1}(\mathbb{R}) = X$  is not open in  $\mathbb{R}$ .  $\square$

**Definition 9.9.** The function  $f : X \rightarrow \mathbb{R}$  is **continuous** (at  $x \in X$ ) if for every region  $R$  containing  $f(x)$ , there exists an open set  $S$  containing  $x$  such that  $S \cap X \subset f^{-1}(R)$ .

**Theorem 9.10.** The function  $f : X \rightarrow \mathbb{R}$  is continuous if and only if it is continuous at every  $x \in X$ .

*Proof.* Suppose first that  $f$  is continuous, and suppose for the sake of contradiction that  $f$  is not continuous at every  $x \in X$ . Then by Definition 9.9, there exists some  $x \in X$  such that  $f$  is not continuous at  $x$ . Thus, there exists a region  $R$  with  $f(x) \in R$  such that for all open sets  $S$  containing  $x$ ,  $S \cap X \not\subset f^{-1}(R)$ . Since  $f$  is continuous by hypothesis and  $R$  is open by Corollary 4.11,  $f^{-1}(R)$  is open in  $X$ . It follows by Definition 8.11 that  $f^{-1}(R) = X \cap S$  for some open set  $S$ . But this implies that  $f^{-1}(R) \not\subset f^{-1}(R)$ , a contradiction.

Now suppose that  $f$  is continuous at every  $x \in X$ . To prove that  $f$  is continuous, Definition 9.4 tells us that it will suffice to show that for every open set  $U \subset \mathbb{R}$ , the preimage  $f^{-1}(U)$  is open in  $X$ . We divide into two cases ( $f^{-1}(U) = \emptyset$  and  $f^{-1}(U) \neq \emptyset$ ). If  $f^{-1}(U) = \emptyset$ , then since  $\emptyset \cap X = \emptyset$  by Script 1 where  $\emptyset$  is open by Theorem 5.1, Definition 8.11 tells us that  $\emptyset = f^{-1}(U)$  is open in  $X$ , as desired. On the other hand, if  $f^{-1}(U) \neq \emptyset$ , Definition 8.11 tells us that it will suffice to show that  $f^{-1}(U) = S \cap X$  where  $S$  is an open set. We first seek to show that for every  $x \in f^{-1}(U)$ , there exists an open set  $S_x$  containing  $x$  such that  $S_x \cap X \subset f^{-1}(U)$ . Let  $x$  be an arbitrary element of  $f^{-1}(U)$ . It follows by Definition 1.18 that  $f(x) \in U$ . Thus, since  $U$  is open, we have by Theorem 4.10 that there exists a region  $R$  such that  $f(x) \in R$  and  $R \subset U$ . Consequently, since  $R$  is open by Corollary 4.11, we have by Definition 9.9 that there exists an open set  $S_x$  containing  $x$  such that  $S_x \cap X \subset f^{-1}(R)$ . Additionally, Script 1 tells us based off of the fact that  $R \subset U$  that  $f^{-1}(R) \subset f^{-1}(U)$ . Thus, by subset transitivity,  $S_x \cap X \subset f^{-1}(U)$ . At this point, let  $S = \bigcup_{x \in f^{-1}(U)} S_x$ . It follows immediately from Corollary 4.18 that  $S$  is open. Additionally, since the intersection of each set in the union with  $X$  is a subset of  $f^{-1}(U)$ , it follows by Script 1 that  $S \cap X \subset f^{-1}(U)$ . Furthermore, for all  $x \in f^{-1}(U)$ , Definition 1.18 asserts that  $x \in X$ . In addition, we have defined an  $S_x$  such that  $x \in S_x$ . These last two results combined demonstrate by Definition 1.6 that  $x \in S \cap X$ . Thus, by Definition 1.3,  $f^{-1}(U) \subset S \cap X$ . Consequently, by Theorem 1.7,  $f^{-1}(U) = S \cap X$ . Since  $f^{-1}(U)$  is the intersection of  $X$  with an open set, Definition 8.11 asserts that it is open in  $X$ , as desired.  $\square$

2/18: **Theorem 9.11.** Suppose that  $f : X \rightarrow \mathbb{R}$  is continuous. If  $X$  is connected, then  $f(X)$  is connected.

*Proof.* This will be a proof by contrapositive; as such, suppose that  $f(X)$  is disconnected. Then by Definition 4.22,  $f(X) = A \cup B$  where  $A, B$  are nonempty, disjoint sets that are open in  $f(X)$ . It follows from the last condition by Definition 8.11 that  $A = G \cap f(X)$  and  $B = H \cap f(X)$ , where  $G, H$  are open sets. Since for all  $x \in X$ ,  $f(x) \in A$  or  $f(x) \in B$ , Definitions 1.2 and 1.6 imply that for all  $x \in X$ ,  $f(x) \in G$  and  $f(x) \in H$ . Thus, by Script 1,  $X \subset f^{-1}(G) \cup f^{-1}(H)$ . Additionally, we know by Definition 1.18 that for all  $x \in f^{-1}(G) \cup f^{-1}(H)$ ,  $x \in X$ . Thus, by Definition 1.3,  $f^{-1}(G) \cup f^{-1}(H) \subset X$ . Consequently, by Theorem 1.7, we have that  $X = f^{-1}(G) \cup f^{-1}(H)$ .

To show that  $f^{-1}(G), f^{-1}(H)$  are disjoint, Definition 1.9 tells us that it will suffice to verify that  $f^{-1}(G) \cap f^{-1}(H) = \emptyset$ . As such, suppose for the sake of contradiction that  $x \in f^{-1}(G) \cap f^{-1}(H)$ . Then by consecutive applications of Definition 1.6,  $x \in f^{-1}(G)$  and  $x \in f^{-1}(H)$ . Thus, by multiple applications of Definition 1.18,  $x \in X$ ,  $f(x) \in G$ , and  $f(x) \in H$ . It follows from the first condition by Definition 1.18 that  $f(x) \in f(X)$ . The facts that  $f(x) \in f(X)$  and  $f(x) \in G$  imply by Definitions 1.6 and 1.2 that  $f(x) \in A$ . Similarly,  $f(x) \in B$ . But these last two statements imply by Definition 1.6 that  $f(x) \in A \cap B$ , a contradiction.

To show that  $f^{-1}(G), f^{-1}(H)$  are nonempty, Definition 1.8 tells us that it will suffice to find an element of each set. As previously mentioned,  $A, B$  are nonempty. Thus, by consecutive applications of Definition 1.8, there exist  $f(x) \in A$  and  $f(y) \in B$ . Consequently, by Definitions 1.2 and 1.6,  $f(x) \in G$  and  $f(y) \in H$ . Therefore, by consecutive applications of Definition 1.18,  $x \in f^{-1}(G)$  and  $y \in f^{-1}(H)$ , as desired.

To show that  $f^{-1}(G)$  and  $f^{-1}(H)$  are open, Definition 9.4 tells us that it will suffice to verify (since  $f$  is continuous by hypothesis) that  $G, H$  are open subsets of  $\mathbb{R}$ . But by definition, they are exactly that.  $\square$

**Exercise 9.12.** Use Theorem 9.11 to prove that if  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, then for every point  $p$  between  $f(a)$  and  $f(b)$ , there exists  $c$  such that  $a < c < b$  and  $f(c) = p$ .

*Proof.* Suppose that  $a < b$ . Then by Lemma 8.3,  $[a, b]$  is an interval. Thus, by Theorem 8.15,  $[a, b]$  is connected. It follows by Theorem 9.11 that  $f([a, b])$  is connected. Consequently, by Theorem 8.15,  $f([a, b])$  is an interval. We divide into three cases ( $f(a) < f(b)$ ,  $f(a) > f(b)$ , and  $f(a) = f(b)$ ).

First, suppose that  $f(a) < f(b)$ , and let  $p$  be an arbitrary point between  $f(a)$  and  $f(b)$  (we know that at least one such point exists by Theorem 5.2). Then by Definition 3.6,  $f(a) < p < f(b)$ . Now  $a, b \in [a, b]$  by Equations 8.1, so by Definition 1.18,  $f(a), f(b) \in f([a, b])$ . It follows by Definition 8.2 since  $f([a, b])$  is an interval that  $[f(a), f(b)] \subset f([a, b])$ . Thus, since  $f(a) < p < f(b)$  implies  $p \in [f(a), f(b)]$  by Equations 8.1, Definition 1.3 asserts that  $p \in f([a, b])$ . Consequently, by Definition 1.18,  $p = f(c)$  for some  $c \in [a, b]$ . Additionally, since  $f(a) < p < f(b)$ , we know that  $p \neq f(a)$  and  $p \neq f(b)$ . It follows that  $p = f(c)$  for some  $c \in (a, b)$ , as desired.

The proof of the second case is symmetric to that of the first.

Third, suppose that  $f(a) = f(b)$ . This implies that there are no points  $p$  between  $f(a)$  and  $f(b)$  by Definition 3.6, so the statement is vacuously true in this case.  $\square$

**Lemma 9.13.** If  $f : (a, b) \rightarrow \mathbb{R}$  is continuous and injective, then  $f$  is either strictly increasing or strictly decreasing on  $(a, b)$ .

*Proof.* Suppose that  $a < b$ . Then by Corollary 5.3, we can pick two distinct points  $c, d \in (a, b)$ . Since  $f$  is injective, we have from the fact that  $x \neq y$  by Definition 1.20 that  $f(c) \neq f(d)$ . We divide into two cases ( $f(c) < f(d)$  and  $f(c) > f(d)$ ).

Suppose first that  $f(c) < f(d)$ . To prove that  $f$  is strictly increasing, Definition 8.16 tells us that it will suffice to show that for all  $x, y \in (a, b)$  with  $x < y$ ,  $f(x) < f(y)$ . Let  $x, y$  be arbitrary elements of  $(a, b)$  that satisfy  $x < y$ , and suppose for the sake of contradiction that  $f(x) \geq f(y)$ . If  $f(x) = f(y)$ , then by Definition 1.20,  $x = y$ , a contradiction. If  $f(x) > f(y)$ , then we divide into five cases ( $x < y < c < d$ ,  $x < c < y < d$ ,  $x < c < d < y$ ,  $c < x < d < y$ , and  $c < d < x < y$ ).

Let  $x < y < c < d$ . If  $f(x) > f(c)$ , let  $p_1$  be a point between  $f(x)$  and  $f(c)$  and let  $p_2$  be a point between  $f(d)$  and  $f(c)$ . Let  $p = \min(p_1, p_2)$ . Since  $f(c) < p < f(x)$ , Exercise 9.12 implies that there exists  $e \in (x, c)$  such that  $f(e) = p$ . Similarly, there exists  $e' \in (c, d)$  such that  $f(e') = p$ . Since  $(x, c)$  and  $(c, d)$  are clearly disjoint,  $e \neq e'$ . But by Definition 1.20,  $f(e) = p = f(e')$  implies that  $e = e'$ , a contradiction. If  $f(x) < f(c)$ , then we can arrive at a similar contradiction by considering values in the regions  $(x, y)$  and  $(y, c)$ .

The proof is symmetric in the second case if we consider values in the regions  $(c, y)$  and  $(y, d)$  when  $f(y)$  is not between  $f(c)$  and  $f(d)$ , and values in the regions  $(x, c)$  and  $(c, y)$  when  $f(y)$  is.

The proof is symmetric in the third case if we consider values in the regions  $(x, c)$  and  $(c, d)$  when  $f(x) > f(c)$ , and values in the regions  $(x, d)$  and  $(d, y)$  when  $f(x) < f(c)$ .

The proof is symmetric in the fourth case if we consider values in the regions  $(c, x)$  and  $(x, d)$  when  $f(x) > f(d)$ , and values in the regions  $(x, d)$  and  $(d, y)$  when  $f(x) < f(d)$ .

The proof is symmetric in the fifth case if we consider values in the regions  $(c, d)$  and  $(d, x)$  when  $f(d) > f(x)$ , and values in the regions  $(d, x)$  and  $(x, y)$  when  $f(d) < f(x)$ .  $\square$

2/23: **Theorem 9.14.** If  $f : (a, b) \rightarrow \mathbb{R}$  is continuous and injective, then the inverse function  $g : f((a, b)) \rightarrow (a, b)$  is continuous.

**Lemma.** Let  $f : (a, b) \rightarrow \mathbb{R}$  be continuous and injective, and let  $(x, y) \subset (a, b)$  be a region. Then  $f((x, y))$  is also a region.

*Proof.* Since  $f : (a, b) \rightarrow \mathbb{R}$  is continuous and injective, Lemma 9.13 implies that  $f$  is either strictly increasing or strictly decreasing on  $(a, b)$ . We now divide into two cases.

Suppose first that  $f$  is strictly increasing. To prove that  $f((x, y))$  is a region, Definition 3.10 tells us that it will suffice to show that  $f((x, y)) = (f(x), f(y))$ . To show this, Definition 1.2 tells us that it will suffice to verify that every  $p \in f((x, y))$  is an element of  $(f(x), f(y))$  and vice versa. Let  $p$  be an arbitrary element of  $f((x, y))$ . Then by Definition 1.18,  $p = f(z)$  for some  $z \in (x, y)$ . Since  $z \in (x, y)$ , we have by Equations 8.1 that  $x < z < y$ . Since  $f$  is strictly increasing on  $(a, b)$ , by Definition 8.16,  $x < z < y$  implies that  $f(x) < f(z) < f(y)$ . But this implies by Equations 8.1 that  $f(z) = p$  is an element of  $(f(x), f(y))$ , as desired. Now let  $p$  be an arbitrary element of  $(f(x), f(y))$ . Then by Equations 8.1,  $f(x) < p < f(y)$ . We seek to prove that  $[x, y] \subset (a, b)$ . Let  $q$  be an arbitrary element of  $[x, y]$ . Then by Equations 8.1,  $x \leq q \leq y$ . Additionally, since  $x, y \in (a, b)$ , Equations 8.1 imply that  $a < x < b$  and  $a < y < b$ . Thus,  $a < x \leq q \leq y < b$ , meaning by Equations 8.1 that  $q \in (a, b)$ . Consequently, by Definition 1.3,  $[x, y] \subset (a, b)$ . If we now consider the restriction  $f|_{[x, y]}$ , we have by Proposition 9.7 that  $f|_{[x, y]}$  is continuous. Thus, since  $f|_{[x, y]} : [x, y] \rightarrow \mathbb{R}$  is continuous and  $f|_{[x, y]}(x) = f(x) < p < f(y) = f|_{[x, y]}(y)$  (by Definition 9.6), Exercise 9.12 implies that there exists  $c \in (x, y)$  such that  $f|_{[x, y]}(c) = f(c) = p$ . But by Definition 1.18, this implies that  $p \in f((x, y))$ .

The proof is symmetric in the other case.  $\square$

*Proof of Theorem 9.14.* We first show that  $g$  exists, and then show that it is continuous.

To prove that  $g$  is a function, Definition 1.16 tells us that it will suffice to show that for all  $y \in f((a, b))$ , there exists a unique  $x \in (a, b)$  such that  $g(y) = x$ . We will first show that for each  $y$ , such an element exists, and then show that it is unique. Let  $y$  be an arbitrary element of  $f((a, b))$ . Then by Definition 1.18,  $y = f(x)$  for some  $x \in (a, b)$ . Thus, since we require that  $g(f(x')) = x'$  and  $f(g(y')) = y'$  for  $g$  to be an inverse function, we assign  $g(y) = x$ . Now suppose that  $g(y) = x_1$  and  $g(y) = x_2$ . Then by the definition of  $g$ ,  $f(x_1) = y$  and  $f(x_2) = y$ . It follows that  $f(x_1) = f(x_2)$ , implying since  $f$  is injective by Definition 1.20 that  $x_1 = x_2$ , as desired.

To prove that  $g$  is continuous, Definition 9.15 tells us that it will suffice to show that for every  $U \subset (a, b)$  that is open in  $(a, b)$ , the preimage  $g^{-1}(U)$  is open in  $f((a, b))$ . Let  $U$  be an arbitrary subset of  $(a, b)$  that is open in  $(a, b)$ . To show that  $g^{-1}(U)$  is open in  $f((a, b))$ , Definition 8.11 tells us that it will suffice to confirm that  $g^{-1}(U) = f((a, b)) \cap G$ , where  $G$  is an open set.

To begin, we have

$$g^{-1}(U) = \{y \in f((a, b)) \mid g(y) \in U\} \quad \text{Definition 1.18}$$

$$= \{f(x) \in f((a, b)) \mid g(f(x)) \in U\} \quad \text{Definition 1.18}$$

By the definition of  $g$ , we have  $g(f(x)) = x$ .

$$\begin{aligned} &= \{f(x) \in f((a, b)) \mid x \in U\} \\ &= \{f(x) \in \{f(x') \in \mathbb{R} \mid x' \in (a, b)\} \mid x \in U\} \end{aligned} \quad \text{Definition 1.18}$$

This next transition is mostly notational in nature.  $f(x)$  being an element of the set of all  $f(x') \in \mathbb{R}$  that meet a certain condition means that  $f(x) \in \mathbb{R}$ . Additionally, since that condition is  $x' \in (a, b)$ , we know that  $x \in (a, b)$ . But if  $x \in (a, b)$  and (from the condition in the original set)  $x \in U$ , we have by Definition 1.6 that  $x \in U \cap (a, b)$ .

$$\begin{aligned} &= \{f(x) \in \mathbb{R} \mid x \in U \cap (a, b)\} \\ &= f(U \cap (a, b)) \end{aligned} \quad \text{Definition 1.18}$$

By definition,  $U$  is open in  $(a, b)$ . Consequently, by Definition 8.11,  $U = (a, b) \cap V$  where  $V$  is open.

$$\begin{aligned} &= f(((a, b) \cap V) \cap (a, b)) \\ &= f((a, b) \cap V) \quad \text{Definition 1.6} \\ &= f((a, b)) \cap f(V) \quad \text{Additional Exercise 9.2b} \end{aligned}$$

All that's left at this point is to prove that  $f(V)$  is open. By Theorem 4.14,  $V = \bigcup_{\lambda \in I} \{R_\lambda\}$  is a collection of regions. It follows by an extension of Additional Exercise 9.2a that  $f(V) = \bigcup_{\lambda \in I} \{f(R_\lambda)\}$ . Additionally, by the lemma, each  $f(R_\lambda)$  is a region; hence, by Corollary 4.11, each  $f(R_\lambda)$  is open. Thus,  $f(V)$  is the union of a collection of open subsets of  $\mathbb{R}$ , so by Corollary 4.18,  $f(V)$  is open.  $\square$

We denote the inverse function  $g$  by  $f^{-1}$ . In this result,  $g$  has codomain  $(a, b)$  but our definition of continuity (Definition 9.4) only applies to functions with codomain  $\mathbb{R}$ . Our definitions/results are easily adapted. The definitions are as given below and we give a sample theorem. Other results can be adjusted in a similar fashion.

**Definition 9.15.** Let  $X, Y \subset \mathbb{R}$ . A function  $f : X \rightarrow Y$  is **continuous** if for every  $U$  that is open in  $Y$ , the preimage  $f^{-1}(U)$  is open in  $X$ .

**Definition 9.16.** The function  $f : X \rightarrow Y$  is **continuous** (at  $x \in X$ ) if for every region  $R$  containing  $f(x)$ , there exists an open set  $S$  containing  $x$  such that  $S \cap X \subset f^{-1}(R \cap Y)$ .

**Theorem 9.17.** *The function  $f : X \rightarrow Y$  is continuous if and only if it is continuous at every  $x \in X$ .*

## Additional Exercises

2. Let  $X \subset \mathbb{R}$  and let  $f : X \rightarrow \mathbb{R}$ . Let  $A, B \subset \mathbb{R}$ . Either prove or give a counterexample to each of the following:
  - a)  $f(A \cup B) = f(A) \cup f(B)$ .
  - b)  $f(A \cap B) = f(A) \cap f(B)$ .
  - c)  $f(A \setminus B) = f(A) \setminus f(B)$ .

## 9.2 Discussion

- 2/16:     • For Proposition 9.5, do an iff proof — use  $\iff$  steps instead of back-and-forth work.
- 2/18:     • To what extent do we need casework in Theorem 9.14?

## Script 10

# Compactness

### 10.1 Journal

2/23: **Definition 10.1.** We say that a function  $f : A \rightarrow \mathbb{R}$  is **bounded** if  $f(A)$  is a bounded subset of  $\mathbb{R}$ . We say that  $f$  is **bounded above** if  $f(A)$  is bounded above and that  $f$  is **bounded below** if  $f(A)$  is bounded below.

If  $f : A \rightarrow \mathbb{R}$  is bounded above, we say that  $f$  **attains** (its least upper bound) if there is some  $a \in A$  such that  $f(a) = \sup f(A)$ . Similarly, if  $f : A \rightarrow \mathbb{R}$  is bounded below, we say that  $f$  **attains** (its greatest lower bound) if there is some  $a \in A$  such that  $f(a) = \inf f(A)$ .

### 10.2 Discussion

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