Script 11

Limits and Continuity

11.1 Journal

3/4: Throughout this sheet, we let $f, g : A \to \mathbb{R}$ be real-valued functions with domain $A \subset \mathbb{R}$, unless otherwise specified.

Definition 11.1. Let $a \in LP(A) \subset \mathbb{R}$. A **limit** of f at a is a number $L \in \mathbb{R}$ satisfying the following condition: for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Lemma 11.2. *Limits are unique: if L and L' are both limits of f at a point a, then $L = L'$.*

Proof. Let the limit of f at a be L, and suppose for the sake of contradiction that the limit of f at a is also equal to L' where $L \neq L'$. Then by consecutive applications of Definition 11.1, we have that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$; and that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|f(x) - L'| < \epsilon$. If we let $\epsilon = \frac{|L-L'|}{2}$, then $\epsilon > 0$ by Script 8. Thus, choosing only x in the range prescribed by the δ corresponding to this ϵ, we have

$$\begin{aligned}
|L - L'| &= |L - f(x) + f(x) - L'| \\
&\leq |L - f(x)| + |f(x) - L'| && \text{Lemma 8.8} \\
&= |f(x) - L| + |f(x) - L'| && \text{Exercise 8.5} \\
&< 2\epsilon \\
&= |L - L'|
\end{aligned}$$

But $|L - L'| \not< |L - L'|$, so we have a contradiction. ☐

Definition 11.3. If L is the limit of f at a, we write

$$\lim_{x \to a} f(x) = L$$

Exercise 11.4. Give an example of a set $A \subset \mathbb{R}$, a function $f : A \to \mathbb{R}$, and a point $a \in LP(A)$ such that $\lim_{x \to a} f(x)$ does not exist.

Proof. Let $A = \mathbb{R}$, let $f : A \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

and consider $0 \in LP(\mathbb{R})$ (by Corollary 5.4). Now suppose for the sake of contradiction that $\lim_{x \to a} f(x) = L$. Then by Definitions 11.3[1] and 11.1, for all $\epsilon > 0$, there exists some $\delta > 0$ such that if $x \in \mathbb{R}$ and $0 < |x - 0| = |x| < \delta$, then $|f(x) - L| < \epsilon$. If we let $\epsilon = 0.5$, then $\epsilon > 0$. Choosing a corresponding δ, we

[1] I will not cite this definition again for the sake of concision.

have by an extension of Exercise 8.9 that all $x \in (-\delta, 0) \cup (0, \delta)$ satisfy $|f(x) - L| < \epsilon$. This would include objects $y \in (0, \delta)$ and $z \in (-\delta, 0)$. We have by the definition of f that $f(y) = 1$ and $f(z) = 0$; thus, we have

$$\begin{aligned} 1 &= |f(y) - f(z)| \\ &= |f(y) - L + L - f(z)| \\ &\leq |f(y) - L| + |f(z) - L| \\ &< 0.5 + 0.5 \\ &= 1 \end{aligned}$$

But $1 \not< 1$, so we have a contradiction. $\square$

Theorem 11.5. *Let $x \in A$. Then the following are equivalent:*

(*a*) *f is continuous at x.*

(*b*) *For every $\epsilon > 0$, there exists $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$.*

(*c*) *Either $x \notin LP(A)$ or $\lim_{y \to x} f(y) = f(x)$.*

Proof. To illustrate that statements a-c are equivalent, it will suffice to verify that $a \Rightarrow b$, $b \Rightarrow c$, and $c \Rightarrow a$. Note that this foregoes the need for explicit proofs of "backwards implications" such as $b \Rightarrow a$ since that implication, for example, follows from $b \Rightarrow c \Rightarrow a$. Let's begin.

To prove that $a \Rightarrow b$, let $\epsilon > 0$ be arbitrary and look to find a $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then $|f(y) - f(x)|$.

We first locate δ. To do so, begin by defining the region $R = (f(x) - \epsilon, f(x) + \epsilon)$ (clearly R contains $f(x)$). Since R is open by Corollary 4.11 and f is continuous at x, we have by Definition 9.9 that there exists an open set S with $x \in S$ such that $S \cap A \subset f^{-1}(R)$. It follows by Theorem 4.10 that there exists a region (a, b) such that $x \in (a, b)$ and $(a, b) \subset S$. Thus, since (a, b) is an open interval by Corollary 4.11 and Lemma 8.3, we have by Lemma 8.10 that there exists a number $\delta > 0$ such that $(x - \delta, x + \delta) \subset (a, b)$.

As we will now show, this δ satisfies the desired property. Let y be an arbitrary element of A such that $|y - x| < \delta$. Then by Exercise 8.9, $y \in (x - \delta, x + \delta)$. It follows by consecutive applications of Definition 1.3 that $y \in (a, b)$, hence $y \in S$. This result combined with the fact that $y \in A$ by definition implies by Definition 1.6 that $y \in S \cap A$. Thus, by Definition 1.3 again, $y \in f^{-1}(R)$. Consequently, by Definition 1.18, $f(y) \in R$. Therefore, by Exercise 8.9 one more time, $|f(y) - f(x)| < \epsilon$.

To prove that $b \Rightarrow c$, let x be an arbitrary element of $\mathbb{R}$. We divide into two cases ($x \notin LP(A)$ and $x \in LP(A)$). If $x \notin LP(A)$, then we are done. If $x \in LP(A)$, then by the hypothesis, we know that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in A$ and $0 < |y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. It follows by Definition 11.1 that $f(x)$ is the limit of f at x, meaning that $\lim_{y \to x} f(y) = f(x)$, and we are done.

To prove that $c \Rightarrow a$, we divide into two cases ($x \notin LP(A)$ and $\lim_{y \to x} f(y) = f(x)$).

Suppose first that $x \notin LP(A)$. To demonstrate that f is continuous at x, Definition 9.9 tells us that it will suffice to confirm that for every region R containing $f(x)$, there exists an open set S containing x such that $S \cap A \subset f^{-1}(R)$. Let R be an arbitrary region with $f(x) \in R$. Since $x \notin LP(A)$, Definition 3.13 asserts that there exists a region (hence an open set by Corollary 4.11) S such that $x \in S$ and $S \cap (A \setminus \{x\}) = \emptyset$. It follows by Script 1 that $S \cap A = \{x\}$. But since $f(x) \in R$ implies by Definition 1.18 that $x \in f^{-1}(R)$, we have by Definition 1.3 that $S \cap A \subset f^{-1}(R)$. Therefore, S is an open set containing x such that $S \cap A \subset f^{-1}(R)$.

Now suppose that $\lim_{y \to x} f(y) = f(x)$. To demonstrate that f is continuous at x, Definition 9.9 tells us that it will suffice to confirm that for every region (a, b) containing $f(x)$, there exists an open set S containing x such that $S \cap A \subset f^{-1}((a, b))$. Let (a, b) be an arbitrary region with $f(x) \in (a, b)$. Then since (a, b) is an open interval by Lemma 8.3, Lemma 8.10 asserts that there exists $\epsilon > 0$ such that $(f(x) - \epsilon, f(x) + \epsilon) \subset (a, b)$. With regard to this ϵ, since $\lim_{y \to x} f(y) = f(x)$ by hypothesis, we have by Definition 11.1 that there exists a $\delta > 0$ such that if $y \in A$ and $0 < |y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $S = (x - \delta, x + \delta)$. Clearly, S contains x. Additionally, we can confirm that $S \cap A \subset f^{-1}((a, b))$: if we let y be an arbitrary element of $S \cap A$, then Definition 1.6 asserts that $y \in S$ and $y \in A$. It follows from the former condition by Exercise 8.9 that $|y - x| < \delta$. This combined with the fact that $y \in A$ implies that $|f(y) - f(x)| < \epsilon$. Thus, by Exercise 8.9 again, $f(y) \in (f(x) - \epsilon, f(x) + \epsilon)$. Consequently, by Definition 1.3, $f(y) \in (a, b)$. As such, we have by Definition 1.18 that $y \in f^{-1}((a, b))$, as desired. $\square$

Exercise 11.6.

(a) Let $a, b \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = ax + b$. Show that f is continuous at every $x \in \mathbb{R}$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \begin{cases} 1 & x \neq 0 \\ 0 & x = 0 \end{cases}$. Show that f is not continuous at 0.

Proof of a. To prove that f is continuous at every $x \in \mathbb{R}$, let x be an arbitrary element of $\mathbb{R}$; then by Theorem 11.5, it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. We divide into two cases ($a = 0$ and $a \neq 0$). If $a = 0$, then choose $\delta = 1$[2]. This makes it so that for any $y \in A$ such that $|y - x| < \delta = 1$, we have $|f(y) - f(x)| = |b - b| = 0 < \epsilon$, as desired. If $a \neq 0$, then choose $\delta = \frac{\epsilon}{|a|}$. This makes it so that for any $y \in A$ such that $|y - x| < \delta = \frac{\epsilon}{|a|}$, we have

$$\begin{aligned} |a|\,|y - x| &< \epsilon \\ |ay - ax| &< \epsilon \\ |ay + b - (ax + b)| &< \epsilon \\ |f(y) - f(x)| &< \epsilon \end{aligned}$$

as desired. □

Proof of b. To prove that f is not continuous at 0, Theorem 11.5 tells us that it will suffice to show that for some $\epsilon > 0$, no $\delta > 0$ exists such that if $x \in \mathbb{R}$ and $|x - 0| = |x| < \delta$, then $|f(x) - 1| < \epsilon$. Let $\epsilon = 1$, and suppose for the sake of contradiction that $\delta > 0$ is a number such that if $x \in \mathbb{R}$ and $|x| < \delta$, then $|f(x) - 1| < \epsilon$. Clearly, $0 \in \mathbb{R}$ and by the definition of δ and Definition 8.4, $|0| < \delta$. However, $|f(x) - 1| = |0 - 1| = 1 \not< 1 = \epsilon$, a contradiction. □

3/9: **Exercise 11.7.** Show that the absolute value function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ is continuous.

Proof. To prove that the absolute value function is continuous, Theorem 9.10 tells us that it will suffice to show that it is continuous at every $x \in \mathbb{R}$. To do this, let x be an arbitrary element of $\mathbb{R}$; then by Theorem 11.5, it will suffice to verify that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in \mathbb{R}$ and $|y - x| < \delta$, then $||y| - |x|| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Then choose $\delta = \epsilon$. This makes it so that for any $y \in A$ such that $|y - x| < \delta$, we have $||y| - |x|| \leq |y - x| < \delta = \epsilon$ (with the first inequality coming from Lemma 8.8), as desired. □

Given real-valued functions f and g, we define new functions $f + g$, fg, and $\frac{1}{f}$ by

$$(f + g)(x) = f(x) + g(x) \qquad (fg)(x) = f(x) \cdot g(x) \qquad \frac{1}{f}(x) = \frac{1}{f(x)}$$

where $f(x) \neq 0$ in the definition of $\frac{1}{f}$. We wish to understand the limits of $f + g$, fg, and $\frac{1}{f}$ in terms of the limits of f and g.

Lemma 11.8. *If $\lim_{x \to a} f(x) = L > 0$, then there exists a region R with $a \in R$ such that $f(x) > 0$ for all $x \in R \cap A$ such that $x \neq a$. Moreover, if f is continuous at a, then $f(x) > 0$ for all $x \in R \cap A$. The analogous statement is true if $\lim_{x \to a} f(x) = L < 0$.*

Proof. We divide into two cases ($\lim_{x \to a} f(x) = L > 0$ and $\lim_{x \to a} f(x) = L < 0$).

Suppose first that $\lim_{x \to a} f(x) = L > 0$. Choose $\epsilon = L$. Then we have by Definition 11.1 that there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|f(x) - L| < L$. Let $R = (a - \delta, a + \delta)$. Clearly, $a \in R$. Now let x be an arbitrary element of $R \cap A$ such that $x \neq a$. It follows from the first condition by Definition 1.6 that $x \in R$ and $x \in A$. Following from the former condition again, we have by Exercise 8.9 that $|x - a| < \delta$. Additionally, since $x \neq a$, Definition 3.1 asserts that $x > a$ or $x < a$, i.e., $x - a > 0$ or $x - a < 0$; either way, Script 8 implies that $0 < |x - a|$. To recap, we know that $x \in A$ and $0 < |x - a| < \delta$,

[2] This choice is arbitrary; it can be any nonzero value, as we will soon see.

so we have by the initial implication that $|f(x) - L| < L = \epsilon$. Therefore, by the lemma from Exercise 8.9, we have $-L < f(x) - L < L$, i.e., $0 < f(x)$ (which we obtain by adding L to both sides of the inequality as permitted by Definition 7.21).

Moreover, if f is continuous at a, then by Theorem 11.5, $a \notin LP(A)$ or $\lim_{x \to a} f(x) = f(a)$. But by Definition 11.1, $a \in LP(A)$, so we have $\lim_{x \to a} f(x) = f(a)$. The first part of this proof guarantees the existence of a region R such that $f(x) > 0$ for all $x \in R \cap A$ such that $x \neq a$. The fact that $f(a) = L > 0$ takes care of the case where $x = a$.

The proof is symmetric in the other case. $\square$