Script 9

Continuous Functions

9.1 Journal

2/16: **Lemma 9.1.** *Let $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$. If $A, B \subset \mathbb{R}$, then*

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$
$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$
$$f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$$
$$f^{-1}(\mathbb{R}) = X$$

Proof. To prove that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, Definition 1.2 tells us that it will suffice to show that every $x \in f^{-1}(A \cup B)$ is an element of $f^{-1}(A) \cup f^{-1}(B)$ and vice versa. Suppose first that x is an arbitrary element of $f^{-1}(A \cup B)$. Then by Definition 1.18, $f(x) \in A \cup B$. Thus, by Definition 1.5, $f(x) \in A$ or $f(x) \in B$. We now divide into two cases. If $f(x) \in A$, then by Definition 1.18, $x \in f^{-1}(A)$. It follows by Definition 1.5 that $x \in f^{-1}(A) \cup f^{-1}(B)$, as desired. The argument is symmetric in the other case. Now suppose that $x \in f^{-1}(A) \cup f^{-1}(B)$. Then by Definition 1.5, $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. We now divide into two cases. If $x \in f^{-1}(A)$, then by Definition 1.18, $f(x) \in A$. It follows by Definition 1.5 that $f(x) \in A \cup B$. Therefore, by Definition 1.18, $x \in f^{-1}(A \cup B)$. The argument is symmetric in the other case, as desired.

To prove that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$, Definition 1.2 tells us that it will suffice to show that every $x \in f^{-1}(A \cap B)$ is an element of $f^{-1}(A) \cap f^{-1}(B)$ and vice versa. Suppose first that x is an arbitrary element of $f^{-1}(A \cap B)$. Then by Definition 1.18, $f(x) \in A \cap B$. Thus, by Definition 1.6, $f(x) \in A$ and $f(x) \in B$. It follows by consecutive applications of Definition 1.18 that $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$. Therefore, by Definition 1.6, $x \in f^{-1}(A) \cap f^{-1}(B)$, as desired. Now suppose that $x \in f^{-1}(A) \cap f^{-1}(B)$. Then by Definition 1.6, $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$. It follows by consecutive applications of Definition 1.18 that $f(x) \in A$ and $f(x) \in B$. Thus, by Definition 1.6, $f(x) \in A \cap B$. Therefore, by Definition 1.18, $x \in f^{-1}(A \cap B)$, as desired.

To prove that $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$, Definition 1.2 tells us that it will suffice to show that every $x \in f^{-1}(A \setminus B)$ is an element of $f^{-1}(A) \setminus f^{-1}(B)$ and vice versa. Suppose first that x is an arbitrary element of $f^{-1}(A \setminus B)$. Then by Definition 1.18, $f(x) \in A \setminus B$. Thus, by Definition 1.11, $f(x) \in A$ and $f(x) \notin B$. It follows by consecutive applications of Definition 1.18 that $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$. Therefore, by Definition 1.11, $x \in f^{-1}(A) \setminus f^{-1}(B)$, as desired. Now suppose that $x \in f^{-1}(A) \setminus f^{-1}(B)$. Then by Definition 1.11, $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$. It follows by consecutive applications of Definition 1.18 that $f(x) \in A$ and $f(x) \notin B$. Thus, by Definition 1.11, $f(x) \in A \setminus B$. Therefore, by Definition 1.18, $x \in f^{-1}(A \setminus B)$, as desired.

To prove that $f^{-1}(\mathbb{R}) = X$, Definition 1.2 tells us that it will suffice to show that every $x \in f^{-1}(\mathbb{R})$ is an element of X and vice versa. Suppose first that x is an arbitrary element of $f^{-1}(\mathbb{R})$. Then by Definition 1.18, $x \in X$, as desired. Now suppose that $x \in X$. Then by Definition 1.16, $f(x) \in \mathbb{R}$. It follows by Definition 1.18 that $x \in f^{-1}(\mathbb{R})$, as desired. ∎

Exercise 9.2. Let $f : X \to \mathbb{R}$. Let $A \subset X$ and $B \subset \mathbb{R}$. Show that

$$f(f^{-1}(B)) \subset B$$
$$A \subset f^{-1}(f(A))$$

Give examples to show that the inclusions can be proper.

Proof. To prove that $f(f^{-1}(B)) \subset B$, Definition 1.3 tells us that it will suffice to show that every $y \in f(f^{-1}(B))$ is an element of B. Let y be an arbitrary element of $f(f^{-1}(B))$. Then by Definition 1.18, $y = f(x)$ for some $x \in f^{-1}(B)$. By Definition 1.18 again, $f(x) \in B$. Therefore, since $y = f(x)$, it follows that $y \in B$, as desired.

To prove that $A \subset f^{-1}(f(A))$, Definition 1.3 tells us that it will suffice to show that every $x \in A$ is an element of $f^{-1}(f(A))$. Let x be an arbitrary element of A. Then by Definition 1.18, $f(x) \in f(A)$. Therefore, by Definition 1.18, we have $x \in f^{-1}(f(A))$, as desired.

Let $X = \{1, 2\}$ and let $f : X \to \mathbb{R}$ be defined by $f(1) = 3$ and $f(2) = 3$. If we let $B = \{3, 4\}$, then $f(f^{-1}(B)) = \{3\} \subsetneq \{3, 4\}$. Additionally, if we let $A = \{1\}$, then $A \subsetneq f^{-1}(f(A)) = \{1, 2\}$. □

Exercise 9.3. Let $f : X \to \mathbb{R}$. Let $A \subset X$ and $B \subset \mathbb{R}$. Then $f(A) \subset B \iff A \subset f^{-1}(B)$.

Proof. Suppose first that $f(A) \subset B$. To prove that $A \subset f^{-1}(B)$, Definition 1.3 tells us that it will suffice to show that every $x \in A$ is an element of $f^{-1}(B)$. Let x be an arbitrary element of A. Then by Definition 1.18, $f(x) \in f(A)$. It follows by the hypothesis and Definition 1.3 that $f(x) \in B$. Therefore, by Definition 1.18 again, $x \in f^{-1}(B)$.

Now suppose that $A \subset f^{-1}(B)$. To prove that $f(A) \subset B$, Definition 1.3 tells us that it will suffice to show that every $y \in f(A)$ is an element of B. Let y be an arbitrary element of $f(A)$. Then by Definition 1.18, $y = f(x)$ for some $x \in A$. It follows by the hypothesis and Definition 1.3 that $x \in f^{-1}(B)$. Consequently, by Definition 1.18 again, $f(x) \in B$. Therefore, since $y = f(x)$, $y \in B$. □

Definition 9.4. Let $X \subset \mathbb{R}$. A function $f : X \to \mathbb{R}$ is **continuous** if for every open set $U \subset \mathbb{R}$, the preimage $f^{-1}(U)$ is open in X.

Proposition 9.5. *Let $X \subset \mathbb{R}$. A function $f : X \to \mathbb{R}$ is continuous if and only if for every closed set $F \subset \mathbb{R}$, the preimage $f^{-1}(F)$ is closed in X.*

Proof. Suppose first that f is continuous. We seek to prove that for every closed set $F \subset \mathbb{R}$, the preimage $f^{-1}(F)$ is closed in X. Let F be an arbitrary closed subset of $\mathbb{R}$. Then by Definition 4.8, $F = \mathbb{R} \setminus U$ for some open set $U \subset \mathbb{R}$. It follows by Definition 9.4 since f is continuous that $f^{-1}(U)$ is open in X. Additionally, by consecutive applications of Lemma 9.1, $f^{-1}(F) = f^{-1}(\mathbb{R} \setminus U) = f^{-1}(\mathbb{R}) \setminus f^{-1}(U) = X \setminus f^{-1}(U)$. Therefore, since $f^{-1}(U)$ is open in X, Exercise 8.13 implies that $X \setminus f^{-1}(U) = f^{-1}(F)$ is closed in X.

The proof is symmetric in the other direction. □

Definition 9.6. Let $X \subset Y \subset \mathbb{R}$ and let $f : Y \to \mathbb{R}$. Then the **restriction** (of f to X), written $f|_X$ is the function $f|_X : X \to \mathbb{R}$ defined by

$$f|_X(x) = f(x)$$

for all $x \in X$.

Proposition 9.7. *Let $X \subset Y \subset \mathbb{R}$. If $f : Y \to \mathbb{R}$ is continuous, then the restriction of f to X is continuous.*

Proof. To prove that $f|_X$ is continuous, Definition 9.4 tells us that it will suffice to show that for every open set $U \subset \mathbb{R}$, the preimage $f|_X^{-1}(U)$ is open in X. Let U be an open subset of $\mathbb{R}$. Then

$$\begin{aligned}
f|_X^{-1}(U) &= \{x \in X \mid f|_X(x) \in U\} && \text{Definition 1.18}\\
&= \{x \in X \mid f(x) \in U\} && \text{Definition 9.6}\\
&= \{x \in Y \mid f(x) \in U\} \cap X && \text{Script 1}\\
&= f^{-1}(U) \cap X && \text{Definition 1.18}\\
&= (Y \cap G) \cap X && \text{Definitions 9.4 and 8.11}
\end{aligned}$$

$$= X \cap G \qquad \text{Script 1}$$

Since $f|_X^{-1}(U) = X \cap G$ where G is an open set, Definition 8.11 asserts that $f|_X^{-1}(U)$ is open in X. $\square$

Exercise 9.8. Show that for any $X \subsetneq \mathbb{R}$ that is not open and any continuous function $f : X \to \mathbb{R}$, there is an open set U for which $f^{-1}(U)$ is open in X but is not open in $\mathbb{R}$.

Proof. We will prove that $\mathbb{R}$ is an open set for which $f^{-1}(\mathbb{R})$ is open in X but not in $\mathbb{R}$. First, by Theorem 5.1, $\mathbb{R}$ is open. Next, by Lemma 9.1, $f^{-1}(\mathbb{R}) = X$. It follows since $f^{-1}(\mathbb{R}) = X = X \cap \mathbb{R}$ (where $\mathbb{R}$ is an open set) by Definition 8.11 that $f^{-1}(\mathbb{R})$ is open in X. Last, since X is not open (in $\mathbb{R}$) by definition, $f^{-1}(\mathbb{R}) = X$ is not open in $\mathbb{R}$. $\square$

Definition 9.9. The function $f : X \to \mathbb{R}$ is **continuous** (at $x \in X$) if for every region R containing $f(x)$, there exists an open set S containing x such that $S \cap X \subset f^{-1}(R)$.

Theorem 9.10. *The function $f : X \to \mathbb{R}$ is continuous if and only if it is continuous at every $x \in X$.*

Proof. Suppose first that f is continuous, and suppose for the sake of contradiction that f is not continuous at every $x \in X$. Then by Definition 9.9, there exists some $x \in X$ such that f is not continuous at x. Thus, there exists a region R with $f(x) \in R$ such that for all open sets S containing x, $S \cap X \not\subset f^{-1}(R)$. Since f is continuous by hypothesis and R is open by Corollary 4.11, $f^{-1}(R)$ is open in X. It follows by Definition 8.11 that $f^{-1}(R) = X \cap S$ for some open set S. But this implies that $f^{-1}(R) \not\subset f^{-1}(R)$, a contradiction.

Now suppose that f is continuous at every $x \in X$. To prove that f is continuous, Definition 9.4 tells us that it will suffice to show that for every open set $U \subset \mathbb{R}$, the preimage $f^{-1}(U)$ is open in X. We divide into two cases ($f^{-1}(U) = \emptyset$ and $f^{-1}(U) \neq \emptyset$). If $f^{-1}(U) = \emptyset$, then since $\emptyset \cap X = \emptyset$ by Script 1 where $\emptyset$ is open by Theorem 5.1, Definition 8.11 tells us that $\emptyset = f^{-1}(U)$ is open in X, as desired. On the other hand, if $f^{-1}(U) \neq \emptyset$, Definition 8.11 tells us that it will suffice to show that $f^{-1}(U) = S \cap X$ where S is an open set. We first seek to show that for every $x \in f^{-1}(U)$, there exists an open set S_x containing x such that $S_x \cap X \subset f^{-1}(U)$. Let x be an arbitrary element of $f^{-1}(U)$. It follows by Definition 1.18 that $f(x) \in U$. Thus, since U is open, we have by Theorem 4.10 that there exists a region R such that $f(x) \in R$ and $R \subset U$. Consequently, since R is open by Corollary 4.11, we have by Definition 9.9 that there exists an open set S_x containing x such that $S_x \cap X \subset f^{-1}(R)$. Additionally, Script 1 tells us based off of the fact that $R \subset U$ that $f^{-1}(R) \subset f^{-1}(U)$. Thus, by subset transitivity, $S_x \cap X \subset f^{-1}(U)$. At this point, let $S = \bigcup_{x \in f^{-1}(U)} S_x$. It follows immediately from Corollary 4.18 that S is open. Additionally, since the intersection of each set in the union with X is a subset of $f^{-1}(U)$, it follows by Script 1 that $S \cap X \subset f^{-1}(U)$. Furthermore, for all $x \in f^{-1}(U)$, Definition 1.18 asserts that $x \in X$. In addition, we have defined an S_x such that $x \in S_x$. These last two results combined demonstrate by Definition 1.6 that $x \in S \cap X$. Thus, by Definition 1.3, $f^{-1}(U) \subset S \cap X$. Consequently, by Theorem 1.7, $f^{-1}(U) = S \cap X$. Since $f^{-1}(U)$ is the intersection of X with an open set, Definition 8.11 asserts that it is open in X, as desired. $\square$

2/18: **Theorem 9.11.** *Suppose that $f : X \to \mathbb{R}$ is continuous. If X is connected, then $f(X)$ is connected.*

Proof. This will be a proof by contrapositive; as such, suppose that $f(X)$ is disconnected. Then by Definition 4.22, $f(X) = A \cup B$ where A, B are nonempty, disjoint sets that are open in $f(X)$. It follows from the last condition by Definition 8.11 that $A = G \cap f(X)$ and $B = H \cap f(X)$, where G, H are open sets. Since for all $x \in X$, $f(x) \in A$ or $f(x) \in B$, Definitions 1.2 and 1.6 imply that for all $x \in X$, $f(x) \in G$ or $f(x) \in H$. Thus, by Script 1, $X \subset f^{-1}(G) \cup f^{-1}(H)$. Additionally, we know by Definition 1.18 that for all $x \in f^{-1}(G) \cup f^{-1}(H)$, $x \in X$. Thus, by Definition 1.3, $f^{-1}(G) \cup f^{-1}(H) \subset X$. Consequently, by Theorem 1.7, we have that $X = f^{-1}(G) \cup f^{-1}(H)$.

To show that $f^{-1}(G)$ and $f^{-1}(H)$ are disjoint, Definition 1.9 tells us that it will suffice to verify that $f^{-1}(G) \cap f^{-1}(H) = \emptyset$. As such, suppose for the sake of contradiction that $x \in f^{-1}(G) \cap f^{-1}(H)$. Then by Definition 1.6, $x \in f^{-1}(G)$ and $x \in f^{-1}(H)$. Thus, by multiple applications of Definition 1.18, $x \in X$, $f(x) \in G$, and $f(x) \in H$. It follows from the first condition by Definition 1.18 that $f(x) \in f(X)$. The facts that $f(x) \in f(X)$ and $f(x) \in G$ imply by Definitions 1.6 and 1.2 that $f(x) \in A$. Similarly, $f(x) \in B$. But these last two statements imply by Definition 1.6 that $f(x) \in A \cap B$, a contradiction.

To show that $f^{-1}(G)$ and $f^{-1}(H)$ are nonempty, Definition 1.8 tells us that it will suffice to find an element of each set. As previously mentioned, A, B are nonempty. Thus, by consecutive applications of Definition 1.8, there exist $f(x) \in A$ and $f(y) \in B$. Consequently, by Definitions 1.2 and 1.6, $f(x) \in G$ and $f(y) \in H$. Therefore, by consecutive applications of Definition 1.18, $x \in f^{-1}(G)$ and $y \in f^{-1}(H)$, as desired.

To show that $f^{-1}(G)$ and $f^{-1}(H)$ are open in X, Definition 9.4 tells us that it will suffice to verify (since f is continuous by hypothesis) that G, H are open subsets of $\mathbb{R}$. But by definition, they are exactly that. □

Exercise 9.12. Use Theorem 9.11 to prove that if $f : [a,b] \to \mathbb{R}$ is continuous, then for every point p between $f(a)$ and $f(b)$, there exists c such that $a < c < b$ and $f(c) = p$.

Proof. Suppose that $a < b$. Then by Lemma 8.3, $[a,b]$ is an interval. Thus, by Theorem 8.15, $[a,b]$ is connected. It follows by Theorem 9.11 that $f([a,b])$ is connected. Consequently, by Theorem 8.15, $f([a,b])$ is an interval. We divide into three cases ($f(a) < f(b)$, $f(a) > f(b)$, and $f(a) = f(b)$).

First, suppose that $f(a) < f(b)$, and let p be an arbitrary point between $f(a)$ and $f(b)$ (we know that at least one such point exists by Theorem 5.2). Then by Definition 3.6, $f(a) < p < f(b)$. Now $a, b \in [a,b]$ by Equations 8.1, so by Definition 1.18, $f(a), f(b) \in f([a,b])$. It follows by Definition 8.2 since $f([a,b])$ is an interval that $[f(a), f(b)] \subset f([a,b])$. Thus, since $f(a) < p < f(b)$ implies $p \in [f(a), f(b)]$ by Equations 8.1, Definition 1.3 asserts that $p \in f([a,b])$. Consequently, by Definition 1.18, $p = f(c)$ for some $c \in [a,b]$. Additionally, since $f(a) < p < f(b)$, we know that $p \neq f(a)$ and $p \neq f(b)$. It follows that $p = f(c)$ for some $c \in (a,b)$, as desired.

The proof of the second case is symmetric to that of the first.

Third, suppose that $f(a) = f(b)$. This implies that there are no points p between $f(a)$ and $f(b)$ by Definition 3.6, so the statement is vacuously true in this case. □

Lemma 9.13. *If $f : (a,b) \to \mathbb{R}$ is continuous and injective, then f is either strictly increasing or strictly decreasing on (a,b).*

Proof. Suppose for the sake of contradiction that f is neither strictly increasing nor strictly decreasing. Then by Definition 8.16 there exist $x, y, z \in (a,b)$ with $x < y < z$ such that $f(x) \leq f(y)$ and $f(z) \leq f(y)$. Additionally, since f is injective and x, y, z are distinct, we have by Definition 1.20 that $f(x) < f(y)$ and $f(z) < f(y)$.

We divide into two cases ($f(x) < f(z)$ and $f(x) > f(z)$). Suppose first that $f(x) < f(z)$. Since $x, y \in (a,b)$, we have that $[x,y] \subset (a,b)$. It follows by Proposition 9.7 that $f|_{[x,y]} : [x,y] \to \mathbb{R}$ is continuous. Thus, we have by the supposition and Definition 9.6 that $f|_{[x,y]}(x) < f|_{[x,y]}(z) < f|_{[x,y]}(y)$. It follows by Exercise 9.12 that there exists c with $x < c < y$ such that $f|_{[x,y]}(c) = f|_{[x,y]}(z)$. Consequently, by Definition 9.6, $f(c) = f(z)$. Thus, by Definition 1.20, $c = z$. But this implies that $x < z < y$, contradicting the fact that $x < y < z$. The proof is symmetric in the other case. □

2/23: **Theorem 9.14.** *If $f : (a,b) \to \mathbb{R}$ is continuous and injective, then the inverse function $g : f((a,b)) \to (a,b)$ is continuous.*

Lemma. *Let $f : (a,b) \to \mathbb{R}$ be continuous and injective, and let $(x,y) \subset (a,b)$ be a region. Then $f((x,y))$ is also a region.*

Proof. Since $f : (a,b) \to \mathbb{R}$ is continuous and injective, Lemma 9.13 implies that f is either strictly increasing or strictly decreasing on (a,b). We now divide into two cases.

Suppose first that f is strictly increasing. To prove that $f((x,y))$ is a region, Definition 3.10 tells us that it will suffice to show that $f((x,y)) = (f(x), f(y))$. To show this, Definition 1.2 tells us that it will suffice to verify that every $p \in f((x,y))$ is an element of $(f(x), f(y))$ and vice versa. Let p be an arbitrary element of $f((x,y))$. Then by Definition 1.18, $p = f(z)$ for some $z \in (x,y)$. Since $z \in (x,y)$, we have by Equations 8.1 that $x < z < y$. Since f is strictly increasing on (a,b), by Definition 8.16, $x < z < y$ implies that $f(x) < f(z) < f(y)$. But this implies by Equations 8.1 that $f(z) = p$ is an element of $(f(x), f(y))$, as desired. Now let p be an arbitrary element of $(f(x), f(y))$. Then by Equations 8.1, $f(x) < p < f(y)$. We seek to prove that $[x,y] \subset (a,b)$. Let q be an arbitrary element of $[x,y]$. Then by Equations 8.1, $x \leq q \leq y$. Additionally, since $x, y \in (a,b)$, Equations 8.1 imply that $a < x < b$ and $a < y < b$. Thus, $a < x \leq q \leq y < b$, meaning by Equations 8.1 that $q \in (a,b)$. Consequently, by Definition 1.3, $[x,y] \subset (a,b)$. If we now consider the restriction $f|_{[x,y]}$, we have by Proposition 9.7 that $f|_{[x,y]}$ is continuous. Thus, since $f|_{[x,y]} : [x,y] \to \mathbb{R}$ is continuous and $f|_{[x,y]}(x) = f(x) < p < f(y) = f|_{[x,y]}(y)$ (by Definition 9.6), Exercise 9.12 implies that there exists $c \in (x,y)$ such that $f|_{[x,y]}(c) = f(c) = p$. But by Definition 1.18, this implies that $p \in f((x,y))$.

The proof is symmetric in the other case. □

Proof of Theorem 9.14. We first show that g exists, and then show that it is continuous.

To prove that g is a function, Definition 1.16 tells us that it will suffice to show that for all $y \in f((a,b))$, there exists a unique $x \in (a,b)$ such that $g(y) = x$. We will first show that for each y, such an element exists, and then show that it is unique. Let y be an arbitrary element of $f((a,b))$. Then by Definition 1.18, $y = f(x)$ for some $x \in (a,b)$. Thus, since we require that $g(f(x')) = x'$ and $f(g(y')) = y'$ for g to be an inverse function, we assign $g(y) = x$. Now suppose that $g(y) = x_1$ and $g(y) = x_2$. Then by the definition of g, $f(x_1) = y$ and $f(x_2) = y$. It follows that $f(x_1) = f(x_2)$, implying since f is injective by Definition 1.20 that $x_1 = x_2$, as desired.

To prove that g is continuous, Definition 9.15 tells us that it will suffice to show that for every $U \subset (a,b)$ that is open in (a,b), the preimage $g^{-1}(U)$ is open in $f((a,b))$. Let U be an arbitrary subset of (a,b) that is open in (a,b). To show that $g^{-1}(U)$ is open in $f((a,b))$, Definition 8.11 tells us that it will suffice to confirm that $g^{-1}(U) = f((a,b)) \cap G$, where G is an open set.

To begin, we have

$$\begin{aligned} g^{-1}(U) &= \{y \in f((a,b)) \mid g(y) \in U\} && \text{Definition 1.18} \\ &= \{f(x) \in f((a,b)) \mid g(f(x)) \in U\} && \text{Definition 1.18} \end{aligned}$$

By the definition of g, we have $g(f(x)) = x$.

$$\begin{aligned} &= \{f(x) \in f((a,b)) \mid x \in U\} \\ &= \{f(x) \in \{f(x') \in \mathbb{R} \mid x' \in (a,b)\} \mid x \in U\} && \text{Definition 1.18} \end{aligned}$$

This next transition is mostly notational in nature. $f(x)$ being an element of the set of all $f(x') \in \mathbb{R}$ that meet a certain condition means that $f(x) \in \mathbb{R}$. Additionally, since that condition is $x' \in (a,b)$, we know that $x \in (a,b)$. But if $x \in (a,b)$ and (from the condition in the original set) $x \in U$, we have by Definition 1.6 that $x \in U \cap (a,b)$.

$$\begin{aligned} &= \{f(x) \in \mathbb{R} \mid x \in U \cap (a,b)\} \\ &= f(U \cap (a,b)) && \text{Definition 1.18} \end{aligned}$$

By definition, U is open in (a,b). Consequently, by Definition 8.11, $U = (a,b) \cap V$ where V is open.

$$\begin{aligned} &= f(((a,b) \cap V) \cap (a,b)) \\ &= f((a,b) \cap V) && \text{Definition 1.6} \\ &= f((a,b)) \cap f(V) && \text{Additional Exercise 9.2b} \end{aligned}$$

All that's left at this point is to prove that $f(V)$ is open. By Theorem 4.14, $V = \bigcup_{\lambda \in I}\{R_\lambda\}$ is a collection of regions. It follows by an extension of Additional Exercise 9.2a that $f(V) = \bigcup_{\lambda \in I}\{f(R_\lambda)\}$. Additionally, by the lemma, each $f(R_\lambda)$ is a region; hence, by Corollary 4.11, each $f(R_\lambda)$ is open. Thus, $f(V)$ is the union of a collection of open subsets of $\mathbb{R}$, so by Corollary 4.18, $f(V)$ is open. □

We denote the inverse function g by f^{-1}. In this result, g has codomain (a,b) but our definition of continuity (Definition 9.4) only applies to functions with codomain $\mathbb{R}$. Our definitions/results are easily adapted. The definitions are as given below and we give a sample theorem. Other results can be adjusted in a similar fashion.

Definition 9.15. Let $X, Y \subset \mathbb{R}$. A function $f : X \to Y$ is **continuous** if for every U that is open in Y, the preimage $f^{-1}(U)$ is open in X.

Definition 9.16. The function $f : X \to Y$ is **continuous** (at $x \in X$) if for every region R containing $f(x)$, there exists an open set S containing x such that $S \cap X \subset f^{-1}(R \cap Y)$.

Theorem 9.17. *The function $f : X \to Y$ is continuous if and only if it is continuous at every $x \in X$.*

Additional Exercises

2. Let $X \subset \mathbb{R}$ and let $f : X \to \mathbb{R}$. Let $A, B \subset \mathbb{R}$. Either prove or give a counterexample to each of the following:

   a) $f(A \cup B) = f(A) \cup f(B)$.

   b) $f(A \cap B) = f(A) \cap f(B)$.

   c) $f(A \setminus B) = f(A) \setminus f(B)$.