Script 8

Intervals

8.1 Journal

2/9: Now that we have constructed $\mathbb{R}$ and proved the fundamental facts about it, we will work with the real numbers $\mathbb{R}$ instead of an arbitrary continuum C. We will leave behind Dedekind cuts and think of elements of $\mathbb{R}$ as numbers. Accordingly, from now on, we will use lower-case letters like x for real numbers and will write $+$ and $\cdot$ for $\oplus$ and $\otimes$. We will also now use the standard notation (a,b) for the region $\underline{ab} = \{x \in \mathbb{R} \mid a < x < b\}$. Even though the notation is the same, this is *not* the same object as the ordered pair (a,b).

More generally, we adopt the following standard notation:

$$\begin{aligned}
(a,b) &= \{x \in \mathbb{R} \mid a < x < b\} \\
[a,b) &= \{x \in \mathbb{R} \mid a \leq x < b\} \\
(a,b] &= \{x \in \mathbb{R} \mid a < x \leq b\} \\
[a,b] &= \{x \in \mathbb{R} \mid a \leq x \leq b\} \\
(a,\infty) &= \{x \in \mathbb{R} \mid a < x\} \\
[a,\infty) &= \{x \in \mathbb{R} \mid a \leq x\} \\
(-\infty,b) &= \{x \in \mathbb{R} \mid x < b\} \\
(-\infty,b] &= \{x \in \mathbb{R} \mid x \leq b\}
\end{aligned} \tag{8.1}$$

Exercise 8.1. Identify the sets in Equations 8.1 that are open/closed/neither.

Proof. By Corollary 4.11, (a,b) is open.

By an adaptation of Corollary 5.14, $b \in LP([a,b))$ but $b \notin [a,b)$. Since $[a,b)$ doesn't contain all of its limit points, Definition 4.1 tells us that it is not closed. Additionally, since $a \in LP(C \setminus [a,b))$ but $a \notin C \setminus [a,b)$, Definition 4.8 tells us that it is not open. Therefore, it is neither.

The proof that $(a,b]$ is neither is symmetric to the previous case.

By Corollaries 5.15 and 4.7, $[a,b]$ is closed.

By Corollary 4.13, (a,∞) is open.

By Corollary 4.13 and Definition 4.8, $[a,\infty) = C \setminus (-\infty,a)$ is closed.

The proofs that $(-\infty,b)$ and $(-\infty,b]$ are open and closed, respectively, are symmetric to the previous two cases, respectively. ☐

Definition 8.2. A set $I \subset \mathbb{R}$ is an **interval** if for all $x,y \in I$ with $x < y$, $[x,y] \subset I$.

Lemma 8.3. *A proper subset $I \subsetneq \mathbb{R}$ is an interval if and only if it takes one of the eight forms in Equations 8.1.*

Proof. Suppose first that $I \subsetneq \mathbb{R}$ is an interval. If $I = \emptyset$, then $I = (a,a)$ for any $a \in \mathbb{R}$, and we are done. Thus, we will assume for the remainder of the proof of the forward direction that I is nonempty. To address this case, we will prove that the facts that $I \subsetneq \mathbb{R}$, $I \neq \emptyset$, and I is an interval imply that I is bounded above, bounded below, or both. Then in each of these three cases, we will look at whether $\sup I$ and $\inf I$ (if they exist) are elements of the set or not to differentiate between the eight forms in Equations 8.1. Let's begin.

Suppose for the sake of contradiction that there exists a nonempty interval $I \subsetneq \mathbb{R}$ that is neither bounded above nor bounded below. Since $I \subsetneq \mathbb{R}$, we have by Definition 1.3 that there exists a point $p \in \mathbb{R}$ such that $p \notin I$. Additionally, since I is neither bounded above nor below, Definition 5.6 implies that p is neither an upper nor a lower bound on I. Thus, there exist $x, y \in I$ such that $x < p$ and $y > p$. Now by Definition 8.2, $[x, y] \subset I$. But it follows by Definition 1.3 that every point in $[x, y]$, including p, is an element of I, a contradiction.

We now divide into three cases (I is only bounded below, I is only bounded above, and I is bounded below and above).

First, suppose that I is only bounded below. Since I is a nonempty subset of $\mathbb{R}$ that is bounded below, we have by Theorem 5.17 that $\inf I$ exists. We divide into two cases again ($\inf I \in I$ and $\inf I \notin I$).

If $\inf I \in I$, then we can demonstrate that $I = [\inf I, \infty)$. To do this, Definition 1.2 tells us that it will suffice to verify that every $p \in I$ is an element of $[\inf I, \infty)$ and vice versa. Let p be an arbitrary element of I. Then $p \in \mathbb{R}$ and by Definitions 5.7 and 5.6, $\inf I \leq p$. Therefore, $p \in [\inf I, \infty)$, as desired. Now let p be an arbitrary element of $[\inf I, \infty)$. Then $\inf I \leq p$. Additionally, since I is not bounded above, we have by Definition 5.6 that there exists $y \in I$ such that $y > p$. Since $\inf I \in I$, $y \in I$, and $\inf I < y$ (by transitivity), $[\inf I, y] \subset I$ by Definition 8.2. This combined with the fact that $p \in [\inf I, y]$ (we know that $\inf I \leq p < y$, so $\inf I \leq p \leq y$) implies that $p \in I$, as desired.

If $\inf I \notin I$, then we can demonstrate that $I = (\inf I, \infty)$. As before, to do so, it will suffice to verify that every $p \in I$ is an element of $(\inf I, \infty)$ and vice versa. Let p be an arbitrary element of I. Then $p \in \mathbb{R}$ and by Definitions 5.7 and 5.6, $\inf I \leq p$. The additional constraint that $\inf I \notin I$ implies that $\inf I < p$. Therefore, $p \in (\inf I, \infty)$, as desired. Now let p be an arbitrary element of $(\inf I, \infty)$. Then $\inf I < p$. It follows by Lemma 5.11 that there exists a $z \in I$ such that $\inf I \leq z < p$. Additionally, since I is not bounded above, we have by Definition 5.6 that there exists $y \in I$ such that $y > p$. Since $z \in I$, $y \in I$, and $z < y$ (by transitivity), $[z, y] \subset I$ by Definition 8.2. This combined with the fact that $p \in [z, y]$ (we know that $z < p < y$, so $z \leq p \leq y$) implies that $p \in I$, as desired.

Second, suppose that I is only bounded above. The proof of this case is symmetric to that of the first.

Third, suppose that I is bounded below and above. Since I is a nonempty subset of $\mathbb{R}$ that is bounded above and below, we have by consecutive applications of Theorem 5.17 that both $\sup I$ and $\inf I$ exist. We divide into four cases ($\inf I \in I$ and $\sup I \in I$, $\inf I \in I$ and $\sup I \notin I$, $\inf I \notin I$ and $\sup I \in I$, and $\inf I \notin I$ and $\sup I \notin I$).

If $\inf I \in I$ and $\sup I \in I$, then we can demonstrate that $I = [\inf I, \sup I]$. We now divide into two cases again ($\inf I = \sup I$ and $\inf I \neq \sup I$). If $\inf I = \sup I \in I$, then $I = \{\inf I\} = \{\sup I\} = [\inf I, \sup I]$, as desired. On the other hand, if $\inf I \neq \sup I$, we continue. To demonstrate that $I = [\inf I, \sup I]$, Theorem 1.7 tells us that it will suffice to verify that $I \subset [\inf I, \sup I]$ and $[\inf I, \sup I] \subset I$. To verify that the former claim, Definition 1.3 tells us that it will suffice to confirm that every $p \in I$ is an element of $[\inf I, \sup I]$. Let p be an arbitrary element of I. Then $p \in \mathbb{R}$ and by consecutive applications of Definitions 5.7 and 5.6, $\inf I \leq p$ and $p \leq \sup I$. Therefore, $p \in [\inf I, \sup I]$, as desired. On the other hand, since $\inf I \in I$, $\sup I \in I$, and $\inf I < \sup I$ (as follows from Definition 5.7 and the fact that they are unequal), $[\inf I, \sup I] \subset I$ by Definition 8.2, as desired.

If $\inf I \in I$ and $\sup I \notin I$, then we can demonstrate that $I = [\inf I, \sup I)$. To do so, it will suffice to verify that every $p \in I$ is an element of $[\inf I, \sup I)$ and vice versa. Let p be an arbitrary element of I. Then $p \in \mathbb{R}$ and by Definitions 5.7 and 5.6, $\inf I \leq p$ and $p \leq \sup I$. The additional constraint that $\sup I \notin I$ implies that $p < \sup I$. Therefore, $p \in [\inf I, \sup I)$, as desired. Now let p be an arbitrary element of $[\inf I, \sup I)$. Then $\inf I \leq p < \sup I$. It follows by Lemma 5.11 that there exists a $y \in I$ such that $p < y \leq \sup I$. Since $\inf I \in I$, $y \in I$, and $\inf I < y$ (by transitivity), $[\inf I, y] \subset I$ by Definition 8.2. This combined with the fact that $p \in [\inf I, y]$ (we know that $\inf I \leq p < y$, so $\inf I \leq p \leq y$) implies that $p \in I$, as desired.

If $\inf I \notin I$ and $\sup I \in I$, the proof is symmetric to that of the previous case.

If $\inf I \notin I$ and $\sup I \notin I$, then we can demonstrate that $I = (\inf I, \sup I)$. To do so, it will suffice to verify that every $p \in I$ is an element of $(\inf I, \sup I)$ and vice versa. Let p be an arbitrary element of I. Then $p \in \mathbb{R}$ and by Definitions 5.7 and 5.6, $\inf I \leq p$ and $p \leq \sup I$. The additional constraints that $\inf I \notin I$ and

$\sup I \notin I$ imply that $\inf I < p$ and $p < \sup I$, respectively. Therefore, $p \in (\inf I, \sup I)$, as desired. Now let p be an arbitrary element of $(\inf I, \sup I)$. Then $\inf I < p < \sup I$. It follows by consecutive applications of Lemma 5.11 that there exist $x, y \in I$ such that $\inf I \leq x < p$ and $p < y \leq \sup I$. Since $x \in I$, $y \in I$, and $x < y$ (by transitivity), $[x, y] \subset I$ by Definition 8.2. This combined with the fact that $p \in [x, y]$ (we know that $x < p < y$, so $x \leq p \leq y$) implies that $p \in I$, as desired.

Now suppose that $I \subsetneq \mathbb{R}$ takes one of the eight forms in Equations 8.1. To prove that I is an interval, Definition 8.2 tells us that it will suffice to show that for all $x, y \in I$ with $x < y$, $[x, y] \subset I$. Let x, y be arbitrary elements of I with $x < y$. We divide into eight cases (one for each equation in Equations 8.1).

First, suppose that $I = (a, b)$. To demonstrate that $[x, y] \subset I$, Definition 1.3 tells us that it will suffice to confirm that every $z \in [x, y]$ is an element of I. Let z be an arbitrary element of $[x, y]$. Then by Corollary 5.15, $x \leq z \leq y$. But since $a < x < y < b$ by Equations 8.1, the fact that $a < x \leq z \leq y < b$ implies by Equations 8.1 that $z \in (a, b)$, as desired.

Second, suppose that $I = [a, b)$. To demonstrate that $[x, y] \subset I$, Definition 1.3 tells us that it will suffice to confirm that every $z \in [x, y]$ is an element of I. Let z be an arbitrary element of $[x, y]$. Then by Corollary 5.15, $x \leq z \leq y$. But since $a \leq x < y < b$ by Equations 8.1, the fact that $a \leq x \leq z \leq y < b$ implies by Equations 8.1 that $z \in [a, b)$, as desired.

Third, suppose that $I = (a, b]$. To demonstrate that $[x, y] \subset I$, Definition 1.3 tells us that it will suffice to confirm that every $z \in [x, y]$ is an element of I. Let z be an arbitrary element of $[x, y]$. Then by Corollary 5.15, $x \leq z \leq y$. But since $a < x < y \leq b$ by Equations 8.1, the fact that $a < x \leq z \leq y \leq b$ implies by Equations 8.1 that $z \in (a, b]$, as desired.

Fourth, suppose that $I = [a, b]$. To demonstrate that $[x, y] \subset I$, Definition 1.3 tells us that it will suffice to confirm that every $z \in [x, y]$ is an element of I. Let z be an arbitrary element of $[x, y]$. Then by Corollary 5.15, $x \leq z \leq y$. But since $a \leq x < y \leq b$ by Equations 8.1, the fact that $a \leq x \leq z \leq y \leq b$ implies by Equations 8.1 that $z \in [a, b]$, as desired.

Fifth, suppose that $I = (a, \infty)$. To demonstrate that $[x, y] \subset I$, Definition 1.3 tells us that it will suffice to confirm that every $z \in [x, y]$ is an element of I. Let z be an arbitrary element of $[x, y]$. Then by Corollary 5.15, $x \leq z \leq y$. But since $a < x$ by Equations 8.1, the fact that $a < x \leq z$ implies by Equations 8.1 that $z \in (a, \infty)$, as desired.

Sixth, suppose that $I = [a, \infty)$. To demonstrate that $[x, y] \subset I$, Definition 1.3 tells us that it will suffice to confirm that every $z \in [x, y]$ is an element of I. Let z be an arbitrary element of $[x, y]$. Then by Corollary 5.15, $x \leq z \leq y$. But since $a \leq x$ by Equations 8.1, the fact that $a \leq x \leq z$ implies by Equations 8.1 that $z \in [a, \infty)$, as desired.

Seventh, suppose that $I = (-\infty, b)$. To demonstrate that $[x, y] \subset I$, Definition 1.3 tells us that it will suffice to confirm that every $z \in [x, y]$ is an element of I. Let z be an arbitrary element of $[x, y]$. Then by Corollary 5.15, $x \leq z \leq y$. But since $y < b$ by Equations 8.1, the fact that $z \leq y < b$ implies by Equations 8.1 that $z \in (-\infty, b)$, as desired.

Eighth, suppose that $I = (-\infty, b]$. To demonstrate that $[x, y] \subset I$, Definition 1.3 tells us that it will suffice to confirm that every $z \in [x, y]$ is an element of I. Let z be an arbitrary element of $[x, y]$. Then by Corollary 5.15, $x \leq z \leq y$. But since $y \leq b$ by Equations 8.1, the fact that $z \leq y \leq b$ implies by Equations 8.1 that $z \in (-\infty, b]$, as desired. □

Definition 8.4. The **absolute value** of a real number x is the non-negative number $|x|$ defined by

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Exercise 8.5. Show that $|x| = |-x|$ for all $x \in \mathbb{R}$. (Note that this also means that $|x - y| = |y - x|$ for any $x, y \in \mathbb{R}$.)

Proof. Let x be an arbitrary element of $\mathbb{R}$. We divide into two cases ($x \geq 0$ and $x < 0$). Suppose first that $x \geq 0$. Then by Lemma 7.23[1] $-x \leq 0$. Thus, by consecutive applications of Definition 8.4, $|x| = x$ and $|-x| = -(-x)$. Therefore, since $-(-x) = x$ by Corollary 7.11, $|x| = x = |-x|$, as desired. □

[1]And, technically, Theorem 7.47.

Definition 8.6. The **distance** between $x \in \mathbb{R}$ and $y \in \mathbb{R}$ is defined to be $|x - y|$.

Remark 8.7. It follows from Definition 8.6 that $|x|$ is the distance between x and 0.

Lemma 8.8. *For any real numbers x, y, z, we have*

(*a*) $|x + y| \leq |x| + |y|$.

(*b*) $|x - z| \leq |x - y| + |y - z|$.

(*c*) $||x| - |y|| \leq |x - y|$.

Proof of a. We divide into four cases ($x \geq 0$ and $y \geq 0$, $x \geq 0$ and $y < 0$, $x < 0$ and $y \geq 0$, and $x < 0$ and $y < 0$).

First, suppose that $x \geq 0$ and $y \geq 0$. Then by Definition 7.21, $x+y \geq 0$. Thus, by consecutive applications of Definition 8.4, $|x + y| = x + y$, $|x| = x$, and $|y| = y$. Therefore, $|x + y| = x + y \leq x + y = |x| + |y|$, as desired.

Second, suppose that $x \geq 0$ and $y < 0$. By Definition 8.4, $|x| = x$ and $|y| = -y$. We now divide into two cases ($x + y \geq 0$ and $x + y < 0$). If $x + y \geq 0$, then $|x + y| = x + y$. Additionally, since $y > 0$, Lemma 7.23 implies that $0 < -y$. Consequently, by transitivity, $y < -y = |y|$. It follows by Definition 7.21 that $x + y < x + |y|$. Therefore, $|x + y| = x + y < x + |y| = |x| + |y|$, so $|x + y| \leq |x| + |y|$, as desired. On the other hand, if $x + y < 0$, then $|x + y| = -(x + y) = -x + (-y) = -x + |y|$. Additionally, by Lemma 7.23, $x \geq 0$ implies that $-x \leq 0$. It follows by Definition 7.21 since $-x \leq x$ that $-x + |y| \leq x + |y|$. Therefore, $|x + y| = -x + |y| \leq x + |y| = |x| + |y|$, as desired.

The proof of the third case is symmetric to that of the second.

The proof of the fourth case is symmetric to that of the first. □

Proof of b. By part (a), $|x - z| = |x - y + y - z| \leq |x - y| + |y - z|$, as desired. □

Proof of c. To prove that $||x| - |y|| \leq |x - y|$, Definition 8.4 tells us that it will suffice to show that $|x| - |y| \leq |x - y|$ and $-(|x| - |y|) \leq |x - y$. By part (a), $|x| = |x - y + y| \leq |x - y| + |y|$, so $|x| - |y| \leq |x - y|$. Similarly, $|y| - |x| \leq |x - y|$, so $-(|x| - |y|) \leq |x - y|$, as desired. □