

# Script 6

## Construction of the Real Numbers

### 6.1 Journal

1/12: **Definition 6.1.** A subset  $A$  of  $\mathbb{Q}$  is said to be a **cut** (or **Dedekind cut**) if it satisfies the following:

- (a)  $A \neq \emptyset$  and  $A \neq \mathbb{Q}$ .
- (b) If  $r \in A$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in A$ .
- (c)  $A$  does not have a last point; i.e., if  $r \in A$ , then there is some  $s \in A$  with  $s > r$ .

We denote the collection of all cuts by  $\mathbb{R}$ .

**Lemma 6.2.** Let  $A$  be a Dedekind cut and  $x \in \mathbb{Q}$ . Then  $x \notin A$  if and only if  $x$  is an upper bound for  $A$ .

*Proof.* Suppose first that  $x \notin A$ . To prove that  $x$  is an upper bound for  $A$ , Definition 5.6 tells us that it will suffice to show that for all  $r \in A$ ,  $r \leq x$ . Let  $r$  be an arbitrary element of  $A$ . Then by the contrapositive of Definition 6.1b and the hypothesis that  $x \notin A$ , we know that  $r \notin A$ ,  $x \notin \mathbb{Q}$ , or  $x \not\leq r$ . But since  $r \in A$  and  $x \in \mathbb{Q}$ , it must be that  $x \not\leq r$ . Therefore,  $r \leq x$ , as desired.

Now suppose that  $x$  is an upper bound for  $A$ . By Definition 5.6, this implies that for all  $r \in A$ ,  $r \leq x$ . Therefore, since there is no  $r \in A$  with  $r > x$ , by the contrapositive of Definition 6.1c,  $x \notin A$ , as desired.  $\square$

#### Exercise 6.3.

- (a) Prove that for any  $q \in \mathbb{Q}$ ,  $\{x \in \mathbb{Q} \mid x < q\}$  is a Dedekind cut. We then define  $\mathbf{0} = \{x \in \mathbb{Q} \mid x < 0\}$ .
- (b) Prove that  $\{x \in \mathbb{Q} \mid x \leq 0\}$  is not a Dedekind cut.
- (c) Prove that  $\{x \in \mathbb{Q} \mid x < 0\} \cup \{x \in \mathbb{Q} \mid x^2 < 2\}$  is a Dedekind cut.

*Proof of a.* Let  $q$  be an arbitrary element of  $\mathbb{Q}$ . To prove that  $A = \{x \in \mathbb{Q} \mid x < q\}$  is a Dedekind cut, Definition 6.1 tells us that it will suffice to show that  $A \neq \emptyset$ ;  $A \neq \mathbb{Q}$ ; if  $r \in A$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in A$ ; and if  $r \in A$ , then there is some  $s \in A$  with  $s > r$ . We will take this one claim at a time.

To show that  $A \neq \emptyset$ , Definition 1.8 tells us that it will suffice to find an element of  $A$ . By Exercise 3.9d,  $q$  is not the first point of  $\mathbb{Q}$ . Thus, by Definition 3.3, there exists an object  $x \in \mathbb{Q}$  such that  $x < q$ . By the definition of  $A$ , this implies that  $x \in A$ , as desired.

To show that  $A \neq \mathbb{Q}$ , Definition 1.2 tells us that it will suffice to find an element of  $\mathbb{Q}$  that is not an element of  $A$ . By hypothesis,  $q \in \mathbb{Q}$ . By Exercise 3.9d,  $q \not< q$ . Therefore,  $q \in \mathbb{Q}$  but  $q \notin A$ , as desired.

To show that if  $r \in A$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in A$ , we let  $r \in A$  and  $s \in \mathbb{Q}$  be arbitrary elements of their respective sets that satisfy  $s < r$  and seek to verify that  $s \in A$ . Since  $r \in A$ ,  $r < q$ . This combined with the fact that  $s < r$  implies by transitivity that  $s < q$ . Therefore, since  $s \in \mathbb{Q}$  and  $s < q$ ,  $s \in A$ , as desired.

To show that if  $r \in A$ , then there is some  $s \in A$  with  $s > r$ , we let  $r \in A$  and seek to find such an  $s$ . By the definition of  $A$ ,  $r < q$ . Thus, by Additional Exercise 3.1, there exists a point  $s \in \mathbb{Q}$  such that  $r < s < q$ . Since  $s \in \mathbb{Q}$  and  $s < q$ ,  $s \in A$ . It follows that  $s$  is the desired element of  $A$  satisfying  $s > r$ .  $\square$

*Proof of b.* To prove that  $A = \{x \in \mathbb{Q} \mid x \leq 0\}$  is not a Dedekind cut, Definition 6.1 tells us that it will suffice to show that  $A$  does have a last point. To show this, we will demonstrate that 0 is the last point of  $A$ . To demonstrate this, Definition 3.1 tells us that it will suffice to confirm that  $0 \in A$  and for all  $x \in A$ ,  $x \leq 0$ . Since  $0 \leq 0$  and  $0 \in \mathbb{Q}$ ,  $0 \in A$ . Additionally, by the definition of  $A$ , it is true that for all  $x \in A$ ,  $x \leq 0$ .  $\square$

*Proof of c.* Let  $B = \{x \in \mathbb{Q} \mid x < 0\}$  and let  $C = \{x \in \mathbb{Q} \mid x^2 < 2\}$ . To prove that  $A = B \cup C$  is a Dedekind cut, Definition 6.1 tells us that it will suffice to show that  $A \neq \emptyset$ ;  $A \neq \mathbb{Q}$ ; if  $r \in A$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in A$ ; and if  $r \in A$ , then there is some  $s \in A$  with  $s > r$ . We will take this one claim at a time.

To show that  $A \neq \emptyset$ , Definition 1.8 tells us that it will suffice to find an element of  $A$ . Since  $-1 \in \mathbb{Q}$  and  $-1 < 0$ ,  $-1 \in B$ . Therefore, by Definition 1.5,  $-1 \in A$ , as desired.

To show that  $A \neq \mathbb{Q}$ , Definition 1.2 tells us that it will suffice to find an element of  $\mathbb{Q}$  that is not an element of  $A$ . Since  $2 \in \mathbb{Q}$  and  $2 \geq 0$ ,  $2 \notin B$ . Additionally, since  $2^2 \geq 2$ ,  $2 \notin C$ . Therefore, by Definition 1.5,  $2 \notin A$ , as desired.

To show that if  $r \in A$  and  $s \in \mathbb{Q}$  satisfy  $s < r$ , then  $s \in A$ , we let  $r \in A$  and  $s \in \mathbb{Q}$  be arbitrary elements of their respective sets that satisfy  $s < r$  and seek to verify that  $s \in A$ . Since  $r \in A$ , Definition 1.5 tells us that  $r \in B$  or  $r \in C$ . We now divide into two cases. Suppose first that  $r \in B$ . Then  $s < r < 0$ , which implies that  $s \in B$ , meaning that  $s \in A$ . Now suppose that  $r \in C$ . We divide into two cases again ( $r \leq 0$  and  $r > 0$ ). If  $r \leq 0$ , then  $s < r \leq 0$  implies that  $s < 0$ . Thus, by the definition of  $B$ ,  $s \in B$ , implying that  $s \in A$ . On the other hand, if  $r > 0$ , then  $0 < s^2 < r^2 < 2$ . Thus, by the definition of  $C$ ,  $s \in C$ , implying that  $s \in A$ .

To show that  $A$  does not have a last point, suppose for the sake of contradiction that  $A$  has a last point  $p$ . We now divide into two cases ( $p \leq 0$  and  $p > 0$ ). Suppose first that  $p \leq 0$ . Since  $p$  is the last point of  $A$ , Definition 3.3 tells us that  $x \leq p$  for all  $x \in A$ . But  $1 \in A$  (since  $1 \in \mathbb{Q}$  and  $1^2 = 1 < 2$  implies  $1 \in B$ , implies  $1 \in A$ ) and  $1 > 0 \geq p$ , a contradiction. Now suppose that  $p > 0$ . Definition 3.3 tells us that  $p \in A$ , but the condition that  $p > 0$  means  $p \notin B$ , so we must have  $p \in C$ . However, by the proof of Exercise 4.24,  $\frac{2(p+1)}{p+2}$  will be an element of  $B$  (and therefore  $A$ ) that is greater than  $p$  no matter how large  $p$  is, a contradiction.  $\square$