# Script 9

# Continuous Functions

## 9.1  Journal

**Lemma 9.1.** *Let $X \subset \mathbb{R}$ and $f : X \to \mathbb{R}$. If $A, B \subset \mathbb{R}$, then*

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$
$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$
$$f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$$
$$f^{-1}(\mathbb{R}) = X$$

*Proof.* To prove that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, Definition 1.2 tells us that it will suffice to show that every $x \in f^{-1}(A \cup B)$ is an element of $f^{-1}(A) \cup f^{-1}(B)$ and vice versa. Suppose first that $x$ is an arbitrary element of $f^{-1}(A \cup B)$. Then by Definition 1.18, $f(x) \in A \cup B$. Thus, by Definition 1.5, $f(x) \in A$ or $f(x) \in B$. We now divide into two cases. If $f(x) \in A$, then by Definition 1.18, $x \in f^{-1}(A)$. It follows by Definition 1.5 that $x \in f^{-1}(A) \cup f^{-1}(B)$, as desired. The argument is symmetric in the other case. Now suppose that $x \in f^{-1}(A) \cup f^{-1}(B)$. Then by Definition 1.5, $x \in f^{-1}(A)$ or $x \in f^{-1}(B)$. We now divide into two cases. If $x \in f^{-1}(A)$, then by Definition 1.18, $f(x) \in A$. It follows by Definition 1.5 that $f(x) \in A \cup B$. Therefore, by Definition 1.18, $x \in f^{-1}(A \cup B)$. The argument is symmetric in the other case, as desired.

To prove that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$, Definition 1.2 tells us that it will suffice to show that every $x \in f^{-1}(A \cap B)$ is an element of $f^{-1}(A) \cap f^{-1}(B)$ and vice versa. Suppose first that $x$ is an arbitrary element of $f^{-1}(A \cap B)$. Then by Definition 1.18, $f(x) \in A \cap B$. Thus, by Definition 1.6, $f(x) \in A$ and $f(x) \in B$. It follows by consecutive applications of Definition 1.18 that $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$. Therefore, by Definition 1.6, $x \in f^{-1}(A) \cap f^{-1}(B)$, as desired. Now suppose that $x \in f^{-1}(A) \cap f^{-1}(B)$. Then by Definition 1.6, $x \in f^{-1}(A)$ and $x \in f^{-1}(B)$. It follows by consecutive applications of Definition 1.18 that $f(x) \in A$ and $f(x) \in B$. Thus, by Definition 1.6, $f(x) \in A \cap B$. Therefore, by Definition 1.18, $x \in f^{-1}(A \cap B)$, as desired.

To prove that $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$, Definition 1.2 tells us that it will suffice to show that every $x \in f^{-1}(A \setminus B)$ is an element of $f^{-1}(A) \setminus f^{-1}(B)$ and vice versa. Suppose first that $x$ is an arbitrary element of $f^{-1}(A \setminus B)$. Then by Definition 1.18, $f(x) \in A \setminus B$. Thus, by Definition 1.11, $f(x) \in A$ and $f(x) \notin B$. It follows by consecutive applications of Definition 1.18 that $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$. Therefore, by Definition 1.11, $x \in f^{-1}(A) \setminus f^{-1}(B)$, as desired. Now suppose that $x \in f^{-1}(A) \setminus f^{-1}(B)$. Then by Definition 1.11, $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$. It follows by consecutive applications of Definition 1.18 that $f(x) \in A$ and $f(x) \notin B$. Thus, by Definition 1.11, $f(x) \in A \setminus B$. Therefore, by Definition 1.18, $x \in f^{-1}(A \setminus B)$, as desired.

To prove that $f^{-1}(\mathbb{R}) = X$, Definition 1.2 tells us that it will suffice to show that every $x \in f^{-1}(\mathbb{R})$ is an element of $X$ and vice versa. Suppose first that $x$ is an arbitrary element of $f^{-1}(\mathbb{R})$. Then by Definition 1.18, $x \in X$, as desired. Now suppose that $x \in X$. Then by Definition 1.16, $f(x) \in \mathbb{R}$. It follows by Definition 1.18 that $x \in f^{-1}(\mathbb{R})$, as desired. $\qquad\square$

**Exercise 9.2.** Let $f : X \to \mathbb{R}$. Let $A \subset X$ and $B \subset \mathbb{R}$. Show that

$$f(f^{-1}(B)) \subset B$$
$$A \subset f^{-1}(f(A))$$

Give examples to show that the inclusions can be proper.

*Proof.* To prove that $f(f^{-1}(B)) \subset B$, Definition 1.3 tells us that it will suffice to show that every $y \in f(f^{-1}(B))$ is an element of $B$. Let $y$ be an arbitrary element of $f(f^{-1}(B))$. Then by Definition 1.18, $y = f(x)$ for some $x \in f^{-1}(B)$. By Definition 1.18 again, $f(x) \in B$. Therefore, since $y = f(x)$, it follows that $y \in B$, as desired.

To prove that $A \subset f^{-1}(f(A))$, Definition 1.3 tells us that it will suffice to show that every $x \in A$ is an element of $f^{-1}(f(A))$. Let $x$ be an arbitrary element of $A$. Then by Definition 1.18, $f(x) \in f(A)$. Therefore, by Definition 1.18, we have $x \in f^{-1}(f(A))$, as desired.

Let $X = \{1, 2\}$ and let $f : X \to \mathbb{R}$ be defined by $f(1) = 3$ and $f(2) = 3$. If we let $B = \{3, 4\}$, then $f(f^{-1}(B)) = \{3\} \subsetneq \{3, 4\}$. Additionally, if we let $A = \{1\}$, then $A \subsetneq f^{-1}(f(A)) = \{1, 2\}$. □

**Exercise 9.3.** Let $f : X \to \mathbb{R}$. Let $A \subset X$ and $B \subset \mathbb{R}$. Then $f(A) \subset B \Longleftrightarrow A \subset f^{-1}(B)$.

*Proof.* Suppose first that $f(A) \subset B$. To prove that $A \subset f^{-1}(B)$, Definition 1.3 tells us that it will suffice to show that every $x \in A$ is an element of $f^{-1}(B)$. Let $x$ be an arbitrary element of $A$. Then by Definition 1.18, $f(x) \in f(A)$. It follows by the hypothesis and Definition 1.3 that $f(x) \in B$. Therefore, by Definition 1.18 again, $x \in f^{-1}(B)$.

Now suppose that $A \subset f^{-1}(B)$. To prove that $f(A) \subset B$, Definition 1.3 tells us that it will suffice to show that every $y \in f(A)$ is an element of $B$. Let $y$ be an arbitrary element of $f(A)$. Then by Definition 1.18, $y = f(x)$ for some $x \in A$. It follows by the hypothesis and Definition 1.3 that $x \in f^{-1}(B)$. Therefore, by Definition 1.18 again, $y = f(x) \in B$. □

**Definition 9.4.** Let $X \subset \mathbb{R}$. A function $f : X \to \mathbb{R}$ is **continuous** if for every open set $U \subset \mathbb{R}$, the preimage $f^{-1}(U)$ is open in $X$.

**Proposition 9.5.** *Let $X \subset \mathbb{R}$. A function $f : X \to \mathbb{R}$ is continuous if and only if for every closed set $F \subset \mathbb{R}$, the preimage $f^{-1}(F)$ is closed in $X$.*

*Proof.* Suppose first that $f$ is continuous. We seek to prove that for every closed set $F \subset \mathbb{R}$, the preimage $f^{-1}(F)$ is closed in $X$. Let $F$ be an arbitrary closed subset of $\mathbb{R}$. Then by Definition 4.8, $F = \mathbb{R} \setminus U$ for some open set $U \subset \mathbb{R}$. It follows by Definition 9.4 since $f$ is continuous that $f^{-1}(U)$ is open in $X$. Additionally, by consecutive applications of Lemma 9.1, $f^{-1}(F) = f^{-1}(\mathbb{R} \setminus U) = f^{-1}(\mathbb{R}) \setminus f^{-1}(U) = X \setminus f^{-1}(U)$. Therefore, since $f^{-1}(U)$ is open in $X$, Exercise 8.13 implies that $X \setminus f^{-1}(U) = f^{-1}(F)$ is closed in $X$.

The proof is symmetric in the other direction. □

**Definition 9.6.** Let $X \subset Y \subset \mathbb{R}$ and let $f : Y \to \mathbb{R}$. Then the **restriction** (of $f$ to $X$), written $f|_X$ is the function $f|_X : X \to \mathbb{R}$ defined by

$$f|_X(x) = f(x)$$

for all $x \in X$.

**Proposition 9.7.** *Let $X \subset Y \subset \mathbb{R}$. If $f : Y \to \mathbb{R}$ is continuous, then the restriction of $f$ to $X$ is continuous.*

*Proof.* To prove that $f|_X$ is continuous, Definition 9.4 tells us that it will suffice to show that for every open set $U \subset \mathbb{R}$, the preimage $f|_X^{-1}(U)$ is open in $X$. Let $U$ be an open subset of $\mathbb{R}$. Then

$$
\begin{aligned}
f|_X^{-1}(U) &= \{x \in X \mid f|_X(x) \in U\} && \text{Definition 1.18} \\
&= \{x \in X \mid f(x) \in U\} && \text{Definition 9.6} \\
&= \{x \in Y \mid f(x) \in U\} \cap X && \text{Script 1} \\
&= f^{-1}(U) \cap X && \text{Definition 1.18} \\
&= (Y \cap G) \cap X && \text{Definitions 9.4 and 8.11}
\end{aligned}
$$

$$= X \cap G \hspace{4cm} \text{Script 1}$$

Since $f|_X^{-1}(U) = X \cap G$ where $G$ is an open set, Definition 8.11 asserts that $f|_X^{-1}(U)$ is open in $X$. $\hspace{0.5cm}\square$

**Exercise 9.8.** Show that for any $X \subsetneq \mathbb{R}$ that is not open and any continuous function $f : X \to \mathbb{R}$, there is an open set $U$ for which $f^{-1}(U)$ is open in $X$ but is not open in $\mathbb{R}$.

*Proof.* We will prove that $\mathbb{R}$ is an open set for which $f^{-1}(\mathbb{R})$ is open in $X$ but not in $\mathbb{R}$. First, by Theorem 5.1, $\mathbb{R}$ is open. Next, by Lemma 9.1, $f^{-1}(\mathbb{R}) = X$. It follows since $f^{-1}(\mathbb{R}) = X = X \cap \mathbb{R}$ (where $\mathbb{R}$ is an open set) by Definition 8.11 that $f^{-1}(\mathbb{R})$ is open in $X$. Last, since $X$ is not open (in $\mathbb{R}$) by definition, $f^{-1}(\mathbb{R}) = X$ is not open in $\mathbb{R}$. $\hspace{0.5cm}\square$

**Definition 9.9.** The function $f : X \to \mathbb{R}$ is **continuous** (at $x \in X$) if for every region $R$ containing $f(x)$, there exists an open set $S$ containing $x$ such that $S \cap X \subset f^{-1}(R)$.

**Theorem 9.10.** *The function $f : X \to \mathbb{R}$ is continuous if and only if it is continuous at every $x \in X$.*

*Proof.* Suppose first that $f$ is continuous, and suppose for the sake of contradiction that $f$ is not continuous at every $x \in X$. Then by Definition 9.9, there exists some $x \in X$ such that $f$ is not continuous at $x$. Thus, there exists a region $R$ with $f(x) \in R$ such that for all open sets $S$ containing $x$, $S \cap X \not\subset f^{-1}(R)$. Since $f$ is continuous by hypothesis and $R$ is open by Corollary 4.11, $f^{-1}(R)$ is open in $X$. It follows by Definition 8.11 that $f^{-1}(R) = X \cap S$ for some open set $S$. But this implies that $f^{-1}(R) \not\subset f^{-1}(R)$, a contradiction.

Now suppose that $f$ is continuous at every $x \in X$. To prove that $f$ is continuous, Definition 9.4 tells us that it will suffice to show that for every open set $U \subset \mathbb{R}$, the preimage $f^{-1}(U)$ is open in $X$. We divide into two cases ($f^{-1}(U) = \emptyset$ and $f^{-1}(U) \neq \emptyset$). If $f^{-1}(U) = \emptyset$, then since $\emptyset \cap X = \emptyset$ by Script 1 where $\emptyset$ is open by Theorem 5.1, Definition 8.11 tells us that $\emptyset = f^{-1}(U)$ is open in $X$, as desired. On the other hand, if $f^{-1}(U) \neq \emptyset$, Definition 8.11 tells us that it will suffice to show that $f^{-1}(U) = S \cap X$ where $S$ is an open set. We first seek to show that for every $x \in f^{-1}(U)$, there exists an open set $S_x$ containing $x$ such that $S_x \cap X \subset f^{-1}(U)$. Let $x$ be an arbitrary element of $f^{-1}(U)$. It follows by Definition 1.18 that $f(x) \in U$. Thus, since $U$ is open, we have by Theorem 4.10 that there exists a region $R$ such that $f(x) \in R$ and $R \subset U$. Consequently, since $R$ is open by Corollary 4.11, we have by Definition 9.9 that there exists an open set $S_x$ containing $x$ such that $S_x \cap X \subset f^{-1}(R)$. Additionally, Script 1 tells us based off of the fact that $R \subset U$ that $f^{-1}(R) \subset f^{-1}(U)$. Thus, by subset transitivity, $S_x \cap X \subset f^{-1}(U)$. At this point, let $S = \bigcup_{x \in f^{-1}(U)} S_x$. It follows immediately from Corollary 4.18 that $S$ is open. Additionally, since the intersection of each set in the union with $X$ is a subset of $f^{-1}(U)$, it follows by Script 1 that $S \cap X \subset f^{-1}(U)$. Furthermore, for all $x \in f^{-1}(U)$, Definition 1.18 asserts that $x \in X$. In addition, we have defined an $S_x$ such that $x \in S_x$. These last two results combined demonstrate by Definition 1.6 that $x \in S \cap X$. Thus, by Definition 1.3, $f^{-1}(U) \subset S \cap X$. Consequently, by Theorem 1.7, $f^{-1}(U) = S \cap X$. Since $f^{-1}(U)$ is the intersection of $X$ with an open set, Definition 8.11 asserts that it is open in $X$, as desired. $\hspace{0.5cm}\square$

2/18: **Theorem 9.11.** *Suppose that $f : X \to \mathbb{R}$ is continuous. If $X$ is connected, then $f(X)$ is connected.*

*Proof.* This will be a proof by contrapositive; as such, suppose that $f(X)$ is disconnected. Then by Definition 4.22, $f(X) = A \cup B$ where $A, B$ are nonempty, disjoint sets that are open in $f(X)$. It follows from the last condition by Definition 8.11 that $A = G \cap f(X)$ and $B = H \cap f(X)$, where $G, H$ are open sets. Since for all $x \in X$, $f(x) \in A$ or $f(x) \in B$, Definitions 1.2 and 1.6 imply that for all $x \in X$, $f(x) \in G$ and $f(x) \in H$. Thus, by Script 1, $X \subset f^{-1}(G) \cup f^{-1}(H)$. Additionally, we know by Definition 1.18 that for all $x \in f^{-1}(G) \cup f^{-1}(H)$, $x \in X$. Thus, by Definition 1.3, $f^{-1}(G) \cup f^{-1}(H) \subset X$. Consequently, by Theorem 1.7, we have that $X = f^{-1}(G) \cup f^{-1}(H)$.

To show that $f^{-1}(G), f^{-1}(H)$ are disjoint, Definition 1.9 tells us that it will suffice to verify that $f^{-1}(G) \cap f^{-1}(H) = \emptyset$. As such, suppose for the sake of contradiction that $x \in f^{-1}(G) \cap f^{-1}(H)$. Then by consecutive applications of Definition 1.6, $x \in f^{-1}(G)$ and $x \in f^{-1}(H)$. Thus, by multiple applications of Definition 1.18, $x \in X$, $f(x) \in G$, and $f(x) \in H$. It follows from the first condition by Definition 1.18 that $f(x) \in f(X)$. The facts that $f(x) \in f(X)$ and $f(x) \in G$ imply by Definitions 1.6 and 1.2 that $f(x) \in A$. Similarly, $f(x) \in B$. But these last two statements imply by Definition 1.6 that $f(x) \in A \cap B$, a contradiction.

To show that $f^{-1}(G), f^{-1}(H)$ are nonempty, Definition 1.8 tells us that it will suffice to find an element of each set. As previously mentioned, $A, B$ are nonempty. Thus, by consecutive applications of Definition 1.8, there exist $f(x) \in A$ and $f(y) \in B$. Consequently, by Definitions 1.2 and 1.6, $f(x) \in G$ and $f(y) \in H$. Therefore, by consecutive applications of Definition 1.18, $x \in f^{-1}(G)$ and $y \in f^{-1}(H)$, as desired.

To show that $f^{-1}(G)$ and $f^{-1}(H)$ are open, Definition 9.4 tells us that it will suffice to verify (since $f$ is continuous by hypothesis) that $G, H$ are open subsets of $\mathbb{R}$. But by definition, they are exactly that. $\qquad\square$

**Exercise 9.12.** Use Theorem 9.11 to prove that if $f : [a, b] \to \mathbb{R}$ is continuous, then for every point $p$ between $f(a)$ and $f(b)$, there exists $c$ such that $a < c < b$ and $f(c) = p$.

*Proof.* Suppose that $a < b$. Then by Lemma 8.3, $[a, b]$ is an interval. Thus, by Theorem 8.15, $[a, b]$ is connected. It follows by Theorem 9.11 that $f([a, b])$ is connected. Consequently, by Theorem 8.15, $f([a, b])$ is an interval. We divide into three cases ($f(a) < f(b)$, $f(a) > f(b)$, and $f(a) = f(b)$).

First, suppose that $f(a) < f(b)$, and let $p$ be an arbitrary point between $f(a)$ and $f(b)$ (we know that at least one such point exists by Theorem 5.2). Then by Definition 3.6, $f(a) < p < f(b)$. Now $a, b \in [a, b]$ by Equations 8.1, so by Definition 1.18, $f(a), f(b) \in f([a, b])$. It follows by Definition 8.2 since $f([a, b])$ is an interval that $[f(a), f(b)] \subset f([a, b])$. Thus, since $f(a) < p < f(b)$ implies $p \in [f(a), f(b)]$ by Equations 8.1, Definition 1.3 asserts that $p \in f([a, b])$. Consequently, by Definition 1.18, $p = f(c)$ for some $c \in [a, b]$. Additionally, since $f(a) < p < f(b)$, we know that $p \neq f(a)$ and $p \neq f(b)$. It follows that $p = f(c)$ for some $c \in (a, b)$, as desired.

The proof of the second case is symmetric to that of the first.

Third, suppose that $f(a) = f(b)$. This implies that there are no points $p$ between $f(a)$ and $f(b)$ by Definition 3.6, so the statement is vacuously true in this case. $\qquad\square$

**Lemma 9.13.** *If $f : (a, b) \to \mathbb{R}$ is continuous and injective, then $f$ is either strictly increasing or strictly decreasing on $(a, b)$.*

*Proof.* Suppose that $a < b$. Then by Corollary 5.3, we can pick two distinct points $c, d \in (a, b)$. Since $f$ is injective, we have from the fact that $x \neq y$ by Definition 1.20 that $f(c) \neq f(d)$. We divide into two cases ($f(c) < f(d)$ and $f(c) > f(d)$).

Suppose first that $f(c) < f(d)$. To prove that $f$ is strictly increasing, Definition 8.16 tells us that it will suffice to show that for all $x, y \in (a, b)$ with $x < y$, $f(x) < f(y)$. Let $x, y$ be arbitrary elements of $(a, b)$ that satisfy $x < y$, and suppose for the sake of contradiction that $f(x) \geq f(y)$. If $f(x) = f(y)$, then by Definition 1.20, $x = y$, a contradiction. If $f(x) > f(y)$, then we divide into five cases ($x < y < c < d$, $x < c < y < d$, $x < c < d < y$, $c < x < d < y$, and $c < d < x < y$).

Let $x < y < c < d$. If $f(x) > f(c)$, let $p_1$ be a point between $f(x)$ and $f(c)$ and let $p_2$ be a point between $f(d)$ and $f(c)$. Let $p = \min(p_1, p_2)$. Since $f(c) < p < f(x)$, Exercise 9.12 implies that there exists $e \in (x, c)$ such that $f(e) = p$. Similarly, there exists $e' \in (c, d)$ such that $f(e') = p$. Since $(x, c)$ and $(c, d)$ are clearly disjoint, $e \neq e'$. But by Definition 1.20, $f(e) = p = f(e')$ implies that $e = e'$, a contradiction. If $f(x) < f(c)$, then we can arrive at a similar contradiction by considering values in the regions $(x, y)$ and $(y, c)$.

The proof is symmetric in the second case if we consider values in the regions $(c, y)$ and $(y, d)$ when $f(y)$ is not between $f(c)$ and $f(d)$, and values in the regions $(x, c)$ and $(c, y)$ when $f(y)$ is.

The proof is symmetric in the third case if we consider values in the regions $(x, c)$ and $(c, d)$ when $f(x) > f(c)$, and values in the regions $(x, d)$ and $(d, y)$ when $f(x) < f(c)$.

The proof is symmetric in the fourth case if we consider values in the regions $(c, x)$ and $(x, d)$ when $f(x) > f(d)$, and values in the regions $(x, d)$ and $(d, y)$ when $f(x) < f(d)$.

The proof is symmetric in the fifth case if we consider values in the regions $(c, d)$ and $(d, x)$ when $f(d) > f(x)$, and values in the regions $(d, x)$ and $(x, y)$ when $f(d) < f(x)$. $\qquad\square$