Script 7

The Field Axioms

7.1 Journal

1/28: **Definition 7.1.** A **binary operation** on a set X is a function

$$f : X \times X \to X$$

We say that f is **associative** if

$$f(f(x,y),z) = f(x, f(y,z)) \quad \text{for all } x, y, z \in X$$

We say that f is **commutative** if

$$f(x,y) = f(y,x) \quad \text{for all } x, y \in X$$

An **identity element** of a binary operation f is an element $e \in X$ such that

$$f(x,e) = f(e,x) = x \quad \text{for all } x \in X$$

Remark 7.2. Frequently, we denote a binary operation differently. If $* : X \times X \to X$ is the binary operation, we often write $a * b$ in place of $*(a,b)$. We sometimes indicate this same operation by writing $(a,b) \mapsto a * b$.

Exercise 7.3. Rewrite Definition 7.1 using the notation of Remark 7.2.

Answer. A **binary operation** on a set X is a function

$$* : X \times X \to X$$

We say that $*$ is **associative** if

$$(x * y) * z = x * (y * z) \quad \text{for all } x, y, z \in X$$

We say that $*$ is **commutative** if

$$x * y = y * x \quad \text{for all } x, y \in X$$

An **identity element** of a binary operation $*$ is an element $e \in X$ such that

$$x * e = e * x = x \quad \text{for all } x \in X$$

□

Examples 7.4.

1. *The function* $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ *which sends a pair of integers* (m,n) *to* $+(m,n) = m + n$ *is a binary operation on the integers, called addition. Addition is associative, commutative, and has identity element 0.*

2. *The maximum of m and n, denoted $\max(m,n)$, is an associative and commutative binary operation on $\mathbb{Z}$. Is there an identity element for $\max$?*

   Proof. Suppose for the sake of contradiction that there exists an identity element e for max. But $\max(e-1,e) = e \neq e-1$, a contradiction. Therefore, no identity element exists for max. □

3. *Let $\wp(Y)$ be the power set of a set Y. Recall that the power set consists of all subsets of Y. Then the intersection of sets, $(A,B) \mapsto A \cap B$, defines an associative and commutative binary operation on $\wp(Y)$. Is there an identity element for $\cap$?*

   Proof. Clearly, $Y \in \wp(Y)$. By Script 1, $Y \cap A = A \cap Y = A$ where $A \subset Y$. Therefore, Y is an identity element for $\cap$. □

Exercise 7.5. Find a binary operation on a set that is not commutative. Find a binary operation on a set that is not associative.

Proof. We will prove that the subtraction operation on the integers ($- : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$) is neither commutative nor associative. To prove that it's not commutative, Definition 7.1 tells us that it will suffice to show that $x - y \neq y - x$ for some $x, y \in \mathbb{Z}$. Since $2 - 1 = 1$ but $1 - 2 = -1$, we can see that $1, 2 \in \mathbb{Z}$ clearly meet this requirement. To prove that it's not associative, Definition 7.1 tells us that it will suffice to show that $(x-y)-z \neq x-(y-z)$ for some $x,y,z \in \mathbb{Z}$. Since $(3-2)-1 = 0$ but $3-(2-1) = 2$, we can see that $1,2,3 \in \mathbb{Z}$ clearly meet this requirement. □

Exercise 7.6. Let X be a finite set, and let $Y = \{f : X \to X \mid f \text{ is bijective}\}$. Consider the binary operation of composition of functions, denoted $\circ : Y \times Y \to Y$ and defined by $(f \circ g)(x) = f(g(x))$ as seen in Definition 1.25. Decide whether or not composition is commutative and/or associative and whether or not it has an identity.

Proof. To prove that composition is not commutative, Definition 7.1 tells us that it will suffice to find a finite set X paired with two bijections in Y that do not commute. Let $X = \{1,2,3\}$ and consider the bijections $f : X \to X$ (defined by $f(1) = 2$, $f(2) = 3$, $f(3) = 1$) and $g : X \to X$ (defined by $g(1) = 1$, $g(2) = 3$, $g(3) = 2$). In this case, $f \circ g$ would be defined by $f(g(1)) = 2$, $f(g(2)) = 1$, and $f(g(3)) = 3$, but $g \circ f$ would be defined by $g(f(1)) = 3$, $g(f(2)) = 2$, and $g(f(3)) = 1$.

To prove that composition is associative, Definition 7.1 tells us that it will suffice to show that $((f \circ g) \circ h)(x) = (f \circ (g \circ h))(x)$. We may do this with the following algebra.

$$\begin{aligned}
((f \circ g) \circ h)(x) &= (f \circ g)(h(x)) \\
&= f(g(h(x))) \\
&= f((g \circ h)(x)) \\
&= (f \circ (g \circ h))(x)
\end{aligned}$$

With respect to any finite set X, there will always be a bijection $i : X \to X$ defined by $i(x) = x$. To prove that i is an identity element, Definition 7.1 tells us that it will suffice to show that for all $f \in Y$, $f \circ i = i \circ f = f$. We may do this with the following algebra.

$$\begin{aligned}
(f \circ i)(x) &= f(i(x)) \\
&= f(x) \\
&= i(f(x)) \\
&= (i \circ f)(x)
\end{aligned}$$

□

Theorem 7.7. *Identity elements are unique. That is, suppose that f is a binary operation on a set X that has two identity elements e and e'. Then $e = e'$.*

Proof. Let $f : X \times X \to X$ be a binary operation on a set X with two identity elements e, e'. By Definition 7.1, we know that $f(e, e') = e$ and $f(e, e') = e'$. Since f is a well-defined function by definition, it must be that $e = f(e, e') = e'$. $\square$

Definition 7.8. A **field** is a set F with two binary operations on F called addition, denoted $+$, and multiplication, denoted $\cdot$, satisfying the following **field axioms**:

FA1 (Commutativity of Addition) For all $x, y \in F$, $x + y = y + x$.

FA2 (Associativity of Addition) For all $x, y, z \in F$, $(x + y) + z = x + (y + z)$.

FA3 (Additive Identity) There exists an element $0 \in F$ such that $x + 0 = 0 + x = x$ for all $x \in F$.

FA4 (Additive Inverses) For any $x \in F$, there exists $y \in F$ such that $x + y = y + x = 0$, called an additive inverse of x.

FA5 (Commutativity of Multiplication) For all $x, y \in F$, $x \cdot y = y \cdot x$.

FA6 (Associativity of Multiplication) For all $x, y, z \in F$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

FA7 (Multiplicative Identity) There exists an element $1 \in F$ such that $x \cdot 1 = 1 \cdot x = x$ for all $x \in F$.

FA8 (Multiplicative Inverses) For any $x \in F$ such that $x \neq 0$, there exists $y \in F$ such that $x \cdot y = y \cdot x = 1$, called a multiplicative inverse of x.

FA9 (Distributivity of Multiplication over Addition) For all $x, y, z \in F$, $x \cdot (y + z) = x \cdot y + x \cdot z$.

FA10 (Distinct Additive and Multiplicative Identities) $1 \neq 0$.

Exercise 7.9. Consider the set $\mathbb{F}_2 = \{0, 1\}$, and define binary operations $+$ and $\cdot$ on $\mathbb{F}_2$ by

$$\begin{array}{cccc} 0 + 0 = 0 & 0 + 1 = 1 & 1 + 0 = 1 & 1 + 1 = 0 \\ 0 \cdot 0 = 0 & 0 \cdot 1 = 0 & 1 \cdot 0 = 0 & 1 \cdot 1 = 1 \end{array}$$

Show that $\mathbb{F}_2$ is a field.

Proof. To prove that $\mathbb{F}_2$ obeys FA1 from Definition 7.8, it will suffice to show that $0+0 = 0+0$, $0+1 = 1+0$, and $1+1 = 1+1$. The first and third of these are evidently true. For the second, we have $0+1 = 1 = 1+0$, so it is good, too.

To prove that $\mathbb{F}_2$ obeys FA2 from Definition 7.8, the following casework will suffice.

$$\begin{array}{ll} (0+0)+0 = 0 = 0+(0+0) & (0+0)+1 = 1 = 0+(0+1) \\ (0+1)+0 = 1 = 0+(1+0) & (1+0)+0 = 1 = 1+(0+0) \\ (0+1)+1 = 0 = 0+(1+1) & (1+1)+0 = 0 = 1+(1+0) \\ (1+0)+1 = 0 = 1+(0+1) & (1+1)+1 = 1 = 1+(1+1) \end{array}$$

To prove that $\mathbb{F}_2$ obeys FA3 from Definition 7.8, it will suffice to find an element $0 \in \mathbb{F}_2$ such that $x + 0 = 0 + x = x$. Since $0 + 0 = 0$, $1 + 0 = 0$, and by commutativity, it is clear that 0 is an additive identity in $\mathbb{F}_2$.

To prove that $\mathbb{F}_2$ obeys FA4 from Definition 7.8, it will suffice to show that for all $x \in \mathbb{F}_2$, there exists a $y \in \mathbb{F}_2$ such that $x + y = y + x = 0$. For 0, this object is 0 (since $0 + 0 = 0 + 0 = 0$), and for 1, this object is 1 (since $1 + 1 = 1 + 1 = 0$).

To prove that $\mathbb{F}_2$ obeys FA5 from Definition 7.8, it will suffice to show that $0 \cdot 0 = 0 \cdot 0$, $0 \cdot 1 = 1 \cdot 0$, and $1 \cdot 1 = 1 \cdot 1$. The first and third of these are evidently true. For the second, we have $0 \cdot 1 = 0 = 1 \cdot 0$, so it is good, too.

To prove that $\mathbb{F}_2$ obeys FA6 from Definition 7.8, the following casework will suffice.

$$
\begin{aligned}
(0\cdot 0)\cdot 0 &= 0 = 0\cdot(0\cdot 0) & \qquad (0\cdot 0)\cdot 1 &= 0 = 0\cdot(0\cdot 1)\\
(0\cdot 1)\cdot 0 &= 0 = 0\cdot(1\cdot 0) & \qquad (1\cdot 0)\cdot 0 &= 0 = 1\cdot(0\cdot 0)\\
(0\cdot 1)\cdot 1 &= 0 = 0\cdot(1\cdot 1) & \qquad (1\cdot 1)\cdot 0 &= 0 = 1\cdot(1\cdot 0)\\
(1\cdot 0)\cdot 1 &= 0 = 1\cdot(0\cdot 1) & \qquad (1\cdot 1)\cdot 1 &= 1 = 1\cdot(1\cdot 1)
\end{aligned}
$$

To prove that $\mathbb{F}_2$ obeys FA7 from Definition 7.8, it will suffice to find an element $1 \in \mathbb{F}_2$ such that $x \cdot 1 = 1 \cdot x = x$. Since $0 \cdot 1 = 0$, $1 \cdot 1 = 1$, and by commutativity, it is clear that 1 is a multiplicative identity in $\mathbb{F}_2$.

To prove that $\mathbb{F}_2$ obeys FA8 from Definition 7.8, it will suffice to show that for all $x \in \mathbb{F}_2$ such that $x \neq 0$, there exists a $y \in \mathbb{F}_2$ such that $x \cdot y = y \cdot x = 1$. For 1, this object is 1 (since $1 \cdot 1 = 1 \cdot 1 = 1$).

To prove that $\mathbb{F}_2$ obeys FA9 from Definition 7.8, the following casework will suffice.

$$
\begin{aligned}
0\cdot(0+0) &= 0 = 0\cdot 0 + 0\cdot 0 & \qquad 0\cdot(0+1) &= 0 = 0\cdot 0 + 0\cdot 1\\
0\cdot(1+0) &= 0 = 0\cdot 1 + 0\cdot 0 & \qquad 1\cdot(0+0) &= 0 = 1\cdot 0 + 1\cdot 0\\
0\cdot(1+1) &= 0 = 0\cdot 1 + 0\cdot 1 & \qquad 1\cdot(1+0) &= 1 = 1\cdot 1 + 1\cdot 0\\
1\cdot(0+1) &= 1 = 1\cdot 0 + 1\cdot 1 & \qquad 1\cdot(1+1) &= 0 = 1\cdot 1 + 1\cdot 1
\end{aligned}
$$

To prove that $\mathbb{F}_2$ obeys FA10 from Definition 7.8, it will suffice to show that $0 \neq 1$. Clearly this is true. $\square$

Theorem 7.10. *Suppose that F is a field. Then additive inverses are unique. This means: Let $x \in F$. If $y, y' \in F$ satisfy $x + y = 0$ and $x + y' = 0$, then $y = y'$.*

Proof. Let $x, y, y' \in F$ be such that $x + y = 0$ and $x + y' = 0$. From Definition 7.8, we have

$$
\begin{aligned}
y' + (x + y) &= (y' + x) + y && \text{FA2}\\
y' + 0 &= 0 + y && \text{FA4}\\
y' &= y && \text{FA3}
\end{aligned}
$$

$\square$

We usually write $-x$ for the additive inverse of x.

Corollary 7.11. *If $x \in F$, then $-(-x) = x$.*

Proof. Let $x \in F$. Then by consecutive applications of FA4 from Definition 7.8, $-x + (-(-x)) = 0$ and $-x + x = 0$. Therefore, by Theorem 7.10, we have that $-(-x) = x$. $\square$

Theorem 7.12. *Let F be a field, and let $a, b, c \in F$. If $a + b = a + c$, then $b = c$.*

Proof. Let $a, b, c \in F$ be such that $a + b = a + c$. By FA4 from Definition 7.8, there exists $-a \in F$ such that $-a + a = a + (-a) = 0$. Having established that $-a$ exists, we can prove from Definition 7.8 that

$$
\begin{aligned}
-a + (a + b) &= -a + (a + c)\\
(-a + a) + b &= (-a + a) + c && \text{FA2}\\
0 + b &= 0 + c && \text{FA4}\\
b &= c && \text{FA3}
\end{aligned}
$$

$\square$

Theorem 7.13. *Let F be a field. If $a \in F$, then $a \cdot 0 = 0$.*

Proof. Let $a \in F$. From Definition 7.8, we have

$$\begin{aligned} a &= a \cdot 1 && \text{FA7} \\ &= a \cdot (1+0) && \text{FA3} \\ &= a \cdot 1 + a \cdot 0 && \text{FA9} \\ &= a + a \cdot 0 && \text{FA7} \\ 0 &= a \cdot 0 && \text{Theorem 7.12} \end{aligned}$$

□

2/2: **Theorem 7.14.** *Suppose that F is a field. Then multiplicative inverses are unique. This means: Let $x \in F$. If $y, y' \in F$ satisfy $x \cdot y = 1$ and $x \cdot y' = 1$, then $y = y'$.*

Proof. Let $x, y, y' \in F$ be such that $x \cdot y = 1$ and $x \cdot y' = 1$. From Definition 7.8, we have

$$\begin{aligned} (y \cdot x) \cdot y' &= y \cdot (x \cdot y') && \text{FA6} \\ 1 \cdot y' &= y \cdot 1 && \text{FA8} \\ y' &= y && \text{FA7} \end{aligned}$$

□

We usually write x^{-1} or $\frac{1}{x}$ for the multiplicative inverse of x.

Corollary 7.15. *If $x \in F$ and $x \neq 0$, then $(x^{-1})^{-1} = x$.*

Proof. Let $x \in F \setminus \{0\}$. Then by FA8 from Definition 7.8, there exists $x^{-1} \in F$ such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$. It follows from Theorem 7.13 that $x^{-1} \neq 0$ (if $x^{-1} = 0$, then Theorem 7.13 would imply that $1 = x \cdot x^{-1} = 0$, contradicting FA10). Thus, by FA8 from Definition 7.8 again, there exists $(x^{-1})^{-1} \in F$ such that $x^{-1} \cdot (x^{-1})^{-1} = (x^{-1})^{-1} \cdot x^{-1} = 1$. Having established that $(x^{-1})^{-1}$ exists, $x^{-1} \cdot (x^{-1})^{-1} = 1$, and $x^{-1} \cdot x = 1$, we have by Theorem 7.14 that $(x^{-1})^{-1} = x$. □

Theorem 7.16. *Let F be a field, and let $a, b, c \in F$. If $a \cdot b = a \cdot c$ and $a \neq 0$, then $b = c$.*

Proof. Let $a, b, c \in F$ be such that $a \cdot b = a \cdot c$ and $a \neq 0$. By FA8 from Definition 7.8, there exists $a^{-1} \in F$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$. Having established that a^{-1} exists, we can prove from Definition 7.8 that

$$\begin{aligned} a^{-1} \cdot (a \cdot b) &= a^{-1} \cdot (a \cdot c) && \\ (a^{-1} \cdot a) \cdot b &= (a^{-1} \cdot a) \cdot c && \text{FA6} \\ 1 \cdot b &= 1 \cdot c && \text{FA8} \\ b &= c && \text{FA7} \end{aligned}$$

□

Theorem 7.17. *Let F be a field, and let $a, b \in F$. If $a \cdot b = 0$, then $a = 0$ or $b = 0$.*

Proof. Let $a, b \in F$ be such that $a \cdot b = 0$, and suppose for the sake of contradiction that $a \neq 0$ and $b \neq 0$. It follows from the supposition by consecutive applications of FA8 from Definition 7.8 that a^{-1} and b^{-1} exist. Thus, from Definition 7.8, we have

$$\begin{aligned} 1 &= 1 \cdot 1 && \text{FA7} \\ &= (a \cdot a^{-1}) \cdot (b \cdot b^{-1}) && \text{FA8} \\ &= (a \cdot b) \cdot (a^{-1} \cdot b^{-1}) && \text{FA6 and FA7} \\ &= 0 \cdot (a^{-1} \cdot b^{-1}) && \text{Substitution} \\ &= 0 && \text{Theorem 7.13} \end{aligned}$$

But this contradicts FA10 from Definition 7.8. □

Lemma 7.18. *Let F be a field. If $a \in F$, then $-a = (-1)a$.*

Proof. Let $a \in F$. From Definition 7.8, we have

$$\begin{aligned} 0 &= a \cdot 0 && \text{Theorem 7.13} \\ a + (-a) &= a \cdot (1 + (-1)) && \text{FA4} \\ a + (-a) &= a \cdot 1 + a \cdot (-1) && \text{FA9} \\ a + (-a) &= a + a \cdot (-1) && \text{FA7} \\ a + (-a) &= a + (-1)a && \text{FA5} \\ -a &= (-1)a && \text{Theorem 7.12} \end{aligned}$$

□

Lemma 7.19. *Let F be a field. If $a, b \in F$, then $a \cdot (-b) = -(a \cdot b) = (-a) \cdot b$.*

Proof. Let $a, b \in F$. From Definition 7.8, we have

$$\begin{aligned} a \cdot (-b) &= a \cdot ((-1) \cdot b) && \text{Lemma 7.18} \\ &= a \cdot (b \cdot (-1)) && \text{FA5} \\ &= (a \cdot b) \cdot (-1) && \text{FA6} \\ &= (-1) \cdot (a \cdot b) && \text{FA5} \\ &\boxed{= -(a \cdot b)} && \text{Lemma 7.18} \\ &= (-1) \cdot (a \cdot b) && \text{Lemma 7.18} \\ &= ((-1) \cdot a) \cdot b && \text{FA6} \\ &\boxed{= (-a) \cdot b} && \text{Lemma 7.18} \end{aligned}$$

□

Lemma 7.20. *Let F be a field. If $a, b \in F$, then $a \cdot b = (-a) \cdot (-b)$.*

Proof. Let $a, b \in F$. Thus, we have

$$\begin{aligned} (-a) \cdot (-b) &= -(-a) \cdot b && \text{Lemma 7.19} \\ &= a \cdot b && \text{Corollary 7.11} \end{aligned}$$

□

Definition 7.21. An **ordered field** is a field F equipped with an ordering $<$ (satisfying Definition 3.1) such that also:

(a) Addition respects the ordering: if $x < y$, then $x + z < y + z$ for all $z \in F$.

(b) Multiplication respects the ordering: if $0 < x$ and $0 < y$, then $0 < x \cdot y$.

Definition 7.22. Suppose F is an ordered field and $x \in F$. If $0 < x$, we say that x is **positive**. If $x < 0$, we say that x is **negative**.

Lemma 7.23. *Let F be an ordered field, and let $x \in F$. If $0 < x$, then $-x < 0$. Similarly, if $x < 0$, then $0 < -x$.*

Proof. Let $x \in F$ be such that $0 < x$. Then by Definition 7.21a, $0 + (-x) < x + (-x)$. Consequently, from Definition 7.8, we have

$$\begin{aligned} -x &< x + (-x) && \text{FA3} \\ -x &< 0 && \text{FA4} \end{aligned}$$

The proof is symmetric if $x < 0$. □

Lemma 7.24. *Let F be an ordered field, and let $x, y, z \in F$.*

(*a*) *If $x > 0$ and $y < z$, then $x \cdot y < x \cdot z$.*

(*b*) *If $x < 0$ and $y < z$, then $x \cdot z < x \cdot y$.*

Proof of a. Let $x, y, z \in F$ be such that $x > 0$ and $y < z$. It follows from the latter condition by Definition 7.21a that $y + (-y) < z + (-y)$. Thus, by FA4 from Definition 7.8, we have $0 < z + (-y)$. This combined with the fact that $0 < x$ implies by Definition 7.21b that $0 < x \cdot (z + (-y))$. Consequently, from Definition 7.8, we have

$$\begin{aligned}
0 &< x \cdot z + x \cdot (-y) && \text{FA9} \\
0 &< x \cdot z + (-(x \cdot y)) && \text{Lemma 7.19} \\
0 + x \cdot y &< (x \cdot z + (-(x \cdot y))) + x \cdot y && \text{Definition 7.21a} \\
0 + x \cdot y &< x \cdot z + (-(x \cdot y) + x \cdot y) && \text{FA2} \\
0 + x \cdot y &< x \cdot z + 0 && \text{FA4} \\
x \cdot y &< x \cdot z && \text{FA3}
\end{aligned}$$

□

Proof of b. Let $x, y, z \in F$ be such that $x < 0$ and $y < z$. It follows from the former condition by Lemma 7.23 that $0 < -x$. Thus, by Lemma 7.24a, $(-x) \cdot y < (-x) \cdot z$. Consequently, from Definition 7.8, we have

$$\begin{aligned}
-(x \cdot y) &< -(x \cdot z) && \text{Lemma 7.19} \\
-(x \cdot y) + (x \cdot y + x \cdot z) &< -(x \cdot z) + (x \cdot y + x \cdot z) && \text{Definition 7.21a} \\
-(x \cdot y) + (x \cdot y + x \cdot z) &< -(x \cdot z) + (x \cdot z + x \cdot y) && \text{FA1} \\
(-(x \cdot y) + x \cdot y) + x \cdot z &< (-(x \cdot z) + x \cdot z) + x \cdot y && \text{FA2} \\
0 + x \cdot z &< 0 + x \cdot y && \text{FA4} \\
x \cdot z &< x \cdot y && \text{FA3}
\end{aligned}$$

□

Remark 7.25. An immediate consequence of this lemma is the fact that if x and y are both positive or both negative, their product is positive.

Lemma 7.26. *Let F be an ordered field, and let $x \in F$. Then $0 \leq x^2$. Moreover, if $x \neq 0$, then $0 < x^2$.*

Proof. We divide into two cases ($x = 0$ and $x \neq 0$). Suppose first that $x = 0$. Then by Theorem 7.13, $0 \leq 0 = 0 \cdot 0 = 0^2 = x^2$, as desired. Now suppose that $x \neq 0$. We divide into two cases again ($x > 0$ and $x < 0$). If $x > 0$, then by Lemma 7.24a, $x > 0$ and $0 < x$ imply that $x \cdot 0 < x \cdot x$, from which it follows by Theorem 7.13 that $0 < x^2$, as desired. On the other hand, if $x < 0$, then by Lemma 7.24b, $x < 0$ and $x < 0$ imply that $x \cdot 0 < x \cdot x$, from which it follows for the same reason as before that $0 < x^2$, as desired. Both of the original two cases together prove the first statement, while the second original case alone proves the second statement. □

Corollary 7.27. *Let F be an ordered field. Then $0 < 1$.*

Proof. By FA10 from Definition 7.8, $1 \neq 0$. Thus, by Lemma 7.26, $0 < 1^2 = 1$, as desired. □

Theorem 7.28. *If F is an ordered field, then F has no first or last point.*

Proof. Suppose for the sake of contradiction that F has a first point a. By Corollary 7.27, we have that $0 < 1$, which implies by Lemma 7.23 that $-1 < 0$. It follows by Definition 7.21a that $-1 + a < 0 + a$. Thus, by FA3 from Definition 7.8, $-1 + a < a$. Since there exists an object in F (namely $-1 + a$) that is less than a, Definition 3.3 tells us that a is not the first point of F, a contradiction.

The proof is symmetric in the other case. □

Theorem 7.29. *The rational numbers $\mathbb{Q}$ form an ordered field.*

Proof. To prove that $\mathbb{Q}$ forms an ordered field, Definition 7.21 tells us that it will suffice to show that $\mathbb{Q}$ forms a field; has an ordering $<$; satisfies $x+z<y+z$ if $x<y$ for all $z \in \mathbb{Q}$; and satisfies $0<x\cdot y$ if $0<x$ and $0<y$. We will take this one constraint at a time.

To show that $\mathbb{Q}$ forms a field, Definition 7.8 tells us that it will suffice to verify that $\mathbb{Q}$ has two binary operations ($+$ and $\cdot$), and satisfies field axioms 1-10. Define $+$ and $\cdot$ as in Definition 2.7. Under these definitions, parts a-i of Theorem 2.10 guarantee that $\mathbb{Q}$ satisfies FA1-FA9, respectively. As to FA10, to verify that $\left[\frac{1}{1}\right] \neq \left[\frac{0}{1}\right]$, Exercise 2.6 tells us that it will suffice to confirm that $(1,1) \nsim (1,0)$. But since $1\cdot 0 = 0 \neq 1 = 1\cdot 1$, Exercise 2.2e confirms that $(1,1) \nsim (1,0)$, as desired.

$\mathbb{Q}$ has an ordering by Exercise 3.9d, as desired.

To show that $x+z<y+z$ if $x<y$ for all $z \in \mathbb{Q}$, let $\left[\frac{a}{b}\right], \left[\frac{c}{d}\right], \left[\frac{x}{z}\right]$ be arbitrary elements of $\mathbb{Q}$ with positive denominators (we can choose these WLOG by Exercise 3.9b) and the first two satisfying $\left[\frac{a}{b}\right] < \left[\frac{c}{d}\right]$; we seek to verify that $\left[\frac{a}{b}\right] + \left[\frac{x}{z}\right] < \left[\frac{c}{d}\right] + \left[\frac{x}{z}\right]$. Since $\left[\frac{a}{b}\right] < \left[\frac{c}{d}\right]$, we have by Exercise 3.9c that $ad<bc$. It follows by Script 0 that

$$\begin{gathered}
ad < bc \\
adzz < bczz \\
adzz + bdxz < bczz + bdxz \\
azdz + bxdz < bzcz + bzdx \\
(az+bx)(dz) < (bz)(cz+dx)
\end{gathered}$$

Thus, by Exercise 3.9c, $\left[\frac{az+bx}{bz}\right] < \left[\frac{cz+dx}{dz}\right]$. Therefore, by Definition 2.7, $\left[\frac{a}{b}\right] + \left[\frac{x}{z}\right] < \left[\frac{c}{d}\right] + \left[\frac{x}{z}\right]$, as desired.

To show that $0<x\cdot y$ if $0<x$ and $0<y$, let $\left[\frac{a}{b}\right], \left[\frac{c}{d}\right]$ be arbitrary elements of $\mathbb{Q}$ with positive denominators (which we can choose for the same reason as before) and such that $\left[\frac{0}{1}\right] < \left[\frac{a}{b}\right]$ and $\left[\frac{0}{1}\right] < \left[\frac{c}{d}\right]$; we seek to verify that $\left[\frac{0}{1}\right] < \left[\frac{a}{b}\right] \cdot \left[\frac{c}{d}\right]$. Since $\left[\frac{0}{1}\right] < \left[\frac{a}{b}\right]$ and $\left[\frac{0}{1}\right] < \left[\frac{c}{d}\right]$, we have by Exercise 3.9c that $0\cdot b < 1\cdot a$ and $0 \cdot d < 1 \cdot c$. It follows by Script 0 that $0\cdot bd < 1\cdot ac$. Thus, by Exercise 3.9c, $\left[\frac{0}{1}\right] < \left[\frac{ac}{bd}\right]$. Therefore, by Definition 2.7, $\left[\frac{0}{1}\right] < \left[\frac{a}{b}\right] \cdot \left[\frac{c}{d}\right]$, as desired. ☐

2/4: **Definition 7.31.** We define $\oplus$ on $\mathbb{R}$ as follows. Let $A, B \in \mathbb{R}$ be Dedekind cuts. Define

$$A \oplus B = \{a+b \mid a \in A \text{ and } b \in B\}$$

Exercise 7.32.

(a) Prove that $A \oplus B$ is a Dedekind cut.

(b) Prove that $\oplus$ is commutative and associative.

(c) Prove that if $A \in \mathbb{R}$, then $A = \mathbf{0} \oplus A$.

Proof of a. To prove that $A \oplus B$ is a Dedekind cut, Definition 6.1 tells us that it will suffice to show that $A \oplus B \neq \emptyset$; $A \oplus B \neq \mathbb{Q}$; if $r \in A\oplus B$ and $s \in \mathbb{Q}$ satisfy $s<r$, then $s \in A \oplus B$; and if $r \in A \oplus B$, then there is some $s \in A\oplus B$ with $s>r$. We will take this one claim at a time.

To show that $A \oplus B \neq \emptyset$, Definition 1.8 tells us that it will suffice to find an element of $A \oplus B$. Since A, B are Dedekind cuts, Definition 6.1 asserts that they are nonempty. Thus, there exist rational numbers $x \in A$ and $y \in B$. Therefore, by the definition of $A \oplus B$, the sum $x+y \in A \oplus B$, as desired.

To show that $A \oplus B \neq \mathbb{Q}$, Definition 1.2 tells us that it will suffice to find an element of $\mathbb{Q}$ that is not an element of $A \oplus B$. For an analogous reason to before, we can choose $x, y \in \mathbb{Q}$ such that $x \notin A$ and $y \notin B$. It follows by Lemma 6.2 and Definition 5.6 that $x \geq a$ for all $a \in A$ and $y \geq b$ for all $b \in B$. Additionally, since $x \notin A$, we have that $x \neq a$ for any $a \in A$; thus, $x > a$ for all $a \in A$. Similarly, $y > b$ for all $b \in B$. Consequently, by Script 2, $x+y > a+b$ for all $a+b \in A \oplus B$. Therefore, $x + y \notin A \oplus B$, as desired.

To show that if $r \in A \oplus B$ and $s \in \mathbb{Q}$ satisfy $s<r$, then $s \in A \oplus B$, we let $r \in A \oplus B$ and $s \in \mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s<r$ and seek to verify that $s \in A\oplus B$. Since $r \in A \oplus B$, $r = x+y$ for some $x \in A$ and $y \in B$. Additionally, it follows from the fact that $s<r$ that

$s = r - q = x + y - q$ for some $q \in \mathbb{Q}^+$. Since $y \in B$ and $y - q \in \mathbb{Q}$ satisfy $y - q < y$, we have by Definition 6.1b that $y - q \in B$. Therefore, $s = (x) + (y - q)$ is an element of $A \oplus B$, as desired.

To show that if $r \in A \oplus B$, then there is some $s \in A \oplus B$ with $s > r$, we let $r \in A \oplus B$ and seek to find such an s. Since $r \in A \oplus B$, $r = x + y$ for some $x \in A$ and $y \in B$. It follows from the fact that $x \in A$ by Definition 6.1c that there exists a $z \in A$ with $z > x$. Consequently, by Script 0, $z + y > x + y$ is the desired element of $A \oplus B$. □

Proof of b. To prove that $\oplus$ is commutative, Definition 7.1 tells us that it will suffice to show that for all $A, B \in \mathbb{R}$, we have $A \oplus B = B \oplus A$. Let A, B be arbitrary elements of $\mathbb{R}$. Then by Definition 7.31, we clearly have

$$\begin{aligned} A \oplus B &= \{a + b \mid a \in A \text{ and } b \in B\} \\ &= \{b + a \mid b \in B \text{ and } a \in A\} \\ &= B \oplus A \end{aligned}$$

To prove that $\oplus$ is associative, Definition 7.1 tells us that it will suffice to show that for all $A, B, C \in \mathbb{R}$, we have $(A \oplus B) \oplus C = A \oplus (B \oplus C)$. Let A, B, C be arbitrary elements of $\mathbb{R}$. Then by Definition 7.31, we clearly have

$$\begin{aligned} (A \oplus B) \oplus C &= \{a + b \mid a \in A \text{ and } b \in B\} \oplus C \\ &= \{d + c \mid d \in \{a + b \mid a \in A \text{ and } b \in B\} \text{ and } c \in C\} \\ &= \{d + c \mid d = a + b \text{ for some } a \in A \text{ and } b \in B, \text{ and } c \in C\} \\ &= \{a + b + c \mid a \in A \text{ and } b \in B \text{ and } c \in C\} \\ &= \{a + e \mid a \in A, \text{ and } e = b + c \text{ for some } b \in B \text{ and } c \in C\} \\ &= \{a + e \mid c \in C \text{ and } e \in \{b + c \mid b \in B \text{ and } c \in C\}\} \\ &= A \oplus \{b + c \mid b \in B \text{ and } c \in C\} \\ &= A \oplus (B \oplus C) \end{aligned}$$

Note that we also make use of Exercise 7.32a to guarantee $A \oplus B \in \mathbb{R}$, so that we can apply $\oplus$ to $A \oplus B$ and C. We similarly invoke Exercise 7.32a to take the sum of A and $B \oplus C$. □

Proof of c. To prove that for all $A \in \mathbb{R}$, $A = \mathbf{0} \oplus A$, we will show for an arbitrary $A \in \mathbb{R}$ that every element of A is an element of $\mathbf{0} \oplus A$ and vice versa. Let A be an arbitrary element of $\mathbb{R}$. Suppose first that $x \in A$. Then by Definition 6.1c, there exists $y \in A$ such that $y > x$. Let $z = x - y$. Clearly, $z \in \mathbb{Q}$ and $z < 0$, so we know that $z \in \mathbf{0}$. Additionally, since $x - z = y$, we know that $x - z \in A$. Therefore, since $x = (z) + (x - z)$, we have by Definition 7.31 that $x \in \mathbf{0} \oplus A$. Now suppose that $z \in \mathbf{0} \oplus A$. Then by Definition 7.31, $z = x + y$ for some $x \in \mathbf{0}$ and $y \in A$. Since $x \in \mathbf{0}$, we know that $x < 0$, which means that $y > z$. This combined with the fact that $y \in A$ and $z \in \mathbb{Q}$ implies by Definition 6.1b that $z \in A$. □

2/9: **Definition 7.39.** For $A, B \in \mathbb{R}$, $\mathbf{0} < A$, $\mathbf{0} < B$, we define

$$A \otimes B = \{r \in \mathbb{Q} \mid r \leq 0\} \cup \{ab \mid a \in A, b \in B, a > 0, b > 0\}$$

If $A = \mathbf{0}$ or $B = \mathbf{0}$, we define $A \otimes B = \mathbf{0}$. If $A < \mathbf{0}$ but $\mathbf{0} < B$, we replace A with $-A$ and use the definition of multiplication of positive elements. Hence, in this case,

$$A \otimes B = -[(-A) \otimes B]$$

Similarly, if $\mathbf{0} < A$ but $B < \mathbf{0}$, then

$$A \otimes B = -[A \otimes (-B)]$$

and if $A < \mathbf{0}$, $B < \mathbf{0}$, then

$$A \otimes B = (-A) \otimes (-B)$$

Exercise 7.40. [1]

(a) Show that if $A, B \in \mathbb{R}$, then $A \otimes B \in \mathbb{R}$.

(b) Show that $\otimes$ is commutative and associative.

(c) Show that if $A, B \in \mathbb{R}$, $\mathbf{0} < A$, and $\mathbf{0} < B$, then $\mathbf{0} < A \otimes B$.

(d) Let $\mathbf{1} = \{x \in \mathbb{Q} \mid x < 1\}$. Show that if $A \in \mathbb{R}$, then $\mathbf{1} \otimes A = A$.

Proof of a. To prove that $A \otimes B$ where $\mathbf{0} < A, \mathbf{0} < B$ are Dedekind cuts, Definition 6.1 tells us that it will suffice to show that $A \otimes B \neq \emptyset$; $A \otimes B \neq \mathbb{Q}$; if $r \in A \otimes B$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A \otimes B$; and if $r \in A \otimes B$, then there is some $s \in A \otimes B$ with $s > r$. We will take this one claim at a time.

To show that $A \otimes B \neq \emptyset$, Definition 1.8 tells us that it will suffice to find an element of $A \otimes B$. Since $0 \in \mathbb{Q}$ and $0 \leq 0$, $0 \in \{r \in \mathbb{Q} \mid r \leq 0\}$. It follows by Definition 1.5 that $0 \in \{r \in \mathbb{Q} \mid r \leq 0\} \cup \{ab \mid a \in A, b \in B, a > 0, b > 0\}$. Therefore, by Definition 7.39, $0 \in A \otimes B$, as desired.

To show that $A \otimes B \neq \mathbb{Q}$, Definition 1.2 tells us that it will suffice to find an element of $\mathbb{Q}$ that is not an element of $A \otimes B$. Since $\mathbf{0} < A$ and $\mathbf{0} < B$, Definitions 6.4 and 1.3 assert that there exist points $a \in A$ and $b \in B$ such that $a, b \notin \mathbf{0}$, i.e., $a, b \geq 0$. Furthermore, since a, b are not the last points of A, B, respectively, by Definition 6.1c, there exist points $c \in A$ and $d \in B$ such that $c > 0$ and $d > 0$. Now, for an analogous reason to before, we can choose $x, y \in \mathbb{Q}$ such that $x \notin A$ and $y \notin B$. It follows by Lemma 6.2 and Definition 5.6 that $x \geq e$ for all $e \in A$ and $y \geq f$ for all $f \in B$, meaning (when combined with the last result) that $x > 0$ and $y > 0$. Thus, $xy > 0$, so $xy \notin \{r \in \mathbb{Q} \mid r \leq 0\}$. Additionally, we have by Script 2 that $xy > ef$ for all ef formed from the product of positive elements of A and B. Thus, $xy \notin \{ab \mid a \in A, b \in B, a > 0, b > 0\}$. Therefore, by Definitions 1.5 and 7.39, $xy \notin A \otimes B$, as desired.

To show that if $r \in A \otimes B$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in A \otimes B$, we let $r \in A \otimes B$ and $s \in \mathbb{Q}$ be arbitrary elements of their respective sets that satisfy $s < r$ and seek to verify that $s \in A \otimes B$. We divide into two cases ($s \leq 0$ and $s > 0$). Suppose first that $s \leq 0$. Then $s \in \{r \in \mathbb{Q} \mid r \leq 0\}$. It follows by Definition 1.5 that $0 \in \{r \in \mathbb{Q} \mid r \leq 0\} \cup \{ab \mid a \in A, b \in B, a > 0, b > 0\}$. Therefore, by Definition 7.39, $0 \in A \otimes B$. Now suppose that $s > 0$. Then $r > 0$. Since $r \in A \otimes B$ and $r > 0$, $r = xy$ where $x \in A, y \in B, x > 0, y > 0$. Additionally, it follows from the fact that $s < r$ that $s = r - q = xy - q = (x - \frac{q}{y})y$ for some $q \in \mathbb{Q}^+$. Since $x - \frac{q}{y} \in \mathbb{Q}$ and $x - \frac{q}{y} < x$, we have by Definition 6.1b that $x - \frac{q}{y} \in A$. Therefore, $s = (x - \frac{q}{y})(y)$ is an element of $\{ab \mid a \in A, b \in B, a > 0, b > 0\}$, and hence by Definition 7.39, $A \otimes B$.

To show that if $r \in A \otimes B$, then there is some $s \in A \otimes B$ with $s > r$, we let $r \in A \otimes B$ and seek to find such an s. We divide into two cases ($r \leq 0$ and $r > 0$). Suppose first that $r \leq 0$. Then for the same reasons outlined in the proof of the second condition, there exist positive elements of $A \otimes B$ that are greater than r. Now suppose that $r > 0$. This implies that $r = xy$ for some $x \in A, y \in B, x > 0, y > 0$. It follows from the fact that $x \in A$ by Definition 6.1c that there exists a $z \in A$ with $z > x$. Consequently, by Lemma 7.24[2], $zy > xy$ is the desired element of $A \otimes B$. ☐

Proof of b. To prove that $\otimes$ is commutative for $\mathbf{0} < A, \mathbf{0} < B$, Definition 7.1 tells us that it will suffice to show that for all such $A, B \in \mathbb{R}$, we have $A \otimes B = B \otimes A$. Let A, B be arbitrary elements of $\mathbb{R}$ where $\mathbf{0} < A, \mathbf{0} < B$. Then by Definition 7.39, we clearly have

$$\begin{aligned} A \otimes B &= \{r \in \mathbb{Q} \mid r \leq 0\} \cup \{ab \mid a \in A, b \in B, a > 0, b > 0\} \\ &= \{r \in \mathbb{Q} \mid r \leq 0\} \cup \{ba \mid b \in B, a \in A, b > 0, a > 0\} \\ &= B \oplus A \end{aligned}$$

To prove that $\otimes$ is associative for $\mathbf{0} < A, \mathbf{0} < B, \mathbf{0} < C$, Definition 7.1 tells us that it will suffice to show that for all such $A, B, C \in \mathbb{R}$, we have $(A \otimes B) \otimes C = A \otimes (B \otimes C)$. Let A, B, C be arbitrary elements of $\mathbb{R}$ where $\mathbf{0} < A, \mathbf{0} < B, \mathbf{0} < C$. To show that $(A \otimes B) \otimes C = A \otimes (B \otimes C)$, Definition 1.2 tells us that it will suffice to verify that every element of $(A \otimes B) \otimes C$ is an element of $A \otimes (B \otimes C)$ and vice versa. Suppose first that $x \in (A \otimes B) \otimes C$. Then by Definition 7.39, $x \leq 0$ or $x = dc$ where $d \in A \otimes B, c \in C, d > 0, c > 0$. If $x \leq 0$, then by Definition 7.39, $x \in A \otimes (B \otimes C)$ since it's an element of $\{r \in \mathbb{Q} \mid r \leq 0\}$, as desired. On the

[1] Note that the proofs given here only address the case where $\mathbf{0} < A$ and $\mathbf{0} < B$.

[2] And, technically, Theorem 7.29.

other hand, if $x = dc$ where $d \in A \otimes B, c \in C, d > 0, c > 0$, we continue. Now $d \in A \otimes B$ implies that $d \leq 0$ or $d = ab$ where $a \in A, b \in B, a > 0, b > 0$. However, the prior constraint that $d > 0$ guarantees that $d \nleq 0$, so we know that $d = ab$ where a, b satisfy the above conditions. Combining the last two results, we have $x = (ab)(c)$ where $a \in A, b \in B, c \in C, a > 0, b > 0, c > 0$. It follows that we also have $x = (a)(bc)$ under the same conditions. If we let $e = bc$ where $b \in B, c \in C, b > 0, c > 0$, then $e \in \{bc \mid b \in B, c \in C, b > 0, c > 0\}$. Consequently, by Definition 7.31, $e \in B \otimes C$. Additionally, $b > 0, c > 0$ imply by Definition 7.21 that $e > 0$. To recap, at this point we have $x = ae$ where $a \in A, e \in B \otimes C, a > 0, e > 0$. It follows by a similar process to before that $x \in A \otimes (B \otimes C)$. The proof is symmetric in the other direction. □

Proof of c. To prove that $\mathbf{0} < A \otimes B$, Definitions 6.4 and 1.3 tell us that it will suffice to show that every $x \in \mathbf{0}$ is an element of $A \otimes B$ and find an $x \in A \otimes B$ such that $x \notin \mathbf{0}$. Let x be an arbitrary element of $\mathbf{0}$. Then $x \in \mathbb{Q}$ and $x < 0$. But it follows that $x \in \{r \in \mathbb{Q} \mid r \leq 0\}$, which implies by Definition 7.39 that $x \in A \otimes B$. As to the other stipulation, clearly $0 \in \{r \in \mathbb{Q} \mid r \leq 0\}$ but since $0 \nless 0$, $0 \notin \mathbf{0}$. Therefore, by Definition 7.39, $0 \in A \otimes B$, but $0 \notin \mathbf{0}$, as desired. □

Proof of d. To prove that $\mathbf{1} \otimes A = A$, Definition 1.2 tells us that it will suffice to show that every $x \in \mathbf{1} \otimes A$ is an element of A and vice versa.

Let x be an arbitrary element of $\mathbf{1} \otimes A$. Then by Definition 7.39, $x \leq 0$ or $x = da$ where $d \in \mathbf{1}, a \in A, d > 0, a > 0$. We now divide into two cases. Suppose first that $x \leq 0$. We divide into two cases again ($x < 0$ and $x = 0$). If $x < 0$, then $x \in \mathbf{0}$, which implies by Definitions 6.4, 1.3, and the fact that $\mathbf{0} < A$ that $x \in A$, as desired. On the other hand, if $x = 0$, suppose for the sake of contradiction that $x \notin A$. Then by Lemma 6.2 and Definition 5.6, $a \leq x$ for all $a \in A$. This combined with the fact that $x \notin A$ implies that $a < x$ for all $a \in A$. Consequently, since $\mathbf{0} = \{q \in \mathbb{Q} \mid q < 0\}$, it follows that $A \subset \mathbf{0}$. But by Definition 6.4, this implies $A \leq \mathbf{0}$, contradicting the fact that $\mathbf{0} < A$, as desired. Now suppose that $x = da$ where $d \in \mathbf{1}, a \in A, d > 0, a > 0$. Then by Script 2, $d < 1$ implies that $x = da < a$. Therefore, by Definition 6.1b, $x \in A$, as desired.

Let x be an arbitrary element of A. We divide into two cases ($x \leq 0$ and $x > 0$). If $x \leq 0$, then $x \in \{r \in \mathbb{Q} \mid r \leq 0\}$, which implies by Definition 7.39 that $x \in \underline{1} \otimes A$. On the other hand, suppose $x > 0$. Then by Definition 6.1c, there is some $y \in A$ with $y > x$. It follows by Script 2 that $1 > \frac{x}{y} > 0$, so we have that $\frac{x}{y} \in \mathbf{1}$. Thus, since $x = \frac{x}{y} \cdot y$, we know that x is the product of a positive element of $\mathbf{1}$ and a positive element of A (since $y > x > 0$). Therefore, $x \in \{da \mid d \in \mathbf{1}, a \in A, d > 0, a > 0\}$, which implies by Definition 7.39 that $x \in \mathbf{1} \otimes A$. □