# Script 12

# Derivatives

## 12.1 Journal

3/30: Throughout this sheet, we let $f : A \to \mathbb{R}$ be a real valued function with domain $A \subset \mathbb{R}$. We also now assume the domain $A \subset \mathbb{R}$ is open.

**Definition 12.1.** The **derivative** of $f$ at a point $a \in A$ is the number $f'(a)$ defined by the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided that the limit on the right-hand side exists. If $f'(a)$ exists, we say that $f$ is **differentiable** (at $a$). If $f$ is differentiable at all points of its domain, we say that $f$ is **differentiable**. In this case, the values $f'(a)$ define a new function $f' : A \to \mathbb{R}$ called the **derivative** (of $f$).

**Remark 12.2.** If $A$ is not open, the limit in Definition 12.1 may not exist. For example, if $f : [a, b] \to \mathbb{R}$, then we cannot define the derivative at the endpoints. For any $c$ in the domain of $f$, we define the **right-hand derivative** $f'_+(c)$ and the **left-hand derivative** $f'_-(c)$ by

$$f'_+(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \qquad f'_-(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$$

We say that $f$ is **differentiable** (on $[a, b]$) if $f$ is differentiable on $(a, b)$ and $f'_+(a)$ and $f'_-(b)$ exist.

**Lemma 12.3.** *Let $a \in \mathbb{R}$. Then*

$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$$

*assuming that one of the two limits exists. (So if the limit on the left exists, so does the one on the right, and they are equal. Similarly, if the limit on the right exists, then so does the one on the left, and they are equal.)*

*Proof.* Suppose first that $\lim_{x \to a} f(x)$ exists, and let it be equal to $L$. To prove that $\lim_{h \to 0} f(a+h)$ exists and that it equals $\lim_{x \to a} f(x)$, Definition 11.1 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $(h + a) \in A$ and $0 < |h - 0| = |h| < \delta$, then $|f(a+h) - L| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{x \to a} f(x)$ exists, Definition 11.1 implies that there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$. We will choose this $\delta$ to be our $\delta$. Now suppose that $h$ is any number satisfying both $(h + a) \in A$ and $0 < |h| < \delta$; we seek to show that $|f(a+h) - L| < \epsilon$. Since $(h + a) \in A$, $h + a = x$ for some $x \in A$. It follows that $h = x - a$, meaning that $x$ is an object that is both an element of $A$ and that satisfies $0 < |h| = |x - a| < \delta$, so we know that $|f(a+h) - L| = |f(x) - L| < \epsilon$, as desired.

The proof is symmetric in the other direction. $\square$

**Theorem 12.4.** *Let $a \in \mathbb{R}$. Then $f$ is differentiable at $a$ if and only if $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists. Moreover, if $f$ is differentiable at $a$, then the derivative of $f$ at $a$ is given by the limit*

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

*Proof.* Suppose first that $f$ is differentiable at $a$. Then by Definition 12.1, $f'(a)$ exists. It follows that

$$\begin{aligned} f'(a) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} && \text{Definition 12.1} \\ &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{(a+h)-a} \\ &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} && \text{Lemma 12.3} \end{aligned}$$

Note that the substitution in the last step follows from using $\tilde{f}(x) = \frac{f(x)-f(a)}{x-a}$ as the "$f(x)$" function in Lemma 12.3.

The proof is symmetric in the reverse direction. □

**Theorem 12.5.** *If $f$ is differentiable at $a$, then $f$ is continuous at $a$.*

*Proof.* To prove that $f$ is continuous at $a$, Theorem 11.5 tells us that it will suffice to show that $\lim_{x\to a} f(x) = f(a)$. By Definition 12.1, the hypothesis implies that $f'(a)$ exists. Additionally, by Exercise 11.6, $g(x) = x-a$ is continuous at $a$. Thus, by Theorem 11.5, $\lim_{x\to a} g(x)$ exists (and equals $g(a)$). Consequently, knowing that both $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ and $\lim_{x\to a} g(x)$ exist (the former by Theorem 12.4), we have by Theorem 11.9 that the limit of their product exists and equals

$$\lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\cdot(x-a)\right) = \lim_{x\to a}(f(x)-f(a))$$

Moreover, since $g$ is continuous at $a$, $\lim_{x\to a} g(x) = g(a) = a-a = 0$. Thus,

$$\begin{aligned} \lim_{x\to a}(f(x)-f(a)) &= \left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right)\left(\lim_{x\to a}(x-a)\right) \\ &= \left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right)\cdot 0 \\ &= 0 \end{aligned}$$

But then it follows by Theorem 11.9 if we consider $f(a)$ to be a constant function that

$$\begin{aligned} \lim_{x\to a}(f(x)-f(a)) + f(a) &= 0 + f(a) \\ \lim_{x\to a}(f(x)-f(a)+f(a)) &= f(a) \\ \lim_{x\to a} f(x) &= f(a) \end{aligned}$$

as desired. □

**Exercise 12.6.** Show that the converse of Theorem 12.5 is not true.

*Proof.* The converse of Theorem 12.5 asserts that "if $f$ is continuous at $a$, then $f$ is differentiable at $a$." To falsify this statement, we will use the absolute value function $|x|$ as a counterexample. Let's begin.

By Exercise 11.7, $|x|$ is continuous. It follows by Theorem 9.10 that $|x|$ is continuous at 0. However, we can show that $|x|$ is not differentiable at 0.

To do this, Definition 12.1 and Theorem 12.4 tell us that it will suffice to verify that $\lim_{x\to 0}\frac{|x|-|0|}{x-0} = \lim_{x\to 0}\frac{|x|}{x}$ does not exist. Suppose for the sake of contradiction that $\lim_{x\to 0}\frac{|x|}{x} = L$. Then by Definition 11.1, for $\epsilon = 1 > 0$, there exists a $\delta > 0$ such that if $0 < |x-0| = |x| < \delta$, then $|\frac{|x|}{x} - L| < 1$. However, we can show that no such $\delta$ exists. Let $\delta > 0$ be arbitrary. By Theorem 5.2, there exists a number $x \in \mathbb{R}$ such that $0 < x < \delta$. It follows by Definition 8.4 and Exercise 8.5 that $0 < |x| = |-x| < \delta$. Since both $x$ and $-x$ are in the appropriate range, we know that

$$\begin{aligned}
\left|\frac{|x|}{x} - L\right| &= \left|\frac{x}{x} - L\right| & \left|\frac{|-x|}{-x} - L\right| &= \left|\frac{x}{-x} - L\right| && \text{Definition 8.4}\\
&= |1 - L| & &= |-1 - L| && \text{Script 7}\\
&= |L - 1| & &= |L + 1| && \text{Exercise 8.5}\\
&< 1 & &< 1 &&
\end{aligned}$$

By consecutive applications of the lemma from Exercise 8.9, it follows that

$$\begin{aligned}
-1 &< L - 1 < 1 & -1 &< L + 1 < 1\\
0 &< L < 2 & -2 &< L < 0
\end{aligned}$$

But this implies that $L < 0$ and $L > 0$, a contradiction. $\square$

**Exercise 12.7.** Show that for all $n \in \mathbb{N}$,

$$x^n - a^n = (x - a)\left(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1}\right)$$

or equivalently,

$$x^n - a^n = (x - a)\left(\sum_{i=0}^{n-1} x^{n-1-i}a^i\right)$$

*Proof.* By simple algebra (see Script 7), we have

$$\begin{aligned}
(x - a)\left(\sum_{i=0}^{n-1} x^{n-1-i}a^i\right) &= \sum_{i=0}^{n-1}(x - a)x^{n-1-i}a^i\\
&= \sum_{i=0}^{n-1}\left(x^{n-i}a^i - x^{n-1-i}a^{i+1}\right)\\
&= x^n + \sum_{i=1}^{n-1} x^{n-i}a^i - \sum_{i=0}^{n-2} x^{n-1-i}a^{i+1} - a^n\\
&= x^n + \sum_{i=1}^{n-1} x^{n-i}a^i - \sum_{i=0+1}^{n-2+1} x^{n-1-(i-1)}a^{(i-1)+1} - a^n\\
&= x^n + \sum_{i=1}^{n-1} x^{n-i}a^i - \sum_{i=1}^{n-1} x^{n-i}a^i - a^n\\
&= x^n - a^n
\end{aligned}$$

$\square$

**Exercise 12.8.**

(a) Let $n \in \mathbb{N}$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^n$. Use Exercise 12.7 to prove that $f'(a) = na^{n-1}$ for all $a \in \mathbb{R}$.

*Proof.* To prove that $f'(a) = na^{n-1}$ for all $a \in \mathbb{R}$, Theorem 12.4 tell us that it will suffice to show that $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = na^{n-1}$ for all $a \in \mathbb{R}$. Let $a$ be an arbitrary element of $\mathbb{R}$. By Corollary 11.12, the polynomial $\sum_{i=0}^{n-1} x^{n-1-i}a^i$ is continuous. Thus, by Theorem 9.10, it is continuous at $a$. It follows by Theorem 11.5 that $\lim_{x\to a} \sum_{i=0}^{n-1} x^{n-1-i}a^i = \sum_{i=0}^{n-1} a^{n-1-i}a^i$. Therefore,

$$\begin{aligned}
na^{n-1} &= \underbrace{a^{n-1} + \cdots + a^{n-1}}_{n \text{ times}} \\
&= \sum_{i=0}^{n-1} a^{n-1} \\
&= \sum_{i=1}^{n-1} a^{n-1-i}a^i \\
&= \lim_{x\to a} \sum_{i=0}^{n-1} x^{n-1-i}a^i \\
&= \lim_{x\to a} \frac{x-a}{x-a} \cdot \sum_{i=0}^{n-1} x^{n-1-i}a^i
\end{aligned}$$

Note that we can make the above change because $\frac{x-a}{x-a} = 1$ for all $x$ satisfying $0 < |x-a| < \delta$, whatever $\delta$ may be.

$$\begin{aligned}
&= \lim_{x\to a} \frac{x^n - a^n}{x-a} && \text{Exercise 12.7} \\
&= \lim_{x\to a} \frac{f(x) - f(a)}{x-a} \\
&= f'(a) && \text{Theorem 12.4}
\end{aligned}$$

as desired. □

(b) Let $k \in \mathbb{R}$. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = k$, then $f'(a) = 0$ for all $a \in \mathbb{R}$.

*Proof.* To prove that $f'(a) = 0$ for all $a \in \mathbb{R}$, Definition 12.1 tells us that it will suffice to show that $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = 0$ for all $a \in \mathbb{R}$. Let $a$ be an arbitrary element of $\mathbb{R}$. By Exercise 11.6, the function $g(x) = 0$ is continuous. Thus, by Theorem 9.10, it is continuous at $a$. It follows by Theorem 11.5 that $\lim_{h\to 0} g(h) = g(0) = 0$. Therefore,

$$\begin{aligned}
0 &= \lim_{h\to 0} g(h) \\
&= \lim_{h\to 0} 0 \\
&= \lim_{h\to 0} \frac{h}{h} \cdot 0
\end{aligned}$$

Note that we can make the above change for the same reason as part (a).

$$\begin{aligned}
&- \lim_{h\to 0} \frac{0}{h} \\
&= \lim_{h\to 0} \frac{k-k}{h} \\
&= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}
\end{aligned}$$

as desired. □

**Exercise 12.9.** Suppose that $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are differentiable at $a \in A$.

(a) Prove that $f+g$ is differentiable at $a$ and compute $(f+g)'(a)$ in terms of $f'(a)$ and $g'(a)$.

*Proof.* To prove that $f+g$ is differentiable at $a$, Definition 12.1 tells us that it will suffice to show $\lim_{h\to 0} \frac{(f+g)(a+h)-(f+g)(a)}{h}$ exists. Since $f,g$ are differentiable at $a$, we know by Definition 12.1 that $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ and $\lim_{h\to 0} \frac{g(a+h)-g(a)}{h}$ exist. Thus, by Theorem 11.9 the limit of their sum exists and equals

$$\begin{aligned}\lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}+\frac{g(a+h)-g(a)}{h}\right) &= \lim_{h\to 0}\frac{f(a+h)+g(a+h)-f(a)-g(a)}{h}\\ &= \lim_{h\to 0}\frac{(f+g)(a+h)-(f+g)(a)}{h}\end{aligned}$$

as desired. Having established that $\lim_{h\to 0} \frac{(f+g)(a+h)-(f+g)(a)}{h}$ exists, $(f+g)'(a)$ can be computed in terms of $f'(a)$ and $g'(a)$ with the following algebra.

$$\begin{aligned}(f+g)'(a) &= \lim_{h\to 0}\frac{(f+g)(a+h)-(f+g)(a)}{h} && \text{Definition 12.1}\\ &= \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}+\frac{g(a+h)-g(a)}{h}\right)\\ &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}+\lim_{h\to 0}\frac{g(a+h)-g(a)}{h} && \text{Theorem 11.9}\\ &= f'(a)+g'(a) && \text{Definition 12.1}\end{aligned}$$

□

(b) Prove that $fg$ is differentiable and compute $(fg)'(a)$ in terms of $f(a)$, $g(a)$, $f'(a)$, and $g'(a)$.

*Proof.* To prove that $fg$ is differentiable at $a$, Definition 12.1 tells us that it will suffice to show $\lim_{h\to 0} \frac{(fg)(a+h)-(fg)(a)}{h}$ exists. Since $f,g$ are differentiable at $a$, we know by Definition 12.1 that $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ and $\lim_{h\to 0} \frac{g(a+h)-g(a)}{h}$ exist. For the same reason, we know by Theorem 12.5 that $g$ is continuous, i.e., continuous at $a$ by Theorem 9.10. Consequently, by Theorem 11.5, $\lim_{x\to a} g(x)$ exists (and equals $g(a)$). Note that the preceding limit is equal to $\lim_{h\to 0} g(a+h)$ by Lemma 12.3. Lastly, we have by Exercise 11.6 that the constant function $f(a)$ is continuous at 0. Consequently, by Theorem 11.5, $\lim_{h\to 0} f(a)$ exists (and equals $f(a)$). Combining all of these results, consecutive applications of Theorem 11.9 assert that the limits

$$\lim_{h\to 0} g(a+h)\cdot\frac{f(a+h)-f(a)}{h} \qquad \lim_{h\to 0} f(a)\cdot\frac{g(a+h)-g(a)}{h}$$

exist. Furthermore, it asserts that the limit of their sum exists and equals

$$\begin{aligned}&\lim_{h\to 0}\left(g(a+h)\cdot\frac{f(a+h)-f(a)}{h}+f(a)\cdot\frac{g(a+h)-g(a)}{h}\right)\\ &\qquad= \lim_{h\to 0}\frac{g(a+h)(f(a+h)-f(a))+f(a)(g(a+h)-g(a))}{h}\\ &\qquad= \lim_{h\to 0}\frac{f(a+h)g(a+h)-f(a)g(a+h)+f(a)g(a+h)-f(a)g(a)}{h}\\ &\qquad= \lim_{h\to 0}\frac{f(a+h)g(a+h)-f(a)g(a)}{h}\\ &\qquad= \lim_{h\to 0}\frac{(fg)(a+h)-(fg)(a)}{h}\end{aligned}$$

as desired. Having established that $\lim_{h\to 0}\frac{(fg)(a+h)-(fg)(a)}{h}$ exists, $(fg)'(a)$ can be computed in terms of $f(a)$, $g(a)$, $f'(a)$, and $g'(a)$ with the following algebra.

$$\begin{aligned}
(fg)'(a) &= \lim_{h\to 0}\frac{(fg)(a+h)-(fg)(a)}{h} && \text{Definition 12.1}\\
&= \lim_{h\to 0}\left(g(a+h)\cdot\frac{f(a+h)-f(a)}{h}+f(a)\cdot\frac{g(a+h)-g(a)}{h}\right)\\
&= \lim_{h\to 0} g(a+h)\cdot\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}+\lim_{h\to 0} f(a)\cdot\lim_{h\to 0}\frac{g(a+h)-g(a)}{h} && \text{Theorem 11.9}\\
&= g(a)f'(a)+f(a)g'(a)
\end{aligned}$$

□

(c) Prove that $\frac{1}{g}$ is differentiable at $a$ (under an appropriate assumption) and compute $(\frac{1}{g})'(a)$ in terms of $g'(a)$ and $g(a)$. What assumption do you need to make?

*Proof.* Assume, in addition to the fact that $g : A \to \mathbb{R}$ is differentiable at $a \in A$, that $g(a) \neq 0$.

To prove that $\frac{1}{g}$ is differentiable at $a$, Definition 12.1 tells us that it will suffice to show $\lim_{h\to 0}\frac{(\frac{1}{g})(a+h)-(\frac{1}{g})(a)}{h}$ exists. Since $g$ is differentiable at $a$, we know by Definition 12.1 that $\lim_{h\to 0}\frac{g(a+h)-g(a)}{h}$ exists. For the same reason, we know by Theorem 12.5 that $g$ is continuous, i.e., continuous at $a$ by Theorem 9.10. Consequently, by Theorem 11.5, $\lim_{x\to a} g(x)$ exists (and equals $g(a)$). It follows by Lemma 12.3 that the preceding limit is equal to $\lim_{h\to 0} g(a+h)$. Thus, since it is also equal to $g(a)\neq 0$, we have by Theorem 11.9 that $\lim_{h\to 0}\frac{1}{g}(a+h)$ exists (and equals $\frac{1}{g(a)}$). Lastly, we have by Exercise 11.6 that the constant function $-\frac{1}{g(a)}$ is continuous at 0. Consequently, by Theorem 11.5, $\lim_{h\to 0} -\frac{1}{g(a)}$ exists (and equals $-\frac{1}{g(a)}$). Combining this with the previous result, Theorem 11.9 asserts that the limit $\lim_{h\to 0} -\frac{1}{g(a+h)g(a)}$ exists (and equals $-\frac{1}{g(a)^2}$). Furthermore, it asserts that the limit of its product with $\lim_{h\to 0}\frac{(\frac{1}{g})(a+h)-(\frac{1}{g})(a)}{h}$ exists and equals

$$\begin{aligned}
\lim_{h\to 0} -\frac{1}{g(a+h)g(a)}\cdot\frac{g(a+h)-g(a)}{h} &= \lim_{h\to 0}\frac{\frac{g(a)-g(a+h)}{g(a+h)g(a)}}{h}\\
&= \lim_{h\to 0}\frac{\frac{1}{g(a+h)}-\frac{1}{g(a)}}{h}\\
&= \lim_{h\to 0}\frac{(\frac{1}{g})(a+h)-(\frac{1}{g})(a)}{h}
\end{aligned}$$

as desired. Having established that $\lim_{h\to 0}\frac{(\frac{1}{g})(a+h)-(\frac{1}{g})(a)}{h}$ exists, $(\frac{1}{g})'(a)$ can be computed in terms of $g(a)$ and $g'(a)$ with the following algebra.

$$\begin{aligned}
\left(\frac{1}{g}\right)'(a) &= \lim_{h\to 0}\frac{\left(\frac{1}{g}\right)(a+h)-\left(\frac{1}{g}\right)(a)}{h} && \text{Definition 12.1}\\
&= \lim_{h\to 0} -\frac{1}{g(a+h)g(a)}\cdot\frac{g(a+h)-g(a)}{h}\\
&= \lim_{h\to 0} -\frac{1}{g(a+h)g(a)}\cdot\lim_{h\to 0}\frac{g(a+h)-g(a)}{h} && \text{Theorem 11.9}\\
&= -\frac{g'(a)}{g(a)^2}
\end{aligned}$$

□

(d) Prove that $\frac{f}{g}$ is differentiable at $a$ (under an appropriate assumption) and compute $(\frac{f}{g})'(a)$ in terms of $f(a)$, $g(a)$, $f'(a)$, and $g'(a)$. What assumption do you need to make?

*Proof.* Assume, in addition to the fact that $g : A \to \mathbb{R}$ is differentiable at $a \in A$, that $g(a) \neq 0$.

It follows by part (c) that $\frac{1}{g}$ is differentiable at $a$, and then by part (b) that $f \cdot \frac{1}{g} = \frac{f}{g}$ is differentiable at $a$.

Having established that $(\frac{f}{g})'(a)$ exists, it can be computed in terms of $f(a)$, $g(a)$, $f'(a)$, and $g'(a)$ with the following algebra.

$$\begin{aligned}
\left(\frac{f}{g}\right)'(a) &= \left(f \cdot \frac{1}{g}\right)'(a) \\
&= f(a)\left(\frac{1}{g}\right)'(a) + f'(a)\left(\frac{1}{g}\right)(a) \\
&= f(a) \cdot -\frac{g'(a)}{g(a)^2} + \frac{f'(a)g(a)}{g(a)^2} \\
&= \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}
\end{aligned}$$

□

4/1: One of the most important results concerning the differentiation of functions is the rule for the derivative of a composition of functions. Let $f : B \to \mathbb{R}, g : A \to \mathbb{R}$ be functions such that $g(A) \subset B$. The composition $(f \circ g)(x) = f(g(x))$ is defined for all $x \in A$.

**Theorem 12.10.** *Let $a \in A$, $g : A \to \mathbb{R}$, and $f : I \to \mathbb{R}$ where $I$ is an interval containing $g(A)$. Suppose that $g$ is differentiable at $a$ and $f$ is differentiable at $g(a)$. Then $f \circ g$ is differentiable $a$ and*

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$$

*Proof.* To prove that $f \circ g$ is differentiable at $a$, Theorem 12.4 tells us that it will suffice to show that $\lim_{x\to a} \frac{(f\circ g)(x)-(f\circ g)(a)}{x-a}$ exists. To do so, we will define a special function $\varphi$ and prove that it is continuous at $a$. It will follow that $(f \circ g)'(a)$ exists and equals $f'(g(a)) \cdot g'(a)$. Let's begin.

Let $\varphi : I \to \mathbb{R}$ be defined by

$$\varphi(x) = \begin{cases} \frac{f(g(x))-f(g(a))}{g(x)-g(a)} & g(x) \neq g(a) \\ f'(g(a)) & g(x) = g(a) \end{cases}$$

It is clear from the definition that the function is defined for all $x \in A$.

To demonstrate that $\varphi$ is continuous at $a$, Theorem 11.5 tells us that it will suffice to confirm that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $|x-a| < \delta$, then $|\varphi(x)-\varphi(a)| = |\varphi(x)-f'(g(a))| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $f$ is differentiable at $g(a)$, we know that $\lim_{y\to g(a)} \frac{f(y)-f(g(a))}{y-g(a)} = f'(g(a))$. It follows by Definition 11.1 that there is some $\delta' > 0$ such that if $y \in I$ and $0 < |y - g(a)| < \delta'$, then $|\frac{f(y)-f(g(a))}{y-g(a)} - f'(g(a))| < \epsilon$. Additionally, since $g$ is differentiable (hence continuous by Theorem 12.5) at $a$, we have by Theorem 11.5 that there exists a $\delta > 0$ such that if $x \in A$ and $|x-a| < \delta$, then $|g(x)-g(a)| < \delta'$.

Using the above $\delta$, let $x$ be an arbitrary element of $A$ such that $|x-a| < \delta$. We now divide into two cases ($g(x) = g(a)$ and $g(x) \neq g(a)$). If $g(x) = g(a)$, then $|\varphi(x) - f'(g(a))| = |f'(g(a)) - f'(g(a))| = 0 < \epsilon$, as desired. If $g(x) \neq g(a)$, then we continue. Since $|x - a| < \delta$, we have that $|g(x) - g(a)| < \delta'$. This combined with the fact that $g(x) \in I$ and $g(x) \neq g(a)$, i.e., $0 < |g(x)-g(a)|$ illustrates that $|\frac{f(g(x))-f(g(a))}{g(x)-g(a)} - f'(g(a))| = |\varphi(x) - f'(g(a))| < \epsilon$. Therefore, $\varphi$ is continuous at $a$.

It follows by Theorem 11.5 that $\lim_{x\to a} \varphi(x) = \varphi(a) = f'(g(a))$. Additionally, since $g$ is differentiable at $a$, Definition 12.1 and Theorem 12.4 tell us that $\lim_{x\to a} \frac{g(x)-g(a)}{x-a}$ exists (and equals $g'(a)$). The combination of the past two results imply by Theorem 11.9 that the product of the limits exists and equals

$$\begin{aligned}
f'(g(a)) \cdot g'(a) &= \lim_{x \to a} \varphi(x) \cdot \frac{g(x) - g(a)}{x - a} \\
&= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} \\
&= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} \\
&= \lim_{x \to a} \frac{(f \circ g)(x) - (f \circ g)(a)}{x - a} \\
&= (f \circ g)'(a)
\end{aligned}$$

as desired. $\square$