Script 13

Uniform Continuity and Integration

13.1 Journal

4/8: **Definition 13.1.** Let $f : A \to \mathbb{R}$ be a function. We say that f is **uniformly continuous** if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$.

Theorem 13.2. *If f is uniformly continuous, then f is continuous.*

Proof. To prove that f is continuous, Theorem 9.10 tells us that it will suffice to show that f is continuous at every $x \in A$. Let x be an arbitrary element of A. To show that f is continuous at x, Theorem 11.5 tells us that it will suffice to verify that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Then since f is uniformly continuous by hypothesis, Definition 13.1 asserts that there exists a $\delta > 0$ such that for all $y \in A$ satisfying $|y - x| < \delta$, we have $|f(y) - f(x)| < \epsilon$, as desired. □

Exercise 13.3. Determine with proof whether each function f is uniformly continuous on the given interval A.

(a) $f(x) = x^2$ on $A = \mathbb{R}$.

Proof. To prove that f is not uniformly continuous on $\mathbb{R}$, Definition 13.1 tells us that it will suffice to find an $\epsilon > 0$ for which no $\delta > 0$ exists such that for all $x, y \in \mathbb{R}$, if $|y - x| < \delta$, then $|y^2 - x^2| < \epsilon$. Let $\epsilon = 2$, and suppose for the sake of contradiction that $\delta > 0$ is a number such that for all $x, y \in \mathbb{R}$, if $|y - x| < \delta$, then $|y^2 - x^2| < 2$. By Theorem 5.2, there exists a number y such that $0 < y < \delta$. Since $-\delta < 0 < y < \delta$ by Lemma 7.23, it follows by Definitions 3.6 and 3.10 that $y \in (-\delta, \delta)$. Thus, by Exercise 8.9, $|y - 0| = |y| < \delta$. Consequently, $|(y + n) - n| < \delta$. It follows by the above that $|(y + n)^2 - n^2| = |y^2 + 2yn| < 2$. If we now let $n = \frac{1}{y}$, then $|y^2 + 2| < 2$. But since $y > 0$, we have that $y^2 > 0$ by Lemma 7.26. It follows that $y^2 + 2 > 2$ by Definition 7.21. Therefore, by Definition 8.4, we can also show that $|y^2 + 2| > 2$, a contradiction. □

(b) $f(x) = x^2$ on $A = (-2, 2)$.

Proof. To prove that f is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \frac{\epsilon}{4}$, and let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. Since $x, y \in A$, consecutive applications of Equations 8.1 and the lemma from Exercise 8.9 imply that $|x| < 2$ and $|y| < 2$. It follows that $|x| + |y| < 2 + 2 = 4$. Consequently, by Lemma 8.8, $|x + y| < 4$. Additionally, since $0 \leq |y + x|$ by Definition 8.4, we have $|x - y| \cdot |x + y| \leq \frac{\epsilon}{4} \cdot |x + y|$. Combining all of the above results, we have that

$$\begin{aligned} |f(y) - f(x)| &= |y^2 - x^2| \\ &= |y + x| \cdot |y - x| \\ &< 4 \cdot |y - x| \\ &\leq 4 \cdot \frac{\epsilon}{4} \\ &= \epsilon \end{aligned}$$

as desired. □

(c) $f(x) = \frac{1}{x}$ on $A = (0, +\infty)$.

Proof. To prove that f is not uniformly continuous on A, Definition 13.1 tells us that it will suffice to find an $\epsilon > 0$ for which no $\delta > 0$ exists such that for all $x, y \in A$, if $|y - x| < \delta$, then $|\frac{1}{y} - \frac{1}{x}| < \epsilon$. Let $\epsilon = 1$, and suppose for the sake of contradiction that $\delta > 0$ is a number such that for all $x, y \in A$, if $|y - x| < \delta$, then $|\frac{1}{y} - \frac{1}{x}| < 1$. As in part (a), choose $0 < x < \min(\delta, \frac{1}{2})$. Consequently, $|(x + x) - x| < \delta$. It follows by the above that $|\frac{1}{2x} - \frac{1}{x}| < 1$. But this implies that $|\frac{x-2x}{2x^2}| = |\frac{-1}{2x}| = \frac{1}{2x} < 1$. However, $x < \frac{1}{2}$ implies that $1 < \frac{1}{2x}$, a contradiction. □

(d) $f(x) = \frac{1}{x}$ on $A = [1, +\infty)$.

Proof. To prove that f is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \epsilon$, and let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. Since $x, y \in A$, consecutive applications of Equations 8.1 imply that $1 \leq x$ and $1 \leq y$. It follows by Script 7 that $1 \leq |xy|$. This combined with the fact that $|y - x| < \delta = \epsilon$ implies that

$$\begin{aligned} |f(y) - f(x)| &= \left|\frac{1}{y} - \frac{1}{x}\right| \\ &= \left|\frac{x - y}{yx}\right| \\ &= \frac{|y - x|}{|xy|} \\ &< \frac{\epsilon}{|xy|} \\ &\leq \frac{\epsilon}{1} \\ &= \epsilon \end{aligned}$$

as desired. □

(e) $f(x) = \sqrt{x}$ on $A = [1, +\infty)$.

Proof. To prove that f is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \epsilon$, and let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. Since $x, y \in A$, consecutive applications of Equations 8.1 imply that $1 \leq x$ and $1 \leq y$. It follows by Script 7 that $1 \leq \sqrt{x}$ and $1 \leq \sqrt{y}$. Thus, by Script 7 again, $2 \leq |\sqrt{y} + \sqrt{x}|$. Note that it follows that $1 < |\sqrt{y} + \sqrt{x}|$. This combined with the fact that $|y - x| < \delta = \epsilon$ implies that

$$\begin{aligned} |f(y) - f(x)| &= |\sqrt{y} - \sqrt{x}| \\ &< |\sqrt{y} - \sqrt{x}| \cdot |\sqrt{y} + \sqrt{x}| \\ &= |y - x| \\ &= \epsilon \end{aligned}$$

as desired. □

Exercise 13.4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^n$ for $n \in \mathbb{N}$. Show that f is uniformly continuous if and only if $n = 1$.

Proof. Suppose first that $n = 1$. To prove that f is uniformly continuous, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in \mathbb{R}$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \epsilon$. Now let x, y be arbitrary elements of $\mathbb{R}$ that satisfy $|y - x| < \delta$. Then by the definition of f, $|f(y) - f(x)| = |y - x| < \delta = \epsilon$, as desired.

Now suppose that $n > 1$. Additionally, suppose for the sake of contradiction that f is uniformly continuous. Let $\epsilon = 1 > 0$. Then by Definition 13.1, there exists a $\delta > 0$ such that for all $x, y \in \mathbb{R}$, if $|y-x| < \delta$, then $|y^n - x^n| < 1$. Let $x = 0 \in \mathbb{R}$. By Theorem 5.2, there exists a point $y \in \mathbb{R}$ such that $0 < y < \delta$. Additionally, since $\delta > 0$, Lemma 7.23 asserts that $-\delta < 0$. This combined with the previous result demonstrates by transitivity that $-\delta < 0 < y < \delta$, so by the lemma from Exercise 8.9, we have that $|y| < \delta$. Consequently, by Script 7, we know that $|(y+a)-a| < \delta$ for any $a \in \mathbb{R}$. It follows by the above that $|(y+a)^n - a^n| < 1$. Thus, by Additional Exercise 0.7, $|\sum_{k=0}^{n} \binom{n}{k} y^{n-k}a^k - a^n| = |y^n + ny^{n-1}a + \sum_{k=2}^{n-1} \binom{n}{k} y^{n-k}a^k| < 1$. If we now choose $a = \frac{1}{ny^{n-1}}$, Script 7 reduces the above to $|y^n + 1 + \sum_{k=2}^{n-1} y^{n-k}a^k| < 1$. We now seek to reduce the previous statement further to $|y^n + 1| < 1$. To begin, Exercise 12.22 implies that $y^n > 0$ since $y > 0$ and $0^n = 0$, meaning by Script 7 that $y^n + 1 > 0$. Additionally, Script 7 asserts that $\sum_{k=2}^{n-1} y^{n-k}a^k > 0$ since $a > 0$ and $y > 0$. This combined with the previous result implies by Scripts 7 and 8 that $|y^n+1| < |y^n+1+\sum_{k=2}^{n-1} y^{n-k}a^k| < 1$, as desired. However, since $y^n > 0$, Definition 7.21 asserts that $y^n + 1 > 1$. But by Definition 8.4, this implies that $|y^n + 1| > 1$, a contradiction. $\square$

Exercise 13.5. Let f and g be uniformly continuous on $A \subset \mathbb{R}$. Show that

(a) The function $f + g$ is uniformly continuous on A.

(b) For any constant $c \in \mathbb{R}$, the function $c \cdot f$ is uniformly continuous on A.

Proof of a. To prove that $f+g$ is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y-x| < \delta$, then $|(f+g)(y)-(f+g)(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f, g are uniformly continuous on A, consecutive applications of Definition 13.1 reveal that there exist $\delta_1, \delta_2 > 0$ such that for all $x, y \in A$, $|y - x| < \delta_1$ implies $|f(y) - f(x)| < \frac{\epsilon}{2}$ and $|y - x| < \delta_2$ implies $|g(y) - f(x)| < \frac{\epsilon}{2}$. Choose $\delta = \min(\delta_1, \delta_2)$. Let x, y be arbitrary elements of A that satisfy $|y-x| < \delta$. It follows that $|y-x| < \delta_1$ (so $|f(y)-f(x)| < \frac{\epsilon}{2}$), and that $|y-x| < \delta_2$ (so $|g(y)-g(x)| < \frac{\epsilon}{2}$). These two results when combined imply by Script 7 that $|f(y) - f(x)| + |g(y) - g(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$. Therefore, since $|f(y) - f(x) + g(y) - g(x)| \leq |f(y) - f(x)| + |g(y) - g(x)|$ by Lemma 8.8, we have that

$$\begin{aligned}
|(f+g)(y) - (f+g)(x)| &= |f(y) - f(x) + g(y) - g(x)| \\
&\leq |f(y) - f(x)| + |g(y) - g(x)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

as desired. $\square$

Proof of b. To prove that $c \cdot f$ is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y-x| < \delta$, then $|c \cdot f(y) - c \cdot f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. We divide into two cases ($c = 0$ and $c \neq 0$). Suppose first that $c = 0$. Choose $\delta = 1$. Let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. It follows that $|0 \cdot f(y) - 0 \cdot f(x)| = 0 < \epsilon$, as desired. Now suppose that $c \neq 0$. Then since f is uniformly continuous on A, Definition 13.1 tells us that there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \frac{\epsilon}{|c|}$. Choose this δ to be our δ. Let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. Then by the above, we have that $|f(y) - f(x)| < \frac{\epsilon}{|c|}$. Therefore, $|c| \cdot |f(y) - f(x)| < \epsilon$, so we have that $|c \cdot f(y) - c \cdot f(x)| < \epsilon$, as desired. $\square$