Script 16

Series

16.1 Journal

5/20: **Definition 16.1.** Let $N_0 \in \mathbb{N} \cup \{0\}$ and let $(a_n)_{n=N_0}^{\infty}$ be a sequence of real numbers. Then the formal sum

$$\sum_{n=N_0}^{\infty} a_n$$

is called an **infinite series**. (In most instances, we will start the series at $N_0 = 0$ or $N_0 = 1$.)

We will define the **sequence of partial sums** (p_n) of the series by

$$p_n = a_{N_0} + \cdots + a_{N_0+n-1} = \sum_{i=N_0}^{N_0+n-1} a_i$$

Thus, p_n is the sum of the first n terms in the sequence (a_n). We say that the series **converges** if there exists $L \in \mathbb{R}$ such that $\lim_{n\to\infty} p_n = L$. When this is the case, we write this as

$$\sum_{n=N_0}^{\infty} a_n = L$$

and we say that L is the **sum** of the series. When there does not exist such an L, we say that the series **diverges**.

Lemma 16.2. *Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers. Let $N_0 \in \mathbb{N}$. Then $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=N_0}^{\infty} a_n$ converges.*

Lemma. *Let $n \in \mathbb{N}$. Then*

$$\sum_{i=0}^{N_0+n-1} a_i = \sum_{i=0}^{N_0-1} a_i + \sum_{i=N_0}^{N_0+n-1} a_i$$

Proof. This simple result follows immediately from Script 0, so no formal proof will be given. □

Proof of Lemma 16.2. Suppose first that $\sum_{n=0}^{\infty} a_n$ converges, and let $M = \sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=0}^{n-1} a_i$, where the latter equality holds by Definition 16.1. To prove that $\sum_{n=N_0}^{\infty} a_n$ converges, Definition 16.1 tells us that it will suffice to find an $L \in \mathbb{R}$ such that $\lim_{n\to\infty} \sum_{i=N_0}^{N_0+n-1} a_i = L$. Choose $L = M - \sum_{i=0}^{N_0-1} a_i$. To verify that $\lim_{n\to\infty} \sum_{i=N_0}^{N_0+n-1} a_i = L$, Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|\sum_{i=N_0}^{N_0+n-1} a_i - L| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} \sum_{i=0}^{n-1} a_i = M$, Theorem 15.7 implies that there is some $N \in \mathbb{N}$ such that for all $n \geq N$,

$|\sum_{i=0}^{n-1} a_i - M| < \epsilon$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. Since $N_0 + n > n \geq N$, we have by the above that $|\sum_{i=0}^{N_0+n-1} a_i - M| < \epsilon$. Therefore,

$$\begin{aligned}
\left|\sum_{i=N_0}^{N_0+n-1} a_i - L\right| &= \left|\sum_{i=0}^{N_0+n-1} a_i - \sum_{i=0}^{N_0-1} a_i - L\right| && \text{The Lemma} \\
&= \left|\sum_{i=0}^{N_0+n-1} a_i - \left(\sum_{i=0}^{N_0-1} a_i + L\right)\right| \\
&= \left|\sum_{i=0}^{N_0+n-1} a_i - M\right| \\
&< \epsilon
\end{aligned}$$

as desired.

The proof is symmetric in the other direction. □

Exercise 16.3. Prove that $\sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+1})$ converges. What is its sum?

Proof. Let (a_n) be defined by $a_n = \frac{1}{n} - \frac{1}{n+1}$, and let (p_n) be defined by $p_n = \sum_{i=1}^{n} a_i$. Then

$$\begin{aligned}
p_n &= a_1 + a_2 + \cdots + a_n \\
&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \frac{1}{1} - \frac{1}{n+1}
\end{aligned}$$

To prove that $\sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+1}) = 1$, Definition 16.1 tells us that it will suffice to show that $\lim_{n\to\infty} p_n = 1$. By a proof symmetric to that of Exercise 15.6a, we have that $\lim_{n\to\infty} 1 = 1$. By a proof symmetric to that of Exercise 15.6c, we have that $\lim_{n\to\infty} \frac{1}{n+1} = 0$. Therefore, by Theorem 15.9 and the above, we have that

$$\begin{aligned}
\lim_{n\to\infty} p_n &= \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) \\
&= \lim_{n\to\infty} 1 - \lim_{n\to\infty}\frac{1}{n+1} \\
&= 1 - 0 \\
&= 1
\end{aligned}$$

as desired. □

Theorem 16.4 (Divergence Test). *If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.*

Proof. To prove that $\lim_{n\to\infty} a_n = 0$, Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - 0| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\sum_{n=1}^{\infty} a_n$ converges, we have by Theorem 15.19 that there exists an $N \in \mathbb{N}$ such that $|\sum_{i=1}^{n} a_i - \sum_{i=1}^{m} a_i| < \epsilon$ for all $n, m \geq N$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. Then choosing $n, n-1 \geq N$, we have by the above that $|\sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} a_i| < \epsilon$. Therefore,

$$\begin{aligned}
|a_n - 0| &= |a_n| \\
&= \left|\sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} a_i\right| \\
&< \epsilon
\end{aligned}$$

as desired. □

The converse of this theorem, however, is not true, as we see in Theorem 16.6.

Theorem 16.5 (Cauchy Convergence Test)**.** *A series $\sum_{n=1}^{\infty} a_n$ converges if and only if for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|\sum_{k=m+1}^{n} a_k| < \epsilon$ for all $n > m \geq N$.*

Proof. Suppose first that $\sum_{n=1}^{\infty} a_n$ converges. Let $\epsilon > 0$ be arbitrary. By Definition 16.1, (p_n) converges. Thus, by Theorem 15.19, there is some $N \in \mathbb{N}$ such that $|p_n - p_m| < \epsilon$ for all $n, m \geq N$. Choose this N to be our N. Let n, m be two arbitrary natural numbers satisfying $n > m \geq N$. Therefore,

$$\begin{aligned} \left|\sum_{k=m+1}^{n} a_k\right| &= \left|\sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k\right| \\ &= |p_n - p_m| \\ &< \epsilon \end{aligned}$$

as desired.

The proof is symmetric in the other direction. □

Theorem 16.6. *The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.*

Lemma. *For all $N \in \mathbb{N}$, we have*

$$\sum_{n=N+1}^{2N} \frac{1}{n} \geq \frac{1}{2}$$

Proof. We induct on N. For the base case $N = 1$, we have

$$\sum_{n=1+1}^{2\cdot 1} \frac{1}{n} = \frac{1}{2} \geq \frac{1}{2}$$

as desired. Now suppose inductively that we have proven the claim for N. To prove it for $N + 1$, we do the following.

$$\begin{aligned} \sum_{n=N+2}^{2N+2} \frac{1}{n} &= \sum_{n=N+1}^{2N} \frac{1}{n} - \frac{1}{N+1} + \frac{1}{2N+1} + \frac{1}{2(N+1)} \\ &= \sum_{n=N+1}^{2N} \frac{1}{n} + \frac{1}{2(N+1)(2N+1)} \\ &> \sum_{n=N+1}^{2N} \frac{1}{n} \\ &\geq \frac{1}{2} \end{aligned}$$

as desired. □

Proof of Theorem 16.6. To prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, Theorem 16.5 tells us that it will suffice to find an $\epsilon > 0$ such that for all $N \in \mathbb{N}$, there exist $n > m \geq N$ with $|\sum_{k=m+1}^{n} 1/k| \geq \epsilon$. Choose $\epsilon = \frac{1}{2}$. Let N be an arbitrary element of N. If we now choose $n = 2N$ and $m = N$, we will have $n > m \geq N$. It will follow by the lemma that

$$\begin{aligned} \left|\sum_{k=m+1}^{n} \frac{1}{k}\right| &= \left|\sum_{k=N+1}^{2N} \frac{1}{k}\right| \\ &\geq \frac{1}{2} \\ &= \epsilon \end{aligned}$$

as desired. □

5/25: **Theorem 16.7** (Geometric Series Test)**.** *Let* $-1 < x < 1$*. Then*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Proof. Let (a_n) be defined by $a_n = x^n$. Then $p_n = x^0 + x^1 + \cdots + x^{n-1}$ so that

$$\begin{aligned} p_n - xp_n &= x^0 + \cdots + x^{n-1} - x(x^0 + \cdots + x^{n-1}) \\ p_n - xp_n &= 1 - x^n \\ p_n(1-x) &= 1 - x^n \\ p_n &= \frac{1-x^n}{1-x} \end{aligned}$$

Therefore, we have that

$$\begin{aligned} \frac{1}{1-x} &= \frac{1-0}{1-x} \\ &= \frac{1 - \lim_{n\to\infty} x^n}{1-x} && \text{Exercise 15.8b} \\ &= \lim_{n\to\infty} \frac{1-x^n}{1-x} && \text{Theorem 15.9} \\ &= \lim_{n\to\infty} p_n \\ &= \sum_{n=0}^{\infty} x^n && \text{Definition 16.1} \end{aligned}$$

as desired. □

Theorem 16.8. *If* $\sum_{n=1}^{\infty} a_n = L$*,* $\sum_{n=1}^{\infty} b_n = M$*, and* $c \in \mathbb{R}$*, then*

$$\sum_{n=1}^{\infty} (a_n + b_n) = L + M$$

$$\sum_{n=1}^{\infty} (c \cdot a_n) = c \cdot L$$

Proof. For the first claim, we have that

$$\begin{aligned} L + M &= \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} a_i + \lim_{n\to\infty} \sum_{i=1}^{n} b_i && \text{Definition 16.1} \\ &= \lim_{n\to\infty} \left(\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \right) && \text{Theorem 15.9} \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} (a_i + b_i) \\ &= \sum_{n=1}^{\infty} (a_n + b_n) && \text{Definition 16.1} \end{aligned}$$

The proof is symmetric for the second claim. □

Definition 16.9. We say that the series $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

Lemma 16.10. *A series $\sum_{n=1}^{\infty} a_n$ with all $a_n \geq 0$ converges if and only if its sequence of partial sums is bounded.*

Proof. Suppose first that the series $\sum_{n=1}^{\infty} a_n$ with all $a_n \geq 0$ converges. Then by Definition 16.1 its sequence of partial sums (p_n) converges. Therefore, by Theorem 15.13, (p_n) is bounded, as desired.

Now suppose that the sequence of partial sums (p_n) corresponding to a series $\sum_{n=1}^{\infty} a_n$ with all $a_n \geq 0$ is bounded. To prove that $\sum_{n=1}^{\infty} a_n$ converges, Definition 16.1 tells us that it will suffice to show that (p_n) converges. To do so, Theorem 15.14 tells us that it will suffice to verify in addition to the fact that (p_n) is bounded that (p_n) is increasing. To do this, Script 15 tells us that it will suffice to confirm that $p_n \leq p_{n+1}$ for all $n \in \mathbb{N}$. Let n be an arbitrary natural number. By Definition 16.1, $p_{n+1} - p_n = a_{n+1}$. Since $a_{n+1} \geq 0$ by hypothesis, we have by transitivity that $p_{n+1} - p_n \geq 0$, i.e., $p_n \leq p_{n+1}$ by Definition 7.21, as desired. $\square$

Theorem 16.11. *If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges and*

$$\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|$$

Proof. To prove that $\sum_{n=1}^{\infty} a_n$ converges, Theorem 16.5 tells us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|\sum_{k=m+1}^{n} a_k| < \epsilon$ for all $n > m \geq N$. Let $\epsilon > 0$ be arbitrary. Since $\sum_{n=1}^{\infty} a_n$ converges absolutely by hypothesis, we have by Definition 16.9 that $\sum_{n=1}^{\infty} |a_n|$ converges. Thus, by Theorem 16.5, there is some $N \in \mathbb{N}$ such that $|\sum_{k=m+1}^{n} |a_k|| < \epsilon$ for all $n > m \geq N$. Choose this N to be our N. Let n, m be arbitrary natural numbers such that $n > m \geq N$. Therefore,

$$\begin{aligned}
\left|\sum_{k=m+1}^{n} a_k\right| &\leq \sum_{k=m+1}^{n} |a_k| && \text{Lemma 8.8} \\
&= \left|\sum_{k=m+1}^{n} |a_k|\right| \\
&< \epsilon
\end{aligned}$$

as desired.

As to the other part of the claim, to begin, let (b_n) and (c_n) be defined by $b_n = \max(0, a_n)$ and $c_n = \min(0, a_n)$. We will prove a few preliminary results with these definitions that will enable us to tackle the big inequality.

To confirm that $a_n = b_n + c_n$, we divide into two cases ($a_n \geq 0$ and $a_n < 0$). If $a_n \geq 0$, then by their definitions, $b_n = a_n$ and $c_n = 0$. Thus, $a_n = b_n + c_n$ as desired. The argument is symmetric in the other case.

To confirm that $|\sum_{n=1}^{\infty} b_n| + |\sum_{n=1}^{\infty} -c_n| = |\sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} -c_n|$, we can acknowledge that $b_n \geq 0$ and $-c_n \geq 0$ for all $n \in \mathbb{N}$ to demonstrate that

$$\begin{aligned}
\left|\sum_{n=1}^{\infty} b_n\right| + \left|\sum_{n=1}^{\infty} -c_n\right| &= \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} -c_n \\
&= \left|\sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} -c_n\right|
\end{aligned}$$

To confirm that $|a_n| = b_n - c_n$, we divide into two cases ($a_n \geq 0$ and $a_n < 0$). If $a_n \geq 0$, then $b_n = a_n$ and $c_n = 0$. Thus, by Definition 8.4, $|a_n| = a_n = b_n - c_n$, as desired. On the other hand, if $a_n < 0$, then $b_n = 0$ and $c_n = a_n$. Thus, by Definition 8.4 again, $|a_n| = -a_n = b_n - c_n$, as desired.

Having established that $a_n = b_n + c_n$, $|\sum_{n=1}^{\infty} b_n| + |\sum_{n=1}^{\infty} -c_n| = |\sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} -c_n|$, and $|a_n| =$

$b_n - c_n$, we have that

$$\begin{aligned}
\left|\sum_{n=1}^{\infty} a_n\right| &= \left|\sum_{n=1}^{\infty}(b_n + c_n)\right| & \\
&= \left|\sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} c_n\right| & \text{Theorem 16.8} \\
&\leq \left|\sum_{n=1}^{\infty} b_n\right| + \left|\sum_{n=1}^{\infty} c_n\right| & \text{Lemma 8.8} \\
&= \left|\sum_{n=1}^{\infty} b_n\right| + \left|-\sum_{n=1}^{\infty} c_n\right| & \text{Exercise 8.5} \\
&= \left|\sum_{n=1}^{\infty} b_n\right| + \left|\sum_{n=1}^{\infty} -c_n\right| & \text{Theorem 16.8} \\
&= \left|\sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} -c_n\right| & \\
&= \left|\sum_{n=1}^{\infty}(b_n - c_n)\right| & \text{Theorem 16.8} \\
&= \left|\sum_{n=1}^{\infty} |a_n|\right| & \\
&= \sum_{n=1}^{\infty} |a_n| &
\end{aligned}$$

as desired. □

Theorem 16.12 (Alternating Series Test)**.** *Let (a_n) be a decreasing sequence of positive numbers such that $\lim_{n\to\infty} a_n = 0$. Then $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.*

Lemma.

(*a*) *For any natural numbers n, m satisfying $n > m$, we have*

$$\left|\sum_{k=m+1}^{n}(-1)^{k+1}a_k\right| = |a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + a_{m+5} - \cdots \pm a_n|$$

Proof. By Script 0, either $|\sum_{k=m+1}^{n}(-1)^{k+1}a_k| = |a_{m+1} - a_{m+2} + a_{m+3} - \cdots \pm a_n|$ or $|\sum_{k=m+1}^{n}(-1)^{k+1}a_k| = |-a_{m+1} + a_{m+2} - a_{m+3} + \cdots \pm a_n|$. However, by Exercise 8.5, the two results are equal. Thus, we may choose the former WLOG. □

(*b*) *We have*

$$0 \leq a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots \pm a_n$$

Proof. Since (a_n) is a decreasing sequence, we have by Script 15 that $a_{i+1} \leq a_i$ for all $i \in \mathbb{N}$. It follows by Definition 7.21 that $0 \leq a_i - a_{i+1}$ for all $i \in \mathbb{N}$. We now divide into two cases (there are an even number of terms in the sum $a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + a_{m+5} - \cdots \pm a_n$ and there are an odd number of terms in said sum). In the first case, we have that the sum is of the form $a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots + a_{n-1} - a_n$. Thus, since $0 \leq a_{m+1} - a_{m+2}$, $0 \leq a_{m+3} - a_{m+4}$, and on and on, we have by Script 7 that $0 \leq a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots - a_n$, as desired. On the other hand, in the second case, we have that the sum is of the form $a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots + a_{n-2} - a_{n-1} + a_n$. For the same reason as before, we have that $0 \leq a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots + a_{n-2} - a_{n-1}$. However, we need the additional hypothesis that every a_i is positive, i.e., $a_i \geq 0$ for all $i \in \mathbb{N}$ to know that $0 \leq a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots + a_n$, as desired. □

(c) *For all $i \in \mathbb{N}$, we have*

$$-a_i + a_{i+1} \leq 0$$

Proof. Since (a_n) is a decreasing sequence, we have by Script 15 that $a_{i+1} \leq a_i$ for all $i \in \mathbb{N}$. It follows by Definition 7.21 that $-a_i + a_{i+1} \leq 0$ for all $i \in \mathbb{N}$. $\square$

Proof of Theorem 16.12. To prove that $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges, Theorem 16.5 tells us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|\sum_{k=m+1}^{n}(-1)^{k+1}a_k| < \epsilon$ for all $n > m \geq N$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} a_n = 0$ by hypothesis, we have by Theorem 15.7 that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - 0| = |a_n| < \epsilon$, as desired. Choose this N to be our N. Let n, m be arbitrary natural numbers such that $n > m \geq N$.

We divide into two cases ($n-m$ is even [i.e., there are an even number of terms in the sum $\sum_{k=m+1}^{n}(-1)^{k+1}a_k$] and $n - m$ is odd [i.e., there are an odd number of terms in the sum $\sum_{k=m+1}^{n}(-1)^{k+1}a_k$]). If $n - m$ is even, then we have

$$\begin{aligned}
\left|\sum_{k=m+1}^{n}(-1)^{k+1}a_k\right| &= |a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + a_{m+5} - \cdots - a_{n-2} + a_{n-1} - a_n| && \text{Lemma (a)}\\
&= a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + a_{m+5} - \cdots - a_{n-2} + a_{n-1} - a_n && \text{Lemma (b) \& Definition 8.4}\\
&= a_{m+1} + (-a_{m+2} + a_{m+3}) + (-a_{m+4} + a_{m+5}) + \cdots + (-a_{n-2} + a_{n-1}) - a_n\\
&\leq a_{m+1} + (-a_{m+4} + a_{m+5}) + \cdots + (-a_{n-2} + a_{n-1}) - a_n && \text{Lemma (c)}\\
&\leq a_{m+1} + \cdots + (-a_{n-2} + a_{n-1}) - a_n && \text{Lemma (c)}\\
&\vdots && \text{Lemma (c)}\\
&\leq a_{m+1} + (-a_{n-2} + a_{n-1}) - a_n && \text{Lemma (c)}\\
&\leq a_{m+1} - a_n && \text{Lemma (c)}\\
&\leq a_{m+1}\\
&= |a_{m+1}|\\
&< \epsilon
\end{aligned}$$

The argument is symmetric if $n - m$ is odd. $\square$

5/27: The following theorem will be useful to prove more specialized tests for convergence of series.

Theorem 16.13 (Direct Comparison Test). *Let (c_n) be a sequence of positive numbers and let (a_n) be a sequence such that $|a_n| \leq c_n$ for all $n \geq N_0$, where N_0 is some fixed integer. If $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*

Proof. To prove that $\sum_{n=1}^{\infty} a_n$ converges, Theorem 16.11 and Definition 16.9 tell us that it will suffice to show that $\sum_{n=1}^{\infty} |a_n|$ converges. To do this, Theorem 16.5 tells us that it will suffice to verify that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|\sum_{k=m+1}^{n} |a_k|| < \epsilon$ for all $n > m \geq N$. Let $\epsilon > 0$ be arbitrary. Since $\sum_{n=1}^{\infty} c_n$ converges by hypothesis, we have by Theorem 16.5 that there is some $N_1 \in \mathbb{N}$ such that $|\sum_{k=m+1}^{n} c_k| < \epsilon$ for all $n > m \geq N_1$. Choose $N = \max(N_0, N_1)$. Let n, m be arbitrary natural numbers such that $n > m \geq N$. Since $c_k \geq 0$ for all $k \in \mathbb{N}$ by hypothesis, it follows by Script 7 that $\sum_{k=m+1}^{n} c_k \geq 0$. Thus, Definition 8.4 implies that $\sum_{k=m+1}^{n} c_k = |\sum_{k=m+1}^{n} c_k| < \epsilon$. Similarly, we have that $|\sum_{k=m+1}^{n} |a_k|| = \sum_{k=m+1}^{n} |a_k|$. Lastly, since we know by hypothesis that $|a_k| \leq c_k$ for all $k \geq N_0$, i.e., for all $k \geq N$, Script 7 asserts that $\sum_{k=m+1}^{n} |a_k| \leq \sum_{k=m+1}^{n} c_n$. Therefore, combining the last three results, we have that $|\sum_{k=m+1}^{n} |a_k|| = \sum_{k=m+1}^{n} |a_k| \leq \sum_{k=m+1}^{n} c_n < \epsilon$, as desired. $\square$

Lemma 16.14. *Suppose that (b_n) is a sequence of nonnegative numbers with $\lim_{n\to\infty} b_n = L$, where $L < 1$. Then there is some $N \in \mathbb{N}$ such that $0 \leq b_n < \frac{1+L}{2}$ for all $n \geq N$.*

Proof. Choose $\epsilon = \frac{1-L}{2}$; since $L < 1$, we have that $0 < 1 - L$, i.e., $0 < \frac{1-L}{2}$ as needed. It follows by Theorem 15.7 (since $\lim_{n\to\infty} b_n = L$ by hypothesis) that there is some $N \in \mathbb{N}$ such that $|b_n - L| < \frac{1-L}{2}$ for all $n \geq N$. Choose this n to be our N. Let n be an arbitrary natural number such that $n \geq N$. Then

$$\begin{aligned} |b_n - L| &< \frac{1-L}{2} \\ -\frac{1-L}{2} < b_n - L &< \frac{1-L}{2} && \text{Lemma, Exercise 8.9} \\ b_n &< \frac{1+L}{2} \end{aligned}$$

Additionally, since (b_n) is a sequence of nonnegative numbers, $0 \leq b_n$. Therefore, combining the last two results, we have that $0 \leq b_n < \frac{1+L}{2}$, as desired. ☐

Theorem 16.15 (Ratio Test)**.** *Let* (a_n) *be a sequence such that* $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$ *exists. Then*

(*a*) *If* $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$*, then* $\sum_{n=1}^{\infty} a_n$ *converges.*

Proof. Let $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L$. To prove that $\sum_{n=1}^{\infty} a_n$ converges, Theorem 16.13 tells us that it will suffice to find a sequence (c_n) of positive numbers such that $|a_n| \leq c_n$ for all $n \geq N_0$, where N_0 is some fixed integer, for which $\sum_{n=1}^{\infty} c_n$ converges. To begin, since $(|\frac{a_{n+1}}{a_n}|)$ is a sequence of nonnegative numbers with $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L$, where $L < 1$, we have by Lemma 16.14 that there exists an $N_0 \in \mathbb{N}$ such that $0 \leq |\frac{a_{n+1}}{a_n}| < \frac{1+L}{2}$ for all $n \geq N_0$. With this result, we can define (c_n) by $c_n = (\frac{1+L}{2})^{n-N_0} \cdot |a_{N_0}|$ for all $n \in \mathbb{N}$. We will now prove that (c_n) satisfies the necessary properties outlined in the beginning.

First, we must confirm that (c_n) is a sequence of positive numbers, i.e., that $c_n \geq 0$ for all $n \in \mathbb{N}$. Let n be an arbitrary natural number. By the above result from Lemma 16.14 and transitivity, we know that $0 < \frac{1+L}{2}$. It follows by Script 7 that $0 < (\frac{1+L}{2})^{n-N_0}$. Therefore, by Definition 8.4 and Script 7, $0 \leq (\frac{1+L}{2})^{n-N_0} \cdot |a_{N_0}| = c_n$, as desired.

To confirm that $|a_n| \leq c_n$ for all $n \geq N_0$, we induct on n using Additional Exercise 0.2a. For the base case $n = N_0$, we have that

$$|a_{N_0}| = 1 \cdot |a_{N_0}| = \left(\frac{1+L}{2}\right)^{N_0 - N_0} \cdot |a_{N_0}| = c_{N_0}$$

which we may weaken to $|a_{N_0}| \leq c_{N_0}$, as desired. Now suppose inductively that we have demonstrated that $|a_n| \leq c_n$; we now seek to demonstrate that $|a_{n+1}| \leq c_{n+1}$. By hypothesis, $n \geq N_0$, so by the above, we have that $0 \leq |\frac{a_{n+1}}{a_n}| < \frac{1+L}{2}$. It follows by Script 7 that $|a_{n+1}| < \frac{1+L}{2} \cdot |a_n|$. Therefore,

$$\begin{aligned} |a_{n+1}| &< \frac{1+L}{2} \cdot |a_n| \\ &\leq \frac{1+L}{2} \cdot c_n \\ &= \frac{1+L}{2} \cdot \left(\frac{1+L}{2}\right)^{n-N_0} \cdot |a_{N_0}| \\ &= \left(\frac{1+L}{2}\right)^{(n+1)-N_0} \cdot |a_{N_0}| \\ &= c_{n+1} \end{aligned}$$

which we may weaken to $|a_{n+1}| \leq |c_{n+1}|$, as desired.

Lastly, we must confirm that $\sum_{n=1}^{\infty} c_n$ converges. By Script 7, it follows from the hypothesis that $L < 1$ that $1 + L < 2$, which in turn implies that $\frac{1+L}{2} < 1$. Additionally, the above result that $0 < \frac{1+L}{2}$ implies by transitivity that $-1 < \frac{1+L}{2}$. These last two results when combined imply

$\sum_{n=0}^{\infty}(\frac{1+L}{2})^n$ satisfies the constraints of Theorem 16.7, meaning that $\sum_{n=0}^{\infty}(\frac{1+L}{2})^n$ converges. Thus, by Script 0, $\sum_{n=N_0}^{\infty}(\frac{1+L}{2})^{n-N_0}$ converges. Consequently, by consecutive applications of Lemma 16.2, $\sum_{n=1}^{\infty}(\frac{1+L}{2})^{n-N_0}$ converges. It follows by Theorem 16.8 that $\sum_{n=1}^{\infty}(\frac{1+L}{2})^{n-N_0} \cdot |a_{N_0}|$ converges. Therefore, by the definition of c_n, $\sum_{n=1}^{\infty} c_n$ converges, as desired. □

(b) *If* $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| > 1$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Proof. Let $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L$. Suppose for the sake of contradiction that $\lim_{n\to\infty} a_n = 0$. Then by Theorem 15.7, for all $\epsilon_1 > 0$, there is some $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $|a_n| = |a_n - 0| < \epsilon_1$. Similarly, since $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L$, we have that for all $\epsilon_2 > 0$, there is some $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, we have $||\frac{a_{n+1}}{a_n}| - L| < \epsilon_2$. Choose $\epsilon_2 = L - 1$ (it follows from the fact that $L > 1$ by Definition 7.21 that $L - 1 > 0$). Thus, we have that for all $n \geq N_2$,

$$\begin{aligned}
1 &= 1 + L - L + \left|\left|\frac{a_{n+1}}{a_n}\right|\right| - \left|\left|\frac{a_{n+1}}{a_n}\right|\right| \\
&= |L| - \left|\left|\frac{a_{n+1}}{a_n}\right|\right| + 1 - L + \left|\left|\frac{a_{n+1}}{a_n}\right|\right| \\
&\leq \left|L - \left|\frac{a_{n+1}}{a_n}\right|\right| + 1 - L + \left|\left|\frac{a_{n+1}}{a_n}\right|\right| \\
&= \left|\left|\frac{a_{n+1}}{a_n}\right| - L\right| + 1 - L + \left|\left|\frac{a_{n+1}}{a_n}\right|\right| && \text{Exercise 8.5} \\
&< L - 1 + 1 - L + \left|\left|\frac{a_{n+1}}{a_n}\right|\right| \\
&= \left|\frac{a_{n+1}}{a_n}\right|
\end{aligned}$$

which can be rearranged by Lemma 7.24 to demonstrate that $|a_n| < |a_{n+1}|$ for all such n. Now consider the case where $\epsilon_1 = |a_{N_2+1}|$ (note that $|a_{N_2+1}| > |a_{N_2}| \geq 0$ by Definition 8.4). Choose $N = \max(N_1, N_2+2)$. Then by the above, we have by transitivity that $|a_N| > |a_{N-1}| > \cdots > |a_{N_2+1}|$. However, since $N \geq N_1$, we also have that $|a_N| < \epsilon_1 = |a_{N_2+1}|$, a contradiction. Therefore, since $\lim_{n\to\infty} a_n \neq 0$, we have by the contrapositive of Theorem 16.4 that $\sum_{n=1}^{\infty} a_n$ diverges. □

Theorem 16.16 (Root Test)**.** *Let* (a_n) *be a sequence such that* $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ *exists. Then*

(a) *If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, *then* $\sum_{n=1}^{\infty} a_n$ *converges.*

Proof. Let $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$. To prove that $\sum_{n=1}^{\infty} a_n$ converges, Theorem 16.13 tells us that that it will suffice to find a sequence (c_n) of positive numbers such that $|a_n| \leq c_n$ for all $n \geq N_0$, where N_0 is some fixed integer, for which $\sum_{n=1}^{\infty} c_n$ converges. Since $L < 1$ by hypothesis and $0 < 1$ by Corollary 7.27, Theorem 5.2 asserts that there exists a point $x \in \mathbb{R}$ such that $\max(0, L) < x < 1$. We now define (c_n) by $c_n = x^n$ for all $n \in \mathbb{N}$. By Script 7 and the fact that $x \geq 0$, we know that (c_n) is a sequence of positive numbers. Additionally, since $x - L > 0$, Theorem 15.7 implies that there is some $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, $|\sqrt[n]{|a_n|} - L| < x - L$. It follows by Script 7 that $|a_n| \leq x^n = c_n$ for all $n \geq N_0$. Lastly, since $-1 < \max(0, L) < x < 1$, we have by Theorem 16.7 that $\sum_{n=0}^{\infty} c_n$ converges. It follows by Lemma 16.2 that $\sum_{n=1}^{\infty} c_n$ converges, as desired. □

(b) *If* $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Proof. Let $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$. Suppose for the sake of contradiction that $\lim_{n\to\infty} a_n = 0$. Then by Theorem 15.7, for all $\epsilon_1 > 0$, there is some $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $|a_n| = |a_n - 0| < \epsilon_1$. Similarly, since $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$, we have that for all $\epsilon_2 > 0$, there is some $N_2 \in \mathbb{N}$ such that for all

$n \geq N_2$, we have $|\sqrt[n]{|a_n|} - L| < \epsilon_2$. Choose $\epsilon_2 = L - 1$. Thus, we have that for all $n \geq N_2$, $1 < \sqrt[n]{|a_n|}$ (by an argument symmetric to that given in the proof of Theorem 16.15b) which can be rearranged by Script 7 to demonstrate that $|a_n| > 1^n = 1$ for all such n. Now consider the case where $\epsilon_1 = 1$. Choose $N = \max(N_1, N_2)$. Then by the above, the fact that $N \geq N_2$ implies that $|a_N| > 1$. However, since $N \geq N_1$, we also have that $|a_N| < \epsilon_1 = 1$, a contradiction. Therefore, since $\lim_{n\to\infty} a_n \neq 0$, we have by the contrapositive of Theorem 16.4 that $\sum_{n=1}^{\infty} a_n$ diverges. ☐

6/23: **Definition 16.17.** For $n \in \mathbb{N}$, we define the **factorial** of n to be the product of all natural numbers less than or equal to n. We denote this by the formula

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1$$

By convention, we also set $0! = 1$.

Exercise 16.18. Prove that

$$\sum_{n=0}^{\infty} \frac{1}{n!}$$

converges. The number that it converges to is called e.

Proof. To prove that $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges, Theorem 16.13 tells us that it will suffice to find a sequence (c_n) of positive numbers such that $|a_n| \leq c_n$ for all $n \geq N_0$, where N_0 is some fixed integer, for which $\sum_{n=1}^{\infty} c_n$ converges. Define (c_n) by $c_n = (\frac{1}{2})^{n-1}$ for all $n \in \mathbb{N}$. We now address each property in turn.

By Script 7 and the fact that $\frac{1}{2} \geq 0$, we know that (c_n) is a sequence of positive numbers.

Choose $N_0 = 2$. To confirm that $|a_n| \leq c_n$ for all $n \geq N_0$, we induct on n using Additional Exercise 0.2a. For the base case $n = 2$, we have by Definition 16.17 that $|a_2| = |\frac{1}{2!}| = \frac{1}{2} = c_2$, which we may weaken to $|a_2| \leq c_2$, as desired. Now suppose inductively that we have demonstrated that $|a_n| \leq c_n$; we now seek to demonstrate that $|a_{n+1}| \leq c_{n+1}$. But

$$\begin{aligned}
|a_{n+1}| &= \left|\frac{1}{(n+1)!}\right| \\
&= \frac{1}{n+1} \cdot \frac{1}{n!} && \text{Definition 16.17} \\
&= \frac{1}{n+1} \cdot |a_n| \\
&< \frac{1}{2} \cdot |a_n| \\
&\leq \frac{1}{2} \cdot c_n \\
&= \frac{1}{2} \cdot \left(\frac{1}{2}\right)^{n-1} \\
&= \left(\frac{1}{2}\right)^{(n+1)-1} \\
&= c_{n+1}
\end{aligned}$$

which we may weaken to $|a_{n+1}| \leq c_{n+1}$, as desired.

Since $-1 < \frac{1}{2} < 1$, Theorem 16.7 asserts that $\sum_{n=0}^{\infty} c_n$ converges. It follows by Lemma 16.2 that $\sum_{n=1}^{\infty} c_n$ converges, as desired. ☐