MATH 16310 (Honors Calculus III IBL) Notes

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July 7, 2021

Contents

Script 12

Derivatives

12.1 Journal

3/30: Throughout this sheet, we let $f : A \to \mathbb{R}$ be a real valued function with domain $A \subset \mathbb{R}$. We also now assume the domain $A \subset \mathbb{R}$ is open.

Definition 12.1. The **derivative** of f at a point $a \in A$ is the number $f'(a)$ defined by the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided that the limit on the right-hand side exists. If $f'(a)$ exists, we say that f is **differentiable** (at a). If f is differentiable at all points of its domain, we say that f is **differentiable**. In this case, the values $f'(a)$ define a new function $f' : A \to \mathbb{R}$ called the **derivative** (of f).

Remark 12.2. If A is not open, the limit in Definition 12.1 may not exist. For example, if $f : [a, b] \to \mathbb{R}$, then we cannot define the derivative at the endpoints. For any c in the domain of f, we define the **right-hand derivative** $f'_+(c)$ and the **left-hand derivative** $f'_-(c)$ by

$$f'_+(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \qquad\qquad f'_-(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$$

We say that f is **differentiable** (on $[a, b]$) if f is differentiable on (a, b) and $f'_+(a)$ and $f'_-(b)$ exist.

Lemma 12.3. *Let $a \in \mathbb{R}$. Then*

$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$$

assuming that one of the two limits exists. (So if the limit on the left exists, so does the one on the right, and they are equal. Similarly, if the limit on the right exists, then so does the one on the left, and they are equal.)

Proof. Suppose first that $\lim_{x \to a} f(x)$ exists, and let it be equal to L. To prove that $\lim_{h \to 0} f(a+h)$ exists and that it equals $\lim_{x \to a} f(x)$, Definition 11.1 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $(h + a) \in A$ and $0 < |h - 0| = |h| < \delta$, then $|f(a+h) - L| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{x \to a} f(x)$ exists, Definition 11.1 implies that there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$. We will choose this δ to be our δ. Now let h be an arbitrary number satisfying both $(h + a) \in A$ and $0 < |h| < \delta$; we seek to show that $|f(a+h) - L| < \epsilon$. Since $(h + a) \in A$, $h + a = x$ for some $x \in A$. It follows that $h = x - a$, meaning that x is an object that is both an element of A and that satisfies $0 < |h| = |x - a| < \delta$, so we know that $|f(a+h) - L| = |f(x) - L| < \epsilon$, as desired.

The proof is symmetric in the other direction. □

Theorem 12.4. *Let $a \in \mathbb{R}$. Then f is differentiable at a if and only if $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists. Moreover, if f is differentiable at a, then the derivative of f at a is given by the limit*

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Proof. Suppose first that f is differentiable at a. Then by Definition 12.1, $f'(a)$ exists. It follows that

$$\begin{aligned} f'(a) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} && \text{Definition 12.1}\\ &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{(a+h)-a}\\ &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} && \text{Lemma 12.3}\end{aligned}$$

Note that the substitution in the last step follows from using $\tilde{f}(x) = \frac{f(x)-f(a)}{x-a}$ as the "$f(x)$" function in Lemma 12.3.

The proof is symmetric in the reverse direction. □

Theorem 12.5. *If f is differentiable at a, then f is continuous at a.*

Proof. To prove that f is continuous at a, Theorem 11.5 tells us that it will suffice to show that $\lim_{x\to a} f(x) = f(a)$. By Definition 12.1, the hypothesis implies that $f'(a)$ exists. Thus, by Theorem 12.4, we know that $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists. Additionally, by Exercise 11.6, $g(x) = x - a$ is continuous at a. It follows by Theorem 11.5 that either $a \notin LP(A)$ or $\lim_{x\to a} g(x)$ exists (and equals $g(a)$). However, since A is open by hypothesis and $a \in A$, Theorem 4.10 implies that there exists a region R such that $a \in R$ and $R \subset A$. But $a \in R$ implies that $a \in LP(R)$ by Corollary 5.5, and this combined with the fact that $R \subset A$ implies by Theorem 3.14 that $a \in LP(A)$[1]. Thus, we have that $\lim_{x\to a} g(x) = g(a) = a - a = 0$. Combining the last few results, we have

$$\begin{aligned} 0 &= \left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right)\cdot 0\\ &= \left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right)\left(\lim_{x\to a}(x-a)\right)\\ &= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\cdot(x-a)\right) && \text{Theorem 11.9}\\ &= \lim_{x\to a}(f(x)-f(a))\end{aligned}$$

If we now consider $f(a)$ to be a constant function (i.e., $\lim_{x\to a} f(a) = f(a)$ by Exercise 11.6, Theorem 11.5, and the above result that $a \in LP(A)$), it follows from the above that

$$\begin{aligned} \lim_{x\to a}(f(x)-f(a)) + \lim_{x\to a} f(a) &= 0 + f(a)\\ \lim_{x\to a}(f(x)-f(a)+f(a)) &= f(a) && \text{Theorem 11.9}\\ \lim_{x\to a} f(x) &= f(a)\end{aligned}$$

as desired. □

Exercise 12.6. Show that the converse of Theorem 12.5 is not true.

Proof. The converse of Theorem 12.5 asserts that "if f is continuous at a, then f is differentiable at a." To falsify this statement, we will use the absolute value function $|x|$ as a counterexample. Let's begin.

By Exercise 11.7, $|x|$ is continuous. It follows by Theorem 9.10 that $|x|$ is continuous at 0. However, we can show that $|x|$ is not differentiable at 0.

To do this, Definition 12.1 and Theorem 12.4 tell us that it will suffice to verify that $\lim_{x\to 0}\frac{|x|-|0|}{x-0} = \lim_{x\to 0}\frac{|x|}{x}$ does not exist. Suppose for the sake of contradiction that $\lim_{x\to 0}\frac{|x|}{x} = L$. Then by Definition 11.1, for $\epsilon = 1 > 0$, there exists a $\delta > 0$ such that if $0 < |x-0| = |x| < \delta$, then $|\frac{|x|}{x} - L| < 1$. However, we

[1] I will not go through this or similar derivations again, although they may be technically necessary. Indeed, assume moving on that statements analogous to $a \in LP(A)$ hold true.

can show that no such δ exists. Let $\delta > 0$ be arbitrary. By Theorem 5.2, there exists a number $x \in \mathbb{R}$ such that $0 < x < \delta$. It follows by Definition 8.4 and Exercise 8.5 that $0 < |x| = |-x| < \delta$. Since both x and $-x$ are in the appropriate range, we know that

$$\begin{aligned} \left|\frac{|x|}{x} - L\right| &= \left|\frac{x}{x} - L\right| & \left|\frac{|-x|}{-x} - L\right| &= \left|\frac{x}{-x} - L\right| && \text{Definition 8.4} \\ &= |1 - L| & &= |-1 - L| && \text{Script 7} \\ &= |L - 1| & &= |L + 1| && \text{Exercise 8.5} \\ &< 1 & &< 1 \end{aligned}$$

By consecutive applications of the lemma from Exercise 8.9, it follows that

$$\begin{aligned} -1 < L - 1 < 1 \qquad & \qquad -1 < L + 1 < 1 \\ 0 < L < 2 \qquad & \qquad -2 < L < 0 \end{aligned}$$

But this implies that $L < 0$ and $L > 0$, a contradiction. $\square$

Exercise 12.7. Show that for all $n \in \mathbb{N}$,

$$x^n - a^n = (x - a)\left(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1}\right)$$

or equivalently,

$$x^n - a^n = (x - a)\left(\sum_{i=0}^{n-1} x^{n-1-i}a^i\right)$$

Proof. By simple algebra (see Script 7), we have

$$\begin{aligned} (x - a)\left(\sum_{i=0}^{n-1} x^{n-1-i}a^i\right) &= \sum_{i=0}^{n-1}(x - a)x^{n-1-i}a^i \\ &= \sum_{i=0}^{n-1}\left(x^{n-i}a^i - x^{n-1-i}a^{i+1}\right) \\ &= x^n + \sum_{i=1}^{n-1} x^{n-i}a^i - \sum_{i=0}^{n-2} x^{n-1-i}a^{i+1} - a^n \\ &= x^n + \sum_{i=1}^{n-1} x^{n-i}a^i - \sum_{i=0+1}^{n-2+1} x^{n-1-(i-1)}a^{(i-1)+1} - a^n \\ &= x^n + \sum_{i=1}^{n-1} x^{n-i}a^i - \sum_{i=1}^{n-1} x^{n-i}a^i - a^n \\ &= x^n - a^n \end{aligned}$$

as desired. $\square$

Exercise 12.8.

(a) Let $n \in \mathbb{N}$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^n$. Use Exercise 12.7 to prove that $f'(a) = na^{n-1}$ for all $a \in \mathbb{R}$.

Proof. Let a be an arbitrary element of $\mathbb{R}$. To prove that $f'(a) = na^{n-1}$, Theorem 12.4 tell us that it will suffice to show that $\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = na^{n-1}$. By Corollary 11.12, the polynomial $\sum_{i=0}^{n-1} x^{n-1-i}a^i$ is continuous. Thus, by Theorem 9.10, it is continuous at a. It follows by Theorem 11.5 that $\lim_{x \to a} \sum_{i=0}^{n-1} x^{n-1-i}a^i = \sum_{i=0}^{n-1} a^{n-1-i}a^i$.

Additionally, we can demonstrate that $\lim_{x\to a}\frac{x-a}{x-a} = 1$. To verify that $\lim_{h\to 0}\frac{h}{h} = 1$, Definition 11.1 tells us that it will suffice to confirm that for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $h \in \mathbb{R}$ and $0 < |h - 0| = |h| < \delta$, then $|\frac{h}{h} - 1| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = 1$, and let h be an arbitrary element of $\mathbb{R}$ satisfying $0 < |h| < \delta$. It follows by Script 7 that $|\frac{h}{h} - 1| = |1 - 1| = 0 < \epsilon$, as desired. Since $\lim_{h\to 0}\frac{h}{h} = 1$, we know by Script 7 that $\lim_{h\to 0}\frac{(a+h)-a}{(a+h)-a} = 1$. Thus, by Lemma 12.3, $\lim_{x\to a}\frac{x-a}{x-a} = 1$, as desired.

It follows from the above two results that

$$\begin{aligned}
na^{n-1} &= \underbrace{a^{n-1} + \cdots + a^{n-1}}_{n \text{ times}} \\
&= \sum_{i=0}^{n-1} a^{n-1} \\
&= \sum_{i=1}^{n-1} a^{n-1-i}a^i \\
&= \lim_{x\to a}\sum_{i=0}^{n-1} x^{n-1-i}a^i \\
&= 1 \cdot \left(\lim_{x\to a}\sum_{i=0}^{n-1} x^{n-1-i}a^i\right) \\
&= \left(\lim_{x\to a}\frac{x-a}{x-a}\right)\left(\lim_{x\to a}\sum_{i=0}^{n-1} x^{n-1-i}a^i\right) \\
&= \lim_{x\to a}\frac{x-a}{x-a}\cdot\sum_{i=0}^{n-1} x^{n-1-i}a^i && \text{Theorem 11.9} \\
&= \lim_{x\to a}\frac{x^n - a^n}{x-a} && \text{Exercise 12.7} \\
&= \lim_{x\to a}\frac{f(x) - f(a)}{x-a} \\
&= f'(a) && \text{Theorem 12.4}
\end{aligned}$$

as desired. ☐

(b) Let $k \in \mathbb{R}$. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = k$, then $f'(a) = 0$ for all $a \in \mathbb{R}$.

Proof. Let a be an arbitrary element of $\mathbb{R}$. To prove that $f'(a) = 0$, Definition 12.1 tells us that it will suffice to show that $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = 0$. By Exercise 11.6, the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 0$ is continuous at every $x \in \mathbb{R}$, including 0. It follows by Theorem 11.5 that $\lim_{h\to 0} g(h) = g(0) = 0$. Therefore,

$$\begin{aligned}
0 &= \lim_{h\to 0} g(h) \\
&= \lim_{h\to 0} 0 \\
&= \lim_{h\to 0}\frac{0}{h} \\
&= \lim_{h\to 0}\frac{k-k}{h} \\
&= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \\
&= f'(a) && \text{Definition 12.1}
\end{aligned}$$

as desired. ☐

Exercise 12.9. Suppose that $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are differentiable at $a \in A$.

(a) Prove that $f + g$ is differentiable at a and compute $(f + g)'(a)$ in terms of $f'(a)$ and $g'(a)$.

Proof. To prove that $f + g$ is differentiable at a, Definition 12.1 tells us that it will suffice to show $\lim_{h\to 0} \frac{(f+g)(a+h)-(f+g)(a)}{h}$ exists. Since f, g are differentiable at a, we know by Definition 12.1 that $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ and $\lim_{h\to 0} \frac{g(a+h)-g(a)}{h}$ exist. Thus, by Theorem 11.9 the limit of their sum exists and equals

$$\begin{aligned}\lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}+\frac{g(a+h)-g(a)}{h}\right) &= \lim_{h\to 0}\frac{f(a+h)+g(a+h)-f(a)-g(a)}{h}\\ &= \lim_{h\to 0}\frac{(f+g)(a+h)-(f+g)(a)}{h}\end{aligned}$$

as desired. Having established that $\lim_{h\to 0} \frac{(f+g)(a+h)-(f+g)(a)}{h}$ exists, $(f + g)'(a)$ can be computed in terms of $f'(a)$ and $g'(a)$ with the following algebra.

$$\begin{aligned}(f+g)'(a) &= \lim_{h\to 0}\frac{(f+g)(a+h)-(f+g)(a)}{h} && \text{Definition 12.1}\\ &= \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}+\frac{g(a+h)-g(a)}{h}\right)\\ &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}+\lim_{h\to 0}\frac{g(a+h)-g(a)}{h} && \text{Theorem 11.9}\\ &= f'(a)+g'(a) && \text{Definition 12.1}\end{aligned}$$

□

(b) Prove that fg is differentiable and compute $(fg)'(a)$ in terms of $f(a)$, $g(a)$, $f'(a)$, and $g'(a)$.

Proof. To prove that fg is differentiable at a, Definition 12.1 tells us that it will suffice to show $\lim_{h\to 0} \frac{(fg)(a+h)-(fg)(a)}{h}$ exists. Since f, g are differentiable at a, we know by Definition 12.1 that $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ and $\lim_{h\to 0} \frac{g(a+h)-g(a)}{h}$ exist. For the same reason, we know by Theorem 12.5 that g is continuous, i.e., continuous at a by Theorem 9.10. Consequently, by Theorem 11.5, $\lim_{x\to a} g(x)$ exists (and equals $g(a)$). Note that the preceding limit is equal to $\lim_{h\to 0} g(a + h)$ by Lemma 12.3. Lastly, we have by Exercise 11.6 that the constant function $f(a)$ is continuous at 0. Consequently, by Theorem 11.5, $\lim_{h\to 0} f(a)$ exists (and equals $f(a)$). Combining all of these results, consecutive applications of Theorem 11.9 assert that the limits

$$\lim_{h\to 0} g(a+h)\cdot\frac{f(a+h)-f(a)}{h} \qquad\qquad \lim_{h\to 0} f(a)\cdot\frac{g(a+h)-g(a)}{h}$$

exist. Furthermore, it asserts that the limit of their sum exists and equals

$$\begin{aligned}&\lim_{h\to 0}\left(g(a+h)\cdot\frac{f(a+h)-f(a)}{h}+f(a)\cdot\frac{g(a+h)-g(a)}{h}\right)\\ &\qquad= \lim_{h\to 0}\frac{g(a+h)(f(a+h)-f(a))+f(a)(g(a+h)-g(a))}{h}\\ &\qquad= \lim_{h\to 0}\frac{f(a+h)g(a+h)-f(a)g(a+h)+f(a)g(a+h)-f(a)g(a)}{h}\\ &\qquad= \lim_{h\to 0}\frac{f(a+h)g(a+h)-f(a)g(a)}{h}\\ &\qquad= \lim_{h\to 0}\frac{(fg)(a+h)-(fg)(a)}{h}\end{aligned}$$

as desired. Having established that $\lim_{h\to 0}\frac{(fg)(a+h)-(fg)(a)}{h}$ exists, $(fg)'(a)$ can be computed in terms of $f(a)$, $g(a)$, $f'(a)$, and $g'(a)$ with the following algebra.

$$\begin{aligned}
(fg)'(a) &= \lim_{h\to 0}\frac{(fg)(a+h)-(fg)(a)}{h} && \text{Definition 12.1}\\
&= \lim_{h\to 0}\left(g(a+h)\cdot\frac{f(a+h)-f(a)}{h}+f(a)\cdot\frac{g(a+h)-g(a)}{h}\right)\\
&= \lim_{h\to 0} g(a+h)\cdot\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}+\lim_{h\to 0} f(a)\cdot\lim_{h\to 0}\frac{g(a+h)-g(a)}{h} && \text{Theorem 11.9}\\
&= g(a)f'(a)+f(a)g'(a)
\end{aligned}$$

□

(c) Prove that $\frac{1}{g}$ is differentiable at a (under an appropriate assumption) and compute $(\frac{1}{g})'(a)$ in terms of $g'(a)$ and $g(a)$. What assumption do you need to make?

Proof. Assume that $g(a)\neq 0$.

To prove that $\frac{1}{g}$ is differentiable at a, Definition 12.1 tells us that it will suffice to show that the limit $\lim_{h\to 0}\frac{(1/g)(a+h)-(1/g)(a)}{h}$ exists. Since g is differentiable at a, we know by Definition 12.1 that $\lim_{h\to 0}\frac{g(a+h)-g(a)}{h}$ exists. For the same reason, we know by Theorem 12.5 that g is continuous, i.e., continuous at a by Theorem 9.10. Consequently, by Theorem 11.5, $\lim_{x\to a} g(x)$ exists (and equals $g(a)$). It follows by Lemma 12.3 that the preceding limit is equal to $\lim_{h\to 0} g(a+h)$. Thus, since it is also equal to $g(a)\neq 0$, we have by Theorem 11.9 that $\lim_{h\to 0}\frac{1}{g}(a+h)$ exists (and equals $\frac{1}{g(a)}$). Lastly, we have by Exercise 11.6 that the constant function $-\frac{1}{g(a)}$ is continuous at 0. Consequently, by Theorem 11.5, $\lim_{h\to 0}-\frac{1}{g(a)}$ exists (and equals $-\frac{1}{g(a)}$). Combining this with the previous result, Theorem 11.9 asserts that the limit $\lim_{h\to 0}-\frac{1}{g(a+h)g(a)}$ exists (and equals $-\frac{1}{g(a)^2}$). Furthermore, it asserts that the limit of its product with $\lim_{h\to 0}\frac{g(a+h)-g(a)}{h}$ exists and equals

$$\begin{aligned}
\lim_{h\to 0}-\frac{1}{g(a+h)g(a)}\cdot\frac{g(a+h)-g(a)}{h} &= \lim_{h\to 0}\frac{\frac{g(a)-g(a+h)}{g(a+h)g(a)}}{h}\\
&= \lim_{h\to 0}\frac{\frac{1}{g(a+h)}-\frac{1}{g(a)}}{h}\\
&= \lim_{h\to 0}\frac{\frac{1}{g}(a+h)-\frac{1}{g}(a)}{h}
\end{aligned}$$

as desired. Having established that $\lim_{h\to 0}\frac{(1/g)(a+h)-(1/g)(a)}{h}$ exists, $(\frac{1}{g})'(a)$ can be computed in terms of $g(a)$ and $g'(a)$ with the following algebra.

$$\begin{aligned}
\left(\frac{1}{g}\right)'(a) &= \lim_{h\to 0}\frac{\frac{1}{g}(a+h)-\frac{1}{g}(a)}{h} && \text{Definition 12.1}\\
&= \lim_{h\to 0}-\frac{1}{g(a+h)g(a)}\cdot\frac{g(a+h)-g(a)}{h}\\
&= \lim_{h\to 0}-\frac{1}{g(a+h)g(a)}\cdot\lim_{h\to 0}\frac{g(a+h)-g(a)}{h} && \text{Theorem 11.9}\\
&= -\frac{g'(a)}{g(a)^2}
\end{aligned}$$

□

(d) Prove that $\frac{f}{g}$ is differentiable at a (under an appropriate assumption) and compute $(\frac{f}{g})'(a)$ in terms of $f(a)$, $g(a)$, $f'(a)$, and $g'(a)$. What assumption do you need to make?

Proof. Assume that $g(a) \neq 0$.

It follows by part (c) that $\frac{1}{g}$ is differentiable at a, and then by part (b) that $f \cdot \frac{1}{g} = \frac{f}{g}$ is differentiable at a.

Having established that $(\frac{f}{g})'(a)$ exists, it can be computed in terms of $f(a)$, $g(a)$, $f'(a)$, and $g'(a)$ with the following algebra.

$$\begin{aligned}\left(\frac{f}{g}\right)'(a) &= \left(f \cdot \frac{1}{g}\right)'(a)\\ &= f(a)\left(\frac{1}{g}\right)'(a) + f'(a)\left(\frac{1}{g}\right)(a)\\ &= f(a) \cdot -\frac{g'(a)}{g(a)^2} + \frac{f'(a)g(a)}{g(a)^2}\\ &= \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}\end{aligned}$$

□

4/1: One of the most important results concerning the differentiation of functions is the rule for the derivative of a composition of functions. Let $f : B \to \mathbb{R}, g : A \to \mathbb{R}$ be functions such that $g(A) \subset B$. The composition $(f \circ g)(x) = f(g(x))$ is defined for all $x \in A$.

Theorem 12.10. *Let $a \in A$, $g : A \to \mathbb{R}$, and $f : I \to \mathbb{R}$ where I is an interval containing $g(A)$. Suppose that g is differentiable at a and f is differentiable at $g(a)$. Then $f \circ g$ is differentiable a and*

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$$

Proof. To prove that $f \circ g$ is differentiable at a, Theorem 12.4 tells us that it will suffice to show that $\lim_{x\to a} \frac{(f\circ g)(x)-(f\circ g)(a)}{x-a}$ exists. To do so, we will define a special function φ and prove that it is continuous at a. It will follow that $(f \circ g)'(a)$ exists and equals $f'(g(a)) \cdot g'(a)$. Let's begin.

Let $\varphi : I \to \mathbb{R}$ be defined by

$$\varphi(x) = \begin{cases} \frac{f(g(x))-f(g(a))}{g(x)-g(a)} & g(x) \neq g(a) \\ f'(g(a)) & g(x) = g(a) \end{cases}$$

It is clear from the definition that the function is defined for all $x \in A$.

To confirm that φ is continuous at a, Theorem 11.5 tells us that it will suffice to demonstrate that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $|x-a| < \delta$, then $|\varphi(x)-\varphi(a)| = |\varphi(x)-f'(g(a))| < \epsilon$.

Let $\epsilon > 0$ be arbitrary. Since f is differentiable at $g(a)$, Theorem 12.4 asserts that $\lim_{y\to g(a)} \frac{f(y)-f(g(a))}{y-g(a)} = f'(g(a))$. It follows by Definition 11.1 that there is some $\delta' > 0$ such that if $y \in I$ and $0 < |y-g(a)| < \delta'$, then $|\frac{f(y)-f(g(a))}{y-g(a)} - f'(g(a))| < \epsilon$. Additionally, since g is differentiable (hence continuous by Theorem 12.5) at a, we have by Theorem 11.5 that there exists a $\delta > 0$ such that if $x \in A$ and $|x-a| < \delta$, then $|g(x)-g(a)| < \delta'$.

Using the above δ, let x be an arbitrary element of A such that $|x-a| < \delta$. We now divide into two cases ($g(x) = g(a)$ and $g(x) \neq g(a)$). If $g(x) = g(a)$, then $|\varphi(x) - f'(g(a))| = |f'(g(a)) - f'(g(a))| = 0 < \epsilon$, as desired. If $g(x) \neq g(a)$, then we continue. Since $|x-a| < \delta$, we have that $|g(x)-g(a)| < \delta'$. This combined with the fact that $g(x) \in I$ and $g(x) \neq g(a)$, i.e., $0 < |g(x)-g(a)|$ illustrates that $|\frac{f(g(x))-f(g(a))}{g(x)-g(a)} - f'(g(a))| = |\varphi(x) - f'(g(a))| < \epsilon$. Therefore, φ is continuous at a.

It follows by Theorem 11.5 that $\lim_{x\to a} \varphi(x) = \varphi(a) = f'(g(a))$. Additionally, since g is differentiable at a, Definition 12.1 and Theorem 12.4 tell us that $\lim_{x\to a} \frac{g(x)-g(a)}{x-a}$ exists (and equals $g'(a)$). This combined with the previous result implies by Theorem 11.9 that the product of the limits exists and equals $f'(g(a)) \cdot g'(a)$, i.e., we have that $\lim_{x\to a} \varphi(x) \cdot \frac{g(x)-g(a)}{x-a} = f'(g(a)) \cdot g'(a)$.

We now seek to confirm that $\lim_{x\to a} \frac{f(g(x))-f(g(a))}{x-a} = f'(g(a)) \cdot g'(a)$. To do so, Definition 11.1 tells us that it will suffice to demonstrate that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and

$0 < |x-a| < \delta$, then $|\frac{f(g(x))-f(g(a))}{x-a} - f'(g(a)) \cdot g'(a)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{x\to a} \varphi(x) \cdot \frac{g(x)-g(a)}{x-a} = f'(g(a)) \cdot g'(a)$, we have by Definition 11.1 that there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x-a| < \delta$, then $|\varphi(x) \cdot \frac{g(x)-g(a)}{x-a} - f'(g(a)) \cdot g'(a)| < \epsilon$. Choose this δ to be our δ. Let x be an arbitrary element of A that satisfies $0 < |x-a| < \delta$. We divide into two cases ($g(x) = g(a)$ and $g(x) \neq g(a)$). Suppose first that $g(x) = g(a)$. Since $0 < |x-a| < \delta$, we have by the above and the definition of φ that $|f'(g(a)) \cdot \frac{g(x)-g(a)}{x-a} - f'(g(a)) \cdot g'(a)| < \epsilon$. Additionally, it follows from the hypothesis that $g(x) - g(a) = 0$ and $f(g(x)) - f(g(a)) = 0$. Therefore,

$$\begin{aligned}
\left|\frac{f(g(x))-f(g(a))}{x-a} - f'(g(a)) \cdot g'(a)\right| &\leq \left|\frac{f(g(x))-f(g(a))}{x-a} - f'(g(a)) \cdot \frac{g(x)-g(a)}{x-a}\right| && \text{Lemma 8.8} \\
&\quad + \left|f'(g(a)) \cdot \frac{g(x)-g(a)}{x-a} - f'(g(a)) \cdot g'(a)\right| \\
&= \left|\frac{0}{x-a} - f'(g(a)) \cdot \frac{0}{x-a}\right| \\
&\quad + \left|f'(g(a)) \cdot \frac{g(x)-g(a)}{x-a} - f'(g(a)) \cdot g'(a)\right| \\
&= 0 + \left|f'(g(a)) \cdot \frac{g(x)-g(a)}{x-a} - f'(g(a)) \cdot g'(a)\right| \\
&< \epsilon
\end{aligned}$$

as desired. Now suppose that $g(x) \neq g(a)$. Since $0 < |x-a| < \delta$, we have by the above and the definition of φ that $|\frac{f(g(x))-f(g(a))}{g(x)-g(a)} \cdot \frac{g(x)-g(a)}{x-a} - f'(g(a)) \cdot g'(a)| < \epsilon$. Additionally, it follows from the hypothesis that $g(x) - g(a) \neq 0$. Therefore, we have by Script 7 that $|\frac{f(g(x))-f(g(a))}{x-a} - f'(g(a)) \cdot g'(a)| < \epsilon$, as desired.

It follows by Definition 1.25 that $\lim_{x\to a} \frac{(f\circ g)(x)-(f\circ g)(a)}{x-a} = f'(g(a)) \cdot g'(a)$. This proves that $f \circ g$ is differentiable at a. Lastly, it directly follows from Theorem 12.4 that

$$(f \circ g)'(a) = \lim_{x\to a} \frac{(f\circ g)(x) - (f \circ g)(a)}{x-a} = f'(g(a)) \cdot g'(a)$$

as desired. □

We now come to the most important theorem in differential calculus, Corollary 12.16.

Definition 12.11. Let $f : A \to \mathbb{R}$ be a function. If $f(a)$ is the last point of $f(A)$, then $f(a)$ is called the **maximum value** of f. If $f(a)$ is the first point of $f(A)$, then $f(a)$ is the **minimum value** of f. We say that $f(a)$ is a **local maximum value** of f if there exists a region R containing a such that $f(a)$ is the last point of $f(A \cap R)$. We say that $f(a)$ is a **local minimum value** of f if there exists a region R containing a such that $f(a)$ is the first point of $f(A \cap R)$.

Remark 12.12. Equivalently, $f(a)$ is a local maximum (resp. minimum) value of f if there exists U open in A such that $f(a)$ is the last (resp. first) point of $f(U)$.

Theorem 12.13. *Let $f : A \to \mathbb{R}$ be differentiable at a. Suppose that $f(a)$ is the maximum value or minimum value of f. Then $f'(a) = 0$.*

Proof. Suppose first that $f(a)$ is the maximum value of f, and suppose for the sake of contradiction that $f'(a) \neq 0$. Then $f'(a) > 0$ or $f'(a) < 0$. We now divide into two cases. If $f'(a) > 0$, then by Theorem 12.4, $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} > 0$. Thus, by Lemma 11.8, there exists a region R with $a \in R$ such that $\frac{f(x)-f(a)}{x-a} > 0$ for all $x \in R \cap A$. Additionally, since A is open, Theorem 4.10 implies that there exists a region S with $a \in S$ and $S \subset A$. It follows by Theorem 3.18 that $R \cap S = (c, d)$, where (c, d) is a region containing a. Since $a \in (c, d)$, Theorem 5.2 implies that there exists a point $y \in \mathbb{R}$ such that $a < y < d$. Clearly, since $y \in (c, d) = R \cap S$ and $S \subset A$, we have that $y \in R \cap A$. It follows that $\frac{f(y)-f(a)}{y-a} > 0$. Furthermore, $y > a$ implies that $y - a > 0$. Therefore, by Definition 7.21, $f(y) - f(a) = \frac{f(y)-f(a)}{y-a} \cdot (y-a) > 0$. But this means that $f(y) > f(a)$, i.e., that $f(a)$ is not the last point of $f(A)$ (by Definition 3.3), i.e., that $f(a)$ is not the maximum value of f (by Definition 12.11), a contradiction. The argument is symmetric in the other case.

The proof is symmetric in the other case. $\square$

Corollary 12.14. *Let $f : A \to \mathbb{R}$ be differentiable at a. Suppose that $f(a)$ is a local maximum or local minimum value of f. Then $f'(a) = 0$.*

Proof. Suppose first that $f(a)$ is a local maximum of f. Then by Definition 12.11, there exists a region R containing a such that $f(a)$ is the last point of $f(A \cap R)$. Now consider the restriction of f to $A \cap R$. It follows from Definition 9.6 that $f|_{A\cap R}$ is differentiable at a, that $f|_{A\cap R}(A \cap R) = f(A \cap R)$, and that $f|_{A\cap R}(a) = f(a)$ is the last point of $f|_{A\cap R}(A \cap R)$. The latter two results imply by Definition 12.11 that $f|_{A\cap R}(a)$ is the maximum value of $f|_{A\cap R}$. This combined with the fact that $f|_{A\cap R}$ is differentiable at a implies by Theorem 12.13 that $(f|_{A\cap R})'(a) = f'(a) = 0$, as desired. $\square$

Theorem 12.15. *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, differentiable on (a, b), and that $f(a) = f(b) = 0$. Then there exists a point $\lambda \in (a, b)$ such that $f'(\lambda) = 0$.*

Proof. We divide into two cases ($f(x) = 0$ for all $x \in [a, b]$, and $f(x) \neq 0$ for some $x \in [a, b]$).

Suppose first that $f(x) = 0$ for all $x \in [a, b]$. By Theorem 5.2, we can choose a $\lambda \in (a, b)$. It follows from the hypothesis that $f(\lambda) = f(x)$ for all $f(x) \in f([a, b])$. This can be weakened to $f(\lambda) \geq f(x)$ for all $f(x) \in f([a, b])$. Thus, by Definition 3.3, $f(\lambda)$ is the last point of $f([a, b])$. Consequently, by Definition 12.11, $f(\lambda)$ is the maximum value of f. This combined with the fact that f is differentiable at λ (since $\lambda \in (a, b)$ and f is differentiable on (a, b)) implies by Theorem 12.13 that $f'(\lambda) = 0$, as desired.

Now suppose that $f(x) \neq 0$ for some $x \in [a, b]$ which we shall call x_0. We divide into two cases again ($f(x_0) > 0$ and $f(x_0) < 0$). Suppose first that $f(x_0) > 0$. Since $f : [a, b] \to \mathbb{R}$ is continuous, Exercise 10.21 asserts that there exists a point $\lambda \in [a, b]$ such that $f(\lambda) \geq f(x)$ for all $x \in [a, b]$. It follows that $f(\lambda) \geq f(x_0) > 0$, so $f(\lambda) \neq f(a) = f(b)$. Thus, by Definition 1.16, $\lambda \neq a$ and $\lambda \neq b$. This combined with the fact that $\lambda \in [a, b]$ implies by Script 8 that $\lambda \in (a, b)$. Now as before, we can determine from the fact that $f(\lambda) \geq f(x)$ for all $x \in [a, b]$ that $f(\lambda)$ is the maximum value of f. This combined with the fact that f is differentiable at λ (since $\lambda \in (a, b)$ and f is differentiable on (a, b)) implies by Theorem 12.13 that $f'(\lambda) = 0$, as desired. The argument is symmetric in the other case. $\square$

Corollary 12.16. *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point $\lambda \in (a, b)$ such that*

$$f(b) - f(a) = f'(\lambda)(b - a)$$

Proof. Let $h : [a, b] \to \mathbb{R}$ be defined by

$$h(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a)$$

By hypothesis, $f(x)$ is continuous on $[a, b]$. By Exercise 11.6 and Theorem 11.9, $-\frac{f(b)-f(a)}{b-a}(x - a) - f(a)$ is continuous on $[a, b]$. Thus, by Theorem 11.9, their sum (i.e., $h(x)$) is continuous on $[a, b]$. Additionally, by hypothesis, $f(x)$ is differentiable on (a, b). By Exercises 12.8 and 12.9, $-\frac{f(b)-f(a)}{b-a}(x - a) - f(a)$ is differentiable on $[a, b]$. Thus, by Exercise 12.9, their sum (i.e., $h(x)$) is differentiable on (a, b). Furthermore, by simple algebra, we can determine that

$$\begin{aligned} h(a) &= f(a) - \frac{f(b) - f(a)}{b - a}(a - a) - f(a) \\ &= -\frac{f(b) - f(a)}{b - a} \cdot 0 \\ &= 0 \end{aligned} \qquad \begin{aligned} h(b) &= f(b) - \frac{f(b) - f(a)}{b - a}(b - a) - f(a) \\ &= f(b) - (f(b) - f(a)) - f(a) \\ &= 0 \end{aligned}$$

Thus, by Theorem 12.15, there exists a point $\lambda \in (a, b)$ such that $h'(\lambda) = 0$.

We can also calculate $h'(x)$ as follows.

$$\begin{aligned}
h'(x) &= \left(f(x) - \frac{f(b)-f(a)}{b-a}(x-a) - f(a)\right)' \\
&= \left((f(x)) + \left(-\frac{f(b)-f(a)}{b-a}\cdot x\right) + \left(\frac{f(b)-f(a)}{b-a}\cdot a - f(a)\right)\right)' \\
&= f'(x) + \left(-\frac{f(b)-f(a)}{b-a}\cdot x\right)' + \left(\frac{f(b)-f(a)}{b-a}\cdot a - f(a)\right)' && \text{Exercise 12.9} \\
&= f'(x) + \left(-\frac{f(b)-f(a)}{b-a}\right)'\cdot(x) + \left(-\frac{f(b)-f(a)}{b-a}\right)\cdot(x)' + \left(\frac{f(b)-f(a)}{b-a}\cdot a - f(a)\right)' \\
&&& \text{Exercise 12.9} \\
&= f'(x) + 0\cdot x + \frac{f(b)-f(a)}{b-a}\cdot 1x^0 + 0 && \text{Exercise 12.8} \\
&= f'(x) + \frac{f(b)-f(a)}{b-a}
\end{aligned}$$

But it follows that at λ,

$$\begin{aligned}
0 &= f'(\lambda) - \frac{f(b)-f(a)}{b-a} \\
\frac{f(b)-f(a)}{b-a} &= f'(\lambda) \\
f(b)-f(a) &= f'(\lambda)(b-a)
\end{aligned}$$

as desired. ☐

4/6: **Corollary 12.17.** *Suppose that $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then*

(a) *If $f'(x) > 0$ for all $x \in (a,b)$, then f is strictly increasing on $[a,b]$.*

Proof. To prove that f is strictly increasing on $[a,b]$, Definition 8.16 tells us that it will suffice to show that if $x,y \in [a,b]$ with $x < y$, then $f(x) < f(y)$. Let x,y be arbitrary elements of $[a,b]$. WLOG, let $x < y$. Since f is continuous on $[a,b]$ and differentiable on (a,b), Corollary 12.16 asserts that there exists a point $\lambda \in (x,y)$ such that $f(y)-f(x) = f'(\lambda)(y-x)$. But since $f'(\lambda) > 0$ by hypothesis and $y-x > 0$ because $y > x$, we have by Definition 7.21 that $f'(\lambda)(y-x) > 0$. It follows that $f(y)-f(x) > 0$, i.e., that $f(x) < f(y)$, as desired. ☐

(b) *If $f'(x) < 0$ for all $x \in (a,b)$, then f is strictly decreasing on $[a,b]$.*

Proof. The proof is symmetric to that of part (a). ☐

(c) *If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on $[a,b]$.*

Proof. To prove that f is constant on $[a,b]$, it will suffice to show that $f(x) = f(y)$ for all $x,y \in [a,b]$. Let x,y be arbitrary elements of $[a,b]$. WLOG, let $x < y$. Since f is continuous on $[a,b]$ and differentiable on (a,b), Corollary 12.16 asserts that there exists a point $\lambda \in (x,y)$ such that $f(y)-f(x) = f'(\lambda)(y-x)$. But since $f'(\lambda) = 0$ by hypothesis, $f(y)-f(x) = 0$, i.e., $f(y) = f(x)$, as desired. ☐

Remark 12.18. Corollary 12.17 also holds if instead of $[a,b]$, we have an arbitrary interval I; and instead of (a,b), we have the interior of I.

Corollary 12.19. *Suppose that $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ are continuous on $[a,b]$, differentiable on (a,b), and $f'(x) = g'(x)$ for all $x \in (a,b)$. Then there is some $c \in \mathbb{R}$ such that for all $x \in [a,b]$, we have $f(x) = g(x) + c$.*

Proof. Let $h : [a, b] \to \mathbb{R}$ be defined by $h(x) = f(x) - g(x)$. Since f, g are continuous on $[a, b]$, Corollary 11.10 asserts that h is continuous on $[a, b]$. Since f, g are differentiable on (a, b), Exercise 12.9 asserts that h is differentiable on (a, b). Since $f'(x) = g'(x)$ for all $x \in (a, b)$, Exercise 12.9 implies that $h'(x) = f'(x) - g'(x) = 0$ for all $x \in (a, b)$. These three results satisfy the conditions of Corollary 12.17, which means that h is constant on $[a, b]$, i.e., that $h(x) = c$ for all $x \in [a, b]$ where $c \in \mathbb{R}$. But by the definition of h, this implies that for all $x \in [a, b]$, we have $f(x) - g(x) = c$, i.e., $f(x) = g(x) + c$. □

Corollary 12.20. *Suppose that $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b). Then there is a point $\lambda \in (a, b)$ such that*

$$(f(b) - f(a))g'(\lambda) = (g(b) - g(a))f'(\lambda)$$

Proof. Let $h : [a, b] \to \mathbb{R}$ be defined by $h(x) = (g(b) - g(a))f(x) - (f(b) - f(a))g(x) - f(a)g(b) + f(b)g(a)$. For the same reasons as in the proof of Corollary 12.19, h is continuous on $[a, b]$ and differentiable on (a, b). Additionally, we can show with basic algebra that

$$\begin{aligned} h(a) &= (g(b) - g(a))f(a) - (f(b) - f(a))g(a) - f(a)g(b) + f(b)g(a) \\ &= f(a)g(b) - f(a)g(a) - f(b)g(a) + f(a)g(a) - f(a)g(b) + f(b)g(a) \\ &= 0 \end{aligned}$$

$$\begin{aligned} h(b) &= (g(b) - g(a))f(b) - (f(b) - f(a))g(b) - f(a)g(b) + f(b)g(a) \\ &= f(b)g(b) - f(b)g(a) - f(b)g(b) + f(a)g(b) - f(a)g(b) + f(b)g(a) \\ &= 0 \end{aligned}$$

These results satisfy the conditions of Theorem 12.15, which means that there exists a point $\lambda \in (a, b)$ such that $h'(\lambda) = 0$. We can also calculate that $h'(x) = (g(b) - g(a))f'(x) - (f(b) - f(a))g'(x)$ via a similar method to that used in the proof of Corollary 12.16. But it follows that at λ,

$$\begin{aligned} 0 &= (g(b) - g(a))f'(\lambda) - (f(b) - f(a))g'(\lambda) \\ (f(b) - f(a))g'(\lambda) &= (g(b) - g(a))f'(\lambda) \end{aligned}$$

□

Finally, we prove another very important theorem that tells us about inverse functions and their derivatives.

Theorem 12.21. *Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable and that the derivative $f' : (a, b) \to \mathbb{R}$ is continuous. Also suppose that there is a point $p \in (a, b)$ such that $f'(p) \neq 0$. Then there exists a region $R \subset (a, b)$ such that $p \in R$ and f with domain restricted to R is injective. Furthermore, $f^{-1} : f(R) \to R$ is differentiable at the point $f(p)$ and*

$$(f^{-1})'(f(p)) = \frac{1}{f'(p)}$$

Proof. Since $f' : (a, b) \to \mathbb{R}$ is continuous, Theorem 9.10 implies that it is continuous at $p \in (a, b)$. Thus, by Theorem 11.5, $\lim_{x \to p} f'(x) = f'(p)$. This combined with the facts that $f'(p) \neq 0$ (i.e., $f'(p) > 0$ or $f'(p) < 0$) and f' is continuous at p implies by Lemma 11.8 that there exists a region R with $p \in R$ such that $f'(x) > 0$ for all $x \in R \cap (a, b)$ or $f'(x) < 0$ for all $x \in R \cap (a, b)$. Clearly, $R \cap (a, b) = R$. Consequently, since $f'(x) > 0$ for all $x \in R$ or $f'(x) < 0$ for all $x \in R$, Corollary 12.17 asserts that f is strictly increasing or strictly decreasing on R. Since R is an interval by Lemma 8.3, the previous result implies by Lemma 8.17 that f is injective on R. Therefore, we have found an $R = R \cap (a, b) \subset (a, b)$ by Theorem 1.7 with $p \in R$ such that $f|_R$ is injective, as desired.

From now on, we will denote $f|_R : R \to \mathbb{R}$ by f. Since f is differentiable, Definition 12.1 asserts that it is differentiable for all $x \in R$. Thus, by Theorem 12.5, f is continuous for all $x \in R$. Consequently, by Theorem 9.10, f is continuous. This combined with the previously proven fact that f is injective implies by Theorem 9.14 that the inverse function $f^{-1} : f(R) \to R$ exists and is continuous.

To prove that $(f^{-1})'(f(p))$ exists and equals $\frac{1}{f'(p)}$, Theorem 12.4 tells us that it will suffice to show that $\lim_{y\to f(p)} \frac{f^{-1}(y)-f^{-1}(f(p))}{y-f(p)} = \frac{1}{f'(p)}$. To do this, Definition 11.1 tells us that it will suffice to confirm that for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in f(R)$ and $0 < |y - f(p)| < \delta$, then $|\frac{f^{-1}(y)-f^{-1}(f(p))}{y-p} - \frac{1}{f'(p)}| < \epsilon$. Let $\epsilon > 0$ be arbitrary. We have that

$$\begin{aligned}
\frac{1}{f'(p)} &= \frac{1}{\lim_{x\to p} \frac{f(x)-f(p)}{x-p}} && \text{Theorem 12.4}\\
&= \lim_{x\to p} \frac{x-p}{f(x)-f(p)} && \text{Theorem 11.9}\\
&= \lim_{x\to p} \frac{f^{-1}(f(x)) - f^{-1}(f(p))}{f(x)-f(p)} && \text{Definition 1.18}
\end{aligned}$$

Thus, by Definition 11.1, there exists a $\delta' > 0$ such that if $x \in R$ and $0 < |x-p| < \delta'$, then $|\frac{f^{-1}(f(x))-f^{-1}(f(p))}{f(x)-f(p)} - \frac{1}{f'(p)}| < \epsilon$.

The previously proven fact that f^{-1} is continuous implies by Theorem 9.10 that f^{-1} is continuous at $f(p)$. It follows by Theorem 11.5 that either $f(p) \notin LP(f(R))$ or $\lim_{y\to f(p)} f^{-1}(y) = f^{-1}(f(p))$. However, as we will now see, $f(p) \in LP(f(R))$. To begin, the fact that R is a region implies by Lemma 8.3 that R is an interval. It follows by Theorem 8.15 that R is connected, by Theorem 9.11 that $f(R)$ is connected, by Theorem 8.15 again that $f(R)$ is an interval, and finally by extensions of Corollaries 5.5 and 5.14 that $f(p) \in LP(f(R))$. Thus, with this case eliminated, we know that $\lim_{y\to f(p)} f^{-1}(y) = f^{-1}(f(p))$. Consequently, by Definition 1.11, there exists a $\delta > 0$ such that if $y \in f(R)$ and $0 < |y - f(p)| < \delta$, then $|f^{-1}(y) - f^{-1}(f(p))| < \delta'$. With a slight modification from Definition 1.18 (and the definition that $y = f(x)$), we have that there exists a $\delta > 0$ such that if $y \in f(R)$ and $0 < |y - f(p)| < \delta$, then $|x - p| < \delta'$.

Choose the above δ as our δ. Let $y = f(x)$ be an arbitrary element of $f(R)$ satisfying $0 < |y - f(p)| < \delta$. Then $|x - p| < \delta'$. We can also show that $0 < |x - p|$: From Script 8, the fact that $0 < |y - f(p)|$ implies that $f(x) \neq f(p)$; hence by Definition 1.20 and the fact that f is injective, $x \neq p$; hence by Script 8 again, $0 < |x - p|$. Continuing, since $y = f(x) \in f(R)$, we have by Definition 1.18 that $x \in R$. Thus, we have that $x \in R$ and $0 < |x - p| < \delta'$, so we know that $|\frac{f^{-1}(f(x))-f^{-1}(f(p))}{f(x)-f(p)} - \frac{1}{f'(p)}| < \epsilon$. Therefore, with a slight modification from Definition 1.18, we have that $|\frac{f^{-1}(y)-f^{-1}(f(p))}{y-f(p)} - \frac{1}{f'(p)}| < \epsilon$, as desired. □

4/8: **Exercise 12.22.** Consider the function $f(x) = x^n$ for a fixed $n \in \mathbb{N}$. Show that if n is even, then f is strictly increasing on the set of nonnegative real numbers and that if n is odd, then f is strictly increasing on all of $\mathbb{R}$. For a given n, let A be the aforementioned set on which f is strictly increasing. Define the inverse function $f^{-1} : f(A) \to A$ by $f^{-1}(x) = \sqrt[n]{x}$, which we sometimes also denote $f^{-1}(x) = x^{1/n}$. Use Theorem 12.21 to find the points $y \in f(A)$ at which f^{-1} is differentiable, and determine $(f^{-1})'(y)$ at these points.

Proof. For the first part of the question, we must prove that f is strictly increasing on $\mathbb{R}^+$ if n is even and that f is strictly increasing on $\mathbb{R}$ if n is odd. To do so, we induct on n. For the base case $n = 1$, we must confirm that $f(x) = x$ is strictly increasing on $\mathbb{R}$ (since n is odd). By Exercise 12.8, $f'(x) = 1 > 0$ for all $x \in \mathbb{R}$. Thus, by Corollary 12.17 and Remark 12.18, f is strictly increasing on $\mathbb{R}$, as desired. Now suppose inductively that the claim holds for some natural number $n \in \mathbb{N}$; we wish to confirm it for $n + 1$. We divide into two cases ($n + 1$ is even and $n + 1$ is odd).

If $n + 1$ is even, then we must confirm that $f(x) = x^{n+1}$ is strictly increasing on $\mathbb{R}^+$. By Exercise 12.8, $f'(x) = (n+1)x^n$, where n is odd. To verify that f' is strictly increasing on $\mathbb{R}^+$, Definition 8.16 tells us that it will suffice to demonstrate that for all $x, y \in \mathbb{R}^+$ satisfying $x < y$, $f'(x) < f'(y)$. Let x, y be arbitrary elements of $\mathbb{R}^+$ that satisfy $x < y$. Since x^n is strictly increasing on $\mathbb{R}^+ \subset \mathbb{R}$ by the inductive hypothesis, Definition 8.16 tells us that $x^n < y^n$. This combined with the fact that $0 < n + 1$ by Script 0 implies by Lemma 7.24 that $(n+1)x^n < (n+1)y^n$. Thus, by the definition of f', $f'(x) < f'(y)$, as desired. Note also that $f'(0) = (n+1)0^n = 0$. Having established that f' is strictly increasing on $\mathbb{R}^+$ and that $f'(0) = 0$, we have that $f'(x) > 0$ for all $x \in (0, \infty)$ by Definition 8.16 because $0 < x$ implies $0 = f'(0) < f'(x)$. Thus, by Corollary 12.17 and Remark 12.18, f is strictly increasing on $\mathbb{R}^+$, as desired.

If $n+1$ is odd, then we must confirm that $f(x) = x^{n+1}$ is strictly increasing on $\mathbb{R}$. By Exercise 12.8, $f'(x) = (n+1)x^n$, where n is even. With a symmetric argument to that used above, we can verify that f' is strictly increasing on $\mathbb{R}^+$. Since $f'(-x) = (n+1)(-x)^n = (n+1)x^n = f'(x)$ by Script 7, we can similarly prove that f' is strictly *decreasing* on $\mathbb{R}^-$ (essentially, if $x, y \in \mathbb{R}^-$, then $x < y$ implies $-y < -x$ implies $f(-y) < f(-x)$ implies $f(x) > f(y)$, as desired). These two results combined with the fact that we still have $f'(0) = 0$ imply by a symmetric argument to the above that $f'(x) > 0$ for all $x \in (-\infty, 0) \cup (0, \infty)$. But it follows by consecutive applications of Corollary 12.17 and Remark 12.18 that f is strictly increasing on $\mathbb{R}$, as desired.

For the second part of the question, we must find all of the points y where f^{-1} is differentiable and determine the derivative $(f^{-1})'(y)$ at these points for an arbitrary n. Let n be an arbitrary element of $\mathbb{N}$. By Exercise 12.8, f is differentiable, and by both Exercise 12.8 and Corollary 11.12, f' is continuous. We divide into two cases (n is even and n is odd).

Suppose first that n is even. To begin, we will verify that $f(\mathbb{R}^+) = \mathbb{R}^+$. By Definition 1.2, to do so it will suffice to show that every $y \in f(\mathbb{R}^+)$ is an element of $\mathbb{R}^+$ and vice versa. Let y be an arbitrary element of $f(\mathbb{R}^+)$. Then by Definition 1.18, $y = f(x)$ for some $x \in \mathbb{R}^+$. It follows since $x \in \mathbb{R}^+$ that $x \geq 0$. Consequently, since f is strictly increasing on $\mathbb{R}^+$, Definition 8.16 implies that $f(x) \geq f(0) = 0$. But if $y = f(x) \geq 0$, then $y \in \mathbb{R}^+$, as desired. Now let y be an arbitrary element of $\mathbb{R}^+$. We divide into three cases ($y = 0$, $0 < y \leq 1$ and $1 < y$). If $y = 0$, then since $f(0) = 0^n = 0 = y$ and $0 \in \mathbb{R}^+$, $y = f(x)$ for an $x \in \mathbb{R}^+$. Therefore, Definition 1.18 asserts that $y \in f(\mathbb{R}^+)$, as desired. If $0 < y \leq 1$, then since f is continuous (notably on $[0, 2]$) and $f(0) = 0 < y < 2^n = f(2)$ (by Script 7), Exercise 9.12 asserts that there exists a point $x \in \mathbb{R}$ with $0 < x < 2$ such that $f(x) = y$. Therefore, since $y = f(x)$ for an $x \in \mathbb{R}^+$ (we do know that $x > 0$), Definition 1.18 asserts that $y \in f(\mathbb{R}^+)$, as desired. If $y > 1$, then since $f(0) = 0 < 1 < y < y^n = f(y)$ (by Script 7), Exercise 9.12 asserts that there exists a point $x \in \mathbb{R}$ with $0 < x < y$ such that $f(x) = y$. Therefore, since $y = f(x)$ for an $x \in \mathbb{R}^+$ (we do know that $x > 0$), Definition 1.18 asserts that $y \in f(\mathbb{R}^+)$, as desired.

Having established that $f(\mathbb{R}^+) = \mathbb{R}^+$, our task becomes one of finding all points $y \in \mathbb{R}^+$ at which f^{-1} is differentiable, and determining $(f^{-1})'(y)$ at these points. By Theorem 12.21, this means that we need only find all points $f(p)$ corresponding to a p that satisfies $f'(p) \neq 0$. Since $f'(x) = nx^{n-1}$ by Exercise 12.8, Script 7 implies that the only point p where $f'(p) = 0$ is $p = 0$. Thus, we need only exclude $f(0) = 0$ from our set of points at which f^{-1} is differentiable. Therefore, we know that f^{-1} is differentiable at every $y \in \mathbb{R}^+$ such that $y \neq 0$, or more simply, all y in the interval $(0, \infty)$. Additionally, Theorem 12.21 implies that for any $y \in (0, \infty)$,

$$\begin{aligned}
(f^{-1})'(y) &= \frac{1}{f'(f^{-1}(y))} \\
&= \frac{1}{n(f^{-1}(y))^{n-1}} \\
&= \frac{1}{n(y^{1/n})^{n-1}} \\
&= \frac{1}{n} \cdot y^{\frac{1-n}{n}}
\end{aligned}$$

The proof is symmetric if n is odd.

∎

Additional Exercises

3. (a) Suppose $f : [a, b] \to \mathbb{R}$, $x_0 \in (a, b)$, f', f'' exist and are continuouous oon soome interval containing x_0, and $f'(x_0) = 0$, but $f''(x_0) \neq 0$. Prove that
   i) if $f''(x_0) > 0$, then f has a local minimum at x_0;
   ii) if $f''(x_0) < 0$, then f has a local maximum at x_0.

   (b) **Bernoulli's inequality**: Prove that for $x \geq -1$ and $\alpha \geq 1$,

$$(1 + x)^\alpha \geq 1 + \alpha x$$

and for $x \geq -1$ and $0 \leq \alpha \leq 1$,
$$(1+x)^\alpha \leq 1 + \alpha x$$

(c) Prove that $x - \frac{x^3}{6} < \sin x < x$ for all $x > 0$.

12.2 Discussion

3/30:
- We can also do Exercise 12.6 with left- and right-handed limits, as defined in Additional Exercise 11.2.
- We can also do Exercise 12.7 by induction.

4/1:
- We can also do Theorem 12.13 by noting that of the left- and right-hand derivatives, one will be ≤ 0 and the other ≥ 0, but since they must be equal, they must equal 0.
- Include more rigorous restriction bits for Corollary 12.14 as a lemma?

4/6:
- Modify Theorem 12.13 with Lemma 11.8.
- Redo Corollary 12.17c as a direct proof?
- We don't have to be too rigorous with the restriction of f in Corollary 12.17. In fact, we need not even mention it.
- Include a bit more of the basic algebra proving that $h(a) = h(b) = 0$ for Corollary 12.20.
- Potentially for Theorem 12.21, we can use $\varphi(y)$ but modified with f^{-1} to prove the iffy limit transition.
  - Potentially we can apply the chain rule for this proof?
  - Could we use the fact that f is continuous to prove that as $x \to p$, $f(x) \to f(p)$?

Script 13

Uniform Continuity and Integration

13.1 Journal

4/8: **Definition 13.1.** Let $f : A \to \mathbb{R}$ be a function. We say that f is **uniformly continuous** if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$.

Theorem 13.2. *If f is uniformly continuous, then f is continuous.*

Proof. To prove that f is continuous, Theorem 9.10 tells us that it will suffice to show that f is continuous at every $x \in A$. Let x be an arbitrary element of A. To show that f is continuous at x, Theorem 11.5 tells us that it will suffice to verify that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Then since f is uniformly continuous by hypothesis, Definition 13.1 asserts that there exists a $\delta > 0$ such that for all $y \in A$ satisfying $|y - x| < \delta$, we have $|f(y) - f(x)| < \epsilon$, as desired. $\square$

Exercise 13.3. Determine with proof whether each function f is uniformly continuous on the given interval A.

(a) $f(x) = x^2$ on $A = \mathbb{R}$.

Proof. To prove that f is not uniformly continuous on A, Definition 13.1 tells us that it will suffice to find an $\epsilon > 0$ for which no $\delta > 0$ exists such that for all $x, y \in A$, if $|y - x| < \delta$, then $|y^2 - x^2| < \epsilon$. Let $\epsilon = 2$, and suppose for the sake of contradiction that $\delta > 0$ is a number such that for all $x, y \in A$, if $|y - x| < \delta$, then $|y^2 - x^2| < 2$. By Theorem 5.2, there exists a number y such that $0 < y < \delta$. Since $-\delta < 0 < y < \delta$ by Lemma 7.23, we have by the lemma from Exercise 8.9, that $|y| < \delta$. Consequently, $|(y + n) - n| < \delta$. It follows by the above that $|(y + n)^2 - n^2| = |y^2 + 2yn| < 2$. If we now let $n = \frac{1}{y}$, then $|y^2 + 2| < 2$. But since $y > 0$, we have that $y^2 > 0$ by Lemma 7.26. It follows that $y^2 + 2 > 2$ by Definition 7.21. Therefore, by Definition 8.4, we can also show that $|y^2 + 2| > 2$, a contradiction. $\square$

(b) $f(x) = x^2$ on $A = (-2, 2)$.

Proof. To prove that f is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \frac{\epsilon}{4}$, and let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. Since $x, y \in A$, consecutive applications of Equations 8.1 and the lemma from Exercise 8.9 imply that $|x| < 2$ and $|y| < 2$. It follows that $|x| + |y| < 2 + 2 = 4$. Consequently, by Lemma 8.8, $|x + y| < 4$. Additionally, since $0 \leq |y + x|$ by Definition 8.4, we have by Definition 7.21 $|x - y| \cdot |x + y| \leq \frac{\epsilon}{4} \cdot |x + y|$. Combining all of the above results, we have that

$$\begin{aligned} |f(y) - f(x)| &= |y^2 - x^2| \\ &= |y + x| \cdot |y - x| \end{aligned}$$

$$\begin{aligned} &\leq |x+y| \cdot \frac{\epsilon}{4} \\ &< 4 \cdot \frac{\epsilon}{4} \\ &= \epsilon \end{aligned}$$

as desired. □

(c) $f(x) = \frac{1}{x}$ on $A = (0, +\infty)$.

Proof. To prove that f is not uniformly continuous on A, Definition 13.1 tells us that it will suffice to find an $\epsilon > 0$ for which no $\delta > 0$ exists such that for all $x, y \in A$, if $|y - x| < \delta$, then $|\frac{1}{y} - \frac{1}{x}| < \epsilon$. Let $\epsilon = 1$, and suppose for the sake of contradiction that $\delta > 0$ is a number such that for all $x, y \in A$, if $|y - x| < \delta$, then $|\frac{1}{y} - \frac{1}{x}| < 1$. As in part (a), choose $0 < x < \min(\delta, \frac{1}{2})$. Consequently, $|(x + x) - x| < \delta$. It follows by the above that $|\frac{1}{2x} - \frac{1}{x}| < 1$. But this implies that $|\frac{x-2x}{2x^2}| = |\frac{-1}{2x}| = \frac{1}{2x} < 1$. However, $x < \frac{1}{2}$ implies by Lemma 7.24 that $1 < \frac{1}{2x}$, a contradiction. □

(d) $f(x) = \frac{1}{x}$ on $A = [1, +\infty)$.

Proof. To prove that f is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \epsilon$, and let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. Since $x, y \in A$, consecutive applications of Equations 8.1 imply that $1 \leq x$ and $1 \leq y$. It follows by Script 7 that $1 \leq |xy|$. This combined with the fact that $|y - x| < \delta = \epsilon$ implies that

$$\begin{aligned} |f(y) - f(x)| &= \left| \frac{1}{y} - \frac{1}{x} \right| \\ &= \left| \frac{x-y}{yx} \right| \\ &= \frac{|y-x|}{|xy|} \\ &< \frac{\epsilon}{|xy|} \\ &\leq \frac{\epsilon}{1} \\ &= \epsilon \end{aligned}$$

as desired. □

(e) $f(x) = \sqrt{x}$ on $A = [1, +\infty)$.

Proof. To prove that f is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \epsilon$, and let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. Since $x, y \in A$, consecutive applications of Equations 8.1 imply that $1 \leq x$ and $1 \leq y$. It follows by Script 7 that $1 \leq \sqrt{x}$ and $1 \leq \sqrt{y}$. Thus, by Scripts 7 and 8, $2 \leq |\sqrt{y} + \sqrt{x}|$. Note that it follows that $1 < |\sqrt{y} + \sqrt{x}|$. This combined with the fact that $|y - x| < \delta = \epsilon$ implies that

$$\begin{aligned} |f(y) - f(x)| &= |\sqrt{y} - \sqrt{x}| \\ &< |\sqrt{y} - \sqrt{x}| \cdot |\sqrt{y} + \sqrt{x}| \\ &= |y - x| \\ &< \epsilon \end{aligned}$$

as desired. □

Exercise 13.4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^n$ for $n \in \mathbb{N}$. Show that f is uniformly continuous if and only if $n = 1$.

Proof. Suppose first that $n = 1$. To prove that f is uniformly continuous, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in \mathbb{R}$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \epsilon$. Now let x, y be arbitrary elements of $\mathbb{R}$ that satisfy $|y - x| < \delta$. Then by the definition of f, $|f(y) - f(x)| = |y - x| < \delta = \epsilon$, as desired.

Now suppose that $n > 1$. Additionally, suppose for the sake of contradiction that f is uniformly continuous. Let $\epsilon = 1 > 0$. Then by Definition 13.1, there exists a $\delta > 0$ such that for all $x, y \in \mathbb{R}$, if $|y-x| < \delta$, then $|y^n - x^n| < 1$. Let $x = 0 \in \mathbb{R}$. By Theorem 5.2, there exists a point $y \in \mathbb{R}$ such that $0 < y < \delta$. Additionally, since $\delta > 0$, Lemma 7.23 asserts that $-\delta < 0$. This combined with the previous result demonstrates by transitivity that $-\delta < 0 < y < \delta$, so by the lemma from Exercise 8.9, we have that $|y| < \delta$. Consequently, by Script 7, we know that $|(y+a) - a| < \delta$ for any $a \in \mathbb{R}$. It follows by the above that $|(y+a)^n - a^n| < 1$. Thus, by Additional Exercise 0.7, $|\sum_{k=0}^{n} \binom{n}{k} y^{n-k} a^k - a^n| = |y^n + ny^{n-1}a + \sum_{k=2}^{n-1} \binom{n}{k} y^{n-k} a^k| < 1$. If we now choose $a = \frac{1}{ny^{n-1}}$, Script 7 reduces the above to $|y^n + 1 + \sum_{k=2}^{n-1} y^{n-k} a^k| < 1$. We now seek to reduce the previous statement further to $|y^n + 1| < 1$. To begin, Exercise 12.22 implies that $y^n > 0$ since $y > 0$ and $0^n = 0$, meaning by Script 7 that $y^n + 1 > 0$. Additionally, Script 7 asserts that $\sum_{k=2}^{n-1} y^{n-k} a^k > 0$ since $a > 0$ and $y > 0$. This combined with the previous result implies by Scripts 7 and 8 that $|y^n + 1| < |y^n + 1 + \sum_{k=2}^{n-1} y^{n-k} a^k| < 1$, as desired. However, since $y^n > 0$, Definition 7.21 asserts that $y^n + 1 > 1$. But by Definition 8.4, this implies that $|y^n + 1| > 1$, a contradiction. □

Exercise 13.5. Let f and g be uniformly continuous on $A \subset \mathbb{R}$. Show that

(a) The function $f + g$ is uniformly continuous on A.

(b) For any constant $c \in \mathbb{R}$, the function $c \cdot f$ is uniformly continuous on A.

Proof of a. To prove that $f+g$ is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y-x| < \delta$, then $|(f+g)(y) - (f+g)(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f, g are uniformly continuous on A, consecutive applications of Definition 13.1 reveal that there exist $\delta_1, \delta_2 > 0$ such that for all $x, y \in A$, $|y - x| < \delta_1$ implies $|f(y) - f(x)| < \frac{\epsilon}{2}$ and $|y - x| < \delta_2$ implies $|g(y) - g(x)| < \frac{\epsilon}{2}$. Choose $\delta = \min(\delta_1, \delta_2)$. Let x, y be arbitrary elements of A that satisfy $|y-x| < \delta$. It follows that $|y-x| < \delta_1$ (so $|f(y)-f(x)| < \frac{\epsilon}{2}$), and that $|y-x| < \delta_2$ (so $|g(y)-g(x)| < \frac{\epsilon}{2}$). These two results when combined imply by Script 7 that $|f(y) - f(x)| + |g(y) - g(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$. Therefore, since $|f(y) - f(x) + g(y) - g(x)| \leq |f(y) - f(x)| + |g(y) - g(x)|$ by Lemma 8.8, we have that

$$\begin{aligned}
|(f+g)(y) - (f+g)(x)| &= |f(y) - f(x) + g(y) - g(x)| \\
&\leq |f(y) - f(x)| + |g(y) - g(x)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

as desired. □

Proof of b. To prove that $c \cdot f$ is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y-x| < \delta$, then $|c \cdot f(y) - c \cdot f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. We divide into two cases ($c = 0$ and $c \neq 0$). Suppose first that $c = 0$. Choose $\delta = 1$. Let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. It follows that $|0 \cdot f(y) - 0 \cdot f(x)| = 0 < \epsilon$, as desired. Now suppose that $c \neq 0$. Then since f is uniformly continuous on A, Definition 13.1 tells us that there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \frac{\epsilon}{|c|}$. Choose this δ to be our δ. Let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. Then by the above, we have that $|f(y) - f(x)| < \frac{\epsilon}{|c|}$. Therefore, $|c| \cdot |f(y) - f(x)| < \epsilon$, so we have that $|c \cdot f(y) - c \cdot f(x)| < \epsilon$, as desired. □

4/13: **Theorem 13.6.** *Suppose that $X \subset \mathbb{R}$ is compact and $f : X \to \mathbb{R}$ is continuous. Then f is uniformly continuous.*

Proof. To prove that f is uniformly continuous, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f is continuous on X, Theorem 9.10 asserts that f is continuous at every $x \in X$. Thus, by Theorem 11.5, for every $x \in X$, there exists a $\delta_x > 0$ such that if $y \in X$ and $|y - x| < \delta_x$, then $|f(y) - f(x)| < \frac{\epsilon}{2}$. Let $\mathcal{G} = \{(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2}) \mid x \in X\}$. We will now confirm that $\mathcal{G}$ is an open cover of X. To do so, Definition 10.3 tells us that it will suffice to demonstrate that every $x \in X$ is an element of $(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2})$ for some $(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2}) \in \mathcal{G}$. Let x be an arbitrary element of X. We know that $|x - x| = 0 < \frac{\delta_x}{2}$. Thus, by Exercise 8.9, we have that $x \in (x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2})$. Since it follows from the above that $(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2}) \in \mathcal{G}$, we are done.

Having shown that $\mathcal{G}$ is an open cover of X, the fact that X is compact implies by Definition 10.4 that there exists a finite subset $\mathcal{G}'$ of $\mathcal{G}$ that is also an open cover of X. It follows that $\mathcal{G}'$ will be of the form $\{(x_i - \frac{\delta_{x_i}}{2}, x + \frac{\delta_{x_i}}{2}) \mid 1 \leq i \leq n\}$ where n is some natural number. Thus, choose $\delta = \min_{1 \leq i \leq n}(\frac{\delta_{x_i}}{2})$.

Let x, y be arbitrary elements of X that satisfy $|y - x| < \delta$. Since $\mathcal{G}'$ is an open cover of X, Definition 10.3 implies that $x \in (x_i - \frac{\delta_{x_i}}{2}, x_i + \frac{\delta_{x_i}}{2})$ for some $(x_i - \frac{\delta_{x_i}}{2}, x_i + \frac{\delta_{x_i}}{2}) \in \mathcal{G}'$. Considering this x_i more closely, we can determine from the previous result and Exercise 8.9 that $|x - x_i| < \frac{\delta_{x_i}}{2}$. This combined with the hypothesis that $|y - x| < \delta$ implies by Script 7 that $|y - x| + |x - x_i| < \delta + \frac{\delta_{x_i}}{2}$. Additionally, note that by definition, $\delta \leq \frac{\delta_{x_i}}{2}$. Thus, combining the last few results, we have that

$$\begin{aligned}
|y - x_i| &\leq |y - x| + |x - x_i| && \text{Lemma 8.8} \\
&< \delta + \frac{\delta_{x_i}}{2} \\
&\leq \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2} \\
&= \delta_{x_i}
\end{aligned}$$

At this point, we know that $|x - x_i| < \frac{\delta_{x_i}}{2} < \delta_{x_i}$ and that $|y - x_i| < \delta_{x_i}$. It follows by consecutive applications of the above that $|f(x) - f(x_i)| < \frac{\epsilon}{2}$ and $|f(y) - f(x_i)| < \frac{\epsilon}{2}$, respectively. Consequently, we have by Script 7 that $|f(y) - f(x_i)| + |f(x) - f(x_i)| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$. Therefore, if we combine the last several results, we get

$$\begin{aligned}
|f(y) - f(x)| &\leq |f(y) - f(x_i)| + |f(x_i) - f(x)| && \text{Lemma 8.8} \\
&= |f(y) - f(x_i)| + |f(x) - f(x_i)| && \text{Exercise 8.5} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

as desired. □

Exercise 13.7. Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $A = [0, +\infty)$.

Lemma. *Let x, y be arbitrary elements of A. Then $|\sqrt{y} - \sqrt{x}|^2 \leq |(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x})|$.*

Proof. We will first verify that $|\sqrt{y} - \sqrt{x}| \leq |\sqrt{y} + \sqrt{x}|$. To do so, we divide into two cases ($\sqrt{y} \geq \sqrt{x}$ and $\sqrt{y} < \sqrt{x}$). If $\sqrt{y} \geq \sqrt{x}$, then by Definition 7.21, $\sqrt{y} - \sqrt{x} \geq 0$. It follows by Definition 8.4 that $|\sqrt{y} - \sqrt{x}| = \sqrt{y} - \sqrt{x}$. Additionally, we have by an extension of Exercise 12.22 that $\sqrt{x} \geq 0$, implying that $2\sqrt{x} \geq 0$ by Definition 7.21. Thus, combining the last few results, we have that $|\sqrt{y} - \sqrt{x}| = \sqrt{y} - \sqrt{x} \leq \sqrt{y} - \sqrt{x} + 2\sqrt{x} = \sqrt{y} + \sqrt{x}$. Consequently, we know that $0 \leq |\sqrt{y} - \sqrt{x}| \leq \sqrt{y} + \sqrt{x}$, so Definition 8.4 implies that $|\sqrt{y} + \sqrt{x}| = \sqrt{y} + \sqrt{x}$. Therefore, we have that $|\sqrt{y} - \sqrt{x}| \leq \sqrt{y} + \sqrt{x} = |\sqrt{y} + \sqrt{x}|$, as desired. The argument is symmetric in the other case.

Having established that $|\sqrt{y} - \sqrt{x}| \leq |\sqrt{y} + \sqrt{x}|$ and knowing that $0 \leq |\sqrt{y} - \sqrt{x}|$, we have by Lemma 7.24 that $|\sqrt{y} - \sqrt{x}| \cdot |\sqrt{y} - \sqrt{x}| \leq |\sqrt{y} + \sqrt{x}| \cdot |\sqrt{y} - \sqrt{x}|$. It follows by basic algebra that $|\sqrt{y} - \sqrt{x}|^2 \leq |(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x})|$, as desired. □

Proof of Exercise 13.7. To prove that f is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \epsilon^2$. Let x, y be arbitrary elements of X that satisfy $|y - x| < \delta$. Thus, since $(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x}) = y - x$, the lemma asserts that $|\sqrt{y} - \sqrt{x}|^2 \leq |(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x})| = |y - x| < \epsilon^2$. Therefore, by Script 7, $|\sqrt{y} - \sqrt{x}| < \epsilon$, i.e., $|f(y) - f(x)| < \epsilon$, as desired. □

Corollary 13.8. *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Then f is uniformly continuous.*

Proof. By Theorem 10.14, $[a, b]$ is compact. This combined with the hypothesis that f is continuous proves by Theorem 13.6 that f is uniformly continuous. □

Exercise 13.9. Show that if f and g are bounded on A and uniformly continuous on A, then fg is uniformly continuous on A.

Proof. To prove that fg is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|(fg)(y) - (fg)(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary.

Since f is bounded on A, Definition 10.1 implies that $f(A)$ is a bounded subset of $\mathbb{R}$. Thus, by consecutive applications of Definition 5.6, there exist numbers l, u such that for all $f(x) \in f(A)$, $l \leq f(x) \leq u$. Let $a = \max(|l|, |u|) + 1$. It follows by Scripts 7 and 8 that $-a < f(x) < a$ for all $f(x) \in f(A)$. Thus, by the lemma from Exercise 8.9, $|f(x)| < a$ for all $f(x) \in f(A)$. Similarly, there exists a number b such that $|g(x)| < b$ for all $g(x) \in g(A)$.

Since f is uniformly continuous on A, Definition 13.1 implies that there exists a $\delta_1 > 0$ such that for all $x, y \in A$, if $|y - x| < \delta_1$, then $|f(y) - f(x)| < \frac{\epsilon}{2b}$. Similarly, there exists a $\delta_2 > 0$ such that for all $x, y \in A$, if $|y - x| < \delta_2$, then $|g(y) - g(x)| < \frac{\epsilon}{2a}$. Choose $\delta = \min(\delta_1, \delta_2)$. Let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. It follows by consecutive applications of the above that $|f(x)| < a$ and $|g(y)| < b$. Additionally, $|y - x| < \delta \leq \delta_1$ implies that $|f(y) - f(x)| < \frac{\epsilon}{2b}$ and $|y - x| < \delta \leq \delta_2$ implies that $|g(y) - g(x)| < \frac{\epsilon}{2a}$. Therefore, combining the last four results, we have that

$$
\begin{aligned}
|(fg)(y) - (fg)(x)| &= |f(x)g(x) - f(y)g(y)| \\
&= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\
&= |f(x)(g(x) - g(y)) + g(y)(f(x) - f(y))| \\
&\leq |f(x)| \cdot |g(x) - g(y)| + |g(y)| \cdot |f(x) - f(y)| && \text{Lemma 8.8} \\
&< a \cdot \frac{\epsilon}{2a} + b \cdot \frac{\epsilon}{2b} \\
&= \epsilon
\end{aligned}
$$

as desired. □

4/15: **Definition 13.10.** A **partition** of the interval $[a, b]$ is a finite set of points in $[a, b]$ that includes a and b. We usually write partitions as $P = \{t_0, t_1, \dots, t_n\}$, with the convention that

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

If P and Q are partitions of the interval $[a, b]$ and $P \subset Q$, we refer to Q as a **refinement** of P.

Definition 13.11. Suppose that $f : [a, b] \to \mathbb{R}$ is bounded and that $P = \{t_0, \dots, t_n\}$ is a partition of $[a, b]$. Define

$$m_i(f) = \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\} \qquad M_i(f) = \sup\{f(x) \mid t_{i-1} \leq x \leq t_i\}$$

The **lower sum** of f for the partition P is the number

$$L(f, P) = \sum_{i=1}^{n} m_i(f)(t_i - t_{i-1})$$

The **upper sum** of f for the partition P is the number

$$U(f,P)=\sum_{i=1}^{n}M_i(f)(t_i-t_{i-1})$$

Notice that it is always the case that $L(f,P)\leq U(f,P)$.

Lemma 13.12. *Suppose that P and Q are partitions of $[a,b]$ and that Q is a refinement of P. Then $L(f,P)\leq L(f,Q)$ and $U(f,P)\geq U(f,Q)$.*

Lemma. *Let P be a partition of $[a,b]$ and let y be an arbitrary element of $[a,b]\setminus P$. Then $L(f,P)\leq L(f,P\cup\{y\})$ and $U(f,P)\geq U(f,P\cup\{y\})$.*

Proof. We will prove that $L(f,P)\leq L(f,P\cup\{y\})$. The proof will be symmetric in the other case. Let's begin.

By Definition 13.10, P is of the form $\{t_0,\dots,t_n\}$ where $a=t_0<\cdots<t_n=b$. This combined with the hypothesis that $y\in[a,b]\setminus P$ implies by Theorem 3.5 that $a=t_0<\cdots<t_{k-1}<y<t_k<\cdots<t_n=b$. Thus, we have by consecutive applications of Definition 13.11 that

$$\begin{aligned}L(f,P)&=\sum_{i=1}^{n}m_i(f)(t_i-t_{i-1})\\&=\sum_{i=1}^{k-1}m_i(f)(t_i-t_{i-1})+m_k(f)(t_k-t_{k-1})+\sum_{i=k+1}^{n}m_i(f)(t_i-t_{i-1})\end{aligned}$$

and that

$$L(f,P\cup\{y\})=\sum_{i=1}^{k-1}m_i(f)(t_i-t_{i-1})+m_y^-(f)(y-t_{k-1})+m_y^+(f)(t_k-y)+\sum_{i=k+1}^{n}m_i(f)(t_i-t_{i-1})$$

where

$$m_y^-(f)=\inf\{f(x)\mid t_{k-1}\leq x\leq y\}\qquad\qquad m_y^+(f)=\inf\{f(x)\mid y\leq x\leq t_k\}$$

As such, to prove that $L(f,P)\leq L(f,P\cup\{y\})$, it will suffice to show that $m_k(f)(t_k-t_{k-1})\leq m_y^-(f)(y-t_{k-1})+m_y^+(f)(t_k-y)$. To do so, it will suffice to show that $m_k(f)(y-t_{k-1})+m_k(f)(t_k-y)\leq m_y^-(f)(y-t_{k-1})+m_y^+(f)(t_k-y)$, i.e., that $m_k(f)(y-t_{k-1})\leq m_y^-(f)(y-t_{k-1})$ and that $m_k(f)(t_k-y)\leq m_y^+(f)(t_k-y)$, i.e., that $m_k(f)\leq m_y^-(f)$ and that $m_k(f)\leq m_y^+(f)$.

For the sake of proving the first expression, let $A=\{f(x)\mid t_{k-1}<x<t_k\}$ and let $B=\{f(x)\mid t_{k-1}\leq x\leq y\}$. It follows by Definition 13.10 that $m_k(f)=\inf A$ and $m_y^-(f)=\inf B$. Thus, we need only show that $\inf A\leq\inf B$. Since $y<t_k$, we know by Script 1 that $B\subset A$. Thus, since $\inf A$ is a lower bound on A, Script 5 implies that it is also a lower bound on B. Consequently, by Definition 5.7, $\inf A\leq\inf B$, as desired.

The argument is symmetric for the other statement. $\square$

Proof of Lemma 13.12. We will prove that $L(f,P)\leq L(f,Q)$. The proof will be symmetric in the other case. Let's begin.

By Definition 13.10, $P\subset Q$. Thus, by Theorem 1.34, $|P|\leq|Q|$. It follows by Script 1 that $|Q|-|P|=n\in\mathbb{Z}^+$. Thus, to prove the claim for P and Q in general, it will suffice to prove it for each n. To do so, we divide into two cases ($n=0$ and $n\in\mathbb{N}$). If $n=0$, then $|P|=|Q|$. This combined with the fact that $P\subset Q$ implies by Script 1 that $P=Q$. Therefore, $L(f,P)=L(f,Q)$, which we can weaken to $L(f,P)\leq L(f,Q)$, as desired.

On the other hand, if $n\in\mathbb{N}$, then we induct on n. For the base case $n=1$, we have by Script 1 that $Q=P\cup\{y\}$ where $y\notin P$. Therefore, by the lemma, we have that $L(f,P)\leq L(f,P\cup\{y\})=L(f,Q)$, as desired. Now suppose inductively that the claim holds for n; we wish to prove it for $n+1$. Let y be an arbitrary element of Q. Then by Script 1, $|Q\setminus\{y\}|-|P|=n$. Thus, by the inductive hypothesis, $L(f,P)\leq L(f,Q\setminus\{x\})$. Additionally, by the lemma, $L(f,Q\setminus\{x\})\leq L(f,Q)$. Therefore, by transitivity, $L(f,P)\leq L(f,Q)$, as desired. $\square$

Theorem 13.13. *Let P_1 and P_2 be partitions of $[a,b]$ and suppose that $f : [a,b] \to \mathbb{R}$ is bounded. Then $L(f,P_1) \leq U(f,P_2)$.*

Proof. To confirm that $P_1 \cup P_2$ is a partition of $[a,b]$, Definition 13.10 tells us that it will suffice to demonstrate that it is a finite set, that it is a subset of $[a,b]$, and that it includes a and b. Since P_1, P_2 are partitions of $[a,b]$, Definition 13.10 implies that they are finite subsets of $[a,b]$ that contain a,b. It follows by Script 1 that their union is finite, a subset of $[a,b]$, and a set containing a and b. Additionally, we have by Theorem 1.7 that $P_1 \subset P_1 \cup P_2$ and that $P_2 \subset P_1 \cup P_2$. Combining the last two results with consecutive applications of Definition 13.10 reveals that $P_1 \cup P_2$ is a refinement of both P_1 and P_2.

Since P_1 and $P_1 \cup P_2$ are partitions of $[a,b]$ and $P_1 \cup P_2$ is a refinement of P_1, Lemma 13.12 implies that $L(f,P_1) \leq L(f,P_1 \cup P_2)$. Similarly, $U(f,P_1 \cup P_2) \leq U(f,P_2)$. Additionally, we have by Definition 13.11 that $L(f,P_1 \cup P_2) \leq U(f,P_1 \cup P_2)$. Therefore, if we combine the last three results with transitivity, we have that $L(f,P_1) \leq U(f,P_2)$, as desired. □

Definition 13.14. Let $f : [a,b] \to \mathbb{R}$ be bounded. We define

$$L(f) = \sup\{L(f,P) \mid P \text{ is a partition of } [a,b]\} \qquad U(f) = \inf\{U(f,P) \mid P \text{ is a partition of } [a,b]\}$$

to be, respectively, the **lower integral** and **upper integral** of f from a to b.

Exercise 13.15. Why do $L(f)$ and $U(f)$ exist? Find a function f for which $L(f) = U(f)$. Find a function f for which $L(f) \neq U(f)$. Prove that $L(f) \leq U(f)$.

Lemma. *Given $a,b \in \mathbb{R}$ with $a < b$, there exists $p \in \mathbb{R}$ such that $p \notin \mathbb{Q}$ and $a < p < b$.*

Proof. By Definition 7.21, $a + \sqrt{2} < b + \sqrt{2}$. Thus, by Lemma 6.10, there exists a point $\frac{c}{d} \in \mathbb{Q}$ such that $a + \sqrt{2} < \frac{c}{d} < b + \sqrt{2}$. It follows that $a < \frac{c}{d} - \sqrt{2} < b$.

Now suppose for the sake of contradiction that $\frac{c}{d} - \sqrt{2}$ is rational. Then by Script 2, $\frac{c}{d} - \sqrt{2} = \frac{e}{f}$ where $e,f \in \mathbb{Z}$ and $f \neq 0$. It follows by Theorem 2.10 that $\sqrt{2} = \frac{cf-de}{df}$, i.e., that $\sqrt{2}$ is rational. But by the proof of Exercise 4.24, $\sqrt{2}$ is not rational, a contradiction. □

Proof of Exercise 13.15. Let $A = \{L(f,P) \mid P \text{ is a partition of } [a,b]\}$. To prove that $L(f) = \sup A$ exists, Theorem 5.17 tells us that it will suffice to show that A is nonempty and bounded above.

To confirm that A is nonempty, Definition 1.8 tells us that it will suffice to find an element of it. Since $\{a,b\}$ is a finite set of points in $[a,b]$ that includes a and b (by Script 1), Definition 13.10 asserts that $\{a,b\}$ is a partition of $[a,b]$. It follows by Definition 13.11 that $L(f,\{a,b\})$ exists. Therefore, by the definition of A, we have that $L(f,\{a,b\}) \in A$, as desired.

To confirm that A is bounded above, Definition 5.6 tells us that it will suffice to find a point in $u \in \mathbb{R}$ such that for all $L(f,P) \in A$, $L(f,P) \leq u$. Let $u = U(f,\{a,b\})$ (since $\{a,b\}$ is a partition of $[a,b]$ by the above, Definition 13.10 guarantees that $U(f,\{a,b\})$ exists). Now let $L(f,P)$ be an arbitrary element of A. It follows from Theorem 13.13 that $L(f,P) \leq U(f,\{a,b\}) = u$, as desired.

The proof is symmetric for $U(f)$.

Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = 0$. To prove that $L(f) = U(f)$, it will suffice to show that $L(f) = 0$ and $U(f) = 0$. To do this, Script 5 tells us that it will suffice to verify that $\{L(f,P) \mid P \text{ is a partition of } [a,b]\} = \{0\}$ and $\{U(f,P) \mid P \text{ is a partition of } [a,b]\} = \{0\}$. We will start with the first equality.

Let $L(f,P)$ be an arbitrary element of $\{L(f,P) \mid P \text{ is a partition of } [a,b]\}$. Since we have

$$\begin{aligned} m_i(f) &= \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\} \\ &= \inf\{0 \mid t_{i-1} \leq x \leq t_i\} \\ &= \inf\{0\} \\ &= 0 \end{aligned}$$

for all $m_i(f)$, it follows that

$$\begin{aligned} L(f,P) &= \sum_{i=1}^{n} m_i(f)(t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} 0(t_i - t_{i-1}) \\ &= 0 \end{aligned}$$

Therefore, since every element of $\{L(f,P) \mid P \text{ is a partition of } [a,b]\}$ is equal to 0, the set is equal to the singleton set containing 0. The argument is symmetric for the other equality.

Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

To prove that $L(f) \neq U(f)$, it will suffice to show that $L(f) = 0$ and $U(f) = 1$. To do this, Script 5 tells us that it will suffice to verify that $\{L(f,P) \mid P \text{ is a partition of } [a,b]\} = \{0\}$ and $\{U(f,P) \mid P \text{ is a partition of } [a,b]\} = \{1\}$. We will start with the first equality.

Let $L(f,P)$ be an arbitrary element of $\{L(f,P) \mid P \text{ is a partition of } [a,b]\}$. To confirm that $L(f,P) = 0$, Definition 13.11 tells us that it will suffice to demonstrate that $m_i(f) = 0$ for all $m_i(f)$. Let $m_i(f)$ be an arbitrary such object. By Definition 13.10, $m_i(f) = \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\}$. By the lemma, there exists $p \in \mathbb{R}$ such that $p \notin \mathbb{Q}$ and $t_{i-1} \leq p \leq t_i$. Thus, since $f(p) = 0$, $0 \in \{f(x) \mid t_{i-1} \leq x \leq t_i\}$. Additionally, since $f(x) \not< 0$ for any $x \in [0,1]$ by definition, we have that $m_i(f) = 0$. Therefore, since every element of $\{L(f,P) \mid P \text{ is a partition of } [a,b]\}$ is equal to 0, the set is equal to the singleton set containing 0.

As to the other equality, let $U(f,P)$ be an arbitrary element of $\{U(f,P) \mid P \text{ is a partition of } [a,b]\}$. To confirm that $U(f,P) = 1$, Definition 13.11 tells us that we must first demonstrate that $M_i(f) = 1$ for all $M_i(f)$. Let $M_i(f)$ be an arbitrary such object. By Definition 13.10, $M_i(f) = \sup\{f(x) \mid t_{i-1} \leq x \leq t_i\}$. By Lemma 6.10, there exists $p \in \mathbb{Q}$ such that $t_{i-1} \leq p \leq t_i$. Thus, since $f(p) = 1$, $1 \in \{f(x) \mid t_{i-1} \leq x \leq t_i\}$. Additionally, since $f(x) \not> 1$ for any $x \in [0,1]$ by definition, we have that $M_i(f) = 1$. It follows by Definition 13.11 that

$$\begin{aligned} U(f,P) &= \sum_{i=1}^{n} M_i(f)(t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} (t_i - t_{i-1}) \\ &= t_n - t_0 \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

Therefore, since every element of $\{U(f,P) \mid P \text{ is a partition of } [a,b]\}$ is equal to 1, the set is equal to the singleton set containing 1.

Suppose for the sake of contradiction that there exists a function $f : [a,b] \to \mathbb{R}$ for which $U(f) < L(f)$. It follows by consecutive applications of Definition 13.14 and Lemma 5.11 that there exists an $L(f,P_1)$ such that $U(f) < L(f,P_1) \leq L(f)$, and thus that there exists a $U(f,P_2)$ such that $U(f) \leq U(f,P_2) < L(f,P_1)$. But this means that there exist partitions P_1, P_2 of $[a,b]$ such that $L(f,P_1) > U(f,P_2)$, contradicting Theorem 13.13. $\square$

Definition 13.16. Let $f : [a,b] \to \mathbb{R}$ be bounded. We say that f is **integrable** on $[a,b]$ if $L(f) = U(f)$. In this case, the common value $L(f) = U(f)$ is called the **integral** of f from a to b and we write it as

$$\int_a^b f$$

Note that if f is an integrable function on $[a,b]$, it is necessarily bounded.

When we want to display the variable of integration, we write the integral as follows, including the symbol $\mathrm{d}x$ to indicate that variable of integration:

$$\int_a^b f(x)\,\mathrm{d}x$$

For example, if $f(x)=x^2$, we could write $\int_a^b x^2\,\mathrm{d}x$ but not $\int_a^b x^2$.

Exercise 13.17. Fix $c\in\mathbb{R}$ and let $f:[a,b]\to\mathbb{R}$ be defined by $f(x)=c$, for each $x\in[a,b]$. Show that f is integrable on $[a,b]$ and that $\int_a^b f=c(b-a)$.

Proof. To prove that f is integrable on $[a,b]$ and that $\int_a^b f=c(b-a)$, Definition 13.16 tells us that it will suffice to show that f is bounded on $[a,b]$, and that $L(f)=U(f)=c(b-a)$.

To confirm that f is bounded on $[a,b]$, Definition 10.1 tells us that it will suffice to demonstrate that $f([a,b])$ is a bounded subset of $\mathbb{R}$. By Definition 1.18, $f([a,b])=\{f(x)\in\mathbb{R}\mid x\in[a,b]\}$. But since $f(x)=c$ for all $x\in[a,b]$, we have that $c\leq f(x)\leq c$ for all $x\in[a,b]$. It follows by Definition 5.6 that $f([a,b])$ is bounded. Additionally, since $c\in\mathbb{R}$, Definition 1.3 asserts that $f([a,b])=\{c\}\subset\mathbb{R}$.

To confirm that $L(f)=U(f)=c(b-a)$, Definition 13.14 tells us that it will suffice to demonstrate that $L(f,P)=U(f,P)=c(b-a)$ for all partitions P of $[a,b]$. For similar reasons to the above (i.e., $f(x)=c$ for all $x\in[a,b]$), we can show that $m_i(f)=M_i(f)=c$ for all $m_i(f)$ and $M_i(f)$. Therefore, by Definition 13.11 that

$$\begin{aligned} L(f,P) &= \sum_{i=1}^n c(t_i-t_{i-1}) \\ &= c\sum_{i=1}^n (t_{i-1}-t_i) \\ &= c(t_n-t_0) \\ &= c(b-a) \end{aligned} \qquad\qquad \begin{aligned} U(f,P) &= \sum_{i=1}^n c(t_i-t_{i-1}) \\ &= c\sum_{i=1}^n (t_{i-1}-t_i) \\ &= c(t_n-t_0) \\ &= c(b-a) \end{aligned}$$

as desired. □

Theorem 13.18. *Let $f:[a,b]\to\mathbb{R}$ be bounded. Then f is integrable if and only if for every $\epsilon>0$, there exists a partition P of $[a,b]$ such that $U(f,P)-L(f,P)<\epsilon$.*

Proof. Suppose first that f is integrable. Then by Definition 13.16, $L(f)=U(f)$. Let $\epsilon>0$ be arbitrary. By Script 7, $L(f)-\frac{\epsilon}{2}<L(f)$. Thus, by Definition 13.14 and Lemma 5.11, there exists an $L(f,P_1)\in\{L(f,P)\mid P \text{ is a partition of } [a,b]\}$ such that $L(f)-\frac{\epsilon}{2}<L(f,P_1)\leq L(f)$. Similarly, there exists a $U(f,P_2)\in\{U(f,P)\mid P \text{ is a partition of } [a,b]\}$ such that $U(f)\leq U(f,P_2)<U(f)+\frac{\epsilon}{2}$. Now consider $P_1\cup P_2$ (which we will prove is the desired partition). By Theorem 1.7, $P_1\subset P_1\cup P_2$ and $P_2\subset P_1\cup P_2$. It follows by consecutive applications of Definition 13.10 that $P_1\cup P_2$ is a refinement of both P_1 and P_2. Thus, by Lemma 13.12, $L(f,P_1)\leq L(f,P_1\cup P_2)$ and $U(f,P_1\cup P_2)\leq U(f,P_2)$. Combining the last several results with transitivity yields

$$L(f)-\frac{\epsilon}{2}<L(f,P_1)\leq L(f,P_1\cup P_2) \qquad\qquad U(f,P_1\cup P_2)\leq U(f,P_2)<U(f)+\frac{\epsilon}{2}$$

Therefore, knowing that $U(f,P_1\cup P_2)<U(f)+\frac{\epsilon}{2}$ and that $-L(f,P_1\cup P_2)<\frac{\epsilon}{2}-L(f)$ (the latter by Lemma 7.24), we have by Definition 7.21 that

$$\begin{aligned} U(f,P_1\cup P_2)-L(f,P_1\cup P_2) &< U(f)+\frac{\epsilon}{2}+\frac{\epsilon}{2}-L(f) \\ &= \epsilon \end{aligned}$$

as desired.

Now suppose that f is not integrable; we seek to prove that there exists an $\epsilon>0$ such that for all partitions P of $[a,b]$, $U(f,P)-L(f,P)\geq\epsilon$. Since f is not integrable, we have by Definition 13.16 that

$L(f) \neq U(f)$. It follows by Exercise 13.15 that $L(f) < U(f)$. Thus, we can define $\epsilon = \frac{U(f)-L(f)}{2} > 0$. Now let P be an arbitrary partition of $[a,b]$. It follows that $L(f,P) \leq L(f)$ by Definitions 13.14, 5.7, and 5.6. Similarly, $U(f) \leq U(f,P)$. Therefore, knowing that $U(f) \leq U(f,P)$ and that $-L(f) \leq -L(f,P)$ (the latter by Lemma 7.24), we have by Definition 7.21 that $\epsilon = \frac{U(f)-L(f)}{2} < U(f) - L(f) \leq U(f,P) - L(f,P)$, as desired. ☐

4/20: **Theorem 13.19.** *If $f : [a,b] \to \mathbb{R}$ is continuous, then f is integrable.*

Proof. To prove that f is integrable, Theorem 13.18 tells us that it will suffice to show that f is bounded and that for every $\epsilon > 0$, there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$. We will verify the two requirements separately. Let's begin.

To confirm that f is bounded, Definitions 10.1 and 5.6 tell us that it will suffice to find points $l, u \in \mathbb{R}$ such that $l \leq f(x) \leq u$ for all $x \in [a,b]$. But since $f : [a,b] \to \mathbb{R}$ is continuous, consecutive applications of Exercise 10.21 imply that there exist points $c, d \in [a,b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a,b]$, so we can just choose $l = f(c)$ and $u = f(d)$.

As to the other stipulation, let $\epsilon > 0$ be arbitrary. Since $f : [a,b] \to \mathbb{R}$ is continuous, Corollary 13.8 implies that f is uniformly continuous. Thus, by Definition 13.1, there exists a $\delta > 0$ such that for all $x, y \in [a,b]$, if $|y-x| < \delta$, then $|f(y)-f(x)| < \frac{\epsilon}{b-a}$. Considering this δ, we have by Corollary 6.12 that there exist a number $n \in \mathbb{N}$ such that $\frac{2(b-a)}{\delta} < n$. Equipped with this n, we can now define the set $P = \{\frac{b-a}{n} \cdot i + a \mid 0 \leq i \leq n\}$.

We now seek to confirm that P is a partition of $[a,b]$. To do so, Definition 13.10 tells us that it will suffice to demonstrate that P is finite, $P \subset [a,b]$, and $a, b \in P$. By Script 1, P is finite. To demonstrate that $P \subset [a,b]$, Definition 1.3 and Equations 8.1 tell us that it will suffice to show that every $t_i \in P$ satisfies $a \leq t_i \leq b$. But by Script 7, we have that

$$\begin{aligned} 0 &\leq i \leq n \\ 0 &\leq \frac{b-a}{n} \cdot i \leq b-a \\ a &\leq \frac{b-a}{n} \cdot i + a \leq b \end{aligned}$$

as desired. Lastly, consider the elements of P corresponding to $i = 0$ and $i = n$. By consecutive applications of the definition of P, we have that $a = (\frac{b-a}{n} \cdot 0 + a) \in P$ and that $b = b - a + a = (\frac{b-a}{n} \cdot n + a) \in P$.

We now seek to confirm that if $t_i, t_{i-1} \in P$, then $t_i - t_{i-1} < \delta$. Let t_i, t_{i-1} be arbitrary sequential elements of P. By Script 0, we have that $0 < n$. Additionally, we have by hypothesis that $0 < \delta$. It follows by consecutive applications of Lemma 7.24 that the fact that $\frac{2(b-a)}{\delta} < n$ implies that $\frac{2(b-a)}{n} < \delta$. Therefore, we have by Script 7 that

$$\begin{aligned} t_i - t_{i-1} &= \left(\frac{b-a}{n} \cdot i + a\right) - \left(\frac{b-a}{n} \cdot (i-1) + a\right) \\ &= \frac{b-a}{n} \\ &\leq \frac{2(b-a)}{n} \\ &< \delta \end{aligned}$$

as desired.

We now seek to confirm that $M_i(f) - m_i(f) < \frac{\epsilon}{b-a}$ for all i satisfying $1 \leq i \leq n$. Let i be an arbitrary such number, and consider $f|_{[t_{i-1},t_i]}$. Since f is continuous and $[t_{i-1}, t_i] \subset [a,b]$, Proposition 9.7 asserts that $f|_{[t_{i-1},t_i]}$ is continuous. Thus, by Exercise 10.21, there exist $c, d \in [t_{i-1}, t_i]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [t_{i-1}, t_i]$. It follows by consecutive applications of Definitions 13.11 and 3.3 as well as Exercise 5.9 that $m_i(f) = f(c)$ and $M_i(f) = f(d)$. Additionally, since $c, d \in [t_{i-1}, t_i]$, we have by Script 8 that $|d - c| \leq t_i - t_{i-1}$. This combined with the fact that $t_i - t_{i-1} < \delta$ by the above implies by transitivity that

$|d-c|<\delta$. But this implies by the above that

$$\begin{aligned} M_i(f)-m_i(f) &= f(d)-f(c) \\ &= |f(d)-f(c)| \\ &< \frac{\epsilon}{b-a} \end{aligned}$$

as desired.

Having established that $M_i(f)-m_i(f)<\frac{\epsilon}{b-a}$ for all i in the partition P, we have by Definition 13.11 and basic algebra that

$$\begin{aligned} U(f,P)-L(f,P) &= \sum_{i=1}^n M_i(f)(t_i-t_{i-1}) - \sum_{i=1}^n m_i(f)(t_i-t_{i-1}) \\ &= \sum_{i=1}^n (M_i(f)-m_i(f))(t_i-t_{i-1}) \\ &< \sum_{i=1}^n \frac{\epsilon}{b-a}(t_i-t_{i-1}) \\ &= \frac{\epsilon}{b-a}\sum_{i=1}^n (t_i-t_{i-1}) \\ &= \frac{\epsilon}{b-a}(b-a) \\ &= \epsilon \end{aligned}$$

as desired. ☐

Lemma 13.20. *Let $f:[a,b]\to\mathbb{R}$ be bounded. Given $\Omega\in\mathbb{R}$, we have $\Omega=\int_a^b f$ if and only if for all $\epsilon>0$, there is some partition P such that*

$$U(f,P)-\Omega<\epsilon \qquad\qquad \Omega-L(f,P)<\epsilon$$

Proof. Suppose first that $\Omega=\int_a^b f$. Let $\epsilon>0$ be arbitrary. By Theorem 13.18, there exists a partition P of $[a,b]$ such that $U(f,P)-L(f,P)<\epsilon$. Choose this P to be our P. By Definition 13.16, $\Omega=L(f)=U(f)$. Thus, by consecutive applications of Definitions 13.14, 5.7, and 5.6, we have that $L(f,P)\leq L(f)=\Omega$ and $\Omega=U(f)\leq U(f,P)$. With respect to the former result, it follows by Script 7 that $-\Omega\leq -L(f,P)$. Therefore, having established that $\Omega\leq U(f,P)$, $-\Omega\leq -L(f,P)$, and $U(f,P)-L(f,P)<\epsilon$, we have that

$$\begin{aligned} \Omega-L(f,P) &\leq U(f,P)-L(f,P) \\ &< \epsilon \end{aligned} \qquad\qquad \begin{aligned} U(f,P)-\Omega &\leq U(f,P)-L(f,P) \\ &< \epsilon \end{aligned}$$

Now suppose that $\Omega\neq\int_a^b f$; we seek to prove that there exists an $\epsilon>0$ such that for all partitions P, $U(f,P)-\Omega\geq\epsilon$ or $\Omega-L(f,P)\geq\epsilon$. We divide into two cases ($\int_a^b f$ exists and $\int_a^b f$ doesn't exist).

First, suppose that $\int_a^b f$ exists. We divide into two subcases ($\Omega>\int_a^b f$ and $\Omega<\int_a^b f$). If $\Omega>\int_a^b f$, choose $\epsilon=\Omega-\int_a^b f>0$. Let P be an arbitrary partition. As before, we have that $L(f,P)\leq L(f)$. Additionally, Definition 13.16 asserts that $L(f)=\int_a^b f$. Thus, transitivity implies that $L(f,P)\leq\int_a^b f$. It follows by Script 7 that $-\int_a^b f\leq -L(f,P)$. Therefore,

$$\begin{aligned} \epsilon &= \Omega-\int_a^b f \\ &\leq \Omega-L(f,P) \end{aligned}$$

as desired. The argument is symmetric in the other subcase.

Second, suppose that $\int_a^b f$ does not exist. By Exercise 13.15, $L(f)$ and $U(f)$ exist. However, since $\int_a^b f$ does not exist, Definition 13.16 asserts that $L(f) \neq U(f)$. It follows by Exercise 13.15 again that $L(f) < U(f)$. We now divide into three subcases ($\Omega \leq L(f)$, $L(f) < \Omega < U(f)$, and $U(f) \leq \Omega$). If $\Omega \leq L(f)$, choose $\epsilon = U(f) - L(f) > 0$. Let P be an arbitrary partition. As above, $U(f) \leq U(f,P)$. Therefore,

$$\begin{aligned}\epsilon &= U(f) - L(f)\\ &\leq U(f,P) - L(f)\\ &\leq U(f,P) - \Omega\end{aligned}$$

as desired. If $L(f) < \Omega < U(f)$, choose $\epsilon = U(f) - \Omega > 0$. Let P be an arbitrary partition. As above, $U(f) \leq U(f,P)$. Therefore,

$$\begin{aligned}\epsilon &= U(f) - \Omega\\ &\leq U(f,P) - \Omega\end{aligned}$$

as desired. The argument for the last subcase is symmetric to that of the first. □

Exercise 13.21. Define $f : [0,b] \to \mathbb{R}$ by the formula $f(x) = x$. Show that f is integrable on $[0,b]$ and that $\int_0^b f = \frac{b^2}{2}$.

Proof. To prove that f is integrable on $[0,b]$ and that $\int_0^b f = \frac{b^2}{2}$, Lemma 13.20 tells us that it will suffice to show that for all $\epsilon > 0$, there is some partition P such that $U(f,P) - \frac{b^2}{2} < \epsilon$ and $\frac{b^2}{2} - L(f,P) < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\frac{2\epsilon}{b^2}$ is a positive real number by Script 7, Corollary 6.12 asserts that there exists a number $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{2\epsilon}{b^2}$. Equipped with this n, we can now define the set $P = \{\frac{b}{n} \cdot i \mid 0 \leq i \leq n\}$. By a symmetric argument to that used in the proof of Theorem 13.19, we can confirm that P is a partition of $[0,b]$ and that $t_i - t_{i-1} = \frac{b}{n}$.

We now turn our attention strictly to proving that $U(f,P) - \frac{b^2}{2} < \epsilon$; the proof of the other statement will be symmetric. Under the partition P as defined, consider an arbitrary $M_i(f)$. By Definition 13.11, $M_i(f) = \sup\{f(x) \mid t_{i-1} \leq x \leq t_i\}$. Since $f(x) = x$ for all $x \in [t_{i-1}, t_i] \subset [0,b]$, we have by Equations 8.1 that $M_i(f) = \sup[t_{i-1}, t_i]$. Thus, by Script 5, $M_i(f) = t_i = \frac{bi}{n}$. Therefore,

$$\begin{aligned}U(f,P) - \frac{b^2}{2} &= \sum_{i=1}^{n} M_i(f)(t_i - t_{i-1}) - \frac{b^2}{2} && \text{Definition 13.11}\\ &= \sum_{i=1}^{n} \frac{bi}{n}\left(\frac{bi}{n} - \frac{b(i-1)}{n}\right) - \frac{b^2}{2}\\ &= \frac{b^2}{n^2}\sum_{i=1}^{n} i(i-(i-1)) - \frac{b^2}{2}\\ &= \frac{b^2}{n^2}\sum_{i=1}^{n} i - \frac{b^2}{2}\\ &= \frac{b^2}{n^2}\left(\frac{1}{2}n(n+1)\right) - \frac{b^2}{2}\\ &= \frac{b^2}{2} + \frac{b^2}{2n} - \frac{b^2}{2}\\ &= \frac{b^2}{2} \cdot \frac{1}{n}\\ &< \frac{b^2}{2} \cdot \frac{2\epsilon}{b^2}\\ &= \epsilon\end{aligned}$$

as desired. □

Exercise 13.22. Show that the converse of Theorem 13.19 is false in general.

Proof. To prove that even if f is integrable, $f : [a, b] \to \mathbb{R}$ is not necessarily continuous, we need only find an example of an integrable, discontinuous function f. Let $f : [-1, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

To confirm that f is integrable, Theorem 13.18 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $P = \{-1, -\frac{\epsilon}{2}, 0, 1\}$ (clearly P is a partition of $[-1, 1]$ by Definition 13.10). It follows by consecutive applications of Definitions 13.11, 5.7, and 5.6 that

$$\begin{aligned} m_1(f) &= 0 & \qquad M_1(f) &= 0 \\ m_2(f) &= 0 & M_2(f) &= 1 \\ m_3(f) &= 1 & M_3(f) &= 1 \end{aligned}$$

Therefore,

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^{3} M_i(f)(t_i - t_{i-1}) - \sum_{i=1}^{3} m_i(f)(t_i - t_{i-1}) && \text{Definition 13.11} \\ &= \left[0\left(-\frac{\epsilon}{2} - (-1)\right) + 1\left(0 - \left(-\frac{\epsilon}{2}\right)\right) + 1(1 - 0)\right] \\ &\quad - \left[0\left(-\frac{\epsilon}{2} - (-1)\right) + 0\left(0 - \left(-\frac{\epsilon}{2}\right)\right) + 1(1 - 0)\right] \\ &= \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

However, by Corollary 5.5 and Theorem 3.14, $0 \in LP([-1, 1])$. Additionally, by the proof of Exercise 11.4, $\lim_{x \to 0} f(x)$ does not exist. Combining the last two results with Theorem 11.5 reveals that f is not continuous at 0. Therefore, by Theorem 9.10, f is not continuous. $\square$

Theorem 13.23. *Let $a < b < c$. A function $f : [a, c] \to \mathbb{R}$ is integrable on $[a, c]$ if and only if f is integrable on $[a, b]$ and $[b, c]$. When f is integrable on $[a, c]$, we have*

$$\int_a^c f = \int_a^b f + \int_b^c f$$

Lemma. *Let P_1, P_2 be partitions of $[a, b]$ and $[b, c]$, respectively. Define $P' = P_1 \cup P_2$. Then P' is a partition of $[a, c]$, $L(f, P') = L(f, P_1) + L(f, P_2)$, and $U(f, P') = U(f, P_1) + U(f, P_2)$.*

Proof. To prove that P' is a partition of $[a, c]$, Definition 13.10 tells us that it will suffice to show that P' is finite, that $P' \subset [a, c]$, and that $a, c \in P'$. By Definition 13.10, P_1 and P_2 are finite. Thus, by Script 1, their union $P_1 \cup P_2 = P'$ is also finite. To confirm that $P' \subset [a, c]$, Definition 1.3 tells us that it will suffice to demonstrate that every $x \in P'$ is an element of $[a, c]$. Let x be an arbitrary element of P'. Then by Definition 1.5, $x \in P_1$ or $x \in P_2$. We now divide into two cases. If $x \in P_1$, then since $P_1 \subset [a, b]$ by Definition 13.10, Definition 1.3 asserts that $x \in [a, b]$. Thus, by Equations 8.1, $a \leq x \leq b$. Moreover, by hypothesis, we have that $a \leq x \leq b < c$, from which it follows by Equations 8.1 that $x \in [a, c]$, as desired. The argument is symmetric in the other case. Lastly, by consecutive applications of Definition 13.10, $a \in P_1$ and $c \in P_2$. It follows by Definition 1.5 that $a, c \in P'$, as desired.

Additionally, if we express P_1 as containing the objects $a = t_0, \dots, t_n = b$ and P_2 as containing the objects $b = t_n, \dots, t_{n+m} = c$, we have that P' contains every object t_0 through t_{n+m}. Therefore, we have by consecutive applications of Definition 13.11 that

$$\begin{aligned}
L(f,P') &= \sum_{i=1}^{n+m} m_i(f)(t_i - t_{i-1}) \\
&= \sum_{i=1}^{n} m_i(f)(t_i - t_{i-1}) + \sum_{i=n+1}^{n+m} m_i(f)(t_i - t_{i-1}) \\
&= L(f,P_1) + L(f,P_2)
\end{aligned}$$

The proof is symmetric for the other statement. □

Proof of Theorem 13.23. Suppose first that f is integrable on $[a,c]$. To prove that f is integrable on $[a,b]$ and $[b,c]$, Theorem 13.18 tells us that it will suffice to show that for every $\epsilon > 0$, there exist partitions P_1, P_2 of $[a,b]$ and $[b,c]$, respectively, such that $U(f,P_1) - L(f,P_1) < \epsilon$ and $U(f,P_2) - L(f,P_2) < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f is integrable on $[a,c]$, there exists a partition P of $[a,c]$ such that $U(f,P) - L(f,P) < \epsilon$. Now define $P' = P \cup \{b\}$. Since P' is finite (by Script 1), a subset of $[a,c]$ (because $P \subset [a,c]$ by Definition 13.10 and $\{b\} \subset [a,c]$), and contains a, c (because $a, c \in P$ implies $a, c \in P \cup \{b\}$ by Definition 1.5), Definition 13.10 asserts that P' is a partition of $[a,c]$. Furthermore, since $P \subset P'$ by Theorem 1.7, Definition 13.10 implies that P' is a refinement of P. Thus, by Lemma 13.12, $L(f,P) \leq L(f,P')$ and $U(f,P) \geq U(f,P')$. This combined with the fact that $U(f,P) - L(f,P) < \epsilon$ implies by Script 7 that $U(f,P') - L(f,P') \leq U(f,P) - U(f,P) < \epsilon$.

Let $P_1 = P' \cap [a,b]$ and $P_2 = P' \cap [b,c]$. In the same manner as before, we have that P_1 is a partition of $[a,b]$ and P_2 is a partition of $[b,c]$. This combined with the fact that $P_1 \cup P_2 = P' \cap ([a,b] \cup [b,c]) = P'$ by Script 1 implies by the lemma that $L(f,P') = L(f,P_1) + L(f,P_2)$ and $U(f,P') = U(f,P_1) + U(f,P_2)$. Thus, we have that

$$\begin{aligned}
(U(f,P_1) - L(f,P_1)) + (U(f,P_2) - L(f,P_2)) &= U(f,P') - L(f,P') \\
&< \epsilon
\end{aligned}$$

Additionally, we have by consecutive applications of Definition 13.11 that $L(f,P_1) \leq U(f,P_1)$ and $L(f,P_2) \leq U(f,P_2)$. It follows by consecutive applications of Definition 7.21 that $0 \leq U(f,P_1) - L(f,P_1)$ and $0 \leq U(f,P_2) - L(f,P_2)$. This combined with the above result that $(U(f,P_1) - L(f,P_1)) + (U(f,P_2) - L(f,P_2)) < \epsilon$ implies by Script 7 that $U(f,P_1) - L(f,P_1) < \epsilon$ and $U(f,P_2) - L(f,P_2) < \epsilon$.

Now suppose that f is integrable on $[a,b]$ and $[b,c]$. Let $\Omega_1 = \int_a^b f$, $\Omega_2 = \int_b^c f$, and $\Omega = \Omega_1 + \Omega_2$. Thus, to prove that f is integrable on $[a,c]$ and that $\int_a^c f = \int_a^b f + \int_b^c f$, Lemma 13.20 tells us that it will suffice to show that for all $\epsilon > 0$, there is some partition P' of $[a,c]$ such that $U(f,P') - \Omega < \epsilon$ and $\Omega - L(f,P') < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f is integrable on $[a,b]$ and $[b,c]$, we have by consecutive applications of Lemma 13.20 that there exist partitions P_1 of $[a,b]$ and P_2 of $[b,c]$ such that $U(f,P_1) - \Omega_1 < \frac{\epsilon}{2}$, $\Omega_1 - L(f,P_1) < \frac{\epsilon}{2}$, $U(f,P_2) - \Omega_2 < \frac{\epsilon}{2}$, and $\Omega_2 - L(f,P_2) < \frac{\epsilon}{2}$. Choose $P' = P_1 \cup P_2$. By the lemma, P' is a partition of $[a,c]$. Combining all of the above results implies by Script 7 and the lemma that

$$\begin{aligned}
U(f,P') - \Omega &= U(f,P_1) + U(f,P_2) - \Omega_1 - \Omega_2 \\
&= (U(f,P_1) - \Omega_1) + (U(f,P_2) - \Omega_2) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}
\qquad
\begin{aligned}
\Omega - L(f,P') &= \Omega_1 + \Omega_2 - L(f,P_1) - L(f,P_2) \\
&= (\Omega_1 - L(f,P_1)) + (\Omega_2 - L(f,P_2)) \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Note that since the claim technically asks us to prove that $\int_a^c f = \int_a^b f + \int_b^c f$ follows from f being integrable on $[a,c]$, not $[a,b]$ and $[b,c]$, we can do this with the above using the following logic. Let f be integrable on $[a,c]$. Then by the first part of the proof, it is integrable on $[a,b]$ and $[b,c]$. It follows by the second part of the proof that $\int_a^c f = \int_a^b f + \int_b^c f$, as desired. □

4/22: If $b < a$, we define

$$\int_a^b f = -\int_b^a f$$

whenever the latter integral exists. With this notational convention, it follows that the equation

$$\int_a^c f = \int_a^b f + \int_b^c f$$

always holds, regardless of the ordering of a, b, c whenever f is integrable on the largest of the three intervals.

Theorem 13.24. *Suppose that f and g are integrable functions on $[a, b]$ and that $c \in \mathbb{R}$ is a constant. Then $f + g$ and cf are integrable on $[a, b]$ and*

(a) $\int_a^b (f + g) = \int_a^b f + \int_a^b g$.

(b) $\int_a^b cf = c \int_a^b f$.

Lemma.

(a) *Let $P = \{t_0, \dots, t_n\}$ be an arbitrary partition of $[a, b]$. Then for any $i \in [n]$, we have $M_i(f + g) \leq M_i(f) + M_i(g)$ and $m_i(f + g) \geq m_i(f) + m_i(g)$.*

(b) *Let $P = \{t_0, \dots, t_n\}$ be an arbitrary partition of $[a, b]$. Then if $c > 0$, we have $M_i(cf) = c \cdot M_i(f)$ and $m_i(cf) = c \cdot m_i(f)$ for any $i \in [n]$.*

(c) *Let $P = \{t_0, \dots, t_n\}$ be an arbitrary partition of $[a, b]$. Then if $c < 0$, we have $M_i(cf) = c \cdot m_i(f)$ and $m_i(cf) = c \cdot M_i(f)$ for any $i \in [n]$.*

Proof of Lemma (a). Let i be an arbitrary natural number satisfying $1 \leq i \leq n$. By Definitions 13.11, 5.7, and 5.6, $f(x) \leq M_i(f)$ for all $x \in [t_{i-1}, t_i]$. Similarly, $g(x) \leq M_i(g)$ for all $x \in [t_{i-1}, t_i]$. Thus, we have by Definition 7.21 that $(f + g)(x) \leq M_i(f) + M_i(g)$ for all $x \in [t_{i-1}, t_i]$. Consequently, Definition 5.6 asserts that $M_i(f) + M_i(g)$ is an upper bound on $\{(f + g)(x) \mid t_{i-1} \leq x \leq t_i\}$. Therefore, the supremum of that set will be less than or equal to $M_i(f) + M_i(g)$ by Definition 5.7. But since $M_i(f + g)$ *is* said supremum by Definition 13.11, we have that $M_i(f + g) \leq M_i(f) + M_i(g)$ as desired.

The proof is symmetric in the other case. $\square$

Proof of Lemma (b). Suppose for the sake of contradiction that $M_i(cf) \neq c \cdot M_i(f)$. We divide into two cases ($M_i(cf) < c \cdot M_i(f)$ and $M_i(cf) > c \cdot M_i(f)$). If $M_i(cf) < c \cdot M_i(f)$, then since $c > 0$, Lemma 7.24 implies that $\frac{M_i(cf)}{c} < M_i(f)$. It follows by Lemma 5.11 that there exists $f(x) \in \{f(x) \mid t_{i-1} \leq x \leq t_i\}$ such that $\frac{M_i(cf)}{c} < f(x) \leq M_i(f)$, i.e., $M_i(cf) < cf(x)$. But by Definitions 13.11, 5.7, and 5.6, $cf(x) \leq M_i(cf)$ for all $x \in [t_{i-1}, t_i]$, a contradiction. The argument is symmetric in the other case.

The proof is symmetric in the other case. $\square$

Proof of Lemma (c). Suppose for the sake of contradiction that $M_i(cf) \neq c \cdot m_i(f)$. We divide into two cases ($M_i(cf) < c \cdot m_i(f)$ and $M_i(cf) > c \cdot m_i(f)$). If $M_i(cf) < c \cdot m_i(f)$, then since $c < 0$, Lemma 7.24 implies that $\frac{M_i(cf)}{c} > m_i(f)$. It follows by Lemma 5.11 that there exists $f(x) \in \{f(x) \mid t_{i-1} \leq x \leq t_i\}$ such that $\frac{M_i(cf)}{c} > f(x) \geq m_i(f)$, i.e., $M_i(cf) < cf(x)$. But by Definitions 13.11, 5.7, and 5.6, $cf(x) \leq M_i(cf)$ for all $x \in [t_{i-1}, t_i]$, a contradiction. The argument is symmetric in the other case.

The proof is symmetric in the other case. $\square$

Proof of Theorem 13.24a. Let $\Omega_f = \int_a^b f$, $\Omega_g = \int_a^b g$, and $\Omega = \Omega_f + \Omega_g$. To prove that $f + g$ is integrable on $[a, b]$ and that $\int_a^b (f + g) = \int_a^b f + \int_a^b g$, Lemma 13.20 tells us that it will suffice to show that for all $\epsilon > 0$, there is some partition P such that $U(f + g, P) - \Omega < \epsilon$ and $\Omega - L(f + g, P) < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f, g are integrable on $[a, b]$, we have by consecutive applications of Lemma 13.20 that there exist partitions Q, R of $[a, b]$ such that $U(f, Q) - \Omega_f < \frac{\epsilon}{2}$, $\Omega_f - L(f, Q) < \frac{\epsilon}{2}$, $U(g, R) - \Omega_g < \frac{\epsilon}{2}$, and $\Omega_g - L(g, R) < \frac{\epsilon}{2}$. As in previous proofs, $P = Q \cup R$ is also a partition of $[a, b]$ and a refinement of both Q and R. Consequently, we have that $U(f, P) - \Omega_f \leq U(f, Q) - \Omega_f < \frac{\epsilon}{2}$, $\Omega_f - L(f, P) \leq \Omega_f - L(f, Q) < \frac{\epsilon}{2}$,

$U(g,P)-\Omega_g \leq U(g,R)-\Omega_g < \frac{\epsilon}{2}$, and $\Omega_g - L(g,P) \leq \Omega_g - L(g,R) < \frac{\epsilon}{2}$. It follows by consecutive applications of Script 7 that $U(f,P)+U(g,P)-\Omega < \epsilon$ and that $\Omega - (L(f,P)+L(g,P)) < \epsilon$. Therefore, we have that

$$\begin{aligned}
U(f+g,P)-\Omega &= \sum_{i=1}^{n} M_i(f+g)(t_i - t_{i-1}) - \Omega && \text{Definition 13.11} \\
&\leq \sum_{i=1}^{n} (M_i(f)+M_i(g))(t_i - t_{i-1}) - \Omega && \text{Lemma (a)} \\
&= \sum_{i=1}^{n} M_i(f)(t_i - t_{i-1}) + \sum_{i=1}^{n} M_i(g)(t_i - t_{i-1}) - \Omega \\
&= U(f,P)+U(g,P)-\Omega && \text{Definition 13.11} \\
&< \epsilon
\end{aligned}$$

and something similar for $\Omega - L(f+g,P)$. □

Proof of Theorem 13.24b. We divide into three cases ($c=0$, $c>0$, and $c<0$).

If $c=0$, then we have that $cf(x)=0$ for all $x \in [a,b]$. Therefore, we have by Exercise 13.17 that cf is integrable on $[a,b]$ and

$$\begin{aligned}
\int_a^b cf &= 0(b-a) \\
&= 0 \\
&= 0 \cdot \int_a^b f \\
&= c\int_a^b f
\end{aligned}$$

If $c>0$, then let $\Omega = \int_a^b f$. To prove that cf is integrable on $[a,b]$ and that $\int_a^b cf = c\int_a^b f$, Lemma 13.20 tells us that it will suffice to show that for all $\epsilon > 0$, there is some partition P such that $U(cf,P) - c\,\Omega < \epsilon$ and $c\,\Omega - L(cf,P) < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f is integrable on $[a,b]$, we have by Lemma 13.20 that there exists a partition P such that $U(f,P)-\Omega < \frac{\epsilon}{c}$ and $\Omega - L(f,P) < \frac{\epsilon}{c}$. It follows by consecutive applications of Lemma 7.24 that $c\,U(f,P) - c\,\Omega < \epsilon$ and $c\,\Omega - c\,L(f,P) < \epsilon$. Therefore, we have that

$$\begin{aligned}
U(cf,P) - c\,\Omega &= \sum_{i=1}^{n} M_i(cf)(t_i - t_{i-1}) - c\,\Omega && \text{Definition 13.11} \\
&= \sum_{i=1}^{n} c \cdot M_i(f)(t_i - t_{i-1}) - c\,\Omega && \text{Lemma (b)} \\
&= c\sum_{i=1}^{n} M_i(f)(t_i - t_{i-1}) - c\,\Omega \\
&= c\,U(f,P) - c\,\Omega && \text{Definition 13.11} \\
&< \epsilon
\end{aligned}$$

and something similar for $c\,\Omega - L(cf,P)$.

If $c<0$, then let $\Omega = \int_a^b f$. To prove that cf is integrable on $[a,b]$ and that $\int_a^b cf = c\int_a^b f$, Lemma 13.20 tells us that it will suffice to show that for all $\epsilon > 0$, there is some partition P such that $U(cf,P) - c\,\Omega < \epsilon$ and $c\,\Omega - L(cf,P) < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f is integrable on $[a,b]$, we have by Lemma 13.20 that there exists a partition P such that $U(f,P)-\Omega < \frac{\epsilon}{-c}$ and $\Omega - L(f,P) < \frac{\epsilon}{-c}$. It follows by consecutive applications of Lemma 7.24 that $c\,\Omega - c\,U(f,P) < \epsilon$ and $c\,L(f,P) - c\,\Omega < \epsilon$. Therefore, we have that

$$\begin{aligned}
U(cf,P) - c\,\Omega &= \sum_{i=1}^{n} M_i(cf)(t_i - t_{i-1}) - c\,\Omega && \text{Definition 13.11} \\
&= \sum_{i=1}^{n} c \cdot m_i(f)(t_i - t_{i-1}) - c\,\Omega && \text{Lemma (c)} \\
&= c\sum_{i=1}^{n} m_i(f)(t_i - t_{i-1}) - c\,\Omega \\
&= c\,L(f,P) - c\,\Omega && \text{Definition 13.11} \\
&< \epsilon
\end{aligned}$$

and something similar for $c\,\Omega - L(cf,P)$. $\square$

4/27: **Theorem 13.25.** *Suppose that f and g are integrable functions on $[a,b]$ with $f(x) \leq g(x)$ for all $x \in [a,b]$. Then*

$$\int_a^b f \leq \int_a^b g$$

Proof. Suppose for the sake of contradiction that $\int_a^b f > \int_a^b g$. Then by Definition 13.16, $L(f) > L(g)$. It follows by Lemma 5.11 that there exists a $L(f,P) \in \{L(f,P) \mid P \text{ is a partition of } [a,b]\}$ such that $L(f) \geq L(f,P) > L(g)$. Thus, since $L(g,P) \leq L(g)$ by Definitions 13.14, 5.7, and 5.6, we have that $L(g,P) < L(f,P)$. Consequently, by Definition 13.11, $\sum_{i=1}^{n} m_i(g)(t_i - t_{i-1}) < \sum_{i=1}^{n} m_i(f)(t_i - t_{i-1})$. Thus, by Script 7, there exists an i such that $m_i(g) < m_i(f)$. It follows by Lemma 5.11 that there exists a $g(x) \in \{g(x) \mid t_{i-1} \leq x \leq t_i\}$ such that $m_i(g) \leq g(x) < m_i(f)$. But this implies by Definitions 13.11, 5.7, and 5.6 that $g(x) < f(x)$, a contradiction. $\square$

4/29: **Theorem 13.26.** *Suppose that f is an integrable function on $[a,b]$. Then $|f|$ is also integrable on $[a,b]$ and*

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

Lemma. *Let $P = \{t_0, \dots, t_n\}$ be an arbitrary partition of $[a,b]$. Then for any $i \in [n]$, the following inequality holds.*

$$M_i(|f|) - m_i(|f|) \leq M_i(f) - m_i(f)$$

Proof. Let i be an arbitrary natural number satisfying $1 \leq i \leq n$. We divide into three cases ($f(x) \geq 0$ for all $x \in [a,b]$, $f(x) \leq 0$ for all $x \in [a,b]$, and there exist $x,y \in [a,b]$ such that $f(x) < 0 < f(y)$). Let's begin.

First, suppose that $f(x) \geq 0$ for all $x \in [a,b]$. Then by Definition 8.4, $|f(x)| = f(x)$ for all $x \in [a,b]$. It follows that $M_i(|f|) - m_i(|f|) = M_i(f) - m_i(f)$, which can be weakened to $M_i(|f|) - m_i(|f|) \leq M_i(f) - m_i(f)$, as desired.

Second, suppose that $f(x) \leq 0$ for all $x \in [a,b]$. Then by Definition 8.4, $|f(x)| = -f(x)$ for all $x \in [a,b]$. It follows that

$$\begin{aligned}
M_i(|f|) - m_i(|f|) &= M_i(-f) - m_i(-f) \\
&= -m_i(f) - (-M_i(f)) && \text{Lemma (c), Theorem 13.24} \\
&= M_i(f) - m_i(f)
\end{aligned}$$

which can be weakened to $M_i(|f|) - m_i(|f|) \leq M_i(f) - m_i(f)$, as desired.

Third, suppose that there exist $x,y \in [a,b]$ such that $f(x) < 0 < f(y)$. We divide into two subcases ($|M_i(f)| \geq |m_i(f)|$ and $|M_i(f)| < |m_i(f)|$).

Suppose first that $|M_i(f)| \geq |m_i(f)|$. By Definitions 13.11, 5.7, and 5.6 as well as the hypothesis, $M_i(f) \geq f(y) > 0$. Thus, by Definition 8.4, $|M_i(f)| = M_i(f)$. Similarly, $|m_i(f)| = -m_i(f)$. It follows by

Lemma (c) from Theorem 13.24 that $-m_i(f) = M_i(-f)$. Combining the last three results, we have by the hypothesis that $M_i(f) \geq M_i(-f)$. Additionally, we clearly have that $M_i(f) \geq M_i(f)$. Consequently, since $M_i(f) \geq M_i(-f)$ and $M_i(f) \geq M_i(f)$, we have by Script 5 that $M_i(f) \geq M_i(|f|)$. Furthermore, we have by Definitions 13.11, 5.7, 5.6, and 8.4 that $m_i(|f|) \geq 0 > f(x) \geq m_i(f)$. Therefore, since $M_i(|f|) \leq M_i(f)$ and $m_i(f) < m_i(|f|)$, we have that

$$\begin{aligned} M_i(|f|) - m_i(|f|) &\leq M_i(f) - m_i(|f|) \\ &< M_i(f) - m_i(f) \end{aligned}$$

which can be weakened to $M_i(|f|) - m_i(|f|) \leq M_i(f) - m_i(f)$, as desired.

Now suppose that $|M_i(f)| < |m_i(f)|$. As before, $M_i(f) > 0$ and $m_i(|f|) \geq 0$. It follows from the former result by Lemma 7.23 that $-M_i(f) < 0$. This combined with the previous result implies by transitivity that $-M_i(f) \leq m_i(|f|)$. Additionally, we have as before that $-m_i(f) = M_i(-f)$ and $M_i(f) = |M_i(f)| < |m_i(f)| = -m_i(f) = M_i(-f)$. Thus, $M_i(|f|) = M_i(-f) = -m_i(f)$. This combined with the fact that $-M_i(f) \leq m_i(|f|)$ implies by Definition 7.21 that $M_i(|f|) - M_i(f) < m_i(|f|) - m_i(f)$. It follows by consecutive applications of Definition 7.21 that $M_i(|f|) - m_i(|f|) < M_i(f) - m_i(f)$, which can be weakened to $M_i(|f|) - m_i(|f|) \leq M_i(f) - m_i(f)$, as desired. ☐

Proof of Theorem 13.26. To prove that $|f|$ is integrable on $[a, b]$, Theorem 13.18 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a partition P of $[a, b]$ such that $U(|f|, P) - L(|f|, P) < \epsilon$. Let $\epsilon > 0$ be arbitrary. By Theorem 13.18, there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$. Therefore,

$$\begin{aligned} U(|f|, P) - L(|f|, P) &= \sum_{i=1}^{n} M_i(|f|)(t_i - t_{i-1}) - \sum_{i=1}^{n} m_i(|f|)(t_i - t_{i-1}) && \text{Definition 13.11} \\ &= \sum_{i=1}^{n} (M_i(|f|) - m_i(|f|))(t_i - t_{i-1}) \\ &\leq \sum_{i=1}^{n} (M_i(f) - m_i(f))(t_i - t_{i-1}) && \text{The Lemma} \\ &= \sum_{i=1}^{n} M_i(f)(t_i - t_{i-1}) - \sum_{i=1}^{n} m_i(f)(t_i - t_{i-1}) \\ &= U(f, P) - L(f, P) && \text{Definition 13.11} \\ &< \epsilon \end{aligned}$$

as desired.

We now seek to prove that $|\int_a^b f| \leq \int_a^b |f|$. By Script 8, $-|f(x)| \leq f(x) \leq |f(x)|$ for all $x \in [a, b]$. It follows by consecutive applications of Theorem 13.25 that $\int_a^b -|f| \leq \int_a^b f \leq \int_a^b |f|$. Thus, since Theorem 13.24 asserts that $\int_a^b -|f| = -\int_a^b |f|$, we have that $-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|$. Therefore, by the lemma to Exercise 8.9, $|\int_a^b f| \leq \int_a^b |f|$, as desired. ☐

5/4: **Theorem 13.27.** *Suppose that f is integrable on $[a, b]$ and $m \leq f(x) \leq M$ for all $x \in [a, b]$. Then*

$$m(b - a) \leq \int_a^b f \leq M(b - a)$$

Proof. Let $g, h : [a, b] \to \mathbb{R}$ be defined by $g(x) = m$ and $h(x) = M$. By consecutive applications of Exercise 13.17, g and h are integrable on $[a, b]$ with $\int_a^b g = m(b - a)$ and $\int_a^b h = M(b - a)$. Additionally, we have by the definitions of g and h that $g(x) = m \leq f(x) \leq M = h(x)$. This combined with the fact that both g and h are integrable implies by consecutive applications of Theorem 13.25 that $\int_a^b g \leq \int_a^b f \leq \int_a^b h$. But this implies by the above that $m(b - a) \leq \int_a^b f \leq M(b - a)$, as desired. ☐

Theorem 13.28. *Suppose that f is integrable on $[a,b]$. Define $F:[a,b]\to\mathbb{R}$ by*

$$F(x)=\int_a^x f$$

Then F is continuous.

Proof. To prove that F is continuous, Theorem 9.10 tells us that it will suffice to show that F is continuous at every $x\in[a,b]$. Let x be an arbitrary element of $[a,b]$. To show that F is continuous at x, Theorem 11.5 tells us that it will suffice to verify that for every $\epsilon>0$, there exists a $\delta>0$ such that if $y\in[a,b]$ and $|y-x|<\delta$, then $|F(y)-F(x)|<\epsilon$. Let $\epsilon>0$ be arbitrary. By Scripts 1 and 5, the fact that $\{f(x)\mid x\in[a,b]\}$ is nonempty and bounded implies that $\{|f(x)|\mid x\in[a,b]\}$ is nonempty and bounded above. Thus, by Theorem 5.17, $\sup\{|f(x)|\mid x\in[a,b]\}$ exists. As such, we may define $s=\sup\{|f(x)|\mid x\in[a,b]\}$ so that we may choose $\delta=\frac{\epsilon}{s}$. Now let y be an arbitrary element of $[a,b]$ such that $|y-x|<\delta$. Therefore,

$$\begin{aligned}
|F(y)-F(x)| &= \left|\int_a^y f-\int_a^x f\right| && \\
&= \left|\int_x^y f\right| && \text{Theorem 13.23}\\
&\leq \int_x^y |f| && \text{Theorem 13.26}\\
&\leq s(y-x) && \text{Theorem 13.27}\\
&\leq s\cdot|y-x| && \\
&< s\cdot\frac{\epsilon}{s} && \\
&= \epsilon &&
\end{aligned}$$

□

Additional Exercises

2. Suppose that $|f(x)-g(x)|\leq\frac{\epsilon}{2}$ for all $x\in[a,b]$. Let $M_f=\sup\{f(x)\mid x\in[a,b]\}$ and $M_g=\sup\{g(x)\mid x\in[a,b]\}$. Prove that
$$M_f-M_g<\epsilon$$

13.2 Discussion

4/8: • The key to ϵ-δ proofs is to find a way to get $|y-x|$ into the $|f(y)-f(x)|$ expression and then deal with the others.

4/13: • Note that we can also prove Exercise 13.7 with the following procedure:
  - Lemma: If f is uniformly continuous on two intervals I,J whose union $I\cup J$ is also an interval, then f is uniformly continuous on $I\cup J$.
  - Establish that $f(x)=\sqrt{x}$ is uniformly continuous on $[0,1]$ using Theorem 13.6.
    - Note that the continuity of f on $[0,1]$ follows from the fact that f is differentiable on $(0,1)\subset\mathbb{R}^+$ (Exercise 12.22) by Theorems 12.5 and 9.10.
    - Note that the compactness of $[0,1]$ follows from Theorem 10.14.
  - Recall that f is uniformly continuous on $[1,\infty)$ from Exercise 13.3.
  - Apply the lemma.

4/15: • We can just say that the supremum of a singleton set is the element in that set (no proof required).

4/22:

- Theorem 13.24 was originally presented with a Theorem 13.18 ϵ proof preceding the Lemma 13.20 Ω proof. However, this is unnecessary.
  - Lemma (a) can also be proven by contradiction.

5/4:

- For Theorem 13.28, I should more rigorously establish the existence of s, perhaps with Theorem 13.26.

Script 14

Integrals and Derivatives

14.1 Journal

5/4: **Theorem 14.1.** *Suppose that f is integrable on $[a,b]$. Define $F:[a,b]\to\mathbb{R}$ by*

$$F(x)=\int_a^x f$$

If f is continuous at $p\in(a,b)$, then F is differentiable at p and

$$F'(p)=f(p)$$

If f is continuous at a, then $F'_+(a)$ exists and equals $f(a)$. Similarly, if f is continuous at b, $F'_-(b)$ exists and equals $f(b)$.

Proof. To prove that F is differentiable at p and $F'(p)=f(p)$, Definition 12.1 tells us that it will suffice to show that $\lim_{h\to0^+}\frac{F(p+h)-F(p)}{h}=\lim_{h\to0^-}\frac{F(p+h)-F(p)}{h}=f(p)$. We will tackle the right-handed limit first. To do so, Definition 11.1 tells us that it will suffice to verify that for every $\epsilon>0$, there exists a $\delta>0$ such that if $(p+h)\in[a,b]$ and $0<h<\delta$, then $|\frac{F(p+h)-F(p)}{h}-f(p)|<\epsilon$. Let $\epsilon>0$ be arbitrary. Since f is continuous at p, Theorem 11.5 asserts that there exists a $\delta>0$ such that if $x\in[a,b]$ and $|x-p|<\delta$, then $|f(x)-f(p)|<\frac{\epsilon}{2}$. Choose this δ to be our δ. Let h be an arbitrary number satisfying $(p+h)\in[a,b]$ and $0<h<\delta$. Therefore,

$$\begin{aligned}
\left|\frac{F(p+h)-F(p)}{h}-f(p)\right| &= \left|\frac{\int_a^{p+h}f-\int_a^p f}{h}-f(p)\right| \\
&= \left|\frac{\int_p^{p+h}f}{h}-f(p)\right| && \text{Theorem 13.23} \\
&= \left|\frac{\int_p^{p+h}f-hf(p)}{h}\right| \\
&= \left|\frac{\int_p^{p+h}f-f(p)((p+h)-p)}{h}\right| \\
&= \left|\frac{\int_p^{p+h}f-\int_p^{p+h}f(p)\,\mathrm{d}x}{h}\right| && \text{Exercise 13.17} \\
&= \left|\frac{1}{h}\int_p^{p+h}(f(x)-f(p))\,\mathrm{d}x\right| && \text{Theorem 13.24}
\end{aligned}$$

$$\begin{aligned}
&\leq \left|\frac{1}{h}\right| \int_p^{p+h} |f(x) - f(p)| \, \mathrm{d}x && \text{Theorem 13.26} \\
&\leq \left|\frac{1}{h}\right| \frac{\epsilon}{2}((p+h) - p) && \text{Theorem 13.27} \\
&= \frac{\epsilon}{2} \\
&< \epsilon
\end{aligned}$$

The proof is symmetric for the left-handed limit. These proofs can also be applied to the endpoints. ☐

Remark 14.2. Thus, we have that if f is continuous on $[a,b]$, F is differentiable on $[a,b]$ and $F'(p) = f(p)$ for all $p \in [a,b]$ (where at the endpoints, we understand that the derivative should be interpreted as the one-sided derivative).

Lemma 14.3. *Suppose that $f : [a,b] \to \mathbb{R}$ is integrable and that Ω is a number satisfying $L(f,P) \leq \Omega \leq U(f,P)$ for all partitions P of $[a,b]$. Then*

$$\int_a^b f = \Omega$$

Proof. Suppose for the sake of contradiction that $\int_a^b f \neq \Omega$. We divide into two cases ($\int_a^b f < \Omega$ and $\int_a^b f > \Omega$). If $\int_a^b f < \Omega$, then by Definition 13.16, $U(f) = \int_a^b f < \Omega$. It follows by Definition 13.14 and 5.11 that there exists an object $U(f,P) \in \{U(f,P) \mid P \text{ is a partition of } [a,b]\}$ such that $U(f) \leq U(f,P) < \Omega$. But this contradicts the hypothesis that $U(f,P) \geq \Omega$ for all partitions P of $[a,b]$. The argument is symmetric in the other case. ☐

5/6: **Theorem 14.4.** *Let f be integrable on $[a,b]$. Suppose that there is a function G that is continuous on $[a,b]$ and differentiable on (a,b) and such that $f = G'$ on (a,b). Then*

$$\int_a^b f = G(b) - G(a)$$

Proof. To prove that $\int_a^b f = G(b) - G(a)$, Lemma 14.3 tells us that it will suffice to show that for all partitions $P = \{t_0, \dots, t_n\}$ of $[a,b]$, $L(f,P) \leq G(b) - G(a) \leq U(f,P)$. Let P be an arbitrary partition of $[a,b]$. If t_i, t_{i-1} are two sequential elements of P, then since G is continuous on $[t_{i-1}, t_i] \subset [a,b]$ and differentiable on $(t_{i-1}, t_i) \subset (a,b)$, Corollary 12.16 asserts that there exists a point $\lambda \in (t_{i-1}, t_i)$ such that $G(t_i) - G(t_{i-1}) = G'(\lambda)(t_i - t_{i-1})$. It follows since $f = G'$ on (a,b) that

$$G(t_i) - G(t_{i-1}) = f(\lambda)(t_i - t_{i-1})$$

Thus, since we have proven the above statement for an arbitrary i, we can apply it to all i and sum to get

$$\begin{aligned}
\sum_{i=1}^n f(\lambda)(t_i - t_{i-1}) &= \sum_{i=1}^n G(t_i) - G(t_{i-1}) \\
&= G(b) - G(a)
\end{aligned}$$

But by Definitions 13.11, 5.7, and 5.6, we have that $m_i(f) \leq f(\lambda) \leq M_i(f)$ for all i. It follows that

$$\sum_{i=1}^n m_i(f)(t_i - t_{i-1}) \leq \sum_{i=1}^n f(\lambda)(t_i - t_{i-1}) \leq \sum_{i=1}^n M_i(f)(t_i - t_{i-1})$$

Therefore, we have by Definition 13.11 and substitution that

$$L(f,P) \leq G(b) - G(a) \leq U(f,P)$$

as desired. ☐

Corollary 14.5. *Let f, g be functions defined on some open interval containing $[a,b]$ such that f' and g' exist and are continuous on $[a,b]$. Then*

$$\int_a^b fg' = [f(b)g(b) - f(a)g(a)] - \int_a^b f'g$$

Proof. Since f and g are differentiable on $[a,b]$, Exercise 12.9 implies that fg is differentiable on $[a,b]$ with $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ for all $x \in [a,b]$. We now seek to prove that $f'g + fg'$ is integrable on $[a,b]$. By hypothesis, f' and g' are continuous on $[a,b]$. Additionally, since f and g are differentiable on $[a,b]$, Theorem 12.5 asserts that they are continuous on $[a,b]$. Thus, since f, g, f', and g' are continuous on $[a,b]$, we have by consecutive applications of Corollary 11.10 that $f'g + fg'$ is continuous on $[a,b]$. Consequently, by Theorem 13.19, $f'g + fg'$ is integrable on $[a,b]$, as desired. Furthermore, in a similar manner to the above, we can show that $f'g$ and fg' are integrable on $[a,b]$. Lastly, it follows from the fact that f and g are continuous on $[a,b]$ by Corollary 11.10 that fg is continuous on $[a,b]$.

Having established that $f'g + fg'$ is integrable on $[a,b]$, that fg is a function that is continuous on $[a,b]$, differentiable on $(a,b) \subset [a,b]$, and such that $f'g + fg' = (fg)'$ on (a,b), and that $f'g$ and fg' are integrable on $[a,b]$, we have that

$$\begin{aligned}
\int_a^b (f'g + fg') &= (fg)(b) - (fg)(a) && \text{Theorem 14.4}\\
\int_a^b f'g + \int_a^b fg' &= f(b)g(b) - f(a)g(a) && \text{Theorem 13.24}\\
\int_a^b fg' &= [f(b)g(b) - f(a)g(a)] - \int_a^b f'g
\end{aligned}$$

as desired. □

Corollary 14.6. *Let g be a function defined on some interval containing $[a,b]$ such that g' is continuous on $[a,b]$. Suppose that $g([a,b]) \subset [c,d]$ and $f : [c,d] \to \mathbb{R}$ is continuous. Define $F : [c,d] \to \mathbb{R}$ by $F(x) = \int_c^x f$. Then*

$$\int_a^b f(g(x)) \cdot g'(x)\,\mathrm{d}x = F(g(b)) - F(g(a))$$

Proof. To prove that $\int_a^b f(g(x)) \cdot g'(x)\,\mathrm{d}x = F(g(b)) - F(g(a))$, Theorem 14.4 tells us that it will suffice to show that $(f \circ g) \cdot g'$ is integrable on $[a,b]$, $F \circ g$ is continuous on $[a,b]$ and differentiable on (a,b), and $(f \circ g) \cdot g' = (F \circ g)'$ on (a,b). We will confirm each requirement in turn. Let's begin.

To confirm that $(f \circ g) \cdot g'$ is integrable on $[a,b]$, Theorem 13.19 tells us that it will suffice to demonstrate that $(f \circ g) \cdot g'$ is continuous on $[a,b]$. By hypothesis, f is continuous on $[c,d]$. Additionally, since g' is defined on $[a,b]$, we know that g is differentiable on $[a,b]$, which implies by Theorem 12.5 that g is continuous on $[a,b]$. The combination of the previous two results implies by Corollary 11.15 that $f \circ g$ is continuous on $[a,b]$. This combined with the hypothesis that g' is continuous on $[a,b]$ implies by Corollary 11.10 that $(f \circ g) \cdot g'$ is continuous on $[a,b]$.

To confirm that $F \circ g$ is continuous on $[a,b]$, Corollary 11.15 tells us that it will suffice to demonstrate that F is continuous on $[c,d]$ and g is continuous on $[a,b]$. By Theorem 13.28, F is continuous on $[c,d]$. Additionally, we know by the above that g is continuous on $[a,b]$.

To confirm that $F \circ g$ is differentiable on (a,b), Theorem 12.10 tells us that it will suffice to demonstrate that F is differentiable on (c,d) and g is differentiable on (a,b). Since f is continuous on $(c,d) \subset [c,d]$, we have by Theorem 14.1 that F is differentiable on (c,d). Additionally, we know by the above that g is differentiable on $(a,b) \subset [a,b]$.

Since $F \circ g$ is differentiable on (a,b), we have by Theorem 12.10 again that $(F \circ g)' = (F' \circ g) \cdot g'$ for all $x \in (a,b)$. Thus, since $F' = f$ by Theorem 14.1, we have that $(f \circ g) \cdot g' = (F \circ g)'$ on (a,b), as desired. □

14.2 Discussion

5/4:
- We can't use Exercise 10.21 because it requires that f be continuous on the whole interval $[p-\delta, p+\delta]$.
  - For my proof: If all of the values on the interval are within $\frac{\epsilon}{2}$, then the whole thing is within ϵ?

5/6:
- Lemma for Theorem 14.4? Should continuous derivative be in the hypothesis?
- Use Lemma 14.3 and the MVT to prove Theorem 14.4.
- Modify Corollary 14.5 with Theorem 14.4 first, i.e., in the mold of Corollary 14.6?

Script 15

Sequences

15.1 Journal

5/6: **Definition 15.1.** A **sequence** (of real numbers) is a function $a : \mathbb{N} \to \mathbb{R}$.

By setting $a_n = a(n)$, we can think of a sequence as a list $a_1, a_2, a_3, \dots$ of real numbers. We use the notation $(a_n)_{n=1}^{\infty}$ for such a sequence, or if there is no possibility of confusion, we sometimes abbreviate this and write simply (a_n). More generally, we also use the term sequence to refer to the function defined on $\{n \in \mathbb{N} \cup \{0\} \mid n \geq n_0\}$ for any fixed $n_0 \in \mathbb{N} \cup \{0\}$. We write $(a_n)_{n=n_0}^{\infty}$ for such a sequence.

Definition 15.2. We say that a sequence (a_n) **converges** to a point $p \in \mathbb{R}$ if for every open interval I containing p, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in I$. If a sequence converges to some point, we say it is **convergent**. If (a_n) does not converge to any point, we say that the sequence **diverges** or is **divergent**.

Exercise 15.3. Show that a sequence (a_n) converges to p if and only if any region containing p contains all but finitely many terms of the sequence.

Proof. Suppose first that (a_n) converges to p. Let R be an arbitrary region containing p. By Corollary 4.11 and Lemma 8.3, R is an open interval. Thus, by Definition 15.2, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in R$. To prove that R contains all but finitely many terms of the sequence, it will suffice to show that the set $A = \{a_n \mid a_n \notin R\}$ is finite. Since $a_n \in R$ for all $n \geq N$, it follows that $a_n \in R$ only if $n < N$. Thus, by Script 1, $A \subset \{a_n \mid 0 \leq n < N\}$. Since the latter set is clearly finite, it follows by Script 1 that A is finite.

Now suppose that any region containing p contains all but finitely many terms (a_n). To prove that (a_n) converges to p, Definition 15.2 tells us that it will suffice to show that for every open interval I containing p, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in I$. Let I be an arbitrary open interval containing p. Then by Theorem 4.10, there exists a region R containing p such that $R \subset I$. It follows by the hypothesis that $A = \{a_n \mid a_n \notin R\}$ is finite. We divide into two cases ($|A| = 0$ and $|A| \in \mathbb{N}$). Suppose first that $|A| = 0$. Choose $N = n_0$. It follows that if $n \geq N$, then $a_n \notin A$, so $a_n \in R$, so $a_n \in I$, as desired. Now suppose that $|A| \in \mathbb{N}$. By Definition 1.18, $a^{-1}(A) \subset \mathbb{N}$. Consequently, by Lemma 3.4, $a^{-1}(A)$ has a last point $N - 1$. Choose $N = (N-1) + 1$. It follows that if $n \geq N$, then $n \notin a^{-1}(A)$, so $a_n \notin A$, so $a_n \in R$, so $a_n \in I$, as desired. □

Theorem 15.4. *Suppose that (a_n) converges to both p and to p'. Then $p = p'$.*

Proof. Suppose for the sake of contradiction that $p \neq p'$. Then by Theorem 3.22, there exist disjoint regions R, R' containing p, p', respectively. Additionally, by consecutive applications of Corollary 4.11 and Lemma 8.3, R, R' are open intervals. Combining these last two results, we have by consecutive applications of Definition 15.2 that there exist $N, N' \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in R$ and if $n \geq N'$, then $a_n \in R'$. Let $M = \max(N, N')$. It follows that $M \geq N$ and $M \geq N'$. Thus, by the above, $a_M \in R$ and $a_M \in R'$. But this implies by Definition 1.6 that $a_M \in R \cap R'$. Therefore, by Definition 1.9, R and R' are not disjoint, a contradiction. □

Definition 15.5. If a sequence (a_n) converges to $p \in \mathbb{R}$, we call p the **limit** of (a_n) and write

$$\lim_{n\to\infty} a_n = p$$

Exercise 15.6. Which of the following sequences (a_n) converge? Which diverge? For each that converges, what is the limit? Give proofs for your answers.

(a) $a_n = 5$.

Proof. To prove that this sequence converges with limit $\lim_{n\to\infty} a_n = 5$, Definition 15.5 tells us that it will suffice to show that (a_n) converges to 5. To do so, Definition 15.2 tells us that it will suffice to verify that for every open interval I containing 5, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in I$. Let's begin.

Let I be an arbitrary open interval containing 5. Choose $N = 1$. Let n be an arbitrary natural number such that $n \geq N$. It follows by the definition of the sequence that $a_n = 5 \in I$, as desired. □

(b) $a_n = n$.

Proof. To prove that this sequence diverges, Definition 15.2 tells us that it will suffice to show that for every point $p \in \mathbb{R}$, there exists an open interval I containing p such that for all $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \notin I$. Let's begin.

Let p be an arbitrary element of $\mathbb{R}$. Choose $I = (p-1, p+1)$. Clearly $p \in I$. Let N be an arbitrary natural number. By Corollary 6.12, there exists a natural number N' such that $p + 1 < N'$. Choose $M = \max(N, N')$. Thus, $M \geq N$. Additionally, it follows by the definition of the sequence that $a_M = M$. But this implies that $a_M \geq N' > p + 1$, i.e., $a_M \notin I$ by Equations 8.1. □

(c) $a_n = \frac{1}{n}$.

Proof. To prove that this sequence converges with limit $\lim_{n\to\infty} a_n = 0$, Definitions 15.5 and 15.2 tell us that it will suffice to show that for every open interval I containing 0, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in I$. Let's begin.

Let I be an arbitrary interval containing 0. By Lemma 8.10, there exists a region (a, b) containing 0 such that $(a, b) \subset I$. By Corollary 6.12, there exists a natural number N such that $\frac{1}{b} < N$. Choose this N to be our N. Now let n be an arbitrary natural number such that $n \geq N$. It follows that $\frac{1}{b} < n$. Thus, since $0 < b$ and $0 < n$, we have by consecutive applications of Lemma 7.24 that $0 < \frac{1}{n} < b$. Consequently, since we also know that $a < 0$ and $a_n = \frac{1}{n}$, we have by transitivity and substitution that $a < a_n < b$. It follows by Equations 8.1 that $a_n \in (a, b)$. Therefore, by Definition 1.3, $a_n \in I$, as desired. □

(d) $a_n = (-1)^n$.

Proof. To prove that this sequence diverges, Definition 15.2 tells us that it will suffice to show that for every point $p \in \mathbb{R}$, there exists an open interval I containing p such that for all $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \notin I$. Let's begin.

Let p be an arbitrary element of R. Choose $I = (p-1, p+1)$. Clearly $p \in I$. Let N be an arbitrary natural number. By Script 0, either N is even and $N+1$ is odd or vice versa. Thus, let N be even (the case where N is odd is symmetric). It follows that $N \geq N$ yields $a_N = (-1)^{2m} = ((-1)^2)^m = 1^m = 1$ and that $N+1 \geq N$ yields $a_{N+1} = (-1)^{2m+1} = 1(-1)^1 = -1$. Now suppose for the sake of contradiction that $a_N \in I$ and $a_{N+1} \in I$. Since $a_N = 1 \in I$, we have by Equations 8.1 that $p - 1 < 1 < p + 1$. It follows by Definition 7.21 that $p - 3 < -1 < p - 1$. But $-1 < p - 1$ implies by Equations 8.1 that $a_{N+1} = -1 \notin I$, a contradiction. Therefore, $N+1 \geq N$ is a number such that $a_{N+1} \notin I$, as desired. □

5/11: **Theorem 15.7.** *A sequence (a_n) converges to $p \in \mathbb{R}$ if and only if for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - p| < \epsilon$.*

Proof. Suppose first that (a_n) converges to p. Let $\epsilon > 0$ be arbitrary. Consider the p-containing region $R = (p - \epsilon, p + \epsilon)$. By Corollary 4.11 and 8.3, R is an open interval. Thus, by Definition 15.2, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in R$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. Then $a_n \in (p - \epsilon, p + \epsilon)$. Therefore, by Exercise 8.9, $|a_n - p| < \epsilon$, as desired.

Now suppose that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - p| < \epsilon$. To prove that (a_n) converges to p, Definition 15.2 tells us that it will suffice to show that for every open interval I containing p, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in I$. Let I be an arbitrary open interval that satisfies $p \in I$. It follows by Lemma 8.10 that there exists a number $\epsilon > 0$ such that $(p - \epsilon, p + \epsilon) \subset I$. With respect to this ϵ, we have by hypothesis that there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - p| < \epsilon$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. Then $|a_n - p| < \epsilon$. Consequently, by Exercise 8.9, $a_n \in (p - \epsilon, p + \epsilon)$. Therefore, by Definition 1.3, $a_n \in I$, as desired. ☐

Exercise 15.8.

(a) Prove that $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

Proof. To prove that $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$, Definition 15.5 and Theorem 15.7 tell us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - 0| = |a_n| < \epsilon$. Let $\epsilon > 0$ be arbitrary. By Corollary 6.12, there exists a natural number N such that $\frac{1}{\epsilon} < N$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. It follows by transitivity that $\frac{1}{\epsilon} < n$. Thus, since $0 < n$ and $0 < \epsilon$, we have by consecutive applications of Lemma 7.24 that $0 < \frac{1}{n} < \epsilon$. Additionally, since $(-1)^n = 1$ or $(-1)^n = -1$ for all $n \in \mathbb{N}$ by Script 0, we have by Definition 8.4 that $|\frac{(-1)^n}{n}| = |\frac{1}{n}| = \frac{1}{n}$. Consequently, we know that $|\frac{(-1)^n}{n}| < \epsilon$. But since $a_n = \frac{(-1)^n}{n}$, we have that $|a_n| < \epsilon$, as desired. ☐

(b) Let $x \in \mathbb{R}$ with $|x| < 1$. Prove that $\lim_{n\to\infty} x^n = 0$.

Lemma. *If $|y| > 1$ and n is a natural number, then $|y|^n \geq n(|y| - 1) + 1$.*

Proof. Define $1 + x = |y|$. It follows by Definition 7.21 that $x > 0 > -1$, which can be weakened to $x \geq -1$. Additionally, since n is a natural number, $n \geq 1$ by Script 0. Thus, since $x \geq -1$ and $n \geq 1$, we have by Additional Exercise 12.3b that $(1+x)^n \geq 1+nx$. Substituting, we have $|y|^n \geq n(|y|-1)+1$, as desired. ☐

Proof of Exercise 15.8b. To prove that $\lim_{n\to\infty} x^n = 0$, Definition 15.5 and Theorem 15.7 tell us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - 0| = |a_n| < \epsilon$. Let $\epsilon > 0$ be arbitrary. We divide into two cases ($x = 0$ and $x \neq 0$). If $x = 0$, then choose $N = 1$. Let n be an arbitrary natural number such that $n \geq N$. Since $0^n = 0$ by Script 7, we have $|a_n| = |0^n| = 0 < \epsilon$, as desired. On the other hand, if $x \neq 0$, then we continue. By Corollary 6.12, there exists a natural number N such that $\frac{1}{\epsilon(|\frac{1}{x}|-1)} < N$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. Therefore,

$$\begin{aligned}
|x^n| &= |x^n| \cdot \frac{1}{\epsilon\left(\left|\frac{1}{x}\right| - 1\right)} \cdot \epsilon\left(\left|\frac{1}{x}\right| - 1\right) \\
&< |x^n| \cdot N \cdot \epsilon\left(\left|\frac{1}{x}\right| - 1\right) \\
&\leq |x^n| \cdot n \cdot \epsilon\left(\left|\frac{1}{x}\right| - 1\right) \\
&< \epsilon \cdot |x^n| \cdot n\left(\left|\frac{1}{x}\right| - 1\right) + 1
\end{aligned}$$

$$\leq \epsilon \cdot |x^n| \cdot \left|\frac{1}{x}\right|^n \qquad \text{The Lemma}$$
$$= \epsilon \cdot |x^n| \cdot \frac{1}{|x^n|}$$
$$= \epsilon$$

as desired. □

Theorem 15.9. *If* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$ *both exist, then*

(a) $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.

Proof. Let $a = \lim_{n\to\infty} a_n$ and let $b = \lim_{n\to\infty} b_n$. To prove that $\lim_{n\to\infty}(a_n + b_n)$ exists and equals $\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$, Definition 15.5 and Theorem 15.7 tell us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|(a_n + b_n) - (a + b)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. It follows by consecutive applications of Definition 15.5 and Theorem 15.7 that there exist natural numbers N_a, N_b such that for all $n \geq N_a$, we have $|a_n - a| < \frac{\epsilon}{2}$ and for all $n \geq N_b$, we have $|b_n - b| < \frac{\epsilon}{2}$. Now choose $N = \max(N_a, N_b)$. Let n be an arbitrary natural number such that $n \geq N$. It follows that $n \geq N \geq N_a$, so we know that $|a_n - a| < \frac{\epsilon}{2}$. Similarly, $|b_n - b| < \frac{\epsilon}{2}$. Therefore, we have that

$$\begin{aligned} |(a_n + b_n) - (a + b)| &\leq |a_n - a| + |b_n - b| && \text{Lemma 8.8} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

□

(b) $\lim_{n\to\infty}(a_n \cdot b_n) = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$.

Proof. Let $a = \lim_{n\to\infty} a_n$ and let $b = \lim_{n\to\infty} b_n$. To prove that $\lim_{n\to\infty}(a_n \cdot b_n)$ exists and equals $(\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$, Definition 15.5 and Theorem 15.7 tell us that it will suffice too show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n \cdot b_n - a \cdot b| < \epsilon$. Let $\epsilon > 0$ be arbitrary. By Theorem 15.13[1], (a_n) is bounded. Thus, by Definition 15.12, $\{a_n \mid n \in \mathbb{N}\}$ is bounded. Consequently, by the proof of Exercise 13.9, there exists a number M_a such that $|a_n| < M_a$ for all $n \in \mathbb{N}$. Now define $M = \max(M_a, b)$. Using this M (which by definition is positive since it's greater than $|a_n|$, which is at least 0) as well as our previously defined arbitrary ϵ, we have by consecutive applications of Definition 15.5 and Theorem 15.7 that there exist natural numbers N_a, N_b such that for all $n \geq N_a$, we have $|a_n - a| < \frac{\epsilon}{2M}$ and for all $n \geq N_b$, we have $|b_n - b| < \frac{\epsilon}{2M}$. Now choose $N = \max(N_a, N_b)$. Let n be an arbitrary natural number such that $n \geq N$. It follows that $n \geq N \geq N_a$, so we know that $|a_n - a| < \frac{\epsilon}{2M}$. Similarly, $|b_n - b| < \frac{\epsilon}{2M}$. Therefore,

$$\begin{aligned} |a_n b_n - ab| &= |a_n(b_n - b) + b(a_n - a)| \\ &\leq |a_n| \cdot |b_n - b| + |b| \cdot |a_n - a| && \text{Lemma 8.8} \\ &\leq |M| \cdot |b_n - b| + |M| \cdot |a_n - a| \\ &< M \cdot \frac{\epsilon}{2M} + M \cdot \frac{\epsilon}{2M} \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

□

[1]The proof of Theorem 15.13 does not depend on any following results, so its use here is not circular logic.

Moreover, if $\lim_{n\to\infty} b_n \neq 0$, *then*

(*c*) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$.

Lemma. *Let* $\lim_{n\to\infty} b_n = b \neq 0$. *Then there exists* $m \in \mathbb{R}^+$ *such that* $m \leq |b|$ *and* $N \in \mathbb{N}$ *such that if* $n \geq N$, *then* $m \leq |b_n|$.

Proof. Since $b \neq 0$, it follows from Definition 8.4 that $0 < |b|$. Thus, by Theorem 5.2, there exists a point $m \in \mathbb{R}$ such that $0 < m < |b|$. It follows from the fact that $0 < m$ that $m \in \mathbb{R}^+$, and from the fact that $m < |b|$ that $m \leq |b|$, as desired.

As to the other part of the proof, we divide into two cases ($b > 0$ and $b < 0$).

Suppose first that $b > 0$. By Exercise 6.5, Axiom 3, and Definition 3.3, there exists a number y such that $b < y$. Now consider the region (m, y). Since $m < |b| = b < y$, Equations 8.1 assert that $b \in (m, y)$. Additionally, by Corollary 4.11 and Lemma 8.3, (m, y) is an open interval. Thus, by Definitions 15.5 and 15.2, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $b_n \in (m, y)$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. Then $b_n \in (m, y)$. It follows by Equations 8.1 that $m < b_n < y$, which can be weakened to $m \leq b_n$. Since $0 < m \leq b_n$, Definition 8.4 asserts that $m \leq |b_n|$, as desired.

Now suppose that $b < 0$. By Exercise 6.5, Axiom 3, and Definition 3.3, there exists a number x such that $x < b$. Now consider the region $(x, -m)$. Since $m < |b| = -b$, we have by Lemma 7.24 that $b < -m$. This combined with the fact that $x < b$ implies by Equations 8.1 that $b \in (x, -m)$. Additionally, by Corollary 4.11 and Lemma 8.3, $(x, -m)$ is an open interval. Thus, by Definitions 15.5 and 15.2, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $b_n \in (x, -m)$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. Then $b_n \in (x, -m)$. It follows by Equations 8.1 that $x < b_n < -m$, which can be weakened to $b_n \leq -m$. Consequently, by Lemma 7.24, $m \leq -b_n$. Since $0 < m \leq -b_n$, Definition 8.4 asserts that $m \leq |b_n|$, as desired. □

Proof of Theorem 15.9c. Let $a = \lim_{n\to\infty} a_n$ and let $b = \lim_{n\to\infty} b_n$. To prove that $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and equals $\frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, Definition 15.5 and Theorem 15.7 tell us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|\frac{a_n}{b_n} - \frac{a}{b}| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $M = \max(|a|, |b|)$. Additionally, by the lemma, choose $m \in \mathbb{R}, N' \in \mathbb{N}$ such that $m \leq |b|$ and if $n \geq N'$, then $m \leq |b_n|$. Using this M and m (which, again, by definition are both positive and nonzero) as well as our previously defined arbitrary ϵ, we have by consecutive applications of Definition 15.5 and Theorem 15.7 that there exist natural numbers N_a, N_b such that for all $n \geq N_a$, we have $|a_n - a| < \frac{\epsilon m^2}{M}$ and for all $n \geq N_b$, we have $|b_n - b| < \frac{\epsilon m^2}{M}$. Now choose $N = \max(N', N_a, N_b)$. Let n be an arbitrary natural number such that $n \geq N$. It follows that $n \geq N \geq N_a$, so we know that $|a_n - a| < \frac{\epsilon m^2}{M}$. Additionally, since $n \geq N'$, $m \leq |b|$ and $m \leq |b_n|$. Similarly, $|b_n - b| < \frac{\epsilon m^2}{M}$. Therefore,

$$
\begin{aligned}
\left|\frac{a_n}{b_n} - \frac{a}{b}\right| &= \left|\frac{a_n b - b_n a}{b_n b}\right| \\
&= \frac{|b(a_n - a) + a(b - b_n)|}{|b_n| \cdot |b|} \\
&\leq \frac{|b| \cdot |a_n - a| + |a| \cdot |b - b_n|}{|b_n| \cdot |b|} && \text{Lemma 8.8} \\
&\leq \frac{M \cdot |a_n - a| + M \cdot |b_n - b|}{m \cdot m} \\
&= \frac{M}{m^2} \cdot |a_n - a| + \frac{M}{m^2} \cdot |b_n - b| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}
$$

□

5/13: **Exercise 15.10.** Which of the following sequences (a_n) converge? Which diverge? For each that converges, what is the limit? Give proofs for your answers.

(a) $a_n = (-1)^n \cdot n$.

Proof. Suppose for the sake of contradiction that $\lim_{n\to\infty} a_n$ converges. Then since (b_n) defined by $b_n = \frac{1}{n}$ converges by Exercise 15.6c, we have by Theorem 15.9 that

$$\begin{aligned}\left(\lim_{n\to\infty}(-1)^n \cdot n\right)\cdot\left(\lim_{n\to\infty}\frac{1}{n}\right) &= \lim_{n\to\infty}(-1)^n\cdot n\cdot\frac{1}{n}\\ &= \lim_{n\to\infty}(-1)^n\end{aligned}$$

But by Exercise 15.6d, $\lim_{n\to\infty}(-1)^n$ diverges, a contradiction. □

(b) $a_n = \frac{1}{n^2+1}(2+\frac{1}{n})$.

Proof. To prove that $\lim_{n\to\infty}\frac{1}{n^2+1}(2+\frac{1}{n}) = 0$, we will first confirm that $\lim_{n\to\infty}\frac{1}{n^2+1} = 0$ and $\lim_{n\to\infty} 2 = 2$. These results can be tied together with the fact that $\lim_{n\to\infty}\frac{1}{n} = 0$ (by Exercise 15.6c) to prove the desired result with Theorem 15.9. Let's begin.

To confirm that $\lim_{n\to\infty}\frac{1}{n^2+1} = 0$, Theorem 15.7 tells us that it will suffice to demonstrate that for all $\epsilon > 0$, there is some $N\in\mathbb{N}$ such that for all $n\geq N$, we have $|\frac{1}{n^2+1} - 0| = |\frac{1}{n^2+1}| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{n\to\infty}\frac{1}{n} = 0$ by Exercise 15.6c, we have by Theorem 15.7 that there exists an $N\in\mathbb{N}$ such that for all $n\geq N$, we have $|\frac{1}{n}| < \epsilon$. Choose this N to be our N. Let n be an arbitrary natural number such that $n\geq N$. Then since $n\leq n^2 < n^2+1$ by Script 7, we have that

$$\begin{aligned}\left|\frac{1}{n^2+1}\right| &= \frac{1}{n^2+1} && \text{Definition 8.4}\\ &< \frac{1}{n^2}\\ &\leq \frac{1}{n}\\ &= \left|\frac{1}{n}\right| && \text{Definition 8.4}\\ &< \epsilon\end{aligned}$$

as desired.

The proof that $\lim_{n\to\infty} 2 = 2$ is symmetric to that of Exercise 15.6a.

Having established that $\lim_{n\to\infty}\frac{1}{n^2+1} = 0$, $\lim_{n\to\infty} 2 = 2$, and $\lim_{n\to\infty}\frac{1}{n} = 0$, we have by consecutive applications of Theorem 15.9 that

$$\begin{aligned}0 &= 0\cdot(2+0)\\ &= \left(\lim_{n\to\infty}\frac{1}{n^2+1}\right)\cdot\left(\lim_{n\to\infty} 2 + \lim_{n\to\infty}\frac{1}{n}\right)\\ &= \lim_{n\to\infty}\frac{1}{n^2+1}\left(2+\frac{1}{n}\right)\end{aligned}$$

as desired. □

(c) $a_n = \frac{5n+1}{2n+3}$.

Proof. The proof that $\lim_{n\to\infty} 3 = 3$ is symmetric to that of Exercise 15.6a. Additionally, by Exercise 15.6a, the proof of Exercise 15.10b, and Exercise 15.6c, we know that $\lim_{n\to\infty} 5 = 5$, $\lim_{n\to\infty} 2 = 2$, and $\lim_{n\to\infty} \frac{1}{n} = 0$, respectively. Therefore, by consecutive applications of Theorem 15.9, we have that

$$\begin{aligned}\frac{5}{2} &= \frac{5+0}{2+3\cdot 0}\\ &= \frac{\lim_{n\to\infty} 5 + \lim_{n\to\infty}\frac{1}{n}}{\lim_{n\to\infty} 2 + (\lim_{n\to\infty} 3)\cdot\left(\lim_{n\to\infty}\frac{1}{n}\right)}\\ &= \lim_{n\to\infty}\frac{5+\frac{1}{n}}{2+3\cdot\frac{1}{n}}\\ &= \lim_{n\to\infty}\frac{5n+1}{2n+3}\end{aligned}$$

as desired. □

(d) $a_n = \frac{(-1)^n+1}{n}$.

Proof. By Exercises 15.8a and 15.6c, we know that $\lim_{n\to\infty}\frac{(-1)^n}{n} = 0$ and $\lim_{n\to\infty}\frac{1}{n} = 0$, respectively. Therefore, by Theorem 15.9,

$$\begin{aligned}0 &= 0+0\\ &= \lim_{n\to\infty}\frac{(-1)^n}{n} + \lim_{n\to\infty}\frac{1}{n}\\ &= \lim_{n\to\infty}\left(\frac{(-1)^n}{n}+\frac{1}{n}\right)\\ &= \lim_{n\to\infty}\frac{(-1)^n+1}{n}\end{aligned}$$

as desired. □

We've used the word "limit" in two contexts now: The limit of a point in a set, and the limit of a sequence. The definitions of these two terms may seem similar. Is there a formal connection? Theorem 15.11 alludes to an answer.

Theorem 15.11. *Let $A \subset \mathbb{R}$. Then $p \in \overline{A}$ if and only if there exists a sequence (a_n), with each $a_n \in A$, that converges to p.*

Proof. Suppose first that $p \in \overline{A}$. Then by Definitions 4.4 and 1.5, $p \in A$ or $p \in LP(A)$. We now divide into two cases. If $p \in A$, then define (a_n) by $a_n = p$ for all $n \in \mathbb{N}$. Clearly, each $a_n \in A$ since $p \in A$, and $\lim_{n\to\infty} a_n = p$ by a proof symmetric to that of Exercise 15.6a, as desired. If $p \in LP(A)$, then define $R_n = (p - \frac{1}{n}, p + \frac{1}{n})$ for all $n \in \mathbb{N}$. Since $p \in LP(A)$, we have by Definition 3.13 that $R_n \cap (A \setminus \{p\}) \neq \emptyset$ for all $n \in \mathbb{N}$. It follows by the axiom of choice that we can choose a point a_n in $R_n \cap (A \setminus \{p\})$ for all $n \in \mathbb{N}$. Thus, by Definitions 1.6 and 1.11, each $a_n \in A$ (as desired) and $a_n \in R_n$ for all $n \in \mathbb{N}$. We now seek to prove that (a_n) converges to p; to do so, Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - p| < \epsilon$. Let $\epsilon > 0$ be arbitrary. By Corollary 6.12, there exists an $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. Then by consecutive applications of Lemma 7.24, $\frac{1}{n} \leq \frac{1}{N}$. Consequently, by Script 1 as well as the definitions of R_n and R_N, $R_n \subset R_N$. It follows by Definition 1.3 since $a_n \in R_n$ that $a_n \in R_N$. Therefore, by Exercise 8.9, $|a_n - p| < \frac{1}{N} < \epsilon$, as desired.

Now suppose that there exists a sequence (a_n) with each $a_n \in A$ that converges to p. We divide into two cases ($p \in A$ and $p \notin A$). If $p \in A$, then by Definitions 1.5 and 4.4, $p \in \overline{A}$, as desired. If $p \notin A$, then to prove that $p \in \overline{A}$, Definitions 4.4 and 1.5 tell us that we must show that $p \in LP(A)$. To do so, Definition 3.13 tells us that it will suffice to verify that for all regions R containing p, $R \cap (A \setminus \{p\}) \neq \emptyset$. Let R be an arbitrary region R with $p \in R$. By Corollary 4.11 and 8.3, R is an open interval. Thus, by Definition 15.2, there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in R$. It follows that $a_N \in R$. Additionally, by hypothesis, $a_N \in A$. These results combined with the fact that $A = A \setminus \{p\}$ (since $p \notin A$) imply by Definition 1.6 that $a_N \in R \cap (A \setminus \{p\})$. Therefore, by Definition 1.8, $R \cap (A \setminus \{p\}) \neq \emptyset$, as desired. □

Definition 15.12. A sequence (a_n) is **bounded** if its image $\{a_n \mid n \in \mathbb{N}\}$ is bounded.

Theorem 15.13. *Every convergent sequence is bounded.*

Proof. Let (a_n) be a sequence that converges to p. To prove that (a_n) is bounded, Definitions 15.12 and 5.6 tell us that it will suffice to find numbers l, u such that $l \leq a_n \leq u$ for all a_n. Let (x, y) be a region that contains p. By Corollary 4.11 and Lemma 8.3, (x, y) is an open interval. Thus, by Exercise 15.3, we have that (x, y) contains all but finitely many terms of the sequence, i.e., $\{a_n \mid a_n \notin (x, y)\}$ is finite. We divide into two cases ($\{a_n \mid a_n \notin (x, y)\} = \emptyset$ and $\{a_n \mid a_n \notin (x, y) \neq \emptyset$). If $\{a_n \mid a_n \notin (x, y)\} = \emptyset$, then $a_n \in (x, y)$ for all a_n. It follows by Equations 8.1 that $x < a_n < y$ for all a_n. If we now choose $l = x$ and $u = y$, we can weaken the previous statement to $l = x \leq a_n \leq y = u$, as desired. On the other hand, if $\{a_n \mid a_n \notin (x, y)\} \neq \emptyset$, then by Lemma 3.4, $\{a_n \mid a_n \notin (x, y)\}$ has a first and a last point. It follows by Exercise 5.9 that $\{a_n \mid a_n \notin (x, y)\}$ is bounded by $\inf\{a_n \mid a_n \notin (x, y)\}$ and $\sup\{a_n \mid a_n \notin (x, y)\}$. Choose $l = \min(x, \inf\{a_n \mid a_n \notin (x, y)\})$ and $u = \max(y, \sup\{a_n \mid a_n \notin (x, y)\})$. Let a_n be an arbitrary term in the sequence. We divide into two subcases ($a_n \in (x, y)$ and $a_n \notin (x, y)$). If $a_n \in (x, y)$, then $l \leq x < a_n < y \leq u$, as desired. On the other hand, if $a_n \notin (x, y)$, then $l \leq \inf\{a_n \mid a_n \notin (x, y)\} \leq a_n \leq \sup\{a_n \mid a_n \notin (x, y)\} \leq u$, as desired. □

The converse is not true, but there are important partial converses. For the first, Theorem 15.14, we recall Definition 8.16 along with Definition 15.1, which say that (a_n) is an increasing sequence if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$, and (a_n) is decreasing if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. The definitions for strictly increasing/strictly decreasing are similar but with strict inequalities.

Theorem 15.14. *Every bounded increasing sequence converges to the supremum of its image. Every bounded decreasing sequence converges to the infimum of its image.*

Proof. We will only address the first part of the theorem; the proof of the second part is symmetric.

Let (a_n) be a bounded increasing sequence and let $p = \sup\{a_n \mid n \in \mathbb{N}\}$. To prove that (a_n) converges to p, Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - p| < \epsilon$. Let $\epsilon > 0$ be arbitrary. By Lemma 5.11, there exists $a_N \in \{a_n \mid n \in \mathbb{N}\}$ such that $p - \epsilon < a_N \leq p$. Choose N to be the natural number that generates a_N. Let n be an arbitrary natural number such that $n \geq N$. Then since $a_N \leq a_{N+1} \leq \cdots \leq a_{n-1} \leq a_n$, we have by transitivity that $a_N \leq a_n$. Additionally, since $a_n \in \{a_n \mid n \in \mathbb{N}\}$, we have by Definitions 5.7 and 5.6 that $a_n \leq p$. Thus, since $p - \epsilon < a_N \leq a_n \leq p < p + \epsilon$, we have by Equations 8.1 that $a_n \in (p - \epsilon, p + \epsilon)$. Therefore, by Exercise 8.9, $|a_n - p| < \epsilon$, as desired. □

To discuss the second partial converse, Theorem 15.18, we need another definition.

Definition 15.15. Let (a_n) be a sequence. A **subsequence** of (a_n) is a sequence $b : \mathbb{N} \to \mathbb{R}$ defined by the composition $b = a \circ i$, where $i : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function. If (a_n) has a subsequence with limit p, we call p a **subsequential limit** of (a_n).

We can write $b_k = a(i(k)) = a_{i(k)} = a_{i_k}$, so that (b_k) is the sequence $b_1, b_2, b_3, \ldots$, which is equal to the sequence $a_{i_1}, a_{i_2}, a_{i_3}, \ldots$, where $i_1 < i_2 < i_3 < \cdots$.

Theorem 15.16. *If (a_n) converges to p, then so do all of its subsequences.*

Proof. Let (b_n) be an arbitrary subsequence of (a_n). To prove that (b_n) converges to p, Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|b_n - p| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since (a_n) converges to p, Theorem 15.7 implies that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - p| < \epsilon$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. By Definition 15.15 and Script 1, $i(n) \geq n$. Therefore, we have by the above that $|b_n - p| = |a_{i_n} - p| < \epsilon$, as desired. □

5/18: **Exercise 15.17.** Construct a sequence with two subsequential limits. Construct a sequence with infinitely many subsequential limits.

Proof. Let (a_n) be a sequence defined by $a_n = (-1)^n$ for all $n \in \mathbb{N}$. Let (b_n) be a subsequence of (a_n) defined be defined by $b_n = a_{2n}$ for all $n \in \mathbb{N}$. Then by the proof of Exercise 15.6d, $b_n = 1$ for all $n \in \mathbb{N}$. It follows by the proof of Exercise 15.6a that $\lim_{n\to\infty} b_n = 1$. Similarly, if we let (c_n) be defined by $c_n = a_{2n+1}$, then $\lim_{n\to\infty} c_n = -1$.

Now let (a_n) be defined by $a_1 = 1$; $a_2 = 1$ and $a_3 = 2$; $a_4 = 1$, $a_5 = 2$, and $a_6 = 3$; $a_7 = 1$, $a_8 = 2$, $a_9 = 3$, and $a_{10} = 4$; and so on. Clearly there will be infinitely many terms that evaluate to each natural number in this sequence. Therefore, each of the infinitely many natural numbers is a subsequential limit of this sequence. As one example of this, consider the subsequence defined by $b_n = a_{\frac{1}{2}n^2 - \frac{1}{2}n + 1}$. This will include points $a_1, a_2, a_4, a_7, a_{11}, \dots$, which are all of the terms that equal 1. As such, by a proof symmetric to that used for Exercise 15.6a, we can prove that $\lim_{n\to\infty} b_n = 1$ is a subsequential limit of (a_n). ☐

Theorem 15.18. *Every bounded sequence has a convergent subsequence.*

Proof. Let (a_n) be an arbitrary bounded sequence, and define $A = \{a_n \mid n \in \mathbb{N}\}$. We divide into two cases (A is finite and A is infinite).

Suppose first that A is finite. Suppose for the sake of contradiction that for all $a \in A$, the set $\{n \in \mathbb{N} \mid a_n = a\}$ is finite. Then by Script 1, $\bigcup_{a\in A}\{n \in \mathbb{N} \mid a_n = a\} = \{n \in \mathbb{N} \mid a_n \in A\}$ is finite. But since $\{n \in \mathbb{N} \mid a_n \in A\} = \mathbb{N}$ by the definition of A, we have that $\mathbb{N}$ is finite, contradicting Definition 1.35's assertion that $\mathbb{N}$ is infinite.

Now suppose that A is infinite. By Definition 15.12, A is bounded. Additionally, by Script 6, A is a subset of $\mathbb{R}$. This combined with the previous result implies by Theorem 10.18 that there exists a limit point p of A. It follows by Definition 3.13 that for all regions R containing p, $R \cap (A \setminus \{p\}) \neq \emptyset$. Thus, since $R_n = (p - \frac{1}{n}, p + \frac{1}{n})$ is a region containing p for all $n \in \mathbb{N}$, we have that $R_n \cap (A \setminus \{p\}) \neq \emptyset$ for all $n \in \mathbb{N}$. We are now ready to define our subsequence (a_{i_n}). By the axiom of choice, choose $a_{i_n} \in (p - \frac{1}{n}, p + \frac{1}{n})$ where $i_{n-1} < i_n$ for all $n \in \mathbb{N}$ (for a_{i_1}, just choose any element of $(p-1, p+1)$). Note that we are guaranteed that there are elements within $(p - \frac{1}{n}, p + \frac{1}{n})$ arbitrarily far down the sequence (i.e., that we can choose a_{i_n} in $(p - \frac{1}{n}, p + \frac{1}{n})$ *with* $i_{n-1} < i_n$) since if there were not, that would imply that there are only finitely many terms of the sequence within $R_n \cap (A \setminus \{p\})$. But then in this case, we would be able to choose an R sufficiently small such that $R \cap (A \setminus \{p\}) = \emptyset$, a contradiction. Now to prove that (a_{i_n}) converges, Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_{i_n} - p| < \epsilon$. Let $\epsilon > 0$ be arbitrary. By Corollary 6.12, there exists an $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. It follows by the definition of (a_{i_n}) that $a_{i_n} \in (p - \frac{1}{n}, p + \frac{1}{n})$. Thus, since $(p - \frac{1}{n}, p + \frac{1}{n}) \subset (p - \frac{1}{N}, p + \frac{1}{N})$ by Script 1, we have by Definition 1.3 that $a_n \in (p - \frac{1}{N}, p + \frac{1}{N})$. Therefore, by Exercise 8.9, we have that $|a_n - p| < \frac{1}{N} < \epsilon$, as desired. ☐

We are now able to prove a useful characterization of convergent sequences.

Theorem 15.19. *A sequence (a_n) of real numbers converges if and only if for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for all $n, m \geq N$.*

Proof. Suppose first that (a_n) converges to p. Let $\epsilon > 0$ be arbitrary. Then by Theorem 15.7, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - p| < \frac{\epsilon}{2}$. Choose this N to be our N. Let n, m be arbitrary natural numbers such that $n, m \geq N$. Then $|a_n - p| < \frac{\epsilon}{2}$ and $|a_m - p| < \frac{\epsilon}{2}$. Therefore,

$$\begin{aligned}
|a_n - a_m| &\leq |a_n - p| + |p - a_m| && \text{Lemma 8.8} \\
&= |a_n - p| + |a_m - p| && \text{Exercise 8.5} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

as desired.

Now suppose that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for all $n, m \geq N$. To prove that (a_n) converges, we will first show that it is bounded. It will then follow by Theorem 15.18 that (a_n)

has a subsequence (b_n) that converges to p. The existence of (b_n) combined with the hypothesis will suffice to show that (a_n) converges to p. Let's begin.

To confirm that (a_n) is bounded, Definitions 15.12 and 5.6 tell us that it will suffice to find numbers l, u such that $l \leq a_n \leq u$ for all a_n. Since $1 > 0$ by Corollary 7.27, we have by the hypothesis that there is some $N \in \mathbb{N}$ such that $|a_n - a_m| < 1$ for all $n, m \geq N$. Thus, we know that $|a_N - a_n| < 1$ for all $n \geq N$. Consequently, by Exercise 8.9, $a_n \in (a_N - 1, a_N + 1)$ for all $n \geq N$. It follows by Script 1 that $\{a_n \mid a_n \notin (a_N - 1, a_N + 1)$ is finite, since it can contain at most $N - 1$ terms. If we now divide into two cases and evaluate them in a symmetric fashion to the way we did in the proof of Theorem 15.13, we can establish that (a_n) is bounded, as desired.

Since (a_n) is bounded, we have by Theorem 15.18 that there exists a convergent subsequence (b_n) of (a_n). Let $\lim_{n\to\infty} b_n = p$.

To prove that (a_n) converges to p, Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - p| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since (b_n) converges to p, Theorem 15.7 asserts that there is some $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $|b_n - p| < \frac{\epsilon}{2}$. Additionally, we have by the hypothesis that there is some $N_2 \in \mathbb{N}$ such that $|a_n - a_m| < \frac{\epsilon}{2}$ for all $n, m \geq N_2$. Choose $N = \max(N_1, N_2)$. Let n be an arbitrary natural number such that $n \geq N$. By Corollary 6.13, there exists a $b_m = a_{i_m}$ with $i_m \geq N$. Thus, since $i_m \geq N \geq N_1$, we have that $|a_{i_m} - p| < \frac{\epsilon}{2}$. Additionally, since $n \geq N \geq N_2$ and $i_m \geq N \geq N_2$, we have that $|a_n - a_{i_m}| < \frac{\epsilon}{2}$. Therefore,

$$\begin{aligned} |a_n - p| &\leq |a_n - a_{i_m}| + |a_{i_m} - p| && \text{Lemma 8.8} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

as desired. □

Theorem 15.20. *Use Theorem 15.19 to show that the sequence in Exercise 15.10a does not converge.*

Proof. Suppose for the sake of contradiction that $a_n = (-1)^n \cdot n$ converges to some $p \in \mathbb{R}$. Then if we choose $\epsilon = 1$, we have by Theorem 15.19 that there exists some $N \in \mathbb{N}$ such that $|a_n - a_m| < 1$ for all $n, m \geq N$. But we also have that $|a_N - a_{N+1}| = 2N + 1 > 1$, regardless of which natural number N is, a contradiction. □

15.2 Discussion

5/6:
- Using Lemma 8.10 in Exercise 15.3 is overly strong; use Theorem 4.10 instead.

5/11:
- Exercise 15.8b requires that we consider the case where $x = 0$ and where $x \neq 0$.
- In Theorem 15.9, use a and b instead of p and q.
  - Put $(a_n + b_n)$ and derivatives in parentheses.
  - For part c, we must choose N such that $b_n \neq 0$ for all b_n past N. Just choose a point $q < y$ and consider the region $(0, y)$ (symmetric if q is negative). Let M be a bound on all values of a_n, b_n past N including a, b. Then if we assert $|a_n - a| < \frac{\epsilon}{2M}$ and $|b_n - b| < \frac{\epsilon}{2M}$, we have

$$\begin{aligned} |a_n b_n - ab| &= |a_n(b_n - b) + b(a_n - a)| \\ &\leq |a_n| \cdot |b_n - b| + |b| \cdot |a_n - a| \\ &\leq |M| \cdot |b_n - b| + |M| \cdot |a_n - a| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

- For part (c): Let M be a positive upper bound on both sequences, m be a positive lower bound on $|b_n|$ and q.

$$\begin{aligned}
\left|\frac{a_n}{b_n} - \frac{p}{q}\right| &= \left|\frac{a_n q - b_n p}{b_n q}\right| \\
&= \frac{|q(a_n - p) + p(q - b_n)|}{|b_n| \cdot |q|} \\
&\leq \frac{|q| \cdot |a_n - p| + |p| \cdot |q - b_n|}{|b_n| \cdot |q|} \\
&\leq \frac{M \cdot |a_n - p| + M \cdot |q - b_n|}{m \cdot m} \\
&= \frac{M}{m^2} \cdot |a_n - p| + \frac{M}{m^2} \cdot |b_n - q| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

- We should draw upon both Theorem 11.9 and Exercise 13.9 to prove this result.

5/13:

- Label by letter the results from Exercise 15.6.
- Additional exercise 11.1 is the squeeze theorem (for 15.10b?).
- Cite Theorem 15.13 ahead of time?
  - This is fine.
- Call the image $a(\mathbb{N})$?
  - This is not.
- Script 14 journals due Monday.

5/18:

- Rigor in Exercise 15.17?
  - Give an example of a subsequence in the closed form.
- How rigorously do you want us to show that a subsequence is a subsequence?
  - You need to make sure your sequence is in order, not just like a_4 then a_7 then a_3.
  - Something off of the well-ordering principle.
  - Not using theorem 15.11 but just always choosing something closer to p. Always choose a point within $\frac{1}{n}$ of p.
- Alternate Exercise 15.17:
  - Let $\mathbb{Q}^+$ denote the set of nonnegative rational numbers. Thus, $\mathbb{Q}^+ \subset \mathbb{Q}$. Theorem 2.11: $\mathbb{Q}$ is countable. Exercise 1.37: $\mathbb{Q}^+$ is countable. Definition 1.35 and 1.28: There exists a bijection $f : \mathbb{N} \to \mathbb{Q}^+$. Define $g : \mathbb{Q}^+ \to \mathbb{N}$ by $g([0,1)) = 1$, and recursively: $g([p-1,p)) = g(p-2)+1$ for all $p \in \mathbb{N}$. Better yet, define $g([p-1,p)) = p$ for all $p \in \mathbb{N}$. $a = g \circ f : \mathbb{N} \to \mathbb{N}$ is a sequence that has infinitely many 1's, 2's, 3's, etc.

Script 16

Series

16.1 Journal

5/20: **Definition 16.1.** Let $N_0 \in \mathbb{N} \cup \{0\}$ and let $(a_n)_{n=N_0}^{\infty}$ be a sequence of real numbers. Then the formal sum

$$\sum_{n=N_0}^{\infty} a_n$$

is called an **infinite series**. (In most instances, we will start the series at $N_0 = 0$ or $N_0 = 1$.)

We will define the **sequence of partial sums** (p_n) of the series by

$$p_n = a_{N_0} + \cdots + a_{N_0+n-1} = \sum_{i=N_0}^{N_0+n-1} a_i$$

Thus, p_n is the sum of the first n terms in the sequence (a_n). We say that the series **converges** if there exists $L \in \mathbb{R}$ such that $\lim_{n\to\infty} p_n = L$. When this is the case, we write this as

$$\sum_{n=N_0}^{\infty} a_n = L$$

and we say that L is the **sum** of the series. When there does not exist such an L, we say that the series **diverges**.

Lemma 16.2. *Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers. Let $N_0 \in \mathbb{N}$. Then $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=N_0}^{\infty} a_n$ converges.*

Lemma. *Let $n \in \mathbb{N}$. Then*

$$\sum_{i=0}^{N_0+n-1} a_i = \sum_{i=0}^{N_0-1} a_i + \sum_{i=N_0}^{N_0+n-1} a_i$$

Proof. This simple result follows immediately from Script 0, so no formal proof will be given. □

Proof of Lemma 16.2. Suppose first that $\sum_{n=0}^{\infty} a_n$ converges, and let $M = \sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=0}^{n-1} a_i$, where the latter equality holds by Definition 16.1. To prove that $\sum_{n=N_0}^{\infty} a_n$ converges, Definition 16.1 tells us that it will suffice to find an $L \in \mathbb{R}$ such that $\lim_{n\to\infty} \sum_{i=N_0}^{N_0+n-1} a_i = L$. Choose $L = M - \sum_{i=0}^{N_0-1} a_i$. To verify that $\lim_{n\to\infty} \sum_{i=N_0}^{N_0+n-1} a_i = L$, Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|\sum_{i=N_0}^{N_0+n-1} a_i - L| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} \sum_{i=0}^{n-1} a_i = M$, Theorem 15.7 implies that there is some $N \in \mathbb{N}$ such that for all $n \geq N$,

$|\sum_{i=0}^{n-1} a_i - M| < \epsilon$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. Since $N_0 + n > n \geq N$, we have by the above that $|\sum_{i=0}^{N_0+n-1} a_i - M| < \epsilon$. Therefore,

$$\begin{aligned}
\left|\sum_{i=N_0}^{N_0+n-1} a_i - L\right| &= \left|\sum_{i=0}^{N_0+n-1} a_i - \sum_{i=0}^{N_0-1} a_i - L\right| && \text{The Lemma} \\
&= \left|\sum_{i=0}^{N_0+n-1} a_i - \left(\sum_{i=0}^{N_0-1} a_i + L\right)\right| \\
&= \left|\sum_{i=0}^{N_0+n-1} a_i - M\right| \\
&< \epsilon
\end{aligned}$$

as desired.

The proof is symmetric in the other direction. □

Exercise 16.3. Prove that $\sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+1})$ converges. What is its sum?

Proof. Let (a_n) be defined by $a_n = \frac{1}{n} - \frac{1}{n+1}$, and let (p_n) be defined by $p_n = \sum_{i=1}^{n} a_i$. Then

$$\begin{aligned}
p_n &= a_1 + a_2 + \cdots + a_n \\
&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \frac{1}{1} - \frac{1}{n+1}
\end{aligned}$$

To prove that $\sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+1}) = 1$, Definition 16.1 tells us that it will suffice to show that $\lim_{n\to\infty} p_n = 1$. By a proof symmetric to that of Exercise 15.6a, we have that $\lim_{n\to\infty} 1 = 1$. By a proof symmetric to that of Exercise 15.6c, we have that $\lim_{n\to\infty} \frac{1}{n+1} = 0$. Therefore, by Theorem 15.9 and the above, we have that

$$\begin{aligned}
\lim_{n\to\infty} p_n &= \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) \\
&= \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n+1} \\
&= 1 - 0 \\
&= 1
\end{aligned}$$

as desired. □

Theorem 16.4. *If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.*

Proof. To prove that $\lim_{n\to\infty} a_n = 0$, Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - 0| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\sum_{n=1}^{\infty} a_n$ converges, we have by Theorem 15.19 that there exists an $N \in \mathbb{N}$ such that $|\sum_{i=1}^{n} a_i - \sum_{i=1}^{m} a_i| < \epsilon$ for all $n, m \geq N$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. Then choosing $n, n-1 \geq N$, we have by the above that $|\sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} a_i| < \epsilon$. Therefore,

$$\begin{aligned}
|a_n - 0| &= |a_n| \\
&= \left|\sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} a_i\right| \\
&< \epsilon
\end{aligned}$$

as desired. □

The converse of this theorem, however, is not true, as we see in Theorem 16.6.

Theorem 16.5. *A series* $\sum_{n=1}^{\infty} a_n$ *converges if and only if for all* $\epsilon > 0$*, there is some* $N \in \mathbb{N}$ *such that* $|\sum_{k=m+1}^{n} a_k| < \epsilon$ *for all* $n > m \geq N$*.*

Proof. Suppose first that $\sum_{n=1}^{\infty} a_n$ converges. Let $\epsilon > 0$ be arbitrary. By Definition 16.1, (p_n) converges. Thus, by Theorem 15.19, there is some $N \in \mathbb{N}$ such that $|p_n - p_m| < \epsilon$ for all $n, m \geq N$. Choose this N to be our N. Let n, m be two arbitrary natural numbers satisfying $n > m \geq N$. Therefore,

$$\begin{aligned} \left|\sum_{k=m+1}^{n} a_k\right| &= \left|\sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k\right| \\ &= |p_n - p_m| \\ &< \epsilon \end{aligned}$$

as desired.

The proof is symmetric in the other direction. □

Theorem 16.6. *The series* $\sum_{n=1}^{\infty} \frac{1}{n}$ *diverges.*

Lemma. *For all* $N \in \mathbb{N}$*, we have*

$$\sum_{n=N+1}^{2N} \frac{1}{n} \geq \frac{1}{2}$$

Proof. We induct on N. For the base case $N = 1$, we have

$$\sum_{n=1+1}^{2\cdot 1} \frac{1}{n} = \frac{1}{2} \geq \frac{1}{2}$$

as desired. Now suppose inductively that we have proven the claim for N. To prove it for $N + 1$, we do the following.

$$\begin{aligned} \sum_{n=N+2}^{2N+2} \frac{1}{n} &= \sum_{n=N+1}^{2N} \frac{1}{n} - \frac{1}{N+1} + \frac{1}{2N+1} + \frac{1}{2(N+1)} \\ &= \sum_{n=N+1}^{2N} \frac{1}{n} + \frac{1}{2(N+1)(2N+1)} \\ &> \sum_{n=N+1}^{2N} \frac{1}{n} \\ &\geq \frac{1}{2} \end{aligned}$$

as desired. □

Proof of Theorem 16.6. To prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, Theorem 16.5 tells us that it will suffice to find an $\epsilon > 0$ such that for all $N \in \mathbb{N}$, there exist $n > m \geq N$ with $|\sum_{k=m+1}^{n} 1/k| \geq \epsilon$. Choose $\epsilon = \frac{1}{2}$. Let N be an arbitrary element of N. If we now choose $n = 2N$ and $m = N$, we will have $n > m \geq N$. It will follow by the lemma that

$$\begin{aligned} \left|\sum_{k=m+1}^{n} \frac{1}{k}\right| &= \left|\sum_{k=N+1}^{2N} \frac{1}{k}\right| \\ &\geq \frac{1}{2} \\ &= \epsilon \end{aligned}$$

as desired. □

5/25: **Theorem 16.7.** *Let* $-1 < x < 1$. *Then*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Proof. Let (a_n) be defined by $a_n = x^n$. Then $p_n = x^0 + x^1 + \cdots + x^{n-1}$ so that

$$\begin{aligned}
p_n - xp_n &= x^0 + \cdots + x^{n-1} - x(x^0 + \cdots + x^{n-1}) \\
p_n - xp_n &= 1 - x^n \\
p_n(1-x) &= 1 - x^n \\
p_n &= \frac{1-x^n}{1-x}
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
\frac{1}{1-x} &= \frac{1-0}{1-x} \\
&= \frac{1 - \lim_{n\to\infty} x^n}{1-x} && \text{Exercise 15.8b} \\
&= \lim_{n\to\infty} \frac{1-x^n}{1-x} && \text{Theorem 15.9} \\
&= \lim_{n\to\infty} p_n \\
&= \sum_{n=0}^{\infty} x^n && \text{Definition 16.1}
\end{aligned}$$

as desired. □

Theorem 16.8. *If* $\sum_{n=1}^{\infty} a_n = L$, $\sum_{n=1}^{\infty} b_n = M$, *and* $c \in \mathbb{R}$, *then*

$$\sum_{n=1}^{\infty} (a_n + b_n) = L + M$$

$$\sum_{n=1}^{\infty} (c \cdot a_n) = c \cdot L$$

Proof. For the first claim, we have that

$$\begin{aligned}
L + M &= \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} a_i + \lim_{n\to\infty} \sum_{i=1}^{n} b_i && \text{Definition 16.1} \\
&= \lim_{n\to\infty} \left(\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \right) && \text{Theorem 15.9} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} (a_i + b_i) \\
&= \sum_{n=1}^{\infty} (a_n + b_n) && \text{Definition 16.1}
\end{aligned}$$

The proof is symmetric for the second claim. □

Definition 16.9. We say that the series $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

Lemma 16.10. *A series $\sum_{n=1}^{\infty} a_n$ with all $a_n \geq 0$ converges if and only if its sequence of partial sums is bounded.*

Proof. Suppose first that the series $\sum_{n=1}^{\infty} a_n$ with all $a_n \geq 0$ converges. Then by Definition 16.1 its sequence of partial sums (p_n) converges. Therefore, by Theorem 15.13, (p_n) is bounded, as desired.

Now suppose that the sequence of partial sums (p_n) corresponding to a series $\sum_{n=1}^{\infty} a_n$ with all $a_n \geq 0$ is bounded. To prove that $\sum_{n=1}^{\infty} a_n$ converges, Definition 16.1 tells us that it will suffice to show that (p_n) converges. To do so, Theorem 15.14 tells us that it will suffice to verify in addition to the fact that (p_n) is bounded that (p_n) is increasing. To do this, Script 15 tells us that it will suffice to confirm that $p_n \leq p_{n+1}$ for all $n \in \mathbb{N}$. Let n be an arbitrary natural number. By Definition 16.1, $p_{n+1} - p_n = a_{n+1}$. Since $a_{n+1} \geq 0$ by hypothesis, we have by transitivity that $p_{n+1} - p_n \geq 0$, i.e., $p_n \leq p_{n+1}$ by Definition 7.21, as desired. ☐

Theorem 16.11. *If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges and*

$$\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|$$

Proof. To prove that $\sum_{n=1}^{\infty} a_n$ converges, Theorem 16.5 tells us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|\sum_{k=m+1}^{n} a_k| < \epsilon$ for all $n > m \geq N$. Let $\epsilon > 0$ be arbitrary. Since $\sum_{n=1}^{\infty} a_n$ converges absolutely by hypothesis, we have by Definition 16.9 that $\sum_{n=1}^{\infty} |a_n|$ converges. Thus, by Theorem 16.5, there is some $N \in \mathbb{N}$ such that $|\sum_{k=m+1}^{n} |a_k|| < \epsilon$ for all $n > m \geq N$. Choose this N to be our N. Let n, m be arbitrary natural numbers such that $n > m \geq N$. Therefore,

$$\begin{aligned}
\left|\sum_{k=m+1}^{n} a_k\right| &\leq \sum_{k=m+1}^{n} |a_k| && \text{Lemma 8.8} \\
&= \left|\sum_{k=m+1}^{n} |a_k|\right| \\
&< \epsilon
\end{aligned}$$

as desired.

As to the other part of the claim, to begin, let (b_n) and (c_n) be defined by $b_n = \max(0, a_n)$ and $c_n = \min(0, a_n)$. We will prove a few preliminary results with these definitions that will enable us to tackle the big inequality.

To confirm that $a_n = b_n + c_n$, we divide into two cases ($a_n \geq 0$ and $a_n < 0$). If $a_n \geq 0$, then by their definitions, $b_n = a_n$ and $c_n = 0$. Thus, $a_n = b_n + c_n$ as desired. The argument is symmetric in the other case.

To confirm that $|\sum_{n=1}^{\infty} b_n| + |\sum_{n=1}^{\infty} -c_n| = |\sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} -c_n|$, we can acknowledge that $b_n \geq 0$ and $-c_n \geq 0$ for all $n \in \mathbb{N}$ to demonstrate that

$$\begin{aligned}
\left|\sum_{n=1}^{\infty} b_n\right| + \left|\sum_{n=1}^{\infty} -c_n\right| &= \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} -c_n \\
&= \left|\sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} -c_n\right|
\end{aligned}$$

To confirm that $|a_n| = b_n - c_n$, we divide into two cases ($a_n \geq 0$ and $a_n < 0$). If $a_n \geq 0$, then $b_n = a_n$ and $c_n = 0$. Thus, by Definition 8.4, $|a_n| = a_n = b_n - c_n$, as desired. On the other hand, if $a_n < 0$, then $b_n = 0$ and $c_n = a_n$. Thus, by Definition 8.4 again, $|a_n| = -a_n = b_n - c_n$, as desired.

Having established that $a_n = b_n + c_n$, $|\sum_{n=1}^{\infty} b_n| + |\sum_{n=1}^{\infty} -c_n| = |\sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} -c_n|$, and $|a_n| =$

$b_n - c_n$, we have that

$$\begin{aligned}
\left|\sum_{n=1}^{\infty} a_n\right| &= \left|\sum_{n=1}^{\infty}(b_n + c_n)\right| \\
&= \left|\sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} c_n\right| && \text{Theorem 16.8} \\
&\leq \left|\sum_{n=1}^{\infty} b_n\right| + \left|\sum_{n=1}^{\infty} c_n\right| && \text{Lemma 8.8} \\
&= \left|\sum_{n=1}^{\infty} b_n\right| + \left|-\sum_{n=1}^{\infty} c_n\right| && \text{Exercise 8.5} \\
&= \left|\sum_{n=1}^{\infty} b_n\right| + \left|\sum_{n=1}^{\infty} -c_n\right| && \text{Theorem 16.8} \\
&= \left|\sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} -c_n\right| \\
&= \left|\sum_{n=1}^{\infty}(b_n - c_n)\right| && \text{Theorem 16.8} \\
&= \left|\sum_{n=1}^{\infty} |a_n|\right| \\
&= \sum_{n=1}^{\infty} |a_n|
\end{aligned}$$

as desired. □

Theorem 16.12. *Let (a_n) be a decreasing sequence of positive numbers such that $\lim_{n\to\infty} a_n = 0$. Then $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.*

Lemma.

(*a*) *For any natural numbers n, m satisfying $n > m$, we have*

$$\left|\sum_{k=m+1}^{n}(-1)^{k+1}a_k\right| = |a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + a_{m+5} - \cdots \pm a_n|$$

Proof. By Script 0, either $|\sum_{k=m+1}^{n}(-1)^{k+1}a_k| = |a_{m+1} - a_{m+2} + a_{m+3} - \cdots \pm a_n|$ or $|\sum_{k=m+1}^{n}(-1)^{k+1}a_k| = |-a_{m+1} + a_{m+2} - a_{m+3} + \cdots \pm a_n|$. However, by Exercise 8.5, the two results are equal. Thus, we may choose the former WLOG. □

(*b*) *We have*

$$0 \leq a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots \pm a_n$$

Proof. Since (a_n) is a decreasing sequence, we have by Script 15 that $a_{i+1} \leq a_i$ for all $i \in \mathbb{N}$. It follows by Definition 7.21 that $0 \leq a_i - a_{i+1}$ for all $i \in \mathbb{N}$. We now divide into two cases (there are an even number of terms in the sum $a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + a_{m+5} - \cdots \pm a_n$ and there are an odd number of terms in said sum). In the first case, we have that the sum is of the form $a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots + a_{n-1} - a_n$. Thus, since $0 \leq a_{m+1} - a_{m+2}$, $0 \leq a_{m+3} - a_{m+4}$, and on and on, we have by Script 7 that $0 \leq a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots - a_n$, as desired. On the other hand, in the second case, we have that the sum is of the form $a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots + a_{n-2} - a_{n-1} + a_n$. For the same reason as before, we have that $0 \leq a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots + a_{n-2} - a_{n-1}$. However, we need the additional hypothesis that every a_i is positive, i.e., $a_i \geq 0$ for all $i \in \mathbb{N}$ to know that $0 \leq a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots + a_n$, as desired. □

(c) *For all $i \in \mathbb{N}$, we have*

$$-a_i + a_{i+1} \leq 0$$

Proof. Since (a_n) is a decreasing sequence, we have by Script 15 that $a_{i+1} \leq a_i$ for all $i \in \mathbb{N}$. It follows by Definition 7.21 that $-a_i + a_{i+1} \leq 0$ for all $i \in \mathbb{N}$. $\square$

Proof of Theorem 16.12. To prove that $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges, Theorem 16.5 tells us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|\sum_{k=m+1}^{n}(-1)^{k+1}a_k| < \epsilon$ for all $n > m \geq N$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} a_n = 0$ by hypothesis, we have by Theorem 15.7 that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - 0| = |a_n| < \epsilon$, as desired. Choose this N to be our N. Let n, m be arbitrary natural numbers such that $n > m \geq N$.

We divide into two cases ($n-m$ is even [i.e., there are an even number of terms in the sum $\sum_{k=m+1}^{n}(-1)^{k+1}a_k$] and $n-m$ is odd [i.e., there are an odd number of terms in the sum $\sum_{k=m+1}^{n}(-1)^{k+1}a_k$]). If $n-m$ is even, then we have

$$\begin{aligned}
\left|\sum_{k=m+1}^{n}(-1)^{k+1}a_k\right| &= |a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + a_{m+5} - \cdots - a_{n-2} + a_{n-1} - a_n| && \text{Lemma (a)}\\
&= a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + a_{m+5} - \cdots - a_{n-2} + a_{n-1} - a_n && \text{Lemma (b) \& Definition 8.4}\\
&= a_{m+1} + (-a_{m+2} + a_{m+3}) + (-a_{m+4} + a_{m+5}) + \cdots + (-a_{n-2} + a_{n-1}) - a_n\\
&\leq a_{m+1} + (-a_{m+4} + a_{m+5}) + \cdots + (-a_{n-2} + a_{n-1}) - a_n && \text{Lemma (c)}\\
&\leq a_{m+1} + \cdots + (-a_{n-2} + a_{n-1}) - a_n && \text{Lemma (c)}\\
&\vdots && \text{Lemma (c)}\\
&\leq a_{m+1} + (-a_{n-2} + a_{n-1}) - a_n && \text{Lemma (c)}\\
&\leq a_{m+1} - a_n && \text{Lemma (c)}\\
&\leq a_{m+1}\\
&= |a_{m+1}|\\
&< \epsilon
\end{aligned}$$

The argument is symmetric if $n-m$ is odd. $\square$

5/27: The following theorem will be useful to prove more specialized tests for convergence of series.

Theorem 16.13. *Let (c_n) be a sequence of positive numbers and let (a_n) be a sequence such that $|a_n| \leq c_n$ for all $n \geq N_0$, where N_0 is some fixed integer. If $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*

Proof. To prove that $\sum_{n=1}^{\infty} a_n$ converges, Theorem 16.11 and Definition 16.9 tell us that it will suffice to show that $\sum_{n=1}^{\infty} |a_n|$ converges. To do this, Theorem 16.5 tells us that it will suffice to verify that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|\sum_{k=m+1}^{n} |a_k|| < \epsilon$ for all $n > m \geq N$. Let $\epsilon > 0$ be arbitrary. Since $\sum_{n=1}^{\infty} c_n$ converges by hypothesis, we have by Theorem 16.5 that there is some $N_1 \in \mathbb{N}$ such that $|\sum_{k=m+1}^{n} c_k| < \epsilon$ for all $n > m \geq N_1$. Choose $N = \max(N_0, N_1)$. Let n, m be arbitrary natural numbers such that $n > m \geq N$. Since $c_k \geq 0$ for all $k \in \mathbb{N}$ by hypothesis, it follows by Script 7 that $\sum_{k=m+1}^{n} c_k \geq 0$. Thus, Definition 8.4 implies that $\sum_{k=m+1}^{n} c_k = |\sum_{k=m+1}^{n} c_k| < \epsilon$. Similarly, we have that $|\sum_{k=m+1}^{n} |a_k|| = \sum_{k=m+1}^{n} |a_k|$. Lastly, since we know by hypothesis that $|a_k| \leq c_k$ for all $k \geq N_0$, i.e., for all $k \geq N$, Script 7 asserts that $\sum_{k=m+1}^{n} |a_k| \leq \sum_{k=m+1}^{n} c_n$. Therefore, combining the last three results, we have that $|\sum_{k=m+1}^{n} |a_k|| = \sum_{k=m+1}^{n} |a_k| \leq \sum_{k=m+1}^{n} c_n < \epsilon$, as desired. $\square$

Lemma 16.14. *Suppose that (b_n) is a sequence of nonnegative numbers with $\lim_{n\to\infty} b_n = L$, where $L < 1$. Then there is some $N \in \mathbb{N}$ such that $0 \leq b_n < \frac{1+L}{2}$ for all $n \geq N$.*

Proof. Choose $\epsilon = \frac{1-L}{2}$; since $L < 1$, we have that $0 < 1 - L$, i.e., $0 < \frac{1-L}{2}$ as needed. It follows by Theorem 15.7 (since $\lim_{n\to\infty} b_n = L$ by hypothesis) that there is some $N \in \mathbb{N}$ such that $|b_n - L| < \frac{1-L}{2}$ for all $n \geq N$. Choose this n to be our N. Let n be an arbitrary natural number such that $n \geq N$. Then

$$\begin{aligned} |b_n - L| &< \frac{1-L}{2} \\ -\frac{1-L}{2} < b_n - L &< \frac{1-L}{2} && \text{Lemma, Exercise 8.9} \\ b_n &< \frac{1+L}{2} \end{aligned}$$

Additionally, since (b_n) is a sequence of nonnegative numbers, $0 \leq b_n$. Therefore, combining the last two results, we have that $0 \leq b_n < \frac{1+L}{2}$, as desired. ☐

Theorem 16.15. *Let (a_n) be a sequence such that $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$ exists. Then*

(a) *If $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.*

Proof. Let $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L$. To prove that $\sum_{n=1}^{\infty} a_n$ converges, Theorem 16.13 tells us that it will suffice to find a sequence (c_n) of positive numbers such that $|a_n| \leq c_n$ for all $n \geq N_0$, where N_0 is some fixed integer, for which $\sum_{n=1}^{\infty} c_n$ converges. To begin, since $(|\frac{a_{n+1}}{a_n}|)$ is a sequence of nonnegative numbers with $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L$, where $L < 1$, we have by Lemma 16.14 that there exists an $N_0 \in \mathbb{N}$ such that $0 \leq |\frac{a_{n+1}}{a_n}| < \frac{1+L}{2}$ for all $n \geq N_0$. With this result, we can define (c_n) by $c_n = (\frac{1+L}{2})^{n-N_0} \cdot |a_{N_0}|$ for all $n \in \mathbb{N}$. We will now prove that (c_n) satisfies the necessary properties outlined in the beginning.

First, we must confirm that (c_n) is a sequence of positive numbers, i.e., that $c_n \geq 0$ for all $n \in \mathbb{N}$. Let n be an arbitrary natural number. By the above result from Lemma 16.14 and transitivity, we know that $0 < \frac{1+L}{2}$. It follows by Script 7 that $0 < (\frac{1+L}{2})^{n-N_0}$. Therefore, by Definition 8.4 and Script 7, $0 \leq (\frac{1+L}{2})^{n-N_0} \cdot |a_{N_0}| = c_n$, as desired.

To confirm that $|a_n| \leq c_n$ for all $n \geq N_0$, we induct on n using Additional Exercise 0.2a. For the base case $n = N_0$, we have that

$$|a_{N_0}| = 1 \cdot |a_{N_0}| = \left(\frac{1+L}{2}\right)^{N_0 - N_0} \cdot |a_{N_0}| = c_{N_0}$$

which we may weaken to $|a_{N_0}| \leq c_{N_0}$, as desired. Now suppose inductively that we have demonstrated that $|a_n| \leq c_n$; we now seek to demonstrate that $|a_{n+1}| \leq c_{n+1}$. By hypothesis, $n \geq N_0$, so by the above, we have that $0 \leq |\frac{a_{n+1}}{a_n}| < \frac{1+L}{2}$. It follows by Script 7 that $|a_{n+1}| < \frac{1+L}{2} \cdot |a_n|$. Therefore,

$$\begin{aligned} |a_{n+1}| &< \frac{1+L}{2} \cdot |a_n| \\ &\leq \frac{1+L}{2} \cdot c_n \\ &= \frac{1+L}{2} \cdot \left(\frac{1+L}{2}\right)^{n-N_0} \cdot |a_{N_0}| \\ &= \left(\frac{1+L}{2}\right)^{(n+1)-N_0} \cdot |a_{N_0}| \\ &= c_{n+1} \end{aligned}$$

which we may weaken to $|a_{n+1}| \leq |c_{n+1}|$, as desired.

Lastly, we must confirm that $\sum_{n=1}^{\infty} c_n$ converges. By Script 7, it follows from the hypothesis that $L < 1$ that $1 + L < 2$, which in turn implies that $\frac{1+L}{2} < 1$. Additionally, the above result that $0 < \frac{1+L}{2}$ implies by transitivity that $-1 < \frac{1+L}{2}$. These last two results when combined imply

$\sum_{n=0}^{\infty}(\frac{1+L}{2})^n$ satisfies the constraints of Theorem 16.7, meaning that $\sum_{n=0}^{\infty}(\frac{1+L}{2})^n$ converges. Thus, by Script 0, $\sum_{n=N_0}^{\infty}(\frac{1+L}{2})^{n-N_0}$ converges. Consequently, by consecutive applications of Lemma 16.2, $\sum_{n=1}^{\infty}(\frac{1+L}{2})^{n-N_0}$ converges. It follows by Theorem 16.8 that $\sum_{n=1}^{\infty}(\frac{1+L}{2})^{n-N_0}\cdot|a_{N_0}|$ converges. Therefore, by the definition of c_n, $\sum_{n=1}^{\infty}c_n$ converges, as desired. ☐

(b) *If* $\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}|>1$, *then* $\sum_{n=1}^{\infty}a_n$ *diverges.*

Proof. Let $\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}|=L$. Suppose for the sake of contradiction that $\lim_{n\to\infty}a_n=0$. Then by Theorem 15.7, for all $\epsilon_1>0$, there is some $N_1\in\mathbb{N}$ such that for all $n\geq N_1$, we have $|a_n|=|a_n-0|<\epsilon_1$. Similarly, since $\lim_{n\to\infty}|\frac{a_{n+1}}{a_n}|=L$, we have that for all $\epsilon_2>0$, there is some $N_2\in\mathbb{N}$ such that for all $n\geq N_2$, we have $||\frac{a_{n+1}}{a_n}|-L|<\epsilon_2$. Choose $\epsilon_2=L-1$ (it follows from the fact that $L>1$ by Definition 7.21 that $L-1>0$). Thus, we have that for all $n\geq N_2$,

$$\begin{aligned}
1 &= 1+L-L+\left|\left|\frac{a_{n+1}}{a_n}\right|\right|-\left|\left|\frac{a_{n+1}}{a_n}\right|\right| \\
&= |L|-\left|\left|\frac{a_{n+1}}{a_n}\right|\right|+1-L+\left|\left|\frac{a_{n+1}}{a_n}\right|\right| \\
&\leq \left|L-\left|\frac{a_{n+1}}{a_n}\right|\right|+1-L+\left|\left|\frac{a_{n+1}}{a_n}\right|\right| \\
&= \left|\left|\frac{a_{n+1}}{a_n}\right|-L\right|+1-L+\left|\left|\frac{a_{n+1}}{a_n}\right|\right| && \text{Exercise 8.5} \\
&< L-1+1-L+\left|\left|\frac{a_{n+1}}{a_n}\right|\right| \\
&= \left|\frac{a_{n+1}}{a_n}\right|
\end{aligned}$$

which can be rearranged by Lemma 7.24 to demonstrate that $|a_n|<|a_{n+1}|$ for all such n. Now consider the case where $\epsilon_1=|a_{N_2+1}|$ (note that $|a_{N_2+1}|>|a_{N_2}|\geq 0$ by Definition 8.4). Choose $N=\max(N_1,N_2+2)$. Then by the above, we have by transitivity that $|a_N|>|a_{N-1}|>\cdots>|a_{N_2+1}|$. However, since $N\geq N_1$, we also have that $|a_N|<\epsilon_1=|a_{N_2+1}|$, a contradiction. Therefore, since $\lim_{n\to\infty}a_n\neq 0$, we have by the contrapositive of Theorem 16.4 that $\sum_{n=1}^{\infty}a_n$ diverges. ☐

Theorem 16.16. *Let* (a_n) *be a sequence such that* $\lim_{n\to\infty}\sqrt[n]{|a_n|}$ *exists. Then*

(a) *If* $\lim_{n\to\infty}\sqrt[n]{|a_n|}<1$, *then* $\sum_{n=1}^{\infty}a_n$ *converges.*

Proof. Let $\lim_{n\to\infty}\sqrt[n]{|a_n|}=L$. To prove that $\sum_{n=1}^{\infty}a_n$ converges, Theorem 16.13 tells us that that it will suffice to find a sequence (c_n) of positive numbers such that $|a_n|\leq c_n$ for all $n\geq N_0$, where N_0 is some fixed integer, for which $\sum_{n=1}^{\infty}c_n$ converges. Since $L<1$ by hypothesis and $0<1$ by Corollary 7.27, Theorem 5.2 asserts that there exists a point $x\in\mathbb{R}$ such that $\max(0,L)<x<1$. We now define (c_n) by $c_n=x^n$ for all $n\in\mathbb{N}$. By Script 7 and the fact that $x\geq 0$, we know that (c_n) is a sequence of positive numbers. Additionally, since $x-L>0$, Theorem 15.7 implies that there is some $N_0\in\mathbb{N}$ such that for all $n\geq N_0$, $|\sqrt[n]{|a_n|}-L|<x-L$. It follows by Script 7 that $|a_n|\leq x^n=c_n$ for all $n\geq N_0$. Lastly, since $-1<\max(0,L)<x<1$, we have by Theorem 16.7 that $\sum_{n=0}^{\infty}c_n$ converges. It follows by Lemma 16.2 that $\sum_{n=1}^{\infty}c_n$ converges, as desired. ☐

(b) *If* $\lim_{n\to\infty}\sqrt[n]{|a_n|}>1$, *then* $\sum_{n=1}^{\infty}a_n$ *diverges.*

Proof. Let $\lim_{n\to\infty}\sqrt[n]{|a_n|}=L$. Suppose for the sake of contradiction that $\lim_{n\to\infty}a_n=0$. Then by Theorem 15.7, for all $\epsilon_1>0$, there is some $N_1\in\mathbb{N}$ such that for all $n\geq N_1$, we have $|a_n|=|a_n-0|<\epsilon_1$. Similarly, since $\lim_{n\to\infty}\sqrt[n]{|a_n|}=L$, we have that for all $\epsilon_2>0$, there is some $N_2\in\mathbb{N}$ such that for all

$n \geq N_2$, we have $|\sqrt[n]{|a_n|} - L| < \epsilon_2$. Choose $\epsilon_2 = L - 1$. Thus, we have that for all $n \geq N_2$, $1 < \sqrt[n]{|a_n|}$ (by an argument symmetric to that given in the proof of Theorem 16.15b) which can be rearranged by Script 7 to demonstrate that $|a_n| > 1^n = 1$ for all such n. Now consider the case where $\epsilon_1 = 1$. Choose $N = \max(N_1, N_2)$. Then by the above, the fact that $N \geq N_2$ implies that $|a_N| > 1$. However, since $N \geq N_1$, we also have that $|a_N| < \epsilon_1 = 1$, a contradiction. Therefore, since $\lim_{n\to\infty} a_n \neq 0$, we have by the contrapositive of Theorem 16.4 that $\sum_{n=1}^{\infty} a_n$ diverges. ☐

6/23: **Definition 16.17.** For $n \in \mathbb{N}$, we define the **factorial** of n to be the product of all natural numbers less than or equal to n. We denote this by the formula

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 3 \cdot 2 \cdot 1$$

By convention, we also set $0! = 1$.

Exercise 16.18. Prove that

$$\sum_{n=0}^{\infty} \frac{1}{n!}$$

converges. The number that it converges to is called e.

Proof. To prove that $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges, Theorem 16.13 tells us that it will suffice to find a sequence (c_n) of positive numbers such that $|a_n| \leq c_n$ for all $n \geq N_0$, where N_0 is some fixed integer, for which $\sum_{n=1}^{\infty} c_n$ converges. Define (c_n) by $c_n = (\frac{1}{2})^{n-1}$ for all $n \in \mathbb{N}$. We now address each property in turn.

By Script 7 and the fact that $\frac{1}{2} \geq 0$, we know that (c_n) is a sequence of positive numbers.

Choose $N_0 = 2$. To confirm that $|a_n| \leq c_n$ for all $n \geq N_0$, we induct on n using Additional Exercise 0.2a. For the base case $n = 2$, we have by Definition 16.17 that $|a_2| = |\frac{1}{2!}| = \frac{1}{2} = c_2$, which we may weaken to $|a_2| \leq c_2$, as desired. Now suppose inductively that we have demonstrated that $|a_n| \leq c_n$; we now seek to demonstrate that $|a_{n+1}| \leq c_{n+1}$. But

$$\begin{aligned}
|a_{n+1}| &= \left|\frac{1}{(n+1)!}\right| \\
&= \frac{1}{n+1} \cdot \frac{1}{n!} && \text{Definition 16.17} \\
&= \frac{1}{n+1} \cdot |a_n| \\
&< \frac{1}{2} \cdot |a_n| \\
&\leq \frac{1}{2} \cdot c_n \\
&= \frac{1}{2} \cdot \left(\frac{1}{2}\right)^{n-1} \\
&= \left(\frac{1}{2}\right)^{(n+1)-1} \\
&= c_{n+1}
\end{aligned}$$

which we may weaken to $|a_{n+1}| \leq c_{n+1}$, as desired.

Since $-1 < \frac{1}{2} < 1$, Theorem 16.7 asserts that $\sum_{n=0}^{\infty} c_n$ converges. It follows by Lemma 16.2 that $\sum_{n=1}^{\infty} c_n$ converges, as desired. ☐

16.2 Discussion

5/20: • We could also prove Exercise 16.3 for an arbitrary N_0 and then use Lemma 16.2, but this would be overly circuitous.

5/27:

- Lemma 16.14:
  - I don't need the first part. I could choose $|b_n - L| < \frac{1-L}{2}$. I could also choose any region with a value less than L, i.e., $L-1$ or, most simply, $\frac{L}{2}$.
- Written part: 48 hours Sunday evening to Tuesday evening.
- There are scripts up to Script 20.

Final-Specific Questions

6/2: 1. For any $a, b \in \mathbb{R}$ with $a \leq b$, define $\mathcal{F}_{[a,b]}$ to be the set of all bounded functions $f : [a,b] \to \mathbb{R}$. Suppose that for all a, b there exists a function $\mathbb{S}_a^b : \mathcal{F}_{[a,b]} \to \mathbb{R}$ which satisfies the following properties:

(i) $\mathbb{S}_a^b(f+g) = \mathbb{S}_a^b(f) + \mathbb{S}_a^b(g)$.

(ii) For any $c \in \mathbb{R}$, $\mathbb{S}_a^b(cf) = c\,\mathbb{S}_a^b(f)$.

(iii) For any $c \in \mathbb{R}$, if $f(x) = c$ for all $x \in [a,b]$, then $\mathbb{S}_a^b(f) = c(b-a)$.

(iv) If $f(x) \geq g(x)$ for all $x \in [a,b]$, then $\mathbb{S}_a^b(f) \geq \mathbb{S}_a^b(g)$.

(v) If $c \in [a,b]$, then $\mathbb{S}_a^b(f) = \mathbb{S}_a^c(f) + \mathbb{S}_c^b(f)$.

Use these properties to prove the following:

(a) Show that if $m \leq f(x) \leq M$ for all $x \in [a,b]$, then $m(b-a) \leq \mathbb{S}_a^b(f) \leq M(b-a)$.

Proof. Let $g, h : [a,b] \to \mathbb{R}$ be defined by $g(x) = m$ and $h(x) = M$ for all $x \in [a,b]$. Thus, since $m \leq g(x) \leq m$ and $M \leq h(x) \leq M$ for all $x \in [a,b]$ by Script 0, we have by consecutive applications of Definitions 5.6 and 10.1 that g and h are bounded functions. It follows by the definition of $\mathcal{F}_{[a,b]}$ that $g, h \in \mathcal{F}_{[a,b]}$. Consequently, consecutive applications of Property (iii) imply that $\mathbb{S}_a^b(g) = m(b-a)$ and $\mathbb{S}_a^b(h) = M(b-a)$. Thus, since $g(x) \leq f(x) \leq h(x)$ for all $x \in [a,b]$ by hypothesis, we have by consecutive applications of Property (iv) that $\mathbb{S}_a^b(g) \leq \mathbb{S}_a^b(f) \leq \mathbb{S}_a^b(h)$. Therefore, by the above, we have that $m(b-a) \leq \mathbb{S}_a^b(f) \leq M(b-a)$, as desired. ☐

(b) Show that if $f = g$ on (a,b), then $\mathbb{S}_a^b(f) = \mathbb{S}_a^b(g)$.

Proof. We will first deal with the trivial case where $a = b$. In this case, $f(x) = f(a)$ and $g(x) = g(b)$ for all $x \in [a,b]$. Thus, by consecutive applications of Property (iii), we have that $\mathbb{S}_a^b(f) = f(a) \cdot (b-a) = f(a) \cdot (a-a) = 0$ and similarly that $\mathbb{S}_a^b(g) = 0$. Therefore, $\mathbb{S}_a^b(f) = \mathbb{S}_a^b(g)$.

Having dealt with the trivial case, we can assume from now on that $a < b$. Let $h : [a,b] \to \mathbb{R}$ be defined by $h(x) = f(x) - g(x)$ for all $x \in [a,b]$. It follows by the hypothesis that $f = g$ for all $x \in (a,b)$ that $h(x) = 0$ for all $x \in (a,b)$. Thus, by Definition 1.18, $h([a,b]) = \{h(a), 0, h(b)\}$, so we clearly have by Definitions 5.6 and 10.1 that h is bounded. It follows that h is in the domain of $\mathbb{S}_a^b$. Having established this, we now seek to verify that $\mathbb{S}_a^b(h) = 0$ by dividing into four cases ($h(a) = h(b) = 0$, $h(a) \neq 0 = h(b)$, $h(a) = 0 \neq h(b)$, and $h(a) \neq 0 \neq h(b)$). Let's begin.

In the first case, we have by the definition of h that $h(x) = 0$ for all $x \in [a,b]$. Therefore, by Property (iii), we have that $\mathbb{S}_a^b(h) = 0(b-a) = 0$, as desired.

In the second case, we divide into two subcases ($h(a) > 0$ and $h(a) < 0$).

Suppose first that $h(a) > 0$. Let c be an arbitrary element of (a,b). It follows by Property (v) that $\mathbb{S}_a^b(h) = \mathbb{S}_a^c(h) + \mathbb{S}_c^b(h)$. By an argument symmetric to that of the first case verified herein, we have that $\mathbb{S}_c^b(h) = 0$. Thus, $\mathbb{S}_a^b(h) = \mathbb{S}_a^c(h)$. We now take a closer look at $\mathbb{S}_a^c(h)$. Since $0 \leq h(x) \leq h(a)$ for all $x \in [a,c]$, part (a) asserts that $0(c-a) \leq \mathbb{S}_a^c(h) \leq h(a) \cdot (c-a)$. It follows from the first inequality that $0 \leq \mathbb{S}_a^c(h)$. As such, to confirm that $\mathbb{S}_a^c(h) = 0$, suppose for the sake of contradiction that $0 < \mathbb{S}_a^c(h)$. Then since $h(a) > 0$ by supposition, Lemma 7.24 asserts that

$0 < \frac{\mathbb{S}_a^c(h)}{h(a)}$. Consequently, by Definition 7.21, $a < a + \frac{\mathbb{S}_a^c(h)}{h(a)}$. Thus, by Theorem 5.2, there exists a point c such that $a < c < \min(b, \frac{\mathbb{S}_a^c(h)}{h(a)})$[1]. By Definition 7.21 and Lemma 7.24 again, we have that $0 < h(a) \cdot (c-a) < \mathbb{S}_a^b(h)$. However, since $a < c < b$, Equations 8.1 imply that $c \in (a,b)$. Thus, by the above, $\mathbb{S}_a^b(h) \leq h(a) \cdot (c-a)$, a contradiction. Therefore, $\mathbb{S}_a^c(h) = 0$, so we have by the above that $\mathbb{S}_a^b(h) = \mathbb{S}_a^c(h) = 0$, as desired.

The argument is symmetric in the other subcase.

In the third case, the verification is symmetric to that of the second.

In the fourth case, begin by letting $h_-, h^+ : [a,b] \to \mathbb{R}$ be defined by

$$h_-(x) = \begin{cases} h(a) & x = a \\ 0 & x \neq a \end{cases}$$

$$h^+(x) = \begin{cases} 0 & x \neq b \\ h(b) & x = b \end{cases}$$

It follows by the definition of h that $h = h_- + h^+$. Additionally, we have by the second case that $\mathbb{S}_a^b(h_-) = 0$ and by the third case that $\mathbb{S}_a^b(h^+) = 0$. Therefore, by Property (i), $\mathbb{S}_a^b(h) = \mathbb{S}_a^b(h_-) + \mathbb{S}_a^b(h^+) = 0 + 0 = 0$, as desired.

Having established that $\mathbb{S}_a^b(h) = 0$ in any case, we can show that

$$\begin{aligned} 0 &= \mathbb{S}_a^b(h) \\ &= \mathbb{S}_a^b(f) + \mathbb{S}_a^b(-g) && \text{Property (i)} \\ &= \mathbb{S}_a^b(f) - \mathbb{S}_a^b(g) && \text{Property (ii)} \end{aligned}$$

Therefore, by Script 7, $\mathbb{S}_a^b(f) = \mathbb{S}_a^b(g)$. □

(c) Let $P = \{t_0, \ldots, t_n\}$ be a partition of $[a,b]$. Suppose that for each i we have that $f(x) = f_i$ for all $x \in (t_{i-1}, t_i)$. Show that

$$\mathbb{S}_a^b(f) = \sum_{i=1}^{n} f_i \cdot (t_i - t_{i-1})$$

Proof. Let i be an arbitrary natural number between 1 and n, and let $h_i : [t_{i-1}, t_i] \to \mathbb{R}$ be defined by $h_i(x) = f_i$. Therefore, applying the above definitions for each i, we have that

$$\begin{aligned} \mathbb{S}_a^b(f) &= \mathbb{S}_{t_0}^{t_1}(f) + \mathbb{S}_{t_1}^{t_2} + \cdots + \mathbb{S}_{t_{n-1}}^{t_n}(f) && \text{Property (v)} \\ &= \sum_{i=1}^{n} \mathbb{S}_{t_{i-1}}^{t_i}(f) \\ &= \sum_{i=1}^{n} \mathbb{S}_{t_{i-1}}^{t_i}(h_i) && \text{Part (b)} \\ &= \sum_{i=1}^{n} f_i \cdot (t_i - t_{i-1}) && \text{Property (iii)} \end{aligned}$$

as desired. □

(d) For $f(x) = x$, show that $\mathbb{S}_0^b(f) = \frac{b^2}{2}$.

[1] Note that it is right here that we make use of the condition that $a < b$; this is why we consider the trivial case where $a = b$ separately at the beginning.

Proof. Suppose for the sake of contradiction that $\mathbb{S}_0^b(f) \neq \frac{b^2}{2}$. We divide into two cases ($\mathbb{S}_0^b(f) < \frac{b^2}{2}$ and $\mathbb{S}_0^b(f) > \frac{b^2}{2}$). Let's begin.

Suppose first that $\mathbb{S}_0^b(f) < \frac{b^2}{2}$. By Exercise 13.21, $\int_0^b f = \frac{b^2}{2}$. Thus, if we define $\epsilon = \frac{b^2}{2} - \mathbb{S}_0^b(f)$, we have by Lemma 13.20 that there is some partition $P = \{t_0, \dots, t_n\}$ such that $U(f,P) - \frac{b^2}{2} < \epsilon$ and $\frac{b^2}{2} - L(f,P) < \epsilon$. It follows from the latter result by the definition of ϵ and Definition 7.21 that $-L(f,P) < -\mathbb{S}_0^b(f)$. Consequently, by Lemma 7.24, we have that $\mathbb{S}_0^b(f) < L(f,P)$. Switching gears for a moment, we have by Definition 13.11 that $L(f,P) = \sum_{i=1}^n m_i(f)(t_i - t_{i-1})$. Additionally, if we define $m : [0,b] \to \mathbb{R}$ by $m(x) = m_i(f)$ for all $x \in (t_{i-1}, t_i)$ for all $i \in [n]$ and $m(t_i) = f(t_i)$, we will have by Part (c) that $\mathbb{S}_0^b(m) = \sum_{i=1}^n m_i(f)(t_i - t_{i-1})$. Combining these last two results with transitivity, we have that $\mathbb{S}_0^b(f) < L(f,P) = \mathbb{S}_0^b(m)$. However, by the definition of m, we also have that $m(x) \leq f(x)$ for all $x \in [0,b]$. but it follows from this by Property (iv) that $\mathbb{S}_0^b(m) \leq \mathbb{S}_0^b(f)$, a contradiction.

The proof is symmetric in the other case. ☐

2. Suppose that f is positive and decreasing on $[1,\infty)$. Suppose also that f is integrable on $[1,n]$ for all $n \in \mathbb{N}$ and define the sequences (a_n) and (b_n) by

$$a_n = f(n)$$
$$b_n = \int_1^n f$$

Show that $\sum_{n=1}^\infty a_n$ converges if and only if $\lim_{n\to\infty} b_n$ exists.

Proof. Suppose first that $\sum_{n=1}^\infty a_n$ converges. To prove that $\lim_{n\to\infty} b_n$ exists, Theorem 15.19 tells us that it will suffice to show that for all $\epsilon > 0$, theer is some $N \in \mathbb{N}$ such that $|b_n - b_m| < \epsilon$ for all $n, m \geq N$. Let $\epsilon > 0$ be arbitrary. By Theorem 16.5, there is some $(N-1) \in \mathbb{N}$ such that $|\sum_{k=m+1}^n a_k| < \epsilon$ for all $n > m \geq N$. Choose $N = (N-1)+1$ to be our N. Let n, m be arbitrary natural numbers such that $n, m \geq N$. We divide into three cases ($n > m$, $n = m$, and $n < m$).

If $n > m$, then we will need a few preliminary results. First, we seek to confirm that $\int_m^n f \geq 0$. Since f is positive, we know that $f(x) \geq 0$ for all $x \in [1,\infty)$. Thus, by Theorem 13.27, $0 = 0(n-m) \leq \int_m^n f$, as desired. Next, we seek to confirm that $\int_i^{i+1} f \leq a_i$. To begin, $a_i = f(i)$. Additionally, we have by Definition 8.16 that $f(i) \geq f(x)$ for all $x > i$. Thus, by Theorem 13.27, $\int_i^{i+1} \leq a_i((i+1)-i) = a_i$, as desired. Lastly, we seek to confirm that $\sum_{k=(m-1)+1}^{n-1} a_k \geq 0$. Since f is positive, we have by the definition of a_i that $a_i \geq 0$ for all i. Thus, by Script 7, $\sum_{k=(m-1)+1}^{n-1} a_k \geq 0$, as desired. Note that all three of these results will be used in the main inequality here, and results one and three will additionally be used in the inequality used for the reverse direction of the proof. Anyway, without further ado, we have

$$\begin{aligned}
|b_n - b_m| &= \left| \int_1^n f - \int_1^m f \right| \\
&= \left| \int_m^n f \right| && \text{Theorem 13.23} \\
&= \int_m^n f && \text{Definition 8.4} \\
&= \int_m^{m+1} f + \int_{m+1}^{m+2} f + \dots + \int_{n-1}^n f && \text{Theorem 13.23} \\
&\leq a_m + a_{m+1} + \dots + a_{n-1} \\
&= \sum_{k=(m-1)+1}^{n-1} a_k
\end{aligned}$$

$$
\begin{aligned}
&= \left| \sum_{k=(m-1)+1}^{n-1} a_k \right| && \text{Definition 8.4} \\
&< \epsilon
\end{aligned}
$$

as desired.

If $n = m$, then we have that $|b_n - b_m| = 0 < \epsilon$, as desired.

If $n < m$, then the argument is symmetric to that of the first case.

Now suppose that $\lim_{n\to\infty} b_n$ exists. To prove that $\sum_{n=1}^{\infty} a_n$ converges, Theorem 16.5 tells us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|\sum_{k=m+1}^{n} a_k| < \epsilon$ for all $n > m \geq N$. Let $\epsilon > 0$ be arbitrary. Then by Theorem 15.19, there is some $N \in \mathbb{N}$ such that $|b_n - b_m| < \epsilon$ for all $n, m \geq N$. Choose this N to be our N. Let n, m be arbitrary natural numbers that satisfy $n > m \geq N$. In addition to the two aforementioned preliminary results, we need one more, i.e., we now seek to confirm that $a_{i+1} \leq \int_i^{i+1} f$. To begin, $a_i = f(i)$. Additionally, we have by Definition 8.16 that $f(x) \geq f(i)$ for all $x < i$. Thus, by Theorem 13.27, $a_{i+1} = a_{i+1}((i+1) - i) \leq \int_i^{i+1} f$, as desired. With these results, we have that

$$
\begin{aligned}
\left| \sum_{k=m+1}^{n} a_k \right| &= \sum_{k=m+1}^{n} a_k && \text{Definition 8.4} \\
&= a_{m+1} + a_{m+2} + \cdots + a_n \\
&\leq \int_m^{m+1} f + \int_{m+1}^{m+2} f + \cdots + \int_{n-1}^{n} f \\
&= \int_m^n f && \text{Theorem 13.23} \\
&= \left| \int_m^n f \right| && \text{Definition 8.4} \\
&= \left| \int_1^n f - \int_1^m f \right| && \text{Theorem 13.23} \\
&= |b_n - b_m| \\
&< \epsilon
\end{aligned}
$$

as desired. □

3. Let $f : [a, b] \to \mathbb{R}$.

   (a) Show that if f is differentiable on $[a, b]$ and f' is bounded on $[a, b]$, then f is uniformly continuous on $[a, b]$.

   Proof. Since f is differentiable on $[a, b]$, we have by Theorem 12.5 that f is continuous on $[a, b]$. It follows by Theorem 9.10 that $f : [a, b] \to \mathbb{R}$ is continuous. Therefore, by Corollary 13.8, f is uniformly continuous on $[a, b]$, as desired. □

   (b) Use Theorem 12.15 to show that a polynomial of degree $n \geq 1$ has at most n distinct roots.

   Proof. We begin by stating two preliminary and previously proven results that will allow us to invoke Theorem 12.15 for any polynomial. First off, we have by Corollary 11.12 that polynomials are continuous. Second, we have by Exercises 12.8 and 12.9 that polynomials are differentiable. Having established these two facts, we are ready to begin in earnest.

   To prove the claim, we induct on n.

   For the base case $n = 1$, let p be a polynomial of degree n. Suppose for the sake of contradiction that p has m distinct roots where $m > n$. Choose roots r_1 and r_2. By Theorem 12.15, there exists a point $\lambda \in (r_1, r_2)$ such that $p'(\lambda) = 0$. Additionally, we have by Exercise 12.8 that $p'(x) = a$, where $a \in \mathbb{R}$, for all $x \in \mathbb{R}$. These last two results when combined necessitate that $p'(x) = 0$ for all $x \in \mathbb{R}$. Thus, by Corollary 12.17, p is constant on $\mathbb{R}$. But by Definition 11.11, this implies that p has degree 0, a contradiction.

Now suppose inductively that we have proven that a polynomial of degree n has at most n distinct roots; we wish to prove that a polynomial of degree $n+1$ has at most $n+1$ distinct roots. To do so, suppose for the sake of contradiction that p has m distinct roots where $m > n+1$. By Theorem 3.5, we may name the roots $r_1, \dots, r_m$ such that $r_1 < r_2 < \cdots < r_m$. It follows by consecutive applications of Theorem 12.15 that there exists a point $\lambda_i \in (r_i, r_{i+1})$ such that $p'(\lambda_i) = 0$ for all $i \in [m-1]$. Consequently, since each λ_i is in a disjoint region from all other λ_i's, we know that each λ_i is distinct. Thus, we know that p' has at least $m-1$ distinct roots. More notably, since we defined $m > n+1$, we have by Definition 7.21 that $m-1 > n$, meaning that p' has more than n distinct roots. However, by Exercise 12.8, p' has degree n. But this means by the inductive hypothesis implies that p' has at most n distinct roots, a contradiction. □

4. Let $f : \mathbb{R} \to \mathbb{R}$. Suppose that $|f(y) - f(x)| \leq c|y - x|$ for all $x, y \in \mathbb{R}$, where $c < 1$ is a constant.

   (a) Show that f is continuous.

   Proof. To prove that f is continuous, Theorem 13.2 tells us that it will suffice to show that f is uniformly continuous. To do this, Definition 13.1 tells us that it will suffice to verify that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in \mathbb{R}$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \epsilon$. Let x, y be arbitrary real numbers such that $|y - x| < \delta$. Then by the supposition,

$$\begin{aligned} |f(y) - f(x)| &\leq c|y - x| \\ &< |y - x| \\ &< \delta \\ &= \epsilon \end{aligned}$$

   as desired. □

   (b) Show that there is at most one point $x \in \mathbb{R}$ where $f(x) = x$.

   Proof. Suppose for the sake of contradiction that there exist multiple points $x \in \mathbb{R}$ where $f(x) = x$. Let $x, y \in \mathbb{R}$ be two such points. Then by the supposition,

$$\begin{aligned} |y - x| &= |f(y) - f(x)| \\ &\leq c|y - x| \\ &< |y - x| \end{aligned}$$

   a contradiction. □

   (c) Show that there exists a point $x \in \mathbb{R}$ where $f(x) = x$.

   Proof. Let (a_n) be defined by $a_n = f^{n-1}(0)$ for all $n \in \mathbb{N}$. To prove that (a_n) converges, Theorem 15.19 tells us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for all $n, m \geq N$. Let $\epsilon > 0$ be arbitrary. Since $c < 1$, by Exercise 15.8, $\lim_{n\to\infty} c^n = 0$. Thus, by Theorem 15.7, there is some $N \in \mathbb{N}$ such that $|c^n| < \epsilon$ for all $n \geq N$. Choose this N to be our N. Let n, m be arbitrary natural numbers such that $n, m \geq N$ and WLOG let $n \geq m$. Then by consecutive applications of the supposition, $|a_n - a_m| = |f^{n-1}(0) - f^{m-1}(0)| \leq c^{m-1}|f^{n-m}(0) - 0| < \epsilon|f^{n-m}(0)| < \epsilon$, as desired.

   □

Script 17

Sequences and Series of Functions

6/23: **Definition 17.1.** Let $A \subset \mathbb{R}$, and consider $X = \{f : A \to \mathbb{R}\}$, the collection of real-valued functions on A. A **sequence of functions** (on A) is an ordered list $(f_1, f_2, f_3, \dots)$ which we will denote (f_n), where each $f_n \in X$. (More formally, we can think of the sequence as a function $F : \mathbb{N} \to X$, where $f_n = F(n)$, for each $n \in \mathbb{N}$, but this degree of formality is not particularly helpful.)

We can take the sequence to start at any $n_0 \in \mathbb{Z}$ and not just at 1, just like we did for sequences of real numbers.

Definition 17.2. The sequence (f_n) **converges pointwise** to a function $f : A \to \mathbb{R}$ if for all $x \in A$ and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$. In other words, we have that for all $x \in A$, $\lim_{n\to\infty} f_n(x) = f(x)$.

Definition 17.3. The sequence (f_n) **converges uniformly** to a function $f : A \to \mathbb{R}$ if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$ for every $x \in A$.

Equivalently, the sequence (f_n) **converges uniformly** to a function $f : A \to \mathbb{R}$ if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $\sup_{x\in A} |f_n(x) - f(x)| < \epsilon$.

Exercise 17.4. Suppose that a sequence (f_n) converges pointwise to a function f. Prove that if (f_n) converges uniformly to a function g, then $f = g$.

Proof. To prove that $f = g$, Definition 1.16 tells us that it will suffice to show that $f(x) = g(x)$ for all $x \in A$. Suppose for the sake of contradiction that $f(x) \neq g(x)$ for some $x \in A$. Since (f_n) converges pointwise to f by hypothesis, Definition 17.2 implies that for all $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then $|f_n(x) - f(x)| < \epsilon$. Additionally, since (f_n) converges uniformly to g by hypothesis, Definition 17.3 asserts that for all $\epsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that if $n \geq N_2$, then $|f_n(x) - g(x)| < \epsilon$.

WLOG, let $f(x) > g(x)$. Choose $\epsilon = \frac{f(x)-g(x)}{2}$, and let $N = \max(N_1, N_2)$. Since $N \geq N_1$, $|f_N(x) - f(x)| < \frac{f(x)-g(x)}{2}$. Similarly, $|f_N(x) - g(x)| < \frac{f(x)-g(x)}{2}$. But this implies that

$$\begin{aligned}
f(x) - g(x) &= |f(x) - f_N(x) + f_N(x) - g(x)| \\
&\leq |f(x) - f_N(x)| + |f_N(x) - g(x)| && \text{Lemma 8.8} \\
&= |f_N(x) - f(x)| + |f_N(x) - g(x)| && \text{Exercise 8.5} \\
&< \frac{f(x) - g(x)}{2} + \frac{f(x) - g(x)}{2} \\
&= f(x) - g(x)
\end{aligned}$$

a contradiction. □

Exercise 17.5. For each of the following sequences of functions, determine what function the sequence (f_n) converges to pointwise. Does the sequence converge uniformly to this function?

(a) For $n \in \mathbb{N}$, let $f_n : [0, 1] \to \mathbb{R}$ be given by $f_n(x) = x^n$.

Answer. Converges to the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$$

Does not converge uniformly. □

(b) For $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be given by $f_n(x) = \frac{\sin(nx)}{n}$. (For the purposes of this example, you may assume basic knowledge of sine.)

Answer. Converges to the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$. Does converge uniformly. □

(c) For $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} n^2 x & 0 \leq x \leq \frac{1}{n} \\ n(2 - nx) & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \frac{2}{n} \leq x \leq 1 \end{cases}$$

Answer. Converges to the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = 0$. Does not converge uniformly. □

Theorem 17.6. *Let (f_n) be a sequence of functions, and suppose that each $f_n : A \to \mathbb{R}$ is continuous. If (f_n) converges uniformly to $f : A \to \mathbb{R}$, then f is continuous.*

Proof. To prove that f is continuous, Theorem 9.10 tells us that it will suffice to show that f is continuous at every $x \in A$. Let x be an arbitrary element of A. To show that f is continuous at x, Theorem 11.5 tells us that it will suffice to verify that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary, and also let y be an arbitrary element of A. Since (f_n) converges uniformly, Definition 17.3 implies that there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(a) - f(a)| < \frac{\epsilon}{3}$ for all $a \in A$. Thus, $|f_N(x) - f(x)| < \frac{\epsilon}{3}$ and $|f_N(y) - f(y)| < \frac{\epsilon}{3}$. Additionally, since each f_n is continuous, Theorems 9.10 and 11.5 assert that there exists $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then $|f_N(y) - f_N(x)| < \frac{\epsilon}{3}$. Choose this δ to be our δ. Therefore,

$$\begin{aligned} |f(y) - f(x)| &\leq |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| && \text{Lemma 8.8} \\ &= |f_N(y) - f(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| && \text{Exercise 8.5} \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

as desired. □

6/26: **Theorem 17.7.** *Suppose that (f_n) is a sequence of integrable functions on $[a,b]$ and suppose that (f_n) converges uniformly to $f : [a,b] \to \mathbb{R}$. Then*

$$\int_a^b f = \lim_{n \to \infty} \int_a^b f_n$$

Lemma. *f is integrable on $[a,b]$.*

Proof. To prove that f is integrable on $[a,b]$, Theorem 13.18 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since (f_n) converges uniformly to f by hypothesis, Definition 17.3 asserts that there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|f_n(x) - f(x)| < \frac{\epsilon}{6(b-a)}$[1] for all $x \in [a,b]$. This statement will be useful in the verification of the three following results.

[1] For the purposes of this proof, we will assume that $a < b$, on the basis of the fact that the proof of the case where $a = b$ is trivial.

To confirm that $|U(f_N,P)-L(f_N,P)|<\frac{\epsilon}{3}$, we first invoke Theorem 13.18, which tells us that since f_N is integrable by hypothesis, there exists a partition P of $[a,b]$ such that $U(f_N,P)-L(f_N,P)<\frac{\epsilon}{3}$. Additionally, since $L(f_N,P)\leq U(f_N,P)$ by Theorem 13.13, we have by Definition 8.4 that $U(f_N,P)-L(f_N,P)=|U(f_N,P)-L(f_N,P)|$. Therefore, we have by transitivity that $|U(f_N,P)-L(f_N,P)|=U(f_N,P)-L(f_N,P)<\frac{\epsilon}{3}$, as desired.

To confirm that $|U(f,P)-U(f_N,P)|<\frac{\epsilon}{3}$, we begin with the following contradiction argument[2].

Suppose for the sake of contradiction that $|M_i(f)-M_i(f_N)|\geq\frac{\epsilon}{3(b-a)}$. We divide into two cases ($M_i(f)-M_i(f_N)\geq\frac{\epsilon}{3(b-a)}$ and $M_i(f_N)-M_i(f)\geq\frac{\epsilon}{3(b-a)}$). Suppose first that $M_i(f)-M_i(f_N)\geq\frac{\epsilon}{3(b-a)}$. By Lemma 5.11, there exists $f(x)\in\{f(x)\mid t_{i-1}\leq x\leq t_i\}$ such that $M_i(f)-\frac{\epsilon}{6(b-a)}<f(x)\leq M_i(f)$. Similarly, there exists $f_N(x)\in\{f_N(x)\mid t_{i-1}\leq x\leq t_i\}$ such that $M_i(f_N)-\frac{\epsilon}{6(b-a)}<f_N(x)\leq M_i(f_N)$. Thus, we have that

$$f(x)>M_i(f)-\frac{\epsilon}{6(b-a)}>M_i(f)-\frac{\epsilon}{3(b-a)}\geq M_i(f_N)\geq f_N(x)$$

It follows that

$$\begin{aligned}
|f(x)-f_N(x)| &= f(x)-f_N(x)\\
&> \left(M_i(f)-\frac{\epsilon}{6(b-a)}\right)-f_N(x)\\
&\geq \left(M_i(f)-\frac{\epsilon}{6(b-a)}\right)-M_i(f_N)\\
&= M_i(f)-M_i(f_N)-\frac{\epsilon}{6(b-a)}\\
&\geq \frac{\epsilon}{3(b-a)}-\frac{\epsilon}{6(b-a)}\\
&= \frac{\epsilon}{6(b-a)}
\end{aligned}$$

But this contradicts the previously proven fact that $|f(x)-f_N(x)|=|f_N(x)-f(x)|<\frac{\epsilon}{6(b-a)}$. The argument is symmetric in the other case.

Thus, we know that $|M_i(f)-M_i(f_N)|<\frac{\epsilon}{3(b-a)}$. Therefore, we have that

$$\begin{aligned}
|U(f,P)-U(f_N,P)| &= \left|\sum_{i=1}^{k}M_i(f)(t_i-t_{i-1})-\sum_{i=1}^{k}M_i(f_N)(t_i-t_{i-1})\right| && \text{Definition 13.10}\\
&= \left|\sum_{i=1}^{k}(M_i(f)-M_i(f_N)(t_i-t_{i-1})\right|\\
&< \left|\sum_{i=1}^{k}\frac{\epsilon}{3(b-a)}(t_i-t_{i-1})\right|\\
&= \frac{\epsilon}{3(b-a)}(b-a)\\
&= \frac{\epsilon}{3}
\end{aligned}$$

as desired.

The verification of the statement that $|L(f_N,P)-L(f,P)|<\frac{\epsilon}{3}$ is symmetric to the previous argument.

Having established that $|U(f_N,P)-L(f_N,P)|<\frac{\epsilon}{3}$, $|U(f,P)-U(f_N,P)|<\frac{\epsilon}{3}$, and $|L(f_N,P)-L(f,P)|<$

[2]Note that this argument is analogous to the proof of Additional Exercise 13.2.

$\frac{\epsilon}{3}$, we can now show that

$$\begin{aligned} U(f,P)-L(f,P) &= |U(f,P)-L(f,P)| && \text{Theorem 13.13} \\ &\leq |U(f,P)-U(f_N,P)|+|U(f_N,P)-L(f_N,P)|+|L(f_N,P)-L(f,P)| \\ &< \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

as desired. □

Proof of Theorem 17.7. To prove that $\lim_{n\to\infty}\int_a^b f_n = \int_a^b f$, Theorem 15.7 tells us that it will suffice to show that for all $\epsilon>0$, there exists $N\in\mathbb{N}$ such that for all $n\geq N$, we have $|\int_a^b f_n - \int_a^b f| < \epsilon$. Let $\epsilon>0$ be arbitrary. Since (f_n) converges uniformly to f, we have by Definition 17.3 that there exists $N\in\mathbb{N}$ such that if $n\geq N$, then $|f_n(x)-f(x)|<\frac{\epsilon}{b-a}$ for all $x\in[a,b]$. Choose this N to be our N. Let n be an arbitrary natural number such that $n\geq N$. It follows from the lemma to Exercise 8.9 that $-\frac{\epsilon}{b-a} < f_n(x)-f(x) < \frac{\epsilon}{b-a}$ for all $x\in[a,b]$. Additionally, since f_n is integrable on $[a,b]$ by hypothesis and f is integrable on $[a,b]$ by the lemma, Theorem 13.24 implies that f_n-f is integrable on $[a,b]$. Combining these last two results, we have by Theorem 13.27 that $-\frac{\epsilon}{b-a}(b-a) < \int_a^b(f_n-f) < \frac{\epsilon}{b-a}(b-a)$. Consequently, by Script 7 and the lemma to Exercise 8.9, we have that $|\int_a^b(f_n-f)|<\epsilon$. Therefore, by Theorem 13.24, we have that $|\int_a^b f_n - \int_a^b f|<\epsilon$, as desired. □

Theorem 17.8. *Let (f_n) be a sequence of functions defined on an open interval containing $[a,b]$ such that each f_n is differentiable on $[a,b]$ and f_n' is integrable on $[a,b]$. Suppose further that (f_n) converges pointwise to f on $[a,b]$ and that (f_n') converges uniformly to a continuous function g on $[a,b]$. Then f is differentiable at every $x\in[a,b]$ and*

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

Proof. Let x be an arbitrary element of $[a,b]$. Since (f_n') converges uniformly to g, Definition 17.3 and Theorem 15.7 imply that $\lim_{n\to\infty} f_n'(x) = g(x)$. Additionally, we have that

$$\begin{aligned} \int_a^x g &= \lim_{n\to\infty}\int_a^x f_n' && \text{Theorem 17.7} \\ &= \lim_{n\to\infty}(f_n(x)-f_n(a)) && \text{Theorem 14.4} \\ &= \lim_{n\to\infty} f_n(x) - \lim_{n\to\infty} f_n(a) && \text{Theorem 15.9} \\ &= f(x)-f(a) && \text{Definition 17.2} \end{aligned}$$

This combined with the fact that g is continuous (hence continuous at x by Theorem 9.10) implies that

$$\begin{aligned} g(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(f(x)-f(a)) && \text{Theorem 14.1} \\ &= \frac{\mathrm{d}}{\mathrm{d}x}(f(x)) - \frac{\mathrm{d}}{\mathrm{d}x}(f(a)) && \text{Exercise 12.9} \\ &= f'(x) && \text{Exercise 12.8} \end{aligned}$$

Therefore, we have by transitivity that $f'(x)=\lim_{n\to\infty} f_n'(x)$, as desired. □

Theorem 17.9. *Let (f_n) be a sequence of functions defined on a set A. Then the following are equivalent.*

(*a*) *There is some function f such that (f_n) converges uniformly to f on A.*

(*b*) *For all $\epsilon>0$, there exists an $N\in\mathbb{N}$ such that when $m,n\geq N$, $|f_n(x)-f_m(x)|<\epsilon$ for all $x\in A$.*

Proof. Suppose first that there is some function f to which (f_n) converges uniformly on A. Let $\epsilon > 0$ be arbitrary. By Definition 17.3, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ for all $x \in A$. Choose this N to be our N. Let n, m be arbitrary natural numbers such that $n, m \geq N$. Then $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ and $|f_m(x) - f(x)| < \frac{\epsilon}{2}$ for all $x \in A$. Therefore, we have that

$$\begin{aligned}
|f_n(x) - f_m(x)| &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| && \text{Lemma 8.8} \\
&= |f_n(x) - f(x)| + |f_m(x) - f(x)| && \text{Exercise 8.5} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

for all $x \in A$, as desired.

Now suppose that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that when $n, m \geq N$, $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in A$. It follows by Theorem 15.19 that $(f_n(x))$ converges for all $x \in A$, i.e., for all $x \in A$, there exists a point $f(x) \in \mathbb{R}$ to which $(f_n(x))$ converges. Let $f : A \to \mathbb{R}$ be defined by $f(x) = \lim_{n\to\infty} f_n(x)$.

To prove that (f_n) converges uniformly to f, Definition 17.3 tells us that it will suffice to show that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$ for all $x \in A$. Let $\epsilon > 0$ be arbitrary. By the hypothesis, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|f_n(x) - f_m(x)| < \frac{\epsilon}{2}$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$, and let x be an arbitrary element of A. Since $(f_m(x))$ converges to $f(x)$, Theorem 15.7 asserts that there exists an $N' \in \mathbb{N}$ such that for all $m \geq N'$, $|f_m(x) - f(x)| < \frac{\epsilon}{2}$. Choose $M = \max(N, N')$. It follows that

$$\begin{aligned}
|f_n(x) - f(x)| &\leq |f_n(x) - f_M(x)| + |f_M(x) - f(x)| && \text{Lemma 8.8} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

as desired. □

Definition 17.10. We define series of functions the same way we defined series of numbers. That is, given a sequence (f_n), define the sequence of partial sums (p_n) by $p_n(x) = f_1(x) + \cdots + f_n(x)$ and say that $\sum_{n=1}^{\infty} f_n$ converges pointwise or converges uniformly to f if the sequence (p_n) does.

Theorem 17.11. *Suppose that $f_n : A \to \mathbb{R}$ is a sequence of functions and that there exists a sequence of positive real numbers (M_n) such that for all $x \in A$, we have $|f_n(x)| \leq M_n$. If $\sum_{n=1}^{\infty} M_n$ converges, then for each $x \in A$, the series of numbers $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely. Furthermore, $\sum_{n=1}^{\infty} f_n$ converges uniformly to the function f defined by $f(x) = \sum_{n=1}^{\infty} f_n(x)$.*

Proof. Let x be an arbitrary element of A. To prove that $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely, Definition 16.9 tells us that it will suffice to show that $\sum_{n=1}^{\infty} |f_n(x)|$ converges. Since (M_n) is a sequence of positive numbers and $|f_n(x)| \leq M_n$ for all $n \geq 1$, the proof of Theorem 16.13 asserts that $\sum_{n=1}^{\infty} |f_n(x)|$ converges.

To prove that $\sum_{n=1}^{\infty} f_n$ converges uniformly to f, Definition 17.10 tells us that it will suffice to show that the sequence of partial sums (p_n) defined by $p_k(x) = \sum_{n=1}^{k} f_n(x)$ converges uniformly to f. To do this, Definition 17.3 tells us that it will suffice to verify that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $j \geq N$, then $|\sum_{n=1}^{j} f_n(x) - f(x)| < \epsilon$ for all $x \in A$. Let $\epsilon > 0$ be arbitrary. By Definition 16.1, $\sum_{n=1}^{\infty} M_n = \lim_{k\to\infty} \sum_{n=1}^{k} M_n$. Thus, by Theorem 15.7, there is some $N \in \mathbb{N}$ such that for all $j \geq N$, $|\sum_{n=1}^{j} M_n - \sum_{n=1}^{\infty} M_n| < \epsilon$. Choose this N to be our N. Let j be an arbitrary natural number such that $j \geq N$. It follows by Script 16 that $|\sum_{n=j+1}^{\infty} M_n| < \epsilon$. Additionally, since (M_n) is a sequence of positive numbers, $\sum_{n=j+1}^{\infty} M_n = |\sum_{n=j+1}^{\infty} M_n|$. Therefore, combining the last several results and letting x be an arbitrary element of A, we have that

$$\begin{aligned}
\left|\sum_{n=1}^{j} f_n(x) - f(x)\right| &= \left|\sum_{n=1}^{j} f_n(x) - \sum_{n=1}^{\infty} f_n(x)\right| \\
&= \left|\sum_{n=j+1}^{\infty} f_n(x)\right| \\
&\leq \sum_{n=j+1}^{\infty} |f_n(x)| && \text{Theorem 16.11} \\
&\leq \sum_{n=j+1}^{\infty} M_n \\
&= \left|\sum_{n=j+1}^{\infty} M_n\right| \\
&< \epsilon
\end{aligned}$$

as desired. □

6/30: **Definition 17.12.** A function of the form $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$, where $c_n \in \mathbb{R}$ is called a **power series**. The power series is **centered** at a, and the numbers c_n are called the **coefficients**.

Theorem 17.13. *Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$ be a power series centered at 0. Suppose that x_0 is a real number such that the series $f(x_0) = \sum_{n=0}^{\infty} c_n x_0^n$ converges. Let r be any number such that $0 < r < |x_0|$. Then the following series of functions converges uniformly on $[-r, r]$ (and absolutely for each $x \in [-r, r]$):*

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad g(x) = \sum_{n=0}^{\infty} n c_n x^{n-1} \qquad h(x) = \sum_{n=0}^{\infty} c_n \cdot \frac{x^{n+1}}{n+1}$$

Furthermore, f is differentiable on $[-r, r]$ and $f' = g$. Also, h is differentiable on $[-r, r]$ and $h' = f$.

We may paraphrase this theorem as follows: If a (zero-centered) power series converges at x_0, then it may be differentiated and anti-differentiated term-by-term on $(-|x_0|, |x_0|)$ to obtain power series representations of the derivative and antiderivative of f.

Proof. Let (f_k) be defined by $f_k(x) = \sum_{n=0}^{k} c_n x^n$ for each $k \in \mathbb{N}$. To prove that (f_k) converges uniformly on $[-r, r]$ and that $(f_k(x))$ converges absolutely for each $x \in [-r, r]$, Theorem 17.11 tells us that it will suffice to find a sequence of positive real numbers (M_n) such that for all $x \in [-r, r]$, we have $|c_n x^n| \leq M_n$ and such that $\sum_{n=1}^{\infty} M_n$ converges.

To begin, we will show that there exists a number M such that $|c_n x_0^n| \leq M$ for all $n \in \mathbb{N}$. By the hypothesis, $f(x_0) = \sum_{n=0}^{\infty} c_n x_0^n$ converges. Thus, by Theorem 16.4, $\lim_{n\to\infty} c_n x_0^n = 0$. Consequently, by Theorem 15.13, $(c_n x_0^n)$ is bounded. It follows by Definition 15.12 that $\{c_n x_0^n \mid n \in \mathbb{N}\}$ is bounded. Thus, by Definition 5.6, there exist numbers $l, u \in \mathbb{R}$ such that $l \leq c_n x_0^n \leq u$ for all $n \in \mathbb{N}$. Let $M = \max(|l|, |u|)$. It follows by Script 0 that $-M \leq l \leq c_n x_0^n \leq u \leq M$ for all $n \in \mathbb{N}$. Therefore, by the lemma to Exercise 8.9, $|c_n x_0^n| \leq M$ for all $n \in \mathbb{N}$, as desired.

We can now define (M_n): Let (M_n) be defined by $M_n = M(\frac{r}{|x_0|})^n$ for all $n \in \mathbb{N}$.

Next, we will show that for all $x \in [-r, r]$, $|c_n x^n| \leq M_n$ for all $n \in \mathbb{N}$. Let x be an arbitrary element of $[-r, r]$. It follows by Equations 8.1 that $-r \leq x \leq r$. Thus, by the lemma to Exercise 8.9, $|x| \leq r$.

Consequently, by Exercise 12.22, $|x^n| \leq |r^n|$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} |c_n x^n| &\leq |c_n r^n| \\ &= |c_n x_0^n| \left(\frac{|r^n|}{|x_0^n|} \right) \\ &\leq M \left(\frac{r}{|x_0|} \right)^n \\ &= M_n \end{aligned}$$

for all $n \in \mathbb{N}$, as desired. Note that this result also implies by Definition 8.4 that (M_n) is a sequence of positive real numbers.

Lastly, we will show that $\sum_{n=1}^{\infty} M_n$ converges. Since $0 < r < |x_0|$ by hypothesis, Script 7 implies that $-1 < \frac{r}{|x_0|} < 1$. Thus, by Theorem 16.7, $\sum_{n=0}^{\infty} (\frac{r}{|x_0|})^n$ converges. Consequently, by Lemma 16.2, $\sum_{n=1}^{\infty} (\frac{r}{|x_0|})^n$ converges. Therefore, by Theorem 16.8, $\sum_{n=1}^{\infty} M(\frac{r}{|x_0|})^n$ (i.e., $\sum_{n=1}^{\infty} M_n$) converges, as desired.

Let (g_k) be defined by $g_k(x) = \sum_{n=0}^{k} nc_n x^{n-1}$ for each $k \in \mathbb{N}$. To prove that (g_k) converges uniformly on $[-r, r]$ and that $(g_k(x))$ converges absolutely for each $x \in [-r, r]$, Theorem 17.11 tells us that it will suffice to find a sequence of positive real numbers (M_n) such that for all $x \in [-r, r]$, we have $|nc_n x^{n-1}| \leq M_n$ and such that $\sum_{n=1}^{\infty} M_n$ converges.

To begin, we define (M_n) and prove its basic properties. Let (M_n) be defined by $M_n = \frac{Mn}{|r|} |\frac{r}{x_0}|^n$ for all $n \in \mathbb{N}$, where M is the same constant defined above. We now show that for all $x \in [-r, r]$, we have $|nc_n x^{n-1} \leq M_n$ for all $n \in \mathbb{N}$. Let x be an arbitrary element of $[-r, r]$. It follows as before that $|x^{n-1}| \leq |r^{n-1}|$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} |nc_n x^{n-1}| &= n|c_n||x^{n-1}| \\ &\leq n|c_n||r^{n-1}| \\ &= \frac{|c_n|}{|r|} |x_0|^n n \left| \frac{r}{x_0} \right|^n \\ &\leq \frac{Mn}{|r|} \left| \frac{r}{x_0} \right|^n \\ &= M_n \end{aligned}$$

for all $n \in \mathbb{N}$, as desired. Note that as before, this result also implies that (M_n) is a sequence of positive real numbers.

Next, we will show that $\sum_{n=1}^{\infty} M_n$ converges. To do so, Theorem 16.15 tells us that it will suffice to show that $\lim_{n\to\infty} |\frac{M_{n+1}}{M_n}| < 1$. As before, $|\frac{r}{x_0}| < 1$. Additionally, by an argument symmetric to that used in Exercise 15.6a, we know that $\lim_{n\to\infty} |\frac{r}{x_0}|$ converges to $|\frac{r}{x_0}|$. Furthermore, by an argument symmetric to that used in Exercise 15.10c, we have that $\lim_{n\to\infty} |\frac{n+1}{n}|$ converges to 1. Combining these last three results, we have that

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{M_{n+1}}{M_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{M(n+1)}{|r|} \left| \frac{r}{x_0} \right|^{n+1}}{\frac{Mn}{|r|} \left| \frac{r}{x_0} \right|^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{n+1}{n} \left| \frac{r}{x_0} \right| \right| \\ &= \left(\lim_{n\to\infty} \left| \frac{r}{x_0} \right| \right) \left(\lim_{n\to\infty} \left| \frac{n+1}{n} \right| \right) && \text{Theorem 15.9} \\ &= \left| \frac{r}{x_0} \right| \cdot 1 \\ &< 1 \end{aligned}$$

as desired.

The argument for (h_k) defined by $h_k(x) = \sum_{n=0}^{k} c_n \cdot \frac{x^{n+1}}{n+1}$ is symmetric to that for (g_k).

To prove that f is differentiable on $[-r,r]$ and $f' = g$, Theorem 14.1 tells us that it will suffice to show that g is integrable on $[-r,r]$, that $f(x) + c = \int_{-r}^{x} g$ where $c \in \mathbb{R}$ is a constant, and that g is continuous on $[-r,r]$. We will verify each constraint in order.

Let k be an arbitrary natural number. Thus, by the definition of (g_k), $g_k(x) = \sum_{n=0}^{k} nc_n x^{n-1}$. Consequently, by Definition 11.11, g_k is a polynomial. It follows by Corollary 11.12 that g_k is continuous. Thus, by Theorem 13.19, g_k is integrable. Therefore, since (g_k) is a sequence of integrable functions on $[-r,r]$, the lemma to Theorem 17.7 asserts that g is integrable on $[-r,r]$, as desired.

It follows from the above that

$$\begin{aligned}
\int_{-r}^{x} g &= \lim_{k\to\infty} \int_{-r}^{x} g_k && \text{Theorem 17.7} \\
&= \lim_{k\to\infty} \int_{-r}^{x} \sum_{n=0}^{k} nc_n t^{n-1}\,\mathrm{d}t \\
&= \lim_{k\to\infty} \sum_{n=0}^{k} \int_{-r}^{x} nc_n t^{n-1}\,\mathrm{d}t && \text{Theorem 13.24} \\
&= \lim_{k\to\infty} \sum_{n=0}^{k} (c_n x^n - c_n(-r)^n)^{[3]} && \text{Theorem 14.4} \\
&= \sum_{n=0}^{\infty} (c_n x^n - c_n(-r)^n) && \text{Definition 16.1} \\
&= \sum_{n=0}^{\infty} c_n x^n - \sum_{n=0}^{\infty} c_n(-r)^n && \text{Theorem 16.8} \\
&= f(x) + c
\end{aligned}$$

as desired.

Lastly, since each g_k is continuous and (g_k) converges uniformly to g, Theorem 17.6 asserts that g is continuous on $[-r,r]$, as desired.

The argument for that h is differentiable on $[-r,r]$ and $h' = f$ is symmetric to the above. □

[3]Note that in order to verify the equality of the previous equation and this one, we must technically prove the power rule (analogous to Exercise 13.21) in general for arbitrary domains and n. However, as this would be a Script 13 proof in nature, we omit it.