Script 18

The Euclidean Space $\mathbb{R}^n$

7/7: For the next three sheets, we will be studying multivariable calculus, that is "calculus on $\mathbb{R}^n$." First, we need to understand the space $\mathbb{R}^n$.

Definition 18.1. The **Euclidean n-space** $\mathbb{R}^n$ is the n-fold Cartesian product of $\mathbb{R}$. Symbolically,

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{R}\}$$

is the set of n-tuples of real numbers. We often write

$$\mathbf{x} = (x_1, \ldots, x_n)$$

to denote an element, which is also referred to as a **vector**, in $\mathbb{R}^n$ and

$$\mathbf{0} = (0, \ldots, 0)$$

Definition 18.2. Let $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. We define the following operations.

(a) (Addition) $\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_n + y_n)$.

(b) (Scalar Multiplication) $\lambda\mathbf{x} = (\lambda x_1, \ldots, \lambda x_n)$.

Exercise 18.3. Prove that the addition on $\mathbb{R}^n$ satisfies FA1-FA4 (see Definition 7.8). Moreover, prove that

VS1. (Associativity of Scalar Multiplication) If $\lambda, \mu \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, then $(\lambda\mu)\mathbf{x} = \lambda(\mu\mathbf{x})$.

VS2. (Distributivity of Scalars) If $\lambda, \mu \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, then $(\lambda + \mu)\mathbf{x} = \lambda\mathbf{x} + \mu\mathbf{x}$.

VS3. (Distributivity of Vectors) If $\lambda \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then $\lambda(\mathbf{x} + \mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y}$.

VS4. (Scalar Multiplicative Identity) If $\mathbf{x} \in \mathbb{R}^n$, then $1\mathbf{x} = \mathbf{x}$.

These eight properties together are called the **vector space axioms**.

Proof. To prove that $\mathbb{R}^n$ obeys FA1 from Definition 7.8, it will suffice to show that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$. Let $\mathbf{x}, \mathbf{y}$ be arbitrary elements of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}\mathbf{x} + \mathbf{y} &= (x_1 + y_1, \ldots, x_n + y_n)\\ &= (y_1 + x_1, \ldots, y_n + x_n)\\ &= \mathbf{y} + \mathbf{x}\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys FA2 from Definition 7.8, it will suffice to show that for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$. Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be arbitrary elements of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}(\mathbf{x} + \mathbf{y}) + \mathbf{z} &= (x_1 + y_1, \dots, x_n + y_n) + \mathbf{z}\\ &= ((x_1 + y_1) + z_1, \dots, (x_n + y_n) + z_n)\\ &= (x_1 + (y_1 + z_1), \dots, x_n + (y_n + z_n))\\ &= \mathbf{x} + (y_1 + z_1, \dots, y_n + z_n)\\ &= \mathbf{x} + (\mathbf{y} + \mathbf{z})\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys FA3 from Definition 7.8, it will suffice to find an element $0 \in \mathbb{R}^n$ such that $\mathbf{x} + 0 = 0 + \mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Choose $\mathbf{0}$ to be our 0. Let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}\mathbf{x} + \mathbf{0} &= (x_1 + 0, \dots, x_n + 0)\\ &= (x_1, \dots, x_n)\\ &= \mathbf{x}\\ &= (0 + x_1, \dots, 0 + x_n)\\ &= \mathbf{0} + \mathbf{x}\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys FA4 from Definition 7.8, it will suffice to show that for all $\mathbf{x} \in \mathbb{R}^n$, there exists $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x} = 0$. Let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Choose $\mathbf{y} = (-x_1, \dots, -x_n)$. Then by Definition 18.2,

$$\begin{aligned}\mathbf{x} + \mathbf{y} &= (x_1 + (-x_1), \dots, x_n + (-x_n))\\ &= (0, \dots, 0)\\ &= \mathbf{0}\\ &= ((-x_1) + x_1, \dots, (-x_n) + x_n)\\ &= \mathbf{y} + \mathbf{x}\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys VS1, it will suffice to show that for all $\lambda, \mu \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, we have $(\lambda\mu)\mathbf{x} = \lambda(\mu\mathbf{x})$. Let λ, μ be arbitrary elements of $\mathbb{R}$, and let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}(\lambda\mu)\mathbf{x} &= ((\lambda\mu)x_1, \dots, (\lambda\mu)x_n)\\ &= (\lambda(\mu x_1), \dots, \lambda(\mu x_n))\\ &= \lambda(\mu\mathbf{x})\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys VS2, it will suffice to show that for all $\lambda, \mu \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, we have $(\lambda + \mu)\mathbf{x} = \lambda\mathbf{x} + \mu\mathbf{x}$. Let λ, μ be arbitrary elements of $\mathbb{R}$, and let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}(\lambda + \mu)\mathbf{x} &= ((\lambda + \mu)x_1, \dots, (\lambda + \mu)x_n)\\ &= (\lambda x_1 + \mu x_1, \dots, \lambda x_n + \mu x_n)\\ &= (\lambda x_1, \dots, \lambda x_n) + (\mu x_1, \dots, \mu x_n)\\ &= \lambda\mathbf{x} + \mu\mathbf{x}\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys VS3, it will suffice to show that for all $\lambda \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have $\lambda(\mathbf{x}+\mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y}$. Let λ be an arbitrary element of $\mathbb{R}$, and let $\mathbf{x}, \mathbf{y}$ be arbitrary elements of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}
\lambda(\mathbf{x}+\mathbf{y}) &= \lambda(x_1 + y_1, \dots, x_n + y_n) \\
&= (\lambda(x_1 + y_1), \dots, \lambda(x_n + y_n)) \\
&= (\lambda x_1 + \lambda y_1, \dots, \lambda x_n + \lambda y_n) \\
&= (\lambda x_1, \dots, \lambda x_n) + (\lambda y_1, \dots, \lambda y_2) \\
&= \lambda\mathbf{x} + \lambda\mathbf{y}
\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys VS4, it will suffice to show that for all $\mathbf{x} \in \mathbb{R}^n$, we have $1\mathbf{x} = \mathbf{x}$. Let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}
1\mathbf{x} &= (1x_1, \dots, 1x_n) \\
&= (x_1, \dots, x_n) \\
&= \mathbf{x}
\end{aligned}$$

as desired. □

Remark 18.4. Since $\mathbb{R}^n$ with the two operations defined as above satisfies these eight axioms, we call $\mathbb{R}^n$ a **vector space**.

Exercise 18.5. Prove that if $\mathbf{x} \in \mathbb{R}^n$, then $0\mathbf{x} = \mathbf{0}$.

Proof. By Definition 18.2, we have that

$$\begin{aligned}
0\mathbf{x} &= (0x_1, \dots, 0x_n) \\
&= (0, \dots, 0) \\
&= \mathbf{0}
\end{aligned}$$

as desired. □

Definition 18.6. Let $\mathbf{x} \in \mathbb{R}^n$. The **norm** of $\mathbf{x}$ is defined as

$$\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

Definition 18.7. We call $\|\mathbf{y} - \mathbf{x}\|$ the **distance** between $\mathbf{x}$ and $\mathbf{y}$.

Remark 18.8. If $n = 1$, the norm coincides with the definition of the absolute value in $\mathbb{R}$.

Lemma 18.9.

(a) *If $x, y \in \mathbb{R}$, then $xy \leq \frac{x^2+y^2}{2}$.*

Proof. Let x, y be arbitrary elements of $\mathbb{R}$. Then by Lemma 7.26, $0 \leq (x-y)^2$. Therefore, we have that

$$\begin{aligned}
xy &= \frac{2xy + 0}{2} \\
&\leq \frac{2xy + (x-y)^2}{2} \\
&= \frac{2xy + x^2 - 2xy + y^2}{2} \\
&= \frac{x^2 + y^2}{2}
\end{aligned}$$

as desired. □

(*b*) *If* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, *then* $|x_1y_1 + \cdots + x_ny_n| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|$.

Proof. Suppose first that $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$. Then by Definition 18.6, $\|\mathbf{x}\| = 1 = \sqrt{x_1^2 + \cdots + x_n^2}$, from which it follows that $1 = x_1^2 + \cdots + x_n^2$. Therefore, we have that

$$\begin{aligned}
|x_1y_1 + \cdots + x_ny_n| &\leq |x_1y_1| + \cdots + |x_ny_n| && \text{Lemma 8.8}\\
&= |x_1||y_1| + \cdots + |x_n||y_n|\\
&\leq \frac{|x_1|^2 + |y_1|^2}{2} + \cdots + \frac{|x_n|^2 + |y_n|^2}{2} && \text{Lemma 18.9a}\\
&= \frac{x_1^2 + y_1^2}{2} + \cdots + \frac{x_n^2 + y_n^2}{2}\\
&= \frac{(x_1^2 + \cdots + x_n^2) + (y_1^2 + \cdots + y_n^2)}{2}\\
&= \frac{1+1}{2}\\
&= 1
\end{aligned}$$

as desired.

Now let $\mathbf{x}, \mathbf{y}$ be arbitrary elements of $\mathbb{R}^n$. Consider the vectors $\mathbf{u}_\mathbf{x}, \mathbf{u}_\mathbf{y}$ defined by $\mathbf{u}_\mathbf{x} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$ and $\mathbf{u}_\mathbf{y} = \frac{\mathbf{y}}{\|\mathbf{y}\|}$. By the proof of the first case, we have that

$$\begin{aligned}
|x_1y_1 + \cdots + x_ny_n| &= \|\mathbf{x}\| \cdot \|\mathbf{y}\| \cdot \left| \frac{x_1y_1}{\|\mathbf{x}\| \cdot \|\mathbf{y}\|} + \cdots + \frac{x_ny_n}{\|\mathbf{x}\| \cdot \|\mathbf{y}\|} \right|\\
&= \|\mathbf{x}\| \cdot \|\mathbf{y}\| \cdot |u_{\mathbf{x}_1}u_{\mathbf{y}_1} + \cdots + u_{\mathbf{x}_n}u_{\mathbf{y}_n}|\\
&\leq \|\mathbf{x}\| \cdot \|\mathbf{y}\| \cdot 1\\
&= \|\mathbf{x}\| \cdot \|\mathbf{y}\|
\end{aligned}$$

as desired. □

Theorem 18.10. *If* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}$, *then*

(*a*) $\|\mathbf{x}\| \geq 0$. *Moreover,* $\|\mathbf{x}\| = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$.

Proof. Let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$.

We first prove that $\|\mathbf{x}\| \geq 0$. By Lemma 7.26, $x_i^2 \geq 0$ for all $i \in [n]$. Thus, by Definition 7.21, $x_1^2 + \cdots + x_n^2 \geq 0$. Therefore, we have by Definition 18.6 that $\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2} \geq 0$, as desired.

We now prove that $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$. Suppose first that $\|\mathbf{x}\| = 0$. Then by Definition 18.6 and Script 7, $x_1^2 + \cdots + x_n^2 = 0$. Now suppose for the sake of contradiction that $\mathbf{x} \neq \mathbf{0}$. Then there exists an x_i such that $x_i \neq 0$. Thus, by Lemma 7.26, $x_i^2 > 0$. Additionally, $x_j^2 \geq 0$ for all $j \in [n]$. Thus, we have that $0 < x_i^2 \leq x_1^2 + \cdots + x_n^2$. But by Definition 3.1, this implies that $x_1^2 + \cdots + x_n^2 \neq 0$, a contradiction.

Now suppose that $\mathbf{x} = \mathbf{0}$. Then by Definition 18.6, $\|\mathbf{x}\| = \sqrt{0^2 + \cdots + 0^2} = 0$, as desired. □

(*b*) $\|\lambda\mathbf{x}\| = |\lambda| \cdot \|\mathbf{x}\|$.

Proof. Let λ be an arbitrary element of $\mathbb{R}$, and let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Then we have that

$$\begin{aligned}
\|\lambda\mathbf{x}\| &= \sqrt{(\lambda x_1)^2 + \cdots + (\lambda x_n)^2} && \text{Definition 18.6}\\
&= |\lambda| \cdot \sqrt{x_1^2 + \cdots + x_n^2}\\
&= |\lambda| \cdot \|\mathbf{x}\| && \text{Definition 18.6}
\end{aligned}$$

as desired. □

(c) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

Proof. Let $\mathbf{x}, \mathbf{y}$ be arbitrary elements of $\mathbb{R}^n$. Then we have that

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\| &= \sqrt{(x_1+y_1)^2 + \cdots + (x_n+y_n)^2} && \text{Definition 18.6} \\
&= \sqrt{(x_1^2 + \cdots + x_n^2) + (2x_1y_1 + \cdots + 2x_ny_n) + (y_1^2 + \cdots + y_n^2)} \\
&\leq \sqrt{\|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2} && \text{Lemma 18.9} \\
&= \sqrt{(\|\mathbf{x}\| + \|\mathbf{y}\|)^2} \\
&= \|\mathbf{x}\| + \|\mathbf{y}\|
\end{aligned}$$

as desired. □

Corollary 18.11. *If $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, then*

(a) $\|\mathbf{x}-\mathbf{z}\| \leq \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{y}-\mathbf{z}\|$.

(b) $|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq \|\mathbf{x}-\mathbf{y}\|$.

Proof. The proofs are symmetric to those of Lemma 8.8. □

7/10: The next goal is to "topologize" $\mathbb{R}^n$. To discuss topology on $\mathbb{R}^n$, we first need to introduce notions for $\mathbb{R}^n$ that are analogous to open and closed intervals for $\mathbb{R}$.

Remark 18.12. For $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\mathbf{y} = (y_1, \dots, y_m) \in \mathbb{R}^m$, we identify $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m$ with $(x_1, \dots, x_n, y_1, \dots, y_m) \in \mathbb{R}^{n+m}$. So if $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$, we can consider $A \times B$ to be a subset of $\mathbb{R}^{n+m}$.

If also $C \in \mathbb{R}^k$, then $(A \times B) \times C$ and $A \times (B \times C)$ correspond to the same subset of $\mathbb{R}^{n+m+k}$ under this identification; we write $A \times B \times C$ for this set.

Definition 18.13. An **open rectangle** in $\mathbb{R}^n$ is a set of the form $(a_1, b_1) \times \cdots \times (a_n, b_n)$, a product of open intervals. Similarly, a **closed rectangle** in $\mathbb{R}^n$ is a set of the form $[a_1, b_1] \times \cdots \times [a_n, b_n]$. We allow the possibility that $a_j = b_j$ (where $[a_j, a_j] = \{a_j\}$). If there is at least one j with $a_j = b_j$, then we say that the rectangle is **degenerate**; otherwise, we say that the rectangle is **non-degenerate**.

Definition 18.14. A subset $U \subset \mathbb{R}^n$ is **open** if for all $\mathbf{x} \in U$, there exists an open rectangle R such that $\mathbf{x} \in R \subset U$. A subset $C \in \mathbb{R}^n$ is **closed** if its compliment is open.

Exercise 18.15. Decide whether each of the following is an open set in $\mathbb{R}^2$.

(a) $\{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R},\ x_1 > 0,\ x_2 > 0\}$.

Proof. To prove that $U = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R},\ x_1 > 0,\ x_2 > 0\}$ is open, Definition 18.14 tells us that it will suffice to show that for all $\mathbf{x} \in U$, there exists an open rectangle R such that $\mathbf{x} \in R \subset U$. Let $\mathbf{x}$ be an arbitrary element of U. Then by the definition of U, $0 < x_1$ and $0 < x_2$. It follows by Theorem 5.2 and Corollary 6.12, there exist a_1, b_1, a_2, b_2 such that $0 < a_1 < x_1 < b_1$ and $0 < a_2 < x_2 < b_2$. Thus, by Equations 8.1, $x_1 \in (a_1, b_1)$ and $x_2 \in (a_2, b_2)$. Consequently, if we let $R = (a_1, b_1) \times (a_2, b_2)$, Definition 18.13 guarantees that R is an open rectangle. Additionally, Definition 1.15 asserts that $(x_1, x_2) = \mathbf{x} \in R$, as desired. Additionally, if $\mathbf{y}$ is any vector in R, then by the definition of R, $0 < a_1 < y_1$ and $0 < a_2 < y_2$. Thus, by transitivity, $\mathbf{y} \in U$. Therefore, by Definition 1.3, $R \subset U$, as desired. □

(b) $\{(x, 0) \mid x \in \mathbb{R}\}$.

Proof. To prove that $U = \{(x, 0) \mid x \in \mathbb{R}\}$ is not open, Definition 18.14 tells us that it will suffice to find an $\mathbf{x} \in U$ such that for all open rectangles R containing $\mathbf{x}$, $R \not\subset U$. Let $\mathbf{x} = (0, 0)$, and let R be an arbitrary open rectangle containing $\mathbf{x}$. By Definitions 18.13 and 1.15 along with Equations 8.1, $a_1 < 0 < b_1$ and $a_2 < 0 < b_2$. Thus, by consecutive applications of Theorem 5.2, there exist points $y_1, y_2 \in \mathbb{R}$ such that $a_1 < y_1 < 0$ and $a_2 < y_2 < 0$. It follows that $\mathbf{y} = (y_1, y_2) \in R$. However, since $y_2 \neq 0$ by Definition 3.1, $\mathbf{y} \notin U$. Therefore, by Definition 1.3, $R \not\subset U$, as desired. □

Exercise 18.16. Show that if $R_1, \dots, R_m$ are open rectangles containing $\mathbf{x} \in \mathbb{R}^n$, then $R = R_1 \cap \cdots \cap R_m$ is an open rectangle containing $\mathbf{x} \in \mathbb{R}^n$. If $R = (a_1, b_1) \times \cdots \times (a_n, b_n)$, derive formulas for a_i and b_i in terms of the corresponding quantities for $R_1, \dots, R_m$.

Proof. Let $R_i = (r_{ij}, s_{ij})_{j=1}^n$ for all $i \in [m]$. To prove that $R = \bigcap_{i=1}^m R_i$ is an open rectangle containing $\mathbf{x}$, Definitions 18.13 and 1.15 tell us that it will suffice to show that R is the Cartesian product of open intervals, each containing its respective x_j. Since $\mathbf{x} \in R_i$ for all $i \in [m]$, we have by Definition 1.15 that $x_j \in (r_{ij}, s_{ij})$ for all $i \in [m]$, $j \in [n]$. Thus, by Corollary 3.19, $\bigcap_{i=1}^m (r_{ij}, s_{ij})$ is a region (hence an open interval by Corollary 4.11 and Lemma 8.3) containing x_j for all $j \in [n]$. Therefore, since $R = \bigcap_{i=1}^m R_i = \prod_{i=1}^n (\bigcap_{i=1}^m (r_{ij}, s_{ij}))$ by Script 1, we have that R is the Cartesian product of open intervals, each containing its respective x_j, as desired.

Let $a_j = \max_{i=1}^m (r_{ij})$ and let $b_j = \min_{i=1}^m (s_{ij})$ for all $j \in [n]$. To prove that $R = (a_j, b_j)_{j=1}^n$, Definition 1.2 tells us that it will suffice to show that every $\mathbf{x} \in R$ is an element of $(a_j, b_j)_{j=1}^n$ and vice versa. Suppose first that $\mathbf{x}$ is an arbitrary element of R. Then by Definition 1.6, $\mathbf{x} \in R_i$ for all $i \in [m]$. It follows by Definition 1.15 that $x_j \in (r_{ij}, s_{ij})$ for all $i \in [m]$, $j \in [n]$, including the j, j' for which r_{ij} is at its maximum and $s_{ij'}$ is at its minimum. In other words, $x_j \in (a_j, b_j)$ for all $j \in [n]$. Therefore, by Definition 1.15, $\mathbf{x} \in (a_j, b_j)_{j=1}^n$, as desired. The proof is symmetric in the other direction. ☐

Definition 18.17. The **open ball** (in $\mathbb{R}^n$ with center $\mathbf{p}$ and radius $r > 0$) is defined as

$$B(\mathbf{p}, r) = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{p}\| < r\}$$

The **closed ball** (in $\mathbb{R}^n$ with center $\mathbf{p}$ and radius $r > 0$) is defined as

$$\overline{B}(\mathbf{p}, r) = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{p}\| \leq r\}$$

Remark 18.18. In $\mathbb{R}^1$, an open rectangle is also an open ball, and vice versa.

The following results illustrate how open rectangles and open balls in $\mathbb{R}^n$ are "compatible" with each other.

Lemma 18.19. *Fix* $\mathbf{x} \in \mathbb{R}$.

(*a*) *If R is an open rectangle containing* $\mathbf{x}$*, then there exists $r > 0$ such that $B(\mathbf{x}, r) \subset R$.*

Proof. Since $\mathbf{x} \in R$, Definitions 18.13 and 1.15 tell us that that $x_i \in (a_i, b_i)$ for all $i \in [n]$. Additionally, we know by Corollary 4.11 and Lemma 8.3 that each (a_i, b_i) is an open interval. Combining the last two results, we have by Lemma 8.10 that for each $i \in [n]$, there exists $\delta_i > 0$ such that $(x_i - \delta_i, x_i + \delta_i) \subset (a_i, b_i)$. Let $r = \min\{\delta_i\}_{i=1}^n$.

To prove that $B(\mathbf{x}, r) \subset R$, Definition 1.3 tells us that it will suffice to show that every $\mathbf{y} \in B(\mathbf{x}, r)$ is an element of R. Let $\mathbf{y}$ be an arbitrary element of $B(\mathbf{x}, r)$. Then by Definition 18.17, $\|\mathbf{y} - \mathbf{x}\| < r$. It follows that

$$\begin{aligned}
|y_i - x_i| &= \sqrt{(y_i - x_i)^2} \\
&\leq \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2} && \text{Lemma 7.26} \\
&= \|\mathbf{y} - \mathbf{x}\| && \text{Definition 18.6} \\
&< r
\end{aligned}$$

for all $i \in [n]$. Thus, by the definition of r, $|y_i - x_i| \leq \delta_i$ for all $i \in [n]$. Consequently, by Exercise 8.9 and Definition 1.3, $y_i \in (a_i, b_i)$ for all $i \in [n]$. Therefore, by Definitions 1.15 and 18.13, $\mathbf{y} \in R$, as desired. ☐

(*b*) *If B is an open ball containing* $\mathbf{x}$*, then there exists an open rectangle R such that $\mathbf{x} \in R \subset B$.*

Lemma. *If $\mathbf{x} \in \mathbb{R}^n$, then $\|\mathbf{x}\| \leq \sum_{i=1}^n |x_i|$.*

Proof. By Definition 18.2, we can decompose $\mathbf{x}$ into the sum of n unit vectors $\mathbf{u_i}$ (where $\mathbf{u_i}$ points one unit in the i^{th} direction), each scaled by x_i; symbolically, let $\mathbf{x} = \sum_{i=1}^{n} x_i\mathbf{u_i}$. Therefore,

$$\begin{aligned}
\|\mathbf{x}\| &= \left\|\sum_{i=1}^{n} x_i\mathbf{u_i}\right\| \\
&= \sum_{i=1}^{n} \|x_i\mathbf{u_i}\| && \text{Theorem 18.10c} \\
&= \sum_{i=1}^{n} |x_i| \cdot \|\mathbf{u_i}\| && \text{Theorem 18.10b} \\
&= \sum_{i=1}^{n} |x_i| \cdot \sqrt{1^2} && \text{Definition 18.6} \\
&= \sum_{i=1}^{n} |x_i|
\end{aligned}$$

as desired. □

Proof of Lemma 18.19b. Suppose $\mathbf{x} \in B(\mathbf{y}, r)$. Then by Definition 18.17, $\|\mathbf{x} - \mathbf{y}\| < r$. Thus, we can define $r' = r - \|\mathbf{x} - \mathbf{y}\|$ such that $r' > 0$. With this term defined, we can let $R = (x_i - \frac{r'}{n}, x_i + \frac{r'}{n})_{i=1}^{n}$.

To prove that $\mathbf{x} \in R$, Definition 18.13 tells us that it will suffice to show that $x_i \in (x_i - \frac{r'}{n}, x_i + \frac{r'}{n})$ for all $i \in [n]$. But since $|x_i - x_i| = 0 < \frac{r'}{n}$ for all $i \in [n]$, Exercise 8.9 asserts that this is true.

To prove that $R \subset B$, Definition 1.3 tells us that it will suffice to show that every $\mathbf{z} \in R$ is an element of B. Let $\mathbf{z}$ be an arbitrary element of R. Then by Definition 18.13, $z_i \in (x_i - \frac{r'}{n}, x_i + \frac{r'}{n})$ for all $i \in [n]$. It follows by Exercise 8.9 that $|z_i - x_i| < \frac{r'}{n}$ for all $i \in [n]$. Consequently,

$$\begin{aligned}
\|\mathbf{z} - \mathbf{y}\| &\leq \|\mathbf{z} - \mathbf{x}\| + \|\mathbf{x} - \mathbf{y}\| && \text{Corollary 18.11} \\
&\leq \sum_{i=1}^{n} |z_i - x_i| + \|\mathbf{x} - \mathbf{y}\| && \text{Lemma} \\
&< \sum_{i=1}^{n} \frac{r'}{n} + \|\mathbf{x} - \mathbf{y}\| \\
&= r' + \|\mathbf{x} - \mathbf{y}\| \\
&= r - \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{x} - \mathbf{y}\| \\
&= r
\end{aligned}$$

Therefore, by Definition 18.17, $\mathbf{z} \in B$, as desired. □

Corollary 18.20. *A set $U \subset \mathbb{R}^n$ is open if and only if for every $\mathbf{x} \in U$, there exists $r > 0$ such that $B(\mathbf{x}, r) \subset U$.*

Proof. Suppose first that $U \subset \mathbb{R}^n$ is open. Let $\mathbf{x}$ be an arbitrary element of U. By Definition 18.14, there exists an open rectangle R such that $\mathbf{x} \in R \subset U$. Therefore, by Lemma 18.19, there exists $r > 0$ such that $B(\mathbf{x}, r) \subset R \subset U$, as desired.

Now suppose that for all $\mathbf{x} \in U$, there exists $r > 0$ such that $B(\mathbf{x}, r) \subset U$. To prove that U is open, Definition 18.14 tells us that it will suffice to show that for all $\mathbf{x} \in U$, there exists an open rectangle R such that $\mathbf{x} \in R \subset U$. Let $\mathbf{x}$ be an arbitrary element of U. Then there exists $r > 0$ such that $B(\mathbf{x}, r) \subset U$. Therefore, by Lemma 18.19, there exists an open rectangle R such that $\mathbf{x} \in R \subset B \subset U$, as desired. □

7/14: **Corollary 18.21.** *Open balls are open and closed balls are closed.*

Proof. We will take this one claim at a time.

Let $B(\mathbf{x}, r)$ be an arbitrary open ball. To prove that B is open, Definition 18.14 tells us that it will suffice to show that for all $\mathbf{y} \in B$, there exists an open rectangle R such that $\mathbf{y} \in R \subset B$. But by Lemma 18.19, this is true.

Let $\overline{B}(\mathbf{x}, r)$ be an arbitrary closed ball. To prove that $\overline{B}$ is closed, Definition 18.14 tells us that it will suffice to show that $\mathbb{R}^n \setminus \overline{B}$ is open. To do this, Definition 18.14 tells us again that it will suffice to verify that for all $\mathbf{y} \in \mathbb{R}^n \setminus \overline{B}$, there exists an open rectangle R such that $\mathbf{y} \in R \subset \mathbb{R}^n \setminus \overline{B}$. Let $\mathbf{y}$ be an arbitrary element of $\mathbb{R}^n \setminus \overline{B}$. Then by Definition 18.17, $\|\mathbf{y} - \mathbf{x}\| > r$. Thus, $\|\mathbf{y} - \mathbf{x}\| - r > 0$, so we may define $r' = \|\mathbf{y} - \mathbf{x}\| - r$. Now consider $B(\mathbf{y}, r')$. By Lemma 18.19, there exists an open rectangle R such that $\mathbf{y} \in R \subset B$. Consequently, by Script 1, the only thing left to do to verify that $R \subset \mathbb{R}^n \setminus \overline{B}$ is to show that $B \cap \overline{B} = \emptyset$. As such, suppose for the sake of contradiction that $B \cap \overline{B} \neq \emptyset$. Then there exists $\mathbf{z} \in \mathbb{R}^n$ such that $\mathbf{z} \in B$ and $z \in \overline{B}$. It follows by consecutive applications of Definition 18.17 that $\|\mathbf{z} - \mathbf{y}\| < r'$ and $\|\mathbf{z} - \mathbf{x}\| \leq r$. But then we have that

$$
\begin{aligned}
\|\mathbf{x} - \mathbf{y}\| &\leq \|\mathbf{x} - \mathbf{z}\| + \|\mathbf{z} - \mathbf{y}\| && \text{Corollary 18.11} \\
&< r' + r \\
&= \|\mathbf{y} - \mathbf{x}\| - r + r \\
&= \|\mathbf{x} - \mathbf{y}\|
\end{aligned}
$$

a contradiction, as desired. □

Proposition 18.22. *Let $U \subset \mathbb{R}^n$. The following are equivalent:*

(*a*) *U is open.*

(*b*) *U is a (possibly empty) union of open balls.*

(*c*) *U is a (possibly empty) union of open rectangles.*

Proof. As in Theorem 11.5, to prove that statements a-c are equivalent, it will suffice to verify that $a \Rightarrow b$, $b \Rightarrow c$, and $c \Rightarrow a$. Let's begin.

First, suppose that U is open. Then by Corollary 18.20, for every $\mathbf{x} \in U$, there exists $r > 0$ such that $B_{\mathbf{x}}(\mathbf{x}, r) \subset U$. Therefore, $U = \bigcup_{\mathbf{x} \in U} B_{\mathbf{x}}$, as desired.

Second, suppose that U is a union of open balls. Then for every open ball $B(\mathbf{x}, r)$ comprising U, Lemma 18.19 asserts that for every $\mathbf{y} \in B$, there exists an open rectangle $R_{\mathbf{y}}$ such that $\mathbf{y} \in R_{\mathbf{y}} \subset B$. Therefore, $U = \bigcup_{\mathbf{y} \in U} R_{\mathbf{y}}$, as desired.

Third, suppose that U is a union of open rectangles. Then for every $\mathbf{x} \in U$, there exists an open rectangle R such that $\mathbf{x} \in R \subset U$. Therefore, by Definition 18.14, U is open, as desired. □

Remark 18.23. If $X \subset \mathbb{R}^n$, then X is also a topolotical space with the **subspace topology**. That is, $A \subset X$ is **open** (in X) if there exists an open set $U \subset \mathbb{R}^n$ such that $X \cap U = A$. (See Script 8.)

We now discuss functions between Euclidean spaces.

Definition 18.24. Let $A \subset \mathbb{R}^n$ and let $f : A \to \mathbb{R}$. Define the **graph** of f by

$$
\operatorname{graph}(f) = \{(x_1, \ldots, x_n, f(x_1, \ldots, x_n)) \in \mathbb{R}^{n+1} \mid (x_1, \ldots, x_n) \in A\}
$$

Exercise 18.25. For each of the following functions, describe the graph as a subset of $\mathbb{R}^3$.

(a) $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = 2$ for all $(x, y) \in \mathbb{R}^2$.

Description. For this function, we have $\operatorname{graph}(f) = \{(x, y, 2) \in \mathbb{R}^3 \mid (x, y) \in \mathbb{R}^2\}$. This makes the graph equal to the set of all points in $\mathbb{R}^3$ with $z = 2$, which will be a planar, constant, infinite subspace of $\mathbb{R}^3$. □

(b) $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x + y + 1$ for all $(x, y) \in \mathbb{R}^2$.

Description. For this function, we have $\text{graph}(f) = \{(x, y, x + y + 1) \in \mathbb{R}^3 \mid (x, y) \in \mathbb{R}^2\}$. Thus, the graph will be a planar, sloped, infinite subspace of $\mathbb{R}^3$ with gradient pointing in the $\hat{\imath} + \hat{\jmath}$ direction. $\square$

(c) $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2 + y^2$ for all $(x, y) \in \mathbb{R}^2$.

Description. For this function, we have $\text{graph}(f) = \{(x, y, x^2 + y^2) \in \mathbb{R}^3 \mid (x, y) \in \mathbb{R}^2\}$. Thus, the graph will be the paraboloid centered at the origin. $\square$

In Script 9, we gave a definition of continuity that we can generalize to this case:

Definition 18.26. Let X, Y be topological spaces. A function $f : X \to Y$ is **continuous** if for every open set $U \subset Y$, the preimage $f^{-1}(U)$ is open in X.

The function $f : X \to Y$ is **continuous** at $x \in X$ if for every open set $U \subset Y$ containing $f(x)$, the preimage $f^{-1}(U)$ is open in X.

Theorem 18.27.

(a) *A function $f : X \to Y$ is continuous if and only if it is continuous at every $x \in X$.*

(b) *A function $f : X \to Y$ is continuous if and only if $f^{-1}(B)$ is closed in X whenever B is closed in Y.*

Proof. The proofs are symmetric to those of Theorem 9.10 and Proposition 9.5, respectively. $\square$