

# Script 14

## Integrals and Derivatives

### 14.1 Journal

5/4: **Theorem 14.1.** Suppose that  $f$  is integrable on  $[a, b]$ . Define  $F : [a, b] \rightarrow \mathbb{R}$  by

$$F(x) = \int_a^x f$$

If  $f$  is continuous at  $p \in (a, b)$ , then  $F$  is differentiable at  $p$  and

$$F'(p) = f(p)$$

If  $f$  is continuous at  $a$ , then  $F'_+(a)$  exists and equals  $f(a)$ . Similarly, if  $f$  is continuous at  $b$ ,  $F'_-(b)$  exists and equals  $f(b)$ .

*Proof.* To prove that  $F$  is differentiable at  $p$  and  $F'(p) = f(p)$ , Definition 12.1 tells us that it will suffice to show that  $\lim_{h \rightarrow 0^+} \frac{F(p+h) - F(p)}{h} = \lim_{h \rightarrow 0^-} \frac{F(p+h) - F(p)}{h} = f(p)$ . We will tackle the right-handed limit first. To do so, Definition 11.1 tells us that it will suffice to verify that for every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that if  $(p+h) \in [a, b]$  and  $0 < h < \delta$ , then  $|\frac{F(p+h) - F(p)}{h} - f(p)| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Since  $f$  is continuous at  $p$ , Theorem 11.5 asserts that there exists a  $\delta > 0$  such that if  $x \in [a, b]$  and  $|x - p| < \delta$ , then  $|f(x) - f(p)| < \frac{\epsilon}{2}$ . Choose this  $\delta$  to be our  $\delta$ . Let  $h$  be an arbitrary number satisfying  $(p+h) \in [a, b]$  and  $0 < h < \delta$ . Therefore,

$$\begin{aligned} \left| \frac{F(p+h) - F(p)}{h} - f(p) \right| &= \left| \frac{\int_a^{p+h} f - \int_a^p f}{h} - f(p) \right| \\ &= \left| \frac{\int_p^{p+h} f}{h} - f(p) \right| && \text{Theorem 13.23} \\ &= \left| \frac{\int_p^{p+h} f - hf(p)}{h} \right| \\ &= \left| \frac{\int_p^{p+h} f - f(p)((p+h) - p)}{h} \right| \\ &= \left| \frac{\int_p^{p+h} f - \int_p^{p+h} f(p) \, dx}{h} \right| && \text{Exercise 13.17} \\ &= \left| \frac{1}{h} \int_p^{p+h} (f(x) - f(p)) \, dx \right| && \text{Theorem 13.24} \end{aligned}$$

$$\leq \left| \frac{1}{h} \right| \int_p^{p+h} |f(x) - f(p)| \, dx \quad \text{Theorem 13.26}$$

$$\leq \left| \frac{1}{h} \right| \frac{\epsilon}{2} ((p+h) - p) \quad \text{Theorem 13.27}$$

$$= \frac{\epsilon}{2} \\ < \epsilon$$

The proof is symmetric for the left-handed limit. These proofs can also be applied to the endpoints.  $\square$

**Remark 14.2.** Thus, we have that if  $f$  is continuous on  $[a, b]$ ,  $F$  is differentiable on  $[a, b]$  and  $F'(p) = f(p)$  for all  $p \in [a, b]$  (where at the endpoints, we understand that the derivative should be interpreted as the one-sided derivative).

**Lemma 14.3.** Suppose that  $f : [a, b] \rightarrow \mathbb{R}$  is integrable and that  $\Omega$  is a number satisfying  $L(f, P) \leq \Omega \leq U(f, P)$  for all partitions  $P$  of  $[a, b]$ . Then

$$\int_a^b f = \Omega$$

*Proof.* Suppose for the sake of contradiction that  $\int_a^b f \neq \Omega$ . We divide into two cases ( $\int_a^b f < \Omega$  and  $\int_a^b f > \Omega$ ). If  $\int_a^b f < \Omega$ , then by Definition 13.16,  $U(f) = \int_a^b f < \Omega$ . It follows by Definition 13.14 and 5.11 that there exists an object  $U(f, P) \in \{U(f, P) \mid P \text{ is a partition of } [a, b]\}$  such that  $U(f) \leq U(f, P) < \Omega$ . But this contradicts the hypothesis that  $U(f, P) \geq \Omega$  for all partitions  $P$  of  $[a, b]$ . The argument is symmetric in the other case.  $\square$

5/6: **Theorem 14.4.** Let  $f$  be integrable on  $[a, b]$ . Suppose that there is a function  $G$  that is continuous on  $[a, b]$  and differentiable on  $(a, b)$  and such that  $f = G'$  on  $(a, b)$ . Then

$$\int_a^b f = G(b) - G(a)$$

*Proof.* To prove that  $\int_a^b f = G(b) - G(a)$ , Lemma 14.3 tells us that it will suffice to show that for all partitions  $P = \{t_0, \dots, t_n\}$  of  $[a, b]$ ,  $L(f, P) \leq G(b) - G(a) \leq U(f, P)$ . Let  $P$  be an arbitrary partition of  $[a, b]$ . If  $t_i, t_{i-1}$  are two sequential elements of  $P$ , then since  $G$  is continuous on  $[t_{i-1}, t_i] \subset [a, b]$  and differentiable on  $(t_{i-1}, t_i) \subset (a, b)$ , Corollary 12.16 asserts that there exists a point  $\lambda \in (t_{i-1}, t_i)$  such that  $G(t_i) - G(t_{i-1}) = G'(\lambda)(t_i - t_{i-1})$ . It follows since  $f = G'$  on  $(a, b)$  that

$$G(t_i) - G(t_{i-1}) = f(\lambda)(t_i - t_{i-1})$$

Thus, since we have proven the above statement for an arbitrary  $i$ , we can apply it to all  $i$  and sum to get

$$\begin{aligned} \sum_{i=1}^n f(\lambda)(t_i - t_{i-1}) &= \sum_{i=1}^n G(t_i) - G(t_{i-1}) \\ &= G(b) - G(a) \end{aligned}$$

But by Definitions 13.11, 5.7, and 5.6, we have that  $m_i(f) \leq f(\lambda) \leq M_i(f)$  for all  $i$ . It follows that

$$\sum_{i=1}^n m_i(f)(t_i - t_{i-1}) \leq \sum_{i=1}^n f(\lambda)(t_i - t_{i-1}) \leq \sum_{i=1}^n M_i(f)(t_i - t_{i-1})$$

Therefore, we have by Definition 13.11 and substitution that

$$L(f, P) \leq G(b) - G(a) \leq U(f, P)$$

as desired.  $\square$

**Corollary 14.5.** *Let  $f, g$  be functions defined on some open interval containing  $[a, b]$  such that  $f'$  and  $g'$  exist and are continuous on  $[a, b]$ . Then*

$$\int_a^b f g' = [f(b)g(b) - f(a)g(a)] - \int_a^b f' g$$

*Proof.* Since  $f$  and  $g$  are differentiable on  $[a, b]$ , Exercise 12.9 implies that  $fg$  is differentiable on  $[a, b]$  with  $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$  for all  $x \in [a, b]$ . We now seek to prove that  $f'g + fg'$  is integrable on  $[a, b]$ . By hypothesis,  $f'$  and  $g'$  are continuous on  $[a, b]$ . Additionally, since  $f$  and  $g$  are differentiable on  $[a, b]$ , Theorem 12.5 asserts that they are continuous on  $[a, b]$ . Thus, since  $f, g, f'$ , and  $g'$  are continuous on  $[a, b]$ , we have by consecutive applications of Corollary 11.10 that  $f'g + fg'$  is continuous on  $[a, b]$ . Consequently, by Theorem 13.19,  $f'g + fg'$  is integrable on  $[a, b]$ , as desired. Furthermore, in a similar manner to the above, we can show that  $f'g$  and  $fg'$  are integrable on  $[a, b]$ . Lastly, it follows from the fact that  $f$  and  $g$  are continuous on  $[a, b]$  by Corollary 11.10 that  $fg$  is continuous on  $[a, b]$ .

Having established that  $f'g + fg'$  is integrable on  $[a, b]$ , that  $fg$  is a function that is continuous on  $[a, b]$ , differentiable on  $(a, b) \subset [a, b]$ , and such that  $f'g + fg' = (fg)'$  on  $(a, b)$ , and that  $f'g$  and  $fg'$  are integrable on  $[a, b]$ , we have that

$$\int_a^b (f'g + fg') = (fg)(b) - (fg)(a) \quad \text{Theorem 14.4}$$

$$\int_a^b f'g + \int_a^b fg' = f(b)g(b) - f(a)g(a) \quad \text{Theorem 13.24}$$

$$\int_a^b fg' = [f(b)g(b) - f(a)g(a)] - \int_a^b f'g$$

as desired.  $\square$

**Corollary 14.6.** *Let  $g$  be a function defined on some interval containing  $[a, b]$  such that  $g'$  is continuous on  $[a, b]$ . Suppose that  $g([a, b]) \subset [c, d]$  and  $f : [c, d] \rightarrow \mathbb{R}$  is continuous. Define  $F : [c, d] \rightarrow \mathbb{R}$  by  $F(x) = \int_c^x f$ . Then*

$$\int_a^b f(g(x)) \cdot g'(x) dx = F(g(b)) - F(g(a))$$

*Proof.* To prove that  $\int_a^b f(g(x)) \cdot g'(x) dx = F(g(b)) - F(g(a))$ , Theorem 14.4 tells us that it will suffice to show that  $(f \circ g) \cdot g'$  is integrable on  $[a, b]$ ,  $F \circ g$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , and  $(f \circ g) \cdot g' = (F \circ g)'$  on  $(a, b)$ . We will confirm each requirement in turn. Let's begin.

To confirm that  $(f \circ g) \cdot g'$  is integrable on  $[a, b]$ , Theorem 13.19 tells us that it will suffice to demonstrate that  $(f \circ g) \cdot g'$  is continuous on  $[a, b]$ . By hypothesis,  $f$  is continuous on  $[c, d]$ . Additionally, since  $g'$  is defined on  $[a, b]$ , we know that  $g$  is differentiable on  $[a, b]$ , which implies by Theorem 12.5 that  $g$  is continuous on  $[a, b]$ . The combination of the previous two results implies by Corollary 11.15 that  $f \circ g$  is continuous on  $[a, b]$ . This combined with the hypothesis that  $g'$  is continuous on  $[a, b]$  implies by Corollary 11.10 that  $(f \circ g) \cdot g'$  is continuous on  $[a, b]$ .

To confirm that  $F \circ g$  is continuous on  $[a, b]$ , Corollary 11.15 tells us that it will suffice to demonstrate that  $F$  is continuous on  $[c, d]$  and  $g$  is continuous on  $[a, b]$ . By Theorem 13.28,  $F$  is continuous on  $[c, d]$ . Additionally, we know by the above that  $g$  is continuous on  $[a, b]$ .

To confirm that  $F \circ g$  is differentiable on  $(a, b)$ , Theorem 12.10 tells us that it will suffice to demonstrate that  $F$  is differentiable on  $(c, d)$  and  $g$  is differentiable on  $(a, b)$ . Since  $f$  is continuous on  $(c, d) \subset [c, d]$ , we have by Theorem 14.1 that  $F$  is differentiable on  $(c, d)$ . Additionally, we know by the above that  $g$  is differentiable on  $(a, b) \subset [a, b]$ .

Since  $F \circ g$  is differentiable on  $(a, b)$ , we have by Theorem 12.10 again that  $(F \circ g)' = (F' \circ g) \cdot g'$  for all  $x \in (a, b)$ . Thus, since  $F' = f$  by Theorem 14.1, we have that  $(f \circ g) \cdot g' = (F \circ g)'$  on  $(a, b)$ , as desired.  $\square$