Script 18

The Euclidean Space $\mathbb{R}^n$

7/7: For the next three sheets, we will be studying multivariable calculus, that is "calculus on $\mathbb{R}^n$." First, we need to understand the space $\mathbb{R}^n$.

Definition 18.1. The **Euclidean *n*-space** $\mathbb{R}^n$ is the n-fold Cartesian product of $\mathbb{R}$. Symbolically,

$$\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{R}\}$$

is the set of n-tuples of real numbers. We often write

$$\mathbf{x} = (x_1, \dots, x_n)$$

to denote an element, which is also referred to as a **vector**, in $\mathbb{R}^n$ and

$$\mathbf{0} = (0, \dots, 0)$$

Definition 18.2. Let $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. We define the following operations.

(a) (Addition) $\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_n + y_n)$.

(b) (Scalar Multiplication) $\lambda\mathbf{x} = (\lambda x_1, \dots, \lambda x_n)$.

Exercise 18.3. Prove that the addition on $\mathbb{R}^n$ satisfies FA1-FA4 (see Definition 7.8). Moreover, prove that

VS1. (Associativity of Scalar Multiplication) If $\lambda, \mu \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, then $(\lambda\mu)\mathbf{x} = \lambda(\mu\mathbf{x})$.

VS2. (Distributivity of Scalars) If $\lambda, \mu \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, then $(\lambda + \mu)\mathbf{x} = \lambda\mathbf{x} + \mu\mathbf{x}$.

VS3. (Distributivity of Vectors) If $\lambda \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then $\lambda(\mathbf{x} + \mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y}$.

VS4. (Scalar Multiplicative Identity) If $\mathbf{x} \in \mathbb{R}^n$, then $1\mathbf{x} = \mathbf{x}$.

These eight properties together are called the **vector space axioms**.

Proof. To prove that $\mathbb{R}^n$ obeys FA1 from Definition 7.8, it will suffice to show that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$. Let $\mathbf{x}, \mathbf{y}$ be arbitrary elements of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}\mathbf{x} + \mathbf{y} &= (x_1 + y_1, \dots, x_n + y_n)\\ &= (y_1 + x_1, \dots, y_n + x_n)\\ &= \mathbf{y} + \mathbf{x}\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys FA2 from Definition 7.8, it will suffice to show that for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$. Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be arbitrary elements of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}
(\mathbf{x} + \mathbf{y}) + \mathbf{z} &= (x_1 + y_1, \dots, x_n + y_n) + \mathbf{z} \\
&= ((x_1 + y_1) + z_1, \dots, (x_n + y_n) + z_n) \\
&= (x_1 + (y_1 + z_1), \dots, x_n + (y_n + z_n)) \\
&= \mathbf{x} + (y_1 + z_1, \dots, y_n + z_n) \\
&= \mathbf{x} + (\mathbf{y} + \mathbf{z})
\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys FA3 from Definition 7.8, it will suffice to find an element $0 \in \mathbb{R}^n$ such that $\mathbf{x} + 0 = 0 + \mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Choose $\mathbf{0}$ to be our 0. Let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}
\mathbf{x} + \mathbf{0} &= (x_1 + 0, \dots, x_n + 0) \\
&= (x_1, \dots, x_n) \\
&= \mathbf{x} \\
&= (0 + x_1, \dots, 0 + x_n) \\
&= \mathbf{0} + \mathbf{x}
\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys FA4 from Definition 7.8, it will suffice to show that for all $\mathbf{x} \in \mathbb{R}^n$, there exists $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x} = 0$. Let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Choose $\mathbf{y} = (-x_1, \dots, -x_n)$. Then by Definition 18.2,

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= (x_1 + (-x_1), \dots, x_n + (-x_n)) \\
&= (0, \dots, 0) \\
&= \mathbf{0} \\
&= ((-x_1) + x_1, \dots, (-x_n) + x_n) \\
&= \mathbf{y} + \mathbf{x}
\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys VS1, it will suffice to show that for all $\lambda, \mu \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, we have $(\lambda\mu)\mathbf{x} = \lambda(\mu\mathbf{x})$. Let λ, μ be arbitrary elements of $\mathbb{R}$, and let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}
(\lambda\mu)\mathbf{x} &= ((\lambda\mu)x_1, \dots, (\lambda\mu)x_n) \\
&= (\lambda(\mu x_1), \dots, \lambda(\mu x_n)) \\
&= \lambda(\mu\mathbf{x})
\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys VS2, it will suffice to show that for all $\lambda, \mu \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, we have $(\lambda + \mu)\mathbf{x} = \lambda\mathbf{x} + \mu\mathbf{x}$. Let λ, μ be arbitrary elements of $\mathbb{R}$, and let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}
(\lambda + \mu)\mathbf{x} &= ((\lambda + \mu)x_1, \dots, (\lambda + \mu)x_n) \\
&= (\lambda x_1 + \mu x_1, \dots, \lambda x_n + \mu x_n) \\
&= (\lambda x_1, \dots, \lambda x_n) + (\mu x_1, \dots, \mu x_n) \\
&= \lambda\mathbf{x} + \mu\mathbf{x}
\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys VS3, it will suffice to show that for all $\lambda \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have $\lambda(\mathbf{x}+\mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y}$. Let λ be an arbitrary element of $\mathbb{R}$, and let $\mathbf{x}, \mathbf{y}$ be arbitrary elements of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}
\lambda(\mathbf{x}+\mathbf{y}) &= \lambda(x_1+y_1, \dots, x_n+y_n) \\
&= (\lambda(x_1+y_1), \dots, \lambda(x_n+y_n)) \\
&= (\lambda x_1 + \lambda y_1, \dots, \lambda x_n + \lambda y_n) \\
&= (\lambda x_1, \dots, \lambda x_n) + (\lambda y_1, \dots, \lambda y_2) \\
&= \lambda\mathbf{x} + \lambda\mathbf{y}
\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys VS4, it will suffice to show that for all $\mathbf{x} \in \mathbb{R}^n$, we have $1\mathbf{x} = \mathbf{x}$. Let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}
1\mathbf{x} &= (1x_1, \dots, 1x_n) \\
&= (x_1, \dots, x_n) \\
&= \mathbf{x}
\end{aligned}$$

as desired. □

Remark 18.4. Since $\mathbb{R}^n$ with the two operations defined as above satisfies these eight axioms, we call $\mathbb{R}^n$ a **vector space**.

Exercise 18.5. Prove that if $\mathbf{x} \in \mathbb{R}^n$, then $0\mathbf{x} = \mathbf{0}$.

Proof. By Definition 18.2, we have that

$$\begin{aligned}
0\mathbf{x} &= (0x_1, \dots, 0x_n) \\
&= (0, \dots, 0) \\
&= \mathbf{0}
\end{aligned}$$

as desired. □

Definition 18.6. Let $\mathbf{x} \in \mathbb{R}^n$. The **norm** of $\mathbf{x}$ is defined as

$$\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

Definition 18.7. We call $\|\mathbf{y} - \mathbf{x}\|$ the **distance** between $\mathbf{x}$ and $\mathbf{y}$.

Remark 18.8. If $n = 1$, the norm coincides with the definition of the absolute value in $\mathbb{R}$.

Lemma 18.9.

(a) *If $x, y \in \mathbb{R}$, then $xy \leq \frac{x^2+y^2}{2}$.*

Proof. Let x, y be arbitrary elements of $\mathbb{R}$. Then by Lemma 7.26, $0 \leq (x-y)^2$. Therefore, we have that

$$\begin{aligned}
xy &= \frac{2xy + 0}{2} \\
&\leq \frac{2xy + (x-y)^2}{2} \\
&= \frac{2xy + x^2 - 2xy + y^2}{2} \\
&= \frac{x^2+y^2}{2}
\end{aligned}$$

as desired. □

(*b*) *If* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$*, then* $|x_1y_1 + \cdots + x_ny_n| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|$.

Proof. Suppose first that $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$. Then by Definition 18.6, $\|\mathbf{x}\| = 1 = \sqrt{x_1^2 + \cdots + x_n^2}$, from which it follows that $1 = x_1^2 + \cdots + x_n^2$. Therefore, we have that

$$\begin{aligned}
|x_1y_1 + \cdots + x_ny_n| &\leq |x_1y_1| + \cdots + |x_ny_n| && \text{Lemma 8.8} \\
&= |x_1||y_1| + \cdots + |x_n||y_n| \\
&\leq \frac{|x_1|^2 + |y_1|^2}{2} + \cdots + \frac{|x_n|^2 + |y_n|^2}{2} && \text{Lemma 18.9a} \\
&= \frac{x_1^2 + y_1^2}{2} + \cdots + \frac{x_n^2 + y_n^2}{2} \\
&= \frac{(x_1^2 + \cdots + x_n^2) + (y_1^2 + \cdots + y_n^2)}{2} \\
&= \frac{1+1}{2} \\
&= 1
\end{aligned}$$

as desired.

Now let $\mathbf{x}, \mathbf{y}$ be arbitrary elements of $\mathbb{R}^n$. Consider the vectors $\mathbf{u_x}, \mathbf{u_y}$ defined by $\mathbf{u_x} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$ and $\mathbf{u_y} = \frac{\mathbf{y}}{\|\mathbf{y}\|}$. By the proof of the first case, we have that

$$\begin{aligned}
|x_1y_1 + \cdots + x_ny_n| &= \|\mathbf{x}\| \cdot \|\mathbf{y}\| \cdot \left| \frac{x_1y_1}{\|\mathbf{x}\| \cdot \|\mathbf{y}\|} + \cdots + \frac{x_ny_n}{\|\mathbf{x}\| \cdot \|\mathbf{y}\|} \right| \\
&= \|\mathbf{x}\| \cdot \|\mathbf{y}\| \cdot |u_{\mathbf{x}_1}u_{\mathbf{y}_1} + \cdots + u_{\mathbf{x}_n}u_{\mathbf{y}_n}| \\
&\leq \|\mathbf{x}\| \cdot \|\mathbf{y}\| \cdot 1 \\
&= \|\mathbf{x}\| \cdot \|\mathbf{y}\|
\end{aligned}$$

as desired. ☐

Theorem 18.10. *If* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}$*, then*

(*a*) $\|\mathbf{x}\| \geq 0$*. Moreover,* $\|\mathbf{x}\| = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$.

Proof. Let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$.

We first prove that $\|\mathbf{x}\| \geq 0$. By Lemma 7.26, $x_i^2 \geq 0$ for all $i \in [n]$. Thus, by Definition 7.21, $x_1^2 + \cdots + x_n^2 \geq 0$. Therefore, we have by Definition 18.6 that $\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2} \geq 0$, as desired.

We now prove that $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$. Suppose first that $\|\mathbf{x}\| = 0$. Then by Definition 18.6 and Script 7, $x_1^2 + \cdots + x_n^2 = 0$. Now suppose for the sake of contradiction that $\mathbf{x} \neq \mathbf{0}$. Then there exists an x_i such that $x_i \neq 0$. Thus, by Lemma 7.26, $x_i^2 > 0$. Additionally, $x_j^2 \geq 0$ for all $j \in [n]$. Thus, we have that $0 < x_i^2 \leq x_1^2 + \cdots + x_n^2$. But by Definition 3.1, this implies that $x_1^2 + \cdots + x_n^2 \neq 0$, a contradiction.

Now suppose that $\mathbf{x} = \mathbf{0}$. Then by Definition 18.6, $\|\mathbf{x}\| = \sqrt{0^2 + \cdots + 0^2} = 0$, as desired. ☐

(*b*) $\|\lambda\mathbf{x}\| = |\lambda| \cdot \|\mathbf{x}\|$.

Proof. Let λ be an arbitrary element of $\mathbb{R}$, and let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Then we have that

$$\begin{aligned}
\|\lambda\mathbf{x}\| &= \sqrt{(\lambda x_1)^2 + \cdots + (\lambda x_n)^2} && \text{Definition 18.6} \\
&= |\lambda| \cdot \sqrt{x_1^2 + \cdots + x_n^2} \\
&= |\lambda| \cdot \|\mathbf{x}\| && \text{Definition 18.6}
\end{aligned}$$

as desired. ☐

(c) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

Proof. Let $\mathbf{x}, \mathbf{y}$ be arbitrary elements of $\mathbb{R}^n$. Then we have that

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\| &= \sqrt{(x_1+y_1)^2 + \cdots + (x_n+y_n)^2} && \text{Definition 18.6}\\
&= \sqrt{(x_1^2 + \cdots + x_n^2) + (2x_1y_1 + \cdots + 2x_ny_n) + (y_1^2 + \cdots + y_n^2)}\\
&\leq \sqrt{\|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2} && \text{Lemma 18.9}\\
&= \sqrt{(\|\mathbf{x}\| + \|\mathbf{y}\|)^2}\\
&= \|\mathbf{x}\| + \|\mathbf{y}\|
\end{aligned}$$

as desired. □

Corollary 18.11. *If $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, then*

(a) $\|\mathbf{x}-\mathbf{z}\| \leq \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{y}-\mathbf{z}\|$.

(b) $|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq \|\mathbf{x}-\mathbf{y}\|$.

Proof. The proofs are symmetric to those of Lemma 8.8. □

7/10: The next goal is to "topologize" $\mathbb{R}^n$. To discuss topology on $\mathbb{R}^n$, we first need to introduce notions for $\mathbb{R}^n$ that are analogous to open and closed intervals for $\mathbb{R}$.

Remark 18.12. For $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\mathbf{y} = (y_1, \dots, y_m) \in \mathbb{R}^m$, we identify $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m$ with $(x_1, \dots, x_n, y_1, \dots, y_m) \in \mathbb{R}^{n+m}$. So if $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$, we can consider $A \times B$ to be a subset of $\mathbb{R}^{n+m}$.

If also $C \in \mathbb{R}^k$, then $(A \times B) \times C$ and $A \times (B \times C)$ correspond to the same subset of $\mathbb{R}^{n+m+k}$ under this identification; we write $A \times B \times C$ for this set.

Definition 18.13. An **open rectangle** in $\mathbb{R}^n$ is a set of the form $(a_1, b_1) \times \cdots \times (a_n, b_n)$, a product of open intervals. Similarly, a **closed rectangle** in $\mathbb{R}^n$ is a set of the form $[a_1, b_1] \times \cdots \times [a_n, b_n]$. We allow the possibility that $a_j = b_j$ (where $[a_j, a_j] = \{a_j\}$). If there is at least one j with $a_j = b_j$, then we say that the rectangle is **degenerate**; otherwise, we say that the rectangle is **non-degenerate**.

Definition 18.14. A subset $U \subset \mathbb{R}^n$ is **open** if for all $\mathbf{x} \in U$, there exists an open rectangle R such that $\mathbf{x} \in R \subset U$. A subset $C \in \mathbb{R}^n$ is **closed** if its compliment is open.

Exercise 18.15. Decide whether each of the following is an open set in $\mathbb{R}^2$.

(a) $\{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R},\ x_1 > 0,\ x_2 > 0\}$.

Proof. To prove that $U = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R},\ x_1 > 0,\ x_2 > 0\}$ is open, Definition 18.14 tells us that it will suffice to show that for all $\mathbf{x} \in U$, there exists an open rectangle R such that $\mathbf{x} \in R \subset U$. Let $\mathbf{x}$ be an arbitrary element of U. Then by the definition of U, $0 < x_1$ and $0 < x_2$. It follows by Theorem 5.2 and Corollary 6.12, there exist a_1, b_1, a_2, b_2 such that $0 < a_1 < x_1 < b_1$ and $0 < a_2 < x_2 < b_2$. Thus, by Equations 8.1, $x_1 \in (a_1, b_1)$ and $x_2 \in (a_2, b_2)$. Consequently, if we let $R = (a_1, b_1) \times (a_2, b_2)$, Definition 18.13 guarantees that R is an open rectangle. Additionally, Definition 1.15 asserts that $(x_1, x_2) = \mathbf{x} \in R$, as desired. Additionally, if $\mathbf{y}$ is any vector in R, then by the definition of R, $0 < a_1 < y_1$ and $0 < a_2 < y_2$. Thus, by transitivity, $\mathbf{y} \in U$. Therefore, by Definition 1.3, $R \subset U$, as desired. □

(b) $\{(x, 0) \mid x \in \mathbb{R}\}$.

Proof. To prove that $U = \{(x, 0) \mid x \in \mathbb{R}\}$ is not open, Definition 18.14 tells us that it will suffice to find an $\mathbf{x} \in U$ such that for all open rectangles R containing $\mathbf{x}$, $R \not\subset U$. Let $\mathbf{x} = (0, 0)$, and let R be an arbitrary open rectangle containing $\mathbf{x}$. By Definitions 18.13 and 1.15 along with Equations 8.1, $a_1 < 0 < b_1$ and $a_2 < 0 < b_2$. Thus, by consecutive applications of Theorem 5.2, there exist points $y_1, y_2 \in \mathbb{R}$ such that $a_1 < y_1 < 0$ and $a_2 < y_2 < 0$. It follows that $\mathbf{y} = (y_1, y_2) \in R$. However, since $y_2 \neq 0$ by Definition 3.1, $\mathbf{y} \notin U$. Therefore, by Definition 1.3, $R \not\subset U$, as desired. □

Exercise 18.16. Show that if $R_1, \dots, R_m$ are open rectangles containing $\mathbf{x} \in \mathbb{R}^n$, then $R = R_1 \cap \cdots \cap R_m$ is an open rectangle containing $\mathbf{x} \in \mathbb{R}^n$. If $R = (a_1, b_1) \times \cdots \times (a_n, b_n)$, derive formulas for a_i and b_i in terms of the corresponding quantities for $R_1, \dots, R_m$.

Proof. Let $R_i = (r_{ij}, s_{ij})_{j=1}^n$ for all $i \in [m]$. To prove that $R = \bigcap_{i=1}^m R_i$ is an open rectangle containing $\mathbf{x}$, Definitions 18.13 and 1.15 tell us that it will suffice to show that R is the Cartesian product of open intervals, each containing its respective x_j. Since $\mathbf{x} \in R_i$ for all $i \in [m]$, we have by Definition 1.15 that $x_j \in (r_{ij}, s_{ij})$ for all $i \in [m]$, $j \in [n]$. Thus, by Corollary 3.19, $\bigcap_{i=1}^m (r_{ij}, s_{ij})$ is a region (hence an open interval by Corollary 4.11 and Lemma 8.3) containing x_j for all $j \in [n]$. Therefore, since $R = \bigcap_{i=1}^m R_i = \prod_{i=1}^n (\bigcap_{i=1}^m (r_{ij}, s_{ij}))$ by Script 1, we have that R is the Cartesian product of open intervals, each containing its respective x_j, as desired.

Let $a_j = \max_{i=1}^m (r_{ij})$ and let $b_j = \min_{i=1}^m (s_{ij})$ for all $j \in [n]$. To prove that $R = (a_j, b_j)_{j=1}^n$, Definition 1.2 tells us that it will suffice to show that every $\mathbf{x} \in R$ is an element of $(a_j, b_j)_{j=1}^n$ and vice versa. Suppose first that $\mathbf{x}$ is an arbitrary element of R. Then by Definition 1.6, $\mathbf{x} \in R_i$ for all $i \in [m]$. It follows by Definition 1.15 that $x_j \in (r_{ij}, s_{ij})$ for all $i \in [m]$, $j \in [n]$, including the j, j' for which r_{ij} is at its maximum and $s_{ij'}$ is at its minimum. In other words, $x_j \in (a_j, b_j)$ for all $j \in [n]$. Therefore, by Definition 1.15, $\mathbf{x} \in (a_j, b_j)_{j=1}^n$, as desired. The proof is symmetric in the other direction. ☐

Definition 18.17. The **open ball** (in $\mathbb{R}^n$ with center $\mathbf{p}$ and radius $r > 0$) is defined as

$$B(\mathbf{p}, r) = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{p}\| < r\}$$

The **closed ball** (in $\mathbb{R}^n$ with center $\mathbf{p}$ and radius $r > 0$) is defined as

$$\overline{B}(\mathbf{p}, r) = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{p}\| \leq r\}$$

Remark 18.18. In $\mathbb{R}^1$, an open rectangle is also an open ball, and vice versa.

The following results illustrate how open rectangles and open balls in $\mathbb{R}^n$ are "compatible" with each other.

Lemma 18.19. *Fix* $\mathbf{x} \in \mathbb{R}$.

(*a*) *If R is an open rectangle containing* $\mathbf{x}$*, then there exists $r > 0$ such that $B(\mathbf{x}, r) \subset R$.*

Proof. Since $\mathbf{x} \in R$, Definitions 18.13 and 1.15 tell us that that $x_i \in (a_i, b_i)$ for all $i \in [n]$. Additionally, we know by Corollary 4.11 and Lemma 8.3 that each (a_i, b_i) is an open interval. Combining the last two results, we have by Lemma 8.10 that for each $i \in [n]$, there exists $\delta_i > 0$ such that $(x_i - \delta_i, x_i + \delta_i) \subset (a_i, b_i)$. Let $r = \min\{\delta_i\}_{i=1}^n$.

To prove that $B(\mathbf{x}, r) \subset R$, Definition 1.3 tells us that it will suffice to show that every $\mathbf{y} \in B(\mathbf{x}, r)$ is an element of R. Let $\mathbf{y}$ be an arbitrary element of $B(\mathbf{x}, r)$. Then by Definition 18.17, $\|\mathbf{y} - \mathbf{x}\| < r$. It follows that

$$\begin{aligned}
|y_i - x_i| &= \sqrt{(y_i - x_i)^2} \\
&\leq \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2} && \text{Lemma 7.26} \\
&= \|\mathbf{y} - \mathbf{x}\| && \text{Definition 18.6} \\
&< r
\end{aligned}$$

for all $i \in [n]$. Thus, by the definition of r, $|y_i - x_i| \leq \delta_i$ for all $i \in [n]$. Consequently, by Exercise 8.9 and Definition 1.3, $y_i \in (a_i, b_i)$ for all $i \in [n]$. Therefore, by Definitions 1.15 and 18.13, $\mathbf{y} \in R$, as desired. ☐

(*b*) *If B is an open ball containing* $\mathbf{x}$*, then there exists an open rectangle R such that $\mathbf{x} \in R \subset B$.*

Lemma. *If $\mathbf{x} \in \mathbb{R}^n$, then $\|\mathbf{x}\| \leq \sum_{i=1}^n |x_i|$.*

Proof. By Definition 18.2, we can decompose $\mathbf{x}$ into the sum of n unit vectors $\mathbf{u_i}$ (where $\mathbf{u_i}$ points one unit in the i^{th} direction), each scaled by x_i; symbolically, let $\mathbf{x} = \sum_{i=1}^n x_i\mathbf{u_i}$. Therefore,

$$\begin{aligned}
\|\mathbf{x}\| &= \left\|\sum_{i=1}^n x_i\mathbf{u_i}\right\| \\
&= \sum_{i=1}^n \|x_i\mathbf{u_i}\| && \text{Theorem 18.10c} \\
&= \sum_{i=1}^n |x_i| \cdot \|\mathbf{u_i}\| && \text{Theorem 18.10b} \\
&= \sum_{i=1}^n |x_i| \cdot \sqrt{1^2} && \text{Definition 18.6} \\
&= \sum_{i=1}^n |x_i|
\end{aligned}$$

as desired. $\square$

Proof of Lemma 18.19b. Suppose $\mathbf{x} \in B(\mathbf{y}, r)$. Then by Definition 18.17, $\|\mathbf{x} - \mathbf{y}\| < r$. Thus, we can define $r' = r - \|\mathbf{x} - \mathbf{y}\|$ such that $r' > 0$. With this term defined, we can let $R = (x_i - \frac{r'}{n}, x_i + \frac{r'}{n})_{i=1}^n$.

To prove that $\mathbf{x} \in R$, Definition 18.13 tells us that it will suffice to show that $x_i \in (x_i - \frac{r'}{n}, x_i + \frac{r'}{n})$ for all $i \in [n]$. But since $|x_i - x_i| = 0 < \frac{r'}{n}$ for all $i \in [n]$, Exercise 8.9 asserts that this is true.

To prove that $R \subset B$, Definition 1.3 tells us that it will suffice to show that every $\mathbf{z} \in R$ is an element of B. Let $\mathbf{z}$ be an arbitrary element of R. Then by Definition 18.13, $z_i \in (x_i - \frac{r'}{n}, x_i + \frac{r'}{n})$ for all $i \in [n]$. It follows by Exercise 8.9 that $|z_i - x_i| < \frac{r'}{n}$ for all $i \in [n]$. Consequently,

$$\begin{aligned}
\|\mathbf{z} - \mathbf{y}\| &\leq \|\mathbf{z} - \mathbf{x}\| + \|\mathbf{x} - \mathbf{y}\| && \text{Corollary 18.11} \\
&\leq \sum_{i=1}^n |z_i - x_i| + \|\mathbf{x} - \mathbf{y}\| && \text{Lemma} \\
&< \sum_{i=1}^n \frac{r'}{n} + \|\mathbf{x} - \mathbf{y}\| \\
&= r' + \|\mathbf{x} - \mathbf{y}\| \\
&= r - \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{x} - \mathbf{y}\| \\
&= r
\end{aligned}$$

Therefore, by Definition 18.17, $\mathbf{z} \in B$, as desired. $\square$

Corollary 18.20. *A set $U \subset \mathbb{R}^n$ is open if and only if for every $\mathbf{x} \in U$, there exists $r > 0$ such that $B(\mathbf{x}, r) \subset U$.*

Proof. Suppose first that $U \subset \mathbb{R}^n$ is open. Let $\mathbf{x}$ be an arbitrary element of U. By Definition 18.14, there exists an open rectangle R such that $\mathbf{x} \in R \subset U$. Therefore, by Lemma 18.19, there exists $r > 0$ such that $B(\mathbf{x}, r) \subset R \subset U$, as desired.

Now suppose that for all $\mathbf{x} \in U$, there exists $r > 0$ such that $B(\mathbf{x}, r) \subset U$. To prove that U is open, Definition 18.14 tells us that it will suffice to show that for all $\mathbf{x} \in U$, there exists an open rectangle R such that $\mathbf{x} \in R \subset U$. Let $\mathbf{x}$ be an arbitrary element of U. Then there exists $r > 0$ such that $B(\mathbf{x}, r) \subset U$. Therefore, by Lemma 18.19, there exists an open rectangle R such that $\mathbf{x} \in R \subset B \subset U$, as desired. $\square$

7/14: **Corollary 18.21.** *Open balls are open and closed balls are closed.*

Proof. We will take this one claim at a time.

Let $B(\mathbf{x}, r)$ be an arbitrary open ball. To prove that B is open, Definition 18.14 tells us that it will suffice to show that for all $\mathbf{y} \in B$, there exists an open rectangle R such that $\mathbf{y} \in R \subset B$. But by Lemma 18.19, this is true.

Let $\overline{B}(\mathbf{x}, r)$ be an arbitrary closed ball. To prove that $\overline{B}$ is closed, Definition 18.14 tells us that it will suffice to show that $\mathbb{R}^n \setminus \overline{B}$ is open. To do this, Definition 18.14 tells us again that it will suffice to verify that for all $\mathbf{y} \in \mathbb{R}^n \setminus \overline{B}$, there exists an open rectangle R such that $\mathbf{y} \in R \subset \mathbb{R}^n \setminus \overline{B}$. Let $\mathbf{y}$ be an arbitrary element of $\mathbb{R}^n \setminus \overline{B}$. Then by Definition 18.17, $\|\mathbf{y} - \mathbf{x}\| > r$. Thus, $\|\mathbf{y} - \mathbf{x}\| - r > 0$, so we may define $r' = \|\mathbf{y} - \mathbf{x}\| - r$. Now consider $B(\mathbf{y}, r')$. By Lemma 18.19, there exists an open rectangle R such that $\mathbf{y} \in R \subset B$. Consequently, by Script 1, the only thing left to do to verify that $R \subset \mathbb{R}^n \setminus \overline{B}$ is to show that $B \cap \overline{B} = \emptyset$. As such, suppose for the sake of contradiction that $B \cap \overline{B} \neq \emptyset$. Then there exists $\mathbf{z} \in \mathbb{R}^n$ such that $\mathbf{z} \in B$ and $z \in \overline{B}$. It follows by consecutive applications of Definition 18.17 that $\|\mathbf{z} - \mathbf{y}\| < r'$ and $\|\mathbf{z} - \mathbf{x}\| \leq r$. But then we have that

$$\begin{aligned} \|\mathbf{x} - \mathbf{y}\| &\leq \|\mathbf{x} - \mathbf{z}\| + \|\mathbf{z} - \mathbf{y}\| && \text{Corollary 18.11} \\ &< r' + r \\ &= \|\mathbf{y} - \mathbf{x}\| - r + r \\ &= \|\mathbf{x} - \mathbf{y}\| \end{aligned}$$

a contradiction, as desired. □

Proposition 18.22. *Let $U \subset \mathbb{R}^n$. The following are equivalent:*

(*a*) *U is open.*

(*b*) *U is a (possibly empty) union of open balls.*

(*c*) *U is a (possibly empty) union of open rectangles.*

Proof. As in Theorem 11.5, to prove that statements a-c are equivalent, it will suffice to verify that $a \Rightarrow b$, $b \Rightarrow c$, and $c \Rightarrow a$. Let's begin.

First, suppose that U is open. Then by Corollary 18.20, for every $\mathbf{x} \in U$, there exists $r > 0$ such that $B_{\mathbf{x}}(\mathbf{x}, r) \subset U$. Therefore, $U = \bigcup_{\mathbf{x} \in U} B_{\mathbf{x}}$, as desired.

Second, suppose that U is a union of open balls. Then for every open ball $B(\mathbf{x}, r)$ comprising U, Lemma 18.19 asserts that for every $\mathbf{y} \in B$, there exists an open rectangle $R_{\mathbf{y}}$ such that $\mathbf{y} \in R_{\mathbf{y}} \subset B$. Therefore, $U = \bigcup_{\mathbf{y} \in U} R_{\mathbf{y}}$, as desired.

Third, suppose that U is a union of open rectangles. Then for every $\mathbf{x} \in U$, there exists an open rectangle R such that $\mathbf{x} \in R \subset U$. Therefore, by Definition 18.14, U is open, as desired. □

Remark 18.23. If $X \subset \mathbb{R}^n$, then X is also a topolotical space with the **subspace topology**. That is, $A \subset X$ is **open** (in X) if there exists an open set $U \subset \mathbb{R}^n$ such that $X \cap U = A$. (See Script 8.)

We now discuss functions between Euclidean spaces.

Definition 18.24. Let $A \subset \mathbb{R}^n$ and let $f : A \to \mathbb{R}$. Define the **graph** of f by

$$\operatorname{graph}(f) = \{(x_1, \dots, x_n, f(x_1, \dots, x_n)) \in \mathbb{R}^{n+1} \mid (x_1, \dots, x_n) \in A\}$$

Exercise 18.25. For each of the following functions, describe the graph as a subset of $\mathbb{R}^3$.

(a) $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = 2$ for all $(x, y) \in \mathbb{R}^2$.

Description. For this function, we have $\operatorname{graph}(f) = \{(x, y, 2) \in \mathbb{R}^3 \mid (x, y) \in \mathbb{R}^2\}$. This makes the graph equal to the set of all points in $\mathbb{R}^3$ with $z = 2$, which will be a planar, constant, infinite subspace of $\mathbb{R}^3$. □

(b) $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x + y + 1$ for all $(x, y) \in \mathbb{R}^2$.

Description. For this function, we have $\text{graph}(f) = \{(x, y, x + y + 1) \in \mathbb{R}^3 \mid (x, y) \in \mathbb{R}^2\}$. Thus, the graph will be a planar, sloped, infinite subspace of $\mathbb{R}^3$ with gradient pointing in the $\hat{\imath} + \hat{\jmath}$ direction. □

(c) $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = x^2 + y^2$ for all $(x, y) \in \mathbb{R}^2$.

Description. For this function, we have $\text{graph}(f) = \{(x, y, x^2 + y^2) \in \mathbb{R}^3 \mid (x, y) \in \mathbb{R}^2\}$. Thus, the graph will be the paraboloid centered at the origin. □

In Script 9, we gave a definition of continuity that we can generalize to this case:

Definition 18.26. Let X, Y be topological spaces. A function $f : X \to Y$ is **continuous** if for every open set $U \subset Y$, the preimage $f^{-1}(U)$ is open in X.

The function $f : X \to Y$ is **continuous** at $x \in X$ if for every open set $U \subset Y$ containing $f(x)$, the preimage $f^{-1}(U)$ is open in X.

Theorem 18.27.

(a) *A function $f : X \to Y$ is continuous if and only if it is continuous at every $x \in X$.*

(b) *A function $f : X \to Y$ is continuous if and only if $f^{-1}(B)$ is closed in X whenever B is closed in Y.*

Proof. The proofs are symmetric to those of Theorem 9.10 and Proposition 9.5, respectively. □

7/17: **Remark 18.28.** There is also a characterization of continuity in terms of limits, as in one variable, as we shall now see. First we need the definitions of limit point and limit.

Definition 18.29. Let $A \subset \mathbb{R}^n$.

(a) We say that $\mathbf{x}$ is a **limit point** of A if for every open set U containing $\mathbf{x}$, $A \cap (U \setminus \{\mathbf{x}\}) \neq \emptyset$.

(b) Let $\mathbf{x} \in LP(A)$ and $f : A \to \mathbb{R}^m$. We say $\mathbf{L} \in \mathbb{R}^m$ is the **limit** (of f at $\mathbf{x}$) if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\mathbf{y} \in A$ and $0 < \|\mathbf{y} - \mathbf{x}\| < \delta$, then $\|f(\mathbf{x}) - \mathbf{L}\| < \epsilon$. As in one variable, we can show that limits are unique. If $\mathbf{L}$ is the limit of f at $\mathbf{x}$, we write $\lim_{\mathbf{y} \to \mathbf{x}} f(\mathbf{x}) = \mathbf{L}$.

Exercise 18.30. Compute the following limits if they exist, or prove that the limit does not exist.

Lemma. *Let $\mathbf{x} = (x_1, \dots, x_n)$ be an arbitrary element of $\mathbb{R}^n$. Then $\|\mathbf{x}\| < \delta$ implies that $|x_i| < \delta$ for all $1 \leq i \leq n$.*

Proof. Suppose for the sake of contradiction that for some $1 \leq i \leq n$, $|x_i| \geq \delta$. Note that since $0 \leq \|\mathbf{x}\| < \delta$ by Theorem 18.10, $|\delta| = \delta$ by Definition 8.4. Then

$$\begin{aligned}
\|\mathbf{x}\| &= \sqrt{x_1^2 + \cdots + x_{i-1}^2 + x_i^2 + x_{i+1}^2 + \cdots + x_n^2} && \text{Definition 18.6} \\
&\geq \sqrt{x_1^2 + \cdots + x_{i-1}^2 + \delta^2 + x_{i+1}^2 + \cdots + x_n^2} \\
&\geq \sqrt{\delta^2} \\
&= \delta \\
&> \|\mathbf{x}\|
\end{aligned}$$

a contradiction. □

(a) $\lim_{(x,y) \to (a,b)} 4xy$.

Proof. To prove that $\lim_{(x,y) \to (a,b)} 4xy = 4ab$, Definition 18.29 tells us that it will suffice to show that for every $\epsilon > 0$, there exists $\delta > 0$ such that if $(x, y) \in \mathbb{R}^2$ and $0 < \|(x, y) - (a, b)\| < \delta$, then $\|4xy - 4ab\| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \min(\min(\frac{\epsilon}{8(|b|+1)}, 1), \frac{\epsilon}{8(|a|-1)})$. Then since $\|(x - a, y - b)\| < \delta$ by hypothesis and Definition 18.2, the lemma asserts that $|x - a| < \delta$ and $|y - b| < \delta$. It follows that $|x - a| < \min(\frac{\epsilon}{8(|b|+1)}, 1)$ and $|y - b| < \frac{\epsilon}{8(|a|-1)}$. Consequently, by an argument symmetric to the proof of Theorem 11.9, $|xy - ab| < \frac{\epsilon}{4}$. Therefore, $\|4xy - 4ab\| = |4xy - 4ab| < 4 \cdot \frac{\epsilon}{4} = \epsilon$, as desired. □

(b) $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$.

Proof. To ensure that $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$ is well-defined, Definition 18.29 tells us that we must show that $(0,0) \in LP(\mathbb{R}^2 \setminus \{(0,0)\})$, assuming that $\mathbb{R}^2 \setminus \{(0,0)\}$ is the domain of $\frac{x^3-y^3}{x^2+y^2}$ since the domain is not explicitly specified. To do so, Definition 18.29 tells us again that it will suffice to verify that for every open set U containing $(0,0)$, $(\mathbb{R}^2 \setminus \{(0,0)\}) \cap (U \setminus \{(0,0)\}) \neq \emptyset$. Let U be an arbitrary open set containing $(0,0)$. By Definition 18.14, there exists an open rectangle R such that $(0,0) \in R \subset U$. By Definition 18.13, R is not a singleton set. Thus, there exist at least one point in R, i.e., in U that is not equal to $(0,0)$ and is (naturally) in $\mathbb{R}^2$, as desired.

To prove that $\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0$, Definition 18.29 tells us that it will suffice to show that for every $\epsilon > 0$, there exists $\delta > 0$ such that if $(x,y) \in \mathbb{R}^2$ and $0 < \|(x,y) - (0,0)\| < \delta$, then $||\frac{x^3-y^3}{x^2+y^2} - 0|| = |\frac{x^3-y^3}{x^2+y^2}| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \frac{\epsilon}{3}$. Then from previous results, we can prove two important bounds on combinations of x and y that will be useful in the final inequality. Let's begin.

First, since $0 < \|(x,y)\|$, Theorem 18.10 implies that $x \neq 0$ or $y \neq 0$. Thus, $x^2 + y^2 \neq 0$. Consequently, we may argue in a well-defined manner that

$$\begin{aligned}
\left|\frac{xy}{x^2+y^2}\right| &= |xy| \cdot \left|\frac{1}{x^2+y^2}\right| \\
&\leq \frac{x^2+y^2}{2} \cdot \left|\frac{1}{x^2+y^2}\right| && \text{Lemma 18.9} \\
&= \frac{1}{2}
\end{aligned}$$

Second, since we know from the lemma that $|x| < \frac{\epsilon}{3}$ and $|y| < \frac{\epsilon}{3}$, we have that

$$\begin{aligned}
|x-y| &\leq |x| + |-y| && \text{Lemma 8.8} \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \frac{2\epsilon}{3}
\end{aligned}$$

Therefore, combining the last two results, we have that

$$\begin{aligned}
\left|\frac{x^3-y^3}{x^2+y^2}\right| &= \left|\frac{(x-y)(x^2+xy+y^2)}{x^2+y^2}\right| \\
&= |x-y| \cdot \left|\frac{xy}{x^2+y^2} + 1\right| \\
&\leq |x-y| \cdot \left|\frac{xy}{x^2+y^2}\right| + |x-y| && \text{Lemma 8.8} \\
&< \frac{2\epsilon}{3} \cdot \frac{1}{2} + \frac{2\epsilon}{3} \\
&= \epsilon
\end{aligned}$$

as desired. ☐

(c) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$.

Proof. To prove that $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist, Definition 18.29 tells us that it will suffice to show that for every $L \in \mathbb{R}$, there exists an $\epsilon > 0$ such that for all $\delta > 0$, there exists $(x,y) \in \mathbb{R}^2$ satisfying $0 < \|(x,y) - (0,0)\| < \delta$ such that $||\frac{x^2-y^2}{x^2+y^2} - L|| \geq \epsilon$. Let L be an arbitrary element of $\mathbb{R}$.

We divide into two cases ($L \geq 0$ and $L < 0$). Suppose first that $L \geq 0$. Choose $\epsilon = 1$. Let $\delta > 0$ be arbitrary. Choose $(0, \frac{\delta}{2}) \in \mathbb{R}^2$. By Definition 18.6, $0 < \|(0, \frac{\delta}{2})\| = \sqrt{\delta^2/4} = \frac{\delta}{2} < \delta$. Additionally,

$$\begin{aligned} \left\| \frac{0^2 - \left(\frac{\delta}{2}\right)^2}{0^2 + \left(\frac{\delta}{2}\right)^2} - L \right\| &= \left| \frac{-1}{1} - L \right| \\ &= |-1 - L| \\ &\geq 1 - |L| \\ &\geq 1 \\ &= \epsilon \end{aligned}$$

as desired. The proof is symmetric in the other case. □

Theorem 18.31. *Let $A \subset \mathbb{R}^n$ and $\mathbf{x} \in A$. Let $f : A \to \mathbb{R}^m$. Then the following are equivalent:*

(*a*) *f is continuous at $\mathbf{x}$.*

(*b*) *For every $\epsilon > 0$, there exists $\delta > 0$ such that if $\mathbf{y} \in A$ and $\|\mathbf{y} - \mathbf{x}\| < \delta$, then $\|f(\mathbf{y}) - f(\mathbf{x})\| < \epsilon$.*

(*c*) *Either $\mathbf{x} \notin LP(A)$ or $\lim_{\mathbf{y} \to \mathbf{x}} f(\mathbf{y}) = f(\mathbf{x})$.*

Proof. The proof is symmetric to that of Theorem 11.9. □

Exercise 18.32. For each of the following, prove that f is continuous at every point in its domain.

(a) $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ is a constant function.

Proof. Since f is a constant function, we may let $f(\mathbf{x}) = \mathbf{c}$ for all $\mathbf{x} \in A$. To prove that f is continuous at every $\mathbf{x} \in A$, let $\mathbf{x}$ be an arbitrary element of A; then Theorem 18.31 tells us that it will suffice to show that for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\mathbf{y} \in A$ and $\|\mathbf{y} - \mathbf{x}\| < \delta$, then $\|f(\mathbf{y}) - f(\mathbf{x})\| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = 1$. Let $\mathbf{y}$ be an arbitrary element of A satisfying $\|\mathbf{y} - \mathbf{x}\| < \delta$. Then

$$\begin{aligned} \|f(\mathbf{y}) - f(\mathbf{x})\| &= \|\mathbf{c} - \mathbf{c}\| \\ &= \|\mathbf{0}\| \\ &= 0 && \text{Theorem 18.10} \\ &< \epsilon \end{aligned}$$

as desired. □

(b) Fix $\mathbf{a} \in \mathbb{R}^m$. Define $f : \mathbb{R} \to \mathbb{R}^m$ by $f(h) = h\mathbf{a}$.

Proof. We divide into two cases ($\mathbf{a} = 0$ and $\mathbf{a} \neq 0$). If $\mathbf{a} = 0$, then by Exercise 18.32a, f is continuous at every point in its domain. If $\mathbf{a} \neq 0$, we continue.

Let x be an arbitrary element of $\mathbb{R}$. To prove that f is continuous at x, Theorem 18.31 tells us that it will suffice to show that for every $\epsilon > 0$, there exists $\delta > 0$ such that if $y \in \mathbb{R}$ and $\|y - x\| = |y - x| < \delta$, then $\|f(y) - f(x)\| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \frac{\epsilon}{\|\mathbf{a}\|}$ (by Theorem 18.10 and the supposition that $\mathbf{a} \neq 0$, we know that $\|\mathbf{a}\| \neq 0$). Let y be an arbitrary element of $\mathbb{R}$ satisfying $|y - x| < \delta$. Then

$$\begin{aligned} \|f(y) - f(x)\| &= \|y\mathbf{a} - x\mathbf{a}\| \\ &= \|(y - x)\mathbf{a}\| \\ &= |y - x| \cdot \|\mathbf{a}\| && \text{Theorem 18.10} \\ &< \frac{\epsilon}{\|\mathbf{a}\|} \cdot \|\mathbf{a}\| \\ &= \epsilon \end{aligned}$$

as desired. □

(c) Fix $\mathbf{x} \in \mathbb{R}^n$. Define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(\mathbf{y}) = \|\mathbf{y} - \mathbf{x}\|$.

Proof. Let $\mathbf{y}$ be an arbitrary element of $\mathbb{R}^n$. To prove that f is continuous at y, Theorem 18.31 tells us that it will suffice to show that for all $\epsilon > 0$, there exists $\delta > 0$ such that if $\mathbf{z} \in \mathbb{R}^n$ and $\|\mathbf{z} - \mathbf{y}\| < \delta$, then $\|f(\mathbf{z}) - f(\mathbf{y})\| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \epsilon$. Let $\mathbf{z}$ be an arbitrary element of $\mathbb{R}^n$ satisfying $\|\mathbf{z} - \mathbf{y}\| < \delta$. Then

$$\begin{aligned} \|f(\mathbf{z}) - f(\mathbf{y})\| &= \|\|\mathbf{z} - \mathbf{x}\| - \|\mathbf{y} - \mathbf{x}\|\| \\ &\leq \|(\mathbf{z} - \mathbf{x}) - (\mathbf{y} - \mathbf{x})\| && \text{Corollary 18.11} \\ &= \|\mathbf{z} - \mathbf{y}\| \\ &< \epsilon \end{aligned}$$

as desired. □

(d) $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = 4xy$.

Proof. Let (a, b) be an arbitrary element of $\mathbb{R}^2$. To prove that f is continuous at (a, b), Theorem 18.31 tells us that it will suffice to show that either $(a, b) \notin LP(\mathbb{R}^2)$ or $\lim_{(x,y)\to(a,b)} 4xy = 4ab$. But by Exercise 18.30a, $\lim_{(x,y)\to(a,b)} 4xy = 4ab$, as desired. □

Exercise 18.33. Consider the function $f : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}$ given by $f(x, y) = \frac{x^3 - y^3}{x^2 + y^2}$ (see Exercise 18.30b). It can be shown that this function is continuous on its domain. Can you extend this function continuously to $\mathbb{R}^2$? More specifically, can you define a continuous function $g : \mathbb{R}^2 \to \mathbb{R}$ such that $g(x, y) = f(x, y)$ for all $(x, y) \neq (0, 0)$?

Proof. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$g(x, y) = \begin{cases} f(x, y) & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

By the continuity of f on $\mathbb{R}^2 \setminus \{(0, 0)\}$, g is continuous on $\mathbb{R}^2 \setminus \{(0, 0)\}$. Additionally, by Exercise 18.30c, $\lim_{(x,y)\to(0,0)} g(x, y) = 0 = g(0, 0)$. Thus, by Theorem 18.31, g is continuous at $(0, 0)$. Therefore, g is continuous on $(\mathbb{R}^2 \setminus \{(0, 0)\}) \cup \{(0, 0)\} = \mathbb{R}^2$, as desired. □

7/21: **Definition 18.34.** Let $m \in \mathbb{N}$. Suppose $I = \{i_1, \dots, i_k\} \subset [m]$ with $i_1 < \cdots < i_k$. We define the **projection function** $\pi_I : \mathbb{R}^m \to \mathbb{R}^k$ as

$$\pi_I(\mathbf{x}) = (x_{i_1}, \dots, x_{i_k})$$

If $I = \{i\}$ has only one element, we write π_i instead of $\pi_{\{i\}}$.

Exercise 18.35. Prove that each π_I is continuous.

Proof. Let I be an arbitrary subset of $[m]$, and let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^m$. To prove that π_I is continuous at $\mathbf{x}$, Theorem 18.31 tells us that it will suffice to show that for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\mathbf{y} \in \mathbb{R}^m$ and $\|\mathbf{y} - \mathbf{x}\| < \delta$, then $\|\pi_I(\mathbf{y}) - \pi_I(\mathbf{x})\| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \epsilon$. Let $\mathbf{y}$ be an arbitrary element of $\mathbb{R}^m$ such that $\|\mathbf{y} - \mathbf{x}\| < \delta$. Then

$$\begin{aligned} \|\pi_I(\mathbf{y}) - \pi_I(\mathbf{x})\| &= \|(y_{i_1} - x_{i_1}, \dots, y_{i_k} - x_{i_k})\| && \text{Definition 18.2} \\ &= \sqrt{(y_{i_1} - x_{i_1})^2 + \cdots + (y_{i_k} - x_{i_k})^2} && \text{Definition 18.6} \\ &\leq \sqrt{(y_1 - x_1)^2 + \cdots + (y_m - x_m)^2} \\ &= \|(y_1 - x_1, \dots, y_m - x_m)\| && \text{Definition 18.6} \\ &= \|\mathbf{y} - \mathbf{x}\| && \text{Definition 18.2} \\ &< \epsilon \end{aligned}$$

as desired.[1] □

[1] This can also be done, without much difficulty, with the open preimage form of continuity.

Remark 18.36. Let $A \subset \mathbb{R}^n$ be a rectangle (open or closed). Then

$$A = \pi_1(A) \times \cdots \times \pi_n(A)$$

Definition 18.37. Let $f : A \to \mathbb{R}^m$. Its i^{th} component function $f_i : A \to \mathbb{R}$ is defined as

$$f_i = \pi_i \circ f$$

In other words,

$$f(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_m(\mathbf{x}))$$

Theorem 18.38. *Let $A \subset \mathbb{R}^n$ and let $\mathbf{x}$ be a limit point of A. Suppose $f : A \to \mathbb{R}^m$. If $\lim_{\mathbf{y}\to\mathbf{x}} f(\mathbf{y}) = \mathbf{z}$ exists (with $\mathbf{z} = (z_1, \dots, z_m)$), then for all $i \in [m]$, $\lim_{\mathbf{y}\to\mathbf{x}} f_i(\mathbf{y})$ exists and equals z_i. Conversely, if for all $i \in [m]$, $\lim_{\mathbf{y}\to\mathbf{x}} f_i(\mathbf{y}) = z_i$, then $\lim_{\mathbf{y}\to\mathbf{x}} f(\mathbf{y})$ exists and equals $\mathbf{z} = (z_1, \dots, z_m)$.*

Proof. Suppose first that $\lim_{\mathbf{y}\to\mathbf{x}} f(\mathbf{y}) = \mathbf{z}$. Let i be an arbitrary element of $[m]$. To prove that $\lim_{\mathbf{y}\to\mathbf{x}} f_i(\mathbf{y}) = z_i$, Definition 18.29 tells us that it will suffice to show that for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\mathbf{y} \in A$ and $0 < \|\mathbf{y} - \mathbf{x}\| < \delta$, then $\|f_i(\mathbf{y}) - z_i\| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{\mathbf{y}\to\mathbf{x}} f(\mathbf{y}) = \mathbf{z}$, Definition 18.29 asserts that there exists $\delta > 0$ such that if $\mathbf{y} \in A$ and $0 < \|\mathbf{y} - \mathbf{x}\| < \delta$, then $\|f(\mathbf{y}) - \mathbf{z}\| < \epsilon$. Choose this δ to be our δ. Let $\mathbf{y}$ be an arbitrary element of A satisfying $0 < \|\mathbf{y} - \mathbf{x}\| < \delta$. Then

$$\begin{aligned}
\|f_i(\mathbf{y}) - z_i\| &= \sqrt{(f_i(\mathbf{y}) - z_i)^2} && \text{Definition 18.6} \\
&\leq \sqrt{(f_1(\mathbf{y}) - z_1)^2 + \cdots + (f_m(\mathbf{y}) - z_m)^2} \\
&= \|(f_1(\mathbf{y}), \dots, f_m(\mathbf{y})) - (z_1, \dots, z_m)\| && \text{Definition 18.6} \\
&= \|f(\mathbf{y}) - \mathbf{z}\| && \text{Definition 18.37} \\
&< \epsilon
\end{aligned}$$

as desired.

Now suppose that for all $i \in [m]$, $\lim_{\mathbf{y}\to\mathbf{x}} f_i(\mathbf{y}) = z_i$. To prove that $\lim_{\mathbf{y}\to\mathbf{x}} f(\mathbf{y}) = \mathbf{z}$, Definition 18.29 tells us that it will suffice to show that for all $\epsilon > 0$, there exists $\delta > 0$ such that if $\mathbf{y} \in A$ and $0 < \|\mathbf{y} - \mathbf{x}\| < \delta$, then $\|f(\mathbf{y}) - \mathbf{z}\| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{\mathbf{y}\to\mathbf{x}} f_i(\mathbf{y}) = z_i$ for all $i \in [m]$, Definition 18.29 asserts that for all $i \in [m]$, there exists $\delta_i > 0$ such that if $\mathbf{y} \in A$ and $0 < \|\mathbf{y} - \mathbf{x}\| < \delta$, then $\|f_i(\mathbf{y}) - z_i\| < \frac{\epsilon}{m}$. Choose $\delta = \min(\delta_1, \dots, \delta_m)$. Let $\mathbf{y}$ be an arbitrary element of A satisfying $0 < \|\mathbf{y} - \mathbf{x}\| < \delta$. Then

$$\begin{aligned}
\|f(\mathbf{y}) - \mathbf{z}\| &= \|(f_1(\mathbf{y}) - z_1) + \cdots + (f_m(\mathbf{y}) - z_m)\| \\
&\leq \|f_1(\mathbf{y}) - z_1\| + \cdots + \|f_m(\mathbf{y}) - z_m\| && \text{Theorem 18.10} \\
&< \underbrace{\frac{\epsilon}{m} + \cdots + \frac{\epsilon}{m}}_{m \text{ times}} \\
&= \epsilon
\end{aligned}$$

as desired. ☐

Corollary 18.39. *Let $A \subset \mathbb{R}^n$. A function $f : A \to \mathbb{R}^m$ is continuous if and only if $f_1, \dots, f_m$ are all continuous.*

Proof. Suppose first that f is continuous. Let $\mathbf{x}$ be an arbitrary element of A, and let i be an arbitrary element of $[m]$. To prove that f_i is continuous at $\mathbf{x}$, Theorem 18.31 tells us that it will suffice to show that either $\mathbf{x} \notin LP(A)$ or $\lim_{\mathbf{y}\to\mathbf{x}} f_i(\mathbf{y}) = f_i(\mathbf{x})$. Since f is continuous at $\mathbf{x}$ by hypothesis, Theorem 18.31 asserts that either $\mathbf{x} \notin LP(A)$ or $\lim_{\mathbf{y}\to\mathbf{x}} f(\mathbf{y}) = f(\mathbf{x})$. We now divide into two cases. If $\mathbf{x} \notin LP(A)$, then we are done. On the other hand, if $\lim_{\mathbf{y}\to\mathbf{x}} f(\mathbf{y}) = f(\mathbf{x})$, then by Theorem 18.38 and Definition 18.37, $\lim_{\mathbf{y}\to\mathbf{x}} f_i(\mathbf{y}) = f_i(\mathbf{x})$, as desired.

The proof is symmetric in the other direction. ☐

7/28: Now we revisit compactness, but in $\mathbb{R}^n$. For our purposes, the key result is Corollary 18.48.

Definition 18.40. Let $A \subset \mathbb{R}^n$. Then A is **compact** if every open cover $\mathcal{G}$ of A has a finite subcover.

Proposition 18.41. *Let $A \subset \mathbb{R}^n$. Then A is compact if and only if every open cover $\mathcal{G}$ of A consisting solely of open rectangles has a finite subcover.*

Proof. Suppose first that A is compact. Let $\mathcal{G}$ be an arbitrary open cover of A consisting solely of open rectangles. Then since A is compact, by Definition 18.40, $\mathcal{G}$ has a finite subcover.

Now suppose that every open cover $\mathcal{G}$ of A consisting solely of open rectangles has a finite subcover. To prove that A is compact, Definition 18.40 tells us that it will suffice to show that every open cover $\mathcal{G}$ of A has a finite subcover. Let $\mathcal{G} = \{G_\lambda \mid \lambda \in \Lambda\}$ be an arbitrary open cover of A, and let G_λ be an arbitrary element of $\mathcal{G}$. By Definition 10.3, G_λ is open. Thus, by Proposition 18.22, $G_\lambda = \bigcup_{\gamma \in \Gamma_\lambda} R_{\lambda_\gamma}$, where each R_{λ_γ} is an open rectangle. Now let $\mathcal{H} = \{R_{\lambda_\gamma} \mid \lambda \in \Lambda, \gamma \in \Gamma_\lambda\}$. It follows by Script 1 that $\mathcal{G} = \mathcal{H}$. Additionally, by the hypothesis, there exists a finite subcover $\mathcal{H}' \subset \mathcal{H}$ of A. Finally, if $R_{\lambda_\gamma} \in \mathcal{H}'$, let $G_\lambda \in \mathcal{G}'$. It follows that $\mathcal{G}'$ is a finite subcover of $\mathcal{G}$, as desired. $\square$

Definition 18.42. Let $A \subset \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$. We say that f is **uniformly continuous** if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $\mathbf{x}, \mathbf{y} \in A$ and $\|\mathbf{x} - \mathbf{y}\| < \delta$, then $\|f(\mathbf{x}) - f(\mathbf{y})\| < \epsilon$.

Theorem 18.43. *Let $A \subset \mathbb{R}^n$ be compact and $f : A \to \mathbb{R}^m$ be continuous. Then f is uniformly continuous.*

Proof. The proof is symmetric to that of Theorem 13.6. $\square$

Theorem 18.44. *If $A \subset \mathbb{R}^n$ is compact and $f : A \to \mathbb{R}^m$ is continuous, then $f(A)$ is compact.*

Proof. The proof is symmetric to that of Theorem 10.19. $\square$

Corollary 18.45. *Let $\mathbf{x} \in \mathbb{R}^n$. If B is a compact subset of $\mathbb{R}^m$, then $\{\mathbf{x}\} \times B$ is a compact subset of $\mathbb{R}^{n+m}$.*

Proof. Let $f : B \to \mathbb{R}^{n+m}$ be defined by $f(\mathbf{y}) = (x_1, \dots, x_n, y_1, \dots, y_m)$ for all $\mathbf{y} \in B$. Let $\mathbf{y}$ be an arbitrary element of B. To prove that f is continuous at $\mathbf{y}$, Theorem 18.31 tells us that it will suffice to show that for every $\epsilon > 0$, there exists δ such that if $\mathbf{z} \in B$ and $\|\mathbf{z} - \mathbf{y}\| < \delta$, then $\|f(\mathbf{z}) - f(\mathbf{y})\| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \epsilon$. Let $\mathbf{z}$ be an arbitrary element of B satisfying $\|\mathbf{z} - \mathbf{y}\| < \delta$. Then

$$\begin{aligned}
\|f(\mathbf{z}) - f(\mathbf{y})\| &= \|(x_1 - x_1, \dots, x_n - x_n, z_1 - y_1, \dots, z_m - y_m)\| \\
&= \|(z_1 - y_1, \dots, z_m - y_m)\| \\
&= \|\mathbf{z} - \mathbf{y}\| \\
&< \epsilon
\end{aligned}$$

as desired. Therefore, since $B \subset \mathbb{R}^m$ is compact and $f : B \to \mathbb{R}^{n+m}$ is continuous, Theorem 18.44 asserts that $f(B)$ is compact. Naturally, $f(B) = \{\mathbf{x}\} \times B$, so the latter set is compact, too, as desired. $\square$

Lemma 18.46. *Let $\mathbf{x} \in \mathbb{R}^n$ and $B \subset \mathbb{R}^m$. If $\mathcal{G}$ is a finite set of open rectangles that covers $\{\mathbf{x}\} \times B \subset \mathbb{R}^{n+m}$, then there exists an open rectangle $R \subset \mathbb{R}^n$ containing $\mathbf{x}$ such that $\mathcal{G}$ covers $R \times B$.*

Proof. Let $\mathcal{G} = \{R_i \mid i \in [k]\}$. To begin, we will show that every $\pi_{[n]}(R_i)$ is an open rectangle containing $\mathbf{x}$. It will follow that the intersection of all $\pi_{[n]}(R_i)$ is an open rectangle R containing $\mathbf{x}$. Thus, since this R is a subset of each R_i in dimensions 1 through n, we will be able to show that $\mathcal{G}$ covers $R \times B$. Let's begin.

First, we will show that every $\pi_{[n]}(R_i)$ is an open rectangle. Let i be an arbitrary element of $[k]$, and let $R_i = (r_{i_j}, s_{i_j})_{j=1}^{n+m}$. To show that $\pi_{[n]}(R_i)$ is an open rectangle, Definition 18.13 tells us that it will suffice to verify that $\pi_{[n]}(R_i) = (r_{i_j}, s_{i_j})_{j=1}^{n}$. Let $\mathbf{y}$ be an arbitrary element of $\pi_{[n]}(R_i)$. By Definition 1.18, $\mathbf{y} = \pi_{[n]}(\mathbf{z})$ for some $\mathbf{z} \in R_i$. Thus, by Definition 18.34, $y_j = z_j$ for all $j \in [n]$. Consequently, since $z_j \in (r_{i_j}, s_{i_j})$ for all $j \in [n]$ by Definition 18.13, we have that $y_j \in (r_{i_j}, s_{i_j})$ for all $j \in [n]$. Therefore, by Definition 1.15, $\mathbf{y} \in (r_{i_j}, s_{i_j})_{j=1}^{n}$. The argument is symmetric in the other direction. Both arguments, when combined, imply by Definition 1.2 that $\pi_{[n]}(R_i) = (r_{i_j}, s_{i_j})_{j=1}^{n}$, as desired.

Next, we will show that every $\pi_{[n]}(R_i)$ contains $\mathbf{x}$. Let i be an arbitrary element of $[k]$, and let $\mathbf{y}$ be an arbitrary element of $\{\mathbf{x}\} \times B$ satisfying $\mathbf{y} \in R_i$ (Definition 10.3 guarantees that $\mathbf{y}$ is in some R_i). Thus, by

Definition 1.18, $\pi_{[n]}(\mathbf{y}) \in \pi_{[n]}(R_i)$. Additionally, by Definition 1.15, $\mathbf{y} = (x_1, \dots, x_n, y_1, \dots, y_m)$. It follows by Definition 18.34 that $\pi_{[n]}(\mathbf{y}) = \mathbf{x}$. Therefore, $\mathbf{x} \in \pi_{[n]}(R_i)$, as desired.

Let $R = \bigcap_{i \in [k]} \pi_{[n]}(R_i)$. Consequently, by Exercise 18.16, R is an open rectangle containing $\mathbf{x}$.

To prove that $\mathcal{G}$ covers $R \times B$, Definition 10.3 tells us that it will suffice to show that for all $\mathbf{y} \in R \times B$, $\mathbf{y} \in R_i$ for some $R_i \in \mathcal{G}$. Let $\mathbf{y} = (y_1, \dots, y_{n+m})$ be an arbitrary element of $R \times B$. By Definition 1.15, $(y_1, \dots, y_n) \in R$ and $(y_{n+1}, \dots, y_{n+m}) \in B$. It follows from the latter statement and the fact that $\mathcal{G}$ is a cover of $\{\mathbf{x}\} \times B$ that $(x_1, \dots, x_n, y_{n+1}, \dots, y_{n+m}) \in R_i$ for some $i \in [k]$. Consider this R_i; we will confirm that $\mathbf{y}$ is an element of it. To do so, Definition 18.13 tells us that it will suffice to demonstrate that $y_j \in (r_{i_j}, s_{i_j})$ for all $j \in [n+m]$. We divide into two cases ($j \in [n]$ and $j \in [n+1 : m]$). Suppose first that $j \in [n]$. Then since $R = \bigcap_{i \in [k]} \pi_{[n]}(R_i)$, Theorem 1.7 asserts that $R \subset \pi_{[n]}(R_i)$. Thus, since $(y_1, \dots, y_n) \in R$ by the above, Definition 1.3 implies that $(y_1, \dots, y_n) \in \pi_{[n]}(R_i)$. Additionally, by the above, $\pi_{[n]}(R_i)$ can be written in the form $(r_{i_j}, s_{i_j})_{j=1}^n$. Combining the last two results, we have by Definition 1.2 that $(y_1, \dots, y_n) \in (r_{i_j}, s_{i_j})_{j=1}^n$. Therefore, by Definition 18.13, $y_j \in (r_{i_j}, s_{i_j})$ for all $j \in [n]$, as desired. Now suppose that $j \in [n+1 : m]$. By the above, $(x_1, \dots, x_n, y_{n+1}, \dots, y_{n+m}) \in R_i$. Therefore, by Definition 18.13, $y_j \in (r_{i_j}, s_{i_j})$ for all $j \in [n+1 : m]$, as desired. □

Theorem 18.47. *If $A \subset \mathbb{R}^n$ and $B \subset R^m$ are compact, then $A \times B \subset \mathbb{R}^{n+m}$ is also compact.*

Proof. To prove that $A \times B$ is compact, Proposition 18.41 tells us that it will suffice to show that every open cover $\mathcal{G}$ of $A \times B$ consisting solely of open rectangles has a finite subcover. Let $\mathcal{G}$ be an arbitrary open cover of $A \times B$ consisting solely of open rectangles. By Corollary 18.45, for all $\mathbf{x} \in A$, $\{\mathbf{x}\} \times B$ is compact. Thus, by Definition 18.40, for all $\mathbf{x} \in A$, there is a finite subcover $\mathcal{G}_{\mathbf{x}} \subset \mathcal{G}$ that covers $\{\mathbf{x}\} \times B$. Since $\mathcal{G}_{\mathbf{x}}$ is a finite set of open rectangles that covers $\{\mathbf{x}\} \times B$, it follows by Lemma 18.46 that for each $\mathcal{G}_{\mathbf{x}}$, there is an open rectangle $R_{\mathbf{x}}$ containing $\mathbf{x}$ such that $\mathcal{G}_{\mathbf{x}}$ covers $R_{\mathbf{x}} \times B$. Additionally, since each $R_{\mathbf{x}}$ is open and $\mathbf{x} \in R_{\mathbf{x}}$ for all $\mathbf{x} \in A$, Definition 10.3 asserts that $\{R_{\mathbf{x}} \mid \mathbf{x} \in A\}$ is an open cover of A. But since A is compact, there exists a finite subcover $\{R_{\mathbf{x}} \mid \mathbf{x} \in I\} \subset \{R_{\mathbf{x}} \mid \mathbf{x} \in A\}$ of A, where $I \subset A$. We are now ready to define our finite subcover $\mathcal{G}' \subset \mathcal{G}$ of $A \times B$, and verify that it is such.

Let $\mathcal{G}' = \bigcup_{\mathbf{x} \in I} \mathcal{G}_{\mathbf{x}}$. Since $\mathcal{G}'$ is the union of finitely many finite subsets of $\mathcal{G}$, Script 1 guarantees that $\mathcal{G}'$ is, itself, a finite subset of $\mathcal{G}$. To confirm that $\mathcal{G}'$ is an open cover of $A \times B$, Definition 10.3 tells us that it will suffice to show that every $\mathbf{y} \in A \times B$ is an element of G for some $G \in \mathcal{G}'$. Let $\mathbf{y}$ be an arbitrary element of $A \times B$. By Definition 1.15, $\mathbf{y} = (a_1, \dots, a_n, b_1, \dots, b_m)$, where $(a_1, \dots, a_n) \in A$ and $(b_1, \dots, b_m) \in B$. It follows from the former statement and the definition of $\{R_{\mathbf{x}} \mid \mathbf{x} \in I\}$ that $(a_1, \dots, a_n) \in R_{\mathbf{x}}$ for some $\mathbf{x} \in I$. This combined with the latter statement implies by Definition 1.15 that $\mathbf{y} \in R_{\mathbf{x}} \times B$. Thus, since $\mathcal{G}_{\mathbf{x}}$ covers $R_{\mathbf{x}} \times B$, there exists $G \in \mathcal{G}_{\mathbf{x}}$ such that $y \in G$. Additionally, Theorem 1.7 implies that $\mathcal{G}_{\mathbf{x}} \subset \mathcal{G}$, so we have by Definition 1.3 that $G \in \mathcal{G}'$. Therefore, $\mathbf{y} \in G$ for some $G \in \mathcal{G}'$, as desired. □

Corollary 18.48. *If $A_1, \dots, A_n$ are all compact, then so is $A_1 \times \cdots \times A_n$. In particular, a closed rectangle is compact.*

Proof. We induct on n. For the base case $n = 1$, if A_1 is compact, then $\prod_{i=1}^1 A_i = A_i$ is trivially compact. Now suppose inductively that we have proven the claim for n; we now seek to prove it for $n+1$. Let $A_1, \dots, A_{n+1}$ be compact. By hypothesis, $\prod_{i=1}^n A_i$ is compact. Thus, by Theorem 18.47, $\prod_{i=1}^{n+1} A_i = (\prod_{i=1}^n A_i) \times A_{n+1}$ is compact, as desired.

Let R be an arbitrary closed rectangle. By Definition 18.13, $R = [a_i, b_i]_{i=1}^n$. Additionally, by Theorem 10.14, every $[a_i, b_i]$ is compact. Thus, since R is the Cartesian product of n compact sets, we have by the above that R is compact, as desired. □

Theorem 18.49. *If $A \subset X \subset \mathbb{R}^n$ with X compact and A closed in $\mathbb{R}^n$, then A is compact.*

Proof. The proof is symmetric to that of Theorem 10.15. □

Theorem 18.50. *Closed balls are compact.*

Proof. Let $\overline{B}(\mathbf{x}, r)$ be an arbitrary closed ball. By Definition 18.13, $R = \prod_{i=1}^n [x_i - r, x_i + r]$ is a closed rectangle. Thus, to prove that $\overline{B}(\mathbf{x}, r)$ is compact, Theorem 18.49 tells us that it will suffice to show that $\overline{B} \subset R$, that R is compact, and that $\overline{B}$ is closed. Let's begin.

To prove that $\overline{B} \subset R$, Definition 1.3 tells us that it will suffice to show that every $\mathbf{y} \in \overline{B}$ is an element of R. Let $\mathbf{y}$ be an arbitrary element of $\overline{B}$. Then by Definition 18.17, $\|\mathbf{y} - \mathbf{x}\| \leq r$. It follows by the lemma to Exercise 18.30 that $|y_i - x_i| \leq r$ for all $1 \leq i \leq n$. Thus, by Exercise 8.9, $y_i \in [x_i - r, x_i + r]$ for all $1 \leq i \leq n$. Consequently, by Definition 18.13, $\mathbf{y} \in R$, as desired.

By Corollary 18.48, R is compact, as desired.

By Corollary 18.21, $\overline{B}$ is closed, as desired. $\square$

Definition 18.51. A subset A of $\mathbb{R}^n$ is bounded if there exists a closed rectangle R such that $A \subset R$.

Theorem 18.52 (The Heine-Borel theorem in $\mathbb{R}^n$)**.** *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. Suppose first that X is a compact subset of $\mathbb{R}^n$.

The proof that X is closed is symmetric to the proof of Theorem 10.11.

To prove that X is bounded, Definition 18.51 tells us that it will suffice to show that there exists a closed rectangle R such that $X \subset R$. Let $\mathcal{G} = \{\prod_{i=1}^n (x_i, x_i + 2) \mid x_1, \dots, x_n \in \mathbb{Z}\}$.

To confirm that $\mathcal{G}$ is an open cover of X, Definition 10.3 tells us that it will suffice to show that for all $\mathbf{y} \in X$, $\mathbf{y} \in \prod_{i=1}^n (x_i, x_i+2)$ for some $\prod_{i=1}^n (x_i, x_i+2) \in \mathcal{G}$. Let $\mathbf{y}$ be an arbitrary element of X. By consecutive applications of Corollary 6.14, there exist n integers $x_i + 2 \in \mathbb{Z}$ such that $(x_i + 2) - 1 \leq y_i < (x_i + 2)$ for each $i \in [n]$. Thus, $x_i < x_i + 1 \leq y_i < x_i + 2$ for all $i \in [n]$. It follows by Equations 8.1 that $y_i \in (x_i, x_i + 2)$ for all $i \in [n]$. Consequently, by Definition 1.15, we have that $\mathbf{y} \in \prod_{i=1}^n (x_i, x_i + 2)$, where $\prod_{i=1}^n (x_i, x_i + 2) \in \mathcal{G}$ by definition, as desired.

Having established that $\mathcal{G}$ is an open cover, we know by Definition 18.40, since X is compact by hypothesis, that there exists a finite subcover $\mathcal{G}' \subset \mathcal{G}$ of X. It follows by the definition of $\mathcal{G}$ that $\mathcal{G}'$ is of the form $\mathcal{G}' = \{\prod_{i=1}^n (x_{ij}, x_{ij} + 2) \mid x_{ij} \in \mathbb{Z}, i \in [n], j \in [m]\}$ for some natural number m. Let R be defined by $R = \prod_{i=1}^n (x_{i_1}, x_{i_2} + 2)$, where $x_{i_1} = \min_j \{x_{ij}\}$ and $x_{i_2} = \max_j \{x_{ij}\}$ for all $j \in [m]$.

To confirm that $X \subset R$, Definition 1.3 tells us that it will suffice to show that every $\mathbf{y} \in X$ is an element of R. Let $\mathbf{y}$ be an arbitrary element of X. Then by the definition of $\mathcal{G}'$, $\mathbf{y} \in \prod_{i=1}^n (x_{ij}, x_{ij} + 2)$ for some $j \in [m]$. Thus, by Definition 1.15 and Equations 8.1, $x_{ij} < y_i < x_{ij} + 2$ for all $i \in [n]$. It follows that $x_{i_1} \leq x_{ij} < y_i < x_{ij} + 2 \leq x_{i_2} + 2$. Consequently, by Equations 8.1 and Definition 1.15, $\mathbf{y} \in \prod_{i=1}^n (x_{i_1}, x_{i_2} + 2)$. Therefore, $\mathbf{y} \in R$, as desired.

Now suppose that X is a closed and bounded subset of $\mathbb{R}^n$. Since X is bounded, Definition 18.51 implies that there exists a closed rectangle R such that $X \subset R$. Additionally, since R is a closed rectangle, Corollary 18.48 implies that R is compact. Thus, since $X \subset R$ with R compact and X closed, Theorem 18.49 asserts that X is compact. $\square$