

# Script 15

## Sequences

### 15.1 Journal

5/6: **Definition 15.1.** A **sequence** (of real numbers) is a function  $a : \mathbb{N} \rightarrow \mathbb{R}$ .

By setting  $a_n = a(n)$ , we can think of a sequence as a list  $a_1, a_2, a_3, \dots$  of real numbers. We use the notation  $(a_n)_{n=1}^\infty$  for such a sequence, or if there is no possibility of confusion, we sometimes abbreviate this and write simply  $(a_n)$ . More generally, we also use the term sequence to refer to the function defined on  $\{n \in \mathbb{N} \cup \{0\} \mid n \geq n_0\}$  for any fixed  $n_0 \in \mathbb{N} \cup \{0\}$ . We write  $(a_n)_{n=n_0}^\infty$  for such a sequence.

**Definition 15.2.** We say that a sequence  $(a_n)$  **converges** to a point  $p \in \mathbb{R}$  if for every open interval  $I$  containing  $p$ , there exists  $N \in \mathbb{N}$  such that if  $n \geq N$ , then  $a_n \in I$ . If a sequence converges to some point, we say it is **convergent**. If  $(a_n)$  does not converge to any point, we say that the sequence **diverges** or is **divergent**.

**Exercise 15.3.** Show that a sequence  $(a_n)$  converges to  $p$  if and only if any region containing  $p$  contains all but finitely many terms of the sequence.

*Proof.* Suppose first that  $(a_n)$  converges to  $p$ . Let  $R$  be an arbitrary region containing  $p$ . By Corollary 4.11 and Lemma 8.3,  $R$  is an open interval. Thus, by Definition 15.2, there exists  $N \in \mathbb{N}$  such that if  $n \geq N$ , then  $a_n \in R$ . To prove that  $R$  contains all but finitely many terms of the sequence, it will suffice to show that the set  $A = \{a_n \mid a_n \notin R\}$  is finite. Since  $a_n \in R$  for all  $n \geq N$ , it follows that  $a_n \in R$  only if  $n < N$ . Thus, by Script 1,  $A \subset \{a_n \mid 0 \leq n < N\}$ . Since the latter set is clearly finite, it follows by Script 1 that  $A$  is finite.

Now suppose that any region containing  $p$  contains all but finitely many terms  $(a_n)$ . To prove that  $(a_n)$  converges to  $p$ , Definition 15.2 tells us that it will suffice to show that for every open interval  $I$  containing  $p$ , there exists  $N \in \mathbb{N}$  such that if  $n \geq N$ , then  $a_n \in I$ . Let  $I$  be an arbitrary open interval containing  $p$ . Then by Theorem 4.10, there exists a region  $R$  containing  $p$  such that  $R \subset I$ . It follows by the hypothesis that  $A = \{a_n \mid a_n \notin R\}$  is finite. We divide into two cases ( $|A| = 0$  and  $|A| \in \mathbb{N}$ ). Suppose first that  $|A| = 0$ . Choose  $N = n_0$ . It follows that if  $n \geq N$ , then  $a_n \notin A$ , so  $a_n \in R$ , so  $a_n \in I$ , as desired. Now suppose that  $|A| \in \mathbb{N}$ . By Definition 1.18,  $a^{-1}(A) \subset \mathbb{N}$ . Consequently, by Lemma 3.4,  $a^{-1}(A)$  has a last point  $N - 1$ . Choose  $N = (N - 1) + 1$ . It follows that if  $n \geq N$ , then  $n \notin a^{-1}(A)$ , so  $a_n \notin A$ , so  $a_n \in R$ , so  $a_n \in I$ , as desired.  $\square$

**Theorem 15.4.** Suppose that  $(a_n)$  converges to both  $p$  and to  $p'$ . Then  $p = p'$ .

*Proof.* Suppose for the sake of contradiction that  $p \neq p'$ . Then by Theorem 3.22, there exist disjoint regions  $R, R'$  containing  $p, p'$ , respectively. Additionally, by consecutive applications of Corollary 4.11 and Lemma 8.3,  $R, R'$  are open intervals. Combining these last two results, we have by consecutive applications of Definition 15.2 that there exist  $N, N' \in \mathbb{N}$  such that if  $n \geq N$ , then  $a_n \in R$  and if  $n \geq N'$ , then  $a_n \in R'$ . Let  $M = \max(N, N')$ . It follows that  $M \geq N$  and  $M \geq N'$ . Thus, by the above,  $a_M \in R$  and  $a_M \in R'$ . But this implies by Definition 1.6 that  $a_M \in R \cap R'$ . Therefore, by Definition 1.9,  $R$  and  $R'$  are not disjoint, a contradiction.  $\square$

**Definition 15.5.** If a sequence  $(a_n)$  converges to  $p \in \mathbb{R}$ , we call  $p$  the **limit** of  $(a_n)$  and write

$$\lim_{n \rightarrow \infty} a_n = p$$

**Exercise 15.6.** Which of the following sequences  $(a_n)$  converge? Which diverge? For each that converges, what is the limit? Give proofs for your answers.

(a)  $a_n = 5$ .

*Proof.* To prove that this sequence converges with limit  $\lim_{n \rightarrow \infty} a_n = 5$ , Definition 15.5 tells us that it will suffice to show that  $(a_n)$  converges to 5. To do so, Definition 15.2 tells us that it will suffice to verify that for every open interval  $I$  containing 5, there exists  $N \in \mathbb{N}$  such that if  $n \geq N$ , then  $a_n \in I$ . Let's begin.

Let  $I$  be an arbitrary open interval containing 5. Choose  $N = 1$ . Let  $n$  be an arbitrary natural number such that  $n \geq N$ . It follows by the definition of the sequence that  $a_n = 5 \in I$ , as desired.  $\square$

(b)  $a_n = n$ .

*Proof.* To prove that this sequence diverges, Definition 15.2 tells us that it will suffice to show that for every point  $p \in \mathbb{R}$ , there exists an open interval  $I$  containing  $p$  such that for all  $N \in \mathbb{N}$ , there exists an  $n \geq N$  such that  $a_n \notin I$ . Let's begin.

Let  $p$  be an arbitrary element of  $\mathbb{R}$ . Choose  $I = (p - 1, p + 1)$ . Clearly  $p \in I$ . Let  $N$  be an arbitrary natural number. By Corollary 6.12, there exists a natural number  $N'$  such that  $p + 1 < N'$ . Choose  $M = \max(N, N')$ . Thus,  $M \geq N$ . Additionally, it follows by the definition of the sequence that  $a_M = M$ . But this implies that  $a_M \geq N' > p + 1$ , i.e.,  $a_M \notin I$  by Equations 8.1.  $\square$

(c)  $a_n = \frac{1}{n}$ .

*Proof.* To prove that this sequence converges with limit  $\lim_{n \rightarrow \infty} a_n = 0$ , Definitions 15.5 and 15.2 tell us that it will suffice to show that for every open interval  $I$  containing 0, there exists  $N \in \mathbb{N}$  such that if  $n \geq N$ , then  $a_n \in I$ . Let's begin.

Let  $I$  be an arbitrary interval containing 0. By Lemma 8.10, there exists a region  $(a, b)$  containing 0 such that  $(a, b) \subset I$ . By Corollary 6.12, there exists a natural number  $N$  such that  $\frac{1}{b} < N$ . Choose this  $N$  to be our  $N$ . Now let  $n$  be an arbitrary natural number such that  $n \geq N$ . It follows that  $\frac{1}{b} < n$ . Thus, since  $0 < b$  and  $0 < n$ , we have by consecutive applications of Lemma 7.24 that  $0 < \frac{1}{n} < b$ . Consequently, since we also know that  $a < 0$  and  $a_n = \frac{1}{n}$ , we have by transitivity and substitution that  $a < a_n < b$ . It follows by Equations 8.1 that  $a_n \in (a, b)$ . Therefore, by Definition 1.3,  $a_n \in I$ , as desired.  $\square$

(d)  $a_n = (-1)^n$ .

*Proof.* To prove that this sequence diverges, Definition 15.2 tells us that it will suffice to show that for every point  $p \in \mathbb{R}$ , there exists an open interval  $I$  containing  $p$  such that for all  $N \in \mathbb{N}$ , there exists an  $n \geq N$  such that  $a_n \notin I$ . Let's begin.

Let  $p$  be an arbitrary element of  $\mathbb{R}$ . Choose  $I = (p - 1, p + 1)$ . Clearly  $p \in I$ . Let  $N$  be an arbitrary natural number. By Script 0, either  $N$  is even and  $N + 1$  is odd or vice versa. Thus, let  $N$  be even (the case where  $N$  is odd is symmetric). It follows that  $N \geq N$  yields  $a_N = (-1)^{2m} = ((-1)^2)^m = 1^m = 1$  and that  $N + 1 \geq N$  yields  $a_{N+1} = (-1)^{2m+1} = 1(-1)^1 = -1$ . Now suppose for the sake of contradiction that  $a_N \in I$  and  $a_{N+1} \in I$ . Since  $a_N = 1 \in I$ , we have by Equations 8.1 that  $p - 1 < 1 < p + 1$ . It follows by Definition 7.21 that  $p - 3 < -1 < p - 1$ . But  $-1 < p - 1$  implies by Equations 8.1 that  $a_{N+1} = -1 \notin I$ , a contradiction. Therefore,  $N + 1 \geq N$  is a number such that  $a_{N+1} \notin I$ , as desired.  $\square$

5/11: **Theorem 15.7.** A sequence  $(a_n)$  converges to  $p \in \mathbb{R}$  if and only if for all  $\epsilon > 0$ , there is some  $N \in \mathbb{N}$  such that for all  $n \geq N$ , we have  $|a_n - p| < \epsilon$ .

*Proof.* Suppose first that  $(a_n)$  converges to  $p$ . Let  $\epsilon > 0$  be arbitrary. Consider the  $p$ -containing region  $R = (p - \epsilon, p + \epsilon)$ . By Corollary 4.11 and 8.3,  $R$  is an open interval. Thus, by Definition 15.2, there exists  $N \in \mathbb{N}$  such that if  $n \geq N$ , then  $a_n \in R$ . Choose this  $N$  to be our  $N$ . Let  $n$  be an arbitrary natural number such that  $n \geq N$ . Then  $a_n \in (p - \epsilon, p + \epsilon)$ . Therefore, by Exercise 8.9,  $|a_n - p| < \epsilon$ , as desired.

Now suppose that for all  $\epsilon > 0$ , there is some  $N \in \mathbb{N}$  such that for all  $n \geq N$ , we have  $|a_n - p| < \epsilon$ . To prove that  $(a_n)$  converges to  $p$ , Definition 15.2 tells us that it will suffice to show that for every open interval  $I$  containing  $p$ , there exists  $N \in \mathbb{N}$  such that if  $n \geq N$ , then  $a_n \in I$ . Let  $I$  be an arbitrary open interval that satisfies  $p \in I$ . It follows by Lemma 8.10 that there exists a number  $\epsilon > 0$  such that  $(p - \epsilon, p + \epsilon) \subset I$ . With respect to this  $\epsilon$ , we have by hypothesis that there is some  $N \in \mathbb{N}$  such that for all  $n \geq N$ , we have  $|a_n - p| < \epsilon$ . Choose this  $N$  to be our  $N$ . Let  $n$  be an arbitrary natural number such that  $n \geq N$ . Then  $|a_n - p| < \epsilon$ . Consequently, by Exercise 8.9,  $a_n \in (p - \epsilon, p + \epsilon)$ . Therefore, by Definition 1.3,  $a_n \in I$ , as desired.  $\square$

### Exercise 15.8.

(a) Prove that  $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$ .

*Proof.* To prove that  $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$ , Definition 15.5 and Theorem 15.7 tell us that it will suffice to show that for all  $\epsilon > 0$ , there is some  $N \in \mathbb{N}$  such that for all  $n \geq N$ , we have  $|a_n - 0| = |a_n| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. By Corollary 6.12, there exists a natural number  $N$  such that  $\frac{1}{\epsilon} < N$ . Choose this  $N$  to be our  $N$ . Let  $n$  be an arbitrary natural number such that  $n \geq N$ . It follows by transitivity that  $\frac{1}{\epsilon} < n$ . Thus, since  $0 < n$  and  $0 < \epsilon$ , we have by consecutive applications of Lemma 7.24 that  $0 < \frac{1}{n} < \epsilon$ . Additionally, since  $(-1)^n = 1$  or  $(-1)^n = -1$  for all  $n \in \mathbb{N}$  by Script 0, we have by Definition 8.4 that  $|\frac{(-1)^n}{n}| = |\frac{1}{n}| = \frac{1}{n}$ . Consequently, we know that  $|\frac{(-1)^n}{n}| < \epsilon$ . But since  $a_n = \frac{(-1)^n}{n}$ , we have that  $|a_n| < \epsilon$ , as desired.  $\square$

(b) Let  $x \in \mathbb{R}$  with  $|x| < 1$ . Prove that  $\lim_{n \rightarrow \infty} x^n = 0$ .

**Lemma.** If  $|y| > 1$  and  $n$  is a natural number, then  $|y|^n \geq n(|y| - 1) + 1$ .

*Proof.* Define  $1 + x = |y|$ . It follows by Definition 7.21 that  $x > 0 > -1$ , which can be weakened to  $x \geq -1$ . Additionally, since  $n$  is a natural number,  $n \geq 1$  by Script 0. Thus, since  $x \geq -1$  and  $n \geq 1$ , we have by Additional Exercise 12.3b that  $(1 + x)^n \geq 1 + nx$ . Substituting, we have  $|y|^n \geq n(|y| - 1) + 1$ , as desired.  $\square$

*Proof of Exercise 15.8b.* To prove that  $\lim_{n \rightarrow \infty} x^n = 0$ , Definition 15.5 and Theorem 15.7 tell us that it will suffice to show that for all  $\epsilon > 0$ , there is some  $N \in \mathbb{N}$  such that for all  $n \geq N$ , we have  $|a_n - 0| = |a_n| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. We divide into two cases ( $x = 0$  and  $x \neq 0$ ). If  $x = 0$ , then choose  $N = 1$ . Let  $n$  be an arbitrary natural number such that  $n \geq N$ . Since  $0^n = 0$  by Script 7, we have  $|a_n| = |0^n| = 0 < \epsilon$ , as desired. On the other hand, if  $x \neq 0$ , then we continue. By Corollary 6.12, there exists a natural number  $N$  such that  $\frac{1}{\epsilon(|\frac{1}{x}| - 1)} < N$ . Choose this  $N$  to be our  $N$ . Let  $n$  be an arbitrary natural number such that  $n \geq N$ . Therefore,

$$\begin{aligned} |x^n| &= |x^n| \cdot \frac{1}{\epsilon \left( \left| \frac{1}{x} \right| - 1 \right)} \cdot \epsilon \left( \left| \frac{1}{x} \right| - 1 \right) \\ &< |x^n| \cdot N \cdot \epsilon \left( \left| \frac{1}{x} \right| - 1 \right) \\ &\leq |x^n| \cdot n \cdot \epsilon \left( \left| \frac{1}{x} \right| - 1 \right) \\ &< \epsilon \cdot |x^n| \cdot n \left( \left| \frac{1}{x} \right| - 1 \right) + 1 \end{aligned}$$

$$\begin{aligned}
&\leq \epsilon \cdot |x^n| \cdot \left| \frac{1}{x} \right|^n \\
&= \epsilon \cdot |x^n| \cdot \frac{1}{|x^n|} \\
&= \epsilon
\end{aligned}$$

The Lemma

as desired. □**Theorem 15.9.** *If  $\lim_{n \rightarrow \infty} a_n = a$  and  $\lim_{n \rightarrow \infty} b_n = b$  both exist, then*

$$(a) \lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n.$$

*Proof.* Let  $a = \lim_{n \rightarrow \infty} a_n$  and let  $b = \lim_{n \rightarrow \infty} b_n$ . To prove that  $\lim_{n \rightarrow \infty} (a_n + b_n)$  exists and equals  $\lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$ , Definition 15.5 and Theorem 15.7 tell us that it will suffice to show that for all  $\epsilon > 0$ , there is some  $N \in \mathbb{N}$  such that for all  $n \geq N$ , we have  $|(a_n + b_n) - (a + b)| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. It follows by consecutive applications of Definition 15.5 and Theorem 15.7 that there exist natural numbers  $N_a, N_b$  such that for all  $n \geq N_a$ , we have  $|a_n - a| < \frac{\epsilon}{2}$  and for all  $n \geq N_b$ , we have  $|b_n - b| < \frac{\epsilon}{2}$ . Now choose  $N = \max(N_a, N_b)$ . Let  $n$  be an arbitrary natural number such that  $n \geq N$ . It follows that  $n \geq N \geq N_a$ , so we know that  $|a_n - a| < \frac{\epsilon}{2}$ . Similarly,  $|b_n - b| < \frac{\epsilon}{2}$ . Therefore, we have that

$$\begin{aligned}
|(a_n + b_n) - (a + b)| &\leq |a_n - a| + |b_n - b| && \text{Lemma 8.8} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

□

$$(b) \lim_{n \rightarrow \infty} (a_n \cdot b_n) = (\lim_{n \rightarrow \infty} a_n) \cdot (\lim_{n \rightarrow \infty} b_n).$$

*Proof.* Let  $a = \lim_{n \rightarrow \infty} a_n$  and let  $b = \lim_{n \rightarrow \infty} b_n$ . To prove that  $\lim_{n \rightarrow \infty} (a_n \cdot b_n)$  exists and equals  $(\lim_{n \rightarrow \infty} a_n) \cdot (\lim_{n \rightarrow \infty} b_n)$ , Definition 15.5 and Theorem 15.7 tell us that it will suffice to show that for all  $\epsilon > 0$ , there is some  $N \in \mathbb{N}$  such that for all  $n \geq N$ , we have  $|a_n \cdot b_n - a \cdot b| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. By Theorem 15.13<sup>[1]</sup>,  $(a_n)$  is bounded. Thus, by Definition 15.12,  $\{a_n \mid n \in \mathbb{N}\}$  is bounded. Consequently, by the proof of Exercise 13.9, there exists a number  $M_a$  such that  $|a_n| < M_a$  for all  $n \in \mathbb{N}$ . Now define  $M = \max(M_a, b)$ . Using this  $M$  (which by definition is positive since it's greater than  $|a_n|$ , which is at least 0) as well as our previously defined arbitrary  $\epsilon$ , we have by consecutive applications of Definition 15.5 and Theorem 15.7 that there exist natural numbers  $N_a, N_b$  such that for all  $n \geq N_a$ , we have  $|a_n - a| < \frac{\epsilon}{2M}$  and for all  $n \geq N_b$ , we have  $|b_n - b| < \frac{\epsilon}{2M}$ . Now choose  $N = \max(N_a, N_b)$ . Let  $n$  be an arbitrary natural number such that  $n \geq N$ . It follows that  $n \geq N \geq N_a$ , so we know that  $|a_n - a| < \frac{\epsilon}{2M}$ . Similarly,  $|b_n - b| < \frac{\epsilon}{2M}$ . Therefore,

$$\begin{aligned}
|a_n b_n - ab| &= |a_n(b_n - b) + b(a_n - a)| \\
&\leq |a_n| \cdot |b_n - b| + |b| \cdot |a_n - a| && \text{Lemma 8.8} \\
&\leq |M| \cdot |b_n - b| + |M| \cdot |a_n - a| \\
&< M \cdot \frac{\epsilon}{2M} + M \cdot \frac{\epsilon}{2M} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

□

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<sup>1</sup>The proof of Theorem 15.13 does not depend on any following results, so its use here is not circular logic.

Moreover, if  $\lim_{n \rightarrow \infty} b_n \neq 0$ , then

$$(c) \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}.$$

**Lemma.** Let  $\lim_{n \rightarrow \infty} b_n = b \neq 0$ . Then there exists  $m \in \mathbb{R}^+$  such that  $m \leq |b|$  and  $N \in \mathbb{N}$  such that if  $n \geq N$ , then  $m \leq |b_n|$ .

*Proof.* Since  $b \neq 0$ , it follows from Definition 8.4 that  $0 < |b|$ . Thus, by Theorem 5.2, there exists a point  $m \in \mathbb{R}$  such that  $0 < m < |b|$ . It follows from the fact that  $0 < m$  that  $m \in \mathbb{R}^+$ , and from the fact that  $m < |b|$  that  $m \leq |b|$ , as desired.

As to the other part of the proof, we divide into two cases ( $b > 0$  and  $b < 0$ ).

Suppose first that  $b > 0$ . By Exercise 6.5, Axiom 3, and Definition 3.3, there exists a number  $y$  such that  $b < y$ . Now consider the region  $(m, y)$ . Since  $m < |b| = b < y$ , Equations 8.1 assert that  $b \in (m, y)$ . Additionally, by Corollary 4.11 and Lemma 8.3,  $(m, y)$  is an open interval. Thus, by Definitions 15.5 and 15.2, there exists  $N \in \mathbb{N}$  such that if  $n \geq N$ , then  $b_n \in (m, y)$ . Choose this  $N$  to be our  $N$ . Let  $n$  be an arbitrary natural number such that  $n \geq N$ . Then  $b_n \in (m, y)$ . It follows by Equations 8.1 that  $m < b_n < y$ , which can be weakened to  $m \leq b_n$ . Since  $0 < m \leq b_n$ , Definition 8.4 asserts that  $m \leq |b_n|$ , as desired.

Now suppose that  $b < 0$ . By Exercise 6.5, Axiom 3, and Definition 3.3, there exists a number  $x$  such that  $x < b$ . Now consider the region  $(x, -m)$ . Since  $m < |b| = -b$ , we have by Lemma 7.24 that  $b < -m$ . This combined with the fact that  $x < b$  implies by Equations 8.1 that  $b \in (x, -m)$ . Additionally, by Corollary 4.11 and Lemma 8.3,  $(x, -m)$  is an open interval. Thus, by Definitions 15.5 and 15.2, there exists  $N \in \mathbb{N}$  such that if  $n \geq N$ , then  $b_n \in (x, -m)$ . Choose this  $N$  to be our  $N$ . Let  $n$  be an arbitrary natural number such that  $n \geq N$ . Then  $b_n \in (x, -m)$ . It follows by Equations 8.1 that  $x < b_n < -m$ , which can be weakened to  $b_n \leq -m$ . Consequently, by Lemma 7.24,  $m \leq -b_n$ . Since  $0 < m \leq -b_n$ , Definition 8.4 asserts that  $m \leq |b_n|$ , as desired.  $\square$

*Proof of Theorem 15.9c.* Let  $a = \lim_{n \rightarrow \infty} a_n$  and let  $b = \lim_{n \rightarrow \infty} b_n$ . To prove that  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$  exists and equals  $\frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}$ , Definition 15.5 and Theorem 15.7 tell us that it will suffice to show that for all  $\epsilon > 0$ , there is some  $N \in \mathbb{N}$  such that for all  $n \geq N$ , we have  $|\frac{a_n}{b_n} - \frac{a}{b}| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Choose  $M = \max(|a|, |b|)$ . Additionally, by the lemma, choose  $m \in \mathbb{R}$ ,  $N' \in \mathbb{N}$  such that  $m \leq |b|$  and if  $n \geq N'$ , then  $m \leq |b_n|$ . Using this  $M$  and  $m$  (which, again, by definition are both positive and nonzero) as well as our previously defined arbitrary  $\epsilon$ , we have by consecutive applications of Definition 15.5 and Theorem 15.7 that there exist natural numbers  $N_a, N_b$  such that for all  $n \geq N_a$ , we have  $|a_n - a| < \frac{\epsilon m^2}{M}$  and for all  $n \geq N_b$ , we have  $|b_n - b| < \frac{\epsilon m^2}{M}$ . Now choose  $N = \max(N', N_a, N_b)$ . Let  $n$  be an arbitrary natural number such that  $n \geq N$ . It follows that  $n \geq N \geq N_a$ , so we know that  $|a_n - a| < \frac{\epsilon m^2}{M}$ . Additionally, since  $n \geq N'$ ,  $m \leq |b|$  and  $m \leq |b_n|$ . Similarly,  $|b_n - b| < \frac{\epsilon m^2}{M}$ . Therefore,

$$\begin{aligned} \left| \frac{a_n}{b_n} - \frac{a}{b} \right| &= \left| \frac{a_n b - b_n a}{b_n b} \right| \\ &= \frac{|b(a_n - a) + a(b - b_n)|}{|b_n| \cdot |b|} \\ &\leq \frac{|b| \cdot |a_n - a| + |a| \cdot |b - b_n|}{|b_n| \cdot |b|} && \text{Lemma 8.8} \\ &\leq \frac{M \cdot |a_n - a| + M \cdot |b_n - b|}{m \cdot m} \\ &= \frac{M}{m^2} \cdot |a_n - a| + \frac{M}{m^2} \cdot |b_n - b| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

$\square$

5/13: **Exercise 15.10.** Which of the following sequences  $(a_n)$  converge? Which diverge? For each that converges, what is the limit? Give proofs for your answers.

(a)  $a_n = (-1)^n \cdot n.$

*Proof.* Suppose for the sake of contradiction that  $\lim_{n \rightarrow \infty} a_n$  converges. Then since  $(b_n)$  defined by  $b_n = \frac{1}{n}$  converges by Exercise 15.6c, we have by Theorem 15.9 that

$$\begin{aligned} \left( \lim_{n \rightarrow \infty} (-1)^n \cdot n \right) \cdot \left( \lim_{n \rightarrow \infty} \frac{1}{n} \right) &= \lim_{n \rightarrow \infty} (-1)^n \cdot n \cdot \frac{1}{n} \\ &= \lim_{n \rightarrow \infty} (-1)^n \end{aligned}$$

But by Exercise 15.6d,  $\lim_{n \rightarrow \infty} (-1)^n$  diverges, a contradiction.  $\square$

(b)  $a_n = \frac{1}{n^2+1} \left( 2 + \frac{1}{n} \right).$

*Proof.* To prove that  $\lim_{n \rightarrow \infty} \frac{1}{n^2+1} \left( 2 + \frac{1}{n} \right) = 0$ , we will first confirm that  $\lim_{n \rightarrow \infty} \frac{1}{n^2+1} = 0$  and  $\lim_{n \rightarrow \infty} 2 = 2$ . These results can be tied together with the fact that  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$  (by Exercise 15.6c) to prove the desired result with Theorem 15.9. Let's begin.

To confirm that  $\lim_{n \rightarrow \infty} \frac{1}{n^2+1} = 0$ , Theorem 15.7 tells us that it will suffice to demonstrate that for all  $\epsilon > 0$ , there is some  $N \in \mathbb{N}$  such that for all  $n \geq N$ , we have  $|\frac{1}{n^2+1} - 0| = |\frac{1}{n^2+1}| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Since  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$  by Exercise 15.6c, we have by Theorem 15.7 that there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$ , we have  $|\frac{1}{n}| < \epsilon$ . Choose this  $N$  to be our  $N$ . Let  $n$  be an arbitrary natural number such that  $n \geq N$ . Then since  $n \leq n^2 < n^2 + 1$  by Script 7, we have that

$$\begin{aligned} \left| \frac{1}{n^2+1} \right| &= \frac{1}{n^2+1} && \text{Definition 8.4} \\ &< \frac{1}{n^2} \\ &\leq \frac{1}{n} \\ &= \left| \frac{1}{n} \right| && \text{Definition 8.4} \\ &< \epsilon \end{aligned}$$

as desired.

The proof that  $\lim_{n \rightarrow \infty} 2 = 2$  is symmetric to that of Exercise 15.6a.

Having established that  $\lim_{n \rightarrow \infty} \frac{1}{n^2+1} = 0$ ,  $\lim_{n \rightarrow \infty} 2 = 2$ , and  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ , we have by consecutive applications of Theorem 15.9 that

$$\begin{aligned} 0 &= 0 \cdot (2 + 0) \\ &= \left( \lim_{n \rightarrow \infty} \frac{1}{n^2+1} \right) \cdot \left( \lim_{n \rightarrow \infty} 2 + \lim_{n \rightarrow \infty} \frac{1}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^2+1} \left( 2 + \frac{1}{n} \right) \end{aligned}$$

as desired.  $\square$

(c)  $a_n = \frac{5n+1}{2n+3}.$

*Proof.* The proof that  $\lim_{n \rightarrow \infty} 3 = 3$  is symmetric to that of Exercise 15.6a. Additionally, by Exercise 15.6a, the proof of Exercise 15.10b, and Exercise 15.6c, we know that  $\lim_{n \rightarrow \infty} 5 = 5$ ,  $\lim_{n \rightarrow \infty} 2 = 2$ , and  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ , respectively. Therefore, by consecutive applications of Theorem 15.9, we have that

$$\begin{aligned} \frac{5}{2} &= \frac{5 + 0}{2 + 3 \cdot 0} \\ &= \frac{\lim_{n \rightarrow \infty} 5 + \lim_{n \rightarrow \infty} \frac{1}{n}}{\lim_{n \rightarrow \infty} 2 + (\lim_{n \rightarrow \infty} 3) \cdot (\lim_{n \rightarrow \infty} \frac{1}{n})} \\ &= \lim_{n \rightarrow \infty} \frac{5 + \frac{1}{n}}{2 + 3 \cdot \frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} \frac{5n + 1}{2n + 3} \end{aligned}$$

as desired. □

(d)  $a_n = \frac{(-1)^{n+1}}{n}$ .

*Proof.* By Exercises 15.8a and 15.6c, we know that  $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$  and  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ , respectively. Therefore, by Theorem 15.9,

$$\begin{aligned} 0 &= 0 + 0 \\ &= \lim_{n \rightarrow \infty} \frac{(-1)^n}{n} + \lim_{n \rightarrow \infty} \frac{1}{n} \\ &= \lim_{n \rightarrow \infty} \left( \frac{(-1)^n}{n} + \frac{1}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{(-1)^n + 1}{n} \end{aligned}$$

as desired. □

We've used the word "limit" in two contexts now: The limit of a point in a set, and the limit of a sequence. The definitions of these two terms may seem similar. Is there a formal connection? Theorem 15.11 alludes to an answer.

**Theorem 15.11.** *Let  $A \subset \mathbb{R}$ . Then  $p \in \overline{A}$  if and only if there exists a sequence  $(a_n)$ , with each  $a_n \in A$ , that converges to  $p$ .*

*Proof.* Suppose first that  $p \in \overline{A}$ . Then by Definitions 4.4 and 1.5,  $p \in A$  or  $p \in LP(A)$ . We now divide into two cases. If  $p \in A$ , then define  $(a_n)$  by  $a_n = p$  for all  $n \in \mathbb{N}$ . Clearly, each  $a_n \in A$  since  $p \in A$ , and  $\lim_{n \rightarrow \infty} a_n = p$  by a proof symmetric to that of Exercise 15.6a, as desired. If  $p \in LP(A)$ , then define  $R_n = (p - \frac{1}{n}, p + \frac{1}{n})$  for all  $n \in \mathbb{N}$ . Since  $p \in LP(A)$ , we have by Definition 3.13 that  $R_n \cap (A \setminus \{p\}) \neq \emptyset$  for all  $n \in \mathbb{N}$ . It follows by the axiom of choice that we can choose a point  $a_n$  in  $R_n \cap (A \setminus \{p\})$  for all  $n \in \mathbb{N}$ . Thus, by Definitions 1.6 and 1.11, each  $a_n \in A$  (as desired) and  $a_n \in R_n$  for all  $n \in \mathbb{N}$ . We now seek to prove that  $(a_n)$  converges to  $p$ ; to do so, Theorem 15.7 tells us that it will suffice to show that for all  $\epsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|a_n - p| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. By Corollary 6.12, there exists an  $N \in \mathbb{N}$  such that  $\frac{1}{N} < \epsilon$ . Choose this  $N$  to be our  $N$ . Let  $n$  be an arbitrary natural number such that  $n \geq N$ . Then by consecutive applications of Lemma 7.24,  $\frac{1}{n} \leq \frac{1}{N}$ . Consequently, by Script 1 as well as the definitions of  $R_n$  and  $R_N$ ,  $R_n \subset R_N$ . It follows by Definition 1.3 since  $a_n \in R_n$  that  $a_n \in R_N$ . Therefore, by Exercise 8.9,  $|a_n - p| < \frac{1}{N} < \epsilon$ , as desired.

Now suppose that there exists a sequence  $(a_n)$  with each  $a_n \in A$  that converges to  $p$ . We divide into two cases ( $p \in A$  and  $p \notin A$ ). If  $p \in A$ , then by Definitions 1.5 and 4.4,  $p \in \overline{A}$ , as desired. If  $p \notin A$ , then to prove that  $p \in \overline{A}$ , Definitions 4.4 and 1.5 tell us that we must show that  $p \in LP(A)$ . To do so, Definition 3.13 tells us that it will suffice to verify that for all regions  $R$  containing  $p$ ,  $R \cap (A \setminus \{p\}) \neq \emptyset$ . Let  $R$  be an arbitrary region  $R$  with  $p \in R$ . By Corollary 4.11 and 8.3,  $R$  is an open interval. Thus, by Definition 15.2,

there exists an  $N \in \mathbb{N}$  such that if  $n \geq N$ , then  $a_n \in R$ . It follows that  $a_N \in R$ . Additionally, by hypothesis,  $a_N \in A$ . These results combined with the fact that  $A = A \setminus \{p\}$  (since  $p \notin A$ ) imply by Definition 1.6 that  $a_N \in R \cap (A \setminus \{p\})$ . Therefore, by Definition 1.8,  $R \cap (A \setminus \{p\}) \neq \emptyset$ , as desired.  $\square$

**Definition 15.12.** A sequence  $(a_n)$  is **bounded** if its image  $\{a_n \mid n \in \mathbb{N}\}$  is bounded.

**Theorem 15.13.** *Every convergent sequence is bounded.*

*Proof.* Let  $(a_n)$  be a sequence that converges to  $p$ . To prove that  $(a_n)$  is bounded, Definitions 15.12 and 5.6 tell us that it will suffice to find numbers  $l, u$  such that  $l \leq a_n \leq u$  for all  $a_n$ . Let  $(x, y)$  be a region that contains  $p$ . By Corollary 4.11 and Lemma 8.3,  $(x, y)$  is an open interval. Thus, by Exercise 15.3, we have that  $(x, y)$  contains all but finitely many terms of the sequence, i.e.,  $\{a_n \mid a_n \notin (x, y)\}$  is finite. We divide into two cases ( $\{a_n \mid a_n \notin (x, y)\} = \emptyset$  and  $\{a_n \mid a_n \notin (x, y)\} \neq \emptyset$ ). If  $\{a_n \mid a_n \notin (x, y)\} = \emptyset$ , then  $a_n \in (x, y)$  for all  $a_n$ . It follows by Equations 8.1 that  $x < a_n < y$  for all  $a_n$ . If we now choose  $l = x$  and  $u = y$ , we can weaken the previous statement to  $l = x \leq a_n \leq y = u$ , as desired. On the other hand, if  $\{a_n \mid a_n \notin (x, y)\} \neq \emptyset$ , then by Lemma 3.4,  $\{a_n \mid a_n \notin (x, y)\}$  has a first and a last point. It follows by Exercise 5.9 that  $\{a_n \mid a_n \notin (x, y)\}$  is bounded by  $\inf\{a_n \mid a_n \notin (x, y)\}$  and  $\sup\{a_n \mid a_n \notin (x, y)\}$ . Choose  $l = \min(x, \inf\{a_n \mid a_n \notin (x, y)\})$  and  $u = \max(y, \sup\{a_n \mid a_n \notin (x, y)\})$ . Let  $a_n$  be an arbitrary term in the sequence. We divide into two subcases ( $a_n \in (x, y)$  and  $a_n \notin (x, y)$ ). If  $a_n \in (x, y)$ , then  $l \leq x < a_n < y \leq u$ , as desired. On the other hand, if  $a_n \notin (x, y)$ , then  $l \leq \inf\{a_n \mid a_n \notin (x, y)\} \leq a_n \leq \sup\{a_n \mid a_n \notin (x, y)\} \leq u$ , as desired.  $\square$

The converse is not true, but there are important partial converses. For the first, Theorem 15.14, we recall Definition 8.16 along with Definition 15.1, which say that  $(a_n)$  is an increasing sequence if  $a_n \leq a_{n+1}$  for all  $n \in \mathbb{N}$ , and  $(a_n)$  is decreasing if  $a_n \geq a_{n+1}$  for all  $n \in \mathbb{N}$ . The definitions for strictly increasing/strictly decreasing are similar but with strict inequalities.

**Theorem 15.14.** *Every bounded increasing sequence converges to the supremum of its image. Every bounded decreasing sequence converges to the infimum of its image.*

*Proof.* We will only address the first part of the theorem; the proof of the second part is symmetric.

Let  $(a_n)$  be a bounded increasing sequence and let  $p = \sup\{a_n \mid n \in \mathbb{N}\}$ . To prove that  $(a_n)$  converges to  $p$ , Theorem 15.7 tells us that it will suffice to show that for all  $\epsilon > 0$ , there is some  $N \in \mathbb{N}$  such that for all  $n \geq N$ , we have  $|a_n - p| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. By Lemma 5.11, there exists  $a_N \in \{a_n \mid n \in \mathbb{N}\}$  such that  $p - \epsilon < a_N \leq p$ . Choose  $N$  to be the natural number that generates  $a_N$ . Let  $n$  be an arbitrary natural number such that  $n \geq N$ . Then since  $a_N \leq a_{N+1} \leq \dots \leq a_{n-1} \leq a_n$ , we have by transitivity that  $a_N \leq a_n$ . Additionally, since  $a_n \in \{a_n \mid n \in \mathbb{N}\}$ , we have by Definitions 5.7 and 5.6 that  $a_n \leq p$ . Thus, since  $p - \epsilon < a_N \leq a_n \leq p < p + \epsilon$ , we have by Equations 8.1 that  $a_n \in (p - \epsilon, p + \epsilon)$ . Therefore, by Exercise 8.9,  $|a_n - p| < \epsilon$ , as desired.  $\square$

To discuss the second partial converse, Theorem 15.18, we need another definition.

**Definition 15.15.** Let  $(a_n)$  be a sequence. A **subsequence** of  $(a_n)$  is a sequence  $b : \mathbb{N} \rightarrow \mathbb{R}$  defined by the composition  $b = a \circ i$ , where  $i : \mathbb{N} \rightarrow \mathbb{N}$  is a strictly increasing function. If  $(a_n)$  has a subsequence with limit  $p$ , we call  $p$  a **subsequential limit** of  $(a_n)$ .

We can write  $b_k = a(i(k)) = a_{i(k)} = a_{i_k}$ , so that  $(b_k)$  is the sequence  $b_1, b_2, b_3, \dots$ , which is equal to the sequence  $a_{i_1}, a_{i_2}, a_{i_3}, \dots$ , where  $i_1 < i_2 < i_3 < \dots$ .

**Theorem 15.16.** *If  $(a_n)$  converges to  $p$ , then so do all of its subsequences.*

*Proof.* Let  $(b_n)$  be an arbitrary subsequence of  $(a_n)$ . To prove that  $(b_n)$  converges to  $p$ , Theorem 15.7 tells us that it will suffice to show that for all  $\epsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$ , we have  $|b_n - p| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Since  $(a_n)$  converges to  $p$ , Theorem 15.7 implies that there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$ , we have  $|a_n - p| < \epsilon$ . Choose this  $N$  to be our  $N$ . Let  $n$  be an arbitrary natural number such that  $n \geq N$ . By Definition 15.15 and Script 1,  $i(n) \geq n$ . Therefore, we have by the above that  $|b_n - p| = |a_{i_n} - p| < \epsilon$ , as desired.  $\square$



5/18: **Exercise 15.17.** Construct a sequence with two subsequential limits. Construct a sequence with infinitely many subsequential limits.

*Proof.* Let  $(a_n)$  be a sequence defined by  $a_n = (-1)^n$  for all  $n \in \mathbb{N}$ . Let  $(b_n)$  be a subsequence of  $(a_n)$  defined by  $b_n = a_{2n}$  for all  $n \in \mathbb{N}$ . Then by the proof of Exercise 15.6d,  $b_n = 1$  for all  $n \in \mathbb{N}$ . It follows by the proof of Exercise 15.6a that  $\lim_{n \rightarrow \infty} b_n = 1$ . Similarly, if we let  $(c_n)$  be defined by  $c_n = a_{2n+1}$ , then  $\lim_{n \rightarrow \infty} c_n = -1$ .

Now let  $(a_n)$  be defined by  $a_1 = 1$ ;  $a_2 = 1$  and  $a_3 = 2$ ;  $a_4 = 1$ ,  $a_5 = 2$ , and  $a_6 = 3$ ;  $a_7 = 1$ ,  $a_8 = 2$ ,  $a_9 = 3$ , and  $a_{10} = 4$ ; and so on. Clearly there will be infinitely many terms that evaluate to each natural number in this sequence. Therefore, each of the infinitely many natural numbers is a subsequential limit of this sequence. As one example of this, consider the subsequence defined by  $b_n = a_{\frac{1}{2}n^2 - \frac{1}{2}n + 1}$ . This will include points  $a_1, a_2, a_4, a_7, a_{11}, \dots$ , which are all of the terms that equal 1. As such, by a proof symmetric to that used for Exercise 15.6a, we can prove that  $\lim_{n \rightarrow \infty} b_n = 1$  is a subsequential limit of  $(a_n)$ .  $\square$

**Theorem 15.18.** *Every bounded sequence has a convergent subsequence.*

*Proof.* Let  $(a_n)$  be an arbitrary bounded sequence, and define  $A = \{a_n \mid n \in \mathbb{N}\}$ . We divide into two cases ( $A$  is finite and  $A$  is infinite).

Suppose first that  $A$  is finite. Suppose for the sake of contradiction that for all  $a \in A$ , the set  $\{n \in \mathbb{N} \mid a_n = a\}$  is finite. Then by Script 1,  $\bigcup_{a \in A} \{n \in \mathbb{N} \mid a_n = a\} = \{n \in \mathbb{N} \mid a_n \in A\}$  is finite. But since  $\{n \in \mathbb{N} \mid a_n \in A\} = \mathbb{N}$  by the definition of  $A$ , we have that  $\mathbb{N}$  is finite, contradicting Definition 1.35's assertion that  $\mathbb{N}$  is infinite.

Now suppose that  $A$  is infinite. By Definition 15.12,  $A$  is bounded. Additionally, by Script 6,  $A$  is a subset of  $\mathbb{R}$ . This combined with the previous result implies by Theorem 10.18 that there exists a limit point  $p$  of  $A$ . It follows by Definition 3.13 that for all regions  $R$  containing  $p$ ,  $R \cap (A \setminus \{p\}) \neq \emptyset$ . Thus, since  $R_n = (p - \frac{1}{n}, p + \frac{1}{n})$  is a region containing  $p$  for all  $n \in \mathbb{N}$ , we have that  $R_n \cap (A \setminus \{p\}) \neq \emptyset$  for all  $n \in \mathbb{N}$ . We are now ready to define our subsequence  $(a_{i_n})$ . By the axiom of choice, choose  $a_{i_n} \in (p - \frac{1}{n}, p + \frac{1}{n})$  where  $i_{n-1} < i_n$  for all  $n \in \mathbb{N}$  (for  $a_{i_1}$ , just choose any element of  $(p - 1, p + 1)$ ). Note that we are guaranteed that there are elements within  $(p - \frac{1}{n}, p + \frac{1}{n})$  arbitrarily far down the sequence (i.e., that we can choose  $a_{i_n}$  in  $(p - \frac{1}{n}, p + \frac{1}{n})$  with  $i_{n-1} < i_n$ ) since if there were not, that would imply that there are only finitely many terms of the sequence within  $R_n \cap (A \setminus \{p\})$ . But then in this case, we would be able to choose an  $R$  sufficiently small such that  $R \cap (A \setminus \{p\}) = \emptyset$ , a contradiction. Now to prove that  $(a_{i_n})$  converges, Theorem 15.7 tells us that it will suffice to show that for all  $\epsilon > 0$ , there is some  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|a_{i_n} - p| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. By Corollary 6.12, there exists an  $N \in \mathbb{N}$  such that  $\frac{1}{N} < \epsilon$ . Choose this  $N$  to be our  $N$ . Let  $n$  be an arbitrary natural number such that  $n \geq N$ . It follows by the definition of  $(a_{i_n})$  that  $a_{i_n} \in (p - \frac{1}{n}, p + \frac{1}{n})$ . Thus, since  $(p - \frac{1}{n}, p + \frac{1}{n}) \subset (p - \frac{1}{N}, p + \frac{1}{N})$  by Script 1, we have by Definition 1.3 that  $a_n \in (p - \frac{1}{N}, p + \frac{1}{N})$ . Therefore, by Exercise 8.9, we have that  $|a_n - p| < \frac{1}{N} < \epsilon$ , as desired.  $\square$

We are now able to prove a useful characterization of convergent sequences.

**Theorem 15.19.** *A sequence  $(a_n)$  of real numbers converges if and only if for all  $\epsilon > 0$ , there is some  $N \in \mathbb{N}$  such that  $|a_n - a_m| < \epsilon$  for all  $n, m \geq N$ .*

*Proof.* Suppose first that  $(a_n)$  converges to  $p$ . Let  $\epsilon > 0$  be arbitrary. Then by Theorem 15.7, there is some  $N \in \mathbb{N}$  such that for all  $n \geq N$ , we have  $|a_n - p| < \frac{\epsilon}{2}$ . Choose this  $N$  to be our  $N$ . Let  $n, m$  be arbitrary natural numbers such that  $n, m \geq N$ . Then  $|a_n - p| < \frac{\epsilon}{2}$  and  $|a_m - p| < \frac{\epsilon}{2}$ . Therefore,

$$\begin{aligned} |a_n - a_m| &\leq |a_n - p| + |p - a_m| && \text{Lemma 8.8} \\ &= |a_n - p| + |a_m - p| && \text{Exercise 8.5} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

as desired.

Now suppose that for all  $\epsilon > 0$ , there is some  $N \in \mathbb{N}$  such that  $|a_n - a_m| < \epsilon$  for all  $n, m \geq N$ . To prove that  $(a_n)$  converges, we will first show that it is bounded. It will then follow by Theorem 15.18 that  $(a_n)$

has a subsequence  $(b_n)$  that converges to  $p$ . The existence of  $(b_n)$  combined with the hypothesis will suffice to show that  $(a_n)$  converges to  $p$ . Let's begin.

To confirm that  $(a_n)$  is bounded, Definitions 15.12 and 5.6 tell us that it will suffice to find numbers  $l, u$  such that  $l \leq a_n \leq u$  for all  $a_n$ . Since  $1 > 0$  by Corollary 7.27, we have by the hypothesis that there is some  $N \in \mathbb{N}$  such that  $|a_n - a_m| < 1$  for all  $n, m \geq N$ . Thus, we know that  $|a_N - a_n| < 1$  for all  $n \geq N$ . Consequently, by Exercise 8.9,  $a_n \in (a_N - 1, a_N + 1)$  for all  $n \geq N$ . It follows by Script 1 that  $\{a_n \mid a_n \notin (a_N - 1, a_N + 1)\}$  is finite, since it can contain at most  $N - 1$  terms. If we now divide into two cases and evaluate them in a symmetric fashion to the way we did in the proof of Theorem 15.13, we can establish that  $(a_n)$  is bounded, as desired.

Since  $(a_n)$  is bounded, we have by Theorem 15.18 that there exists a convergent subsequence  $(b_n)$  of  $(a_n)$ . Let  $\lim_{n \rightarrow \infty} b_n = p$ .

To prove that  $(a_n)$  converges to  $p$ , Theorem 15.7 tells us that it will suffice to show that for all  $\epsilon > 0$ , there is some  $N \in \mathbb{N}$  such that for all  $n \geq N$ , we have  $|a_n - p| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Since  $(b_n)$  converges to  $p$ , Theorem 15.7 asserts that there is some  $N_1 \in \mathbb{N}$  such that for all  $n \geq N_1$ , we have  $|b_n - p| < \frac{\epsilon}{2}$ . Additionally, we have by the hypothesis that there is some  $N_2 \in \mathbb{N}$  such that  $|a_n - a_m| < \frac{\epsilon}{2}$  for all  $n, m \geq N_2$ . Choose  $N = \max(N_1, N_2)$ . Let  $n$  be an arbitrary natural number such that  $n \geq N$ . By Corollary 6.13, there exists a  $b_m = a_{i_m}$  with  $i_m \geq N$ . Thus, since  $i_m \geq N \geq N_1$ , we have that  $|a_{i_m} - p| < \frac{\epsilon}{2}$ . Additionally, since  $n \geq N \geq N_2$  and  $i_m \geq N \geq N_2$ , we have that  $|a_n - a_{i_m}| < \frac{\epsilon}{2}$ . Therefore,

$$\begin{aligned} |a_n - p| &\leq |a_n - a_{i_m}| + |a_{i_m} - p| && \text{Lemma 8.8} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

as desired. □

**Theorem 15.20.** *Use Theorem 15.19 to show that the sequence in Exercise 15.10a does not converge.*

*Proof.* Suppose for the sake of contradiction that  $a_n = (-1)^n \cdot n$  converges to some  $p \in \mathbb{R}$ . Then if we choose  $\epsilon = 1$ , we have by Theorem 15.19 that there exists some  $N \in \mathbb{N}$  such that  $|a_n - a_m| < 1$  for all  $n, m \geq N$ . But we also have that  $|a_N - a_{N+1}| = 2N + 1 > 1$ , regardless of which natural number  $N$  is, a contradiction. □