

Script 15

Sequences

15.1 Journal

5/6: **Definition 15.1.** A **sequence** (of real numbers) is a function $a : \mathbb{N} \rightarrow \mathbb{R}$.

By setting $a_n = a(n)$, we can think of a sequence as a list a_1, a_2, a_3, \dots of real numbers. We use the notation $(a_n)_{n=1}^\infty$ for such a sequence, or if there is no possibility of confusion, we sometimes abbreviate this and write simply (a_n) . More generally, we also use the term sequence to refer to the function defined on $\{n \in \mathbb{N} \cup \{0\} \mid n \geq n_0\}$ for any fixed $n_0 \in \mathbb{N} \cup \{0\}$. We write $(a_n)_{n=n_0}^\infty$ for such a sequence.

Definition 15.2. We say that a sequence (a_n) **converges** to a point $p \in \mathbb{R}$ if for every open interval I containing p , there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in I$. If a sequence converges to some point, we say it is **convergent**. If (a_n) does not converge to any point, we say that the sequence **diverges** or is **divergent**.

Exercise 15.3. Show that a sequence (a_n) converges to p if and only if any region containing p contains all but finitely many terms of the sequence.

Proof. Suppose first that (a_n) converges to p . Let R be an arbitrary region containing p . By Corollary 4.11 and Lemma 8.3, R is an open interval. Thus, by Definition 15.2, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in R$. To prove that R contains all but finitely many terms of the sequence, it will suffice to show that the set $A = \{a_n \mid a_n \notin R\}$ is finite. Since $a_n \in R$ for all $n \geq N$, it follows that $a_n \in R$ only if $n < N$. Thus, by Script 1, $A \subset \{a_n \mid 0 \leq n < N\}$. Since the latter set is clearly finite, it follows by Script 1 that A is finite.

Now suppose that any region containing p contains all but finitely many terms (a_n) . To prove that (a_n) converges to p , Definition 15.2 tells us that it will suffice to show that for every open interval I containing p , there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in I$. Let I be an arbitrary open interval containing p . Then by Theorem 4.10, there exists a region R containing p such that $R \subset I$. It follows by the hypothesis that $A = \{a_n \mid a_n \notin R\}$ is finite. We divide into two cases ($|A| = 0$ and $|A| \in \mathbb{N}$). Suppose first that $|A| = 0$. Choose $N = n_0$. It follows that if $n \geq N$, then $a_n \notin A$, so $a_n \in R$, so $a_n \in I$, as desired. Now suppose that $|A| \in \mathbb{N}$. By Definition 1.18, $a^{-1}(A) \subset \mathbb{N}$. Consequently, by Lemma 3.4, $a^{-1}(A)$ has a last point $N - 1$. Choose $N = (N - 1) + 1$. It follows that if $n \geq N$, then $n \notin a^{-1}(A)$, so $a_n \notin A$, so $a_n \in R$, so $a_n \in I$, as desired. \square

Theorem 15.4. Suppose that (a_n) converges to both p and to p' . Then $p = p'$.

Proof. Suppose for the sake of contradiction that $p \neq p'$. Then by Theorem 3.22, there exist disjoint regions R, R' containing p, p' , respectively. Additionally, by consecutive applications of Corollary 4.11 and Lemma 8.3, R, R' are open intervals. Combining these last two results, we have by consecutive applications of Definition 15.2 that there exist $N, N' \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in R$ and if $n \geq N'$, then $a_n \in R'$. Let $M = \max(N, N')$. It follows that $M \geq N$ and $M \geq N'$. Thus, by the above, $a_M \in R$ and $a_M \in R'$. But this implies by Definition 1.6 that $a_M \in R \cap R'$. Therefore, by Definition 1.9, R and R' are not disjoint, a contradiction. \square

Definition 15.5. If a sequence (a_n) converges to $p \in \mathbb{R}$, we call p the **limit** of (a_n) and write

$$\lim_{n \rightarrow \infty} a_n = p$$

Exercise 15.6. Which of the following sequences (a_n) converge? Which diverge? For each that converges, what is the limit? Give proofs for your answers.

(a) $a_n = 5$.

Proof. To prove that this sequence converges with limit $\lim_{n \rightarrow \infty} a_n = 5$, Definition 15.5 tells us that it will suffice to show that (a_n) converges to 5. To do so, Definition 15.2 tells us that it will suffice to verify that for every open interval I containing 5, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in I$. Let's begin.

Let I be an arbitrary open interval containing 5. Choose $N = 1$. Let n be an arbitrary natural number such that $n \geq N$. It follows by the definition of the sequence that $a_n = 5 \in I$, as desired. \square

(b) $a_n = n$.

Proof. To prove that this sequence diverges, Definition 15.2 tells us that it will suffice to show that for every point $p \in \mathbb{R}$, there exists an open interval I containing p such that for all $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \notin I$. Let's begin.

Let p be an arbitrary element of \mathbb{R} . Choose $I = (p - 1, p + 1)$. Clearly $p \in I$. Let N be an arbitrary natural number. By Corollary 6.12, there exists a natural number N' such that $p + 1 < N'$. Choose $M = \max(N, N')$. Thus, $M \geq N$. Additionally, it follows by the definition of the sequence that $a_M = M$. But this implies that $a_M \geq N' > p + 1$, i.e., $a_M \notin I$ by Equations 8.1. \square

(c) $a_n = \frac{1}{n}$.

Proof. To prove that this sequence converges with limit $\lim_{n \rightarrow \infty} a_n = 0$, Definitions 15.5 and 15.2 tell us that it will suffice to show that for every open interval I containing 0, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in I$. Let's begin.

Let I be an arbitrary interval containing 0. By Lemma 8.10, there exists a region (a, b) containing 0 such that $(a, b) \subset I$. By Corollary 6.12, there exists a natural number N such that $\frac{1}{b} < N$. Choose this N to be our N . Now let n be an arbitrary natural number such that $n \geq N$. It follows that $\frac{1}{b} < n$. Thus, since $0 < b$ and $0 < n$, we have by consecutive applications of Lemma 7.24 that $0 < \frac{1}{n} < b$. Consequently, since we also know that $a < 0$ and $a_n = \frac{1}{n}$, we have by transitivity and substitution that $a < a_n < b$. It follows by Equations 8.1 that $a_n \in (a, b)$. Therefore, by Definition 1.3, $a_n \in I$, as desired. \square

(d) $a_n = (-1)^n$.

Proof. To prove that this sequence diverges, Definition 15.2 tells us that it will suffice to show that for every point $p \in \mathbb{R}$, there exists an open interval I containing p such that for all $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \notin I$. Let's begin.

Let p be an arbitrary element of \mathbb{R} . Choose $I = (p - 1, p + 1)$. Clearly $p \in I$. Let N be an arbitrary natural number. By Script 0, either N is even and $N + 1$ is odd or vice versa. Thus, let N be even (the case where N is odd is symmetric). It follows that $N \geq N$ yields $a_N = (-1)^{2m} = ((-1)^2)^m = 1^m = 1$ and that $N + 1 \geq N$ yields $a_{N+1} = (-1)^{2m+1} = 1(-1)^1 = -1$. Now suppose for the sake of contradiction that $a_N \in I$ and $a_{N+1} \in I$. Since $a_N = 1 \in I$, we have by Equations 8.1 that $p - 1 < 1 < p + 1$. It follows by Definition 7.21 that $p - 3 < -1 < p - 1$. But $-1 < p - 1$ implies by Equations 8.1 that $a_{N+1} = -1 \notin I$, a contradiction. Therefore, $N + 1 \geq N$ is a number such that $a_{N+1} \notin I$, as desired. \square

5/11: **Theorem 15.7.** *A sequence (a_n) converges to $p \in \mathbb{R}$ if and only if for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - p| < \epsilon$.*

Proof. Suppose first that (a_n) converges to p . Let $\epsilon > 0$ be arbitrary. Consider the p -containing region $R = (p - \epsilon, p + \epsilon)$. By Corollary 4.11 and 8.3, R is an open interval. Thus, by Definition 15.2, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in R$. Choose this N to be our N . Let n be an arbitrary natural number such that $n \geq N$. Then $a_n \in (p - \epsilon, p + \epsilon)$. Therefore, by Exercise 8.9, $|a_n - p| < \epsilon$, as desired.

Now suppose that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - p| < \epsilon$. To prove that (a_n) converges to p , Definition 15.2 tells us that it will suffice to show that for every open interval I containing p , there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in I$. Let I be an arbitrary open interval that satisfies $p \in I$. It follows by Lemma 8.10 that there exists a number $\epsilon > 0$ such that $(p - \epsilon, p + \epsilon) \subset I$. With respect to this ϵ , we have by hypothesis that there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - p| < \epsilon$. Choose this N to be our N . Let n be an arbitrary natural number such that $n \geq N$. Then $|a_n - p| < \epsilon$. Consequently, by Exercise 8.9, $a_n \in (p - \epsilon, p + \epsilon)$. Therefore, by Definition 1.3, $a_n \in I$, as desired. \square

Exercise 15.8.

(a) Prove that $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$.

Proof. To prove that $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$, Definition 15.5 and Theorem 15.7 tell us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - 0| = |a_n| < \epsilon$. Let $\epsilon > 0$ be arbitrary. By Corollary 6.12, there exists a natural number N such that $\frac{1}{\epsilon} < N$. Choose this N to be our N . Let n be an arbitrary natural number such that $n \geq N$. It follows by transitivity that $\frac{1}{\epsilon} < n$. Thus, since $0 < n$ and $0 < \epsilon$, we have by consecutive applications of Lemma 7.24 that $0 < \frac{1}{n} < \epsilon$. Additionally, since $(-1)^n = 1$ or $(-1)^n = -1$ for all $n \in \mathbb{N}$ by Script 0, we have by Definition 8.4 that $|\frac{(-1)^n}{n}| = |\frac{1}{n}| = \frac{1}{n}$. Consequently, we know that $|\frac{(-1)^n}{n}| < \epsilon$. But since $a_n = \frac{(-1)^n}{n}$, we have that $|a_n| < \epsilon$, as desired. \square

(b) Let $x \in \mathbb{R}$ with $|x| < 1$. Prove that $\lim_{n \rightarrow \infty} x^n = 0$.

Lemma. *If $|y| > 1$ and n is a natural number, then $|y|^n \geq n(|y| - 1) + 1$.*

Proof. Define $1 + x = |y|$. It follows by Definition 7.21 that $x > 0 > -1$, which can be weakened to $x \geq -1$. Additionally, since n is a natural number, $n \geq 1$ by Script 0. Thus, since $x \geq -1$ and $n \geq 1$, we have by Additional Exercise 12.3b that $(1 + x)^n \geq 1 + nx$. Substituting, we have $|y|^n \geq n(|y| - 1) + 1$, as desired. \square

Proof of Exercise 15.8b. To prove that $\lim_{n \rightarrow \infty} x^n = 0$, Definition 15.5 and Theorem 15.7 tell us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - 0| = |a_n| < \epsilon$. Let $\epsilon > 0$ be arbitrary. We divide into two cases ($x = 0$ and $x \neq 0$). If $x = 0$, then choose $N = 1$. Let n be an arbitrary natural number such that $n \geq N$. Since $0^n = 0$ by Script 7, we have $|a_n| = |0^n| = 0 < \epsilon$, as desired. On the other hand, if $x \neq 0$, then we continue. By Corollary 6.12, there exists a natural number N such that $\frac{1}{\epsilon(|\frac{1}{x}| - 1)} < N$. Choose this N to be our N . Let n be an arbitrary natural number such that $n \geq N$. Therefore,

$$\begin{aligned} |x^n| &= |x^n| \cdot \frac{1}{\epsilon \left(\left| \frac{1}{x} \right| - 1 \right)} \cdot \epsilon \left(\left| \frac{1}{x} \right| - 1 \right) \\ &< |x^n| \cdot N \cdot \epsilon \left(\left| \frac{1}{x} \right| - 1 \right) \\ &\leq |x^n| \cdot n \cdot \epsilon \left(\left| \frac{1}{x} \right| - 1 \right) \\ &< \epsilon \cdot |x^n| \cdot n \left(\left| \frac{1}{x} \right| - 1 \right) + 1 \end{aligned}$$

$$\begin{aligned}
&\leq \epsilon \cdot |x^n| \cdot \left| \frac{1}{x} \right|^n \\
&= \epsilon \cdot |x^n| \cdot \frac{1}{|x^n|} \\
&= \epsilon
\end{aligned}$$

The Lemma

as desired. □**Theorem 15.9.** *If $\lim_{n \rightarrow \infty} a_n = a$ and $\lim_{n \rightarrow \infty} b_n = b$ both exist, then*

$$(a) \lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n.$$

Proof. Let $a = \lim_{n \rightarrow \infty} a_n$ and let $b = \lim_{n \rightarrow \infty} b_n$. To prove that $\lim_{n \rightarrow \infty} (a_n + b_n)$ exists and equals $\lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n$, Definition 15.5 and Theorem 15.7 tell us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|(a_n + b_n) - (a + b)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. It follows by consecutive applications of Definition 15.5 and Theorem 15.7 that there exist natural numbers N_a, N_b such that for all $n \geq N_a$, we have $|a_n - a| < \frac{\epsilon}{2}$ and for all $n \geq N_b$, we have $|b_n - b| < \frac{\epsilon}{2}$. Now choose $N = \max(N_a, N_b)$. Let n be an arbitrary natural number such that $n \geq N$. It follows that $n \geq N \geq N_a$, so we know that $|a_n - a| < \frac{\epsilon}{2}$. Similarly, $|b_n - b| < \frac{\epsilon}{2}$. Therefore, we have that

$$\begin{aligned}
|(a_n + b_n) - (a + b)| &\leq |a_n - a| + |b_n - b| && \text{Lemma 8.8} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

□

$$(b) \lim_{n \rightarrow \infty} (a_n \cdot b_n) = (\lim_{n \rightarrow \infty} a_n) \cdot (\lim_{n \rightarrow \infty} b_n).$$

Proof. Let $a = \lim_{n \rightarrow \infty} a_n$ and let $b = \lim_{n \rightarrow \infty} b_n$. To prove that $\lim_{n \rightarrow \infty} (a_n \cdot b_n)$ exists and equals $(\lim_{n \rightarrow \infty} a_n) \cdot (\lim_{n \rightarrow \infty} b_n)$, Definition 15.5 and Theorem 15.7 tell us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n \cdot b_n - a \cdot b| < \epsilon$. Let $\epsilon > 0$ be arbitrary. By Theorem 15.13^[1], (a_n) is bounded. Thus, by Definition 15.12, $\{a_n \mid n \in \mathbb{N}\}$ is bounded. Consequently, by the proof of Exercise 13.9, there exists a number M_a such that $|a_n| < M_a$ for all $n \in \mathbb{N}$. Now define $M = \max(M_a, b)$. Using this M (which by definition is positive since it's greater than $|a_n|$, which is at least 0) as well as our previously defined arbitrary ϵ , we have by consecutive applications of Definition 15.5 and Theorem 15.7 that there exist natural numbers N_a, N_b such that for all $n \geq N_a$, we have $|a_n - a| < \frac{\epsilon}{2M}$ and for all $n \geq N_b$, we have $|b_n - b| < \frac{\epsilon}{2M}$. Now choose $N = \max(N_a, N_b)$. Let n be an arbitrary natural number such that $n \geq N$. It follows that $n \geq N \geq N_a$, so we know that $|a_n - a| < \frac{\epsilon}{2M}$. Similarly, $|b_n - b| < \frac{\epsilon}{2M}$. Therefore,

$$\begin{aligned}
|a_n b_n - ab| &= |a_n(b_n - b) + b(a_n - a)| \\
&\leq |a_n| \cdot |b_n - b| + |b| \cdot |a_n - a| && \text{Lemma 8.8} \\
&\leq |M| \cdot |b_n - b| + |M| \cdot |a_n - a| \\
&< M \cdot \frac{\epsilon}{2M} + M \cdot \frac{\epsilon}{2M} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

□

¹The proof of Theorem 15.13 does not depend on any following results, so its use here is not circular logic.

Moreover, if $\lim_{n \rightarrow \infty} b_n \neq 0$, then

$$(c) \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}.$$

Lemma. Let $\lim_{n \rightarrow \infty} b_n = b \neq 0$. Then there exists $m \in \mathbb{R}^+$ such that $m \leq |b|$ and $N \in \mathbb{N}$ such that if $n \geq N$, then $m \leq |b_n|$.

Proof. Since $b \neq 0$, it follows from Definition 8.4 that $0 < |b|$. Thus, by Theorem 5.2, there exists a point $m \in \mathbb{R}$ such that $0 < m < |b|$. It follows from the fact that $0 < m$ that $m \in \mathbb{R}^+$, and from the fact that $m < |b|$ that $m \leq |b|$, as desired.

As to the other part of the proof, we divide into two cases ($b > 0$ and $b < 0$).

Suppose first that $b > 0$. By Exercise 6.5, Axiom 3, and Definition 3.3, there exists a number y such that $b < y$. Now consider the region (m, y) . Since $m < |b| = b < y$, Equations 8.1 assert that $b \in (m, y)$. Additionally, by Corollary 4.11 and Lemma 8.3, (m, y) is an open interval. Thus, by Definitions 15.5 and 15.2, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $b_n \in (m, y)$. Choose this N to be our N . Let n be an arbitrary natural number such that $n \geq N$. Then $b_n \in (m, y)$. It follows by Equations 8.1 that $m < b_n < y$, which can be weakened to $m \leq b_n$. Since $0 < m \leq b_n$, Definition 8.4 asserts that $m \leq |b_n|$, as desired.

Now suppose that $b < 0$. By Exercise 6.5, Axiom 3, and Definition 3.3, there exists a number x such that $x < b$. Now consider the region $(x, -m)$. Since $m < |b| = -b$, we have by Lemma 7.24 that $b < -m$. This combined with the fact that $x < b$ implies by Equations 8.1 that $b \in (x, -m)$. Additionally, by Corollary 4.11 and Lemma 8.3, $(x, -m)$ is an open interval. Thus, by Definitions 15.5 and 15.2, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $b_n \in (x, -m)$. Choose this N to be our N . Let n be an arbitrary natural number such that $n \geq N$. Then $b_n \in (x, -m)$. It follows by Equations 8.1 that $x < b_n < -m$, which can be weakened to $b_n \leq -m$. Consequently, by Lemma 7.24, $m \leq -b_n$. Since $0 < m \leq -b_n$, Definition 8.4 asserts that $m \leq |b_n|$, as desired. \square

Proof of Theorem 15.9c. Let $a = \lim_{n \rightarrow \infty} a_n$ and let $b = \lim_{n \rightarrow \infty} b_n$. To prove that $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ exists and equals $\frac{\lim_{n \rightarrow \infty} a_n}{\lim_{n \rightarrow \infty} b_n}$, Definition 15.5 and Theorem 15.7 tell us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|\frac{a_n}{b_n} - \frac{a}{b}| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $M = \max(|a|, |b|)$. Additionally, by the lemma, choose $m \in \mathbb{R}$, $N' \in \mathbb{N}$ such that $m \leq |b|$ and if $n \geq N'$, then $m \leq |b_n|$. Using this M and m (which, again, by definition are both positive and nonzero) as well as our previously defined arbitrary ϵ , we have by consecutive applications of Definition 15.5 and Theorem 15.7 that there exist natural numbers N_a, N_b such that for all $n \geq N_a$, we have $|a_n - a| < \frac{\epsilon m^2}{M}$ and for all $n \geq N_b$, we have $|b_n - b| < \frac{\epsilon m^2}{M}$. Now choose $N = \max(N', N_a, N_b)$. Let n be an arbitrary natural number such that $n \geq N$. It follows that $n \geq N \geq N_a$, so we know that $|a_n - a| < \frac{\epsilon m^2}{M}$. Additionally, since $n \geq N'$, $m \leq |b|$ and $m \leq |b_n|$. Similarly, $|b_n - b| < \frac{\epsilon m^2}{M}$. Therefore,

$$\begin{aligned} \left| \frac{a_n}{b_n} - \frac{a}{b} \right| &= \left| \frac{a_n b - b_n a}{b_n b} \right| \\ &= \frac{|b(a_n - a) + a(b - b_n)|}{|b_n| \cdot |b|} \\ &\leq \frac{|b| \cdot |a_n - a| + |a| \cdot |b - b_n|}{|b_n| \cdot |b|} && \text{Lemma 8.8} \\ &\leq \frac{M \cdot |a_n - a| + M \cdot |b_n - b|}{m \cdot m} \\ &= \frac{M}{m^2} \cdot |a_n - a| + \frac{M}{m^2} \cdot |b_n - b| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

\square

5/13: **Exercise 15.10.** Which of the following sequences (a_n) converge? Which diverge? For each that converges, what is the limit? Give proofs for your answers.

(a) $a_n = (-1)^n \cdot n$.

Proof. Suppose for the sake of contradiction that $\lim_{n \rightarrow \infty} a_n$ converges. Then since (b_n) defined by $b_n = \frac{1}{n}$ converges by Exercise 15.6c, we have by Theorem 15.9 that

$$\begin{aligned} \left(\lim_{n \rightarrow \infty} (-1)^n \cdot n \right) \cdot \left(\lim_{n \rightarrow \infty} \frac{1}{n} \right) &= \lim_{n \rightarrow \infty} (-1)^n \cdot n \cdot \frac{1}{n} \\ &= \lim_{n \rightarrow \infty} (-1)^n \end{aligned}$$

But by Exercise 15.6d, $\lim_{n \rightarrow \infty} (-1)^n$ diverges, a contradiction. \square

(b) $a_n = \frac{1}{n^2+1} \left(2 + \frac{1}{n} \right)$.

Proof. To prove that $\lim_{n \rightarrow \infty} \frac{1}{n^2+1} \left(2 + \frac{1}{n} \right) = 0$, we will first confirm that $\lim_{n \rightarrow \infty} \frac{1}{n^2+1} = 0$ and $\lim_{n \rightarrow \infty} 2 = 2$. These results can be tied together with the fact that $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ (by Exercise 15.6c) to prove the desired result with Theorem 15.9. Let's begin.

To confirm that $\lim_{n \rightarrow \infty} \frac{1}{n^2+1} = 0$, Theorem 15.7 tells us that it will suffice to demonstrate that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|\frac{1}{n^2+1} - 0| = |\frac{1}{n^2+1}| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ by Exercise 15.6c, we have by Theorem 15.7 that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|\frac{1}{n}| < \epsilon$. Choose this N to be our N . Let n be an arbitrary natural number such that $n \geq N$. Then since $n \leq n^2 < n^2 + 1$ by Script 7, we have that

$$\begin{aligned} \left| \frac{1}{n^2+1} \right| &= \frac{1}{n^2+1} && \text{Definition 8.4} \\ &< \frac{1}{n^2} \\ &\leq \frac{1}{n} \\ &= \left| \frac{1}{n} \right| && \text{Definition 8.4} \\ &< \epsilon \end{aligned}$$

as desired.

The proof that $\lim_{n \rightarrow \infty} 2 = 2$ is symmetric to that of Exercise 15.6a.

Having established that $\lim_{n \rightarrow \infty} \frac{1}{n^2+1} = 0$, $\lim_{n \rightarrow \infty} 2 = 2$, and $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$, we have by consecutive applications of Theorem 15.9 that

$$\begin{aligned} 0 &= 0 \cdot (2 + 0) \\ &= \left(\lim_{n \rightarrow \infty} \frac{1}{n^2+1} \right) \cdot \left(\lim_{n \rightarrow \infty} 2 + \lim_{n \rightarrow \infty} \frac{1}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^2+1} \left(2 + \frac{1}{n} \right) \end{aligned}$$

as desired. \square

(c) $a_n = \frac{5n+1}{2n+3}$.

Proof. The proof that $\lim_{n \rightarrow \infty} 3 = 3$ is symmetric to that of Exercise 15.6a. Additionally, by Exercise 15.6a, the proof of Exercise 15.10b, and Exercise 15.6c, we know that $\lim_{n \rightarrow \infty} 5 = 5$, $\lim_{n \rightarrow \infty} 2 = 2$, and $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$, respectively. Therefore, by consecutive applications of Theorem 15.9, we have that

$$\begin{aligned} \frac{5}{2} &= \frac{5 + 0}{2 + 3 \cdot 0} \\ &= \frac{\lim_{n \rightarrow \infty} 5 + \lim_{n \rightarrow \infty} \frac{1}{n}}{\lim_{n \rightarrow \infty} 2 + (\lim_{n \rightarrow \infty} 3) \cdot (\lim_{n \rightarrow \infty} \frac{1}{n})} \\ &= \lim_{n \rightarrow \infty} \frac{5 + \frac{1}{n}}{2 + 3 \cdot \frac{1}{n}} \\ &= \lim_{n \rightarrow \infty} \frac{5n + 1}{2n + 3} \end{aligned}$$

as desired. □

(d) $a_n = \frac{(-1)^{n+1}}{n}$.

Proof. By Exercises 15.8a and 15.6c, we know that $\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$ and $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$, respectively. Therefore, by Theorem 15.9,

$$\begin{aligned} 0 &= 0 + 0 \\ &= \lim_{n \rightarrow \infty} \frac{(-1)^n}{n} + \lim_{n \rightarrow \infty} \frac{1}{n} \\ &= \lim_{n \rightarrow \infty} \left(\frac{(-1)^n}{n} + \frac{1}{n} \right) \\ &= \lim_{n \rightarrow \infty} \frac{(-1)^n + 1}{n} \end{aligned}$$

as desired. □

We've used the word "limit" in two contexts now: The limit of a point in a set, and the limit of a sequence. The definitions of these two terms may seem similar. Is there a formal connection? Theorem 15.11 alludes to an answer.

Theorem 15.11. *Let $A \subset \mathbb{R}$. Then $p \in \overline{A}$ if and only if there exists a sequence (a_n) , with each $a_n \in A$, that converges to p .*

Proof. Suppose first that $p \in \overline{A}$. Then by Definitions 4.4 and 1.5, $p \in A$ or $p \in LP(A)$. We now divide into two cases. If $p \in A$, then define (a_n) by $a_n = p$ for all $n \in \mathbb{N}$. Clearly, each $a_n \in A$ since $p \in A$, and $\lim_{n \rightarrow \infty} a_n = p$ by a proof symmetric to that of Exercise 15.6a, as desired. If $p \in LP(A)$, then define $R_n = (p - \frac{1}{n}, p + \frac{1}{n})$ for all $n \in \mathbb{N}$. Since $p \in LP(A)$, we have by Definition 3.13 that $R_n \cap (A \setminus \{p\}) \neq \emptyset$ for all $n \in \mathbb{N}$. It follows by the axiom of choice that we can choose a point a_n in $R_n \cap (A \setminus \{p\})$ for all $n \in \mathbb{N}$. Thus, by Definitions 1.6 and 1.11, each $a_n \in A$ (as desired) and $a_n \in R_n$ for all $n \in \mathbb{N}$. We now seek to prove that (a_n) converges to p ; to do so, Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - p| < \epsilon$. Let $\epsilon > 0$ be arbitrary. By Corollary 6.12, there exists an $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Choose this N to be our N . Let n be an arbitrary natural number such that $n \geq N$. Then by consecutive applications of Lemma 7.24, $\frac{1}{n} \leq \frac{1}{N}$. Consequently, by Script 1 as well as the definitions of R_n and R_N , $R_n \subset R_N$. It follows by Definition 1.3 since $a_n \in R_n$ that $a_n \in R_N$. Therefore, by Exercise 8.9, $|a_n - p| < \frac{1}{N} < \epsilon$, as desired.

Now suppose that there exists a sequence (a_n) with each $a_n \in A$ that converges to p . We divide into two cases ($p \in A$ and $p \notin A$). If $p \in A$, then by Definitions 1.5 and 4.4, $p \in \overline{A}$, as desired. If $p \notin A$, then to prove that $p \in \overline{A}$, Definitions 4.4 and 1.5 tell us that we must show that $p \in LP(A)$. To do so, Definition 3.13 tells us that it will suffice to verify that for all regions R containing p , $R \cap (A \setminus \{p\}) \neq \emptyset$. Let R be an arbitrary region R with $p \in R$. By Corollary 4.11 and 8.3, R is an open interval. Thus, by Definition 15.2,

there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in R$. It follows that $a_N \in R$. Additionally, by hypothesis, $a_N \in A$. These results combined with the fact that $A = A \setminus \{p\}$ (since $p \notin A$) imply by Definition 1.6 that $a_N \in R \cap (A \setminus \{p\})$. Therefore, by Definition 1.8, $R \cap (A \setminus \{p\}) \neq \emptyset$, as desired. \square

Definition 15.12. A sequence (a_n) is **bounded** if its image $\{a_n \mid n \in \mathbb{N}\}$ is bounded.

Theorem 15.13. *Every convergent sequence is bounded.*

Proof. Let (a_n) be a sequence that converges to p . To prove that (a_n) is bounded, Definitions 15.12 and 5.6 tell us that it will suffice to find numbers l, u such that $l \leq a_n \leq u$ for all a_n . Let (x, y) be a region that contains p . By Corollary 4.11 and Lemma 8.3, (x, y) is an open interval. Thus, by Exercise 15.3, we have that (x, y) contains all but finitely many terms of the sequence, i.e., $\{a_n \mid a_n \notin (x, y)\}$ is finite. We divide into two cases ($\{a_n \mid a_n \notin (x, y)\} = \emptyset$ and $\{a_n \mid a_n \notin (x, y)\} \neq \emptyset$). If $\{a_n \mid a_n \notin (x, y)\} = \emptyset$, then $a_n \in (x, y)$ for all a_n . It follows by Equations 8.1 that $x < a_n < y$ for all a_n . If we now choose $l = x$ and $u = y$, we can weaken the previous statement to $l = x \leq a_n \leq y = u$, as desired. On the other hand, if $\{a_n \mid a_n \notin (x, y)\} \neq \emptyset$, then by Lemma 3.4, $\{a_n \mid a_n \notin (x, y)\}$ has a first and a last point. It follows by Exercise 5.9 that $\{a_n \mid a_n \notin (x, y)\}$ is bounded by $\inf\{a_n \mid a_n \notin (x, y)\}$ and $\sup\{a_n \mid a_n \notin (x, y)\}$. Choose $l = \min(x, \inf\{a_n \mid a_n \notin (x, y)\})$ and $u = \max(y, \sup\{a_n \mid a_n \notin (x, y)\})$. Let a_n be an arbitrary term in the sequence. We divide into two subcases ($a_n \in (x, y)$ and $a_n \notin (x, y)$). If $a_n \in (x, y)$, then $l \leq x < a_n < y \leq u$, as desired. On the other hand, if $a_n \notin (x, y)$, then $l \leq \inf\{a_n \mid a_n \notin (x, y)\} \leq a_n \leq \sup\{a_n \mid a_n \notin (x, y)\} \leq u$, as desired. \square

The converse is not true, but there are important partial converses. For the first, Theorem 15.14, we recall Definition 8.16 along with Definition 15.1, which say that (a_n) is an increasing sequence if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$, and (a_n) is decreasing if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. The definitions for strictly increasing/strictly decreasing are similar but with strict inequalities.

Theorem 15.14 (Monotone Convergence Theorem). *Every bounded increasing sequence converges to the supremum of its image. Every bounded decreasing sequence converges to the infimum of its image.*

Proof. We will only address the first part of the theorem; the proof of the second part is symmetric.

Let (a_n) be a bounded increasing sequence and let $p = \sup\{a_n \mid n \in \mathbb{N}\}$. To prove that (a_n) converges to p , Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - p| < \epsilon$. Let $\epsilon > 0$ be arbitrary. By Lemma 5.11, there exists $a_N \in \{a_n \mid n \in \mathbb{N}\}$ such that $p - \epsilon < a_N \leq p$. Choose N to be the natural number that generates a_N . Let n be an arbitrary natural number such that $n \geq N$. Then since $a_N \leq a_{N+1} \leq \dots \leq a_{n-1} \leq a_n$, we have by transitivity that $a_N \leq a_n$. Additionally, since $a_n \in \{a_n \mid n \in \mathbb{N}\}$, we have by Definitions 5.7 and 5.6 that $a_n \leq p$. Thus, since $p - \epsilon < a_N \leq a_n \leq p < p + \epsilon$, we have by Equations 8.1 that $a_n \in (p - \epsilon, p + \epsilon)$. Therefore, by Exercise 8.9, $|a_n - p| < \epsilon$, as desired. \square

To discuss the second partial converse, Theorem 15.18, we need another definition.

Definition 15.15. Let (a_n) be a sequence. A **subsequence** of (a_n) is a sequence $b : \mathbb{N} \rightarrow \mathbb{R}$ defined by the composition $b = a \circ i$, where $i : \mathbb{N} \rightarrow \mathbb{N}$ is a strictly increasing function. If (a_n) has a subsequence with limit p , we call p a **subsequential limit** of (a_n) .

We can write $b_k = a(i(k)) = a_{i(k)} = a_{i_k}$, so that (b_k) is the sequence b_1, b_2, b_3, \dots , which is equal to the sequence $a_{i_1}, a_{i_2}, a_{i_3}, \dots$, where $i_1 < i_2 < i_3 < \dots$.

Theorem 15.16. *If (a_n) converges to p , then so do all of its subsequences.*

Proof. Let (b_n) be an arbitrary subsequence of (a_n) . To prove that (b_n) converges to p , Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|b_n - p| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since (a_n) converges to p , Theorem 15.7 implies that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - p| < \epsilon$. Choose this N to be our N . Let n be an arbitrary natural number such that $n \geq N$. By Definition 15.15 and Script 1, $i(n) \geq n$. Therefore, we have by the above that $|b_n - p| = |a_{i_n} - p| < \epsilon$, as desired. \square

5/18: **Exercise 15.17.** Construct a sequence with two subsequential limits. Construct a sequence with infinitely many subsequential limits.

Proof. Let (a_n) be a sequence defined by $a_n = (-1)^n$ for all $n \in \mathbb{N}$. Let (b_n) be a subsequence of (a_n) defined by $b_n = a_{2n}$ for all $n \in \mathbb{N}$. Then by the proof of Exercise 15.6d, $b_n = 1$ for all $n \in \mathbb{N}$. It follows by the proof of Exercise 15.6a that $\lim_{n \rightarrow \infty} b_n = 1$. Similarly, if we let (c_n) be defined by $c_n = a_{2n+1}$, then $\lim_{n \rightarrow \infty} c_n = -1$.

Now let (a_n) be defined by $a_1 = 1$; $a_2 = 1$ and $a_3 = 2$; $a_4 = 1$, $a_5 = 2$, and $a_6 = 3$; $a_7 = 1$, $a_8 = 2$, $a_9 = 3$, and $a_{10} = 4$; and so on. Clearly there will be infinitely many terms that evaluate to each natural number in this sequence. Therefore, each of the infinitely many natural numbers is a subsequential limit of this sequence. As one example of this, consider the subsequence defined by $b_n = a_{\frac{1}{2}n^2 - \frac{1}{2}n + 1}$. This will include points $a_1, a_2, a_4, a_7, a_{11}, \dots$, which are all of the terms that equal 1. As such, by a proof symmetric to that used for Exercise 15.6a, we can prove that $\lim_{n \rightarrow \infty} b_n = 1$ is a subsequential limit of (a_n) . \square

Theorem 15.18 (Bolzano-Weierstrass Theorem). *Every bounded sequence has a convergent subsequence.*

Proof. Let (a_n) be an arbitrary bounded sequence, and define $A = \{a_n \mid n \in \mathbb{N}\}$. We divide into two cases (A is finite and A is infinite).

Suppose first that A is finite. Suppose for the sake of contradiction that for all $a \in A$, the set $\{n \in \mathbb{N} \mid a_n = a\}$ is finite. Then by Script 1, $\bigcup_{a \in A} \{n \in \mathbb{N} \mid a_n = a\} = \{n \in \mathbb{N} \mid a_n \in A\}$ is finite. But since $\{n \in \mathbb{N} \mid a_n \in A\} = \mathbb{N}$ by the definition of A , we have that \mathbb{N} is finite, contradicting Definition 1.35's assertion that \mathbb{N} is infinite.

Now suppose that A is infinite. By Definition 15.12, A is bounded. Additionally, by Script 6, A is a subset of \mathbb{R} . This combined with the previous result implies by Theorem 10.18 that there exists a limit point p of A . It follows by Definition 3.13 that for all regions R containing p , $R \cap (A \setminus \{p\}) \neq \emptyset$. Thus, since $R_n = (p - \frac{1}{n}, p + \frac{1}{n})$ is a region containing p for all $n \in \mathbb{N}$, we have that $R_n \cap (A \setminus \{p\}) \neq \emptyset$ for all $n \in \mathbb{N}$. We are now ready to define our subsequence (a_{i_n}) . By the axiom of choice, choose $a_{i_n} \in (p - \frac{1}{n}, p + \frac{1}{n})$ where $i_{n-1} < i_n$ for all $n \in \mathbb{N}$ (for a_{i_1} , just choose any element of $(p - 1, p + 1)$). Note that we are guaranteed that there are elements within $(p - \frac{1}{n}, p + \frac{1}{n})$ arbitrarily far down the sequence (i.e., that we can choose a_{i_n} in $(p - \frac{1}{n}, p + \frac{1}{n})$ with $i_{n-1} < i_n$) since if there were not, that would imply that there are only finitely many terms of the sequence within $R_n \cap (A \setminus \{p\})$. But then in this case, we would be able to choose an R sufficiently small such that $R \cap (A \setminus \{p\}) = \emptyset$, a contradiction. Now to prove that (a_{i_n}) converges, Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_{i_n} - p| < \epsilon$. Let $\epsilon > 0$ be arbitrary. By Corollary 6.12, there exists an $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Choose this N to be our N . Let n be an arbitrary natural number such that $n \geq N$. It follows by the definition of (a_{i_n}) that $a_{i_n} \in (p - \frac{1}{n}, p + \frac{1}{n})$. Thus, since $(p - \frac{1}{n}, p + \frac{1}{n}) \subset (p - \frac{1}{N}, p + \frac{1}{N})$ by Script 1, we have by Definition 1.3 that $a_n \in (p - \frac{1}{N}, p + \frac{1}{N})$. Therefore, by Exercise 8.9, we have that $|a_n - p| < \frac{1}{N} < \epsilon$, as desired. \square

We are now able to prove a useful characterization of convergent sequences.

Theorem 15.19 (Cauchy Completeness Theorem). *A sequence (a_n) of real numbers converges if and only if for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for all $n, m \geq N$.*

Proof. Suppose first that (a_n) converges to p . Let $\epsilon > 0$ be arbitrary. Then by Theorem 15.7, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - p| < \frac{\epsilon}{2}$. Choose this N to be our N . Let n, m be arbitrary natural numbers such that $n, m \geq N$. Then $|a_n - p| < \frac{\epsilon}{2}$ and $|a_m - p| < \frac{\epsilon}{2}$. Therefore,

$$\begin{aligned} |a_n - a_m| &\leq |a_n - p| + |p - a_m| && \text{Lemma 8.8} \\ &= |a_n - p| + |a_m - p| && \text{Exercise 8.5} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

as desired.

Now suppose that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for all $n, m \geq N$. To prove that (a_n) converges, we will first show that it is bounded. It will then follow by Theorem 15.18 that (a_n)

has a subsequence (b_n) that converges to p . The existence of (b_n) combined with the hypothesis will suffice to show that (a_n) converges to p . Let's begin.

To confirm that (a_n) is bounded, Definitions 15.12 and 5.6 tell us that it will suffice to find numbers l, u such that $l \leq a_n \leq u$ for all a_n . Since $1 > 0$ by Corollary 7.27, we have by the hypothesis that there is some $N \in \mathbb{N}$ such that $|a_n - a_m| < 1$ for all $n, m \geq N$. Thus, we know that $|a_N - a_n| < 1$ for all $n \geq N$. Consequently, by Exercise 8.9, $a_n \in (a_N - 1, a_N + 1)$ for all $n \geq N$. It follows by Script 1 that $\{a_n \mid a_n \notin (a_N - 1, a_N + 1)\}$ is finite, since it can contain at most $N - 1$ terms. If we now divide into two cases and evaluate them in a symmetric fashion to the way we did in the proof of Theorem 15.13, we can establish that (a_n) is bounded, as desired.

Since (a_n) is bounded, we have by Theorem 15.18 that there exists a convergent subsequence (b_n) of (a_n) . Let $\lim_{n \rightarrow \infty} b_n = p$.

To prove that (a_n) converges to p , Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - p| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since (b_n) converges to p , Theorem 15.7 asserts that there is some $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $|b_n - p| < \frac{\epsilon}{2}$. Additionally, we have by the hypothesis that there is some $N_2 \in \mathbb{N}$ such that $|a_n - a_m| < \frac{\epsilon}{2}$ for all $n, m \geq N_2$. Choose $N = \max(N_1, N_2)$. Let n be an arbitrary natural number such that $n \geq N$. By Corollary 6.13, there exists a $b_m = a_{i_m}$ with $i_m \geq N$. Thus, since $i_m \geq N \geq N_1$, we have that $|a_{i_m} - p| < \frac{\epsilon}{2}$. Additionally, since $n \geq N \geq N_2$ and $i_m \geq N \geq N_2$, we have that $|a_n - a_{i_m}| < \frac{\epsilon}{2}$. Therefore,

$$\begin{aligned} |a_n - p| &\leq |a_n - a_{i_m}| + |a_{i_m} - p| && \text{Lemma 8.8} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

as desired. □

Theorem 15.20. *Use Theorem 15.19 to show that the sequence in Exercise 15.10a does not converge.*

Proof. Suppose for the sake of contradiction that $a_n = (-1)^n \cdot n$ converges to some $p \in \mathbb{R}$. Then if we choose $\epsilon = 1$, we have by Theorem 15.19 that there exists some $N \in \mathbb{N}$ such that $|a_n - a_m| < 1$ for all $n, m \geq N$. But we also have that $|a_N - a_{N+1}| = 2N + 1 > 1$, regardless of which natural number N is, a contradiction. □