Script 14

Integrals and Derivatives

14.1 Journal

5/4: **Theorem 14.1.** Suppose that f is integrable on [a,b]. Define $F:[a,b] \to \mathbb{R}$ by

$$F(x) = \int_{a}^{x} f$$

If f is continuous at $p \in (a,b)$, then F is differentiable at p and

$$F'(p) = f(p)$$

If f is continuous at a, then $F'_{+}(a)$ exists and equals f(a). Similarly, if f is continuous at b, $F'_{-}(b)$ exists and equals f(b).

Proof. To prove that F is differentiable at p and F'(p) = f(p), Definition 12.1 tells us that it will suffice to show that $\lim_{h\to 0^+} \frac{F(p+h)-F(p)}{h} = \lim_{h\to 0^-} \frac{F(p+h)-F(p)}{h} = f(p)$. We will tackle the right-handed limit first. To do so, Definition 11.1 tells us that it will suffice to verify that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $(p+h) \in [a,b]$ and $0 < h < \delta$, then $|\frac{F(p+h)-F(p)}{h} - f(p)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f is continuous at p, Theorem 11.5 asserts that there exists a $\delta > 0$ such that if $x \in [a,b]$ and $|x-p| < \delta$, then $|f(x)-f(p)| < \frac{\epsilon}{2}$. Choose this δ to be our δ . Let h be an arbitrary number satisfying $(p+h) \in [a,b]$ and $0 < h < \delta$. Therefore,

$$\left| \frac{F(p+h) - F(p)}{h} - f(p) \right| = \left| \frac{\int_a^{p+h} f - \int_a^p f}{h} - f(p) \right|$$

$$= \left| \frac{\int_p^{p+h} f}{h} - f(p) \right|$$

$$= \left| \frac{\int_p^{p+h} f - hf(p)}{h} \right|$$

$$= \left| \frac{\int_p^{p+h} f - f(p)((p+h) - p)}{h} \right|$$

$$= \left| \frac{\int_p^{p+h} f - \int_p^{p+h} f(p) dx}{h} \right|$$
Exercise 13.17
$$= \left| \frac{1}{h} \int_p^{p+h} (f(x) - f(p)) dx \right|$$
Theorem 13.24

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$$\leq \left| \frac{1}{h} \right| \int_{p}^{p+h} |f(x) - f(p)| \, \mathrm{d}x \qquad \text{Theorem } 13.26$$

$$\leq \left| \frac{1}{h} \right| \frac{\epsilon}{2} ((p+h) - p) \qquad \text{Theorem } 13.27$$

$$= \frac{\epsilon}{2}$$

$$\leq \epsilon$$

The proof is symmetric for the left-handed limit. These proofs can also be applied to the endpoints. \Box

Remark 14.2. Thus, we have that if f is continuous on [a, b], F is differentiable on [a, b] and F'(p) = f(p) for all $p \in [a, b]$ (where at the endpoints, we understand that the derivative should be interpreted as the one-sided derivative).

Lemma 14.3. Suppose that $f:[a,b] \to \mathbb{R}$ is integrable and that Ω is a number satisfying $L(f,P) \le \Omega \le U(f,P)$ for all partitions P of [a,b]. Then

$$\int_{a}^{b} f = \Omega$$

Proof. Suppose for the sake of contradiction that $\int_a^b f \neq \Omega$. We divide into two cases $(\int_a^b f < \Omega)$ and $\int_a^b f > \Omega$. If $\int_a^b f < \Omega$, then by Definition 13.16, $U(f) = \int_a^b f < \Omega$. It follows by Definition 13.14 and 5.11 that there exists an object $U(f,P) \in \{U(f,P) \mid P \text{ is a partition of } [a,b] \}$ such that $U(f) \leq U(f,P) < \Omega$. But this contradicts the hypothesis that $U(f,P) \geq \Omega$ for all partitions P of [a,b]. The argument is symmetric in the other case.

5/6: **Theorem 14.4.** Let f be integrable on [a,b]. Suppose that there is a function G that is continuous on [a,b] and differentiable on (a,b) and such that f=G' on (a,b). Then

$$\int_{a}^{b} f = G(b) - G(a)$$

Proof. By Script 12, the hypothesis that G is continuous on [a,b] and differentiable on (a,b) implies that G'=f is continuous on (a,b). Now define $F:[a,b]\to\mathbb{R}$ by $F(x)=\int_a^x f$. It follows by Theorem 13.28 that F is continuous on [a,b]. Additionally, the fact that f is continuous on (a,b) implies by Theorem 14.1 that F is differentiable on (a,b) with F'=f. Thus, since F,G are continuous on [a,b], differentiable on (a,b), and F'(x)=f(x)=G'(x) for all $x\in(a,b)$, Corollary 12.19 asserts that there is some $c\in\mathbb{R}$ such that F(x)=G(x)+c for all $x\in[a,b]$. Consequently, we have that G(b)-G(a)=(F(b)-c)-(F(a)-c)=F(b)-F(a). Furthermore, Theorem 13.23 implies that $F(b)-F(a)=\int_a^b f-\int_a^a f=\int_a^b f$. Therefore, we have by transitivity that $\int_a^b f=F(b)-F(a)=G(b)-G(a)$, as desired.

Corollary 14.5. Let f, g be functions defined on some open interval containing [a, b] such that f' and g' exist and are continuous on [a, b]. Then

$$\int_{a}^{b} fg' = [f(b)g(b) - f(a)g(a)] - \int_{a}^{b} f'g$$

Proof. Since f and g are differentiable on [a, b], Exercise 12.9 implies that fg is differentiable on [a, b] with (fg)'(x) = f'(x)g(x) + f(x)g'(x) for all $x \in [a, b]$. We now seek to prove that f'g + fg' is integrable on [a, b]. By hypothesis, f' and g' are continuous on [a, b]. Additionally, since f and g are differentiable on [a, b], Theorem 12.5 asserts that they are continuous on [a, b]. Thus, since f, g, f', and g' are continuous on [a, b], we have by consecutive applications of Corollary 11.10 that f'g + fg' is continuous on [a, b]. Consequently, by Theorem 13.19, f'g + fg' is integrable on [a, b], as desired. Furthermore, in a similar manner to the above, we can show that f'g and fg' are integrable on [a, b]. Lastly, it follows from the fact that f and g are continuous on [a, b] by Corollary 11.10 that fg is continuous on [a, b].

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Having established that f'g + fg' is integrable on [a, b], that fg is a function that is continuous on [a, b], differentiable on $(a, b) \subset [a, b]$, and such that f'g + fg' = (fg)' on (a, b), and that f'g and fg' are integrable on [a, b], we have that

$$\int_{a}^{b} (f'g + fg') = (fg)(b) - (fg)(a)$$
 Theorem 14.4
$$\int_{a}^{b} f'g + \int_{a}^{b} fg' = f(b)g(b) - f(a)g(a)$$
 Theorem 13.24
$$\int_{a}^{b} fg' = [f(b)g(b) - f(a)g(a)] - \int_{a}^{b} f'g$$

as desired. \Box

Corollary 14.6. Let g be a function defined on some interval containing [a,b] such that g' is continuous on [a,b]. Suppose that $g([a,b]) \subset [c,d]$ and $f:[c,d] \to \mathbb{R}$ is continuous. Define $F:[c,d] \to \mathbb{R}$ by $F(x) = \int_c^x f$. Then

$$\int_a^b f(g(x)) \cdot g'(x) \, \mathrm{d}x = F(g(b)) - F(g(a))$$

Proof. To prove that $\int_a^b f(g(x)) \cdot g'(x) dx = F(g(b)) - F(g(a))$, Theorem 14.4 tells us that it will suffice to show that $(f \circ g) \cdot g'$ is integrable on [a,b], $F \circ g$ is continuous on [a,b] and differentiable on (a,b), and $(f \circ g) \cdot g' = (F \circ g)'$ on (a,b). We will confirm each requirement in turn. Let's begin.

To confirm that $(f \circ g) \cdot g'$ is integrable on [a, b], Theorem 13.19 tells us that it will suffice to demonstrate that $(f \circ g) \cdot g'$ is continuous on [a, b]. By hypothesis, f is continuous on [c, d]. Additionally, since g' is defined on [a, b], we know that g is differentiable on [a, b], which implies by Theorem 12.5 that g is continuous on [a, b]. The combination of the previous two results implies by Corollary 11.15 that $f \circ g$ is continuous on [a, b]. This combined with the hypothesis that g' is continuous on [a, b] implies by Corollary 11.10 that $(f \circ g) \cdot g'$ is continuous on [a, b].

To confirm that $F \circ g$ is continuous on [a, b], Corollary 11.15 tells us that it will suffice to demonstrate that F is continuous on [c, d] and g is continuous on [a, b]. By Theorem 13.28, F is continuous on [c, d]. Additionally, we know by the above that g is continuous on [a, b].

To confirm that $F \circ g$ is differentiable on (a,b), Theorem 12.10 tells us that it will suffice to demonstrate that F is differentiable on (c,d) and g is differentiable on (a,b). Since f is continuous on $(c,d) \subset [c,d]$, we have by Theorem 14.1 that F is differentiable on (c,d). Additionally, we know by the above that g is differentiable on $(a,b) \subset [a,b]$.

Since $F \circ g$ is differentiable on (a,b), we have by Theorem 12.10 again that $(F \circ g)' = (F' \circ g) \cdot g'$ for all $x \in (a,b)$. Thus, since F' = f by Theorem 14.1, we have that $(f \circ g) \cdot g' = (F \circ g)'$ on (a,b), as desired. \square