Script 14

Integrals and Derivatives

14.1 Journal

5/4: **Theorem 14.1.** *Suppose that f is integrable on $[a,b]$. Define $F:[a,b]\to\mathbb{R}$ by*

$$F(x)=\int_a^x f$$

If f is continuous at $p\in(a,b)$, then F is differentiable at p and

$$F'(p)=f(p)$$

If f is continuous at a, then $F'_+(a)$ exists and equals $f(a)$. Similarly, if f is continuous at b, $F'_-(b)$ exists and equals $f(b)$.

Proof. To prove that F is differentiable at p and $F'(p)=f(p)$, Definition 12.1 tells us that it will suffice to show that $\lim_{h\to 0^+}\frac{F(p+h)-F(p)}{h}=\lim_{h\to 0^-}\frac{F(p+h)-F(p)}{h}=f(p)$. We will tackle the right-handed limit first. To do so, Definition 11.1 tells us that it will suffice to verify that for every $\epsilon>0$, there exists a $\delta>0$ such that if $(p+h)\in[a,b]$ and $0<h<\delta$, then $|\frac{F(p+h)-F(p)}{h}-f(p)|<\epsilon$. Let $\epsilon>0$ be arbitrary. Since f is continuous at p, Theorem 11.5 asserts that there exists a $\delta>0$ such that if $x\in[a,b]$ and $|x-p|<\delta$, then $|f(x)-f(p)|<\frac{\epsilon}{2}$. Choose this δ to be our δ. Let h be an arbitrary number satisfying $(p+h)\in[a,b]$ and $0<h<\delta$. Therefore,

$$\begin{aligned}
\left|\frac{F(p+h)-F(p)}{h}-f(p)\right| &= \left|\frac{\int_a^{p+h}f-\int_a^p f}{h}-f(p)\right| \\
&= \left|\frac{\int_p^{p+h}f}{h}-f(p)\right| && \text{Theorem 13.23} \\
&= \left|\frac{\int_p^{p+h}f-hf(p)}{h}\right| \\
&= \left|\frac{\int_p^{p+h}f-f(p)((p+h)-p)}{h}\right| \\
&= \left|\frac{\int_p^{p+h}f-\int_p^{p+h}f(p)\,\mathrm{d}x}{h}\right| && \text{Exercise 13.17} \\
&= \left|\frac{1}{h}\int_p^{p+h}(f(x)-f(p))\,\mathrm{d}x\right| && \text{Theorem 13.24}
\end{aligned}$$

$$\begin{aligned}
&\leq \left|\frac{1}{h}\right| \int_p^{p+h} |f(x) - f(p)| \, \mathrm{d}x && \text{Theorem 13.26} \\
&\leq \left|\frac{1}{h}\right| \frac{\epsilon}{2}((p+h) - p) && \text{Theorem 13.27} \\
&= \frac{\epsilon}{2} \\
&< \epsilon
\end{aligned}$$

The proof is symmetric for the left-handed limit. These proofs can also be applied to the endpoints. □

Remark 14.2. Thus, we have that if f is continuous on $[a, b]$, F is differentiable on $[a, b]$ and $F'(p) = f(p)$ for all $p \in [a, b]$ (where at the endpoints, we understand that the derivative should be interpreted as the one-sided derivative).

Lemma 14.3. *Suppose that $f : [a, b] \to \mathbb{R}$ is integrable and that Ω is a number satisfying $L(f, P) \leq \Omega \leq U(f, P)$ for all partitions P of $[a, b]$. Then*

$$\int_a^b f = \Omega$$

Proof. Suppose for the sake of contradiction that $\int_a^b f \neq \Omega$. We divide into two cases ($\int_a^b f < \Omega$ and $\int_a^b f > \Omega$). If $\int_a^b f < \Omega$, then by Definition 13.16, $U(f) = \int_a^b f < \Omega$. It follows by Definition 13.14 and 5.11 that there exists an object $U(f, P) \in \{U(f, P) \mid P \text{ is a partition of } [a, b]\}$ such that $U(f) \leq U(f, P) < \Omega$. But this contradicts the hypothesis that $U(f, P) \geq \Omega$ for all partitions P of $[a, b]$. The argument is symmetric in the other case. □