Script 13

Uniform Continuity and Integration

13.1 Journal

4/8: **Definition 13.1.** Let $f : A \to \mathbb{R}$ be a function. We say that f is **uniformly continuous** if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$.

Theorem 13.2. *If f is uniformly continuous, then f is continuous.*

Proof. To prove that f is continuous, Theorem 9.10 tells us that it will suffice to show that f is continuous at every $x \in A$. Let x be an arbitrary element of A. To show that f is continuous at x, Theorem 11.5 tells us that it will suffice to verify that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Then since f is uniformly continuous by hypothesis, Definition 13.1 asserts that there exists a $\delta > 0$ such that for all $y \in A$ satisfying $|y - x| < \delta$, we have $|f(y) - f(x)| < \epsilon$, as desired. $\square$

Exercise 13.3. Determine with proof whether each function f is uniformly continuous on the given interval A.

(a) $f(x) = x^2$ on $A = \mathbb{R}$.

Proof. To prove that f is not uniformly continuous on $\mathbb{R}$, Definition 13.1 tells us that it will suffice to find an $\epsilon > 0$ for which no $\delta > 0$ exists such that for all $x, y \in \mathbb{R}$, if $|y - x| < \delta$, then $|y^2 - x^2| < \epsilon$. Let $\epsilon = 2$, and suppose for the sake of contradiction that $\delta > 0$ is a number such that for all $x, y \in \mathbb{R}$, if $|y - x| < \delta$, then $|y^2 - x^2| < 2$. By Theorem 5.2, there exists a number y such that $0 < y < \delta$. Since $-\delta < 0 < y < \delta$ by Lemma 7.23, it follows by Definitions 3.6 and 3.10 that $y \in (-\delta, \delta)$. Thus, by Exercise 8.9, $|y - 0| = |y| < \delta$. Consequently, $|(y + n) - n| < \delta$. It follows by the above that $|(y + n)^2 - n^2| = |y^2 + 2yn| < 2$. If we now let $n = \frac{1}{y}$, then $|y^2 + 2| < 2$. But since $y > 0$, we have that $y^2 > 0$ by Lemma 7.26. It follows that $y^2 + 2 > 2$ by Definition 7.21. Therefore, by Definition 8.4, we can also show that $|y^2 + 2| > 2$, a contradiction. $\square$

(b) $f(x) = x^2$ on $A = (-2, 2)$.

Proof. To prove that f is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \frac{\epsilon}{4}$, and let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. Since $x, y \in A$, consecutive applications of Equations 8.1 and the lemma from Exercise 8.9 imply that $|x| < 2$ and $|y| < 2$. It follows that $|x| + |y| < 2 + 2 = 4$. Consequently, by Lemma 8.8, $|x + y| < 4$. Additionally, since $0 \leq |y + x|$ by Definition 8.4, we have $|x - y| \cdot |x + y| \leq \frac{\epsilon}{4} \cdot |x + y|$. Combining all of the above results, we have that

$$\begin{aligned}|f(y)-f(x)| &= |y^2-x^2| \\ &= |y+x|\cdot|y-x| \\ &< 4\cdot|y-x| \\ &\leq 4\cdot\frac{\epsilon}{4} \\ &= \epsilon\end{aligned}$$

as desired. □

(c) $f(x)=\frac{1}{x}$ on $A=(0,+\infty)$.

Proof. To prove that f is not uniformly continuous on A, Definition 13.1 tells us that it will suffice to find an $\epsilon>0$ for which no $\delta>0$ exists such that for all $x,y\in A$, if $|y-x|<\delta$, then $|\frac{1}{y}-\frac{1}{x}|<\epsilon$. Let $\epsilon=1$, and suppose for the sake of contradiction that $\delta>0$ is a number such that for all $x,y\in A$, if $|y-x|<\delta$, then $|\frac{1}{y}-\frac{1}{x}|<1$. As in part (a), choose $0<x<\min(\delta,\frac{1}{2})$. Consequently, $|(x+x)-x|<\delta$. It follows by the above that $|\frac{1}{2x}-\frac{1}{x}|<1$. But this implies that $|\frac{x-2x}{2x^2}|=|\frac{-1}{2x}|=\frac{1}{2x}<1$. However, $x<\frac{1}{2}$ implies that $1<\frac{1}{2x}$, a contradiction. □

(d) $f(x)=\frac{1}{x}$ on $A=[1,+\infty)$.

Proof. To prove that f is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon>0$, there exists a $\delta>0$ such that for all $x,y\in A$, if $|y-x|<\delta$, then $|f(y)-f(x)|<\epsilon$. Let $\epsilon>0$ be arbitrary. Choose $\delta=\epsilon$, and let x,y be arbitrary elements of A that satisfy $|y-x|<\delta$. Since $x,y\in A$, consecutive applications of Equations 8.1 imply that $1\leq x$ and $1\leq y$. It follows by Script 7 that $1\leq|xy|$. This combined with the fact that $|y-x|<\delta=\epsilon$ implies that

$$\begin{aligned}|f(y)-f(x)| &= \left|\frac{1}{y}-\frac{1}{x}\right| \\ &= \left|\frac{x-y}{yx}\right| \\ &= \frac{|y-x|}{|xy|} \\ &< \frac{\epsilon}{|xy|} \\ &\leq \frac{\epsilon}{1} \\ &= \epsilon\end{aligned}$$

as desired. □

(e) $f(x)=\sqrt{x}$ on $A=[1,+\infty)$.

Proof. To prove that f is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon>0$, there exists a $\delta>0$ such that for all $x,y\in A$, if $|y-x|<\delta$, then $|f(y)-f(x)|<\epsilon$. Let $\epsilon>0$ be arbitrary. Choose $\delta=\epsilon$, and let x,y be arbitrary elements of A that satisfy $|y-x|<\delta$. Since $x,y\in A$, consecutive applications of Equations 8.1 imply that $1\leq x$ and $1\leq y$. It follows by Script 7 that $1\leq\sqrt{x}$ and $1\leq\sqrt{y}$. Thus, by Script 7 again, $2\leq|\sqrt{y}+\sqrt{x}|$. Note that it follows that $1<|\sqrt{y}+\sqrt{x}|$. This combined with the fact that $|y-x|<\delta=\epsilon$ implies that

$$\begin{aligned}|f(y)-f(x)| &= |\sqrt{y}-\sqrt{x}| \\ &< |\sqrt{y}-\sqrt{x}|\cdot|\sqrt{y}+\sqrt{x}| \\ &= |y-x| \\ &= \epsilon\end{aligned}$$

as desired. □

Exercise 13.4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^n$ for $n \in \mathbb{N}$. Show that f is uniformly continuous if and only if $n = 1$.

Proof. Suppose first that $n = 1$. To prove that f is uniformly continuous, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in \mathbb{R}$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \epsilon$. Now let x, y be arbitrary elements of $\mathbb{R}$ that satisfy $|y - x| < \delta$. Then by the definition of f, $|f(y) - f(x)| = |y - x| < \delta = \epsilon$, as desired.

Now suppose that $n > 1$. Additionally, suppose for the sake of contradiction that f is uniformly continuous. Let $\epsilon = 1 > 0$. Then by Definition 13.1, there exists a $\delta > 0$ such that for all $x, y \in \mathbb{R}$, if $|y-x| < \delta$, then $|y^n - x^n| < 1$. Let $x = 0 \in \mathbb{R}$. By Theorem 5.2, there exists a point $y \in \mathbb{R}$ such that $0 < y < \delta$. Additionally, since $\delta > 0$, Lemma 7.23 asserts that $-\delta < 0$. This combined with the previous result demonstrates by transitivity that $-\delta < 0 < y < \delta$, so by the lemma from Exercise 8.9, we have that $|y| < \delta$. Consequently, by Script 7, we know that $|(y+a)-a| < \delta$ for any $a \in \mathbb{R}$. It follows by the above that $|(y+a)^n - a^n| < 1$. Thus, by Additional Exercise 0.7, $|\sum_{k=0}^{n} \binom{n}{k} y^{n-k}a^k - a^n| = |y^n + ny^{n-1}a + \sum_{k=2}^{n-1} \binom{n}{k} y^{n-k}a^k| < 1$. If we now choose $a = \frac{1}{ny^{n-1}}$, Script 7 reduces the above to $|y^n + 1 + \sum_{k=2}^{n-1} y^{n-k}a^k| < 1$. We now seek to reduce the previous statement further to $|y^n + 1| < 1$. To begin, Exercise 12.22 implies that $y^n > 0$ since $y > 0$ and $0^n = 0$, meaning by Script 7 that $y^n + 1 > 0$. Additionally, Script 7 asserts that $\sum_{k=2}^{n-1} y^{n-k}a^k > 0$ since $a > 0$ and $y > 0$. This combined with the previous result implies by Scripts 7 and 8 that $|y^n + 1| < |y^n + 1 + \sum_{k=2}^{n-1} y^{n-k}a^k| < 1$, as desired. However, since $y^n > 0$, Definition 7.21 asserts that $y^n + 1 > 1$. But by Definition 8.4, this implies that $|y^n + 1| > 1$, a contradiction. $\square$

Exercise 13.5. Let f and g be uniformly continuous on $A \subset \mathbb{R}$. Show that

(a) The function $f + g$ is uniformly continuous on A.

(b) For any constant $c \in \mathbb{R}$, the function $c \cdot f$ is uniformly continuous on A.

Proof of a. To prove that $f+g$ is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y-x| < \delta$, then $|(f+g)(y) - (f+g)(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f, g are uniformly continuous on A, consecutive applications of Definition 13.1 reveal that there exist $\delta_1, \delta_2 > 0$ such that for all $x, y \in A$, $|y - x| < \delta_1$ implies $|f(y) - f(x)| < \frac{\epsilon}{2}$ and $|y - x| < \delta_2$ implies $|g(y) - f(x)| < \frac{\epsilon}{2}$. Choose $\delta = \min(\delta_1, \delta_2)$. Let x, y be arbitrary elements of A that satisfy $|y-x| < \delta$. It follows that $|y-x| < \delta_1$ (so $|f(y)-f(x)| < \frac{\epsilon}{2}$), and that $|y-x| < \delta_2$ (so $|g(y)-g(x)| < \frac{\epsilon}{2}$). These two results when combined imply by Script 7 that $|f(y) - f(x)| + |g(y) - g(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$. Therefore, since $|f(y) - f(x) + g(y) - g(x)| \leq |f(y) - f(x)| + |g(y) - g(x)|$ by Lemma 8.8, we have that

$$\begin{aligned} |(f+g)(y) - (f+g)(x)| &= |f(y) - f(x) + g(y) - g(x)| \\ &\leq |f(y) - f(x)| + |g(y) - g(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

as desired. $\square$

Proof of b. To prove that $c \cdot f$ is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y-x| < \delta$, then $|c \cdot f(y) - c \cdot f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. We divide into two cases ($c = 0$ and $c \neq 0$). Suppose first that $c = 0$. Choose $\delta = 1$. Let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. It follows that $|0 \cdot f(y) - 0 \cdot f(x)| = 0 < \epsilon$, as desired. Now suppose that $c \neq 0$. Then since f is uniformly continuous on A, Definition 13.1 tells us that there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \frac{\epsilon}{|c|}$. Choose this δ to be our δ. Let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. Then by the above, we have that $|f(y) - f(x)| < \frac{\epsilon}{|c|}$. Therefore, $|c| \cdot |f(y) - f(x)| < \epsilon$, so we have that $|c \cdot f(y) - c \cdot f(x)| < \epsilon$, as desired. $\square$

4/13: **Theorem 13.6.** *Suppose that $X \subset \mathbb{R}$ is compact and $f : X \to \mathbb{R}$ is continuous. Then f is uniformly continuous.*

Proof. To prove that f is uniformly continuous, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f is continuous on X, Theorem 9.10 asserts that f is continuous at every $x \in X$. Thus, by Theorem 11.5, for every $x \in X$, there exists a $\delta_x > 0$ such that if $y \in X$ and $|y - x| < \delta_x$, then $|f(y) - f(x)| < \frac{\epsilon}{2}$. Let $\mathcal{G} = \{(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2}) \mid x \in X\}$. We will now confirm that $\mathcal{G}$ is an open cover of X. To do so, Definition 10.3 tells us that it will suffice to demonstrate that every $x \in X$ is an element of $(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2})$ for some $(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2}) \in \mathcal{G}$. Let x be an arbitrary element of X. We know that $|x - x| = 0 < \frac{\delta_x}{2}$. Thus, by Exercise 8.9, we have that $x \in (x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2})$. Since it follows from the above that $(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2}) \in \mathcal{G}$, we are done.

Having shown that $\mathcal{G}$ is an open cover of X, the fact that X is compact implies by Definition 10.4 that there exists a finite subset $\mathcal{G}'$ of $\mathcal{G}$ that is also an open cover of X. It follows that $\mathcal{G}'$ will be of the form $\{(x_i - \frac{\delta_{x_i}}{2}, x + \frac{\delta_{x_i}}{2}) \mid 1 \leq i \leq n\}$ where n is some natural number. Thus, choose $\delta = \min_{1 \leq i \leq n}(\frac{\delta_{x_i}}{2})$.

Let x, y be arbitrary elements of X that satisfy $|y - x| < \delta$. Since $\mathcal{G}'$ is an open cover of X, Definition 10.3 implies that $x \in (x_i - \frac{\delta_{x_i}}{2}, x_i + \frac{\delta_{x_i}}{2})$ for some $(x_i - \frac{\delta_{x_i}}{2}, x_i + \frac{\delta_{x_i}}{2}) \in \mathcal{G}'$. Considering this x_i more closely, we can determine from the previous result and Exercise 8.9 that $|x - x_i| < \frac{\delta_{x_i}}{2}$. This combined with the hypothesis that $|y - x| < \delta$ implies by Script 7 that $|y - x| + |x - x_i| < \delta + \frac{\delta_{x_i}}{2}$. Additionally, note that by definition, $\delta \leq \frac{\delta_{x_i}}{2}$. Thus, combining the last few results, we have that

$$\begin{aligned}
|y - x_i| &\leq |y - x| + |x - x_i| && \text{Lemma 8.8}\\
&< \delta + \frac{\delta_{x_i}}{2}\\
&\leq \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2}\\
&= \delta_{x_i}
\end{aligned}$$

At this point, we know that $|x - x_i| < \frac{\delta_{x_i}}{2} < \delta_{x_i}$ and that $|y - x_i| < \delta_{x_i}$. It follows by consecutive applications of the above that $|f(x) - f(x_i)| < \frac{\epsilon}{2}$ and $|f(y) - f(x_i)| < \frac{\epsilon}{2}$, respectively. Consequently, we have by Script 7 that $|f(y) - f(x_i)| + |f(x) - f(x_i)| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$. Therefore, if we combine the last several results, we get

$$\begin{aligned}
|f(y) - f(x)| &\leq |f(y) - f(x_i)| + |f(x_i) - f(x)| && \text{Lemma 8.8}\\
&= |f(y) - f(x_i)| + |f(x) - f(x_i)| && \text{Exercise 8.5}\\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2}\\
&= \epsilon
\end{aligned}$$

as desired. □

Exercise 13.7. Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $A = [0, +\infty)$.

Lemma. *Let x, y be arbitrary elements of A. Then $|\sqrt{y} - \sqrt{x}|^2 \leq |(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x})|$.*

Proof. We will first verify that $|\sqrt{y} - \sqrt{x}| \leq |\sqrt{y} + \sqrt{x}|$. To do so, we divide into two cases ($\sqrt{y} \geq \sqrt{x}$ and $\sqrt{y} < \sqrt{x}$). If $\sqrt{y} \geq \sqrt{x}$, then by Definition 7.21, $\sqrt{y} - \sqrt{x} \geq 0$. It follows by Definition 8.4 that $|\sqrt{y} - \sqrt{x}| = \sqrt{y} - \sqrt{x}$. Additionally, we have by an extension of Exercise 12.22 that $\sqrt{x} \geq 0$, implying that $2\sqrt{x} \geq 0$ by Definition 7.21. Thus, combining the last few results, we have that $|\sqrt{y} - \sqrt{x}| = \sqrt{y} - \sqrt{x} \leq \sqrt{y} - \sqrt{x} + 2\sqrt{x} = \sqrt{y} + \sqrt{x}$. Consequently, we know that $0 \leq |\sqrt{y} - \sqrt{x}| \leq \sqrt{y} + \sqrt{x}$, so Definition 8.4 implies that $|\sqrt{y} + \sqrt{x}| = \sqrt{y} + \sqrt{x}$. Therefore, we have that $|\sqrt{y} - \sqrt{x}| \leq \sqrt{y} + \sqrt{x} = |\sqrt{y} + \sqrt{x}|$, as desired. The argument is symmetric in the other case.

Having established that $|\sqrt{y} - \sqrt{x}| \leq |\sqrt{y} + \sqrt{x}|$ and knowing that $0 \leq |\sqrt{y} - \sqrt{x}|$, we have by Lemma 7.24 that $|\sqrt{y} - \sqrt{x}| \cdot |\sqrt{y} - \sqrt{x}| \leq |\sqrt{y} + \sqrt{x}| \cdot |\sqrt{y} - \sqrt{x}|$. It follows by basic algebra that $|\sqrt{y} - \sqrt{x}|^2 \leq |(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x})|$, as desired. □

Proof of Exercise 13.7. To prove that f is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \epsilon^2$. Let x, y be arbitrary elements of X that satisfy $|y - x| < \delta$. Thus, since $(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x}) = y - x$, the lemma asserts that $|\sqrt{y} - \sqrt{x}|^2 \leq |(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x})| = |y - x| < \epsilon^2$. Therefore, by Script 7, $|\sqrt{y} - \sqrt{x}| < \epsilon$, i.e., $|f(y) - f(x)| < \epsilon$, as desired. □

Corollary 13.8. *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Then f is uniformly continuous.*

Proof. By Theorem 10.14, $[a, b]$ is compact. This combined with the hypothesis that f is continuous proves by Theorem 13.6 that f is uniformly continuous. □

Exercise 13.9. Show that if f and g are bounded on A and uniformly continuous on A, then fg is uniformly continuous on A.

Proof. To prove that fg is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|(fg)(y) - (fg)(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary.

Since f is bounded on A, Definition 10.1 implies that $f(A)$ is a bounded subset of $\mathbb{R}$. Thus, by consecutive applications of Definition 5.6, there exist numbers l, u such that for all $f(x) \in f(A)$, $l \leq f(x) \leq u$. Let $a = \max(|l|, |u|) + 1$. It follows by Scripts 7 and 8 that $-a < f(x) < a$ for all $f(x) \in f(A)$. Thus, by the lemma from Exercise 8.9, $|f(x)| < a$ for all $f(x) \in f(A)$. Similarly, there exists a number b such that $|g(x)| < b$ for all $g(x) \in g(A)$.

Since f is uniformly continuous on A, Definition 13.1 implies that there exists a $\delta_1 > 0$ such that for all $x, y \in A$, if $|y - x| < \delta_1$, then $|f(y) - f(x)| < \frac{\epsilon}{2b}$. Similarly, there exists a $\delta_2 > 0$ such that for all $x, y \in A$, if $|y - x| < \delta_2$, then $|g(y) - g(x)| < \frac{\epsilon}{2a}$. Choose $\delta = \min(\delta_1, \delta_2)$. Let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. It follows by consecutive applications of the above that $|f(x)| < a$ and $|g(y)| < b$. Additionally, $|y - x| < \delta \leq \delta_1$ implies that $|f(y) - f(x)| < \frac{\epsilon}{2b}$ and $|y - x| < \delta \leq \delta_2$ implies that $|g(y) - g(x)| < \frac{\epsilon}{2a}$. Therefore, combining the last four results, we have that

$$\begin{aligned}
|(fg)(y) - (fg)(x)| &= |f(x)g(x) - f(y)g(y)| \\
&= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\
&= |f(x)(g(x) - g(y)) + g(y)(f(x) - f(y))| \\
&\leq |f(x)| \cdot |g(x) - g(y)| + |g(y)| \cdot |f(x) - f(y)| && \text{Lemma 8.8} \\
&< a \cdot \frac{\epsilon}{2a} + b \cdot \frac{\epsilon}{2b} \\
&= \epsilon
\end{aligned}$$

as desired. □

4/15: **Definition 13.10.** A **partition** of the interval $[a, b]$ is a finite set of points in $[a, b]$ that includes a and b. We usually write partitions as $P = \{t_0, t_1, \dots, t_n\}$, with the convention that

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

If P and Q are partitions of the interval $[a, b]$ and $P \subset Q$, we refer to Q as a **refinement** of P.

Definition 13.11. Suppose that $f : [a, b] \to \mathbb{R}$ is bounded and that $P = \{t_0, \dots, t_n\}$ is a partition of $[a, b]$. Define

$$m_i(f) = \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\} \qquad M_i(f) = \sup\{f(x) \mid t_{i-1} \leq x \leq t_i\}$$

The **lower sum** of f for the partition P is the number

$$L(f, P) = \sum_{i=1}^{n} m_i(f)(t_i - t_{i-1})$$

The **upper sum** of f for the partition P is the number

$$U(f,P) = \sum_{i=1}^{n} M_i(f)(t_i - t_{i-1})$$

Notice that it is always the case that $L(f,P) \leq U(f,P)$.

Lemma 13.12. *Suppose that P and Q are partitions of $[a,b]$ and that Q is a refinement of P. Then $L(f,P) \leq L(f,Q)$ and $U(f,P) \geq U(f,Q)$.*

Lemma. *Let P be a partition of $[a,b]$ and let y be an arbitrary element of $[a,b] \setminus P$. Then $L(f,P) \leq L(f,P \cup \{y\})$ and $U(f,P) \geq U(f,P \cup \{y\})$.*

Proof. We will prove that $L(f,P) \leq L(f,P \cup \{y\})$. The proof will be symmetric in the other case. Let's begin.

By Definition 13.10, P is of the form $\{t_0,\dots,t_n\}$ where $a = t_0 < \cdots < t_n = b$. This combined with the hypothesis that $y \in [a,b] \setminus P$ implies by Theorem 3.5 that $a = t_0 < \cdots < t_{k-1} < y < t_k < \cdots < t_n = b$. Thus, we have by consecutive applications of Definition 13.11 that

$$\begin{aligned} L(f,P) &= \sum_{i=1}^{n} m_i(f)(t_i - t_{i-1}) \\ &= \sum_{i=1}^{k-1} m_i(f)(t_i - t_{i-1}) + m_k(f)(t_k - t_{k-1}) + \sum_{i=k+1}^{n} m_i(f)(t_i - t_{i-1}) \end{aligned}$$

and that

$$L(f,P \cup \{y\}) = \sum_{i=1}^{k-1} m_i(f)(t_i - t_{i-1}) + m_y^-(f)(y - t_{k-1}) + m_y^+(f)(t_k - y) + \sum_{i=k+1}^{n} m_i(f)(t_i - t_{i-1})$$

where

$$m_y^-(f) = \inf\{f(x) \mid t_{k-1} \leq x \leq y\} \qquad m_y^+(f) = \inf\{f(x) \mid y \leq x \leq t_k\}$$

As such, to prove that $L(f,P) \leq L(f,P \cup \{y\})$, it will suffice to show that $m_k(f)(t_k - t_{k-1}) \leq m_y^-(f)(y - t_{k-1}) + m_y^+(f)(t_k - y)$. To do so, it will suffice to show that $m_k(f)(y - t_{k-1}) + m_k(f)(t_k - y) \leq m_y^-(f)(y - t_{k-1}) + m_y^+(f)(t_k - y)$, i.e., that $m_k(f)(y - t_{k-1}) \leq m_y^-(f)(y - t_{k-1})$ and that $m_k(f)(t_k - y) \leq m_y^+(f)(t_k - y)$, i.e., that $m_k(f) \leq m_y^-(f)$ and that $m_k(f) \leq m_y^+(f)$.

For the sake of proving the first expression, let $A = \{f(x) \mid t_{k-1} < x < t_k\}$ and let $B = \{f(x) \mid t_{k-1} \leq x \leq y\}$. It follows by Definition 13.10 that $m_k(f) = \inf A$ and $m_y^-(f) = \inf B$. Thus, we need only show that $\inf A \leq \inf B$. Since $y < t_k$, we know by Script 1 that $B \subset A$. Thus, since $\inf A$ is a lower bound on A, Script 5 implies that it is also a lower bound on B. Consequently, by Definition 5.7, $\inf A \leq \inf B$, as desired.

The argument is symmetric for the other statement. □

Proof of Lemma 13.12. We will prove that $L(f,P) \leq L(f,Q)$. The proof will be symmetric in the other case. Let's begin.

By Definition 13.10, $P \subset Q$. Thus, by Theorem **??**, $|P| \leq |Q|$. It follows by Script 1 that $|Q| - |P| = n \in \mathbb{Z}^+$. Thus, to prove the claim for P and Q in general, it will suffice to prove it for each n. To do so, we divide into two cases ($n = 0$ and $n \in \mathbb{N}$). If $n = 0$, then $|P| = |Q|$. This combined with the fact that $P \subset Q$ implies by Script 1 that $P = Q$. Therefore, $L(f,P) = L(f,Q)$, which we can weaken to $L(f,P) \leq L(f,Q)$, as desired.

On the other hand, if $n \in \mathbb{N}$, then we induct on n. For the base case $n = 1$, we have by Script 1 that $Q = P \cup \{y\}$ where $y \notin P$. Therefore, by the lemma, we have that $L(f,P) \leq L(f,P \cup \{y\}) = L(f,Q)$, as desired. Now suppose inductively that the claim holds or n; we wish to prove it for $n+1$. Let y be an arbitrary element of Q. Then by Script 1, $|Q \setminus \{y\}| - |P| = n$. Thus, by the inductive hypothesis, $L(f,P) \leq L(f,Q \setminus \{x\})$. Additionally, by the lemma, $L(f,Q \setminus \{x\}) \leq L(f,Q)$. Therefore, by transitivity, $L(f,P) \leq L(f,Q)$, as desired. □

Theorem 13.13. *Let P_1 and P_2 be partitions of $[a,b]$ and suppose that $f:[a,b]\to\mathbb{R}$ is bounded. Then $L(f,P_1)\leq U(f,P_2)$.*

Proof. To confirm that $P_1\cup P_2$ is a partition of $[a,b]$, Definition 13.10 tells us that it will suffice to demonstrate that it is a finite set, that it is a subset of $[a,b]$, and that it includes a and b. Since P_1,P_2 are partitions of $[a,b]$, Definition 13.10 implies that they are finite subsets of $[a,b]$ that contain a,b. It follows by Script 1 that their union is finite, a subset of $[a,b]$, and a set containing a and b. Additionally, we have by Theorem 1.7 that $P_1\subset P_1\cup P_2$ and that $P_2\subset P_1\cup P_2$. Combining the last two results with consecutive applications of Definition 13.10 reveals that $P_1\cup P_2$ is a refinement of both P_1 and P_2.

Since P_1 and $P_1\cup P_2$ are partitions of $[a,b]$ and $P_1\cup P_2$ is a refinement of P_1, Lemma 13.12 implies that $L(f,P_1)\leq L(f,P_1\cup P_2)$. Similarly, $U(f,P_1\cup P_2)\leq U(f,P_2)$. Additionally, we have by Definition 13.11 that $L(f,P_1\cup P_2)\leq U(f,P_1\cup P_2)$. Therefore, if we combine the last three results with transitivity, we have that $L(f,P_1)\leq U(f,P_2)$, as desired. ☐

Definition 13.14. Let $f:[a,b]\to\mathbb{R}$ be bounded. We define

$$L(f)=\sup\{L(f,P)\mid P\text{ is a partition of }[a,b]\}\qquad U(f)=\inf\{U(f,P)\mid P\text{ is a partition of }[a,b]\}$$

to be, respectively, the **lower integral** and **upper integral** of f from a to b.

Exercise 13.15. Why do $L(f)$ and $U(f)$ exist? Find a function f for which $L(f)=U(f)$. Find a function f for which $L(f)\neq U(f)$. Prove that $L(f)\leq U(f)$.

Lemma. *Given $a,b\in\mathbb{R}$ with $a<b$, there exists $p\in\mathbb{R}$ such that $p\notin\mathbb{Q}$ and $a<p<b$.*

Proof. By Definition 7.21, $a+\sqrt{2}<b+\sqrt{2}$. Thus, by Lemma 6.10, there exists a point $\frac{c}{d}\in\mathbb{Q}$ such that $a+\sqrt{2}<\frac{c}{d}<b+\sqrt{2}$. It follows that $a<\frac{c}{d}-\sqrt{2}<b$.

Now suppose for the sake of contradiction that $\frac{c}{d}-\sqrt{2}$ is rational. Then by Script 2, $\frac{c}{d}-\sqrt{2}=\frac{e}{f}$ where $e,f\in\mathbb{Z}$ and $f\neq 0$. It follows by Theorem 2.10 that $\sqrt{2}=\frac{cf-de}{df}$, i.e., that $\sqrt{2}$ is rational. But by the proof of Exercise 4.24, $\sqrt{2}$ is not rational, a contradiction. ☐

Proof of Exercise 13.15. Let $A=\{L(f,P)\mid P\text{ is a partition of }[a,b]\}$. To prove that $L(f)=\sup A$ exists, Theorem 5.17 tells us that it will suffice to show that A is nonempty and bounded above.

To confirm that A is nonempty, Definition 1.8 tells us that it will suffice to find an element of it. Since $\{a,b\}$ is a finite set of points in $[a,b]$ that includes a and b (by Script 1), Definition 13.10 asserts that $\{a,b\}$ is a partition of $[a,b]$. It follows by Definition 13.11 that $L(f,\{a,b\})$ exists. Therefore, by the definition of A, we have that $L(f,\{a,b\})\in A$, as desired.

To confirm that A is bounded above, Definition 5.6 tells us that it will suffice to find a point in $u\in\mathbb{R}$ such that for all $L(f,P)\in A$, $L(f,P)\leq u$. Let $u=U(f,\{a,b\})$ (since $\{a,b\}$ is a partition of $[a,b]$ by the above, Definition 13.10 guarantees that $U(f,\{a,b\})$ exists). Now let $L(f,P)$ be an arbitrary element of A. It follows from Theorem 13.13 that $L(f,P)\leq U(f,\{a,b\})=u$, as desired.

The proof is symmetric for $U(f)$.

Let $f:[0,1]\to\mathbb{R}$ be defined by $f(x)=0$. To prove that $L(f)=U(f)$, it will suffice to show that $L(f)=0$ and $U(f)=0$. To do this, Script 5 tells us that it will suffice to verify that $\{L(f,P)\mid P\text{ is a partition of }[a,b]\}=\{0\}$ and $\{U(f,P)\mid P\text{ is a partition of }[a,b]\}=\{0\}$. We will start with the first equality.

Let $L(f,P)$ be an arbitrary element of $\{L(f,P)\mid P\text{ is a partition of }[a,b]\}$. Since we have

$$\begin{aligned}m_i(f)&=\inf\{f(x)\mid t_{i-1}\leq x\leq t_i\}\\&=\inf\{0\mid t_{i-1}\leq x\leq t_i\}\\&=\inf\{0\}\\&=0\end{aligned}$$

for all $m_i(f)$, it follows that

$$\begin{aligned} L(f,P) &= \sum_{i=1}^{n} m_i(f)(t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} 0(t_i - t_{i-1}) \\ &= 0 \end{aligned}$$

Therefore, since every element of $\{L(f,P) \mid P \text{ is a partition of } [a,b]\}$ is equal to 0, the set is equal to the singleton set containing 0. The argument is symmetric for the other equality.

Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

To prove that $L(f) \neq U(f)$, it will suffice to show that $L(f) = 0$ and $U(f) = 1$. To do this, Script 5 tells us that it will suffice to verify that $\{L(f,P) \mid P \text{ is a partition of } [a,b]\} = \{0\}$ and $\{U(f,P) \mid P \text{ is a partition of } [a,b]\} = \{1\}$. We will start with the first equality.

Let $L(f,P)$ be an arbitrary element of $\{L(f,P) \mid P \text{ is a partition of } [a,b]\}$. To confirm that $L(f,P) = 0$, Definition 13.11 tells us that it will suffice to demonstrate that $m_i(f) = 0$ for all $m_i(f)$. Let $m_i(f)$ be an arbitrary such object. By Definition 13.10, $m_i(f) = \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\}$. By the lemma, there exists $p \in \mathbb{R}$ such that $p \notin \mathbb{Q}$ and $t_{i-1} \leq p \leq t_i$. Thus, since $f(p) = 0$, $0 \in \{f(x) \mid t_{i-1} \leq x \leq t_i\}$. Additionally, since $f(x) \not< 0$ for any $x \in [0,1]$ by definition, we have that $m_i(f) = 0$. Therefore, since every element of $\{L(f,P) \mid P \text{ is a partition of } [a,b]\}$ is equal to 0, the set is equal to the singleton set containing 0.

As to the other equality, let $U(f,P)$ be an arbitrary element of $\{U(f,P) \mid P \text{ is a partition of } [a,b]\}$. To confirm that $U(f,P) = 1$, Definition 13.11 tells us that we must first demonstrate that $M_i(f) = 1$ for all $M_i(f)$. Let $M_i(f)$ be an arbitrary such object. By Definition 13.10, $M_i(f) = \sup\{f(x) \mid t_{i-1} \leq x \leq t_i\}$. By Lemma 6.10, there exists $p \in \mathbb{Q}$ such that $t_{i-1} \leq p \leq t_i$. Thus, since $f(p) = 1$, $1 \in \{f(x) \mid t_{i-1} \leq x \leq t_i\}$. Additionally, since $f(x) \not> 1$ for any $x \in [0,1]$ by definition, we have that $M_i(f) = 1$. It follows by Definition 13.11 that

$$\begin{aligned} U(f,P) &= \sum_{i=1}^{n} M_i(f)(t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} (t_i - t_{i-1}) \\ &= t_n - t_0 \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

Therefore, since every element of $\{U(f,P) \mid P \text{ is a partition of } [a,b]\}$ is equal to 1, the set is equal to the singleton set containing 1.

Suppose for the sake of contradiction that there exists a function $f : [a,b] \to \mathbb{R}$ for which $U(f) < L(f)$. It follows by consecutive applications of Definition 13.14 and Lemma 5.11 that there exists an $L(f,P_1)$ such that $U(f) < L(f,P_1) \leq L(f)$, and thus that there exists a $U(f,P_2)$ such that $U(f) \leq U(f,P_2) < L(f,P_1)$. But this means that there exist partitions P_1, P_2 of $[a,b]$ such that $L(f,P_1) > U(f,P_2)$, contradicting Theorem 13.13. □

Definition 13.16. Let $f : [a,b] \to \mathbb{R}$ be bounded. We say that f is **integrable** on $[a,b]$ if $L(f) = U(f)$. In this case, the common value $L(f) = U(f)$ is called the **integral** of f from a to b and we write it as

$$\int_a^b f$$

Note that if f is an integrable function on $[a, b]$, it is necessarily bounded.

When we want to display the variable of integration, we write the integral as follows, including the symbol $\mathrm{d}x$ to indicate that variable of integration:

$$\int_a^b f(x)\,\mathrm{d}x$$

For example, if $f(x) = x^2$, we could write $\int_a^b x^2\,\mathrm{d}x$ but not $\int_a^b x^2$.

Exercise 13.17. Fix $c \in \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be defined by $f(x) = c$, for each $x \in [a, b]$. Show that f is integrable on $[a, b]$ and that $\int_a^b f = c(b - a)$.

Proof. To prove that f is integrable on $[a, b]$ and that $\int_a^b f = c(b - a)$, Definition 13.16 tells us that it will suffice to show that f is bounded on $[a, b]$, and that $L(f) = U(f) = c(b - a)$.

To confirm that f is bounded on $[a, b]$, Definition 10.1 tells us that it will suffice to demonstrate that $f([a, b])$ is a bounded subset of $\mathbb{R}$. By Definition 1.18, $f([a, b]) = \{f(x) \in \mathbb{R} \mid x \in [a, b]\}$. But since $f(x) = c$ for all $x \in [a, b]$, $f([a, b]) = \{c\}$. Thus, since $c \leq c \leq c$, Definition 5.6 implies that $f([a, b])$ is bounded. Additionally, since $c \in \mathbb{R}$, Definition 1.3 asserts that $f([a, b]) = \{c\} \subset \mathbb{R}$.

To confirm that $L(f) = U(f) = c(b - a)$, Definition 13.14 tells us that it will suffice to demonstrate that $L(f, P) = U(f, P) = c(b - a)$ for all partitions P of $[a, b]$. For similar reasons to the above (i.e., $f(x) = c$ for all $x \in [a, b]$), we can show that $m_i(f) = M_i(f) = c$ for all $m_i(f)$ and $M_i(f)$. Therefore, by Definition 13.11 that

$$\begin{aligned} L(f, P) &= \sum_{i=1}^{n} c(t_i - t_{i-1}) \\ &= c \sum_{i=1}^{n} (t_{i-1} - t_i) \\ &= c(t_n - t_0) \\ &= c(b - a) \end{aligned} \qquad\qquad \begin{aligned} U(f, P) &= \sum_{i=1}^{n} c(t_i - t_{i-1}) \\ &= c \sum_{i=1}^{n} (t_{i-1} - t_i) \\ &= c(t_n - t_0) \\ &= c(b - a) \end{aligned}$$

as desired. □

Theorem 13.18. *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is integrable if and only if for every $\epsilon > 0$, there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$.*

Proof. Suppose first that f is integrable. Then by Definition 13.16, $L(f) = U(f)$. Let $\epsilon > 0$ be arbitrary. By Script 7, $L(f) - \frac{\epsilon}{2} < L(f)$. Thus, by Definition 13.14 and Lemma 5.11, there exists an $L(f, P_1) \in \{L(f, P) \mid P \text{ is a partition of } [a, b]\}$ such that $L(f) - \frac{\epsilon}{2} < L(f, P_1) \leq L(f)$. Similarly, there exists a $U(f, P_2) \in \{U(f, P) \mid P \text{ is a partition of } [a, b]\}$ such that $U(f) \leq U(f, P_2) < U(f) + \frac{\epsilon}{2}$. Now consider $P_1 \cup P_2$ (which we will prove is the desired partition). By Theorem 1.7, $P_1 \subset P_1 \cup P_2$ and $P_2 \subset P_1 \cup P_2$. It follows by consecutive applications of Definition 13.10 that $P_1 \cup P_2$ is a refinement of both P_1 and P_2. Thus, by Lemma 13.12, $L(f, P_1) \leq L(f, P_1 \cup P_2)$ and $U(f, P_1 \cup P_2) \leq U(f, P_2)$. Combining the last several results with transitivity yields

$$L(f) - \frac{\epsilon}{2} < L(f, P_1) \leq L(f, P_1 \cup P_2) \qquad\qquad U(f, P_1 \cup P_2) \leq U(f, P_2) < U(f) + \frac{\epsilon}{2}$$

Therefore, knowing that $U(f, P_1 \cup P_2) < U(f) + \frac{\epsilon}{2}$ and that $-L(f, P_1 \cup P_2) < \frac{\epsilon}{2} - L(f)$ (the latter by Lemma 7.24), we have by Definition 7.21 that

$$\begin{aligned} U(f, P_1 \cup P_2) - L(f, P_1 \cup P_2) &< U(f) + \frac{\epsilon}{2} + \frac{\epsilon}{2} - L(f) \\ &= \epsilon \end{aligned}$$

as desired.

Now suppose that f is not integrable; we seek to prove that there exists an $\epsilon > 0$ such that for all partitions P of $[a, b]$, $U(f, P) - L(f, P) \geq \epsilon$. Since f is not integrable, we have by Definition 13.16 that

$L(f) \neq U(f)$. It follows by Exercise 13.15 that $L(f) < U(f)$. Thus, we can define $\epsilon = \frac{U(f)-L(f)}{2} > 0$. Now let P be an arbitrary partition of $[a, b]$. It follows that $L(f, P) \leq L(f)$ by Definitions 13.14, 5.7, and 5.6. Similarly, $U(f) \leq U(f, P)$. Therefore, knowing that $U(f) \leq U(f, P)$ and that $-L(f) \leq -L(f, P)$ (the latter by Lemma 7.24), we have by Definition 7.21 that $\epsilon < U(f) - L(f) \leq U(f, P) - L(f, P)$, as desired. $\square$