Script 18

The Euclidean Space $\mathbb{R}^n$

7/7: For the next three sheets, we will be studying multivariable calculus, that is "calculus on $\mathbb{R}^n$." First, we need to understand the space $\mathbb{R}^n$.

Definition 18.1. The **Euclidean *n*-space** $\mathbb{R}^n$ is the n-fold Cartesian product of $\mathbb{R}$. Symbolically,

$$\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{R}\}$$

is the set of n-tuples of real numbers. We often write

$$\mathbf{x} = (x_1, \dots, x_n)$$

to denote an element, which is also referred to as a **vector**, in $\mathbb{R}^n$ and

$$\mathbf{0} = (0, \dots, 0)$$

Definition 18.2. Let $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. We define the following operations.

(a) (Addition) $\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_n + y_n)$.

(b) (Scalar Multiplication) $\lambda\mathbf{x} = (\lambda x_1, \dots, \lambda x_n)$.

Exercise 18.3. Prove that the addition on $\mathbb{R}^n$ satisfies FA1-FA4 (see Definition 7.8). Moreover, prove that

VS1. (Associativity of Scalar Multiplication) If $\lambda, \mu \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, then $(\lambda\mu)\mathbf{x} = \lambda(\mu\mathbf{x})$.

VS2. (Distributivity of Scalars) If $\lambda, \mu \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, then $(\lambda + \mu)\mathbf{x} = \lambda\mathbf{x} + \mu\mathbf{x}$.

VS3. (Distributivity of Vectors) If $\lambda \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then $\lambda(\mathbf{x} + \mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y}$.

VS4. (Scalar Multiplicative Identity) If $\mathbf{x} \in \mathbb{R}^n$, then $1\mathbf{x} = \mathbf{x}$.

These eight properties together are called the **vector space axioms**.

Proof. To prove that $\mathbb{R}^n$ obeys FA1 from Definition 7.8, it will suffice to show that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$. Let $\mathbf{x}, \mathbf{y}$ be arbitrary elements of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}\mathbf{x} + \mathbf{y} &= (x_1 + y_1, \dots, x_n + y_n)\\ &= (y_1 + x_1, \dots, y_n + x_n)\\ &= \mathbf{y} + \mathbf{x}\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys FA2 from Definition 7.8, it will suffice to show that for all $\mathbf{x},\mathbf{y},\mathbf{z} \in \mathbb{R}^n$, $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$. Let $\mathbf{x},\mathbf{y},\mathbf{z}$ be arbitrary elements of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}(\mathbf{x}+\mathbf{y})+\mathbf{z} &= (x_1+y_1,\dots,x_n+y_n)+\mathbf{z}\\ &= ((x_1+y_1)+z_1,\dots,(x_n+y_n)+z_n)\\ &= (x_1+(y_1+z_1),\dots,x_n+(y_n+z_n))\\ &= \mathbf{x}+(y_1+z_1,\dots,y_n+z_n)\\ &= \mathbf{x}+(\mathbf{y}+\mathbf{z})\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys FA3 from Definition 7.8, it will suffice to find an element $0 \in \mathbb{R}^n$ such that $\mathbf{x}+0 = 0+\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Choose $\mathbf{0}$ to be our 0. Let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}\mathbf{x}+\mathbf{0} &= (x_1+0,\dots,x_n+0)\\ &= (x_1,\dots,x_n)\\ &= \mathbf{x}\\ &= (0+x_1,\dots,0+x_n)\\ &= \mathbf{0}+\mathbf{x}\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys FA4 from Definition 7.8, it will suffice to show that for all $\mathbf{x} \in \mathbb{R}^n$, there exists $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x} = 0$. Let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Choose $\mathbf{y} = (-x_1,\dots,-x_n)$. Then by Definition 18.2,

$$\begin{aligned}\mathbf{x}+\mathbf{y} &= (x_1+(-x_1),\dots,x_n+(-x_n))\\ &= (0,\dots,0)\\ &= \mathbf{0}\\ &= ((-x_1)+x_1,\dots,(-x_n)+x_n)\\ &= \mathbf{y}+\mathbf{x}\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys VS1, it will suffice to show that for all $\lambda,\mu \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, we have $(\lambda\mu)\mathbf{x} = \lambda(\mu\mathbf{x})$. Let λ,μ be arbitrary elements of $\mathbb{R}$, and let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}(\lambda\mu)\mathbf{x} &= ((\lambda\mu)x_1,\dots,(\lambda\mu)x_n)\\ &= (\lambda(\mu x_1),\dots,\lambda(\mu x_n))\\ &= \lambda(\mu\mathbf{x})\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys VS2, it will suffice to show that for all $\lambda,\mu \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, we have $(\lambda+\mu)\mathbf{x} = \lambda\mathbf{x}+\mu\mathbf{x}$. Let λ,μ be arbitrary elements of $\mathbb{R}$, and let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}(\lambda+\mu)\mathbf{x} &= ((\lambda+\mu)x_1,\dots,(\lambda+\mu)x_n)\\ &= (\lambda x_1+\mu x_1,\dots,\lambda x_n+\mu x_n)\\ &= (\lambda x_1,\dots,\lambda x_n)+(\mu x_1,\dots,\mu x_n)\\ &= \lambda\mathbf{x}+\mu\mathbf{x}\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys VS3, it will suffice to show that for all $\lambda \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have $\lambda(\mathbf{x}+\mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y}$. Let λ be an arbitrary element of $\mathbb{R}$, and let $\mathbf{x}, \mathbf{y}$ be arbitrary elements of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}\lambda(\mathbf{x}+\mathbf{y}) &= \lambda(x_1+y_1, \dots, x_n+y_n)\\ &= (\lambda(x_1+y_1), \dots, \lambda(x_n+y_n))\\ &= (\lambda x_1 + \lambda y_1, \dots, \lambda x_n + \lambda y_n)\\ &= (\lambda x_1, \dots, \lambda x_n) + (\lambda y_1, \dots, \lambda y_2)\\ &= \lambda\mathbf{x} + \lambda\mathbf{y}\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys VS4, it will suffice to show that for all $\mathbf{x} \in \mathbb{R}^n$, we have $1\mathbf{x} = \mathbf{x}$. Let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}1\mathbf{x} &= (1x_1, \dots, 1x_n)\\ &= (x_1, \dots, x_n)\\ &= \mathbf{x}\end{aligned}$$

as desired. ☐

Remark 18.4. Since $\mathbb{R}^n$ with the two operations defined as above satisfies these eight axioms, we call $\mathbb{R}^n$ a **vector space**.

Exercise 18.5. Prove that if $\mathbf{x} \in \mathbb{R}^n$, then $0\mathbf{x} = \mathbf{0}$.

Proof. By Definition 18.2, we have that

$$\begin{aligned}0\mathbf{x} &= (0x_1, \dots, 0x_n)\\ &= (0, \dots, 0)\\ &= \mathbf{0}\end{aligned}$$

as desired. ☐

Definition 18.6. Let $\mathbf{x} \in \mathbb{R}^n$. The **norm** of $\mathbf{x}$ is defined as

$$\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

Definition 18.7. We call $\|\mathbf{y} - \mathbf{x}\|$ the **distance** between $\mathbf{x}$ and $\mathbf{y}$.

Remark 18.8. If $n = 1$, the norm coincides with the definition of the absolute value in $\mathbb{R}$.

Lemma 18.9.

(a) *If $x, y \in \mathbb{R}$, then $xy \leq \frac{x^2+y^2}{2}$.*

Proof. Let x, y be arbitrary elements of $\mathbb{R}$. Then by Lemma 7.26, $0 \leq (x-y)^2$. Therefore, we have that

$$\begin{aligned}xy &= \frac{2xy+0}{2}\\ &\leq \frac{2xy+(x-y)^2}{2}\\ &= \frac{2xy+x^2-2xy+y^2}{2}\\ &= \frac{x^2+y^2}{2}\end{aligned}$$

as desired. ☐

(b) *If* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, *then* $|x_1y_1 + \cdots + x_ny_n| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|$.

Proof. Suppose first that $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$. Then by Definition 18.6, $\|\mathbf{x}\| = 1 = \sqrt{x_1^2 + \cdots + x_n^2}$, from which it follows that $1 = x_1^2 + \cdots + x_n^2$. Therefore, we have that

$$\begin{aligned}
|x_1y_1 + \cdots + x_ny_n| &\leq |x_1y_1| + \cdots + |x_ny_n| && \text{Lemma 8.8} \\
&\leq \frac{x_1^2 + y_1^2}{2} + \cdots + \frac{x_n^2 + y_n^2}{2} \\
&= \frac{(x_1^2 + \cdots + x_n^2) + (y_1^2 + \cdots + y_n^2)}{2} \\
&= \frac{1+1}{2} \\
&= 1
\end{aligned}$$

as desired.

Now let $\mathbf{x}, \mathbf{y}$ be arbitrary elements of $\mathbb{R}^n$. Consider the vectors $\mathbf{u_x}, \mathbf{u_y}$ defined by $\mathbf{u_x} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$ and $\mathbf{u_y} = \frac{\mathbf{y}}{\|\mathbf{y}\|}$. By the proof of the first case, we have that

$$\begin{aligned}
|x_1y_1 + \cdots + x_ny_n| &= \|\mathbf{x}\| \cdot \|\mathbf{y}\| \cdot \left| \frac{x_1y_1}{\|\mathbf{x}\| \cdot \|\mathbf{y}\|} + \cdots + \frac{x_ny_n}{\|\mathbf{x}\| \cdot \|\mathbf{y}\|} \right| \\
&= \|\mathbf{x}\| \cdot \|\mathbf{y}\| \cdot |u_{\mathbf{x}_1}u_{\mathbf{y}_1} + \cdots + u_{\mathbf{x}_n}u_{\mathbf{y}_n}| \\
&\leq \|\mathbf{x}\| \cdot \|\mathbf{y}\| \cdot 1 \\
&= \|\mathbf{x}\| \cdot \|\mathbf{y}\|
\end{aligned}$$

as desired. □

Theorem 18.10. *If* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}$, *then*

(a) $\|\mathbf{x}\| \geq 0$. *Moreover,* $\|\mathbf{x}\| = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$.

Proof. Let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$.

We first prove that $\|\mathbf{x}\| \geq 0$. By Lemma 7.26, $x_i^2 \geq 0$ for all $i \in [n]$. Thus, by Definition 7.21, $x_1^2 + \cdots + x_n^2 \geq 0$. Therefore, we have by Definition 18.6 that $\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2} \geq 0$, as desired.

We now prove that $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$. Suppose first that $\|\mathbf{x}\| = 0$. Then by Definition 18.6 and Script 7, $x_1^2 + \cdots + x_n^2 = 0$. Now suppose for the sake of contradiction that $\mathbf{x} \neq \mathbf{0}$. Then there exists an x_i such that $x_i \neq 0$. Thus, by Lemma 7.26, $x_i^2 > 0$. Additionally, $x_j^2 \geq 0$ for all $j \in [n]$. Thus, we have that $0 < x_i^2 \leq x_1^2 + \cdots + x_n^2$. But by Definition 3.1, this implies that $x_1^2 + \cdots + x_n^2 \neq 0$, a contradiction.

Now suppose that $\mathbf{x} = \mathbf{0}$. Then by Definition 18.6, $\|\mathbf{x}\| = \sqrt{0^2 + \cdots + 0^2} = 0$, as desired. □

(b) $\|\lambda\mathbf{x}\| = |\lambda| \cdot \|\mathbf{x}\|$.

Proof. Let λ be an arbitrary element of $\mathbb{R}$, and let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Then we have that

$$\begin{aligned}
\|\lambda\mathbf{x}\| &= \sqrt{(\lambda x_1)^2 + \cdots + (\lambda x_n)^2} && \text{Definition 18.6} \\
&= |\lambda| \cdot \sqrt{x_1^2 + \cdots + x_n^2} \\
&= |\lambda| \cdot \|\mathbf{x}\| && \text{Definition 18.6}
\end{aligned}$$

as desired. □

(c) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\|+\|\mathbf{y}\|$.

Proof. Let $\mathbf{x}, \mathbf{y}$ be arbitrary elements of $\mathbb{R}^n$. Then we have that

$$\begin{aligned}
\|\mathbf{x}+\mathbf{y}\| &= \sqrt{(x_1+y_1)^2+\cdots+(x_n+y_n)^2} && \text{Definition 18.6}\\
&= \sqrt{(x_1^2+\cdots+x_n^2)+(2x_1y_1+\cdots+2x_ny_n)+(y_1^2+\cdots+y_n^2)}\\
&\leq \sqrt{\|\mathbf{x}\|^2+2\|\mathbf{x}\|\|\mathbf{y}\|+\|y\|^2} && \text{Lemma 18.9}\\
&= \sqrt{(\|\mathbf{x}\|+\|\mathbf{y}\|)^2}\\
&= \|\mathbf{x}\|+\|\mathbf{y}\|
\end{aligned}$$

as desired. □

Corollary 18.11. *If $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, then*

(a) $\|\mathbf{x}-\mathbf{z}\| \leq \|\mathbf{x}-\mathbf{y}\|+\|\mathbf{y}-\mathbf{z}\|$.

(b) $|\|\mathbf{x}\|-\|\mathbf{y}\|| \leq \|x-y\|$.

Proof. The proofs are symmetric to those of Lemma 8.8. □

7/10: The next goal is to "topologize" $\mathbb{R}^n$. To discuss topology on $\mathbb{R}^n$, we first need to introduce notions for $\mathbb{R}^n$ that are analogous to open and closed intervals for $\mathbb{R}$.

Remark 18.12. For $\mathbf{x}=(x_1,\dots,x_n) \in \mathbb{R}^n$ and $\mathbf{y}=(y_1,\dots,y_m) \in \mathbb{R}^m$, we identify $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m$ with $(x_1,\dots,x_n,y_1,\dots,y_m) \in \mathbb{R}^{n+m}$. So if $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$, we can consider $A \times B$ to be a subset of $\mathbb{R}^{n+m}$.

If also $C \in \mathbb{R}^k$, then $(A\times B)\times C$ and $A\times(B\times C)$ correspond to the same subset of $\mathbb{R}^{n+m+k}$ under this identification; we write $A\times B\times C$ for this set.

Definition 18.13. An **open rectangle** in $\mathbb{R}^n$ is a set of the form $(a_1,b_1)\times\cdots\times(a_n,b_n)$, a product of open intervals. Similarly, a **closed rectangle** in $\mathbb{R}^n$ is a set of the form $[a_1,b_1]\times\cdots\times[a_n,b_n]$. We allow the possibility that $a_j=b_j$ (where $[a_j,a_j]=\{a_j\}$). If there is at least one j with $a_j=b_j$, then we say that the rectangle is **degenerate**; otherwise, we say that the rectangle is **non-degenerate**.

Definition 18.14. A subset $U \subset \mathbb{R}^n$ is **open** if for all $\mathbf{a} \in U$, there exists an open rectangle R such that $\mathbf{a} \in R \subset U$. A subset $C \in \mathbb{R}^n$ is closed if its compliment is open.

Exercise 18.15. Decide whether each of the following is an open set in $\mathbb{R}^2$.

(a) $\{(x_1,x_2) \mid x_1,x_2 \in \mathbb{R},\ x_1>0,\ x_2>0\}$.

Proof. To prove that $U=\{(x_1,x_2) \mid x_1,x_2 \in \mathbb{R},\ x_1>0,\ x_2>0\}$ is open, Definition 18.14 tells us that it will suffice to show that for all $\mathbf{x} \in U$, there exists an open rectangle R such that $\mathbf{x} \in R \subset U$. Let $\mathbf{x}$ be an arbitrary element of U. Then by the definition of U, $0<x_1$ and $0<x_2$. It follows by Theorem 5.2 and Corollary 6.12, there exist a_1,b_1,a_2,b_2 such that $0<a_1<x_1<b_1$ and $0<a_2<x_2<b_2$. Thus, by Equations 8.1, $x_1 \in (a_1,b_1)$ and $x_2 \in (a_2,b_2)$. Consequently, if we let $R=(a_1,b_1)\times(a_2,b_2)$, Definition 18.13 guarantees that R is an open rectangle. Additionally, Definition 1.15 asserts that $(x_1,x_2)=\mathbf{x} \in R$, as desired. Additionally, if $\mathbf{y}$ is any vector in R, then by the definition of R, $0<a_1<y_1$ and $0<a_2<y_2$. Thus, by transitivity, $\mathbf{y} \in U$. Therefore, by Definition 1.3, $R \subset U$, as desired. □

(b) $\{(x,0) \mid x \in \mathbb{R}\}$.

Proof. To prove that $U=\{(x,0) \mid x \in \mathbb{R}\}$ is not open, Definition 18.14 tells us that it will suffice to find an $\mathbf{x} \in U$ such that for all open rectangles R containing $\mathbf{x}$, $R \not\subset U$. Let $\mathbf{x}=(0,0)$, and let R be an arbitrary open rectangle containing $\mathbf{x}$. By Definitions 18.13 and 1.15 along with Equations 8.1, $a_1<0<b_1$ and $a_2<0<b_2$. Thus, by consecutive applications of Theorem 5.2, there exist points $y_1,y_2 \in \mathbb{R}$ such that $a_1<y_1<0$ and $a_2<y_2<0$. It follows that $\mathbf{y}=(y_1,y_2) \in R$. However, since $y_2 \neq 0$ by Definition 3.1, $\mathbf{y} \notin U$. Therefore, by Definition 1.3, $R \not\subset U$, as desired. □

Exercise 18.16. Show that if $R_1, \dots, R_m$ are open rectangles containing $\mathbf{x} \in \mathbb{R}^n$, then $R = R_1 \cap \cdots \cap R_m$ is an open rectangle containing $\mathbf{x} \in \mathbb{R}^n$. If $R = (a_1, b_1) \times \cdots \times (a_n, b_n)$, derive formulas for a_i and b_i in terms of the corresponding quantities for $R_1, \dots, R_m$.

Proof. Let $R_i = (r_{ij}, s_{ij})_{j=1}^n$ for all $i \in [m]$. To prove that $R = \bigcap_{i=1}^m R_i$ is an open rectangle containing $\mathbf{x}$, Definitions 18.13 and 1.15 tell us that it will suffice to show that R is the Cartesian product of open intervals, each containing its respective x_j. Since $\mathbf{x} \in R_i$ for all $i \in [m]$, we have by Definition 1.15 that $x_j \in (r_{ij}, s_{ij})$ for all $i \in [m],\ j \in [n]$. Thus, by Corollary 3.19, $\bigcap_{i=1}^m (r_{ij}, s_{ij})$ is a region (hence an open interval by Corollary 4.11 and Lemma 8.3) containing x_j for all $j \in [n]$. Therefore, since $R = \bigcap_{i=1}^m R_i = \prod_{i=1}^n (\bigcap_{i=1}^m (r_{ij}, s_{ij}))$ by Script 1, we have that R is the Cartesian product of open intervals, each containing its respective x_j, as desired.

Let $a_j = \max_{i=1}^m (r_{ij})$ and let $b_j = \min_{i=1}^m (s_{ij})$ for all $j \in [n]$. To prove that $R = (a_j, b_j)_{j=1}^n$, Definition 1.2 tells us that it will suffice to show that every $\mathbf{x} \in R$ is an element of $(a_j, b_j)_{j=1}^n$ and vice versa. Suppose first that $\mathbf{x}$ is an arbitrary element of R. Then by the definition of R, $\mathbf{x} \in R_i$ for all $i \in [m]$. It follows by Definition 1.15 that $x_j \in (r_{ij}, s_{ij})$ for all $i \in [m],\ j \in [n]$, including the j, j' for which r_{ij} is at its maximum and $s_{ij'}$ is at its minimum. In other words, $x_j \in (a_j, b_j)$ for all $j \in [n]$. Therefore, by Definition 1.15, $\mathbf{x} \in (a_j, b_j)_{j=1}^n$, as desired. The proof is symmetric in the other direction. $\square$