Script 16

Series

16.1 Journal

5/20: **Definition 16.1.** Let $N_0 \in \mathbb{N} \cup \{0\}$ and let $(a_n)_{n=N_0}^{\infty}$ be a sequence of real numbers. Then the formal sum

$$\sum_{n=N_0}^{\infty} a_n$$

is called an **infinite series**. (In most instances, we will start the series at $N_0 = 0$ or $N_0 = 1$.)

We will define the **sequence of partial sums** (p_n) of the series by

$$p_n = a_{N_0} + \cdots + a_{N_0+n-1} = \sum_{i=N_0}^{N_0+n-1} a_i$$

Thus, p_n is the sum of the first n terms in the sequence (a_n). We say that the series **converges** if there exists $L \in \mathbb{R}$ such that $\lim_{n\to\infty} p_n = L$. When this is the case, we write this as

$$\sum_{n=N_0}^{\infty} a_n = L$$

and we say that L is the **sum** of the series. When there does not exist such an L, we say that the series **diverges**.

Lemma 16.2. *Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers. Let $N_0 \in \mathbb{N}$. Then $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=N_0}^{\infty} a_n$ converges.*

Lemma. *Let $n \in \mathbb{N}$. Then*

$$\sum_{i=0}^{N_0+n-1} a_i = \sum_{i=0}^{N_0-1} a_i + \sum_{i=N_0}^{N_0+n-1} a_i$$

Proof. This simple result follows immediately from Script 0, so no formal proof will be given. □

Proof of Lemma 16.2. Suppose first that $\sum_{n=0}^{\infty} a_n$ converges, and let $M = \sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=0}^{n-1} a_i$, where the latter equality holds by Definition 16.1. To prove that $\sum_{n=N_0}^{\infty} a_n$ converges, Definition 16.1 tells us that it will suffice to find an $L \in \mathbb{R}$ such that $\lim_{n\to\infty} \sum_{i=N_0}^{N_0+n-1} a_i = L$. Choose $L = M - \sum_{i=0}^{N_0-1} a_i$. To verify that $\lim_{n\to\infty} \sum_{i=N_0}^{N_0+n-1} a_i = L$, Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|\sum_{i=N_0}^{N_0+n-1} a_i - L| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} \sum_{i=0}^{n-1} a_i = M$, Theorem 15.7 implies that there is some $N \in \mathbb{N}$ such that for all $n \geq N$,

$|\sum_{i=0}^{n-1} a_i - M| < \epsilon$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. Since $N_0 + n > n \geq N$, we have by the above that $|\sum_{i=0}^{N_0+n-1} a_i - M| < \epsilon$. Therefore,

$$\begin{aligned}
\left|\sum_{i=N_0}^{N_0+n-1} a_i - L\right| &= \left|\sum_{i=0}^{N_0+n-1} a_i - \sum_{i=0}^{N_0-1} a_i - L\right| && \text{The Lemma} \\
&= \left|\sum_{i=0}^{N_0+n-1} a_i - \left(\sum_{i=0}^{N_0-1} a_i + L\right)\right| \\
&= \left|\sum_{i=0}^{N_0+n-1} a_i - M\right| \\
&< \epsilon
\end{aligned}$$

as desired.

The proof is symmetric in the other direction. □

Exercise 16.3. Prove that $\sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+1})$ converges. What is its sum?

Proof. Let (a_n) be defined by $a_n = \frac{1}{n} - \frac{1}{n+1}$, and let (p_n) be defined by $p_n = \sum_{i=1}^{n} a_i$. Then

$$\begin{aligned}
p_n &= a_1 + a_2 + \cdots + a_n \\
&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \frac{1}{1} - \frac{1}{n+1}
\end{aligned}$$

To prove that $\sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+1}) = 1$, Definition 16.1 tells us that it will suffice to show that $\lim_{n\to\infty} p_n = 1$. By a proof symmetric to that of Exercise 15.6a, we have that $\lim_{n\to\infty} 1 = 1$. By a proof symmetric to that of Exercise 15.6c, we have that $\lim_{n\to\infty} \frac{1}{n+1} = 0$. Therefore, by Theorem 15.9 and the above, we have that

$$\begin{aligned}
\lim_{n\to\infty} p_n &= \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) \\
&= \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n+1} \\
&= 1 - 0 \\
&= 1
\end{aligned}$$

as desired. □

Theorem 16.4. *If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.*

Proof. To prove that $\lim_{n\to\infty} a_n = 0$, Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - 0| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\sum_{n=1}^{\infty} a_n$ converges, we have by Theorem 15.19 that there exists an $N \in \mathbb{N}$ such that $|\sum_{i=1}^{n} a_i - \sum_{i=1}^{m} a_i| < \epsilon$ for all $n, m \geq N$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. Then choosing $n, n-1 \geq N$, we have by the above that $|\sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} a_i| < \epsilon$. Therefore,

$$\begin{aligned}
|a_n - 0| &= |a_n| \\
&= \left|\sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} a_i\right| \\
&< \epsilon
\end{aligned}$$

as desired. □

The converse of this theorem, however, is not true, as we see in Theorem 16.6.

Theorem 16.5. *A series $\sum_{n=1}^{\infty} a_n$ converges if and only if for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|\sum_{k=m+1}^{n} a_k| < \epsilon$ for all $n > m \geq N$.*

Proof. Suppose first that $\sum_{n=1}^{\infty} a_n$ converges. Let $\epsilon > 0$ be arbitrary. By Definition 16.1, (p_n) converges. Thus, by Theorem 15.19, there is some $N \in \mathbb{N}$ such that $|p_n - p_m| < \epsilon$ for all $n, m \geq N$. Choose this N to be our N. Let n, m be two arbitrary natural numbers satisfying $n > m \geq N$. Therefore,

$$\begin{aligned} \left|\sum_{k=m+1}^{n} a_k\right| &= \left|\sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k\right| \\ &= |p_n - p_m| \\ &< \epsilon \end{aligned}$$

as desired.

The proof is symmetric in the other direction. □

Theorem 16.6. *The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.*

Lemma. *For all $N \in \mathbb{N}$, we have*

$$\sum_{n=N+1}^{2N} \frac{1}{n} \geq \frac{1}{2}$$

Proof. We induct on N. For the base case $N = 1$, we have

$$\sum_{n=1+1}^{2\cdot 1} \frac{1}{n} = \frac{1}{2} \geq \frac{1}{2}$$

as desired. Now suppose inductively that we have proven the claim for N. To prove it for $N + 1$, we do the following.

$$\begin{aligned} \sum_{n=N+2}^{2N+2} \frac{1}{n} &= \sum_{n=N+1}^{2N} \frac{1}{n} - \frac{1}{N+1} + \frac{1}{2N+1} + \frac{1}{2(N+1)} \\ &= \sum_{n=N+1}^{2N} \frac{1}{n} + \frac{1}{2(N+1)(2N+1)} \\ &> \sum_{n=N+1}^{2N} \frac{1}{n} \\ &\geq \frac{1}{2} \end{aligned}$$

as desired. □

Proof of Theorem 16.6. To prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, Theorem 16.5 tells us that it will suffice to find an $\epsilon > 0$ such that for all $N \in \mathbb{N}$, there exist $n > m \geq N$ with $|\sum_{k=m+1}^{n} 1/k| \geq \epsilon$. Choose $\epsilon = \frac{1}{2}$. Let N be an arbitrary element of N. If we now choose $n = 2N$ and $m = N$, we will have $n > m \geq N$. It will follow by the lemma that

$$\begin{aligned} \left|\sum_{k=m+1}^{n} \frac{1}{k}\right| &= \left|\sum_{k=N+1}^{2N} \frac{1}{k}\right| \\ &\geq \frac{1}{2} \\ &= \epsilon \end{aligned}$$

as desired. □

5/25: **Theorem 16.7.** *Let $-1 < x < 1$. Then*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Proof. Let (a_n) be defined by $a_n = x^n$. Then $p_n = x^0 + x^1 + \cdots + x^{n-1}$ so that

$$\begin{aligned} p_n - xp_n &= x^0 + \cdots + x^{n-1} - x(x^0 + \cdots + x^{n-1}) \\ p_n - xp_n &= 1 - x^n \\ p_n(1-x) &= 1 - x^n \\ p_n &= \frac{1-x^n}{1-x} \end{aligned}$$

Therefore, we have that

$$\begin{aligned} \frac{1}{1-x} &= \frac{1-0}{1-x} \\ &= \frac{1 - \lim_{n\to\infty} x^n}{1-x} && \text{Exercise 15.8b} \\ &= \lim_{n\to\infty} \frac{1-x^n}{1-x} && \text{Theorem 15.9} \\ &= \lim_{n\to\infty} p_n \\ &= \sum_{n=0}^{\infty} x^n && \text{Definition 16.1} \end{aligned}$$

as desired. □

Theorem 16.8. *If $\sum_{n=1}^{\infty} a_n = L$, $\sum_{n=1}^{\infty} b_n = M$, and $c \in \mathbb{R}$, then*

$$\sum_{n=1}^{\infty} (a_n + b_n) = L + M$$

$$\sum_{n=1}^{\infty} (c \cdot a_n) = c \cdot L$$

Proof. For the first claim, we have that

$$\begin{aligned} L + M &= \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} a_i + \lim_{n\to\infty} \sum_{i=1}^{n} b_i && \text{Definition 16.1} \\ &= \lim_{n\to\infty} \left(\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \right) && \text{Theorem 15.9} \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} (a_i + b_i) \\ &= \sum_{n=1}^{\infty} (a_n + b_n) && \text{Definition 16.1} \end{aligned}$$

The proof is symmetric for the second claim. □

Definition 16.9. We say that the series $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

Lemma 16.10. *A series $\sum_{n=1}^{\infty} a_n$ with all $a_n \geq 0$ converges if and only if its sequence of partial sums is bounded.*

Proof. Suppose first that the series $\sum_{n=1}^{\infty} a_n$ with all $a_n \geq 0$ converges. Then by Definition 16.1 its sequence of partial sums (p_n) converges. Therefore, by Theorem 15.13, (p_n) is bounded, as desired.

Now suppose that the sequence of partial sums (p_n) corresponding to a series $\sum_{n=1}^{\infty} a_n$ with all $a_n \geq 0$ is bounded. To prove that $\sum_{n=1}^{\infty} a_n$ converges, Definition 16.1 tells us that it will suffice to show that (p_n) converges. To do so, Theorem 15.14 tells us that it will suffice to verify in addition to the fact that (p_n) is bounded that (p_n) is increasing. To do this, Script 15 tells us that it will suffice to confirm that $p_n \leq p_{n+1}$ for all $n \in \mathbb{N}$. Let n be an arbitrary natural number. By Definition 16.1, $p_{n+1} - p_n = a_{n+1}$. Since $a_{n+1} \geq 0$ by hypothesis, we have by transitivity that $p_{n+1} - p_n \geq 0$, i.e., $p_n \leq p_{n+1}$ by Definition 7.21, as desired. □

Theorem 16.11. *If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges and*

$$\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|$$

Proof. To prove that $\sum_{n=1}^{\infty} a_n$ converges, Theorem 16.5 tells us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|\sum_{k=m+1}^{n} a_k| < \epsilon$ for all $n > m \geq N$. Let $\epsilon > 0$ be arbitrary. Since $\sum_{n=1}^{\infty} a_n$ converges absolutely by hypothesis, we have by Definition 16.9 that $\sum_{n=1}^{\infty} |a_n|$ converges. Thus, by Theorem 16.5, there is some $N \in \mathbb{N}$ such that $|\sum_{k=m+1}^{n} |a_k|| < \epsilon$ for all $n > m \geq N$. Choose this N to be our N. Let n, m be arbitrary natural numbers such that $n > m \geq N$. Therefore,

$$\begin{aligned} \left|\sum_{k=m+1}^{n} a_k\right| &\leq \sum_{k=m+1}^{n} |a_k| && \text{Lemma 8.8} \\ &= \left|\sum_{k=m+1}^{n} |a_k|\right| \\ &< \epsilon \end{aligned}$$

as desired.

As to the other part of the claim, to begin, let (b_n) and (c_n) be defined by $b_n = \max(0, a_n)$ and $c_n = \min(0, a_n)$. We will prove a few preliminary results with these definitions that will enable us to tackle the big inequality.

To confirm that $a_n = b_n + c_n$, we divide into two cases ($a_n \geq 0$ and $a_n < 0$). If $a_n \geq 0$, then by their definitions, $b_n = a_n$ and $c_n = 0$. Thus, $a_n = b_n + c_n$ as desired. The argument is symmetric in the other case.

To confirm that $|\sum_{n=1}^{\infty} b_n| + |\sum_{n=1}^{\infty} -c_n| = |\sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} -c_n|$, we can acknowledge that $b_n \geq 0$ and $-c_n \geq 0$ for all $n \in \mathbb{N}$ to demonstrate that

$$\begin{aligned} \left|\sum_{n=1}^{\infty} b_n\right| + \left|\sum_{n=1}^{\infty} -c_n\right| &= \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} -c_n \\ &= \left|\sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} -c_n\right| \end{aligned}$$

To confirm that $|a_n| = b_n - c_n$, we divide into two cases ($a_n \geq 0$ and $a_n < 0$). If $a_n \geq 0$, then $b_n = a_n$ and $c_n = 0$. Thus, by Definition 8.4, $|a_n| = a_n = b_n - c_n$, as desired. On the other hand, if $a_n < 0$, then $b_n = 0$ and $c_n = a_n$. Thus, by Definition 8.4 again, $|a_n| = -a_n = b_n - c_n$, as desired.

Having established that $a_n = b_n + c_n$, $|\sum_{n=1}^{\infty} b_n| + |\sum_{n=1}^{\infty} -c_n| = |\sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} -c_n|$, and $|a_n| =$

$b_n - c_n$, we have that

$$\begin{aligned}
\left|\sum_{n=1}^{\infty} a_n\right| &= \left|\sum_{n=1}^{\infty}(b_n + c_n)\right| \\
&= \left|\sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} c_n\right| && \text{Theorem 16.8} \\
&\leq \left|\sum_{n=1}^{\infty} b_n\right| + \left|\sum_{n=1}^{\infty} c_n\right| && \text{Lemma 8.8} \\
&= \left|\sum_{n=1}^{\infty} b_n\right| + \left|-\sum_{n=1}^{\infty} c_n\right| && \text{Exercise 8.5} \\
&= \left|\sum_{n=1}^{\infty} b_n\right| + \left|\sum_{n=1}^{\infty} -c_n\right| && \text{Theorem 16.8} \\
&= \left|\sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} -c_n\right| \\
&= \left|\sum_{n=1}^{\infty}(b_n - c_n)\right| && \text{Theorem 16.8} \\
&= \left|\sum_{n=1}^{\infty} |a_n|\right| \\
&= \sum_{n=1}^{\infty} |a_n|
\end{aligned}$$

as desired. □

Theorem 16.12. *Let (a_n) be a decreasing sequence of positive numbers such that $\lim_{n\to\infty} a_n = 0$. Then $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.*

Lemma.

(*a*) *For any natural numbers n, m satisfying $n > m$, we have*

$$\left|\sum_{k=m+1}^{n}(-1)^{k+1}a_k\right| = |a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + a_{m+5} - \cdots \pm a_n|$$

Proof. By Script 0, either $|\sum_{k=m+1}^{n}(-1)^{k+1}a_k| = |a_{m+1} - a_{m+2} + a_{m+3} - \cdots \pm a_n|$ or $|\sum_{k=m+1}^{n}(-1)^{k+1}a_k| = |-a_{m+1} + a_{m+2} - a_{m+3} + \cdots \pm a_n|$. However, by Exercise 8.5, the two results are equal. Thus, we may choose the former WLOG. □

(*b*) *We have*

$$0 \leq a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots \pm a_n$$

Proof. Since (a_n) is a decreasing sequence, we have by Script 15 that $a_{i+1} \leq a_i$ for all $i \in \mathbb{N}$. It follows by Definition 7.21 that $0 \leq a_i - a_{i+1}$ for all $i \in \mathbb{N}$. We now divide into two cases (there are an even number of terms in the sum $a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + a_{m+5} - \cdots \pm a_n$ and there are an odd number of terms in said sum). In the first case, we have that the sum is of the form $a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots + a_{n-1} - a_n$. Thus, since $0 \leq a_{m+1} - a_{m+2}$, $0 \leq a_{m+3} - a_{m+4}$, and on and on, we have by Script 7 that $0 \leq a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots - a_n$, as desired. On the other hand, in the second case, we have that the sum is of the form $a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots + a_{n-2} - a_{n-1} + a_n$. For the same reason as before, we have that $0 \leq a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots + a_{n-2} - a_{n-1}$. However, we need the additional hypothesis that every a_i is positive, i.e., $a_i \geq 0$ for all $i \in \mathbb{N}$ to know that $0 \leq a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots + a_n$, as desired. □

(c) *For all $i \in \mathbb{N}$, we have*

$$-a_i + a_{i+1} \leq 0$$

Proof. Since (a_n) is a decreasing sequence, we have by Script 15 that $a_{i+1} \leq a_i$ for all $i \in \mathbb{N}$. It follows by Definition 7.21 that $-a_i + a_{i+1} \leq 0$ for all $i \in \mathbb{N}$. $\square$

Proof of Theorem 16.12. To prove that $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges, Theorem 16.5 tells us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|\sum_{k=m+1}^{n}(-1)^{k+1}a_k| < \epsilon$ for all $n > m \geq N$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} a_n = 0$ by hypothesis, we have by Theorem 15.7 that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - 0| = |a_n| < \epsilon$, as desired. Choose this N to be our N. Let n, m be arbitrary natural numbers such that $n > m \geq N$.

We divide into two cases ($n-m$ is even [i.e., there are an even number of terms in the sum $\sum_{k=m+1}^{n}(-1)^{k+1}a_k$] and $n-m$ is odd [i.e., there are an odd number of terms in the sum $\sum_{k=m+1}^{n}(-1)^{k+1}a_k$]). If $n-m$ is even, then we have

$$\begin{aligned}
\left|\sum_{k=m+1}^{n}(-1)^{k+1}a_k\right| &= |a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + a_{m+5} - \cdots - a_{n-2} + a_{n-1} - a_n| && \text{Lemma (a)} \\
&= a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + a_{m+5} - \cdots - a_{n-2} + a_{n-1} - a_n \\
& && \text{Lemma (b) \& Definition 8.4} \\
&= a_{m+1} + (-a_{m+2} + a_{m+3}) + (-a_{m+4} + a_{m+5}) + \cdots + (-a_{n-2} + a_{n-1}) - a_n \\
&\leq a_{m+1} + (-a_{m+4} + a_{m+5}) + \cdots + (-a_{n-2} + a_{n-1}) - a_n && \text{Lemma (c)} \\
&\leq a_{m+1} + \cdots + (-a_{n-2} + a_{n-1}) - a_n && \text{Lemma (c)} \\
&\vdots && \text{Lemma (c)} \\
&\leq a_{m+1} + (-a_{n-2} + a_{n-1}) - a_n && \text{Lemma (c)} \\
&\leq a_{m+1} - a_n && \text{Lemma (c)} \\
&\leq a_{m+1} \\
&= |a_{m+1}| \\
&< \epsilon
\end{aligned}$$

The argument is symmetric if $n - m$ is odd. $\square$