Script 14

Integrals and Derivatives

14.1 Journal

5/4: **Theorem 14.1** (First Fundamental Theorem of Calculus — Derivative of Integrals)**.** *Suppose that f is integrable on $[a,b]$. Define $F:[a,b]\to\mathbb{R}$ by*

$$F(x)=\int_a^x f$$

If f is continuous at $p\in(a,b)$, then F is differentiable at p and

$$F'(p)=f(p)$$

If f is continuous at a, then $F'_+(a)$ exists and equals $f(a)$. Similarly, if f is continuous at b, $F'_-(b)$ exists and equals $f(b)$.

Proof. To prove that F is differentiable at p and $F'(p)=f(p)$, Definition 12.1 tells us that it will suffice to show that $\lim_{h\to 0^+}\frac{F(p+h)-F(p)}{h}=\lim_{h\to 0^-}\frac{F(p+h)-F(p)}{h}=f(p)$. We will tackle the right-handed limit first. To do so, Definition 11.1 tells us that it will suffice to verify that for every $\epsilon>0$, there exists a $\delta>0$ such that if $(p+h)\in[a,b]$ and $0<h<\delta$, then $|\frac{F(p+h)-F(p)}{h}-f(p)|<\epsilon$. Let $\epsilon>0$ be arbitrary. Since f is continuous at p, Theorem 11.5 asserts that there exists a $\delta>0$ such that if $x\in[a,b]$ and $|x-p|<\delta$, then $|f(x)-f(p)|<\frac{\epsilon}{2}$. Choose this δ to be our δ. Let h be an arbitrary number satisfying $(p+h)\in[a,b]$ and $0<h<\delta$. Therefore,

$$\begin{aligned}
\left|\frac{F(p+h)-F(p)}{h}-f(p)\right| &= \left|\frac{\int_a^{p+h}f-\int_a^p f}{h}-f(p)\right| \\
&= \left|\frac{\int_p^{p+h}f}{h}-f(p)\right| && \text{Theorem 13.23} \\
&= \left|\frac{\int_p^{p+h}f-hf(p)}{h}\right| \\
&= \left|\frac{\int_p^{p+h}f-f(p)((p+h)-p)}{h}\right| \\
&= \left|\frac{\int_p^{p+h}f-\int_p^{p+h}f(p)\,\mathrm{d}x}{h}\right| && \text{Exercise 13.17} \\
&= \left|\frac{1}{h}\int_p^{p+h}(f(x)-f(p))\,\mathrm{d}x\right| && \text{Theorem 13.24}
\end{aligned}$$

$$
\begin{aligned}
&\leq \left|\frac{1}{h}\right| \int_p^{p+h} |f(x) - f(p)|\,\mathrm{d}x && \text{Theorem 13.26} \\
&\leq \left|\frac{1}{h}\right| \frac{\epsilon}{2}((p+h) - p) && \text{Theorem 13.27} \\
&= \frac{\epsilon}{2} \\
&< \epsilon
\end{aligned}
$$

The proof is symmetric for the left-handed limit. These proofs can also be applied to the endpoints. □

Remark 14.2. Thus, we have that if f is continuous on $[a,b]$, F is differentiable on $[a,b]$ and $F'(p) = f(p)$ for all $p \in [a,b]$ (where at the endpoints, we understand that the derivative should be interpreted as the one-sided derivative).

Lemma 14.3. *Suppose that $f : [a,b] \to \mathbb{R}$ is integrable and that Ω is a number satisfying $L(f,P) \leq \Omega \leq U(f,P)$ for all partitions P of $[a,b]$. Then*

$$\int_a^b f = \Omega$$

Proof. Suppose for the sake of contradiction that $\int_a^b f \neq \Omega$. We divide into two cases ($\int_a^b f < \Omega$ and $\int_a^b f > \Omega$). If $\int_a^b f < \Omega$, then by Definition 13.16, $U(f) = \int_a^b f < \Omega$. It follows by Definition 13.14 and 5.11 that there exists an object $U(f,P) \in \{U(f,P) \mid P \text{ is a partition of } [a,b]\}$ such that $U(f) \leq U(f,P) < \Omega$. But this contradicts the hypothesis that $U(f,P) \geq \Omega$ for all partitions P of $[a,b]$. The argument is symmetric in the other case. □

5/6: **Theorem 14.4** (Second Fundamental Theorem of Calculus — Integral of Derivatives)**.** *Let f be integrable on $[a,b]$. Suppose that there is a function G that is continuous on $[a,b]$ and differentiable on (a,b) and such that $f = G'$ on (a,b). Then*

$$\int_a^b f = G(b) - G(a)$$

Proof. To prove that $\int_a^b f = G(b) - G(a)$, Lemma 14.3 tells us that it will suffice to show that for all partitions $P = \{t_0, \dots, t_n\}$ of $[a,b]$, $L(f,P) \leq G(b) - G(a) \leq U(f,P)$. Let P be an arbitrary partition of $[a,b]$. If t_i, t_{i-1} are two sequential elements of P, then since G is continuous on $[t_{i-1}, t_i] \subset [a,b]$ and differentiable on $(t_{i-1}, t_i) \subset (a,b)$, Corollary 12.16 asserts that there exists a point $\lambda \in (t_{i-1}, t_i)$ such that $G(t_i) - G(t_{i-1}) = G'(\lambda)(t_i - t_{i-1})$. It follows since $f = G'$ on (a,b) that

$$G(t_i) - G(t_{i-1}) = f(\lambda)(t_i - t_{i-1})$$

Thus, since we have proven the above statement for an arbitrary i, we can apply it to all i and sum to get

$$
\begin{aligned}
\sum_{i=1}^n f(\lambda)(t_i - t_{i-1}) &= \sum_{i=1}^n G(t_i) - G(t_{i-1}) \\
&= G(b) - G(a)
\end{aligned}
$$

But by Definitions 13.11, 5.7, and 5.6, we have that $m_i(f) \leq f(\lambda) \leq M_i(f)$ for all i. It follows that

$$\sum_{i=1}^n m_i(f)(t_i - t_{i-1}) \leq \sum_{i=1}^n f(\lambda)(t_i - t_{i-1}) \leq \sum_{i=1}^n M_i(f)(t_i - t_{i-1})$$

Therefore, we have by Definition 13.11 and substitution that

$$L(f,P) \leq G(b) - G(a) \leq U(f,P)$$

as desired. □

Corollary 14.5 (Integration by Parts). *Let f, g be functions defined on some open interval containing $[a, b]$ such that f' and g' exist and are continuous on $[a, b]$. Then*

$$\int_a^b fg' = [f(b)g(b) - f(a)g(a)] - \int_a^b f'g$$

Proof. Since f and g are differentiable on $[a, b]$, Exercise 12.9 implies that fg is differentiable on $[a, b]$ with $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ for all $x \in [a, b]$. We now seek to prove that $f'g + fg'$ is integrable on $[a, b]$. By hypothesis, f' and g' are continuous on $[a, b]$. Additionally, since f and g are differentiable on $[a, b]$, Theorem 12.5 asserts that they are continuous on $[a, b]$. Thus, since f, g, f', and g' are continuous on $[a, b]$, we have by consecutive applications of Corollary 11.10 that $f'g + fg'$ is continuous on $[a, b]$. Consequently, by Theorem 13.19, $f'g + fg'$ is integrable on $[a, b]$, as desired. Furthermore, in a similar manner to the above, we can show that $f'g$ and fg' are integrable on $[a, b]$. Lastly, it follows from the fact that f and g are continuous on $[a, b]$ by Corollary 11.10 that fg is continuous on $[a, b]$.

Having established that $f'g + fg'$ is integrable on $[a, b]$, that fg is a function that is continuous on $[a, b]$, differentiable on $(a, b) \subset [a, b]$, and such that $f'g + fg' = (fg)'$ on (a, b), and that $f'g$ and fg' are integrable on $[a, b]$, we have that

$$\begin{aligned}
\int_a^b (f'g + fg') &= (fg)(b) - (fg)(a) && \text{Theorem 14.4}\\
\int_a^b f'g + \int_a^b fg' &= f(b)g(b) - f(a)g(a) && \text{Theorem 13.24}\\
\int_a^b fg' &= [f(b)g(b) - f(a)g(a)] - \int_a^b f'g
\end{aligned}$$

as desired. ☐

Corollary 14.6 (Substitution). *Let g be a function defined on some interval containing $[a, b]$ such that g' is continuous on $[a, b]$. Suppose that $g([a, b]) \subset [c, d]$ and $f : [c, d] \to \mathbb{R}$ is continuous. Define $F : [c, d] \to \mathbb{R}$ by $F(x) = \int_c^x f$. Then*

$$\int_a^b f(g(x)) \cdot g'(x) \, \mathrm{d}x = F(g(b)) - F(g(a))$$

Proof. To prove that $\int_a^b f(g(x)) \cdot g'(x) \, \mathrm{d}x = F(g(b)) - F(g(a))$, Theorem 14.4 tells us that it will suffice to show that $(f \circ g) \cdot g'$ is integrable on $[a, b]$, $F \circ g$ is continuous on $[a, b]$ and differentiable on (a, b), and $(f \circ g) \cdot g' = (F \circ g)'$ on (a, b). We will confirm each requirement in turn. Let's begin.

To confirm that $(f \circ g) \cdot g'$ is integrable on $[a, b]$, Theorem 13.19 tells us that it will suffice to demonstrate that $(f \circ g) \cdot g'$ is continuous on $[a, b]$. By hypothesis, f is continuous on $[c, d]$. Additionally, since g' is defined on $[a, b]$, we know that g is differentiable on $[a, b]$, which implies by Theorem 12.5 that g is continuous on $[a, b]$. The combination of the previous two results implies by Corollary 11.15 that $f \circ g$ is continuous on $[a, b]$. This combined with the hypothesis that g' is continuous on $[a, b]$ implies by Corollary 11.10 that $(f \circ g) \cdot g'$ is continuous on $[a, b]$.

To confirm that $F \circ g$ is continuous on $[a, b]$, Corollary 11.15 tells us that it will suffice to demonstrate that F is continuous on $[c, d]$ and g is continuous on $[a, b]$. By Theorem 13.28, F is continuous on $[c, d]$. Additionally, we know by the above that g is continuous on $[a, b]$.

To confirm that $F \circ g$ is differentiable on (a, b), Theorem 12.10 tells us that it will suffice to demonstrate that F is differentiable on (c, d) and g is differentiable on (a, b). Since f is continuous on $(c, d) \subset [c, d]$, we have by Theorem 14.1 that F is differentiable on (c, d). Additionally, we know by the above that g is differentiable on $(a, b) \subset [a, b]$.

Since $F \circ g$ is differentiable on (a, b), we have by Theorem 12.10 again that $(F \circ g)' = (F' \circ g) \cdot g'$ for all $x \in (a, b)$. Thus, since $F' = f$ by Theorem 14.1, we have that $(f \circ g) \cdot g' = (F \circ g)'$ on (a, b), as desired. ☐