

Script 16

Series

16.1 Journal

5/20: **Definition 16.1.** Let $N_0 \in \mathbb{N} \cup \{0\}$ and let $(a_n)_{n=N_0}^\infty$ be a sequence of real numbers. Then the formal sum

$$\sum_{n=N_0}^{\infty} a_n$$

is called an **infinite series**. (In most instances, we will start the series at $N_0 = 0$ or $N_0 = 1$.)

We will define the **sequence of partial sums** (p_n) of the series by

$$p_n = a_{N_0} + \cdots + a_{N_0+n-1} = \sum_{i=N_0}^{N_0+n-1} a_i$$

Thus, p_n is the sum of the first n terms in the sequence (a_n) . We say that the series **converges** if there exists $L \in \mathbb{R}$ such that $\lim_{n \rightarrow \infty} p_n = L$. When this is the case, we write this as

$$\sum_{n=N_0}^{\infty} a_n = L$$

and we say that L is the **sum** of the series. When there does not exist such an L , we say that the series **diverges**.

Lemma 16.2. Let $(a_n)_{n=0}^\infty$ be a sequence of real numbers. Let $N_0 \in \mathbb{N}$. Then $\sum_{n=0}^\infty a_n$ converges if and only if $\sum_{n=N_0}^\infty a_n$ converges.

Lemma. Let $n \in \mathbb{N}$. Then

$$\sum_{i=0}^{N_0+n-1} a_i = \sum_{i=0}^{N_0-1} a_i + \sum_{i=N_0}^{N_0+n-1} a_i$$

Proof. This simple result follows immediately from Script 0, so no formal proof will be given. \square

Proof of Lemma 16.2. Suppose first that $\sum_{n=0}^\infty a_n$ converges, and let $M = \sum_{n=0}^\infty a_n = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} a_i$, where the latter equality holds by Definition 16.1. To prove that $\sum_{n=N_0}^\infty a_n$ converges, Definition 16.1 tells us that it will suffice to find an $L \in \mathbb{R}$ such that $\lim_{n \rightarrow \infty} \sum_{i=N_0}^{N_0+n-1} a_i = L$. Choose $L = M - \sum_{i=0}^{N_0-1} a_i$. To verify that $\lim_{n \rightarrow \infty} \sum_{i=N_0}^{N_0+n-1} a_i = L$, Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|\sum_{i=N_0}^{N_0+n-1} a_i - L| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} a_i = M$, Theorem 15.7 implies that there is some $N \in \mathbb{N}$ such that for all $n \geq N$,

$|\sum_{i=0}^{n-1} a_i - M| < \epsilon$. Choose this N to be our N . Let n be an arbitrary natural number such that $n \geq N$. Since $N_0 + n > n \geq N$, we have by the above that $|\sum_{i=0}^{N_0+n-1} a_i - M| < \epsilon$. Therefore,

$$\begin{aligned} \left| \sum_{i=N_0}^{N_0+n-1} a_i - L \right| &= \left| \sum_{i=0}^{N_0+n-1} a_i - \sum_{i=0}^{N_0-1} a_i - L \right| && \text{The Lemma} \\ &= \left| \sum_{i=0}^{N_0+n-1} a_i - \left(\sum_{i=0}^{N_0-1} a_i + L \right) \right| \\ &= \left| \sum_{i=0}^{N_0+n-1} a_i - M \right| \\ &< \epsilon \end{aligned}$$

as desired.

The proof is symmetric in the other direction. □

Exercise 16.3. Prove that $\sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{n+1})$ converges. What is its sum?

Proof. Let (a_n) be defined by $a_n = \frac{1}{n} - \frac{1}{n+1}$, and let (p_n) be defined by $p_n = \sum_{i=1}^n a_i$. Then

$$\begin{aligned} p_n &= a_1 + a_2 + \cdots + a_n \\ &= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= \frac{1}{1} - \frac{1}{n+1} \end{aligned}$$

To prove that $\sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{n+1}) = 1$, Definition 16.1 tells us that it will suffice to show that $\lim_{n \rightarrow \infty} p_n = 1$. By a proof symmetric to that of Exercise 15.6a, we have that $\lim_{n \rightarrow \infty} 1 = 1$. By a proof symmetric to that of Exercise 15.6c, we have that $\lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$. Therefore, by Theorem 15.9 and the above, we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} p_n &= \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1} \right) \\ &= \lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} \frac{1}{n+1} \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

as desired. □

Theorem 16.4. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n \rightarrow \infty} a_n = 0$.

Proof. To prove that $\lim_{n \rightarrow \infty} a_n = 0$, Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - 0| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\sum_{n=1}^{\infty} a_n$ converges, we have by Theorem 15.19 that there exists an $N \in \mathbb{N}$ such that $|\sum_{i=1}^n a_i - \sum_{i=1}^m a_i| < \epsilon$ for all $n, m \geq N$. Choose this N to be our N . Let n be an arbitrary natural number such that $n \geq N$. Then choosing $n, n-1 \geq N$, we have by the above that $|\sum_{i=1}^n a_i - \sum_{i=1}^{n-1} a_i| < \epsilon$. Therefore,

$$\begin{aligned} |a_n - 0| &= |a_n| \\ &= \left| \sum_{i=1}^n a_i - \sum_{i=1}^{n-1} a_i \right| \\ &< \epsilon \end{aligned}$$

as desired. □

The converse of this theorem, however, is not true, as we see in Theorem 16.6.

Theorem 16.5. *A series $\sum_{n=1}^{\infty} a_n$ converges if and only if for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|\sum_{k=m+1}^n a_k| < \epsilon$ for all $n > m \geq N$.*

Proof. Suppose first that $\sum_{n=1}^{\infty} a_n$ converges. Let $\epsilon > 0$ be arbitrary. By Definition 16.1, (p_n) converges. Thus, by Theorem 15.19, there is some $N \in \mathbb{N}$ such that $|p_n - p_m| < \epsilon$ for all $n, m \geq N$. Choose this N to be our N . Let n, m be two arbitrary natural numbers satisfying $n > m \geq N$. Therefore,

$$\begin{aligned} \left| \sum_{k=m+1}^n a_k \right| &= \left| \sum_{k=1}^n a_k - \sum_{k=1}^m a_k \right| \\ &= |p_n - p_m| \\ &< \epsilon \end{aligned}$$

as desired.

The proof is symmetric in the other direction. □

Theorem 16.6. *The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.*

Lemma. *For all $N \in \mathbb{N}$, we have*

$$\sum_{n=N+1}^{2N} \frac{1}{n} \geq \frac{1}{2}$$

Proof. We induct on N . For the base case $N = 1$, we have

$$\sum_{n=1+1}^{2 \cdot 1} \frac{1}{n} = \frac{1}{2} \geq \frac{1}{2}$$

as desired. Now suppose inductively that we have proven the claim for N . To prove it for $N + 1$, we do the following.

$$\begin{aligned} \sum_{n=N+2}^{2N+2} \frac{1}{n} &= \sum_{n=N+1}^{2N} \frac{1}{n} - \frac{1}{N+1} + \frac{1}{2N+1} + \frac{1}{2(N+1)} \\ &= \sum_{n=N+1}^{2N} \frac{1}{n} + \frac{1}{2(N+1)(2N+1)} \\ &> \sum_{n=N+1}^{2N} \frac{1}{n} \\ &\geq \frac{1}{2} \end{aligned}$$

as desired. □

Proof of Theorem 16.6. To prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, Theorem 16.5 tells us that it will suffice to find an $\epsilon > 0$ such that for all $N \in \mathbb{N}$, there exist $n > m \geq N$ with $|\sum_{k=m+1}^n 1/k| \geq \epsilon$. Choose $\epsilon = \frac{1}{2}$. Let N be an arbitrary element of \mathbb{N} . If we now choose $n = 2N$ and $m = N$, we will have $n > m \geq N$. It will follow by the lemma that

$$\begin{aligned} \left| \sum_{k=m+1}^n \frac{1}{k} \right| &= \left| \sum_{k=N+1}^{2N} \frac{1}{k} \right| \\ &\geq \frac{1}{2} \\ &= \epsilon \end{aligned}$$

as desired. □

5/25: **Theorem 16.7.** Let $-1 < x < 1$. Then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

Proof. Let (a_n) be defined by $a_n = x^n$. Then $p_n = x^0 + x^1 + \cdots + x^{n-1}$ so that

$$\begin{aligned} p_n - xp_n &= x^0 + \cdots + x^{n-1} - x(x^0 + \cdots + x^{n-1}) \\ p_n - xp_n &= 1 - x^n \\ p_n(1-x) &= 1 - x^n \\ p_n &= \frac{1-x^n}{1-x} \end{aligned}$$

Therefore, we have that

$$\begin{aligned} \frac{1}{1-x} &= \frac{1-0}{1-x} \\ &= \frac{1 - \lim_{n \rightarrow \infty} x^n}{1-x} && \text{Exercise 15.8b} \\ &= \lim_{n \rightarrow \infty} \frac{1-x^n}{1-x} && \text{Theorem 15.9} \\ &= \lim_{n \rightarrow \infty} p_n \\ &= \sum_{n=0}^{\infty} x^n && \text{Definition 16.1} \end{aligned}$$

as desired. □

Theorem 16.8. If $\sum_{n=1}^{\infty} a_n = L$, $\sum_{n=1}^{\infty} b_n = M$, and $c \in \mathbb{R}$, then

$$\begin{aligned} \sum_{n=1}^{\infty} (a_n + b_n) &= L + M \\ \sum_{n=1}^{\infty} (c \cdot a_n) &= c \cdot L \end{aligned}$$

Proof. For the first claim, we have that

$$\begin{aligned} L + M &= \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i + \lim_{n \rightarrow \infty} \sum_{i=1}^n b_i && \text{Definition 16.1} \\ &= \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n a_i + \sum_{i=1}^n b_i \right) && \text{Theorem 15.9} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n (a_i + b_i) \\ &= \sum_{n=1}^{\infty} (a_n + b_n) && \text{Definition 16.1} \end{aligned}$$

The proof is symmetric for the second claim. □

Definition 16.9. We say that the series $\sum_{n=1}^{\infty} a_n$ **converges absolutely** if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

Lemma 16.10. *A series $\sum_{n=1}^{\infty} a_n$ with all $a_n \geq 0$ converges if and only if its sequence of partial sums is bounded.*

Proof. Suppose first that the series $\sum_{n=1}^{\infty} a_n$ with all $a_n \geq 0$ converges. Then by Definition 16.1 its sequence of partial sums (p_n) converges. Therefore, by Theorem 15.13, (p_n) is bounded, as desired.

Now suppose that the sequence of partial sums (p_n) corresponding to a series $\sum_{n=1}^{\infty} a_n$ with all $a_n \geq 0$ is bounded. To prove that $\sum_{n=1}^{\infty} a_n$ converges, Definition 16.1 tells us that it will suffice to show that (p_n) converges. To do so, Theorem 15.14 tells us that it will suffice to verify in addition to the fact that (p_n) is bounded that (p_n) is increasing. To do this, Script 15 tells us that it will suffice to confirm that $p_n \leq p_{n+1}$ for all $n \in \mathbb{N}$. Let n be an arbitrary natural number. By Definition 16.1, $p_{n+1} - p_n = a_{n+1}$. Since $a_{n+1} \geq 0$ by hypothesis, we have by transitivity that $p_{n+1} - p_n \geq 0$, i.e., $p_n \leq p_{n+1}$ by Definition 7.21, as desired. \square

Theorem 16.11. *If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges and*

$$\left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} |a_n|$$

Proof. To prove that $\sum_{n=1}^{\infty} a_n$ converges, Theorem 16.5 tells us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|\sum_{k=m+1}^n a_k| < \epsilon$ for all $n > m \geq N$. Let $\epsilon > 0$ be arbitrary. Since $\sum_{n=1}^{\infty} a_n$ converges absolutely by hypothesis, we have by Definition 16.9 that $\sum_{n=1}^{\infty} |a_n|$ converges. Thus, by Theorem 16.5, there is some $N \in \mathbb{N}$ such that $|\sum_{k=m+1}^n |a_k|| < \epsilon$ for all $n > m \geq N$. Choose this N to be our N . Let n, m be arbitrary natural numbers such that $n > m \geq N$. Therefore,

$$\begin{aligned} \left| \sum_{k=m+1}^n a_k \right| &\leq \sum_{k=m+1}^n |a_k| && \text{Lemma 8.8} \\ &= \left| \sum_{k=m+1}^n |a_k| \right| \\ &< \epsilon \end{aligned}$$

as desired.

As to the other part of the claim, to begin, let (b_n) and (c_n) be defined by $b_n = \max(0, a_n)$ and $c_n = \min(0, a_n)$. We will prove a few preliminary results with these definitions that will enable us to tackle the big inequality.

To confirm that $a_n = b_n + c_n$, we divide into two cases ($a_n \geq 0$ and $a_n < 0$). If $a_n \geq 0$, then by their definitions, $b_n = a_n$ and $c_n = 0$. Thus, $a_n = b_n + c_n$ as desired. The argument is symmetric in the other case.

To confirm that $|\sum_{n=1}^{\infty} b_n| + |\sum_{n=1}^{\infty} -c_n| = |\sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} -c_n|$, we can acknowledge that $b_n \geq 0$ and $-c_n \geq 0$ for all $n \in \mathbb{N}$ to demonstrate that

$$\begin{aligned} \left| \sum_{n=1}^{\infty} b_n \right| + \left| \sum_{n=1}^{\infty} -c_n \right| &= \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} -c_n \\ &= \left| \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} -c_n \right| \end{aligned}$$

To confirm that $|a_n| = b_n - c_n$, we divide into two cases ($a_n \geq 0$ and $a_n < 0$). If $a_n \geq 0$, then $b_n = a_n$ and $c_n = 0$. Thus, by Definition 8.4, $|a_n| = a_n = b_n - c_n$, as desired. On the other hand, if $a_n < 0$, then $b_n = 0$ and $c_n = a_n$. Thus, by Definition 8.4 again, $|a_n| = -a_n = b_n - c_n$, as desired.

Having established that $a_n = b_n + c_n$, $|\sum_{n=1}^{\infty} b_n| + |\sum_{n=1}^{\infty} -c_n| = |\sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} -c_n|$, and $|a_n| =$

$b_n - c_n$, we have that

$$\begin{aligned}
 \left| \sum_{n=1}^{\infty} a_n \right| &= \left| \sum_{n=1}^{\infty} (b_n + c_n) \right| && \text{Theorem 16.8} \\
 &= \left| \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} c_n \right| \\
 &\leq \left| \sum_{n=1}^{\infty} b_n \right| + \left| \sum_{n=1}^{\infty} c_n \right| && \text{Lemma 8.8} \\
 &= \left| \sum_{n=1}^{\infty} b_n \right| + \left| - \sum_{n=1}^{\infty} c_n \right| && \text{Exercise 8.5} \\
 &= \left| \sum_{n=1}^{\infty} b_n \right| + \left| \sum_{n=1}^{\infty} -c_n \right| && \text{Theorem 16.8} \\
 &= \left| \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} -c_n \right| \\
 &= \left| \sum_{n=1}^{\infty} (b_n - c_n) \right| && \text{Theorem 16.8} \\
 &= \left| \sum_{n=1}^{\infty} |a_n| \right| \\
 &= \sum_{n=1}^{\infty} |a_n|
 \end{aligned}$$

as desired. \square

Theorem 16.12. Let (a_n) be a decreasing sequence of positive numbers such that $\lim_{n \rightarrow \infty} a_n = 0$. Then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Lemma.

(a) For any natural numbers n, m satisfying $n > m$, we have

$$\left| \sum_{k=m+1}^n (-1)^{k+1} a_k \right| = |a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + a_{m+5} - \cdots \pm a_n|$$

Proof. By Script 0, either $|\sum_{k=m+1}^n (-1)^{k+1} a_k| = |a_{m+1} - a_{m+2} + a_{m+3} - \cdots \pm a_n|$ or $|\sum_{k=m+1}^n (-1)^{k+1} a_k| = |-a_{m+1} + a_{m+2} - a_{m+3} + \cdots \pm a_n|$. However, by Exercise 8.5, the two results are equal. Thus, we may choose the former WLOG. \square

(b) We have

$$0 \leq a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots \pm a_n$$

Proof. Since (a_n) is a decreasing sequence, we have by Script 15 that $a_{i+1} \leq a_i$ for all $i \in \mathbb{N}$. It follows by Definition 7.21 that $0 \leq a_i - a_{i+1}$ for all $i \in \mathbb{N}$. We now divide into two cases (there are an even number of terms in the sum $a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + a_{m+5} - \cdots \pm a_n$ and there are an odd number of terms in said sum). In the first case, we have that the sum is of the form $a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots + a_{n-1} - a_n$. Thus, since $0 \leq a_{m+1} - a_{m+2}$, $0 \leq a_{m+3} - a_{m+4}$, and on and on, we have by Script 7 that $0 \leq a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots - a_n$, as desired. On the other hand, in the second case, we have that the sum is of the form $a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots + a_{n-2} - a_{n-1} + a_n$. For the same reason as before, we have that $0 \leq a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots + a_{n-2} - a_{n-1}$. However, we need the additional hypothesis that every a_i is positive, i.e., $a_i \geq 0$ for all $i \in \mathbb{N}$ to know that $0 \leq a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots + a_n$, as desired. \square

(c) For all $i \in \mathbb{N}$, we have

$$-a_i + a_{i+1} \leq 0$$

Proof. Since (a_n) is a decreasing sequence, we have by Script 15 that $a_{i+1} \leq a_i$ for all $i \in \mathbb{N}$. It follows by Definition 7.21 that $-a_i + a_{i+1} \leq 0$ for all $i \in \mathbb{N}$. \square

Proof of Theorem 16.12. To prove that $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges, Theorem 16.5 tells us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|\sum_{k=m+1}^n (-1)^{k+1} a_k| < \epsilon$ for all $n > m \geq N$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{n \rightarrow \infty} a_n = 0$ by hypothesis, we have by Theorem 15.7 that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - 0| = |a_n| < \epsilon$, as desired. Choose this N to be our N . Let n, m be arbitrary natural numbers such that $n > m \geq N$.

We divide into two cases ($n-m$ is even [i.e., there are an even number of terms in the sum $\sum_{k=m+1}^n (-1)^{k+1} a_k$] and $n-m$ is odd [i.e., there are an odd number of terms in the sum $\sum_{k=m+1}^n (-1)^{k+1} a_k$]). If $n-m$ is even, then we have

$$\begin{aligned} \left| \sum_{k=m+1}^n (-1)^{k+1} a_k \right| &= |a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + a_{m+5} - \cdots - a_{n-2} + a_{n-1} - a_n| && \text{Lemma (a)} \\ &= a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + a_{m+5} - \cdots - a_{n-2} + a_{n-1} - a_n && \text{Lemma (b) \& Definition 8.4} \\ &= a_{m+1} + (-a_{m+2} + a_{m+3}) + (-a_{m+4} + a_{m+5}) + \cdots + (-a_{n-2} + a_{n-1}) - a_n \\ &\leq a_{m+1} + (-a_{m+4} + a_{m+5}) + \cdots + (-a_{n-2} + a_{n-1}) - a_n && \text{Lemma (c)} \\ &\leq a_{m+1} + \cdots + (-a_{n-2} + a_{n-1}) - a_n && \text{Lemma (c)} \\ &\vdots && \text{Lemma (c)} \\ &\leq a_{m+1} + (-a_{n-2} + a_{n-1}) - a_n && \text{Lemma (c)} \\ &\leq a_{m+1} - a_n && \text{Lemma (c)} \\ &\leq a_{m+1} \\ &= |a_{m+1}| \\ &< \epsilon \end{aligned}$$

The argument is symmetric if $n-m$ is odd. \square

5/27: The following theorem will be useful to prove more specialized tests for convergence of series.

Theorem 16.13. Let (c_n) be a sequence of positive numbers and let (a_n) be a sequence such that $|a_n| \leq c_n$ for all $n \geq N_0$, where N_0 is some fixed integer. If $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof. To prove that $\sum_{n=1}^{\infty} a_n$ converges, Theorem 16.11 and Definition 16.9 tell us that it will suffice to show that $\sum_{n=1}^{\infty} |a_n|$ converges. To do this, Theorem 16.5 tells us that it will suffice to verify that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|\sum_{k=m+1}^n |a_k|| < \epsilon$ for all $n > m \geq N$. Let $\epsilon > 0$ be arbitrary. Since $\sum_{n=1}^{\infty} c_n$ converges by hypothesis, we have by Theorem 16.5 that there is some $N_1 \in \mathbb{N}$ such that $|\sum_{k=m+1}^n c_k| < \epsilon$ for all $n > m \geq N_1$. Choose $N = \max(N_0, N_1)$. Let n, m be arbitrary natural numbers such that $n > m \geq N$. Since $c_k \geq 0$ for all $k \in \mathbb{N}$ by hypothesis, it follows by Script 7 that $\sum_{k=m+1}^n c_k \geq 0$. Thus, Definition 8.4 implies that $\sum_{k=m+1}^n c_k = |\sum_{k=m+1}^n c_k| < \epsilon$. Similarly, we have that $|\sum_{k=m+1}^n |a_k|| = \sum_{k=m+1}^n |a_k|$. Lastly, since we know by hypothesis that $|a_k| \leq c_k$ for all $k \geq N_0$, i.e., for all $k \geq N$, Script 7 asserts that $\sum_{k=m+1}^n |a_k| \leq \sum_{k=m+1}^n c_n$. Therefore, combining the last three results, we have that $|\sum_{k=m+1}^n |a_k|| = \sum_{k=m+1}^n |a_k| \leq \sum_{k=m+1}^n c_n < \epsilon$, as desired. \square

Lemma 16.14. Suppose that (b_n) is a sequence of nonnegative numbers with $\lim_{n \rightarrow \infty} b_n = L$, where $L < 1$. Then there is some $N \in \mathbb{N}$ such that $0 \leq b_n < \frac{1+L}{2}$ for all $n \geq N$.

Proof. Suppose for the sake of contradiction that $L < 0$. By Exercise 6.5, choose $a < L$. Since $a < L < 0$, Equations 8.1 allow us to define the L -containing region $(a, 0)$. Thus, since $(a, 0)$ is an open interval by Corollary 4.11 and Lemma 8.3, respectively, we have by Definition 15.2 that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $n \in (a, 0)$. Specifically, this implies that $b_N \in (a, 0)$. But by Equations 8.1, this means that $b_N < 0$, contradicting the hypothesis that all b_n are nonnegative.

Additionally, it will be useful later on to know that $L < \frac{1+L}{2}$, as we can demonstrate with the following.

$$\begin{aligned} L &= 2 \cdot \frac{L}{2} \\ &= \frac{L+L}{2} \\ &< \frac{1+L}{2} \end{aligned}$$

Having established that $0 \leq L$ and that $L < \frac{1+L}{2}$, we know that implies that $L \in R$, where $R = (-1, \frac{1+L}{2})$. Additionally, by Corollary 4.11 and Lemma 8.3, R is an open interval. It follows by Definition 15.2 that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $b_n \in R$. Choose this N to be our N . Let b_n be an arbitrary element of the sequence with $n \geq N$. Since b_n is nonnegative, we know that $0 \leq b_n$. Additionally, since $b_n \in R$, we have by Equations 8.1 that $b_n < \frac{1+L}{2}$. Therefore, we have that $0 \leq b_n < \frac{1+L}{2}$, as desired. \square