Script 16

Series

16.1 Journal

5/20: **Definition 16.1.** Let $N_0 \in \mathbb{N} \cup \{0\}$ and let $(a_n)_{n=N_0}^{\infty}$ be a sequence of real numbers. Then the formal sum

$$\sum_{n=N_0}^{\infty} a_n$$

is called an **infinite series**. (In most instances, we will start the series at $N_0 = 0$ or $N_0 = 1$.)

We will define the **sequence of partial sums** (p_n) of the series by

$$p_n = a_{N_0} + \cdots + a_{N_0+n-1} = \sum_{i=N_0}^{N_0+n-1} a_i$$

Thus, p_n is the sum of the first n terms in the sequence (a_n). We say that the series **converges** if there exists $L \in \mathbb{R}$ such that $\lim_{n\to\infty} p_n = L$. When this is the case, we write this as

$$\sum_{n=N_0}^{\infty} a_n = L$$

and we say that L is the **sum** of the series. When there does not exist such an L, we say that the series **diverges**.

Lemma 16.2. *Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers. Let $N_0 \in \mathbb{N}$. Then $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=N_0}^{\infty} a_n$ converges.*

Lemma. *Let $n \in \mathbb{N}$. Then*

$$\sum_{i=0}^{N_0+n-1} a_i = \sum_{i=0}^{N_0-1} a_i + \sum_{i=N_0}^{N_0+n-1} a_i$$

Proof. This simple result follows immediately from Script 0, so no formal proof will be given. $\square$

Proof of Lemma 16.2. Suppose first that $\sum_{n=0}^{\infty} a_n$ converges, and let $M = \sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=0}^{n-1} a_i$, where the latter equality holds by Definition 16.1. To prove that $\sum_{n=N_0}^{\infty} a_n$ converges, Definition 16.1 tells us that it will suffice to find an $L \in \mathbb{R}$ such that $\lim_{n\to\infty} \sum_{i=N_0}^{N_0+n-1} a_i = L$. Choose $L = M - \sum_{i=0}^{N_0-1} a_i$. To verify that $\lim_{n\to\infty} \sum_{i=N_0}^{N_0+n-1} a_i = L$, Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|\sum_{i=N_0}^{N_0+n-1} a_i - L| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} \sum_{i=0}^{n-1} a_i = M$, Theorem 15.7 implies that there is some $N \in \mathbb{N}$ such that for all $n \geq N$,

$|\sum_{i=0}^{n-1} a_i - M| < \epsilon$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. Since $N_0 + n > n \geq N$, we have by the above that $|\sum_{i=0}^{N_0+n-1} a_i - M| < \epsilon$. Therefore,

$$\begin{aligned}
\left|\sum_{i=N_0}^{N_0+n-1} a_i - L\right| &= \left|\sum_{i=0}^{N_0+n-1} a_i - \sum_{i=0}^{N_0-1} a_i - L\right| && \text{The Lemma} \\
&= \left|\sum_{i=0}^{N_0+n-1} a_i - \left(\sum_{i=0}^{N_0-1} a_i + L\right)\right| \\
&= \left|\sum_{i=0}^{N_0+n-1} a_i - M\right| \\
&< \epsilon
\end{aligned}$$

as desired.

The proof is symmetric in the other direction. □

Exercise 16.3. Prove that $\sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+1})$ converges. What is its sum?

Proof. Let (a_n) be defined by $a_n = \frac{1}{n} - \frac{1}{n+1}$, and let (p_n) be defined by $p_n = \sum_{i=1}^{n} a_i$. Then

$$\begin{aligned}
p_n &= a_1 + a_2 + \cdots + a_n \\
&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\
&= \frac{1}{1} - \frac{1}{n+1}
\end{aligned}$$

To prove that $\sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+1}) = 1$, Definition 16.1 tells us that it will suffice to show that $\lim_{n\to\infty} p_n = 1$. By a proof symmetric to that of Exercise 15.6a, we have that $\lim_{n\to\infty} 1 = 1$. By a proof symmetric to that of Exercise 15.6c, we have that $\lim_{n\to\infty} \frac{1}{n+1} = 0$. Therefore, by Theorem 15.9 and the above, we have that

$$\begin{aligned}
\lim_{n\to\infty} p_n &= \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) \\
&= \lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n+1} \\
&= 1 - 0 \\
&= 1
\end{aligned}$$

as desired. □

Theorem 16.4. *If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.*

Proof. To prove that $\lim_{n\to\infty} a_n = 0$, Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - 0| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\sum_{n=1}^{\infty} a_n$ converges, we have by Theorem 15.19 that there exists an $N \in \mathbb{N}$ such that $|\sum_{i=1}^{n} a_i - \sum_{i=1}^{m} a_i| < \epsilon$ for all $n, m \geq N$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. Then choosing $n, n-1 \geq N$, we have by the above that $|\sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} a_i| < \epsilon$. Therefore,

$$\begin{aligned}
|a_n - 0| &= |a_n| \\
&= \left|\sum_{i=1}^{n} a_i - \sum_{i=1}^{n-1} a_i\right| \\
&< \epsilon
\end{aligned}$$

as desired. □

The converse of this theorem, however, is not true, as we see in Theorem 16.6.

Theorem 16.5. *A series $\sum_{n=1}^{\infty} a_n$ converges if and only if for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|\sum_{k=m+1}^{n} a_k| < \epsilon$ for all $n > m \geq N$.*

Proof. Suppose first that $\sum_{n=1}^{\infty} a_n$ converges. Let $\epsilon > 0$ be arbitrary. By Definition 16.1, (p_n) converges. Thus, by Theorem 15.19, there is some $N \in \mathbb{N}$ such that $|p_n - p_m| < \epsilon$ for all $n, m \geq N$. Choose this N to be our N. Let n, m be two arbitrary natural numbers satisfying $n > m \geq N$. Therefore,

$$\begin{aligned} \left| \sum_{k=m+1}^{n} a_k \right| &= \left| \sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k \right| \\ &= |p_n - p_m| \\ &< \epsilon \end{aligned}$$

as desired.

The proof is symmetric in the other direction. □

Theorem 16.6. *The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.*

Lemma. *For all $N \in \mathbb{N}$, we have*

$$\sum_{n=N+1}^{2N} \frac{1}{n} \geq \frac{1}{2}$$

Proof. We induct on N. For the base case $N = 1$, we have

$$\sum_{n=1+1}^{2\cdot 1} \frac{1}{n} = \frac{1}{2} \geq \frac{1}{2}$$

as desired. Now suppose inductively that we have proven the claim for N. To prove it for $N + 1$, we do the following.

$$\begin{aligned} \sum_{n=N+2}^{2N+2} \frac{1}{n} &= \sum_{n=N+1}^{2N} \frac{1}{n} - \frac{1}{N+1} + \frac{1}{2N+1} + \frac{1}{2(N+1)} \\ &= \sum_{n=N+1}^{2N} \frac{1}{n} + \frac{1}{2(N+1)(2N+1)} \\ &> \sum_{n=N+1}^{2N} \frac{1}{n} \\ &\geq \frac{1}{2} \end{aligned}$$

as desired. □

Proof of Theorem 16.6. To prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, Theorem 16.5 tells us that it will suffice to find an $\epsilon > 0$ such that for all $N \in \mathbb{N}$, there exist $n > m \geq N$ with $|\sum_{k=m+1}^{n} 1/k| \geq \epsilon$. Choose $\epsilon = \frac{1}{2}$. Let N be an arbitrary element of N. If we now choose $n = 2N$ and $m = N$, we will have $n > m \geq N$. It will follow by the lemma that

$$\begin{aligned} \left| \sum_{k=m+1}^{n} \frac{1}{k} \right| &= \left| \sum_{k=N+1}^{2N} \frac{1}{k} \right| \\ &\geq \frac{1}{2} \\ &= \epsilon \end{aligned}$$

as desired. □