

Script 13

Uniform Continuity and Integration

13.1 Journal

4/8: **Definition 13.1.** Let $f : A \rightarrow \mathbb{R}$ be a function. We say that f is **uniformly continuous** if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$.

Theorem 13.2. *If f is uniformly continuous, then f is continuous.*

Proof. To prove that f is continuous, Theorem 9.10 tells us that it will suffice to show that f is continuous at every $x \in A$. Let x be an arbitrary element of A . To show that f is continuous at x , Theorem 11.5 tells us that it will suffice to verify that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Then since f is uniformly continuous by hypothesis, Definition 13.1 asserts that there exists a $\delta > 0$ such that for all $y \in A$ satisfying $|y - x| < \delta$, we have $|f(y) - f(x)| < \epsilon$, as desired. \square

Exercise 13.3. Determine with proof whether each function f is uniformly continuous on the given interval A .

- (a) $f(x) = x^2$ on $A = \mathbb{R}$.

Proof. To prove that f is not uniformly continuous on \mathbb{R} , Definition 13.1 tells us that it will suffice to find an $\epsilon > 0$ for which no $\delta > 0$ exists such that for all $x, y \in \mathbb{R}$, if $|y - x| < \delta$, then $|y^2 - x^2| < \epsilon$. Let $\epsilon = 2$, and suppose for the sake of contradiction that $\delta > 0$ is a number such that for all $x, y \in \mathbb{R}$, if $|y - x| < \delta$, then $|y^2 - x^2| < 2$. By Theorem 5.2, there exists a number y such that $0 < y < \delta$. Since $-\delta < 0 < y < \delta$ by Lemma 7.23, it follows by Definitions 3.6 and 3.10 that $y \in (-\delta, \delta)$. Thus, by Exercise 8.9, $|y - 0| = |y| < \delta$. Consequently, $|(y + n) - n| < \delta$. It follows by the above that $|(y + n)^2 - n^2| = |y^2 + 2yn| < 2$. If we now let $n = \frac{1}{y}$, then $|y^2 + 2| < 2$. But since $y > 0$, we have that $y^2 > 0$ by Lemma 7.26. It follows that $y^2 + 2 > 2$ by Definition 7.21. Therefore, by Definition 8.4, we can also show that $|y^2 + 2| > 2$, a contradiction. \square

- (b) $f(x) = x^2$ on $A = (-2, 2)$.

Proof. To prove that f is uniformly continuous on A , Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \frac{\epsilon}{4}$, and let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. Since $x, y \in A$, consecutive applications of Equations 8.1 and the lemma from Exercise 8.9 imply that $|x| < 2$ and $|y| < 2$. It follows that $|x| + |y| < 2 + 2 = 4$. Consequently, by Lemma 8.8, $|x + y| < 4$. Additionally, since $0 \leq |y - x|$ by Definition 8.4, we have $|x - y| \cdot |x + y| \leq \frac{\epsilon}{4} \cdot |x + y|$. Combining all

of the above results, we have that

$$\begin{aligned}
 |f(y) - f(x)| &= |y^2 - x^2| \\
 &= |y + x| \cdot |y - x| \\
 &< 4 \cdot |y - x| \\
 &\leq 4 \cdot \frac{\epsilon}{4} \\
 &= \epsilon
 \end{aligned}$$

as desired. □

(c) $f(x) = \frac{1}{x}$ on $A = (0, +\infty)$.

Proof. To prove that f is not uniformly continuous on A , Definition 13.1 tells us that it will suffice to find an $\epsilon > 0$ for which no $\delta > 0$ exists such that for all $x, y \in A$, if $|y - x| < \delta$, then $|\frac{1}{y} - \frac{1}{x}| < \epsilon$. Let $\epsilon = 1$, and suppose for the sake of contradiction that $\delta > 0$ is a number such that for all $x, y \in A$, if $|y - x| < \delta$, then $|\frac{1}{y} - \frac{1}{x}| < 1$. As in part (a), choose $0 < x < \min(\delta, \frac{1}{2})$. Consequently, $|(x+x) - x| < \delta$. It follows by the above that $|\frac{1}{2x} - \frac{1}{x}| < 1$. But this implies that $|\frac{x-2x}{2x^2}| = |\frac{-1}{2x}| = \frac{1}{2x} < 1$. However, $x < \frac{1}{2}$ implies that $1 < \frac{1}{2x}$, a contradiction. □

(d) $f(x) = \frac{1}{x}$ on $A = [1, +\infty)$.

Proof. To prove that f is uniformly continuous on A , Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \epsilon$, and let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. Since $x, y \in A$, consecutive applications of Equations 8.1 imply that $1 \leq x$ and $1 \leq y$. It follows by Script 7 that $1 \leq |xy|$. This combined with the fact that $|y - x| < \delta = \epsilon$ implies that

$$\begin{aligned}
 |f(y) - f(x)| &= \left| \frac{1}{y} - \frac{1}{x} \right| \\
 &= \left| \frac{x - y}{xy} \right| \\
 &= \frac{|y - x|}{|xy|} \\
 &< \frac{\epsilon}{|xy|} \\
 &\leq \frac{\epsilon}{1} \\
 &= \epsilon
 \end{aligned}$$

as desired. □

(e) $f(x) = \sqrt{x}$ on $A = [1, +\infty)$.

Proof. To prove that f is uniformly continuous on A , Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \epsilon$, and let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. Since $x, y \in A$, consecutive applications of Equations 8.1 imply that $1 \leq x$ and $1 \leq y$. It follows by Script 7 that $1 \leq \sqrt{x}$ and $1 \leq \sqrt{y}$. Thus, by Script 7 again, $2 \leq |\sqrt{y} + \sqrt{x}|$. Note that it follows that $1 < |\sqrt{y} + \sqrt{x}|$. This combined with the fact that $|y - x| < \delta = \epsilon$ implies that

$$\begin{aligned}
 |f(y) - f(x)| &= |\sqrt{y} - \sqrt{x}| \\
 &< |\sqrt{y} - \sqrt{x}| \cdot |\sqrt{y} + \sqrt{x}| \\
 &= |y - x| \\
 &= \epsilon
 \end{aligned}$$

as desired. □

Exercise 13.4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^n$ for $n \in \mathbb{N}$. Show that f is uniformly continuous if and only if $n = 1$.

Proof. Suppose first that $n = 1$. To prove that f is uniformly continuous, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in \mathbb{R}$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \epsilon$. Now let x, y be arbitrary elements of \mathbb{R} that satisfy $|y - x| < \delta$. Then by the definition of f , $|f(y) - f(x)| = |y - x| < \delta = \epsilon$, as desired.

Now suppose that $n > 1$. Additionally, suppose for the sake of contradiction that f is uniformly continuous. Let $\epsilon = 1 > 0$. Then by Definition 13.1, there exists a $\delta > 0$ such that for all $x, y \in \mathbb{R}$, if $|y - x| < \delta$, then $|y^n - x^n| < 1$. Let $x = 0 \in \mathbb{R}$. By Theorem 5.2, there exists a point $y \in \mathbb{R}$ such that $0 < y < \delta$. Additionally, since $\delta > 0$, Lemma 7.23 asserts that $-\delta < 0$. This combined with the previous result demonstrates by transitivity that $-\delta < 0 < y < \delta$, so by the lemma from Exercise 8.9, we have that $|y| < \delta$. Consequently, by Script 7, we know that $|(y + a) - a| < \delta$ for any $a \in \mathbb{R}$. It follows by the above that $|(y + a)^n - a^n| < 1$. Thus, by Additional Exercise 0.7, $|\sum_{k=0}^n \binom{n}{k} y^{n-k} a^k - a^n| = |y^n + ny^{n-1}a + \sum_{k=2}^{n-1} \binom{n}{k} y^{n-k} a^k| < 1$. If we now choose $a = \frac{1}{ny^{n-1}}$, Script 7 reduces the above to $|y^n + 1 + \sum_{k=2}^{n-1} y^{n-k} a^k| < 1$. We now seek to reduce the previous statement further to $|y^n + 1| < 1$. To begin, Exercise 12.22 implies that $y^n > 0$ since $y > 0$ and $0^n = 0$, meaning by Script 7 that $y^n + 1 > 0$. Additionally, Script 7 asserts that $\sum_{k=2}^{n-1} y^{n-k} a^k > 0$ since $a > 0$ and $y > 0$. This combined with the previous result implies by Scripts 7 and 8 that $|y^n + 1| < |y^n + 1 + \sum_{k=2}^{n-1} y^{n-k} a^k| < 1$, as desired. However, since $y^n > 0$, Definition 7.21 asserts that $y^n + 1 > 1$. But by Definition 8.4, this implies that $|y^n + 1| > 1$, a contradiction. \square

Exercise 13.5. Let f and g be uniformly continuous on $A \subset \mathbb{R}$. Show that

- (a) The function $f + g$ is uniformly continuous on A .
- (b) For any constant $c \in \mathbb{R}$, the function $c \cdot f$ is uniformly continuous on A .

Proof of a. To prove that $f + g$ is uniformly continuous on A , Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|(f + g)(y) - (f + g)(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f, g are uniformly continuous on A , consecutive applications of Definition 13.1 reveal that there exist $\delta_1, \delta_2 > 0$ such that for all $x, y \in A$, $|y - x| < \delta_1$ implies $|f(y) - f(x)| < \frac{\epsilon}{2}$ and $|y - x| < \delta_2$ implies $|g(y) - g(x)| < \frac{\epsilon}{2}$. Choose $\delta = \min(\delta_1, \delta_2)$. Let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. It follows that $|y - x| < \delta_1$ (so $|f(y) - f(x)| < \frac{\epsilon}{2}$), and that $|y - x| < \delta_2$ (so $|g(y) - g(x)| < \frac{\epsilon}{2}$). These two results when combined imply by Script 7 that $|f(y) - f(x)| + |g(y) - g(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$. Therefore, since $|f(y) - f(x) + g(y) - g(x)| \leq |f(y) - f(x)| + |g(y) - g(x)|$ by Lemma 8.8, we have that

$$\begin{aligned} |(f + g)(y) - (f + g)(x)| &= |f(y) - f(x) + g(y) - g(x)| \\ &\leq |f(y) - f(x)| + |g(y) - g(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

as desired. \square

Proof of b. To prove that $c \cdot f$ is uniformly continuous on A , Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|c \cdot f(y) - c \cdot f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. We divide into two cases ($c = 0$ and $c \neq 0$). Suppose first that $c = 0$. Choose $\delta = 1$. Let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. It follows that $|0 \cdot f(y) - 0 \cdot f(x)| = 0 < \epsilon$, as desired. Now suppose that $c \neq 0$. Then since f is uniformly continuous on A , Definition 13.1 tells us that there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \frac{\epsilon}{|c|}$. Choose this δ to be our δ . Let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. Then by the above, we have that $|f(y) - f(x)| < \frac{\epsilon}{|c|}$. Therefore, $|c| \cdot |f(y) - f(x)| < \epsilon$, so we have that $|c \cdot f(y) - c \cdot f(x)| < \epsilon$, as desired. \square

4/13: **Theorem 13.6.** Suppose that $X \subset \mathbb{R}$ is compact and $f : X \rightarrow \mathbb{R}$ is continuous. Then f is uniformly continuous.

Proof. To prove that f is uniformly continuous, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f is continuous on X , Theorem 9.10 asserts that f is continuous at every $x \in X$. Thus, by Theorem 11.5, for every $x \in X$, there exists a $\delta_x > 0$ such that if $y \in X$ and $|y - x| < \delta_x$, then $|f(y) - f(x)| < \frac{\epsilon}{2}$. Let $\mathcal{G} = \{(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2}) \mid x \in X\}$. We will now confirm that \mathcal{G} is an open cover of X . To do so, Definition 10.3 tells us that it will suffice to demonstrate that every $x \in X$ is an element of $(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2})$ for some $(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2}) \in \mathcal{G}$. Let x be an arbitrary element of X . We know that $|x - x| = 0 < \frac{\delta_x}{2}$. Thus, by Exercise 8.9, we have that $x \in (x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2})$. Since it follows from the above that $(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2}) \in \mathcal{G}$, we are done.

Having shown that \mathcal{G} is an open cover of X , the fact that X is compact implies by Definition 10.4 that there exists a finite subset \mathcal{G}' of \mathcal{G} that is also an open cover of X . It follows that \mathcal{G}' will be of the form $\{(x_i - \frac{\delta_{x_i}}{2}, x_i + \frac{\delta_{x_i}}{2}) \mid 1 \leq i \leq n\}$ where n is some natural number. Thus, choose $\delta = \min_{1 \leq i \leq n} (\frac{\delta_{x_i}}{2})$.

Let x, y be arbitrary elements of X that satisfy $|y - x| < \delta$. Since \mathcal{G}' is an open cover of X , Definition 10.3 implies that $x \in (x_i - \frac{\delta_{x_i}}{2}, x_i + \frac{\delta_{x_i}}{2})$ for some $(x_i - \frac{\delta_{x_i}}{2}, x_i + \frac{\delta_{x_i}}{2}) \in \mathcal{G}'$. Considering this x_i more closely, we can determine from the previous result and Exercise 8.9 that $|x - x_i| < \frac{\delta_{x_i}}{2}$. This combined with the hypothesis that $|y - x| < \delta$ implies by Script 7 that $|y - x| + |x - x_i| < \delta + \frac{\delta_{x_i}}{2}$. Additionally, note that by definition, $\delta \leq \frac{\delta_{x_i}}{2}$. Thus, combining the last few results, we have that

$$\begin{aligned} |y - x_i| &\leq |y - x| + |x - x_i| && \text{Lemma 8.8} \\ &< \delta + \frac{\delta_{x_i}}{2} \\ &\leq \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2} \\ &= \delta_{x_i} \end{aligned}$$

At this point, we know that $|x - x_i| < \frac{\delta_{x_i}}{2} < \delta_{x_i}$ and that $|y - x_i| < \delta_{x_i}$. It follows by consecutive applications of the above that $|f(x) - f(x_i)| < \frac{\epsilon}{2}$ and $|f(y) - f(x_i)| < \frac{\epsilon}{2}$, respectively. Consequently, we have by Script 7 that $|f(y) - f(x_i)| + |f(x) - f(x_i)| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$. Therefore, if we combine the last several results, we get

$$\begin{aligned} |f(y) - f(x)| &\leq |f(y) - f(x_i)| + |f(x_i) - f(x)| && \text{Lemma 8.8} \\ &= |f(y) - f(x_i)| + |f(x) - f(x_i)| && \text{Exercise 8.5} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

as desired. □

Exercise 13.7. Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $A = [0, +\infty)$.

Lemma. Let x, y be arbitrary elements of A . Then $|\sqrt{y} - \sqrt{x}|^2 \leq |(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x})|$.

Proof. We will first verify that $|\sqrt{y} - \sqrt{x}| \leq |\sqrt{y} + \sqrt{x}|$. To do so, we divide into two cases ($\sqrt{y} \geq \sqrt{x}$ and $\sqrt{y} < \sqrt{x}$). If $\sqrt{y} \geq \sqrt{x}$, then by Definition 7.21, $\sqrt{y} - \sqrt{x} \geq 0$. It follows by Definition 8.4 that $|\sqrt{y} - \sqrt{x}| = \sqrt{y} - \sqrt{x}$. Additionally, we have by an extension of Exercise 12.22 that $\sqrt{x} \geq 0$, implying that $2\sqrt{x} \geq 0$ by Definition 7.21. Thus, combining the last few results, we have that $|\sqrt{y} - \sqrt{x}| = \sqrt{y} - \sqrt{x} \leq \sqrt{y} - \sqrt{x} + 2\sqrt{x} = \sqrt{y} + \sqrt{x}$. Consequently, we know that $0 \leq |\sqrt{y} - \sqrt{x}| \leq \sqrt{y} + \sqrt{x}$, so Definition 8.4 implies that $|\sqrt{y} + \sqrt{x}| = \sqrt{y} + \sqrt{x}$. Therefore, we have that $|\sqrt{y} - \sqrt{x}| \leq \sqrt{y} + \sqrt{x} = |\sqrt{y} + \sqrt{x}|$, as desired. The argument is symmetric in the other case.

Having established that $|\sqrt{y} - \sqrt{x}| \leq |\sqrt{y} + \sqrt{x}|$ and knowing that $0 \leq |\sqrt{y} - \sqrt{x}|$, we have by Lemma 7.24 that $|\sqrt{y} - \sqrt{x}| \cdot |\sqrt{y} - \sqrt{x}| \leq |\sqrt{y} + \sqrt{x}| \cdot |\sqrt{y} - \sqrt{x}|$. It follows by basic algebra that $|\sqrt{y} - \sqrt{x}|^2 \leq |(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x})|$, as desired. □

Proof of Exercise 13.7. To prove that f is uniformly continuous on A , Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \epsilon^2$. Let x, y be arbitrary elements of X that satisfy $|y - x| < \delta$. Thus, since $(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x}) = y - x$, the lemma asserts that $|\sqrt{y} - \sqrt{x}|^2 \leq |(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x})| = |y - x| < \epsilon^2$. Therefore, by Script 7, $|\sqrt{y} - \sqrt{x}| < \epsilon$, i.e., $|f(y) - f(x)| < \epsilon$, as desired. \square

Corollary 13.8. Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is continuous. Then f is uniformly continuous.

Proof. By Theorem 10.14, $[a, b]$ is compact. This combined with the hypothesis that f is continuous proves by Theorem 13.6 that f is uniformly continuous. \square

Exercise 13.9. Show that if f and g are bounded on A and uniformly continuous on A , then fg is uniformly continuous on A .

Proof. To prove that fg is uniformly continuous on A , Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|(fg)(y) - (fg)(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary.

Since f is bounded on A , Definition 10.1 implies that $f(A)$ is a bounded subset of \mathbb{R} . Thus, by consecutive applications of Definition 5.6, there exist numbers l, u such that for all $f(x) \in f(A)$, $l \leq f(x) \leq u$. Let $a = \max(|l|, |u|) + 1$. It follows by Scripts 7 and 8 that $-a < f(x) < a$ for all $f(x) \in f(A)$. Thus, by the lemma from Exercise 8.9, $|f(x)| < a$ for all $f(x) \in f(A)$. Similarly, there exists a number b such that $|g(x)| < b$ for all $g(x) \in g(A)$.

Since f is uniformly continuous on A , Definition 13.1 implies that there exists a $\delta_1 > 0$ such that for all $x, y \in A$, if $|y - x| < \delta_1$, then $|f(y) - f(x)| < \frac{\epsilon}{2b}$. Similarly, there exists a $\delta_2 > 0$ such that for all $x, y \in A$, if $|y - x| < \delta_2$, then $|g(y) - g(x)| < \frac{\epsilon}{2a}$. Choose $\delta = \min(\delta_1, \delta_2)$. Let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. It follows by consecutive applications of the above that $|f(x)| < a$ and $|g(y)| < b$. Additionally, $|y - x| < \delta \leq \delta_1$ implies that $|f(y) - f(x)| < \frac{\epsilon}{2b}$ and $|y - x| < \delta \leq \delta_2$ implies that $|g(y) - g(x)| < \frac{\epsilon}{2a}$. Therefore, combining the last four results, we have that

$$\begin{aligned}
 |(fg)(y) - (fg)(x)| &= |f(x)g(x) - f(y)g(y)| \\
 &= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\
 &= |f(x)(g(x) - g(y)) + g(y)(f(x) - f(y))| \\
 &\leq |f(x)| \cdot |g(x) - g(y)| + |g(y)| \cdot |f(x) - f(y)| \quad \text{Lemma 8.8} \\
 &< a \cdot \frac{\epsilon}{2a} + b \cdot \frac{\epsilon}{2b} \\
 &= \epsilon
 \end{aligned}$$

as desired. \square

4/15: **Definition 13.10.** A **partition** of the interval $[a, b]$ is a finite set of points in $[a, b]$ that includes a and b . We usually write partitions as $P = \{t_0, t_1, \dots, t_n\}$, with the convention that

$$a = t_0 < t_1 < \dots < t_{n-1} < t_n = b$$

If P and Q are partitions of the interval $[a, b]$ and $P \subset Q$, we refer to Q as a **refinement** of P .

Definition 13.11. Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is bounded and that $P = \{t_0, \dots, t_n\}$ is a partition of $[a, b]$. Define

$$m_i(f) = \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\} \quad M_i(f) = \sup\{f(x) \mid t_{i-1} \leq x \leq t_i\}$$

The **lower sum** of f for the partition P is the number

$$L(f, P) = \sum_{i=1}^n m_i(f)(t_i - t_{i-1})$$

The **upper sum** of f for the partition P is the number

$$U(f, P) = \sum_{i=1}^n M_i(f)(t_i - t_{i-1})$$

Notice that it is always the case that $L(f, P) \leq U(f, P)$.

Lemma 13.12. *Suppose that P and Q are partitions of $[a, b]$ and that Q is a refinement of P . Then $L(f, P) \leq L(f, Q)$ and $U(f, P) \geq U(f, Q)$.*

Lemma. *Let P be a partition of $[a, b]$ and let y be an arbitrary element of $[a, b] \setminus P$. Then $L(f, P) \leq L(f, P \cup \{y\})$ and $U(f, P) \geq U(f, P \cup \{y\})$.*

Proof. We will prove that $L(f, P) \leq L(f, P \cup \{y\})$. The proof will be symmetric in the other case. Let's begin.

By Definition 13.10, P is of the form $\{t_0, \dots, t_n\}$ where $a = t_0 < \dots < t_n = b$. This combined with the hypothesis that $y \in [a, b] \setminus P$ implies by Theorem 3.5 that $a = t_0 < \dots < t_{k-1} < y < t_k < \dots < t_n = b$. Thus, we have by consecutive applications of Definition 13.11 that

$$\begin{aligned} L(f, P) &= \sum_{i=1}^n m_i(f)(t_i - t_{i-1}) \\ &= \sum_{i=1}^{k-1} m_i(f)(t_i - t_{i-1}) + m_k(f)(t_k - t_{k-1}) + \sum_{i=k+1}^n m_i(f)(t_i - t_{i-1}) \end{aligned}$$

and that

$$L(f, P \cup \{y\}) = \sum_{i=1}^{k-1} m_i(f)(t_i - t_{i-1}) + m_y^-(f)(y - t_{k-1}) + m_y^+(f)(t_k - y) + \sum_{i=k+1}^n m_i(f)(t_i - t_{i-1})$$

where

$$m_y^-(f) = \inf\{f(x) \mid t_{k-1} \leq x \leq y\} \quad m_y^+(f) = \inf\{f(x) \mid y \leq x \leq t_k\}$$

As such, to prove that $L(f, P) \leq L(f, P \cup \{y\})$, it will suffice to show that $m_k(f)(t_k - t_{k-1}) \leq m_y^-(f)(y - t_{k-1}) + m_y^+(f)(t_k - y)$. To do so, it will suffice to show that $m_k(f)(y - t_{k-1}) + m_k(f)(t_k - y) \leq m_y^-(f)(y - t_{k-1}) + m_y^+(f)(t_k - y)$, i.e., that $m_k(f)(y - t_{k-1}) \leq m_y^-(f)(y - t_{k-1})$ and that $m_k(f)(t_k - y) \leq m_y^+(f)(t_k - y)$, i.e., that $m_k(f) \leq m_y^-(f)$ and that $m_k(f) \leq m_y^+(f)$.

For the sake of proving the first expression, let $A = \{f(x) \mid t_{k-1} < x < t_k\}$ and let $B = \{f(x) \mid t_{k-1} \leq x \leq y\}$. It follows by Definition 13.10 that $m_k(f) = \inf A$ and $m_y^-(f) = \inf B$. Thus, we need only show that $\inf A \leq \inf B$. Since $y < t_k$, we know by Script 1 that $B \subset A$. Thus, since $\inf A$ is a lower bound on A , Script 5 implies that it is also a lower bound on B . Consequently, by Definition 5.7, $\inf A \leq \inf B$, as desired.

The argument is symmetric for the other statement. \square

Proof of Lemma 13.12. We will prove that $L(f, P) \leq L(f, Q)$. The proof will be symmetric in the other case. Let's begin.

By Definition 13.10, $P \subset Q$. Thus, by Theorem ??, $|P| \leq |Q|$. It follows by Script 1 that $|Q| - |P| = n \in \mathbb{Z}^+$. Thus, to prove the claim for P and Q in general, it will suffice to prove it for each n . To do so, we divide into two cases ($n = 0$ and $n \in \mathbb{N}$). If $n = 0$, then $|P| = |Q|$. This combined with the fact that $P \subset Q$ implies by Script 1 that $P = Q$. Therefore, $L(f, P) = L(f, Q)$, which we can weaken to $L(f, P) \leq L(f, Q)$, as desired.

On the other hand, if $n \in \mathbb{N}$, then we induct on n . For the base case $n = 1$, we have by Script 1 that $Q = P \cup \{y\}$ where $y \notin P$. Therefore, by the lemma, we have that $L(f, P) \leq L(f, P \cup \{y\}) = L(f, Q)$, as desired. Now suppose inductively that the claim holds on n ; we wish to prove it for $n + 1$. Let y be an arbitrary element of Q . Then by Script 1, $|Q \setminus \{y\}| - |P| = n$. Thus, by the inductive hypothesis, $L(f, P) \leq L(f, Q \setminus \{y\})$. Additionally, by the lemma, $L(f, Q \setminus \{y\}) \leq L(f, Q)$. Therefore, by transitivity, $L(f, P) \leq L(f, Q)$, as desired. \square

Theorem 13.13. Let P_1 and P_2 be partitions of $[a, b]$ and suppose that $f : [a, b] \rightarrow \mathbb{R}$ is bounded. Then $L(f, P_1) \leq U(f, P_2)$.

Proof. To confirm that $P_1 \cup P_2$ is a partition of $[a, b]$, Definition 13.10 tells us that it will suffice to demonstrate that it is a finite set, that it is a subset of $[a, b]$, and that it includes a and b . Since P_1, P_2 are partitions of $[a, b]$, Definition 13.10 implies that they are finite subsets of $[a, b]$ that contain a, b . It follows by Script 1 that their union is finite, a subset of $[a, b]$, and a set containing a and b . Additionally, we have by Theorem 1.7 that $P_1 \subset P_1 \cup P_2$ and that $P_2 \subset P_1 \cup P_2$. Combining the last two results with consecutive applications of Definition 13.10 reveals that $P_1 \cup P_2$ is a refinement of both P_1 and P_2 .

Since P_1 and $P_1 \cup P_2$ are partitions of $[a, b]$ and $P_1 \cup P_2$ is a refinement of P_1 , Lemma 13.12 implies that $L(f, P_1) \leq L(f, P_1 \cup P_2)$. Similarly, $U(f, P_1 \cup P_2) \leq U(f, P_2)$. Additionally, we have by Definition 13.11 that $L(f, P_1 \cup P_2) \leq U(f, P_1 \cup P_2)$. Therefore, if we combine the last three results with transitivity, we have that $L(f, P_1) \leq U(f, P_2)$, as desired. \square

Definition 13.14. Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded. We define

$$L(f) = \sup\{L(f, P) \mid P \text{ is a partition of } [a, b]\} \quad U(f) = \inf\{U(f, P) \mid P \text{ is a partition of } [a, b]\}$$

to be, respectively, the **lower integral** and **upper integral** of f from a to b .

Exercise 13.15. Why do $L(f)$ and $U(f)$ exist? Find a function f for which $L(f) = U(f)$. Find a function f for which $L(f) \neq U(f)$. Prove that $L(f) \leq U(f)$.

Lemma. Given $a, b \in \mathbb{R}$ with $a < b$, there exists $p \in \mathbb{R}$ such that $p \notin \mathbb{Q}$ and $a < p < b$.

Proof. By Definition 7.21, $a + \sqrt{2} < b + \sqrt{2}$. Thus, by Lemma 6.10, there exists a point $\frac{c}{d} \in \mathbb{Q}$ such that $a + \sqrt{2} < \frac{c}{d} < b + \sqrt{2}$. It follows that $a < \frac{c}{d} - \sqrt{2} < b$.

Now suppose for the sake of contradiction that $\frac{c}{d} - \sqrt{2}$ is rational. Then by Script 2, $\frac{c}{d} - \sqrt{2} = \frac{e}{f}$ where $e, f \in \mathbb{Z}$ and $f \neq 0$. It follows by Theorem 2.10 that $\sqrt{2} = \frac{cf - de}{df}$, i.e., that $\sqrt{2}$ is rational. But by the proof of Exercise 4.24, $\sqrt{2}$ is not rational, a contradiction. \square

Proof of Exercise 13.15. Let $A = \{L(f, P) \mid P \text{ is a partition of } [a, b]\}$. To prove that $L(f) = \sup A$ exists, Theorem 5.17 tells us that it will suffice to show that A is nonempty and bounded above.

To confirm that A is nonempty, Definition 1.8 tells us that it will suffice to find an element of it. Since $\{a, b\}$ is a finite set of points in $[a, b]$ that includes a and b (by Script 1), Definition 13.10 asserts that $\{a, b\}$ is a partition of $[a, b]$. It follows by Definition 13.11 that $L(f, \{a, b\})$ exists. Therefore, by the definition of A , we have that $L(f, \{a, b\}) \in A$, as desired.

To confirm that A is bounded above, Definition 5.6 tells us that it will suffice to find a point in $u \in \mathbb{R}$ such that for all $L(f, P) \in A$, $L(f, P) \leq u$. Let $u = U(f, \{a, b\})$ (since $\{a, b\}$ is a partition of $[a, b]$ by the above, Definition 13.10 guarantees that $U(f, \{a, b\})$ exists). Now let $L(f, P)$ be an arbitrary element of A . It follows from Theorem 13.13 that $L(f, P) \leq U(f, \{a, b\}) = u$, as desired.

The proof is symmetric for $U(f)$.

Let $f : [0, 1] \rightarrow \mathbb{R}$ be defined by $f(x) = 0$. To prove that $L(f) = U(f)$, it will suffice to show that $L(f) = 0$ and $U(f) = 0$. To do this, Script 5 tells us that it will suffice to verify that $\{L(f, P) \mid P \text{ is a partition of } [a, b]\} = \{0\}$ and $\{U(f, P) \mid P \text{ is a partition of } [a, b]\} = \{0\}$. We will start with the first equality.

Let $L(f, P)$ be an arbitrary element of $\{L(f, P) \mid P \text{ is a partition of } [a, b]\}$. Since we have

$$\begin{aligned} m_i(f) &= \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\} \\ &= \inf\{0 \mid t_{i-1} \leq x \leq t_i\} \\ &= \inf\{0\} \\ &= 0 \end{aligned}$$

for all $m_i(f)$, it follows that

$$\begin{aligned} L(f, P) &= \sum_{i=1}^n m_i(f)(t_i - t_{i-1}) \\ &= \sum_{i=1}^n 0(t_i - t_{i-1}) \\ &= 0 \end{aligned}$$

Therefore, since every element of $\{L(f, P) \mid P \text{ is a partition of } [a, b]\}$ is equal to 0, the set is equal to the singleton set containing 0. The argument is symmetric for the other equality.

Let $f : [0, 1] \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

To prove that $L(f) \neq U(f)$, it will suffice to show that $L(f) = 0$ and $U(f) = 1$. To do this, Script 5 tells us that it will suffice to verify that $\{L(f, P) \mid P \text{ is a partition of } [a, b]\} = \{0\}$ and $\{U(f, P) \mid P \text{ is a partition of } [a, b]\} = \{1\}$. We will start with the first equality.

Let $L(f, P)$ be an arbitrary element of $\{L(f, P) \mid P \text{ is a partition of } [a, b]\}$. To confirm that $L(f, P) = 0$, Definition 13.11 tells us that it will suffice to demonstrate that $m_i(f) = 0$ for all $m_i(f)$. Let $m_i(f)$ be an arbitrary such object. By Definition 13.10, $m_i(f) = \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\}$. By the lemma, there exists $p \in \mathbb{R}$ such that $p \notin \mathbb{Q}$ and $t_{i-1} \leq p \leq t_i$. Thus, since $f(p) = 0$, $0 \in \{f(x) \mid t_{i-1} \leq x \leq t_i\}$. Additionally, since $f(x) \not\leq 0$ for any $x \in [0, 1]$ by definition, we have that $m_i(f) = 0$. Therefore, since every element of $\{L(f, P) \mid P \text{ is a partition of } [a, b]\}$ is equal to 0, the set is equal to the singleton set containing 0.

As to the other equality, let $U(f, P)$ be an arbitrary element of $\{U(f, P) \mid P \text{ is a partition of } [a, b]\}$. To confirm that $U(f, P) = 1$, Definition 13.11 tells us that we must first demonstrate that $M_i(f) = 1$ for all $M_i(f)$. Let $M_i(f)$ be an arbitrary such object. By Definition 13.10, $M_i(f) = \sup\{f(x) \mid t_{i-1} \leq x \leq t_i\}$. By Lemma 6.10, there exists $p \in \mathbb{Q}$ such that $t_{i-1} \leq p \leq t_i$. Thus, since $f(p) = 1$, $1 \in \{f(x) \mid t_{i-1} \leq x \leq t_i\}$. Additionally, since $f(x) \not\geq 1$ for any $x \in [0, 1]$ by definition, we have that $M_i(f) = 1$. It follows by Definition 13.11 that

$$\begin{aligned} U(f, P) &= \sum_{i=1}^n M_i(f)(t_i - t_{i-1}) \\ &= \sum_{i=1}^n (t_i - t_{i-1}) \\ &= t_n - t_0 \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

Therefore, since every element of $\{U(f, P) \mid P \text{ is a partition of } [a, b]\}$ is equal to 1, the set is equal to the singleton set containing 1.

Suppose for the sake of contradiction that there exists a function $f : [a, b] \rightarrow \mathbb{R}$ for which $U(f) < L(f)$. It follows by consecutive applications of Definition 13.14 and Lemma 5.11 that there exists an $L(f, P_1)$ such that $U(f) < L(f, P_1) \leq L(f)$, and thus that there exists a $U(f, P_2)$ such that $U(f) \leq U(f, P_2) < L(f, P_1)$. But this means that there exist partitions P_1, P_2 of $[a, b]$ such that $L(f, P_1) > U(f, P_2)$, contradicting Theorem 13.13. \square

Definition 13.16. Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded. We say that f is **integrable** on $[a, b]$ if $L(f) = U(f)$. In this case, the common value $L(f) = U(f)$ is called the **integral** of f from a to b and we write it as

$$\int_a^b f$$

Note that if f is an integrable function on $[a, b]$, it is necessarily bounded.

When we want to display the variable of integration, we write the integral as follows, including the symbol dx to indicate that variable of integration:

$$\int_a^b f(x) dx$$

For example, if $f(x) = x^2$, we could write $\int_a^b x^2 dx$ but not $\int_a^b x^2$.

Exercise 13.17. Fix $c \in \mathbb{R}$ and let $f : [a, b] \rightarrow \mathbb{R}$ be defined by $f(x) = c$, for each $x \in [a, b]$. Show that f is integrable on $[a, b]$ and that $\int_a^b f = c(b - a)$.

Proof. To prove that f is integrable on $[a, b]$ and that $\int_a^b f = c(b - a)$, Definition 13.16 tells us that it will suffice to show that f is bounded on $[a, b]$, and that $L(f) = U(f) = c(b - a)$.

To confirm that f is bounded on $[a, b]$, Definition 10.1 tells us that it will suffice to demonstrate that $f([a, b])$ is a bounded subset of \mathbb{R} . By Definition 1.18, $f([a, b]) = \{f(x) \in \mathbb{R} \mid x \in [a, b]\}$. But since $f(x) = c$ for all $x \in [a, b]$, $f([a, b]) = \{c\}$. Thus, since $c \leq c \leq c$, Definition 5.6 implies that $f([a, b])$ is bounded. Additionally, since $c \in \mathbb{R}$, Definition 1.3 asserts that $f([a, b]) = \{c\} \subset \mathbb{R}$.

To confirm that $L(f) = U(f) = c(b - a)$, Definition 13.14 tells us that it will suffice to demonstrate that $L(f, P) = U(f, P) = c(b - a)$ for all partitions P of $[a, b]$. For similar reasons to the above (i.e., $f(x) = c$ for all $x \in [a, b]$), we can show that $m_i(f) = M_i(f) = c$ for all $m_i(f)$ and $M_i(f)$. Therefore, by Definition 13.11 that

$$\begin{aligned} L(f, P) &= \sum_{i=1}^n c(t_i - t_{i-1}) & U(f, P) &= \sum_{i=1}^n c(t_i - t_{i-1}) \\ &= c \sum_{i=1}^n (t_{i-1} - t_i) & &= c \sum_{i=1}^n (t_{i-1} - t_i) \\ &= c(t_n - t_0) & &= c(t_n - t_0) \\ &= c(b - a) & &= c(b - a) \end{aligned}$$

as desired. □

Theorem 13.18. Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded. Then f is integrable if and only if for every $\epsilon > 0$, there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$.

Proof. Suppose first that f is integrable. Then by Definition 13.16, $L(f) = U(f)$. Let $\epsilon > 0$ be arbitrary. By Script 7, $L(f) - \frac{\epsilon}{2} < L(f)$. Thus, by Definition 13.14 and Lemma 5.11, there exists an $L(f, P_1) \in \{L(f, P) \mid P \text{ is a partition of } [a, b]\}$ such that $L(f) - \frac{\epsilon}{2} < L(f, P_1) \leq L(f)$. Similarly, there exists a $U(f, P_2) \in \{U(f, P) \mid P \text{ is a partition of } [a, b]\}$ such that $U(f) \leq U(f, P_2) < U(f) + \frac{\epsilon}{2}$. Now consider $P_1 \cup P_2$ (which we will prove is the desired partition). By Theorem 1.7, $P_1 \subset P_1 \cup P_2$ and $P_2 \subset P_1 \cup P_2$. It follows by consecutive applications of Definition 13.10 that $P_1 \cup P_2$ is a refinement of both P_1 and P_2 . Thus, by Lemma 13.12, $L(f, P_1) \leq L(f, P_1 \cup P_2)$ and $U(f, P_1 \cup P_2) \leq U(f, P_2)$. Combining the last several results with transitivity yields

$$L(f) - \frac{\epsilon}{2} < L(f, P_1) \leq L(f, P_1 \cup P_2) \qquad U(f, P_1 \cup P_2) \leq U(f, P_2) < U(f) + \frac{\epsilon}{2}$$

Therefore, knowing that $U(f, P_1 \cup P_2) < U(f) + \frac{\epsilon}{2}$ and that $-L(f, P_1 \cup P_2) < \frac{\epsilon}{2} - L(f)$ (the latter by Lemma 7.24), we have by Definition 7.21 that

$$\begin{aligned} U(f, P_1 \cup P_2) - L(f, P_1 \cup P_2) &< U(f) + \frac{\epsilon}{2} + \frac{\epsilon}{2} - L(f) \\ &= \epsilon \end{aligned}$$

as desired.

Now suppose that f is not integrable; we seek to prove that there exists an $\epsilon > 0$ such that for all partitions P of $[a, b]$, $U(f, P) - L(f, P) \geq \epsilon$. Since f is not integrable, we have by Definition 13.16 that

$L(f) \neq U(f)$. It follows by Exercise 13.15 that $L(f) < U(f)$. Thus, we can define $\epsilon = \frac{U(f) - L(f)}{2} > 0$. Now let P be an arbitrary partition of $[a, b]$. It follows that $L(f, P) \leq L(f)$ by Definitions 13.14, 5.7, and 5.6. Similarly, $U(f) \leq U(f, P)$. Therefore, knowing that $U(f) \leq U(f, P)$ and that $-L(f) \leq -L(f, P)$ (the latter by Lemma 7.24), we have by Definition 7.21 that $\epsilon < U(f) - L(f) \leq U(f, P) - L(f, P)$, as desired. \square

4/20: **Theorem 13.19.** *If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, then f is integrable.*

Proof. To prove that f is integrable, Theorem 13.18 tells us that it will suffice to show that f is bounded and that for every $\epsilon > 0$, there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$. We will verify the two requirements separately. Let's begin.

To confirm that f is bounded, Definitions 10.1 and 5.6 tell us that it will suffice to find points $l, u \in \mathbb{R}$ such that $l \leq f(x) \leq u$ for all $x \in [a, b]$. But since $f : [a, b] \rightarrow \mathbb{R}$ is continuous, consecutive applications of Exercise 10.21 imply that there exist points $c, d \in [a, b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$, so we can just choose $l = f(c)$ and $u = f(d)$.

As to the other stipulation, let $\epsilon > 0$ be arbitrary. Since $f : [a, b] \rightarrow \mathbb{R}$ is continuous, Corollary 13.8 implies that f is uniformly continuous. Thus, by Definition 13.1, there exists a $\delta > 0$ such that for all $x, y \in [a, b]$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \frac{\epsilon}{b-a}$. Considering this δ , we have by Corollary 6.12 that there exist a number $n \in \mathbb{N}$ such that $\frac{2(b-a)}{\delta} < n$. Equipped with this n , we can now define the set $P = \{\frac{b-a}{n} \cdot i + a \mid 0 \leq i \leq n\}$.

We now seek to confirm that P is a partition of $[a, b]$. To do so, Definition 13.10 tells us that it will suffice to demonstrate that P is finite, $P \subset [a, b]$, and $a, b \in P$. By Script 1, P is finite. To demonstrate that $P \subset [a, b]$, Definition 1.3 and Equations 8.1 tell us that it will suffice to show that every $t_i \in P$ satisfies $a \leq t_i \leq b$. But by Script 7, we have that

$$\begin{aligned} 0 &\leq i \leq n \\ 0 &\leq \frac{b-a}{n} \cdot i \leq b-a \\ a &\leq \frac{b-a}{n} \cdot i + a \leq b \end{aligned}$$

as desired. Lastly, consider the elements of P corresponding to $i = 0$ and $i = n$. By consecutive applications of the definition of P , we have that $a = (\frac{b-a}{n} \cdot 0 + a) \in P$ and that $b = b - a + a = (\frac{b-a}{n} \cdot n + a) \in P$.

We now seek to confirm that if $t_i, t_{i-1} \in P$, then $t_i - t_{i-1} < \delta$. Let t_i, t_{i-1} be arbitrary sequential elements of P . By Script 0, we have that $0 < n$. Additionally, we have by hypothesis that $0 < \delta$. It follows by consecutive applications of Lemma 7.24 that the fact that $\frac{2(b-a)}{\delta} < n$ implies that $\frac{2(b-a)}{n} < \delta$. Therefore, we have by Script 7 that

$$\begin{aligned} t_i - t_{i-1} &= \left(\frac{b-a}{n} \cdot i + a \right) - \left(\frac{b-a}{n} \cdot (i-1) + a \right) \\ &= \frac{b-a}{n} \\ &\leq \frac{2(b-a)}{n} \\ &< \delta \end{aligned}$$

as desired.

We now seek to confirm that $M_i(f) - m_i(f) < \frac{\epsilon}{b-a}$ for all i satisfying $1 \leq i \leq n$. Let i be an arbitrary such number, and consider $f|_{[t_{i-1}, t_i]}$. Since f is continuous and $[t_{i-1}, t_i] \subset [a, b]$, Proposition 9.7 asserts that $f|_{[t_{i-1}, t_i]}$ is continuous. Thus, by Exercise 10.21, there exist $c, d \in [t_{i-1}, t_i]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [t_{i-1}, t_i]$. It follows by consecutive applications of Definitions 13.11 and 3.3 as well as Exercise 5.9 that $m_i(f) = f(c)$ and $M_i(f) = f(d)$. Additionally, since $c, d \in [t_{i-1}, t_i]$, we have by Script 8 that $|d - c| \leq t_i - t_{i-1}$. This combined with the fact that $t_i - t_{i-1} < \delta$ by the above implies by transitivity that

$|d - c| < \delta$. But this implies by the above that

$$\begin{aligned} M_i(f) - m_i(f) &= f(d) - f(c) \\ &\leq |f(d) - f(c)| \\ &< \frac{\epsilon}{b-a} \end{aligned}$$

as desired.

Having established that $M_i(f) - m_i(f) < \frac{\epsilon}{b-a}$ for all i in the partition P , we have by Definition 13.11 and basic algebra that

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^n M_i(f)(t_i - t_{i-1}) - \sum_{i=1}^n m_i(f)(t_i - t_{i-1}) \\ &= \sum_{i=1}^n (M_i(f) - m_i(f))(t_i - t_{i-1}) \\ &< \sum_{i=1}^n \frac{\epsilon}{b-a} (t_i - t_{i-1}) \\ &= \frac{\epsilon}{b-a} \sum_{i=1}^n (t_i - t_{i-1}) \\ &= \frac{\epsilon}{b-a} (b-a) \\ &= \epsilon \end{aligned}$$

as desired. □

Lemma 13.20. *Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded. Given $\Omega \in \mathbb{R}$, we have $\Omega = \int_a^b f$ if and only if for all $\epsilon > 0$, there is some partition P such that*

$$U(f, P) - \Omega < \epsilon \qquad \Omega - L(f, P) < \epsilon$$

Proof. Suppose first that $\Omega = \int_a^b f$. Let $\epsilon > 0$ be arbitrary. By Theorem 13.18, there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$. Choose this P to be our P . By Definition 13.16, $\Omega = L(f) = U(f)$. Thus, by consecutive applications of Definitions 13.14, 5.7, and 5.6, we have that $L(f, P) \leq L(f) = \Omega$ and $\Omega = U(f) \leq U(f, P)$. With respect to the former result, it follows by Script 7 that $-\Omega \leq -L(f, P)$. Therefore, having established that $\Omega \leq U(f, P)$, $-\Omega \leq -L(f, P)$, and $U(f, P) - L(f, P) < \epsilon$, we have that

$$\begin{aligned} \Omega - L(f, P) &\leq U(f, P) - L(f, P) & U(f, P) - \Omega &\leq U(f, P) - L(f, P) \\ &< \epsilon & &< \epsilon \end{aligned}$$

Now suppose that $\Omega \neq \int_a^b f$; we seek to prove that there exists an $\epsilon > 0$ such that for all partitions P , $U(f, P) - \Omega \geq \epsilon$ or $\Omega - L(f, P) \geq \epsilon$. We divide into two cases ($\int_a^b f$ exists and $\int_a^b f$ doesn't exist).

First, suppose that $\int_a^b f$ exists. We divide into two subcases ($\Omega > \int_a^b f$ and $\Omega < \int_a^b f$). If $\Omega > \int_a^b f$, choose $\epsilon = \Omega - \int_a^b f > 0$. Let P be an arbitrary partition. As before, we have that $L(f, P) \leq L(f)$. Additionally, Definition 13.16 asserts that $L(f) = \int_a^b f$. Thus, transitivity implies that $L(f, P) \leq \int_a^b f$. It follows by Script 7 that $-\int_a^b f \leq -L(f, P)$. Therefore,

$$\begin{aligned} \epsilon &= \Omega - \int_a^b f \\ &\leq \Omega - L(f, P) \end{aligned}$$

as desired. The argument is symmetric in the other subcase.

Second, suppose that $\int_a^b f$ does not exist. By Exercise 13.15, $L(f)$ and $U(f)$ exist. However, since $\int_a^b f$ does not exist, Definition 13.16 asserts that $L(f) \neq U(f)$. It follows by Exercise 13.15 again that $L(f) < U(f)$. We now divide into three subcases ($\Omega \leq L(f)$, $L(f) < \Omega < U(f)$, and $U(f) \leq \Omega$). If $\Omega \leq L(f)$, choose $\epsilon = U(f) - L(f) > 0$. Let P be an arbitrary partition. As above, $U(f) \leq U(f, P)$. Therefore,

$$\begin{aligned}\epsilon &= U(f) - L(f) \\ &\leq U(f, P) - L(f) \\ &\leq U(f, P) - \Omega\end{aligned}$$

as desired. If $L(f) < \Omega < U(f)$, choose $\epsilon = U(f) - \Omega > 0$. Let P be an arbitrary partition. As above, $U(f) \leq U(f, P)$. Therefore,

$$\begin{aligned}\epsilon &= U(f) - \Omega \\ &\leq U(f, P) - \Omega\end{aligned}$$

as desired. The argument for the last subcase is symmetric to that of the first. \square

Exercise 13.21. Define $f : [0, b] \rightarrow \mathbb{R}$ by the formula $f(x) = x$. Show that f is integrable on $[0, b]$ and that $\int_0^b f = \frac{b^2}{2}$.

Proof. To prove that f is integrable on $[0, b]$ and that $\int_0^b f = \frac{b^2}{2}$, Lemma 13.20 tells us that it will suffice to show that for all $\epsilon > 0$, there is some partition P such that $U(f, P) - \frac{b^2}{2} < \epsilon$ and $\frac{b^2}{2} - L(f, P) < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\frac{2\epsilon}{b^2}$ is a positive real number by Script 7, Corollary 6.12 asserts that there exists a number $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{2\epsilon}{b^2}$. Equipped with this n , we can now define the set $P = \{\frac{b}{n} \cdot i \mid 0 \leq i \leq n\}$. By a symmetric argument to that used in the proof of Theorem 13.19, we can confirm that P is a partition of $[a, b]$ and that $t_i - t_{i-1} = \frac{b}{n}$.

We now turn our attention strictly to proving that $U(f, P) - \frac{b^2}{2} < \epsilon$; the proof of the other statement will be symmetric. Under the partition P as defined, consider an arbitrary $M_i(f)$. By Definition 13.11, $M_i(f) = \sup\{f(x) \mid t_{i-1} \leq x \leq t_i\}$. Since $f(x) = x$ for all $x \in [t_{i-1}, t_i] \subset [0, b]$, we have by Equations 8.1 that $M_i(f) = \sup[t_{i-1}, t_i]$. Thus, by Script 5, $M_i(f) = t_i = \frac{bi}{n}$. Therefore,

$$\begin{aligned}U(f, P) - \frac{b^2}{2} &= \sum_{i=1}^n M_i(f)(t_i - t_{i-1}) - \frac{b^2}{2} && \text{Definition 13.11} \\ &= \sum_{i=1}^n \frac{bi}{n} \left(\frac{bi}{n} - \frac{b(i-1)}{n} \right) - \frac{b^2}{2} \\ &= \frac{b^2}{n^2} \sum_{i=1}^n i(i - (i-1)) - \frac{b^2}{2} \\ &= \frac{b^2}{n^2} \sum_{i=1}^n i - \frac{b^2}{2} \\ &= \frac{b^2}{n^2} \left(\frac{1}{2}n(n+1) \right) - \frac{b^2}{2} \\ &= \frac{b^2}{2} + \frac{b^2}{2n} - \frac{b^2}{2} \\ &= \frac{b^2}{2} \cdot \frac{1}{n} \\ &< \frac{b^2}{2} \cdot \frac{2\epsilon}{b^2} \\ &= \epsilon\end{aligned}$$

as desired. \square

Exercise 13.22. Show that the converse of Theorem 13.19 is false in general.

Proof. To prove that even if f is integrable, $f : [a, b] \rightarrow \mathbb{R}$ is not necessarily continuous, we need only find an example of an integrable, discontinuous function f . Let $f : [-1, 1] \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

To confirm that f is integrable, Theorem 13.18 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $P = \{-1, -\frac{\epsilon}{2}, 0, 1\}$ (clearly P is a partition of $[-1, 1]$ by Definition 13.10). It follows by consecutive applications of Definitions 13.11, 5.7, and 5.6 that

$$\begin{array}{ll} m_1(f) = 0 & M_1(f) = 0 \\ m_2(f) = 0 & M_2(f) = 1 \\ m_3(f) = 1 & M_3(f) = 1 \end{array}$$

Therefore,

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^3 M_i(f)(t_i - t_{i-1}) - \sum_{i=1}^3 m_i(f)(t_i - t_{i-1}) && \text{Definition 13.11} \\ &= \left[0 \left(-\frac{\epsilon}{2} - (-1) \right) + 1 \left(0 - \left(-\frac{\epsilon}{2} \right) \right) + 1(1 - 0) \right] \\ &\quad - \left[0 \left(-\frac{\epsilon}{2} - (-1) \right) + 0 \left(0 - \left(-\frac{\epsilon}{2} \right) \right) + 1(1 - 0) \right] \\ &= \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

However, by Corollary 5.5 and Theorem 3.14, $0 \in LP([-1, 1])$. Additionally, by the proof of Exercise 11.4, $\lim_{x \rightarrow 0} f(x)$ does not exist. Combining the last two results with Theorem 11.5 reveals that f is not continuous at 0. Therefore, by Theorem 9.10, f is not continuous. \square

Theorem 13.23. Let $a < b < c$. A function $f : [a, c] \rightarrow \mathbb{R}$ is integrable on $[a, c]$ if and only if f is integrable on $[a, b]$ and $[b, c]$. When f is integrable on $[a, c]$, we have

$$\int_a^c f = \int_a^b f + \int_b^c f$$

Lemma. Let P_1, P_2 be partitions of $[a, b]$ and $[b, c]$, respectively. Define $P' = P_1 \cup P_2$. Then P' is a partition of $[a, c]$, $L(f, P') = L(f, P_1) + L(f, P_2)$, and $U(f, P') = U(f, P_1) + U(f, P_2)$.

Proof. To prove that P' is a partition of $[a, c]$, Definition 13.10 tells us that it will suffice to show that P' is finite, that $P' \subset [a, c]$, and that $a, c \in P'$. By Definition 13.10, P_1 and P_2 are finite. Thus, by Script 1, their union $P_1 \cup P_2 = P'$ is also finite. To confirm that $P' \subset [a, c]$, Definition 1.3 tells us that it will suffice to demonstrate that every $x \in P'$ is an element of $[a, c]$. Let x be an arbitrary element of P' . Then by Definition 1.5, $x \in P_1$ or $x \in P_2$. We now divide into two cases. If $x \in P_1$, then since $P_1 \subset [a, b]$ by Definition 13.10, Definition 1.3 asserts that $x \in [a, b]$. Thus, by Equations 8.1, $a \leq x \leq b$. Moreover, by hypothesis, we have that $a \leq x \leq b < c$, from which it follows by Equations 8.1 that $x \in [a, c]$, as desired. The argument is symmetric in the other case. Lastly, by consecutive applications of Definition 13.10, $a \in P_1$ and $c \in P_2$. It follows by Definition 1.5 that $a, c \in P'$, as desired.

Additionally, if we express P_1 as containing the objects $a = t_0, \dots, t_n = b$ and P_2 as containing the objects $b = t_n, \dots, t_{n+m} = c$, we have that P' contains every object t_0 through t_{n+m} . Therefore, we have by

consecutive applications of Definition 13.11 that

$$\begin{aligned}
 L(f, P') &= \sum_{i=1}^{n+m} m_i(f)(t_i - t_{i-1}) \\
 &= \sum_{i=1}^n m_i(f)(t_i - t_{i-1}) + \sum_{i=n+1}^{n+m} m_i(f)(t_i - t_{i-1}) \\
 &= L(f, P_1) + L(f, P_2)
 \end{aligned}$$

The proof is symmetric for the other statement. \square

Proof of Theorem 13.23. Suppose first that f is integrable on $[a, c]$. To prove that f is integrable on $[a, b]$ and $[b, c]$, Theorem 13.18 tells us that it will suffice to show that for every $\epsilon > 0$, there exist partitions P_1, P_2 of $[a, b]$ and $[b, c]$, respectively, such that $U(f, P_1) - L(f, P_1) < \epsilon$ and $U(f, P_2) - L(f, P_2) < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f is integrable on $[a, c]$, there exists a partition P of $[a, c]$ such that $U(f, P) - L(f, P) < \epsilon$. Now define $P' = P \cup \{b\}$. Since P' is finite (by Script 1), a subset of $[a, c]$ (because $P \subset [a, c]$ by Definition 13.10 and $\{b\} \subset [a, c]$), and contains a, c (because $a, c \in P$ implies $a, c \in P \cup \{b\}$ by Definition 1.5), Definition 13.10 asserts that P' is a partition of $[a, c]$. Furthermore, since $P \subset P'$ by Theorem 1.7, Definition 13.10 implies that P' is a refinement of P . Thus, by Lemma 13.12, $L(f, P) \leq L(f, P')$ and $U(f, P) \geq U(f, P')$. This combined with the fact that $U(f, P) - L(f, P) < \epsilon$ implies by Script 7 that $U(f, P') - L(f, P') \leq U(f, P) - L(f, P) < \epsilon$.

Let $P_1 = P' \cap [a, b]$ and $P_2 = P' \cap [b, c]$. In the same manner as before, we have that P_1 is a partition of $[a, b]$ and P_2 is a partition of $[b, c]$. This combined with the fact that $P_1 \cup P_2 = P' \cap ([a, b] \cup [b, c]) = P'$ by Script 1 implies by the lemma that $L(f, P') = L(f, P_1) + L(f, P_2)$ and $U(f, P') = U(f, P_1) + U(f, P_2)$. Thus, we have that

$$\begin{aligned}
 (U(f, P_1) - L(f, P_1)) + (U(f, P_2) - L(f, P_2)) &= U(f, P') - L(f, P') \\
 &< \epsilon
 \end{aligned}$$

Additionally, we have by consecutive applications of Definition 13.11 that $L(f, P_1) \leq U(f, P_1)$ and $L(f, P_2) \leq U(f, P_2)$. It follows by consecutive applications of Definition 7.21 that $0 \leq U(f, P_1) - L(f, P_1)$ and $0 \leq U(f, P_2) - L(f, P_2)$. This combined with the above result that $(U(f, P_1) - L(f, P_1)) + (U(f, P_2) - L(f, P_2)) < \epsilon$ implies by Script 7 that $U(f, P_1) - L(f, P_1) < \epsilon$ and $U(f, P_2) - L(f, P_2) < \epsilon$.

Now suppose that f is integrable on $[a, b]$ and $[b, c]$. Let $\Omega_1 = \int_a^b f$, $\Omega_2 = \int_b^c f$, and $\Omega = \Omega_1 + \Omega_2$. Thus, to prove that f is integrable on $[a, c]$ and that $\int_a^c f = \int_a^b f + \int_b^c f$, Lemma 13.20 tells us that it will suffice to show that for all $\epsilon > 0$, there is some partition P' of $[a, c]$ such that $U(f, P') - \Omega < \epsilon$ and $\Omega - L(f, P') < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f is integrable on $[a, b]$ and $[b, c]$, we have by consecutive applications of Lemma 13.20 that there exist partitions P_1 of $[a, b]$ and P_2 of $[b, c]$ such that $U(f, P_1) - \Omega_1 < \frac{\epsilon}{2}$, $\Omega_1 - L(f, P_1) < \frac{\epsilon}{2}$, $U(f, P_2) - \Omega_2 < \frac{\epsilon}{2}$, and $\Omega_2 - L(f, P_2) < \frac{\epsilon}{2}$. Choose $P' = P_1 \cup P_2$. By the lemma, P' is a partition of $[a, c]$. Combining all of the above results implies by Script 7 and the lemma that

$$\begin{aligned}
 U(f, P') - \Omega &= U(f, P_1) + U(f, P_2) - \Omega_1 - \Omega_2 & \Omega - L(f, P') &= \Omega_1 + \Omega_2 - L(f, P_1) - L(f, P_2) \\
 &= (U(f, P_1) - \Omega_1) + (U(f, P_2) - \Omega_2) & &= (\Omega_1 - L(f, P_1)) + (\Omega_2 - L(f, P_2)) \\
 &< \frac{\epsilon}{2} + \frac{\epsilon}{2} & &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
 &= \epsilon & &< \epsilon
 \end{aligned}$$

Note that since the claim technically asks us to prove that $\int_a^c f = \int_a^b f + \int_b^c f$ follows from f being integrable on $[a, c]$, not $[a, b]$ and $[b, c]$, we can do this with the above using the following logic. Let f be integrable on $[a, c]$. Then by the first part of the proof, it is integrable on $[a, b]$ and $[b, c]$. It follows by the second part of the proof that $\int_a^c f = \int_a^b f + \int_b^c f$, as desired. \square