Script 14

Integrals and Derivatives

14.1 Journal

5/4: **Theorem 14.1.** *Suppose that f is integrable on $[a, b]$. Define $F : [a, b] \to \mathbb{R}$ by*

$$F(x) = \int_a^x f$$

If f is continuous at $p \in (a, b)$, then F is differentiable at p and

$$F'(p) = f(p)$$

If f is continuous at a, then $F'_+(a)$ exists and equals $f(a)$. Similarly, if f is continuous at b, $F'_-(b)$ exists and equals $f(b)$.

Proof. To prove that F is differentiable at p and $F'(p) = f(p)$, Definition 12.1 and Theorem 12.4 tell us that it will suffice to show that $\lim_{x \to p} \frac{F(x)-F(p)}{x-p} = f(p)$. To do this, Definition 11.1 tells us that it will suffice to verify that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in [a, b]$ and $0 < |x - p| < \delta$, then $|\frac{F(x)-F(p)}{x-p} - f(p)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f is continuous at p, Theorem 11.5 asserts that there exists a $\delta_1 > 0$ such that if $x \in [a, b]$ and $|x - p| < \delta_1$, $|f(x) - f(p)| < \epsilon$. Additionally, since $p \in (a, b)$, we have by Theorem 4.10 that there exists a region R containing p such that $R \subset (a, b)$. It follows by Corollary 4.11 and Lemma 8.3 that R is an open interval. Thus, by Lemma 8.10, there exists a $\delta_2 > 0$ such that $(p - \delta_2, p + \delta_2) \subset R$.

Choose $\delta = \min(\frac{\delta_1}{2}, \delta_2)$. Before we can prove the desired inequality, we need a few preliminary results. First off, we can show that $[p - \delta, p + \delta] \subset [a, b]$ by Script 1 and the fact that $\delta \leq \delta_2$. Additionally, Exercise 10.21 implies that there exists a point $c \in [p - \delta, p + \delta]$ such that $f(c) \geq f(x)$ for all $x \in [p - \delta, p + \delta]$. By the previous result, $c \in [p - \delta, p + \delta]$ implies that $c \in [a, b]$. Furthermore, we have by an extension of Lemma 8.9 that $|c - p| \leq \delta \leq \frac{\delta_1}{2} < \delta_1$. This combined with the previous result implies by the above that $|f(c) - f(p)| < \epsilon$. Now let x be an arbitrary element of $[a, b]$ that satisfies $0 < |x - p| < \delta$. However, before we go into the inequality, we have one final result to confirm: that $\int_p^x f \leq f(c)(x - p)$. Since $f(y) \leq f(c)$ for all $y \in [p - \delta, p + \delta]$, we naturally have that $f(y) \leq f(c)$ for all $y \in [p, x] \cup [x, p] \subset [p - \delta, p + \delta]$. Thus, by Theorem 13.27, $\int_p^x f \leq f(c)(x - p)$ as desired. Therefore,

$$\begin{aligned}
\left|\frac{F(x) - F(p)}{x - p} - f(p)\right| &= \left|\frac{\int_a^x f - \int_a^p f}{x - p} - f(p)\right| \\
&= \left|\frac{\int_p^x f}{x - p} - f(p)\right| && \text{Theorem 13.23} \\
&\leq \left|\frac{f(c)(x - p)}{x - p} - f(p)\right|
\end{aligned}$$

$$
\begin{aligned}
&= |f(c) - f(p)| \\
&< \epsilon
\end{aligned}
$$

as desired.

The proof is symmetric in the other two cases, with the help of Remark 12.2. □