Script 17

Sequences and Series of Functions

6/23: **Definition 17.1.** Let $A \subset \mathbb{R}$, and consider $X = \{f : A \to \mathbb{R}\}$, the collection of real-valued functions on A. A **sequence of functions** (on A) is an ordered list $(f_1, f_2, f_3, \dots)$ which we will denote (f_n), where each $f_n \in X$. (More formally, we can think of the sequence as a function $F : \mathbb{N} \to X$, where $f_n = F(n)$, for each $n \in \mathbb{N}$, but this degree of formality is not particularly helpful.)

We can take the sequence to start at any $n_0 \in \mathbb{Z}$ and not just at 1, just like we did for sequences of real numbers.

Definition 17.2. The sequence (f_n) **converges pointwise** to a function $f : A \to \mathbb{R}$ if for all $x \in A$ and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$. In other words, we have that for all $x \in A$, $\lim_{n\to\infty} f_n(x) = f(x)$.

Definition 17.3. The sequence (f_n) **converges uniformly** to a function $f : A \to \mathbb{R}$ if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$ for every $x \in A$.

Equivalently, the sequence (f_n) **converges uniformly** to a function $f : A \to \mathbb{R}$ if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $\sup_{x\in A} |f_n(x) - f(x)| < \epsilon$.

Exercise 17.4. Suppose that a sequence (f_n) converges pointwise to a function f. Prove that if (f_n) converges uniformly to a function g, then $f = g$.

Proof. To prove that $f = g$, Definition 1.16 tells us that it will suffice to show that $f(x) = g(x)$ for all $x \in A$. Suppose for the sake of contradiction that $f(x) \neq g(x)$ for some $x \in A$. Since (f_n) converges pointwise to f by hypothesis, Definition 17.2 implies that for all $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then $|f_n(x) - f(x)| < \epsilon$. Additionally, since (f_n) converges uniformly to g by hypothesis, Definition 17.3 asserts that for all $\epsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that if $n \geq N_2$, then $|f_n(x) - g(x)| < \epsilon$.

WLOG, let $f(x) > g(x)$. Choose $\epsilon = \frac{f(x)-g(x)}{2}$, and let $N = \max(N_1, N_2)$. Since $N \geq N_1$, $|f_N(x) - f(x)| < \frac{f(x)-g(x)}{2}$. Similarly, $|f_N(x) - g(x)| < \frac{f(x)-g(x)}{2}$. But this implies that

$$\begin{aligned}
f(x) - g(x) &= |f(x) - f_N(x) + f_N(x) - g(x)| \\
&\leq |f(x) - f_N(x)| + |f_N(x) - g(x)| && \text{Lemma 8.8} \\
&= |f_N(x) - f(x)| + |f_N(x) - g(x)| && \text{Exercise 8.5} \\
&< \frac{f(x)-g(x)}{2} + \frac{f(x)-g(x)}{2} \\
&= f(x) - g(x)
\end{aligned}$$

a contradiction. $\square$

Exercise 17.5. For each of the following sequences of functions, determine what function the sequence (f_n) converges to pointwise. Does the sequence converge uniformly to this function?

(a) For $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be given by $f_n(x) = x^n$.

Answer. Converges to the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$$

Does not converge uniformly. ☐

(b) For $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be given by $f_n(x) = \frac{\sin(nx)}{n}$. (For the purposes of this example, you may assume basic knowledge of sine.)

Answer. Converges to the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$. Does converge uniformly. ☐

(c) For $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} n^2 x & 0 \leq x \leq \frac{1}{n} \\ n(2 - nx) & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \frac{2}{n} \leq x \leq 1 \end{cases}$$

Answer. Converges to the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = 0$. Does not converge uniformly. ☐

Theorem 17.6. *Let (f_n) be a sequence of functions, and suppose that each $f_n : A \to \mathbb{R}$ is continuous. If (f_n) converges uniformly to $f : A \to \mathbb{R}$, then f is continuous.*

Proof. To prove that f is continuous, Theorem 9.10 tells us that it will suffice to show that f is continuous at every $x \in A$. Let x be an arbitrary element of A. To show that f is continuous at x, Theorem 11.5 tells us that it will suffice to verify that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary, and also let y be an arbitrary element of A. Since (f_n) converges uniformly, Definition 17.3 implies that there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(a) - f(a)| < \frac{\epsilon}{3}$ for all $a \in A$. Thus, $|f_N(x) - f(x)| < \frac{\epsilon}{3}$ and $|f_N(y) - f(y)| < \frac{\epsilon}{3}$. Additionally, since each f_n is continuous, Theorems 9.10 and 11.5 assert that there exists $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then $|f_N(y) - f_N(x)| < \frac{\epsilon}{3}$. Choose this δ to be our δ. Therefore,

$$\begin{aligned} |f(y) - f(x)| &\leq |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| && \text{Lemma 8.8} \\ &= |f_N(y) - f(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| && \text{Exercise 8.5} \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

as desired. ☐

6/26: **Theorem 17.7.** *Suppose that (f_n) is a sequence of integrable functions on $[a,b]$ and suppose that (f_n) converges uniformly to $f : [a,b] \to \mathbb{R}$. Then*

$$\int_a^b f = \lim_{n \to \infty} \int_a^b f_n$$

Lemma. *f is integrable on $[a,b]$.*

Proof. To prove that f is integrable on $[a,b]$, Theorem 13.18 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since (f_n) converges uniformly to f by hypothesis, Definition 17.3 asserts that there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|f_n(x) - f(x)| < \frac{\epsilon}{6(b-a)}$[1] for all $x \in [a,b]$. This statement will be useful in the verification of the three following results.

[1] For the purposes of this proof, we will assume that $a < b$, on the basis of the fact that the proof of the case where $a = b$ is trivial.

To confirm that $|U(f_N,P)-L(f_N,P)|<\frac{\epsilon}{3}$, we first invoke Theorem 13.18, which tells us that since f_N is integrable by hypothesis, there exists a partition P of $[a,b]$ such that $U(f_N,P)-L(f_N,P)<\frac{\epsilon}{3}$. Additionally, since $L(f_N,P)\leq U(f_N,P)$ by Theorem 13.13, we have by Definition 8.4 that $U(f_N,P)-L(f_N,P)=|U(f_N,P)-L(f_N,P)|$. Therefore, we have by transitivity that $|U(f_N,P)-L(f_N,P)|=U(f_N,P)-L(f_N,P)<\frac{\epsilon}{3}$, as desired.

To confirm that $|U(f,P)-U(f_N,P)|<\frac{\epsilon}{3}$, we begin with the following contradiction argument[2].

Suppose for the sake of contradiction that $|M_i(f)-M_i(f_N)|\geq\frac{\epsilon}{3(b-a)}$. We divide into two cases ($M_i(f)-M_i(f_N)\geq\frac{\epsilon}{3(b-a)}$ and $M_i(f_N)-M_i(f)\geq\frac{\epsilon}{3(b-a)}$). Suppose first that $M_i(f)-M_i(f_N)\geq\frac{\epsilon}{3(b-a)}$. By Lemma 5.11, there exists $f(x)\in\{f(x)\mid t_{i-1}\leq x\leq t_i\}$ such that $M_i(f)-\frac{\epsilon}{6(b-a)}<f(x)\leq M_i(f)$. Similarly, there exists $f_N(x)\in\{f_N(x)\mid t_{i-1}\leq x\leq t_i\}$ such that $M_i(f_N)-\frac{\epsilon}{6(b-a)}<f_N(x)\leq M_i(f_N)$. Thus, we have that

$$f(x)>M_i(f)-\frac{\epsilon}{6(b-a)}>M_i(f)-\frac{\epsilon}{3(b-a)}\geq M_i(f_N)\geq f_N(x)$$

It follows that

$$\begin{aligned}
|f(x)-f_N(x)| &= f(x)-f_N(x)\\
&> \left(M_i(f)-\frac{\epsilon}{6(b-a)}\right)-f_N(x)\\
&\geq \left(M_i(f)-\frac{\epsilon}{6(b-a)}\right)-M_i(f_N)\\
&= M_i(f)-M_i(f_N)-\frac{\epsilon}{6(b-a)}\\
&\geq \frac{\epsilon}{3(b-a)}-\frac{\epsilon}{6(b-a)}\\
&= \frac{\epsilon}{6(b-a)}
\end{aligned}$$

But this contradicts the previously proven fact that $|f(x)-f_N(x)|=|f_N(x)-f(x)|<\frac{\epsilon}{6(b-a)}$. The argument is symmetric in the other case.

Thus, we know that $|M_i(f)-M_i(f_N)|<\frac{\epsilon}{3(b-a)}$. Therefore, we have that

$$\begin{aligned}
|U(f,P)-U(f_N,P)| &= \left|\sum_{i=1}^k M_i(f)(t_i-t_{i-1})-\sum_{i=1}^k M_i(f_N)(t_i-t_{i-1})\right| && \text{Definition 13.10}\\
&= \left|\sum_{i=1}^k (M_i(f)-M_i(f_N)(t_i-t_{i-1})\right|\\
&< \left|\sum_{i=1}^k \frac{\epsilon}{3(b-a)}(t_i-t_{i-1})\right|\\
&= \frac{\epsilon}{3(b-a)}(b-a)\\
&= \frac{\epsilon}{3}
\end{aligned}$$

as desired.

The verification of the statement that $|L(f_N,P)-L(f,P)|<\frac{\epsilon}{3}$ is symmetric to the previous argument.

Having established that $|U(f_N,P)-L(f_N,P)|<\frac{\epsilon}{3}$, $|U(f,P)-U(f_N,P)|<\frac{\epsilon}{3}$, and $|L(f_N,P)-L(f,P)|<$

[2]Note that this argument is analogous to the proof of Additional Exercise 13.2.

$\frac{\epsilon}{3}$, we can now show that

$$\begin{aligned}
U(f,P)-L(f,P) &= |U(f,P)-L(f,P)| && \text{Theorem 13.13}\\
&\leq |U(f,P)-U(f_N,P)|+|U(f_N,P)-L(f_N,P)|+|L(f_N,P)-L(f,P)|\\
&< \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}\\
&= \epsilon
\end{aligned}$$

as desired. □

Proof of Theorem 17.7. To prove that $\lim_{n\to\infty}\int_a^b f_n = \int_a^b f$, Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|\int_a^b f_n - \int_a^b f| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since (f_n) converges uniformly to f, we have by Definition 17.3 that there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$ for all $x \in [a,b]$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. It follows from the lemma to Exercise 8.9 that $-\frac{\epsilon}{b-a} < f_n(x) - f(x) < \frac{\epsilon}{b-a}$ for all $x \in [a,b]$. Additionally, since f_n is integrable on $[a,b]$ by hypothesis and f is integrable on $[a,b]$ by the lemma, Theorem 13.24 implies that $f_n - f$ is integrable on $[a,b]$. Combining these last two results, we have by Theorem 13.27 that $-\frac{\epsilon}{b-a}(b-a) < \int_a^b (f_n - f) < \frac{\epsilon}{b-a}(b-a)$. Consequently, by Script 7 and the lemma to Exercise 8.9, we have that $|\int_a^b (f_n - f)| < \epsilon$. Therefore, by Theorem 13.24, we have that $|\int_a^b f_n - \int_a^b f| < \epsilon$, as desired. □

Theorem 17.8. *Let (f_n) be a sequence of functions defined on an open interval containing $[a,b]$ such that each f_n is differentiable on $[a,b]$ and f_n' is integrable on $[a,b]$. Suppose further that (f_n) converges pointwise to f on $[a,b]$ and that (f_n') converges uniformly to a continuous function g on $[a,b]$. Then f is differentiable at every $x \in [a,b]$ and*

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

Proof. Let x be an arbitrary element of $[a,b]$. Since (f_n') converges uniformly to g, Definition 17.3 and Theorem 15.7 imply that $\lim_{n\to\infty} f_n'(x) = g(x)$. Additionally, we have that

$$\begin{aligned}
\int_a^x g &= \lim_{n\to\infty}\int_a^x f_n' && \text{Theorem 17.7}\\
&= \lim_{n\to\infty}(f_n(x) - f_n(a)) && \text{Theorem 14.4}\\
&= \lim_{n\to\infty} f_n(x) - \lim_{n\to\infty} f_n(a) && \text{Theorem 15.9}\\
&= f(x) - f(a) && \text{Definition 17.2}
\end{aligned}$$

This combined with the fact that g is continuous (hence continuous at x by Theorem 9.10) implies that

$$\begin{aligned}
g(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(f(x) - f(a)) && \text{Theorem 14.1}\\
&= \frac{\mathrm{d}}{\mathrm{d}x}(f(x)) - \frac{\mathrm{d}}{\mathrm{d}x}(f(a)) && \text{Exercise 12.9}\\
&= f'(x) && \text{Exercise 12.8}
\end{aligned}$$

Therefore, we have by transitivity that $f'(x) = \lim_{n\to\infty} f_n'(x)$, as desired. □

Theorem 17.9. *Let (f_n) be a sequence of functions defined on a set A. Then the following are equivalent.*

(a) *There is some function f such that (f_n) converges uniformly to f on A.*

(b) *For all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that when $m, n \geq N$, $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in A$.*

Proof. Suppose first that there is some function f to which (f_n) converges uniformly on A. Let $\epsilon > 0$ be arbitrary. By Definition 17.3, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ for all $x \in A$. Choose this N to be our N. Let n, m be arbitrary natural numbers such that $n, m \geq N$. Then $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ and $|f_m(x) - f(x)| < \frac{\epsilon}{2}$ for all $x \in A$. Therefore, we have that

$$\begin{aligned}
|f_n(x) - f_m(x)| &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| && \text{Lemma 8.8} \\
&= |f_n(x) - f(x)| + |f_m(x) - f(x)| && \text{Exercise 8.5} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

for all $x \in A$, as desired.

Now suppose that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that when $n, m \geq N$, $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in A$. It follows by Theorem 15.19 that $(f_n(x))$ converges for all $x \in A$, i.e., for all $x \in A$, there exists a point $f(x) \in \mathbb{R}$ to which $(f_n(x))$ converges. Let $f : A \to \mathbb{R}$ be defined by $f(x) = \lim_{n\to\infty} f_n(x)$.

To prove that (f_n) converges uniformly to f, Definition 17.3 tells us that it will suffice to show that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$ for all $x \in A$. Let $\epsilon > 0$ be arbitrary. By the hypothesis, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|f_n(x) - f_m(x)| < \frac{\epsilon}{2}$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$, and let x be an arbitrary element of A. Since $(f_m(x))$ converges to $f(x)$, Theorem 15.7 asserts that there exists an $N' \in \mathbb{N}$ such that for all $m \geq N'$, $|f_m(x) - f(x)| < \frac{\epsilon}{2}$. Choose $M = \max(N, N')$. It follows that

$$\begin{aligned}
|f_n(x) - f(x)| &\leq |f_n(x) - f_M(x)| + |f_M(x) - f(x)| && \text{Lemma 8.8} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

as desired. □

Definition 17.10. We define series of functions the same way we defined series of numbers. That is, given a sequence (f_n), define the sequence of partial sums (p_n) by $p_n(x) = f_1(x) + \cdots + f_n(x)$ and say that $\sum_{n=1}^{\infty} f_n$ converges pointwise or converges uniformly to f if the sequence (p_n) does.

Theorem 17.11. *Suppose that $f_n : A \to \mathbb{R}$ is a sequence of functions and that there exists a sequence of positive real numbers (M_n) such that for all $x \in A$, we have $|f_n(x)| \leq M_n$. If $\sum_{n=1}^{\infty} M_n$ converges, then for each $x \in A$, the series of numbers $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely. Furthermore, $\sum_{n=1}^{\infty} f_n$ converges uniformly to the function f defined by $f(x) = \sum_{n=1}^{\infty} f_n(x)$.*

Proof. Let x be an arbitrary element of A. To prove that $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely, Definition 16.9 tells us that it will suffice to show that $\sum_{n=1}^{\infty} |f_n(x)|$ converges. Since (M_n) is a sequence of positive numbers and $|f_n(x)| \leq M_n$ for all $n \geq 1$, the proof of Theorem 16.13 asserts that $\sum_{n=1}^{\infty} |f_n(x)|$ converges.

To prove that $\sum_{n=1}^{\infty} f_n$ converges uniformly to f, Definition 17.10 tells us that it will suffice to show that the sequence of partial sums (p_n) defined by $p_k(x) = \sum_{n=1}^{k} f_n(x)$ converges uniformly to f. To do this, Definition 17.3 tells us that it will suffice to verify that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $j \geq N$, then $|\sum_{n=1}^{j} f_n(x) - f(x)| < \epsilon$ for all $x \in A$. Let $\epsilon > 0$ be arbitrary. By Definition 16.1, $\sum_{n=1}^{\infty} M_n = \lim_{k\to\infty} \sum_{n=1}^{k} M_n$. Thus, by Theorem 15.7, there is some $N \in \mathbb{N}$ such that for all $j \geq N$, $|\sum_{n=1}^{j} M_n - \sum_{n=1}^{\infty} M_n| < \epsilon$. Choose this N to be our N. Let j be an arbitrary natural number such that $j \geq N$. It follows by Script 16 that $|\sum_{n=j+1}^{\infty} M_n| < \epsilon$. Additionally, since (M_n) is a sequence of positive numbers, $\sum_{n=j+1}^{\infty} M_n = |\sum_{n=j+1}^{\infty} M_n|$. Therefore, combining the last several results and letting x be an arbitrary element of A, we have that

$$\begin{aligned}
\left|\sum_{n=1}^{j} f_n(x) - f(x)\right| &= \left|\sum_{n=1}^{j} f_n(x) - \sum_{n=1}^{\infty} f_n(x)\right| \\
&= \left|\sum_{n=j+1}^{\infty} f_n(x)\right| \\
&\leq \sum_{n=j+1}^{\infty} |f_n(x)| && \text{Theorem 16.11} \\
&\leq \sum_{n=j+1}^{\infty} M_n \\
&= \left|\sum_{n=j+1}^{\infty} M_n\right| \\
&< \epsilon
\end{aligned}$$

as desired. $\square$