

# Script 18

## The Euclidean Space $\mathbb{R}^n$

7/7: For the next three sheets, we will be studying multivariable calculus, that is “calculus on  $\mathbb{R}^n$ .” First, we need to understand the space  $\mathbb{R}^n$ .

**Definition 18.1.** The **Euclidean  $n$ -space**  $\mathbb{R}^n$  is the  $n$ -fold Cartesian product of  $\mathbb{R}$ . Symbolically,

$$\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{R}\}$$

is the set of  $n$ -tuples of real numbers. We often write

$$\mathbf{x} = (x_1, \dots, x_n)$$

to denote an element, which is also referred to as a **vector**, in  $\mathbb{R}^n$  and

$$\mathbf{0} = (0, \dots, 0)$$

**Definition 18.2.** Let  $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$  and  $\lambda \in \mathbb{R}$ . We define the following operations.

- (a) (Addition)  $\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_n + y_n)$ .
- (b) (Scalar Multiplication)  $\lambda \mathbf{x} = (\lambda x_1, \dots, \lambda x_n)$ .

**Exercise 18.3.** Prove that the addition on  $\mathbb{R}^n$  satisfies FA1-FA4 (see Definition 7.8). Moreover, prove that

VS1. (Associativity of Scalar Multiplication) If  $\lambda, \mu \in \mathbb{R}$  and  $\mathbf{x} \in \mathbb{R}^n$ , then  $(\lambda\mu)\mathbf{x} = \lambda(\mu\mathbf{x})$ .

VS2. (Distributivity of Scalars) If  $\lambda, \mu \in \mathbb{R}$  and  $\mathbf{x} \in \mathbb{R}^n$ , then  $(\lambda + \mu)\mathbf{x} = \lambda\mathbf{x} + \mu\mathbf{x}$ .

VS3. (Distributivity of Vectors) If  $\lambda \in \mathbb{R}$  and  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ , then  $\lambda(\mathbf{x} + \mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y}$ .

VS4. (Scalar Multiplicative Identity) If  $\mathbf{x} \in \mathbb{R}^n$ , then  $1\mathbf{x} = \mathbf{x}$ .

These eight properties together are called the **vector space axioms**.

*Proof.* To prove that  $\mathbb{R}^n$  obeys FA1 from Definition 7.8, it will suffice to show that for all  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ ,  $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ . Let  $\mathbf{x}, \mathbf{y}$  be arbitrary elements of  $\mathbb{R}^n$ . Then by Definition 18.2,

$$\begin{aligned}\mathbf{x} + \mathbf{y} &= (x_1 + y_1, \dots, x_n + y_n) \\ &= (y_1 + x_1, \dots, y_n + x_n) \\ &= \mathbf{y} + \mathbf{x}\end{aligned}$$

as desired.

To prove that  $\mathbb{R}^n$  obeys FA2 from Definition 7.8, it will suffice to show that for all  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ ,  $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ . Let  $\mathbf{x}, \mathbf{y}, \mathbf{z}$  be arbitrary elements of  $\mathbb{R}^n$ . Then by Definition 18.2,

$$\begin{aligned} (\mathbf{x} + \mathbf{y}) + \mathbf{z} &= (x_1 + y_1, \dots, x_n + y_n) + \mathbf{z} \\ &= ((x_1 + y_1) + z_1, \dots, (x_n + y_n) + z_n) \\ &= (x_1 + (y_1 + z_1), \dots, x_n + (y_n + z_n)) \\ &= \mathbf{x} + (y_1 + z_1, \dots, y_n + z_n) \\ &= \mathbf{x} + (\mathbf{y} + \mathbf{z}) \end{aligned}$$

as desired.

To prove that  $\mathbb{R}^n$  obeys FA3 from Definition 7.8, it will suffice to find an element  $\mathbf{0} \in \mathbb{R}^n$  such that  $\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$  for all  $\mathbf{x} \in \mathbb{R}^n$ . Choose  $\mathbf{0}$  to be our  $\mathbf{0}$ . Let  $\mathbf{x}$  be an arbitrary element of  $\mathbb{R}^n$ . Then by Definition 18.2,

$$\begin{aligned} \mathbf{x} + \mathbf{0} &= (x_1 + 0, \dots, x_n + 0) \\ &= (x_1, \dots, x_n) \\ &= \mathbf{x} \\ &= (0 + x_1, \dots, 0 + x_n) \\ &= \mathbf{0} + \mathbf{x} \end{aligned}$$

as desired.

To prove that  $\mathbb{R}^n$  obeys FA4 from Definition 7.8, it will suffice to show that for all  $\mathbf{x} \in \mathbb{R}^n$ , there exists  $\mathbf{y} \in \mathbb{R}^n$  such that  $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x} = \mathbf{0}$ . Let  $\mathbf{x}$  be an arbitrary element of  $\mathbb{R}^n$ . Choose  $\mathbf{y} = (-x_1, \dots, -x_n)$ . Then by Definition 18.2,

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1 + (-x_1), \dots, x_n + (-x_n)) \\ &= (0, \dots, 0) \\ &= \mathbf{0} \\ &= ((-x_1) + x_1, \dots, (-x_n) + x_n) \\ &= \mathbf{y} + \mathbf{x} \end{aligned}$$

as desired.

To prove that  $\mathbb{R}^n$  obeys VS1, it will suffice to show that for all  $\lambda, \mu \in \mathbb{R}$  and  $\mathbf{x} \in \mathbb{R}^n$ , we have  $(\lambda\mu)\mathbf{x} = \lambda(\mu\mathbf{x})$ . Let  $\lambda, \mu$  be arbitrary elements of  $\mathbb{R}$ , and let  $\mathbf{x}$  be an arbitrary element of  $\mathbb{R}^n$ . Then by Definition 18.2,

$$\begin{aligned} (\lambda\mu)\mathbf{x} &= ((\lambda\mu)x_1, \dots, (\lambda\mu)x_n) \\ &= (\lambda(\mu x_1), \dots, \lambda(\mu x_n)) \\ &= \lambda(\mu\mathbf{x}) \end{aligned}$$

as desired.

To prove that  $\mathbb{R}^n$  obeys VS2, it will suffice to show that for all  $\lambda, \mu \in \mathbb{R}$  and  $\mathbf{x} \in \mathbb{R}^n$ , we have  $(\lambda + \mu)\mathbf{x} = \lambda\mathbf{x} + \mu\mathbf{x}$ . Let  $\lambda, \mu$  be arbitrary elements of  $\mathbb{R}$ , and let  $\mathbf{x}$  be an arbitrary element of  $\mathbb{R}^n$ . Then by Definition 18.2,

$$\begin{aligned} (\lambda + \mu)\mathbf{x} &= ((\lambda + \mu)x_1, \dots, (\lambda + \mu)x_n) \\ &= (\lambda x_1 + \mu x_1, \dots, \lambda x_n + \mu x_n) \\ &= (\lambda x_1, \dots, \lambda x_n) + (\mu x_1, \dots, \mu x_n) \\ &= \lambda\mathbf{x} + \mu\mathbf{x} \end{aligned}$$

as desired.

To prove that  $\mathbb{R}^n$  obeys VS3, it will suffice to show that for all  $\lambda \in \mathbb{R}$  and  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ , we have  $\lambda(\mathbf{x} + \mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y}$ . Let  $\lambda$  be an arbitrary element of  $\mathbb{R}$ , and let  $\mathbf{x}, \mathbf{y}$  be arbitrary elements of  $\mathbb{R}^n$ . Then by Definition 18.2,

$$\begin{aligned}\lambda(\mathbf{x} + \mathbf{y}) &= \lambda(x_1 + y_1, \dots, x_n + y_n) \\ &= (\lambda(x_1 + y_1), \dots, \lambda(x_n + y_n)) \\ &= (\lambda x_1 + \lambda y_1, \dots, \lambda x_n + \lambda y_n) \\ &= (\lambda x_1, \dots, \lambda x_n) + (\lambda y_1, \dots, \lambda y_n) \\ &= \lambda\mathbf{x} + \lambda\mathbf{y}\end{aligned}$$

as desired.

To prove that  $\mathbb{R}^n$  obeys VS4, it will suffice to show that for all  $\mathbf{x} \in \mathbb{R}^n$ , we have  $1\mathbf{x} = \mathbf{x}$ . Let  $\mathbf{x}$  be an arbitrary element of  $\mathbb{R}^n$ . Then by Definition 18.2,

$$\begin{aligned}1\mathbf{x} &= (1x_1, \dots, 1x_n) \\ &= (x_1, \dots, x_n) \\ &= \mathbf{x}\end{aligned}$$

as desired. □

**Remark 18.4.** Since  $\mathbb{R}^n$  with the two operations defined as above satisfies these eight axioms, we call  $\mathbb{R}^n$  a **vector space**.

**Exercise 18.5.** Prove that if  $\mathbf{x} \in \mathbb{R}^n$ , then  $0\mathbf{x} = \mathbf{0}$ .

*Proof.* By Definition 18.2, we have that

$$\begin{aligned}0\mathbf{x} &= (0x_1, \dots, 0x_n) \\ &= (0, \dots, 0) \\ &= \mathbf{0}\end{aligned}$$

as desired. □

**Definition 18.6.** Let  $\mathbf{x} \in \mathbb{R}^n$ . The **norm** of  $\mathbf{x}$  is defined as

$$\|\mathbf{x}\| = \sqrt{x_1^2 + \dots + x_n^2}$$

**Definition 18.7.** We call  $\|\mathbf{y} - \mathbf{x}\|$  the **distance** between  $\mathbf{x}$  and  $\mathbf{y}$ .

**Remark 18.8.** If  $n = 1$ , the norm coincides with the definition of the absolute value in  $\mathbb{R}$ .

**Lemma 18.9.**

(a) If  $x, y \in \mathbb{R}$ , then  $xy \leq \frac{x^2 + y^2}{2}$ .

*Proof.* Let  $x, y$  be arbitrary elements of  $\mathbb{R}$ . Then by Lemma 7.26,  $0 \leq (x - y)^2$ . Therefore, we have that

$$\begin{aligned}xy &= \frac{2xy + 0}{2} \\ &\leq \frac{2xy + (x - y)^2}{2} \\ &= \frac{2xy + x^2 - 2xy + y^2}{2} \\ &= \frac{x^2 + y^2}{2}\end{aligned}$$

as desired. □

(b) If  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ , then  $|x_1y_1 + \cdots + x_ny_n| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|$ .

*Proof.* Suppose first that  $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$ . Then by Definition 18.6,  $\|\mathbf{x}\| = 1 = \sqrt{x_1^2 + \cdots + x_n^2}$ , from which it follows that  $1 = x_1^2 + \cdots + x_n^2$ . Therefore, we have that

$$\begin{aligned}
 |x_1y_1 + \cdots + x_ny_n| &\leq |x_1y_1| + \cdots + |x_ny_n| && \text{Lemma 8.8} \\
 &= |x_1||y_1| + \cdots + |x_n||y_n| \\
 &\leq \frac{|x_1|^2 + |y_1|^2}{2} + \cdots + \frac{|x_n|^2 + |y_n|^2}{2} && \text{Lemma 18.9a} \\
 &= \frac{x_1^2 + y_1^2}{2} + \cdots + \frac{x_n^2 + y_n^2}{2} \\
 &= \frac{(x_1^2 + \cdots + x_n^2) + (y_1^2 + \cdots + y_n^2)}{2} \\
 &= \frac{1 + 1}{2} \\
 &= 1
 \end{aligned}$$

as desired.

Now let  $\mathbf{x}, \mathbf{y}$  be arbitrary elements of  $\mathbb{R}^n$ . Consider the vectors  $\mathbf{u}_\mathbf{x}, \mathbf{u}_\mathbf{y}$  defined by  $\mathbf{u}_\mathbf{x} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$  and  $\mathbf{u}_\mathbf{y} = \frac{\mathbf{y}}{\|\mathbf{y}\|}$ . By the proof of the first case, we have that

$$\begin{aligned}
 |x_1y_1 + \cdots + x_ny_n| &= \|\mathbf{x}\| \cdot \|\mathbf{y}\| \cdot \left| \frac{x_1y_1}{\|\mathbf{x}\| \cdot \|\mathbf{y}\|} + \cdots + \frac{x_ny_n}{\|\mathbf{x}\| \cdot \|\mathbf{y}\|} \right| \\
 &= \|\mathbf{x}\| \cdot \|\mathbf{y}\| \cdot |u_{\mathbf{x}_1}u_{\mathbf{y}_1} + \cdots + u_{\mathbf{x}_n}u_{\mathbf{y}_n}| \\
 &\leq \|\mathbf{x}\| \cdot \|\mathbf{y}\| \cdot 1 \\
 &= \|\mathbf{x}\| \cdot \|\mathbf{y}\|
 \end{aligned}$$

as desired. □

**Theorem 18.10.** If  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  and  $\lambda \in \mathbb{R}$ , then

(a)  $\|\mathbf{x}\| \geq 0$ . Moreover,  $\|\mathbf{x}\| = 0$  if and only if  $\mathbf{x} = \mathbf{0}$ .

*Proof.* Let  $\mathbf{x}$  be an arbitrary element of  $\mathbb{R}^n$ .

We first prove that  $\|\mathbf{x}\| \geq 0$ . By Lemma 7.26,  $x_i^2 \geq 0$  for all  $i \in [n]$ . Thus, by Definition 7.21,  $x_1^2 + \cdots + x_n^2 \geq 0$ . Therefore, we have by Definition 18.6 that  $\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2} \geq 0$ , as desired.

We now prove that  $\|\mathbf{x}\| = 0$  if and only if  $\mathbf{x} = \mathbf{0}$ . Suppose first that  $\|\mathbf{x}\| = 0$ . Then by Definition 18.6 and Script 7,  $x_1^2 + \cdots + x_n^2 = 0$ . Now suppose for the sake of contradiction that  $\mathbf{x} \neq \mathbf{0}$ . Then there exists an  $x_i$  such that  $x_i \neq 0$ . Thus, by Lemma 7.26,  $x_i^2 > 0$ . Additionally,  $x_j^2 \geq 0$  for all  $j \in [n]$ . Thus, we have that  $0 < x_i^2 \leq x_1^2 + \cdots + x_n^2$ . But by Definition 3.1, this implies that  $x_1^2 + \cdots + x_n^2 \neq 0$ , a contradiction.

Now suppose that  $\mathbf{x} = \mathbf{0}$ . Then by Definition 18.6,  $\|\mathbf{x}\| = \sqrt{0^2 + \cdots + 0^2} = 0$ , as desired. □

(b)  $\|\lambda\mathbf{x}\| = |\lambda| \cdot \|\mathbf{x}\|$ .

*Proof.* Let  $\lambda$  be an arbitrary element of  $\mathbb{R}$ , and let  $\mathbf{x}$  be an arbitrary element of  $\mathbb{R}^n$ . Then we have that

$$\begin{aligned}
 \|\lambda\mathbf{x}\| &= \sqrt{(\lambda x_1)^2 + \cdots + (\lambda x_n)^2} && \text{Definition 18.6} \\
 &= |\lambda| \cdot \sqrt{x_1^2 + \cdots + x_n^2} \\
 &= |\lambda| \cdot \|\mathbf{x}\| && \text{Definition 18.6}
 \end{aligned}$$

as desired. □

$$(c) \|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

*Proof.* Let  $\mathbf{x}, \mathbf{y}$  be arbitrary elements of  $\mathbb{R}^n$ . Then we have that

$$\begin{aligned} \|\mathbf{x} + \mathbf{y}\| &= \sqrt{(x_1 + y_1)^2 + \cdots + (x_n + y_n)^2} && \text{Definition 18.6} \\ &= \sqrt{x_1^2 + \cdots + x_n^2 + 2x_1y_1 + \cdots + 2x_ny_n + y_1^2 + \cdots + y_n^2} \\ &\leq \sqrt{\|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|\mathbf{y}\|^2} && \text{Lemma 18.9} \\ &= \sqrt{(\|\mathbf{x}\| + \|\mathbf{y}\|)^2} \\ &= \|\mathbf{x}\| + \|\mathbf{y}\| \end{aligned}$$

as desired.  $\square$

**Corollary 18.11.** If  $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$  and  $\lambda \in \mathbb{R}$ , then

$$(a) \|\mathbf{x} - \mathbf{z}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{z}\|.$$

$$(b) \left| \|\mathbf{x}\| - \|\mathbf{y}\| \right| \leq \|\mathbf{x} - \mathbf{y}\|.$$

*Proof.* The proofs are symmetric to those of Lemma 8.8.  $\square$

7/10: The next goal is to “topologize”  $\mathbb{R}^n$ . To discuss topology on  $\mathbb{R}^n$ , we first need to introduce notions for  $\mathbb{R}^n$  that are analogous to open and closed intervals for  $\mathbb{R}$ .

**Remark 18.12.** For  $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$  and  $\mathbf{y} = (y_1, \dots, y_m) \in \mathbb{R}^m$ , we identify  $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^m$  with  $(x_1, \dots, x_n, y_1, \dots, y_m) \in \mathbb{R}^{n+m}$ . So if  $A \subset \mathbb{R}^n$  and  $B \subset \mathbb{R}^m$ , we can consider  $A \times B$  to be a subset of  $\mathbb{R}^{n+m}$ .

If also  $C \in \mathbb{R}^k$ , then  $(A \times B) \times C$  and  $A \times (B \times C)$  correspond to the same subset of  $\mathbb{R}^{n+m+k}$  under this identification; we write  $A \times B \times C$  for this set.

**Definition 18.13.** An **open rectangle** in  $\mathbb{R}^n$  is a set of the form  $(a_1, b_1) \times \cdots \times (a_n, b_n)$ , a product of open intervals. Similarly, a **closed rectangle** in  $\mathbb{R}^n$  is a set of the form  $[a_1, b_1] \times \cdots \times [a_n, b_n]$ . We allow the possibility that  $a_j = b_j$  (where  $[a_j, a_j] = \{a_j\}$ ). If there is at least one  $j$  with  $a_j = b_j$ , then we say that the rectangle is **degenerate**; otherwise, we say that the rectangle is **non-degenerate**.

**Definition 18.14.** A subset  $U \subset \mathbb{R}^n$  is **open** if for all  $\mathbf{x} \in U$ , there exists an open rectangle  $R$  such that  $\mathbf{x} \in R \subset U$ . A subset  $C \subset \mathbb{R}^n$  is **closed** if its complement is open.

**Exercise 18.15.** Decide whether each of the following is an open set in  $\mathbb{R}^2$ .

$$(a) \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}, x_1 > 0, x_2 > 0\}.$$

*Proof.* To prove that  $U = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}, x_1 > 0, x_2 > 0\}$  is open, Definition 18.14 tells us that it will suffice to show that for all  $\mathbf{x} \in U$ , there exists an open rectangle  $R$  such that  $\mathbf{x} \in R \subset U$ . Let  $\mathbf{x}$  be an arbitrary element of  $U$ . Then by the definition of  $U$ ,  $0 < x_1$  and  $0 < x_2$ . It follows by Theorem 5.2 and Corollary 6.12, there exist  $a_1, b_1, a_2, b_2$  such that  $0 < a_1 < x_1 < b_1$  and  $0 < a_2 < x_2 < b_2$ . Thus, by Equations 8.1,  $x_1 \in (a_1, b_1)$  and  $x_2 \in (a_2, b_2)$ . Consequently, if we let  $R = (a_1, b_1) \times (a_2, b_2)$ , Definition 18.13 guarantees that  $R$  is an open rectangle. Additionally, Definition 1.15 asserts that  $(x_1, x_2) = \mathbf{x} \in R$ , as desired. Additionally, if  $\mathbf{y}$  is any vector in  $R$ , then by the definition of  $R$ ,  $0 < a_1 < y_1$  and  $0 < a_2 < y_2$ . Thus, by transitivity,  $\mathbf{y} \in U$ . Therefore, by Definition 1.3,  $R \subset U$ , as desired.  $\square$

$$(b) \{(x, 0) \mid x \in \mathbb{R}\}.$$

*Proof.* To prove that  $U = \{(x, 0) \mid x \in \mathbb{R}\}$  is not open, Definition 18.14 tells us that it will suffice to find an  $\mathbf{x} \in U$  such that for all open rectangles  $R$  containing  $\mathbf{x}$ ,  $R \not\subset U$ . Let  $\mathbf{x} = (0, 0)$ , and let  $R$  be an arbitrary open rectangle containing  $\mathbf{x}$ . By Definitions 18.13 and 1.15 along with Equations 8.1,  $a_1 < 0 < b_1$  and  $a_2 < 0 < b_2$ . Thus, by consecutive applications of Theorem 5.2, there exist points  $y_1, y_2 \in \mathbb{R}$  such that  $a_1 < y_1 < 0$  and  $a_2 < y_2 < 0$ . It follows that  $\mathbf{y} = (y_1, y_2) \in R$ . However, since  $y_2 \neq 0$  by Definition 3.1,  $\mathbf{y} \notin U$ . Therefore, by Definition 1.3,  $R \not\subset U$ , as desired.  $\square$

**Exercise 18.16.** Show that if  $R_1, \dots, R_m$  are open rectangles containing  $\mathbf{x} \in \mathbb{R}^n$ , then  $R = R_1 \cap \dots \cap R_m$  is an open rectangle containing  $\mathbf{x} \in \mathbb{R}^n$ . If  $R = (a_1, b_1) \times \dots \times (a_n, b_n)$ , derive formulas for  $a_i$  and  $b_i$  in terms of the corresponding quantities for  $R_1, \dots, R_m$ .

*Proof.* Let  $R_i = (r_{ij}, s_{ij})_{j=1}^n$  for all  $i \in [m]$ . To prove that  $R = \bigcap_{i=1}^m R_i$  is an open rectangle containing  $\mathbf{x}$ , Definitions 18.13 and 1.15 tell us that it will suffice to show that  $R$  is the Cartesian product of open intervals, each containing its respective  $x_j$ . Since  $\mathbf{x} \in R_i$  for all  $i \in [m]$ , we have by Definition 1.15 that  $x_j \in (r_{ij}, s_{ij})$  for all  $i \in [m]$ ,  $j \in [n]$ . Thus, by Corollary 3.19,  $\bigcap_{i=1}^m (r_{ij}, s_{ij})$  is a region (hence an open interval by Corollary 4.11 and Lemma 8.3) containing  $x_j$  for all  $j \in [n]$ . Therefore, since  $R = \bigcap_{i=1}^m R_i = \prod_{i=1}^n (\bigcap_{i=1}^m (r_{ij}, s_{ij}))$  by Script 1, we have that  $R$  is the Cartesian product of open intervals, each containing its respective  $x_j$ , as desired.

Let  $a_j = \max_{i=1}^m (r_{ij})$  and let  $b_j = \min_{i=1}^m (s_{ij})$  for all  $j \in [n]$ . To prove that  $R = (a_j, b_j)_{j=1}^n$ , Definition 1.2 tells us that it will suffice to show that every  $\mathbf{x} \in R$  is an element of  $(a_j, b_j)_{j=1}^n$  and vice versa. Suppose first that  $\mathbf{x}$  is an arbitrary element of  $R$ . Then by Definition 1.6,  $\mathbf{x} \in R_i$  for all  $i \in [m]$ . It follows by Definition 1.15 that  $x_j \in (r_{ij}, s_{ij})$  for all  $i \in [m]$ ,  $j \in [n]$ , including the  $j, j'$  for which  $r_{ij}$  is at its maximum and  $s_{ij'}$  is at its minimum. In other words,  $x_j \in (a_j, b_j)$  for all  $j \in [n]$ . Therefore, by Definition 1.15,  $\mathbf{x} \in (a_j, b_j)_{j=1}^n$ , as desired. The proof is symmetric in the other direction.  $\square$

**Definition 18.17.** The **open ball** (in  $\mathbb{R}^n$  with center  $\mathbf{p}$  and radius  $r > 0$ ) is defined as

$$B(\mathbf{p}, r) = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{p}\| < r\}$$

The **closed ball** (in  $\mathbb{R}^n$  with center  $\mathbf{p}$  and radius  $r > 0$ ) is defined as

$$\overline{B}(\mathbf{p}, r) = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{p}\| \leq r\}$$

**Remark 18.18.** In  $\mathbb{R}^1$ , an open rectangle is also an open ball, and vice versa.

The following results illustrate how open rectangles and open balls in  $\mathbb{R}^n$  are “compatible” with each other.

**Lemma 18.19.** Fix  $\mathbf{x} \in \mathbb{R}^n$ .

(a) If  $R$  is an open rectangle containing  $\mathbf{x}$ , then there exists  $r > 0$  such that  $B(\mathbf{x}, r) \subset R$ .

*Proof.* Since  $\mathbf{x} \in R$ , Definitions 18.13 and 1.15 tell us that  $x_i \in (a_i, b_i)$  for all  $i \in [n]$ . Additionally, we know by Corollary 4.11 and Lemma 8.3 that each  $(a_i, b_i)$  is an open interval. Combining the last two results, we have by Lemma 8.10 that for each  $i \in [n]$ , there exists  $\delta_i > 0$  such that  $(x_i - \delta_i, x_i + \delta_i) \subset (a_i, b_i)$ . Let  $r = \min\{\delta_i\}_{i=1}^n$ .

To prove that  $B(\mathbf{x}, r) \subset R$ , Definition 1.3 tells us that it will suffice to show that every  $\mathbf{y} \in B(\mathbf{x}, r)$  is an element of  $R$ . Let  $\mathbf{y}$  be an arbitrary element of  $B(\mathbf{x}, r)$ . Then by Definition 18.17,  $\|\mathbf{y} - \mathbf{x}\| < r$ . It follows that

$$\begin{aligned} |y_i - x_i| &= \sqrt{(y_i - x_i)^2} \\ &\leq \sqrt{(y_1 - x_1)^2 + \dots + (y_n - x_n)^2} && \text{Lemma 7.26} \\ &= \|\mathbf{y} - \mathbf{x}\| && \text{Definition 18.6} \\ &< r \end{aligned}$$

for all  $i \in [n]$ . Thus, by the definition of  $r$ ,  $|y_i - x_i| \leq \delta_i$  for all  $i \in [n]$ . Consequently, by Exercise 8.9 and Definition 1.3,  $y_i \in (a_i, b_i)$  for all  $i \in [n]$ . Therefore, by Definitions 1.15 and 18.13,  $\mathbf{y} \in R$ , as desired.  $\square$

(b) If  $B$  is an open ball containing  $\mathbf{x}$ , then there exists an open rectangle  $R$  such that  $\mathbf{x} \in R \subset B$ .

**Lemma.** If  $\mathbf{x} \in \mathbb{R}^n$ , then  $\|\mathbf{x}\| \leq \sum_{i=1}^n |x_i|$ .

*Proof.* By Definition 18.2, we can decompose  $\mathbf{x}$  into the sum of  $n$  unit vectors  $\mathbf{u}_i$  (where  $\mathbf{u}_i$  points one unit in the  $i^{\text{th}}$  direction), each scaled by  $x_i$ ; symbolically, let  $\mathbf{x} = \sum_{i=1}^n x_i \mathbf{u}_i$ . Therefore,

$$\begin{aligned}
 \|\mathbf{x}\| &= \left\| \sum_{i=1}^n x_i \mathbf{u}_i \right\| \\
 &= \sum_{i=1}^n \|x_i \mathbf{u}_i\| && \text{Theorem 18.10c} \\
 &= \sum_{i=1}^n |x_i| \cdot \|\mathbf{u}_i\| && \text{Theorem 18.10b} \\
 &= \sum_{i=1}^n |x_i| \cdot \sqrt{1^2} && \text{Definition 18.6} \\
 &= \sum_{i=1}^n |x_i|
 \end{aligned}$$

as desired.  $\square$

*Proof of Lemma 18.19b.* Suppose  $\mathbf{x} \in B(\mathbf{y}, r)$ . Then by Definition 18.17,  $\|\mathbf{x} - \mathbf{y}\| < r$ . Thus, we can define  $r' = r - \|\mathbf{x} - \mathbf{y}\|$  such that  $r' > 0$ . With this term defined, we can let  $R = (x_i - \frac{r'}{n}, x_i + \frac{r'}{n})_{i=1}^n$ .

To prove that  $\mathbf{x} \in R$ , Definition 18.13 tells us that it will suffice to show that  $x_i \in (x_i - \frac{r'}{n}, x_i + \frac{r'}{n})$  for all  $i \in [n]$ . But since  $|x_i - x_i| = 0 < \frac{r'}{n}$  for all  $i \in [n]$ , Exercise 8.9 asserts that this is true.

To prove that  $R \subset B$ , Definition 1.3 tells us that it will suffice to show that every  $\mathbf{z} \in R$  is an element of  $B$ . Let  $\mathbf{z}$  be an arbitrary element of  $R$ . Then by Definition 18.13,  $z_i \in (x_i - \frac{r'}{n}, x_i + \frac{r'}{n})$  for all  $i \in [n]$ . It follows by Exercise 8.9 that  $|z_i - x_i| < \frac{r'}{n}$  for all  $i \in [n]$ . Consequently,

$$\begin{aligned}
 \|\mathbf{z} - \mathbf{y}\| &\leq \|\mathbf{z} - \mathbf{x}\| + \|\mathbf{x} - \mathbf{y}\| && \text{Corollary 18.11} \\
 &\leq \sum_{i=1}^n |z_i - x_i| + \|\mathbf{x} - \mathbf{y}\| && \text{Lemma} \\
 &< \sum_{i=1}^n \frac{r'}{n} + \|\mathbf{x} - \mathbf{y}\| \\
 &= r' + \|\mathbf{x} - \mathbf{y}\| \\
 &= r - \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{x} - \mathbf{y}\| \\
 &= r
 \end{aligned}$$

Therefore, by Definition 18.17,  $\mathbf{z} \in B$ , as desired.  $\square$

**Corollary 18.20.** A set  $U \subset \mathbb{R}^n$  is open if and only if for every  $\mathbf{x} \in U$ , there exists  $r > 0$  such that  $B(\mathbf{x}, r) \subset U$ .

*Proof.* Suppose first that  $U \subset \mathbb{R}^n$  is open. Let  $\mathbf{x}$  be an arbitrary element of  $U$ . By Definition 18.14, there exists an open rectangle  $R$  such that  $\mathbf{x} \in R \subset U$ . Therefore, by Lemma 18.19, there exists  $r > 0$  such that  $B(\mathbf{x}, r) \subset R \subset U$ , as desired.

Now suppose that for all  $\mathbf{x} \in U$ , there exists  $r > 0$  such that  $B(\mathbf{x}, r) \subset U$ . To prove that  $U$  is open, Definition 18.14 tells us that it will suffice to show that for all  $\mathbf{x} \in U$ , there exists an open rectangle  $R$  such that  $\mathbf{x} \in R \subset U$ . Let  $\mathbf{x}$  be an arbitrary element of  $U$ . Then there exists  $r > 0$  such that  $B(\mathbf{x}, r) \subset U$ . Therefore, by Lemma 18.19, there exists an open rectangle  $R$  such that  $\mathbf{x} \in R \subset B \subset U$ , as desired.  $\square$

7/14: **Corollary 18.21.** Open balls are open and closed balls are closed.

*Proof.* We will take this one claim at a time.

Let  $B(\mathbf{x}, r)$  be an arbitrary open ball. To prove that  $B$  is open, Definition 18.14 tells us that it will suffice to show that for all  $\mathbf{y} \in B$ , there exists an open rectangle  $R$  such that  $\mathbf{y} \in R \subset B$ . But by Lemma 18.19, this is true.

Let  $\overline{B}(\mathbf{x}, r)$  be an arbitrary closed ball. To prove that  $\overline{B}$  is closed, Definition 18.14 tells us that it will suffice to show that  $\mathbb{R}^n \setminus \overline{B}$  is open. To do this, Definition 18.14 tells us again that it will suffice to verify that for all  $\mathbf{y} \in \mathbb{R}^n \setminus \overline{B}$ , there exists an open rectangle  $R$  such that  $\mathbf{y} \in R \subset \mathbb{R}^n \setminus \overline{B}$ . Let  $\mathbf{y}$  be an arbitrary element of  $\mathbb{R}^n \setminus \overline{B}$ . Then by Definition 18.17,  $\|\mathbf{y} - \mathbf{x}\| > r$ . Thus,  $\|\mathbf{y} - \mathbf{x}\| - r > 0$ , so we may define  $r' = \|\mathbf{y} - \mathbf{x}\| - r$ . Now consider  $B(\mathbf{y}, r')$ . By Lemma 18.19, there exists an open rectangle  $R$  such that  $\mathbf{y} \in R \subset B$ . Consequently, by Script 1, the only thing left to do to verify that  $R \subset \mathbb{R}^n \setminus \overline{B}$  is to show that  $B \cap \overline{B} = \emptyset$ . As such, suppose for the sake of contradiction that  $B \cap \overline{B} \neq \emptyset$ . Then there exists  $\mathbf{z} \in \mathbb{R}^n$  such that  $\mathbf{z} \in B$  and  $\mathbf{z} \in \overline{B}$ . It follows by consecutive applications of Definition 18.17 that  $\|\mathbf{z} - \mathbf{y}\| < r'$  and  $\|\mathbf{z} - \mathbf{x}\| \leq r$ . But then we have that

$$\begin{aligned} \|\mathbf{x} - \mathbf{y}\| &\leq \|\mathbf{x} - \mathbf{z}\| + \|\mathbf{z} - \mathbf{y}\| && \text{Corollary 18.11} \\ &< r' + r \\ &= \|\mathbf{y} - \mathbf{x}\| - r + r \\ &= \|\mathbf{x} - \mathbf{y}\| \end{aligned}$$

a contradiction, as desired. □

**Proposition 18.22.** *Let  $U \subset \mathbb{R}^n$ . The following are equivalent:*

- (a)  $U$  is open.
- (b)  $U$  is a (possibly empty) union of open balls.
- (c)  $U$  is a (possibly empty) union of open rectangles.

*Proof.* As in Theorem 11.5, to prove that statements a-c are equivalent, it will suffice to verify that  $a \Rightarrow b$ ,  $b \Rightarrow c$ , and  $c \Rightarrow a$ . Let's begin.

First, suppose that  $U$  is open. Then by Corollary 18.20, for every  $\mathbf{x} \in U$ , there exists  $r > 0$  such that  $B_{\mathbf{x}}(\mathbf{x}, r) \subset U$ . Therefore,  $U = \bigcup_{\mathbf{x} \in U} B_{\mathbf{x}}$ , as desired.

Second, suppose that  $U$  is a union of open balls. Then for every open ball  $B(\mathbf{x}, r)$  comprising  $U$ , Lemma 18.19 asserts that for every  $\mathbf{y} \in B$ , there exists an open rectangle  $R_{\mathbf{y}}$  such that  $\mathbf{y} \in R_{\mathbf{y}} \subset B$ . Therefore,  $U = \bigcup_{\mathbf{y} \in U} R_{\mathbf{y}}$ , as desired.

Third, suppose that  $U$  is a union of open rectangles. Then for every  $\mathbf{x} \in U$ , there exists an open rectangle  $R$  such that  $\mathbf{x} \in R \subset U$ . Therefore, by Definition 18.14,  $U$  is open, as desired. □

**Remark 18.23.** If  $X \subset \mathbb{R}^n$ , then  $X$  is also a topological space with the **subspace topology**. That is,  $A \subset X$  is **open** (in  $X$ ) if there exists an open set  $U \subset \mathbb{R}^n$  such that  $X \cap U = A$ . (See Script 8.)

We now discuss functions between Euclidean spaces.

**Definition 18.24.** Let  $A \subset \mathbb{R}^n$  and let  $f : A \rightarrow \mathbb{R}$ . Define the **graph** of  $f$  by

$$\text{graph}(f) = \{(x_1, \dots, x_n, f(x_1, \dots, x_n)) \in \mathbb{R}^{n+1} \mid (x_1, \dots, x_n) \in A\}$$

**Exercise 18.25.** For each of the following functions, describe the graph as a subset of  $\mathbb{R}^3$ .

- (a)  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  given by  $f(x, y) = 2$  for all  $(x, y) \in \mathbb{R}^2$ .

*Description.* For this function, we have  $\text{graph}(f) = \{(x, y, 2) \in \mathbb{R}^3 \mid (x, y) \in \mathbb{R}^2\}$ . This makes the graph equal to the set of all points in  $\mathbb{R}^3$  with  $z = 2$ , which will be a planar, constant, infinite subspace of  $\mathbb{R}^3$ . □

- (b)  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  given by  $f(x, y) = x + y + 1$  for all  $(x, y) \in \mathbb{R}^2$ .



*Description.* For this function, we have  $\text{graph}(f) = \{(x, y, x + y + 1) \in \mathbb{R}^3 \mid (x, y) \in \mathbb{R}^2\}$ . Thus, the graph will be a planar, sloped, infinite subspace of  $\mathbb{R}^3$  with gradient pointing in the  $\hat{i} + \hat{j}$  direction.  $\square$

(c)  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  given by  $f(x, y) = x^2 + y^2$  for all  $(x, y) \in \mathbb{R}^2$ .

*Description.* For this function, we have  $\text{graph}(f) = \{(x, y, x^2 + y^2) \in \mathbb{R}^3 \mid (x, y) \in \mathbb{R}^2\}$ . Thus, the graph will be the paraboloid centered at the origin.  $\square$

In Script 9, we gave a definition of continuity that we can generalize to this case:

**Definition 18.26.** Let  $X, Y$  be topological spaces. A function  $f : X \rightarrow Y$  is **continuous** if for every open set  $U \subset Y$ , the preimage  $f^{-1}(U)$  is open in  $X$ .

The function  $f : X \rightarrow Y$  is **continuous** at  $x \in X$  if for every open set  $U \subset Y$  containing  $f(x)$ , the preimage  $f^{-1}(U)$  is open in  $X$ .

**Theorem 18.27.**

(a) A function  $f : X \rightarrow Y$  is continuous if and only if it is continuous at every  $x \in X$ .

(b) A function  $f : X \rightarrow Y$  is continuous if and only if  $f^{-1}(B)$  is closed in  $X$  whenever  $B$  is closed in  $Y$ .

*Proof.* The proofs are symmetric to those of Theorem 9.10 and Proposition 9.5, respectively.  $\square$

7/17: **Remark 18.28.** There is also a characterization of continuity in terms of limits, as in one variable, as we shall now see. First we need the definitions of limit point and limit.

**Definition 18.29.** Let  $A \subset \mathbb{R}^n$ .

(a) We say that  $\mathbf{x}$  is a **limit point** of  $A$  if for every open set  $U$  containing  $\mathbf{x}$ ,  $A \cap (U \setminus \{\mathbf{x}\}) \neq \emptyset$ .

(b) Let  $\mathbf{x} \in LP(A)$  and  $f : A \rightarrow \mathbb{R}^m$ . We say  $\mathbf{L} \in \mathbb{R}^m$  is the **limit** (of  $f$  at  $\mathbf{x}$ ) if for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $\mathbf{y} \in A$  and  $0 < \|\mathbf{y} - \mathbf{x}\| < \delta$ , then  $\|f(\mathbf{y}) - \mathbf{L}\| < \epsilon$ . As in one variable, we can show that limits are unique. If  $\mathbf{L}$  is the limit of  $f$  at  $\mathbf{x}$ , we write  $\lim_{\mathbf{y} \rightarrow \mathbf{x}} f(\mathbf{y}) = \mathbf{L}$ .

**Exercise 18.30.** Compute the following limits if they exist, or prove that the limit does not exist.

**Lemma.** Let  $\mathbf{x} = (x_1, \dots, x_n)$  be an arbitrary element of  $\mathbb{R}^n$ . Then  $\|\mathbf{x}\| < \delta$  implies that  $|x_i| < \delta$  for all  $1 \leq i \leq n$ .

*Proof.* Suppose for the sake of contradiction that for some  $1 \leq i \leq n$ ,  $|x_i| \geq \delta$ . Note that since  $0 \leq \|\mathbf{x}\| < \delta$  by Theorem 18.10,  $|\delta| = \delta$  by Definition 8.4. Then

$$\begin{aligned} \|\mathbf{x}\| &= \sqrt{x_1^2 + \dots + x_{i-1}^2 + x_i^2 + x_{i+1}^2 + \dots + x_n^2} && \text{Definition 18.6} \\ &\geq \sqrt{x_1^2 + \dots + x_{i-1}^2 + \delta^2 + x_{i+1}^2 + \dots + x_n^2} \\ &\geq \sqrt{\delta^2} \\ &= \delta \\ &> \|\mathbf{x}\| \end{aligned}$$

a contradiction.  $\square$

(a)  $\lim_{(x,y) \rightarrow (a,b)} 4xy$ .

*Proof.* To prove that  $\lim_{(x,y) \rightarrow (a,b)} 4xy = 4ab$ , Definition 18.29 tells us that it will suffice to show that for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $(x, y) \in \mathbb{R}^2$  and  $0 < \|(x, y) - (a, b)\| < \delta$ , then  $\|4xy - 4ab\| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Choose  $\delta = \min(\min(\frac{\epsilon}{8(|b|+1)}, 1), \frac{\epsilon}{8(|a|-1)})$ . Then since  $\|(x - a, y - b)\| < \delta$  by hypothesis and Definition 18.2, the lemma asserts that  $|x - a| < \delta$  and  $|y - b| < \delta$ . It follows that  $|x - a| < \min(\frac{\epsilon}{8(|b|+1)}, 1)$  and  $|y - b| < \frac{\epsilon}{8(|a|-1)}$ . Consequently, by an argument symmetric to the proof of Theorem 11.9,  $|xy - ab| < \frac{\epsilon}{4}$ . Therefore,  $\|4xy - 4ab\| = |4xy - 4ab| < 4 \cdot \frac{\epsilon}{4} = \epsilon$ , as desired.  $\square$

(b)  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - y^3}{x^2 + y^2}.$

*Proof.* To ensure that  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - y^3}{x^2 + y^2}$  is well-defined, Definition 18.29 tells us that we must show that  $(0,0) \in LP(\mathbb{R}^2 \setminus \{(0,0)\})$ , assuming that  $\mathbb{R}^2 \setminus \{(0,0)\}$  is the domain of  $\frac{x^3 - y^3}{x^2 + y^2}$  since the domain is not explicitly specified. To do so, Definition 18.29 tells us again that it will suffice to verify that for every open set  $U$  containing  $(0,0)$ ,  $(\mathbb{R}^2 \setminus \{(0,0)\}) \cap (U \setminus \{(0,0)\}) \neq \emptyset$ . Let  $U$  be an arbitrary open set containing  $(0,0)$ . By Definition 18.14, there exists an open rectangle  $R$  such that  $(0,0) \in R \subset U$ . By Definition 18.13,  $R$  is not a singleton set. Thus, there exist at least one point in  $R$ , i.e., in  $U$  that is not equal to  $(0,0)$  and is (naturally) in  $\mathbb{R}^2$ , as desired.

To prove that  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^3 - y^3}{x^2 + y^2} = 0$ , Definition 18.29 tells us that it will suffice to show that for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $(x,y) \in \mathbb{R}^2$  and  $0 < \|(x,y) - (0,0)\| < \delta$ , then  $\|\frac{x^3 - y^3}{x^2 + y^2} - 0\| = |\frac{x^3 - y^3}{x^2 + y^2}| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Choose  $\delta = \frac{\epsilon}{3}$ . Then from previous results, we can prove two important bounds on combinations of  $x$  and  $y$  that will be useful in the final inequality. Let's begin.

First, since  $0 < \|(x,y)\|$ , Theorem 18.10 implies that  $x \neq 0$  or  $y \neq 0$ . Thus,  $x^2 + y^2 \neq 0$ . Consequently, we may argue in a well-defined manner that

$$\begin{aligned} \left| \frac{xy}{x^2 + y^2} \right| &= |xy| \cdot \left| \frac{1}{x^2 + y^2} \right| \\ &\leq \frac{x^2 + y^2}{2} \cdot \left| \frac{1}{x^2 + y^2} \right| \\ &= \frac{1}{2} \end{aligned} \quad \text{Lemma 18.9}$$

Second, since we know from the lemma that  $|x| < \frac{\epsilon}{3}$  and  $|y| < \frac{\epsilon}{3}$ , we have that

$$\begin{aligned} |x - y| &\leq |x| + |-y| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \frac{2\epsilon}{3} \end{aligned} \quad \text{Lemma 8.8}$$

Therefore, combining the last two results, we have that

$$\begin{aligned} \left| \frac{x^3 - y^3}{x^2 + y^2} \right| &= \left| \frac{(x - y)(x^2 + xy + y^2)}{x^2 + y^2} \right| \\ &= |x - y| \cdot \left| \frac{xy}{x^2 + y^2} + 1 \right| \\ &\leq |x - y| \cdot \left| \frac{xy}{x^2 + y^2} \right| + |x - y| \\ &< \frac{2\epsilon}{3} \cdot \frac{1}{2} + \frac{2\epsilon}{3} \\ &= \epsilon \end{aligned} \quad \text{Lemma 8.8}$$

as desired. □

(c)  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}.$

*Proof.* To prove that  $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$  does not exist, Definition 18.29 tells us that it will suffice to show that for every  $L \in \mathbb{R}$ , there exists an  $\epsilon > 0$  such that for all  $\delta > 0$ , there exists  $(x,y) \in \mathbb{R}^2$  satisfying  $0 < \|(x,y) - (0,0)\| < \delta$  such that  $\|\frac{x^2 - y^2}{x^2 + y^2} - L\| \geq \epsilon$ . Let  $L$  be an arbitrary element of  $\mathbb{R}$ .

We divide into two cases ( $L \geq 0$  and  $L < 0$ ). Suppose first that  $L \geq 0$ . Choose  $\epsilon = 1$ . Let  $\delta > 0$  be arbitrary. Choose  $(0, \frac{\delta}{2}) \in \mathbb{R}^2$ . By Definition 18.6,  $0 < \|(0, \frac{\delta}{2})\| = \sqrt{\delta^2/4} = \frac{\delta}{2} < \delta$ . Additionally,

$$\begin{aligned} \left\| \frac{0^2 - (\frac{\delta}{2})^2}{0^2 + (\frac{\delta}{2})^2} - L \right\| &= \left| \frac{-1}{1} - L \right| \\ &= |-1 - L| \\ &\geq 1 - |L| \\ &\geq 1 \\ &= \epsilon \end{aligned}$$

as desired. The proof is symmetric in the other case.  $\square$

**Theorem 18.31.** Let  $A \subset \mathbb{R}^n$  and  $\mathbf{x} \in A$ . Let  $f : A \rightarrow \mathbb{R}^m$ . Then the following are equivalent:

- (a)  $f$  is continuous at  $\mathbf{x}$ .
- (b) For every  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $\mathbf{y} \in A$  and  $\|\mathbf{y} - \mathbf{x}\| < \delta$ , then  $\|f(\mathbf{y}) - f(\mathbf{x})\| < \epsilon$ .
- (c) Either  $\mathbf{x} \notin \text{LP}(A)$  or  $\lim_{\mathbf{y} \rightarrow \mathbf{x}} f(\mathbf{y}) = f(\mathbf{x})$ .

*Proof.* The proof is symmetric to that of Theorem 11.9.  $\square$

**Exercise 18.32.** For each of the following, prove that  $f$  is continuous at every point in its domain.

- (a)  $A \subset \mathbb{R}^n$  and  $f : A \rightarrow \mathbb{R}^m$  is a constant function.

*Proof.* Since  $f$  is a constant function, we may let  $f(\mathbf{x}) = \mathbf{c}$  for all  $\mathbf{x} \in A$ . To prove that  $f$  is continuous at every  $\mathbf{x} \in A$ , let  $\mathbf{x}$  be an arbitrary element of  $A$ ; then Theorem 18.31 tells us that it will suffice to show that for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $\mathbf{y} \in A$  and  $\|\mathbf{y} - \mathbf{x}\| < \delta$ , then  $\|f(\mathbf{y}) - f(\mathbf{x})\| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Choose  $\delta = 1$ . Let  $\mathbf{y}$  be an arbitrary element of  $A$  satisfying  $\|\mathbf{y} - \mathbf{x}\| < \delta$ . Then

$$\begin{aligned} \|f(\mathbf{y}) - f(\mathbf{x})\| &= \|\mathbf{c} - \mathbf{c}\| \\ &= \|\mathbf{0}\| \\ &= 0 \\ &< \epsilon \end{aligned} \quad \text{Theorem 18.10}$$

as desired.  $\square$

- (b) Fix  $\mathbf{a} \in \mathbb{R}^m$ . Define  $f : \mathbb{R} \rightarrow \mathbb{R}^m$  by  $f(h) = h\mathbf{a}$ .

*Proof.* We divide into two cases ( $\mathbf{a} = 0$  and  $\mathbf{a} \neq 0$ ). If  $\mathbf{a} = 0$ , then by Exercise 18.32a,  $f$  is continuous at every point in its domain. If  $\mathbf{a} \neq 0$ , we continue.

Let  $x$  be an arbitrary element of  $\mathbb{R}$ . To prove that  $f$  is continuous at  $x$ , Theorem 18.31 tells us that it will suffice to show that for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $y \in \mathbb{R}$  and  $\|y - x\| = |y - x| < \delta$ , then  $\|f(y) - f(x)\| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Choose  $\delta = \frac{\epsilon}{\|\mathbf{a}\|}$  (by Theorem 18.10 and the supposition that  $\mathbf{a} \neq 0$ , we know that  $\|\mathbf{a}\| \neq 0$ ). Let  $y$  be an arbitrary element of  $\mathbb{R}$  satisfying  $|y - x| < \delta$ . Then

$$\begin{aligned} \|f(y) - f(x)\| &= \|y\mathbf{a} - x\mathbf{a}\| \\ &= \|(y - x)\mathbf{a}\| \\ &= |y - x| \cdot \|\mathbf{a}\| \\ &< \frac{\epsilon}{\|\mathbf{a}\|} \cdot \|\mathbf{a}\| \\ &= \epsilon \end{aligned} \quad \text{Theorem 18.10}$$

as desired.  $\square$

(c) Fix  $\mathbf{x} \in \mathbb{R}^n$ . Define  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  by  $f(\mathbf{y}) = \|\mathbf{y} - \mathbf{x}\|$ .

*Proof.* Let  $\mathbf{y}$  be an arbitrary element of  $\mathbb{R}^n$ . To prove that  $f$  is continuous at  $\mathbf{y}$ , Theorem 18.31 tells us that it will suffice to show that for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $\mathbf{z} \in \mathbb{R}^n$  and  $\|\mathbf{z} - \mathbf{y}\| < \delta$ , then  $\|f(\mathbf{z}) - f(\mathbf{y})\| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Choose  $\delta = \epsilon$ . Let  $\mathbf{z}$  be an arbitrary element of  $\mathbb{R}^n$  satisfying  $\|\mathbf{z} - \mathbf{y}\| < \delta$ . Then

$$\begin{aligned} \|f(\mathbf{z}) - f(\mathbf{y})\| &= \left| \|\mathbf{z} - \mathbf{x}\| - \|\mathbf{y} - \mathbf{x}\| \right| \\ &\leq \|(\mathbf{z} - \mathbf{x}) - (\mathbf{y} - \mathbf{x})\| && \text{Corollary 18.11} \\ &= \|\mathbf{z} - \mathbf{y}\| \\ &< \epsilon \end{aligned}$$

as desired. □

(d)  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  given by  $f(x, y) = 4xy$ .

*Proof.* Let  $(a, b)$  be an arbitrary element of  $\mathbb{R}^2$ . To prove that  $f$  is continuous at  $(a, b)$ , Theorem 18.31 tells us that it will suffice to show that either  $(a, b) \notin LP(\mathbb{R}^2)$  or  $\lim_{(x,y) \rightarrow (a,b)} 4xy = 4ab$ . But by Exercise 18.30a,  $\lim_{(x,y) \rightarrow (a,b)} 4xy = 4ab$ , as desired. □

**Exercise 18.33.** Consider the function  $f : \mathbb{R}^2 \setminus \{(0, 0)\} \rightarrow \mathbb{R}$  given by  $f(x, y) = \frac{x^3 - y^3}{x^2 + y^2}$  (see Exercise 18.30b). It can be shown that this function is continuous on its domain. Can you extend this function continuously to  $\mathbb{R}^2$ ? More specifically, can you define a continuous function  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  such that  $g(x, y) = f(x, y)$  for all  $(x, y) \neq (0, 0)$ ?

*Proof.* Let  $g : \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined by

$$g(x, y) = \begin{cases} f(x, y) & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

By the continuity of  $f$  on  $\mathbb{R}^2 \setminus \{(0, 0)\}$ ,  $g$  is continuous on  $\mathbb{R}^2 \setminus \{(0, 0)\}$ . Additionally, by Exercise 18.30c,  $\lim_{(x,y) \rightarrow (0,0)} g(x, y) = 0 = g(0, 0)$ . Thus, by Theorem 18.31,  $g$  is continuous at  $(0, 0)$ . Therefore,  $g$  is continuous on  $(\mathbb{R}^2 \setminus \{(0, 0)\}) \cup \{(0, 0)\} = \mathbb{R}^2$ , as desired. □

7/21: **Definition 18.34.** Let  $m \in \mathbb{N}$ . Suppose  $I = \{i_1, \dots, i_k\} \subset [m]$  with  $i_1 < \dots < i_k$ . We define the **projection function**  $\pi_I : \mathbb{R}^m \rightarrow \mathbb{R}^k$  as

$$\pi_I(\mathbf{x}) = (x_{i_1}, \dots, x_{i_k})$$

If  $I = \{i\}$  has only one element, we write  $\pi_i$  instead of  $\pi_{\{i\}}$ .

**Exercise 18.35.** Prove that each  $\pi_I$  is continuous.

*Proof.* Let  $I$  be an arbitrary subset of  $[m]$ , and let  $\mathbf{x}$  be an arbitrary element of  $\mathbb{R}^m$ . To prove that  $\pi_I$  is continuous at  $\mathbf{x}$ , Theorem 18.31 tells us that it will suffice to show that for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $\mathbf{y} \in \mathbb{R}^m$  and  $\|\mathbf{y} - \mathbf{x}\| < \delta$ , then  $\|\pi_I(\mathbf{y}) - \pi_I(\mathbf{x})\| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Choose  $\delta = \epsilon$ . Let  $\mathbf{y}$  be an arbitrary element of  $\mathbb{R}^m$  such that  $\|\mathbf{y} - \mathbf{x}\| < \delta$ . Then

$$\begin{aligned} \|\pi_I(\mathbf{y}) - \pi_I(\mathbf{x})\| &= \|(y_{i_1} - x_{i_1}, \dots, y_{i_k} - x_{i_k})\| && \text{Definition 18.2} \\ &= \sqrt{(y_{i_1} - x_{i_1})^2 + \dots + (y_{i_k} - x_{i_k})^2} && \text{Definition 18.6} \\ &\leq \sqrt{(y_1 - x_1)^2 + \dots + (y_m - x_m)^2} \\ &= \|(y_1 - x_1, \dots, y_m - x_m)\| && \text{Definition 18.6} \\ &= \|\mathbf{y} - \mathbf{x}\| && \text{Definition 18.2} \\ &< \epsilon \end{aligned}$$

as desired.<sup>[1]</sup> □

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<sup>1</sup>This can also be done, without much difficulty, with the open preimage form of continuity.

**Remark 18.36.** Let  $A \subset \mathbb{R}^n$  be a rectangle (open or closed). Then

$$A = \pi_1(A) \times \cdots \times \pi_n(A)$$

**Definition 18.37.** Let  $f : A \rightarrow \mathbb{R}^m$ . Its  $i^{\text{th}}$  component function  $f_i : A \rightarrow \mathbb{R}$  is defined as

$$f_i = \pi_i \circ f$$

In other words,

$$f(\mathbf{x}) = (f_1(\mathbf{x}), \dots, f_m(\mathbf{x}))$$

**Theorem 18.38.** Let  $A \subset \mathbb{R}^n$  and let  $\mathbf{x}$  be a limit point of  $A$ . Suppose  $f : A \rightarrow \mathbb{R}^m$ . If  $\lim_{\mathbf{y} \rightarrow \mathbf{x}} f(\mathbf{y}) = \mathbf{z}$  exists (with  $\mathbf{z} = (z_1, \dots, z_m)$ ), then for all  $i \in [m]$ ,  $\lim_{\mathbf{y} \rightarrow \mathbf{x}} f_i(\mathbf{y})$  exists and equals  $z_i$ . Conversely, if for all  $i \in [m]$ ,  $\lim_{\mathbf{y} \rightarrow \mathbf{x}} f_i(\mathbf{y}) = z_i$ , then  $\lim_{\mathbf{y} \rightarrow \mathbf{x}} f(\mathbf{y})$  exists and equals  $\mathbf{z} = (z_1, \dots, z_m)$ .

*Proof.* Suppose first that  $\lim_{\mathbf{y} \rightarrow \mathbf{x}} f(\mathbf{y}) = \mathbf{z}$ . Let  $i$  be an arbitrary element of  $[m]$ . To prove that  $\lim_{\mathbf{y} \rightarrow \mathbf{x}} f_i(\mathbf{y}) = z_i$ , Definition 18.29 tells us that it will suffice to show that for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $\mathbf{y} \in A$  and  $0 < \|\mathbf{y} - \mathbf{x}\| < \delta$ , then  $\|f_i(\mathbf{y}) - z_i\| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Since  $\lim_{\mathbf{y} \rightarrow \mathbf{x}} f(\mathbf{y}) = \mathbf{z}$ , Definition 18.29 asserts that there exists  $\delta > 0$  such that if  $\mathbf{y} \in A$  and  $0 < \|\mathbf{y} - \mathbf{x}\| < \delta$ , then  $\|f(\mathbf{y}) - \mathbf{z}\| < \epsilon$ . Choose this  $\delta$  to be our  $\delta$ . Let  $\mathbf{y}$  be an arbitrary element of  $A$  satisfying  $0 < \|\mathbf{y} - \mathbf{x}\| < \delta$ . Then

$$\begin{aligned} \|f_i(\mathbf{y}) - z_i\| &= \sqrt{(f_i(\mathbf{y}) - z_i)^2} && \text{Definition 18.6} \\ &\leq \sqrt{(f_1(\mathbf{y}) - z_1)^2 + \cdots + (f_m(\mathbf{y}) - z_m)^2} \\ &= \|(f_1(\mathbf{y}), \dots, f_m(\mathbf{y})) - (z_1, \dots, z_m)\| && \text{Definition 18.6} \\ &= \|f(\mathbf{y}) - \mathbf{z}\| && \text{Definition 18.37} \\ &< \epsilon \end{aligned}$$

as desired.

Now suppose that for all  $i \in [m]$ ,  $\lim_{\mathbf{y} \rightarrow \mathbf{x}} f_i(\mathbf{y}) = z_i$ . To prove that  $\lim_{\mathbf{y} \rightarrow \mathbf{x}} f(\mathbf{y}) = \mathbf{z}$ , Definition 18.29 tells us that it will suffice to show that for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $\mathbf{y} \in A$  and  $0 < \|\mathbf{y} - \mathbf{x}\| < \delta$ , then  $\|f(\mathbf{y}) - \mathbf{z}\| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Since  $\lim_{\mathbf{y} \rightarrow \mathbf{x}} f_i(\mathbf{y}) = z_i$  for all  $i \in [m]$ , Definition 18.29 asserts that for all  $i \in [m]$ , there exists  $\delta_i > 0$  such that if  $\mathbf{y} \in A$  and  $0 < \|\mathbf{y} - \mathbf{x}\| < \delta_i$ , then  $\|f_i(\mathbf{y}) - z_i\| < \frac{\epsilon}{m}$ . Choose  $\delta = \min(\delta_1, \dots, \delta_m)$ . Let  $\mathbf{y}$  be an arbitrary element of  $A$  satisfying  $0 < \|\mathbf{y} - \mathbf{x}\| < \delta$ . Then

$$\begin{aligned} \|f(\mathbf{y}) - \mathbf{z}\| &= \|(f_1(\mathbf{y}) - z_1) + \cdots + (f_m(\mathbf{y}) - z_m)\| \\ &\leq \|f_1(\mathbf{y}) - z_1\| + \cdots + \|f_m(\mathbf{y}) - z_m\| && \text{Theorem 18.10} \\ &< \underbrace{\frac{\epsilon}{m} + \cdots + \frac{\epsilon}{m}}_{m \text{ times}} \\ &= \epsilon \end{aligned}$$

as desired. □

**Corollary 18.39.** Let  $A \subset \mathbb{R}^n$ . A function  $f : A \rightarrow \mathbb{R}^m$  is continuous if and only if  $f_1, \dots, f_m$  are all continuous.

*Proof.* Suppose first that  $f$  is continuous. Let  $\mathbf{x}$  be an arbitrary element of  $A$ , and let  $i$  be an arbitrary element of  $[m]$ . To prove that  $f_i$  is continuous at  $\mathbf{x}$ , Theorem 18.31 tells us that it will suffice to show that either  $\mathbf{x} \notin LP(A)$  or  $\lim_{\mathbf{y} \rightarrow \mathbf{x}} f_i(\mathbf{y}) = f_i(\mathbf{x})$ . Since  $f$  is continuous at  $\mathbf{x}$  by hypothesis, Theorem 18.31 asserts that either  $\mathbf{x} \notin LP(A)$  or  $\lim_{\mathbf{y} \rightarrow \mathbf{x}} f(\mathbf{y}) = f(\mathbf{x})$ . We now divide into two cases. If  $\mathbf{x} \notin LP(A)$ , then we are done. On the other hand, if  $\lim_{\mathbf{y} \rightarrow \mathbf{x}} f(\mathbf{y}) = f(\mathbf{x})$ , then by Theorem 18.38 and Definition 18.37,  $\lim_{\mathbf{y} \rightarrow \mathbf{x}} f_i(\mathbf{y}) = f_i(\mathbf{x})$ , as desired.

The proof is symmetric in the other direction. □

Now we revisit compactness, but in  $\mathbb{R}^n$ . For our purposes, the key result is Corollary ??.

**Definition 18.40.** Let  $A \subset \mathbb{R}^n$ . Then  $A$  is **compact** if every open cover  $\mathcal{G}$  of  $A$  has a finite subcover.

**Proposition 18.41.** Let  $A \subset \mathbb{R}^n$ . Then  $A$  is compact if and only if every open cover  $\mathcal{G}$  of  $A$  consisting solely of open rectangles has a finite subcover.

*Proof.* Suppose first that  $A$  is compact. Let  $\mathcal{G}$  be an arbitrary open cover of  $A$  consisting solely of open rectangles. Then since  $A$  is compact, by Definition 18.40,  $\mathcal{G}$  has a finite subcover.

Now suppose that every open cover  $\mathcal{G}$  of  $A$  consisting solely of open rectangles has a finite subcover. To prove that  $A$  is compact, Definition 18.40 tells us that it will suffice to show that every open cover  $\mathcal{G}$  of  $A$  has a finite subcover. Let  $\mathcal{G} = \{G_\lambda \mid \lambda \in \Lambda\}$  be an arbitrary open cover of  $A$ , and let  $G_\lambda$  be an arbitrary element of  $\mathcal{G}$ . By Definition 10.3,  $G_\lambda$  is open. Thus, by Proposition 18.22,  $G_\lambda = \bigcup_{\gamma \in \Gamma_\lambda} R_{\lambda_\gamma}$ , where each  $R_{\lambda_\gamma}$  is an open rectangle. Now let  $\mathcal{H} = \{R_{\lambda_\gamma} \mid \lambda \in \Lambda, \gamma \in \Gamma_\lambda\}$ . It follows by Script 1 that  $\mathcal{G} = \mathcal{H}$ . Additionally, by the hypothesis, there exists a finite subcover  $\mathcal{H}' \subset \mathcal{H}$  of  $A$ . Finally, if  $R_{\lambda_\gamma} \in \mathcal{H}'$ , let  $G_\lambda \in \mathcal{G}'$ . It follows that  $\mathcal{G}'$  is a finite subcover of  $\mathcal{G}$ , as desired.  $\square$

**Definition 18.42.** Let  $A \subset \mathbb{R}^n$  and  $f : A \rightarrow \mathbb{R}^m$ . We say that  $f$  is **uniformly continuous** if for every  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $\mathbf{x}, \mathbf{y} \in A$  and  $\|\mathbf{x} - \mathbf{y}\| < \delta$ , then  $\|f(\mathbf{x}) - f(\mathbf{y})\| < \epsilon$ .

**Theorem 18.43.** Let  $A \subset \mathbb{R}^n$  be compact and  $f : A \rightarrow \mathbb{R}^m$  be continuous. Then  $f$  is uniformly continuous.

*Proof.* The proof is symmetric to that of Theorem 13.6.  $\square$

**Theorem 18.44.** If  $A \subset \mathbb{R}^n$  is compact and  $f : A \rightarrow \mathbb{R}^m$  is continuous, then  $f(A)$  is compact.

*Proof.* The proof is symmetric to that of Theorem 10.19.  $\square$

**Corollary 18.45.** Let  $\mathbf{x} \in \mathbb{R}^n$ . If  $B$  is a compact subset of  $\mathbb{R}^m$ , then  $\{\mathbf{x}\} \times B$  is a compact subset of  $\mathbb{R}^{n+m}$ .

*Proof.* Let  $f : B \rightarrow \mathbb{R}^{n+m}$  be defined by  $f(\mathbf{y}) = (x_1, \dots, x_n, y_1, \dots, y_m)$  for all  $\mathbf{y} \in B$ . Let  $\mathbf{y}$  be an arbitrary element of  $B$ . To prove that  $f$  is continuous at  $\mathbf{y}$ , Theorem 18.31 tells us that it will suffice to show that for every  $\epsilon > 0$ , there exists  $\delta$  such that if  $\mathbf{z} \in B$  and  $\|\mathbf{z} - \mathbf{y}\| < \delta$ , then  $\|f(\mathbf{z}) - f(\mathbf{y})\| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Choose  $\delta = \epsilon$ . Let  $\mathbf{z}$  be an arbitrary element of  $B$  satisfying  $\|\mathbf{z} - \mathbf{y}\| < \delta$ . Then

$$\begin{aligned} \|f(\mathbf{z}) - f(\mathbf{y})\| &= \|(x_1 - x_1, \dots, x_n - x_n, z_1 - y_1, \dots, z_m - y_m)\| \\ &= \|(z_1 - y_1, \dots, z_m - y_m)\| \\ &= \|\mathbf{z} - \mathbf{y}\| \\ &< \epsilon \end{aligned}$$

as desired. Therefore, since  $B \subset \mathbb{R}^m$  is compact and  $f : B \rightarrow \mathbb{R}^{n+m}$  is continuous, Theorem 18.44 asserts that  $f(B)$  is compact. Naturally,  $f(B) = \{\mathbf{x}\} \times B$ , so the latter set is compact, too, as desired.  $\square$

**Lemma 18.46.** Let  $\mathbf{x} \in \mathbb{R}^n$  and  $B \subset \mathbb{R}^m$ . If  $\mathcal{G}$  is a finite set of open rectangles that covers  $\{\mathbf{x}\} \times B \subset \mathbb{R}^{n+m}$ , then there exists an open rectangle  $R \subset \mathbb{R}^n$  containing  $\mathbf{x}$  such that  $\mathcal{G}$  covers  $R \times B$ .

*Proof.* Let  $\mathcal{G} = \{R_i \mid i \in [k]\}$ . To begin, we will show that every  $\pi_{[n]}(R_i)$  is an open rectangle containing  $\mathbf{x}$ . It will follow that the intersection of all  $\pi_{[n]}(R_i)$  is an open rectangle  $R$  containing  $\mathbf{x}$ . Thus, since this  $R$  is a subset of each  $R_i$  in dimensions 1 through  $n$ , we will be able to show that  $\mathcal{G}$  covers  $R \times B$ . Let's begin.

First, we will show that every  $\pi_{[n]}(R_i)$  is an open rectangle. Let  $i$  be an arbitrary element of  $[k]$ , and let  $R_i = (r_{ij}, s_{ij})_{j=1}^{n+m}$ . To show that  $\pi_{[n]}(R_i)$  is an open rectangle, Definition 18.13 tells us that it will suffice to verify that  $\pi_{[n]}(R_i) = (r_{ij}, s_{ij})_{j=1}^n$ . Let  $\mathbf{y}$  be an arbitrary element of  $\pi_{[n]}(R_i)$ . By Definition 1.18,  $\mathbf{y} = \pi_{[n]}(\mathbf{z})$  for some  $\mathbf{z} \in R_i$ . Thus, by Definition 18.34,  $y_j = z_j$  for all  $j \in [n]$ . Consequently, since  $z_j \in (r_{ij}, s_{ij})$  for all  $j \in [n]$  by Definition 18.13, we have that  $y_j \in (r_{ij}, s_{ij})$  for all  $j \in [n]$ . Therefore, by Definition 1.15,  $\mathbf{y} \in (r_{ij}, s_{ij})_{j=1}^n$ . The argument is symmetric in the other direction. Both arguments, when combined, imply by Definition 1.2 that  $\pi_{[n]}(R_i) = (r_{ij}, s_{ij})_{j=1}^n$ , as desired.

Next, we will show that every  $\pi_{[n]}(R_i)$  contains  $\mathbf{x}$ . Let  $i$  be an arbitrary element of  $[k]$ , and let  $\mathbf{y}$  be an arbitrary element of  $\{\mathbf{x}\} \times B$  satisfying  $\mathbf{y} \in R_i$  (Definition 10.3 guarantees that  $\mathbf{y}$  is in some  $R_i$ ). Thus, by

Definition 1.18,  $\pi_{[n]}(\mathbf{y}) \in \pi_{[n]}(R_i)$ . Additionally, by Definition 1.15,  $\mathbf{y} = (x_1, \dots, x_n, y_1, \dots, y_m)$ . It follows by Definition 18.34 that  $\pi_{[n]}(\mathbf{y}) = \mathbf{x}$ . Therefore,  $\mathbf{x} \in \pi_{[n]}(R_i)$ , as desired.

Let  $R = \bigcap_{i \in [k]} \pi_{[n]}(R_i)$ . Consequently, by Exercise 18.16,  $R$  is an open rectangle containing  $\mathbf{x}$ .

To prove that  $\mathcal{G}$  covers  $R \times B$ , Definition 10.3 tells us that it will suffice to show that for all  $\mathbf{y} \in R \times B$ ,  $\mathbf{y} \in R_i$  for some  $R_i \in \mathcal{G}$ . Let  $\mathbf{y} = (y_1, \dots, y_{n+m})$  be an arbitrary element of  $R \times B$ . By Definition 1.15,  $(y_1, \dots, y_n) \in R$  and  $(y_{n+1}, \dots, y_{n+m}) \in B$ . It follows from the latter statement and the fact that  $\mathcal{G}$  is a cover of  $\{\mathbf{x}\} \times B$  that  $(x_1, \dots, x_n, y_{n+1}, \dots, y_{n+m}) \in R_i$  for some  $i \in [k]$ . Consider this  $R_i$ ; we will confirm that  $\mathbf{y}$  is an element of it. To do so, Definition 18.13 tells us that it will suffice to demonstrate that  $y_j \in (r_{i_j}, s_{i_j})$  for all  $j \in [n+m]$ . We divide into two cases ( $j \in [n]$  and  $j \in [n+1 : m]$ ). Suppose first that  $j \in [n]$ . Then since  $R = \bigcap_{i \in [k]} \pi_{[n]}(R_i)$ , Theorem 1.7 asserts that  $R \subset \pi_{[n]}(R_i)$ . Thus, since  $(y_1, \dots, y_n) \in R$  by the above, Definition 1.3 implies that  $(y_1, \dots, y_n) \in \pi_{[n]}(R_i)$ . Additionally, by the above,  $\pi_{[n]}(R_i)$  can be written in the form  $(r_{i_j}, s_{i_j})_{j=1}^n$ . Combining the last two results, we have by Definition 1.2 that  $(y_1, \dots, y_n) \in (r_{i_j}, s_{i_j})_{j=1}^n$ . Therefore, by Definition 18.13,  $y_j \in (r_{i_j}, s_{i_j})$  for all  $j \in [n]$ , as desired. Now suppose that  $j \in [n+1 : m]$ . By the above,  $(x_1, \dots, x_n, y_{n+1}, \dots, y_{n+m}) \in R_i$ . Therefore, by Definition 18.13,  $y_j \in (r_{i_j}, s_{i_j})$  for all  $j \in [n+1 : m]$ , as desired.  $\square$