Script 15

Sequences

15.1 Journal

5/6: **Definition 15.1.** A **sequence** (of real numbers) is a function $a : \mathbb{N} \to \mathbb{R}$.

By setting $a_n = a(n)$, we can think of a sequence as a list $a_1, a_2, a_3, \dots$ of real numbers. We use the notation $(a_n)_{n=1}^{\infty}$ for such a sequence, or if there is no possibility of confusion, we sometimes abbreviate this and write simply (a_n). More generally, we also use the term sequence to refer to the function defined on $\{n \in \mathbb{N} \cup \{0\} \mid n \geq n_0\}$ for any fixed $n_0 \in \mathbb{N} \cup \{0\}$. We write $(a_n)_{n=n_0}^{\infty}$ for such a sequence.

Definition 15.2. We say that a sequence (a_n) **converges** to a point $p \in \mathbb{R}$ if for every open interval I containing p, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in I$. If a sequence converges to some point, we say it is **convergent**. If (a_n) does not converge to any point, we say that the sequence **diverges** or is **divergent**.

Exercise 15.3. Show that a sequence (a_n) converges to p if and only if any region containing p contains all but finitely many terms of the sequence.

Proof. Suppose first that (a_n) converges to p. Let R be an arbitrary region containing p. By Corollary 4.11 and Lemma 8.3, R is an open interval. Thus, by Definition 15.2, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in R$. To prove that R contains all but finitely many terms of the sequence, it will suffice to show that the set $A = \{a_n \mid a_n \notin R\}$ is finite. Since $a_n \in R$ for all $n \geq N$, it follows that $a_n \in R$ only if $n < N$. Thus, by Script 1, $A \subset \{a_n \mid 0 \leq n < N\}$. Since the latter set is clearly finite, it follows by Script 1 that A is finite.

Now suppose that any region containing p contains all but finitely many terms (a_n). To prove that (a_n) converges to p, Definition 15.2 tells us that it will suffice to show that for every open interval I containing p, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in I$. Let I be an arbitrary open interval containing p. Then by Theorem 4.10, there exists a region R containing p such that $R \subset I$. It follows by the hypothesis that $A = \{a_n \mid a_n \notin R\}$ is finite. We divide into two cases ($|A| = 0$ and $|A| \in \mathbb{N}$). Suppose first that $|A| = 0$. Choose $N = n_0$. It follows that if $n \geq N$, then $a_n \notin A$, so $a_n \in R$, so $a_n \in I$, as desired. Now suppose that $|A| \in \mathbb{N}$. By Definition 1.18, $a^{-1}(A) \subset \mathbb{N}$. Consequently, by Lemma 3.4, $a^{-1}(A)$ has a last point $N - 1$. Choose $N = (N - 1) + 1$. It follows that if $n \geq N$, then $n \notin a^{-1}(A)$, so $a_n \notin A$, so $a_n \in R$, so $a_n \in I$, as desired. □

Theorem 15.4. *Suppose that (a_n) converges to both p and to p'. Then $p = p'$.*

Proof. Suppose for the sake of contradiction that $p \neq p'$. Then by Theorem 3.22, there exist disjoint regions R, R' containing p, p', respectively. Additionally, by consecutive applications of Corollary 4.11 and Lemma 8.3, R, R' are open intervals. Combining these last two results, we have by consecutive applications of Definition 15.2 that there exist $N, N' \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in R$ and if $n \geq N'$, then $a_n \in R'$. Let $M = \max(N, N')$. It follows that $M \geq N$ and $M \geq N'$. Thus, by the above, $a_M \in R$ and $a_M \in R'$. But this implies by Definition 1.6 that $a_M \in R \cap R'$. Therefore, by Definition 1.9, R and R' are not disjoint, a contradiction. □

Definition 15.5. If a sequence (a_n) converges to $p \in \mathbb{R}$, we call p the **limit** of (a_n) and write

$$\lim_{n\to\infty} a_n = p$$

Exercise 15.6. Which of the following sequences (a_n) converge? Which diverge? For each that converges, what is the limit? Give proofs for your answers.

(a) $a_n = 5$.

Proof. To prove that this sequence converges with limit $\lim_{n\to\infty} a_n = 5$, Definition 15.5 tells us that it will suffice to show that (a_n) converges to 5. To do so, Definition 15.2 tells us that it will suffice to verify that for every open interval I containing 5, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in I$. Let's begin.

Let I be an arbitrary open interval containing 5. Choose $N = 1$. Let n be an arbitrary natural number such that $n \geq N$. It follows by the definition of the sequence that $a_n = 5 \in I$, as desired. $\square$

(b) $a_n = n$.

Proof. To prove that this sequence diverges, Definition 15.2 tells us that it will suffice to show that for every point $p \in \mathbb{R}$, there exists an open interval I containing p such that for all $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \notin I$. Let's begin.

Let p be an arbitrary element of $\mathbb{R}$. Choose $I = (p-1, p+1)$. Clearly $p \in I$. Let N be an arbitrary natural number. By Corollary 6.12, there exists a natural number N' such that $p + 1 < N'$. Choose $M = \max(N, N')$. Thus, $M \geq N$. Additionally, it follows by the definition of the sequence that $a_M = M$. But this implies that $a_M \geq N' > p + 1$, i.e., $a_M \notin I$ by Equations 8.1. $\square$

(c) $a_n = \frac{1}{n}$.

Proof. To prove that this sequence converges with limit $\lim_{n\to\infty} a_n = 0$, Definitions 15.5 and 15.2 tell us that it will suffice to show that for every open interval I containing 0, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in I$. Let's begin.

Let I be an arbitrary interval containing 0. By Lemma 8.10, there exists a region (a, b) containing 0 such that $(a, b) \subset I$. By Corollary 6.12, there exists a natural number N such that $\frac{1}{b} < N$. Choose this N to be our N. Now let n be an arbitrary natural number such that $n \geq N$. It follows that $\frac{1}{b} < n$. Thus, since $0 < b$ and $0 < n$, we have by consecutive applications of Lemma 7.24 that $0 < \frac{1}{n} < b$. Consequently, since we also know that $a < 0$ and $a_n = \frac{1}{n}$, we have by transitivity and substitution that $a < a_n < b$. It follows by Equations 8.1 that $a_n \in (a, b)$. Therefore, by Definition 1.3, $a_n \in I$, as desired. $\square$

(d) $a_n = (-1)^n$.

Proof. To prove that this sequence diverges, Definition 15.2 tells us that it will suffice to show that for every point $p \in \mathbb{R}$, there exists an open interval I containing p such that for all $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \notin I$. Let's begin.

Let p be an arbitrary element of R. Choose $I = (p-1, p+1)$. Clearly $p \in I$. Let N be an arbitrary natural number. By Script 0, either N is even and $N+1$ is odd or vice versa. Thus, let N be even (the case where N is odd is symmetric). It follows that $N \geq N$ yields $a_N = (-1)^{2m} = ((-1)^2)^m = 1^m = 1$ and that $N+1 \geq N$ yields $a_{N+1} = (-1)^{2m+1} = 1(-1)^1 = -1$. Now suppose for the sake of contradiction that $a_N \in I$ and $a_{N+1} \in I$. Since $a_N = 1 \in I$, we have by Equations 8.1 that $p - 1 < 1 < p + 1$. It follows by Definition 7.21 that $p - 3 < -1 < p - 1$. But $-1 < p - 1$ implies by Equations 8.1 that $a_{N+1} = -1 \notin I$, a contradiction. Therefore, $N+1 \geq N$ is a number such that $a_{N+1} \notin I$, as desired. $\square$

5/11: **Theorem 15.7.** *A sequence (a_n) converges to $p \in \mathbb{R}$ if and only if for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - p| < \epsilon$.*

Proof. Suppose first that (a_n) converges to p. Let $\epsilon > 0$ be arbitrary. Consider the p-containing region $R = (p - \epsilon, p + \epsilon)$. By Corollary 4.11 and 8.3, R is an open interval. Thus, by Definition 15.2, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in R$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. Then $a_n \in (p - \epsilon, p + \epsilon)$. Therefore, by Exercise 8.9, $|a_n - p| < \epsilon$, as desired.

Now suppose that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - p| < \epsilon$. To prove that (a_n) converges to p, Definition 15.2 tells us that it will suffice to show that for every open interval I containing p, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in I$. Let I be an arbitrary open interval that satisfies $p \in I$. It follows by Lemma 8.10 that there exists a number $\epsilon > 0$ such that $(p - \epsilon, p + \epsilon) \subset I$. With respect to this ϵ, we have by hypothesis that there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - p| < \epsilon$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. Then $|a_n - p| < \epsilon$. Consequently, by Exercise 8.9, $a_n \in (p - \epsilon, p + \epsilon)$. Therefore, by Definition 1.3, $a_n \in I$, as desired. □

Exercise 15.8.

(a) Prove that $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

Proof. To prove that $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$, Definition 15.5 and Theorem 15.7 tell us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - 0| = |a_n| < \epsilon$. Let $\epsilon > 0$ be arbitrary. By Corollary 6.12, there exists a natural number N such that $\frac{1}{\epsilon} < N$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. It follows by transitivity that $\frac{1}{\epsilon} < n$. Thus, since $0 < n$ and $0 < \epsilon$, we have by consecutive applications of Lemma 7.24 that $0 < \frac{1}{n} < \epsilon$. Additionally, since $(-1)^n = 1$ or $(-1)^n = -1$ for all $n \in \mathbb{N}$ by Script 0, we have by Definition 8.4 that $|\frac{(-1)^n}{n}| = |\frac{1}{n}| = \frac{1}{n}$. Consequently, we know that $|\frac{(-1)^n}{n}| < \epsilon$. But since $a_n = \frac{(-1)^n}{n}$, we have that $|a_n| < \epsilon$, as desired. □

(b) Let $x \in \mathbb{R}$ with $|x| < 1$. Prove that $\lim_{n\to\infty} x^n = 0$.

Lemma. *If $|y| > 1$ and n is a natural number, then $|y|^n \geq n(|y| - 1) + 1$.*

Proof. Define $1 + x = |y|$. It follows by Definition 7.21 that $x > 0 > -1$, which can be weakened to $x \geq -1$. Additionally, since n is a natural number, $n \geq 1$ by Script 0. Thus, since $x \geq -1$ and $n \geq 1$, we have by Additional Exercise 12.3b that $(1+x)^n \geq 1+nx$. Substituting, we have $|y|^n \geq n(|y|-1)+1$, as desired. □

Proof of Exercise 15.8b. To prove that $\lim_{n\to\infty} x^n = 0$, Definition 15.5 and Theorem 15.7 tell us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - 0| = |a_n| < \epsilon$. Let $\epsilon > 0$ be arbitrary. By Corollary 6.12, there exists a natural number N such that $\frac{1}{\epsilon(|\frac{1}{x}|-1)} < N$. Let n be an arbitrary natural number such that $n \geq N$. It follows by Script 7, the lemma, and the fact that $\frac{1}{|x|} > 1$ (since $1 > |x|$) that

$$\begin{aligned}
|x^n| &= |x^n| \cdot \frac{1}{\epsilon\left(\left|\frac{1}{x}\right| - 1\right)} \cdot \epsilon\left(\left|\frac{1}{x}\right| - 1\right) \\
&< |x^n| \cdot n \cdot \epsilon\left(\left|\frac{1}{x}\right| - 1\right) \\
&< \epsilon \cdot |x^n| \cdot n\left(\left|\frac{1}{x}\right| - 1\right) + 1 \\
&\leq \epsilon \cdot |x^n| \cdot \left|\frac{1}{x}\right|^n \\
&= \epsilon \cdot |x^n| \cdot \frac{1}{|x^n|} \\
&= \epsilon
\end{aligned}$$

as desired. □

Theorem 15.9. *If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ both exist, then*[1]

(a) $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.

Proof. Let $p = \lim_{n\to\infty} a_n$ and let $q = \lim_{n\to\infty} b_n$. To prove that $\lim_{n\to\infty}(a_n + b_n)$ exists and equals $\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$, Definition 15.5 and Theorem 15.7 tell us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n + b_n - (p+q)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. It follows by consecutive applications of Definition 15.5 and Theorem 15.7 that there exist natural numbers N_a, N_b such that for all $n \geq N_a$, we have $|a_n - p| < \frac{\epsilon}{2}$ and for all $n \geq N_b$, we have $|b_n - q| < \frac{\epsilon}{2}$. Now choose $N = \max(N_a, N_b)$. Let n be an arbitrary natural number such that $n \geq N$. It follows that $n \geq N \geq N_a$, so we know that $|a_n - p| < \frac{\epsilon}{2}$. Similarly, $|b_n - q| < \frac{\epsilon}{2}$. Therefore, we have that

$$
\begin{aligned}
|a_n + b_n - (p+q)| &\leq |a_n - p| + |b_n - q| && \text{Lemma 8.8} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}
$$

□

(b) $\lim_{n\to\infty}(a_n \cdot b_n) = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$.

Proof. Let $p = \lim_{n\to\infty} a_n$ and let $q = \lim_{n\to\infty} b_n$. To prove that $\lim_{n\to\infty}(a_n \cdot b_n)$ exists and equals $(\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$, Definition 15.5 and Theorem 15.7 tell us that it will suffice too show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n \cdot b_n - p \cdot q| < \epsilon$. Let $\epsilon > 0$ be arbitrary. It follows by consecutive applications of Definition 15.5 and Theorem 15.7 that there exist natural numbers N_a, N_b such that for all $n \geq N_a$, we have $|a_n - p| < \min(\frac{\epsilon}{2(|q|+1)}, 1)$ and for all $n \geq N_b$, we have $|b_n - q| < \frac{\epsilon}{2(|p|+1)}$. Now choose $N = \max(N_a, N_b)$. Let n be an arbitrary natural number such that $n \geq N$. It follows that $n \geq N \geq N_a$, so we know that $|a_n - p| < \min(\frac{\epsilon}{2(|q|+1)}, 1)$. Similarly, $|b_n - q| < \frac{\epsilon}{2(|p|+1)}$. As a last note before we launch into the main inequality, observe that $|a_n| - |p| \leq |a_n - p| < \min(\frac{\epsilon}{2(|p|+1)}, 1) \leq 1$, i.e., that $|a_n| < 1 + |p|$. Therefore, we have that

$$
\begin{aligned}
|a_n \cdot b_n - p \cdot q| &= |a_n(b_n - q) + q(a_n - p)| \\
&\leq |a_n| \cdot |b_n - q| + |q| \cdot |a_n - p| && \text{Lemma 8.8} \\
&< (1 + |p|) \cdot \frac{\epsilon}{2(|p|+1)} + |q| \cdot \frac{\epsilon}{2(|q|+1)} \\
&= \frac{\epsilon}{2} \cdot \frac{1+|p|}{1+|p|} + \frac{\epsilon}{2} \cdot \frac{|q|}{|q|+1} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}
$$

□

Moreover, if $\lim_{n\to\infty} b_n \neq 0$, then

(c) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$.

Proof. Let $p = \lim_{n\to\infty} a_n$ and let $q = \lim_{n\to\infty} b_n$. To prove that $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and equals $\frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, Definition 15.5 and Theorem 15.7 tell us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|\frac{a_n}{b_n} - \frac{p}{q}| < \epsilon$. Let $\epsilon > 0$ be arbitrary. It follows by consecutive applications of Definition 15.5 and Theorem 15.7 that there exist natural numbers N_a, N_b

[1]Note that these proofs are entirely symmetric to those in Theorem 11.9.

such that for all $n \geq N_a$, we have $|a_n - p| < \frac{\epsilon|q|}{4}$ and for all $n \geq N_b$, we have $|b_n - q| < \min(\frac{|q|}{2}, \frac{\epsilon|q|^2}{4|p|})$. Now choose $N = \max(N_a, N_b)$. Let n be an arbitrary natural number such that $n \geq N$. It follows that $n \geq N \geq N_a$, so we know that $|a_n - p| < \frac{\epsilon|q|}{4}$. Similarly, $|b_n - q| < \min(\frac{|q|}{2}, \frac{\epsilon|q|^2}{4|p|})$.

Before we get into the body of the proof, we need a preliminary result: it follows from the fact that $|b_n - q| < \frac{|q|}{2}$ that

$$\begin{aligned}
|q| &= 2|q| - |q| \\
&= 2(|q| - |b_n| + |b_n|) - |q| \\
&\leq 2(|q - b_n| + |b_n|) - |q| \\
&= 2(|b_n - q| + |b_n|) - |q| \\
&< 2\left(\frac{|q|}{2} + |b_n|\right) - |q| \\
&= |q| + 2|b_n| - |q| \\
&= 2|b_n|
\end{aligned}$$

With this result, we are ready to introduce the main inequality:

$$\begin{aligned}
\left|\frac{a_n}{b_n} - \frac{p}{q}\right| &= \left|\frac{a_n q - b_n p}{b_n q}\right| \\
&= \frac{|q(a_n - p) + p(q - b_n)|}{|b_n| \cdot |q|} \\
&\leq \frac{|q| \cdot |a_n - p| + |p| \cdot |q - b_n|}{|b_n| \cdot |q|} \\
&< \frac{|q| \cdot \frac{\epsilon|q|}{4} + |p| \cdot \frac{\epsilon|q|^2}{4|p|}}{|b_n| \cdot |q|} \\
&= \frac{\epsilon|q|}{2|b_n|} \\
&< \frac{\epsilon|q|}{|q|} \\
&= \epsilon
\end{aligned}$$

□