Script 17

Sequences and Series of Functions

6/23: **Definition 17.1.** Let $A \subset \mathbb{R}$, and consider $X = \{f : A \to \mathbb{R}\}$, the collection of real-valued functions on A. A **sequence of functions** (on A) is an ordered list $(f_1, f_2, f_3, \dots)$ which we will denote (f_n), where each $f_n \in X$. (More formally, we can think of the sequence as a function $F : \mathbb{N} \to X$, where $f_n = F(n)$, for each $n \in \mathbb{N}$, but this degree of formality is not particularly helpful.)

We can take the sequence to start at any $n_0 \in \mathbb{Z}$ and not just at 1, just like we did for sequences of real numbers.

Definition 17.2. The sequence (f_n) **converges pointwise** to a function $f : A \to \mathbb{R}$ if for all $x \in A$ and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$. In other words, we have that for all $x \in A$, $\lim_{n\to\infty} f_n(x) = f(x)$.

Definition 17.3. The sequence (f_n) **converges uniformly** to a function $f : A \to \mathbb{R}$ if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$ for every $x \in A$.

Equivalently, the sequence (f_n) **converges uniformly** to a function $f : A \to \mathbb{R}$ if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $\sup_{x\in A} |f_n(x) - f(x)| < \epsilon$.

Exercise 17.4. Suppose that a sequence (f_n) converges pointwise to a function f. Prove that if (f_n) converges uniformly to a function g, then $f = g$.

Proof. To prove that $f = g$, Definition 1.16 tells us that it will suffice to show that $f(x) = g(x)$ for all $x \in A$. Suppose for the sake of contradiction that $f(x) \neq g(x)$ for some $x \in A$. Since (f_n) converges pointwise to f by hypothesis, Definition 17.2 implies that for all $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then $|f_n(x) - f(x)| < \epsilon$. Additionally, since (f_n) converges uniformly to g by hypothesis, Definition 17.3 asserts that for all $\epsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that if $n \geq N_2$, then $|f_n(x) - g(x)| < \epsilon$.

WLOG, let $f(x) > g(x)$. Choose $\epsilon = \frac{f(x)-g(x)}{2}$, and let $N = \max(N_1, N_2)$. Since $N \geq N_1$, $|f_N(x) - f(x)| < \frac{f(x)-g(x)}{2}$. Similarly, $|f_N(x) - g(x)| < \frac{f(x)-g(x)}{2}$. But this implies that

$$\begin{aligned}
f(x) - g(x) &= |f(x) - f_N(x) + f_N(x) - g(x)| \\
&\leq |f(x) - f_N(x)| + |f_N(x) - g(x)| && \text{Lemma 8.8} \\
&= |f_N(x) - f(x)| + |f_N(x) - g(x)| && \text{Exercise 8.5} \\
&< \frac{f(x) - g(x)}{2} + \frac{f(x) - g(x)}{2} \\
&= f(x) - g(x)
\end{aligned}$$

a contradiction. ☐

Exercise 17.5. For each of the following sequences of functions, determine what function the sequence (f_n) converges to pointwise. Does the sequence converge uniformly to this function?

(a) For $n \in \mathbb{N}$, let $f_n : [0, 1] \to \mathbb{R}$ be given by $f_n(x) = x^n$.

Answer. Converges to the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$$

Does not converge uniformly. □

(b) For $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be given by $f_n(x) = \frac{\sin(nx)}{n}$. (For the purposes of this example, you may assume basic knowledge of sine.)

Answer. Converges to the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$. Does converge uniformly. □

(c) For $n \in \mathbb{N}$, let $f_n : [0, 1] \to \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} n^2 x & 0 \leq x \leq \frac{1}{n} \\ n(2 - nx) & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \frac{2}{n} \leq x \leq 1 \end{cases}$$

Answer. Converges to the function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = 0$. Does not converge uniformly. □

Theorem 17.6. *Let (f_n) be a sequence of functions, and suppose that each $f_n : A \to \mathbb{R}$ is continuous. If (f_n) converges uniformly to $f : A \to \mathbb{R}$, then f is continuous.*

Proof. To prove that f is continuous, Theorem 9.10 tells us that it will suffice to show that f is continuous at every $x \in A$. Let x be an arbitrary element of A. To show that f is continuous at x, Theorem 11.5 tells us that it will suffice to verify that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary, and also let y be an arbitrary element of A. Since (f_n) converges uniformly, Definition 17.3 implies that there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(a) - f(a)| < \frac{\epsilon}{3}$ for all $a \in A$. Thus, $|f_N(x) - f(x)| < \frac{\epsilon}{3}$ and $|f_N(y) - f(y)| < \frac{\epsilon}{3}$. Additionally, since each f_n is continuous, Theorems 9.10 and 11.5 assert that there exists $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then $|f_N(y) - f_N(x)| < \frac{\epsilon}{3}$. Choose this δ to be our δ. Therefore,

$$\begin{aligned} |f(y) - f(x)| &\leq |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| && \text{Lemma 8.8} \\ &= |f_N(y) - f(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| && \text{Exercise 8.5} \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

as desired. □

6/26: **Theorem 17.7.** *Suppose that (f_n) is a sequence of integrable functions on $[a, b]$ and suppose that (f_n) converges uniformly to $f : [a, b] \to \mathbb{R}$. Then*

$$\int_a^b f = \lim_{n \to \infty} \int_a^b f_n$$

Lemma. *f is integrable on $[a, b]$.*

Proof. To prove that f is integrable on $[a, b]$, Theorem 13.18 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since (f_n) converges uniformly to f by hypothesis, Definition 17.3 asserts that there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|f_n(x) - f(x)| < \frac{\epsilon}{6(b-a)}$[1] for all $x \in [a, b]$. This statement will be useful in the verification of the three following results.

[1] For the purposes of this proof, we will assume that $a < b$, on the basis of the fact that the proof of the case where $a = b$ is trivial.

To confirm that $|U(f_N, P) - L(f_N, P)| < \frac{\epsilon}{3}$, we first invoke Theorem 13.18, which tells us that since f_N is integrable by hypothesis, there exists a partition P of $[a, b]$ such that $U(f_N, P) - L(f_N, P) < \frac{\epsilon}{3}$. Additionally, since $L(f_N, P) \leq U(f_N, P)$ by Theorem 13.13, we have by Definition 8.4 that $U(f_N, P) - L(f_N, P) = |U(f_N, P) - L(f_N, P)|$. Therefore, we have by transitivity that $|U(f_N, P) - L(f_N, P)| = U(f_N, P) - L(f_N, P) < \frac{\epsilon}{3}$, as desired.

To confirm that $|U(f, P) - U(f_N, P)| < \frac{\epsilon}{3}$, we begin with the following contradiction argument[2].

Suppose for the sake of contradiction that $|M_i(f) - M_i(f_N)| \geq \frac{\epsilon}{3(b-a)}$. We divide into two cases ($M_i(f) - M_i(f_N) \geq \frac{\epsilon}{3(b-a)}$ and $M_i(f_N) - M_i(f) \geq \frac{\epsilon}{3(b-a)}$). Suppose first that $M_i(f) - M_i(f_N) \geq \frac{\epsilon}{3(b-a)}$. By Lemma 5.11, there exists $f(x) \in \{f(x) \mid t_{i-1} \leq x \leq t_i\}$ such that $M_i(f) - \frac{\epsilon}{6(b-a)} < f(x) \leq M_i(f)$. Similarly, there exists $f_N(x) \in \{f_N(x) \mid t_{i-1} \leq x \leq t_i\}$ such that $M_i(f_N) - \frac{\epsilon}{6(b-a)} < f_N(x) \leq M_i(f_N)$. Thus, we have that

$$f(x) > M_i(f) - \frac{\epsilon}{6(b-a)} > M_i(f) - \frac{\epsilon}{3(b-a)} \geq M_i(f_N) \geq f_N(x)$$

It follows that

$$\begin{aligned}
|f(x) - f_N(x)| &= f(x) - f_N(x) \\
&> \left(M_i(f) - \frac{\epsilon}{6(b-a)}\right) - f_N(x) \\
&\geq \left(M_i(f) - \frac{\epsilon}{6(b-a)}\right) - M_i(f_N) \\
&= M_i(f) - M_i(f_N) - \frac{\epsilon}{6(b-a)} \\
&\geq \frac{\epsilon}{3(b-a)} - \frac{\epsilon}{6(b-a)} \\
&= \frac{\epsilon}{6(b-a)}
\end{aligned}$$

But this contradicts the previously proven fact that $|f(x) - f_N(x)| = |f_N(x) - f(x)| < \frac{\epsilon}{6(b-a)}$. The argument is symmetric in the other case.

Thus, we know that $|M_i(f) - M_i(f_N)| < \frac{\epsilon}{3(b-a)}$. Therefore, we have that

$$\begin{aligned}
|U(f, P) - U(f_N, P)| &= \left|\sum_{i=1}^{k} M_i(f)(t_i - t_{i-1}) - \sum_{i=1}^{k} M_i(f_N)(t_i - t_{i-1})\right| && \text{Definition 13.10} \\
&= \left|\sum_{i=1}^{k} (M_i(f) - M_i(f_N)(t_i - t_{i-1})\right| \\
&< \left|\sum_{i=1}^{k} \frac{\epsilon}{3(b-a)}(t_i - t_{i-1})\right| \\
&= \frac{\epsilon}{3(b-a)}(b-a) \\
&= \frac{\epsilon}{3}
\end{aligned}$$

as desired.

The verification of the statement that $|L(f_N, P) - L(f, P)| < \frac{\epsilon}{3}$ is symmetric to the previous argument.

Having established that $|U(f_N, P) - L(f_N, P)| < \frac{\epsilon}{3}$, $|U(f, P) - U(f_N, P)| < \frac{\epsilon}{3}$, and $|L(f_N, P) - L(f, P)| <$

[2]Note that this argument is analogous to the proof of Additional Exercise 13.2.

$\frac{\epsilon}{3}$, we can now show that

$$\begin{aligned}
U(f,P) - L(f,P) &= |U(f,P) - L(f,P)| && \text{Theorem 13.13} \\
&\leq |U(f,P) - U(f_N,P)| + |U(f_N,P) - L(f_N,P)| + |L(f_N,P) - L(f,P)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon
\end{aligned}$$

as desired. □

Proof of Theorem 17.7. To prove that $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$, Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|\int_a^b f_n - \int_a^b f| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since (f_n) converges uniformly to f, we have by Definition 17.3 that there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$ for all $x \in [a,b]$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. It follows from the lemma to Exercise 8.9 that $-\frac{\epsilon}{b-a} < f_n(x) - f(x) < \frac{\epsilon}{b-a}$ for all $x \in [a,b]$. Additionally, since f_n is integrable on $[a,b]$ by hypothesis and f is integrable on $[a,b]$ by the lemma, Theorem 13.24 implies that $f_n - f$ is integrable on $[a,b]$. Combining these last two results, we have by Theorem 13.27 that $-\frac{\epsilon}{b-a}(b-a) < \int_a^b (f_n - f) < \frac{\epsilon}{b-a}(b-a)$. Consequently, by Script 7 and the lemma to Exercise 8.9, we have that $|\int_a^b (f_n - f)| < \epsilon$. Therefore, by Theorem 13.24, we have that $|\int_a^b f_n - \int_a^b f| < \epsilon$, as desired. □

Theorem 17.8. *Let (f_n) be a sequence of functions defined on an open interval containing $[a,b]$ such that each f_n is differentiable on $[a,b]$ and f_n' is integrable on $[a,b]$. Suppose further that (f_n) converges pointwise to f on $[a,b]$ and that (f_n') converges uniformly to a continuous function g on $[a,b]$. Then f is differentiable at every $x \in [a,b]$ and*

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

Proof. Let x be an arbitrary element of $[a,b]$. Since (f_n') converges uniformly to g, Definition 17.3 and Theorem 15.7 imply that $\lim_{n\to\infty} f_n'(x) = g(x)$. Additionally, we have that

$$\begin{aligned}
\int_a^x g &= \lim_{n\to\infty} \int_a^x f_n' && \text{Theorem 17.7} \\
&= \lim_{n\to\infty} (f_n(x) - f_n(a)) && \text{Theorem 14.4} \\
&= \lim_{n\to\infty} f_n(x) - \lim_{n\to\infty} f_n(a) && \text{Theorem 15.9} \\
&= f(x) - f(a) && \text{Definition 17.2}
\end{aligned}$$

This combined with the fact that g is continuous (hence continuous at x by Theorem 9.10) implies that

$$\begin{aligned}
g(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(f(x) - f(a)) && \text{Theorem 14.1} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}(f(x)) - \frac{\mathrm{d}}{\mathrm{d}x}(f(a)) && \text{Exercise 12.9} \\
&= f'(x) && \text{Exercise 12.8}
\end{aligned}$$

Therefore, we have by transitivity that $f'(x) = \lim_{n\to\infty} f_n'(x)$, as desired. □

Theorem 17.9. *Let (f_n) be a sequence of functions defined on a set A. Then the following are equivalent.*

(*a*) *There is some function f such that (f_n) converges uniformly to f on A.*

(*b*) *For all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that when $m, n \geq N$, $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in A$.*

Proof. Suppose first that there is some function f to which (f_n) converges uniformly on A. Let $\epsilon > 0$ be arbitrary. By Definition 17.3, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ for all $x \in A$. Choose this N to be our N. Let n, m be arbitrary natural numbers such that $n, m \geq N$. Then $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ and $|f_m(x) - f(x)| < \frac{\epsilon}{2}$ for all $x \in A$. Therefore, we have that

$$\begin{aligned}
|f_n(x) - f_m(x)| &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| && \text{Lemma 8.8} \\
&= |f_n(x) - f(x)| + |f_m(x) - f(x)| && \text{Exercise 8.5} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

for all $x \in A$, as desired.

Now suppose that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that when $n, m \geq N$, $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in A$. It follows by Theorem 15.19 that $(f_n(x))$ converges for all $x \in A$, i.e., for all $x \in A$, there exists a point $f(x) \in \mathbb{R}$ to which $(f_n(x))$ converges. Let $f : A \to \mathbb{R}$ be defined by $f(x) = \lim_{n\to\infty} f_n(x)$.

To prove that (f_n) converges uniformly to f, Definition 17.3 tells us that it will suffice to show that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$ for all $x \in A$. Let $\epsilon > 0$ be arbitrary. By the hypothesis, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|f_n(x) - f_m(x)| < \frac{\epsilon}{2}$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$, and let x be an arbitrary element of A. Since $(f_m(x))$ converges to $f(x)$, Theorem 15.7 asserts that there exists an $N' \in \mathbb{N}$ such that for all $m \geq N'$, $|f_m(x) - f(x)| < \frac{\epsilon}{2}$. Choose $M = \max(N, N')$. It follows that

$$\begin{aligned}
|f_n(x) - f(x)| &\leq |f_n(x) - f_M(x)| + |f_M(x) - f(x)| && \text{Lemma 8.8} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

as desired. ☐

Definition 17.10. We define series of functions the same way we defined series of numbers. That is, given a sequence (f_n), define the sequence of partial sums (p_n) by $p_n(x) = f_1(x) + \cdots + f_n(x)$ and say that $\sum_{n=1}^{\infty} f_n$ converges pointwise or converges uniformly to f if the sequence (p_n) does.

Theorem 17.11. *Suppose that $f_n : A \to \mathbb{R}$ is a sequence of functions and that there exists a sequence of positive real numbers (M_n) such that for all $x \in A$, we have $|f_n(x)| \leq M_n$. If $\sum_{n=1}^{\infty} M_n$ converges, then for each $x \in A$, the series of numbers $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely. Furthermore, $\sum_{n=1}^{\infty} f_n$ converges uniformly to the function f defined by $f(x) = \sum_{n=1}^{\infty} f_n(x)$.*

Proof. Let x be an arbitrary element of A. To prove that $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely, Definition 16.9 tells us that it will suffice to show that $\sum_{n=1}^{\infty} |f_n(x)|$ converges. Since (M_n) is a sequence of positive numbers and $|f_n(x)| \leq M_n$ for all $n \geq 1$, the proof of Theorem 16.13 asserts that $\sum_{n=1}^{\infty} |f_n(x)|$ converges.

To prove that $\sum_{n=1}^{\infty} f_n$ converges uniformly to f, Definition 17.10 tells us that it will suffice to show that the sequence of partial sums (p_n) defined by $p_k(x) = \sum_{n=1}^{k} f_n(x)$ converges uniformly to f. To do this, Definition 17.3 tells us that it will suffice to verify that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $j \geq N$, then $|\sum_{n=1}^{j} f_n(x) - f(x)| < \epsilon$ for all $x \in A$. Let $\epsilon > 0$ be arbitrary. By Definition 16.1, $\sum_{n=1}^{\infty} M_n = \lim_{k\to\infty} \sum_{n=1}^{k} M_n$. Thus, by Theorem 15.7, there is some $N \in \mathbb{N}$ such that for all $j \geq N$, $|\sum_{n=1}^{j} M_n - \sum_{n=1}^{\infty} M_n| < \epsilon$. Choose this N to be our N. Let j be an arbitrary natural number such that $j \geq N$. It follows by Script 16 that $|\sum_{n=j+1}^{\infty} M_n| < \epsilon$. Additionally, since (M_n) is a sequence of positive numbers, $\sum_{n=j+1}^{\infty} M_n = |\sum_{n=j+1}^{\infty} M_n|$. Therefore, combining the last several results and letting x be an arbitrary element of A, we have that

$$\begin{aligned}\left|\sum_{n=1}^{j} f_n(x) - f(x)\right| &= \left|\sum_{n=1}^{j} f_n(x) - \sum_{n=1}^{\infty} f_n(x)\right| \\ &= \left|\sum_{n=j+1}^{\infty} f_n(x)\right| \\ &\leq \sum_{n=j+1}^{\infty} |f_n(x)| && \text{Theorem 16.11} \\ &\leq \sum_{n=j+1}^{\infty} M_n \\ &= \left|\sum_{n=j+1}^{\infty} M_n\right| \\ &< \epsilon\end{aligned}$$

as desired. □

6/30: **Definition 17.12.** A function of the form $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$, where $c_n \in \mathbb{R}$ is called a **power series**. The power series is **centered** at a, and the numbers c_n are called the **coefficients**.

Theorem 17.13. *Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$ be a power series centered at 0. Suppose that x_0 is a real number such that the series $f(x_0) = \sum_{n=0}^{\infty} c_n x_0^n$ converges. Let r be any number such that $0 < r < |x_0|$. Then the following series of functions converges uniformly on $[-r, r]$ (and absolutely for each $x \in [-r, r]$):*

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad g(x) = \sum_{n=0}^{\infty} n c_n x^{n-1} \qquad h(x) = \sum_{n=0}^{\infty} c_n \cdot \frac{x^{n+1}}{n+1}$$

Furthermore, f is differentiable on $[-r, r]$ and $f' = g$. Also, h is differentiable on $[-r, r]$ and $h' = f$.

We may paraphrase this theorem as follows: If a (zero-centered) power series converges at x_0, then it may be differentiated and anti-differentiated term-by-term on $(-|x_0|, |x_0|)$ to obtain power series representations of the derivative and antiderivative of f.

Proof. Let (f_k) be defined by $f_k(x) = \sum_{n=0}^{k} c_n x^n$ for each $k \in \mathbb{N}$. To prove that (f_k) converges uniformly on $[-r, r]$ and that $(f_k(x))$ converges absolutely for each $x \in [-r, r]$, Theorem 17.11 tells us that it will suffice to find a sequence of positive real numbers (M_n) such that for all $x \in [-r, r]$, we have $|c_n x^n| \leq M_n$ and such that $\sum_{n=1}^{\infty} M_n$ converges.

To begin, we will show that there exists a number M such that $|c_n x_0^n| \leq M$ for all $n \in \mathbb{N}$. By the hypothesis, $f(x_0) = \sum_{n=0}^{\infty} c_n x_0^n$ converges. Thus, by Theorem 16.4, $\lim_{n\to\infty} c_n x_0^n = 0$. Consequently, by Theorem 15.13, $(c_n x_0^n)$ is bounded. It follows by Definition 15.12 that $\{c_n x_0^n \mid n \in \mathbb{N}\}$ is bounded. Thus, by Definition 5.6, there exist numbers $l, u \in \mathbb{R}$ such that $l \leq c_n x_0^n \leq u$ for all $n \in \mathbb{N}$. Let $M = \max(|l|, |u|)$. It follows by Script 0 that $-M \leq l \leq c_n x_0^n \leq u \leq M$ for all $n \in \mathbb{N}$. Therefore, by the lemma to Exercise 8.9, $|c_n x_0^n| \leq M$ for all $n \in \mathbb{N}$, as desired.

We can now define (M_n): Let (M_n) be defined by $M_n = M(\frac{r}{|x_0|})^n$ for all $n \in \mathbb{N}$.

Next, we will show that for all $x \in [-r, r]$, $|c_n x^n| \leq M_n$ for all $n \in \mathbb{N}$. Let x be an arbitrary element of $[-r, r]$. It follows by Equations 8.1 that $-r \leq x \leq r$. Thus, by the lemma to Exercise 8.9, $|x| \leq r$.

Consequently, by Exercise 12.22, $|x^n| \leq |r^n|$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} |c_n x^n| &\leq |c_n r^n| \\ &= |c_n x_0^n| \left(\frac{|r^n|}{|x_0^n|} \right) \\ &\leq M \left(\frac{r}{|x_0|} \right)^n \\ &= M_n \end{aligned}$$

for all $n \in \mathbb{N}$, as desired. Note that this result also implies by Definition 8.4 that (M_n) is a sequence of positive real numbers.

Lastly, we will show that $\sum_{n=1}^{\infty} M_n$ converges. Since $0 < r < |x_0|$ by hypothesis, Script 7 implies that $-1 < \frac{r}{|x_0|} < 1$. Thus, by Theorem 16.7, $\sum_{n=0}^{\infty} (\frac{r}{|x_0|})^n$ converges. Consequently, by Lemma 16.2, $\sum_{n=1}^{\infty} (\frac{r}{|x_0|})^n$ converges. Therefore, by Theorem 16.8, $\sum_{n=1}^{\infty} M(\frac{r}{|x_0|})^n$ (i.e., $\sum_{n=1}^{\infty} M_n$) converges, as desired.

Let (g_k) be defined by $g_k(x) = \sum_{n=0}^{k} nc_n x^{n-1}$ for each $k \in \mathbb{N}$. To prove that (g_k) converges uniformly on $[-r, r]$ and that $(g_k(x))$ converges absolutely for each $x \in [-r, r]$, Theorem 17.11 tells us that it will suffice to find a sequence of positive real numbers (M_n) such that for all $x \in [-r, r]$, we have $|nc_n x^{n-1}| \leq M_n$ and such that $\sum_{n=1}^{\infty} M_n$ converges.

To begin, we define (M_n) and prove its basic properties. Let (M_n) be defined by $M_n = \frac{Mn}{|r|}|\frac{r}{x_0}|^n$ for all $n \in \mathbb{N}$, where M is the same constant defined above. We now show that for all $x \in [-r, r]$, we have $|nc_n x^{n-1} \leq M_n$ for all $n \in \mathbb{N}$. Let x be an arbitrary element of $[-r, r]$. It follows as before that $|x^{n-1}| \leq |r^{n-1}|$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} |nc_n x^{n-1}| &= n|c_n||x^{n-1}| \\ &\leq n|c_n||r^{n-1}| \\ &= \frac{|c_n|}{|r|}|x_0|^n n \left| \frac{r}{x_0} \right|^n \\ &\leq \frac{Mn}{|r|} \left| \frac{r}{x_0} \right|^n \\ &= M_n \end{aligned}$$

for all $n \in \mathbb{N}$, as desired. Note that as before, this result also implies that (M_n) is a sequence of positive real numbers.

Next, we will show that $\sum_{n=1}^{\infty} M_n$ converges. To do so, Theorem 16.15 tells us that it will suffice to show that $\lim_{n\to\infty} |\frac{M_{n+1}}{M_n}| < 1$. As before, $|\frac{r}{x_0}| < 1$. Additionally, by an argument symmetric to that used in Exercise 15.6a, we know that $\lim_{n\to\infty} |\frac{r}{x_0}|$ converges to $|\frac{r}{x_0}|$. Furthermore, by an argument symmetric to that used in Exercise 15.10c, we have that $\lim_{n\to\infty} |\frac{n+1}{n}|$ converges to 1. Combining these last three results, we have that

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{M_{n+1}}{M_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{M(n+1)}{|r|} \left| \frac{r}{x_0} \right|^{n+1}}{\frac{Mn}{|r|} \left| \frac{r}{x_0} \right|^n} \right| \\ &= \lim_{n\to\infty} \left| \frac{n+1}{n} \left| \frac{r}{x_0} \right| \right| \\ &= \left(\lim_{n\to\infty} \left| \frac{r}{x_0} \right| \right) \left(\lim_{n\to\infty} \left| \frac{n+1}{n} \right| \right) && \text{Theorem 15.9} \\ &= \left| \frac{r}{x_0} \right| \cdot 1 \\ &< 1 \end{aligned}$$

as desired.

The argument for (h_k) defined by $h_k(x) = \sum_{n=0}^{k} c_n \cdot \frac{x^{n+1}}{n+1}$ is symmetric to that for (g_k).

To prove that f is differentiable on $[-r, r]$ and $f' = g$, Theorem 14.1 tells us that it will suffice to show that g is integrable on $[-r, r]$, that $f(x) + c = \int_{-r}^{x} g$ where $c \in \mathbb{R}$ is a constant, and that g is continuous on $[-r, r]$. We will verify each constraint in order.

Let k be an arbitrary natural number. Thus, by the definition of (g_k), $g_k(x) = \sum_{n=0}^{k} nc_n x^{n-1}$. Consequently, by Definition 11.11, g_k is a polynomial. It follows by Corollary 11.12 that g_k is continuous. Thus, by Theorem 13.19, g_k is integrable. Therefore, since (g_k) is a sequence of integrable functions on $[-r, r]$, the lemma to Theorem 17.7 asserts that g is integrable on $[-r, r]$, as desired.

It follows from the above that

$$\begin{aligned}
\int_{-r}^{x} g &= \lim_{k\to\infty} \int_{-r}^{x} g_k && \text{Theorem 17.7} \\
&= \lim_{k\to\infty} \int_{-r}^{x} \sum_{n=0}^{k} nc_n t^{n-1}\,\mathrm{d}t \\
&= \lim_{k\to\infty} \sum_{n=0}^{k} \int_{-r}^{x} nc_n t^{n-1}\,\mathrm{d}t && \text{Theorem 13.24} \\
&= \lim_{k\to\infty} \sum_{n=0}^{k} (c_n x^n - c_n(-r)^n)^{[3]} && \text{Theorem 14.4} \\
&= \sum_{n=0}^{\infty} (c_n x^n - c_n(-r)^n) && \text{Definition 16.1} \\
&= \sum_{n=0}^{\infty} c_n x^n - \sum_{n=0}^{\infty} c_n(-r)^n && \text{Theorem 16.8} \\
&= f(x) + c
\end{aligned}$$

as desired.

Lastly, since each g_k is continuous and (g_k) converges uniformly to g, Theorem 17.6 asserts that g is continuous on $[-r, r]$, as desired.

The argument for that h is differentiable on $[-r, r]$ and $h' = f$ is symmetric to the above. □

[3]Note that in order to verify the equality of the previous equation and this one, we must technically prove the power rule (analogous to Exercise 13.21) in general for arbitrary domains and n. However, as this would be a Script 13 proof in nature, we omit it.