MATH 16310 (Honors Calculus III IBL) Notes

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Contents

Script 12

Derivatives

12.1 Journal

3/30: Throughout this sheet, we let $f : A \to \mathbb{R}$ be a real valued function with domain $A \subset \mathbb{R}$. We also now assume the domain $A \subset \mathbb{R}$ is open.

Definition 12.1. The **derivative** of f at a point $a \in A$ is the number $f'(a)$ defined by the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided that the limit on the right-hand side exists. If $f'(a)$ exists, we say that f is **differentiable** (at a). If f is differentiable at all points of its domain, we say that f is **differentiable**. In this case, the values $f'(a)$ define a new function $f' : A \to \mathbb{R}$ called the **derivative** (of f).

Remark 12.2. If A is not open, the limit in Definition 12.1 may not exist. For example, if $f : [a, b] \to \mathbb{R}$, then we cannot define the derivative at the endpoints. For any c in the domain of f, we define the **right-hand derivative** $f'_+(c)$ and the **left-hand derivative** $f'_-(c)$ by

$$f'_+(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \qquad f'_-(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$$

We say that f is **differentiable** (on $[a, b]$) if f is differentiable on (a, b) and $f'_+(a)$ and $f'_-(b)$ exist.

Lemma 12.3. *Let $a \in \mathbb{R}$. Then*

$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$$

assuming that one of the two limits exists. (So if the limit on the left exists, so does the one on the right, and they are equal. Similarly, if the limit on the right exists, then so does the one on the left, and they are equal.)

Proof. Suppose first that $\lim_{x \to a} f(x)$ exists, and let it be equal to L. To prove that $\lim_{h \to 0} f(a+h)$ exists and that it equals $\lim_{x \to a} f(x)$, Definition 11.1 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $(h + a) \in A$ and $0 < |h - 0| = |h| < \delta$, then $|f(a+h) - L| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{x \to a} f(x)$ exists, Definition 11.1 implies that there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$. We will choose this δ to be our δ. Now let h be an arbitrary number satisfying both $(h + a) \in A$ and $0 < |h| < \delta$; we seek to show that $|f(a+h) - L| < \epsilon$. Since $(h + a) \in A$, $h + a = x$ for some $x \in A$. It follows that $h = x - a$, meaning that x is an object that is both an element of A and that satisfies $0 < |h| = |x - a| < \delta$, so we know that $|f(a+h) - L| = |f(x) - L| < \epsilon$, as desired.

The proof is symmetric in the other direction. ☐

Theorem 12.4. *Let $a \in \mathbb{R}$. Then f is differentiable at a if and only if $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$ exists. Moreover, if f is differentiable at a, then the derivative of f at a is given by the limit*

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Proof. Suppose first that f is differentiable at a. Then by Definition 12.1, $f'(a)$ exists. It follows that

$$\begin{aligned} f'(a) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} && \text{Definition 12.1}\\ &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{(a+h)-a}\\ &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} && \text{Lemma 12.3}\end{aligned}$$

Note that the substitution in the last step follows from using $\tilde{f}(x)=\frac{f(x)-f(a)}{x-a}$ as the "$f(x)$" function in Lemma 12.3.

The proof is symmetric in the reverse direction. □

Theorem 12.5. *If f is differentiable at a, then f is continuous at a.*

Proof. To prove that f is continuous at a, Theorem 11.5 tells us that it will suffice to show that $\lim_{x\to a} f(x) = f(a)$. By Definition 12.1, the hypothesis implies that $f'(a)$ exists. Thus, by Theorem 12.4, we know that $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists. Additionally, by Exercise 11.6, $g(x)=x-a$ is continuous at a. It follows by Theorem 11.5 that either $a\notin LP(A)$ or $\lim_{x\to a} g(x)$ exists (and equals $g(a)$). However, since A is open by hypothesis and $a\in A$, Theorem 4.10 implies that there exists a region R such that $a\in R$ and $R\subset A$. But $a\in R$ implies that $a\in LP(R)$ by Corollary 5.5, and this combined with the fact that $R\subset A$ implies by Theorem 3.14 that $a\in LP(A)$[1]. Thus, we have that $\lim_{x\to a} g(x)=g(a)=a-a=0$. Combining the last few results, we have

$$\begin{aligned} 0 &= \left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right)\cdot 0\\ &= \left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right)\left(\lim_{x\to a}(x-a)\right)\\ &= \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\cdot(x-a)\right) && \text{Theorem 11.9}\\ &= \lim_{x\to a}(f(x)-f(a))\end{aligned}$$

If we now consider $f(a)$ to be a constant function (i.e., $\lim_{x\to a} f(a)=f(a)$ by Exercise 11.6, Theorem 11.5, and the above result that $a\in LP(A)$), it follows from the above that

$$\begin{aligned} \lim_{x\to a}(f(x)-f(a))+\lim_{x\to a} f(a) &= 0+f(a)\\ \lim_{x\to a}(f(x)-f(a)+f(a)) &= f(a) && \text{Theorem 11.9}\\ \lim_{x\to a} f(x) &= f(a)\end{aligned}$$

as desired. □

Exercise 12.6. Show that the converse of Theorem 12.5 is not true.

Proof. The converse of Theorem 12.5 asserts that "if f is continuous at a, then f is differentiable at a." To falsify this statement, we will use the absolute value function $|x|$ as a counterexample. Let's begin.

By Exercise 11.7, $|x|$ is continuous. It follows by Theorem 9.10 that $|x|$ is continuous at 0. However, we can show that $|x|$ is not differentiable at 0.

To do this, Definition 12.1 and Theorem 12.4 tell us that it will suffice to verify that $\lim_{x\to 0}\frac{|x|-|0|}{x-0}=\lim_{x\to 0}\frac{|x|}{x}$ does not exist. Suppose for the sake of contradiction that $\lim_{x\to 0}\frac{|x|}{x}=L$. Then by Definition 11.1, for $\epsilon=1>0$, there exists a $\delta>0$ such that if $0<|x-0|=|x|<\delta$, then $|\frac{|x|}{x}-L|<1$. However, we

[1] I will not go through this or similar derivations again, although they may be technically necessary. Indeed, assume moving on that statements analogous to $a\in LP(A)$ hold true.

can show that no such δ exists. Let $\delta > 0$ be arbitrary. By Theorem 5.2, there exists a number $x \in \mathbb{R}$ such that $0 < x < \delta$. It follows by Definition 8.4 and Exercise 8.5 that $0 < |x| = |-x| < \delta$. Since both x and $-x$ are in the appropriate range, we know that

$$\begin{aligned}
\left|\frac{|x|}{x} - L\right| &= \left|\frac{x}{x} - L\right| & \left|\frac{|-x|}{-x} - L\right| &= \left|\frac{x}{-x} - L\right| && \text{Definition 8.4}\\
&= |1 - L| & &= |-1 - L| && \text{Script 7}\\
&= |L - 1| & &= |L + 1| && \text{Exercise 8.5}\\
&< 1 & &< 1
\end{aligned}$$

By consecutive applications of the lemma from Exercise 8.9, it follows that

$$\begin{aligned}
-1 < L - 1 &< 1 & -1 < L + 1 &< 1\\
0 < L &< 2 & -2 < L &< 0
\end{aligned}$$

But this implies that $L < 0$ and $L > 0$, a contradiction. $\square$

Exercise 12.7. Show that for all $n \in \mathbb{N}$,

$$x^n - a^n = (x - a)\left(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1}\right)$$

or equivalently,

$$x^n - a^n = (x - a)\left(\sum_{i=0}^{n-1} x^{n-1-i}a^i\right)$$

Proof. By simple algebra (see Script 7), we have

$$\begin{aligned}
(x - a)\left(\sum_{i=0}^{n-1} x^{n-1-i}a^i\right) &= \sum_{i=0}^{n-1}(x - a)x^{n-1-i}a^i\\
&= \sum_{i=0}^{n-1}\left(x^{n-i}a^i - x^{n-1-i}a^{i+1}\right)\\
&= x^n + \sum_{i=1}^{n-1} x^{n-i}a^i - \sum_{i=0}^{n-2} x^{n-1-i}a^{i+1} - a^n\\
&= x^n + \sum_{i=1}^{n-1} x^{n-i}a^i - \sum_{i=0+1}^{n-2+1} x^{n-1-(i-1)}a^{(i-1)+1} - a^n\\
&= x^n + \sum_{i=1}^{n-1} x^{n-i}a^i - \sum_{i=1}^{n-1} x^{n-i}a^i - a^n\\
&= x^n - a^n
\end{aligned}$$

as desired. $\square$

Exercise 12.8.

(a) Let $n \in \mathbb{N}$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^n$. Use Exercise 12.7 to prove that $f'(a) = na^{n-1}$ for all $a \in \mathbb{R}$.

Proof. Let a be an arbitrary element of $\mathbb{R}$. To prove that $f'(a) = na^{n-1}$, Theorem 12.4 tell us that it will suffice to show that $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = na^{n-1}$. By Corollary 11.12, the polynomial $\sum_{i=0}^{n-1} x^{n-1-i}a^i$ is continuous. Thus, by Theorem 9.10, it is continuous at a. It follows by Theorem 11.5 that $\lim_{x\to a} \sum_{i=0}^{n-1} x^{n-1-i}a^i = \sum_{i=0}^{n-1} a^{n-1-i}a^i$.

Additionally, we can demonstrate that $\lim_{x\to a}\frac{x-a}{x-a}=1$. To verify that $\lim_{h\to 0}\frac{h}{h}=1$, Definition 11.1 tells us that it will suffice to confirm that for all $\epsilon>0$, there exists a $\delta>0$ such that if $h\in\mathbb{R}$ and $0<|h-0|=|h|<\delta$, then $|\frac{h}{h}-1|<\epsilon$. Let $\epsilon>0$ be arbitrary. Choose $\delta=1$, and let h be an arbitrary element of $\mathbb{R}$ satisfying $0<|h|<\delta$. It follows by Script 7 that $|\frac{h}{h}-1|=|1-1|=0<\epsilon$, as desired. Since $\lim_{h\to 0}\frac{h}{h}=1$, we know by Script 7 that $\lim_{h\to 0}\frac{(a+h)-a}{(a+h)-a}=1$. Thus, by Lemma 12.3, $\lim_{x\to a}\frac{x-a}{x-a}=1$, as desired.

It follows from the above two results that

$$\begin{aligned}
na^{n-1} &= \underbrace{a^{n-1}+\cdots+a^{n-1}}_{n\text{ times}} \\
&= \sum_{i=0}^{n-1} a^{n-1} \\
&= \sum_{i=1}^{n-1} a^{n-1-i}a^i \\
&= \lim_{x\to a}\sum_{i=0}^{n-1} x^{n-1-i}a^i \\
&= 1\cdot\left(\lim_{x\to a}\sum_{i=0}^{n-1} x^{n-1-i}a^i\right) \\
&= \left(\lim_{x\to a}\frac{x-a}{x-a}\right)\left(\lim_{x\to a}\sum_{i=0}^{n-1} x^{n-1-i}a^i\right) \\
&= \lim_{x\to a}\frac{x-a}{x-a}\cdot\sum_{i=0}^{n-1} x^{n-1-i}a^i && \text{Theorem 11.9} \\
&= \lim_{x\to a}\frac{x^n-a^n}{x-a} && \text{Exercise 12.7} \\
&= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} \\
&= f'(a) && \text{Theorem 12.4}
\end{aligned}$$

as desired. □

(b) Let $k\in\mathbb{R}$. Prove that if $f:\mathbb{R}\to\mathbb{R}$ is given by $f(x)=k$, then $f'(a)=0$ for all $a\in\mathbb{R}$.

Proof. Let a be an arbitrary element of $\mathbb{R}$. To prove that $f'(a)=0$, Definition 12.1 tells us that it will suffice to show that $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=0$. By Exercise 11.6, the function $g:\mathbb{R}\to\mathbb{R}$ defined by $g(x)=0$ is continuous at every $x\in\mathbb{R}$, including 0. It follows by Theorem 11.5 that $\lim_{h\to 0}g(h)=g(0)=0$. Therefore,

$$\begin{aligned}
0 &= \lim_{h\to 0} g(h) \\
&= \lim_{h\to 0} 0 \\
&= \lim_{h\to 0}\frac{0}{h} \\
&= \lim_{h\to 0}\frac{k-k}{h} \\
&= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \\
&= f'(a) && \text{Definition 12.1}
\end{aligned}$$

as desired. □

Exercise 12.9. Suppose that $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are differentiable at $a \in A$.

(a) Prove that $f + g$ is differentiable at a and compute $(f + g)'(a)$ in terms of $f'(a)$ and $g'(a)$.

Proof. To prove that $f + g$ is differentiable at a, Definition 12.1 tells us that it will suffice to show $\lim_{h\to 0} \frac{(f+g)(a+h)-(f+g)(a)}{h}$ exists. Since f, g are differentiable at a, we know by Definition 12.1 that $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ and $\lim_{h\to 0} \frac{g(a+h)-g(a)}{h}$ exist. Thus, by Theorem 11.9 the limit of their sum exists and equals

$$\begin{aligned}
\lim_{h\to 0} \left(\frac{f(a+h)-f(a)}{h} + \frac{g(a+h)-g(a)}{h} \right) &= \lim_{h\to 0} \frac{f(a+h)+g(a+h)-f(a)-g(a)}{h} \\
&= \lim_{h\to 0} \frac{(f+g)(a+h)-(f+g)(a)}{h}
\end{aligned}$$

as desired. Having established that $\lim_{h\to 0} \frac{(f+g)(a+h)-(f+g)(a)}{h}$ exists, $(f + g)'(a)$ can be computed in terms of $f'(a)$ and $g'(a)$ with the following algebra.

$$\begin{aligned}
(f+g)'(a) &= \lim_{h\to 0} \frac{(f+g)(a+h)-(f+g)(a)}{h} && \text{Definition 12.1} \\
&= \lim_{h\to 0} \left(\frac{f(a+h)-f(a)}{h} + \frac{g(a+h)-g(a)}{h} \right) \\
&= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} + \lim_{h\to 0} \frac{g(a+h)-g(a)}{h} && \text{Theorem 11.9} \\
&= f'(a) + g'(a) && \text{Definition 12.1}
\end{aligned}$$

□

(b) Prove that fg is differentiable and compute $(fg)'(a)$ in terms of $f(a)$, $g(a)$, $f'(a)$, and $g'(a)$.

Proof. To prove that fg is differentiable at a, Definition 12.1 tells us that it will suffice to show $\lim_{h\to 0} \frac{(fg)(a+h)-(fg)(a)}{h}$ exists. Since f, g are differentiable at a, we know by Definition 12.1 that $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ and $\lim_{h\to 0} \frac{g(a+h)-g(a)}{h}$ exist. For the same reason, we know by Theorem 12.5 that g is continuous, i.e., continuous at a by Theorem 9.10. Consequently, by Theorem 11.5, $\lim_{x\to a} g(x)$ exists (and equals $g(a)$). Note that the preceding limit is equal to $\lim_{h\to 0} g(a + h)$ by Lemma 12.3. Lastly, we have by Exercise 11.6 that the constant function $f(a)$ is continuous at 0. Consequently, by Theorem 11.5, $\lim_{h\to 0} f(a)$ exists (and equals $f(a)$). Combining all of these results, consecutive applications of Theorem 11.9 assert that the limits

$$\lim_{h\to 0} g(a+h) \cdot \frac{f(a+h)-f(a)}{h} \qquad\qquad \lim_{h\to 0} f(a) \cdot \frac{g(a+h)-g(a)}{h}$$

exist. Furthermore, it asserts that the limit of their sum exists and equals

$$\begin{aligned}
\lim_{h\to 0} &\left(g(a+h) \cdot \frac{f(a+h)-f(a)}{h} + f(a) \cdot \frac{g(a+h)-g(a)}{h} \right) \\
&= \lim_{h\to 0} \frac{g(a+h)(f(a+h)-f(a)) + f(a)(g(a+h)-g(a))}{h} \\
&= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a+h) + f(a)g(a+h) - f(a)g(a)}{h} \\
&= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h} \\
&= \lim_{h\to 0} \frac{(fg)(a+h)-(fg)(a)}{h}
\end{aligned}$$

as desired. Having established that $\lim_{h\to 0}\frac{(fg)(a+h)-(fg)(a)}{h}$ exists, $(fg)'(a)$ can be computed in terms of $f(a)$, $g(a)$, $f'(a)$, and $g'(a)$ with the following algebra.

$$\begin{aligned}
(fg)'(a) &= \lim_{h\to 0}\frac{(fg)(a+h)-(fg)(a)}{h} && \text{Definition 12.1}\\
&= \lim_{h\to 0}\left(g(a+h)\cdot\frac{f(a+h)-f(a)}{h}+f(a)\cdot\frac{g(a+h)-g(a)}{h}\right)\\
&= \lim_{h\to 0} g(a+h)\cdot\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}+\lim_{h\to 0} f(a)\cdot\lim_{h\to 0}\frac{g(a+h)-g(a)}{h} && \text{Theorem 11.9}\\
&= g(a)f'(a)+f(a)g'(a)
\end{aligned}$$

□

(c) Prove that $\frac{1}{g}$ is differentiable at a (under an appropriate assumption) and compute $(\frac{1}{g})'(a)$ in terms of $g'(a)$ and $g(a)$. What assumption do you need to make?

Proof. Assume that $g(a)\neq 0$.

To prove that $\frac{1}{g}$ is differentiable at a, Definition 12.1 tells us that it will suffice to show that the limit $\lim_{h\to 0}\frac{(1/g)(a+h)-(1/g)(a)}{h}$ exists. Since g is differentiable at a, we know by Definition 12.1 that $\lim_{h\to 0}\frac{g(a+h)-g(a)}{h}$ exists. For the same reason, we know by Theorem 12.5 that g is continuous, i.e., continuous at a by Theorem 9.10. Consequently, by Theorem 11.5, $\lim_{x\to a} g(x)$ exists (and equals $g(a)$). It follows by Lemma 12.3 that the preceding limit is equal to $\lim_{h\to 0} g(a+h)$. Thus, since it is also equal to $g(a)\neq 0$, we have by Theorem 11.9 that $\lim_{h\to 0}\frac{1}{g}(a+h)$ exists (and equals $\frac{1}{g(a)}$). Lastly, we have by Exercise 11.6 that the constant function $-\frac{1}{g(a)}$ is continuous at 0. Consequently, by Theorem 11.5, $\lim_{h\to 0}-\frac{1}{g(a)}$ exists (and equals $-\frac{1}{g(a)}$). Combining this with the previous result, Theorem 11.9 asserts that the limit $\lim_{h\to 0}-\frac{1}{g(a+h)g(a)}$ exists (and equals $-\frac{1}{g(a)^2}$). Furthermore, it asserts that the limit of its product with $\lim_{h\to 0}\frac{g(a+h)-g(a)}{h}$ exists and equals

$$\begin{aligned}
\lim_{h\to 0}-\frac{1}{g(a+h)g(a)}\cdot\frac{g(a+h)-g(a)}{h} &= \lim_{h\to 0}\frac{\frac{g(a)-g(a+h)}{g(a+h)g(a)}}{h}\\
&= \lim_{h\to 0}\frac{\frac{1}{g(a+h)}-\frac{1}{g(a)}}{h}\\
&= \lim_{h\to 0}\frac{\frac{1}{g}(a+h)-\frac{1}{g}(a)}{h}
\end{aligned}$$

as desired. Having established that $\lim_{h\to 0}\frac{(1/g)(a+h)-(1/g)(a)}{h}$ exists, $(\frac{1}{g})'(a)$ can be computed in terms of $g(a)$ and $g'(a)$ with the following algebra.

$$\begin{aligned}
\left(\frac{1}{g}\right)'(a) &= \lim_{h\to 0}\frac{\frac{1}{g}(a+h)-\frac{1}{g}(a)}{h} && \text{Definition 12.1}\\
&= \lim_{h\to 0}-\frac{1}{g(a+h)g(a)}\cdot\frac{g(a+h)-g(a)}{h}\\
&= \lim_{h\to 0}-\frac{1}{g(a+h)g(a)}\cdot\lim_{h\to 0}\frac{g(a+h)-g(a)}{h} && \text{Theorem 11.9}\\
&= -\frac{g'(a)}{g(a)^2}
\end{aligned}$$

□

(d) Prove that $\frac{f}{g}$ is differentiable at a (under an appropriate assumption) and compute $(\frac{f}{g})'(a)$ in terms of $f(a)$, $g(a)$, $f'(a)$, and $g'(a)$. What assumption do you need to make?

Proof. Assume that $g(a) \neq 0$.

It follows by part (c) that $\frac{1}{g}$ is differentiable at a, and then by part (b) that $f \cdot \frac{1}{g} = \frac{f}{g}$ is differentiable at a.

Having established that $(\frac{f}{g})'(a)$ exists, it can be computed in terms of $f(a)$, $g(a)$, $f'(a)$, and $g'(a)$ with the following algebra.

$$\begin{aligned}\left(\frac{f}{g}\right)'(a) &= \left(f \cdot \frac{1}{g}\right)'(a) \\ &= f(a)\left(\frac{1}{g}\right)'(a) + f'(a)\left(\frac{1}{g}\right)(a) \\ &= f(a) \cdot -\frac{g'(a)}{g(a)^2} + \frac{f'(a)g(a)}{g(a)^2} \\ &= \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}\end{aligned}$$

□

4/1: One of the most important results concerning the differentiation of functions is the rule for the derivative of a composition of functions. Let $f : B \to \mathbb{R}, g : A \to \mathbb{R}$ be functions such that $g(A) \subset B$. The composition $(f \circ g)(x) = f(g(x))$ is defined for all $x \in A$.

Theorem 12.10. *Let $a \in A$, $g : A \to \mathbb{R}$, and $f : I \to \mathbb{R}$ where I is an interval containing $g(A)$. Suppose that g is differentiable at a and f is differentiable at $g(a)$. Then $f \circ g$ is differentiable a and*

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$$

Proof. To prove that $f \circ g$ is differentiable at a, Theorem 12.4 tells us that it will suffice to show that $\lim_{x\to a} \frac{(f\circ g)(x)-(f\circ g)(a)}{x-a}$ exists. To do so, we will define a special function φ and prove that it is continuous at a. It will follow that $(f \circ g)'(a)$ exists and equals $f'(g(a)) \cdot g'(a)$. Let's begin.

Let $\varphi : I \to \mathbb{R}$ be defined by

$$\varphi(x) = \begin{cases} \frac{f(g(x))-f(g(a))}{g(x)-g(a)} & g(x) \neq g(a) \\ f'(g(a)) & g(x) = g(a) \end{cases}$$

It is clear from the definition that the function is defined for all $x \in A$.

To confirm that φ is continuous at a, Theorem 11.5 tells us that it will suffice to demonstrate that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $|x-a| < \delta$, then $|\varphi(x)-\varphi(a)| = |\varphi(x)-f'(g(a))| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f is differentiable at $g(a)$, Theorem 12.4 asserts that $\lim_{y\to g(a)} \frac{f(y)-f(g(a))}{y-g(a)} = f'(g(a))$. It follows by Definition 11.1 that there is some $\delta' > 0$ such that if $y \in I$ and $0 < |y-g(a)| < \delta'$, then $|\frac{f(y)-f(g(a))}{y-g(a)} - f'(g(a))| < \epsilon$. Additionally, since g is differentiable (hence continuous by Theorem 12.5) at a, we have by Theorem 11.5 that there exists a $\delta > 0$ such that if $x \in A$ and $|x-a| < \delta$, then $|g(x)-g(a)| < \delta'$.

Using the above δ, let x be an arbitrary element of A such that $|x-a| < \delta$. We now divide into two cases ($g(x) = g(a)$ and $g(x) \neq g(a)$). If $g(x) = g(a)$, then $|\varphi(x) - f'(g(a))| = |f'(g(a)) - f'(g(a))| = 0 < \epsilon$, as desired. If $g(x) \neq g(a)$, then we continue. Since $|x - a| < \delta$, we have that $|g(x) - g(a)| < \delta'$. This combined with the fact that $g(x) \in I$ and $g(x) \neq g(a)$, i.e., $0 < |g(x)-g(a)|$ illustrates that $|\frac{f(g(x))-f(g(a))}{g(x)-g(a)} - f'(g(a))| = |\varphi(x) - f'(g(a))| < \epsilon$. Therefore, φ is continuous at a.

It follows by Theorem 11.5 that $\lim_{x\to a} \varphi(x) = \varphi(a) = f'(g(a))$. Additionally, since g is differentiable at a, Definition 12.1 and Theorem 12.4 tell us that $\lim_{x\to a} \frac{g(x)-g(a)}{x-a}$ exists (and equals $g'(a)$). The combination of the past two results imply by Theorem 11.9 that the product of the limits exists and equals

$$\begin{aligned} f'(g(a)) \cdot g'(a) &= \lim_{x\to a} \varphi(x) \cdot \frac{g(x)-g(a)}{x-a} \\ &= \lim_{x\to a} \frac{f(g(x))-f(g(a))}{g(x)-g(a)} \cdot \frac{g(x)-g(a)}{x-a} \\ &= \lim_{x\to a} \frac{f(g(x))-f(g(a))}{x-a} \\ &= \lim_{x\to a} \frac{(f\circ g)(x)-(f\circ g)(a)}{x-a} \\ &= (f\circ g)'(a) \end{aligned}$$

as desired. $\square$

We now come to the most important theorem in differential calculus, Corollary 12.16.

Definition 12.11. Let $f : A \to \mathbb{R}$ be a function. If $f(a)$ is the last point of $f(A)$, then $f(a)$ is called the **maximum value** of f. If $f(a)$ is the first point of $f(A)$, then $f(a)$ is the **minimum value** of f. We say that $f(a)$ is a **local maximum value** of f if there exists a region R containing a such that $f(a)$ is the last point of $f(A \cap R)$. We say that $f(a)$ is a **local minimum value** of f if there exists a region R containing a such that $f(a)$ is the first point of $f(A \cap R)$.

Remark 12.12. Equivalently, $f(a)$ is a local maximum (resp. minimum) value of f if there exists U open in A such that $f(a)$ is the last (resp. first) point of $f(U)$.

Theorem 12.13. *Let $f : A \to \mathbb{R}$ be differentiable at a. Suppose that $f(a)$ is the maximum value or minimum value of f. Then $f'(a) = 0$.*

Proof. Suppose first that $f(a)$ is the maximum value of f, and suppose for the sake of contradiction that $f'(a) \neq 0$. Then $f'(a) > 0$ or $f'(a) < 0$. We now divide into two cases. If $f'(a) > 0$, then by Theorem 12.4, $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} > 0$. Thus, by Lemma 11.8, there exists a region R with $a \in \mathbb{R}$ such that $\frac{f(x)-f(a)}{x-a} > 0$ for all $x \in R \cap A$. Additionally, since A is open, Theorem 4.10 implies that there exists a region S with $a \in S$ and $S \subset A$. It follows by Theorem 3.18 that $R \cap S = (c, d)$, where (c, d) is a region containing a. Since $a \in (c, d)$, Theorem 5.2 implies that there exists a point $y \in \mathbb{R}$ such that $a < y < d$. Clearly, since $y \in (c, d) = R \cap S$ and $S \subset A$, we have that $y \in R \cap A$. It follows that $\frac{f(y)-f(a)}{y-a} > 0$. Furthermore, $y > a$ implies that $y - a > 0$. Therefore, by Definition 7.21, $f(y) - f(a) = \frac{f(y)-f(a)}{y-a} \cdot (y - a) > 0$. But this means that $f(y) > f(a)$, i.e., that $f(a)$ is not the last point of $f(A)$ (by Definition 3.3), i.e., that $f(a)$ is not the maximum value of f (by Definition 12.11), a contradiction. The argument is symmetric in the other case.

The proof is symmetric in the other case. $\square$

Corollary 12.14. *Let $f : A \to \mathbb{R}$ be differentiable at a. Suppose that $f(a)$ is a local maximum or local minimum value of f. Then $f'(a) = 0$.*

Proof. Suppose first that $f(a)$ is a local maximum of f. Then by Definition 12.11, there exists a region R containing a such that $f(a)$ is the last point of $f(A \cap R)$. Now consider the restriction of f to $A \cap R$. It follows from Definition 9.6 that $f|_{A\cap R}$ is differentiable at a, that $f|_{A\cap R}(A \cap R) = f(A \cap R)$, and that $f|_{A\cap R}(a) = f(a)$ is the last point of $f|_{A\cap R}(A \cap R)$. The latter two results imply by Definition 12.11 that $f|_{A\cap R}(a)$ is the maximum value of $f|_{A\cap R}$. This combined with the fact that $f|_{A\cap R}$ is differentiable at a implies by Theorem 12.13 that $(f|_{A\cap R})'(a) = f'(a) = 0$, as desired. $\square$

Theorem 12.15. *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, differentiable on (a, b), and that $f(a) = f(b) = 0$. Then there exists a point $\lambda \in (a, b)$ such that $f'(\lambda) = 0$.*

Proof. We divide into two cases ($f(x) = 0$ for all $x \in [a, b]$, and $f(x) \neq 0$ for some $x \in [a, b]$).

Suppose first that $f(x) = 0$ for all $x \in [a, b]$. By Theorem 5.2, we can choose a $\lambda \in (a, b)$. It follows from the hypothesis that $f(\lambda) = f(x)$ for all $f(x) \in f([a, b])$. This can be weakened to $f(\lambda) \geq f(x)$ for all

$f(x) \in f([a,b])$. Thus, by Definition 3.3, $f(\lambda)$ is the last point of $f([a,b])$. Consequently, by Definition 12.11, $f(\lambda)$ is the maximum value of f. This combined with the fact that f is differentiable at λ (since $\lambda \in (a,b)$ and f is differentiable on (a,b)) implies by Theorem 12.13 that $f'(\lambda) = 0$, as desired.

Now suppose that $f(x) \neq 0$ for some $x \in [a,b]$ which we shall call x_0. We divide into two cases again ($f(x_0) > 0$ and $f(x_0) < 0$). Suppose first that $f(x_0) > 0$. Since $f : [a,b] \to \mathbb{R}$ is continuous, Exercise 10.21 asserts that there exists a point $\lambda \in [a,b]$ such that $f(\lambda) \geq f(x)$ for all $x \in [a,b]$. It follows that $f(\lambda) \geq f(x_0) > 0$, so $f(\lambda) \neq f(a) = f(b)$. Thus, by Definition 1.16, $\lambda \neq a$ and $\lambda \neq b$. This combined with the fact that $\lambda \in [a,b]$ implies by Script 8 that $\lambda \in (a,b)$. Now as before, we can determine from the fact that $f(\lambda) \geq f(x)$ for all $x \in [a,b]$ that $f(\lambda)$ is the maximum value of f. This combined with the fact that f is differentiable at λ (since $\lambda \in (a,b)$ and f is differentiable on (a,b)) implies by Theorem 12.13 that $f'(\lambda) = 0$, as desired. The argument is symmetric in the other case. □

Corollary 12.16. *Suppose that $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then there exists a point $\lambda \in (a,b)$ such that*

$$f(b) - f(a) = f'(\lambda)(b-a)$$

Proof. Let $h : [a,b] \to \mathbb{R}$ be defined by

$$h(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a) - f(a)$$

By hypothesis, $f(x)$ is continuous on $[a,b]$. By Exercise 11.6 and Theorem 11.9, $-\frac{f(b)-f(a)}{b-a}(x-a) - f(a)$ is continuous on $[a,b]$. Thus, by Theorem 11.9, their sum (i.e., $h(x)$) is continuous on $[a,b]$. Additionally, by hypothesis, $f(x)$ is differentiable on (a,b). By Exercises 12.8 and 12.9, $-\frac{f(b)-f(a)}{b-a}(x-a) - f(a)$ is differentiable on $[a,b]$. Thus, by Exercise 12.9, their sum (i.e., $h(x)$) is differentiable on (a,b). Furthermore, by simple algebra, we can determine that

$$\begin{aligned} h(a) &= f(a) - \frac{f(b)-f(a)}{b-a}(a-a) - f(a) \\ &= -\frac{f(b)-f(a)}{b-a} \cdot 0 \\ &= 0 \end{aligned} \qquad \begin{aligned} h(b) &= f(b) - \frac{f(b)-f(a)}{b-a}(b-a) - f(a) \\ &= f(b) - (f(b)-f(a)) - f(a) \\ &= 0 \end{aligned}$$

Thus, by Theorem 12.15, there exists a point $\lambda \in (a,b)$ such that $h'(\lambda) = 0$.

We can also calculate $h'(x)$ as follows.

$$\begin{aligned} h'(x) &= \left(f(x) - \frac{f(b)-f(a)}{b-a}(x-a) - f(a) \right)' \\ &= \left((f(x)) + \left(-\frac{f(b)-f(a)}{b-a} \cdot x \right) + \left(\frac{f(b)-f(a)}{b-a} \cdot a - f(a) \right) \right)' \\ &= f'(x) + \left(-\frac{f(b)-f(a)}{b-a} \cdot x \right)' + \left(\frac{f(b)-f(a)}{b-a} \cdot a - f(a) \right)' && \text{Exercise 12.9} \\ &= f'(x) + \left(-\frac{f(b)-f(a)}{b-a} \right)' \cdot (x) + \left(-\frac{f(b)-f(a)}{b-a} \right) \cdot (x)' + \left(\frac{f(b)-f(a)}{b-a} \cdot a - f(a) \right)' \\ & && \text{Exercise 12.9} \\ &= f'(x) + 0 \cdot x + \frac{f(b)-f(a)}{b-a} \cdot 1x^0 + 0 && \text{Exercise 12.8} \\ &= f'(x) + \frac{f(b)-f(a)}{b-a} \end{aligned}$$

But it follows that at λ,

$$\begin{aligned}
0 &= f'(\lambda) - \frac{f(b)-f(a)}{b-a} \\
\frac{f(b)-f(a)}{b-a} &= f'(\lambda) \\
f(b)-f(a) &= f'(\lambda)(b-a)
\end{aligned}$$

as desired. ☐

4/6: **Corollary 12.17.** *Suppose that $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then*

(a) *If $f'(x) > 0$ for all $x \in (a,b)$, then f is strictly increasing on $[a,b]$.*

Proof. To prove that f is strictly increasing on $[a,b]$, Definition 8.16 tells us that it will suffice to show that if $x,y \in [a,b]$ with $x<y$, then $f(x)<f(y)$. Let x,y be arbitrary elements of $[a,b]$. WLOG, let $x<y$. Since f is continuous on $[a,b]$ and differentiable on (a,b), Corollary 12.16 asserts that there exists a point $\lambda \in (x,y)$ such that $f(y)-f(x) = f'(\lambda)(y-x)$. But since $f'(\lambda) > 0$ by hypothesis and $y-x>0$ because $y>x$, we have by Definition 7.21 that $f'(\lambda)(y-x) > 0$. It follows that $f(y)-f(x)>0$, i.e., that $f(x)<f(y)$, as desired. ☐

(b) *If $f'(x) < 0$ for all $x \in (a,b)$, then f is strictly decreasing on $[a,b]$.*

Proof. The proof is symmetric to that of part (a). ☐

(c) *If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant on $[a,b]$.*

Proof. To prove that f is constant on $[a,b]$, it will suffice to show that $f(x)=f(y)$ for all $x,y \in [a,b]$. Let x,y be arbitrary elements of $[a,b]$. WLOG, let $x<y$. Since f is continuous on $[a,b]$ and differentiable on (a,b), Corollary 12.16 asserts that there exists a point $\lambda \in (x,y)$ such that $f(y)-f(x) = f'(\lambda)(y-x)$. But since $f'(\lambda)=0$ by hypothesis, $f(y)-f(x)=0$, i.e., $f(y)=f(x)$, as desired. ☐

Remark 12.18. Corollary 12.17 also holds if instead of $[a,b]$, we have an arbitrary interval I; and instead of (a,b), we have the interior of I.

Corollary 12.19. *Suppose that $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ are continuous on $[a,b]$, differentiable on (a,b), and $f'(x)=g'(x)$ for all $x\in(a,b)$. Then there is some $c\in\mathbb{R}$ such that for all $x\in[a,b]$, we have $f(x)=g(x)+c$.*

Proof. Let $h:[a,b]\to\mathbb{R}$ be defined by $h(x)=f(x)-g(x)$. Since f,g are continuous on $[a,b]$, Corollary 11.10 asserts that h is continuous on $[a,b]$. Since f,g are differentiable on (a,b), Exercise 12.9 asserts that h is differentiable on (a,b). Since $f'(x)=g'(x)$ for all $x\in(a,b)$, Exercise 12.9 implies that $h'(x) = f'(x)-g'(x)=0$ for all $x\in(a,b)$. These three results satisfy the conditions of Corollary 12.17, which means that h is constant on $[a,b]$, i.e., that $h(x)=c$ for all $x\in[a,b]$ where $c\in\mathbb{R}$. But by the definition of h, this implies that for all $x\in[a,b]$, we have $f(x)-g(x)=c$, i.e., $f(x)=g(x)+c$. ☐

Corollary 12.20. *Suppose that $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ are continuous on $[a,b]$ and differentiable on (a,b). Then there is a point $\lambda\in(a,b)$ such that*

$$(f(b)-f(a))g'(\lambda) = (g(b)-g(a))f'(\lambda)$$

Proof. Let $h : [a,b] \to \mathbb{R}$ be defined by $h(x) = (g(b) - g(a))f(x) - (f(b) - f(a))g(x) - f(a)g(b) + f(b)g(a)$. For the same reasons as in the proof of Corollary 12.19, h is continuous on $[a,b]$ and differentiable on (a,b). Additionally, we can show with basic algebra that

$$\begin{aligned} h(a) &= (g(b) - g(a))f(a) - (f(b) - f(a))g(a) - f(a)g(b) + f(b)g(a) \\ &= f(a)g(b) - f(a)g(a) - f(b)g(a) + f(a)g(a) - f(a)g(b) + f(b)g(a) \\ &= 0 \end{aligned}$$

$$\begin{aligned} h(b) &= (g(b) - g(a))f(b) - (f(b) - f(a))g(b) - f(a)g(b) + f(b)g(a) \\ &= f(b)g(b) - f(b)g(a) - f(b)g(b) + f(a)g(b) - f(a)g(b) + f(b)g(a) \\ &= 0 \end{aligned}$$

These results satisfy the conditions of Theorem 12.15, which means that there exists a point $\lambda \in (a,b)$ such that $h'(\lambda) = 0$. We can also calculate that $h'(x) = (g(b) - g(a))f'(x) - (f(b) - f(a))g'(x)$ via a similar method to that used in the proof of Corollary 12.16. But it follows that at λ,

$$\begin{aligned} 0 &= (g(b) - g(a))f'(\lambda) - (f(b) - f(a))g'(\lambda) \\ (f(b) - f(a))g'(\lambda) &= (g(b) - g(a))f'(\lambda) \end{aligned}$$

□

Finally, we prove another very important theorem that tells us about inverse functions and their derivatives.

Theorem 12.21. *Suppose that $f : (a,b) \to \mathbb{R}$ is differentiable and that the derivative $f' : (a,b) \to \mathbb{R}$ is continuous. Also suppose that there is a point $p \in (a,b)$ such that $f'(p) \neq 0$. Then there exists a region $R \subset (a,b)$ such that $p \in R$ and f with domain restricted to R is injective. Furthermore, $f^{-1} : f(R) \to R$ is differentiable at the point $f(p)$ and*

$$(f^{-1})'(f(p)) = \frac{1}{f'(p)}$$

Proof. Since $f' : (a,b) \to \mathbb{R}$ is continuous, Theorem 9.10 implies that it is continuous at $p \in (a,b)$. Thus, by Theorem 11.5, $\lim_{x \to p} f'(x) = f'(p)$. This combined with the facts that $f'(p) \neq 0$ (i.e., $f'(p) > 0$ or $f'(p) < 0$) and f' is continuous at p implies by Lemma 11.8 that there exists a region R with $p \in R$ such that $f'(x) > 0$ for all $x \in R \cap (a,b)$ or $f'(x) < 0$ for all $x \in R \cap (a,b)$. Clearly, $R \cap (a,b) = R$. Consequently, since $f'(x) > 0$ for all $x \in R$ or $f'(x) < 0$ for all $x \in R$, Corollary 12.17 asserts that f is strictly increasing or strictly decreasing on R. Since R is an interval by Lemma 8.3, the previous result implies by Lemma 8.17 that f is injective on R. Therefore, we have found an $R = R \cap (a,b) \subset (a,b)$ by Theorem 1.7 with $p \in R$ such that $f|_R$ is injective, as desired.

From now on, we will denote $f|_R : R \to \mathbb{R}$ by f. Since f is differentiable, Definition 12.1 asserts that it is differentiable for all $x \in R$. Thus, by Theorem 12.5, f is continuous for all $x \in R$. Consequently, by Theorem 9.10, f is continuous. This combined with the previously proven fact that f is injective implies by Theorem 9.14 that the inverse function $f^{-1} : f(R) \to R$ exists and is continuous.

To prove that $(f^{-1})'(f(p))$ exists and equals $\frac{1}{f'(p)}$, Theorem 12.4 tells us that it will suffice to show that $\lim_{y \to f(p)} \frac{f^{-1}(y) - f^{-1}(f(p))}{y - f(p)} = \frac{1}{f'(p)}$. To do this, Definition 11.1 tells us that it will suffice to confirm that for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in f(R)$ and $0 < |y - f(p)| < \delta$, then $|\frac{f^{-1}(y) - f^{-1}(f(p))}{y - p} - \frac{1}{f'(p)}| < \epsilon$. Let $\epsilon > 0$ be arbitrary. We have that

$$\begin{aligned} \frac{1}{f'(p)} &= \frac{1}{\lim_{x \to p} \frac{f(x) - f(p)}{x - p}} && \text{Theorem 12.4} \\ &= \lim_{x \to p} \frac{x - p}{f(x) - f(p)} && \text{Theorem 11.9} \\ &= \lim_{x \to p} \frac{f^{-1}(f(x)) - f^{-1}(f(p))}{f(x) - f(p)} && \text{Definition 1.18} \end{aligned}$$

Thus, by Definition 11.1, there exists a $\delta' > 0$ such that if $x \in R$ and $0 < |x-p| < \delta'$, then $|\frac{f^{-1}(f(x))-f^{-1}(f(p))}{f(x)-f(p)} - \frac{1}{f'(p)}| < \epsilon$.

The previously proven fact that f^{-1} is continuous implies by Theorem 9.10 that f^{-1} is continuous at $f(p)$. It follows by Theorem 11.5 that either $f(p) \notin LP(f(R))$ or $\lim_{y\to f(p)} f^{-1}(y) = f^{-1}(f(p))$. However, as we will now see, $f(p) \in LP(f(R))$. To begin, the fact that R is a region implies by Lemma 8.3 that R is an interval. It follows by Theorem 8.15 that R is connected, by Theorem 9.11 that $f(R)$ is connected, by Theorem 8.15 again that $f(R)$ is an interval, and finally by extensions of Corollaries 5.5 and 5.14 that $f(p) \in LP(f(R))$. Thus, with this case eliminated, we know that $\lim_{y\to f(p)} f^{-1}(y) = f^{-1}(f(p))$. Consequently, by Definition 1.11, there exists a $\delta > 0$ such that if $y \in f(R)$ and $0 < |y-f(p)| < \delta$, then $|f^{-1}(y)-f^{-1}(f(p))| < \delta'$. With a slight modification from Definition 1.18 (and the definition that $y = f(x)$), we have that there exists a $\delta > 0$ such that if $y \in f(R)$ and $0 < |y - f(p)| < \delta$, then $|x - p| < \delta'$.

Choose the above δ as our δ. Let $y = f(x)$ be an arbitrary element of $f(R)$ satisfying $0 < |y - f(p)| < \delta$. Then $|x - p| < \delta'$. We can also show that $0 < |x - p|$: From Script 8, the fact that $0 < |y - f(p)|$ implies that $f(x) \neq f(p)$; hence by Definition 1.20 and the fact that f is injective, $x \neq p$; hence by Script 8 again, $0 < |x - p|$. Continuing, since $y = f(x) \in f(R)$, we have by Definition 1.18 that $x \in R$. Thus, we have that $x \in R$ and $0 < |x - p| < \delta'$, so we know that $|\frac{f^{-1}(f(x))-f^{-1}(f(p))}{f(x)-f(p)} - \frac{1}{f'(p)}| < \epsilon$. Therefore, with a slight modification from Definition 1.18, we have that $|\frac{f^{-1}(y)-f^{-1}(f(p))}{y-f(p)} - \frac{1}{f'(p)}| < \epsilon$, as desired. □

4/8: **Exercise 12.22.** Consider the function $f(x) = x^n$ for a fixed $n \in \mathbb{N}$. Show that if n is even, then f is strictly increasing on the set of nonnegative real numbers and that if n is odd, then f is strictly increasing on all of $\mathbb{R}$. For a given n, let A be the aforementioned set on which f is strictly increasing. Define the inverse function $f^{-1} : f(A) \to A$ by $f^{-1}(x) = \sqrt[n]{x}$, which we sometimes also denote $f^{-1}(x) = x^{1/n}$. Use Theorem 12.21 to find the points $y \in f(A)$ at which f^{-1} is differentiable, and determine $(f^{-1})'(y)$ at these points.

Proof. For the first part of the question, we must prove that f is strictly increasing on $\mathbb{R}^+$ if n is even and that f is strictly increasing on $\mathbb{R}$ if n is odd. To do so, we induct on n. For the base case $n = 1$, we must confirm that $f(x) = x$ is strictly increasing on $\mathbb{R}$ (since n is odd). By Exercise 12.8, $f'(x) = 1 > 0$ for all $x \in \mathbb{R}$. Thus, by Corollary 12.17 and Remark 12.18, f is strictly increasing on $\mathbb{R}$, as desired. Now suppose inductively that the claim holds for some natural number $n \in \mathbb{N}$; we wish to confirm it for $n + 1$. We divide into two cases ($n + 1$ is even and $n + 1$ is odd).

If $n + 1$ is even, then we must confirm that $f(x) = x^{n+1}$ is strictly increasing on $\mathbb{R}^+$. By Exercise 12.8, $f'(x) = (n+1)x^n$, where n is odd. To verify that f' is strictly increasing on $\mathbb{R}^+$, Definition 8.16 tells us that it will suffice to demonstrate that for all $x, y \in \mathbb{R}^+$ satisfying $x < y$, $f'(x) < f'(y)$. Let x, y be arbitrary elements of $\mathbb{R}^+$ that satisfy $x < y$. Since x^n is strictly increasing on $\mathbb{R}^+ \subset \mathbb{R}$ by the inductive hypothesis, Definition 8.16 tells us that $x^n < y^n$. This combined with the fact that $0 < n + 1$ by Script 0 implies by Lemma 7.24 that $(n + 1)x^n < (n + 1)y^n$. Thus, by the definition of f', $f'(x) < f'(y)$, as desired. Note also that $f'(0) = (n + 1)0^n = 0$. Having established that f' is strictly increasing on $\mathbb{R}^+$ and that $f'(0) = 0$, we have that $f'(x) > 0$ for all $x \in (0, \infty)$ by Definition 8.16 because $0 < x$ implies $0 = f'(0) < f'(x)$. Thus, by Corollary 12.17 and Remark 12.18, f is strictly increasing on $\mathbb{R}^+$, as desired.

If $n + 1$ is odd, then we must confirm that $f(x) = x^{n+1}$ is strictly increasing on $\mathbb{R}$. By Exercise 12.8, $f'(x) = (n + 1)x^n$, where n is even. With a symmetric argument to that used above, we can verify that f' is strictly increasing on $\mathbb{R}^+$. Since $f'(-x) = (n + 1)(-x)^n = (n + 1)x^n = f'(x)$ by Script 7, we can similarly prove that f' is strictly *decreasing* on $\mathbb{R}^-$ (essentially, if $x, y \in \mathbb{R}^-$, then $x < y$ implies $-y < -x$ implies $f(-y) < f(-x)$ implies $f(x) > f(y)$, as desired). These two results combined with the fact that we still have $f'(0) = 0$ imply by a symmetric argument to the above that $f'(x) > 0$ for all $x \in (-\infty, 0) \cup (0, \infty)$. But it follows by consecutive applications of Corollary 12.17 and Remark 12.18 that f is strictly increasing on $\mathbb{R}$, as desired.

For the second part of the question, we must find all of the points y where f^{-1} is differentiable and determine the derivative $(f^{-1})'(y)$ at these points for an arbitrary n. Let n be an arbitrary element of $\mathbb{N}$. By Exercise 12.8, f is differentiable, and by both Exercise 12.8 and Corollary 11.12, f' is continuous. We divide into two cases (n is even and n is odd).

Suppose first that n is even. To begin, we will verify that $f(\mathbb{R}^+) = \mathbb{R}^+$. By Definition 1.2, to do so it will suffice to show that every $y \in f(\mathbb{R}^+)$ is an element of $\mathbb{R}^+$ and vice versa. Let y be an arbitrary element of $f(\mathbb{R}^+)$. Then by Definition 1.18, $y = f(x)$ for some $x \in \mathbb{R}^+$. It follows since $x \in \mathbb{R}^+$ that $x \geq 0$. Consequently, since f is strictly increasing on $\mathbb{R}^+$, Definition 8.16 implies that $f(x) \geq f(0) = 0$. But if $y = f(x) \geq 0$, then $y \in \mathbb{R}^+$, as desired. Now let y be an arbitrary element of $\mathbb{R}^+$. We divide into three cases ($y = 0$, $0 < y \leq 1$ and $1 < y$). If $y = 0$, then since $f(0) = 0^n = 0 = y$ and $0 \in \mathbb{R}^+$, $y = f(x)$ for an $x \in \mathbb{R}^+$. Therefore, Definition 1.18 asserts that $y \in f(\mathbb{R}^+)$, as desired. If $0 < y \leq 1$, then since f is continuous (notably on $[0,2]$) and $f(0) = 0 < y < 2^n = f(2)$ (by Script 7), Exercise 9.12 asserts that there exists a point $x \in \mathbb{R}$ with $0 < x < 2$ such that $f(x) = y$. Therefore, since $y = f(x)$ for an $x \in \mathbb{R}^+$ (we do know that $x > 0$), Definition 1.18 asserts that $y \in f(\mathbb{R}^+)$, as desired. If $y > 1$, then since $f(0) = 0 < 1 < y < y^n = f(y)$ (by Script 7), Exercise 9.12 asserts that there exists a point $x \in \mathbb{R}$ with $0 < x < y$ such that $f(x) = y$. Therefore, since $y = f(x)$ for an $x \in \mathbb{R}^+$ (we do know that $x > 0$), Definition 1.18 asserts that $y \in f(\mathbb{R}^+)$, as desired.

Having established that $f(\mathbb{R}^+) = \mathbb{R}^+$, our task becomes one of finding all points $y \in \mathbb{R}^+$ at which f^{-1} is differentiable, and determining $(f^{-1})'(y)$ at these points. By Theorem 12.21, this means that we need only find all points $f(p)$ corresponding to a p that satisfies $f'(p) \neq 0$. Since $f'(x) = nx^{n-1}$ by Exercise 12.8, Script 7 implies that the only point p where $f'(p) = 0$ is $p = 0$. Thus, we need only exclude $f(0) = 0$ from our set of points at which f^{-1} is differentiable. Therefore, we know that f^{-1} is differentiable at every $y \in \mathbb{R}^+$ such that $y \neq 0$, or more simply, all y in the interval $(0, \infty)$. Additionally, Theorem 12.21 implies that for any $y \in (0, \infty)$,

$$\begin{aligned}(f^{-1})'(y) &= \frac{1}{f'(f^{-1}(y))} \\ &= \frac{1}{n(f^{-1}(y))^{n-1}} \\ &= \frac{1}{n(y^{1/n})^{n-1}} \\ &= \frac{1}{n} \cdot y^{\frac{1-n}{n}}\end{aligned}$$

The proof is symmetric if n is odd.

□

12.2 Discussion

3/30:

- We can also do Exercise 12.6 with left- and right-handed limits, as defined in Additional Exercise 11.2.
- We can also do Exercise 12.7 by induction.

4/1:

- We can also do Theorem 12.13 by noting that of the left- and right-hand derivatives, one will be ≤ 0 and the other ≥ 0, but since they must be equal, they must equal 0.
- Include more rigorous restriction bits for Corollary 12.14 as a lemma?

4/6:

- Modify Theorem 12.13 with Lemma 11.8.
- Redo Corollary 12.17c as a direct proof?
- We don't have to be too rigorous with the restriction of f in Corollary 12.17. In fact, we need not even mention it.
- Include a bit more of the basic algebra proving that $h(a) = h(b) = 0$ for Corollary 12.20.
- Potentially for Theorem 12.21, we can use $\varphi(y)$ but modified with f^{-1} to prove the iffy limit transition.
  - Potentially we can apply the chain rule for this proof?
  - Could we use the fact that f is continuous to prove that as $x \to p$, $f(x) \to f(p)$?

Script 13

Uniform Continuity and Integration

13.1 Journal

4/8: **Definition 13.1.** Let $f : A \to \mathbb{R}$ be a function. We say that f is **uniformly continuous** if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$.

Theorem 13.2. *If f is uniformly continuous, then f is continuous.*

Proof. To prove that f is continuous, Theorem 9.10 tells us that it will suffice to show that f is continuous at every $x \in A$. Let x be an arbitrary element of A. To show that f is continuous at x, Theorem 11.5 tells us that it will suffice to verify that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Then since f is uniformly continuous by hypothesis, Definition 13.1 asserts that there exists a $\delta > 0$ such that for all $y \in A$ satisfying $|y - x| < \delta$, we have $|f(y) - f(x)| < \epsilon$, as desired. $\square$

Exercise 13.3. Determine with proof whether each function f is uniformly continuous on the given interval A.

(a) $f(x) = x^2$ on $A = \mathbb{R}$.

Proof. To prove that f is not uniformly continuous on $\mathbb{R}$, Definition 13.1 tells us that it will suffice to find an $\epsilon > 0$ for which no $\delta > 0$ exists such that for all $x, y \in \mathbb{R}$, if $|y - x| < \delta$, then $|y^2 - x^2| < \epsilon$. Let $\epsilon = 2$, and suppose for the sake of contradiction that $\delta > 0$ is a number such that for all $x, y \in \mathbb{R}$, if $|y - x| < \delta$, then $|y^2 - x^2| < 2$. By Theorem 5.2, there exists a number y such that $0 < y < \delta$. Since $-\delta < 0 < y < \delta$ by Lemma 7.23, it follows by Definitions 3.6 and 3.10 that $y \in (-\delta, \delta)$. Thus, by Exercise 8.9, $|y - 0| = |y| < \delta$. Consequently, $|(y + n) - n| < \delta$. It follows by the above that $|(y+n)^2 - n^2| = |y^2 + 2yn| < 2$. If we now let $n = \frac{1}{y}$, then $|y^2 + 2| < 2$. But since $y > 0$, we have that $y^2 > 0$ by Lemma 7.26. It follows that $y^2 + 2 > 2$ by Definition 7.21. Therefore, by Definition 8.4, we can also show that $|y^2 + 2| > 2$, a contradiction. $\square$

(b) $f(x) = x^2$ on $A = (-2, 2)$.

Proof. To prove that f is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \frac{\epsilon}{4}$, and let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. Since $x, y \in A$, consecutive applications of Equations 8.1 and the lemma from Exercise 8.9 imply that $|x| < 2$ and $|y| < 2$. It follows that $|x| + |y| < 2 + 2 = 4$. Consequently, by Lemma 8.8, $|x + y| < 4$. Additionally, since $0 \leq |y + x|$ by Definition 8.4, we have $|x - y| \cdot |x + y| \leq \frac{\epsilon}{4} \cdot |x + y|$. Combining all of the above results, we have that

$$\begin{aligned}|f(y)-f(x)| &= |y^2-x^2| \\ &= |y+x|\cdot|y-x| \\ &< 4\cdot|y-x| \\ &\leq 4\cdot\frac{\epsilon}{4} \\ &= \epsilon\end{aligned}$$

as desired. □

(c) $f(x)=\frac{1}{x}$ on $A=(0,+\infty)$.

Proof. To prove that f is not uniformly continuous on A, Definition 13.1 tells us that it will suffice to find an $\epsilon>0$ for which no $\delta>0$ exists such that for all $x,y\in A$, if $|y-x|<\delta$, then $|\frac{1}{y}-\frac{1}{x}|<\epsilon$. Let $\epsilon=1$, and suppose for the sake of contradiction that $\delta>0$ is a number such that for all $x,y\in A$, if $|y-x|<\delta$, then $|\frac{1}{y}-\frac{1}{x}|<1$. As in part (a), choose $0<x<\min(\delta,\frac{1}{2})$. Consequently, $|(x+x)-x|<\delta$. It follows by the above that $|\frac{1}{2x}-\frac{1}{x}|<1$. But this implies that $|\frac{x-2x}{2x^2}|=|\frac{-1}{2x}|=\frac{1}{2x}<1$. However, $x<\frac{1}{2}$ implies that $1<\frac{1}{2x}$, a contradiction. □

(d) $f(x)=\frac{1}{x}$ on $A=[1,+\infty)$.

Proof. To prove that f is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon>0$, there exists a $\delta>0$ such that for all $x,y\in A$, if $|y-x|<\delta$, then $|f(y)-f(x)|<\epsilon$. Let $\epsilon>0$ be arbitrary. Choose $\delta=\epsilon$, and let x,y be arbitrary elements of A that satisfy $|y-x|<\delta$. Since $x,y\in A$, consecutive applications of Equations 8.1 imply that $1\leq x$ and $1\leq y$. It follows by Script 7 that $1\leq|xy|$. This combined with the fact that $|y-x|<\delta=\epsilon$ implies that

$$\begin{aligned}|f(y)-f(x)| &= \left|\frac{1}{y}-\frac{1}{x}\right| \\ &= \left|\frac{x-y}{yx}\right| \\ &= \frac{|y-x|}{|xy|} \\ &< \frac{\epsilon}{|xy|} \\ &\leq \frac{\epsilon}{1} \\ &= \epsilon\end{aligned}$$

as desired. □

(e) $f(x)=\sqrt{x}$ on $A=[1,+\infty)$.

Proof. To prove that f is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon>0$, there exists a $\delta>0$ such that for all $x,y\in A$, if $|y-x|<\delta$, then $|f(y)-f(x)|<\epsilon$. Let $\epsilon>0$ be arbitrary. Choose $\delta=\epsilon$, and let x,y be arbitrary elements of A that satisfy $|y-x|<\delta$. Since $x,y\in A$, consecutive applications of Equations 8.1 imply that $1\leq x$ and $1\leq y$. It follows by Script 7 that $1\leq\sqrt{x}$ and $1\leq\sqrt{y}$. Thus, by Script 7 again, $2\leq|\sqrt{y}+\sqrt{x}|$. Note that it follows that $1<|\sqrt{y}+\sqrt{x}|$. This combined with the fact that $|y-x|<\delta=\epsilon$ implies that

$$\begin{aligned}|f(y)-f(x)| &= |\sqrt{y}-\sqrt{x}| \\ &< |\sqrt{y}-\sqrt{x}|\cdot|\sqrt{y}+\sqrt{x}| \\ &= |y-x| \\ &= \epsilon\end{aligned}$$

as desired. □

Exercise 13.4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^n$ for $n \in \mathbb{N}$. Show that f is uniformly continuous if and only if $n = 1$.

Proof. Suppose first that $n = 1$. To prove that f is uniformly continuous, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in \mathbb{R}$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \epsilon$. Now let x, y be arbitrary elements of $\mathbb{R}$ that satisfy $|y - x| < \delta$. Then by the definition of f, $|f(y) - f(x)| = |y - x| < \delta = \epsilon$, as desired.

Now suppose that $n > 1$. Additionally, suppose for the sake of contradiction that f is uniformly continuous. Let $\epsilon = 1 > 0$. Then by Definition 13.1, there exists a $\delta > 0$ such that for all $x, y \in \mathbb{R}$, if $|y-x| < \delta$, then $|y^n - x^n| < 1$. Let $x = 0 \in \mathbb{R}$. By Theorem 5.2, there exists a point $y \in \mathbb{R}$ such that $0 < y < \delta$. Additionally, since $\delta > 0$, Lemma 7.23 asserts that $-\delta < 0$. This combined with the previous result demonstrates by transitivity that $-\delta < 0 < y < \delta$, so by the lemma from Exercise 8.9, we have that $|y| < \delta$. Consequently, by Script 7, we know that $|(y+a) - a| < \delta$ for any $a \in \mathbb{R}$. It follows by the above that $|(y+a)^n - a^n| < 1$. Thus, by Additional Exercise 0.7, $|\sum_{k=0}^{n} \binom{n}{k} y^{n-k} a^k - a^n| = |y^n + ny^{n-1}a + \sum_{k=2}^{n-1} \binom{n}{k} y^{n-k} a^k| < 1$. If we now choose $a = \frac{1}{ny^{n-1}}$, Script 7 reduces the above to $|y^n + 1 + \sum_{k=2}^{n-1} y^{n-k} a^k| < 1$. We now seek to reduce the previous statement further to $|y^n + 1| < 1$. To begin, Exercise 12.22 implies that $y^n > 0$ since $y > 0$ and $0^n = 0$, meaning by Script 7 that $y^n + 1 > 0$. Additionally, Script 7 asserts that $\sum_{k=2}^{n-1} y^{n-k} a^k > 0$ since $a > 0$ and $y > 0$. This combined with the previous result implies by Scripts 7 and 8 that $|y^n + 1| < |y^n + 1 + \sum_{k=2}^{n-1} y^{n-k} a^k| < 1$, as desired. However, since $y^n > 0$, Definition 7.21 asserts that $y^n + 1 > 1$. But by Definition 8.4, this implies that $|y^n + 1| > 1$, a contradiction. ☐

Exercise 13.5. Let f and g be uniformly continuous on $A \subset \mathbb{R}$. Show that

(a) The function $f + g$ is uniformly continuous on A.

(b) For any constant $c \in \mathbb{R}$, the function $c \cdot f$ is uniformly continuous on A.

Proof of a. To prove that $f+g$ is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y-x| < \delta$, then $|(f+g)(y) - (f+g)(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f, g are uniformly continuous on A, consecutive applications of Definition 13.1 reveal that there exist $\delta_1, \delta_2 > 0$ such that for all $x, y \in A$, $|y - x| < \delta_1$ implies $|f(y) - f(x)| < \frac{\epsilon}{2}$ and $|y - x| < \delta_2$ implies $|g(y) - f(x)| < \frac{\epsilon}{2}$. Choose $\delta = \min(\delta_1, \delta_2)$. Let x, y be arbitrary elements of A that satisfy $|y-x| < \delta$. It follows that $|y-x| < \delta_1$ (so $|f(y)-f(x)| < \frac{\epsilon}{2}$), and that $|y-x| < \delta_2$ (so $|g(y)-g(x)| < \frac{\epsilon}{2}$). These two results when combined imply by Script 7 that $|f(y) - f(x)| + |g(y) - g(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$. Therefore, since $|f(y) - f(x) + g(y) - g(x)| \leq |f(y) - f(x)| + |g(y) - g(x)|$ by Lemma 8.8, we have that

$$\begin{aligned}
|(f+g)(y) - (f+g)(x)| &= |f(y) - f(x) + g(y) - g(x)| \\
&\leq |f(y) - f(x)| + |g(y) - g(x)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

as desired. ☐

Proof of b. To prove that $c \cdot f$ is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y-x| < \delta$, then $|c \cdot f(y) - c \cdot f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. We divide into two cases ($c = 0$ and $c \neq 0$). Suppose first that $c = 0$. Choose $\delta = 1$. Let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. It follows that $|0 \cdot f(y) - 0 \cdot f(x)| = 0 < \epsilon$, as desired. Now suppose that $c \neq 0$. Then since f is uniformly continuous on A, Definition 13.1 tells us that there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \frac{\epsilon}{|c|}$. Choose this δ to be our δ. Let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. Then by the above, we have that $|f(y) - f(x)| < \frac{\epsilon}{|c|}$. Therefore, $|c| \cdot |f(y) - f(x)| < \epsilon$, so we have that $|c \cdot f(y) - c \cdot f(x)| < \epsilon$, as desired. ☐

4/13: **Theorem 13.6.** *Suppose that $X \subset \mathbb{R}$ is compact and $f : X \to \mathbb{R}$ is continuous. Then f is uniformly continuous.*

Proof. To prove that f is uniformly continuous, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f is continuous on X, Theorem 9.10 asserts that f is continuous at every $x \in X$. Thus, by Theorem 11.5, for every $x \in X$, there exists a $\delta_x > 0$ such that if $y \in X$ and $|y - x| < \delta_x$, then $|f(y) - f(x)| < \frac{\epsilon}{2}$. Let $\mathcal{G} = \{(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2}) \mid x \in X\}$. We will now confirm that $\mathcal{G}$ is an open cover of X. To do so, Definition 10.3 tells us that it will suffice to demonstrate that every $x \in X$ is an element of $(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2})$ for some $(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2}) \in \mathcal{G}$. Let x be an arbitrary element of X. We know that $|x - x| = 0 < \frac{\delta_x}{2}$. Thus, by Exercise 8.9, we have that $x \in (x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2})$. Since it follows from the above that $(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2}) \in \mathcal{G}$, we are done.

Having shown that $\mathcal{G}$ is an open cover of X, the fact that X is compact implies by Definition 10.4 that there exists a finite subset $\mathcal{G}'$ of $\mathcal{G}$ that is also an open cover of X. It follows that $\mathcal{G}'$ will be of the form $\{(x_i - \frac{\delta_{x_i}}{2}, x + \frac{\delta_{x_i}}{2}) \mid 1 \leq i \leq n\}$ where n is some natural number. Thus, choose $\delta = \min_{1 \leq i \leq n}(\frac{\delta_{x_i}}{2})$.

Let x, y be arbitrary elements of X that satisfy $|y - x| < \delta$. Since $\mathcal{G}'$ is an open cover of X, Definition 10.3 implies that $x \in (x_i - \frac{\delta_{x_i}}{2}, x_i + \frac{\delta_{x_i}}{2})$ for some $(x_i - \frac{\delta_{x_i}}{2}, x_i + \frac{\delta_{x_i}}{2}) \in \mathcal{G}'$. Considering this x_i more closely, we can determine from the previous result and Exercise 8.9 that $|x - x_i| < \frac{\delta_{x_i}}{2}$. This combined with the hypothesis that $|y - x| < \delta$ implies by Script 7 that $|y - x| + |x - x_i| < \delta + \frac{\delta_{x_i}}{2}$. Additionally, note that by definition, $\delta \leq \frac{\delta_{x_i}}{2}$. Thus, combining the last few results, we have that

$$\begin{aligned} |y - x_i| &\leq |y - x| + |x - x_i| && \text{Lemma 8.8} \\ &< \delta + \frac{\delta_{x_i}}{2} \\ &\leq \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2} \\ &= \delta_{x_i} \end{aligned}$$

At this point, we know that $|x - x_i| < \frac{\delta_{x_i}}{2} < \delta_{x_i}$ and that $|y - x_i| < \delta_{x_i}$. It follows by consecutive applications of the above that $|f(x) - f(x_i)| < \frac{\epsilon}{2}$ and $|f(y) - f(x_i)| < \frac{\epsilon}{2}$, respectively. Consequently, we have by Script 7 that $|f(y) - f(x_i)| + |f(x) - f(x_i)| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$. Therefore, if we combine the last several results, we get

$$\begin{aligned} |f(y) - f(x)| &\leq |f(y) - f(x_i)| + |f(x_i) - f(x)| && \text{Lemma 8.8} \\ &= |f(y) - f(x_i)| + |f(x) - f(x_i)| && \text{Exercise 8.5} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

as desired. □

Exercise 13.7. Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $A = [0, +\infty)$.

Lemma. *Let x, y be arbitrary elements of A. Then $|\sqrt{y} - \sqrt{x}|^2 \leq |(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x})|$.*

Proof. We will first verify that $|\sqrt{y} - \sqrt{x}| \leq |\sqrt{y} + \sqrt{x}|$. To do so, we divide into two cases ($\sqrt{y} \geq \sqrt{x}$ and $\sqrt{y} < \sqrt{x}$). If $\sqrt{y} \geq \sqrt{x}$, then by Definition 7.21, $\sqrt{y} - \sqrt{x} \geq 0$. It follows by Definition 8.4 that $|\sqrt{y} - \sqrt{x}| = \sqrt{y} - \sqrt{x}$. Additionally, we have by an extension of Exercise 12.22 that $\sqrt{x} \geq 0$, implying that $2\sqrt{x} \geq 0$ by Definition 7.21. Thus, combining the last few results, we have that $|\sqrt{y} - \sqrt{x}| = \sqrt{y} - \sqrt{x} \leq \sqrt{y} - \sqrt{x} + 2\sqrt{x} = \sqrt{y} + \sqrt{x}$. Consequently, we know that $0 \leq |\sqrt{y} - \sqrt{x}| \leq \sqrt{y} + \sqrt{x}$, so Definition 8.4 implies that $|\sqrt{y} + \sqrt{x}| = \sqrt{y} + \sqrt{x}$. Therefore, we have that $|\sqrt{y} - \sqrt{x}| \leq \sqrt{y} + \sqrt{x} = |\sqrt{y} + \sqrt{x}|$, as desired. The argument is symmetric in the other case.

Having established that $|\sqrt{y} - \sqrt{x}| \leq |\sqrt{y} + \sqrt{x}|$ and knowing that $0 \leq |\sqrt{y} - \sqrt{x}|$, we have by Lemma 7.24 that $|\sqrt{y} - \sqrt{x}| \cdot |\sqrt{y} - \sqrt{x}| \leq |\sqrt{y} + \sqrt{x}| \cdot |\sqrt{y} - \sqrt{x}|$. It follows by basic algebra that $|\sqrt{y} - \sqrt{x}|^2 \leq |(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x})|$, as desired. □

Proof of Exercise 13.7. To prove that f is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \epsilon^2$. Let x, y be arbitrary elements of X that satisfy $|y - x| < \delta$. Thus, since $(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x}) = y - x$, the lemma asserts that $|\sqrt{y} - \sqrt{x}|^2 \leq |(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x})| = |y - x| < \epsilon^2$. Therefore, by Script 7, $|\sqrt{y} - \sqrt{x}| < \epsilon$, i.e., $|f(y) - f(x)| < \epsilon$, as desired. □

Corollary 13.8. *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous. Then f is uniformly continuous.*

Proof. By Theorem 10.14, $[a, b]$ is compact. This combined with the hypothesis that f is continuous proves by Theorem 13.6 that f is uniformly continuous. □

Exercise 13.9. Show that if f and g are bounded on A and uniformly continuous on A, then fg is uniformly continuous on A.

Proof. To prove that fg is uniformly continuous on A, Definition 13.1 tells us that it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$, if $|y - x| < \delta$, then $|(fg)(y) - (fg)(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary.

Since f is bounded on A, Definition 10.1 implies that $f(A)$ is a bounded subset of $\mathbb{R}$. Thus, by consecutive applications of Definition 5.6, there exist numbers l, u such that for all $f(x) \in f(A)$, $l \leq f(x) \leq u$. Let $a = \max(|l|, |u|) + 1$. It follows by Scripts 7 and 8 that $-a < f(x) < a$ for all $f(x) \in f(A)$. Thus, by the lemma from Exercise 8.9, $|f(x)| < a$ for all $f(x) \in f(A)$. Similarly, there exists a number b such that $|g(x)| < b$ for all $g(x) \in g(A)$.

Since f is uniformly continuous on A, Definition 13.1 implies that there exists a $\delta_1 > 0$ such that for all $x, y \in A$, if $|y - x| < \delta_1$, then $|f(y) - f(x)| < \frac{\epsilon}{2b}$. Similarly, there exists a $\delta_2 > 0$ such that for all $x, y \in A$, if $|y - x| < \delta_2$, then $|g(y) - g(x)| < \frac{\epsilon}{2a}$. Choose $\delta = \min(\delta_1, \delta_2)$. Let x, y be arbitrary elements of A that satisfy $|y - x| < \delta$. It follows by consecutive applications of the above that $|f(x)| < a$ and $|g(y)| < b$. Additionally, $|y - x| < \delta \leq \delta_1$ implies that $|f(y) - f(x)| < \frac{\epsilon}{2b}$ and $|y - x| < \delta \leq \delta_2$ implies that $|g(y) - g(x)| < \frac{\epsilon}{2a}$. Therefore, combining the last four results, we have that

$$\begin{aligned}
|(fg)(y) - (fg)(x)| &= |f(x)g(x) - f(y)g(y)| \\
&= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\
&= |f(x)(g(x) - g(y)) + g(y)(f(x) - f(y))| \\
&\leq |f(x)| \cdot |g(x) - g(y)| + |g(y)| \cdot |f(x) - f(y)| && \text{Lemma 8.8} \\
&< a \cdot \frac{\epsilon}{2a} + b \cdot \frac{\epsilon}{2b} \\
&= \epsilon
\end{aligned}$$

as desired. □

4/15: **Definition 13.10.** A **partition** of the interval $[a, b]$ is a finite set of points in $[a, b]$ that includes a and b. We usually write partitions as $P = \{t_0, t_1, \dots, t_n\}$, with the convention that

$$a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$$

If P and Q are partitions of the interval $[a, b]$ and $P \subset Q$, we refer to Q as a **refinement** of P.

Definition 13.11. Suppose that $f : [a, b] \to \mathbb{R}$ is bounded and that $P = \{t_0, \dots, t_n\}$ is a partition of $[a, b]$. Define

$$m_i(f) = \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\} \qquad M_i(f) = \sup\{f(x) \mid t_{i-1} \leq x \leq t_i\}$$

The **lower sum** of f for the partition P is the number

$$L(f, P) = \sum_{i=1}^{n} m_i(f)(t_i - t_{i-1})$$

The **upper sum** of f for the partition P is the number

$$U(f,P)=\sum_{i=1}^{n}M_i(f)(t_i-t_{i-1})$$

Notice that it is always the case that $L(f,P)\leq U(f,P)$.

Lemma 13.12. *Suppose that P and Q are partitions of $[a,b]$ and that Q is a refinement of P. Then $L(f,P)\leq L(f,Q)$ and $U(f,P)\geq U(f,Q)$.*

Lemma. *Let P be a partition of $[a,b]$ and let y be an arbitrary element of $[a,b]\setminus P$. Then $L(f,P)\leq L(f,P\cup\{y\})$ and $U(f,P)\geq U(f,P\cup\{y\})$.*

Proof. We will prove that $L(f,P)\leq L(f,P\cup\{y\})$. The proof will be symmetric in the other case. Let's begin.

By Definition 13.10, P is of the form $\{t_0,\dots,t_n\}$ where $a=t_0<\cdots<t_n=b$. This combined with the hypothesis that $y\in[a,b]\setminus P$ implies by Theorem 3.5 that $a=t_0<\cdots<t_{k-1}<y<t_k<\cdots<t_n=b$. Thus, we have by consecutive applications of Definition 13.11 that

$$\begin{aligned}L(f,P)&=\sum_{i=1}^{n}m_i(f)(t_i-t_{i-1})\\&=\sum_{i=1}^{k-1}m_i(f)(t_i-t_{i-1})+m_k(f)(t_k-t_{k-1})+\sum_{i=k+1}^{n}m_i(f)(t_i-t_{i-1})\end{aligned}$$

and that

$$L(f,P\cup\{y\})=\sum_{i=1}^{k-1}m_i(f)(t_i-t_{i-1})+m_y^-(f)(y-t_{k-1})+m_y^+(f)(t_k-y)+\sum_{i=k+1}^{n}m_i(f)(t_i-t_{i-1})$$

where

$$m_y^-(f)=\inf\{f(x)\mid t_{k-1}\leq x\leq y\}\qquad\qquad m_y^+(f)=\inf\{f(x)\mid y\leq x\leq t_k\}$$

As such, to prove that $L(f,P)\leq L(f,P\cup\{y\})$, it will suffice to show that $m_k(f)(t_k-t_{k-1})\leq m_y^-(f)(y-t_{k-1})+m_y^+(f)(t_k-y)$. To do so, it will suffice to show that $m_k(f)(y-t_{k-1})+m_k(f)(t_k-y)\leq m_y^-(f)(y-t_{k-1})+m_y^+(f)(t_k-y)$, i.e., that $m_k(f)(y-t_{k-1})\leq m_y^-(f)(y-t_{k-1})$ and that $m_k(f)(t_k-y)\leq m_y^+(f)(t_k-y)$, i.e., that $m_k(f)\leq m_y^-(f)$ and that $m_k(f)\leq m_y^+(f)$.

For the sake of proving the first expression, let $A=\{f(x)\mid t_{k-1}<x<t_k\}$ and let $B=\{f(x)\mid t_{k-1}\leq x\leq y\}$. It follows by Definition 13.10 that $m_k(f)=\inf A$ and $m_y^-(f)=\inf B$. Thus, we need only show that $\inf A\leq\inf B$. Since $y<t_k$, we know by Script 1 that $B\subset A$. Thus, since $\inf A$ is a lower bound on A, Script 5 implies that it is also a lower bound on B. Consequently, by Definition 5.7, $\inf A\leq\inf B$, as desired.

The argument is symmetric for the other statement. □

Proof of Lemma 13.12. We will prove that $L(f,P)\leq L(f,Q)$. The proof will be symmetric in the other case. Let's begin.

By Definition 13.10, $P\subset Q$. Thus, by Theorem **??**, $|P|\leq|Q|$. It follows by Script 1 that $|Q|-|P|=n\in\mathbb{Z}^+$. Thus, to prove the claim for P and Q in general, it will suffice to prove it for each n. To do so, we divide into two cases ($n=0$ and $n\in\mathbb{N}$). If $n=0$, then $|P|=|Q|$. This combined with the fact that $P\subset Q$ implies by Script 1 that $P=Q$. Therefore, $L(f,P)=L(f,Q)$, which we can weaken to $L(f,P)\leq L(f,Q)$, as desired.

On the other hand, if $n\in\mathbb{N}$, then we induct on n. For the base case $n=1$, we have by Script 1 that $Q=P\cup\{y\}$ where $y\notin P$. Therefore, by the lemma, we have that $L(f,P)\leq L(f,P\cup\{y\})=L(f,Q)$, as desired. Now suppose inductively that the claim holds or n; we wish to prove it for $n+1$. Let y be an arbitrary element of Q. Then by Script 1, $|Q\setminus\{y\}|-|P|=n$. Thus, by the inductive hypothesis, $L(f,P)\leq L(f,Q\setminus\{x\})$. Additionally, by the lemma, $L(f,Q\setminus\{x\})\leq L(f,Q)$. Therefore, by transitivity, $L(f,P)\leq L(f,Q)$, as desired. □

Theorem 13.13. *Let P_1 and P_2 be partitions of $[a,b]$ and suppose that $f:[a,b]\to\mathbb{R}$ is bounded. Then $L(f,P_1)\leq U(f,P_2)$.*

Proof. To confirm that $P_1\cup P_2$ is a partition of $[a,b]$, Definition 13.10 tells us that it will suffice to demonstrate that it is a finite set, that it is a subset of $[a,b]$, and that it includes a and b. Since P_1,P_2 are partitions of $[a,b]$, Definition 13.10 implies that they are finite subsets of $[a,b]$ that contain a,b. It follows by Script 1 that their union is finite, a subset of $[a,b]$, and a set containing a and b. Additionally, we have by Theorem 1.7 that $P_1\subset P_1\cup P_2$ and that $P_2\subset P_1\cup P_2$. Combining the last two results with consecutive applications of Definition 13.10 reveals that $P_1\cup P_2$ is a refinement of both P_1 and P_2.

Since P_1 and $P_1\cup P_2$ are partitions of $[a,b]$ and $P_1\cup P_2$ is a refinement of P_1, Lemma 13.12 implies that $L(f,P_1)\leq L(f,P_1\cup P_2)$. Similarly, $U(f,P_1\cup P_2)\leq U(f,P_2)$. Additionally, we have by Definition 13.11 that $L(f,P_1\cup P_2)\leq U(f,P_1\cup P_2)$. Therefore, if we combine the last three results with transitivity, we have that $L(f,P_1)\leq U(f,P_2)$, as desired. ☐

Definition 13.14. Let $f:[a,b]\to\mathbb{R}$ be bounded. We define

$$L(f)=\sup\{L(f,P)\mid P\text{ is a partition of }[a,b]\}\qquad U(f)=\inf\{U(f,P)\mid P\text{ is a partition of }[a,b]\}$$

to be, respectively, the **lower integral** and **upper integral** of f from a to b.

Exercise 13.15. Why do $L(f)$ and $U(f)$ exist? Find a function f for which $L(f)=U(f)$. Find a function f for which $L(f)\neq U(f)$. Prove that $L(f)\leq U(f)$.

Lemma. *Given $a,b\in\mathbb{R}$ with $a<b$, there exists $p\in\mathbb{R}$ such that $p\notin\mathbb{Q}$ and $a<p<b$.*

Proof. By Definition 7.21, $a+\sqrt{2}<b+\sqrt{2}$. Thus, by Lemma 6.10, there exists a point $\frac{c}{d}\in\mathbb{Q}$ such that $a+\sqrt{2}<\frac{c}{d}<b+\sqrt{2}$. It follows that $a<\frac{c}{d}-\sqrt{2}<b$.

Now suppose for the sake of contradiction that $\frac{c}{d}-\sqrt{2}$ is rational. Then by Script 2, $\frac{c}{d}-\sqrt{2}=\frac{e}{f}$ where $e,f\in\mathbb{Z}$ and $f\neq 0$. It follows by Theorem 2.10 that $\sqrt{2}=\frac{cf-de}{df}$, i.e., that $\sqrt{2}$ is rational. But by the proof of Exercise 4.24, $\sqrt{2}$ is not rational, a contradiction. ☐

Proof of Exercise 13.15. Let $A=\{L(f,P)\mid P\text{ is a partition of }[a,b]\}$. To prove that $L(f)=\sup A$ exists, Theorem 5.17 tells us that it will suffice to show that A is nonempty and bounded above.

To confirm that A is nonempty, Definition 1.8 tells us that it will suffice to find an element of it. Since $\{a,b\}$ is a finite set of points in $[a,b]$ that includes a and b (by Script 1), Definition 13.10 asserts that $\{a,b\}$ is a partition of $[a,b]$. It follows by Definition 13.11 that $L(f,\{a,b\})$ exists. Therefore, by the definition of A, we have that $L(f,\{a,b\})\in A$, as desired.

To confirm that A is bounded above, Definition 5.6 tells us that it will suffice to find a point in $u\in\mathbb{R}$ such that for all $L(f,P)\in A$, $L(f,P)\leq u$. Let $u=U(f,\{a,b\})$ (since $\{a,b\}$ is a partition of $[a,b]$ by the above, Definition 13.10 guarantees that $U(f,\{a,b\})$ exists). Now let $L(f,P)$ be an arbitrary element of A. It follows from Theorem 13.13 that $L(f,P)\leq U(f,\{a,b\})=u$, as desired.

The proof is symmetric for $U(f)$.

Let $f:[0,1]\to\mathbb{R}$ be defined by $f(x)=0$. To prove that $L(f)=U(f)$, it will suffice to show that $L(f)=0$ and $U(f)=0$. To do this, Script 5 tells us that it will suffice to verify that $\{L(f,P)\mid P\text{ is a partition of }[a,b]\}=\{0\}$ and $\{U(f,P)\mid P\text{ is a partition of }[a,b]\}=\{0\}$. We will start with the first equality.

Let $L(f,P)$ be an arbitrary element of $\{L(f,P)\mid P\text{ is a partition of }[a,b]\}$. Since we have

$$\begin{aligned}m_i(f)&=\inf\{f(x)\mid t_{i-1}\leq x\leq t_i\}\\&=\inf\{0\mid t_{i-1}\leq x\leq t_i\}\\&=\inf\{0\}\\&=0\end{aligned}$$

for all $m_i(f)$, it follows that

$$\begin{aligned} L(f,P) &= \sum_{i=1}^{n} m_i(f)(t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} 0(t_i - t_{i-1}) \\ &= 0 \end{aligned}$$

Therefore, since every element of $\{L(f,P) \mid P \text{ is a partition of } [a,b]\}$ is equal to 0, the set is equal to the singleton set containing 0. The argument is symmetric for the other equality.

Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

To prove that $L(f) \neq U(f)$, it will suffice to show that $L(f) = 0$ and $U(f) = 1$. To do this, Script 5 tells us that it will suffice to verify that $\{L(f,P) \mid P \text{ is a partition of } [a,b]\} = \{0\}$ and $\{U(f,P) \mid P \text{ is a partition of } [a,b]\} = \{1\}$. We will start with the first equality.

Let $L(f,P)$ be an arbitrary element of $\{L(f,P) \mid P \text{ is a partition of } [a,b]\}$. To confirm that $L(f,P) = 0$, Definition 13.11 tells us that it will suffice to demonstrate that $m_i(f) = 0$ for all $m_i(f)$. Let $m_i(f)$ be an arbitrary such object. By Definition 13.10, $m_i(f) = \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\}$. By the lemma, there exists $p \in \mathbb{R}$ such that $p \notin \mathbb{Q}$ and $t_{i-1} \leq p \leq t_i$. Thus, since $f(p) = 0$, $0 \in \{f(x) \mid t_{i-1} \leq x \leq t_i\}$. Additionally, since $f(x) \not< 0$ for any $x \in [0,1]$ by definition, we have that $m_i(f) = 0$. Therefore, since every element of $\{L(f,P) \mid P \text{ is a partition of } [a,b]\}$ is equal to 0, the set is equal to the singleton set containing 0.

As to the other equality, let $U(f,P)$ be an arbitrary element of $\{U(f,P) \mid P \text{ is a partition of } [a,b]\}$. To confirm that $U(f,P) = 1$, Definition 13.11 tells us that we must first demonstrate that $M_i(f) = 1$ for all $M_i(f)$. Let $M_i(f)$ be an arbitrary such object. By Definition 13.10, $M_i(f) = \sup\{f(x) \mid t_{i-1} \leq x \leq t_i\}$. By Lemma 6.10, there exists $p \in \mathbb{Q}$ such that $t_{i-1} \leq p \leq t_i$. Thus, since $f(p) = 1$, $1 \in \{f(x) \mid t_{i-1} \leq x \leq t_i\}$. Additionally, since $f(x) \not> 1$ for any $x \in [0,1]$ by definition, we have that $M_i(f) = 1$. It follows by Definition 13.11 that

$$\begin{aligned} U(f,P) &= \sum_{i=1}^{n} M_i(f)(t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} (t_i - t_{i-1}) \\ &= t_n - t_0 \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

Therefore, since every element of $\{U(f,P) \mid P \text{ is a partition of } [a,b]\}$ is equal to 1, the set is equal to the singleton set containing 1.

Suppose for the sake of contradiction that there exists a function $f : [a,b] \to \mathbb{R}$ for which $U(f) < L(f)$. It follows by consecutive applications of Definition 13.14 and Lemma 5.11 that there exists an $L(f,P_1)$ such that $U(f) < L(f,P_1) \leq L(f)$, and thus that there exists a $U(f,P_2)$ such that $U(f) \leq U(f,P_2) < L(f,P_1)$. But this means that there exist partitions P_1, P_2 of $[a,b]$ such that $L(f,P_1) > U(f,P_2)$, contradicting Theorem 13.13. $\square$

Definition 13.16. Let $f : [a,b] \to \mathbb{R}$ be bounded. We say that f is **integrable** on $[a,b]$ if $L(f) = U(f)$. In this case, the common value $L(f) = U(f)$ is called the **integral** of f from a to b and we write it as

$$\int_a^b f$$

Note that if f is an integrable function on $[a, b]$, it is necessarily bounded.

When we want to display the variable of integration, we write the integral as follows, including the symbol $\mathrm{d}x$ to indicate that variable of integration:

$$\int_a^b f(x)\,\mathrm{d}x$$

For example, if $f(x) = x^2$, we could write $\int_a^b x^2\,\mathrm{d}x$ but not $\int_a^b x^2$.

Exercise 13.17. Fix $c \in \mathbb{R}$ and let $f : [a, b] \to \mathbb{R}$ be defined by $f(x) = c$, for each $x \in [a, b]$. Show that f is integrable on $[a, b]$ and that $\int_a^b f = c(b - a)$.

Proof. To prove that f is integrable on $[a, b]$ and that $\int_a^b f = c(b - a)$, Definition 13.16 tells us that it will suffice to show that f is bounded on $[a, b]$, and that $L(f) = U(f) = c(b - a)$.

To confirm that f is bounded on $[a, b]$, Definition 10.1 tells us that it will suffice to demonstrate that $f([a, b])$ is a bounded subset of $\mathbb{R}$. By Definition 1.18, $f([a, b]) = \{f(x) \in \mathbb{R} \mid x \in [a, b]\}$. But since $f(x) = c$ for all $x \in [a, b]$, $f([a, b]) = \{c\}$. Thus, since $c \leq c \leq c$, Definition 5.6 implies that $f([a, b])$ is bounded. Additionally, since $c \in \mathbb{R}$, Definition 1.3 asserts that $f([a, b]) = \{c\} \subset \mathbb{R}$.

To confirm that $L(f) = U(f) = c(b - a)$, Definition 13.14 tells us that it will suffice to demonstrate that $L(f, P) = U(f, P) = c(b - a)$ for all partitions P of $[a, b]$. For similar reasons to the above (i.e., $f(x) = c$ for all $x \in [a, b]$), we can show that $m_i(f) = M_i(f) = c$ for all $m_i(f)$ and $M_i(f)$. Therefore, by Definition 13.11 that

$$\begin{aligned} L(f, P) &= \sum_{i=1}^{n} c(t_i - t_{i-1}) \\ &= c \sum_{i=1}^{n} (t_{i-1} - t_i) \\ &= c(t_n - t_0) \\ &= c(b - a) \end{aligned} \qquad\qquad \begin{aligned} U(f, P) &= \sum_{i=1}^{n} c(t_i - t_{i-1}) \\ &= c \sum_{i=1}^{n} (t_{i-1} - t_i) \\ &= c(t_n - t_0) \\ &= c(b - a) \end{aligned}$$

as desired. □

Theorem 13.18. *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is integrable if and only if for every $\epsilon > 0$, there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$.*

Proof. Suppose first that f is integrable. Then by Definition 13.16, $L(f) = U(f)$. Let $\epsilon > 0$ be arbitrary. By Script 7, $L(f) - \frac{\epsilon}{2} < L(f)$. Thus, by Definition 13.14 and Lemma 5.11, there exists an $L(f, P_1) \in \{L(f, P) \mid P \text{ is a partition of } [a, b]\}$ such that $L(f) - \frac{\epsilon}{2} < L(f, P_1) \leq L(f)$. Similarly, there exists a $U(f, P_2) \in \{U(f, P) \mid P \text{ is a partition of } [a, b]\}$ such that $U(f) \leq U(f, P_2) < U(f) + \frac{\epsilon}{2}$. Now consider $P_1 \cup P_2$ (which we will prove is the desired partition). By Theorem 1.7, $P_1 \subset P_1 \cup P_2$ and $P_2 \subset P_1 \cup P_2$. It follows by consecutive applications of Definition 13.10 that $P_1 \cup P_2$ is a refinement of both P_1 and P_2. Thus, by Lemma 13.12, $L(f, P_1) \leq L(f, P_1 \cup P_2)$ and $U(f, P_1 \cup P_2) \leq U(f, P_2)$. Combining the last several results with transitivity yields

$$L(f) - \frac{\epsilon}{2} < L(f, P_1) \leq L(f, P_1 \cup P_2) \qquad\qquad U(f, P_1 \cup P_2) \leq U(f, P_2) < U(f) + \frac{\epsilon}{2}$$

Therefore, knowing that $U(f, P_1 \cup P_2) < U(f) + \frac{\epsilon}{2}$ and that $-L(f, P_1 \cup P_2) < \frac{\epsilon}{2} - L(f)$ (the latter by Lemma 7.24), we have by Definition 7.21 that

$$\begin{aligned} U(f, P_1 \cup P_2) - L(f, P_1 \cup P_2) &< U(f) + \frac{\epsilon}{2} + \frac{\epsilon}{2} - L(f) \\ &= \epsilon \end{aligned}$$

as desired.

Now suppose that f is not integrable; we seek to prove that there exists an $\epsilon > 0$ such that for all partitions P of $[a, b]$, $U(f, P) - L(f, P) \geq \epsilon$. Since f is not integrable, we have by Definition 13.16 that

$L(f) \neq U(f)$. It follows by Exercise 13.15 that $L(f) < U(f)$. Thus, we can define $\epsilon = \frac{U(f)-L(f)}{2} > 0$. Now let P be an arbitrary partition of $[a,b]$. It follows that $L(f,P) \leq L(f)$ by Definitions 13.14, 5.7, and 5.6. Similarly, $U(f) \leq U(f,P)$. Therefore, knowing that $U(f) \leq U(f,P)$ and that $-L(f) \leq -L(f,P)$ (the latter by Lemma 7.24), we have by Definition 7.21 that $\epsilon < U(f) - L(f) \leq U(f,P) - L(f,P)$, as desired. $\square$

13.2 Discussion

4/8:

- The key to ϵ-δ proofs is to find a way to get $|y-x|$ into the $|f(y)-f(x)|$ expression and then deal with the others.

4/13:

- Note that we can also prove Exercise 13.7 with the following procedure:
  - Lemma: If f is uniformly continuous on two intervals I, J whose union $I \cup J$ is also an interval, then f is uniformly continuous on $I \cup J$.
  - Establish that $f(x) = \sqrt{x}$ is uniformly continuous on $[0,1]$ using Theorem 13.6.
    - Note that the continuity of f on $[0,1]$ follows from the fact that f is differentiable on $(0,1) \subset \mathbb{R}^+$ (Exercise 12.22) by Theorems 12.5 and 9.10.
    - Note that the compactness of $[0,1]$ follows from Theorem 10.14.
  - Recall that f is uniformly continuous on $[1,\infty)$ from Exercise 13.3.
  - Apply the lemma.

4/15:

- We can just say that the supremum of a singleton set is the element in that set (no proof required).