Script 18

The Euclidean Space $\mathbb{R}^n$

7/7: **Definition 18.1.** The **Euclidean n-space** $\mathbb{R}^n$ is the n-fold Cartesian product of $\mathbb{R}$. Symbolically,

$$\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{R}\}$$

is the set of n-tuples of real numbers. We often write

$$\mathbf{x} = (x_1, \dots, x_n)$$

to denote an element, which is also referred to as a **vector**, in $\mathbb{R}^n$ and

$$\mathbf{0} = (0, \dots, 0)$$

Definition 18.2. Let $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. We define the following operations.

(a) (Addition) $\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_n + y_n)$.

(b) (Scalar Multiplication) $\lambda\mathbf{x} = (\lambda x_1, \dots, \lambda x_n)$.

Exercise 18.3. Prove that the addition on $\mathbb{R}^n$ satisfies FA1-FA4 (see Definition 7.8). Moreover, prove that

VS1. (Associativity of Scalar Multiplication) If $\lambda, \mu \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, then $(\lambda\mu)\mathbf{x} = \lambda(\mu\mathbf{x})$.

VS2. (Distributivity of Scalars) If $\lambda, \mu \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, then $(\lambda + \mu)\mathbf{x} = \lambda\mathbf{x} + \mu\mathbf{x}$.

VS3. (Distributivity of Vectors) If $\lambda \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then $\lambda(\mathbf{x} + \mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y}$.

VS4. (Scalar Multiplicative Identity) If $\mathbf{x} \in \mathbb{R}^n$, then $1\mathbf{x} = \mathbf{x}$.

These eight properties together are called the **vector space axioms**.

Proof. To prove that $\mathbb{R}^n$ obeys FA1 from Definition 7.8, it will suffice to show that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$. Let $\mathbf{x}, \mathbf{y}$ be arbitrary elements of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= (x_1 + y_1, \dots, x_n + y_n) \\
&= (y_1 + x_1, \dots, y_n + x_n) \\
&= \mathbf{y} + \mathbf{x}
\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys FA2 from Definition 7.8, it will suffice to show that for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$. Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be arbitrary elements of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}
(\mathbf{x} + \mathbf{y}) + \mathbf{z} &= (x_1 + y_1, \dots, x_n + y_n) + \mathbf{z} \\
&= ((x_1 + y_1) + z_1, \dots, (x_n + y_n) + z_n) \\
&= (x_1 + (y_1 + z_1), \dots, x_n + (y_n + z_n)) \\
&= \mathbf{x} + (y_1 + z_1, \dots, y_n + z_n) \\
&= \mathbf{x} + (\mathbf{y} + \mathbf{z})
\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys FA3 from Definition 7.8, it will suffice to find an element $0 \in \mathbb{R}^n$ such that $\mathbf{x} + 0 = 0 + \mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Choose $\mathbf{0}$ to be our 0. Let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}
\mathbf{x} + \mathbf{0} &= (x_1 + 0, \ldots, x_n + 0) \\
&= (x_1, \ldots, x_n) \\
&= \mathbf{x} \\
&= (0 + x_1, \ldots, 0 + x_n) \\
&= \mathbf{0} + \mathbf{x}
\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys FA4 from Definition 7.8, it will suffice to show that for all $\mathbf{x} \in \mathbb{R}^n$, there exists $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x} = 0$. Let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Choose $\mathbf{y} = (-x_1, \ldots, -x_n)$. Then by Definition 18.2,

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= (x_1 + (-x_1), \ldots, x_n + (-x_n)) \\
&= (0, \ldots, 0) \\
&= \mathbf{0} \\
&= ((-x_1) + x_1, \ldots, (-x_n) + x_n) \\
&= \mathbf{y} + \mathbf{x}
\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys VS1, it will suffice to show that for all $\lambda, \mu \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, we have $(\lambda\mu)\mathbf{x} = \lambda(\mu\mathbf{x})$. Let λ, μ be arbitrary elements of $\mathbb{R}$, and let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}
(\lambda\mu)\mathbf{x} &= ((\lambda\mu)x_1, \ldots, (\lambda\mu)x_n) \\
&= (\lambda(\mu x_1), \ldots, \lambda(\mu x_n)) \\
&= \lambda(\mu\mathbf{x})
\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys VS2, it will suffice to show that for all $\lambda, \mu \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, we have $(\lambda + \mu)\mathbf{x} = \lambda\mathbf{x} + \mu\mathbf{x}$. Let λ, μ be arbitrary elements of $\mathbb{R}$, and let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}
(\lambda + \mu)\mathbf{x} &= ((\lambda + \mu)x_1, \ldots, (\lambda + \mu)x_n) \\
&= (\lambda x_1 + \mu x_1, \ldots, \lambda x_n + \mu x_n) \\
&= (\lambda x_1, \ldots, \lambda x_n) + (\mu x_1, \ldots, \mu x_n) \\
&= \lambda\mathbf{x} + \mu\mathbf{x}
\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys VS3, it will suffice to show that for all $\lambda \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have $\lambda(\mathbf{x} + \mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y}$. Let λ be an arbitrary element of $\mathbb{R}$, and let $\mathbf{x}, \mathbf{y}$ be arbitrary elements of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}
\lambda(\mathbf{x} + \mathbf{y}) &= \lambda(x_1 + y_1, \ldots, x_n + y_n) \\
&= (\lambda(x_1 + y_1), \ldots, \lambda(x_n + y_n)) \\
&= (\lambda x_1 + \lambda y_1, \ldots, \lambda x_n + \lambda y_n) \\
&= (\lambda x_1, \ldots, \lambda x_n) + (\lambda y_1, \ldots, \lambda y_2) \\
&= \lambda\mathbf{x} + \lambda\mathbf{y}
\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys VS4, it will suffice to show that for all $\mathbf{x} \in \mathbb{R}^n$, we have $1\mathbf{x} = \mathbf{x}$. Let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned} 1\mathbf{x} &= (1x_1, \dots, 1x_n) \\ &= (x_1, \dots, x_n) \\ &= \mathbf{x} \end{aligned}$$

as desired. □

Remark 18.4. Since $\mathbb{R}^n$ with the two operations defined as above satisfies these eight axioms, we call $\mathbb{R}^n$ a **vector space**.

Exercise 18.5. Prove that if $\mathbf{x} \in \mathbb{R}^n$, then $0\mathbf{x} = \mathbf{0}$.

Proof. By Definition 18.2, we have that

$$\begin{aligned} 0\mathbf{x} &= (0x_1, \dots, 0x_n) \\ &= (0, \dots, 0) \\ &= \mathbf{0} \end{aligned}$$

as desired. □

Definition 18.6. Let $\mathbf{x} \in \mathbb{R}^n$. The **norm** of $\mathbf{x}$ is defined as

$$\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

Definition 18.7. We call $\|\mathbf{y} - \mathbf{x}\|$ the **distance** between $\mathbf{x}$ and $\mathbf{y}$.

Remark 18.8. If $n = 1$, the norm coincides with the definition of the absolute value in $\mathbb{R}$.

Lemma 18.9.

(a) *If $x, y \in \mathbb{R}$, then $xy \leq \frac{x^2+y^2}{2}$.*

Proof. Let x, y be arbitrary elements of $\mathbb{R}$. Then by Lemma 7.26, $0 \leq (x-y)^2$. Therefore, we have that

$$\begin{aligned} xy &= \frac{2xy + 0}{2} \\ &\leq \frac{2xy + (x-y)^2}{2} \\ &= \frac{2xy + x^2 - 2xy + y^2}{2} \\ &= \frac{x^2 + y^2}{2} \end{aligned}$$

as desired. □

(b) *If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then $|x_1y_1 + \cdots + x_ny_n| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|$.*

Proof. Suppose first that $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$. Then by Definition 18.6, $\|\mathbf{x}\| = 1 = \sqrt{x_1^2 + \cdots + x_n^2}$, from which it follows that $1 = x_1^2 + \cdots + x_n^2$. Therefore, we have that

$$\begin{aligned} |x_1y_1 + \cdots + x_ny_n| &\leq |x_1y_1| + \cdots + |x_ny_n| && \text{Lemma 8.8} \\ &\leq \frac{x_1^2 + y_1^2}{2} + \cdots + \frac{x_n^2 + y_n^2}{2} \\ &= \frac{(x_1^2 + \cdots + x_n^2) + (y_1^2 + \cdots + y_n^2)}{2} \\ &= \frac{1+1}{2} \\ &= 1 \end{aligned}$$

as desired.

Now let $\mathbf{x}, \mathbf{y}$ be arbitrary elements of $\mathbb{R}^n$. Consider the vectors $\mathbf{u_x}, \mathbf{u_y}$ defined by $\mathbf{u_x} = \frac{\mathbf{x}}{\|\mathbf{x}\|}$ and $\mathbf{u_y} = \frac{\mathbf{y}}{\|\mathbf{y}\|}$. By the proof of the first case, we have that

$$\begin{aligned}
|x_1y_1 + \cdots + x_ny_n| &= \|\mathbf{x}\| \cdot \|\mathbf{y}\| \cdot \left| \frac{x_1y_1}{\|\mathbf{x}\| \cdot \|\mathbf{y}\|} + \cdots + \frac{x_ny_n}{\|\mathbf{x}\| \cdot \|\mathbf{y}\|} \right| \\
&= \|\mathbf{x}\| \cdot \|\mathbf{y}\| \cdot |u_{\mathbf{x}_1}u_{\mathbf{y}_1} + \cdots + u_{\mathbf{x}_n}u_{\mathbf{y}_n}| \\
&\leq \|\mathbf{x}\| \cdot \|\mathbf{y}\| \cdot 1 \\
&= \|\mathbf{x}\| \cdot \|\mathbf{y}\|
\end{aligned}$$

as desired. □

Theorem 18.10. *If* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}$, *then*

(*a*) $\|\mathbf{x}\| \geq 0$. *Moreover,* $\|\mathbf{x}\| = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$.

Proof. Let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$.

We first prove that $\|\mathbf{x}\| \geq 0$. By Lemma 7.26, $x_i^2 \geq 0$ for all $i \in [n]$. Thus, by Definition 7.21, $x_1^2 + \cdots + x_n^2 \geq 0$. Therefore, we have by Definition 18.6 that $\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2} \geq 0$, as desired.

We now prove that $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$. Suppose first that $\|\mathbf{x}\| = 0$. Then by Definition 18.6 and Script 7, $x_1^2 + \cdots + x_n^2 = 0$. Now suppose for the sake of contradiction that $\mathbf{x} \neq \mathbf{0}$. Then there exists an x_i such that $x_i \neq 0$. Thus, by Lemma 7.26, $x_i^2 > 0$. Additionally, $x_j^2 \geq 0$ for all $j \in [n]$. Thus, we have that $0 < x_i^2 \leq x_1^2 + \cdots + x_n^2$. But by Definition 3.1, this implies that $x_1^2 + \cdots + x_n^2 \neq 0$, a contradiction.

Now suppose that $\mathbf{x} = \mathbf{0}$. Then by Definition 18.6, $\|\mathbf{x}\| = \sqrt{0^2 + \cdots + 0^2} = 0$, as desired. □

(*b*) $\|\lambda\mathbf{x}\| = |\lambda| \cdot \|\mathbf{x}\|$.

Proof. Let λ be an arbitrary element of $\mathbb{R}$, and let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Then we have that

$$\begin{aligned}
\|\lambda\mathbf{x}\| &= \sqrt{(\lambda x_1)^2 + \cdots + (\lambda x_n)^2} && \text{Definition 18.6} \\
&= |\lambda| \cdot \sqrt{x_1^2 + \cdots + x_n^2} \\
&= |\lambda| \cdot \|\mathbf{x}\| && \text{Definition 18.6}
\end{aligned}$$

as desired. □

(*c*) $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

Proof. Let $\mathbf{x}, \mathbf{y}$ be arbitrary elements of $\mathbb{R}^n$. Then we have that

$$\begin{aligned}
\|\mathbf{x} + \mathbf{y}\| &= \sqrt{(x_1 + y_1)^2 + \cdots + (x_n + y_n)^2} && \text{Definition 18.6} \\
&= \sqrt{(x_1^2 + \cdots + x_n^2) + (2x_1y_1 + \cdots + 2x_ny_n) + (y_1^2 + \cdots + y_n^2)} \\
&\leq \sqrt{\|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\|\mathbf{y}\| + \|y\|^2} && \text{Lemma 18.9} \\
&= \sqrt{(\|\mathbf{x}\| + \|\mathbf{y}\|)^2} \\
&= \|\mathbf{x}\| + \|\mathbf{y}\|
\end{aligned}$$

as desired. □

Corollary 18.11. *If* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ *and* $\lambda \in \mathbb{R}$*, then*

(*a*) $\|\mathbf{x} - \mathbf{z}\| \leq \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{z}\|$.

(*b*) $|\|\mathbf{x}\| - \|\mathbf{y}\|| \leq \|x - y\|$.

Proof. The proofs are symmetric to those of Lemma 8.8. $\square$