Script 15

Sequences

15.1 Journal

5/6: **Definition 15.1.** A **sequence** (of real numbers) is a function $a : \mathbb{N} \to \mathbb{R}$.

By setting $a_n = a(n)$, we can think of a sequence as a list $a_1, a_2, a_3, \dots$ of real numbers. We use the notation $(a_n)_{n=1}^{\infty}$ for such a sequence, or if there is no possibility of confusion, we sometimes abbreviate this and write simply (a_n). More generally, we also use the term sequence to refer to the function defined on $\{n \in \mathbb{N} \cup \{0\} \mid n \geq n_0\}$ for any fixed $n_0 \in \mathbb{N} \cup \{0\}$. We write $(a_n)_{n=n_0}^{\infty}$ for such a sequence.

Definition 15.2. We say that a sequence (a_n) **converges** to a point $p \in \mathbb{R}$ if for every open interval I containing p, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in I$. If a sequence converges to some point, we say it is **convergent**. If (a_n) does not converge to any point, we say that the sequence **diverges** or is **divergent**.

Exercise 15.3. Show that a sequence (a_n) converges to p if and only if any region containing p contains all but finitely many terms of the sequence.

Proof. Suppose first that (a_n) converges to p. Let R be an arbitrary region containing p. By Corollary 4.11 and Lemma 8.3, R is an open interval. Thus, by Definition 15.2, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in R$. To prove that R contains all but finitely many terms of the sequence, it will suffice to show that the set $A = \{a_n \mid a_n \notin R\}$ is finite. Since $a_n \in R$ for all $n \geq N$, it follows that $a_n \in R$ only if $n < N$. Thus, by Script 1, $A \subset \{a_n \mid 0 \leq n < N\}$. Since the latter set is clearly finite, it follows by Script 1 that A is finite.

Now suppose that any region containing p contains all but finitely many terms (a_n). To prove that (a_n) converges to p, Definition 15.2 tells us that it will suffice to show that for every open interval I containing p, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in I$. Let I be an arbitrary open interval containing p. Then by Theorem 4.10, there exists a region R containing p such that $R \subset I$. It follows by the hypothesis that $A = \{a_n \mid a_n \notin R\}$ is finite. We divide into two cases ($|A| = 0$ and $|A| \in \mathbb{N}$). Suppose first that $|A| = 0$. Choose $N = n_0$. It follows that if $n \geq N$, then $a_n \notin A$, so $a_n \in R$, so $a_n \in I$, as desired. Now suppose that $|A| \in \mathbb{N}$. By Definition 1.18, $a^{-1}(A) \subset \mathbb{N}$. Consequently, by Lemma 3.4, $a^{-1}(A)$ has a last point $N - 1$. Choose $N = (N-1)+1$. It follows that if $n \geq N$, then $n \notin a^{-1}(A)$, so $a_n \notin A$, so $a_n \in R$, so $a_n \in I$, as desired. ☐

Theorem 15.4. *Suppose that (a_n) converges to both p and to p'. Then $p = p'$.*

Proof. Suppose for the sake of contradiction that $p \neq p'$. Then by Theorem 3.22, there exist disjoint regions R, R' containing p, p', respectively. Additionally, by consecutive applications of Corollary 4.11 and Lemma 8.3, R, R' are open intervals. Combining these last two results, we have by consecutive applications of Definition 15.2 that there exist $N, N' \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in R$ and if $n \geq N'$, then $a_n \in R'$. Let $M = \max(N, N')$. It follows that $M \geq N$ and $M \geq N'$. Thus, by the above, $a_M \in R$ and $a_M \in R'$. But this implies by Definition 1.6 that $a_M \in R \cap R'$. Therefore, by Definition 1.9, R and R' are not disjoint, a contradiction. ☐

Definition 15.5. If a sequence (a_n) converges to $p \in \mathbb{R}$, we call p the **limit** of (a_n) and write

$$\lim_{n\to\infty} a_n = p$$

Exercise 15.6. Which of the following sequences (a_n) converge? Which diverge? For each that converges, what is the limit? Give proofs for your answers.

(a) $a_n = 5$.

Proof. To prove that this sequence converges with limit $\lim_{n\to\infty} a_n = 5$, Definition 15.5 tells us that it will suffice to show that (a_n) converges to 5. To do so, Definition 15.2 tells us that it will suffice to verify that for every open interval I containing 5, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in I$. Let's begin.

Let I be an arbitrary open interval containing 5. Choose $N = 1$. Let n be an arbitrary natural number such that $n \geq N$. It follows by the definition of the sequence that $a_n = 5 \in I$, as desired. $\square$

(b) $a_n = n$.

Proof. To prove that this sequence diverges, Definition 15.2 tells us that it will suffice to show that for every point $p \in \mathbb{R}$, there exists an open interval I containing p such that for all $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \notin I$. Let's begin.

Let p be an arbitrary element of $\mathbb{R}$. Choose $I = (p-1, p+1)$. Clearly $p \in I$. Let N be an arbitrary natural number. By Corollary 6.12, there exists a natural number N' such that $p + 1 < N'$. Choose $M = \max(N, N')$. Thus, $M \geq N$. Additionally, it follows by the definition of the sequence that $a_M = M$. But this implies that $a_M \geq N' > p + 1$, i.e., $a_M \notin I$ by Equations 8.1. $\square$

(c) $a_n = \frac{1}{n}$.

Proof. To prove that this sequence converges with limit $\lim_{n\to\infty} a_n = 0$, Definitions 15.5 and 15.2 tell us that it will suffice to show that for every open interval I containing 0, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in I$. Let's begin.

Let I be an arbitrary interval containing 0. By Lemma 8.10, there exists a region (a, b) containing 0 such that $(a, b) \subset I$. By Corollary 6.12, there exists a natural number N such that $\frac{1}{b} < N$. Choose this N to be our N. Now let n be an arbitrary natural number such that $n \geq N$. It follows that $\frac{1}{b} < n$. Thus, since $0 < b$ and $0 < n$, we have by consecutive applications of Lemma 7.24 that $0 < \frac{1}{n} < b$. Consequently, since we also know that $a < 0$ and $a_n = \frac{1}{n}$, we have by transitivity and substitution that $a < a_n < b$. It follows by Equations 8.1 that $a_n \in (a, b)$. Therefore, by Definition 1.3, $a_n \in I$, as desired. $\square$

(d) $a_n = (-1)^n$.

Proof. To prove that this sequence diverges, Definition 15.2 tells us that it will suffice to show that for every point $p \in \mathbb{R}$, there exists an open interval I containing p such that for all $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \notin I$. Let's begin.

Let p be an arbitrary element of R. Choose $I = (p-1, p+1)$. Clearly $p \in I$. Let N be an arbitrary natural number. By Script 0, either N is even and $N+1$ is odd or vice versa. Thus, let N be even (the case where N is odd is symmetric). It follows that $N \geq N$ yields $a_N = (-1)^{2m} = ((-1)^2)^m = 1^m = 1$ and that $N+1 \geq N$ yields $a_{N+1} = (-1)^{2m+1} = 1(-1)^1 = -1$. Now suppose for the sake of contradiction that $a_N \in I$ and $a_{N+1} \in I$. Since $a_N = 1 \in I$, we have by Equations 8.1 that $p - 1 < 1 < p + 1$. It follows by Definition 7.21 that $p - 3 < -1 < p - 1$. But $-1 < p - 1$ implies by Equations 8.1 that $a_{N+1} = -1 \notin I$, a contradiction. Therefore, $N+1 \geq N$ is a number such that $a_{N+1} \notin I$, as desired. $\square$

5/11: **Theorem 15.7.** *A sequence (a_n) converges to $p \in \mathbb{R}$ if and only if for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - p| < \epsilon$.*

Proof. Suppose first that (a_n) converges to p. Let $\epsilon > 0$ be arbitrary. Consider the p-containing region $R = (p - \epsilon, p + \epsilon)$. By Corollary 4.11 and 8.3, R is an open interval. Thus, by Definition 15.2, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in R$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. Then $a_n \in (p - \epsilon, p + \epsilon)$. Therefore, by Exercise 8.9, $|a_n - p| < \epsilon$, as desired.

Now suppose that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - p| < \epsilon$. To prove that (a_n) converges to p, Definition 15.2 tells us that it will suffice to show that for every open interval I containing p, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in I$. Let I be an arbitrary open interval that satisfies $p \in I$. It follows by Lemma 8.10 that there exists a number $\epsilon > 0$ such that $(p - \epsilon, p + \epsilon) \subset I$. With respect to this ϵ, we have by hypothesis that there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - p| < \epsilon$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. Then $|a_n - p| < \epsilon$. Consequently, by Exercise 8.9, $a_n \in (p - \epsilon, p + \epsilon)$. Therefore, by Definition 1.3, $a_n \in I$, as desired. ☐

Exercise 15.8.

(a) Prove that $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

Proof. To prove that $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$, Definition 15.5 and Theorem 15.7 tell us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - 0| = |a_n| < \epsilon$. Let $\epsilon > 0$ be arbitrary. By Corollary 6.12, there exists a natural number N such that $\frac{1}{\epsilon} < N$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. It follows by transitivity that $\frac{1}{\epsilon} < n$. Thus, since $0 < n$ and $0 < \epsilon$, we have by consecutive applications of Lemma 7.24 that $0 < \frac{1}{n} < \epsilon$. Additionally, since $(-1)^n = 1$ or $(-1)^n = -1$ for all $n \in \mathbb{N}$ by Script 0, we have by Definition 8.4 that $|\frac{(-1)^n}{n}| = |\frac{1}{n}| = \frac{1}{n}$. Consequently, we know that $|\frac{(-1)^n}{n}| < \epsilon$. But since $a_n = \frac{(-1)^n}{n}$, we have that $|a_n| < \epsilon$, as desired. ☐

(b) Let $x \in \mathbb{R}$ with $|x| < 1$. Prove that $\lim_{n\to\infty} x^n = 0$.

Lemma. *If $|y| > 1$ and n is a natural number, then $|y|^n \geq n(|y| - 1) + 1$.*

Proof. Define $1 + x = |y|$. It follows by Definition 7.21 that $x > 0 > -1$, which can be weakened to $x \geq -1$. Additionally, since n is a natural number, $n \geq 1$ by Script 0. Thus, since $x \geq -1$ and $n \geq 1$, we have by Additional Exercise 12.3b that $(1+x)^n \geq 1+nx$. Substituting, we have $|y|^n \geq n(|y|-1)+1$, as desired. ☐

Proof of Exercise 15.8b. To prove that $\lim_{n\to\infty} x^n = 0$, Definition 15.5 and Theorem 15.7 tell us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - 0| = |a_n| < \epsilon$. Let $\epsilon > 0$ be arbitrary. We divide into two cases ($x = 0$ and $x \neq 0$). If $x = 0$, then choose $N = 1$. Let n be an arbitrary natural number such that $n \geq N$. Since $0^n = 0$ by Script 7, we have $|a_n| = |0^n| = 0 < \epsilon$, as desired. On the other hand, if $x \neq 0$, then we continue. By Corollary 6.12, there exists a natural number N such that $\frac{1}{\epsilon(|\frac{1}{x}|-1)} < N$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. Therefore,

$$\begin{aligned}
|x^n| &= |x^n| \cdot \frac{1}{\epsilon\left(\left|\frac{1}{x}\right| - 1\right)} \cdot \epsilon\left(\left|\frac{1}{x}\right| - 1\right) \\
&< |x^n| \cdot N \cdot \epsilon\left(\left|\frac{1}{x}\right| - 1\right) \\
&\leq |x^n| \cdot n \cdot \epsilon\left(\left|\frac{1}{x}\right| - 1\right) \\
&< \epsilon \cdot |x^n| \cdot n\left(\left|\frac{1}{x}\right| - 1\right) + 1
\end{aligned}$$

$$
\begin{aligned}
&\leq \epsilon \cdot |x^n| \cdot \left|\frac{1}{x}\right|^n && \text{The Lemma} \\
&= \epsilon \cdot |x^n| \cdot \frac{1}{|x^n|} \\
&= \epsilon
\end{aligned}
$$

as desired. □

Theorem 15.9. *If* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$ *both exist, then*

(a) $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$.

Proof. Let $a = \lim_{n\to\infty} a_n$ and let $b = \lim_{n\to\infty} b_n$. To prove that $\lim_{n\to\infty}(a_n + b_n)$ exists and equals $\lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$, Definition 15.5 and Theorem 15.7 tell us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|(a_n + b_n) - (a + b)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. It follows by consecutive applications of Definition 15.5 and Theorem 15.7 that there exist natural numbers N_a, N_b such that for all $n \geq N_a$, we have $|a_n - a| < \frac{\epsilon}{2}$ and for all $n \geq N_b$, we have $|b_n - b| < \frac{\epsilon}{2}$. Now choose $N = \max(N_a, N_b)$. Let n be an arbitrary natural number such that $n \geq N$. It follows that $n \geq N \geq N_a$, so we know that $|a_n - a| < \frac{\epsilon}{2}$. Similarly, $|b_n - b| < \frac{\epsilon}{2}$. Therefore, we have that

$$
\begin{aligned}
|(a_n + b_n) - (a + b)| &\leq |a_n - a| + |b_n - b| && \text{Lemma 8.8} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}
$$

□

(b) $\lim_{n\to\infty}(a_n \cdot b_n) = (\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$.

Proof. Let $a = \lim_{n\to\infty} a_n$ and let $b = \lim_{n\to\infty} b_n$. To prove that $\lim_{n\to\infty}(a_n \cdot b_n)$ exists and equals $(\lim_{n\to\infty} a_n) \cdot (\lim_{n\to\infty} b_n)$, Definition 15.5 and Theorem 15.7 tell us that it will suffice too show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n \cdot b_n - a \cdot b| < \epsilon$. Let $\epsilon > 0$ be arbitrary. By Theorem 15.13[1], (a_n) is bounded. Thus, by Definition 15.12, $\{a_n \mid n \in \mathbb{N}\}$ is bounded. Consequently, by the proof of Exercise 13.9, there exists a number M_a such that $|a_n| < M_a$ for all $n \in \mathbb{N}$. Now define $M = \max(M_a, b)$. Using this M (which by definition is positive since it's greater than $|a_n|$, which is at least 0) as well as our previously defined arbitrary ϵ, we have by consecutive applications of Definition 15.5 and Theorem 15.7 that there exist natural numbers N_a, N_b such that for all $n \geq N_a$, we have $|a_n - a| < \frac{\epsilon}{2M}$ and for all $n \geq N_b$, we have $|b_n - b| < \frac{\epsilon}{2M}$. Now choose $N = \max(N_a, N_b)$. Let n be an arbitrary natural number such that $n \geq N$. It follows that $n \geq N \geq N_a$, so we know that $|a_n - a| < \frac{\epsilon}{2M}$. Similarly, $|b_n - b| < \frac{\epsilon}{2M}$. Therefore,

$$
\begin{aligned}
|a_n b_n - ab| &= |a_n(b_n - b) + b(a_n - a)| \\
&\leq |a_n| \cdot |b_n - b| + |b| \cdot |a_n - a| && \text{Lemma 8.8} \\
&\leq |M| \cdot |b_n - b| + |M| \cdot |a_n - a| \\
&< M \cdot \frac{\epsilon}{2M} + M \cdot \frac{\epsilon}{2M} \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}
$$

□

[1] The proof of Theorem 15.13 does not depend on any following results, so its use here is not circular logic.

Moreover, if $\lim_{n\to\infty} b_n \neq 0$*, then*

(c) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$.

Lemma. *Let* $\lim_{n\to\infty} b_n = b \neq 0$*. Then there exists* $m \in \mathbb{R}^+$ *such that* $m \leq |b|$ *and* $N \in \mathbb{N}$ *such that if* $n \geq N$*, then* $m \leq |b_n|$*.*

Proof. Since $b \neq 0$, it follows from Definition 8.4 that $0 < |b|$. Thus, by Theorem 5.2, there exists a point $m \in \mathbb{R}$ such that $0 < m < |b|$. It follows from the fact that $0 < m$ that $m \in \mathbb{R}^+$, and from the fact that $m < |b|$ that $m \leq |b|$, as desired.

As to the other part of the proof, we divide into two cases ($b > 0$ and $b < 0$).

Suppose first that $b > 0$. By Exercise 6.5, Axiom 3, and Definition 3.3, there exists a number y such that $b < y$. Now consider the region (m, y). Since $m < |b| = b < y$, Equations 8.1 assert that $b \in (m, y)$. Additionally, by Corollary 4.11 and Lemma 8.3, (m, y) is an open interval. Thus, by Definitions 15.5 and 15.2, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $b_n \in (m, y)$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. Then $b_n \in (m, y)$. It follows by Equations 8.1 that $m < b_n < y$, which can be weakened to $m \leq b_n$. Since $0 < m \leq b_n$, Definition 8.4 asserts that $m \leq |b_n|$, as desired.

Now suppose that $b < 0$. By Exercise 6.5, Axiom 3, and Definition 3.3, there exists a number x such that $x < b$. Now consider the region $(x, -m)$. Since $m < |b| = -b$, we have by Lemma 7.24 that $b < -m$. This combined with the fact that $x < b$ implies by Equations 8.1 that $b \in (x, -m)$. Additionally, by Corollary 4.11 and Lemma 8.3, $(x, -m)$ is an open interval. Thus, by Definitions 15.5 and 15.2, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $b_n \in (x, -m)$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. Then $b_n \in (x, -m)$. It follows by Equations 8.1 that $x < b_n < -m$, which can be weakened to $b_n \leq -m$. Consequently, by Lemma 7.24, $m \leq -b_n$. Since $0 < m \leq -b_n$, Definition 8.4 asserts that $m \leq |b_n|$, as desired. □

Proof of Theorem 15.9c. Let $a = \lim_{n\to\infty} a_n$ and let $b = \lim_{n\to\infty} b_n$. To prove that $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and equals $\frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, Definition 15.5 and Theorem 15.7 tell us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|\frac{a_n}{b_n} - \frac{a}{b}| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $M = \max(|a|, |b|)$. Additionally, by the lemma, choose $m \in \mathbb{R}, N' \in \mathbb{N}$ such that $m \leq |b|$ and if $n \geq N'$, then $m \leq |b_n|$. Using this M and m (which, again, by definition are both positive and nonzero) as well as our previously defined arbitrary ϵ, we have by consecutive applications of Definition 15.5 and Theorem 15.7 that there exist natural numbers N_a, N_b such that for all $n \geq N_a$, we have $|a_n - a| < \frac{\epsilon m^2}{M}$ and for all $n \geq N_b$, we have $|b_n - b| < \frac{\epsilon m^2}{M}$. Now choose $N = \max(N', N_a, N_b)$. Let n be an arbitrary natural number such that $n \geq N$. It follows that $n \geq N \geq N_a$, so we know that $|a_n - a| < \frac{\epsilon m^2}{M}$. Additionally, since $n \geq N'$, $m \leq |b|$ and $m \leq |b_n|$. Similarly, $|b_n - b| < \frac{\epsilon m^2}{M}$. Therefore,

$$\begin{aligned}
\left|\frac{a_n}{b_n} - \frac{a}{b}\right| &= \left|\frac{a_n b - b_n a}{b_n b}\right| \\
&= \frac{|b(a_n - a) + a(b - b_n)|}{|b_n| \cdot |b|} \\
&\leq \frac{|b| \cdot |a_n - a| + |a| \cdot |b - b_n|}{|b_n| \cdot |b|} && \text{Lemma 8.8} \\
&\leq \frac{M \cdot |a_n - a| + M \cdot |b_n - b|}{m \cdot m} \\
&= \frac{M}{m^2} \cdot |a_n - a| + \frac{M}{m^2} \cdot |b_n - b| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

□

5/13: **Exercise 15.10.** Which of the following sequences (a_n) converge? Which diverge? For each that converges, what is the limit? Give proofs for your answers.

(a) $a_n = (-1)^n \cdot n$.

Proof. Suppose for the sake of contradiction that $\lim_{n\to\infty} a_n$ converges. Then since (b_n) defined by $b_n = \frac{1}{n}$ converges by Exercise 15.6c, we have by Theorem 15.9 that

$$\begin{aligned}\left(\lim_{n\to\infty}(-1)^n \cdot n\right)\cdot\left(\lim_{n\to\infty}\frac{1}{n}\right) &= \lim_{n\to\infty}(-1)^n \cdot n \cdot \frac{1}{n}\\ &= \lim_{n\to\infty}(-1)^n\end{aligned}$$

But by Exercise 15.6d, $\lim_{n\to\infty}(-1)^n$ diverges, a contradiction. ☐

(b) $a_n = \frac{1}{n^2+1}(2+\frac{1}{n})$.

Proof. To prove that $\lim_{n\to\infty}\frac{1}{n^2+1}(2+\frac{1}{n}) = 0$, we will first confirm that $\lim_{n\to\infty}\frac{1}{n^2+1} = 0$ and $\lim_{n\to\infty} 2 = 2$. These results can be tied together with the fact that $\lim_{n\to\infty}\frac{1}{n} = 0$ (by Exercise 15.6c) to prove the desired result with Theorem 15.9. Let's begin.

To confirm that $\lim_{n\to\infty}\frac{1}{n^2+1} = 0$, Theorem 15.7 tells us that it will suffice to demonstrate that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|\frac{1}{n^2+1} - 0| = |\frac{1}{n^2+1}| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{n\to\infty}\frac{1}{n} = 0$ by Exercise 15.6c, we have by Theorem 15.7 that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|\frac{1}{n}| < \epsilon$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. Then since $n \leq n^2 < n^2+1$ by Script 7, we have that

$$\begin{aligned}\left|\frac{1}{n^2+1}\right| &= \frac{1}{n^2+1} && \text{Definition 8.4}\\ &< \frac{1}{n^2}\\ &\leq \frac{1}{n}\\ &= \left|\frac{1}{n}\right| && \text{Definition 8.4}\\ &< \epsilon\end{aligned}$$

as desired.

The proof that $\lim_{n\to\infty} 2 = 2$ is symmetric to that of Exercise 15.6a.

Having established that $\lim_{n\to\infty}\frac{1}{n^2+1} = 0$, $\lim_{n\to\infty} 2 = 2$, and $\lim_{n\to\infty}\frac{1}{n} = 0$, we have by consecutive applications of Theorem 15.9 that

$$\begin{aligned}0 &= 0\cdot(2+0)\\ &= \left(\lim_{n\to\infty}\frac{1}{n^2+1}\right)\cdot\left(\lim_{n\to\infty} 2 + \lim_{n\to\infty}\frac{1}{n}\right)\\ &= \lim_{n\to\infty}\frac{1}{n^2+1}\left(2+\frac{1}{n}\right)\end{aligned}$$

as desired. ☐

(c) $a_n = \frac{5n+1}{2n+3}$.

Proof. The proof that $\lim_{n\to\infty} 3 = 3$ is symmetric to that of Exercise 15.6a. Additionally, by Exercise 15.6a, the proof of Exercise 15.10b, and Exercise 15.6c, we know that $\lim_{n\to\infty} 5 = 5$, $\lim_{n\to\infty} 2 = 2$, and $\lim_{n\to\infty} \frac{1}{n} = 0$, respectively. Therefore, by consecutive applications of Theorem 15.9, we have that

$$\begin{aligned}\frac{5}{2} &= \frac{5+0}{2+3\cdot 0}\\ &= \frac{\lim_{n\to\infty} 5 + \lim_{n\to\infty} \frac{1}{n}}{\lim_{n\to\infty} 2 + (\lim_{n\to\infty} 3)\cdot\left(\lim_{n\to\infty}\frac{1}{n}\right)}\\ &= \lim_{n\to\infty} \frac{5+\frac{1}{n}}{2+3\cdot\frac{1}{n}}\\ &= \lim_{n\to\infty}\frac{5n+1}{2n+3}\end{aligned}$$

as desired. ☐

(d) $a_n = \frac{(-1)^n+1}{n}$.

Proof. By Exercises 15.8a and 15.6c, we know that $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$ and $\lim_{n\to\infty} \frac{1}{n} = 0$, respectively. Therefore, by Theorem 15.9,

$$\begin{aligned}0 &= 0+0\\ &= \lim_{n\to\infty}\frac{(-1)^n}{n} + \lim_{n\to\infty}\frac{1}{n}\\ &= \lim_{n\to\infty}\left(\frac{(-1)^n}{n}+\frac{1}{n}\right)\\ &= \lim_{n\to\infty}\frac{(-1)^n+1}{n}\end{aligned}$$

as desired. ☐

We've used the word "limit" in two contexts now: The limit of a point in a set, and the limit of a sequence. The definitions of these two terms may seem similar. Is there a formal connection? Theorem 15.11 alludes to an answer.

Theorem 15.11. *Let $A \subset \mathbb{R}$. Then $p \in \overline{A}$ if and only if there exists a sequence (a_n), with each $a_n \in A$, that converges to p.*

Proof. Suppose first that $p \in \overline{A}$. Then by Definitions 4.4 and 1.5, $p \in A$ or $p \in LP(A)$. We now divide into two cases. If $p \in A$, then define (a_n) by $a_n = p$ for all $n \in \mathbb{N}$. Clearly, each $a_n \in A$ since $p \in A$, and $\lim_{n\to\infty} a_n = p$ by a proof symmetric to that of Exercise 15.6a, as desired. If $p \in LP(A)$, then define $R_n = (p - \frac{1}{n}, p + \frac{1}{n})$ for all $n \in \mathbb{N}$. Since $p \in LP(A)$, we have by Definition 3.13 that $R_n \cap (A \setminus \{p\}) \neq \emptyset$ for all $n \in \mathbb{N}$. It follows by the axiom of choice that we can choose a point a_n in $R_n \cap (A \setminus \{p\})$ for all $n \in \mathbb{N}$. Thus, by Definitions 1.6 and 1.11, each $a_n \in A$ (as desired) and $a_n \in R_n$ for all $n \in \mathbb{N}$. We now seek to prove that (a_n) converges to p; to do so, Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - p| < \epsilon$. Let $\epsilon > 0$ be arbitrary. By Corollary 6.12, there exists an $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. Then by consecutive applications of Lemma 7.24, $\frac{1}{n} \leq \frac{1}{N}$. Consequently, by Script 1 as well as the definitions of R_n and R_N, $R_n \subset R_N$. It follows by Definition 1.3 since $a_n \in R_n$ that $a_n \in R_N$. Therefore, by Exercise 8.9, $|a_n - p| < \frac{1}{N} < \epsilon$, as desired.

Now suppose that there exists a sequence (a_n) with each $a_n \in A$ that converges to p. We divide into two cases ($p \in A$ and $p \notin A$). If $p \in A$, then by Definitions 1.5 and 4.4, $p \in \overline{A}$, as desired. If $p \notin A$, then to prove that $p \in \overline{A}$, Definitions 4.4 and 1.5 tell us that we must show that $p \in LP(A)$. To do so, Definition 3.13 tells us that it will suffice to verify that for all regions R containing p, $R \cap (A \setminus \{p\}) \neq \emptyset$. Let R be an arbitrary region R with $p \in R$. By Corollary 4.11 and 8.3, R is an open interval. Thus, by Definition 15.2, there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in R$. It follows that $a_N \in R$. Additionally, by hypothesis, $a_N \in A$. These results combined with the fact that $A = A \setminus \{p\}$ (since $p \notin A$) imply by Definition 1.6 that $a_N \in R \cap (A \setminus \{p\})$. Therefore, by Definition 1.8, $R \cap (A \setminus \{p\}) \neq \emptyset$, as desired. □

Definition 15.12. A sequence (a_n) is **bounded** if its image $\{a_n \mid n \in \mathbb{N}\}$ is bounded.

Theorem 15.13. *Every convergent sequence is bounded.*

Proof. Let (a_n) be a sequence that converges to p. To prove that (a_n) is bounded, Definitions 15.12 and 5.6 tell us that it will suffice to find numbers l, u such that $l \leq a_n \leq u$ for all a_n. Let (x, y) be a region that contains p. By Corollary 4.11 and Lemma 8.3, (x, y) is an open interval. Thus, by Exercise 15.3, we have that (x, y) contains all but finitely many terms of the sequence, i.e., $\{a_n \mid a_n \notin (x, y)\}$ is finite. We divide into two cases ($\{a_n \mid a_n \notin (x, y)\} = \emptyset$ and $\{a_n \mid a_n \notin (x, y) \neq \emptyset$). If $\{a_n \mid a_n \notin (x, y)\} = \emptyset$, then $a_n \in (x, y)$ for all a_n. It follows by Equations 8.1 that $x < a_n < y$ for all a_n. If we now choose $l = x$ and $u = y$, we can weaken the previous statement to $l = x \leq a_n \leq y = u$, as desired. On the other hand, if $\{a_n \mid a_n \notin (x, y)\} \neq \emptyset$, then by Lemma 3.4, $\{a_n \mid a_n \notin (x, y)\}$ has a first and a last point. It follows by Exercise 5.9 that $\{a_n \mid a_n \notin (x, y)\}$ is bounded by $\inf\{a_n \mid a_n \notin (x, y)\}$ and $\sup\{a_n \mid a_n \notin (x, y)\}$. Choose $l = \min(x, \inf\{a_n \mid a_n \notin (x, y)\})$ and $u = \max(y, \sup\{a_n \mid a_n \notin (x, y)\})$. Let a_n be an arbitrary term in the sequence. We divide into two subcases ($a_n \in (x, y)$ and $a_n \notin (x, y)$). If $a_n \in (x, y)$, then $l \leq x < a_n < y \leq u$, as desired. On the other hand, if $a_n \notin (x, y)$, then $l \leq \inf\{a_n \mid a_n \notin (x, y)\} \leq a_n \leq \sup\{a_n \mid a_n \notin (x, y)\} \leq u$, as desired. □

The converse is not true, but there are important partial converses. For the first, Theorem 15.14, we recall Definition 8.16 along with Definition 15.1, which say that (a_n) is an increasing sequence if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$, and (a_n) is decreasing if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. The definitions for strictly increasing/strictly decreasing are similar but with strict inequalities.

Theorem 15.14. *Every bounded increasing sequence converges to the supremum of its image. Every bounded decreasing sequence converges to the infimum of its image.*

Proof. We will only address the first part of the theorem; the proof of the second part is symmetric.

Let (a_n) be a bounded increasing sequence and let $p = \sup\{a_n \mid n \in \mathbb{N}\}$. To prove that (a_n) converges to p, Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - p| < \epsilon$. Let $\epsilon > 0$ be arbitrary. By Lemma 5.11, there exists $a_N \in \{a_n \mid n \in \mathbb{N}\}$ such that $p - \epsilon < a_N \leq p$. Choose N to be the natural number that generates a_N. Let n be an arbitrary natural number such that $n \geq N$. Then since $a_N \leq a_{N+1} \leq \cdots \leq a_{n-1} \leq a_n$, we have by transitivity that $a_N \leq a_n$. Additionally, since $a_n \in \{a_n \mid n \in \mathbb{N}\}$, we have by Definitions 5.7 and 5.6 that $a_n \leq p$. Thus, since $p - \epsilon < a_N \leq a_n \leq p < p + \epsilon$, we have by Equations 8.1 that $a_n \in (p - \epsilon, p + \epsilon)$. Therefore, by Exercise 8.9, $|a_n - p| < \epsilon$, as desired. □

To discuss the second partial converse, Theorem 15.18, we need another definition.

Definition 15.15. Let (a_n) be a sequence. A **subsequence** of (a_n) is a sequence $b : \mathbb{N} \to \mathbb{R}$ defined by the composition $b = a \circ i$, where $i : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function. If (a_n) has a subsequence with limit p, we call p a **subsequential limit** of (a_n).

We can write $b_k = a(i(k)) = a_{i(k)} = a_{i_k}$, so that (b_k) is the sequence $b_1, b_2, b_3, \ldots$, which is equal to the sequence $a_{i_1}, a_{i_2}, a_{i_3}, \ldots$, where $i_1 < i_2 < i_3 < \cdots$.

Theorem 15.16. *If (a_n) converges to p, then so do all of its subsequences.*

Proof. Let (b_n) be an arbitrary subsequence of (a_n). To prove that (b_n) converges to p, Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|b_n - p| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since (a_n) converges to p, Theorem 15.7 implies that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - p| < \epsilon$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. By Definition 15.15 and Script 1, $i(n) \geq n$. Therefore, we have by the above that $|b_n - p| = |a_{i_n} - p| < \epsilon$, as desired. □

5/18: **Exercise 15.17.** Construct a sequence with two subsequential limits. Construct a sequence with infinitely many subsequential limits.

Proof. Let (a_n) be a sequence defined by $a_n = (-1)^n$ for all $n \in \mathbb{N}$. Let (b_n) be a subsequence of (a_n) defined be defined by $b_n = a_{2n}$ for all $n \in \mathbb{N}$. Then by the proof of Exercise 15.6d, $b_n = 1$ for all $n \in \mathbb{N}$. It follows by the proof of Exercise 15.6a that $\lim_{n\to\infty} b_n = 1$. Similarly, if we let (c_n) be defined by $c_n = a_{2n+1}$, then $\lim_{n\to\infty} c_n = -1$.

Now let (a_n) be defined by $a_1 = 1$; $a_2 = 1$ and $a_3 = 2$; $a_4 = 1$, $a_5 = 2$, and $a_6 = 3$; $a_7 = 1$, $a_8 = 2$, $a_9 = 3$, and $a_{10} = 4$; and so on. Clearly there will be infinitely many terms that evaluate to each natural number in this sequence. Therefore, each of the infinitely many natural numbers is a subsequential limit of this sequence. ☐

Theorem 15.18. *Every bounded sequence has a convergent subsequence.*

Proof. Let (a_n) be an arbitrary bounded sequence, and define $A = \{a_n \mid n \in \mathbb{N}\}$. By Definition 15.12, A is bounded. Additionally, by Script 1, A is infinite. Furthermore, by Script 6, A is a subset of $\mathbb{R}$. Combining these last three results, we have by Theorem 10.18 that there exists a limit point p of A. It follows by Definition 4.4 that $p \in \overline{A}$. Thus, by Theorem 15.11, there exists a sequence (b_n) with each $b_n \in A$ that converges to p. Since each b_n equals an a_n, we have by Definition 15.15 that (b_n) is a subsequence of (a_n). Therefore, we have found a convergent subsequence of (a_n), as desired. ☐

We are now able to prove a useful characterization of convergent sequences.

Theorem 15.19. *A sequence (a_n) of real numbers converges if and only if for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for all $n, m \geq N$.*

Proof. Suppose first that (a_n) converges to p. Let $\epsilon > 0$ be arbitrary. Then by Theorem 15.7, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - p| < \frac{\epsilon}{2}$. Choose this N to be our N. Let n, m be arbitrary natural numbers such that $n, m \geq N$. Then $|a_n - p| < \frac{\epsilon}{2}$ and $|a_m - p| < \frac{\epsilon}{2}$. Therefore,

$$\begin{aligned}
|a_n - a_m| &\leq |a_n - p| + |p - a_m| && \text{Lemma 8.8} \\
&= |a_n - p| + |a_m - p| && \text{Exercise 8.5} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

as desired.

Now suppose that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for all $n, m \geq N$. To prove that (a_n) converges, we will first show that it is bounded. It will then follow by Theorem 15.18 that (a_n) has a subsequence (b_n) that converges to p. The existence of (b_n) combined with the hypothesis will suffice to show that (a_n) converges to p. Let's begin.

To confirm that (a_n) is bounded, Definitions 15.12 and 5.6 tell us that it will suffice to find numbers l, u such that $l \leq a_n \leq u$ for all a_n. Since $1 > 0$ by Corollary 7.27, we have by the hypothesis that there is some $N \in \mathbb{N}$ such that $|a_n - a_m| < 1$ for all $n, m \geq N$. Thus, we know that $|a_N - a_n| < 1$ for all $n \geq N$. Consequently, by Exercise 8.9, $a_n \in (a_N - 1, a_N + 1)$ for all $n \geq N$. It follows by Script 1 that $\{a_n \mid a_n \notin (a_N - 1, a_N + 1)$ is finite, since it can contain at most $N - 1$ terms. If we now divide into two cases and evaluate them in a symmetric fashion to the way we did in the proof of Theorem 15.13, we can establish that (a_n) is bounded, as desired.

Since (a_n) is bounded, we have by Theorem 15.18 that there exists a convergent subsequence (b_n) of (a_n). Let $\lim_{n\to\infty} b_n = p$.

To prove that (a_n) converges to p, Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|a_n - p| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since (b_n) converges to p, Theorem 15.7 asserts that there is some $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $|b_n - p| < \frac{\epsilon}{2}$. Additionally, we have by the hypothesis that there is some $N_2 \in \mathbb{N}$ such that $|a_n - a_m| < \frac{\epsilon}{2}$ for all $n, m \geq N_2$. Choose $N = \max(N_1, N_2)$. Let n be an arbitrary natural number such that $n \geq N$. By Corollary 6.13, there exists a $b_m = a_{i_m}$ with $i_m \geq N$. Thus, since $i_m \geq N \geq N_1$, we have that $|a_{i_m} - p| < \frac{\epsilon}{2}$. Additionally, since $n \geq N \geq N_2$ and $i_m \geq N \geq N_2$, we have that $|a_n - a_{i_m}| < \frac{\epsilon}{2}$. Therefore,

$$\begin{aligned} |a_n - p| &\leq |a_n - a_{i_m}| + |a_{i_m} - p| && \text{Lemma 8.8} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

as desired. □

Theorem 15.20. *Use Theorem 15.19 to show that the sequence in Exercise 15.10a does not converge.*

Proof. Suppose for the sake of contradiction that $a_n = (-1)^n \cdot n$ converges to some $p \in \mathbb{R}$. Then if we choose $\epsilon = 1$, we have by Theorem 15.19 that there exists some $N \in \mathbb{N}$ such that $|a_n - a_m| < 1$ for all $n, m \geq N$. But we also have that $|a_N - a_{N+1}| = 2N + 1 > 1$, regardless of which natural number N is, a contradiction. □