

# Script 16

## Series

### 16.1 Journal

5/20: **Definition 16.1.** Let  $N_0 \in \mathbb{N} \cup \{0\}$  and let  $(a_n)_{n=N_0}^\infty$  be a sequence of real numbers. Then the formal sum

$$\sum_{n=N_0}^{\infty} a_n$$

is called an **infinite series**. (In most instances, we will start the series at  $N_0 = 0$  or  $N_0 = 1$ .)

We will define the **sequence of partial sums**  $(p_n)$  of the series by

$$p_n = a_{N_0} + \cdots + a_{N_0+n-1} = \sum_{i=N_0}^{N_0+n-1} a_i$$

Thus,  $p_n$  is the sum of the first  $n$  terms in the sequence  $(a_n)$ . We say that the series **converges** if there exists  $L \in \mathbb{R}$  such that  $\lim_{n \rightarrow \infty} p_n = L$ . When this is the case, we write this as

$$\sum_{n=N_0}^{\infty} a_n = L$$

and we say that  $L$  is the **sum** of the series. When there does not exist such an  $L$ , we say that the series **diverges**.

**Lemma 16.2.** Let  $(a_n)_{n=0}^\infty$  be a sequence of real numbers. Let  $N_0 \in \mathbb{N}$ . Then  $\sum_{n=0}^\infty a_n$  converges if and only if  $\sum_{n=N_0}^\infty a_n$  converges.

**Lemma.** Let  $n \in \mathbb{N}$ . Then

$$\sum_{i=0}^{N_0+n-1} a_i = \sum_{i=0}^{N_0-1} a_i + \sum_{i=N_0}^{N_0+n-1} a_i$$

*Proof.* This simple result follows immediately from Script 0, so no formal proof will be given.  $\square$

*Proof of Lemma 16.2.* Suppose first that  $\sum_{n=0}^\infty a_n$  converges, and let  $M = \sum_{n=0}^\infty a_n = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} a_i$ , where the latter equality holds by Definition 16.1. To prove that  $\sum_{n=N_0}^\infty a_n$  converges, Definition 16.1 tells us that it will suffice to find an  $L \in \mathbb{R}$  such that  $\lim_{n \rightarrow \infty} \sum_{i=N_0}^{N_0+n-1} a_i = L$ . Choose  $L = M - \sum_{i=0}^{N_0-1} a_i$ . To verify that  $\lim_{n \rightarrow \infty} \sum_{i=N_0}^{N_0+n-1} a_i = L$ , Theorem 15.7 tells us that it will suffice to show that for all  $\epsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|\sum_{i=N_0}^{N_0+n-1} a_i - L| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Since  $\lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} a_i = M$ , Theorem 15.7 implies that there is some  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,

$|\sum_{i=0}^{n-1} a_i - M| < \epsilon$ . Choose this  $N$  to be our  $N$ . Let  $n$  be an arbitrary natural number such that  $n \geq N$ . Since  $N_0 + n > n \geq N$ , we have by the above that  $|\sum_{i=0}^{N_0+n-1} a_i - M| < \epsilon$ . Therefore,

$$\begin{aligned} \left| \sum_{i=N_0}^{N_0+n-1} a_i - L \right| &= \left| \sum_{i=0}^{N_0+n-1} a_i - \sum_{i=0}^{N_0-1} a_i - L \right| && \text{The Lemma} \\ &= \left| \sum_{i=0}^{N_0+n-1} a_i - \left( \sum_{i=0}^{N_0-1} a_i + L \right) \right| \\ &= \left| \sum_{i=0}^{N_0+n-1} a_i - M \right| \\ &< \epsilon \end{aligned}$$

as desired.

The proof is symmetric in the other direction. □

**Exercise 16.3.** Prove that  $\sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{n+1})$  converges. What is its sum?

*Proof.* Let  $(a_n)$  be defined by  $a_n = \frac{1}{n} - \frac{1}{n+1}$ , and let  $(p_n)$  be defined by  $p_n = \sum_{i=1}^n a_i$ . Then

$$\begin{aligned} p_n &= a_1 + a_2 + \cdots + a_n \\ &= \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \cdots + \left( \frac{1}{n} - \frac{1}{n+1} \right) \\ &= \frac{1}{1} - \frac{1}{n+1} \end{aligned}$$

To prove that  $\sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{n+1}) = 1$ , Definition 16.1 tells us that it will suffice to show that  $\lim_{n \rightarrow \infty} p_n = 1$ . By a proof symmetric to that of Exercise 15.6a, we have that  $\lim_{n \rightarrow \infty} 1 = 1$ . By a proof symmetric to that of Exercise 15.6c, we have that  $\lim_{n \rightarrow \infty} \frac{1}{n+1} = 0$ . Therefore, by Theorem 15.9 and the above, we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} p_n &= \lim_{n \rightarrow \infty} \left( 1 - \frac{1}{n+1} \right) \\ &= \lim_{n \rightarrow \infty} 1 - \lim_{n \rightarrow \infty} \frac{1}{n+1} \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

as desired. □

**Theorem 16.4.** If  $\sum_{n=1}^{\infty} a_n$  converges, then  $\lim_{n \rightarrow \infty} a_n = 0$ .

*Proof.* To prove that  $\lim_{n \rightarrow \infty} a_n = 0$ , Theorem 15.7 tells us that it will suffice to show that for all  $\epsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|a_n - 0| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Since  $\sum_{n=1}^{\infty} a_n$  converges, we have by Theorem 15.19 that there exists an  $N \in \mathbb{N}$  such that  $|\sum_{i=1}^n a_i - \sum_{i=1}^m a_i| < \epsilon$  for all  $n, m \geq N$ . Choose this  $N$  to be our  $N$ . Let  $n$  be an arbitrary natural number such that  $n \geq N$ . Then choosing  $n, n-1 \geq N$ , we have by the above that  $|\sum_{i=1}^n a_i - \sum_{i=1}^{n-1} a_i| < \epsilon$ . Therefore,

$$\begin{aligned} |a_n - 0| &= |a_n| \\ &= \left| \sum_{i=1}^n a_i - \sum_{i=1}^{n-1} a_i \right| \\ &< \epsilon \end{aligned}$$

as desired. □

The converse of this theorem, however, is not true, as we see in Theorem 16.6.

**Theorem 16.5.** *A series  $\sum_{n=1}^{\infty} a_n$  converges if and only if for all  $\epsilon > 0$ , there is some  $N \in \mathbb{N}$  such that  $|\sum_{k=m+1}^n a_k| < \epsilon$  for all  $n > m \geq N$ .*

*Proof.* Suppose first that  $\sum_{n=1}^{\infty} a_n$  converges. Let  $\epsilon > 0$  be arbitrary. By Definition 16.1,  $(p_n)$  converges. Thus, by Theorem 15.19, there is some  $N \in \mathbb{N}$  such that  $|p_n - p_m| < \epsilon$  for all  $n, m \geq N$ . Choose this  $N$  to be our  $N$ . Let  $n, m$  be two arbitrary natural numbers satisfying  $n > m \geq N$ . Therefore,

$$\begin{aligned} \left| \sum_{k=m+1}^n a_k \right| &= \left| \sum_{k=1}^n a_k - \sum_{k=1}^m a_k \right| \\ &= |p_n - p_m| \\ &< \epsilon \end{aligned}$$

as desired.

The proof is symmetric in the other direction. □

**Theorem 16.6.** *The series  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges.*

**Lemma.** *For all  $N \in \mathbb{N}$ , we have*

$$\sum_{n=N+1}^{2N} \frac{1}{n} \geq \frac{1}{2}$$

*Proof.* We induct on  $N$ . For the base case  $N = 1$ , we have

$$\sum_{n=1+1}^{2 \cdot 1} \frac{1}{n} = \frac{1}{2} \geq \frac{1}{2}$$

as desired. Now suppose inductively that we have proven the claim for  $N$ . To prove it for  $N + 1$ , we do the following.

$$\begin{aligned} \sum_{n=N+2}^{2N+2} \frac{1}{n} &= \sum_{n=N+1}^{2N} \frac{1}{n} - \frac{1}{N+1} + \frac{1}{2N+1} + \frac{1}{2(N+1)} \\ &= \sum_{n=N+1}^{2N} \frac{1}{n} + \frac{1}{2(N+1)(2N+1)} \\ &> \sum_{n=N+1}^{2N} \frac{1}{n} \\ &\geq \frac{1}{2} \end{aligned}$$

as desired. □

*Proof of Theorem 16.6.* To prove that  $\sum_{n=1}^{\infty} \frac{1}{n}$  diverges, Theorem 16.5 tells us that it will suffice to find an  $\epsilon > 0$  such that for all  $N \in \mathbb{N}$ , there exist  $n > m \geq N$  with  $|\sum_{k=m+1}^n 1/k| \geq \epsilon$ . Choose  $\epsilon = \frac{1}{2}$ . Let  $N$  be an arbitrary element of  $\mathbb{N}$ . If we now choose  $n = 2N$  and  $m = N$ , we will have  $n > m \geq N$ . It will follow by the lemma that

$$\begin{aligned} \left| \sum_{k=m+1}^n \frac{1}{k} \right| &= \left| \sum_{k=N+1}^{2N} \frac{1}{k} \right| \\ &\geq \frac{1}{2} \\ &= \epsilon \end{aligned}$$

as desired. □

5/25: **Theorem 16.7.** Let  $-1 < x < 1$ . Then

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

*Proof.* Let  $(a_n)$  be defined by  $a_n = x^n$ . Then  $p_n = x^0 + x^1 + \cdots + x^{n-1}$  so that

$$\begin{aligned} p_n - xp_n &= x^0 + \cdots + x^{n-1} - x(x^0 + \cdots + x^{n-1}) \\ p_n - xp_n &= 1 - x^n \\ p_n(1-x) &= 1 - x^n \\ p_n &= \frac{1-x^n}{1-x} \end{aligned}$$

Therefore, we have that

$$\begin{aligned} \frac{1}{1-x} &= \frac{1-0}{1-x} \\ &= \frac{1-\lim_{n \rightarrow \infty} x^n}{1-x} \\ &= \lim_{n \rightarrow \infty} \frac{1-x^n}{1-x} \\ &= \lim_{n \rightarrow \infty} p_n \\ &= \sum_{n=0}^{\infty} x^n \end{aligned}$$

Exercise 15.8b

Theorem 15.9

Definition 16.1

as desired. □

**Theorem 16.8.** If  $\sum_{n=1}^{\infty} a_n = L$ ,  $\sum_{n=1}^{\infty} b_n = M$ , and  $c \in \mathbb{R}$ , then

$$\begin{aligned} \sum_{n=1}^{\infty} (a_n + b_n) &= L + M \\ \sum_{n=1}^{\infty} (c \cdot a_n) &= c \cdot L \end{aligned}$$

*Proof.* For the first claim, we have that

$$\begin{aligned} L + M &= \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i + \lim_{n \rightarrow \infty} \sum_{i=1}^n b_i && \text{Definition 16.1} \\ &= \lim_{n \rightarrow \infty} \left( \sum_{i=1}^n a_i + \sum_{i=1}^n b_i \right) && \text{Theorem 15.9} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n (a_i + b_i) \\ &= \sum_{n=1}^{\infty} (a_n + b_n) && \text{Definition 16.1} \end{aligned}$$

The proof is symmetric for the second claim. □

**Definition 16.9.** We say that the series  $\sum_{n=1}^{\infty} a_n$  **converges absolutely** if the series  $\sum_{n=1}^{\infty} |a_n|$  converges.

**Lemma 16.10.** *A series  $\sum_{n=1}^{\infty} a_n$  with all  $a_n \geq 0$  converges if and only if its sequence of partial sums is bounded.*

*Proof.* Suppose first that the series  $\sum_{n=1}^{\infty} a_n$  with all  $a_n \geq 0$  converges. Then by Definition 16.1 its sequence of partial sums  $(p_n)$  converges. Therefore, by Theorem 15.13,  $(p_n)$  is bounded, as desired.

Now suppose that the sequence of partial sums  $(p_n)$  corresponding to a series  $\sum_{n=1}^{\infty} a_n$  with all  $a_n \geq 0$  is bounded. To prove that  $\sum_{n=1}^{\infty} a_n$  converges, Definition 16.1 tells us that it will suffice to show that  $(p_n)$  converges. To do so, Theorem 15.14 tells us that it will suffice to verify in addition to the fact that  $(p_n)$  is bounded that  $(p_n)$  is increasing. To do this, Script 15 tells us that it will suffice to confirm that  $p_n \leq p_{n+1}$  for all  $n \in \mathbb{N}$ . Let  $n$  be an arbitrary natural number. By Definition 16.1,  $p_{n+1} - p_n = a_{n+1}$ . Since  $a_{n+1} \geq 0$  by hypothesis, we have by transitivity that  $p_{n+1} - p_n \geq 0$ , i.e.,  $p_n \leq p_{n+1}$  by Definition 7.21, as desired.  $\square$

**Theorem 16.11.** *If  $\sum_{n=1}^{\infty} a_n$  converges absolutely, then  $\sum_{n=1}^{\infty} a_n$  converges and*

$$\left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} |a_n|$$

*Proof.* To prove that  $\sum_{n=1}^{\infty} a_n$  converges, Theorem 16.5 tells us that it will suffice to show that for all  $\epsilon > 0$ , there is some  $N \in \mathbb{N}$  such that  $|\sum_{k=m+1}^n a_k| < \epsilon$  for all  $n > m \geq N$ . Let  $\epsilon > 0$  be arbitrary. Since  $\sum_{n=1}^{\infty} a_n$  converges absolutely by hypothesis, we have by Definition 16.9 that  $\sum_{n=1}^{\infty} |a_n|$  converges. Thus, by Theorem 16.5, there is some  $N \in \mathbb{N}$  such that  $|\sum_{k=m+1}^n |a_k|| < \epsilon$  for all  $n > m \geq N$ . Choose this  $N$  to be our  $N$ . Let  $n, m$  be arbitrary natural numbers such that  $n > m \geq N$ . Therefore,

$$\begin{aligned} \left| \sum_{k=m+1}^n a_k \right| &\leq \sum_{k=m+1}^n |a_k| && \text{Lemma 8.8} \\ &= \left| \sum_{k=m+1}^n |a_k| \right| \\ &< \epsilon \end{aligned}$$

as desired.

As to the other part of the claim, to begin, let  $(b_n)$  and  $(c_n)$  be defined by  $b_n = \max(0, a_n)$  and  $c_n = \min(0, a_n)$ . We will prove a few preliminary results with these definitions that will enable us to tackle the big inequality.

To confirm that  $a_n = b_n + c_n$ , we divide into two cases ( $a_n \geq 0$  and  $a_n < 0$ ). If  $a_n \geq 0$ , then by their definitions,  $b_n = a_n$  and  $c_n = 0$ . Thus,  $a_n = b_n + c_n$  as desired. The argument is symmetric in the other case.

To confirm that  $|\sum_{n=1}^{\infty} b_n| + |\sum_{n=1}^{\infty} -c_n| = |\sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} -c_n|$ , we can acknowledge that  $b_n \geq 0$  and  $-c_n \geq 0$  for all  $n \in \mathbb{N}$  to demonstrate that

$$\begin{aligned} \left| \sum_{n=1}^{\infty} b_n \right| + \left| \sum_{n=1}^{\infty} -c_n \right| &= \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} -c_n \\ &= \left| \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} -c_n \right| \end{aligned}$$

To confirm that  $|a_n| = b_n - c_n$ , we divide into two cases ( $a_n \geq 0$  and  $a_n < 0$ ). If  $a_n \geq 0$ , then  $b_n = a_n$  and  $c_n = 0$ . Thus, by Definition 8.4,  $|a_n| = a_n = b_n - c_n$ , as desired. On the other hand, if  $a_n < 0$ , then  $b_n = 0$  and  $c_n = a_n$ . Thus, by Definition 8.4 again,  $|a_n| = -a_n = b_n - c_n$ , as desired.

Having established that  $a_n = b_n + c_n$ ,  $|\sum_{n=1}^{\infty} b_n| + |\sum_{n=1}^{\infty} -c_n| = |\sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} -c_n|$ , and  $|a_n| =$

$b_n - c_n$ , we have that

$$\begin{aligned}
 \left| \sum_{n=1}^{\infty} a_n \right| &= \left| \sum_{n=1}^{\infty} (b_n + c_n) \right| && \text{Theorem 16.8} \\
 &= \left| \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} c_n \right| \\
 &\leq \left| \sum_{n=1}^{\infty} b_n \right| + \left| \sum_{n=1}^{\infty} c_n \right| && \text{Lemma 8.8} \\
 &= \left| \sum_{n=1}^{\infty} b_n \right| + \left| - \sum_{n=1}^{\infty} c_n \right| && \text{Exercise 8.5} \\
 &= \left| \sum_{n=1}^{\infty} b_n \right| + \left| \sum_{n=1}^{\infty} -c_n \right| && \text{Theorem 16.8} \\
 &= \left| \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} -c_n \right| \\
 &= \left| \sum_{n=1}^{\infty} (b_n - c_n) \right| && \text{Theorem 16.8} \\
 &= \left| \sum_{n=1}^{\infty} |a_n| \right| \\
 &= \sum_{n=1}^{\infty} |a_n|
 \end{aligned}$$

as desired.  $\square$

**Theorem 16.12.** Let  $(a_n)$  be a decreasing sequence of positive numbers such that  $\lim_{n \rightarrow \infty} a_n = 0$ . Then  $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$  converges.

**Lemma.**

(a) For any natural numbers  $n, m$  satisfying  $n > m$ , we have

$$\left| \sum_{k=m+1}^n (-1)^{k+1} a_k \right| = |a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + a_{m+5} - \cdots \pm a_n|$$

*Proof.* By Script 0, either  $|\sum_{k=m+1}^n (-1)^{k+1} a_k| = |a_{m+1} - a_{m+2} + a_{m+3} - \cdots \pm a_n|$  or  $|\sum_{k=m+1}^n (-1)^{k+1} a_k| = |-a_{m+1} + a_{m+2} - a_{m+3} + \cdots \pm a_n|$ . However, by Exercise 8.5, the two results are equal. Thus, we may choose the former WLOG.  $\square$

(b) We have

$$0 \leq a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots \pm a_n$$

*Proof.* Since  $(a_n)$  is a decreasing sequence, we have by Script 15 that  $a_{i+1} \leq a_i$  for all  $i \in \mathbb{N}$ . It follows by Definition 7.21 that  $0 \leq a_i - a_{i+1}$  for all  $i \in \mathbb{N}$ . We now divide into two cases (there are an even number of terms in the sum  $a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + a_{m+5} - \cdots \pm a_n$  and there are an odd number of terms in said sum). In the first case, we have that the sum is of the form  $a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots + a_{n-1} - a_n$ . Thus, since  $0 \leq a_{m+1} - a_{m+2}$ ,  $0 \leq a_{m+3} - a_{m+4}$ , and on and on, we have by Script 7 that  $0 \leq a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots - a_n$ , as desired. On the other hand, in the second case, we have that the sum is of the form  $a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots + a_{n-2} - a_{n-1} + a_n$ . For the same reason as before, we have that  $0 \leq a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots + a_{n-2} - a_{n-1}$ . However, we need the additional hypothesis that every  $a_i$  is positive, i.e.,  $a_i \geq 0$  for all  $i \in \mathbb{N}$  to know that  $0 \leq a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + \cdots + a_n$ , as desired.  $\square$

(c) For all  $i \in \mathbb{N}$ , we have

$$-a_i + a_{i+1} \leq 0$$

*Proof.* Since  $(a_n)$  is a decreasing sequence, we have by Script 15 that  $a_{i+1} \leq a_i$  for all  $i \in \mathbb{N}$ . It follows by Definition 7.21 that  $-a_i + a_{i+1} \leq 0$  for all  $i \in \mathbb{N}$ .  $\square$

*Proof of Theorem 16.12.* To prove that  $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$  converges, Theorem 16.5 tells us that it will suffice to show that for all  $\epsilon > 0$ , there is some  $N \in \mathbb{N}$  such that  $|\sum_{k=m+1}^n (-1)^{k+1} a_k| < \epsilon$  for all  $n > m \geq N$ . Let  $\epsilon > 0$  be arbitrary. Since  $\lim_{n \rightarrow \infty} a_n = 0$  by hypothesis, we have by Theorem 15.7 that there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$ , we have  $|a_n - 0| = |a_n| < \epsilon$ , as desired. Choose this  $N$  to be our  $N$ . Let  $n, m$  be arbitrary natural numbers such that  $n > m \geq N$ .

We divide into two cases ( $n-m$  is even [i.e., there are an even number of terms in the sum  $\sum_{k=m+1}^n (-1)^{k+1} a_k$ ] and  $n-m$  is odd [i.e., there are an odd number of terms in the sum  $\sum_{k=m+1}^n (-1)^{k+1} a_k$ ]). If  $n-m$  is even, then we have

$$\begin{aligned} \left| \sum_{k=m+1}^n (-1)^{k+1} a_k \right| &= |a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + a_{m+5} - \cdots - a_{n-2} + a_{n-1} - a_n| && \text{Lemma (a)} \\ &= a_{m+1} - a_{m+2} + a_{m+3} - a_{m+4} + a_{m+5} - \cdots - a_{n-2} + a_{n-1} - a_n && \text{Lemma (b) \& Definition 8.4} \\ &= a_{m+1} + (-a_{m+2} + a_{m+3}) + (-a_{m+4} + a_{m+5}) + \cdots + (-a_{n-2} + a_{n-1}) - a_n \\ &\leq a_{m+1} + (-a_{m+4} + a_{m+5}) + \cdots + (-a_{n-2} + a_{n-1}) - a_n && \text{Lemma (c)} \\ &\leq a_{m+1} + \cdots + (-a_{n-2} + a_{n-1}) - a_n && \text{Lemma (c)} \\ &\vdots && \text{Lemma (c)} \\ &\leq a_{m+1} + (-a_{n-2} + a_{n-1}) - a_n && \text{Lemma (c)} \\ &\leq a_{m+1} - a_n && \text{Lemma (c)} \\ &\leq a_{m+1} \\ &= |a_{m+1}| \\ &< \epsilon \end{aligned}$$

The argument is symmetric if  $n-m$  is odd.  $\square$

5/27: The following theorem will be useful to prove more specialized tests for convergence of series.

**Theorem 16.13.** Let  $(c_n)$  be a sequence of positive numbers and let  $(a_n)$  be a sequence such that  $|a_n| \leq c_n$  for all  $n \geq N_0$ , where  $N_0$  is some fixed integer. If  $\sum_{n=1}^{\infty} c_n$  converges, then  $\sum_{n=1}^{\infty} a_n$  converges.

*Proof.* To prove that  $\sum_{n=1}^{\infty} a_n$  converges, Theorem 16.11 and Definition 16.9 tell us that it will suffice to show that  $\sum_{n=1}^{\infty} |a_n|$  converges. To do this, Theorem 16.5 tells us that it will suffice to verify that for all  $\epsilon > 0$ , there exists an  $N \in \mathbb{N}$  such that  $|\sum_{k=m+1}^n |a_k|| < \epsilon$  for all  $n > m \geq N$ . Let  $\epsilon > 0$  be arbitrary. Since  $\sum_{n=1}^{\infty} c_n$  converges by hypothesis, we have by Theorem 16.5 that there is some  $N_1 \in \mathbb{N}$  such that  $|\sum_{k=m+1}^n c_k| < \epsilon$  for all  $n > m \geq N_1$ . Choose  $N = \max(N_0, N_1)$ . Let  $n, m$  be arbitrary natural numbers such that  $n > m \geq N$ . Since  $c_k \geq 0$  for all  $k \in \mathbb{N}$  by hypothesis, it follows by Script 7 that  $\sum_{k=m+1}^n c_k \geq 0$ . Thus, Definition 8.4 implies that  $\sum_{k=m+1}^n c_k = |\sum_{k=m+1}^n c_k| < \epsilon$ . Similarly, we have that  $|\sum_{k=m+1}^n |a_k|| = \sum_{k=m+1}^n |a_k|$ . Lastly, since we know by hypothesis that  $|a_k| \leq c_k$  for all  $k \geq N_0$ , i.e., for all  $k \geq N$ , Script 7 asserts that  $\sum_{k=m+1}^n |a_k| \leq \sum_{k=m+1}^n c_n$ . Therefore, combining the last three results, we have that  $|\sum_{k=m+1}^n |a_k|| = \sum_{k=m+1}^n |a_k| \leq \sum_{k=m+1}^n c_n < \epsilon$ , as desired.  $\square$

**Lemma 16.14.** Suppose that  $(b_n)$  is a sequence of nonnegative numbers with  $\lim_{n \rightarrow \infty} b_n = L$ , where  $L < 1$ . Then there is some  $N \in \mathbb{N}$  such that  $0 \leq b_n < \frac{1+L}{2}$  for all  $n \geq N$ .

*Proof.* Choose  $\epsilon = \frac{1-L}{2}$ ; since  $L < 1$ , we have that  $0 < 1-L$ , i.e.,  $0 < \frac{1-L}{2}$  as needed. It follows by Theorem 15.7 (since  $\lim_{n \rightarrow \infty} b_n = L$  by hypothesis) that there is some  $N \in \mathbb{N}$  such that  $|b_n - L| < \frac{1-L}{2}$  for all  $n \geq N$ . Choose this  $n$  to be our  $N$ . Let  $n$  be an arbitrary natural number such that  $n \geq N$ . Then

$$\begin{aligned} |b_n - L| &< \frac{1-L}{2} \\ -\frac{1-L}{2} &< b_n - L < \frac{1-L}{2} \\ b_n &< \frac{1+L}{2} \end{aligned} \quad \text{Lemma, Exercise 8.9}$$

Additionally, since  $(b_n)$  is a sequence of nonnegative numbers,  $0 \leq b_n$ . Therefore, combining the last two results, we have that  $0 \leq b_n < \frac{1+L}{2}$ , as desired.  $\square$

**Theorem 16.15.** Let  $(a_n)$  be a sequence such that  $\lim_{n \rightarrow \infty} |\frac{a_{n+1}}{a_n}|$  exists. Then

(a) If  $\lim_{n \rightarrow \infty} |\frac{a_{n+1}}{a_n}| < 1$ , then  $\sum_{n=1}^{\infty} a_n$  converges.

*Proof.* Let  $\lim_{n \rightarrow \infty} |\frac{a_{n+1}}{a_n}| = L$ . To prove that  $\sum_{n=1}^{\infty} a_n$  converges, Theorem 16.13 tells us that it will suffice to find a sequence  $(c_n)$  of positive numbers such that  $|a_n| \leq c_n$  for all  $n \geq N_0$ , where  $N_0$  is some fixed integer, for which  $\sum_{n=1}^{\infty} c_n$  converges. To begin, since  $(|\frac{a_{n+1}}{a_n}|)$  is a sequence of nonnegative numbers with  $\lim_{n \rightarrow \infty} |\frac{a_{n+1}}{a_n}| = L$ , where  $L < 1$ , we have by Lemma 16.14 that there exists an  $N_0 \in \mathbb{N}$  such that  $0 \leq |\frac{a_{n+1}}{a_n}| < \frac{1+L}{2}$  for all  $n \geq N_0$ . With this result, we can define  $(c_n)$  by  $c_n = (\frac{1+L}{2})^{n-N_0} \cdot |a_{N_0}|$  for all  $n \in \mathbb{N}$ . We will now prove that  $(c_n)$  satisfies the necessary properties outlined in the beginning.

First, we must confirm that  $(c_n)$  is a sequence of positive numbers, i.e., that  $c_n \geq 0$  for all  $n \in \mathbb{N}$ . Let  $n$  be an arbitrary natural number. By the above result from Lemma 16.14 and transitivity, we know that  $0 < \frac{1+L}{2}$ . It follows by Script 7 that  $0 < (\frac{1+L}{2})^{n-N_0}$ . Therefore, by Definition 8.4 and Script 7,  $0 \leq (\frac{1+L}{2})^{n-N_0} \cdot |a_{N_0}| = c_n$ , as desired.

To confirm that  $|a_n| \leq c_n$  for all  $n \geq N_0$ , we induct on  $n$  using Additional Exercise 0.2a. For the base case  $n = N_0$ , we have that

$$|a_{N_0}| = 1 \cdot |a_{N_0}| = \left(\frac{1+L}{2}\right)^{N_0-N_0} \cdot |a_{N_0}| = c_{N_0}$$

which we may weaken to  $|a_{N_0}| \leq c_{N_0}$ , as desired. Now suppose inductively that we have demonstrated that  $|a_n| \leq c_n$ ; we now seek to demonstrate that  $|a_{n+1}| \leq c_{n+1}$ . By hypothesis,  $n \geq N_0$ , so by the above, we have that  $0 \leq |\frac{a_{n+1}}{a_n}| < \frac{1+L}{2}$ . It follows by Script 7 that  $|a_{n+1}| < \frac{1+L}{2} \cdot |a_n|$ . Therefore,

$$\begin{aligned} |a_{n+1}| &< \frac{1+L}{2} \cdot |a_n| \\ &\leq \frac{1+L}{2} \cdot c_n \\ &= \frac{1+L}{2} \cdot \left(\frac{1+L}{2}\right)^{n-N_0} \cdot |a_{N_0}| \\ &= \left(\frac{1+L}{2}\right)^{(n+1)-N_0} \cdot |a_{N_0}| \\ &= c_{n+1} \end{aligned}$$

which we may weaken to  $|a_{n+1}| \leq c_{n+1}$ , as desired.

Lastly, we must confirm that  $\sum_{n=1}^{\infty} c_n$  converges. By Script 7, it follows from the hypothesis that  $L < 1$  that  $1+L < 2$ , which in turn implies that  $\frac{1+L}{2} < 1$ . Additionally, the above result that  $0 < \frac{1+L}{2}$  implies by transitivity that  $-1 < \frac{1+L}{2}$ . These last two results when combined imply



$\sum_{n=0}^{\infty} (\frac{1+L}{2})^n$  satisfies the constraints of Theorem 16.7, meaning that  $\sum_{n=0}^{\infty} (\frac{1+L}{2})^n$  converges. Thus, by Script 0,  $\sum_{n=N_0}^{\infty} (\frac{1+L}{2})^{n-N_0}$  converges. Consequently, by consecutive applications of Lemma 16.2,  $\sum_{n=1}^{\infty} (\frac{1+L}{2})^{n-N_0}$  converges. It follows by Theorem 16.8 that  $\sum_{n=1}^{\infty} (\frac{1+L}{2})^{n-N_0} \cdot |a_{N_0}|$  converges. Therefore, by the definition of  $c_n$ ,  $\sum_{n=1}^{\infty} c_n$  converges, as desired.  $\square$

(b) If  $\lim_{n \rightarrow \infty} |\frac{a_{n+1}}{a_n}| > 1$ , then  $\sum_{n=1}^{\infty} a_n$  diverges.

*Proof.* Let  $\lim_{n \rightarrow \infty} |\frac{a_{n+1}}{a_n}| = L$ . Suppose for the sake of contradiction that  $\lim_{n \rightarrow \infty} a_n = 0$ . Then by Theorem 15.7, for all  $\epsilon_1 > 0$ , there is some  $N_1 \in \mathbb{N}$  such that for all  $n \geq N_1$ , we have  $|a_n| = |a_n - 0| < \epsilon_1$ . Similarly, since  $\lim_{n \rightarrow \infty} |\frac{a_{n+1}}{a_n}| = L$ , we have that for all  $\epsilon_2 > 0$ , there is some  $N_2 \in \mathbb{N}$  such that for all  $n \geq N_2$ , we have  $||\frac{a_{n+1}}{a_n}| - L| < \epsilon_2$ . Choose  $\epsilon_2 = L - 1$  (it follows from the fact that  $L > 1$  by Definition 7.21 that  $L - 1 > 0$ ). Thus, we have that for all  $n \geq N_2$ ,

$$\begin{aligned} 1 &= 1 + L - L + \left| \frac{a_{n+1}}{a_n} \right| - \left| \frac{a_{n+1}}{a_n} \right| \\ &= |L| - \left| \frac{a_{n+1}}{a_n} \right| + 1 - L + \left| \frac{a_{n+1}}{a_n} \right| \\ &\leq \left| L - \left| \frac{a_{n+1}}{a_n} \right| \right| + 1 - L + \left| \frac{a_{n+1}}{a_n} \right| \\ &= \left| \left| \frac{a_{n+1}}{a_n} \right| - L \right| + 1 - L + \left| \frac{a_{n+1}}{a_n} \right| \\ &< L - 1 + 1 - L + \left| \frac{a_{n+1}}{a_n} \right| \\ &= \left| \frac{a_{n+1}}{a_n} \right| \end{aligned} \quad \text{Exercise 8.5}$$

which can be rearranged by Lemma 7.24 to demonstrate that  $|a_n| < |a_{n+1}|$  for all such  $n$ . Now consider the case where  $\epsilon_1 = |a_{N_2+1}|$  (note that  $|a_{N_2+1}| > |a_{N_2}| \geq 0$  by Definition 8.4). Choose  $N = \max(N_1, N_2 + 2)$ . Then by the above, we have by transitivity that  $|a_N| > |a_{N-1}| > \dots > |a_{N_2+1}|$ . However, since  $N \geq N_1$ , we also have that  $|a_N| < \epsilon_1 = |a_{N_2+1}|$ , a contradiction. Therefore, since  $\lim_{n \rightarrow \infty} a_n \neq 0$ , we have by the contrapositive of Theorem 16.4 that  $\sum_{n=1}^{\infty} a_n$  diverges.  $\square$

**Theorem 16.16.** Let  $(a_n)$  be a sequence such that  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}$  exists. Then

(a) If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} < 1$ , then  $\sum_{n=1}^{\infty} a_n$  converges.

*Proof.* Let  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L$ . To prove that  $\sum_{n=1}^{\infty} a_n$  converges, Theorem 16.13 tells us that that it will suffice to find a sequence  $(c_n)$  of positive numbers such that  $|a_n| \leq c_n$  for all  $n \geq N_0$ , where  $N_0$  is some fixed integer, for which  $\sum_{n=1}^{\infty} c_n$  converges. Since  $L < 1$  by hypothesis and  $0 < 1$  by Corollary 7.27, Theorem 5.2 asserts that there exists a point  $x \in \mathbb{R}$  such that  $\max(0, L) < x < 1$ . We now define  $(c_n)$  by  $c_n = x^n$  for all  $n \in \mathbb{N}$ . By Script 7 and the fact that  $x \geq 0$ , we know that  $(c_n)$  is a sequence of positive numbers. Additionally, since  $x - L > 0$ , Theorem 15.7 implies that there is some  $N_0 \in \mathbb{N}$  such that for all  $n \geq N_0$ ,  $|\sqrt[n]{|a_n|} - L| < x - L$ . It follows by Script 7 that  $|a_n| \leq x^n = c_n$  for all  $n \geq N_0$ . Lastly, since  $-1 < \max(0, L) < x < 1$ , we have by Theorem 16.7 that  $\sum_{n=1}^{\infty} c_n$  converges, as desired.  $\square$

(b) If  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} > 1$ , then  $\sum_{n=1}^{\infty} a_n$  diverges.

*Proof.* Let  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L$ . Suppose for the sake of contradiction that  $\lim_{n \rightarrow \infty} a_n = 0$ . Then by Theorem 15.7, for all  $\epsilon_1 > 0$ , there is some  $N_1 \in \mathbb{N}$  such that for all  $n \geq N_1$ , we have  $|a_n| = |a_n - 0| < \epsilon_1$ . Similarly, since  $\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L$ , we have that for all  $\epsilon_2 > 0$ , there is some  $N_2 \in \mathbb{N}$  such that for all  $n \geq N_2$ , we have  $|\sqrt[n]{|a_n|} - L| < \epsilon_2$ . Choose  $\epsilon_2 = L - 1$ . Thus, we have that for all  $n \geq N_2$ ,  $1 < \sqrt[n]{|a_n|}$

(by an argument symmetric to that given in the proof of Theorem 16.15b) which can be rearranged by Script 7 to demonstrate that  $|a_n| > 1^n = 1$  for all such  $n$ . Now consider the case where  $\epsilon_1 = 1$ . Choose  $N = \max(N_1, N_2)$ . Then by the above, the fact that  $N \geq N_2$  implies that  $|a_N| > 1$ . However, since  $N \geq N_1$ , we also have that  $|a_N| < \epsilon_1 = 1$ , a contradiction. Therefore, since  $\lim_{n \rightarrow \infty} a_n \neq 0$ , we have by the contrapositive of Theorem 16.4 that  $\sum_{n=1}^{\infty} a_n$  diverges.  $\square$