Script 17

Sequences and Series of Functions

6/23: **Definition 17.1.** Let $A \subset \mathbb{R}$, and consider $X = \{f : A \to \mathbb{R}\}$, the collection of real-valued functions on A. A **sequence of functions** (on A) is an ordered list $(f_1, f_2, f_3, \dots)$ which we will denote (f_n), where each $f_n \in X$. (More formally, we can think of the sequence as a function $F : \mathbb{N} \to X$, where $f_n = F(n)$, for each $n \in \mathbb{N}$, but this degree of formality is not particularly helpful.)

We can take the sequence to start at any $n_0 \in \mathbb{Z}$ and not just at 1, just like we did for sequences of real numbers.

Definition 17.2. The sequence (f_n) **converges pointwise** to a function $f : A \to \mathbb{R}$ if for all $x \in A$ and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$. In other words, we have that for all $x \in A$, $\lim_{n\to\infty} f_n(x) = f(x)$.

Definition 17.3. The sequence (f_n) **converges uniformly** to a function $f : A \to \mathbb{R}$ if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$ for every $x \in A$.

Equivalently, the sequence (f_n) **converges uniformly** to a function $f : A \to \mathbb{R}$ if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $\sup_{x\in A} |f_n(x) - f(x)| < \epsilon$.

Exercise 17.4. Suppose that a sequence (f_n) converges pointwise to a function f. Prove that if (f_n) converges uniformly to a function g, then $f = g$.

Proof. To prove that $f = g$, Definition 1.16 tells us that it will suffice to show that $f(x) = g(x)$ for all $x \in A$. Suppose for the sake of contradiction that $f(x) \neq g(x)$ for some $x \in A$. Since (f_n) converges pointwise to f by hypothesis, Definition 17.2 implies that for all $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that if $n \geq N_1$, then $|f_n(x) - f(x)| < \epsilon$. Additionally, since (f_n) converges uniformly to g by hypothesis, Definition 17.3 asserts that for all $\epsilon > 0$, there exists $N_2 \in \mathbb{N}$ such that if $n \geq N_2$, then $|f_n(x) - g(x)| < \epsilon$.

WLOG, let $f(x) > g(x)$. Choose $\epsilon = \frac{f(x)-g(x)}{2}$, and let $N = \max(N_1, N_2)$. Since $N \geq N_1$, $|f_N(x) - f(x)| < \frac{f(x)-g(x)}{2}$. Similarly, $|f_N(x) - g(x)| < \frac{f(x)-g(x)}{2}$. But this implies that

$$
\begin{aligned}
f(x) - g(x) &= |f(x) - f_N(x) + f_N(x) - g(x)| \\
&\leq |f(x) - f_N(x)| + |f_N(x) - g(x)| && \text{Lemma 8.8} \\
&= |f_N(x) - f(x)| + |f_N(x) - g(x)| && \text{Exercise 8.5} \\
&< \frac{f(x) - g(x)}{2} + \frac{f(x) - g(x)}{2} \\
&= f(x) - g(x)
\end{aligned}
$$

a contradiction. $\square$

Exercise 17.5. For each of the following sequences of functions, determine what function the sequence (f_n) converges to pointwise. Does the sequence converge uniformly to this function?

(a) For $n \in \mathbb{N}$, let $f_n : [0, 1] \to \mathbb{R}$ be given by $f_n(x) = x^n$.

Answer. The sequence (f_n) converges pointwise, but not uniformly, to the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & x < 1 \\ 1 & x = 1 \end{cases}$$

$\square$

Proof. To prove that (f_n) converges pointwise to f, Definition 17.2 tells us that it will suffice to show that for all $x \in [0,1]$, $\lim_{n\to\infty} f_n(x) = f(x)$. We divide into two cases ($x \in [0,1)$ and $x = 1$). If $x \in [0,1)$, then by Script 8, $|x| < 1$. Therefore,

$$\begin{aligned} \lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty} x^n \\ &= 0 && \text{Exercise 15.8b} \\ &= f(x) \end{aligned}$$

as desired. On the other hand, if $x = 1$, then

$$\begin{aligned} \lim_{n\to\infty} f_n(1) &= \lim_{n\to\infty} 1^n \\ &= \lim_{n\to\infty} 1 \\ &= 1 && \text{Exercise 15.6a} \\ &= f(1) \end{aligned}$$

as desired.

Suppose for the sake of contradiction that (f_n) converges uniformly to f. Then by Definition 17.3, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \frac{1}{2}$ for all $x \in [0,1]$. Let $x = \frac{1}{2^{1/N}}$. We have that $0 < \frac{1}{2^{1/N}}$ and that

$$\begin{aligned} \frac{1}{2^{1/N}} &= \frac{1^{1/N}}{2^{1/N}} \\ &< \frac{2^{1/N}}{2^{1/N}} \\ &= 1 \end{aligned}$$

for all $N \in \mathbb{N}$, so this is an acceptable x. However, we have that

$$\begin{aligned} |f_N(x) - f(x)| &= \left| \left(\frac{1}{2^{1/N}} \right)^N - 0 \right| \\ &= \frac{1}{2} \end{aligned}$$

a contradiction[1]. $\square$

(b) For $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be given by $f_n(x) = \frac{\sin(nx)}{n}$. (For the purposes of this example, you may assume basic knowledge of sine.)

Answer. The sequence (f_n) converges uniformly to the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 0$. $\square$

[1]Note that as an alternative to this second contradiction argument, we can prove that (f_n) does not converge uniformly to f via the contrapositive of Theorem 17.6. Indeed, since f has a discontinuity at 1 while each f_n is continuous by Corollary 11.12, the contrapositive of Theorem 17.6 implies that (f_n) cannot converge uniformly to f.

Proof. To prove that (f_n) converges uniformly to f, Definition 17.3 tells us that it will suffice to show that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$ for all $x \in \mathbb{R}$. Let $\epsilon > 0$ be arbitrary. By Exercise 15.6c and Theorem 15.7, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|\frac{1}{n} - 0| < \epsilon$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. Before constructing the main inequality, recall from our basic knowledge of sine that $|\sin(nx)| \leq 1$ for all $x \in \mathbb{R}$. Therefore,

$$\begin{aligned} |f_n(x) - f(x)| &= \left|\frac{\sin(nx)}{n} - 0\right| \\ &\leq \left|\frac{1}{n}\right| \\ &< \epsilon \end{aligned}$$

as desired. ☐

(c) For $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be given by

$$f_n(x) = \begin{cases} n^2 x & 0 \leq x \leq \frac{1}{n} \\ n(2 - nx) & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \frac{2}{n} \leq x \leq 1 \end{cases}$$

Answer. The sequence (f_n) converges pointwise, but not uniformly, to the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = 0$. ☐

Proof. To prove that (f_n) converges pointwise to f, Definition 17.2 tells us that it will suffice to show that for all $x \in [0,1]$ and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. We divide into two cases ($x = 0$ and $x \in (0,1]$). Suppose first that $x = 0$. Choose $N = 1$. Let n be an arbitrary natural number such that $n \geq N$. Therefore,

$$\begin{aligned} |f_n(0) - f(0)| &= |n^2 \cdot 0 - 0| \\ &= 0 \\ &< \epsilon \end{aligned}$$

as desired. Now suppose that $x \in (0,1]$. Choose $N = \frac{2}{x}$. Let n be an arbitrary natural number such that $n \geq N$. It follows that

$$\begin{aligned} \frac{2}{n} &\leq \frac{2}{N} \\ &= x \end{aligned}$$

Therefore,

$$\begin{aligned} |f_n(x) - f(x)| &= |0 - 0| \\ &= 0 \\ &< \epsilon \end{aligned}$$

as desired.

Suppose for the sake of contradiction that (f_n) converges uniformly to f. Then by Definition 17.3, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| < 1$ for all $x \in [0,1]$. Let $x = \frac{1}{N}$. By Script 7, $0 \leq \frac{1}{N} \leq 1$ for all $N \in \mathbb{N}$, so this is an acceptable x. However, we have that

$$\begin{aligned} |f_N(x) - f(x)| &= |N^2 \cdot \frac{1}{N} - 0| \\ &= N \\ &\geq 1 \end{aligned}$$

a contradiction. ☐

Theorem 17.6 (Uniform Limit Theorem)**.** *Let (f_n) be a sequence of functions, and suppose that each $f_n : A \to \mathbb{R}$ is continuous. If (f_n) converges uniformly to $f : A \to \mathbb{R}$, then f is continuous.*

Proof. To prove that f is continuous, Theorem 9.10 tells us that it will suffice to show that f is continuous at every $x \in A$. Let x be an arbitrary element of A. To show that f is continuous at x, Theorem 11.5 tells us that it will suffice to verify that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary, and also let y be an arbitrary element of A. Since (f_n) converges uniformly, Definition 17.3 implies that there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(a) - f(a)| < \frac{\epsilon}{3}$ for all $a \in A$. Thus, $|f_N(x) - f(x)| < \frac{\epsilon}{3}$ and $|f_N(y) - f(y)| < \frac{\epsilon}{3}$. Additionally, since each f_n is continuous, Theorems 9.10 and 11.5 assert that there exists $\delta > 0$ such that if $y \in A$ and $|y - x| < \delta$, then $|f_N(y) - f_N(x)| < \frac{\epsilon}{3}$. Choose this δ to be our δ. Therefore,

$$\begin{aligned}
|f(y) - f(x)| &\leq |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| && \text{Lemma 8.8} \\
&= |f_N(y) - f(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| && \text{Exercise 8.5} \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon
\end{aligned}$$

as desired. ☐

6/26: **Theorem 17.7.** *Suppose that (f_n) is a sequence of integrable functions on $[a, b]$ and suppose that (f_n) converges uniformly to $f : [a, b] \to \mathbb{R}$. Then*

$$\int_a^b f = \lim_{n \to \infty} \int_a^b f_n$$

Lemma. *f is integrable on $[a, b]$.*

Proof. To prove that f is integrable on $[a, b]$, Theorem 13.18 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since (f_n) converges uniformly to f by hypothesis, Definition 17.3 asserts that there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|f_n(x) - f(x)| < \frac{\epsilon}{6(b-a)}$[2] for all $x \in [a, b]$. This statement will be useful in the verification of the three following results.

To confirm that $|U(f_N, P) - L(f_N, P)| < \frac{\epsilon}{3}$, we first invoke Theorem 13.18, which tells us that since f_N is integrable by hypothesis, there exists a partition P of $[a, b]$ such that $U(f_N, P) - L(f_N, P) < \frac{\epsilon}{3}$. Additionally, since $L(f_N, P) \leq U(f_N, P)$ by Theorem 13.13, we have by Definition 8.4 that $U(f_N, P) - L(f_N, P) = |U(f_N, P) - L(f_N, P)|$. Therefore, we have by transitivity that $|U(f_N, P) - L(f_N, P)| = U(f_N, P) - L(f_N, P) < \frac{\epsilon}{3}$, as desired.

To confirm that $|U(f, P) - U(f_N, P)| < \frac{\epsilon}{3}$, we begin with the following contradiction argument[3].

Suppose for the sake of contradiction that $|M_i(f) - M_i(f_N)| \geq \frac{\epsilon}{3(b-a)}$. We divide into two cases ($M_i(f) - M_i(f_N) \geq \frac{\epsilon}{3(b-a)}$ and $M_i(f_N) - M_i(f) \geq \frac{\epsilon}{3(b-a)}$). Suppose first that $M_i(f) - M_i(f_N) \geq \frac{\epsilon}{3(b-a)}$. By Lemma 5.11, there exists $f(x) \in \{f(x) \mid t_{i-1} \leq x \leq t_i\}$ such that $M_i(f) - \frac{\epsilon}{6(b-a)} < f(x) \leq M_i(f)$. Similarly, there exists $f_N(x) \in \{f_N(x) \mid t_{i-1} \leq x \leq t_i\}$ such that $M_i(f_N) - \frac{\epsilon}{6(b-a)} < f_N(x) \leq M_i(f_N)$. Thus, we have that

$$f(x) > M_i(f) - \frac{\epsilon}{6(b-a)} > M_i(f) - \frac{\epsilon}{3(b-a)} \geq M_i(f_N) \geq f_N(x)$$

[2]For the purposes of this proof, we will assume that $a < b$, on the basis of the fact that the proof of the case where $a = b$ is trivial.

[3]Note that this argument is analogous to the proof of Additional Exercise 13.2.

It follows that

$$\begin{aligned}
|f(x) - f_N(x)| &= f(x) - f_N(x) \\
&> \left(M_i(f) - \frac{\epsilon}{6(b-a)}\right) - f_N(x) \\
&\geq \left(M_i(f) - \frac{\epsilon}{6(b-a)}\right) - M_i(f_N) \\
&= M_i(f) - M_i(f_N) - \frac{\epsilon}{6(b-a)} \\
&\geq \frac{\epsilon}{3(b-a)} - \frac{\epsilon}{6(b-a)} \\
&= \frac{\epsilon}{6(b-a)}
\end{aligned}$$

But this contradicts the previously proven fact that $|f(x)-f_N(x)| = |f_N(x)-f(x)| < \frac{\epsilon}{6(b-a)}$. The argument is symmetric in the other case.

Thus, we know that $|M_i(f) - M_i(f_N)| < \frac{\epsilon}{3(b-a)}$. Therefore, we have that

$$\begin{aligned}
|U(f,P) - U(f_N,P)| &= \left|\sum_{i=1}^{k} M_i(f)(t_i - t_{i-1}) - \sum_{i=1}^{k} M_i(f_N)(t_i - t_{i-1})\right| && \text{Definition 13.10} \\
&= \left|\sum_{i=1}^{k} (M_i(f) - M_i(f_N)(t_i - t_{i-1})\right| \\
&< \left|\sum_{i=1}^{k} \frac{\epsilon}{3(b-a)}(t_i - t_{i-1})\right| \\
&= \frac{\epsilon}{3(b-a)}(b-a) \\
&= \frac{\epsilon}{3}
\end{aligned}$$

as desired.

The verification of the statement that $|L(f_N,P) - L(f,P)| < \frac{\epsilon}{3}$ is symmetric to the previous argument.

Having established that $|U(f_N,P)-L(f_N,P)| < \frac{\epsilon}{3}$, $|U(f,P)-U(f_N,P)| < \frac{\epsilon}{3}$, and $|L(f_N,P)-L(f,P)| < \frac{\epsilon}{3}$, we can now show that

$$\begin{aligned}
U(f,P) - L(f,P) &= |U(f,P) - L(f,P)| && \text{Theorem 13.13} \\
&\leq |U(f,P) - U(f_N,P)| + |U(f_N,P) - L(f_N,P)| + |L(f_N,P) - L(f,P)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&= \epsilon
\end{aligned}$$

as desired. □

Proof of Theorem 17.7. To prove that $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$, Theorem 15.7 tells us that it will suffice to show that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|\int_a^b f_n - \int_a^b f| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since (f_n) converges uniformly to f, we have by Definition 17.3 that there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$ for all $x \in [a,b]$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$. It follows from the lemma to Exercise 8.9 that $-\frac{\epsilon}{b-a} < f_n(x) - f(x) < \frac{\epsilon}{b-a}$ for all $x \in [a,b]$. Additionally, since f_n is integrable on $[a,b]$ by hypothesis and f is integrable on $[a,b]$ by the lemma, Theorem 13.24 implies that $f_n - f$ is integrable on $[a,b]$. Combining these last two results, we have by Theorem 13.27 that $-\frac{\epsilon}{b-a}(b-a) < \int_a^b (f_n - f) < \frac{\epsilon}{b-a}(b-a)$. Consequently, by Script 7 and the lemma to Exercise 8.9, we have that $|\int_a^b (f_n - f)| < \epsilon$. Therefore, by Theorem 13.24, we have that $|\int_a^b f_n - \int_a^b f| < \epsilon$, as desired. □

Theorem 17.8. *Let (f_n) be a sequence of functions defined on an open interval containing $[a,b]$ such that each f_n is differentiable on $[a,b]$ and f_n' is integrable on $[a,b]$. Suppose further that (f_n) converges pointwise to f on $[a,b]$ and that (f_n') converges uniformly to a continuous function g on $[a,b]$. Then f is differentiable at every $x \in [a,b]$ and*

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

Proof. Let x be an arbitrary element of $[a,b]$. Since (f_n') converges uniformly to g, Definition 17.3 and Theorem 15.7 imply that $\lim_{n\to\infty} f_n'(x) = g(x)$. Additionally, we have that

$$\begin{aligned}
\int_a^x g &= \lim_{n\to\infty} \int_a^x f_n' && \text{Theorem 17.7}\\
&= \lim_{n\to\infty} (f_n(x) - f_n(a)) && \text{Theorem 14.4}\\
&= \lim_{n\to\infty} f_n(x) - \lim_{n\to\infty} f_n(a) && \text{Theorem 15.9}\\
&= f(x) - f(a) && \text{Definition 17.2}
\end{aligned}$$

This combined with the fact that g is continuous (hence continuous at x by Theorem 9.10) implies that

$$\begin{aligned}
g(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(f(x) - f(a)) && \text{Theorem 14.1}\\
&= \frac{\mathrm{d}}{\mathrm{d}x}(f(x)) - \frac{\mathrm{d}}{\mathrm{d}x}(f(a)) && \text{Exercise 12.9}\\
&= f'(x) && \text{Exercise 12.8}
\end{aligned}$$

Therefore, we have by transitivity that $f'(x) = \lim_{n\to\infty} f_n'(x)$, as desired. $\square$

Theorem 17.9 (Uniformly Cauchy Completeness Theorem)**.** *Let (f_n) be a sequence of functions defined on a set A. Then the following are equivalent.*

(*a*) *There is some function f such that (f_n) converges uniformly to f on A.*

(*b*) *For all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that when $m,n \geq N$, $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in A$.*

Proof. Suppose first that there is some function f to which (f_n) converges uniformly on A. Let $\epsilon > 0$ be arbitrary. By Definition 17.3, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ for all $x \in A$. Choose this N to be our N. Let n,m be arbitrary natural numbers such that $n,m \geq N$. Then $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ and $|f_m(x) - f(x)| < \frac{\epsilon}{2}$ for all $x \in A$. Therefore, we have that

$$\begin{aligned}
|f_n(x) - f_m(x)| &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| && \text{Lemma 8.8}\\
&= |f_n(x) - f(x)| + |f_m(x) - f(x)| && \text{Exercise 8.5}\\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2}\\
&= \epsilon
\end{aligned}$$

for all $x \in A$, as desired.

Now suppose that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that when $n,m \geq N$, $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in A$. It follows by Theorem 15.19 that $(f_n(x))$ converges for all $x \in A$, i.e., for all $x \in A$, there exists a point $f(x) \in \mathbb{R}$ to which $(f_n(x))$ converges. Let $f : A \to \mathbb{R}$ be defined by $f(x) = \lim_{n\to\infty} f_n(x)$.

To prove that (f_n) converges uniformly to f, Definition 17.3 tells us that it will suffice to show that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$ for all $x \in A$. Let $\epsilon > 0$ be arbitrary. By the hypothesis, there exists $N \in \mathbb{N}$ such that for all $n,m \geq N$, $|f_n(x) - f_m(x)| < \frac{\epsilon}{2}$. Choose this N to be our N. Let n be an arbitrary natural number such that $n \geq N$, and let x be an arbitrary element of A. Since $(f_m(x))$ converges to $f(x)$, Theorem 15.7 asserts that there exists an $N' \in \mathbb{N}$ such that for all $m \geq N'$,

$|f_m(x) - f(x)| < \frac{\epsilon}{2}$. Choose $M = \max(N, N')$. It follows that

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_M(x)| + |f_M(x) - f(x)| && \text{Lemma 8.8} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

as desired. $\square$

Definition 17.10. We define series of functions the same way we defined series of numbers. That is, given a sequence (f_n), define the sequence of partial sums (p_n) by $p_n(x) = f_1(x) + \cdots + f_n(x)$ and say that $\sum_{n=1}^{\infty} f_n$ converges pointwise or converges uniformly to f if the sequence (p_n) does.

Theorem 17.11 (Weierstrass M-Test)**.** *Suppose that $f_n : A \to \mathbb{R}$ is a sequence of functions and that there exists a sequence of positive real numbers (M_n) such that for all $x \in A$, we have $|f_n(x)| \leq M_n$. If $\sum_{n=1}^{\infty} M_n$ converges, then for each $x \in A$, the series of numbers $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely. Furthermore, $\sum_{n=1}^{\infty} f_n$ converges uniformly to the function f defined by $f(x) = \sum_{n=1}^{\infty} f_n(x)$.*

Proof. Let x be an arbitrary element of A. To prove that $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely, Definition 16.9 tells us that it will suffice to show that $\sum_{n=1}^{\infty} |f_n(x)|$ converges. Since (M_n) is a sequence of positive numbers and $|f_n(x)| \leq M_n$ for all $n \geq 1$, the proof of Theorem 16.13 asserts that $\sum_{n=1}^{\infty} |f_n(x)|$ converges.

To prove that $\sum_{n=1}^{\infty} f_n$ converges uniformly to f, Definition 17.10 tells us that it will suffice to show that the sequence of partial sums (p_n) defined by $p_k(x) = \sum_{n=1}^{k} f_n(x)$ converges uniformly to f. To do this, Definition 17.3 tells us that it will suffice to verify that for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that if $j \geq N$, then $|\sum_{n=1}^{j} f_n(x) - f(x)| < \epsilon$ for all $x \in A$. Let $\epsilon > 0$ be arbitrary. By Definition 16.1, $\sum_{n=1}^{\infty} M_n = \lim_{k\to\infty} \sum_{n=1}^{k} M_n$. Thus, by Theorem 15.7, there is some $N \in \mathbb{N}$ such that for all $j \geq N$, $|\sum_{n=1}^{j} M_n - \sum_{n=1}^{\infty} M_n| < \epsilon$. Choose this N to be our N. Let j be an arbitrary natural number such that $j \geq N$. It follows by Script 16 that $|\sum_{n=j+1}^{\infty} M_n| < \epsilon$. Additionally, since (M_n) is a sequence of positive numbers, $\sum_{n=j+1}^{\infty} M_n = |\sum_{n=j+1}^{\infty} M_n|$. Therefore, combining the last several results and letting x be an arbitrary element of A, we have that

$$\begin{aligned} \left|\sum_{n=1}^{j} f_n(x) - f(x)\right| &= \left|\sum_{n=1}^{j} f_n(x) - \sum_{n=1}^{\infty} f_n(x)\right| \\ &= \left|\sum_{n=j+1}^{\infty} f_n(x)\right| \\ &\leq \sum_{n=j+1}^{\infty} |f_n(x)| && \text{Theorem 16.11} \\ &\leq \sum_{n=j+1}^{\infty} M_n \\ &= \left|\sum_{n=j+1}^{\infty} M_n\right| \\ &< \epsilon \end{aligned}$$

as desired. $\square$

6/30: **Definition 17.12.** A function of the form $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$, where $c_n \in \mathbb{R}$ is called a **power series**. The power series is **centered** at a, and the numbers c_n are called the **coefficients**.

Theorem 17.13. *Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$ be a power series centered at 0. Suppose that x_0 is a real number such that the series $f(x_0) = \sum_{n=0}^{\infty} c_n x_0^n$ converges. Let r be any number such that $0 < r < |x_0|$. Then the following series of functions converges uniformly on $[-r, r]$ (and absolutely for each $x \in [-r, r]$):*

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \qquad g(x) = \sum_{n=0}^{\infty} n c_n x^{n-1} \qquad h(x) = \sum_{n=0}^{\infty} c_n \cdot \frac{x^{n+1}}{n+1}$$

Furthermore, f is differentiable on $[-r, r]$ and $f' = g$. Also, h is differentiable on $[-r, r]$ and $h' = f$.

We may paraphrase this theorem as follows: If a (zero-centered) power series converges at x_0, then it may be differentiated and anti-differentiated term-by-term on $(-|x_0|, |x_0|)$ to obtain power series representations of the derivative and antiderivative of f.

Lemma. *Let $n \in \mathbb{N}$. Suppose $f : [a, b] \to \mathbb{R}$ is given by $f(x) = nx^{n-1}$. Then f is integrable and $\int_a^b f = b^n - a^n$.*

Proof. By Definition 11.11, f is a polynomial. Thus, by Corollary 11.12, f is continuous. Consequently, by Theorem 13.19, f is integrable.

To prove that $\int_a^b f = b^n - a^n$, Theorem 14.4 tells us that it will suffice to show that the function $G : [a, b] \to \mathbb{R}$ defined by $G(x) = x^n$ is continuous on $[a, b]$, differentiable on (a, b), and such that $f = G'$. By Definition 11.11, G is a polynomial. Thus, by Corollary 11.12, G is continuous (notably on $[a, b]$), as desired. Additionally, by Exercise 12.8, G is differentiable (notably on (a, b)) and $G'(x) = nx^{n-1} = f(x)$ for all $x \in \mathbb{R}$ (notably all $x \in [a, b]$), as desired. □

Proof of Theorem 17.13. Let (f_k) be defined by $f_k(x) = \sum_{n=0}^{k} c_n x^n$ for each $k \in \mathbb{N}$. To prove that (f_k) converges uniformly on $[-r, r]$ and that $(f_k(x))$ converges absolutely for each $x \in [-r, r]$, Theorem 17.11 tells us that it will suffice to find a sequence of positive real numbers (M_n) such that for all $x \in [-r, r]$, we have $|c_n x^n| \leq M_n$ and such that $\sum_{n=1}^{\infty} M_n$ converges.

To begin, we will show that there exists a number M such that $|c_n x_0^n| \leq M$ for all $n \in \mathbb{N}$. By the hypothesis, $f(x_0) = \sum_{n=0}^{\infty} c_n x_0^n$ converges. Thus, by Theorem 16.4, $\lim_{n\to\infty} c_n x_0^n = 0$. Consequently, by Theorem 15.13, $(c_n x_0^n)$ is bounded. It follows by Definition 15.12 that $\{c_n x_0^n \mid n \in \mathbb{N}\}$ is bounded. Thus, by Definition 5.6, there exist numbers $l, u \in \mathbb{R}$ such that $l \leq c_n x_0^n \leq u$ for all $n \in \mathbb{N}$. Let $M = \max(|l|, |u|)$. It follows by Script 0 that $-M \leq l \leq c_n x_0^n \leq u \leq M$ for all $n \in \mathbb{N}$. Therefore, by the lemma to Exercise 8.9, $|c_n x_0^n| \leq M$ for all $n \in \mathbb{N}$, as desired.

We can now define (M_n): Let (M_n) be defined by $M_n = M(\frac{r}{|x_0|})^n$ for all $n \in \mathbb{N}$.

Next, we will show that for all $x \in [-r, r]$, $|c_n x^n| \leq M_n$ for all $n \in \mathbb{N}$. Let x be an arbitrary element of $[-r, r]$. It follows by Equations 8.1 that $-r \leq x \leq r$. Thus, by the lemma to Exercise 8.9, $|x| \leq r$. Consequently, by Exercise 12.22, $|x^n| \leq |r^n|$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} |c_n x^n| &\leq |c_n r^n| \\ &= |c_n x_0^n| \left(\frac{|r^n|}{|x_0^n|} \right) \\ &\leq M \left(\frac{r}{|x_0|} \right)^n \\ &= M_n \end{aligned}$$

for all $n \in \mathbb{N}$, as desired. Note that this result also implies by Definition 8.4 that (M_n) is a sequence of positive real numbers.

Lastly, we will show that $\sum_{n=1}^{\infty} M_n$ converges. Since $0 < r < |x_0|$ by hypothesis, Script 7 implies that $-1 < \frac{r}{|x_0|} < 1$. Thus, by Theorem 16.7, $\sum_{n=0}^{\infty} (\frac{r}{|x_0|})^n$ converges. Consequently, by Lemma 16.2, $\sum_{n=1}^{\infty} (\frac{r}{|x_0|})^n$ converges. Therefore, by Theorem 16.8, $\sum_{n=1}^{\infty} M(\frac{r}{|x_0|})^n$ (i.e., $\sum_{n=1}^{\infty} M_n$) converges, as desired.

Let (g_k) be defined by $g_k(x) = \sum_{n=0}^{k} n c_n x^{n-1}$ for each $k \in \mathbb{N}$. To prove that (g_k) converges uniformly on $[-r, r]$ and that $(g_k(x))$ converges absolutely for each $x \in [-r, r]$, Theorem 17.11 tells us that it will suffice to find a sequence of positive real numbers (M_n) such that for all $x \in [-r,r]$, we have $|nc_nx^{n-1}| \leq M_n$ and such that $\sum_{n=1}^{\infty} M_n$ converges.

To begin, we define (M_n) and prove its basic properties. Let (M_n) be defined by $M_n = \frac{Mn}{|r|}|\frac{r}{x_0}|^n$ for all $n \in \mathbb{N}$, where M is the same constant defined above. We now show that for all $x \in [-r,r]$, we have $|nc_nx^{n-1} \leq M_n$ for all $n \in \mathbb{N}$. Let x be an arbitrary element of $[-r,r]$. It follows as before that $|x^{n-1}| \leq |r^{n-1}|$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}
|nc_nx^{n-1}| &= n|c_n||x^{n-1}| \\
&\leq n|c_n||r^{n-1}| \\
&= \frac{|c_n|}{|r|}|x_0|^n n \left|\frac{r}{x_0}\right|^n \\
&\leq \frac{Mn}{|r|}\left|\frac{r}{x_0}\right|^n \\
&= M_n
\end{aligned}$$

for all $n \in \mathbb{N}$, as desired. Note that as before, this result also implies that (M_n) is a sequence of positive real numbers.

Next, we will show that $\sum_{n=1}^{\infty} M_n$ converges. To do so, Theorem 16.15 tells us that it will suffice to show that $\lim_{n\to\infty}|\frac{M_{n+1}}{M_n}| < 1$. As before, $|\frac{r}{x_0}| < 1$. Additionally, by an argument symmetric to that used in Exercise 15.6a, we know that $\lim_{n\to\infty}|\frac{r}{x_0}|$ converges to $|\frac{r}{x_0}|$. Furthermore, by an argument symmetric to that used in Exercise 15.10c, we have that $\lim_{n\to\infty}|\frac{n+1}{n}|$ converges to 1. Combining these last three results, we have that

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{M_{n+1}}{M_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{M(n+1)}{|r|}\left|\frac{r}{x_0}\right|^{n+1}}{\frac{Mn}{|r|}\left|\frac{r}{x_0}\right|^n}\right| \\
&= \lim_{n\to\infty}\left|\frac{n+1}{n}\left|\frac{r}{x_0}\right|\right| \\
&= \left(\lim_{n\to\infty}\left|\frac{r}{x_0}\right|\right)\left(\lim_{n\to\infty}\left|\frac{n+1}{n}\right|\right) && \text{Theorem 15.9} \\
&= \left|\frac{r}{x_0}\right| \cdot 1 \\
&< 1
\end{aligned}$$

as desired.

The argument for (h_k) defined by $h_k(x) = \sum_{n=0}^{k} c_n \cdot \frac{x^{n+1}}{n+1}$ is symmetric to that for (g_k).

To prove that f is differentiable on $[-r,r]$ and $f' = g$, Theorem 14.1 tells us that it will suffice to show that g is integrable on $[-r,r]$, that $f(x) + c = \int_{-r}^{x} g$ where $c \in \mathbb{R}$ is a constant, and that g is continuous on $[-r,r]$. We will verify each constraint in order.

Let k be an arbitrary natural number. Thus, by the definition of (g_k), $g_k(x) = \sum_{n=0}^{k} nc_nx^{n-1}$. Consequently, by Definition 11.11, g_k is a polynomial. It follows by Corollary 11.12 that g_k is continuous. Thus, by Theorem 13.19, g_k is integrable. Therefore, since (g_k) is a sequence of integrable functions on $[-r,r]$, the lemma to Theorem 17.7 asserts that g is integrable on $[-r,r]$, as desired.

It follows from the above that

$$\begin{aligned}
\int_{-r}^{x} g &= \lim_{k\to\infty} \int_{-r}^{x} g_k && \text{Theorem 17.7} \\
&= \lim_{k\to\infty} \int_{-r}^{x} \sum_{n=0}^{k} nc_n t^{n-1}\,\mathrm{d}t \\
&= \lim_{k\to\infty} \sum_{n=0}^{k} \int_{-r}^{x} nc_n t^{n-1}\,\mathrm{d}t && \text{Theorem 13.24} \\
&= \lim_{k\to\infty} \sum_{n=0}^{k} c_n \int_{-r}^{x} nt^{n-1}\,\mathrm{d}t && \text{Theorem 13.24} \\
&= \lim_{k\to\infty} \sum_{n=0}^{k} c_n(x^n - (-r)^n) && \text{Lemma} \\
&= \lim_{k\to\infty} \sum_{n=0}^{k} (c_n x^n - c_n(-r)^n) \\
&= \sum_{n=0}^{\infty} (c_n x^n - c_n(-r)^n) && \text{Definition 16.1} \\
&= \sum_{n=0}^{\infty} c_n x^n - \sum_{n=0}^{\infty} c_n(-r)^n && \text{Theorem 16.8} \\
&= f(x) + c
\end{aligned}$$

as desired.

Lastly, since each g_k is continuous and (g_k) converges uniformly to g, Theorem 17.6 asserts that g is continuous on $[-r, r]$, as desired.

The argument for that h is differentiable on $[-r, r]$ and $h' = f$ is symmetric to the above. □