MATH 16310 (Honors Calculus III IBL) Notes

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Contents

Script 12

Derivatives

12.1 Journal

3/30: Throughout this sheet, we let $f : A \to \mathbb{R}$ be a real valued function with domain $A \subset \mathbb{R}$. We also now assume the domain $A \subset \mathbb{R}$ is open.

Definition 12.1. The **derivative** of f at a point $a \in A$ is the number $f'(a)$ defined by the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided that the limit on the right-hand side exists. If $f'(a)$ exists, we say that f is **differentiable** (at a). If f is differentiable at all points of its domain, we say that f is **differentiable**. In this case, the values $f'(a)$ define a new function $f' : A \to \mathbb{R}$ called the **derivative** (of f).

Remark 12.2. If A is not open, the limit in Definition 12.1 may not exist. For example, if $f : [a, b] \to \mathbb{R}$, then we cannot define the derivative at the endpoints. For any c in the domain of f, we define the **right-hand derivative** $f'_+(c)$ and the **left-hand derivative** $f'_-(c)$ by

$$f'_+(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \qquad f'_-(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$$

We say that f is **differentiable** (on $[a, b]$) if f is differentiable on (a, b) and $f'_+(a)$ and $f'_-(b)$ exist.

Lemma 12.3. *Let $a \in \mathbb{R}$. Then*

$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$$

assuming that one of the two limits exists. (So if the limit on the left exists, so does the one on the right, and they are equal. Similarly, if the limit on the right exists, then so does the one on the left, and they are equal.)

Proof. Suppose first that $\lim_{x \to a} f(x)$ exists, and let it be equal to L. To prove that $\lim_{h \to 0} f(a+h)$ exists and that it equals $\lim_{x \to a} f(x)$, Definition 11.1 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $(h + a) \in A$ and $0 < |h - 0| = |h| < \delta$, then $|f(a + h) - L| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{x \to a} f(x)$ exists, Definition 11.1 implies that there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$. We will choose this δ to be our δ. Now suppose that h is any number satisfying both $(h + a) \in A$ and $0 < |h| < \delta$; we seek to show that $|f(a + h) - L| < \epsilon$. Since $(h + a) \in A$, $h + a = x$ for some $x \in A$. It follows that $h = x - a$, meaning that x is an object that is both an element of A and that satisfies $0 < |h| = |x - a| < \delta$, so we know that $|f(a + h) - L| = |f(x) - L| < \epsilon$, as desired.

The proof is symmetric in the other direction. ☐

Theorem 12.4. *Let $a \in \mathbb{R}$. Then f is differentiable at a if and only if $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists. Moreover, if f is differentiable at a, then the derivative of f at a is given by the limit*

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Proof. Suppose first that f is differentiable at a. Then by Definition 12.1, $f'(a)$ exists. It follows that

$$\begin{aligned} f'(a) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} && \text{Definition 12.1}\\ &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{(a+h)-a}\\ &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a} && \text{Lemma 12.3}\end{aligned}$$

Note that the substitution in the last step follows from using $\tilde{f}(x)=\frac{f(x)-f(a)}{x-a}$ as the "$f(x)$" function in Lemma 12.3.

The proof is symmetric in the reverse direction. □

Theorem 12.5. *If f is differentiable at a, then f is continuous at a.*

Proof. To prove that f is continuous at a, Theorem 11.5 tells us that it will suffice to show that $\lim_{x\to a} f(x) = f(a)$. By Definition 12.1, the hypothesis implies that $f'(a)$ exists. Additionally, by Exercise 11.6, $g(x)=x-a$ is continuous at a. Thus, by Theorem 11.5, $\lim_{x\to a} g(x)$ exists (and equals $g(a)$). Consequently, knowing that both $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ and $\lim_{x\to a} g(x)$ exist (the former by Theorem 12.4), we have by Theorem 11.9 that the limit of their product exists and equals

$$\lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}\cdot(x-a)\right)=\lim_{x\to a}(f(x)-f(a))$$

Moreover, since g is continuous at a, $\lim_{x\to a} g(x)=g(a)=a-a=0$. Thus,

$$\begin{aligned}\lim_{x\to a}(f(x)-f(a)) &= \left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right)\left(\lim_{x\to a}(x-a)\right)\\ &= \left(\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right)\cdot 0\\ &= 0\end{aligned}$$

But then it follows by Theorem 11.9 if we consider $f(a)$ to be a constant function that

$$\begin{aligned}\lim_{x\to a}(f(x)-f(a))+f(a) &= 0+f(a)\\ \lim_{x\to a}(f(x)-f(a)+f(a)) &= f(a)\\ \lim_{x\to a} f(x) &= f(a)\end{aligned}$$

as desired. □

Exercise 12.6. Show that the converse of Theorem 12.5 is not true.

Proof. The converse of Theorem 12.5 asserts that "if f is continuous at a, then f is differentiable at a." To falsify this statement, we will use the absolute value function $|x|$ as a counterexample. Let's begin.

By Exercise 11.7, $|x|$ is continuous. It follows by Theorem 9.10 that $|x|$ is continuous at 0. However, we can show that $|x|$ is not differentiable at 0.

To do this, Definition 12.1 and Theorem 12.4 tell us that it will suffice to verify that $\lim_{x\to 0}\frac{|x|-|0|}{x-0}=\lim_{x\to 0}\frac{|x|}{x}$ does not exist. Suppose for the sake of contradiction that $\lim_{x\to 0}\frac{|x|}{x}=L$. Then by Definition 11.1, for $\epsilon=1>0$, there exists a $\delta>0$ such that if $0<|x-0|=|x|<\delta$, then $|\frac{|x|}{x}-L|<1$. However, we can show that no such δ exists. Let $\delta>0$ be arbitrary. By Theorem 5.2, there exists a number $x\in\mathbb{R}$ such that $0 < x < \delta$. It follows by Definition 8.4 and Exercise 8.5 that $0 < |x| = |-x| < \delta$. Since both x and $-x$ are in the appropriate range, we know that

$$\begin{aligned}
\left|\frac{|x|}{x} - L\right| &= \left|\frac{x}{x} - L\right| & \left|\frac{|-x|}{-x} - L\right| &= \left|\frac{x}{-x} - L\right| && \text{Definition 8.4}\\
&= |1 - L| & &= |-1 - L| && \text{Script 7}\\
&= |L - 1| & &= |L + 1| && \text{Exercise 8.5}\\
&< 1 & &< 1 &&
\end{aligned}$$

By consecutive applications of the lemma from Exercise 8.9, it follows that

$$\begin{aligned}
-1 < L - 1 &< 1 & -1 < L + 1 &< 1\\
0 < L &< 2 & -2 < L &< 0
\end{aligned}$$

But this implies that $L < 0$ and $L > 0$, a contradiction. $\square$

Exercise 12.7. Show that for all $n \in \mathbb{N}$,

$$x^n - a^n = (x - a)\left(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1}\right)$$

or equivalently,

$$x^n - a^n = (x - a)\left(\sum_{i=0}^{n-1} x^{n-1-i}a^i\right)$$

Proof. By simple algebra (see Script 7), we have

$$\begin{aligned}
(x - a)\left(\sum_{i=0}^{n-1} x^{n-1-i}a^i\right) &= \sum_{i=0}^{n-1}(x - a)x^{n-1-i}a^i\\
&= \sum_{i=0}^{n-1}\left(x^{n-i}a^i - x^{n-1-i}a^{i+1}\right)\\
&= x^n + \sum_{i=1}^{n-1} x^{n-i}a^i - \sum_{i=0}^{n-2} x^{n-1-i}a^{i+1} - a^n\\
&= x^n + \sum_{i=1}^{n-1} x^{n-i}a^i - \sum_{i=0+1}^{n-2+1} x^{n-1-(i-1)}a^{(i-1)+1} - a^n\\
&= x^n + \sum_{i=1}^{n-1} x^{n-i}a^i - \sum_{i=1}^{n-1} x^{n-i}a^i - a^n\\
&= x^n - a^n
\end{aligned}$$

$\square$

Exercise 12.8.

(a) Let $n \in \mathbb{N}$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^n$. Use Exercise 12.7 to prove that $f'(a) = na^{n-1}$ for all $a \in \mathbb{R}$.

Proof. To prove that $f'(a) = na^{n-1}$ for all $a \in \mathbb{R}$, Theorem 12.4 tell us that it will suffice to show that $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = na^{n-1}$ for all $a \in \mathbb{R}$. Let a be an arbitrary element of $\mathbb{R}$. By Corollary 11.12, the polynomial $\sum_{i=0}^{n-1} x^{n-1-i}a^i$ is continuous. Thus, by Theorem 9.10, it is continuous at a. It follows by Theorem 11.5 that $\lim_{x\to a}\sum_{i=0}^{n-1} x^{n-1-i}a^i = \sum_{i=0}^{n-1} a^{n-1-i}a^i$. Therefore,

$$\begin{aligned}
na^{n-1} &= \underbrace{a^{n-1}+\cdots+a^{n-1}}_{n\text{ times}}\\
&= \sum_{i=0}^{n-1} a^{n-1}\\
&= \sum_{i=1}^{n-1} a^{n-1-i}a^i\\
&= \lim_{x\to a}\sum_{i=0}^{n-1} x^{n-1-i}a^i\\
&= \lim_{x\to a}\frac{x-a}{x-a}\cdot\sum_{i=0}^{n-1} x^{n-1-i}a^i
\end{aligned}$$

Note that we can make the above change because $\frac{x-a}{x-a}=1$ for all x satisfying $0<|x-a|<\delta$, whatever δ may be.

$$\begin{aligned}
&= \lim_{x\to a}\frac{x^n-a^n}{x-a} && \text{Exercise 12.7}\\
&= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\\
&= f'(a) && \text{Theorem 12.4}
\end{aligned}$$

as desired. ☐

(b) Let $k\in\mathbb{R}$. Prove that if $f:\mathbb{R}\to\mathbb{R}$ is given by $f(x)=k$, then $f'(a)=0$ for all $a\in\mathbb{R}$.

Proof. To prove that $f'(a)=0$ for all $a\in\mathbb{R}$, Definition 12.1 tells us that it will suffice to show that $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=0$ for all $a\in\mathbb{R}$. Let a be an arbitrary element of $\mathbb{R}$. By Exercise 11.6, the function $g(x)=0$ is continuous. Thus, by Theorem 9.10, it is continuous at a. It follows by Theorem 11.5 that $\lim_{h\to 0} g(h)=g(0)=0$. Therefore,

$$\begin{aligned}
0 &= \lim_{h\to 0} g(h)\\
&= \lim_{h\to 0} 0\\
&= \lim_{h\to 0}\frac{h}{h}\cdot 0
\end{aligned}$$

Note that we can make the above change for the same reason as part (a).

$$\begin{aligned}
&- \lim_{h\to 0}\frac{0}{h}\\
&= \lim_{h\to 0}\frac{k-k}{h}\\
&= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}
\end{aligned}$$

as desired. ☐

Exercise 12.9. Suppose that $f:A\to\mathbb{R}$ and $g:A\to\mathbb{R}$ are differentiable at $a\in A$.

(a) Prove that $f+g$ is differentiable at a and compute $(f+g)'(a)$ in terms of $f'(a)$ and $g'(a)$.

Proof. To prove that $f+g$ is differentiable at a, Definition 12.1 tells us that it will suffice to show $\lim_{h\to 0}\frac{(f+g)(a+h)-(f+g)(a)}{h}$ exists. Since f,g are differentiable at a, we know by Definition 12.1 that $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ and $\lim_{h\to 0}\frac{g(a+h)-g(a)}{h}$ exist. Thus, by Theorem 11.9 the limit of their sum exists and equals

$$\begin{aligned}\lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}+\frac{g(a+h)-g(a)}{h}\right) &= \lim_{h\to 0}\frac{f(a+h)+g(a+h)-f(a)-g(a)}{h}\\ &= \lim_{h\to 0}\frac{(f+g)(a+h)-(f+g)(a)}{h}\end{aligned}$$

as desired. Having established that $\lim_{h\to 0}\frac{(f+g)(a+h)-(f+g)(a)}{h}$ exists, $(f+g)'(a)$ can be computed in terms of $f'(a)$ and $g'(a)$ with the following algebra.

$$\begin{aligned}(f+g)'(a) &= \lim_{h\to 0}\frac{(f+g)(a+h)-(f+g)(a)}{h} && \text{Definition 12.1}\\ &= \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}+\frac{g(a+h)-g(a)}{h}\right)\\ &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}+\lim_{h\to 0}\frac{g(a+h)-g(a)}{h} && \text{Theorem 11.9}\\ &= f'(a)+g'(a) && \text{Definition 12.1}\end{aligned}$$

□

(b) Prove that fg is differentiable and compute $(fg)'(a)$ in terms of $f(a)$, $g(a)$, $f'(a)$, and $g'(a)$.

Proof. To prove that fg is differentiable at a, Definition 12.1 tells us that it will suffice to show $\lim_{h\to 0}\frac{(fg)(a+h)-(fg)(a)}{h}$ exists. Since f,g are differentiable at a, we know by Definition 12.1 that $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ and $\lim_{h\to 0}\frac{g(a+h)-g(a)}{h}$ exist. For the same reason, we know by Theorem 12.5 that g is continuous, i.e., continuous at a by Theorem 9.10. Consequently, by Theorem 11.5, $\lim_{x\to a} g(x)$ exists (and equals $g(a)$). Note that the preceding limit is equal to $\lim_{h\to 0} g(a+h)$ by Lemma 12.3. Lastly, we have by Exercise 11.6 that the constant function $f(a)$ is continuous at 0. Consequently, by Theorem 11.5, $\lim_{h\to 0} f(a)$ exists (and equals $f(a)$). Combining all of these results, consecutive applications of Theorem 11.9 assert that the limits

$$\lim_{h\to 0} g(a+h)\cdot\frac{f(a+h)-f(a)}{h} \qquad\qquad \lim_{h\to 0} f(a)\cdot\frac{g(a+h)-g(a)}{h}$$

exist. Furthermore, it asserts that the limit of their sum exists and equals

$$\begin{aligned}&\lim_{h\to 0}\left(g(a+h)\cdot\frac{f(a+h)-f(a)}{h}+f(a)\cdot\frac{g(a+h)-g(a)}{h}\right)\\ &\qquad= \lim_{h\to 0}\frac{g(a+h)(f(a+h)-f(a))+f(a)(g(a+h)-g(a))}{h}\\ &\qquad= \lim_{h\to 0}\frac{f(a+h)g(a+h)-f(a)g(a+h)+f(a)g(a+h)-f(a)g(a)}{h}\\ &\qquad= \lim_{h\to 0}\frac{f(a+h)g(a+h)-f(a)g(a)}{h}\\ &\qquad= \lim_{h\to 0}\frac{(fg)(a+h)-(fg)(a)}{h}\end{aligned}$$

as desired. Having established that $\lim_{h\to 0}\frac{(fg)(a+h)-(fg)(a)}{h}$ exists, $(fg)'(a)$ can be computed in terms of $f(a)$, $g(a)$, $f'(a)$, and $g'(a)$ with the following algebra.

$$\begin{aligned}
(fg)'(a) &= \lim_{h\to 0}\frac{(fg)(a+h)-(fg)(a)}{h} && \text{Definition 12.1}\\
&= \lim_{h\to 0}\left(g(a+h)\cdot\frac{f(a+h)-f(a)}{h}+f(a)\cdot\frac{g(a+h)-g(a)}{h}\right)\\
&= \lim_{h\to 0}g(a+h)\cdot\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}+\lim_{h\to 0}f(a)\cdot\lim_{h\to 0}\frac{g(a+h)-g(a)}{h} && \text{Theorem 11.9}\\
&= g(a)f'(a)+f(a)g'(a)
\end{aligned}$$

□

(c) Prove that $\frac{1}{g}$ is differentiable at a (under an appropriate assumption) and compute $(\frac{1}{g})'(a)$ in terms of $g'(a)$ and $g(a)$. What assumption do you need to make?

Proof. Assume, in addition to the fact that $g : A \to \mathbb{R}$ is differentiable at $a \in A$, that $g(a) \neq 0$.

To prove that $\frac{1}{g}$ is differentiable at a, Definition 12.1 tells us that it will suffice to show $\lim_{h\to 0}\frac{(\frac{1}{g})(a+h)-(\frac{1}{g})(a)}{h}$ exists. Since g is differentiable at a, we know by Definition 12.1 that $\lim_{h\to 0}\frac{g(a+h)-g(a)}{h}$ exists. For the same reason, we know by Theorem 12.5 that g is continuous, i.e., continuous at a by Theorem 9.10. Consequently, by Theorem 11.5, $\lim_{x\to a} g(x)$ exists (and equals $g(a)$). It follows by Lemma 12.3 that the preceding limit is equal to $\lim_{h\to 0} g(a+h)$. Thus, since it is also equal to $g(a) \neq 0$, we have by Theorem 11.9 that $\lim_{h\to 0}\frac{1}{g}(a+h)$ exists (and equals $\frac{1}{g(a)}$). Lastly, we have by Exercise 11.6 that the constant function $-\frac{1}{g(a)}$ is continuous at 0. Consequently, by Theorem 11.5, $\lim_{h\to 0} -\frac{1}{g(a)}$ exists (and equals $-\frac{1}{g(a)}$). Combining this with the previous result, Theorem 11.9 asserts that the limit $\lim_{h\to 0} -\frac{1}{g(a+h)g(a)}$ exists (and equals $-\frac{1}{g(a)^2}$). Furthermore, it asserts that the limit of its product with $\lim_{h\to 0}\frac{(\frac{1}{g})(a+h)-(\frac{1}{g})(a)}{h}$ exists and equals

$$\begin{aligned}
\lim_{h\to 0} -\frac{1}{g(a+h)g(a)}\cdot\frac{g(a+h)-g(a)}{h} &= \lim_{h\to 0}\frac{\frac{g(a)-g(a+h)}{g(a+h)g(a)}}{h}\\
&= \lim_{h\to 0}\frac{\frac{1}{g(a+h)}-\frac{1}{g(a)}}{h}\\
&= \lim_{h\to 0}\frac{(\frac{1}{g})(a+h)-(\frac{1}{g})(a)}{h}
\end{aligned}$$

as desired. Having established that $\lim_{h\to 0}\frac{(\frac{1}{g})(a+h)-(\frac{1}{g})(a)}{h}$ exists, $(\frac{1}{g})'(a)$ can be computed in terms of $g(a)$ and $g'(a)$ with the following algebra.

$$\begin{aligned}
\left(\frac{1}{g}\right)'(a) &= \lim_{h\to 0}\frac{\left(\frac{1}{g}\right)(a+h)-\left(\frac{1}{g}\right)(a)}{h} && \text{Definition 12.1}\\
&= \lim_{h\to 0} -\frac{1}{g(a+h)g(a)}\cdot\frac{g(a+h)-g(a)}{h}\\
&= \lim_{h\to 0} -\frac{1}{g(a+h)g(a)}\cdot\lim_{h\to 0}\frac{g(a+h)-g(a)}{h} && \text{Theorem 11.9}\\
&= -\frac{g'(a)}{g(a)^2}
\end{aligned}$$

□

(d) Prove that $\frac{f}{g}$ is differentiable at a (under an appropriate assumption) and compute $(\frac{f}{g})'(a)$ in terms of $f(a)$, $g(a)$, $f'(a)$, and $g'(a)$. What assumption do you need to make?

Proof. Assume, in addition to the fact that $g : A \to \mathbb{R}$ is differentiable at $a \in A$, that $g(a) \neq 0$.

It follows by part (c) that $\frac{1}{g}$ is differentiable at a, and then by part (b) that $f \cdot \frac{1}{g} = \frac{f}{g}$ is differentiable at a.

Having established that $(\frac{f}{g})'(a)$ exists, it can be computed in terms of $f(a)$, $g(a)$, $f'(a)$, and $g'(a)$ with the following algebra.

$$\begin{aligned}\left(\frac{f}{g}\right)'(a) &= \left(f \cdot \frac{1}{g}\right)'(a)\\ &= f(a)\left(\frac{1}{g}\right)'(a) + f'(a)\left(\frac{1}{g}\right)(a)\\ &= f(a) \cdot -\frac{g'(a)}{g(a)^2} + \frac{f'(a)g(a)}{g(a)^2}\\ &= \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}\end{aligned}$$

□

4/1: One of the most important results concerning the differentiation of functions is the rule for the derivative of a composition of functions. Let $f : B \to \mathbb{R}, g : A \to \mathbb{R}$ be functions such that $g(A) \subset B$. The composition $(f \circ g)(x) = f(g(x))$ is defined for all $x \in A$.

Theorem 12.10. *Let $a \in A$, $g : A \to \mathbb{R}$, and $f : I \to \mathbb{R}$ where I is an interval containing $g(A)$. Suppose that g is differentiable at a and f is differentiable at $g(a)$. Then $f \circ g$ is differentiable a and*

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$$

Proof. To prove that $f \circ g$ is differentiable at a, Theorem 12.4 tells us that it will suffice to show that $\lim_{x\to a} \frac{(f\circ g)(x)-(f\circ g)(a)}{x-a}$ exists. To do so, we will define a special function φ and prove that it is continuous at a. It will follow that $(f \circ g)'(a)$ exists and equals $f'(g(a)) \cdot g'(a)$. Let's begin.

Let $\varphi : I \to \mathbb{R}$ be defined by

$$\varphi(x) = \begin{cases} \frac{f(g(x))-f(g(a))}{g(x)-g(a)} & g(x) \neq g(a) \\ f'(g(a)) & g(x) = g(a) \end{cases}$$

It is clear from the definition that the function is defined for all $x \in A$.

To demonstrate that φ is continuous at a, Theorem 11.5 tells us that it will suffice to confirm that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $|x-a| < \delta$, then $|\varphi(x) - \varphi(a)| = |\varphi(x) - f'(g(a))| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f is differentiable at $g(a)$, we know that $\lim_{y\to g(a)} \frac{f(y)-f(g(a))}{y-g(a)} = f'(g(a))$. It follows by Definition 11.1 that there is some $\delta' > 0$ such that if $y \in I$ and $0 < |y - g(a)| < \delta'$, then $|\frac{f(y)-f(g(a))}{y-g(a)} - f'(g(a))| < \epsilon$. Additionally, since g is differentiable (hence continuous by Theorem 12.5) at a, we have by Theorem 11.5 that there exists a $\delta > 0$ such that if $x \in A$ and $|x-a| < \delta$, then $|g(x) - g(a)| < \delta'$.

Using the above δ, let x be an arbitrary element of A such that $|x-a| < \delta$. We now divide into two cases ($g(x) = g(a)$ and $g(x) \neq g(a)$). If $g(x) = g(a)$, then $|\varphi(x) - f'(g(a))| = |f'(g(a)) - f'(g(a))| = 0 < \epsilon$, as desired. If $g(x) \neq g(a)$, then we continue. Since $|x - a| < \delta$, we have that $|g(x) - g(a)| < \delta'$. This combined with the fact that $g(x) \in I$ and $g(x) \neq g(a)$, i.e., $0 < |g(x)-g(a)|$ illustrates that $|\frac{f(g(x))-f(g(a))}{g(x)-g(a)} - f'(g(a))| = |\varphi(x) - f'(g(a))| < \epsilon$. Therefore, φ is continuous at a.

It follows by Theorem 11.5 that $\lim_{x\to a} \varphi(x) = \varphi(a) = f'(g(a))$. Additionally, since g is differentiable at a, Definition 12.1 and Theorem 12.4 tell us that $\lim_{x\to a} \frac{g(x)-g(a)}{x-a}$ exists (and equals $g'(a)$). The combination of the past two results imply by Theorem 11.9 that the product of the limits exists and equals

$$\begin{aligned} f'(g(a)) \cdot g'(a) &= \lim_{x \to a} \varphi(x) \cdot \frac{g(x) - g(a)}{x - a} \\ &= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x - a} \\ &= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} \\ &= \lim_{x \to a} \frac{(f \circ g)(x) - (f \circ g)(a)}{x - a} \\ &= (f \circ g)'(a) \end{aligned}$$

as desired. □

We now come to the most important theorem in differential calculus, Corollary 12.16.

Definition 12.11. Let $f : A \to \mathbb{R}$ be a function. If $f(a)$ is the last point of $f(A)$, then $f(a)$ is called the **maximum value** of f. If $f(a)$ is the first point of $f(A)$, then $f(a)$ is the **minimum value** of f. We say that $f(a)$ is a **local maximum value** of f if there exists a region R containing a such that $f(a)$ is the last point of $f(A \cap R)$. We say that $f(a)$ is a **local minimum value** of f if there exists a region R containing a such that $f(a)$ is the first point of $f(A \cap R)$.

Remark 12.12. Equivalently, $f(a)$ is a local maximum (resp. minimum) value of f if there exists U open in A such that $f(a)$ is the last (resp. first) point of $f(U)$.

Theorem 12.13. *Let $f : A \to \mathbb{R}$ be differentiable at a. Suppose that $f(a)$ is the maximum value or minimum value of f. Then $f'(a) = 0$.*

Proof. Suppose first that $f(a)$ is the maximum value of f, and suppose for the sake of contradiction that $f'(a) \neq 0$. Then $f'(a) > 0$ or $f'(a) < 0$. We now divide into two cases. If $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a) > 0$, then for $\epsilon = f'(a)$, we have by Definition 11.1 that there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|\frac{f(x) - f(a)}{x - a} - f'(a)| < f'(a)$. Let $x_0 \in A$ satisfy $0 < x - a < \delta$. Then by Definition 8.4, $0 < |x_0 - a| < \delta$. Consequently, $|\frac{f(x_0) - f(a)}{x_0 - a} - f'(a)| < f'(a)$. It follows by the lemma from Exercise 8.9 that $-f'(a) < \frac{f(x_0) - f(a)}{x_0 - a} - f'(a) < f'(a)$. Thus, by Script 7, we have that $0 < \frac{f(x_0) - f(a)}{x_0 - a}$, implying since $x - a > 0$ that $0 < f(x_0) - f(a)$. But this means that $f(x_0) > f(a)$, i.e., that $f(a)$ is not the last point of $f(A)$ (by Definition 3.3), i.e., that $f(a)$ is not the maximum value of f (by Definition 12.11), a contradiction. The argument is symmetric in the other case.

The proof is symmetric in the other case. □

Corollary 12.14. *Let $f : A \to \mathbb{R}$ be differentiable at a. Suppose that $f(a)$ is a local maximum or local minimum value of f. Then $f'(a) = 0$.*

Proof. Suppose first that $f(a)$ is a local maximum of f. Then by Definition 12.11, there exists a region R containing a such that $f(a)$ is the last point of $f(A \cap R)$. Now consider the restriction of f to $A \cap R$. It follows from Definition 9.6 that $f|_{A \cap R}$ is differentiable at a, that $f|_{A \cap R}(A \cap R) = f(A \cap R)$, and that $f_{A \cap R}(a) = f(a)$ is the last point of $f|_{A \cap R}(A \cap R)$. The latter two results imply by Definition 12.11 that $f_{A \cap R}(a)$ is the maximum value of $f|_{A \cap R}$. This combined with the fact that $f|_{A \cap R}$ is differentiable at a implies by Theorem 12.13 that $f|'_{A \cap R}(a) = f'(a) = 0$, as desired. □

Theorem 12.15. *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, differentiable on (a, b), and that $f(a) = f(b) = 0$. Then there exists a point $\lambda \in (a, b)$ such that $f'(\lambda) = 0$.*

Proof. We divide into two cases ($f(x) = 0$ for all $x \in [a, b]$, and $f(x) \neq 0$ for some $x \in [a, b]$).

Suppose first that $f(x) = 0$ for all $x \in [a, b]$. By Theorem 5.2, we can choose a $\lambda \in (a, b)$. It follows from the hypothesis that $f(\lambda) = f(x)$ for all $f(x) \in f([a, b])$. This can be weakened to $f(\lambda) \geq f(x)$ for all $f(x) \in f([a, b])$. Thus, by Definition 3.3, $f(\lambda)$ is the last point of $f([a, b])$. Consequently, by Definition 12.11,

$f(\lambda)$ is the maximum value of f. This combined with the fact that f is differentiable at λ (since $\lambda \in (a,b)$ and f is differentiable on (a,b)) implies by Theorem 12.13 that $f'(\lambda) = 0$, as desired.

Now suppose that $f(x) \neq 0$ for some $x \in [a,b]$ which we shall call x_0. We divide into two cases again ($f(x_0) > 0$ and $f(x_0) < 0$). If $f(x_0) > 0$, then by Exercise 10.21, there exists a point $\lambda \in [a,b]$ such that $f(\lambda) \geq f(x)$ for all $x \in [a,b]$. It follows that $f(\lambda) \geq f(x_0) > 0$, so $f(\lambda) \neq f(a) = f(b)$. Thus, by Definition 1.16, $\lambda \neq a$ and $\lambda \neq b$. This combined with the fact that $\lambda \in [a,b]$ implies by Script 8 that $\lambda \in (a,b)$. Now as before, we can determine from the fact that $f(\lambda) \geq f(x)$ for all $x \in [a,b]$ that $f(\lambda)$ is the maximum value of f. This combined with the fact that f is differentiable at λ (since $\lambda \in (a,b)$ and f is differentiable on (a,b)) implies by Theorem 12.13 that $f'(\lambda) = 0$, as desired. The argument is symmetric in the other case. □

Corollary 12.16. *Suppose that $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then there exists a point $\lambda \in (a,b)$ such that*

$$f(b) - f(a) = f'(\lambda)(b-a)$$

Proof. Let $h : [a,b] \to \mathbb{R}$ be defined by

$$h(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a) - f(a)$$

By hypothesis, $f(x)$ is continuous on $[a,b]$. By Exercise 11.6 and Theorem 11.9, $-\frac{f(b)-f(a)}{b-a}(x-a) - f(a)$ is continuous on $[a,b]$. Thus, by Theorem 11.9, their sum (i.e., $h(x)$) is continuous on $[a,b]$. Additionally, by hypothesis, $f(x)$ is differentiable on (a,b). By Exercises 12.8 and 12.9, $-\frac{f(b)-f(a)}{b-a}(x-a) - f(a)$ is differentiable on $[a,b]$. Thus, by Exercise 12.9, their sum (i.e., $h(x)$) is differentiable on (a,b). Furthermore, by simple algebra, we can determine that

$$\begin{aligned} h(a) &= f(a) - \frac{f(b)-f(a)}{b-a}(a-a) - f(a) \\ &= -\frac{f(b)-f(a)}{b-a} \cdot 0 \\ &= 0 \end{aligned} \qquad \begin{aligned} h(b) &= f(b) - \frac{f(b)-f(a)}{b-a}(b-a) - f(a) \\ &= f(b) - (f(b)-f(a)) - f(a) \\ &= 0 \end{aligned}$$

Thus, by Theorem 12.15, there exists a point $\lambda \in (a,b)$ such that $h'(\lambda) = 0$.

We can also calculate $h'(x)$ as follows.

$$\begin{aligned} h'(x) &= \left(f(x) - \frac{f(b)-f(a)}{b-a}(x-a) - f(a) \right)' \\ &= \left((f(x)) + \left(-\frac{f(b)-f(a)}{b-a} \cdot x \right) + \left(\frac{f(b)-f(a)}{b-a} \cdot a - f(a) \right) \right)' \\ &= f'(x) + \left(-\frac{f(b)-f(a)}{b-a} \cdot x \right)' + \left(\frac{f(b)-f(a)}{b-a} \cdot a - f(a) \right)' && \text{Exercise 12.9} \\ &= f'(x) + \left(-\frac{f(b)-f(a)}{b-a} \right)' \cdot (x) + \left(-\frac{f(b)-f(a)}{b-a} \right) \cdot (x)' + \left(\frac{f(b)-f(a)}{b-a} \cdot a - f(a) \right)' \\ & && \text{Exercise 12.9} \\ &= f'(x) + 0 \cdot x + \frac{f(b)-f(a)}{b-a} \cdot 1x^0 + 0 && \text{Exercise 12.8} \\ &= f'(x) + \frac{f(b)-f(a)}{b-a} \end{aligned}$$

But it follows that at λ,

$$\begin{aligned} 0 &= f'(\lambda) - \frac{f(b)-f(a)}{b-a} \\ \frac{f(b)-f(a)}{b-a} &= f'(\lambda) \\ f(b) - f(a) &= f'(\lambda)(b-a) \end{aligned}$$

as desired. □

12.2 Discussion

3/30:

- We can also do Exercise 12.6 with left- and right-handed limits, as defined in Additional Exercise 11.2.
- We can also do Exercise 12.7 by induction.

4/1:

- We can also do Theorem 12.13 by noting that of the left- and right-hand derivatives, one will be ≤ 0 and the other ≥ 0, but since they must be equal, they must equal 0.
- Include more rigorous restriction bits for Corollary 12.14 as a lemma?