

## Script 13

# Uniform Continuity and Integration

### 13.1 Journal

4/8: **Definition 13.1.** Let  $f : A \rightarrow \mathbb{R}$  be a function. We say that  $f$  is **uniformly continuous** if for all  $\epsilon > 0$ , there exists a  $\delta > 0$  such that for all  $x, y \in A$ , if  $|y - x| < \delta$ , then  $|f(y) - f(x)| < \epsilon$ .

**Theorem 13.2.** *If  $f$  is uniformly continuous, then  $f$  is continuous.*

*Proof.* To prove that  $f$  is continuous, Theorem 9.10 tells us that it will suffice to show that  $f$  is continuous at every  $x \in A$ . Let  $x$  be an arbitrary element of  $A$ . To show that  $f$  is continuous at  $x$ , Theorem 11.5 tells us that it will suffice to verify that for every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that if  $y \in A$  and  $|y - x| < \delta$ , then  $|f(y) - f(x)| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Then since  $f$  is uniformly continuous by hypothesis, Definition 13.1 asserts that there exists a  $\delta > 0$  such that for all  $y \in A$  satisfying  $|y - x| < \delta$ , we have  $|f(y) - f(x)| < \epsilon$ , as desired.  $\square$

**Exercise 13.3.** Determine with proof whether each function  $f$  is uniformly continuous on the given interval  $A$ .

- (a)  $f(x) = x^2$  on  $A = \mathbb{R}$ .

*Proof.* To prove that  $f$  is not uniformly continuous on  $\mathbb{R}$ , Definition 13.1 tells us that it will suffice to find an  $\epsilon > 0$  for which no  $\delta > 0$  exists such that for all  $x, y \in \mathbb{R}$ , if  $|y - x| < \delta$ , then  $|y^2 - x^2| < \epsilon$ . Let  $\epsilon = 2$ , and suppose for the sake of contradiction that  $\delta > 0$  is a number such that for all  $x, y \in \mathbb{R}$ , if  $|y - x| < \delta$ , then  $|y^2 - x^2| < 2$ . By Theorem 5.2, there exists a number  $y$  such that  $0 < y < \delta$ . Since  $-\delta < 0 < y < \delta$  by Lemma 7.23, it follows by Definitions 3.6 and 3.10 that  $y \in (-\delta, \delta)$ . Thus, by Exercise 8.9,  $|y - 0| = |y| < \delta$ . Consequently,  $|(y + n) - n| < \delta$ . It follows by the above that  $|(y + n)^2 - n^2| = |y^2 + 2yn| < 2$ . If we now let  $n = \frac{1}{y}$ , then  $|y^2 + 2| < 2$ . But since  $y > 0$ , we have that  $y^2 > 0$  by Lemma 7.26. It follows that  $y^2 + 2 > 2$  by Definition 7.21. Therefore, by Definition 8.4, we can also show that  $|y^2 + 2| > 2$ , a contradiction.  $\square$

- (b)  $f(x) = x^2$  on  $A = (-2, 2)$ .

*Proof.* To prove that  $f$  is uniformly continuous on  $A$ , Definition 13.1 tells us that it will suffice to show that for all  $\epsilon > 0$ , there exists a  $\delta > 0$  such that for all  $x, y \in A$ , if  $|y - x| < \delta$ , then  $|f(y) - f(x)| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Choose  $\delta = \frac{\epsilon}{4}$ , and let  $x, y$  be arbitrary elements of  $A$  that satisfy  $|y - x| < \delta$ . Since  $x, y \in A$ , consecutive applications of Equations 8.1 and the lemma from Exercise 8.9 imply that  $|x| < 2$  and  $|y| < 2$ . It follows that  $|x| + |y| < 2 + 2 = 4$ . Consequently, by Lemma 8.8,  $|x + y| < 4$ . Additionally, since  $0 \leq |y - x|$  by Definition 8.4, we have  $|x - y| \cdot |x + y| \leq \frac{\epsilon}{4} \cdot |x + y|$ . Combining all

of the above results, we have that

$$\begin{aligned}
 |f(y) - f(x)| &= |y^2 - x^2| \\
 &= |y + x| \cdot |y - x| \\
 &< 4 \cdot |y - x| \\
 &\leq 4 \cdot \frac{\epsilon}{4} \\
 &= \epsilon
 \end{aligned}$$

as desired. □

(c)  $f(x) = \frac{1}{x}$  on  $A = (0, +\infty)$ .

*Proof.* To prove that  $f$  is not uniformly continuous on  $A$ , Definition 13.1 tells us that it will suffice to find an  $\epsilon > 0$  for which no  $\delta > 0$  exists such that for all  $x, y \in A$ , if  $|y - x| < \delta$ , then  $|\frac{1}{y} - \frac{1}{x}| < \epsilon$ . Let  $\epsilon = 1$ , and suppose for the sake of contradiction that  $\delta > 0$  is a number such that for all  $x, y \in A$ , if  $|y - x| < \delta$ , then  $|\frac{1}{y} - \frac{1}{x}| < 1$ . As in part (a), choose  $0 < x < \min(\delta, \frac{1}{2})$ . Consequently,  $|(x+x) - x| < \delta$ . It follows by the above that  $|\frac{1}{2x} - \frac{1}{x}| < 1$ . But this implies that  $|\frac{x-2x}{2x^2}| = |\frac{-1}{2x}| = \frac{1}{2x} < 1$ . However,  $x < \frac{1}{2}$  implies that  $1 < \frac{1}{2x}$ , a contradiction. □

(d)  $f(x) = \frac{1}{x}$  on  $A = [1, +\infty)$ .

*Proof.* To prove that  $f$  is uniformly continuous on  $A$ , Definition 13.1 tells us that it will suffice to show that for all  $\epsilon > 0$ , there exists a  $\delta > 0$  such that for all  $x, y \in A$ , if  $|y - x| < \delta$ , then  $|f(y) - f(x)| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Choose  $\delta = \epsilon$ , and let  $x, y$  be arbitrary elements of  $A$  that satisfy  $|y - x| < \delta$ . Since  $x, y \in A$ , consecutive applications of Equations 8.1 imply that  $1 \leq x$  and  $1 \leq y$ . It follows by Script 7 that  $1 \leq |xy|$ . This combined with the fact that  $|y - x| < \delta = \epsilon$  implies that

$$\begin{aligned}
 |f(y) - f(x)| &= \left| \frac{1}{y} - \frac{1}{x} \right| \\
 &= \left| \frac{x - y}{xy} \right| \\
 &= \frac{|y - x|}{|xy|} \\
 &< \frac{\epsilon}{|xy|} \\
 &\leq \frac{\epsilon}{1} \\
 &= \epsilon
 \end{aligned}$$

as desired. □

(e)  $f(x) = \sqrt{x}$  on  $A = [1, +\infty)$ .

*Proof.* To prove that  $f$  is uniformly continuous on  $A$ , Definition 13.1 tells us that it will suffice to show that for all  $\epsilon > 0$ , there exists a  $\delta > 0$  such that for all  $x, y \in A$ , if  $|y - x| < \delta$ , then  $|f(y) - f(x)| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Choose  $\delta = \epsilon$ , and let  $x, y$  be arbitrary elements of  $A$  that satisfy  $|y - x| < \delta$ . Since  $x, y \in A$ , consecutive applications of Equations 8.1 imply that  $1 \leq x$  and  $1 \leq y$ . It follows by Script 7 that  $1 \leq \sqrt{x}$  and  $1 \leq \sqrt{y}$ . Thus, by Script 7 again,  $2 \leq |\sqrt{y} + \sqrt{x}|$ . Note that it follows that  $1 < |\sqrt{y} + \sqrt{x}|$ . This combined with the fact that  $|y - x| < \delta = \epsilon$  implies that

$$\begin{aligned}
 |f(y) - f(x)| &= |\sqrt{y} - \sqrt{x}| \\
 &< |\sqrt{y} - \sqrt{x}| \cdot |\sqrt{y} + \sqrt{x}| \\
 &= |y - x| \\
 &= \epsilon
 \end{aligned}$$

as desired. □

**Exercise 13.4.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = x^n$  for  $n \in \mathbb{N}$ . Show that  $f$  is uniformly continuous if and only if  $n = 1$ .

*Proof.* Suppose first that  $n = 1$ . To prove that  $f$  is uniformly continuous, Definition 13.1 tells us that it will suffice to show that for all  $\epsilon > 0$ , there exists a  $\delta > 0$  such that for all  $x, y \in \mathbb{R}$ , if  $|y - x| < \delta$ , then  $|f(y) - f(x)| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Choose  $\delta = \epsilon$ . Now let  $x, y$  be arbitrary elements of  $\mathbb{R}$  that satisfy  $|y - x| < \delta$ . Then by the definition of  $f$ ,  $|f(y) - f(x)| = |y - x| < \delta = \epsilon$ , as desired.

Now suppose that  $n > 1$ . Additionally, suppose for the sake of contradiction that  $f$  is uniformly continuous. Let  $\epsilon = 1 > 0$ . Then by Definition 13.1, there exists a  $\delta > 0$  such that for all  $x, y \in \mathbb{R}$ , if  $|y - x| < \delta$ , then  $|y^n - x^n| < 1$ . Let  $x = 0 \in \mathbb{R}$ . By Theorem 5.2, there exists a point  $y \in \mathbb{R}$  such that  $0 < y < \delta$ . Additionally, since  $\delta > 0$ , Lemma 7.23 asserts that  $-\delta < 0$ . This combined with the previous result demonstrates by transitivity that  $-\delta < 0 < y < \delta$ , so by the lemma from Exercise 8.9, we have that  $|y| < \delta$ . Consequently, by Script 7, we know that  $|(y + a) - a| < \delta$  for any  $a \in \mathbb{R}$ . It follows by the above that  $|(y + a)^n - a^n| < 1$ . Thus, by Additional Exercise 0.7,  $|\sum_{k=0}^n \binom{n}{k} y^{n-k} a^k - a^n| = |y^n + ny^{n-1}a + \sum_{k=2}^{n-1} \binom{n}{k} y^{n-k} a^k| < 1$ . If we now choose  $a = \frac{1}{ny^{n-1}}$ , Script 7 reduces the above to  $|y^n + 1 + \sum_{k=2}^{n-1} y^{n-k} a^k| < 1$ . We now seek to reduce the previous statement further to  $|y^n + 1| < 1$ . To begin, Exercise 12.22 implies that  $y^n > 0$  since  $y > 0$  and  $0^n = 0$ , meaning by Script 7 that  $y^n + 1 > 0$ . Additionally, Script 7 asserts that  $\sum_{k=2}^{n-1} y^{n-k} a^k > 0$  since  $a > 0$  and  $y > 0$ . This combined with the previous result implies by Scripts 7 and 8 that  $|y^n + 1| < |y^n + 1 + \sum_{k=2}^{n-1} y^{n-k} a^k| < 1$ , as desired. However, since  $y^n > 0$ , Definition 7.21 asserts that  $y^n + 1 > 1$ . But by Definition 8.4, this implies that  $|y^n + 1| > 1$ , a contradiction.  $\square$

**Exercise 13.5.** Let  $f$  and  $g$  be uniformly continuous on  $A \subset \mathbb{R}$ . Show that

- (a) The function  $f + g$  is uniformly continuous on  $A$ .
- (b) For any constant  $c \in \mathbb{R}$ , the function  $c \cdot f$  is uniformly continuous on  $A$ .

*Proof of a.* To prove that  $f + g$  is uniformly continuous on  $A$ , Definition 13.1 tells us that it will suffice to show that for all  $\epsilon > 0$ , there exists a  $\delta > 0$  such that for all  $x, y \in A$ , if  $|y - x| < \delta$ , then  $|(f + g)(y) - (f + g)(x)| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Since  $f, g$  are uniformly continuous on  $A$ , consecutive applications of Definition 13.1 reveal that there exist  $\delta_1, \delta_2 > 0$  such that for all  $x, y \in A$ ,  $|y - x| < \delta_1$  implies  $|f(y) - f(x)| < \frac{\epsilon}{2}$  and  $|y - x| < \delta_2$  implies  $|g(y) - g(x)| < \frac{\epsilon}{2}$ . Choose  $\delta = \min(\delta_1, \delta_2)$ . Let  $x, y$  be arbitrary elements of  $A$  that satisfy  $|y - x| < \delta$ . It follows that  $|y - x| < \delta_1$  (so  $|f(y) - f(x)| < \frac{\epsilon}{2}$ ), and that  $|y - x| < \delta_2$  (so  $|g(y) - g(x)| < \frac{\epsilon}{2}$ ). These two results when combined imply by Script 7 that  $|f(y) - f(x)| + |g(y) - g(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$ . Therefore, since  $|f(y) - f(x) + g(y) - g(x)| \leq |f(y) - f(x)| + |g(y) - g(x)|$  by Lemma 8.8, we have that

$$\begin{aligned} |(f + g)(y) - (f + g)(x)| &= |f(y) - f(x) + g(y) - g(x)| \\ &\leq |f(y) - f(x)| + |g(y) - g(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

as desired.  $\square$

*Proof of b.* To prove that  $c \cdot f$  is uniformly continuous on  $A$ , Definition 13.1 tells us that it will suffice to show that for all  $\epsilon > 0$ , there exists a  $\delta > 0$  such that for all  $x, y \in A$ , if  $|y - x| < \delta$ , then  $|c \cdot f(y) - c \cdot f(x)| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. We divide into two cases ( $c = 0$  and  $c \neq 0$ ). Suppose first that  $c = 0$ . Choose  $\delta = 1$ . Let  $x, y$  be arbitrary elements of  $A$  that satisfy  $|y - x| < \delta$ . It follows that  $|0 \cdot f(y) - 0 \cdot f(x)| = 0 < \epsilon$ , as desired. Now suppose that  $c \neq 0$ . Then since  $f$  is uniformly continuous on  $A$ , Definition 13.1 tells us that there exists a  $\delta > 0$  such that for all  $x, y \in A$ , if  $|y - x| < \delta$ , then  $|f(y) - f(x)| < \frac{\epsilon}{|c|}$ . Choose this  $\delta$  to be our  $\delta$ . Let  $x, y$  be arbitrary elements of  $A$  that satisfy  $|y - x| < \delta$ . Then by the above, we have that  $|f(y) - f(x)| < \frac{\epsilon}{|c|}$ . Therefore,  $|c| \cdot |f(y) - f(x)| < \epsilon$ , so we have that  $|c \cdot f(y) - c \cdot f(x)| < \epsilon$ , as desired.  $\square$

4/13: **Theorem 13.6.** Suppose that  $X \subset \mathbb{R}$  is compact and  $f : X \rightarrow \mathbb{R}$  is continuous. Then  $f$  is uniformly continuous.

*Proof.* To prove that  $f$  is uniformly continuous, Definition 13.1 tells us that it will suffice to show that for all  $\epsilon > 0$ , there exists a  $\delta > 0$  such that for all  $x, y \in A$ , if  $|y - x| < \delta$ , then  $|f(y) - f(x)| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Since  $f$  is continuous on  $X$ , Theorem 9.10 asserts that  $f$  is continuous at every  $x \in X$ . Thus, by Theorem 11.5, for every  $x \in X$ , there exists a  $\delta_x > 0$  such that if  $y \in X$  and  $|y - x| < \delta_x$ , then  $|f(y) - f(x)| < \frac{\epsilon}{2}$ . Let  $\mathcal{G} = \{(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2}) \mid x \in X\}$ . We will now confirm that  $\mathcal{G}$  is an open cover of  $X$ . To do so, Definition 10.3 tells us that it will suffice to demonstrate that every  $x \in X$  is an element of  $(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2})$  for some  $(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2}) \in \mathcal{G}$ . Let  $x$  be an arbitrary element of  $X$ . We know that  $|x - x| = 0 < \frac{\delta_x}{2}$ . Thus, by Exercise 8.9, we have that  $x \in (x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2})$ . Since it follows from the above that  $(x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2}) \in \mathcal{G}$ , we are done.

Having shown that  $\mathcal{G}$  is an open cover of  $X$ , the fact that  $X$  is compact implies by Definition 10.4 that there exists a finite subset  $\mathcal{G}'$  of  $\mathcal{G}$  that is also an open cover of  $X$ . It follows that  $\mathcal{G}'$  will be of the form  $\{(x_i - \frac{\delta_{x_i}}{2}, x_i + \frac{\delta_{x_i}}{2}) \mid 1 \leq i \leq n\}$  where  $n$  is some natural number. Thus, choose  $\delta = \min_{1 \leq i \leq n} (\frac{\delta_{x_i}}{2})$ .

Let  $x, y$  be arbitrary elements of  $X$  that satisfy  $|y - x| < \delta$ . Since  $\mathcal{G}'$  is an open cover of  $X$ , Definition 10.3 implies that  $x \in (x_i - \frac{\delta_{x_i}}{2}, x_i + \frac{\delta_{x_i}}{2})$  for some  $(x_i - \frac{\delta_{x_i}}{2}, x_i + \frac{\delta_{x_i}}{2}) \in \mathcal{G}'$ . Considering this  $x_i$  more closely, we can determine from the previous result and Exercise 8.9 that  $|x - x_i| < \frac{\delta_{x_i}}{2}$ . This combined with the hypothesis that  $|y - x| < \delta$  implies by Script 7 that  $|y - x| + |x - x_i| < \delta + \frac{\delta_{x_i}}{2}$ . Additionally, note that by definition,  $\delta \leq \frac{\delta_{x_i}}{2}$ . Thus, combining the last few results, we have that

$$\begin{aligned} |y - x_i| &\leq |y - x| + |x - x_i| && \text{Lemma 8.8} \\ &< \delta + \frac{\delta_{x_i}}{2} \\ &\leq \frac{\delta_{x_i}}{2} + \frac{\delta_{x_i}}{2} \\ &= \delta_{x_i} \end{aligned}$$

At this point, we know that  $|x - x_i| < \frac{\delta_{x_i}}{2} < \delta_{x_i}$  and that  $|y - x_i| < \delta_{x_i}$ . It follows by consecutive applications of the above that  $|f(x) - f(x_i)| < \frac{\epsilon}{2}$  and  $|f(y) - f(x_i)| < \frac{\epsilon}{2}$ , respectively. Consequently, we have by Script 7 that  $|f(y) - f(x_i)| + |f(x) - f(x_i)| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$ . Therefore, if we combine the last several results, we get

$$\begin{aligned} |f(y) - f(x)| &\leq |f(y) - f(x_i)| + |f(x_i) - f(x)| && \text{Lemma 8.8} \\ &= |f(y) - f(x_i)| + |f(x) - f(x_i)| && \text{Exercise 8.5} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

as desired. □

**Exercise 13.7.** Prove that  $f(x) = \sqrt{x}$  is uniformly continuous on  $A = [0, +\infty)$ .

**Lemma.** Let  $x, y$  be arbitrary elements of  $A$ . Then  $|\sqrt{y} - \sqrt{x}|^2 \leq |(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x})|$ .

*Proof.* We will first verify that  $|\sqrt{y} - \sqrt{x}| \leq |\sqrt{y} + \sqrt{x}|$ . To do so, we divide into two cases ( $\sqrt{y} \geq \sqrt{x}$  and  $\sqrt{y} < \sqrt{x}$ ). If  $\sqrt{y} \geq \sqrt{x}$ , then by Definition 7.21,  $\sqrt{y} - \sqrt{x} \geq 0$ . It follows by Definition 8.4 that  $|\sqrt{y} - \sqrt{x}| = \sqrt{y} - \sqrt{x}$ . Additionally, we have by an extension of Exercise 12.22 that  $\sqrt{x} \geq 0$ , implying that  $2\sqrt{x} \geq 0$  by Definition 7.21. Thus, combining the last few results, we have that  $|\sqrt{y} - \sqrt{x}| = \sqrt{y} - \sqrt{x} \leq \sqrt{y} - \sqrt{x} + 2\sqrt{x} = \sqrt{y} + \sqrt{x}$ . Consequently, we know that  $0 \leq |\sqrt{y} - \sqrt{x}| \leq \sqrt{y} + \sqrt{x}$ , so Definition 8.4 implies that  $|\sqrt{y} + \sqrt{x}| = \sqrt{y} + \sqrt{x}$ . Therefore, we have that  $|\sqrt{y} - \sqrt{x}| \leq \sqrt{y} + \sqrt{x} = |\sqrt{y} + \sqrt{x}|$ , as desired. The argument is symmetric in the other case.

Having established that  $|\sqrt{y} - \sqrt{x}| \leq |\sqrt{y} + \sqrt{x}|$  and knowing that  $0 \leq |\sqrt{y} - \sqrt{x}|$ , we have by Lemma 7.24 that  $|\sqrt{y} - \sqrt{x}| \cdot |\sqrt{y} - \sqrt{x}| \leq |\sqrt{y} + \sqrt{x}| \cdot |\sqrt{y} - \sqrt{x}|$ . It follows by basic algebra that  $|\sqrt{y} - \sqrt{x}|^2 \leq |(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x})|$ , as desired. □

*Proof of Exercise 13.7.* To prove that  $f$  is uniformly continuous on  $A$ , Definition 13.1 tells us that it will suffice to show that for all  $\epsilon > 0$ , there exists a  $\delta > 0$  such that for all  $x, y \in A$ , if  $|y - x| < \delta$ , then  $|f(y) - f(x)| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Choose  $\delta = \epsilon^2$ . Let  $x, y$  be arbitrary elements of  $X$  that satisfy  $|y - x| < \delta$ . Thus, since  $(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x}) = y - x$ , the lemma asserts that  $|\sqrt{y} - \sqrt{x}|^2 \leq |(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x})| = |y - x| < \epsilon^2$ . Therefore, by Script 7,  $|\sqrt{y} - \sqrt{x}| < \epsilon$ , i.e.,  $|f(y) - f(x)| < \epsilon$ , as desired.  $\square$

**Corollary 13.8.** Suppose that  $f : [a, b] \rightarrow \mathbb{R}$  is continuous. Then  $f$  is uniformly continuous.

*Proof.* By Theorem 10.14,  $[a, b]$  is compact. This combined with the hypothesis that  $f$  is continuous proves by Theorem 13.6 that  $f$  is uniformly continuous.  $\square$

**Exercise 13.9.** Show that if  $f$  and  $g$  are bounded on  $A$  and uniformly continuous on  $A$ , then  $fg$  is uniformly continuous on  $A$ .

*Proof.* To prove that  $fg$  is uniformly continuous on  $A$ , Definition 13.1 tells us that it will suffice to show that for all  $\epsilon > 0$ , there exists a  $\delta > 0$  such that for all  $x, y \in A$ , if  $|y - x| < \delta$ , then  $|(fg)(y) - (fg)(x)| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary.

Since  $f$  is bounded on  $A$ , Definition 10.1 implies that  $f(A)$  is a bounded subset of  $\mathbb{R}$ . Thus, by consecutive applications of Definition 5.6, there exist numbers  $l, u$  such that for all  $f(x) \in f(A)$ ,  $l \leq f(x) \leq u$ . Let  $a = \max(|l|, |u|) + 1$ . It follows by Scripts 7 and 8 that  $-a < f(x) < a$  for all  $f(x) \in f(A)$ . Thus, by the lemma from Exercise 8.9,  $|f(x)| < a$  for all  $f(x) \in f(A)$ . Similarly, there exists a number  $b$  such that  $|g(x)| < b$  for all  $g(x) \in g(A)$ .

Since  $f$  is uniformly continuous on  $A$ , Definition 13.1 implies that there exists a  $\delta_1 > 0$  such that for all  $x, y \in A$ , if  $|y - x| < \delta_1$ , then  $|f(y) - f(x)| < \frac{\epsilon}{2b}$ . Similarly, there exists a  $\delta_2 > 0$  such that for all  $x, y \in A$ , if  $|y - x| < \delta_2$ , then  $|g(y) - g(x)| < \frac{\epsilon}{2a}$ . Choose  $\delta = \min(\delta_1, \delta_2)$ . Let  $x, y$  be arbitrary elements of  $A$  that satisfy  $|y - x| < \delta$ . It follows by consecutive applications of the above that  $|f(x)| < a$  and  $|g(y)| < b$ . Additionally,  $|y - x| < \delta \leq \delta_1$  implies that  $|f(y) - f(x)| < \frac{\epsilon}{2b}$  and  $|y - x| < \delta \leq \delta_2$  implies that  $|g(y) - g(x)| < \frac{\epsilon}{2a}$ . Therefore, combining the last four results, we have that

$$\begin{aligned}
 |(fg)(y) - (fg)(x)| &= |f(x)g(x) - f(y)g(y)| \\
 &= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\
 &= |f(x)(g(x) - g(y)) + g(y)(f(x) - f(y))| \\
 &\leq |f(x)| \cdot |g(x) - g(y)| + |g(y)| \cdot |f(x) - f(y)| \quad \text{Lemma 8.8} \\
 &< a \cdot \frac{\epsilon}{2a} + b \cdot \frac{\epsilon}{2b} \\
 &= \epsilon
 \end{aligned}$$

as desired.  $\square$

4/15: **Definition 13.10.** A **partition** of the interval  $[a, b]$  is a finite set of points in  $[a, b]$  that includes  $a$  and  $b$ . We usually write partitions as  $P = \{t_0, t_1, \dots, t_n\}$ , with the convention that

$$a = t_0 < t_1 < \dots < t_{n-1} < t_n = b$$

If  $P$  and  $Q$  are partitions of the interval  $[a, b]$  and  $P \subset Q$ , we refer to  $Q$  as a **refinement** of  $P$ .

**Definition 13.11.** Suppose that  $f : [a, b] \rightarrow \mathbb{R}$  is bounded and that  $P = \{t_0, \dots, t_n\}$  is a partition of  $[a, b]$ . Define

$$m_i(f) = \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\} \qquad M_i(f) = \sup\{f(x) \mid t_{i-1} \leq x \leq t_i\}$$

The **lower sum** of  $f$  for the partition  $P$  is the number

$$L(f, P) = \sum_{i=1}^n m_i(f)(t_i - t_{i-1})$$

The **upper sum** of  $f$  for the partition  $P$  is the number

$$U(f, P) = \sum_{i=1}^n M_i(f)(t_i - t_{i-1})$$

Notice that it is always the case that  $L(f, P) \leq U(f, P)$ .

**Lemma 13.12.** *Suppose that  $P$  and  $Q$  are partitions of  $[a, b]$  and that  $Q$  is a refinement of  $P$ . Then  $L(f, P) \leq L(f, Q)$  and  $U(f, P) \geq U(f, Q)$ .*

**Lemma.** *Let  $P$  be a partition of  $[a, b]$  and let  $y$  be an arbitrary element of  $[a, b] \setminus P$ . Then  $L(f, P) \leq L(f, P \cup \{y\})$  and  $U(f, P) \geq U(f, P \cup \{y\})$ .*

*Proof.* We will prove that  $L(f, P) \leq L(f, P \cup \{y\})$ . The proof will be symmetric in the other case. Let's begin.

By Definition 13.10,  $P$  is of the form  $\{t_0, \dots, t_n\}$  where  $a = t_0 < \dots < t_n = b$ . This combined with the hypothesis that  $y \in [a, b] \setminus P$  implies by Theorem 3.5 that  $a = t_0 < \dots < t_{k-1} < y < t_k < \dots < t_n = b$ . Thus, we have by consecutive applications of Definition 13.11 that

$$\begin{aligned} L(f, P) &= \sum_{i=1}^n m_i(f)(t_i - t_{i-1}) \\ &= \sum_{i=1}^{k-1} m_i(f)(t_i - t_{i-1}) + m_k(f)(t_k - t_{k-1}) + \sum_{i=k+1}^n m_i(f)(t_i - t_{i-1}) \end{aligned}$$

and that

$$L(f, P \cup \{y\}) = \sum_{i=1}^{k-1} m_i(f)(t_i - t_{i-1}) + m_y^-(f)(y - t_{k-1}) + m_y^+(f)(t_k - y) + \sum_{i=k+1}^n m_i(f)(t_i - t_{i-1})$$

where

$$m_y^-(f) = \inf\{f(x) \mid t_{k-1} \leq x \leq y\} \quad m_y^+(f) = \inf\{f(x) \mid y \leq x \leq t_k\}$$

As such, to prove that  $L(f, P) \leq L(f, P \cup \{y\})$ , it will suffice to show that  $m_k(f)(t_k - t_{k-1}) \leq m_y^-(f)(y - t_{k-1}) + m_y^+(f)(t_k - y)$ . To do so, it will suffice to show that  $m_k(f)(y - t_{k-1}) + m_k(f)(t_k - y) \leq m_y^-(f)(y - t_{k-1}) + m_y^+(f)(t_k - y)$ , i.e., that  $m_k(f)(y - t_{k-1}) \leq m_y^-(f)(y - t_{k-1})$  and that  $m_k(f)(t_k - y) \leq m_y^+(f)(t_k - y)$ , i.e., that  $m_k(f) \leq m_y^-(f)$  and that  $m_k(f) \leq m_y^+(f)$ .

For the sake of proving the first expression, let  $A = \{f(x) \mid t_{k-1} < x < t_k\}$  and let  $B = \{f(x) \mid t_{k-1} \leq x \leq y\}$ . It follows by Definition 13.10 that  $m_k(f) = \inf A$  and  $m_y^-(f) = \inf B$ . Thus, we need only show that  $\inf A \leq \inf B$ . Since  $y < t_k$ , we know by Script 1 that  $B \subset A$ . Thus, since  $\inf A$  is a lower bound on  $A$ , Script 5 implies that it is also a lower bound on  $B$ . Consequently, by Definition 5.7,  $\inf A \leq \inf B$ , as desired.

The argument is symmetric for the other statement.  $\square$

*Proof of Lemma 13.12.* We will prove that  $L(f, P) \leq L(f, Q)$ . The proof will be symmetric in the other case. Let's begin.

By Definition 13.10,  $P \subset Q$ . Thus, by Theorem ??,  $|P| \leq |Q|$ . It follows by Script 1 that  $|Q| - |P| = n \in \mathbb{Z}^+$ . Thus, to prove the claim for  $P$  and  $Q$  in general, it will suffice to prove it for each  $n$ . To do so, we divide into two cases ( $n = 0$  and  $n \in \mathbb{N}$ ). If  $n = 0$ , then  $|P| = |Q|$ . This combined with the fact that  $P \subset Q$  implies by Script 1 that  $P = Q$ . Therefore,  $L(f, P) = L(f, Q)$ , which we can weaken to  $L(f, P) \leq L(f, Q)$ , as desired.

On the other hand, if  $n \in \mathbb{N}$ , then we induct on  $n$ . For the base case  $n = 1$ , we have by Script 1 that  $Q = P \cup \{y\}$  where  $y \notin P$ . Therefore, by the lemma, we have that  $L(f, P) \leq L(f, P \cup \{y\}) = L(f, Q)$ , as desired. Now suppose inductively that the claim holds on  $n$ ; we wish to prove it for  $n + 1$ . Let  $y$  be an arbitrary element of  $Q$ . Then by Script 1,  $|Q \setminus \{y\}| - |P| = n$ . Thus, by the inductive hypothesis,  $L(f, P) \leq L(f, Q \setminus \{x\})$ . Additionally, by the lemma,  $L(f, Q \setminus \{x\}) \leq L(f, Q)$ . Therefore, by transitivity,  $L(f, P) \leq L(f, Q)$ , as desired.  $\square$

**Theorem 13.13.** Let  $P_1$  and  $P_2$  be partitions of  $[a, b]$  and suppose that  $f : [a, b] \rightarrow \mathbb{R}$  is bounded. Then  $L(f, P_1) \leq U(f, P_2)$ .

*Proof.* To confirm that  $P_1 \cup P_2$  is a partition of  $[a, b]$ , Definition 13.10 tells us that it will suffice to demonstrate that it is a finite set, that it is a subset of  $[a, b]$ , and that it includes  $a$  and  $b$ . Since  $P_1, P_2$  are partitions of  $[a, b]$ , Definition 13.10 implies that they are finite subsets of  $[a, b]$  that contain  $a, b$ . It follows by Script 1 that their union is finite, a subset of  $[a, b]$ , and a set containing  $a$  and  $b$ . Additionally, we have by Theorem 1.7 that  $P_1 \subset P_1 \cup P_2$  and that  $P_2 \subset P_1 \cup P_2$ . Combining the last two results with consecutive applications of Definition 13.10 reveals that  $P_1 \cup P_2$  is a refinement of both  $P_1$  and  $P_2$ .

Since  $P_1$  and  $P_1 \cup P_2$  are partitions of  $[a, b]$  and  $P_1 \cup P_2$  is a refinement of  $P_1$ , Lemma 13.12 implies that  $L(f, P_1) \leq L(f, P_1 \cup P_2)$ . Similarly,  $U(f, P_1 \cup P_2) \leq U(f, P_2)$ . Additionally, we have by Definition 13.11 that  $L(f, P_1 \cup P_2) \leq U(f, P_1 \cup P_2)$ . Therefore, if we combine the last three results with transitivity, we have that  $L(f, P_1) \leq U(f, P_2)$ , as desired.  $\square$

**Definition 13.14.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be bounded. We define

$$L(f) = \sup\{L(f, P) \mid P \text{ is a partition of } [a, b]\} \quad U(f) = \inf\{U(f, P) \mid P \text{ is a partition of } [a, b]\}$$

to be, respectively, the **lower integral** and **upper integral** of  $f$  from  $a$  to  $b$ .

**Exercise 13.15.** Why do  $L(f)$  and  $U(f)$  exist? Find a function  $f$  for which  $L(f) = U(f)$ . Find a function  $f$  for which  $L(f) \neq U(f)$ . Prove that  $L(f) \leq U(f)$ .

**Lemma.** Given  $a, b \in \mathbb{R}$  with  $a < b$ , there exists  $p \in \mathbb{R}$  such that  $p \notin \mathbb{Q}$  and  $a < p < b$ .

*Proof.* By Definition 7.21,  $a + \sqrt{2} < b + \sqrt{2}$ . Thus, by Lemma 6.10, there exists a point  $\frac{c}{d} \in \mathbb{Q}$  such that  $a + \sqrt{2} < \frac{c}{d} < b + \sqrt{2}$ . It follows that  $a < \frac{c}{d} - \sqrt{2} < b$ .

Now suppose for the sake of contradiction that  $\frac{c}{d} - \sqrt{2}$  is rational. Then by Script 2,  $\frac{c}{d} - \sqrt{2} = \frac{e}{f}$  where  $e, f \in \mathbb{Z}$  and  $f \neq 0$ . It follows by Theorem 2.10 that  $\sqrt{2} = \frac{cf - de}{df}$ , i.e., that  $\sqrt{2}$  is rational. But by the proof of Exercise 4.24,  $\sqrt{2}$  is not rational, a contradiction.  $\square$

*Proof of Exercise 13.15.* Let  $A = \{L(f, P) \mid P \text{ is a partition of } [a, b]\}$ . To prove that  $L(f) = \sup A$  exists, Theorem 5.17 tells us that it will suffice to show that  $A$  is nonempty and bounded above.

To confirm that  $A$  is nonempty, Definition 1.8 tells us that it will suffice to find an element of it. Since  $\{a, b\}$  is a finite set of points in  $[a, b]$  that includes  $a$  and  $b$  (by Script 1), Definition 13.10 asserts that  $\{a, b\}$  is a partition of  $[a, b]$ . It follows by Definition 13.11 that  $L(f, \{a, b\})$  exists. Therefore, by the definition of  $A$ , we have that  $L(f, \{a, b\}) \in A$ , as desired.

To confirm that  $A$  is bounded above, Definition 5.6 tells us that it will suffice to find a point in  $u \in \mathbb{R}$  such that for all  $L(f, P) \in A$ ,  $L(f, P) \leq u$ . Let  $u = U(f, \{a, b\})$  (since  $\{a, b\}$  is a partition of  $[a, b]$  by the above, Definition 13.10 guarantees that  $U(f, \{a, b\})$  exists). Now let  $L(f, P)$  be an arbitrary element of  $A$ . It follows from Theorem 13.13 that  $L(f, P) \leq U(f, \{a, b\}) = u$ , as desired.

The proof is symmetric for  $U(f)$ .

Let  $f : [0, 1] \rightarrow \mathbb{R}$  be defined by  $f(x) = 0$ . To prove that  $L(f) = U(f)$ , it will suffice to show that  $L(f) = 0$  and  $U(f) = 0$ . To do this, Script 5 tells us that it will suffice to verify that  $\{L(f, P) \mid P \text{ is a partition of } [a, b]\} = \{0\}$  and  $\{U(f, P) \mid P \text{ is a partition of } [a, b]\} = \{0\}$ . We will start with the first equality.

Let  $L(f, P)$  be an arbitrary element of  $\{L(f, P) \mid P \text{ is a partition of } [a, b]\}$ . Since we have

$$\begin{aligned} m_i(f) &= \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\} \\ &= \inf\{0 \mid t_{i-1} \leq x \leq t_i\} \\ &= \inf\{0\} \\ &= 0 \end{aligned}$$

for all  $m_i(f)$ , it follows that

$$\begin{aligned} L(f, P) &= \sum_{i=1}^n m_i(f)(t_i - t_{i-1}) \\ &= \sum_{i=1}^n 0(t_i - t_{i-1}) \\ &= 0 \end{aligned}$$

Therefore, since every element of  $\{L(f, P) \mid P \text{ is a partition of } [a, b]\}$  is equal to 0, the set is equal to the singleton set containing 0. The argument is symmetric for the other equality.

Let  $f : [0, 1] \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

To prove that  $L(f) \neq U(f)$ , it will suffice to show that  $L(f) = 0$  and  $U(f) = 1$ . To do this, Script 5 tells us that it will suffice to verify that  $\{L(f, P) \mid P \text{ is a partition of } [a, b]\} = \{0\}$  and  $\{U(f, P) \mid P \text{ is a partition of } [a, b]\} = \{1\}$ . We will start with the first equality.

Let  $L(f, P)$  be an arbitrary element of  $\{L(f, P) \mid P \text{ is a partition of } [a, b]\}$ . To confirm that  $L(f, P) = 0$ , Definition 13.11 tells us that it will suffice to demonstrate that  $m_i(f) = 0$  for all  $m_i(f)$ . Let  $m_i(f)$  be an arbitrary such object. By Definition 13.10,  $m_i(f) = \inf\{f(x) \mid t_{i-1} \leq x \leq t_i\}$ . By the lemma, there exists  $p \in \mathbb{R}$  such that  $p \notin \mathbb{Q}$  and  $t_{i-1} \leq p \leq t_i$ . Thus, since  $f(p) = 0$ ,  $0 \in \{f(x) \mid t_{i-1} \leq x \leq t_i\}$ . Additionally, since  $f(x) \not\leq 0$  for any  $x \in [0, 1]$  by definition, we have that  $m_i(f) = 0$ . Therefore, since every element of  $\{L(f, P) \mid P \text{ is a partition of } [a, b]\}$  is equal to 0, the set is equal to the singleton set containing 0.

As to the other equality, let  $U(f, P)$  be an arbitrary element of  $\{U(f, P) \mid P \text{ is a partition of } [a, b]\}$ . To confirm that  $U(f, P) = 1$ , Definition 13.11 tells us that we must first demonstrate that  $M_i(f) = 1$  for all  $M_i(f)$ . Let  $M_i(f)$  be an arbitrary such object. By Definition 13.10,  $M_i(f) = \sup\{f(x) \mid t_{i-1} \leq x \leq t_i\}$ . By Lemma 6.10, there exists  $p \in \mathbb{Q}$  such that  $t_{i-1} \leq p \leq t_i$ . Thus, since  $f(p) = 1$ ,  $1 \in \{f(x) \mid t_{i-1} \leq x \leq t_i\}$ . Additionally, since  $f(x) \not\geq 1$  for any  $x \in [0, 1]$  by definition, we have that  $M_i(f) = 1$ . It follows by Definition 13.11 that

$$\begin{aligned} U(f, P) &= \sum_{i=1}^n M_i(f)(t_i - t_{i-1}) \\ &= \sum_{i=1}^n (t_i - t_{i-1}) \\ &= t_n - t_0 \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

Therefore, since every element of  $\{U(f, P) \mid P \text{ is a partition of } [a, b]\}$  is equal to 1, the set is equal to the singleton set containing 1.

Suppose for the sake of contradiction that there exists a function  $f : [a, b] \rightarrow \mathbb{R}$  for which  $U(f) < L(f)$ . It follows by consecutive applications of Definition 13.14 and Lemma 5.11 that there exists an  $L(f, P_1)$  such that  $U(f) < L(f, P_1) \leq L(f)$ , and thus that there exists a  $U(f, P_2)$  such that  $U(f) \leq U(f, P_2) < L(f, P_1)$ . But this means that there exist partitions  $P_1, P_2$  of  $[a, b]$  such that  $L(f, P_1) > U(f, P_2)$ , contradicting Theorem 13.13.  $\square$

**Definition 13.16.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be bounded. We say that  $f$  is **integrable** on  $[a, b]$  if  $L(f) = U(f)$ . In this case, the common value  $L(f) = U(f)$  is called the **integral** of  $f$  from  $a$  to  $b$  and we write it as

$$\int_a^b f$$



Note that if  $f$  is an integrable function on  $[a, b]$ , it is necessarily bounded.

When we want to display the variable of integration, we write the integral as follows, including the symbol  $dx$  to indicate that variable of integration:

$$\int_a^b f(x) dx$$

For example, if  $f(x) = x^2$ , we could write  $\int_a^b x^2 dx$  but not  $\int_a^b x^2$ .

**Exercise 13.17.** Fix  $c \in \mathbb{R}$  and let  $f : [a, b] \rightarrow \mathbb{R}$  be defined by  $f(x) = c$ , for each  $x \in [a, b]$ . Show that  $f$  is integrable on  $[a, b]$  and that  $\int_a^b f = c(b - a)$ .

*Proof.* To prove that  $f$  is integrable on  $[a, b]$  and that  $\int_a^b f = c(b - a)$ , Definition 13.16 tells us that it will suffice to show that  $f$  is bounded on  $[a, b]$ , and that  $L(f) = U(f) = c(b - a)$ .

To confirm that  $f$  is bounded on  $[a, b]$ , Definition 10.1 tells us that it will suffice to demonstrate that  $f([a, b])$  is a bounded subset of  $\mathbb{R}$ . By Definition 1.18,  $f([a, b]) = \{f(x) \in \mathbb{R} \mid x \in [a, b]\}$ . But since  $f(x) = c$  for all  $x \in [a, b]$ ,  $f([a, b]) = \{c\}$ . Thus, since  $c \leq c \leq c$ , Definition 5.6 implies that  $f([a, b])$  is bounded. Additionally, since  $c \in \mathbb{R}$ , Definition 1.3 asserts that  $f([a, b]) = \{c\} \subset \mathbb{R}$ .

To confirm that  $L(f) = U(f) = c(b - a)$ , Definition 13.14 tells us that it will suffice to demonstrate that  $L(f, P) = U(f, P) = c(b - a)$  for all partitions  $P$  of  $[a, b]$ . For similar reasons to the above (i.e.,  $f(x) = c$  for all  $x \in [a, b]$ ), we can show that  $m_i(f) = M_i(f) = c$  for all  $m_i(f)$  and  $M_i(f)$ . Therefore, by Definition 13.11 that

$$\begin{aligned} L(f, P) &= \sum_{i=1}^n c(t_i - t_{i-1}) & U(f, P) &= \sum_{i=1}^n c(t_i - t_{i-1}) \\ &= c \sum_{i=1}^n (t_{i-1} - t_i) & &= c \sum_{i=1}^n (t_{i-1} - t_i) \\ &= c(t_n - t_0) & &= c(t_n - t_0) \\ &= c(b - a) & &= c(b - a) \end{aligned}$$

as desired. □

**Theorem 13.18.** Let  $f : [a, b] \rightarrow \mathbb{R}$  be bounded. Then  $f$  is integrable if and only if for every  $\epsilon > 0$ , there exists a partition  $P$  of  $[a, b]$  such that  $U(f, P) - L(f, P) < \epsilon$ .

*Proof.* Suppose first that  $f$  is integrable. Then by Definition 13.16,  $L(f) = U(f)$ . Let  $\epsilon > 0$  be arbitrary. By Script 7,  $L(f) - \frac{\epsilon}{2} < L(f)$ . Thus, by Definition 13.14 and Lemma 5.11, there exists an  $L(f, P_1) \in \{L(f, P) \mid P \text{ is a partition of } [a, b]\}$  such that  $L(f) - \frac{\epsilon}{2} < L(f, P_1) \leq L(f)$ . Similarly, there exists a  $U(f, P_2) \in \{U(f, P) \mid P \text{ is a partition of } [a, b]\}$  such that  $U(f) \leq U(f, P_2) < U(f) + \frac{\epsilon}{2}$ . Now consider  $P_1 \cup P_2$  (which we will prove is the desired partition). By Theorem 1.7,  $P_1 \subset P_1 \cup P_2$  and  $P_2 \subset P_1 \cup P_2$ . It follows by consecutive applications of Definition 13.10 that  $P_1 \cup P_2$  is a refinement of both  $P_1$  and  $P_2$ . Thus, by Lemma 13.12,  $L(f, P_1) \leq L(f, P_1 \cup P_2)$  and  $U(f, P_1 \cup P_2) \leq U(f, P_2)$ . Combining the last several results with transitivity yields

$$L(f) - \frac{\epsilon}{2} < L(f, P_1) \leq L(f, P_1 \cup P_2) \qquad U(f, P_1 \cup P_2) \leq U(f, P_2) < U(f) + \frac{\epsilon}{2}$$

Therefore, knowing that  $U(f, P_1 \cup P_2) < U(f) + \frac{\epsilon}{2}$  and that  $-L(f, P_1 \cup P_2) < \frac{\epsilon}{2} - L(f)$  (the latter by Lemma 7.24), we have by Definition 7.21 that

$$\begin{aligned} U(f, P_1 \cup P_2) - L(f, P_1 \cup P_2) &< U(f) + \frac{\epsilon}{2} + \frac{\epsilon}{2} - L(f) \\ &= \epsilon \end{aligned}$$

as desired.

Now suppose that  $f$  is not integrable; we seek to prove that there exists an  $\epsilon > 0$  such that for all partitions  $P$  of  $[a, b]$ ,  $U(f, P) - L(f, P) \geq \epsilon$ . Since  $f$  is not integrable, we have by Definition 13.16 that

$L(f) \neq U(f)$ . It follows by Exercise 13.15 that  $L(f) < U(f)$ . Thus, we can define  $\epsilon = \frac{U(f) - L(f)}{2} > 0$ . Now let  $P$  be an arbitrary partition of  $[a, b]$ . It follows that  $L(f, P) \leq L(f)$  by Definitions 13.14, 5.7, and 5.6. Similarly,  $U(f) \leq U(f, P)$ . Therefore, knowing that  $U(f) \leq U(f, P)$  and that  $-L(f) \leq -L(f, P)$  (the latter by Lemma 7.24), we have by Definition 7.21 that  $\epsilon < U(f) - L(f) \leq U(f, P) - L(f, P)$ , as desired.  $\square$

4/20: **Theorem 13.19.** *If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, then  $f$  is integrable.*

*Proof.* To prove that  $f$  is integrable, Theorem 13.18 tells us that it will suffice to show that  $f$  is bounded and that for every  $\epsilon > 0$ , there exists a partition  $P$  of  $[a, b]$  such that  $U(f, P) - L(f, P) < \epsilon$ . We will verify the two requirements separately. Let's begin.

To confirm that  $f$  is bounded, Definitions 10.1 and 5.6 tell us that it will suffice to find points  $l, u \in \mathbb{R}$  such that  $l \leq f(x) \leq u$  for all  $x \in [a, b]$ . But since  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, consecutive applications of Exercise 10.21 imply that there exist points  $c, d \in [a, b]$  such that  $f(c) \leq f(x) \leq f(d)$  for all  $x \in [a, b]$ , so we can just choose  $l = f(c)$  and  $u = f(d)$ .

As to the other stipulation, let  $\epsilon > 0$  be arbitrary. Since  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, Corollary 13.8 implies that  $f$  is uniformly continuous. Thus, by Definition 13.1, there exists a  $\delta > 0$  such that for all  $x, y \in [a, b]$ , if  $|y - x| < \delta$ , then  $|f(y) - f(x)| < \frac{\epsilon}{b-a}$ . Considering this  $\delta$ , we have by Corollary 6.12 that there exist a number  $n \in \mathbb{N}$  such that  $\frac{2(b-a)}{\delta} < n$ . Equipped with this  $n$ , we can now define the set  $P = \{\frac{b-a}{n} \cdot i + a \mid 0 \leq i \leq n\}$ .

We now seek to confirm that  $P$  is a partition of  $[a, b]$ . To do so, Definition 13.10 tells us that it will suffice to demonstrate that  $P$  is finite,  $P \subset [a, b]$ , and  $a, b \in P$ . By Script 1,  $P$  is finite. To demonstrate that  $P \subset [a, b]$ , Definition 1.3 and Equations 8.1 tell us that it will suffice to show that every  $t_i \in P$  satisfies  $a \leq t_i \leq b$ . But by Script 7, we have that

$$\begin{aligned} 0 &\leq i \leq n \\ 0 &\leq \frac{b-a}{n} \cdot i \leq b-a \\ a &\leq \frac{b-a}{n} \cdot i + a \leq b \end{aligned}$$

as desired. Lastly, consider the elements of  $P$  corresponding to  $i = 0$  and  $i = n$ . By consecutive applications of the definition of  $P$ , we have that  $a = (\frac{b-a}{n} \cdot 0 + a) \in P$  and that  $b = \frac{b-a}{n} \cdot n + a = (\frac{b-a}{n} \cdot n + a) \in P$ .

We now seek to confirm that if  $t_i, t_{i-1} \in P$ , then  $t_i - t_{i-1} < \delta$ . Let  $t_i, t_{i-1}$  be arbitrary sequential elements of  $P$ . By Script 0, we have that  $0 < n$ . Additionally, we have by hypothesis that  $0 < \delta$ . It follows by consecutive applications of Lemma 7.24 that the fact that  $\frac{2(b-a)}{\delta} < n$  implies that  $\frac{2(b-a)}{n} < \delta$ . Therefore, we have by Script 7 that

$$\begin{aligned} t_i - t_{i-1} &= \left( \frac{b-a}{n} \cdot i + a \right) - \left( \frac{b-a}{n} \cdot (i-1) + a \right) \\ &= \frac{b-a}{n} \\ &\leq \frac{2(b-a)}{n} \\ &< \delta \end{aligned}$$

as desired.

We now seek to confirm that  $M_i(f) - m_i(f) < \frac{\epsilon}{b-a}$  for all  $i$  satisfying  $1 \leq i \leq n$ . Let  $i$  be an arbitrary such number, and consider  $f|_{[t_{i-1}, t_i]}$ . Since  $f$  is continuous and  $[t_{i-1}, t_i] \subset [a, b]$ , Proposition 9.7 asserts that  $f|_{[t_{i-1}, t_i]}$  is continuous. Thus, by Exercise 10.21, there exist  $c, d \in [t_{i-1}, t_i]$  such that  $f(c) \leq f(x) \leq f(d)$  for all  $x \in [t_{i-1}, t_i]$ . It follows by consecutive applications of Definitions 13.11 and 3.3 as well as Exercise 5.9 that  $m_i(f) = f(c)$  and  $M_i(f) = f(d)$ . Additionally, since  $c, d \in [t_{i-1}, t_i]$ , we have by Script 8 that  $|d - c| \leq t_i - t_{i-1}$ . This combined with the fact that  $t_i - t_{i-1} < \delta$  by the above implies by transitivity that

$|d - c| < \delta$ . But this implies by the above that

$$\begin{aligned} M_i(f) - m_i(f) &= f(d) - f(c) \\ &\leq |f(d) - f(c)| \\ &< \frac{\epsilon}{b-a} \end{aligned}$$

as desired.

Having established that  $M_i(f) - m_i(f) < \frac{\epsilon}{b-a}$  for all  $i$  in the partition  $P$ , we have by Definition 13.11 and basic algebra that

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^n M_i(f)(t_i - t_{i-1}) - \sum_{i=1}^n m_i(f)(t_i - t_{i-1}) \\ &= \sum_{i=1}^n (M_i(f) - m_i(f))(t_i - t_{i-1}) \\ &< \sum_{i=1}^n \frac{\epsilon}{b-a} (t_i - t_{i-1}) \\ &= \frac{\epsilon}{b-a} \sum_{i=1}^n (t_i - t_{i-1}) \\ &= \frac{\epsilon}{b-a} (b-a) \\ &= \epsilon \end{aligned}$$

as desired. □

**Lemma 13.20.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be bounded. Given  $\Omega \in \mathbb{R}$ , we have  $\Omega = \int_a^b f$  if and only if for all  $\epsilon > 0$ , there is some partition  $P$  such that*

$$U(f, P) - \Omega < \epsilon \qquad \Omega - L(f, P) < \epsilon$$

*Proof.* Suppose first that  $\Omega = \int_a^b f$ . Let  $\epsilon > 0$  be arbitrary. By Theorem 13.18, there exists a partition  $P$  of  $[a, b]$  such that  $U(f, P) - L(f, P) < \epsilon$ . Choose this  $P$  to be our  $P$ . By Definition 13.16,  $\Omega = L(f) = U(f)$ . Thus, by consecutive applications of Definitions 13.14, 5.7, and 5.6, we have that  $L(f, P) \leq L(f) = \Omega$  and  $\Omega = U(f) \leq U(f, P)$ . With respect to the former result, it follows by Script 7 that  $-\Omega \leq -L(f, P)$ . Therefore, having established that  $\Omega \leq U(f, P)$ ,  $-\Omega \leq -L(f, P)$ , and  $U(f, P) - L(f, P) < \epsilon$ , we have that

$$\begin{aligned} \Omega - L(f, P) &\leq U(f, P) - L(f, P) & U(f, P) - \Omega &\leq U(f, P) - L(f, P) \\ &< \epsilon & &< \epsilon \end{aligned}$$

Now suppose that  $\Omega \neq \int_a^b f$ ; we seek to prove that there exists an  $\epsilon > 0$  such that for all partitions  $P$ ,  $U(f, P) - \Omega \geq \epsilon$  or  $\Omega - L(f, P) \geq \epsilon$ . We divide into two cases ( $\int_a^b f$  exists and  $\int_a^b f$  doesn't exist).

First, suppose that  $\int_a^b f$  exists. We divide into two subcases ( $\Omega > \int_a^b f$  and  $\Omega < \int_a^b f$ ). If  $\Omega > \int_a^b f$ , choose  $\epsilon = \Omega - \int_a^b f > 0$ . Let  $P$  be an arbitrary partition. As before, we have that  $L(f, P) \leq L(f)$ . Additionally, Definition 13.16 asserts that  $L(f) = \int_a^b f$ . Thus, transitivity implies that  $L(f, P) \leq \int_a^b f$ . It follows by Script 7 that  $-\int_a^b f \leq -L(f, P)$ . Therefore,

$$\begin{aligned} \epsilon &= \Omega - \int_a^b f \\ &\leq \Omega - L(f, P) \end{aligned}$$

as desired. The argument is symmetric in the other subcase.

Second, suppose that  $\int_a^b f$  does not exist. By Exercise 13.15,  $L(f)$  and  $U(f)$  exist. However, since  $\int_a^b f$  does not exist, Definition 13.16 asserts that  $L(f) \neq U(f)$ . It follows by Exercise 13.15 again that  $L(f) < U(f)$ . We now divide into three subcases ( $\Omega \leq L(f)$ ,  $L(f) < \Omega < U(f)$ , and  $U(f) \leq \Omega$ ). If  $\Omega \leq L(f)$ , choose  $\epsilon = U(f) - L(f) > 0$ . Let  $P$  be an arbitrary partition. As above,  $U(f) \leq U(f, P)$ . Therefore,

$$\begin{aligned}\epsilon &= U(f) - L(f) \\ &\leq U(f, P) - L(f) \\ &\leq U(f, P) - \Omega\end{aligned}$$

as desired. If  $L(f) < \Omega < U(f)$ , choose  $\epsilon = U(f) - \Omega > 0$ . Let  $P$  be an arbitrary partition. As above,  $U(f) \leq U(f, P)$ . Therefore,

$$\begin{aligned}\epsilon &= U(f) - \Omega \\ &\leq U(f, P) - \Omega\end{aligned}$$

as desired. The argument for the last subcase is symmetric to that of the first.  $\square$

**Exercise 13.21.** Define  $f : [0, b] \rightarrow \mathbb{R}$  by the formula  $f(x) = x$ . Show that  $f$  is integrable on  $[0, b]$  and that  $\int_0^b f = \frac{b^2}{2}$ .

*Proof.* To prove that  $f$  is integrable on  $[0, b]$  and that  $\int_0^b f = \frac{b^2}{2}$ , Lemma 13.20 tells us that it will suffice to show that for all  $\epsilon > 0$ , there is some partition  $P$  such that  $U(f, P) - \frac{b^2}{2} < \epsilon$  and  $\frac{b^2}{2} - L(f, P) < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Since  $\frac{2\epsilon}{b^2}$  is a positive real number by Script 7, Corollary 6.12 asserts that there exists a number  $n \in \mathbb{N}$  such that  $\frac{1}{n} < \frac{2\epsilon}{b^2}$ . Equipped with this  $n$ , we can now define the set  $P = \{\frac{b}{n} \cdot i \mid 0 \leq i \leq n\}$ . By a symmetric argument to that used in the proof of Theorem 13.19, we can confirm that  $P$  is a partition of  $[a, b]$  and that  $t_i - t_{i-1} = \frac{b}{n}$ .

We now turn our attention strictly to proving that  $U(f, P) - \frac{b^2}{2} < \epsilon$ ; the proof of the other statement will be symmetric. Under the partition  $P$  as defined, consider an arbitrary  $M_i(f)$ . By Definition 13.11,  $M_i(f) = \sup\{f(x) \mid t_{i-1} \leq x \leq t_i\}$ . Since  $f(x) = x$  for all  $x \in [t_{i-1}, t_i] \subset [0, b]$ , we have by Equations 8.1 that  $M_i(f) = \sup[t_{i-1}, t_i]$ . Thus, by Script 5,  $M_i(f) = t_i = \frac{bi}{n}$ . Therefore,

$$\begin{aligned}U(f, P) - \frac{b^2}{2} &= \sum_{i=1}^n M_i(f)(t_i - t_{i-1}) - \frac{b^2}{2} && \text{Definition 13.11} \\ &= \sum_{i=1}^n \frac{bi}{n} \left( \frac{bi}{n} - \frac{b(i-1)}{n} \right) - \frac{b^2}{2} \\ &= \frac{b^2}{n^2} \sum_{i=1}^n i(i - (i-1)) - \frac{b^2}{2} \\ &= \frac{b^2}{n^2} \sum_{i=1}^n i - \frac{b^2}{2} \\ &= \frac{b^2}{n^2} \left( \frac{1}{2}n(n+1) \right) - \frac{b^2}{2} \\ &= \frac{b^2}{2} + \frac{b^2}{2n} - \frac{b^2}{2} \\ &= \frac{b^2}{2} \cdot \frac{1}{n} \\ &< \frac{b^2}{2} \cdot \frac{2\epsilon}{b^2} \\ &= \epsilon\end{aligned}$$

as desired.  $\square$

**Exercise 13.22.** Show that the converse of Theorem 13.19 is false in general.

*Proof.* To prove that even if  $f$  is integrable,  $f : [a, b] \rightarrow \mathbb{R}$  is not necessarily continuous, we need only find an example of an integrable, discontinuous function  $f$ . Let  $f : [-1, 1] \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

To confirm that  $f$  is integrable, Theorem 13.18 tells us that it will suffice to show that for every  $\epsilon > 0$ , there exists a partition  $P$  of  $[a, b]$  such that  $U(f, P) - L(f, P) < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Choose  $P = \{-1, -\frac{\epsilon}{2}, 0, 1\}$  (clearly  $P$  is a partition of  $[-1, 1]$  by Definition 13.10). It follows by consecutive applications of Definitions 13.11, 5.7, and 5.6 that

$$\begin{array}{ll} m_1(f) = 0 & M_1(f) = 0 \\ m_2(f) = 0 & M_2(f) = 1 \\ m_3(f) = 1 & M_3(f) = 1 \end{array}$$

Therefore,

$$\begin{aligned} U(f, P) - L(f, P) &= \sum_{i=1}^3 M_i(f)(t_i - t_{i-1}) - \sum_{i=1}^3 m_i(f)(t_i - t_{i-1}) && \text{Definition 13.11} \\ &= \left[ 0 \left( -\frac{\epsilon}{2} - (-1) \right) + 1 \left( 0 - \left( -\frac{\epsilon}{2} \right) \right) + 1(1 - 0) \right] \\ &\quad - \left[ 0 \left( -\frac{\epsilon}{2} - (-1) \right) + 0 \left( 0 - \left( -\frac{\epsilon}{2} \right) \right) + 1(1 - 0) \right] \\ &= \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

However, by Corollary 5.5 and Theorem 3.14,  $0 \in LP([-1, 1])$ . Additionally, by the proof of Exercise 11.4,  $\lim_{x \rightarrow 0} f(x)$  does not exist. Combining the last two results with Theorem 11.5 reveals that  $f$  is not continuous at 0. Therefore, by Theorem 9.10,  $f$  is not continuous.  $\square$

**Theorem 13.23.** Let  $a < b < c$ . A function  $f : [a, c] \rightarrow \mathbb{R}$  is integrable on  $[a, c]$  if and only if  $f$  is integrable on  $[a, b]$  and  $[b, c]$ . When  $f$  is integrable on  $[a, c]$ , we have

$$\int_a^c f = \int_a^b f + \int_b^c f$$

**Lemma.** Let  $P_1, P_2$  be partitions of  $[a, b]$  and  $[b, c]$ , respectively. Define  $P' = P_1 \cup P_2$ . Then  $P'$  is a partition of  $[a, c]$ ,  $L(f, P') = L(f, P_1) + L(f, P_2)$ , and  $U(f, P') = U(f, P_1) + U(f, P_2)$ .

*Proof.* To prove that  $P'$  is a partition of  $[a, c]$ , Definition 13.10 tells us that it will suffice to show that  $P'$  is finite, that  $P' \subset [a, c]$ , and that  $a, c \in P'$ . By Definition 13.10,  $P_1$  and  $P_2$  are finite. Thus, by Script 1, their union  $P_1 \cup P_2 = P'$  is also finite. To confirm that  $P' \subset [a, c]$ , Definition 1.3 tells us that it will suffice to demonstrate that every  $x \in P'$  is an element of  $[a, c]$ . Let  $x$  be an arbitrary element of  $P'$ . Then by Definition 1.5,  $x \in P_1$  or  $x \in P_2$ . We now divide into two cases. If  $x \in P_1$ , then since  $P_1 \subset [a, b]$  by Definition 13.10, Definition 1.3 asserts that  $x \in [a, b]$ . Thus, by Equations 8.1,  $a \leq x \leq b$ . Moreover, by hypothesis, we have that  $a \leq x \leq b < c$ , from which it follows by Equations 8.1 that  $x \in [a, c]$ , as desired. The argument is symmetric in the other case. Lastly, by consecutive applications of Definition 13.10,  $a \in P_1$  and  $c \in P_2$ . It follows by Definition 1.5 that  $a, c \in P'$ , as desired.

Additionally, if we express  $P_1$  as containing the objects  $a = t_0, \dots, t_n = b$  and  $P_2$  as containing the objects  $b = t_n, \dots, t_{n+m} = c$ , we have that  $P'$  contains every object  $t_0$  through  $t_{n+m}$ . Therefore, we have by

consecutive applications of Definition 13.11 that

$$\begin{aligned}
 L(f, P') &= \sum_{i=1}^{n+m} m_i(f)(t_i - t_{i-1}) \\
 &= \sum_{i=1}^n m_i(f)(t_i - t_{i-1}) + \sum_{i=n+1}^{n+m} m_i(f)(t_i - t_{i-1}) \\
 &= L(f, P_1) + L(f, P_2)
 \end{aligned}$$

The proof is symmetric for the other statement.  $\square$

*Proof of Theorem 13.23.* Suppose first that  $f$  is integrable on  $[a, c]$ . To prove that  $f$  is integrable on  $[a, b]$  and  $[b, c]$ , Theorem 13.18 tells us that it will suffice to show that for every  $\epsilon > 0$ , there exist partitions  $P_1, P_2$  of  $[a, b]$  and  $[b, c]$ , respectively, such that  $U(f, P_1) - L(f, P_1) < \epsilon$  and  $U(f, P_2) - L(f, P_2) < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Since  $f$  is integrable on  $[a, c]$ , there exists a partition  $P$  of  $[a, c]$  such that  $U(f, P) - L(f, P) < \epsilon$ . Now define  $P' = P \cup \{b\}$ . Since  $P'$  is finite (by Script 1), a subset of  $[a, c]$  (because  $P \subset [a, c]$  by Definition 13.10 and  $\{b\} \subset [a, c]$ ), and contains  $a, c$  (because  $a, c \in P$  implies  $a, c \in P \cup \{b\}$  by Definition 1.5), Definition 13.10 asserts that  $P'$  is a partition of  $[a, c]$ . Furthermore, since  $P \subset P'$  by Theorem 1.7, Definition 13.10 implies that  $P'$  is a refinement of  $P$ . Thus, by Lemma 13.12,  $L(f, P) \leq L(f, P')$  and  $U(f, P) \geq U(f, P')$ . This combined with the fact that  $U(f, P) - L(f, P) < \epsilon$  implies by Script 7 that  $U(f, P') - L(f, P') \leq U(f, P) - L(f, P) < \epsilon$ .

Let  $P_1 = P' \cap [a, b]$  and  $P_2 = P' \cap [b, c]$ . In the same manner as before, we have that  $P_1$  is a partition of  $[a, b]$  and  $P_2$  is a partition of  $[b, c]$ . This combined with the fact that  $P_1 \cup P_2 = P' \cap ([a, b] \cup [b, c]) = P'$  by Script 1 implies by the lemma that  $L(f, P') = L(f, P_1) + L(f, P_2)$  and  $U(f, P') = U(f, P_1) + U(f, P_2)$ . Thus, we have that

$$\begin{aligned}
 (U(f, P_1) - L(f, P_1)) + (U(f, P_2) - L(f, P_2)) &= U(f, P') - L(f, P') \\
 &< \epsilon
 \end{aligned}$$

Additionally, we have by consecutive applications of Definition 13.11 that  $L(f, P_1) \leq U(f, P_1)$  and  $L(f, P_2) \leq U(f, P_2)$ . It follows by consecutive applications of Definition 7.21 that  $0 \leq U(f, P_1) - L(f, P_1)$  and  $0 \leq U(f, P_2) - L(f, P_2)$ . This combined with the above result that  $(U(f, P_1) - L(f, P_1)) + (U(f, P_2) - L(f, P_2)) < \epsilon$  implies by Script 7 that  $U(f, P_1) - L(f, P_1) < \epsilon$  and  $U(f, P_2) - L(f, P_2) < \epsilon$ .

Now suppose that  $f$  is integrable on  $[a, b]$  and  $[b, c]$ . Let  $\Omega_1 = \int_a^b f$ ,  $\Omega_2 = \int_b^c f$ , and  $\Omega = \Omega_1 + \Omega_2$ . Thus, to prove that  $f$  is integrable on  $[a, c]$  and that  $\int_a^c f = \int_a^b f + \int_b^c f$ , Lemma 13.20 tells us that it will suffice to show that for all  $\epsilon > 0$ , there is some partition  $P'$  of  $[a, c]$  such that  $U(f, P') - \Omega < \epsilon$  and  $\Omega - L(f, P') < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Since  $f$  is integrable on  $[a, b]$  and  $[b, c]$ , we have by consecutive applications of Lemma 13.20 that there exist partitions  $P_1$  of  $[a, b]$  and  $P_2$  of  $[b, c]$  such that  $U(f, P_1) - \Omega_1 < \frac{\epsilon}{2}$ ,  $\Omega_1 - L(f, P_1) < \frac{\epsilon}{2}$ ,  $U(f, P_2) - \Omega_2 < \frac{\epsilon}{2}$ , and  $\Omega_2 - L(f, P_2) < \frac{\epsilon}{2}$ . Choose  $P' = P_1 \cup P_2$ . By the lemma,  $P'$  is a partition of  $[a, c]$ . Combining all of the above results implies by Script 7 and the lemma that

$$\begin{aligned}
 U(f, P') - \Omega &= U(f, P_1) + U(f, P_2) - \Omega_1 - \Omega_2 & \Omega - L(f, P') &= \Omega_1 + \Omega_2 - L(f, P_1) - L(f, P_2) \\
 &= (U(f, P_1) - \Omega_1) + (U(f, P_2) - \Omega_2) & &= (\Omega_1 - L(f, P_1)) + (\Omega_2 - L(f, P_2)) \\
 &< \frac{\epsilon}{2} + \frac{\epsilon}{2} & &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
 &= \epsilon & &< \epsilon
 \end{aligned}$$

Note that since the claim technically asks us to prove that  $\int_a^c f = \int_a^b f + \int_b^c f$  follows from  $f$  being integrable on  $[a, c]$ , not  $[a, b]$  and  $[b, c]$ , we can do this with the above using the following logic. Let  $f$  be integrable on  $[a, c]$ . Then by the first part of the proof, it is integrable on  $[a, b]$  and  $[b, c]$ . It follows by the second part of the proof that  $\int_a^c f = \int_a^b f + \int_b^c f$ , as desired.  $\square$

4/22: If  $b < a$ , we define

$$\int_a^b f = - \int_b^a f$$

whenever the latter integral exists. With this notational convention, it follows that the equation

$$\int_a^c f = \int_a^b f + \int_b^c f$$

always holds, regardless of the ordering of  $a, b, c$  whenever  $f$  is integrable on the largest of the three intervals.

**Theorem 13.24.** Suppose that  $f$  and  $g$  are integrable functions on  $[a, b]$  and that  $c \in \mathbb{R}$  is a constant. Then  $f + g$  and  $cf$  are integrable on  $[a, b]$  and

$$(a) \int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

$$(b) \int_a^b cf = c \int_a^b f.$$

**Lemma.**

(a) Let  $P = \{t_0, \dots, t_n\}$  be an arbitrary partition of  $[a, b]$ . Then for any  $i \in [n]$ , we have  $M_i(f + g) \leq M_i(f) + M_i(g)$  and  $m_i(f + g) \geq m_i(f) + m_i(g)$ .

(b) Let  $P = \{t_0, \dots, t_n\}$  be an arbitrary partition of  $[a, b]$ . Then if  $c > 0$ , we have  $M_i(cf) = c \cdot M_i(f)$  and  $m_i(cf) = c \cdot m_i(f)$  for any  $i \in [n]$ .

(c) Let  $P = \{t_0, \dots, t_n\}$  be an arbitrary partition of  $[a, b]$ . Then if  $c < 0$ , we have  $M_i(cf) = c \cdot m_i(f)$  and  $m_i(cf) = c \cdot M_i(f)$  for any  $i \in [n]$ .

*Proof of Lemma (a).* Let  $i$  be an arbitrary natural number satisfying  $1 \leq i \leq n$ . By Definitions 13.11, 5.7, and 5.6,  $f(x) \leq M_i(f)$  for all  $x \in [t_{i-1}, t_i]$ . Similarly,  $g(x) \leq M_i(g)$  for all  $x \in [t_{i-1}, t_i]$ . Thus, we have by Definition 7.21 that  $(f + g)(x) \leq M_i(f) + M_i(g)$  for all  $x \in [t_{i-1}, t_i]$ . Consequently, Definition 5.6 asserts that  $M_i(f) + M_i(g)$  is an upper bound on  $\{(f + g)(x) \mid t_{i-1} \leq x \leq t_i\}$ . Therefore, the supremum of that set will be less than or equal to  $M_i(f) + M_i(g)$  by Definition 5.7. But since  $M_i(f + g)$  is said supremum by Definition 13.11, we have that  $M_i(f + g) \leq M_i(f) + M_i(g)$  as desired.

The proof is symmetric in the other case.  $\square$

*Proof of Lemma (b).* Suppose for the sake of contradiction that  $M_i(cf) \neq c \cdot M_i(f)$ . We divide into two cases ( $M_i(cf) < c \cdot M_i(f)$  and  $M_i(cf) > c \cdot M_i(f)$ ). If  $M_i(cf) < c \cdot M_i(f)$ , then since  $c > 0$ , Lemma 7.24 implies that  $\frac{M_i(cf)}{c} < M_i(f)$ . It follows by Lemma 5.11 that there exists  $f(x) \in \{f(x) \mid t_{i-1} \leq x \leq t_i\}$  such that  $\frac{M_i(cf)}{c} < f(x) \leq M_i(f)$ , i.e.,  $M_i(cf) < cf(x)$ . But by Definitions 13.11, 5.7, and 5.6,  $cf(x) \leq M_i(cf)$  for all  $x \in [t_{i-1}, t_i]$ , a contradiction. The argument is symmetric in the other case.

The proof is symmetric in the other case.  $\square$

*Proof of Lemma (c).* Suppose for the sake of contradiction that  $M_i(cf) \neq c \cdot m_i(f)$ . We divide into two cases ( $M_i(cf) < c \cdot m_i(f)$  and  $M_i(cf) > c \cdot m_i(f)$ ). If  $M_i(cf) < c \cdot m_i(f)$ , then since  $c < 0$ , Lemma 7.24 implies that  $\frac{M_i(cf)}{c} > m_i(f)$ . It follows by Lemma 5.11 that there exists  $f(x) \in \{f(x) \mid t_{i-1} \leq x \leq t_i\}$  such that  $\frac{M_i(cf)}{c} > f(x) \geq m_i(f)$ , i.e.,  $M_i(cf) < cf(x)$ . But by Definitions 13.11, 5.7, and 5.6,  $cf(x) \leq M_i(cf)$  for all  $x \in [t_{i-1}, t_i]$ , a contradiction. The argument is symmetric in the other case.

The proof is symmetric in the other case.  $\square$

*Proof of Theorem 13.24a.* Let  $\Omega_f = \int_a^b f$ ,  $\Omega_g = \int_a^b g$ , and  $\Omega = \Omega_f + \Omega_g$ . To prove that  $f + g$  is integrable on  $[a, b]$  and that  $\int_a^b (f + g) = \int_a^b f + \int_a^b g$ , Lemma 13.20 tells us that it will suffice to show that for all  $\epsilon > 0$ , there is some partition  $P$  such that  $U(f + g, P) - \Omega < \epsilon$  and  $\Omega - L(f + g, P) < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Since  $f, g$  are integrable on  $[a, b]$ , we have by consecutive applications of Lemma 13.20 that there exist partitions  $Q, R$  of  $[a, b]$  such that  $U(f, Q) - \Omega_f < \frac{\epsilon}{2}$ ,  $\Omega_f - L(f, Q) < \frac{\epsilon}{2}$ ,  $U(g, R) - \Omega_g < \frac{\epsilon}{2}$ , and  $\Omega_g - L(g, R) < \frac{\epsilon}{2}$ . As in previous proofs,  $P = Q \cup R$  is also a partition of  $[a, b]$  and a refinement of both  $Q$  and  $R$ . Consequently, we have that  $U(f, P) - \Omega_f \leq U(f, Q) - \Omega_f < \frac{\epsilon}{2}$ ,  $\Omega_f - L(f, P) \leq \Omega_f - L(f, Q) < \frac{\epsilon}{2}$ ,

$U(g, P) - \Omega_g \leq U(g, R) - \Omega_g < \frac{\epsilon}{2}$ , and  $\Omega_g - L(g, P) \leq \Omega_g - L(g, R) < \frac{\epsilon}{2}$ . It follows by consecutive applications of Script 7 that  $U(f, P) + U(g, P) - \Omega < \epsilon$  and that  $\Omega - (L(f, P) + L(g, P)) < \epsilon$ . Therefore, we have that

$$\begin{aligned}
 U(f + g, P) - \Omega &= \sum_{i=1}^n M_i(f + g)(t_i - t_{i-1}) - \Omega && \text{Definition 13.11} \\
 &\leq \sum_{i=1}^n (M_i(f) + M_i(g))(t_i - t_{i-1}) - \Omega && \text{Lemma (a)} \\
 &= \sum_{i=1}^n M_i(f)(t_i - t_{i-1}) + \sum_{i=1}^n M_i(g)(t_i - t_{i-1}) - \Omega \\
 &= U(f, P) + U(g, P) - \Omega && \text{Definition 13.11} \\
 &< \epsilon
 \end{aligned}$$

and something similar for  $\Omega - L(f + g, P)$ . □

*Proof of Theorem 13.24b.* We divide into three cases ( $c = 0$ ,  $c > 0$ , and  $c < 0$ ).

If  $c = 0$ , then we have that  $cf(x) = 0$  for all  $x \in [a, b]$ . Therefore, we have by Exercise 13.17 that  $cf$  is integrable on  $[a, b]$  and

$$\begin{aligned}
 \int_a^b cf &= 0(b - a) \\
 &= 0 \\
 &= 0 \cdot \int_a^b f \\
 &= c \int_a^b f
 \end{aligned}$$

If  $c > 0$ , then let  $\Omega = \int_a^b f$ . To prove that  $cf$  is integrable on  $[a, b]$  and that  $\int_a^b cf = c \int_a^b f$ , Lemma 13.20 tells us that it will suffice to show that for all  $\epsilon > 0$ , there is some partition  $P$  such that  $U(cf, P) - c\Omega < \epsilon$  and  $c\Omega - L(cf, P) < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Since  $f$  is integrable on  $[a, b]$ , we have by Lemma 13.20 that there exists a partition  $P$  such that  $U(f, P) - \Omega < \frac{\epsilon}{c}$  and  $\Omega - L(f, P) < \frac{\epsilon}{c}$ . It follows by consecutive applications of Lemma 7.24 that  $cU(f, P) - c\Omega < \epsilon$  and  $c\Omega - cL(f, P) < \epsilon$ . Therefore, we have that

$$\begin{aligned}
 U(cf, P) - c\Omega &= \sum_{i=1}^n M_i(cf)(t_i - t_{i-1}) - c\Omega && \text{Definition 13.11} \\
 &= \sum_{i=1}^n c \cdot M_i(f)(t_i - t_{i-1}) - c\Omega && \text{Lemma (b)} \\
 &= c \sum_{i=1}^n M_i(f)(t_i - t_{i-1}) - c\Omega \\
 &= cU(f, P) - c\Omega && \text{Definition 13.11} \\
 &< \epsilon
 \end{aligned}$$

and something similar for  $c\Omega - L(cf, P)$ .

If  $c < 0$ , then let  $\Omega = \int_a^b f$ . To prove that  $cf$  is integrable on  $[a, b]$  and that  $\int_a^b cf = c \int_a^b f$ , Lemma 13.20 tells us that it will suffice to show that for all  $\epsilon > 0$ , there is some partition  $P$  such that  $U(cf, P) - c\Omega < \epsilon$  and  $c\Omega - L(cf, P) < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Since  $f$  is integrable on  $[a, b]$ , we have by Lemma 13.20 that there exists a partition  $P$  such that  $U(f, P) - \Omega < \frac{\epsilon}{-c}$  and  $\Omega - L(f, P) < \frac{\epsilon}{-c}$ . It follows by consecutive



applications of Lemma 7.24 that  $c\Omega - cU(f, P) < \epsilon$  and  $cL(f, P) - c\Omega < \epsilon$ . Therefore, we have that

$$\begin{aligned}
 U(cf, P) - c\Omega &= \sum_{i=1}^n M_i(cf)(t_i - t_{i-1}) - c\Omega && \text{Definition 13.11} \\
 &= \sum_{i=1}^n c \cdot m_i(f)(t_i - t_{i-1}) - c\Omega && \text{Lemma (c)} \\
 &= c \sum_{i=1}^n m_i(f)(t_i - t_{i-1}) - c\Omega \\
 &= cL(f, P) - c\Omega && \text{Definition 13.11} \\
 &< \epsilon
 \end{aligned}$$

and something similar for  $c\Omega - L(cf, P)$ . □

4/27: **Theorem 13.25.** Suppose that  $f$  and  $g$  are integrable functions on  $[a, b]$  with  $f(x) \leq g(x)$  for all  $x \in [a, b]$ . Then

$$\int_a^b f \leq \int_a^b g$$

*Proof.* Suppose for the sake of contradiction that  $\int_a^b f > \int_a^b g$ . Then by Definition 13.16,  $L(f) > L(g)$ . It follows by Lemma 5.11 that there exists a  $L(f, P) \in \{L(f, P) \mid P \text{ is a partition of } [a, b]\}$  such that  $L(f) \geq L(f, P) > L(g)$ . Thus, since  $L(g, P) \leq L(g)$  by Definitions 13.14, 5.7, and 5.6, we have that  $L(g, P) < L(f, P)$ . Consequently, by Definition 13.11,  $\sum_{i=1}^n m_i(g)(t_i - t_{i-1}) < \sum_{i=1}^n m_i(f)(t_i - t_{i-1})$ . Thus, by Script 7, there exists an  $i$  such that  $m_i(g) < m_i(f)$ . It follows by Lemma 5.11 that there exists a  $g(x) \in \{g(x) \mid t_{i-1} \leq x \leq t_i\}$  such that  $m_i(g) \leq g(x) < m_i(f)$ . But this implies by Definitions 13.11, 5.7, and 5.6 that  $g(x) < f(x)$ , a contradiction. □

4/29: **Theorem 13.26.** Suppose that  $f$  is an integrable function on  $[a, b]$ . Then  $|f|$  is also integrable on  $[a, b]$  and

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

**Lemma.** Let  $P = \{t_0, \dots, t_n\}$  be an arbitrary partition of  $[a, b]$ . Then for any  $i \in [n]$ , the following inequality holds.

$$M_i(|f|) - m_i(|f|) \leq M_i(f) - m_i(f)$$

*Proof.* Let  $i$  be an arbitrary natural number satisfying  $1 \leq i \leq n$ . We divide into three cases ( $f(x) \geq 0$  for all  $x \in [a, b]$ ,  $f(x) \leq 0$  for all  $x \in [a, b]$ , and there exist  $x, y \in [a, b]$  such that  $f(x) < 0 < f(y)$ ). Let's begin.

First, suppose that  $f(x) \geq 0$  for all  $x \in [a, b]$ . Then by Definition 8.4,  $|f(x)| = f(x)$  for all  $x \in [a, b]$ . It follows that  $M_i(|f|) - m_i(|f|) = M_i(f) - m_i(f)$ , which can be weakened to  $M_i(|f|) - m_i(|f|) \leq M_i(f) - m_i(f)$ , as desired.

Second, suppose that  $f(x) \leq 0$  for all  $x \in [a, b]$ . Then by Definition 8.4,  $|f(x)| = -f(x)$  for all  $x \in [a, b]$ . It follows that

$$\begin{aligned}
 M_i(|f|) - m_i(|f|) &= M_i(-f) - m_i(-f) \\
 &= -m_i(f) - (-M_i(f)) && \text{Lemma (c), Theorem 13.24} \\
 &= M_i(f) - m_i(f)
 \end{aligned}$$

which can be weakened to  $M_i(|f|) - m_i(|f|) \leq M_i(f) - m_i(f)$ , as desired.

Third, suppose that there exist  $x, y \in [a, b]$  such that  $f(x) < 0 < f(y)$ . We divide into two subcases ( $|M_i(f)| \geq |m_i(f)|$  and  $|M_i(f)| < |m_i(f)|$ ).

Suppose first that  $|M_i(f)| \geq |m_i(f)|$ . By Definitions 13.11, 5.7, and 5.6 as well as the hypothesis,  $M_i(f) \geq f(y) > 0$ . Thus, by Definition 8.4,  $|M_i(f)| = M_i(f)$ . Similarly,  $|m_i(f)| = -m_i(f)$ . It follows by

Lemma (c) from Theorem 13.24 that  $-m_i(f) = M_i(-f)$ . This combined with the fact that  $|M_i(f)| = M_i(f)$  implies by the hypothesis that  $M_i(f) \geq M_i(-f)$ . Additionally, we clearly have that  $M_i(f) \geq M_i(f)$ . Consequently, since  $M_i(f) \geq M_i(-f)$  and  $M_i(f) \geq M_i(f)$ , we have by Script 5 that  $M_i(f) \geq M_i(|f|)$ . Furthermore, we have by Definitions 13.11, 5.7, 5.6, and 8.4 that  $m_i(|f|) \geq 0 > f(x) \geq m_i(f)$ . Therefore, since  $M_i(|f|) \leq M_i(f)$  and  $m_i(f) < m_i(|f|)$ , we have that

$$\begin{aligned} M_i(|f|) - m_i(|f|) &\leq M_i(f) - m_i(|f|) \\ &< M_i(f) - m_i(f) \end{aligned}$$

which can be weakened to  $M_i(|f|) - m_i(|f|) \leq M_i(f) - m_i(f)$ , as desired.

Now suppose that  $|M_i(f)| < |m_i(f)|$ . As before,  $M_i(f) > 0$  and  $m_i(|f|) \geq 0$ . It follows from the former result by Lemma 7.23 that  $-M_i(f) < 0$ . This combined with the previous result implies by transitivity that  $-M_i(f) \leq m_i(|f|)$ . Additionally, we have as before that  $-m_i(f) = M_i(-f)$  and  $M_i(f) = |M_i(f)| < |m_i(f)| = -m_i(f) = M_i(-f)$ . Thus,  $M_i(|f|) = M_i(-f) = -m_i(f)$ . This combined with the fact that  $-M_i(f) \leq m_i(|f|)$  implies by Definition 7.21 that  $M_i(|f|) - M_i(f) < m_i(|f|) - m_i(f)$ . It follows by consecutive applications of Definition 7.21 that  $M_i(|f|) - m_i(|f|) < M_i(f) - m_i(f)$ , which can be weakened to  $M_i(|f|) - m_i(|f|) \leq M_i(f) - m_i(f)$ , as desired.  $\square$

*Proof of Theorem 13.26.* To prove that  $|f|$  is integrable on  $[a, b]$ , Theorem 13.18 tells us that it will suffice to show that for every  $\epsilon > 0$ , there exists a partition  $P$  of  $[a, b]$  such that  $U(|f|, P) - L(|f|, P) < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. By Theorem 13.18, there exists a partition  $P$  of  $[a, b]$  such that  $U(f, P) - L(f, P) < \epsilon$ . Therefore,

$$\begin{aligned} U(|f|, P) - L(|f|, P) &= \sum_{i=1}^n M_i(|f|)(t_i - t_{i-1}) - \sum_{i=1}^n m_i(|f|)(t_i - t_{i-1}) && \text{Definition 13.11} \\ &= \sum_{i=1}^n (M_i(|f|) - m_i(|f|))(t_i - t_{i-1}) \\ &\leq \sum_{i=1}^n (M_i(f) - m_i(f))(t_i - t_{i-1}) && \text{The Lemma} \\ &= \sum_{i=1}^n M_i(f)(t_i - t_{i-1}) - \sum_{i=1}^n m_i(f)(t_i - t_{i-1}) \\ &= U(f, P) - L(f, P) && \text{Definition 13.11} \\ &< \epsilon \end{aligned}$$

as desired.

We now seek to prove that  $|\int_a^b f| \leq \int_a^b |f|$ . By Script 8,  $-|f(x)| \leq f(x) \leq |f(x)|$  for all  $x \in [a, b]$ . It follows by consecutive applications of Theorem 13.25 that  $\int_a^b -|f| \leq \int_a^b f \leq \int_a^b |f|$ . Thus, since Theorem 13.24 asserts that  $\int_a^b -|f| = -\int_a^b |f|$ , we have that  $-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|$ . Therefore, by the lemma to Exercise 8.9,  $|\int_a^b f| \leq \int_a^b |f|$ , as desired.  $\square$

5/4: **Theorem 13.27.** Suppose that  $f$  is integrable on  $[a, b]$  and  $m \leq f(x) \leq M$  for all  $x \in [a, b]$ . Then

$$m(b-a) \leq \int_a^b f \leq M(b-a)$$

*Proof.* Let  $g, h : [a, b] \rightarrow \mathbb{R}$  be defined by  $g(x) = m$  and  $h(x) = M$ . By consecutive applications of Exercise 13.17,  $g$  and  $h$  are integrable on  $[a, b]$  with  $\int_a^b g = m(b-a)$  and  $\int_a^b h = M(b-a)$ . Additionally, we have by the definitions of  $g$  and  $h$  that  $g(x) = m \leq f(x) \leq M = h(x)$ . This combined with the fact that both  $g$  and  $h$  are integrable implies by consecutive applications of Theorem 13.25 that  $\int_a^b g \leq \int_a^b f \leq \int_a^b h$ . But this implies by the above that  $m(b-a) \leq \int_a^b f \leq M(b-a)$ , as desired.  $\square$

**Theorem 13.28.** Suppose that  $f$  is integrable on  $[a, b]$ . Define  $F : [a, b] \rightarrow \mathbb{R}$  by

$$F(x) = \int_a^x f$$

Then  $F$  is continuous.

*Proof.* To prove that  $F$  is continuous, Theorem 9.10 tells us that it will suffice to show that  $F$  is continuous at every  $x \in [a, b]$ . Let  $x$  be an arbitrary element of  $[a, b]$ . To show that  $F$  is continuous at  $x$ , Theorem 11.5 tells us that it will suffice to verify that for every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that if  $y \in [a, b]$  and  $|y - x| < \delta$ , then  $|F(y) - F(x)| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. By Scripts 1 and 5, the fact that  $\{f(x) \mid x \in [a, b]\}$  is nonempty and bounded implies that  $\{|f(x)| \mid x \in [a, b]\}$  is nonempty and bounded above. Thus, by Theorem 5.17,  $\sup\{|f(x)| \mid x \in [a, b]\}$  exists. As such, we may define  $s = \sup\{|f(x)| \mid x \in [a, b]\}$  so that we may choose  $\delta = \frac{\epsilon}{s}$ . Now let  $y$  be an arbitrary element of  $[a, b]$  such that  $|y - x| < \delta$ . Therefore,

$$\begin{aligned} |F(y) - F(x)| &= \left| \int_a^y f - \int_a^x f \right| \\ &= \left| \int_x^y f \right| && \text{Theorem 13.23} \\ &\leq \int_x^y |f| && \text{Theorem 13.26} \\ &\leq s(y - x) && \text{Theorem 13.27} \\ &< s \cdot \frac{\epsilon}{s} \\ &= \epsilon \end{aligned}$$

□