# MATH 16310 (Honors Calculus III IBL) Notes

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# Contents

# Script 12

# Derivatives

## 12.1 Journal

3/30: Throughout this sheet, we let $f : A \to \mathbb{R}$ be a real valued function with domain $A \subset \mathbb{R}$. We also now assume the domain $A \subset \mathbb{R}$ is open.

**Definition 12.1.** The **derivative** of $f$ at a point $a \in A$ is the number $f'(a)$ defined by the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided that the limit on the right-hand side exists. If $f'(a)$ exists, we say that $f$ is **differentiable** (at $a$). If $f$ is differentiable at all points of its domain, we say that $f$ is **differentiable**. In this case, the values $f'(a)$ define a new function $f' : A \to \mathbb{R}$ called the **derivative** (of $f$).

**Remark 12.2.** If $A$ is not open, the limit in Definition 12.1 may not exist. For example, if $f : [a,b] \to \mathbb{R}$, then we cannot define the derivative at the endpoints. For any $c$ in the domain of $f$, we define the **right-hand derivative** $f'_+(c)$ and the **left-hand derivative** $f'_-(c)$ by

$$f'_+(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \qquad f'_-(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$$

We say that $f$ is **differentiable** (on $[a,b]$) if $f$ is differentiable on $(a,b)$ and $f'_+(a)$ and $f'_-(b)$ exist.

**Lemma 12.3.** *Let $a \in \mathbb{R}$. Then*

$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$$

*assuming that one of the two limits exists. (So if the limit on the left exists, so does the one on the right, and they are equal. Similarly, if the limit on the right exists, then so does the one on the left, and they are equal.)*

*Proof.* Suppose first that $\lim_{x \to a} f(x)$ exists, and let it be equal to $L$. To prove that $\lim_{h \to 0} f(a+h)$ exists and that it equals $\lim_{x \to a} f(x)$, Definition 11.1 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $(h+a) \in A$ and $0 < |h - 0| = |h| < \delta$, then $|f(a+h) - L| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{x \to a} f(x)$ exists, Definition 11.1 implies that there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$. We will choose this $\delta$ to be our $\delta$. Now suppose that $h$ is any number satisfying both $(h+a) \in A$ and $0 < |h| < \delta$; we seek to show that $|f(a+h) - L| < \epsilon$. Since $(h+a) \in A$, $h + a = x$ for some $x \in A$. It follows that $h = x - a$, meaning that $x$ is an object that is both an element of $A$ and that satisfies $0 < |h| = |x - a| < \delta$, so we know that $|f(a+h) - L| = |f(x) - L| < \epsilon$, as desired.

The proof is symmetric in the other direction. $\square$

## 12.2 Discussion

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