Script 12

Derivatives

12.1 Journal

Throughout this sheet, we let $f : A \to \mathbb{R}$ be a real valued function with domain $A \subset \mathbb{R}$. We also now assume the domain $A \subset \mathbb{R}$ is open.

Definition 12.1. The **derivative** of f at a point $a \in A$ is the number $f'(a)$ defined by the limit

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided that the limit on the right-hand side exists. If $f'(a)$ exists, we say that f is **differentiable** (at a). If f is differentiable at all points of its domain, we say that f is **differentiable**. In this case, the values $f'(a)$ define a new function $f' : A \to \mathbb{R}$ called the **derivative** (of f).

Remark 12.2. If A is not open, the limit in Definition 12.1 may not exist. For example, if $f : [a, b] \to \mathbb{R}$, then we cannot define the derivative at the endpoints. For any c in the domain of f, we define the **right-hand derivative** $f'_+(c)$ and the **left-hand derivative** $f'_-(c)$ by

$$f'_+(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \qquad\qquad f'_-(c) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$$

We say that f is **differentiable** (on $[a, b]$) if f is differentiable on (a, b) and $f'_+(a)$ and $f'_-(b)$ exist.

Lemma 12.3. *Let $a \in \mathbb{R}$. Then*

$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a + h)$$

assuming that one of the two limits exists. (So if the limit on the left exists, so does the one on the right, and they are equal. Similarly, if the limit on the right exists, then so does the one on the left, and they are equal.)

Proof. Suppose first that $\lim_{x \to a} f(x)$ exists, and let it be equal to L. To prove that $\lim_{h \to 0} f(a + h)$ exists and that it equals $\lim_{x \to a} f(x)$, Definition 11.1 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $(h + a) \in A$ and $0 < |h - 0| = |h| < \delta$, then $|f(a + h) - L| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{x \to a} f(x)$ exists, Definition 11.1 implies that there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$. We will choose this δ to be our δ. Now let h be an arbitrary number satisfying both $(h + a) \in A$ and $0 < |h| < \delta$; we seek to show that $|f(a + h) - L| < \epsilon$. Since $(h + a) \in A$, $h + a = x$ for some $x \in A$. It follows that $h = x - a$, meaning that x is an object that is both an element of A and that satisfies $0 < |h| = |x - a| < \delta$, so we know that $|f(a + h) - L| = |f(x) - L| < \epsilon$, as desired.

The proof is symmetric in the other direction. $\square$

Theorem 12.4. *Let $a \in \mathbb{R}$. Then f is differentiable at a if and only if $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists. Moreover, if f is differentiable at a, then the derivative of f at a is given by the limit*

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Proof. Suppose first that f is differentiable at a. Then by Definition 12.1, $f'(a)$ exists. It follows that

$$
\begin{aligned}
f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} & \text{Definition 12.1} \\
&= \lim_{h \to 0} \frac{f(a+h) - f(a)}{(a+h) - a} \\
&= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} & \text{Lemma 12.3}
\end{aligned}
$$

Note that the substitution in the last step follows from using $\tilde{f}(x) = \frac{f(x) - f(a)}{x - a}$ as the "$f(x)$" function in Lemma 12.3.

The proof is symmetric in the reverse direction. $\qquad \square$

Theorem 12.5. *If f is differentiable at a, then f is continuous at a.*

Proof. To prove that f is continuous at a, Theorem 11.5 tells us that it will suffice to show that $\lim_{x \to a} f(x) = f(a)$. By Definition 12.1, the hypothesis implies that $f'(a)$ exists. Thus, by Theorem 12.4, we know that $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists. Additionally, by Exercise 11.6, $g(x) = x - a$ is continuous at a. It follows by Theorem 11.5 that either $a \notin LP(A)$ or $\lim_{x \to a} g(x)$ exists (and equals $g(a)$). However, since A is open by hypothesis and $a \in A$, Theorem 4.10 implies that there exists a region R such that $a \in R$ and $R \subset A$. But $a \in R$ implies that $a \in LP(R)$ by Corollary 5.5, and this combined with the fact that $R \subset A$ implies by Theorem 3.14 that $a \in LP(A)$[1]. Thus, we have that $\lim_{x \to a} g(x) = g(a) = a - a = 0$. Combining the last few results, we have

$$
\begin{aligned}
0 &= \left(\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right) \cdot 0 \\
&= \left(\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right) \left(\lim_{x \to a} (x - a) \right) \\
&= \lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} \cdot (x - a) \right) & \text{Theorem 11.9} \\
&= \lim_{x \to a} (f(x) - f(a))
\end{aligned}
$$

If we now consider $f(a)$ to be a constant function (i.e., $\lim_{x \to a} f(a) = f(a)$ by Exercise 11.6, Theorem 11.5, and the above result that $a \in LP(A)$), it follows from the above that

$$
\begin{aligned}
\lim_{x \to a} (f(x) - f(a)) + \lim_{x \to a} f(a) &= 0 + f(a) \\
\lim_{x \to a} (f(x) - f(a) + f(a)) &= f(a) & \text{Theorem 11.9} \\
\lim_{x \to a} f(x) &= f(a)
\end{aligned}
$$

as desired. $\qquad \square$

Exercise 12.6. Show that the converse of Theorem 12.5 is not true.

Proof. The converse of Theorem 12.5 asserts that "if f is continuous at a, then f is differentiable at a." To falsify this statement, we will use the absolute value function $|x|$ as a counterexample. Let's begin.

By Exercise 11.7, $|x|$ is continuous. It follows by Theorem 9.10 that $|x|$ is continuous at 0. However, we can show that $|x|$ is not differentiable at 0.

To do this, Definition 12.1 and Theorem 12.4 tell us that it will suffice to verify that $\lim_{x \to 0} \frac{|x| - |0|}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}$ does not exist. Suppose for the sake of contradiction that $\lim_{x \to 0} \frac{|x|}{x} = L$. Then by Definition 11.1, for $\epsilon = 1 > 0$, there exists a $\delta > 0$ such that if $0 < |x - 0| = |x| < \delta$, then $|\frac{|x|}{x} - L| < 1$. However, we

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[1] I will not go through this or similar derivations again, although they may be technically necessary. Indeed, assume moving on that statements analogous to $a \in LP(A)$ hold true.

can show that no such δ exists. Let $\delta > 0$ be arbitrary. By Theorem 5.2, there exists a number $x \in \mathbb{R}$ such that $0 < x < \delta$. It follows by Definition 8.4 and Exercise 8.5 that $0 < |x| = |-x| < \delta$. Since both x and $-x$ are in the appropriate range, we know that

$$\left|\frac{|x|}{x} - L\right| = \left|\frac{x}{x} - L\right| \qquad\qquad \left|\frac{|-x|}{-x} - L\right| = \left|\frac{x}{-x} - L\right| \qquad\qquad \text{Definition 8.4}$$

$$= |1 - L| \qquad\qquad\qquad = |-1 - L| \qquad\qquad \text{Script 7}$$

$$= |L - 1| \qquad\qquad\qquad = |L + 1| \qquad\qquad \text{Exercise 8.5}$$

$$< 1 \qquad\qquad\qquad\qquad < 1$$

By consecutive applications of the lemma from Exercise 8.9, it follows that

$$-1 < L - 1 < 1 \qquad\qquad\qquad -1 < L + 1 < 1$$
$$0 < L < 2 \qquad\qquad\qquad\qquad -2 < L < 0$$

But this implies that $L < 0$ and $L > 0$, a contradiction. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Exercise 12.7. Show that for all $n \in \mathbb{N}$,

$$x^n - a^n = (x - a)\left(x^{n-1} + ax^{n-2} + a^2 x^{n-3} + \cdots + a^{n-2}x + a^{n-1}\right)$$

or equivalently,

$$x^n - a^n = (x - a)\left(\sum_{i=0}^{n-1} x^{n-1-i}a^i\right)$$

Proof. By simple algebra (see Script 7), we have

$$(x - a)\left(\sum_{i=0}^{n-1} x^{n-1-i}a^i\right) = \sum_{i=0}^{n-1}(x - a)x^{n-1-i}a^i$$

$$= \sum_{i=0}^{n-1}\left(x^{n-i}a^i - x^{n-1-i}a^{i+1}\right)$$

$$= x^n + \sum_{i=1}^{n-1}x^{n-i}a^i - \sum_{i=0}^{n-2}x^{n-1-i}a^{i+1} - a^n$$

$$= x^n + \sum_{i=1}^{n-1}x^{n-i}a^i - \sum_{i=0+1}^{n-2+1}x^{n-1-(i-1)}a^{(i-1)+1} - a^n$$

$$= x^n + \sum_{i=1}^{n-1}x^{n-i}a^i - \sum_{i=1}^{n-1}x^{n-i}a^i - a^n$$

$$= x^n - a^n$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Exercise 12.8 (Power Rule).

(a) Let $n \in \mathbb{N}$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^n$. Use Exercise 12.7 to prove that $f'(a) = na^{n-1}$ for all $a \in \mathbb{R}$.

Proof. Let a be an arbitrary element of $\mathbb{R}$. To prove that $f'(a) = na^{n-1}$, Theorem 12.4 tell us that it will suffice to show that $\lim_{x \to a}\frac{f(x)-f(a)}{x-a} = na^{n-1}$. By Corollary 11.12, the polynomial $\sum_{i=0}^{n-1}x^{n-1-i}a^i$ is continuous. Thus, by Theorem 9.10, it is continuous at a. It follows by Theorem 11.5 that $\lim_{x \to a}\sum_{i=0}^{n-1}x^{n-1-i}a^i = \sum_{i=0}^{n-1}a^{n-1-i}a^i$.

Additionally, we can demonstrate that $\lim_{x \to a} \frac{x-a}{x-a} = 1$. To verify that $\lim_{h \to 0} \frac{h}{h} = 1$, Definition 11.1 tells us that it will suffice to confirm that for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $h \in \mathbb{R}$ and $0 < |h - 0| = |h| < \delta$, then $|\frac{h}{h} - 1| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = 1$, and let h be an arbitrary element of $\mathbb{R}$ satisfying $0 < |h| < \delta$. It follows by Script 7 that $|\frac{h}{h} - 1| = |1 - 1| = 0 < \epsilon$, as desired. Since $\lim_{h \to 0} \frac{h}{h} = 1$, we know by Script 7 that $\lim_{h \to 0} \frac{(a+h)-a}{(a+h)-a} = 1$. Thus, by Lemma 12.3, $\lim_{x \to a} \frac{x-a}{x-a} = 1$, as desired.

It follows from the above two results that

$$na^{n-1} = \underbrace{a^{n-1} + \cdots + a^{n-1}}_{n \text{ times}}$$

$$= \sum_{i=0}^{n-1} a^{n-1}$$

$$= \sum_{i=1}^{n-1} a^{n-1-i} a^i$$

$$= \lim_{x \to a} \sum_{i=0}^{n-1} x^{n-1-i} a^i$$

$$= 1 \cdot \left(\lim_{x \to a} \sum_{i=0}^{n-1} x^{n-1-i} a^i \right)$$

$$= \left(\lim_{x \to a} \frac{x-a}{x-a} \right) \left(\lim_{x \to a} \sum_{i=0}^{n-1} x^{n-1-i} a^i \right)$$

$$= \lim_{x \to a} \frac{x-a}{x-a} \cdot \sum_{i=0}^{n-1} x^{n-1-i} a^i \qquad \text{Theorem 11.9}$$

$$= \lim_{x \to a} \frac{x^n - a^n}{x-a} \qquad \text{Exercise 12.7}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{x-a}$$

$$= f'(a) \qquad \text{Theorem 12.4}$$

as desired. $\qquad \square$

(b) Let $k \in \mathbb{R}$. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = k$, then $f'(a) = 0$ for all $a \in \mathbb{R}$.

Proof. Let a be an arbitrary element of $\mathbb{R}$. To prove that $f'(a) = 0$, Definition 12.1 tells us that it will suffice to show that $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = 0$. By Exercise 11.6, the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 0$ is continuous at every $x \in \mathbb{R}$, including 0. It follows by Theorem 11.5 that $\lim_{h \to 0} g(h) = g(0) = 0$. Therefore,

$$0 = \lim_{h \to 0} g(h)$$

$$= \lim_{h \to 0} 0$$

$$= \lim_{h \to 0} \frac{0}{h}$$

$$= \lim_{h \to 0} \frac{k-k}{h}$$

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= f'(a) \qquad \text{Definition 12.1}$$

as desired. $\qquad \square$

Exercise 12.9 (Sum, Product, and Quotient Rules). Suppose that $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are differentiable at $a \in A$.

(a) Prove that $f + g$ is differentiable at a and compute $(f + g)'(a)$ in terms of $f'(a)$ and $g'(a)$.

Proof. To prove that $f + g$ is differentiable at a, Definition 12.1 tells us that it will suffice to show $\lim_{h \to 0} \frac{(f+g)(a+h)-(f+g)(a)}{h}$ exists. Since f, g are differentiable at a, we know by Definition 12.1 that $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ and $\lim_{h \to 0} \frac{g(a+h)-g(a)}{h}$ exist. Thus, by Theorem 11.9 the limit of their sum exists and equals

$$\lim_{h \to 0} \left(\frac{f(a+h)-f(a)}{h} + \frac{g(a+h)-g(a)}{h} \right) = \lim_{h \to 0} \frac{f(a+h)+g(a+h)-f(a)-g(a)}{h}$$
$$= \lim_{h \to 0} \frac{(f+g)(a+h)-(f+g)(a)}{h}$$

as desired. Having established that $\lim_{h \to 0} \frac{(f+g)(a+h)-(f+g)(a)}{h}$ exists, $(f + g)'(a)$ can be computed in terms of $f'(a)$ and $g'(a)$ with the following algebra.

$$
\begin{aligned}
(f + g)'(a) &= \lim_{h \to 0} \frac{(f+g)(a+h)-(f+g)(a)}{h} && \text{Definition 12.1} \\
&= \lim_{h \to 0} \left(\frac{f(a+h)-f(a)}{h} + \frac{g(a+h)-g(a)}{h} \right) \\
&= \lim_{h \to 0} \frac{f(a+h)-f(a)}{h} + \lim_{h \to 0} \frac{g(a+h)-g(a)}{h} && \text{Theorem 11.9} \\
&= f'(a) + g'(a) && \text{Definition 12.1}
\end{aligned}
$$

$\square$

(b) Prove that fg is differentiable and compute $(fg)'(a)$ in terms of $f(a)$, $g(a)$, $f'(a)$, and $g'(a)$.

Proof. To prove that fg is differentiable at a, Definition 12.1 tells us that it will suffice to show $\lim_{h \to 0} \frac{(fg)(a+h)-(fg)(a)}{h}$ exists. Since f, g are differentiable at a, we know by Definition 12.1 that $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ and $\lim_{h \to 0} \frac{g(a+h)-g(a)}{h}$ exist. For the same reason, we know by Theorem 12.5 that g is continuous, i.e., continuous at a by Theorem 9.10. Consequently, by Theorem 11.5, $\lim_{x \to a} g(x)$ exists (and equals $g(a)$). Note that the preceding limit is equal to $\lim_{h \to 0} g(a + h)$ by Lemma 12.3. Lastly, we have by Exercise 11.6 that the constant function $f(a)$ is continuous at 0. Consequently, by Theorem 11.5, $\lim_{h \to 0} f(a)$ exists (and equals $f(a)$). Combining all of these results, consecutive applications of Theorem 11.9 assert that the limits

$$\lim_{h \to 0} g(a+h) \cdot \frac{f(a+h)-f(a)}{h} \qquad\qquad \lim_{h \to 0} f(a) \cdot \frac{g(a+h)-g(a)}{h}$$

exist. Furthermore, it asserts that the limit of their sum exists and equals

$$
\begin{aligned}
\lim_{h \to 0} \left(g(a+h) \cdot \frac{f(a+h)-f(a)}{h} + f(a) \cdot \frac{g(a+h)-g(a)}{h} \right) \\
= \lim_{h \to 0} \frac{g(a+h)(f(a+h)-f(a)) + f(a)(g(a+h)-g(a))}{h} \\
= \lim_{h \to 0} \frac{f(a+h)g(a+h)-f(a)g(a+h)+f(a)g(a+h)-f(a)g(a)}{h} \\
= \lim_{h \to 0} \frac{f(a+h)g(a+h)-f(a)g(a)}{h} \\
= \lim_{h \to 0} \frac{(fg)(a+h)-(fg)(a)}{h}
\end{aligned}
$$

as desired. Having established that $\lim_{h\to 0} \frac{(fg)(a+h)-(fg)(a)}{h}$ exists, $(fg)'(a)$ can be computed in terms of $f(a)$, $g(a)$, $f'(a)$, and $g'(a)$ with the following algebra.

$$\begin{aligned}
(fg)'(a) &= \lim_{h\to 0} \frac{(fg)(a+h)-(fg)(a)}{h} && \text{Definition 12.1}\\
&= \lim_{h\to 0} \left(g(a+h) \cdot \frac{f(a+h)-f(a)}{h} + f(a) \cdot \frac{g(a+h)-g(a)}{h} \right)\\
&= \lim_{h\to 0} g(a+h) \cdot \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} + \lim_{h\to 0} f(a) \cdot \lim_{h\to 0} \frac{g(a+h)-g(a)}{h} && \text{Theorem 11.9}\\
&= g(a)f'(a) + f(a)g'(a)
\end{aligned}$$

$\square$

(c) Prove that $\frac{1}{g}$ is differentiable at a (under an appropriate assumption) and compute $(\frac{1}{g})'(a)$ in terms of $g'(a)$ and $g(a)$. What assumption do you need to make?

Proof. Assume that $g(a) \neq 0$.

To prove that $\frac{1}{g}$ is differentiable at a, Definition 12.1 tells us that it will suffice to show that the limit $\lim_{h\to 0} \frac{(1/g)(a+h)-(1/g)(a)}{h}$ exists. Since g is differentiable at a, we know by Definition 12.1 that $\lim_{h\to 0} \frac{g(a+h)-g(a)}{h}$ exists. For the same reason, we know by Theorem 12.5 that g is continuous, i.e., continuous at a by Theorem 9.10. Consequently, by Theorem 11.5, $\lim_{x\to a} g(x)$ exists (and equals $g(a)$). It follows by Lemma 12.3 that the preceding limit is equal to $\lim_{h\to 0} g(a+h)$. Thus, since it is also equal to $g(a) \neq 0$, we have by Theorem 11.9 that $\lim_{h\to 0} \frac{1}{g}(a+h)$ exists (and equals $\frac{1}{g(a)}$). Lastly, we have by Exercise 11.6 that the constant function $-\frac{1}{g(a)}$ is continuous at 0. Consequently, by Theorem 11.5, $\lim_{h\to 0} -\frac{1}{g(a)}$ exists (and equals $-\frac{1}{g(a)}$). Combining this with the previous result, Theorem 11.9 asserts that the limit $\lim_{h\to 0} -\frac{1}{g(a+h)g(a)}$ exists (and equals $-\frac{1}{g(a)^2}$). Furthermore, it asserts that the limit of its product with $\lim_{h\to 0} \frac{g(a+h)-g(a)}{h}$ exists and equals

$$\begin{aligned}
\lim_{h\to 0} -\frac{1}{g(a+h)g(a)} \cdot \frac{g(a+h)-g(a)}{h} &= \lim_{h\to 0} \frac{\frac{g(a)-g(a+h)}{g(a+h)g(a)}}{h}\\
&= \lim_{h\to 0} \frac{\frac{1}{g(a+h)} - \frac{1}{g(a)}}{h}\\
&= \lim_{h\to 0} \frac{\frac{1}{g}(a+h) - \frac{1}{g}(a)}{h}
\end{aligned}$$

as desired. Having established that $\lim_{h\to 0} \frac{(1/g)(a+h)-(1/g)(a)}{h}$ exists, $(\frac{1}{g})'(a)$ can be computed in terms of $g(a)$ and $g'(a)$ with the following algebra.

$$\begin{aligned}
\left(\frac{1}{g}\right)'(a) &= \lim_{h\to 0} \frac{\frac{1}{g}(a+h) - \frac{1}{g}(a)}{h} && \text{Definition 12.1}\\
&= \lim_{h\to 0} -\frac{1}{g(a+h)g(a)} \cdot \frac{g(a+h)-g(a)}{h}\\
&= \lim_{h\to 0} -\frac{1}{g(a+h)g(a)} \cdot \lim_{h\to 0} \frac{g(a+h)-g(a)}{h} && \text{Theorem 11.9}\\
&= -\frac{g'(a)}{g(a)^2}
\end{aligned}$$

$\square$

(d) Prove that $\frac{f}{g}$ is differentiable at a (under an appropriate assumption) and compute $(\frac{f}{g})'(a)$ in terms of $f(a)$, $g(a)$, $f'(a)$, and $g'(a)$. What assumption do you need to make?

Proof. Assume that $g(a) \neq 0$.

It follows by part (c) that $\frac{1}{g}$ is differentiable at a, and then by part (b) that $f \cdot \frac{1}{g} = \frac{f}{g}$ is differentiable at a.

Having established that $(\frac{f}{g})'(a)$ exists, it can be computed in terms of $f(a)$, $g(a)$, $f'(a)$, and $g'(a)$ with the following algebra.

$$\left(\frac{f}{g}\right)'(a) = \left(f \cdot \frac{1}{g}\right)'(a)$$

$$= f(a)\left(\frac{1}{g}\right)'(a) + f'(a)\left(\frac{1}{g}\right)(a)$$

$$= f(a) \cdot -\frac{g'(a)}{g(a)^2} + \frac{f'(a)g(a)}{g(a)^2}$$

$$= \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$$

$\square$

4/1: One of the most important results concerning the differentiation of functions is the rule for the derivative of a composition of functions. Let $f : B \to \mathbb{R}, g : A \to \mathbb{R}$ be functions such that $g(A) \subset B$. The composition $(f \circ g)(x) = f(g(x))$ is defined for all $x \in A$.

Theorem 12.10 (Chain Rule). *Let $a \in A$, $g : A \to \mathbb{R}$, and $f : I \to \mathbb{R}$ where I is an interval containing $g(A)$. Suppose that g is differentiable at a and f is differentiable at $g(a)$. Then $f \circ g$ is differentiable a and*

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$$

Proof. To prove that $f \circ g$ is differentiable at a, Theorem 12.4 tells us that it will suffice to show that $\lim_{x \to a} \frac{(f \circ g)(x) - (f \circ g)(a)}{x - a}$ exists. To do so, we will define a special function φ and prove that it is continuous at a. It will follow that $(f \circ g)'(a)$ exists and equals $f'(g(a)) \cdot g'(a)$. Let's begin.

Let $\varphi : I \to \mathbb{R}$ be defined by

$$\varphi(x) = \begin{cases} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} & g(x) \neq g(a) \\ f'(g(a)) & g(x) = g(a) \end{cases}$$

It is clear from the definition that the function is defined for all $x \in A$.

To confirm that φ is continuous at a, Theorem 11.5 tells us that it will suffice to demonstrate that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $|x - a| < \delta$, then $|\varphi(x) - \varphi(a)| = |\varphi(x) - f'(g(a))| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f is differentiable at $g(a)$, Theorem 12.4 asserts that $\lim_{y \to g(a)} \frac{f(y) - f(g(a))}{y - g(a)} = f'(g(a))$. It follows by Definition 11.1 that there is some $\delta' > 0$ such that if $y \in I$ and $0 < |y - g(a)| < \delta'$, then $|\frac{f(y) - f(g(a))}{y - g(a)} - f'(g(a))| < \epsilon$. Additionally, since g is differentiable (hence continuous by Theorem 12.5) at a, we have by Theorem 11.5 that there exists a $\delta > 0$ such that if $x \in A$ and $|x - a| < \delta$, then $|g(x) - g(a)| < \delta'$.

Using the above δ, let x be an arbitrary element of A such that $|x - a| < \delta$. We now divide into two cases ($g(x) = g(a)$ and $g(x) \neq g(a)$). If $g(x) = g(a)$, then $|\varphi(x) - f'(g(a))| = |f'(g(a)) - f'(g(a))| = 0 < \epsilon$, as desired. If $g(x) \neq g(a)$, then we continue. Since $|x - a| < \delta$, we have that $|g(x) - g(a)| < \delta'$. This combined with the fact that $g(x) \in I$ and $g(x) \neq g(a)$, i.e., $0 < |g(x) - g(a)|$ illustrates that $|\frac{f(g(x)) - f(g(a))}{g(x) - g(a)} - f'(g(a))| = |\varphi(x) - f'(g(a))| < \epsilon$. Therefore, φ is continuous at a.

It follows by Theorem 11.5 that $\lim_{x \to a} \varphi(x) = \varphi(a) = f'(g(a))$. Additionally, since g is differentiable at a, Definition 12.1 and Theorem 12.4 tell us that $\lim_{x \to a} \frac{g(x) - g(a)}{x - a}$ exists (and equals $g'(a)$). This combined with the previous result implies by Theorem 11.9 that the product of the limits exists and equals $f'(g(a)) \cdot g'(a)$, i.e., we have that $\lim_{x \to a} \varphi(x) \cdot \frac{g(x) - g(a)}{x - a} = f'(g(a)) \cdot g'(a)$.

We now seek to confirm that $\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} = f'(g(a)) \cdot g'(a)$. To do so, Definition 11.1 tells us that it will suffice to demonstrate that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and

$0 < |x - a| < \delta$, then $|\frac{f(g(x)) - f(g(a))}{x-a} - f'(g(a)) \cdot g'(a)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{x \to a} \varphi(x) \cdot \frac{g(x) - g(a)}{x-a} = f'(g(a)) \cdot g'(a)$, we have by Definition 11.1 that there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - a| < \delta$, then $|\varphi(x) \cdot \frac{g(x) - g(a)}{x-a} - f'(g(a)) \cdot g'(a)| < \epsilon$. Choose this δ to be our δ. Let x be an arbitrary element of A that satisfies $0 < |x - a| < \delta$. We divide into two cases ($g(x) = g(a)$ and $g(x) \neq g(a)$). Suppose first that $g(x) = g(a)$. Since $0 < |x - a| < \delta$, we have by the above and the definition of φ that $|f'(g(a)) \cdot \frac{g(x) - g(a)}{x-a} - f'(g(a)) \cdot g'(a)| < \epsilon$. Additionally, it follows from the hypothesis that $g(x) - g(a) = 0$ and $f(g(x)) - f(g(a)) = 0$. Therefore,

$$
\begin{aligned}
\left| \frac{f(g(x)) - f(g(a))}{x-a} - f'(g(a)) \cdot g'(a) \right| &\leq \left| \frac{f(g(x)) - f(g(a))}{x-a} - f'(g(a)) \cdot \frac{g(x) - g(a)}{x-a} \right| \\
&\quad + \left| f'(g(a)) \cdot \frac{g(x) - g(a)}{x-a} - f'(g(a)) \cdot g'(a) \right| \\
&= \left| \frac{0}{x-a} - f'(g(a)) \cdot \frac{0}{x-a} \right| \\
&\quad + \left| f'(g(a)) \cdot \frac{g(x) - g(a)}{x-a} - f'(g(a)) \cdot g'(a) \right| \\
&= 0 + \left| f'(g(a)) \cdot \frac{g(x) - g(a)}{x-a} - f'(g(a)) \cdot g'(a) \right| \\
&< \epsilon
\end{aligned}
$$

Lemma 8.8

as desired. Now suppose that $g(x) \neq g(a)$. Since $0 < |x - a| < \delta$, we have by the above and the definition of φ that $|\frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \cdot \frac{g(x) - g(a)}{x-a} - f'(g(a)) \cdot g'(a)| < \epsilon$. Additionally, it follows from the hypothesis that $g(x) - g(a) \neq 0$. Therefore, we have by Script 7 that $|\frac{f(g(x)) - f(g(a))}{x-a} - f'(g(a)) \cdot g'(a)| < \epsilon$, as desired.

It follows by Definition 1.25 that $\lim_{x \to a} \frac{(f \circ g)(x) - (f \circ g)(a)}{x-a} = f'(g(a)) \cdot g'(a)$. This proves that $f \circ g$ is differentiable at a. Lastly, it directly follows from Theorem 12.4 that

$$
(f \circ g)'(a) = \lim_{x \to a} \frac{(f \circ g)(x) - (f \circ g)(a)}{x-a} = f'(g(a)) \cdot g'(a)
$$

as desired. $\qquad \square$

We now come to the most important theorem in differential calculus, Corollary 12.16.

Definition 12.11. Let $f : A \to \mathbb{R}$ be a function. If $f(a)$ is the last point of $f(A)$, then $f(a)$ is called the **maximum value** of f. If $f(a)$ is the first point of $f(A)$, then $f(a)$ is the **minimum value** of f. We say that $f(a)$ is a **local maximum value** of f if there exists a region R containing a such that $f(a)$ is the last point of $f(A \cap R)$. We say that $f(a)$ is a **local minimum value** of f if there exists a region R containing a such that $f(a)$ is the first point of $f(A \cap R)$.

Remark 12.12. Equivalently, $f(a)$ is a local maximum (resp. minimum) value of f if there exists U open in A such that $f(a)$ is the last (resp. first) point of $f(U)$.

Theorem 12.13. *Let $f : A \to \mathbb{R}$ be differentiable at a. Suppose that $f(a)$ is the maximum value or minimum value of f. Then $f'(a) = 0$.*

Proof. Suppose first that $f(a)$ is the maximum value of f, and suppose for the sake of contradiction that $f'(a) \neq 0$. Then $f'(a) > 0$ or $f'(a) < 0$. We now divide into two cases. If $f'(a) > 0$, then by Theorem 12.4, $\lim_{x \to a} \frac{f(x) - f(a)}{x-a} > 0$. Thus, by Lemma 11.8, there exists a region R with $a \in R$ such that $\frac{f(x) - f(a)}{x-a} > 0$ for all $x \in R \cap A$. Additionally, since A is open, Theorem 4.10 implies that there exists a region S with $a \in S$ and $S \subset A$. It follows by Theorem 3.18 that $R \cap S = (c, d)$, where (c, d) is a region containing a. Since $a \in (c, d)$, Theorem 5.2 implies that there exists a point $y \in \mathbb{R}$ such that $a < y < d$. Clearly, since $y \in (c, d) = R \cap S$ and $S \subset A$, we have that $y \in R \cap A$. It follows that $\frac{f(y) - f(a)}{y-a} > 0$. Furthermore, $y > a$ implies that $y - a > 0$. Therefore, by Definition 7.21, $f(y) - f(a) = \frac{f(y) - f(a)}{y-a} \cdot (y - a) > 0$. But this means

that $f(y) > f(a)$, i.e., that $f(a)$ is not the last point of $f(A)$ (by Definition 3.3), i.e., that $f(a)$ is not the maximum value of f (by Definition 12.11), a contradiction. The argument is symmetric in the other case.

The proof is symmetric in the other case. $\qquad\square$

Corollary 12.14. *Let $f : A \to \mathbb{R}$ be differentiable at a. Suppose that $f(a)$ is a local maximum or local minimum value of f. Then $f'(a) = 0$.*

Proof. Suppose first that $f(a)$ is a local maximum of f. Then by Definition 12.11, there exists a region R containing a such that $f(a)$ is the last point of $f(A \cap R)$. Now consider the restriction of f to $A \cap R$. It follows from Definition 9.6 that $f|_{A \cap R}$ is differentiable at a, that $f|_{A \cap R}(A \cap R) = f(A \cap R)$, and that $f|_{A \cap R}(a) = f(a)$ is the last point of $f|_{A \cap R}(A \cap R)$. The latter two results imply by Definition 12.11 that $f|_{A \cap R}(a)$ is the maximum value of $f|_{A \cap R}$. This combined with the fact that $f|_{A \cap R}$ is differentiable at a implies by Theorem 12.13 that $(f|_{A \cap R})'(a) = f'(a) = 0$, as desired. $\qquad\square$

Theorem 12.15 (Rolle's Theorem). *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, differentiable on (a, b), and that $f(a) = f(b) = 0$. Then there exists a point $\lambda \in (a, b)$ such that $f'(\lambda) = 0$.*

Proof. We divide into two cases ($f(x) = 0$ for all $x \in [a, b]$, and $f(x) \neq 0$ for some $x \in [a, b]$).

Suppose first that $f(x) = 0$ for all $x \in [a, b]$. By Theorem 5.2, we can choose a $\lambda \in (a, b)$. It follows from the hypothesis that $f(\lambda) = f(x)$ for all $f(x) \in f([a, b])$. This can be weakened to $f(\lambda) \geq f(x)$ for all $f(x) \in f([a, b])$. Thus, by Definition 3.3, $f(\lambda)$ is the last point of $f([a, b])$. Consequently, by Definition 12.11, $f(\lambda)$ is the maximum value of f. This combined with the fact that f is differentiable at λ (since $\lambda \in (a, b)$ and f is differentiable on (a, b)) implies by Theorem 12.13 that $f'(\lambda) = 0$, as desired.

Now suppose that $f(x) \neq 0$ for some $x \in [a, b]$ which we shall call x_0. We divide into two cases again ($f(x_0) > 0$ and $f(x_0) < 0$). Suppose first that $f(x_0) > 0$. Since $f : [a, b] \to \mathbb{R}$ is continuous, Exercise 10.21 asserts that there exists a point $\lambda \in [a, b]$ such that $f(\lambda) \geq f(x)$ for all $x \in [a, b]$. It follows that $f(\lambda) \geq f(x_0) > 0$, so $f(\lambda) \neq f(a) = f(b)$. Thus, by Definition 1.16, $\lambda \neq a$ and $\lambda \neq b$. This combined with the fact that $\lambda \in [a, b]$ implies by Script 8 that $\lambda \in (a, b)$. Now as before, we can determine from the fact that $f(\lambda) \geq f(x)$ for all $x \in [a, b]$ that $f(\lambda)$ is the maximum value of f. This combined with the fact that f is differentiable at λ (since $\lambda \in (a, b)$ and f is differentiable on (a, b)) implies by Theorem 12.13 that $f'(\lambda) = 0$, as desired. The argument is symmetric in the other case. $\qquad\square$

Corollary 12.16 (Mean Value Theorem). *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point $\lambda \in (a, b)$ such that*

$$f(b) - f(a) = f'(\lambda)(b - a)$$

Proof. Let $h : [a, b] \to \mathbb{R}$ be defined by

$$h(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a)$$

By hypothesis, $f(x)$ is continuous on $[a, b]$. By Exercise 11.6 and Theorem 11.9, $-\frac{f(b) - f(a)}{b - a}(x - a) - f(a)$ is continuous on $[a, b]$. Thus, by Theorem 11.9, their sum (i.e., $h(x)$) is continuous on $[a, b]$. Additionally, by hypothesis, $f(x)$ is differentiable on (a, b). By Exercises 12.8 and 12.9, $-\frac{f(b) - f(a)}{b - a}(x - a) - f(a)$ is differentiable on $[a, b]$. Thus, by Exercise 12.9, their sum (i.e., $h(x)$) is differentiable on (a, b). Furthermore, by simple algebra, we can determine that

$$
\begin{aligned}
h(a) &= f(a) - \frac{f(b) - f(a)}{b - a}(a - a) - f(a) \qquad\qquad & h(b) &= f(b) - \frac{f(b) - f(a)}{b - a}(b - a) - f(a) \\
&= -\frac{f(b) - f(a)}{b - a} \cdot 0 & &= f(b) - (f(b) - f(a)) - f(a) \\
&= 0 & &= 0
\end{aligned}
$$

Thus, by Theorem 12.15, there exists a point $\lambda \in (a, b)$ such that $h'(\lambda) = 0$.

We can also calculate $h'(x)$ as follows.

$$h'(x) = \left(f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a) \right)'$$

$$= \left((f(x)) + \left(-\frac{f(b) - f(a)}{b - a} \cdot x \right) + \left(\frac{f(b) - f(a)}{b - a} \cdot a - f(a) \right) \right)'$$

$$= f'(x) + \left(-\frac{f(b) - f(a)}{b - a} \cdot x \right)' + \left(\frac{f(b) - f(a)}{b - a} \cdot a - f(a) \right)' \qquad \text{Exercise 12.9}$$

$$= f'(x) + \left(-\frac{f(b) - f(a)}{b - a} \right)' \cdot (x) + \left(-\frac{f(b) - f(a)}{b - a} \right) \cdot (x)' + \left(\frac{f(b) - f(a)}{b - a} \cdot a - f(a) \right)'$$

$$\qquad \text{Exercise 12.9}$$

$$= f'(x) + 0 \cdot x + \frac{f(b) - f(a)}{b - a} \cdot 1x^0 + 0 \qquad \text{Exercise 12.8}$$

$$= f'(x) + \frac{f(b) - f(a)}{b - a}$$

But it follows that at λ,

$$0 = f'(\lambda) - \frac{f(b) - f(a)}{b - a}$$

$$\frac{f(b) - f(a)}{b - a} = f'(\lambda)$$

$$f(b) - f(a) = f'(\lambda)(b - a)$$

as desired. $\qquad \square$

4/6: **Corollary 12.17** (First Derivative Test). *Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then*

(a) *If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on $[a, b]$.*

Proof. To prove that f is strictly increasing on $[a, b]$, Definition 8.16 tells us that it will suffice to show that if $x, y \in [a, b]$ with $x < y$, then $f(x) < f(y)$. Let x, y be arbitrary elements of $[a, b]$. WLOG, let $x < y$. Since f is continuous on $[a, b]$ and differentiable on (a, b), Corollary 12.16 asserts that there exists a point $\lambda \in (x, y)$ such that $f(y) - f(x) = f'(\lambda)(y - x)$. But since $f'(\lambda) > 0$ by hypothesis and $y - x > 0$ because $y > x$, we have by Definition 7.21 that $f'(\lambda)(y - x) > 0$. It follows that $f(y) - f(x) > 0$, i.e., that $f(x) < f(y)$, as desired. $\qquad \square$

(b) *If $f'(x) < 0$ for all $x \in (a, b)$, then f is strictly decreasing on $[a, b]$.*

Proof. The proof is symmetric to that of part (a). $\qquad \square$

(c) *If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.*

Proof. To prove that f is constant on $[a, b]$, it will suffice to show that $f(x) = f(y)$ for all $x, y \in [a, b]$. Let x, y be arbitrary elements of $[a, b]$. WLOG, let $x < y$. Since f is continuous on $[a, b]$ and differentiable on (a, b), Corollary 12.16 asserts that there exists a point $\lambda \in (x, y)$ such that $f(y) - f(x) = f'(\lambda)(y - x)$. But since $f'(\lambda) = 0$ by hypothesis, $f(y) - f(x) = 0$, i.e., $f(y) = f(x)$, as desired. $\qquad \square$

Remark 12.18. Corollary 12.17 also holds if instead of $[a, b]$, we have an arbitrary interval I; and instead of (a, b), we have the interior of I.

Corollary 12.19. *Suppose that $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$, differentiable on (a, b), and $f'(x) = g'(x)$ for all $x \in (a, b)$. Then there is some $c \in \mathbb{R}$ such that for all $x \in [a, b]$, we have $f(x) = g(x) + c$.*

Proof. Let $h : [a, b] \to \mathbb{R}$ be defined by $h(x) = f(x) - g(x)$. Since f, g are continuous on $[a, b]$, Corollary 11.10 asserts that h is continuous on $[a, b]$. Since f, g are differentiable on (a, b), Exercise 12.9 asserts that h is differentiable on (a, b). Since $f'(x) = g'(x)$ for all $x \in (a, b)$, Exercise 12.9 implies that $h'(x) = f'(x) - g'(x) = 0$ for all $x \in (a, b)$. These three results satisfy the conditions of Corollary 12.17, which means that h is constant on $[a, b]$, i.e., that $h(x) = c$ for all $x \in [a, b]$ where $c \in \mathbb{R}$. But by the definition of h, this implies that for all $x \in [a, b]$, we have $f(x) - g(x) = c$, i.e., $f(x) = g(x) + c$. □

Corollary 12.20. *Suppose that $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b). Then there is a point $\lambda \in (a, b)$ such that*

$$(f(b) - f(a))g'(\lambda) = (g(b) - g(a))f'(\lambda)$$

Proof. Let $h : [a, b] \to \mathbb{R}$ be defined by $h(x) = (g(b) - g(a))f(x) - (f(b) - f(a))g(x) - f(a)g(b) + f(b)g(a)$. For the same reasons as in the proof of Corollary 12.19, h is continuous on $[a, b]$ and differentiable on (a, b). Additionally, we can show with basic algebra that

$$\begin{aligned}
h(a) &= (g(b) - g(a))f(a) - (f(b) - f(a))g(a) - f(a)g(b) + f(b)g(a) \\
&= f(a)g(b) - f(a)g(a) - f(b)g(a) + f(a)g(a) - f(a)g(b) + f(b)g(a) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
h(b) &= (g(b) - g(a))f(b) - (f(b) - f(a))g(b) - f(a)g(b) + f(b)g(a) \\
&= f(b)g(b) - f(b)g(a) - f(b)g(b) + f(a)g(b) - f(a)g(b) + f(b)g(a) \\
&= 0
\end{aligned}$$

These results satisfy the conditions of Theorem 12.15, which means that there exists a point $\lambda \in (a, b)$ such that $h'(\lambda) = 0$. We can also calculate that $h'(x) = (g(b) - g(a))f'(x) - (f(b) - f(a))g'(x)$ via a similar method to that used in the proof of Corollary 12.16. But it follows that at λ,

$$0 = (g(b) - g(a))f'(\lambda) - (f(b) - f(a))g'(\lambda)$$
$$(f(b) - f(a))g'(\lambda) = (g(b) - g(a))f'(\lambda)$$

□

Finally, we prove another very important theorem that tells us about inverse functions and their derivatives.

Theorem 12.21 (Inverse Function Theorem). *Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable and that the derivative $f' : (a, b) \to \mathbb{R}$ is continuous. Also suppose that there is a point $p \in (a, b)$ such that $f'(p) \neq 0$. Then there exists a region $R \subset (a, b)$ such that $p \in R$ and f with domain restricted to R is injective. Furthermore, $f^{-1} : f(R) \to R$ is differentiable at the point $f(p)$ and*

$$(f^{-1})'(f(p)) = \frac{1}{f'(p)}$$

Proof. Since $f' : (a, b) \to \mathbb{R}$ is continuous, Theorem 9.10 implies that it is continuous at $p \in (a, b)$. Thus, by Theorem 11.5, $\lim_{x \to p} f'(x) = f'(p)$. This combined with the facts that $f'(p) \neq 0$ (i.e., $f'(p) > 0$ or $f'(p) < 0$) and f' is continuous at p implies by Lemma 11.8 that there exists a region R with $p \in R$ such that $f'(x) > 0$ for all $x \in R \cap (a, b)$ or $f'(x) < 0$ for all $x \in R \cap (a, b)$. Clearly, $R \cap (a, b) = R$. Consequently, since $f'(x) > 0$ for all $x \in R$ or $f'(x) < 0$ for all $x \in R$, Corollary 12.17 asserts that f is strictly increasing or strictly decreasing on R. Since R is an interval by Lemma 8.3, the previous result implies by Lemma 8.17 that f is injective on R. Therefore, we have found an $R = R \cap (a, b) \subset (a, b)$ by Theorem 1.7 with $p \in R$ such that $f|_R$ is injective, as desired.

From now on, we will denote $f|_R : R \to \mathbb{R}$ by f. Since f is differentiable, Definition 12.1 asserts that it is differentiable for all $x \in R$. Thus, by Theorem 12.5, f is continuous for all $x \in R$. Consequently, by Theorem 9.10, f is continuous. This combined with the previously proven fact that f is injective implies by Theorem 9.14 that the inverse function $f^{-1} : f(R) \to R$ exists and is continuous.

To prove that $(f^{-1})'(f(p))$ exists and equals $\frac{1}{f'(p)}$, Theorem 12.4 tells us that it will suffice to show that $\lim_{y \to f(p)} \frac{f^{-1}(y) - f^{-1}(f(p))}{y - f(p)} = \frac{1}{f'(p)}$. To do this, Definition 11.1 tells us that it will suffice to confirm that for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $y \in f(R)$ and $0 < |y - f(p)| < \delta$, then $|\frac{f^{-1}(y) - f^{-1}(f(p))}{y - p} - \frac{1}{f'(p)}| < \epsilon$. Let $\epsilon > 0$ be arbitrary. We have that

$$\frac{1}{f'(p)} = \frac{1}{\lim_{x \to p} \frac{f(x) - f(p)}{x - p}} \qquad \text{Theorem 12.4}$$

$$= \lim_{x \to p} \frac{x - p}{f(x) - f(p)} \qquad \text{Theorem 11.9}$$

$$= \lim_{x \to p} \frac{f^{-1}(f(x)) - f^{-1}(f(p))}{f(x) - f(p)} \qquad \text{Definition 1.18}$$

Thus, by Definition 11.1, there exists a $\delta' > 0$ such that if $x \in R$ and $0 < |x - p| < \delta'$, then $|\frac{f^{-1}(f(x)) - f^{-1}(f(p))}{f(x) - f(p)} - \frac{1}{f'(p)}| < \epsilon$.

The previously proven fact that f^{-1} is continuous implies by Theorem 9.10 that f^{-1} is continuous at $f(p)$. It follows by Theorem 11.5 that either $f(p) \notin LP(f(R))$ or $\lim_{y \to f(p)} f^{-1}(y) = f^{-1}(f(p))$. However, as we will now see, $f(p) \in LP(f(R))$. To begin, the fact that R is a region implies by Lemma 8.3 that R is an interval. It follows by Theorem 8.15 that R is connected, by Theorem 9.11 that $f(R)$ is connected, by Theorem 8.15 again that $f(R)$ is an interval, and finally by extensions of Corollaries 5.5 and 5.14 that $f(p) \in LP(f(R))$. Thus, with this case eliminated, we know that $\lim_{y \to f(p)} f^{-1}(y) = f^{-1}(f(p))$. Consequently, by Definition 1.11, there exists a $\delta > 0$ such that if $y \in f(R)$ and $0 < |y - f(p)| < \delta$, then $|f^{-1}(y) - f^{-1}(f(p))| < \delta'$. With a slight modification from Definition 1.18 (and the definition that $y = f(x)$), we have that there exists a $\delta > 0$ such that if $y \in f(R)$ and $0 < |y - f(p)| < \delta$, then $|x - p| < \delta'$.

Choose the above δ as our δ. Let $y = f(x)$ be an arbitrary element of $f(R)$ satisfying $0 < |y - f(p)| < \delta$. Then $|x - p| < \delta'$. We can also show that $0 < |x - p|$: From Script 8, the fact that $0 < |y - f(p)|$ implies that $f(x) \neq f(p)$; hence by Definition 1.20 and the fact that f is injective, $x \neq p$; hence by Script 8 again, $0 < |x - p|$. Continuing, since $y = f(x) \in f(R)$, we have by Definition 1.18 that $x \in R$. Thus, we have that $x \in R$ and $0 < |x - p| < \delta'$, so we know that $|\frac{f^{-1}(f(x)) - f^{-1}(f(p))}{f(x) - f(p)} - \frac{1}{f'(p)}| < \epsilon$. Therefore, with a slight modification from Definition 1.18, we have that $|\frac{f^{-1}(y) - f^{-1}(f(p))}{y - f(p)} - \frac{1}{f'(p)}| < \epsilon$, as desired. $\qquad \square$

4/8: **Exercise 12.22.** Consider the function $f(x) = x^n$ for a fixed $n \in \mathbb{N}$. Show that if n is even, then f is strictly increasing on the set of nonnegative real numbers and that if n is odd, then f is strictly increasing on all of $\mathbb{R}$. For a given n, let A be the aforementioned set on which f is strictly increasing. Define the inverse function $f^{-1} : f(A) \to A$ by $f^{-1}(x) = \sqrt[n]{x}$, which we sometimes also denote $f^{-1}(x) = x^{1/n}$. Use Theorem 12.21 to find the points $y \in f(A)$ at which f^{-1} is differentiable, and determine $(f^{-1})'(y)$ at these points.

Proof. For the first part of the question, we must prove that f is strictly increasing on $\mathbb{R}^+$ if n is even and that f is strictly increasing on $\mathbb{R}$ if n is odd. To do so, we induct on n. For the base case $n = 1$, we must confirm that $f(x) = x$ is strictly increasing on $\mathbb{R}$ (since n is odd). By Exercise 12.8, $f'(x) = 1 > 0$ for all $x \in \mathbb{R}$. Thus, by Corollary 12.17 and Remark 12.18, f is strictly increasing on $\mathbb{R}$, as desired. Now suppose inductively that the claim holds for some natural number $n \in \mathbb{N}$; we wish to confirm it for $n + 1$. We divide into two cases ($n + 1$ is even and $n + 1$ is odd).

If $n + 1$ is even, then we must confirm that $f(x) = x^{n+1}$ is strictly increasing on $\mathbb{R}^+$. By Exercise 12.8, $f'(x) = (n+1)x^n$, where n is odd. To verify that f' is strictly increasing on $\mathbb{R}^+$, Definition 8.16 tells us that it will suffice to demonstrate that for all $x, y \in \mathbb{R}^+$ satisfying $x < y$, $f'(x) < f'(y)$. Let x, y be arbitrary elements of $\mathbb{R}^+$ that satisfy $x < y$. Since x^n is strictly increasing on $\mathbb{R}^+ \subset \mathbb{R}$ by the inductive hypothesis, Definition 8.16 tells us that $x^n < y^n$. This combined with the fact that $0 < n + 1$ by Script 0 implies by Lemma 7.24 that $(n+1)x^n < (n+1)y^n$. Thus, by the definition of f', $f'(x) < f'(y)$, as desired. Note also

that $f'(0) = (n+1)0^n = 0$. Having established that f' is strictly increasing on $\mathbb{R}^+$ and that $f'(0) = 0$, we have that $f'(x) > 0$ for all $x \in (0, \infty)$ by Definition 8.16 because $0 < x$ implies $0 = f'(0) < f'(x)$. Thus, by Corollary 12.17 and Remark 12.18, f is strictly increasing on $\mathbb{R}^+$, as desired.

If $n + 1$ is odd, then we must confirm that $f(x) = x^{n+1}$ is strictly increasing on $\mathbb{R}$. By Exercise 12.8, $f'(x) = (n+1)x^n$, where n is even. With a symmetric argument to that used above, we can verify that f' is strictly increasing on $\mathbb{R}^+$. Since $f'(-x) = (n+1)(-x)^n = (n+1)x^n = f'(x)$ by Script 7, we can similarly prove that f' is strictly *decreasing* on $\mathbb{R}^-$ (essentially, if $x, y \in \mathbb{R}^-$, then $x < y$ implies $-y < -x$ implies $f(-y) < f(-x)$ implies $f(x) > f(y)$, as desired). These two results combined with the fact that we still have $f'(0) = 0$ imply by a symmetric argument to the above that $f'(x) > 0$ for all $x \in (-\infty, 0) \cup (0, \infty)$. But it follows by consecutive applications of Corollary 12.17 and Remark 12.18 that f is strictly increasing on $\mathbb{R}$, as desired.

For the second part of the question, we must find all of the points y where f^{-1} is differentiable and determine the derivative $(f^{-1})'(y)$ at these points for an arbitrary n. Let n be an arbitrary element of $\mathbb{N}$. By Exercise 12.8, f is differentiable, and by both Exercise 12.8 and Corollary 11.12, f' is continuous. We divide into two cases (n is even and n is odd).

Suppose first that n is even. To begin, we will verify that $f(\mathbb{R}^+) = \mathbb{R}^+$. By Definition 1.2, to do so it will suffice to show that every $y \in f(\mathbb{R}^+)$ is an element of $\mathbb{R}^+$ and vice versa. Let y be an arbitrary element of $f(\mathbb{R}^+)$. Then by Definition 1.18, $y = f(x)$ for some $x \in \mathbb{R}^+$. It follows since $x \in \mathbb{R}^+$ that $x \geq 0$. Consequently, since f is strictly increasing on $\mathbb{R}^+$, Definition 8.16 implies that $f(x) \geq f(0) = 0$. But if $y = f(x) \geq 0$, then $y \in \mathbb{R}^+$, as desired. Now let y be an arbitrary element of $\mathbb{R}^+$. We divide into three cases ($y = 0$, $0 < y \leq 1$ and $1 < y$). If $y = 0$, then since $f(0) = 0^n = 0 = y$ and $0 \in \mathbb{R}^+$, $y = f(x)$ for an $x \in \mathbb{R}^+$. Therefore, Definition 1.18 asserts that $y \in f(\mathbb{R}^+)$, as desired. If $0 < y \leq 1$, then since f is continuous (notably on $[0, 2]$) and $f(0) = 0 < y < 2^n = f(2)$ (by Script 7), Exercise 9.12 asserts that there exists a point $x \in \mathbb{R}$ with $0 < x < 2$ such that $f(x) = y$. Therefore, since $y = f(x)$ for an $x \in \mathbb{R}^+$ (we do know that $x > 0$), Definition 1.18 asserts that $y \in f(\mathbb{R}^+)$, as desired. If $y > 1$, then since $f(0) = 0 < 1 < y < y^n = f(y)$ (by Script 7), Exercise 9.12 asserts that there exists a point $x \in \mathbb{R}$ with $0 < x < y$ such that $f(x) = y$. Therefore, since $y = f(x)$ for an $x \in \mathbb{R}^+$ (we do know that $x > 0$), Definition 1.18 asserts that $y \in f(\mathbb{R}^+)$, as desired.

Having established that $f(\mathbb{R}^+) = \mathbb{R}^+$, our task becomes one of finding all points $y \in \mathbb{R}^+$ at which f^{-1} is differentiable, and determining $(f^{-1})'(y)$ at these points. By Theorem 12.21, this means that we need only find all points $f(p)$ corresponding to a p that satisfies $f'(p) \neq 0$. Since $f'(x) = nx^{n-1}$ by Exercise 12.8, Script 7 implies that the only point p where $f'(p) = 0$ is $p = 0$. Thus, we need only exclude $f(0) = 0$ from our set of points at which f^{-1} is differentiable. Therefore, we know that f^{-1} is differentiable at every $y \in \mathbb{R}^+$ such that $y \neq 0$, or more simply, all y in the interval $(0, \infty)$. Additionally, Theorem 12.21 implies that for any $y \in (0, \infty)$,

$$
\begin{aligned}
(f^{-1})'(y) &= \frac{1}{f'(f^{-1}(y))} \\
&= \frac{1}{n(f^{-1}(y))^{n-1}} \\
&= \frac{1}{n(y^{1/n})^{n-1}} \\
&= \frac{1}{n} \cdot y^{\frac{1-n}{n}}
\end{aligned}
$$

The proof is symmetric if n is odd.

$\square$

Additional Exercises

3. (a) Suppose $f : [a, b] \to \mathbb{R}$, $x_0 \in (a, b)$, f', f'' exist and are continuoous oon soome interval containing x_0, and $f'(x_0) = 0$, but $f''(x_0) \neq 0$. Prove that

    i) if $f''(x_0) > 0$, then f has a local minimum at x_0;

ii) if $f''(x_0) < 0$, then f has a local maximum at x_0.

(b) **Bernoulli's inequality**: Prove that for $x \geq -1$ and $\alpha \geq 1$,

$$(1+x)^\alpha \geq 1 + \alpha x$$

and for $x \geq -1$ and $0 \leq \alpha \leq 1$,

$$(1+x)^\alpha \leq 1 + \alpha x$$

(c) Prove that $x - \frac{x^3}{6} < \sin x < x$ for all $x > 0$.