Script 15

Sequences

15.1 Journal

5/6: **Definition 15.1.** A **sequence** (of real numbers) is a function $a : \mathbb{N} \to \mathbb{R}$.

By setting $a_n = a(n)$, we can think of a sequence as a list $a_1, a_2, a_3, \ldots$ of real numbers. We use the notation $(a_n)_{n=1}^{\infty}$ for such a sequence, or if there is no possibility of confusion, we sometimes abbreviate this and write simply (a_n). More generally, we also use the term sequence to refer to the function defined on $\{n \in \mathbb{N} \cup \{0\} \mid n \geq n_0\}$ for any fixed $n_0 \in \mathbb{N} \cup \{0\}$. We write $(a_n)_{n=n_0}^{\infty}$ for such a sequence.

Definition 15.2. We say that a sequence (a_n) **converges** to a point $p \in \mathbb{R}$ if for every open interval I containing p, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in I$. If a sequence converges to some point, we say it is **convergent**. If (a_n) does not converge to any point, we say that the sequence **diverges** or is **divergent**.

Exercise 15.3. Show that a sequence (a_n) converges to p if and only if any region containing p contains all but finitely many terms of the sequence.

Proof. Suppose first that (a_n) converges to p. Let R be an arbitrary region containing p. By Corollary 4.11 and Lemma 8.3, R is an open interval. Thus, by Definition 15.2, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in R$. To prove that R contains all but finitely many terms of the sequence, it will suffice to show that the set $A = \{a_n \mid a_n \notin R\}$ is finite. Since $a_n \in R$ for all $n \geq N$, it follows that $a_n \in R$ only if $n < N$. Thus, by Script 1, $A \subset \{a_n \mid 0 \leq n < N\}$. Since the latter set is clearly finite, it follows by Script 1 that A is finite.

Now suppose that any region containing p contains all but finitely many terms (a_n). To prove that (a_n) converges to p, Definition 15.2 tells us that it will suffice to show that for every open interval I containing p, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in I$. Let I be an arbitrary open interval containing p. Then by Lemma 8.10, there exists a region R containing p such that $R \subset I$. It follows by the hypothesis that $A = \{a_n \mid a_n \notin R\}$ is finite. We divide into two cases ($|A| = 0$ and $|A| \in \mathbb{N}$). Suppose first that $|A| = 0$. Choose $N = n_0$. It follows that if $n \geq N$, then $a_n \notin A$, so $a_n \in R$, so $a_n \in I$, as desired. Now suppose that $|A| \in \mathbb{N}$. By Definition 1.18, $a^{-1}(A) \subset \mathbb{N}$. Consequently, by Lemma 3.4, $a^{-1}(A)$ has a last point $N - 1$. Choose $N = (N - 1) + 1$. It follows that if $n \geq N$, then $n \notin a^{-1}(A)$, so $a_n \notin A$, so $a_n \in R$, so $a_n \in I$, as desired. $\square$

Theorem 15.4. *Suppose that (a_n) converges to both p and to p'. Then $p = p'$.*

Proof. Suppose for the sake of contradiction that $p \neq p'$. Then by Theorem 3.22, there exist disjoint regions R, R' containing p, p', respectively. Additionally, by consecutive applications of Corollary 4.11 and Lemma 8.3, R, R' are open intervals. Combining these last two results, we have by consecutive applications of Definition 15.2 that there exist $N, N' \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in R$ and if $n \geq N'$, then $a_n \in R'$. Let $M = \max(N, N')$. It follows that $M \geq N$ and $M \geq N'$. Thus, by the above, $a_M \in R$ and $a_M \in R'$. But this implies by Definition 1.6 that $a_M \in R \cap R'$. Therefore, by Definition 1.9, R and R' are not disjoint, a contradiction. $\square$

Definition 15.5. If a sequence (a_n) converges to $p \in \mathbb{R}$, we call p the **limit** of (a_n) and write

$$\lim_{n\to\infty} a_n = p$$

Exercise 15.6. Which of the following sequences (a_n) converge? Which diverge? For each that converges, what is the limit? Give proofs for your answers.

(a) $a_n = 5$.

Proof. To prove that this sequence converges with limit $\lim_{n\to\infty} a_n = 5$, Definition 15.5 tells us that it will suffice to show that (a_n) converges to 5. To do so, Definition 15.2 tells us that it will suffice to verify that for every open interval I containing 5, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in I$. Let's begin.

Let I be an arbitrary open interval containing 5. Choose $N = 1$. Let n be an arbitrary natural number such that $n \geq N$. It follows by the definition of the sequence that $a_n = 5 \in I$, as desired. $\square$

(b) $a_n = n$.

Proof. To prove that this sequence diverges, Definition 15.2 tells us that it will suffice to show that for every point $p \in \mathbb{R}$, there exists an open interval I containing p such that for all $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \notin I$. Let's begin.

Let p be an arbitrary element of $\mathbb{R}$. Choose $I = (p-1, p+1)$. Clearly $p \in I$. Let N be an arbitrary natural number. By Corollary 6.12, there exists a natural number N' such that $p + 1 < N'$. Choose $M = \max(N, N')$. Thus, $M \geq N$. Additionally, it follows by the definition of the sequence that $a_M = M$. But this implies that $a_M \geq N' > p + 1$, i.e., $a_M \notin I$ by Equations 8.1. $\square$

(c) $a_n = \frac{1}{n}$.

Proof. To prove that this sequence converges with limit $\lim_{n\to\infty} a_n = 0$, Definitions 15.5 and 15.2 tell us that it will suffice to show that for every open interval I containing 0, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $a_n \in I$. Let's begin.

Let I be an arbitrary interval containing 0. By Lemma 8.10, there exists a region (a, b) containing 0 such that $(a, b) \subset I$. By Corollary 6.12, there exists a natural number N such that $\frac{1}{b} < N$. Choose this N to be our N. Now let n be an arbitrary natural number such that $n \geq N$. It follows that $\frac{1}{b} < n$. Thus, since $0 < b$ and $0 < n$, we have by consecutive applications of Lemma 7.24 that $0 < \frac{1}{n} < b$. Consequently, since we also know that $a < 0$ and $a_n = \frac{1}{n}$, we have by transitivity and substitution that $a < a_n < b$. It follows by Equations 8.1 that $a_n \in (a, b)$. Therefore, by Definition 1.3, $a_n \in I$, as desired. $\square$

(d) $a_n = (-1)^n$.

Proof. To prove that this sequence diverges, Definition 15.2 tells us that it will suffice to show that for every point $p \in \mathbb{R}$, there exists an open interval I containing p such that for all $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \notin I$. Let's begin.

Let p be an arbitrary element of R. Choose $I = (p-1, p+1)$. Clearly $p \in I$. Let N be an arbitrary natural number. By Script 0, either N is even and $N+1$ is odd or vice versa. Thus, let N be even (the case where N is odd is symmetric). It follows that $N \geq N$ yields $a_N = (-1)^{2m} = ((-1)^2)^m = 1^m = 1$ and that $N+1 \geq N$ yields $a_{N+1} = (-1)^{2m+1} = 1(-1)^1 = -1$. Now suppose for the sake of contradiction that $a_N \in I$ and $a_{N+1} \in I$. Since $a_N = 1 \in I$, we have by Equations 8.1 that $p - 1 < 1 < p + 1$. It follows by Definition 7.21 that $p - 3 < -1 < p - 1$. But $-1 < p - 1$ implies by Equations 8.1 that $a_{N+1} = -1 \notin I$, a contradiction. Therefore, $N+1 \geq N$ is a number such that $a_{N+1} \notin I$, as desired. $\square$