Final-Specific Questions

6/2: 1. For any $a, b \in \mathbb{R}$ with $a \leq b$, define $\mathcal{F}_{[a,b]}$ to be the set of all bounded functions $f : [a, b] \to \mathbb{R}$. Suppose that for all a, b there exists a function $\mathbb{S}_a^b : \mathcal{F}_{[a,b]} \to \mathbb{R}$ which satisfies the following properties:

(i) $\mathbb{S}_a^b(f + g) = \mathbb{S}_a^b(f) + \mathbb{S}_a^b(g)$.

(ii) For any $c \in \mathbb{R}$, $\mathbb{S}_a^b(cf) = c\,\mathbb{S}_a^b(f)$.

(iii) For any $c \in \mathbb{R}$, if $f(x) = c$ for all $x \in [a, b]$, then $\mathbb{S}_a^b(f) = c(b - a)$.

(iv) If $f(x) \geq g(x)$ for all $x \in [a, b]$, then $\mathbb{S}_a^b(f) \geq \mathbb{S}_a^b(g)$.

(v) If $c \in [a, b]$, then $\mathbb{S}_a^b(f) = \mathbb{S}_a^c(f) + \mathbb{S}_c^b(f)$.

Use these properties to prove the following:

(a) Show that if $m \leq f(x) \leq M$ for all $x \in [a, b]$, then $m(b - a) \leq \mathbb{S}_a^b(f) \leq M(b - a)$.

Proof. Let $g, h : [a, b] \to \mathbb{R}$ be defined by $g(x) = m$ and $h(x) = M$ for all $x \in [a, b]$. Thus, since $m \leq g(x) \leq m$ and $M \leq h(x) \leq M$ for all $x \in [a, b]$ by Script 0, we have by consecutive applications of Definitions 5.6 and 10.1 that g and h are bounded functions. It follows by the definition of $\mathcal{F}_{[a,b]}$ that $g, h \in \mathcal{F}_{[a,b]}$. Consequently, consecutive applications of Property (iii) imply that $\mathbb{S}_a^b(g) = m(b - a)$ and $\mathbb{S}_a^b(h) = M(b - a)$. Thus, since $g(x) \leq f(x) \leq h(x)$ for all $x \in [a, b]$ by hypothesis, we have by consecutive applications of Property (iv) that $\mathbb{S}_a^b(g) \leq \mathbb{S}_a^b(f) \leq \mathbb{S}_a^b(h)$. Therefore, by the above, we have that $m(b - a) \leq \mathbb{S}_a^b(f) \leq M(b - a)$, as desired. □

(b) Show that if $f = g$ on (a, b), then $\mathbb{S}_a^b(f) = \mathbb{S}_a^b(g)$.

Proof. We will first deal with the trivial case where $a = b$. In this case, $f(x) = f(a)$ and $g(x) = g(b)$ for all $x \in [a, b]$. Thus, by consecutive applications of Property (iii), we have that $\mathbb{S}_a^b(f) = f(a) \cdot (b - a) = f(a) \cdot (a - a) = 0$ and similarly that $\mathbb{S}_a^b(g) = 0$. Therefore, $\mathbb{S}_a^b(f) = \mathbb{S}_a^b(g)$.

Having dealt with the trivial case, we can assume from now on that $a < b$. Let $h : [a, b] \to \mathbb{R}$ be defined by $h(x) = f(x) - g(x)$ for all $x \in [a, b]$. It follows by the hypothesis that $f = g$ for all $x \in (a, b)$ that $h(x) = 0$ for all $x \in (a, b)$. Thus, by Definition 1.18, $h([a, b]) = \{h(a), 0, h(b)\}$, so we clearly have by Definitions 5.6 and 10.1 that h is bounded. It follows that h is in the domain of $\mathbb{S}_a^b$. Having established this, we now seek to verify that $\mathbb{S}_a^b(h) = 0$ by dividing into four cases ($h(a) = h(b) = 0$, $h(a) \neq 0 = h(b)$, $h(a) = 0 \neq h(b)$, and $h(a) \neq 0 \neq h(b)$). Let's begin.

In the first case, we have by the definition of h that $h(x) = 0$ for all $x \in [a, b]$. Therefore, by Property (iii), we have that $\mathbb{S}_a^b(h) = 0(b - a) = 0$, as desired.

In the second case, we divide into two subcases ($h(a) > 0$ and $h(a) < 0$).

Suppose first that $h(a) > 0$. Let c be an arbitrary element of (a, b). It follows by Property (v) that $\mathbb{S}_a^b(h) = \mathbb{S}_a^c(h) + \mathbb{S}_c^b(h)$. By an argument symmetric to that of the first case verified herein, we have that $\mathbb{S}_c^b(h) = 0$. Thus, $\mathbb{S}_a^b(h) = \mathbb{S}_a^c(h)$. We now take a closer look at $\mathbb{S}_a^c(h)$. Since $0 \leq h(x) \leq h(a)$ for all $x \in [a, c]$, part (a) asserts that $0(c - a) \leq \mathbb{S}_a^c(h) \leq h(a) \cdot (c - a)$. It follows from the first inequality that $0 \leq \mathbb{S}_a^c(h)$. As such, to confirm that $\mathbb{S}_a^c(h) = 0$, suppose for the sake of contradiction that $0 < \mathbb{S}_a^c(h)$. Then since $h(a) > 0$ by supposition, Lemma 7.24 asserts that

$0 < \frac{\mathbb{S}_a^c(h)}{h(a)}$. Consequently, by Definition 7.21, $a < a + \frac{\mathbb{S}_a^c(h)}{h(a)}$. Thus, by Theorem 5.2, there exists a point c such that $a < c < \min(b, \frac{\mathbb{S}_a^c(h)}{h(a)})$[1]. By Definition 7.21 and Lemma 7.24 again, we have that $0 < h(a) \cdot (c - a) < \mathbb{S}_a^b(h)$. However, since $a < c < b$, Equations 8.1 imply that $c \in (a, b)$. Thus, by the above, $\mathbb{S}_a^b(h) \leq h(a) \cdot (c - a)$, a contradiction. Therefore, $\mathbb{S}_a^c(h) = 0$, so we have by the above that $\mathbb{S}_a^b(h) = \mathbb{S}_a^c(h) = 0$, as desired.

The argument is symmetric in the other subcase.

In the third case, the verification is symmetric to that of the second.

In the fourth case, begin by letting $h_-, h^+ : [a, b] \to \mathbb{R}$ be defined by

$$h_-(x) = \begin{cases} h(a) & x = a \\ 0 & x \neq a \end{cases}$$

$$h^+(x) = \begin{cases} 0 & x \neq b \\ h(b) & x = b \end{cases}$$

It follows by the definition of h that $h = h_- + h^+$. Additionally, we have by the second case that $\mathbb{S}_a^b(h_-) = 0$ and by the third case that $\mathbb{S}_a^b(h^+) = 0$. Therefore, by Property (i), $\mathbb{S}_a^b(h) = \mathbb{S}_a^b(h_-) + \mathbb{S}_a^b(h^+) = 0 + 0 = 0$, as desired.

Having established that $\mathbb{S}_a^b(h) = 0$ in any case, we can show that

$$\begin{aligned} 0 &= \mathbb{S}_a^b(h) \\ &= \mathbb{S}_a^b(f) + \mathbb{S}_a^b(-g) && \text{Property (i)} \\ &= \mathbb{S}_a^b(f) - \mathbb{S}_a^b(g) && \text{Property (ii)} \end{aligned}$$

Therefore, by Script 7, $\mathbb{S}_a^b(f) = \mathbb{S}_a^b(g)$. □

(c) Let $P = \{t_0, \dots, t_n\}$ be a partition of $[a, b]$. Suppose that for each i we have that $f(x) = f_i$ for all $x \in (t_{i-1}, t_i)$. Show that

$$\mathbb{S}_a^b(f) = \sum_{i=1}^{n} f_i \cdot (t_i - t_{i-1})$$

Proof. Let i be an arbitrary natural number between 1 and n, and let $h_i : [t_{i-1}, t_i] \to \mathbb{R}$ be defined by $h_i(x) = f_i$. Therefore, applying the above definitions for each i, we have that

$$\begin{aligned} \mathbb{S}_a^b(f) &= \mathbb{S}_{t_0}^{t_1}(f) + \mathbb{S}_{t_1}^{t_2} + \cdots + \mathbb{S}_{t_{n-1}}^{t_n}(f) && \text{Property (v)} \\ &= \sum_{i=1}^{n} \mathbb{S}_{t_{i-1}}^{t_i}(f) \\ &= \sum_{i=1}^{n} \mathbb{S}_{t_{i-1}}^{t_i}(h_i) && \text{Part (b)} \\ &= \sum_{i=1}^{n} f_i \cdot (t_i - t_{i-1}) && \text{Property (iii)} \end{aligned}$$

as desired. □

(d) For $f(x) = x$, show that $\mathbb{S}_0^b(f) = \frac{b^2}{2}$.

[1]Note that it is right here that we make use of the condition that $a < b$; this is why we consider the trivial case where $a = b$ separately at the beginning.

Proof. Suppose for the sake of contradiction that $\mathbb{S}_0^b(f) \neq \frac{b^2}{2}$. We divide into two cases ($\mathbb{S}_0^b(f) < \frac{b^2}{2}$ and $\mathbb{S}_0^b(f) > \frac{b^2}{2}$). Let's begin.

Suppose first that $\mathbb{S}_0^b(f) < \frac{b^2}{2}$. By Exercise 13.21, $\int_0^b f = \frac{b^2}{2}$. Thus, if we define $\epsilon = \frac{b^2}{2} - \mathbb{S}_0^b(f)$, we have by Lemma 13.20 that there is some partition $P = \{t_0, \dots, t_n\}$ such that $U(f,P) - \frac{b^2}{2} < \epsilon$ and $\frac{b^2}{2} - L(f,P) < \epsilon$. It follows from the latter result by the definition of ϵ and Definition 7.21 that $-L(f,P) < -\mathbb{S}_0^b(f)$. Consequently, by Lemma 7.24, we have that $\mathbb{S}_0^b(f) < L(f,P)$. Switching gears for a moment, we have by Definition 13.11 that $L(f,P) = \sum_{i=1}^n m_i(f)(t_i - t_{i-1})$. Additionally, if we define $m : [0,b] \to \mathbb{R}$ by $m(x) = m_i(f)$ for all $x \in (t_{i-1}, t_i)$ for all $i \in [n]$ and $m(t_i) = f(t_i)$, we will have by Part (c) that $\mathbb{S}_0^b(m) = \sum_{i=1}^n m_i(f)(t_i - t_{i-1})$. Combining these last two results with transitivity, we have that $\mathbb{S}_0^b(f) < L(f,P) = \mathbb{S}_0^b(m)$. However, by the definition of m, we also have that $m(x) \leq f(x)$ for all $x \in [0,b]$. but it follows from this by Property (iv) that $\mathbb{S}_0^b(m) \leq \mathbb{S}_0^b(f)$, a contradiction.

The proof is symmetric in the other case. $\square$

2. Suppose that f is positive and decreasing on $[1,\infty)$. Suppose also that f is integrable on $[1,n]$ for all $n \in \mathbb{N}$ and define the sequences (a_n) and (b_n) by

$$\begin{aligned} a_n &= f(n) \\ b_n &= \int_1^n f \end{aligned}$$

Show that $\sum_{n=1}^\infty a_n$ converges if and only if $\lim_{n\to\infty} b_n$ exists.

Proof. Suppose first that $\sum_{n=1}^\infty a_n$ converges. To prove that $\lim_{n\to\infty} b_n$ exists, Theorem 15.19 tells us that it will suffice to show that for all $\epsilon > 0$, theer is some $N \in \mathbb{N}$ such that $|b_n - b_m| < \epsilon$ for all $n,m \geq N$. Let $\epsilon > 0$ be arbitrary. By Theorem 16.5, there is some $(N-1) \in \mathbb{N}$ such that $|\sum_{k=m+1}^n a_k| < \epsilon$ for all $n > m \geq N$. Choose $N = (N-1)+1$ to be our N. Let n,m be arbitrary natural numbers such that $n,m \geq N$. We divide into three cases ($n > m$, $n = m$, and $n < m$).

If $n > m$, then we will need a few preliminary results. First, we seek to confirm that $\int_m^n f \geq 0$. Since f is positive, we know that $f(x) \geq 0$ for all $x \in [1,\infty)$. Thus, by Theorem 13.27, $0 = 0(n-m) \leq \int_m^n f$, as desired. Next, we seek to confirm that $\int_i^{i+1} f \leq a_i$. To begin, $a_i = f(i)$. Additionally, we have by Definition 8.16 that $f(i) \geq f(x)$ for all $x > i$. Thus, by Theorem 13.27, $\int_i^{i+1} \leq a_i((i+1)-i) = a_i$, as desired. Lastly, we seek to confirm that $\sum_{k=(m-1)+1}^{n-1} a_k \geq 0$. Since f is positive, we have by the definition of a_i that $a_i \geq 0$ for all i. Thus, by Script 7, $\sum_{k=(m-1)+1}^{n-1} a_k \geq 0$, as desired. Note that all three of these results will be used in the main inequality here, and results one and three will additionally be used in the inequality used for the reverse direction of the proof. Anyway, without further ado, we have

$$\begin{aligned} |b_n - b_m| &= \left| \int_1^n f - \int_1^m f \right| \\ &= \left| \int_m^n f \right| && \text{Theorem 13.23} \\ &= \int_m^n f && \text{Definition 8.4} \\ &= \int_m^{m+1} f + \int_{m+1}^{m+2} f + \cdots + \int_{n-1}^n f && \text{Theorem 13.23} \\ &\leq a_m + a_{m+1} + \cdots + a_{n-1} \\ &= \sum_{k=(m-1)+1}^{n-1} a_k \end{aligned}$$

$$\begin{aligned}
&= \left| \sum_{k=(m-1)+1}^{n-1} a_k \right| && \text{Definition 8.4} \\
&< \epsilon
\end{aligned}$$

as desired.

If $n = m$, then we have that $|b_n - b_m| = 0 < \epsilon$, as desired.

If $n < m$, then the argument is symmetric to that of the first case.

Now suppose that $\lim_{n\to\infty} b_n$ exists. To prove that $\sum_{n=1}^{\infty} a_n$ converges, Theorem 16.5 tells us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|\sum_{k=m+1}^{n} a_k| < \epsilon$ for all $n > m \geq N$. Let $\epsilon > 0$ be arbitrary. Then by Theorem 15.19, there is some $N \in \mathbb{N}$ such that $|b_n - b_m| < \epsilon$ for all $n, m \geq N$. Choose this N to be our N. Let n, m be arbitrary natural numbers that satisfy $n > m \geq N$. In addition to the two aforementioned preliminary results, we need one more, i.e., we now seek to confirm that $a_{i+1} \leq \int_i^{i+1} f$. To begin, $a_i = f(i)$. Additionally, we have by Definition 8.16 that $f(x) \geq f(i)$ for all $x < i$. Thus, by Theorem 13.27, $a_{i+1} = a_{i+1}((i+1) - i) \leq \int_i^{i+1} f$, as desired. With these results, we have that

$$\begin{aligned}
\left| \sum_{k=m+1}^{n} a_k \right| &= \sum_{k=m+1}^{n} a_k && \text{Definition 8.4} \\
&= a_{m+1} + a_{m+2} + \cdots + a_n \\
&\leq \int_m^{m+1} f + \int_{m+1}^{m+2} f + \cdots + \int_{n-1}^{n} f \\
&= \int_m^n f && \text{Theorem 13.23} \\
&= \left| \int_m^n f \right| && \text{Definition 8.4} \\
&= \left| \int_1^n f - \int_1^m f \right| && \text{Theorem 13.23} \\
&= |b_n - b_m| \\
&< \epsilon
\end{aligned}$$

as desired. □

3. Let $f : [a, b] \to \mathbb{R}$.

   (a) Show that if f is differentiable on $[a, b]$ and f' is bounded on $[a, b]$, then f is uniformly continuous on $[a, b]$.

   Proof. Since f is differentiable on $[a, b]$, we have by Theorem 12.5 that f is continuous on $[a, b]$. It follows by Theorem 9.10 that $f : [a, b] \to \mathbb{R}$ is continuous. Therefore, by Corollary 13.8, f is uniformly continuous on $[a, b]$, as desired. □

   (b) Use Theorem 12.15 to show that a polynomial of degree $n \geq 1$ has at most n distinct roots.

   Proof. We begin by stating two preliminary and previously proven results that will allow us to invoke Theorem 12.15 for any polynomial. First off, we have by Corollary 11.12 that polynomials are continuous. Second, we have by Exercises 12.8 and 12.9 that polynomials are differentiable. Having established these two facts, we are ready to begin in earnest.

   To prove the claim, we induct on n.

   For the base case $n = 1$, let p be a polynomial of degree n. Suppose for the sake of contradiction that p has m distinct roots where $m > n$. Choose roots r_1 and r_2. By Theorem 12.15, there exists a point $\lambda \in (r_1, r_2)$ such that $p'(\lambda) = 0$. Additionally, we have by Exercise 12.8 that $p'(x) = a$, where $a \in \mathbb{R}$, for all $x \in \mathbb{R}$. These last two results when combined necessitate that $p'(x) = 0$ for all $x \in \mathbb{R}$. Thus, by Corollary 12.17, p is constant on $\mathbb{R}$. But by Definition 11.11, this implies that p has degree 0, a contradiction.

Now suppose inductively that we have proven that a polynomial of degree n has at most n distinct roots; we wish to prove that a polynomial of degree $n+1$ has at most $n+1$ distinct roots. To do so, suppose for the sake of contradiction that p has m distinct roots where $m > n+1$. By Theorem 3.5, we may name the roots $r_1, \dots, r_m$ such that $r_1 < r_2 < \cdots < r_m$. It follows by consecutive applications of Theorem 12.15 that there exists a point $\lambda_i \in (r_i, r_{i+1})$ such that $p'(\lambda_i) = 0$ for all $i \in [m-1]$. Consequently, since each λ_i is in a disjoint region from all other λ_i's, we know that each λ_i is distinct. Thus, we know that p' has at least $m-1$ distinct roots. More notably, since we defined $m > n+1$, we have by Definition 7.21 that $m-1 > n$, meaning that p' has more than n distinct roots. However, by Exercise 12.8, p' has degree n. But this means by the inductive hypothesis implies that p' has at most n distinct roots, a contradiction. □

4. Let $f : \mathbb{R} \to \mathbb{R}$. Suppose that $|f(y) - f(x)| \leq c|y - x|$ for all $x, y \in \mathbb{R}$, where $c < 1$ is a constant.

   (a) Show that f is continuous.

   Proof. To prove that f is continuous, Theorem 13.2 tells us that it will suffice to show that f is uniformly continuous. To do this, Definition 13.1 tells us that it will suffice to verify that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in \mathbb{R}$, if $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Choose $\delta = \epsilon$. Let x, y be arbitrary real numbers such that $|y - x| < \delta$. Then by the supposition,

$$\begin{aligned} |f(y) - f(x)| &\leq c|y - x| \\ &< |y - x| \\ &< \delta \\ &= \epsilon \end{aligned}$$

   as desired. □

   (b) Show that there is at most one point $x \in \mathbb{R}$ where $f(x) = x$.

   Proof. Suppose for the sake of contradiction that there exist multiple points $x \in \mathbb{R}$ where $f(x) = x$. Let $x, y \in \mathbb{R}$ be two such points. Then by the supposition,

$$\begin{aligned} |y - x| &= |f(y) - f(x)| \\ &\leq c|y - x| \\ &< |y - x| \end{aligned}$$

   a contradiction. □

   (c) Show that there exists a point $x \in \mathbb{R}$ where $f(x) = x$.

   Proof. Let (a_n) be defined by $a_n = f^{n-1}(0)$ for all $n \in \mathbb{N}$. To prove that (a_n) converges, Theorem 15.19 tells us that it will suffice to show that for all $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon$ for all $n, m \geq N$. Let $\epsilon > 0$ be arbitrary. Since $c < 1$, by Exercise 15.8, $\lim_{n\to\infty} c^n = 0$. Thus, by Theorem 15.7, there is some $N \in \mathbb{N}$ such that $|c^n| < \epsilon$ for all $n \geq N$. Choose this N to be our N. Let n, m be arbitrary natural numbers such that $n, m \geq N$ and WLOG let $n \geq m$. Then by consecutive applications of the supposition, $|a_n - a_m| = |f^{n-1}(0) - f^{m-1}(0)| \leq c^{m-1}|f^{n-m}(0) - 0| < \epsilon|f^{n-m}(0)| < \epsilon$, as desired.

   □