Script 18

The Euclidean Space $\mathbb{R}^n$

7/7: **Definition 18.1.** The **Euclidean n-space** $\mathbb{R}^n$ is the n-fold Cartesian product of $\mathbb{R}$. Symbolically,

$$\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_1, \dots, x_n \in \mathbb{R}\}$$

is the set of n-tuples of real numbers. We often write

$$\mathbf{x} = (x_1, \dots, x_n)$$

to denote an element, which is also referred to as a **vector**, in $\mathbb{R}^n$ and

$$\mathbf{0} = (0, \dots, 0)$$

Definition 18.2. Let $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. We define the following operations.

(a) (Addition) $\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_n + y_n)$.

(b) (Scalar Multiplication) $\lambda\mathbf{x} = (\lambda x_1, \dots, \lambda x_n)$.

Exercise 18.3. Prove that the addition on $\mathbb{R}^n$ satisfies FA1-FA4 (see Definition 7.8). Moreover, prove that

VS1. (Associativity of Scalar Multiplication) If $\lambda, \mu \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, then $(\lambda\mu)\mathbf{x} = \lambda(\mu\mathbf{x})$.

VS2. (Distributivity of Scalars) If $\lambda, \mu \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, then $(\lambda + \mu)\mathbf{x} = \lambda\mathbf{x} + \mu\mathbf{x}$.

VS3. (Distributivity of Vectors) If $\lambda \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then $\lambda(\mathbf{x} + \mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y}$.

VS4. (Scalar Multiplicative Identity) If $\mathbf{x} \in \mathbb{R}^n$, then $1\mathbf{x} = \mathbf{x}$.

These eight properties together are called the **vector space axioms**.

Proof. To prove that $\mathbb{R}^n$ obeys FA1 from Definition 7.8, it will suffice to show that for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$. Let $\mathbf{x}, \mathbf{y}$ be arbitrary elements of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}
\mathbf{x} + \mathbf{y} &= (x_1 + y_1, \dots, x_n + y_n) \\
&= (y_1 + x_1, \dots, y_n + x_n) \\
&= \mathbf{y} + \mathbf{x}
\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys FA2 from Definition 7.8, it will suffice to show that for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$, $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$. Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ be arbitrary elements of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}
(\mathbf{x} + \mathbf{y}) + \mathbf{z} &= (x_1 + y_1, \dots, x_n + y_n) + \mathbf{z} \\
&= ((x_1 + y_1) + z_1, \dots, (x_n + y_n) + z_n) \\
&= (x_1 + (y_1 + z_1), \dots, x_n + (y_n + z_n)) \\
&= \mathbf{x} + (y_1 + z_1, \dots, y_n + z_n) \\
&= \mathbf{x} + (\mathbf{y} + \mathbf{z})
\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys FA3 from Definition 7.8, it will suffice to find an element $0 \in \mathbb{R}^n$ such that $\mathbf{x} + 0 = 0 + \mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Choose $\mathbf{0}$ to be our 0. Let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}\mathbf{x} + \mathbf{0} &= (x_1 + 0, \dots, x_n + 0)\\ &= (x_1, \dots, x_n)\\ &= \mathbf{x}\\ &= (0 + x_1, \dots, 0 + x_n)\\ &= \mathbf{0} + \mathbf{x}\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys FA4 from Definition 7.8, it will suffice to show that for all $\mathbf{x} \in \mathbb{R}^n$, there exists $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x} = 0$. Let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Choose $\mathbf{y} = (-x_1, \dots, -x_n)$. Then by Definition 18.2,

$$\begin{aligned}\mathbf{x} + \mathbf{y} &= (x_1 + (-x_1), \dots, x_n + (-x_n))\\ &= (0, \dots, 0)\\ &= \mathbf{0}\\ &= ((-x_1) + x_1, \dots, (-x_n) + x_n)\\ &= \mathbf{y} + \mathbf{x}\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys VS1, it will suffice to show that for all $\lambda, \mu \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, we have $(\lambda\mu)\mathbf{x} = \lambda(\mu\mathbf{x})$. Let λ, μ be arbitrary elements of $\mathbb{R}$, and let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}(\lambda\mu)\mathbf{x} &= ((\lambda\mu)x_1, \dots, (\lambda\mu)x_n)\\ &= (\lambda(\mu x_1), \dots, \lambda(\mu x_n))\\ &= \lambda(\mu\mathbf{x})\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys VS2, it will suffice to show that for all $\lambda, \mu \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$, we have $(\lambda + \mu)\mathbf{x} = \lambda\mathbf{x} + \mu\mathbf{x}$. Let λ, μ be arbitrary elements of $\mathbb{R}$, and let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}(\lambda + \mu)\mathbf{x} &= ((\lambda + \mu)x_1, \dots, (\lambda + \mu)x_n)\\ &= (\lambda x_1 + \mu x_1, \dots, \lambda x_n + \mu x_n)\\ &= (\lambda x_1, \dots, \lambda x_n) + (\mu x_1, \dots, \mu x_n)\\ &= \lambda\mathbf{x} + \mu\mathbf{x}\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys VS3, it will suffice to show that for all $\lambda \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have $\lambda(\mathbf{x} + \mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y}$. Let λ be an arbitrary element of $\mathbb{R}$, and let $\mathbf{x}, \mathbf{y}$ be arbitrary elements of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned}\lambda(\mathbf{x} + \mathbf{y}) &= \lambda(x_1 + y_1, \dots, x_n + y_n)\\ &= (\lambda(x_1 + y_1), \dots, \lambda(x_n + y_n))\\ &= (\lambda x_1 + \lambda y_1, \dots, \lambda x_n + \lambda y_n)\\ &= (\lambda x_1, \dots, \lambda x_n) + (\lambda y_1, \dots, \lambda y_2)\\ &= \lambda\mathbf{x} + \lambda\mathbf{y}\end{aligned}$$

as desired.

To prove that $\mathbb{R}^n$ obeys VS4, it will suffice to show that for all $\mathbf{x} \in \mathbb{R}^n$, we have $1\mathbf{x} = \mathbf{x}$. Let $\mathbf{x}$ be an arbitrary element of $\mathbb{R}^n$. Then by Definition 18.2,

$$\begin{aligned} 1\mathbf{x} &= (1x_1, \dots, 1x_n) \\ &= (x_1, \dots, x_n) \\ &= \mathbf{x} \end{aligned}$$

as desired. □

Remark 18.4. Since $\mathbb{R}^n$ with the two operations defined as above satisfies these eight axioms, we call $\mathbb{R}^n$ a **vector space**.

Exercise 18.5. Prove that if $\mathbf{x} \in \mathbb{R}^n$, then $0\mathbf{x} = \mathbf{0}$.

Proof. By Definition 18.2, we have that

$$\begin{aligned} 0\mathbf{x} &= (0x_1, \dots, 0x_n) \\ &= (0, \dots, 0) \\ &= \mathbf{0} \end{aligned}$$

as desired. □