MATH 20410 (Analysis in $\mathbb{R}^n$ II – Accelerated) Problem Sets

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Contents

1 Differentiation

From Rudin (1976).

Chapter 5

1. Let f be defined for all real x, and suppose that

$$|f(y) - f(x)| \leq (y - x)^2$$

for all real x and y. Prove that f is constant.

2. Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b) and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)}$$

for $a < x < b$.

3. Suppose g is a real function on $\mathbb{R}^1$, with bounded derivative (say $|g'| \leq M$). Fix $\epsilon > 0$ and define $f(x) = x + \epsilon g(x)$. Prove that f is one-to-one if ϵ is small enough. (A set of admissable values of ϵ can be determined which depends only on M.)

4. If

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$$

where $C_0, \ldots, C_n$ are real constants, prove that the equation

$$C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 and 1.

5. Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

References

Rudin, W. (1976). *Principles of mathematical analysis* (A. A. Arthur & S. L. Langman, Eds.; Third). McGraw-Hill.