Chapter 8

Some Special Functions

8.1 Notes

3/7:

- Plan:
  1. Go over some of the hits in chapter 8.
  2. Define sine.
  3. Power series.
  4. Exponential functions (log, sin, cos).
- Proposition (power series properties): If $\sum_{n=0}^{\infty} a_n x^n$ converges for all $|x| < R$, and $f : B_R(0) \to \mathbb{R}$ is defined by
$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$
then:
  (a) f is continuous.
    - From the root test, $\sum_{n=0}^{\infty} a_n x^n$ is in fact absolutely convergent on $(-R, R)$. Therefore, on any interval $[-R+\epsilon, R-\epsilon]$ $(0 < \epsilon < R)$, we have
$$|a_n x^n| \leq |a_n||R+\epsilon|^n$$
so by the M-test, $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-R+\epsilon, R-\epsilon]$. Then since $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-R+\epsilon, R-\epsilon]$, we have (a) since all $\sum_{n=0}^{N} a_n x^n$ are continuous.

  (b) f is differentiable with $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.
    - (b) follows similarly to (a) by uniform convergence.
    - Note that $\limsup \sqrt[n]{|na_n|} = \limsup \sqrt[n]{|a_n|}$ (since $\lim_{n\to\infty} \sqrt[n]{n} = 1$).
    - Therefore, $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges on $(-R, R)$.

  (c) More generally, f is infinitely differentiable with
$$f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k}$$
    - Now (c) follows as in the proof of (b).

  (d) We have the identity
$$a_k = \frac{f^{(k)}(0)}{k!}$$
    - (d) follows from (c) by plugging in zero.

- Note that historically, the analysis of power series motivated the development of all of the Chapter 7 theorems; we simply learned those first without motivation to present the proofs in an ordered manner.
- Aside: Consider the exponential function x^y for $x, y \in \mathbb{R}$ with $x \geq 0$.
  - We define it for natural numbers and integers fairly easily, then rationals, and then for reals as the supremum of exponentials of the entries in the Dedekind cut below $x \in \mathbb{R}$.
  - Under this definition, we can confirm our normal exponential rules and then that x^y is continuous.
- Recall that
$$\mathrm{e} = \sum_{n=0}^{\infty} \frac{1}{n!}$$
- So now we are going to construct $E(x)$, $L(x)$, $C(x)$, and $S(x)$ (which are just e^x, $\ln(x)$, $\cos(x)$, and $\sin(x)$).
- Define $E : \mathbb{C} \to \mathbb{C}$ by
$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
  - By the proposition, it converges and is continuous for all $z \in \mathbb{C}$.
  - For the real numbers, E is differentiable. (E is also complex-differentiable, but we won't go into that).
- Proposition: $E(z)E(w) = E(z + w)$ for all $z, w \in \mathbb{C}$.
  - We have by the Cauchy product (Mertens' theorem) that
$$\begin{aligned} E(z)E(w) &= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{w^n}{n!}\right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^k w^{n-k}}{k!(n-k)!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}(z+w)^n \\ &= E(z+w) \end{aligned}$$
- Corollary: $E(z)E(-z) = E(0) = 1$ for all $z \in \mathbb{C}$.
- $E(x) > 0$ for $x \geq 0$.
  - It follows since $E(z + w) = E(z)E(w)$ that $E(x) > 0$ for all $x \in \mathbb{R}$.
- $\mathrm{d}E/\mathrm{d}x = E$; E is the unique, normalized ($E(0) = 1$) function such that this is true.
  - We can prove this from the power series definition.
- $E(x) \to \infty$ as $x \to \infty$ and $E(x) \to 0$ as $x \to -\infty$. (Also from the power series definition.)
- $0 \leq x_1 < x_2$ implies that $E(x_1) < E(x_2)$.
  - Either from $\mathrm{d}E/\mathrm{d}x = E > 0$ or from the power series definition.
  - It follows from $E(z + w) = E(z)E(w)$ that $x_1 < x_2$ implies $E(x_1) < E(x_2)$.