Chapter 6

The Riemann-Stieltjes Integral

6.1 Notes

1/28: - Plan:
    1. Finish up Fundamental Theorem of Calculus proof.
    2. Basic consequences.
    3. Rectifiable curves.
- Recall that we're given $f:[a,b]\to\mathbb{R}$ continuous, $f:[a,b]\to\mathbb{R}$, and $x\mapsto\int_a^x f(t)\,\mathrm{d}t$.
- Goal: Show $F'(x_0)=f(x_0)$.
    - WTS: Find δ such that $|x-x_0|<\delta$ implies

$$\begin{aligned}\left|\frac{1}{x-x_0}\int_{x_0}^x f(t)\,\mathrm{d}t-f(x_0)\right| &= \left|\frac{1}{x-x_0}\int_{x_0}^x f(t)\,\mathrm{d}t-\frac{1}{x-x_0}\int_{x_0}^x f(x_0)\,\mathrm{d}t\right|\\ &= \frac{1}{|x-x_0|}\left|\int_{x_0}^x (f(t)-f(x_0))\,\mathrm{d}t\right|\\ &\leq \frac{1}{|x-x_0|}\int_{x_0}^x |f(t)-f(x_0)|\,\mathrm{d}t\\ &< \epsilon\end{aligned}$$

    - Since f is continuous, there exists δ such that if $|x-x_0|<\delta$, then $|f(x)-f(x_0)|<\epsilon$.
    - Now

$$\begin{aligned}\frac{1}{|x-x_0|}\int_{x_0}^x |f(t)-f(x_0)|\,\mathrm{d}t &< \frac{1}{|x-x_0|}\int_{x_0}^x \epsilon\,\mathrm{d}t\\ &= \epsilon\end{aligned}$$

- Applications:
    1. Theorem (MVT for integration): $f:[a,b]\to\mathbb{R}$ continuous, then there exists $x_0\in[a,b]$ such that

$$f(x_0)=\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

        - Apply MVT to $F(x)=\int_a^x f(t)\,\mathrm{d}t$. Then

$$F'(x_0)=f(x_0)=\frac{F(b)-F(a)}{b-a}$$

        as desired.

2. Theorem (Integration by parts): Let $F, G : [a, b] \to \mathbb{R}$ be differentiable with $F' = f$, $G' = g$ and with f and g both integrable. Then
$$\int_a^b Fg = F(b)G(b) - F(a)G(a) - \int_a^b fG$$
   - Just use the product rule plus the FTC to prove.
   - We have
$$\begin{aligned} \int_a^b (FG)' &= \int_a^b fG + \int_a^b Fg \\ F(b)G(b) - F(a)G(a) &= \int_a^b fG + \int_a^b Fg \\ \int_a^b Fg &= F(b)G(b) - F(a)G(a) - \int_a^b fG \end{aligned}$$
3. Theorem (u-substitution).
   - Follows similarly from the chain rule and FTC.

- Integration of vector-valued functions.
- If $f : [a, b] \to \mathbb{R}^k$, we define $\int_a^b f$ by
$$\int_a^b f = \left(\int_a^b f_1, \dots, \int_a^b f_k \right)$$
- Alternatively, you can define $\int_a^b f$ using P, $U(f, P)$, $L(f, P)$, etc. and then prove that the integral exists iff all f_i are integrable and in this case the above definition holds.
- Rectifiable curves: Let $\gamma : [a, b] \to \mathbb{R}^k$ be a continuous function.
- Plan: Define the length of γ and show that we can compute it with an integral.
   - Idea: For polygonal paths, we know how to define length. So let's approximate γ by polygons and take a limit.
   - Ref: Given a partition P, then define the length of γ with respect to P as $\Lambda(\gamma, P)$. Let the length of γ be $\Lambda(\gamma) = \sup_P \Lambda(\gamma, P)$ if this limit exists in this case, we call γ **rectifiable**.
- Fractals are not rectifiable — their length diverges.
- Theorem: Suppose γ is continuously differentiable (i.e., γ is differentiable and γ' is continuous). Then γ si rectifiable and
$$\Lambda(\gamma) = \int_a^b |\gamma'(t)| \, \mathrm{d}t$$
   - Notice: If $P \leq P'$, then $\Lambda(\gamma, P) \leq \Lambda(\gamma, P')$. (Prove with triangle inequality.)
   - WTS: For all partitions P, $\Lambda(\gamma, P) \leq \int_a^b |\gamma'(t)| \, \mathrm{d}t$ and thus $\Lambda(\gamma) \leq \int_a^b |\gamma'(t)| \, \mathrm{d}t$.
   - We have that
$$\begin{aligned} \Lambda(\gamma, P) &= \sum_{i=1}^n |\gamma(x_i) - \gamma(x_{i-1}) \\ &= \sum_{i=1}^n \left| \int_{x_{i-1}}^{x_i} \gamma'(t) \, \mathrm{d}t \right| \\ &\leq \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |\gamma'(t)| \, \mathrm{d}t \\ &= \int_a^b |\gamma'(t)| \, \mathrm{d}t \end{aligned}$$

- Catch up.
  - I should make up PSets 1-2.
  - Exams have less than Rudin-strength problems.
  - Exams are mostly true/false (and of that, mostly false, provide a counterexample).