4 Sequences and Series of Functions II / Functions of Several Variables

From Rudin (1976).

Chapter 7

2/16: **5.** Let

$$f_n(x) = \begin{cases} 0 & x < \frac{1}{n+1} \\ \sin^2 \frac{\pi}{x} & \frac{1}{n+1} \le x \le \frac{1}{n} \\ 0 & \frac{1}{n} < x \end{cases}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that absolute convergence, even for all x, does not imply uniform convergence.

6. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

8. If

$$I(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$$

if $\{x_n\}$ is a sequence of distinct points of (a, b), and if $\sum |c_n|$ converges, prove that the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n)$$

converges uniformly on $[a, b]$, and that f is continuous for every $x \neq x_n$.

9. Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \to x$ and $x \in E$. Is the converse of this true?

Chapter 9

1. If S is a nonempty subset of a vector space X, prove (as asserted in Section 9.1) that the span of S is a vector space.

2. Prove (as asserted in Section 9.6) that BA is linear if A and B are linear transformations. Prove also that A^{-1} is linear and invertible.

3. Assume $A \in L(X, Y)$ and $A\mathbf{x} = \mathbf{0}$ only when $\mathbf{x} = \mathbf{0}$. Prove that A is then 1-1.

4. Prove (as asserted in Section 9.30) that null spaces and ranges of linear transformations are vector spaces.