Chapter 8

Some Special Functions

8.1 Notes

3/7:

- Plan:
  1. Go over some of the hits in chapter 8.
  2. Define sine.
  3. Power series.
  4. Exponential functions (log, sin, cos).
- Proposition (power series properties): If $\sum_{n=0}^{\infty} a_n x^n$ converges for all $|x| < R$, and $f : B_R(0) \to \mathbb{R}$ is defined by
$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$
then:
  (a) f is continuous.
    - From the root test, $\sum_{n=0}^{\infty} a_n x^n$ is in fact absolutely convergent on $(-R, R)$. Therefore, on any interval $[-R+\epsilon, R-\epsilon]$ $(0 < \epsilon < R)$, we have
$$|a_n x^n| \leq |a_n||R+\epsilon|^n$$
so by the M-test, $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-R+\epsilon, R-\epsilon]$. Then since $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-R+\epsilon, R-\epsilon]$, we have (a) since all $\sum_{n=0}^{N} a_n x^n$ are continuous.

  (b) f is differentiable with $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.
    - (b) follows similarly to (a) by uniform convergence.
    - Note that $\limsup \sqrt[n]{|na_n|} = \limsup \sqrt[n]{|a_n|}$ (since $\lim_{n\to\infty} \sqrt[n]{n} = 1$).
    - Therefore, $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges on $(-R, R)$.

  (c) More generally, f is infinitely differentiable with
$$f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k}$$
    - Now (c) follows as in the proof of (b).

  (d) We have the identity
$$a_k = \frac{f^{(k)}(0)}{k!}$$
    - (d) follows from (c) by plugging in zero.

- Note that historically, the analysis of power series motivated the development of all of the Chapter 7 theorems; we simply learned those first without motivation to present the proofs in an ordered manner.
- Aside: Consider the exponential function x^y for $x, y \in \mathbb{R}$ with $x \geq 0$.
  - We define it for natural numbers and integers fairly easily, then rationals, and then for reals as the supremum of exponentials of the entries in the Dedekind cut below $x \in \mathbb{R}$.
  - Under this definition, we can confirm our normal exponential rules and then that x^y is continuous.
- Recall that

$$\mathrm{e} = \sum_{n=0}^{\infty} \frac{1}{n!}$$

- So now we are going to construct $E(x)$, $L(x)$, $C(x)$, and $S(x)$ (which are just e^x, $\ln(x)$, $\cos(x)$, and $\sin(x)$).
- Define $E : \mathbb{C} \to \mathbb{C}$ by

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

  - By the proposition, it converges and is continuous for all $z \in \mathbb{C}$.
  - For the real numbers, E is differentiable. (E is also complex-differentiable, but we won't go into that).
- Proposition: $E(z)E(w) = E(z + w)$ for all $z, w \in \mathbb{C}$.
  - We have by the Cauchy product (Mertens' theorem) that

$$\begin{aligned}
E(z)E(w) &= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{w^n}{n!}\right) \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^k w^{n-k}}{k!(n-k)!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} (z + w)^n \\
&= E(z + w)
\end{aligned}$$

- Corollary: $E(z)E(-z) = E(0) = 1$ for all $z \in \mathbb{C}$.
- $E(x) > 0$ for $x \geq 0$.
  - It follows since $E(z + w) = E(z)E(w)$ that $E(x) > 0$ for all $x \in \mathbb{R}$.
- $\mathrm{d}E/\mathrm{d}x = E$; E is the unique, normalized ($E(0) = 1$) function such that this is true.
  - We can prove this from the power series definition.
- $E(x) \to \infty$ as $x \to \infty$ and $E(x) \to 0$ as $x \to -\infty$. (Also from the power series definition.)
- $0 \leq x_1 < x_2$ implies that $E(x_1) < E(x_2)$.
  - Either from $\mathrm{d}E/\mathrm{d}x = E > 0$ or from the power series definition.
  - It follows from $E(z + w) = E(z)E(w)$ that $x_1 < x_2$ implies $E(x_1) < E(x_2)$.

3/9: - Plan:
    1. Keep going with E, L, C, and S.
    2. Prove the fundamental theorem of algebra.
- Define
$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
    - Recall that $E(z+w) = E(z)E(w)$.
- Theorem: $E(x) = \mathrm{e}^x$ for all $x \in \mathbb{R}$.
    - $E(1) = \mathrm{e}^1$ (by definition).
    - $E(n) = \mathrm{e}^n$ (by $E(z+w) = E(z)E(w)$).
    - $[E(p/q)]^q = E(p) = \mathrm{e}^p$ (by $E(z+w) = E(z)E(w)$).
    - $E(p/q) = \mathrm{e}^{p/q}$ for all $p/q \in \mathbb{Q}$.
    - $E(x) = \mathrm{e}^x$ for all $x \in \mathbb{R}$ (since both LHS and RHS are continuous functions that agree on $\mathbb{Q}$).
- Briefly: $E : \mathbb{R} \to \mathbb{R}^+$ is a strictly increasing surjective function. Thus, we have an inverse function $L : \mathbb{R}^+ \to \mathbb{R}$.
- Theorem: L is differentiable (and therefore continuous).
    - Since $E' = E > 0$ everywhere, we may apply the inverse function theorem at every point.
- Now by the chain rule, $E(L(x)) = x$ for all $x \in \mathbb{R}^+$, so taking derivatives yields
$$\begin{aligned}
E'(L(x))L'(x) &= 1\\
E(L(x))L'(x) &= 1\\
xL'(x) &= 1\\
L'(x) &= \frac{1}{x}
\end{aligned}$$
- Proposition:
    1. $L(uw) = L(u) + L(w)$.
    2. $L(x) = \int_1^x t^{-1}\,\mathrm{d}t$.
- Trig functions:
$$C(x) = \frac{1}{2}[E(ix) + E(-ix)] \qquad S(x) = \frac{1}{2i}[E(ix) - E(-ix)]$$
    - You can use these definitions to prove trig identities, having derived them geometrically.
- Proposition: If $x \in \mathbb{R}$, then $C(x), S(x) \in \mathbb{R}$.
    - Key observation: $E(\bar{z}) = \overline{E(z)}$.
    - We have
$$\begin{aligned}
\overline{C(x)} &= \frac{1}{2}[\overline{E(ix)} + \overline{E(-ix)}]\\
&= \frac{1}{2}[E(-ix) + E(ix)]\\
&= C(x)
\end{aligned}$$

- Symmetric for $S(x)$.

- Note that we could equally well define C,S by
$$C(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k} \qquad\qquad S(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1}$$
- Proposition: $E(ix)=C(x)+iS(x)$.
- Proposition: C,S are differentiable with
$$C'(x)=-S(x) \qquad\qquad S'(x)=C(x)$$
- Proposition: For all $x\in\mathbb{R}$, $|E(ix)|=1$.
  - We have that
$$|E(ix)|^2=E(ix)\overline{E(ix)}=E(ix)E(-ix)=E(0)=1$$
  - Taking square roots of both sides yields the desired result.
- The above result proves that the imaginary axis maps onto the unit circle in the complex plane.
- We now define π and all that.
  - Goal: Show that for all $z\in\mathbb{C}$ with $|z|=1$, there exists a unique $\theta\in[0,2\pi)$ such that $\mathrm{e}^{i\theta}=z$. Further, $E(ix)$ has period 2π.
- Proposition: $C(x)^2+S(x)^2=1$.
  - Use $E(ix)=C(x)+iS(x)$ and $|E(ix)|=1$.
- Proposition: There exists some positive number x such that $C(x)=0$.
  - Suppose (contradiction): $C(x)>0$ for all $x>0$ (since $C(0)=1$).
  - Thus, $S'(x)>0$ for all $x>0$.
  - Consequently, given $0<x<y$,
$$S(x)(y-x)<\int_x^y S(t)\,\mathrm{d}t=C(x)-C(y)\leq 2$$
  - But we can choose y large enough to make $S(x)(y-x)>2$, a contradiction.
- $\boldsymbol{\pi}$: The real number such that $\pi/2$ is the unique smallest positive real number with $C(\pi/2)=0$.
  - We know that a unique smallest number exists because since $C(0)=1$ and C is continuous, there exists a neighborhood around 0 where C is nonzero.
- Proposition: $S(\pi/2)=1$.
  - We have
$$\begin{aligned}C(\pi/2)^2+S(\pi/2)^2&=1\\ S(\pi/2)&=\pm1\end{aligned}$$
  - Furthermore, since $S(0)=0$ and $S'(x)=C(x)$ is positive on $[0,\pi/2)$, we know that S is increasing and thus $S(\pi/2)=+1$.