6 Functions of Several Variables II

From Rudin (1976).

Chapter 9

2/22: **5.** Prove that to every $A \in L(\mathbb{R}^n, \mathbb{R}^1)$ corresponds a unique $\mathbf{y} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$. Prove also that $\|A\| = |\mathbf{y}|$. (Hint: Under certain conditions, equality holds in the Schwarz inequality.)

6. If

$$f(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \end{cases}$$

prove that $(D_1 f)(x,y)$ and $(D_2 f)(x,y)$ exist at every point of $\mathbb{R}^2$, although f is not continuous at $(0,0)$.

7. Suppose that f is a real-valued function defined in an open set $E \subset \mathbb{R}^n$, and that the partial derivatives $D_1 f, \dots, D_n f$ are bounded on E. Prove that f is continuous in E. (Hint: Proceed as in the proof of Theorem 9.21.)

8. Suppose that f is a differentiable real function in an open set $E \subset \mathbb{R}^n$, and that f has a local maximum at a point $\mathbf{x} \in E$. Prove that $f'(\mathbf{x}) = 0$.

10. If f is a real function defined in a convex open set $E \subset \mathbb{R}^n$, such that $(D_1 f)(\mathbf{x}) = 0$ for every $\mathbf{x} \in E$, prove that $f(\mathbf{x})$ depends only on $x_2, \dots, x_n$. Show that the convexity of E can be replaced by a weaker condition, but that some condition is required. For example, if $n = 2$ and E is shaped like a horseshoe, the statement may be false.

11. If f and g are differentiable real functions in $\mathbb{R}^n$, prove that

$$\nabla(fg) = f\nabla g + g\nabla f$$

and that

$$\nabla\left(\frac{1}{f}\right) = -\frac{\nabla f}{f^2}$$

wherever $f \neq 0$.

17. Let $\mathbf{f} = (f_1, f_2)$ be the mapping of $\mathbb{R}^2$ into $\mathbb{R}^2$ given by

$$f_1(x,y) = \mathrm{e}^x \cos y \qquad\qquad f_2(x,y) = \mathrm{e}^x \sin y$$

(a) What is the range of f?

(b) Show that the Jacobian of f is not zero at any point of $\mathbb{R}^2$. Thus, every point of $\mathbb{R}^2$ has a neighborhood in which f is one-to-one. Nevertheless, f is not one-to-one on $\mathbb{R}^2$.

(c) Put $\mathbf{a} = (0, \pi/3)$, $\mathbf{b} = f(\mathbf{a})$, and let $\mathbf{g}$ be the continuous inverse of $\mathbf{f}$, defined in a neighborhood of $\mathbf{b}$, such that $\mathbf{g}(\mathbf{b}) = \mathbf{a}$. Find an explicit formula for $\mathbf{g}$, compute $\mathbf{f}'(\mathbf{a})$ and $\mathbf{g}'(\mathbf{b})$, and verify that

$$\mathbf{g}'(\mathbf{b}) = [\mathbf{f}'(\mathbf{g}(\mathbf{b}))]^{-1}$$

(d) What are the images under $\mathbf{f}$ of lines parallel to the coordinate axes?