# 7 Functions of Several Variables III

*From Rudin (1976).*

## Chapter 9

3/2: **9.** If $\mathbf{f}$ is a differentiable mapping of a *connected* open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, and if $\mathbf{f}'(\mathbf{x}) = 0$ for every $\mathbf{x} \in E$, prove that $\mathbf{f}$ is constant in $E$.

**13.** Suppose $\mathbf{f}$ is a differentiable mapping of $\mathbb{R}^1$ into $\mathbb{R}^3$ such that $|\mathbf{f}(t)| = 1$ for every $t$. Prove that $\mathbf{f}'(t) \cdot \mathbf{f}(t) = 0$. Interpret this result geometrically.

**14.** Define

$$f(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ \frac{x^3}{x^2+y^2} & (x,y) \neq (0,0) \end{cases}$$

(a) Prove that $D_1 f$ and $D_2 f$ are bounded functions in $\mathbb{R}^2$. (Hence $f$ is continuous.)

(b) Let $\mathbf{u}$ be any unit vector in $\mathbb{R}^2$. Show that the directional derivative $(D_{\mathbf{u}} f)(0,0)$ exists, and that its absolute value is at most 1.

(c) Let $\gamma$ be a differentiable mapping of $\mathbb{R}^1$ into $\mathbb{R}^2$ (in other words, $\gamma$ is a differentiable curve in $\mathbb{R}^2$), with $\gamma(0) = (0,0)$ and $|\gamma'(0)| > 0$. Put $g(t) = f(\gamma(t))$ and prove that $g$ is differentiable for every $t \in \mathbb{R}^1$. If $\gamma \in C^1$, prove that $g \in C^1$.

(d) In spite of this, prove that $f$ is not differentiable at $(0,0)$. (Hint: The formula $(D_{\mathbf{u}} f)(\mathbf{x}) = \sum_{i=1}^n (D_i f)(\mathbf{x}) u_i$ fails.)

**16.** Show that the continuity of $\mathbf{f}'$ at the point $\mathbf{a}$ is needed in the inverse function theorem, even in the case $n = 1$: If

$$f(t) = \begin{cases} t + 2t^2 \sin \frac{1}{t} & t \neq 0 \\ 0 & t = 0 \end{cases}$$

then $f'(0) = 1$ and $f'$ is bounded in $(-1,1)$, but $f$ is not one-to-one in any neighborhood of 0.