# Chapter 9

# Functions of Several Variables

## 9.1 Notes

2/14:

- Plan:
  1. Warm-up with matrices.
  2. The total derivatives of $f : \mathbb{R}^n \to \mathbb{R}^m$ ($n = m = 2$, i.e., $f : \mathbb{C} \to \mathbb{C}$).
  3. Basic properties: Chain rule, relation with partial derivatives, implicit function theorem.
- Let $V, W$ be finite-dimensional vector spaces over $\mathbb{R}$. We let $L(V, W)$ be the vector space of all linear transformations $\phi : V \to W$.
- If we pick bases $\mathbf{v}_1, \dots, \mathbf{v}_n$ of $V$ and $\mathbf{w}_1, \dots, \mathbf{w}_m$ of $W$, then $V \cong \mathbb{R}^n$ and $W \cong \mathbb{R}^m$. It follows that $L(V, W) \cong \mathbb{R}^{mn}$.
- $L(V, W) \times L(W, U) \xrightarrow{\text{compose}} L(V, U)$, i.e., $\mathbb{R}^{mn} \times \mathbb{R}^{nl} \xrightarrow[\text{mult.}]{\text{matrix}} \mathbb{R}^{ml}$.
- Sup norm: If $A$ is an $m \times n$ real matrix, then $\|A\| = \sup_{\substack{\mathbf{x} \in \mathbb{R}^n \\ |\mathbf{x}| = 1}} |A\mathbf{x}|$.
  - Basic properties:
    1. $|A\mathbf{x}| \leq \|A\||x|$.
    2. $\|A\| < \infty$ and all $A : \mathbb{R}^n \to \mathbb{R}^m$ are uniformly continuous.
    3. $\|A\| = 0 \iff A = 0$.
    4. $\|cA\| = |c|\|A\|$.
    5. $\|A + B\| \leq \|A\| + \|B\|$.
    6. $\|AB\| \leq \|A\|\|B\|$.
  - Note that we get a metric space structure on $L(V, W)$ by defining $d(A, B) = \|A - B\|$.
- Proves that 1 and 2 imply the uniform continuity of all $A$ (via Lipschitz continuity).
- **Differentiable** (function $\mathbf{f}$ at $\mathbf{x}_0$): A function $\mathbf{f} : U \to \mathbb{R}^m$ ($U \subset \mathbb{R}^n$) such that to $\mathbf{x}_0 \in U$ there corresponds some linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{|\mathbf{f}(\mathbf{x}_0 - \mathbf{h}) - \mathbf{f}(\mathbf{x}_0) - A\mathbf{h}|}{|\mathbf{h}|} = 0$$

- **Total derivative** (of $\mathbf{f}$ at $\mathbf{x}_0$): The linear transformation $A$ in the above definition. *Denoted by* $\mathbf{f'(x_0)}$, $\boldsymbol{D}\mathbf{f(x_0)}$, $\boldsymbol{d}\mathbf{f(x_0)}$.
- "An proof and progress in mathematics" - Thurston.

- Relating to the old one dimensional derivative.
- A paper we'd find rather impressionistic right now.

- Propositions ahead of us.
  - Proposition: Suppose that $\mathbf{f}$ is differentiable at $\mathbf{x}_0 \in U$ and $A, B$ are both derivatives of $\mathbf{f}$ at $\mathbf{x}_0$. Then $A = B$.
  - Proposition: Differentiable implies continuous.
  - Proposition: Sum rule, product rule, quotient rule.

2/16: • Plan: Derivatives of functions $\mathbf{f} : U \to \mathbb{R}^m$ where $U \subset \mathbb{R}^n$.
  - Basic properties: Differentiability implies continuity, $(\mathbf{f}+\mathbf{g})' = \mathbf{f}' + \mathbf{g}'$, $(c\mathbf{f})' = c\mathbf{f}'$, chain rule, $\mathbf{f}' = 0$ iff $\mathbf{f}$ is constant.
  - Relationship with partial derivatives (how we compute everything and anything).
  - When is $\mathbf{f}$ differentiable?
  - Inverse function theorem.
  - Implicit function theorem.

- **Continuously differentiable** (function $\mathbf{f}$): A function $\mathbf{f} : U \to \mathbb{R}^m$ that is differentiable for all $\mathbf{x}_0 \in U$ and such that $\mathbf{f}' : U \to L(\mathbb{R}^n, \mathbb{R}^m)$ is continuous. *Also known as* $\boldsymbol{\mathscr{C}^1}$.

- Proposition: Let $\mathbf{f} : U \to \mathbb{R}^m$ be differentiable at $\mathbf{x}_0 \in U$. Then $\mathbf{f}$ is continuous at $\mathbf{x}_0$.
  - The proof makes use of the fact that $\mathbf{f}(\mathbf{x}_0 + \mathbf{h}) - \mathbf{f}(\mathbf{x}_0) = \mathbf{f}'(\mathbf{x}_0)\mathbf{h} + \mathbf{r}(\mathbf{h})$.

- Proposition: Given $\mathbf{f}, \mathbf{g} : U \to \mathbb{R}^m$ both differentiable at $\mathbf{x}_0 \in U$, then $\mathbf{f} + \mathbf{g}$ is also differentiable at $\mathbf{x}_0$ with
$$(\mathbf{f}+\mathbf{g})'(\mathbf{x}_0) = \mathbf{f}'(\mathbf{x}_0) + \mathbf{g}'(\mathbf{x}_0)$$
  - The proof is immediate via the triangle inequality.

- Theorem (Chain Rule): Given $\mathbf{f} : U \to \mathbb{R}^m$ and $\mathbf{g} : V \to \mathbb{R}^k$, where $U \subset \mathbb{R}^n$ and $\mathbf{f}(U) \subset V \subset \mathbb{R}^m$, with $\mathbf{f}$ differentiable at $\mathbf{x}_0 \in U$ and $\mathbf{g}$ differentiable at $\mathbf{f}(\mathbf{x}_0)$, the composition $\mathbf{g} \circ \mathbf{f}$ is differentiable at $\mathbf{x}_0$ with
$$(\mathbf{g} \circ \mathbf{f})'(\mathbf{x}_0) = \mathbf{g}'(\mathbf{f}(\mathbf{x}_0)) \cdot \mathbf{f}'(\mathbf{x}_0)$$
  - The proof is rather subtle.

- **Partial derivative** (of $f_i$ wrt. $x_j$ at $\mathbf{x}_0$): The following limit, if it exists, where $f_i : \mathbb{R}^n \to \mathbb{R}$, $1 \leq i \leq m$, and $1 \leq j \leq n$. *Denoted by* $\boldsymbol{(\partial f_i / \partial x_j)(\mathbf{x}_0)}$, $\boldsymbol{(D_j f_i)(\mathbf{x}_0)}$. *Given by*
$$\frac{\partial f_i}{\partial x_j}(\mathbf{x_0}) = \lim_{t \to 0} \frac{f_i(\mathbf{x}_0 + t\mathbf{e}_j) - f_i(\mathbf{x}_0)}{t}$$

- **Directional derivative** (of $f_i$ toward $\mathbf{u} \in \mathbb{R}^n$): The following limit, if it exists, where $f_i : \mathbb{R}^n \to \mathbb{R}$ and $1 \leq i \leq m$. *Denoted by* $\boldsymbol{D_{\mathbf{u}} f_i}$. *Given by*
$$D_{\mathbf{u}} f_i = \lim_{t \to 0} \frac{f_i(\mathbf{x}_0 + t\mathbf{u}) - f_i(\mathbf{x}_0)}{t}$$

- **Jacobian**: The following matrix. *Given by*
$$\left[ \frac{\partial f_i}{\partial x_j}(\mathbf{x}_0) \right]$$

- Theorem: Let $\mathbf{f} = (f_1, \dots, f_m) : U \to \mathbb{R}^m$, where $U \subset \mathbb{R}^n$, be differentiable at some $\mathbf{x}_0 \in U$. Then the partial derivatives $\partial f_i / \partial x_j$ $(1 \leq i \leq m; 1 \leq j \leq n)$ exist at $\mathbf{x}_0$ and, with respect to the usual choice of bases,
$$\mathbf{f}'(\mathbf{x}_0) = \left[\frac{\partial f_i}{\partial x_j}(\mathbf{x}_0)\right]$$

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  - We have that
$$\mathbf{f}(\mathbf{x}_0 + t\mathbf{e}_j) - \mathbf{f}(\mathbf{x}_0) = \mathbf{f}'(\mathbf{x}_0)(t\mathbf{e}_j) + \mathbf{r}(t\mathbf{e}_j)$$
  - Since $\mathbf{f}$ is differentiable at $\mathbf{x}_0$, $\mathbf{f}(t\mathbf{e}_j)/t \to 0$ as $t \to 0$.
  - Additionally, $\mathbf{f}'(\mathbf{x}_0)(t\mathbf{e}_j)/t = \mathbf{f}'(\mathbf{x}_0)(\mathbf{e}_j)$.
  - Therefore,
$$\lim_{t\to 0} \frac{\mathbf{f}(\mathbf{x}_0 + t\mathbf{e}_j) - \mathbf{f}(\mathbf{x}_0)}{t} = \lim_{t\to 0} \frac{\mathbf{f}'(\mathbf{x}_0)(t\mathbf{e}_j) - \mathbf{r}(t\mathbf{e}_j)}{t} = \mathbf{f}'(\mathbf{x}_0)(\mathbf{e}_j) - \lim_{t\to 0} \frac{\mathbf{r}(t\mathbf{e}_j)}{t} = \mathbf{f}'(\mathbf{x}_0)(\mathbf{e}_j)$$
    as desired.
  - Unpacking the definition of the linear transformation as a matrix gives the rest of the proof.

- Today:
  - More on differentiation (recall the Jacobian).
  - Sufficient condition for differentiability.
  - $\mathbf{f}' = 0$ iff $\mathbf{f}$ is constant.
  - State the inverse function theorem.

- It is not true that having all partials exist implies that $\mathbf{f}$ is differentiable at $\mathbf{x}_0$.
- Theorem: $\mathbf{f}$ continuously differentiable at $\mathbf{x}_0$ iff all partials exist and are continuous at $\mathbf{x}_0$.
- Theorem (Inverse function theorem): If $E \subset \mathbb{R}^n$ open, $\mathbf{f} : E \to \mathbb{R}^n$ is differentiable at $\mathbf{x}_0 \in E$, and $\mathbf{f}'(\mathbf{x}_0)$ is invertible, then there exist $U \subset E$ open with $\mathbf{x}_0 \in U$ and $V \subset \mathbb{R}^n$ open with $\mathbf{f}(\mathbf{x}_0) \in V$ such that $\mathbf{f}|_U : U \to V$ is a bijection and $(\mathbf{f}|_U)^{-1}$ is continuously differentiable.

## 9.2 Chapter 9: Functions of Several Variables

*From Rudin (1976).*

2/15:

- Defines a vector space by the closure of its elements under addition and scalar multiplication.
- Defines a linear combination, span, independence and dependence, dimension, basis, coordinates, and the standard basis.
- Theorem 9.2: If $X$ is spanned by $r$ vectors, $\dim X \leq r$.
- Corollary: $\dim \mathbb{R}^n = n$.
- Theorem 9.3: Let $X$ a vector space with $\dim X = n$.
  (a) $E \subset X$ containing $n$ vectors spans $X$ iff $E$ is independent.
  (b) $X$ has a basis, and every basis contains $n$ vectors.
  (c) If $1 \leq r \leq n$ and $\{\mathbf{y}_1, \dots, \mathbf{y}_r\}$ is independent in $X$, then $X$ has a basis containing $\{\mathbf{y}_1, \dots, \mathbf{y}_r\}$.
- Defines linear transformation, linear operator.
- Notes that $A\mathbf{0} = \mathbf{0}$ if $A$ is a linear transformation, and that $A$ is completely determined by its action on any basis.

- **Invertible** (linear operator): A linear operator $A$ that is one-to-one and onto.
- Theorem 9.5: $A$ a linear operator on $X$ finite-dimensional is one-to-one iff it is onto.
- Defines $L(X,Y)$, $L(X)$, the product $BA$ of two linear transformations, and the supremum norm of a linear transformation.
- Theorem 9.7:
  (a) $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ implies $\|A\| < \infty$ and $A : \mathbb{R}^n \to \mathbb{R}^m$ uniformly continuous.
  (b) $A, B \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $c \in \mathbb{C}$ implies
  $$\|A + B\| \leq \|A\| + \|B\| \qquad\qquad \|cA\| = |c|\|A\|$$
  Defining $d(A,B) = \|A - B\|$ makes $L(\mathbb{R}^n, \mathbb{R}^m)$ a metric space.
  (c) $A \in L(\mathbb{R}^n, \mathbb{R}^m)$ and $B \in L(\mathbb{R}^m, \mathbb{R}^k)$ implies
  $$\|BA\| \leq \|B\|\|A\|$$
- Theorem 9.8: Let $\Omega$ be the set of all invertible linear operators on $\mathbb{R}^n$.
  (a) $A \in \Omega$, $B \in L(\mathbb{R}^n)$, and $\|B - A\| \cdot \left\|A^{-1}\right\| < 1$ implies $B \in \Omega$.

  *Proof.* Let $\left\|A^{-1}\right\| = 1/\alpha$, and let $\|B - A\| = \beta$. Then
  $$\begin{aligned} \|B - A\| \cdot \left\|A^{-1}\right\| &< 1 \\ \beta \cdot \frac{1}{\alpha} &< 1 \\ \beta &< \alpha \end{aligned}$$
  To prove that $B \in \Omega$, the definition of invertibility and Theorem 9.5 tell us that it will suffice to show that $B$ is 1-1. To do so, it will suffice to show that $B\mathbf{x} = \mathbf{0}$ iff $\mathbf{x} = \mathbf{0}$. Let's begin. Let $\mathbf{x} \in \mathbb{R}^n$ be arbitrary. Then
  $$\begin{aligned} \alpha|\mathbf{x}| = \alpha|A^{-1}A\mathbf{x}| \leq \alpha\left\|A^{-1}\right\| \cdot |Ax| = |A\mathbf{x}| &\leq |(A - B)\mathbf{x}| + |B\mathbf{x}| \leq \beta|\mathbf{x}| + |B\mathbf{x}| \\ (\alpha - \beta)|\mathbf{x}| &\leq |B\mathbf{x}| \end{aligned}$$
  It follows that if $\mathbf{x} \neq \mathbf{0}$, then $|B\mathbf{x}| > 0$. This combined with the fact that $B\mathbf{0} = \mathbf{0}$ implies the desired result. ☐

  (b) $\Omega$ is open in $L(\mathbb{R}^n)$ and $A \mapsto A^{-1}$ is continuous on $\Omega$.

  *Proof.* To prove that $\Omega$ is open in $L(\mathbb{R}^n)$, it will suffice to show that for all $A \in \Omega$, there exists $N_r(A)$ such that if $\|B - A\| < r$, then $B \in \Omega$. Let's begin. Let $A \in \Omega$ be arbitrary. Choose $N_\alpha(A)$ to be our neighborhood, where $\alpha$ is defined as in part (a). Let $B \in L(\mathbb{R}^n)$ satisfy $\|B - A\| < \alpha$. Then $\|B - A\| \cdot \left\|A^{-1}\right\| < 1$, so $B \in \Omega$ by part (a), as desired.
  To prove that $A \mapsto A^{-1}$ is continuous, it will suffice to show that $\left\|B^{-1} - A^{-1}\right\| \to 0$ as $B \to A$. First off, we have by part (a) and the substitution $\mathbf{x} = B^{-1}\mathbf{y}$ ($\mathbf{y} \in \mathbb{R}^n$) that
  $$\begin{aligned} (\alpha - \beta)|B^{-1}\mathbf{y}| \leq |BB^{-1}\mathbf{y}| &= |\mathbf{y}| \\ \left|B^{-1}\left(\frac{\mathbf{y}}{|\mathbf{y}|}\right)\right| &\leq (\alpha - \beta)^{-1} \end{aligned}$$
  Thus, since $|B^{-1}\mathbf{u}|$ is bounded by $(\alpha - \beta)^{-1}$ for every unit vector $\mathbf{u} \in \mathbb{R}^n$, $\left\|B^{-1}\right\|$ is bounded by $(\alpha - \beta)^{-1}$. This combined with the fact that
  $$\begin{aligned} B^{-1} - A^{-1} &= B^{-1}I - IA^{-1} \\ &= B^{-1}AA^{-1} - B^{-1}BA^{-1} \\ &= B^{-1}(A - B)A^{-1} \end{aligned}$$

implies by Theorem 9.7c that

$$\left\|B^{-1}-A^{-1}\right\| \leq \left\|B^{-1}\right\|\|A-B\|\left\|A^{-1}\right\| \leq (\alpha-\beta)^{-1}\cdot\beta\cdot\frac{1}{\alpha}=\frac{\beta}{\alpha(\alpha-\beta)}$$

Therefore, since $\beta \to 0$ as $B \to A$, the above inequality establishes the desired result. ☐

- Note that the mapping $A \mapsto A^{-1}$ defined in Theorem 9.8b is a 1-1 mapping of $\Omega$ onto $\Omega$ and its own inverse.
- Defines matrices, column vectors, and matrix multiplication.
- From the Schwarz inequality, we can show that

$$\|A\| \leq \left(\sum_{i,j} a_{i,j}^2\right)^{1/2}$$

- "If $S$ is a metric space, if $a_{11},\dots,a_{mn}$ are real continuous functions on $S$, and if for each $p \in S$, $A_p$ is the linear transformation of $\mathbb{R}^n$ into $\mathbb{R}^m$ whose matrix has entries $a_{ij}(p)$, then the mapping $p \to A_p$ is a continuous mapping of $S$ into $L(\mathbb{R}^n,\mathbb{R}^m)$" (Rudin, 1976, p. 211).
- Rudin (1976) spends some time motivating the definition of the total derivative. He also discusses the natural 1-1 correspondence between $\mathbb{R}^1$ and $L(\mathbb{R}^1)$.
- Defines differentiability in $\mathbb{R}^n$.
- Theorem 9.12: $A_1, A_2$ the derivative of $\mathbf{f}$ at $\mathbf{x}$ implies $A_1 = A_2$.
- If $\mathbf{f}: E \to \mathbb{R}^m$ where $E \subset \mathbb{R}^n$, then $\mathbf{f}': E \to L(\mathbb{R}^n,\mathbb{R}^m)$.
- $\mathbf{f}$ differentiable implies $\mathbf{f}$ continuous.
- Example ($\mathbf{f}$ is linear):
  - If $A \in L(\mathbb{R}^n,\mathbb{R}^m)$, then $A'(\mathbf{x}) = A$ for all $\mathbf{x} \in \mathbb{R}^n$. Note that this means that $A': \mathbb{R}^n \to L(\mathbb{R}^n,\mathbb{R}^m)$, as expected.
- Theorem 9.15 (Chain Rule): $E$ open in $\mathbb{R}^n$, $\mathbf{f}: E \to \mathbb{R}^m$ differentiable at $\mathbf{x}_0 \in E$, $I \supset \mathbf{f}(E)$ open in $\mathbb{R}^m$, and $\mathbf{g}: I \to \mathbb{R}^k$ differentiable at $\mathbf{f}(\mathbf{x}_0)$ implies $\mathbf{F}: E \to \mathbb{R}^k$ defined by

$$\mathbf{F}(\mathbf{x}) = \mathbf{g}(\mathbf{f}(\mathbf{x}))$$

is differentiable at $\mathbf{x}_0$ with

$$\mathbf{F}'(\mathbf{x}_0) = \mathbf{g}'(\mathbf{f}(\mathbf{x}_0))\mathbf{f}'(\mathbf{x}_0)$$ [1]

*Proof.* Largely symmetric to that of the one-dimensional chain rule in Chapter 5. ☐

- **Components** (of $\mathbf{f}: \mathbb{R}^n \to \mathbb{R}^m$): The real functions $f_1,\dots,f_m$ defined by

$$\mathbf{f}(\mathbf{x}) = \sum_{i=1}^{m} f_i(\mathbf{x})\mathbf{u}_i$$

for all $\mathbf{x} \in E$ or, equivalently, by $f_i(\mathbf{x}) = f(\mathbf{x})\cdot\mathbf{u}_i$ ($1 \leq i \leq m$), where $\mathbf{u}_1,\dots,\mathbf{u}_m$ is the standard basis of $\mathbb{R}^m$.

[1] Note that the right-hand side of this equation contains the product of two linear transformations.

- Defines partial derivatives.
- Theorem 9.17: $E \subset \mathbb{R}^n$ open and $\mathbf{f} : E \to \mathbb{R}^m$ differentiable at $\mathbf{x} \in E$ imply the partial derivatives $(D_j f_i)(\mathbf{x})$ exist and
$$\mathbf{f}'(\mathbf{x})\mathbf{e}_j = \sum_{i=1}^{m} (D_j f_i)(\mathbf{x})\mathbf{u}_i$$
for $1 \leq j \leq n$.
- It follows that
$$[\mathbf{f}'(\mathbf{x})] = \begin{bmatrix} (D_1 f_1)(\mathbf{x}) & \cdots & (D_n f_1)(\mathbf{x}) \\ \vdots & & \vdots \\ (D_1 f_m)(\mathbf{x}) & \cdots & (D_n f_m)(\mathbf{x}) \end{bmatrix}$$
- Discusses the gradient and the directional derivative.
- Theorem 9.19: $E \subset \mathbb{R}^n$ convex and open, $\mathbf{f} : E \to \mathbb{R}^m$ differentiable in $E$, and there exists $M$ such that
$$\|\mathbf{f}'(\mathbf{x})\| \leq M$$
for all $\mathbf{x} \in E$ implies
$$|\mathbf{f}(\mathbf{b}) - \mathbf{f}(\mathbf{a})| \leq M|\mathbf{b} - \mathbf{a}|$$
for all $\mathbf{a}, \mathbf{b} \in E$.
- Corollary: If, in addition, $\mathbf{f}'(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in E$, then $\mathbf{f}$ is constant.
- **Continuously differentiable** (mapping $\mathbf{f} : E \to \mathbb{R}^m$): A function $\mathbf{f} : E \to \mathbb{R}^m$ such that $\mathbf{f}' : E \to L(\mathbb{R}^n, \mathbb{R}^m)$ is continuous. *Also known as* $\boldsymbol{\mathscr{C}^1}$**-mapping**. *Denoted by* $\mathbf{f} \in \boldsymbol{\mathscr{C}^1(E)}$.
- Theorem 9.21: Let $E \subset \mathbb{R}^n$ open and $\mathbf{f} : E \to \mathbb{R}^m$. Then $\mathbf{f} \in \mathscr{C}^1(E)$ iff the partial derivatives $D_j f_i$ $(1 \leq i \leq m;\ 1 \leq j \leq n)$ exist and are continuous on $E$.

2/20:

- **Contraction** (of $X$ into $X$): A function $\varphi : X \to X$ for which there exists a number $c < 1$ such that
$$d(\varphi(x), \varphi(y)) \leq c \cdot d(x, y)$$
for all $x, y \in X$, where $X$ is a metric space with metric $d$.
- Theorem 9.23: $X$ a complete metric space and $\phi$ a contraction of $X$ into $X$ implies there exists a unique $x \in X$ such that $\varphi(x) = x$.

  *Proof.* Let $x_0 \in X$ be arbitrary. Define $\{x_n\}$ recursively by
$$x_{n+1} = \phi(x_n)$$
for $n = 0, 1, 2, \dots$. Let $c < 1$ be the number corresponding to the contraction $\varphi$. Then for $n \geq 1$, we have
$$d(x_{n+1}, x_n) = d(\varphi(x_n), \varphi(x_{n-1})) \leq c \cdot d(x_n, x_{n-1})$$
or, for $n \geq 0$,
$$d(x_{n+1}, x_n) \leq c^n d(x_1, x_0)$$
by induction. Now to prove that $\{x_n\}$ is Cauchy, it will suffice to show that for all $\epsilon > 0$, there exists $N$ such that $m \geq n \geq N$ implies $d(x_n, x_m) < \epsilon$. But since
$$\begin{aligned} d(x_n, x_m) &\leq \sum_{i=n+1}^{m} d(x_i, x_{i-1}) \\ &\leq (c^n + c^{n+1} + \cdots + c^{m-1}) d(x_1, x_0) \\ &\leq [(1-c)^{-1} d(x_1, x_0)] c^n \end{aligned}$$

we can simply choose $N$ large enough that $[(1-c)^{-1}d(x_1,x_0)]c^N < \epsilon$. Thus, since $\{x_n\}$ is Cauchy and $X$ is complete, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Therefore, since $\varphi$ is Lipschitz continuous, we have that

$$\varphi(x) = \lim_{n\to\infty} \varphi(x_n) = \lim_{n\to\infty} x_{n+1} = x$$

as desired.

Now suppose for the sake of contradiction that there exists $y \neq x$ such that $\varphi(y) = y$. Then since $\varphi$ is a contraction,

$$d(y,x) = d(\varphi(y),\varphi(x)) \leq c \cdot d(y,x) < d(y,x)$$

a contradiction. □

- Theorem 9.24 (Inverse Function Theorem): $E \subset \mathbb{R}^n$ open, $\mathbf{f} : E \to \mathbb{R}^n$ a $\mathscr{C}^1$-mapping, $\mathbf{f}'(\mathbf{a})$ invertible for some $\mathbf{a} \in E$, and $\mathbf{b} = \mathbf{f}(\mathbf{a})$ implies

  (a) There exist $U, V \subset \mathbb{R}^n$ open with $\mathbf{a} \in U$, $\mathbf{b} \in V$ such that $\mathbf{f}$ is 1-1 on $U$ and $\mathbf{f}(U) = V$.

  *Proof.* Let $A = \mathbf{f}'(\mathbf{a})$. Choose $\lambda$ such that

  $$2\lambda \left\|A^{-1}\right\| = 1$$

  Define[2] for each $\mathbf{y} \in \mathbb{R}^n$ a function $\varphi$ by

  $$\varphi(\mathbf{x}) = \mathbf{x} + A^{-1}(\mathbf{y} - \mathbf{f}(\mathbf{x}))$$

  for all $\mathbf{x} \in E$. (Note that a key property of $\varphi$ is that as defined, $\mathbf{f}(\mathbf{x}) = \mathbf{y}$ iff $\mathbf{x}$ is a fixed point of $\mathbf{y}$.) Now since $\mathbf{f} \in \mathscr{C}^1$ and hence $\mathbf{f}'$ is continuous at $\mathbf{a}$, there exists an open ball $B_r(\mathbf{a}) \subset E$ such that

  $$\|\mathbf{f}'(\mathbf{x}) - A\| < \lambda$$

  for all $\mathbf{x} \in B_r(\mathbf{a})$. Let $U = B_r(\mathbf{a})$. Clearly it follows that $U$ is open. Thus, since each $\varphi'(\mathbf{x}) = I - A^{-1}\mathbf{f}'(\mathbf{x}) = A^{-1}(A - \mathbf{f}'(\mathbf{x}))$, we have that

  $$\|\varphi'(\mathbf{x})\| \leq \left\|A^{-1}\right\| \|A - \mathbf{f}'(\mathbf{x})\| < \frac{1}{2\lambda} \cdot \lambda = \frac{1}{2}$$

  Consequently, we have by Theorem 9.19 that for all $\mathbf{x}_1, \mathbf{x}_2 \in U$,

  $$|\varphi(\mathbf{x}_1) - \varphi(\mathbf{x}_2)| \leq \frac{1}{2}|\mathbf{x}_1 - \mathbf{x}_2|$$

  Thus, by the uniqueness argument in the proof of Theorem 9.23, $\varphi$ has at most one fixed point in $U$, so $\mathbf{f}(\mathbf{x}) = \mathbf{y}$ for at most one $\mathbf{x} \in U$. Therefore, $\mathbf{f}$ is 1-1 on $U$.

  Let $V = \mathbf{f}(U)$. To prove that $V$ is open, it will suffice to show that for all $\mathbf{y}_0 \in V$, there exists an open subset of $V$ containing $\mathbf{y}_0$ such that. Let $\mathbf{y}_0 \in V$ be arbitrary. By the definition of $V$ as the image of $U$ under $\mathbf{f}$, there exists $\mathbf{x}_0 \in U$ such that $\mathbf{f}(\mathbf{x}_0) = \mathbf{y}_0$. As such, choose $B_r(\mathbf{x}_0)$ such that $\overline{B} \subset U$. Pick $\mathbf{y}$ satisfying $|\mathbf{y} - \mathbf{y}_0| < \lambda r$. Then

  $$|\varphi(\mathbf{x}_0) - \mathbf{x}_0| = |A^{-1}(\mathbf{y} - \mathbf{y}_0)| < \|A\|\lambda r = \frac{r}{2}$$

  so for all $\mathbf{x} \in \overline{B}$,

  $$\begin{aligned} |\varphi(\mathbf{x}) - \mathbf{x}_0| &\leq |\varphi(\mathbf{x}) - \varphi(\mathbf{x}_0)| + |\varphi(\mathbf{x}_0) - \mathbf{x}_0| \\ &< \frac{1}{2}|\mathbf{x} - \mathbf{x}_0| + \frac{r}{2} \\ &\leq \frac{1}{2} \cdot r + \frac{r}{2} \\ &= r \end{aligned}$$

[2] How do we motivate this definition?

Thus, $\varphi(\mathbf{x}_0) \in B$. Moreover, since $|\varphi(\mathbf{x}_1) - \varphi(\mathbf{x}_2)| \leq \frac{1}{2}|\mathbf{x}_1 - \mathbf{x}_2|$ naturally holds for all $\mathbf{x}_1, \mathbf{x}_2 \in \overline{B} \subset U$, we have that $\varphi$ is a contraction of $\overline{B}$ into $\overline{B}$. Additionally, since $\overline{B} \subset \mathbb{R}^n$ is closed, it is a complete metric space under the Euclidean metric. Thus, Theorem 9.23 implies that $\varphi$ has a fixed point $\mathbf{x} \in \overline{B}$. In particular, $\mathbf{f}(\mathbf{x}) = \mathbf{y}$. Therefore, $\mathbf{y} \in f(\overline{B}) \subset \mathbf{f}(U) = V$, as desired. $\square$

(b) If $\mathbf{g}$ is the inverse of $\mathbf{f}$ on $V$ [which exists by (a)], i.e.,

$$\mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{x}$$

for all $\mathbf{x} \in U$, then $\mathbf{g} \in \mathscr{C}^1(V)$.

*Proof.* We first show that for all $\mathbf{y} \in V$, $\mathbf{g}'(\mathbf{y}) = [\mathbf{f}'(\mathbf{g}(\mathbf{y}))]^{-1}$. Let $\mathbf{y} \in V$ be arbitrary, and choose $\mathbf{k}$ such that $(\mathbf{y} + \mathbf{k}) \in V$. It follows by part (a) that there exist $\mathbf{x}, \mathbf{x} + \mathbf{h} \in U$ such that $\mathbf{y} = \mathbf{f}(\mathbf{x})$ and $\mathbf{y} + \mathbf{k} = \mathbf{f}(\mathbf{x} + \mathbf{h})$. Thus,

$$\varphi(\mathbf{x} + \mathbf{h}) - \varphi(\mathbf{x}) = \mathbf{h} + A^{-1}[\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x} + \mathbf{h})] = \mathbf{h} - A^{-1}\mathbf{k}$$

so

$$|\mathbf{h} - A^{-1}\mathbf{k}| = |\varphi(\mathbf{x} + \mathbf{h}) - \varphi(\mathbf{x})| \leq \frac{1}{2}|\mathbf{x} + \mathbf{h} - \mathbf{x}| = \frac{1}{2}|\mathbf{h}|$$

Consequently, $|A^{-1}\mathbf{k}| \geq \frac{1}{2}|\mathbf{h}|$, so

$$|\mathbf{h}| \leq 2\|A^{-1}\||\mathbf{k}| = \frac{|\mathbf{k}|}{\lambda}$$

Additionally, we know that $\|\mathbf{f}'(\mathbf{x}) - A\|\|A^{-1}\| = 1/2 < 1$, so Theorem 9.8a implies that $\mathbf{f}'(\mathbf{x})$ is invertible with an inverse that we may call $T$. Thus, since

$$\begin{aligned}
\mathbf{g}(\mathbf{y} + \mathbf{k}) - \mathbf{g}(\mathbf{y}) - T\mathbf{k} &= \mathbf{h} - T\mathbf{k} \\
&= -T[(\mathbf{y} + \mathbf{k}) - \mathbf{y}] + T\mathbf{f}'(\mathbf{x})\mathbf{h} \\
&= -T[\mathbf{f}(\mathbf{x} + \mathbf{h}) - \mathbf{f}(\mathbf{x}) - \mathbf{f}'(\mathbf{x})\mathbf{h}]
\end{aligned}$$

we have that

$$\frac{|\mathbf{g}(\mathbf{y} + \mathbf{k}) - \mathbf{g}(\mathbf{y}) - T\mathbf{k}|}{|\mathbf{k}|} \leq \frac{\|T\||\mathbf{f}(\mathbf{x} + \mathbf{h}) - \mathbf{f}(\mathbf{x}) - \mathbf{f}'(\mathbf{x})\mathbf{h}|}{\lambda|\mathbf{h}|}$$

Consequently, $\mathbf{k} \to \mathbf{0}$ implies that $\mathbf{h} \to \mathbf{0}$, which implies that the right side of the above inequality goes to zero, which implies that the left side of the above inequality goes to zero. Thus, $\mathbf{g}'(\mathbf{y}) = T$, so

$$\mathbf{g}'(\mathbf{y}) = [\mathbf{f}'(\mathbf{g}(\mathbf{y}))]^{-1}$$

for all $\mathbf{y} \in V$, as desired.

To prove that $\mathbf{g}'$ is continuous on $V$, Theorem 4.7 and the above equation tell us that it will suffice to show that $\mathbf{g} : V \to U$ is continuous, $\mathbf{f}' : U \to L(\mathbb{R}^n)$ is continuous, and $M \mapsto M^{-1} : L(\mathbb{R}^n) \to L(\mathbb{R}^n)$ is continuous. But we have the first condition since differentiability implies continuity and $\mathbf{g}$ is differentiable, we have the second condition since $\mathbf{f} \in \mathscr{C}^1$ by hypothesis, and we have the third condition by Theorem 9.8b, as desired. $\square$

- Theorem 9.25: $E \subset \mathbb{R}^n$ open, $\mathbf{f} :: E \to \mathbb{R}^n$ a $\mathscr{C}^1$-mapping, and $\mathbf{f}'(\mathbf{x})$ invertible for all $\mathbf{x} \in E$ implies $\mathbf{f}(W)$ open in $\mathbb{R}^n$ for every open $W \subset E$.
  - Note that the hypotheses of this theorem guarantee that $\mathbf{f}$ is locally 1-1 at each $\mathbf{x} \in E$, but it may not be 1-1 in $E$ under these conditions (see Exercise 9.17).