5 Sequences and Series of Functions II / Functions of Several Variables

From Rudin (1976).

Chapter 7

2/16: **5.** Let

$$f_n(x) = \begin{cases} 0 & x < \frac{1}{n+1} \\ \sin^2 \frac{\pi}{x} & \frac{1}{n+1} \leq x \leq \frac{1}{n} \\ 0 & \frac{1}{n} < x \end{cases}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that absolute convergence, even for all x, does not imply uniform convergence.

Proof. To prove that $\{f_n\}$ converges pointwise to the continuous function f defined by $f(x) = 0$ for all $x \in \mathbb{R}$, it will suffice to show that for every $\epsilon > 0$ and for every $x \in \mathbb{R}$, there exists an integer N such that if $n \geq N$, then $|f_n(x)| < \epsilon$. Let $\epsilon > 0$ and $x \in \mathbb{R}$ be arbitrary. We divide into three cases ($x \in \{1/n\}_{n=1}^{\infty}$, $x \in [0,1] \setminus \{1/n\}_{n=1}^{\infty}$, and $x \notin [0,1]$).

If $x \in \{1/n\}_{n=1}^{\infty}$, let $x = 1/k$. Then by the definition of $f_n(x)$, we have that

$$f_i(x) = \begin{cases} 0 & i < k-1 \\ \sin^2 \frac{\pi}{1/k} = \sin^2 k\pi = 0 & i = k-1, k \\ 0 & i > k \end{cases}$$

Thus, choose $N = 1$. It follows that if $n \geq N$, then

$$|f_n(x)| = 0 < \epsilon$$

as desired.

If $x \in [0,1] \setminus \{1/n\}_{n=1}^{\infty}$, let $x \in (1/[(N-1)+1], 1/(N-1))$ where $N \in \mathbb{N}$. Choose this N to be our N. It follows that if $n \geq N$, then

$$\frac{1}{n} \leq \frac{1}{N} = \frac{1}{(N-1)+1} < x$$

so by definition,

$$|f_n(x)| = 0 < \epsilon$$

as desired.

If $x \notin [0,1]$, then either $x < 1/(n+1)$ for all $n \in \mathbb{N}$ or $x > 1/n$ for all $n \in \mathbb{N}$. Either way, we choosing $N = 1$ yields that if $n \geq N$, then

$$|f_n(x)| = 0 < \epsilon$$

as desired.

To prove that $\{f_n\}$ does not converge uniformly to f, Theorem 7.9 tells us that it will suffice to show that if $M_n = \sup_{x \in \mathbb{R}} |f_n(x) - f(x)|$, then $M_n \nrightarrow 0$ as $n \to \infty$. Let $n \in \mathbb{N}$ be arbitrary. Since $n < n + 1/2 < n + 1$ and hence $1/(n+1) \leq 2/(2n+1) \leq 1/n$, we have by the properties of the sine function that

$$f_n(\tfrac{2}{2n+1}) = \sin^2 \left[\frac{\pi}{2/(2n+1)} \right] = \sin^2 \left[\frac{2n+1}{2} \pi \right] = \sin^2 \left[\left(n + \frac{1}{2} \right) \pi \right] = 1$$

and that $f_n(x) \leq 1$ everywhere else. Thus, $M_n = 1$ for all $n \in \mathbb{N}$. But then $M_n \nrightarrow 0$ as $n \to \infty$, as desired.

It follows by an argument symmetric to the above that while $\sum f_n$ converges absolutely to

$$f(x) = \begin{cases} 0 & x \leq 0 \\ \sin^2 \frac{\pi}{x} & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$$

$M_n = 1$ for all $n \in \mathbb{N}$. □

6. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

Proof. Let $[a, b]$ be an arbitrary bounded interval, and let $f_n(x) = (-1)^n \frac{x^2+n}{n^2}$. To prove that the series converges uniformly on $[a, b]$, Theorem 7.8 tells us that it will suffice to show that for every $\epsilon > 0$, there exists an N such that if $n, m \geq N$ (WLOG let $n \leq m$) and $x \in [a, b]$, then

$$\left| \sum_{i=n}^{m} f_i(x) \right| < \epsilon$$

Let $\epsilon > 0$ be arbitrary. Define $m = \max(|a|, |b|)$ (note that since $a \neq b$ by definition, $m > 0$). By consecutive applications of Theorem 3.43, we know that both $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converge. Thus, by consecutive applications of Theorem 3.22, there exist integers N_1, N_2 such that $m \geq n \geq N_1$ implies the left result below and $m \geq n \geq N_2$ implies the right result below.

$$\left| \sum_{k=n}^{m} (-1)^k \frac{1}{k^2} \right| < \frac{\epsilon}{2m^2} \qquad\qquad \left| \sum_{k=n}^{m} (-1)^k \frac{1}{k} \right| < \frac{\epsilon}{2}$$

Choose $N = \max(N_1, N_2)$. Now let $n, m \geq N$ with WLOG $n \leq m$, and let $x \in [a, b]$. It follows that

$$\begin{aligned}
\left| \sum_{k=n}^{m} f_k(x) \right| &= \left| \sum_{k=n}^{m} (-1)^k \frac{x^2 + k}{k^2} \right| \\
&= \left| x^2 \sum_{k=n}^{m} (-1)^k \frac{1}{k^2} + \sum_{k=n}^{m} (-1)^k \frac{1}{k} \right| \\
&\leq |x^2| \cdot \left| \sum_{k=n}^{m} (-1)^k \frac{1}{k^2} \right| + \left| \sum_{k=n}^{m} (-1)^k \frac{1}{k} \right| \\
&\leq m^2 \cdot \left| \sum_{k=n}^{m} (-1)^k \frac{1}{k^2} \right| + \left| \sum_{k=n}^{m} (-1)^k \frac{1}{k} \right| \\
&< m^2 \cdot \frac{\epsilon}{2m^2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

as desired.

To prove that the series does not converge absolutely for any value of x, let $x \in \mathbb{R}$ be arbitrary. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} \left| (-1)^n \frac{x^2 + n}{n^2} \right| &= x^2 \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n} \\
&\geq \sum_{n=1}^{\infty} \frac{1}{n}
\end{aligned}$$

where the latter series diverges by Theorem 3.28, yielding the desired result. □

8. If

$$I(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

if $\{x_n\}$ is a sequence of distinct points of (a, b), and if $\sum |c_n|$ converges, prove that the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n)$$

converges uniformly on $[a, b]$, and that f is continuous for every $x \neq x_n$.

Proof. Let $f_n(x) = c_n I(x - x_n)$ for all $n \in \mathbb{N}$. To prove that f converges uniformly on $[a, b]$, Theorem 7.10 tells us that it will suffice to show that $|f_n(x)| \leq M_n$ for all $x \in [a, b]$ and $\sum M_n$ converges. Let $M_n = c_n$ for all $n \in \mathbb{N}$. Then for any $x \in [a, b]$,

$$|f_n(x)| = c_n I(x - x_n) \leq c_n = M_n$$

as desired. Additionally, $\sum M_n = \sum c_n$ converges, as desired. This completes the proof.

For the second part of the proof, let $x \notin \{x_n\}$. Then every f_n is continuous at x by definition. Thus, f is a uniformly convergent sequence of functions continuous at x, so by Theorem 7.12, f is continuous at x. □

9. Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that

$$\lim_{n \to \infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \to x$ and $x \in E$. Is the converse of this true?

Proof. Let $\{x_n\} \subset E$ be an arbitrary sequence of points that converges to some $x \in E$. To prove that $\lim_{n \to \infty} f_n(x_n) = f(x)$, it will suffice to show that for every $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|f_n(x_n) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\{f_n\}$ is a uniformly convergent sequence of continuous functions, Theorem 7.12 implies that f is a continuous function. Thus, there exists a $\delta > 0$ such that if $y \in E$ and $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon/2$. Additionally, since $x_n \to x$, there exists an N_1 such that if $n \geq N_1$, $|x_n - x| < \delta$. Furthermore, since f_n converges uniformly to f, there exists N_2 such that if $n \geq N_2$, then $|f_n(y) - f(y)| < \epsilon/2$ for all $y \in E$. In particular, $|f_n(x_n) - f(x_n)| < \epsilon/2$. Choose $N = \max(N_1, N_2)$. Let $n \geq N$ be arbitrary. Then

$$\begin{aligned} |f_n(x_n) - f(x)| &\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

as desired.

No, it is not true in general that if $\{f_n\}$ is a sequence of continuous functions for which $\lim_{n \to \infty} f_n(x_n) = f(x)$ for every sequence of points $x_n \in E$ such that $x_n \to x$ and $x \in E$, then f_n converges uniformly. Consider the sequence of functions from Exercise 7.5. This is a sequence of continuous functions for which $\lim_{n \to \infty} f_n(x_n) = f(x)$ for any sequence $\{x_n\}$ of the desired type since we can always choose N large enough so that the moving "hump" and neighborhood of x containing all remaining x_n are separated forever more. Moreover, by Exercise 7.5, $\{f_n\}$ does not converge uniformly, as desired. □

Chapter 9

1. If S is a nonempty subset of a vector space X, prove (as asserted in Section 9.1) that the span of S is a vector space.

Proof. Let $S = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ (the proof is symmetric if S is infinite).

To prove that $\text{span}(S)$ is a vector space, it will suffice to show that $\text{span}(S)$ is nonempty and that for all $\mathbf{x}, \mathbf{y} \in \text{span}(S)$ and $c \in \mathbb{C}$, $(\mathbf{x} + \mathbf{y}) \in \text{span}(S)$ and $c\mathbf{x} \in \text{span}(S)$. Since S is nonempty, there exists $\mathbf{x} \in S$; thus, $1\mathbf{x} \in \text{span}(S)$, so $\text{span}(S)$ is nonempty, as desired. Let $\mathbf{x}, \mathbf{y} \in \text{span}(S)$ and $c \in \mathbb{C}$. There exist $a_1, \dots, a_n, b_1, \dots, b_n$ such that

$$\mathbf{x} = a_1\mathbf{u}_1 + \cdots + a_n\mathbf{u}_n \qquad \mathbf{y} = b_1\mathbf{u}_1 + \cdots + b_n\mathbf{u}_n$$

It follows by the definition of $\text{span}(S)$ that

$$\begin{aligned} (a_1 + b_1)\mathbf{u}_1 + \cdots + (a_n + b_n)\mathbf{u}_n &= \mathbf{x} + \mathbf{y} \in \text{span}(S) \\ ca_1\mathbf{u}_1 + \cdots + ca_n\mathbf{u}_n &= c\mathbf{x} \in \text{span}(S) \end{aligned}$$

as desired. □

2. Prove (as asserted in Section 9.6) that BA is linear if A and B are linear transformations. Prove also that A^{-1} is linear and invertible.

Proof. Let $A \in L(X, Y)$ and $B \in L(Y, Z)$. Then for all $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2 \in X$ and $c \in \mathbb{C}$,

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 \qquad A(c\mathbf{x}) = cA\mathbf{x}$$

and for all $\mathbf{y}, \mathbf{y}_1, \mathbf{y}_2 \in Y$ and $c \in \mathbb{C}$,

$$B(\mathbf{y}_1 + \mathbf{y}_2) = B\mathbf{y}_1 + B\mathbf{y}_2 \qquad B(c\mathbf{y}) = cB\mathbf{y}$$

It follows that for any $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2 \in X$ and $c \in \mathbb{C}$, we have that

$$\begin{aligned} BA(\mathbf{x}_1 + \mathbf{x}_2) &= B(A\mathbf{x}_1 + A\mathbf{x}_2) \\ &= BA\mathbf{x}_1 + BA\mathbf{x}_2 \end{aligned} \qquad \begin{aligned} BA(c\mathbf{x}) &= B(cA\mathbf{x}) \\ &= cBA\mathbf{x} \end{aligned}$$

so BA is a linear transformation, as desired.

Let $A \in L(X, Y)$ be invertible. Since A is a linear transformation, the same equalities from above still apply. Thus,

$$\begin{aligned} \mathbf{x}_1 + \mathbf{x}_2 &= \mathbf{x}_1 + \mathbf{x}_2 \\ I(\mathbf{x}_1 + \mathbf{x}_2) &= I\mathbf{x}_1 + I\mathbf{x}_2 \\ AA^{-1}(\mathbf{x}_1 + \mathbf{x}_2) &= AA^{-1}\mathbf{x}_1 + AA^{-1}\mathbf{x}_2 \\ A(A^{-1}(\mathbf{x}_1 + \mathbf{x}_2)) &= A(A^{-1}\mathbf{x}_1 + A^{-1}\mathbf{x}_2) \\ A^{-1}(\mathbf{x}_1 + \mathbf{x}_2) &= A^{-1}\mathbf{x}_1 + A^{-1}\mathbf{x}_2 \end{aligned} \qquad \begin{aligned} c\mathbf{x} &= c\mathbf{x} \\ I(c\mathbf{x}) &= cI\mathbf{x} \\ AA^{-1}(c\mathbf{x}) &= cAA^{-1}\mathbf{x} \\ A(A^{-1}(c\mathbf{x})) &= A(cA^{-1}\mathbf{x}) \\ A^{-1}(c\mathbf{x}) &= cA^{-1}\mathbf{x} \end{aligned}$$

where we use the fact that A is one-to-one for the last equality in both cases. To prove that A^{-1} is invertible, it will suffice to show that it is one-to-one and onto. Suppose $A^{-1}\mathbf{x} = A^{-1}\mathbf{y}$. Then

$$\begin{aligned} AA^{-1}\mathbf{x} &= AA^{-1}\mathbf{y} \\ I\mathbf{x} &= I\mathbf{y} \\ \mathbf{x} &= \mathbf{y} \end{aligned}$$

proving that A^{-1} is one-to-one, as desired. Now suppose $\mathbf{y} \in X$. Then $A\mathbf{y} = \mathbf{x}$ for some $\mathbf{x} \in X$. It follows that

$$A^{-1}\mathbf{x} = A^{-1}A\mathbf{y} = I\mathbf{y} = \mathbf{y}$$

proving that A^{-1} is onto, as desired. □

3. Assume $A \in L(X, Y)$ and $A\mathbf{x} = \mathbf{0}$ only when $\mathbf{x} = \mathbf{0}$. Prove that A is then 1-1.

Proof. If we suppose that $A\mathbf{x} = A\mathbf{y}$, then by linearity,

$$\begin{aligned} \mathbf{0} &= A\mathbf{x} - A\mathbf{y} \\ &= A(\mathbf{x} - \mathbf{y}) \end{aligned}$$

It follows by hypothesis that $\mathbf{x} - \mathbf{y} = \mathbf{0}$, hence $\mathbf{x} = \mathbf{y}$, proving that A is 1-1, as desired. $\square$

4. Prove (as asserted in Section 9.30) that null spaces and ranges of linear transformations are vector spaces.

Proof. Let $A \in L(X, Y)$.

Suppose $\mathbf{x}_1, \mathbf{x}_2 \in \operatorname{null} A$. Then $A\mathbf{x}_1 = \mathbf{0}$ and $A\mathbf{x}_2 = \mathbf{0}$. It follows that

$$\begin{aligned} \mathbf{0} &= A\mathbf{x}_1 + A\mathbf{x}_2 \\ &= A(\mathbf{x}_1 + \mathbf{x}_2) \end{aligned}$$

so $(\mathbf{x}_1 + \mathbf{x}_2) \in \operatorname{null} A$, as desired.

Suppose $\mathbf{x} \in \operatorname{null} A$ and $c \in \mathbb{C}$. Then $A\mathbf{x} = \mathbf{0}$. It follows that

$$\begin{aligned} \mathbf{0} &= c \cdot \mathbf{0} \\ &= cA\mathbf{x} \\ &= A(c\mathbf{x}) \end{aligned}$$

so $c\mathbf{x} \in \operatorname{null} A$, as desired.

Suppose $\mathbf{y}_1, \mathbf{y}_2 \in \operatorname{range} A$. Then there exist $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that $A\mathbf{x}_1 = \mathbf{y}_1$ and $A\mathbf{x}_2 = \mathbf{y}_2$. It follows that

$$\begin{aligned} A(\mathbf{x}_1 + \mathbf{x}_2) &= A\mathbf{x}_1 + A\mathbf{x}_2 \\ &= \mathbf{y}_1 + \mathbf{y}_2 \end{aligned}$$

so $(\mathbf{y}_1 + \mathbf{y}_2) \in \operatorname{range} A$, as desired.

Suppose $\mathbf{y} \in \operatorname{range} A$ and $c \in \mathbb{C}$. Then there exists $\mathbf{x} \in X$ such that $A\mathbf{x} = \mathbf{y}$. It follows that

$$\begin{aligned} A(c\mathbf{x}) &= cA\mathbf{x} \\ &= c\mathbf{y} \end{aligned}$$

so $c\mathbf{y} \in \operatorname{range} A$, as desired. $\square$