MATH 20410 (Analysis in $\mathbb{R}^n$ II – Accelerated) Problem Sets

Steven Labalme

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Contents

1 Differentiation

From Rudin (1976).

Chapter 5

1. Let f be defined for all real x, and suppose that

$$|f(y) - f(x)| \leq (y - x)^2$$

for all real x and y. Prove that f is constant.

Proof. To prove that f is constant, Theorem 5.11b tells us that it will suffice to show that f is differentiable on $\mathbb{R}$ with derivative $f' = 0$. Let $x \in \mathbb{R}$ be arbitrary. We want to show that for all $\epsilon > 0$, there exists a δ such that if $y \in \mathbb{R}$ and $0 < |y - x| < \delta$, then $|[f(y) - f(x)]/(y - x) - 0| < \epsilon$. Let ϵ be arbitrary. Choose $\delta = \epsilon$. Then we have that

$$\begin{aligned}
\left| \frac{f(y) - f(x)}{y - x} - 0 \right| &= \frac{|f(y) - f(x)|}{|y - x|} \\
&\leq \frac{(y - x)^2}{|y - x|} \\
&\leq |y - x| \\
&< \epsilon
\end{aligned}$$

as desired. $\square$

2. Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b) and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)}$$

for $a < x < b$.

Proof. To prove that f is strictly increasing on (a, b), it will suffice to show that $x < y$ implies $f(x) < f(y)$ for all $x, y \in (a, b)$. Let $x, y \in (a, b)$ satisfy $x < y$. Since f is differentiable on (a, b), it is differentiable on $(x, y) \subset (a, b)$ and (by Theorem 5.2) continuous on $[x, y] \subset (a, b)$. Thus, by the MVT, there exists $c \in (x, y)$ such that

$$f(y) - f(x) = (y - x)f'(c)$$

But since $x < y$, $y - x > 0$. This combined with the fact that $f'(c) > 0$ by definition implies that $(y - x)f'(c) > 0$. Consequently,

$$f(x) < f(x) + (y - x)f'(c) = f(y)$$

as desired.

Since f is strictly increasing (and hence 1-1) on (a, b), we may construct a well-defined inverse function $g : f[(a, b)] \to (a, b)$ for it by

$$g(f(x)) = x$$

for all $f(x) \in f[(a, b)]$. It follows by the fact that $f'(x) > 0$ for all $x \in (a, b)$, the definitions of $f'(x)$ and $g'(f(x))$, and Theorem 3.3d that

$$\frac{1}{f'(x)} = \frac{1}{\lim_{y \to x} \frac{f(y) - f(x)}{y - x}}$$

$$\begin{aligned}
&= \lim_{y \to x} \frac{1}{\frac{f(y)-f(x)}{y-x}} \\
&= \lim_{y \to x} \frac{y-x}{f(y)-f(x)} \\
&= \lim_{y \to x} \frac{g(f(y))-g(f(x))}{f(y)-f(x)} \\
&= g'(f(x))
\end{aligned}$$

as desired. □

3. Suppose g is a real function on $\mathbb{R}^1$, with bounded derivative (say $|g'| \leq M$). Fix $\epsilon > 0$ and define $f(x) = x + \epsilon g(x)$. Prove that f is one-to-one if ϵ is small enough. (A set of admissable values of ϵ can be determined which depends only on M.)

Proof. Neglecting the trivial case where $M = 0$, take $\epsilon = 1/2M$. It follows that

$$\begin{aligned}
0 &< 1 - \frac{1}{2} \\
&= 1 + \frac{1}{2M} \cdot -M \\
&\leq 1 + \epsilon g'(x) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}(x) + \frac{\mathrm{d}}{\mathrm{d}x}(\epsilon g) \\
&= f'(x)
\end{aligned}$$

Therefore, by Problem 5.2, f is strictly increasing and, hence, one-to-one. □

4. If

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$$

where $C_0, \dots, C_n$ are real constants, prove that the equation

$$C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 and 1.

Proof. Consider the polynomial

$$f(x) = C_0 x + \frac{C_1}{2}x^2 + \cdots + \frac{C_n}{n+1}x^{n+1}$$

We have that $f(0) = 0$ (by direct substitution) and $f(1) = 0$ (by the constraint on the coefficients). Thus, since f is continuous on $[0,1]$ and differentiable on $(0,1)$ (as a polynomial), we have by the MVT that there exists $x \in (0,1)$ such that

$$\begin{aligned}
f(1) - f(0) &= (1-0)f'(x) \\
f'(x) &= 0 \\
C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n &= 0
\end{aligned}$$

as desired. □

5. Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

Proof. To prove that $\lim_{x\to\infty} g(x) = 0$, it will suffice to show that for every $\epsilon > 0$, there exists $N > 0$ such that if $x > N$, then $|g(x) - 0| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{x\to\infty} f'(x) = 0$ by hypothesis, we know that there exists $N > 0$ such that if $x > N$, then $|f'(x)| < \epsilon$. Choose this N to be our N. Let $x > N$ be arbitrary. Applying the MVT to f on the interval $[x, x+1]$ proves the existence of a c within that closed interval such that

$$f(x+1) - f(x) = f'(c)(x+1-x) = f'(c)$$

Additionally, since $c > x > N$, we have that $|f'(c)| < \epsilon$. Therefore, we have that

$$\begin{aligned}|g(x)| &= |f(x+1) - f(x)|\\ &= |f'(c)|\\ &< \epsilon\end{aligned}$$

as desired. $\square$

2 Differentiation II / Integration

From Rudin (1976).

Chapter 5

8. Suppose f' is continuous on $[a,b]$ and $\epsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left|\frac{f(t)-f(x)}{t-x} - f'(x)\right| < \epsilon$$

whenever $0 < |t-x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$. (This could be expressed by saying that f is **uniformly differentiable** on $[a,b]$ if f' is continuous on $[a,b]$.) Does this hold for vector-valued functions, too?

Proof. By Theorem 2.40, $[a,b]$ is compact. This combined with the fact that f' is continuous implies by Theorem 4.19 that f' is uniformly continuous. Thus, there exists $\delta > 0$ such that if $x,y \in [a,b]$ and $|y-x| < \delta$, then $|f'(y) - f'(x)| < \epsilon$. Choose this δ to be our δ. Let $x,t \in [a,b]$ be such that $0 < |t-x| < \delta$. Then since f is continuous on $[t,x] \subset [a,b]$ and differentiable on $(t,x) \subset [a,b]$, we have by the MVT that there exists $c \in (t,x)$ such that

$$\begin{aligned} f(t) - f(x) &= (t-x)f'(c) \\ \frac{f(t)-f(x)}{t-x} &= f'(c) \end{aligned}$$

Additionally, since $t < c < x$ and $|t-x| < \delta$, we must have $|c-x| < \delta$, meaning that

$$\left|\frac{f(t)-f(x)}{t-x} - f'(x)\right| = |f'(c) - f'(x)| < \epsilon$$

as desired.

And yes, this does hold for vector-valued functions, which we can treat component-wise. □

17. Suppose f is a real, three times differentiable function on $[-1,1]$ such that

$$f(-1) = 0 \qquad f(0) = 0 \qquad f(1) = 1 \qquad f'(0) = 0$$

Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1,1)$. Note that equality holds for $\frac{1}{2}(x^3 + x^2)$. (Hint: Use Theorem 5.15 with $\alpha = 0$ and $\beta = \pm 1$ to show that there exist $s \in (0,1)$ and $t \in (-1,0)$ such that $f^{(3)}(s) + f^{(3)}(t) = 6$.)

Proof. Since f is three times differentiable on $[-1,1]$, we know that f'' is differentiable on $[-1,1]$. It follows by Theorem 5.2 that f'' is continuous on $[-1,1]$. Thus, since f is defined on $[-1,1]$, $3 \in \mathbb{N}$, f'' is continuous on $[-1,1]$, $f^{(3)}$ is defined on $(-1,1)$, $0, 1 \in [-1,1]$ such that $0 \neq 1$, and we can define

$$P(t) = \sum_{k=0}^{2} \frac{f^{(k)}(0)}{k!}(t-0)^k$$

we have by Taylor's theorem that there exists $s \in (0,1)$ such that

$$\begin{aligned} f(1) &= P(1) + \frac{f^{(3)}(s)}{3!}(1-0)^3 \\ 1 - \left[\frac{f(0)}{0!}(1-0)^0 + \frac{f'(0)}{1!}(1-0)^1 + \frac{f''(0)}{2!}(1-0)^2\right] &= \frac{f^{(3)}(s)}{3!} \\ 1 - \left[\frac{f''(0)}{2}\right] &= \frac{f^{(3)}(s)}{6} \\ 6 - 3f''(0) &= f^{(3)}(s) \end{aligned}$$

Similarly, we have that there exists $t \in (-1,0)$ such that

$$\begin{aligned}
f(-1) &= P(-1) + \frac{f^{(3)}(t)}{3!}(-1-0)^3 \\
0 - \left[\frac{f(0)}{0!}(-1-0)^0 + \frac{f'(0)}{1!}(-1-0)^1 + \frac{f''(0)}{2!}(-1-0)^2 \right] &= -\frac{f^{(3)}(t)}{3!} \\
-\left[\frac{f''(0)}{2} \right] &= -\frac{f^{(3)}(t)}{6} \\
3f''(0) &= f^{(3)}(s)
\end{aligned}$$

Thus,

$$f^{(3)}(s) + f^{(3)}(t) = 3f''(0) + 6 - 3f''(0) = 6$$

Now suppose for the sake of contradiction that for all $x \in (-1,1)$, we have $f^{(3)}(x) < 3$. Then $f^{(3)}(s) < 3$ and $f^{(3)}(t) < 3$. It follows that $f^{(3)}(s) + f^{(3)}(t) < 6$, a contradiction. ☐

25. Suppose f is twice differentiable on $[a,b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geq \delta > 0$, and $0 \leq f''(x) \leq M$ for all $x \in [a,b]$. Let ξ be the unique point in (a,b) at which $f(\xi) = 0$. Complete the details in the following outline of **Newton's method** for computing ξ.

(a) Choose $x_1 \in (\xi, b)$ and define $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Interpret this geometrically, in terms of a tangent to the graph of f.

Answer. Since we can rearrange the above to $0 - f(x_n) = f'(x_n)(x_{n+1} - x_n)$, we know that x_{n+1} is the point at which the tangent to f at x_n crosses the x-axis. In other words, the zero of the tangent line

$$y - f(x_n) = f'(x_n)(x - x_n)$$

is $(x_{n+1}, 0)$. ☐

(b) Prove that $x_{n+1} < x_n$ and that

$$\lim_{n\to\infty} x_n = \xi$$

Proof. To prove that $x_{n+1} < x_n$, it will suffice to show that $f(x_n), f'(x_n) > 0$. Since $f'(x) > 0$ for all $x \in [a,b]$ by hypothesis, we know that $f'(x_n) > 0$. As to $f(x_n)$, suppose for the sake of contradiction that $f(x_n) \leq 0$. We know that $f(\xi) = 0$, $f(b) > 0$, and $\xi < x_n < b$. Since ξ is the *unique* point at which $f(\xi) = 0$ by hypothesis and $x_n \neq \xi$, we know that $f(x_n) \neq 0$. And if $f(x_n) < 0$, we have by the intermediate value theorem for f continuous that there exists $c \in (x_n, b)$ such that $f(c) = 0$. But since $\xi < x_n < c < b$, $c \neq \xi$, and thus we have a contradiction here, too.

Having established that $\{x_n\}$ is a monotonically decreasing sequence, Theorem 3.14 tells us that to show that it converges, it will suffice to show that it is bounded. Clearly, $\{x_n\}$ is bounded above by x_1. And on the bottom side, $\{x_n\}$ is bounded by ξ: If there were $x_n < \xi$, this would imply that $f(x_n) < 0$ by a symmetric argument to the above, meaning that $f(x_n)/f'(x_n) < 0$ and implying that $x_{n+1} > x_n$, a contradiction. Furthermore, we know that the limit (call it x) equals ξ since

$$\begin{aligned}
x &= x - \frac{f(x)}{f'(x)} \\
f(x) &= 0
\end{aligned}$$

so $x = \xi$ by the uniqueness of ξ. ☐

(c) Use Taylor's theorem to show that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

for some $t_n \in (\xi, x_n)$.

Proof. Since f is defined on $[a,b]$, $2 \in \mathbb{N}$, f' is continuous on $[a,b]$, f'' is defined on (a,b), $\xi, x_n \in [a,b]$ with $\xi \neq x_n$, and

$$P(t) = \sum_{k=0}^{1} \frac{f^{(k)}(x_n)}{k!}(t - x_n)^k$$

we have by Taylor's theorem that there exists $t_n \in (\xi, x_n)$ such that

$$\begin{aligned} f(\xi) &= \left[\frac{f(x_n)}{0!}(\xi - x_n)^0 + \frac{f'(x_n)}{1!}(\xi - x_n)^1\right] + \frac{f''(t_n)}{2!}(\xi - x_n)^2 \\ 0 &= f(x_n) - f'(x_n)(x_n - \xi) + \frac{f''(t_n)}{2}(x_n - \xi)^2 \\ x_n - \frac{f(x_n)}{f'(x_n)} - \xi &= \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2 \\ x_{n+1} - \xi &= \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2 \end{aligned}$$

as desired. □

(d) If $A = M/2\delta$, deduce that

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A}[A(x_1 - \xi)]^{2n}$$

(Compare with Chapter 3, Exercises 16 and 18.)

Proof. We have from part (b) that $x_i > \xi$ for all $i \in \mathbb{N}$, so naturally $0 \leq x_{n+1} - \xi$. As to the other part of the question, we induct on n. For the base case $n = 1$, we have that

$$\begin{aligned} x_2 - \xi &= \frac{f''(t_1)}{2f'(x_1)}(x_1 - \xi)^2 \\ &\leq \frac{M}{2\delta}(x_1 - \xi)^2 \\ &= \frac{2\delta}{M}\left[\frac{M}{2\delta}(x_1 - \xi)\right]^2 \\ &= \frac{1}{A}[A(x_1 - \xi)]^{2\cdot 1} \end{aligned}$$

Now suppose inductively that we have proven the claim for $n - 1$; we now seek to prove it for n. Indeed, we have that

$$\begin{aligned} x_{n+1} - \xi &= \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2 \\ &\leq \frac{M}{2\delta}(x_n - \xi)^2 \\ &\leq A\left(\frac{1}{A}[A(x_1 - \xi)]^{2(n-1)}\right)^2 \\ &= \frac{1}{A}[A(x_1 - \xi)]^{2n} \end{aligned}$$

as desired. □

(e) Show that Newton's method amounts to finding a fixed point of the function g defined by

$$g(x) = x - \frac{f(x)}{f'(x)}$$

How does $g'(x)$ behave for x near ξ?

Proof. A fixed point of the function g is a point x such that

$$\begin{aligned} g(x) &= x \\ x - \frac{f(x)}{f'(x)} &= x \\ f(x) &= 0 \end{aligned}$$

Thus, if we want to find a point x where $f(x) = 0$, it is equivalent to find a point x such that $g(x) = x$.
As to the other part of the question, we have by the rules of derivatives that

$$\begin{aligned} g'(x) &= 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{f'(x)^2} \\ &= \frac{f(x)f''(x)}{f'(x)^2} \\ &\leq \frac{M}{\delta^2} f(x) \end{aligned}$$

Thus, since $f(x) \to 0$ as $x \to \xi$, $g'(x) \to 0$ as $x \to \xi$. ☐

(f) Put $f(x) = \sqrt[3]{x}$ on $(-\infty, \infty)$ and try Newton's method. What happens?

Answer. We have by the power rule that

$$f'(x) = \frac{1}{3x^{2/3}}$$

Choose $x_1 = 1$. Then

$$\begin{aligned} x_2 &= 1 - \frac{f(1)}{f'(1)} = -2 \\ x_3 &= 1 - \frac{f(-2)}{f'(-2)} = 7 \\ x_4 &= 1 - \frac{f(7)}{f'(7)} = -20 \\ &\vdots \end{aligned}$$

It appears that the series is diverging to ∞ while alternating from positive to negative. In fact, since $x_3 > x_2$, contrary to part (b), we know that something must be wrong (i.e., one of our hypotheses must not be met). Upon further investigation, we can determine that on $[-1, 1]$, we have $f''(1) = -2/9 < 0$; thus, our last hypothesis is the issue with this function. ☐

Chapter 6

1. Suppose α increases on $[a, b]$, $a \leq x_0 \leq b$, α is continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathscr{R}(\alpha)$ and that $\int f \, d\alpha = 0$.

Proof. Since f is bounded on $[a,b]$ with only one discontinuity on $[a,b]$ and α is continuous at the point at which f is discontinuous, Theorem 6.10 implies that $f \in \mathscr{R}(\alpha)$, as desired. It follows that $\inf U(P,f,\alpha) = \sup L(P,f,\alpha) = \int f \, \mathrm{d}\alpha$. But since $L(P,f,\alpha) = 0$ for all P (there is no infinite interval $[x_i, x_{i+1}] \subset [a,b]$ that does not contain 0, and f is bounded below by 0), we know that

$$\int f \, \mathrm{d}\alpha = \sup L(P,f,\alpha) = 0$$

as desired. $\square$

2. Suppose $f \geq 0$, f is continuous on $[a,b]$, and $\int_a^b f(x) \, \mathrm{d}x = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$. (Compare this with Exercise 1.)

Proof. Suppose for the sake of contradiction that $f(x) \neq 0$ for some x. By the definition of f, this must mean that $f(x) > 0$. It follows since f is continuous that there exists some $N_r(x)$ such that $f(y) > 0$ for all $y \in N_r(x)$. Now consider the partition

$$P = \{a, x - r/2, x + r/2, b\}$$

of $[a,b]$. But since $m_2 > 0$, we have that

$$\begin{aligned} 0 &< m_1[(x - r/2) - a] + m_2[(x + r/2) - (x - r/2)] + m_3[b - (x + r/2)] \\ &= L(P,f) \\ &\leq \int_a^b f(x) \, \mathrm{d}x && \text{Theorem 6.4} \end{aligned}$$

a contradiction. $\square$

4. If $f(x) = 0$ for all irrational x and $f(x) = 1$ for all rational x, prove that $f \notin \mathscr{R}$ on $[a,b]$ for any $a < b$.

Proof. Let P be an arbitrary partition of $[a,b]$. Since the rationals and irrationals are dense in the reals, we know that for any $[x_i, x_{i+1}]$, $f(x) = 0$ for some $x \in [x_i, x_{i+1}]$ and $f(x) = 1$ for some $x \in [x_i, x_{i+1}]$. Thus, we have that $L(P,f) = 0$ and $U(P,f) = b - a$. It follows that if $a < b$,

$$\sup L(P,f) = 0 \neq b - a = \inf U(P,f)$$

so $f \notin \mathscr{R}$, as desired. $\square$

3 Integration II

From Rudin (1976).

Chapter 6

2/2: **3.** Define three functions $\beta_1, \beta_2, \beta_3$ as follows:

$$\beta_1 = \begin{cases} 0 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases} \qquad \beta_2 = \begin{cases} 0 & x < 0 \\ 1 & x = 0 \\ 1 & x > 0 \end{cases} \qquad \beta_3 = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases}$$

Let f be a bounded function on $[-1, 1]$.

(a) Prove that $f \in \mathscr{R}(\beta_1)$ if and only if $f(0+) = f(0)$ and that then

$$\int f \,\mathrm{d}\beta_1 = f(0)$$

Proof. Suppose first that $f \in \mathscr{R}(\beta_1)$ with $\int f \,\mathrm{d}\beta_1 = f(0)$. To prove that $f(0+) = f(0)$, it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in [-1, 1]$ and $0 \leq x < \delta$, then $|f(x) - f(0)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $f \in \mathscr{R}(\beta_1)$ by hypothesis, we have by Theorem 6.6 that there exists a partition P of $[-1, 1]$ such that $U(P, f, \beta_1) - L(P, f, \beta_1) < \epsilon$. Now let $x_i = \min\{x \in P : x > 0\}$; we know that such an object exists since there exist elements of P greater than zero (namely 1) and P is finite. It follows by the definition of β_1 that $\Delta x_i = 1$ and $\Delta x_j = 0$ for $j \neq i$. Thus, $U(P, f, \beta_1) = M_i$ and $L(P, f, \beta_1) = m_i$ (which exist because f is bounded on $[-1, 1]$). At this point, we are ready to choose δ, whcih we take to be $\delta = x_i$. Now to confirm that this δ works: Let $0 \leq x < \delta$. By the definition of x_i, x_{i-1}, $m_i \leq f(x) \leq M_i$ and $m_i \leq f(0) \leq M_i$. But since $M_i - m_i < \epsilon$ as per the above, we have that $|f(x) - f(0)| < \epsilon$, as desired.

Now suppose that $f(0+) = f(0)$. To prove that $f \in \mathscr{R}(\beta_1)$, Theorem 6.6 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a P such that $U(P, f, \beta_1) - L(P, f, \beta_2) < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $f(0+) = f(0)$, we know that there exists a $\delta' > 0$ such that if $x \in [-1, 1]$ and $0 \leq x < \delta'$, then $|f(x) - f(0)| < \epsilon/3$. Let $\delta = \min(\delta'/2, 1)$. Thus, we may define $P = \{-1, 0, \delta, 1\}$. We have

$$U(P, f, \beta_1) = \sum_{i=1}^{3} M_i \Delta\beta_{1_i} \qquad L(P, f, \beta_1) = \sum_{i=1}^{3} m_i \Delta\beta_{1_i}$$
$$= M_2 \qquad\qquad = m_2$$

(which exist because f is bounded on $[-1, 1]$). Consequently, $M_2 \leq f(0) + \epsilon/3$. $m_2 \geq f(0) - \epsilon/3$. Therefore,

$$\begin{aligned} U(P, f, \beta_1) - L(P, f, \beta_1) &= M_2 - m_2 \\ &\leq [f(0) + \tfrac{\epsilon}{3}] - [f(0) - \tfrac{\epsilon}{3}] \\ &= \frac{2\epsilon}{3} \\ &< \epsilon \end{aligned}$$

as desired.

As to proving that $\int f \,\mathrm{d}\beta_1$, we know that $M_2 \leq f(0) + \epsilon/3$ for arbitrarily small ϵ implies $M_2 \leq f(0)$. Similarly, $m_2 \geq f(0)$. Thus,

$$\inf U(P, f, \beta_1) \leq U(P, f, \beta_1) = M_2 \leq f(0) \leq m_2 = L(P, f, \beta_1) \leq \sup L(P, f, \beta_1)$$

But by Theorem 6.5, $\sup L(P,f,\beta_1) \leq \inf U(P,f,\beta_1)$. Therefore,

$$\int_{-1}^{1} f\,\mathrm{d}\beta_1 = \sup L(P,f,\beta_1) = \inf U(P,f,\beta_1) = f(0)$$

as desired. ☐

(b) State and prove a similar result for β_2.

Proof. The result will be $f \in \mathscr{R}(\beta_2)$ if and only if $f(0-) = f(0)$ and that then

$$\int f\,\mathrm{d}\beta = f(0)$$

The proof of this result is entirely symmetric to the proof of the previous result. ☐

(c) Prove that $f \in \mathscr{R}(\beta_3)$ if and only if f is continuous at 0.

Proof. Suppose first that $f \in \mathscr{R}(\beta_3)$. To prove that f is continuous at 0, it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in [-1,1]$ and $|x| < \delta$, then $|f(x) - f(0)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $f \in \mathscr{R}(\beta_3)$ by hypothesis, we have by Theorem 6.6 that there exists a partition P of $[-1,1]$ such that $U(P,f,\beta_3) - L(P,f,\beta_3) < \epsilon/2$. Now let $x_i = \max\{x \in P : x < 0\}$ and let $x_j = \min\{x \in P : x > 0\}$. Choose $\delta = \min\{|x_i|, |x_j|\}$. Let $P^* = P \cup \{-\delta, 0, \delta\}$ be a refinement of P. It follows by the definition of β_3 and a reenumeration of P^* that $U(P^*,f,\beta_3) = (M_{i-1} + M_i)/2$ and $L(P^*,f,\beta_3) = (m_{i-1} + m_i)/2$. Now let $|x| < \delta$. We divide into two cases ($x \geq 0$ and $x < 0$). If $x \geq 0$, then $m_i \leq f(x) \leq M_i$ and $m_i \leq f(0) \leq M_i$. But then we have that

$$\begin{aligned}
|f(x) - f(0)| &\leq M_i - m_i \\
&\leq (M_{i-1} - m_{i-1}) + (M_i - m_i) \\
&= 2\left[\frac{M_{i-1} + M_i}{2} - \frac{m_{i-1} + m_i}{2}\right] \\
&= 2[U(P^*,f,\beta_3) - L(P^*,f,\beta_3)] \\
&< \epsilon
\end{aligned}$$

as desired. The proof is symmetric in the other case.

Now suppose that f is continuous at 0. To prove that $f \in \mathscr{R}(\beta_3)$, Theorem 6.6 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a P such that $U(P,f,\beta_3) - L(P,f,\beta_3) < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f is continuous at 0, we know that there exists a $\delta' > 0$ such that if $x \in [-1,1]$ and $|x| < \delta'$, then $|f(x) - f(0)| < \epsilon/3$. Choose $\delta = \min(\delta'/2, 1)$. Consider $P = \{-1, -\delta/2, \delta/2, 1\}$. It follows as before that $U(P,f,\beta_3) = M_2$ and $L(P,f,\beta_3) = m_2$. Consequently, $M_2 \leq f(0) + \epsilon/3$ and $m_2 \geq f(0) - \epsilon/3$. Therefore,

$$\begin{aligned}
U(P,f,\beta_3) - L(P,f,\beta_3) &= M_2 - m_2 \\
&\leq [f(0) + \tfrac{\epsilon}{3}] - [f(0) - \tfrac{\epsilon}{3}] \\
&= \frac{2\epsilon}{3} \\
&< \epsilon
\end{aligned}$$

as desired. ☐

(d) If f is continuous at 0, prove that

$$\int f\,\mathrm{d}\beta_1 = \int f\,\mathrm{d}\beta_2 = \int f\,\mathrm{d}\beta_3 = f(0)$$

Proof. If f is continuous at 0, then $f(0+) = f(0) = f(0-)$. It follows that

$$\begin{aligned} f(0) &= \int f \,\mathrm{d}\beta_1 && \text{Part (a)} \\ &= \int f \,\mathrm{d}\beta_2 && \text{Part (b)} \\ &= \int f \,\mathrm{d}\beta_3 && \text{Part (c)} \end{aligned}$$

Note that calculating the exact value of $\int f \,\mathrm{d}\beta_3$ is symmetric to the proof in part (a). ☐

5. Suppose f is a bounded real function on $[a,b]$, and $f^2 \in \mathscr{R}$ on $[a,b]$. Does it follow that $f \in \mathscr{R}$? Does the answer change if we assume that $f^3 \in \mathscr{R}$?

Proof. $\underline{f^2 \in \mathscr{R} \not\Rightarrow f \in \mathscr{R}}$: Consider the bounded real function $f : [a,b] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & x \notin \mathbb{Q} \\ -1 & x \in \mathbb{Q} \end{cases}$$

Since $f^2(x) = 1$ for all $x \in [a,b]$, $f^2 \in \mathscr{R}$ as a constant function. However, by Exercise 6.4 and a clever application of Theorem 6.12 (to relate it to the function explicitly considered in Exercise 6.4), we know that $f \notin \mathscr{R}$.

$\underline{f^3 \in \mathscr{R} \Rightarrow f \in \mathscr{R}}$: Let $f : [a,b] \to \mathbb{R}$ be any bounded real function such that $f^3 \in \mathscr{R}$. To prove that $f \in \mathscr{R}$, Theorem 6.11 tells us that it will suffice to show that there exist $m, M \in \mathbb{R}$ such that $m \leq f \leq M$ and that there exists a continuous function $\phi : [m,M] \to \mathbb{R}$ such that $f = \phi \circ f^3$. Since f is bounded by hypothesis, we can pick $m, M \in \mathbb{R}$ such that $m \leq f \leq M$. Now let $\phi : [m,M] \to \mathbb{R}$ be defined by

$$\phi(x) = \sqrt[3]{x}$$

for all $x \in [m,M]$. It is obvious that ϕ is continuous and that $\phi \circ f^3 = f$, as desired. ☐

7. Suppose f is a real function on $(0,1]$ and $f \in \mathscr{R}$ on $[c,1]$ for every $c > 0$. Define

$$\int_0^1 f(x) \,\mathrm{d}x = \lim_{c \to 0} \int_c^1 f(x) \,\mathrm{d}x$$

if this limit exists (and is finite).

(a) If $f \in \mathscr{R}$ on $[0,1]$, show that this definition of the integral agrees with the old one.

Proof. To prove that $\int_0^1 f = \lim_{c \to 0} \int_c^1 f$, it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $c \in (0,1]$ and $c < \delta$, then

$$\left| \int_0^c f \right| = \left| \int_c^1 f - \int_0^1 f \right| < \epsilon$$

Let $\epsilon > 0$ be arbitrary. Since f is integrable, f is bounded, i.e., there exists $M \in \mathbb{R}$ such that $|f(x)| < M$ for all $x \in [0,1]$. Choose $\delta = \epsilon / M$. Let $c \in (0,1]$ be such that $c < \delta$. Then by Theorem 6.12d,

$$\begin{aligned} \left| \int_0^c f \right| &\leq M(c - 0) \\ &< \epsilon \end{aligned}$$

as desired. ☐

(b) Construct a function f such that the above limit exists, although it fails to exist with $|f|$ in place of f.

Proof. Let $f : (0, 1] \to \mathbb{R}$ be defined by

$$f(x) = (-1)^n n$$

for $1/n < x \leq 1/(n-1)$ $(n = 2, 3, \dots)$. It follows since f is a constant function save one terminal discontinuity on each $[1/n, 1/(n-1)]$ that

$$\begin{aligned}\int_{1/n}^{1/(n-1)} f &= (-1)^n n \cdot \left(\frac{1}{n-1} - \frac{1}{n} \right) \\ &= \frac{(-1)^n n}{n(n-1)} \\ &= \frac{(-1)^n}{n-1}\end{aligned}$$

for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned}\int_{1/N}^{1} f &= \sum_{n=2}^{N} \int_{1/n}^{1/(n-1)} f \\ &= \sum_{n=2}^{N} \frac{(-1)^n}{n-1}\end{aligned}$$

Thus,

$$\lim_{c \to 0} \int_{c}^{1} f = \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1}$$

which converges by Theorem 3.43. However, the limit fails to exist if f is replaced by $|f|$, because in that case, the integral is equal to the harmonic series, which diverges to infinity. ☐

8. Suppose $f \in \mathscr{R}$ on $[a, b]$ for every $b > a$ where a is fixed. Define

$$\int_{a}^{\infty} f(x)\,\mathrm{d}x = \lim_{b \to \infty} \int_{a}^{b} f(x)\,\mathrm{d}x$$

if this limit exists (and is finite). In that case, we say that the integral on the left **converges**. If it also converges after f has been replaced by $|f|$, it is said to converge **absolutely**.

Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1, \infty)$. Prove that $\int_1^\infty f(x)\,\mathrm{d}x$ converges if and only if $\sum_{n=1}^{\infty} f(n)$ converges. (This is the so-called "integral test" for convergence of series.)

Proof. To prove the claim, we will show that

$$\sum_{n=2}^{N} f(n) \leq \int_{1}^{N} f \leq \sum_{n=1}^{N-1} f(n) \leq f(1) + \int_{1}^{N-1} f(x)\,\mathrm{d}x$$

It will follow since both the sum and the integral limit are monotonically increasing as $N \to \infty$ $(f \geq 0)$ and both are bounded below and above by (a function of) the other, both converge or diverge together. Let's begin.

Since f is monotonically decreasing on $[1, \infty)$, we know that $f(n) \leq f(x)$ for all $1 \leq x \leq n$ $(n \in \mathbb{N})$. Thus, by Theorem 6.12b,

$$\int_{n-1}^{n} f(n)\,\mathrm{d}x \leq \int_{n-1}^{n} f(x)\,\mathrm{d}x$$

Therefore,

$$\begin{aligned}
\sum_{n=2}^{N} f(n) &= \sum_{n=2}^{N} \int_{n-1}^{n} f(n)\,\mathrm{d}x && \text{Theorem 6.12d}\\
&\leq \sum_{n=2}^{N} \int_{n-1}^{n} f(x)\,\mathrm{d}x\\
&= \int_{1}^{N} f(x)\,\mathrm{d}x && \text{Theorem 6.12c}
\end{aligned}$$

for all $N = 2, 3, 4, \dots$, thereby establishing the left inequality above.

Since f is monotonically decreasing on $[1, \infty)$, we know that $f(x) \leq f(n)$ for all $x \geq n$ ($n \in \mathbb{N}$). Thus, by Theorem 6.12b,

$$\int_{n}^{n+1} f(x)\,\mathrm{d}x \leq \int_{n}^{n+1} f(n)\,\mathrm{d}x$$

Therefore,

$$\begin{aligned}
\int_{1}^{N} f(x)\,\mathrm{d}x &= \sum_{n=1}^{N-1} \left(\int_{n}^{n+1} f(x)\,\mathrm{d}x \right) && \text{Theorem 6.12c}\\
&\leq \sum_{n=1}^{N-1} \left(\int_{n}^{n+1} f(n)\,\mathrm{d}x \right)\\
&= \sum_{n=1}^{N-1} f(n) && \text{Theorem 6.12d}
\end{aligned}$$

for all $N = 2, 3, 4, \dots$, thereby establishing the middle inequality above.

From our statement about $f(n)$ and $f(x)$ from the left inequality, we have by Theorem 6.12b that

$$\int_{n-1}^{n} f(n)\,\mathrm{d}x \leq \int_{n-1}^{n} f(x)\,\mathrm{d}x$$

Therefore,

$$\begin{aligned}
\sum_{n=1}^{N-1} f(n) &= f(1) + \sum_{n=2}^{N-1} \int_{n-1}^{n} f(n)\,\mathrm{d}x && \text{Theorem 6.12d}\\
&\leq f(1) + \sum_{n=2}^{N-1} \int_{n-1}^{n} f(x)\,\mathrm{d}x\\
&= f(1) + \int_{1}^{N-1} f(x)\,\mathrm{d}x && \text{Theorem 6.12c}
\end{aligned}$$

for all $N = 2, 3, 4, \dots$, thereby establishing the right inequality above. □

10. Let p, q be positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

Prove the following statements.

(a) If $u, v \geq 0$, then

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}$$

Equality holds if and only if $u^p = v^q$.

Discussion. To prove the desired inequality, it will suffice to show that

$$0 \leq \frac{u^p}{p} + \frac{v^q}{q} - uv$$

i.e., that for all $u, v \geq 0$, the expression on the right above is nonnegative. To consider all such values at once, we can consider applying our analysis toolbox to $f : [0, \infty)^2 \to \mathbb{R}$ defined by

$$f(u, v) = \frac{u^p}{p} + \frac{v^q}{q} - uv$$

with the goal of proving that it is nonnegative everywhere on its domain. However, since we do not yet know multivariable calculus, it will suffice to fix $u \geq 0$ and analyze $f : [0, \infty) \to \mathbb{R}$ defined by

$$f(v) = \frac{u^p}{p} + \frac{v^q}{q} - uv$$

Let's begin. □

Proof. Fix $u \geq 0$. Let $f : [0, \infty) \to \mathbb{R}$ be defined by

$$f(v) = \frac{u^p}{p} + \frac{v^q}{q} - uv$$

It follows from the definition of f that to prove the desired inequality, it will suffice to show that f is nonnegative everywhere on its domain. Let's begin.

Since f is a polynomial in v, f is differentiable. Thus, we may consider

$$f'(v) = v^{q-1} - u$$

As a function of a positive power ($q/(q-1) = p > 0$ and $q > 0$ imply $q - 1 > 0$) of its variable (minus a constant), f' is strictly increasing. Additionally, we have that

$$\begin{aligned} 0 &= f'(v) \\ u &= v^{q-1} \\ &= v^{q/p} \\ v &= u^{p/q} \end{aligned}$$

Thus, we know that $f' < 0$ on $(0, u^{p/q})$ and $f' > 0$ on $(u^{p/q}, \infty)$. It follows by the strict version of Theorem 5.11 that f is strictly decreasing on $(0, u^{p/q})$ and strictly increasing on $(u^{p/q}, \infty)$. Furthermore, since f is differentiable (hence continuous by Theorem 5.2), we know that $f(0) \geq f(u^{p/q})$. Combining the last several results, we have that $f(u^{p/q})$ is the minimum of f over $[0, \infty)$, and hence equal to the minimum value of f over $[0, \infty)$. But since

$$\begin{aligned} f(u^{p/q}) &= \frac{u^p}{p} + \frac{(u^{p/q})^q}{q} - uu^{p/q} \\ &= \frac{u^p}{p} + \frac{u^p}{q} - u^{p/q+1} \\ &= u^p \left(\frac{1}{p} + \frac{1}{q} \right) - u^p \\ &= 0 \end{aligned}$$

we know that $f(v) \geq 0$ on its domain, as desired.

Additionally, since f is strictly decreasing on $(0, u^{p/q})$ and strictly increasing on $(u^{p/q}, \infty)$, we know that $f(v) = 0$ iff $v = u^{p/q}$, i.e., iff $v^q = u^p$, as desired. □

(b) If $f,g\in\mathscr{R}(\alpha)$, $f,g\geq 0$, and

$$\int_a^b f^p\,\mathrm{d}\alpha = 1 = \int_a^b g^q\,\mathrm{d}\alpha$$

then

$$\int_a^b fg\,\mathrm{d}\alpha \leq 1$$

Proof. By Theorem 6.13a, the hypothesis $f,g\in\mathscr{R}(\alpha)$ implies that $fg\in\mathscr{R}(\alpha)$. Thus, we have that

$$\begin{aligned}
\int_a^b fg\,\mathrm{d}\alpha &\leq \int_a^b \left(\frac{f^p}{p}+\frac{g^q}{q}\right)\mathrm{d}\alpha && \text{Theorem 6.12b}\\
&= \frac{1}{p}\int_a^b f^p\,\mathrm{d}\alpha + \frac{1}{q}\int_a^b g^q\,\mathrm{d}\alpha && \text{Theorem 6.12a}\\
&= \frac{1}{p}+\frac{1}{q}\\
&= 1
\end{aligned}$$

as desired. ☐

(c) If f,g are complex functions in $\mathscr{R}(\alpha)$, then

$$\left|\int_a^b fg\,\mathrm{d}\alpha\right| \leq \left(\int_a^b |f|^p\,\mathrm{d}\alpha\right)^{1/p}\left(\int_a^b |g|^q\,\mathrm{d}\alpha\right)^{1/q}$$

This is **Hölder's inequality**. When $p=q=2$, it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)

Proof. By Theorem 6.11 with $\phi(y)=|y|^p$ (resp. $\phi(y)=|y|^q$), the hypothesis $f,g\in\mathscr{R}(\alpha)$ implies that $|f|^p,|g|^q\in\mathscr{R}(\alpha)$. Thus, we may let

$$I_f = \left(\int_a^b |f|^p\,\mathrm{d}\alpha\right)^{1/p} \qquad\qquad I_g = \left(\int_a^b |g|^q\,\mathrm{d}\alpha\right)^{1/q}$$

We divide into two cases ($I_f=0$ or $I_g=0$, and $I_f,I_g\neq 0$). In the first case, WLOG let $I_f=0$. Then since $0\leq |f|^p$, it follows that $f=0$ on $[a,b]$. Thus

$$\left|\int_a^b fg\,\mathrm{d}\alpha\right| = 0 \leq 0 = I_fI_g = \left(\int_a^b |f|^p\,\mathrm{d}\alpha\right)^{1/p}\left(\int_a^b |g|^q\,\mathrm{d}\alpha\right)^{1/q}$$

as desired. In the other case, it follows that

$$\begin{aligned}
I_f^p &= \int_a^b |f|^p\,\mathrm{d}\alpha & I_g^q &= \int_a^b |g|^q\,\mathrm{d}\alpha\\
1 &= \int_a^b \left|\frac{f}{I_f}\right|^p\mathrm{d}\alpha & 1 &= \int_a^b \left|\frac{g}{I_g}\right|^q\mathrm{d}\alpha && \text{Theorem 6.12a}
\end{aligned}$$

Thus, since $|f/I_f|,|g/I_g|\in\mathscr{R}(\alpha)$ by Theorems 6.12 and 6.13 and $|f/I_f|,|g/I_g|\geq 0$ by the definition of the absolute value, we have that

$$\begin{aligned}
\left|\int_a^b fg\,\mathrm{d}\alpha\right| &\leq \int_a^b |fg|\,\mathrm{d}\alpha && \text{Theorem 6.13b}\\
&= I_fI_g\int_a^b \left|\frac{f}{I_f}\right|\left|\frac{g}{I_g}\right|\mathrm{d}\alpha\\
&\leq I_fI_g\cdot 1 && \text{Part (b)}\\
&= \left(\int_a^b |f|^p\,\mathrm{d}\alpha\right)^{1/p}\left(\int_a^b |g|^q\,\mathrm{d}\alpha\right)^{1/q}
\end{aligned}$$

as desired. ☐

11. Let α be a fixed increasing function on $[a,b]$. For $u\in\mathscr{R}(\alpha)$, define

$$\|u\|_2 = \left(\int_a^b |u|^2\,\mathrm{d}\alpha\right)^{1/2}$$

Suppose $f,g,h\in\mathscr{R}(\alpha)$, and prove the triangle inequality

$$\|f-h\|_2 \leq \|f-g\|_2 + \|g-h\|_2$$

as a consequence of the Schwarz inequality, as in the proof of Theorem 1.37.

Proof. By Theorems 6.12a and 6.13b, the hypothesis that $f,g,h\in\mathscr{R}(\alpha)$ implies that $|f-g|,|g-h|\in\mathscr{R}(\alpha)$. Thus, we have that

$$\begin{aligned}
\|f-h\|_2^2 &= \int_a^b |f-h|^2\,\mathrm{d}\alpha\\
&= \int_a^b |(f-g)+(g-h)|^2\,\mathrm{d}\alpha\\
&= \int_a^b |f-g|^2\,\mathrm{d}\alpha + 2\int_a^b |f-g|\cdot|g-h|\,\mathrm{d}\alpha + \int_a^b |g-h|^2\,\mathrm{d}\alpha\\
&\leq \int_a^b |f-g|^2\,\mathrm{d}\alpha + 2\left(\int_a^b |f-g|^2\,\mathrm{d}\alpha\right)^{1/2}\left(\int_a^b |g-h|^2\,\mathrm{d}\alpha\right)^{1/2} + \int_a^b |g-h|^2\,\mathrm{d}\alpha\\
&= \|f-g\|_2^2 + 2\|f-g\|_2\|g-h\|_2 + \|g-h\|_2^2\\
&= (\|f-g\|_2 + \|g-h\|_2)^2
\end{aligned}$$

Taking square roots of both sides of the inequality yields the desired result. ☐

4 Sequences and Series of Functions

From Rudin (1976).

Chapter 7

2/9: **1.** Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

Proof. Let $\{f_n\}$ be an arbitrary uniformly convergent sequence of bounded functions. To prove that it is uniformly bounded, it will suffice to find a number M such that $|f_n(x)| < M$ for all $x \in E$ and $n \in \mathbb{N}$. Let f be the function such that $f_n \rightrightarrows f$, and let $M_n = \sup_{x \in E} |f_n(x) - f(x)|$ for each $n \in \mathbb{N}$ (the boundedness of each f_n implies that such an M_n always exists). Thus, based on the last two definitions, we can invoke Theorem 7.9 to learn that $M_n \to 0$ as $n \to \infty$. But since $\{M_n\}$ converges, Theorem 3.2c implies that $\{M_n\}$ is bounded, say by $M/2$. Taking M to be our M yields that for an arbitrary $x \in E$ and $n \in \mathbb{N}$,

$$|f_n(x)| \leq M_n \leq \frac{M}{2} < M$$

as desired. ☐

2. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.

Proof. To prove that $\{f_n+g_n\}$ converges uniformly on E to $f+g$, it will suffice to show that for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|(f_n + g_n)(x) - (f + g)(x)| < \epsilon$ for all $x \in E$. Let $\epsilon > 0$ be arbitrary. Since $f_n \to f$ uniformly on E, there exists N_1 such that if $n \geq N_1$, then $|f_n(x) - f(x)| < \epsilon/2$ for all $x \in E$. Similarly, there exists N_2 such that if $n \geq N_2$, then $|g_n(x) - g(x)| < \epsilon/2$ for all $x \in E$. Choose $N = \max(N_1, N_2)$. Now suppose $n \geq N$, and let $x \in E$ be arbitrary. It follows from the first condition that $n \geq N \geq N_1$ and $n \geq N \geq N_2$, so

$$\begin{aligned}
|(f_n + g_n)(x) - (f + g)(x)| &= |f_n(x) - f(x) + g_n(x) - g(x)| \\
&\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

as desired.

To prove that $\{f_n g_n\}$ converges uniformly on E to fg, it will suffice to show that for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|(f_n g_n)(x) - (fg)(x)| < \epsilon$ for all $x \in E$. Let $\epsilon > 0$ be arbitrary. Since f_n, g_n are uniformly convergent sequences of bounded functions, Exercise 1 implies that they are uniformly bounded, i.e., there exists $M^f, M^g \in \mathbb{R}$ such that $|f_n| < M^f$ and $|g_n| < M^g$ for all $n \in \mathbb{N}$. If we take $M = \max(M^f, M^g)$, then we have $|f_n| < M$ and $|g_n| < M$ for all $n \in \mathbb{N}$. Note that the same inequality holds for f and g. Now, as before, we may pick N such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon/2M$ and $|g_n(x) - g(x)| < \epsilon/2M$ for all $x \in E$. It follows that for any $n \geq N$ and $x \in E$,

$$\begin{aligned}
|(f_n g_n)(x) - (fg)(x)| &= |f_n(x) \cdot (g_n(x) - g(x)) + g(x) \cdot (f_n(x) - f(x))| \\
&= |f_n(x)| \cdot |g_n(x) - g(x)| + |g(x)| \cdot |f_n(x) - f(x)| \\
&< M \cdot \frac{\epsilon}{2M} + M \cdot \frac{\epsilon}{2M} \\
&= \epsilon
\end{aligned}$$

as desired. ☐

3. Construct sequences $\{f_n\}, \{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converge on E).

Proof. Let

$$f_n(x) = x \qquad\qquad g_n(x) = \frac{1}{n}$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then $\{f_n\}$ converges uniformly to $f(x) = x$ (by choosing $N = 1$ for any ϵ) and $\{g_n\}$ converges uniformly to $g(x) = 0$ (by choosing $1/N < \epsilon$ with the Archimedean principle). However, while $\{f_n g_n\}$ converges pointwise to $(fg)(x) = 0$ by Theorem 3.3c, it does not converge uniformly since for any n, choosing $x = n$ yields $(f_n g_n)(x) = 1$. □

4. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?

Proof. Absolute convergence values: The series converges absolutely for any

$$x \in (-\infty, -1) \cup \left(\bigcup_{k=1}^{\infty} \left(-\frac{1}{k^2}, -\frac{1}{(k+1)^2} \right) \right) \cup (0, \infty)$$

We prove this via casework as follows.

Let $x \in (0, \infty)$. Then we have

$$\sum_{n=1}^{\infty} \left| \frac{1}{1 + n^2 x} \right| = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x} \leq \sum_{n=1}^{\infty} \frac{1}{n^2 x} = \frac{1}{x} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{c}{x}$$

where $c \in \mathbb{R}$ is finite by Theorem 3.28. Therefore, since the sum is monotonically increasing and bounded, Theorem 3.14 implies that the sum overall converges, as desired.

Let $x \in (-\infty, -1)$. Then we have

$$n^2 x + 1 < n^2 x + n^2 = n^2(x + 1)$$

Since $x < -1$,

$$n^2 x + 1 < 0 \qquad\qquad n^2(x + 1) < 0$$

for all $n \in \mathbb{N}$. Thus,

$$\begin{gathered} n^2 x + 1 < n^2(x + 1) \\ \frac{n^2 x + 1}{n^2(x + 1)} > 1 \\ \frac{1}{n^2(x + 1)} < \frac{1}{n^2 x + 1} \\ \left| \frac{1}{n^2 x + 1} \right| < \left| \frac{1}{n^2(x + 1)} \right| \end{gathered}$$

for all $n \in \mathbb{N}$. It follows that

$$\sum_{n=1}^{\infty} \left| \frac{1}{n^2 x + 1} \right| < \sum_{n=1}^{\infty} \left| \frac{1}{n^2(x + 1)} \right| = \frac{1}{x + 1} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{c}{x + 1}$$

where $c \in \mathbb{R}$ is finite by Theorem 3.28. Therefore, since the sum is monotonically increasing and bounded, Theorem 3.14 implies that the sum overall converges, as desired.

Let $x \in (-1/k^2, -1/(k+1)^2)$. For right now, we consider only the sum for $n \geq \sqrt{2}(k+1)$, leaving finitely many terms out of the sum. Let $\delta = 1/(k+1)^2$. It follows that

$$\begin{aligned} n &\geq \sqrt{2}(k+1) & x &< -\frac{1}{(k+1)^2} \\ n &\geq \sqrt{\frac{2}{1/(k+1)^2}} & -x &> \frac{1}{(k+1)^2} \\ n^2 &\geq \frac{2}{\delta} \\ \frac{\delta}{2} &\geq \frac{1}{n^2} \end{aligned}$$

Additionally, since $n \geq \sqrt{2}(k+1) > k$ (hence $n^2 \geq (k+1)^2$) and $x < -1/(k+1)^2$, we have that

$$\begin{aligned} n^2 x &< (k+1)^2 \cdot -\frac{1}{(k+1)^2} \\ n^2 x &< -1 \\ n^2 x + 1 &< 0 \end{aligned}$$

Thus, for $n \geq \sqrt{2}(k+1)$, we have that

$$\left| \frac{1}{1+n^2 x} \right| = \frac{1}{n^2(-x) - 1} < \frac{1}{n^2 \delta - 1} = \frac{1}{n^2} \cdot \frac{1}{\delta - 1/n^2} \leq \frac{1}{n^2} \cdot \frac{1}{\delta - \delta/2} = \frac{2}{\delta n^2}$$

Therefore, since $|f_n(x)| \leq M_n = 2/\delta n^2$ and $\sum M_n$ converges by Theorem 3.28, the comparison test implies that $\sum |f_n(x)|$ converges, as desired. Adding on the finitely many terms we left out of the summation will not change this fact.

Note that the series diverges for $x = 0$ since each term becomes 1 in this case. Additionally, the series fails to exist for $x = -1/k^2$ ($k \in \mathbb{N}$) since the k^{th} term is undefined in this case.

Uniform convergence intervals: The series converges uniformly on any

$$[a, b] \subset (-\infty, -1) \cup \left(\bigcup_{k=1}^{\infty} \left(-\frac{1}{k^2}, -\frac{1}{(k+1)^2} \right) \right) \cup (0, \infty)$$

This is because any such interval will be a subset of either $(-\infty, -1)$, $(0, \infty)$, or a set of the form $(-1/k^2, -1/(k+1)^2)$ ($k \in \mathbb{N}$). Thus, we may take as $\sum M_n$ the supremum on $[a, b]$ of the appropriate bound derived above (either c/x, $c/(x+1)$, or $2c/\delta$, respectively; all supremums of which will exist by the definition of $[a, b]$) and apply Theorem 7.10.

Non-uniform convergence intervals: Any interval containing one or more of the points in the set $\{0\} \cup \{-1/n^2\}_{n=1}^{\infty}$, by the above.

Points of continuity: The series is continuous at all points at which it converges.

Let x be a point at which f converges. Then by the first part of the proof, x is an element of an open set G. Thus, let $N_{2r}(x) \subset G$, and consider $[x - r, x + r]$. By the above, f converges uniformly on this interval. Additionally, each f_n is continuous on this interval by definition. Thus, by Theorem 7.12, f is continuous at x, as desired.

Boundedness: f is not bounded.

If we suppose for the sake of contradiction that f is bounded by m, we nevertheless find that

$$f(\tfrac{1}{4m^2}) > \sum_{n=1}^{2m} \frac{1}{1+\frac{n^2}{4m^2}} = \sum_{n=1}^{2m} \frac{(2m)^2}{(2m)^2+n^2} \geq \sum_{n=1}^{2m} \frac{(2m)^2}{(2m)^2+(2m)^2} = \sum_{n=1}^{2m} \frac{1}{2} = m$$

□

7. For $n = 1, 2, 3, \dots$ and x real, put

$$f_n(x) = \frac{x}{1+nx^2}$$

Show that $\{f_n\}$ converges uniformly to a function f and that the equation

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

is correct if $x \neq 0$ but false if $x = 0$.

Proof. To prove that $\{f_n\}$ converges uniformly to f defined by $f(x) = 0$ $(x \in \mathbb{R})$, Theorem 7.9 tells us that it will suffice to show that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$ and that the sequence $\{M_n\}$ defined by $M_n = \sup_{x\in\mathbb{R}} |f_n(x)|$ tends to zero as $n \to \infty$. Since

$$f_n(x) = \frac{x}{1+nx^2} < \frac{x}{nx^2} = \frac{1}{x}\cdot\frac{1}{n} \to 0$$

as $n \to \infty$ for all $x \neq 0$ and $f_n(0) = 0$ for all n, $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$, as desired. Additionally, by the Schwarz inequality, if a_1, a_2, b_1, b_2 are real numbers, then

$$|a_1b_1 + a_2b_2|^2 \leq (|a_1|^2 + |a_2|^2)(|b_1|^2 + |b_2|^2)$$

It follows that

$$\begin{aligned}
|2\sqrt{n}x|^2 = |\underbrace{1}_{a_1}\cdot\underbrace{\sqrt{n}x}_{b_1} + \underbrace{\sqrt{n}x}_{a_2}\cdot\underbrace{1}_{b_2}|^2 \leq (|1|^2 + |\sqrt{n}x|^2)(|\sqrt{n}x|^2 + |1|^2) &= (1+nx^2)^2 \\
|2\sqrt{n}x| &\leq |1+nx^2| \\
\frac{1}{|1+nx^2|} &\leq \frac{1}{2\sqrt{n}|x|} \\
\frac{|x|}{|1+nx^2|} &\leq \frac{1}{2\sqrt{n}} \\
\left|\frac{x}{1+nx^2}\right| &\leq \frac{1}{2\sqrt{n}}
\end{aligned}$$

for all $x \neq 0$, $n \in \mathbb{N}$. This combined with the facts that $f_n(0) = 0 < \frac{1}{2\sqrt{n}}$ for all $n \in \mathbb{N}$ and $f_n(1/\sqrt{n}) = 1/2\sqrt{n}$ for all $n \in \mathbb{N}$ implies that $M_n = 1/2\sqrt{n}$. Thus, $M_n \to 0$ as $n \to \infty$, as desired.

$f'(x) = 0$ for all $x \in \mathbb{R}$. Additionally,

$$f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2} \leq \frac{1-nx^2}{(nx^2)^2} = \frac{1}{x^4}\cdot\frac{1}{n^2} - \frac{1}{x^2}\cdot\frac{1}{n} \to 0$$

as $n \to \infty$ for all $x \neq 0$, as desired. However, $f_n'(0) = 1$ for all $n \in \mathbb{N}$, as desired. □

5 Sequences and Series of Functions II / Functions of Several Variables

From Rudin (1976).

Chapter 7

2/16: **5.** Let

$$f_n(x) = \begin{cases} 0 & x < \frac{1}{n+1} \\ \sin^2 \frac{\pi}{x} & \frac{1}{n+1} \leq x \leq \frac{1}{n} \\ 0 & \frac{1}{n} < x \end{cases}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that absolute convergence, even for all x, does not imply uniform convergence.

Proof. To prove that $\{f_n\}$ converges pointwise to the continuous function f defined by $f(x) = 0$ for all $x \in \mathbb{R}$, it will suffice to show that for every $\epsilon > 0$ and for every $x \in \mathbb{R}$, there exists an integer N such that if $n \geq N$, then $|f_n(x)| < \epsilon$. Let $\epsilon > 0$ and $x \in \mathbb{R}$ be arbitrary. We divide into three cases ($x \in \{1/n\}_{n=1}^\infty$, $x \in [0,1] \setminus \{1/n\}_{n=1}^\infty$, and $x \notin [0,1]$).

If $x \in \{1/n\}_{n=1}^\infty$, let $x = 1/k$. Then by the definition of $f_n(x)$, we have that

$$f_i(x) = \begin{cases} 0 & i < k-1 \\ \sin^2 \frac{\pi}{1/k} = \sin^2 k\pi = 0 & i = k-1, k \\ 0 & i > k \end{cases}$$

Thus, choose $N = 1$. It follows that if $n \geq N$, then

$$|f_n(x)| = 0 < \epsilon$$

as desired.

If $x \in [0,1] \setminus \{1/n\}_{n=1}^\infty$, let $x \in (1/[(N-1)+1], 1/(N-1))$ where $N \in \mathbb{N}$. Choose this N to be our N. It follows that if $n \geq N$, then

$$\frac{1}{n} \leq \frac{1}{N} = \frac{1}{(N-1)+1} < x$$

so by definition,

$$|f_n(x)| = 0 < \epsilon$$

as desired.

If $x \notin [0,1]$, then either $x < 1/(n+1)$ for all $n \in \mathbb{N}$ or $x > 1/n$ for all $n \in \mathbb{N}$. Either way, we choosing $N = 1$ yields that if $n \geq N$, then

$$|f_n(x)| = 0 < \epsilon$$

as desired.

To prove that $\{f_n\}$ does not converge uniformly to f, Theorem 7.9 tells us that it will suffice to show that if $M_n = \sup_{x \in \mathbb{R}} |f_n(x) - f(x)|$, then $M_n \nrightarrow 0$ as $n \to \infty$. Let $n \in \mathbb{N}$ be arbitrary. Since $n < n + 1/2 < n + 1$ and hence $1/(n+1) \leq 2/(2n+1) \leq 1/n$, we have by the properties of the sine function that

$$f_n(\tfrac{2}{2n+1}) = \sin^2 \left[\frac{\pi}{2/(2n+1)} \right] = \sin^2 \left[\frac{2n+1}{2} \pi \right] = \sin^2 \left[\left(n + \frac{1}{2} \right) \pi \right] = 1$$

and that $f_n(x) \leq 1$ everywhere else. Thus, $M_n = 1$ for all $n \in \mathbb{N}$. But then $M_n \nrightarrow 0$ as $n \to \infty$, as desired.

It follows by an argument symmetric to the above that while $\sum f_n$ converges absolutely to

$$f(x) = \begin{cases} 0 & x \leq 0 \\ \sin^2 \frac{\pi}{x} & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$$

$M_n = 1$ for all $n \in \mathbb{N}$. ☐

6. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

Proof. Let $[a, b]$ be an arbitrary bounded interval, and let $f_n(x) = (-1)^n \frac{x^2+n}{n^2}$. To prove that the series converges uniformly on $[a, b]$, Theorem 7.8 tells us that it will suffice to show that for every $\epsilon > 0$, there exists an N such that if $n, m \geq N$ (WLOG let $n \leq m$) and $x \in [a, b]$, then

$$\left| \sum_{i=n}^{m} f_i(x) \right| < \epsilon$$

Let $\epsilon > 0$ be arbitrary. Define $m = \max(|a|, |b|)$ (note that since $a \neq b$ by definition, $m > 0$). By consecutive applications of Theorem 3.43, we know that both $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converge. Thus, by consecutive applications of Theorem 3.22, there exist integers N_1, N_2 such that $m \geq n \geq N_1$ implies the left result below and $m \geq n \geq N_2$ implies the right result below.

$$\left| \sum_{k=n}^{m} (-1)^k \frac{1}{k^2} \right| < \frac{\epsilon}{2m^2} \qquad \left| \sum_{k=n}^{m} (-1)^k \frac{1}{k} \right| < \frac{\epsilon}{2}$$

Choose $N = \max(N_1, N_2)$. Now let $n, m \geq N$ with WLOG $n \leq m$, and let $x \in [a, b]$. It follows that

$$\begin{aligned}
\left| \sum_{k=n}^{m} f_k(x) \right| &= \left| \sum_{k=n}^{m} (-1)^k \frac{x^2 + k}{k^2} \right| \\
&= \left| x^2 \sum_{k=n}^{m} (-1)^k \frac{1}{k^2} + \sum_{k=n}^{m} (-1)^k \frac{1}{k} \right| \\
&\leq |x^2| \cdot \left| \sum_{k=n}^{m} (-1)^k \frac{1}{k^2} \right| + \left| \sum_{k=n}^{m} (-1)^k \frac{1}{k} \right| \\
&\leq m^2 \cdot \left| \sum_{k=n}^{m} (-1)^k \frac{1}{k^2} \right| + \left| \sum_{k=n}^{m} (-1)^k \frac{1}{k} \right| \\
&< m^2 \cdot \frac{\epsilon}{2m^2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

as desired.

To prove that the series does not converge absolutely for any value of x, let $x \in \mathbb{R}$ be arbitrary. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} \left| (-1)^n \frac{x^2 + n}{n^2} \right| &= x^2 \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n} \\
&\geq \sum_{n=1}^{\infty} \frac{1}{n}
\end{aligned}$$

where the latter series diverges by Theorem 3.28, yielding the desired result. ☐

8. If

$$I(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$$

if $\{x_n\}$ is a sequence of distinct points of (a,b), and if $\sum |c_n|$ converges, prove that the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n)$$

converges uniformly on $[a,b]$, and that f is continuous for every $x \neq x_n$.

Proof. Let $f_n(x) = c_n I(x - x_n)$ for all $n \in \mathbb{N}$. To prove that f converges uniformly on $[a,b]$, Theorem 7.10 tells us that it will suffice to show that $|f_n(x)| \le M_n$ for all $x \in [a,b]$ and $\sum M_n$ converges. Let $M_n = c_n$ for all $n \in \mathbb{N}$. Then for any $x \in [a,b]$,

$$|f_n(x)| = c_n I(x - x_n) \le c_n = M_n$$

as desired. Additionally, $\sum M_n = \sum c_n$ converges, as desired. This completes the proof.

For the second part of the proof, let $x \notin \{x_n\}$. Then every f_n is continuous at x by definition. Thus, f is a uniformly convergent sequence of functions continuous at x, so by Theorem 7.12, f is continuous at x. □

9. Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \to x$ and $x \in E$. Is the converse of this true?

Proof. Let $\{x_n\} \subset E$ be an arbitrary sequence of points that converges to some $x \in E$. To prove that $\lim_{n\to\infty} f_n(x_n) = f(x)$, it will suffice to show that for every $\epsilon > 0$, there exists an N such that if $n \ge N$, then $|f_n(x_n) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\{f_n\}$ is a uniformly convergent sequence of continuous functions, Theorem 7.12 implies that f is a continuous function. Thus, there exists a $\delta > 0$ such that if $y \in E$ and $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon/2$. Additionally, since $x_n \to x$, there exists an N_1 such that if $n \ge N_1$, $|x_n - x| < \delta$. Furthermore, since f_n converges uniformly to f, there exists N_2 such that if $n \ge N_2$, then $|f_n(y) - f(y)| < \epsilon/2$ for all $y \in E$. In particular, $|f_n(x_n) - f(x_n)| < \epsilon/2$. Choose $N = \max(N_1, N_2)$. Let $n \ge N$ be arbitrary. Then

$$\begin{aligned} |f_n(x_n) - f(x)| &\le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

as desired.

No, it is not true in general that if $\{f_n\}$ is a sequence of continuous functions for which $\lim_{n\to\infty} f_n(x_n) = f(x)$ for every sequence of points $x_n \in E$ such that $x_n \to x$ and $x \in E$, then f_n converges uniformly. Consider the sequence of functions from Exercise 7.5. This is a sequence of continuous functions for which $\lim_{n\to\infty} f_n(x_n) = f(x)$ for any sequence $\{x_n\}$ of the desired type since we can always choose N large enough so that the moving "hump" and neighborhood of x containing all remaining x_n are separated forever more. Moreover, by Exercise 7.5, $\{f_n\}$ does not converge uniformly, as desired. □

Chapter 9

1. If S is a nonempty subset of a vector space X, prove (as asserted in Section 9.1) that the span of S is a vector space.

Proof. Let $S = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ (the proof is symmetric if S is infinite).

To prove that $\operatorname{span}(S)$ is a vector space, it will suffice to show that $\operatorname{span}(S)$ is nonempty and that for all $\mathbf{x}, \mathbf{y} \in \operatorname{span}(S)$ and $c \in \mathbb{C}$, $(\mathbf{x} + \mathbf{y}) \in \operatorname{span}(S)$ and $c\mathbf{x} \in \operatorname{span}(S)$. Since S is nonempty, there exists $\mathbf{x} \in S$; thus, $1\mathbf{x} \in \operatorname{span}(S)$, so $\operatorname{span}(S)$ is nonempty, as desired. Let $\mathbf{x}, \mathbf{y} \in \operatorname{span}(S)$ and $c \in \mathbb{C}$. There exist $a_1, \dots, a_n, b_1, \dots, b_n$ such that

$$\mathbf{x} = a_1\mathbf{u}_1 + \cdots + a_n\mathbf{u}_n \qquad \mathbf{y} = b_1\mathbf{u}_1 + \cdots + b_n\mathbf{u}_n$$

It follows by the definition of $\operatorname{span}(S)$ that

$$\begin{aligned} (a_1 + b_1)\mathbf{u}_1 + \cdots + (a_n + b_n)\mathbf{u}_n &= \mathbf{x} + \mathbf{y} \in \operatorname{span}(S) \\ ca_1\mathbf{u}_1 + \cdots + ca_n\mathbf{u}_n &= c\mathbf{x} \in \operatorname{span}(S) \end{aligned}$$

as desired. □

2. Prove (as asserted in Section 9.6) that BA is linear if A and B are linear transformations. Prove also that A^{-1} is linear and invertible.

Proof. Let $A \in L(X, Y)$ and $B \in L(Y, Z)$. Then for all $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2 \in X$ and $c \in \mathbb{C}$,

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 \qquad A(c\mathbf{x}) = cA\mathbf{x}$$

and for all $\mathbf{y}, \mathbf{y}_1, \mathbf{y}_2 \in Y$ and $c \in \mathbb{C}$,

$$B(\mathbf{y}_1 + \mathbf{y}_2) = B\mathbf{y}_1 + B\mathbf{y}_2 \qquad B(c\mathbf{y}) = cB\mathbf{y}$$

It follows that for any $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2 \in X$ and $c \in \mathbb{C}$, we have that

$$\begin{aligned} BA(\mathbf{x}_1 + \mathbf{x}_2) &= B(A\mathbf{x}_1 + A\mathbf{x}_2) \\ &= BA\mathbf{x}_1 + BA\mathbf{x}_2 \end{aligned} \qquad \begin{aligned} BA(c\mathbf{x}) &= B(cA\mathbf{x}) \\ &= cBA\mathbf{x} \end{aligned}$$

so BA is a linear transformation, as desired.

Let $A \in L(X, Y)$ be invertible. Since A is a linear transformation, the same equalities from above still apply. Thus,

$$\begin{aligned} \mathbf{x}_1 + \mathbf{x}_2 &= \mathbf{x}_1 + \mathbf{x}_2 \\ I(\mathbf{x}_1 + \mathbf{x}_2) &= I\mathbf{x}_1 + I\mathbf{x}_2 \\ AA^{-1}(\mathbf{x}_1 + \mathbf{x}_2) &= AA^{-1}\mathbf{x}_1 + AA^{-1}\mathbf{x}_2 \\ A(A^{-1}(\mathbf{x}_1 + \mathbf{x}_2)) &= A(A^{-1}\mathbf{x}_1 + A^{-1}\mathbf{x}_2) \\ A^{-1}(\mathbf{x}_1 + \mathbf{x}_2) &= A^{-1}\mathbf{x}_1 + A^{-1}\mathbf{x}_2 \end{aligned} \qquad \begin{aligned} c\mathbf{x} &= c\mathbf{x} \\ I(c\mathbf{x}) &= cI\mathbf{x} \\ AA^{-1}(c\mathbf{x}) &= cAA^{-1}\mathbf{x} \\ A(A^{-1}(c\mathbf{x})) &= A(cA^{-1}\mathbf{x}) \\ A^{-1}(c\mathbf{x}) &= cA^{-1}\mathbf{x} \end{aligned}$$

where we use the fact that A is one-to-one for the last equality in both cases. To prove that A^{-1} is invertible, it will suffice to show that it is one-to-one and onto. Suppose $A^{-1}\mathbf{x} = A^{-1}\mathbf{y}$. Then

$$\begin{aligned} AA^{-1}\mathbf{x} &= AA^{-1}\mathbf{y} \\ I\mathbf{x} &= I\mathbf{y} \\ \mathbf{x} &= \mathbf{y} \end{aligned}$$

proving that A^{-1} is one-to-one, as desired. Now suppose $\mathbf{y} \in X$. Then $A\mathbf{y} = \mathbf{x}$ for some $\mathbf{x} \in X$. It follows that

$$A^{-1}\mathbf{x} = A^{-1}A\mathbf{y} = I\mathbf{y} = \mathbf{y}$$

proving that A^{-1} is onto, as desired. □

3. Assume $A \in L(X, Y)$ and $A\mathbf{x} = \mathbf{0}$ only when $\mathbf{x} = \mathbf{0}$. Prove that A is then 1-1.

Proof. If we suppose that $A\mathbf{x} = A\mathbf{y}$, then by linearity,

$$\begin{aligned}\mathbf{0} &= A\mathbf{x} - A\mathbf{y}\\ &= A(\mathbf{x} - \mathbf{y})\end{aligned}$$

It follows by hypothesis that $\mathbf{x} - \mathbf{y} = \mathbf{0}$, hence $\mathbf{x} = \mathbf{y}$, proving that A is 1-1, as desired. □

4. Prove (as asserted in Section 9.30) that null spaces and ranges of linear transformations are vector spaces.

Proof. Let $A \in L(X, Y)$.

Suppose $\mathbf{x}_1, \mathbf{x}_2 \in \operatorname{null} A$. Then $A\mathbf{x}_1 = \mathbf{0}$ and $A\mathbf{x}_2 = \mathbf{0}$. It follows that

$$\begin{aligned}\mathbf{0} &= A\mathbf{x}_1 + A\mathbf{x}_2\\ &= A(\mathbf{x}_1 + \mathbf{x}_2)\end{aligned}$$

so $(\mathbf{x}_1 + \mathbf{x}_2) \in \operatorname{null} A$, as desired.

Suppose $\mathbf{x} \in \operatorname{null} A$ and $c \in \mathbb{C}$. Then $A\mathbf{x} = \mathbf{0}$. It follows that

$$\begin{aligned}\mathbf{0} &= c \cdot \mathbf{0}\\ &= cA\mathbf{x}\\ &= A(c\mathbf{x})\end{aligned}$$

so $c\mathbf{x} \in \operatorname{null} A$, as desired.

Suppose $\mathbf{y}_1, \mathbf{y}_2 \in \operatorname{range} A$. Then there exist $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that $A\mathbf{x}_1 = \mathbf{y}_1$ and $A\mathbf{x}_2 = \mathbf{y}_2$. It follows that

$$\begin{aligned}A(\mathbf{x}_1 + \mathbf{x}_2) &= A\mathbf{x}_1 + A\mathbf{x}_2\\ &= \mathbf{y}_1 + \mathbf{y}_2\end{aligned}$$

so $(\mathbf{y}_1 + \mathbf{y}_2) \in \operatorname{range} A$, as desired.

Suppose $\mathbf{y} \in \operatorname{range} A$ and $c \in \mathbb{C}$. Then there exists $\mathbf{x} \in X$ such that $A\mathbf{x} = \mathbf{y}$. It follows that

$$\begin{aligned}A(c\mathbf{x}) &= cA\mathbf{x}\\ &= c\mathbf{y}\end{aligned}$$

so $c\mathbf{y} \in \operatorname{range} A$, as desired. □

References

Rudin, W. (1976). *Principles of mathematical analysis* (A. A. Arthur & S. L. Langman, Eds.; Third). McGraw-Hill.