

# MATH 20410 (Analysis in $\mathbb{R}^n$ II – Accelerated) Notes

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## Chapter 6

# The Riemann-Stieltjes Integral

### 6.1 Notes

1/28:

- Plan:

1. Finish up Fundamental Theorem of Calculus proof.
2. Basic consequences.
3. Rectifiable curves.

- Recall that we're given  $f : [a, b] \rightarrow \mathbb{R}$  continuous,  $f : [a, b] \rightarrow \mathbb{R}$ , and  $x \mapsto \int_a^x f(t) dt$ .

- Goal: Show  $F'(x_0) = f(x_0)$ .

- WTS: Find  $\delta$  such that  $|x - x_0| < \delta$  implies

$$\begin{aligned} \left| \frac{1}{x - x_0} \int_{x_0}^x f(t) dt - f(x_0) \right| &= \left| \frac{1}{x - x_0} \int_{x_0}^x f(t) dt - \frac{1}{x - x_0} \int_{x_0}^x f(x_0) dt \right| \\ &= \frac{1}{|x - x_0|} \left| \int_{x_0}^x (f(t) - f(x_0)) dt \right| \\ &\leq \frac{1}{|x - x_0|} \int_{x_0}^x |f(t) - f(x_0)| dt \\ &< \epsilon \end{aligned}$$

- Since  $f$  is continuous, there exists  $\delta$  such that if  $|x - x_0| < \delta$ , then  $|f(x) - f(x_0)| < \epsilon$ .

- Now

$$\begin{aligned} \frac{1}{|x - x_0|} \int_{x_0}^x |f(t) - f(x_0)| dt &< \frac{1}{|x - x_0|} \int_{x_0}^x \epsilon dt \\ &= \epsilon \end{aligned}$$

- Applications:

1. Theorem (MVT for integration):  $f : [a, b] \rightarrow \mathbb{R}$  continuous, then there exists  $x_0 \in [a, b]$  such that

$$f(x_0) = \frac{1}{b - a} \int_a^b f(x) dx$$

- Apply MVT to  $F(x) = \int_a^x f(t) dt$ . Then

$$F'(x_0) = f(x_0) = \frac{F(b) - F(a)}{b - a}$$

as desired.

2. Theorem (Integration by parts): Let  $F, G : [a, b] \rightarrow \mathbb{R}$  be differentiable with  $F' = f$ ,  $G' = g$  and with  $f$  and  $g$  both integrable. Then

$$\int_a^b Fg = F(b)G(b) - F(a)G(a) - \int_a^b fG$$

- Just use the product rule plus the FTC to prove.
- We have

$$\begin{aligned} \int_a^b (FG)' &= \int_a^b fG + \int_a^b Fg \\ F(b)G(b) - F(a)G(a) &= \int_a^b fG + \int_a^b Fg \\ \int_a^b Fg &= F(b)G(b) - F(a)G(a) - \int_a^b fG \end{aligned}$$

3. Theorem ( $u$ -substitution).

- Follows similarly from the chain rule and FTC.

- Integration of vector-valued functions.

- If  $f : [a, b] \rightarrow \mathbb{R}^k$ , we define  $\int_a^b f$  by

$$\int_a^b f = \left( \int_a^b f_1, \dots, \int_a^b f_k \right)$$

- Alternatively, you can define  $\int_a^b f$  using  $P$ ,  $U(f, P)$ ,  $L(f, P)$ , etc. and then prove that the integral exists iff all  $f_i$  are integrable and in this case the above definition holds.
- Rectifiable curves: Let  $\gamma : [a, b] \rightarrow \mathbb{R}^k$  be a continuous function.
- Plan: Define the length of  $\gamma$  and show that we can compute it with an integral.
  - Idea: For polygonal paths, we know how to define length. So let's approximate  $\gamma$  by polygons and take a limit.
  - Ref: Given a partition  $P$ , then define the length of  $\gamma$  with respect to  $P$  as  $\Lambda(\gamma, P)$ . Let the length of  $\gamma$  be  $\Lambda(\gamma) = \sup_P \Lambda(\gamma, P)$  if this limit exists in this case, we call  $\gamma$  **rectifiable**.
- Fractals are not rectifiable — their length diverges.
- Theorem: Suppose  $\gamma$  is continuously differentiable (i.e.,  $\gamma$  is differentiable and  $\gamma'$  is continuous). Then  $\gamma$  is rectifiable and

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)| dt$$

- Notice: If  $P \leq P'$ , then  $\Lambda(\gamma, P) \leq \Lambda(\gamma, P')$ . (Prove with triangle inequality.)
- WTS: For all partitions  $P$ ,  $\Lambda(\gamma, P) \leq \int_a^b |\gamma'(t)| dt$  and thus  $\Lambda(\gamma) \leq \int_a^b |\gamma'(t)| dt$ .
- We have that

$$\begin{aligned} \Lambda(\gamma, P) &= \sum_{i=1}^n |\gamma(x_i) - \gamma(x_{i-1})| \\ &= \sum_{i=1}^n \left| \int_{x_{i-1}}^{x_i} \gamma'(t) dt \right| \\ &\leq \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |\gamma'(t)| dt \\ &= \int_a^b |\gamma'(t)| dt \end{aligned}$$

- Catch up.
  - I should make up PSets 1-2.
  - Exams have less than Rudin-strength problems.
  - Exams are mostly true/false (and of that, mostly false, provide a counterexample).

# Chapter 7

## Sequences and Series of Functions

### 7.1 Notes

- 1/31:
- Midterm on differentiation and integration, and a bit of stuff from this week.
  - Plan:
    - Talk about sequences of functions, all with the same domain and range, converging.
    - Address what properties of  $f_n$  remain in the limit (e.g., continuity, differentiability, integrability).
      - The answer depends on what we mean by “convergence.”
      - $f_n \rightarrow f$  pointwise implies basically nothing.
      - $f_n \rightarrow f$  uniformly implies that basically everything works out nicely.
  - We’ll restrict ourselves to real functions because those have all the properties (integrability, differentiability, etc.) that we care about.
  - **Pointwise** (convergent sequence  $\{f_n\}$  to  $f$ ): A sequence of functions  $\{f_n\}$  such that for all  $x \in X$ , the sequence  $\{f_n(x)\}$  converges to  $f(x)$ , where  $f_n : X \rightarrow \mathbb{R}$  for all  $n \in \mathbb{N}$  and  $f : X \rightarrow \mathbb{R}$ . Denoted by  $f_n \rightarrow f$ .
  - Bad functions.
    - Consider  $f_n : [0, 1] \rightarrow \mathbb{R}$  defined by  $x \mapsto x^n$ . Each  $f_n$  is continuous, but  $f$  is not (zero everywhere except  $f(1) = 1$ )<sup>[1]</sup>.
    - Consider  $f_n : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f_n(x) = x^2/(1 + x^2)^n$ , and  $f(x) = \sum_{n=0}^{\infty} f_n(x)$ . As a geometric series,  $f(x) = 1 + x^2$  when  $x \neq 0$  but  $f(0) = 0$ . Thus, the limit exists but is not continuous once again.
    - Consider  $f_m : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $x \mapsto \lim_{n \rightarrow \infty} \cos^{2n}(m!\pi x)$ . Each  $f_m$  is integrable, but the limit  $f$  is the function that’s 1 for rationals and zero for irrationals. In particular,  $f$  is not integrable.
      - We take even powers of the cosine to make it always positive.
      - We use  $\cos^2(x)$  just because it’s always between  $[0, 1]$ , and we know when it is equal to 1.
      - In particular,  $\cos^2(\pi x)$  is equal to 1 at every integer,  $\cos^2(2\pi x)$  is equal to 1 at every half integer.  $\cos^2(6\pi x)$  is equal to 1 at every one-sixth of an integer.
      - Then raising it to the  $n^{\text{th}}$  power just makes it spiky.
  - Aside: Interchanging limits.
    - If all  $f_n$  are continuous, then  $\lim_{x \rightarrow x_0} f_n(x) = f_n(x_0)$ .

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<sup>1</sup>Questions that require counterexamples like this could show up on the midterm!

- The question “is  $f$  continuous” is equivalent to being able to interchange limits:

$$\lim_{x \rightarrow x_0} \lim_{n \rightarrow \infty} f_n(x) = f(x_0) = \lim_{n \rightarrow \infty} \lim_{x \rightarrow x_0} f_n(x)$$

- Sequence example showing we need to be careful interchanging limits:  $s_{n,m} = m/(m+n)$ .
- All of this pathology goes away with the right definition, though.
- **Uniformly** (convergent sequence  $\{f_n\}$  to  $f$ ): A sequence of functions  $\{f_n\}$  such that for all  $\epsilon > 0$ , there exists an  $N$  such that if  $n \geq N$ , then  $|f_n(x) - f(x)| < \epsilon$  for all  $x \in X$ , where  $f_n : X \rightarrow \mathbb{R}$  for all  $n \in \mathbb{N}$  and  $f : X \rightarrow \mathbb{R}$ .
- Proposition (Cauchy criterion for uniform convergence):  $f_n \rightarrow f$  uniformly iff for all  $\epsilon > 0$ , there exists  $N$  such that for all  $m, n \geq N$  and for all  $x \in X$ ,  $|f_n(x) - f_m(x)| < \epsilon$ .
  - Forward direction: Let  $\epsilon > 0$ . Suppose  $f_n \rightarrow f$  uniformly. Choose  $N$  such that the functions are within  $\epsilon/2$ . Then

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

2/2:

- Office hours tomorrow 4-5 PM.
- Plan:
  1. More on uniform convergence.
    - Limit of continuous functions is continuous.
    - Limit of the integral of functions is the integral of the limit.
  2.  $\mathcal{C}(X)$  perspectives on uniform convergence.
- Corollary (Weierstraß M-test): If there exist constants  $M_n \in \mathbb{R}$  such that  $|f_n(x)| \leq M_n$  for all  $x$  and  $\sum_{n=1}^{\infty} M_n$  converges, then  $\sum_{n=1}^{\infty} f_n(x)$  converges uniformly.
- Theorem:  $f_n : X \rightarrow \mathbb{R}$ ,  $f_n$  continuous at  $x_0 \in X$  for all  $n$ , and  $f_n \rightarrow f$  uniformly imply  $f$  continuous at  $x_0$ .
  - Idea:
    - “ $\epsilon/3$  trick”: Find  $\delta$  such that if  $|x - x_0| < \delta$ , then

$$|f(x) - f(x_0)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

- Proof:

- $f_n \rightarrow f$  uniformly implies there exists  $N \in \mathbb{N}$  such that  $|f_N(x) - f(x)| < \epsilon/3$  for all  $x \in X$ .
  - $f_N$  continuous at  $x_0$ : There exists  $\delta$  such that if  $d(x, x_0) < \delta$ , then  $|f_N(x) - f_N(x_0)| < \epsilon/3$ .
  - Thus, by the  $\epsilon/3$  trick, we have the continuity of  $f$ .
- Theorem:  $f_n : [a, b] \rightarrow \mathbb{R}$  integrable and  $f_n \rightarrow f$  uniformly implies  $f$  is integrable and

$$\int_a^b f = \lim_{n \rightarrow \infty} \int_a^b f_n$$

- Plan:

1. Show  $f$  is integrable.
2. Show  $\int f = \lim \int f_n$ .

- Proof:

- Fix some partition  $P$  and write

$$\begin{aligned} U(f_n, P) &= \sum_{i=1}^k M_i^n \Delta x_i & L(f_n, P) &= \sum_{i=1}^k m_i^n \Delta x_i \\ U(f, P) &= \sum_{i=1}^k M_i \Delta x_i & L(f, P) &= \sum_{i=1}^k m_i \Delta x_i \end{aligned}$$

- WTS: For large enough  $n$ , we can make all  $|M_i^n - M_i| < \epsilon$  and  $|m_i^n - m_i| < \epsilon$ .
- We can do the above by uniform convergence.
- We end up with

$$\begin{aligned} |U(f_n, P) - U(f, P)| &\leq \sum_{i=1}^n |M_i^n - M_i| \Delta x_i \\ &< \epsilon(b - a) \end{aligned}$$

- Klug had assumed that the functions achieve their maxima/minima on the same interval and forgets how to finish the proof.

- Defining a norm on  $\mathcal{C}(X)$ .

$$\|f\| = \sup_{x \in X} |f(x)|$$

- This makes  $\mathcal{C}(X)$  into a vector space.
- We can now define our metric  $d(f, g)$  by  $d(f, g) = \|f - g\|$ .
- $f_n \rightarrow f \iff f$  is bounded.
  - $f_n \rightarrow f$  uniformly  $\iff \lim_{n \rightarrow \infty} \sup |f_n(x) - f(x)| = 0 \iff f_n \rightarrow f$  is  $\mathcal{C}(X)$ .
- Corollary to the Weierstraß M-test:  $\mathcal{C}(X)$  is complete (i.e., all uniformly Cauchy sequences converge).
  - Assume  $\{f_n\}$  is Cauchy. Then by the Cauchy criterion for uniform convergence,  $f_n$  converges uniformly to some  $f$ . But this  $f$  must be continuous, too, meaning  $f \in \mathcal{C}(X)$ .