2 Differentiation II / Integration

From Rudin (1976).

Chapter 5

8. Suppose f' is continuous on $[a,b]$ and $\epsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left|\frac{f(t)-f(x)}{t-x} - f'(x)\right| < \epsilon$$

whenever $0 < |t-x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$. (This could be expressed by saying that f is **uniformly differentiable** on $[a,b]$ if f' is continuous on $[a,b]$.) Does this hold for vector-valued functions, too?

Proof. By Theorem 2.40, $[a,b]$ is compact. This combined with the fact that f' is continuous implies by Theorem 4.19 that f' is uniformly continuous. Thus, there exists $\delta > 0$ such that if $x,y \in [a,b]$ and $|y-x| < \delta$, then $|f'(y) - f'(x)| < \epsilon$. Choose this δ to be our δ. Let $x,t \in [a,b]$ be such that $0 < |t-x| < \delta$. Then since f is continuous on $[t,x] \subset [a,b]$ and differentiable on $(t,x) \subset [a,b]$, we have by the MVT that there exists $c \in (t,x)$ such that

$$\begin{aligned} f(t) - f(x) &= (t-x)f'(c) \\ \frac{f(t)-f(x)}{t-x} &= f'(c) \end{aligned}$$

Additionally, since $t < c < x$ and $|t-x| < \delta$, we must have $|c-x| < \delta$, meaning that

$$\left|\frac{f(t)-f(x)}{t-x} - f'(x)\right| = |f'(c) - f'(x)| < \epsilon$$

as desired.

And yes, this does hold for vector-valued functions, which we can treat component-wise. □

17. Suppose f is a real, three times differentiable function on $[-1,1]$ such that

$$f(-1) = 0 \qquad f(0) = 0 \qquad f(1) = 1 \qquad f'(0) = 0$$

Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1,1)$. Note that equality holds for $\frac{1}{2}(x^3 + x^2)$. (Hint: Use Theorem 5.15 with $\alpha = 0$ and $\beta = \pm 1$ to show that there exist $s \in (0,1)$ and $t \in (-1,0)$ such that $f^{(3)}(s) + f^{(3)}(t) = 6$.)

Proof. Since f is three times differentiable on $[-1,1]$, we know that f'' is differentiable on $[-1,1]$. It follows by Theorem 5.2 that f'' is continuous on $[-1,1]$. Thus, since f is defined on $[-1,1]$, $3 \in \mathbb{N}$, f'' is continuous on $[-1,1]$, $f^{(3)}$ is defined on $(-1,1)$, $0,1 \in [-1,1]$ such that $0 \neq 1$, and we can define

$$P(t) = \sum_{k=0}^{2} \frac{f^{(k)}(0)}{k!}(t-0)^k$$

we have by Taylor's theorem that there exists $s \in (0,1)$ such that

$$\begin{aligned} f(1) &= P(1) + \frac{f^{(3)}(s)}{3!}(1-0)^3 \\ 1 - \left[\frac{f(0)}{0!}(1-0)^0 + \frac{f'(0)}{1!}(1-0)^1 + \frac{f''(0)}{2!}(1-0)^2\right] &= \frac{f^{(3)}(s)}{3!} \\ 1 - \left[\frac{f''(0)}{2}\right] &= \frac{f^{(3)}(s)}{6} \\ 6 - 3f''(0) &= f^{(3)}(s) \end{aligned}$$

Similarly, we have that there exists $t \in (-1,0)$ such that

$$\begin{aligned}
f(-1) &= P(-1) + \frac{f^{(3)}(t)}{3!}(-1-0)^3 \\
0 - \left[\frac{f(0)}{0!}(-1-0)^0 + \frac{f'(0)}{1!}(-1-0)^1 + \frac{f''(0)}{2!}(-1-0)^2\right] &= -\frac{f^{(3)}(t)}{3!} \\
-\left[\frac{f''(0)}{2}\right] &= -\frac{f^{(3)}(t)}{6} \\
3f''(0) &= f^{(3)}(s)
\end{aligned}$$

Thus,

$$f^{(3)}(s) + f^{(3)}(t) = 3f''(0) + 6 - 3f''(0) = 6$$

Now suppose for the sake of contradiction that for all $x \in (-1,1)$, we have $f^{(3)}(x) < 3$. Then $f^{(3)}(s) < 3$ and $f^{(3)}(t) < 3$. It follows that $f^{(3)}(s) + f^{(3)}(t) < 6$, a contradiction. $\square$

25. Suppose f is twice differentiable on $[a,b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geq \delta > 0$, and $0 \leq f''(x) \leq M$ for all $x \in [a,b]$. Let ξ be the unique point in (a,b) at which $f(\xi) = 0$. Complete the details in the following outline of **Newton's method** for computing ξ.

(a) Choose $x_1 \in (\xi, b)$ and define $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Interpret this geometrically, in terms of a tangent to the graph of f.

Answer. Since we can rearrange the above to $0 - f(x_n) = f'(x_n)(x_{n+1} - x_n)$, we know that x_{n+1} is the point at which the tangent to f at x_n crosses the x-axis. In other words, the zero of the tangent line

$$y - f(x_n) = f'(x_n)(x - x_n)$$

is $(x_{n+1}, 0)$. $\square$

(b) Prove that $x_{n+1} < x_n$ and that

$$\lim_{n\to\infty} x_n = \xi$$

Proof. To prove that $x_{n+1} < x_n$, it will suffice to show that $f(x_n), f'(x_n) > 0$. Since $f'(x) > 0$ for all $x \in [a,b]$ by hypothesis, we know that $f'(x_n) > 0$. As to $f(x_n)$, suppose for the sake of contradiction that $f(x_n) \leq 0$. We know that $f(\xi) = 0$, $f(b) > 0$, and $\xi < x_n < b$. Since ξ is the *unique* point at which $f(\xi) = 0$ by hypothesis and $x_n \neq \xi$, we know that $f(x_n) \neq 0$. And if $f(x_n) < 0$, we have by the intermediate value theorem for f continuous that there exists $c \in (x_n, b)$ such that $f(c) = 0$. But since $\xi < x_n < c < b$, $c \neq \xi$, and thus we have a contradiction here, too.

Having established that $\{x_n\}$ is a monotonically decreasing sequence, Theorem 3.14 tells us that to show that it converges, it will suffice to show that it is bounded. Clearly, $\{x_n\}$ is bounded above by x_1. And on the bottom side, $\{x_n\}$ is bounded by ξ: If there were $x_n < \xi$, this would imply that $f(x_n) < 0$ by a symmetric argument to the above, meaning that $f(x_n)/f'(x_n) < 0$ and implying that $x_{n+1} > x_n$, a contradiction. Furthermore, we know that the limit (call it x) equals ξ since

$$\begin{aligned}
x &= x - \frac{f(x)}{f'(x)} \\
f(x) &= 0
\end{aligned}$$

so $x = \xi$ by the uniqueness of ξ. $\square$

(c) Use Taylor's theorem to show that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

for some $t_n \in (\xi, x_n)$.

Proof. Since f is defined on $[a,b]$, $2 \in \mathbb{N}$, f' is continuous on $[a,b]$, f'' is defined on (a,b), $\xi, x_n \in [a,b]$ with $\xi \neq x_n$, and

$$P(t) = \sum_{k=0}^{1} \frac{f^{(k)}(x_n)}{k!}(t - x_n)^k$$

we have by Taylor's theorem that there exists $t_n \in (\xi, x_n)$ such that

$$\begin{aligned}
f(\xi) &= \left[\frac{f(x_n)}{0!}(\xi - x_n)^0 + \frac{f'(x_n)}{1!}(\xi - x_n)^1\right] + \frac{f''(t_n)}{2!}(\xi - x_n)^2 \\
0 &= f(x_n) - f'(x_n)(x_n - \xi) + \frac{f''(t_n)}{2}(x_n - \xi)^2 \\
x_n - \frac{f(x_n)}{f'(x_n)} - \xi &= \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2 \\
x_{n+1} - \xi &= \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2
\end{aligned}$$

as desired. □

(d) If $A = M/2\delta$, deduce that

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A}[A(x_1 - \xi)]^{2n}$$

(Compare with Chapter 3, Exercises 16 and 18.)

Proof. We have from part (b) that $x_i > \xi$ for all $i \in \mathbb{N}$, so naturally $0 \leq x_{n+1} - \xi$. As to the other part of the question, we induct on n. For the base case $n = 1$, we have that

$$\begin{aligned}
x_2 - \xi &= \frac{f''(t_1)}{2f'(x_1)}(x_1 - \xi)^2 \\
&\leq \frac{M}{2\delta}(x_1 - \xi)^2 \\
&= \frac{2\delta}{M}\left[\frac{M}{2\delta}(x_1 - \xi)\right]^2 \\
&= \frac{1}{A}[A(x_1 - \xi)]^{2 \cdot 1}
\end{aligned}$$

Now suppose inductively that we have proven the claim for $n - 1$; we now seek to prove it for n. Indeed, we have that

$$\begin{aligned}
x_{n+1} - \xi &= \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2 \\
&\leq \frac{M}{2\delta}(x_n - \xi)^2 \\
&\leq A\left(\frac{1}{A}[A(x_1 - \xi)]^{2(n-1)}\right)^2 \\
&= \frac{1}{A}[A(x_1 - \xi)]^{2n}
\end{aligned}$$

as desired. □

(e) Show that Newton's method amounts to finding a fixed point of the function g defined by

$$g(x) = x - \frac{f(x)}{f'(x)}$$

How does $g'(x)$ behave for x near ξ?

Proof. A fixed point of the function g is a point x such that

$$\begin{aligned} g(x) &= x \\ x - \frac{f(x)}{f'(x)} &= x \\ f(x) &= 0 \end{aligned}$$

Thus, if we want to find a point x where $f(x) = 0$, it is equivalent to find a point x such that $g(x) = x$.
As to the other part of the question, we have by the rules of derivatives that

$$\begin{aligned} g'(x) &= 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{f'(x)^2} \\ &= \frac{f(x)f''(x)}{f'(x)^2} \\ &\leq \frac{M}{\delta^2} f(x) \end{aligned}$$

Thus, since $f(x) \to 0$ as $x \to \xi$, $g'(x) \to 0$ as $x \to \xi$. ☐

(f) Put $f(x) = \sqrt[3]{x}$ on $(-\infty, \infty)$ and try Newton's method. What happens?

Answer. We have by the power rule that

$$f'(x) = \frac{1}{3x^{2/3}}$$

Choose $x_1 = 1$. Then

$$\begin{aligned} x_2 &= 1 - \frac{f(1)}{f'(1)} = -2 \\ x_3 &= 1 - \frac{f(-2)}{f'(-2)} = 7 \\ x_4 &= 1 - \frac{f(7)}{f'(7)} = -20 \\ &\vdots \end{aligned}$$

It appears that the series is diverging to ∞ while alternating from positive to negative. In fact, since $x_3 > x_2$, contrary to part (b), we know that something must be wrong (i.e., one of our hypotheses must not be met). Upon further investigation, we can determine that on $[-1, 1]$, we have $f''(1) = -2/9 < 0$; thus, our last hypothesis is the issue with this function. ☐

Chapter 6

1. Suppose α increases on $[a, b]$, $a \leq x_0 \leq b$, α is continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathscr{R}(\alpha)$ and that $\int f \, d\alpha = 0$.

Proof. Since f is bounded on $[a,b]$ with only one discontinuity on $[a,b]$ and α is continuous at the point at which f is discontinuous, Theorem 6.10 implies that $f \in \mathscr{R}(\alpha)$, as desired. It follows that $\inf U(P,f,\alpha) = \sup L(P,f,\alpha) = \int f \, \mathrm{d}\alpha$. But since $L(P,f,\alpha) = 0$ for all P (there is no infinite interval $[x_i, x_{i+1}] \subset [a,b]$ that does not contain 0, and f is bounded below by 0), we know that

$$\int f \, \mathrm{d}\alpha = \sup L(P,f,\alpha) = 0$$

as desired. □

2. Suppose $f \geq 0$, f is continuous on $[a,b]$, and $\int_a^b f(x) \, \mathrm{d}x = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$. (Compare this with Exercise 1.)

Proof. Suppose for the sake of contradiction that $f(x) \neq 0$ for some x. By the definition of f, this must mean that $f(x) > 0$. It follows since f is continuous that there exists some $N_r(x)$ such that $f(y) > 0$ for all $y \in N_r(x)$. Now consider the partition

$$P = \{a, x - r/2, x + r/2, b\}$$

of $[a,b]$. But since $m_2 > 0$, we have that

$$\begin{aligned}
0 &< m_1[(x - r/2) - a] + m_2[(x + r/2) - (x - r/2)] + m_3[b - (x + r/2)] \\
&= L(P,f) \\
&\leq \int_a^b f(x) \, \mathrm{d}x && \text{Theorem 6.4}
\end{aligned}$$

a contradiction. □

4. If $f(x) = 0$ for all irrational x and $f(x) = 1$ for all rational x, prove that $f \notin \mathscr{R}$ on $[a,b]$ for any $a < b$.

Proof. Let P be an arbitrary partition of $[a,b]$. Since the rationals and irrationals are dense in the reals, we know that for any $[x_i, x_{i+1}]$, $f(x) = 0$ for some $x \in [x_i, x_{i+1}]$ and $f(x) = 1$ for some $x \in [x_i, x_{i+1}]$. Thus, we have that $L(P,f) = 0$ and $U(P,f) = b - a$. It follows that if $a < b$,

$$\sup L(P,f) = 0 \neq b - a = \inf U(P,f)$$

so $f \notin \mathscr{R}$, as desired. □