1 Differentiation II / Integration

From Rudin (1976).

Chapter 5

8. Suppose f' is continuous on $[a,b]$ and $\epsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left|\frac{f(t)-f(x)}{t-x} - f'(x)\right| < \epsilon$$

whenever $0 < |t-x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$. (This could be expressed by saying that f is **uniformly differentiable** on $[a,b]$ if f' is continuous on $[a,b]$.) Does this hold for vector-valued functions, too?

17. Suppose f is a real, three times differentiable function on $[-1,1]$ such that

$$f(-1)=0 \qquad f(0)=0 \qquad f(1)=1 \qquad f'(0)=0$$

Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1,1)$. Note that equality holds for $\frac{1}{2}(x^3+x^2)$. (Hint: Use Theorem 5.15 with $\alpha = 0$ and $\beta = \pm 1$ to show that there exist $s \in (0,1)$ and $t \in (-1,0)$ such that $f^{(3)}(s) + f^{(3)}(t) = 6$.)

25. Suppose f is twice differentiable on $[a,b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geq \delta > 0$, and $0 \leq f''(x) \leq M$ for all $x \in [a,b]$. Let ξ be the unique point in (a,b) at which $f(\xi) = 0$. Complete the details in the following outline of **Newton's method** for computing ξ.

(a) Choose $x_1 \in (\xi, b)$ and define $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Interpret this geometrically, in terms of a tangent to the graph of f.

(b) Prove that $x_{n+1} < x_n$ and that

$$\lim_{n\to\infty} x_n = \xi$$

(c) Use Taylor's theorem to show that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

for some $t_n \in (\xi, x_n)$.

(d) If $A = M/2\delta$, deduce that

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A}[A(x_1 - \xi)]^{2n}$$

(Compare with Chapter 3, Exercises 16 and 18.)

(e) Show that Newton's method amounts to finding a fixed point of the function g defined by

$$g(x) = x - \frac{f(x)}{f'(x)}$$

How does $g'(x)$ behave for x near ξ?

(f) Put $f(x) = \sqrt[3]{x}$ on $(-\infty, \infty)$ and try Newton's method. What happens?

Chapter 6

1. Suppose α increases on $[a,b]$, $a \leq x_0 \leq b$, α is continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathscr{R}(\alpha)$ and that $\int f \, d\alpha = 0$.

2. Suppose $f \geq 0$, f is continuous on $[a,b]$, and $\int_a^b f(x)\,dx = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$. (Compare this with Exercise 1.)

4. If $f(x) = 0$ for all irrational x and $f(x) = 1$ for all rational x, prove that $f \notin \mathscr{R}$ on $[a,b]$ for any $a < b$.