MATH 20410 (Analysis in $\mathbb{R}^n$ II – Accelerated) Notes

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Contents

Chapter 6

The Riemann-Stieltjes Integral

6.1 Notes

1/28:

- Plan:
  1. Finish up Fundamental Theorem of Calculus proof.
  2. Basic consequences.
  3. Rectifiable curves.
- Recall that we're given $f:[a,b]\to\mathbb{R}$ continuous, $f:[a,b]\to\mathbb{R}$, and $x\mapsto\int_a^x f(t)\,\mathrm{d}t$.
- Goal: Show $F'(x_0)=f(x_0)$.
  - WTS: Find δ such that $|x-x_0|<\delta$ implies

$$\begin{aligned}\left|\frac{1}{x-x_0}\int_{x_0}^x f(t)\,\mathrm{d}t-f(x_0)\right| &= \left|\frac{1}{x-x_0}\int_{x_0}^x f(t)\,\mathrm{d}t-\frac{1}{x-x_0}\int_{x_0}^x f(x_0)\,\mathrm{d}t\right|\\ &= \frac{1}{|x-x_0|}\left|\int_{x_0}^x (f(t)-f(x_0))\,\mathrm{d}t\right|\\ &\leq \frac{1}{|x-x_0|}\int_{x_0}^x |f(t)-f(x_0)|\,\mathrm{d}t\\ &< \epsilon\end{aligned}$$

  - Since f is continuous, there exists δ such that if $|x-x_0|<\delta$, then $|f(x)-f(x_0)|<\epsilon$.
  - Now

$$\begin{aligned}\frac{1}{|x-x_0|}\int_{x_0}^x |f(t)-f(x_0)|\,\mathrm{d}t &< \frac{1}{|x-x_0|}\int_{x_0}^x \epsilon\,\mathrm{d}t\\ &= \epsilon\end{aligned}$$

- Applications:
  1. Theorem (MVT for integration): $f:[a,b]\to\mathbb{R}$ continuous, then there exists $x_0\in[a,b]$ such that

$$f(x_0)=\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

     - Apply MVT to $F(x)=\int_a^x f(t)\,\mathrm{d}t$. Then

$$F'(x_0)=f(x_0)=\frac{F(b)-F(a)}{b-a}$$

       as desired.

2. Theorem (Integration by parts): Let $F, G : [a, b] \to \mathbb{R}$ be differentiable with $F' = f$, $G' = g$ and with f and g both integrable. Then

$$\int_a^b Fg = F(b)G(b) - F(a)G(a) - \int_a^b fG$$

   - Just use the product rule plus the FTC to prove.
   - We have

$$\begin{aligned} \int_a^b (FG)' &= \int_a^b fG + \int_a^b Fg \\ F(b)G(b) - F(a)G(a) &= \int_a^b fG + \int_a^b Fg \\ \int_a^b Fg &= F(b)G(b) - F(a)G(a) - \int_a^b fG \end{aligned}$$

3. Theorem (u-substitution).
   - Follows similarly from the chain rule and FTC.

- Integration of vector-valued functions.
- If $f : [a, b] \to \mathbb{R}^k$, we define $\int_a^b f$ by

$$\int_a^b f = \left(\int_a^b f_1, \dots, \int_a^b f_k \right)$$

- Alternatively, you can define $\int_a^b f$ using P, $U(f, P)$, $L(f, P)$, etc. and then prove that the integral exists iff all f_i are integrable and in this case the above definition holds.
- Rectifiable curves: Let $\gamma : [a, b] \to \mathbb{R}^k$ be a continuous function.
- Plan: Define the length of γ and show that we can compute it with an integral.
  - Idea: For polygonal paths, we know how to define length. So let's approximate γ by polygons and take a limit.
  - Ref: Given a partition P, then define the length of γ with respect to P as $\Lambda(\gamma, P)$. Let the length of γ be $\Lambda(\gamma) = \sup_P \Lambda(\gamma, P)$ if this limit exists in this case, we call γ **rectifiable**.
- Fractals are not rectifiable — their length diverges.
- Theorem: Suppose γ is continuously differentiable (i.e., γ is differentiable and γ' is continuous). Then γ si rectifiable and

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)| \, \mathrm{d}t$$

  - Notice: If $P \leq P'$, then $\Lambda(\gamma, P) \leq \Lambda(\gamma, P')$. (Prove with triangle inequality.)
  - WTS: For all partitions P, $\Lambda(\gamma, P) \leq \int_a^b |\gamma'(t)| \, \mathrm{d}t$ and thus $\Lambda(\gamma) \leq \int_a^b |\gamma'(t)| \, \mathrm{d}t$.
  - We have that

$$\begin{aligned} \Lambda(\gamma, P) &= \sum_{i=1}^n |\gamma(x_i) - \gamma(x_{i-1}) \\ &= \sum_{i=1}^n \left| \int_{x_{i-1}}^{x_i} \gamma'(t) \, \mathrm{d}t \right| \\ &\leq \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |\gamma'(t)| \, \mathrm{d}t \\ &= \int_a^b |\gamma'(t)| \, \mathrm{d}t \end{aligned}$$

- Catch up.
  - I should make up PSets 1-2.
  - Exams have less than Rudin-strength problems.
  - Exams are mostly true/false (and of that, mostly false, provide a counterexample).

Chapter 7

Sequences and Series of Functions

7.1 Notes

1/31:

- Midterm on differentiation and integration, and a bit of stuff from this week.
- Plan:
  - Talk about sequences of functions, all with the same domain and range, converging.
  - Address what properties of f_n remain in the limit (e.g., continuity, differentiability, integrability).
    - The answer depends on what we mean by "convergence."
    - $f_n \to f$ pointwise implies basically nothing.
    - $f_n \to f$ uniformly implies that basically everything works out nicely.
- We'll restrict ourselves to real functions because those have all the properties (integrability, differentiability, etc.) that we care about.
- **Pointwise** (convergent sequence $\{f_n\}$ to f): A sequence of functions $\{f_n\}$ such that for all $x \in X$, the sequence $\{f_n(x)\}$ converges to $f(x)$, where $f_n : X \to \mathbb{R}$ for all $n \in \mathbb{N}$ and $f : X \to \mathbb{R}$. *Denoted by* $f_n \to f$.
- Bad functions.
  - Consider $f_n : [0,1] \to \mathbb{R}$ defined by $x \mapsto x^n$. Each f_n is continuous, but f is not (zero everywhere except $f(1) = 1$)[1].
  - Consider $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = x^2/(1+x^2)^n$, and $f(x) = \sum_{n=0}^{\infty} f_n(x)$. As a geometric series, $f(x) = 1 + x^2$ when $x \neq 0$ but $f(0) = 0$. Thus, the limit exists but is not continuous once again.
  - Consider $f_m : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto \lim_{n\to\infty} \cos^{2n}(m!\pi x)$. Each f_m is integrable, but the limit f is the function that's 1 for rationals and zero for irrationals. In particular, f is not integrable.
    - We take even powers of the cosine to make it always positive.
    - We use $\cos^2(x)$ just because its always between $[0,1]$, and we know when it is equal to 1.
    - In particular, $\cos^2(\pi x)$ is equal to 1 at every integer, $\cos^2(2\pi x)$ is equal to 1 at every half integer. $\cos^2(6\pi x)$ is equal to 1 at every one-sixth of an integer.
    - Then raising it to the n^{th} power just makes it spiky.
- Aside: Interchanging limits.
  - If all f_n are continuous, then $\lim_{x\to x_0} f_n(x) = f_n(x_0)$.

[1]Questions that require counterexamples like this could show up on the midterm!

  - The question "is f continuous" is equivalent to being able to interchange limits:

$$\lim_{x \to x_0} \lim_{n \to \infty} f_n(x) = f(x_0) = \lim_{n \to \infty} \lim_{x \to x_0} f_n(x)$$

  - Sequence example showing we need to be careful interchanging limits: $s_{n,m} = m/(m+n)$.

- All of this pathology goes away with the right definition, though.
- **Uniformly** (convergent sequence $\{f_n\}$ to f): A sequence of functions $\{f_n\}$ such that for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$ for all $x \in X$, where $f_n : X \to \mathbb{R}$ for all $n \in \mathbb{N}$ and $f : X \to \mathbb{R}$.
- Proposition (Cauchy criterion for uniform convergence): $f_n \to f$ uniformly iff for all $\epsilon > 0$, there exists N such that for all $m, n \geq N$ and for all $x \in X$, $|f_n(x) - f_m(x)| < \epsilon$.
  - Forward direction: Let $\epsilon > 0$. Suppose $f_n \to f$ uniformly. Choose N such that the functions are within $\epsilon/2$. Then

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$