8 Functions of Several Variables IV / Special Functions

From Rudin (1976).

Chapter 8

3/11: **1.** Define

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Prove that f has derivatives of all orders at $x = 0$, and that $f^{(n)}(0) = 0$ for $n = 1, 2, \ldots$.

6. Suppose $f(x)f(y) = f(x + y)$ for all real x and y.

(a) Assuming that f is differentiable and not zero, prove that

$$f(x) = e^{cx}$$

where c is a constant.

(b) Prove the same thing, assuming only that f is continuous.

10. Prove that $\sum_{p \text{ prime}} 1/p$ diverges. (This shows that the primes form a fairly substantial subset of the positive integers.) (Hint: Given N, let $p_1, \ldots, p_k$ be those primes that divide at least one integer less than or equal to N. Then

$$\begin{aligned} \sum_{n=1}^{N} \frac{1}{n} &\leq \prod_{j=1}^{k} \left(1 + \frac{1}{p_j} + \frac{1}{p_j^2} + \cdots\right) \\ &= \prod_{j=1}^{k} \left(1 - \frac{1}{p_j}\right)^{-1} \\ &\leq \exp\left(\sum_{j=1}^{k} \frac{2}{p_j}\right) \end{aligned}$$

The last inequality holds because

$$(1 - x)^{-1} \leq e^{2x}$$

if $0 \leq x \leq 1/2$.)

Chapter 9

20. Take $n = m = 1$ in the implicit function theorem, and interpret the theorem (as well as its proof) graphically.