

# MATH 20410 (Analysis in $\mathbb{R}^n$ II – Accelerated) Problem Sets

Steven Labalme

February 3, 2022

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# 1 Differentiation

From Rudin (1976).

## Chapter 5

1. Let  $f$  be defined for all real  $x$ , and suppose that

$$|f(y) - f(x)| \leq (y - x)^2$$

for all real  $x$  and  $y$ . Prove that  $f$  is constant.

*Proof.* To prove that  $f$  is constant, Theorem 5.11b tells us that it will suffice to show that  $f$  is differentiable on  $\mathbb{R}$  with derivative  $f' = 0$ . Let  $x \in \mathbb{R}$  be arbitrary. We want to show that for all  $\epsilon > 0$ , there exists a  $\delta$  such that if  $y \in \mathbb{R}$  and  $0 < |y - x| < \delta$ , then  $|(f(y) - f(x))/(y - x) - 0| < \epsilon$ . Let  $\epsilon$  be arbitrary. Choose  $\delta = \epsilon$ . Then we have that

$$\begin{aligned} \left| \frac{f(y) - f(x)}{y - x} - 0 \right| &= \frac{|f(y) - f(x)|}{|y - x|} \\ &\leq \frac{(y - x)^2}{|y - x|} \\ &\leq |y - x| \\ &< \epsilon \end{aligned}$$

as desired. □

2. Suppose  $f'(x) > 0$  in  $(a, b)$ . Prove that  $f$  is strictly increasing in  $(a, b)$  and let  $g$  be its inverse function. Prove that  $g$  is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)}$$

for  $a < x < b$ .

*Proof.* To prove that  $f$  is strictly increasing on  $(a, b)$ , it will suffice to show that  $x < y$  implies  $f(x) < f(y)$  for all  $x, y \in (a, b)$ . Let  $x, y \in (a, b)$  satisfy  $x < y$ . Since  $f$  is differentiable on  $(a, b)$ , it is differentiable on  $(x, y) \subset (a, b)$  and (by Theorem 5.2) continuous on  $[x, y] \subset (a, b)$ . Thus, by the MVT, there exists  $c \in (x, y)$  such that

$$f(y) - f(x) = (y - x)f'(c)$$

But since  $x < y$ ,  $y - x > 0$ . This combined with the fact that  $f'(c) > 0$  by definition implies that  $(y - x)f'(c) > 0$ . Consequently,

$$f(y) < f(x) + (y - x)f'(c) = f(y)$$

as desired.

Since  $f$  is strictly increasing (and hence 1-1) on  $(a, b)$ , we may construct a well-defined inverse function  $g : f[(a, b)] \rightarrow (a, b)$  for it by

$$g(f(x)) = x$$

for all  $f(x) \in f[(a, b)]$ . It follows by the fact that  $f'(x) > 0$  for all  $x \in (a, b)$ , the definitions of  $f'(x)$  and  $g'(f(x))$ , and Theorem 3.3d that

$$\frac{1}{f'(x)} = \frac{1}{\lim_{y \rightarrow x} \frac{f(y) - f(x)}{y - x}}$$

$$\begin{aligned}
&= \lim_{y \rightarrow x} \frac{1}{\frac{f(y)-f(x)}{y-x}} \\
&= \lim_{y \rightarrow x} \frac{y-x}{f(y)-f(x)} \\
&= \lim_{y \rightarrow x} \frac{g(f(y)) - g(f(x))}{f(y) - f(x)} \\
&= g'(f(x))
\end{aligned}$$

as desired.  $\square$

3. Suppose  $g$  is a real function on  $\mathbb{R}^1$ , with bounded derivative (say  $|g'| \leq M$ ). Fix  $\epsilon > 0$  and define  $f(x) = x + \epsilon g(x)$ . Prove that  $f$  is one-to-one if  $\epsilon$  is small enough. (A set of admissible values of  $\epsilon$  can be determined which depends only on  $M$ .)

*Proof.* Neglecting the trivial case where  $M = 0$ , take  $\epsilon = 1/2M$ . It follows that

$$\begin{aligned}
0 &< 1 - \frac{1}{2} \\
&= 1 + \frac{1}{2M} \cdot -M \\
&\leq 1 + \epsilon g'(x) \\
&= \frac{d}{dx}(x) + \frac{d}{dx}(\epsilon g) \\
&= f'(x)
\end{aligned}$$

Therefore, by Problem 5.2,  $f$  is strictly increasing and, hence, one-to-one.  $\square$

4. If

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$$

where  $C_0, \dots, C_n$  are real constants, prove that the equation

$$C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n = 0$$

has at least one real root between 0 and 1.

*Proof.* Consider the polynomial

$$f(x) = C_0x + \frac{C_1}{2}x^2 + \cdots + \frac{C_n}{n+1}x^{n+1}$$

We have that  $f(0) = 0$  (by direct substitution) and  $f(1) = 0$  (by the constraint on the coefficients). Thus, since  $f$  is continuous on  $[0, 1]$  and differentiable on  $(0, 1)$  (as a polynomial), we have by the MVT that there exists  $x \in (0, 1)$  such that

$$\begin{aligned}
f(1) - f(0) &= (1 - 0)f'(x) \\
f'(x) &= 0 \\
C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n &= 0
\end{aligned}$$

as desired.  $\square$

5. Suppose  $f$  is defined and differentiable for every  $x > 0$ , and  $f'(x) \rightarrow 0$  as  $x \rightarrow +\infty$ . Put  $g(x) = f(x+1) - f(x)$ . Prove that  $g(x) \rightarrow 0$  as  $x \rightarrow +\infty$ .

*Proof.* To prove that  $\lim_{x \rightarrow \infty} g(x) = 0$ , it will suffice to show that for every  $\epsilon > 0$ , there exists  $N > 0$  such that if  $x > N$ , then  $|g(x) - 0| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Since  $\lim_{x \rightarrow \infty} f'(x) = 0$  by hypothesis, we know that there exists  $N > 0$  such that if  $x > N$ , then  $|f'(x)| < \epsilon$ . Choose this  $N$  to be our  $N$ . Let  $x > N$  be arbitrary. Applying the MVT to  $f$  on the interval  $[x, x+1]$  proves the existence of a  $c$  within that closed interval such that

$$f(x+1) - f(x) = f'(c)(x+1-x) = f'(c)$$

Additionally, since  $c > x > N$ , we have that  $|f'(c)| < \epsilon$ . Therefore, we have that

$$\begin{aligned} |g(x)| &= |f(x+1) - f(x)| \\ &= |f'(c)| \\ &< \epsilon \end{aligned}$$

as desired. □

## 2 Differentiation II / Integration

From Rudin (1976).

### Chapter 5

8. Suppose  $f'$  is continuous on  $[a, b]$  and  $\epsilon > 0$ . Prove that there exists  $\delta > 0$  such that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \epsilon$$

whenever  $0 < |t - x| < \delta$ ,  $a \leq x \leq b$ ,  $a \leq t \leq b$ . (This could be expressed by saying that  $f$  is **uniformly differentiable** on  $[a, b]$  if  $f'$  is continuous on  $[a, b]$ .) Does this hold for vector-valued functions, too?

*Proof.* By Theorem 2.40,  $[a, b]$  is compact. This combined with the fact that  $f'$  is continuous implies by Theorem 4.19 that  $f'$  is uniformly continuous. Thus, there exists  $\delta > 0$  such that if  $x, y \in [a, b]$  and  $|y - x| < \delta$ , then  $|f'(y) - f'(x)| < \epsilon$ . Choose this  $\delta$  to be our  $\delta$ . Let  $x, t \in [a, b]$  be such that  $0 < |t - x| < \delta$ . Then since  $f$  is continuous on  $[t, x] \subset [a, b]$  and differentiable on  $(t, x) \subset [a, b]$ , we have by the MVT that there exists  $c \in (t, x)$  such that

$$\begin{aligned} f(t) - f(x) &= (t - x)f'(c) \\ \frac{f(t) - f(x)}{t - x} &= f'(c) \end{aligned}$$

Additionally, since  $t < c < x$  and  $|t - x| < \delta$ , we must have  $|c - x| < \delta$ , meaning that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| = |f'(c) - f'(x)| < \epsilon$$

as desired.

And yes, this does hold for vector-valued functions, which we can treat component-wise.  $\square$

17. Suppose  $f$  is a real, three times differentiable function on  $[-1, 1]$  such that

$$f(-1) = 0 \qquad f(0) = 0 \qquad f(1) = 1 \qquad f'(0) = 0$$

Prove that  $f^{(3)}(x) \geq 3$  for some  $x \in (-1, 1)$ . Note that equality holds for  $\frac{1}{2}(x^3 + x^2)$ . (Hint: Use Theorem 5.15 with  $\alpha = 0$  and  $\beta = \pm 1$  to show that there exist  $s \in (0, 1)$  and  $t \in (-1, 0)$  such that  $f^{(3)}(s) + f^{(3)}(t) = 6$ .)

*Proof.* Since  $f$  is three times differentiable on  $[-1, 1]$ , we know that  $f''$  is differentiable on  $[-1, 1]$ . It follows by Theorem 5.2 that  $f''$  is continuous on  $[-1, 1]$ . Thus, since  $f$  is defined on  $[-1, 1]$ ,  $3 \in \mathbb{N}$ ,  $f''$  is continuous on  $[-1, 1]$ ,  $f^{(3)}$  is defined on  $(-1, 1)$ ,  $0, 1 \in [-1, 1]$  such that  $0 \neq 1$ , and we can define

$$P(t) = \sum_{k=0}^2 \frac{f^{(k)}(0)}{k!} (t - 0)^k$$

we have by Taylor's theorem that there exists  $s \in (0, 1)$  such that

$$\begin{aligned} f(1) &= P(1) + \frac{f^{(3)}(s)}{3!} (1 - 0)^3 \\ 1 - \left[ \frac{f(0)}{0!} (1 - 0)^0 + \frac{f'(0)}{1!} (1 - 0)^1 + \frac{f''(0)}{2!} (1 - 0)^2 \right] &= \frac{f^{(3)}(s)}{3!} \\ 1 - \left[ \frac{f''(0)}{2} \right] &= \frac{f^{(3)}(s)}{6} \\ 6 - 3f''(0) &= f^{(3)}(s) \end{aligned}$$

Similarly, we have that there exists  $t \in (-1, 0)$  such that

$$\begin{aligned} f(-1) &= P(-1) + \frac{f^{(3)}(t)}{3!}(-1-0)^3 \\ 0 - \left[ \frac{f(0)}{0!}(-1-0)^0 + \frac{f'(0)}{1!}(-1-0)^1 + \frac{f''(0)}{2!}(-1-0)^2 \right] &= -\frac{f^{(3)}(t)}{3!} \\ - \left[ \frac{f''(0)}{2} \right] &= -\frac{f^{(3)}(t)}{6} \\ 3f''(0) &= f^{(3)}(s) \end{aligned}$$

Thus,

$$f^{(3)}(s) + f^{(3)}(t) = 3f''(0) + 6 - 3f''(0) = 6$$

Now suppose for the sake of contradiction that for all  $x \in (-1, 1)$ , we have  $f^{(3)}(x) < 3$ . Then  $f^{(3)}(s) < 3$  and  $f^{(3)}(t) < 3$ . It follows that  $f^{(3)}(s) + f^{(3)}(t) < 6$ , a contradiction.  $\square$

25. Suppose  $f$  is twice differentiable on  $[a, b]$ ,  $f(a) < 0$ ,  $f(b) > 0$ ,  $f'(x) \geq \delta > 0$ , and  $0 \leq f''(x) \leq M$  for all  $x \in [a, b]$ . Let  $\xi$  be the unique point in  $(a, b)$  at which  $f(\xi) = 0$ . Complete the details in the following outline of **Newton's method** for computing  $\xi$ .

- (a) Choose  $x_1 \in (\xi, b)$  and define  $\{x_n\}$  by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Interpret this geometrically, in terms of a tangent to the graph of  $f$ .

*Answer.* Since we can rearrange the above to  $0 - f(x_n) = f'(x_n)(x_{n+1} - x_n)$ , we know that  $x_{n+1}$  is the point at which the tangent to  $f$  at  $x_n$  crosses the  $x$ -axis. In other words, the zero of the tangent line

$$y - f(x_n) = f'(x_n)(x - x_n)$$

is  $(x_{n+1}, 0)$ .  $\square$

- (b) Prove that  $x_{n+1} < x_n$  and that

$$\lim_{n \rightarrow \infty} x_n = \xi$$

*Proof.* To prove that  $x_{n+1} < x_n$ , it will suffice to show that  $f(x_n), f'(x_n) > 0$ . Since  $f'(x) > 0$  for all  $x \in [a, b]$  by hypothesis, we know that  $f'(x_n) > 0$ . As to  $f(x_n)$ , suppose for the sake of contradiction that  $f(x_n) \leq 0$ . We know that  $f(\xi) = 0$ ,  $f(b) > 0$ , and  $\xi < x_n < b$ . Since  $\xi$  is the *unique* point at which  $f(\xi) = 0$  by hypothesis and  $x_n \neq \xi$ , we know that  $f(x_n) \neq 0$ . And if  $f(x_n) < 0$ , we have by the intermediate value theorem for  $f$  continuous that there exists  $c \in (x_n, b)$  such that  $f(c) = 0$ . But since  $\xi < x_n < c < b$ ,  $c \neq \xi$ , and thus we have a contradiction here, too.

Having established that  $\{x_n\}$  is a monotonically decreasing sequence, Theorem 3.14 tells us that to show that it converges, it will suffice to show that it is bounded. Clearly,  $\{x_n\}$  is bounded above by  $x_1$ . And on the bottom side,  $\{x_n\}$  is bounded by  $\xi$ : If there were  $x_n < \xi$ , this would imply that  $f(x_n) < 0$  by a symmetric argument to the above, meaning that  $f(x_n)/f'(x_n) < 0$  and implying that  $x_{n+1} > x_n$ , a contradiction. Furthermore, we know that the limit (call it  $x$ ) equals  $\xi$  since

$$\begin{aligned} x &= x - \frac{f(x)}{f'(x)} \\ f(x) &= 0 \end{aligned}$$

so  $x = \xi$  by the uniqueness of  $\xi$ .  $\square$

(c) Use Taylor's theorem to show that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

for some  $t_n \in (\xi, x_n)$ .

*Proof.* Since  $f$  is defined on  $[a, b]$ ,  $2 \in \mathbb{N}$ ,  $f'$  is continuous on  $[a, b]$ ,  $f''$  is defined on  $(a, b)$ ,  $\xi, x_n \in [a, b]$  with  $\xi \neq x_n$ , and

$$P(t) = \sum_{k=0}^1 \frac{f^{(k)}(x_n)}{k!}(t - x_n)^k$$

we have by Taylor's theorem that there exists  $t_n \in (\xi, x_n)$  such that

$$\begin{aligned} f(\xi) &= \left[ \frac{f(x_n)}{0!}(\xi - x_n)^0 + \frac{f'(x_n)}{1!}(\xi - x_n)^1 \right] + \frac{f''(t_n)}{2!}(\xi - x_n)^2 \\ 0 &= f(x_n) - f'(x_n)(x_n - \xi) + \frac{f''(t_n)}{2}(x_n - \xi)^2 \\ x_n - \frac{f(x_n)}{f'(x_n)} - \xi &= \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2 \\ x_{n+1} - \xi &= \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2 \end{aligned}$$

as desired. □

(d) If  $A = M/2\delta$ , deduce that

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A}[A(x_1 - \xi)]^{2n}$$

(Compare with Chapter 3, Exercises 16 and 18.)

*Proof.* We have from part (b) that  $x_i > \xi$  for all  $i \in \mathbb{N}$ , so naturally  $0 \leq x_{n+1} - \xi$ . As to the other part of the question, we induct on  $n$ . For the base case  $n = 1$ , we have that

$$\begin{aligned} x_2 - \xi &= \frac{f''(t_1)}{2f'(x_1)}(x_1 - \xi)^2 \\ &\leq \frac{M}{2\delta}(x_1 - \xi)^2 \\ &= \frac{2\delta}{M} \left[ \frac{M}{2\delta}(x_1 - \xi) \right]^2 \\ &= \frac{1}{A}[A(x_1 - \xi)]^{2 \cdot 1} \end{aligned}$$

Now suppose inductively that we have proven the claim for  $n - 1$ ; we now seek to prove it for  $n$ . Indeed, we have that

$$\begin{aligned} x_{n+1} - \xi &= \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2 \\ &\leq \frac{M}{2\delta}(x_n - \xi)^2 \\ &\leq A \left( \frac{1}{A}[A(x_1 - \xi)]^{2(n-1)} \right)^2 \\ &= \frac{1}{A}[A(x_1 - \xi)]^{2n} \end{aligned}$$

as desired. □



- (e) Show that Newton's method amounts to finding a fixed point of the function  $g$  defined by

$$g(x) = x - \frac{f(x)}{f'(x)}$$

How does  $g'(x)$  behave for  $x$  near  $\xi$ ?

*Proof.* A fixed point of the function  $g$  is a point  $x$  such that

$$\begin{aligned} g(x) &= x \\ x - \frac{f(x)}{f'(x)} &= x \\ f(x) &= 0 \end{aligned}$$

Thus, if we want to find a point  $x$  where  $f(x) = 0$ , it is equivalent to find a point  $x$  such that  $g(x) = x$ .

As to the other part of the question, we have by the rules of derivatives that

$$\begin{aligned} g'(x) &= 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{f'(x)^2} \\ &= \frac{f(x)f''(x)}{f'(x)^2} \\ &\leq \frac{M}{\delta^2} f(x) \end{aligned}$$

Thus, since  $f(x) \rightarrow 0$  as  $x \rightarrow \xi$ ,  $g'(x) \rightarrow 0$  as  $x \rightarrow \xi$ . □

- (f) Put  $f(x) = \sqrt[3]{x}$  on  $(-\infty, \infty)$  and try Newton's method. What happens?

*Answer.* We have by the power rule that

$$f'(x) = \frac{1}{3x^{2/3}}$$

Choose  $x_1 = 1$ . Then

$$\begin{aligned} x_2 &= 1 - \frac{f(1)}{f'(1)} = -2 \\ x_3 &= 1 - \frac{f(-2)}{f'(-2)} = 7 \\ x_4 &= 1 - \frac{f(7)}{f'(7)} = -20 \\ &\vdots \end{aligned}$$

It appears that the series is diverging to  $\infty$  while alternating from positive to negative. In fact, since  $x_3 > x_2$ , contrary to part (b), we know that something must be wrong (i.e., one of our hypotheses must not be met). Upon further investigation, we can determine that on  $[-1, 1]$ , we have  $f''(1) = -2/9 < 0$ ; thus, our last hypothesis is the issue with this function. □

## Chapter 6

1. Suppose  $\alpha$  increases on  $[a, b]$ ,  $a \leq x_0 \leq b$ ,  $\alpha$  is continuous at  $x_0$ ,  $f(x_0) = 1$ , and  $f(x) = 0$  if  $x \neq x_0$ . Prove that  $f \in \mathcal{R}(\alpha)$  and that  $\int f d\alpha = 0$ .

*Proof.* Since  $f$  is bounded on  $[a, b]$  with only one discontinuity on  $[a, b]$  and  $\alpha$  is continuous at the point at which  $f$  is discontinuous, Theorem 6.10 implies that  $f \in \mathcal{R}(\alpha)$ , as desired. It follows that  $\inf U(P, f, \alpha) = \sup L(P, f, \alpha) = \int f d\alpha$ . But since  $L(P, f, \alpha) = 0$  for all  $P$  (there is no infinite interval  $[x_i, x_{i+1}] \subset [a, b]$  that does not contain 0, and  $f$  is bounded below by 0), we know that

$$\int f d\alpha = \sup L(P, f, \alpha) = 0$$

as desired. □

2. Suppose  $f \geq 0$ ,  $f$  is continuous on  $[a, b]$ , and  $\int_a^b f(x) dx = 0$ . Prove that  $f(x) = 0$  for all  $x \in [a, b]$ . (Compare this with Exercise 1.)

*Proof.* Suppose for the sake of contradiction that  $f(x) \neq 0$  for some  $x$ . By the definition of  $f$ , this must mean that  $f(x) > 0$ . It follows since  $f$  is continuous that there exists some  $N_r(x)$  such that  $f(y) > 0$  for all  $y \in N_r(x)$ . Now consider the partition

$$P = \{a, x - r/2, x + r/2, b\}$$

of  $[a, b]$ . But since  $m_2 > 0$ , we have that

$$\begin{aligned} 0 &< m_1[(x - r/2) - a] + m_2[(x + r/2) - (x - r/2)] + m_3[b - (x + r/2)] \\ &= L(P, f) \\ &\leq \int_a^b f(x) dx \end{aligned} \quad \text{Theorem 6.4}$$

a contradiction. □

4. If  $f(x) = 0$  for all irrational  $x$  and  $f(x) = 1$  for all rational  $x$ , prove that  $f \notin \mathcal{R}$  on  $[a, b]$  for any  $a < b$ .

*Proof.* Let  $P$  be an arbitrary partition of  $[a, b]$ . Since the rationals and irrationals are dense in the reals, we know that for any  $[x_i, x_{i+1}]$ ,  $f(x) = 0$  for some  $x \in [x_i, x_{i+1}]$  and  $f(x) = 1$  for some  $x \in [x_i, x_{i+1}]$ . Thus, we have that  $L(P, f) = 0$  and  $U(P, f) = b - a$ . It follows that if  $a < b$ ,

$$\sup L(P, f) = 0 \neq b - a = \inf U(P, f)$$

so  $f \notin \mathcal{R}$ , as desired. □

### 3 Integration II

From Rudin (1976).

#### Chapter 6

2/2: 3. Define three functions  $\beta_1, \beta_2, \beta_3$  as follows:

$$\beta_1 = \begin{cases} 0 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases} \quad \beta_2 = \begin{cases} 0 & x < 0 \\ 1 & x = 0 \\ 1 & x > 0 \end{cases} \quad \beta_3 = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases}$$

Let  $f$  be a bounded function on  $[-1, 1]$ .

(a) Prove that  $f \in \mathcal{R}(\beta_1)$  if and only if  $f(0+) = f(0)$  and that then

$$\int f d\beta_1 = f(0)$$

*Proof.* Suppose first that  $f \in \mathcal{R}(\beta_1)$  with  $\int f d\beta_1 = f(0)$ . To prove that  $f(0+) = f(0)$ , it will suffice to show that for every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that if  $x \in [-1, 1]$  and  $0 \leq x < \delta$ , then  $|f(x) - f(0)| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Since  $f \in \mathcal{R}(\beta_1)$  by hypothesis, we have by Theorem 6.6 that there exists a partition  $P$  of  $[-1, 1]$  such that  $U(P, f, \beta_1) - L(P, f, \beta_1) < \epsilon$ . Now let  $x_i = \min\{x \in P : x > 0\}$ ; we know that such an object exists since there exist elements of  $P$  greater than zero (namely 1) and  $P$  is finite. It follows by the definition of  $\beta_1$  that  $\Delta x_i = 1$  and  $\Delta x_j = 0$  for  $j \neq i$ . Thus,  $U(P, f, \beta_1) = M_i$  and  $L(P, f, \beta_1) = m_i$  (which exist because  $f$  is bounded on  $[-1, 1]$ ). At this point, we are ready to choose  $\delta$ , which we take to be  $\delta = x_i$ . Now to confirm that this  $\delta$  works: Let  $0 \leq x < \delta$ . By the definition of  $x_i, x_{i-1}$ ,  $m_i \leq f(x) \leq M_i$  and  $m_i \leq f(0) \leq M_i$ . But since  $M_i - m_i < \epsilon$  as per the above, we have that  $|f(x) - f(0)| < \epsilon$ , as desired.

Now suppose that  $f(0+) = f(0)$ . To prove that  $f \in \mathcal{R}(\beta_1)$ , Theorem 6.6 tells us that it will suffice to show that for every  $\epsilon > 0$ , there exists a  $P$  such that  $U(P, f, \beta_1) - L(P, f, \beta_1) < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Since  $f(0+) = f(0)$ , we know that there exists a  $\delta' > 0$  such that if  $x \in [-1, 1]$  and  $0 \leq x < \delta'$ , then  $|f(x) - f(0)| < \epsilon/3$ . Let  $\delta = \min(\delta'/2, 1)$ . Thus, we may define  $P = \{-1, 0, \delta, 1\}$ . We have

$$\begin{aligned} U(P, f, \beta_1) &= \sum_{i=1}^3 M_i \Delta \beta_{1_i} & L(P, f, \beta_1) &= \sum_{i=1}^3 m_i \Delta \beta_{1_i} \\ &= M_2 & &= m_2 \end{aligned}$$

(which exist because  $f$  is bounded on  $[-1, 1]$ ). Consequently,  $M_2 \leq f(0) + \epsilon/3$ .  $m_2 \geq f(0) - \epsilon/3$ . Therefore,

$$\begin{aligned} U(P, f, \beta_1) - L(P, f, \beta_1) &= M_2 - m_2 \\ &\leq [f(0) + \frac{\epsilon}{3}] - [f(0) - \frac{\epsilon}{3}] \\ &= \frac{2\epsilon}{3} \\ &< \epsilon \end{aligned}$$

as desired.

As to proving that  $\int f d\beta_1$ , we know that  $M_2 \leq f(0) + \epsilon/3$  for arbitrarily small  $\epsilon$  implies  $M_2 \leq f(0)$ . Similarly,  $m_2 \geq f(0)$ . Thus,

$$\inf U(P, f, \beta_1) \leq U(P, f, \beta_1) = M_2 \leq f(0) \leq m_2 = L(P, f, \beta_1) \leq \sup L(P, f, \beta_1)$$

But by Theorem 6.5,  $\sup L(P, f, \beta_1) \leq \inf U(P, f, \beta_1)$ . Therefore,

$$\int_{-1}^1 f \, d\beta_1 = \sup L(P, f, \beta_1) = \inf U(P, f, \beta_1) = f(0)$$

as desired. □

- (b) State and prove a similar result for  $\beta_2$ .

*Proof.* The result will be  $f \in \mathcal{R}(\beta_2)$  if and only if  $f(0-) = f(0)$  and that then

$$\int f \, d\beta = f(0)$$

The proof of this result is entirely symmetric to the proof of the previous result. □

- (c) Prove that  $f \in \mathcal{R}(\beta_3)$  if and only if  $f$  is continuous at 0.

*Proof.* Suppose first that  $f \in \mathcal{R}(\beta_3)$ . To prove that  $f$  is continuous at 0, it will suffice to show that for every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that if  $x \in [-1, 1]$  and  $|x| < \delta$ , then  $|f(x) - f(0)| < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Since  $f \in \mathcal{R}(\beta_3)$  by hypothesis, we have by Theorem 6.6 that there exists a partition  $P$  of  $[-1, 1]$  such that  $U(P, f, \beta_3) - L(P, f, \beta_3) < \epsilon/2$ . Now let  $x_i = \max\{x \in P : x < 0\}$  and let  $x_j = \min\{x \in P : x > 0\}$ . Choose  $\delta = \min\{|x_i|, |x_j|\}$ . Let  $P^* = P \cup \{-\delta, 0, \delta\}$  be a refinement of  $P$ . It follows by the definition of  $\beta_3$  and a reenumeration of  $P^*$  that  $U(P^*, f, \beta_3) = (M_{i-1} + M_i)/2$  and  $L(P^*, f, \beta_3) = (m_{i-1} + m_i)/2$ . Now let  $|x| < \delta$ . We divide into two cases ( $x \geq 0$  and  $x < 0$ ). If  $x \geq 0$ , then  $m_i \leq f(x) \leq M_i$  and  $m_i \leq f(0) \leq M_i$ . But then we have that

$$\begin{aligned} |f(x) - f(0)| &\leq M_i - m_i \\ &\leq (M_{i-1} - m_{i-1}) + (M_i - m_i) \\ &= 2 \left[ \frac{M_{i-1} + M_i}{2} - \frac{m_{i-1} + m_i}{2} \right] \\ &= 2[U(P^*, f, \beta_3) - L(P^*, f, \beta_3)] \\ &< \epsilon \end{aligned}$$

as desired. The proof is symmetric in the other case.

Now suppose that  $f$  is continuous at 0. To prove that  $f \in \mathcal{R}(\beta_3)$ , Theorem 6.6 tells us that it will suffice to show that for every  $\epsilon > 0$ , there exists a  $P$  such that  $U(P, f, \beta_3) - L(P, f, \beta_3) < \epsilon$ . Let  $\epsilon > 0$  be arbitrary. Since  $f$  is continuous at 0, we know that there exists a  $\delta' > 0$  such that if  $x \in [-1, 1]$  and  $|x| < \delta'$ , then  $|f(x) - f(0)| < \epsilon/3$ . Choose  $\delta = \min(\delta'/2, 1)$ . Consider  $P = \{-1, -\delta/2, \delta/2, 1\}$ . It follows as before that  $U(P, f, \beta_3) = M_2$  and  $L(P, f, \beta_3) = m_2$ . Consequently,  $M_2 \leq f(0) + \epsilon/3$  and  $m_2 \geq f(0) - \epsilon/3$ . Therefore,

$$\begin{aligned} U(P, f, \beta_3) - L(P, f, \beta_3) &= M_2 - m_2 \\ &\leq [f(0) + \frac{\epsilon}{3}] - [f(0) - \frac{\epsilon}{3}] \\ &= \frac{2\epsilon}{3} \\ &< \epsilon \end{aligned}$$

as desired. □

- (d) If  $f$  is continuous at 0, prove that

$$\int f \, d\beta_1 = \int f \, d\beta_2 = \int f \, d\beta_3 = f(0)$$

*Proof.* If  $f$  is continuous at 0, then  $f(0+) = f(0) = f(0-)$ . It follows that

$$f(0) = \int f \, d\beta_1 \quad \text{Part (a)}$$

$$= \int f \, d\beta_2 \quad \text{Part (b)}$$

$$= \int f \, d\beta_3 \quad \text{Part (c)}$$

Note that calculating the exact value of  $\int f \, d\beta_3$  is symmetric to the proof in part (a).  $\square$

5. Suppose  $f$  is a bounded real function on  $[a, b]$ , and  $f^2 \in \mathcal{R}$  on  $[a, b]$ . Does it follow that  $f \in \mathcal{R}$ ? Does the answer change if we assume that  $f^3 \in \mathcal{R}$ ?

*Proof.*  $f^2 \in \mathcal{R} \nRightarrow f \in \mathcal{R}$ : Consider the bounded real function  $f : [a, b] \rightarrow \mathbb{R}$  defined by

$$f(x) = \begin{cases} 1 & x \notin \mathbb{Q} \\ -1 & x \in \mathbb{Q} \end{cases}$$

Since  $f^2(x) = 1$  for all  $x \in [a, b]$ ,  $f^2 \in \mathcal{R}$  as a constant function. However, by Exercise 6.4 and a clever application of Theorem 6.12 (to relate it to the function explicitly considered in Exercise 6.4), we know that  $f \notin \mathcal{R}$ .

$f^3 \in \mathcal{R} \Rightarrow f \in \mathcal{R}$ : Let  $f : [a, b] \rightarrow \mathbb{R}$  be any bounded real function such that  $f^3 \in \mathcal{R}$ . To prove that  $f \in \mathcal{R}$ , Theorem 6.11 tells us that it will suffice to show that there exist  $m, M \in \mathbb{R}$  such that  $m \leq f \leq M$  and that there exists a continuous function  $\phi : [m, M] \rightarrow \mathbb{R}$  such that  $f = \phi \circ f^3$ . Since  $f$  is bounded by hypothesis, we can pick  $m, M \in \mathbb{R}$  such that  $m \leq f \leq M$ . Now let  $\phi : [m, M] \rightarrow \mathbb{R}$  be defined by

$$\phi(x) = \sqrt[3]{x}$$

for all  $x \in [m, M]$ . It is obvious that  $\phi$  is continuous and that  $\phi \circ f^3 = f$ , as desired.  $\square$

7. Suppose  $f$  is a real function on  $(0, 1]$  and  $f \in \mathcal{R}$  on  $[c, 1]$  for every  $c > 0$ . Define

$$\int_0^1 f(x) \, dx = \lim_{c \rightarrow 0} \int_c^1 f(x) \, dx$$

if this limit exists (and is finite).

- (a) If  $f \in \mathcal{R}$  on  $[0, 1]$ , show that this definition of the integral agrees with the old one.

*Proof.* To prove that  $\int_0^1 f = \lim_{c \rightarrow 0} \int_c^1 f$ , it will suffice to show that for every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that if  $c \in (0, 1]$  and  $c < \delta$ , then

$$\left| \int_0^c f \right| = \left| \int_c^1 f - \int_0^1 f \right| < \epsilon$$

Let  $\epsilon > 0$  be arbitrary. Since  $f$  is integrable,  $f$  is bounded, i.e., there exists  $M \in \mathbb{R}$  such that  $|f(x)| < M$  for all  $x \in [0, 1]$ . Choose  $\delta = \epsilon/M$ . Let  $c \in (0, 1]$  be such that  $c < \delta$ . Then by Theorem 6.12d,

$$\left| \int_0^c f \right| \leq M(c - 0) < \epsilon$$

as desired.  $\square$

- (b) Construct a function  $f$  such that the above limit exists, although it fails to exist with  $|f|$  in place of  $f$ .

*Proof.* Let  $f : (0, 1] \rightarrow \mathbb{R}$  be defined by

$$f(x) = (-1)^n n$$

for  $1/n < x \leq 1/(n-1)$  ( $n = 2, 3, \dots$ ). It follows since  $f$  is a constant function save one terminal discontinuity on each  $[1/n, 1/(n-1)]$  that

$$\begin{aligned} \int_{1/n}^{1/(n-1)} f &= (-1)^n n \cdot \left( \frac{1}{n-1} - \frac{1}{n} \right) \\ &= \frac{(-1)^n n}{n(n-1)} \\ &= \frac{(-1)^n}{n-1} \end{aligned}$$

for all  $n \in \mathbb{N}$ . It follows that

$$\begin{aligned} \int_{1/N}^1 f &= \sum_{n=2}^N \int_{1/n}^{1/(n-1)} f \\ &= \sum_{n=2}^N \frac{(-1)^n}{n-1} \end{aligned}$$

Thus,

$$\lim_{c \rightarrow 0} \int_c^1 f = \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1}$$

which converges by Theorem 3.43. However, the limit fails to exist if  $f$  is replaced by  $|f|$ , because in that case, the integral is equal to the harmonic series, which diverges to infinity.  $\square$

8. Suppose  $f \in \mathcal{R}$  on  $[a, b]$  for every  $b > a$  where  $a$  is fixed. Define

$$\int_a^{\infty} f(x) \, dx = \lim_{b \rightarrow \infty} \int_a^b f(x) \, dx$$

if this limit exists (and is finite). In that case, we say that the integral on the left **converges**. If it also converges after  $f$  has been replaced by  $|f|$ , it is said to converge **absolutely**.

Assume that  $f(x) \geq 0$  and that  $f$  decreases monotonically on  $[1, \infty)$ . Prove that  $\int_1^{\infty} f(x) \, dx$  converges if and only if  $\sum_{n=1}^{\infty} f(n)$  converges. (This is the so-called “integral test” for convergence of series.)

*Proof.* To prove the claim, we will show that

$$\sum_{n=2}^N f(n) \leq \int_1^N f \leq \sum_{n=1}^{N-1} f(n) \leq f(1) + \int_1^{N-1} f(x) \, dx$$

It will follow since both the sum and the integral limit are monotonically increasing as  $N \rightarrow \infty$  ( $f \geq 0$ ) and both are bounded below and above by (a function of) the other, both converge or diverge together. Let's begin.

Since  $f$  is monotonically decreasing on  $[1, \infty)$ , we know that  $f(n) \leq f(x)$  for all  $1 \leq x \leq n$  ( $n \in \mathbb{N}$ ). Thus, by Theorem 6.12b,

$$\int_{n-1}^n f(n) \, dx \leq \int_{n-1}^n f(x) \, dx$$

Therefore,

$$\sum_{n=2}^N f(n) = \sum_{n=2}^N \int_{n-1}^n f(n) \, dx \quad \text{Theorem 6.12d}$$

$$\leq \sum_{n=2}^N \int_{n-1}^n f(x) \, dx$$

$$= \int_1^N f(x) \, dx \quad \text{Theorem 6.12c}$$

for all  $N = 2, 3, 4, \dots$ , thereby establishing the left inequality above.

Since  $f$  is monotonically decreasing on  $[1, \infty)$ , we know that  $f(x) \leq f(n)$  for all  $x \geq n$  ( $n \in \mathbb{N}$ ). Thus, by Theorem 6.12b,

$$\int_n^{n+1} f(x) \, dx \leq \int_n^{n+1} f(n) \, dx$$

Therefore,

$$\int_1^N f(x) \, dx = \sum_{n=1}^{N-1} \left( \int_n^{n+1} f(x) \, dx \right) \quad \text{Theorem 6.12c}$$

$$\leq \sum_{n=1}^{N-1} \left( \int_n^{n+1} f(n) \, dx \right)$$

$$= \sum_{n=1}^{N-1} f(n) \quad \text{Theorem 6.12d}$$

for all  $N = 2, 3, 4, \dots$ , thereby establishing the middle inequality above.

From our statement about  $f(n)$  and  $f(x)$  from the left inequality, we have by Theorem 6.12b that

$$\int_{n-1}^n f(n) \, dx \leq \int_{n-1}^n f(x) \, dx$$

Therefore,

$$\sum_{n=1}^{N-1} f(n) = f(1) + \sum_{n=2}^{N-1} \int_{n-1}^n f(n) \, dx \quad \text{Theorem 6.12d}$$

$$\leq f(1) + \sum_{n=2}^{N-1} \int_{n-1}^n f(x) \, dx$$

$$= f(1) + \int_1^{N-1} f(x) \, dx \quad \text{Theorem 6.12c}$$

for all  $N = 2, 3, 4, \dots$ , thereby establishing the right inequality above.  $\square$

10. Let  $p, q$  be positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

Prove the following statements.

(a) If  $u, v \geq 0$ , then

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}$$

Equality holds if and only if  $u^p = v^q$ .

*Discussion.* To prove the desired inequality, it will suffice to show that

$$0 \leq \frac{u^p}{p} + \frac{v^q}{q} - uv$$

i.e., that for all  $u, v \geq 0$ , the expression on the right above is nonnegative. To consider all such values at once, we can consider applying our analysis toolbox to  $f : [0, \infty)^2 \rightarrow \mathbb{R}$  defined by

$$f(u, v) = \frac{u^p}{p} + \frac{v^q}{q} - uv$$

with the goal of proving that it is nonnegative everywhere on its domain. However, since we do not yet know multivariable calculus, it will suffice to fix  $u \geq 0$  and analyze  $f : [0, \infty) \rightarrow \mathbb{R}$  defined by

$$f(v) = \frac{u^p}{p} + \frac{v^q}{q} - uv$$

Let's begin. □

*Proof.* Fix  $u \geq 0$ . Let  $f : [0, \infty) \rightarrow \mathbb{R}$  be defined by

$$f(v) = \frac{u^p}{p} + \frac{v^q}{q} - uv$$

It follows from the definition of  $f$  that to prove the desired inequality, it will suffice to show that  $f$  is nonnegative everywhere on its domain. Let's begin.

Since  $f$  is a polynomial in  $v$ ,  $f$  is differentiable. Thus, we may consider

$$f'(v) = v^{q-1} - u$$

As a function of a positive power ( $q/(q-1) = p > 0$  and  $q > 0$  imply  $q-1 > 0$ ) of its variable (minus a constant),  $f'$  is strictly increasing. Additionally, we have that

$$\begin{aligned} 0 &= f'(v) \\ u &= v^{q-1} \\ &= v^{q/p} \\ v &= u^{p/q} \end{aligned}$$

Thus, we know that  $f' < 0$  on  $(0, u^{p/q})$  and  $f' > 0$  on  $(u^{p/q}, \infty)$ . It follows by the strict version of Theorem 5.11 that  $f$  is strictly decreasing on  $(0, u^{p/q})$  and strictly increasing on  $(u^{p/q}, \infty)$ . Furthermore, since  $f$  is differentiable (hence continuous by Theorem 5.2), we know that  $f(0) \geq f(u^{p/q})$ . Combining the last several results, we have that  $f(u^{p/q})$  is the minimum of  $f$  over  $[0, \infty)$ , and hence equal to the minimum value of  $f$  over  $[0, \infty)$ . But since

$$\begin{aligned} f(u^{p/q}) &= \frac{u^p}{p} + \frac{(u^{p/q})^q}{q} - uu^{p/q} \\ &= \frac{u^p}{p} + \frac{u^p}{q} - u^{p/q+1} \\ &= u^p \left( \frac{1}{p} + \frac{1}{q} \right) - u^p \\ &= 0 \end{aligned}$$

we know that  $f(v) \geq 0$  on its domain, as desired.

Additionally, since  $f$  is strictly decreasing on  $(0, u^{p/q})$  and strictly increasing on  $(u^{p/q}, \infty)$ , we know that  $f(v) = 0$  iff  $v = u^{p/q}$ , i.e., iff  $v^q = u^p$ , as desired. □



(b) If  $f, g \in \mathcal{R}(\alpha)$ ,  $f, g \geq 0$ , and

$$\int_a^b f^p d\alpha = 1 = \int_a^b g^q d\alpha$$

then

$$\int_a^b fg d\alpha \leq 1$$

*Proof.* By Theorem 6.13a, the hypothesis  $f, g \in \mathcal{R}(\alpha)$  implies that  $fg \in \mathcal{R}(\alpha)$ . Thus, we have that

$$\int_a^b fg d\alpha \leq \int_a^b \left( \frac{f^p}{p} + \frac{g^q}{q} \right) d\alpha \quad \text{Theorem 6.12b}$$

$$= \frac{1}{p} \int_a^b f^p d\alpha + \frac{1}{q} \int_a^b g^q d\alpha \quad \text{Theorem 6.12a}$$

$$= \frac{1}{p} + \frac{1}{q}$$

$$= 1$$

as desired. □

(c) If  $f, g$  are complex functions in  $\mathcal{R}(\alpha)$ , then

$$\left| \int_a^b fg d\alpha \right| \leq \left( \int_a^b |f|^p d\alpha \right)^{1/p} \left( \int_a^b |g|^q d\alpha \right)^{1/q}$$

This is **Hölder's inequality**. When  $p = q = 2$ , it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)

*Proof.* By Theorem 6.11 with  $\phi(y) = |y|^p$  (resp.  $\phi(y) = |y|^q$ ), the hypothesis  $f, g \in \mathcal{R}(\alpha)$  implies that  $|f|^p, |g|^q \in \mathcal{R}(\alpha)$ . Thus, we may let

$$I_f = \left( \int_a^b |f|^p d\alpha \right)^{1/p} \quad I_g = \left( \int_a^b |g|^q d\alpha \right)^{1/q}$$

We divide into two cases ( $I_f = 0$  or  $I_g = 0$ , and  $I_f, I_g \neq 0$ ). In the first case, WLOG let  $I_f = 0$ . Then since  $0 \leq |f|^p$ , it follows that  $f = 0$  on  $[a, b]$ . Thus

$$\left| \int_a^b fg d\alpha \right| = 0 \leq 0 = I_f I_g = \left( \int_a^b |f|^p d\alpha \right)^{1/p} \left( \int_a^b |g|^q d\alpha \right)^{1/q}$$

as desired. In the other case, it follows that

$$\begin{aligned} I_f^p &= \int_a^b |f|^p d\alpha & I_g^q &= \int_a^b |g|^q d\alpha \\ 1 &= \int_a^b \left| \frac{f}{I_f} \right|^p d\alpha & 1 &= \int_a^b \left| \frac{g}{I_g} \right|^q d\alpha \end{aligned} \quad \text{Theorem 6.12a}$$

Thus, since  $|f/I_f|, |g/I_g| \in \mathcal{R}(\alpha)$  by Theorems 6.12 and 6.13 and  $|f/I_f|, |g/I_g| \geq 0$  by the defini-

tion of the absolute value, we have that

$$\begin{aligned}
 \left| \int_a^b fg \, d\alpha \right| &\leq \int_a^b |fg| \, d\alpha && \text{Theorem 6.13b} \\
 &= I_f I_g \int_a^b \left| \frac{f}{I_f} \right| \left| \frac{g}{I_g} \right| d\alpha \\
 &\leq I_f I_g \cdot 1 && \text{Part (b)} \\
 &= \left( \int_a^b |f|^p \, d\alpha \right)^{1/p} \left( \int_a^b |g|^q \, d\alpha \right)^{1/q}
 \end{aligned}$$

as desired.  $\square$

**11.** Let  $\alpha$  be a fixed increasing function on  $[a, b]$ . For  $u \in \mathcal{R}(\alpha)$ , define

$$\|u\|_2 = \left( \int_a^b |u|^2 \, d\alpha \right)^{1/2}$$

Suppose  $f, g, h \in \mathcal{R}(\alpha)$ , and prove the triangle inequality

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2$$

as a consequence of the Schwarz inequality, as in the proof of Theorem 1.37.

*Proof.* By Theorems 6.12a and 6.13b, the hypothesis that  $f, g, h \in \mathcal{R}(\alpha)$  implies that  $|f - g|, |g - h| \in \mathcal{R}(\alpha)$ . Thus, we have that

$$\begin{aligned}
 \|f - h\|_2^2 &= \int_a^b |f - h|^2 \, d\alpha \\
 &= \int_a^b |(f - g) + (g - h)|^2 \, d\alpha \\
 &= \int_a^b |f - g|^2 \, d\alpha + 2 \int_a^b |f - g| \cdot |g - h| \, d\alpha + \int_a^b |g - h|^2 \, d\alpha \\
 &\leq \int_a^b |f - g|^2 \, d\alpha + 2 \left( \int_a^b |f - g|^2 \, d\alpha \right)^{1/2} \left( \int_a^b |g - h|^2 \, d\alpha \right)^{1/2} + \int_a^b |g - h|^2 \, d\alpha \\
 &= \|f - g\|_2^2 + 2\|f - g\|_2 \|g - h\|_2 + \|g - h\|_2^2 \\
 &= (\|f - g\|_2 + \|g - h\|_2)^2
 \end{aligned}$$

Taking square roots of both sides of the inequality yields the desired result.  $\square$

## References

Rudin, W. (1976). *Principles of mathematical analysis* (A. A. Arthur & S. L. Langman, Eds.; Third). McGraw-Hill.