

## Chapter 6

# The Riemann-Stieltjes Integral

### 6.1 Notes

1/28:

- Plan:

1. Finish up Fundamental Theorem of Calculus proof.
2. Basic consequences.
3. Rectifiable curves.

- Recall that we're given  $f : [a, b] \rightarrow \mathbb{R}$  continuous,  $f : [a, b] \rightarrow \mathbb{R}$ , and  $x \mapsto \int_a^x f(t) dt$ .

- Goal: Show  $F'(x_0) = f(x_0)$ .

- WTS: Find  $\delta$  such that  $|x - x_0| < \delta$  implies

$$\begin{aligned} \left| \frac{1}{x - x_0} \int_{x_0}^x f(t) dt - f(x_0) \right| &= \left| \frac{1}{x - x_0} \int_{x_0}^x f(t) dt - \frac{1}{x - x_0} \int_{x_0}^x f(x_0) dt \right| \\ &= \frac{1}{|x - x_0|} \left| \int_{x_0}^x (f(t) - f(x_0)) dt \right| \\ &\leq \frac{1}{|x - x_0|} \int_{x_0}^x |f(t) - f(x_0)| dt \\ &< \epsilon \end{aligned}$$

- Since  $f$  is continuous, there exists  $\delta$  such that if  $|x - x_0| < \delta$ , then  $|f(x) - f(x_0)| < \epsilon$ .

- Now

$$\begin{aligned} \frac{1}{|x - x_0|} \int_{x_0}^x |f(t) - f(x_0)| dt &< \frac{1}{|x - x_0|} \int_{x_0}^x \epsilon dt \\ &= \epsilon \end{aligned}$$

- Applications:

1. Theorem (MVT for integration):  $f : [a, b] \rightarrow \mathbb{R}$  continuous, then there exists  $x_0 \in [a, b]$  such that

$$f(x_0) = \frac{1}{b - a} \int_a^b f(x) dx$$

- Apply MVT to  $F(x) = \int_a^x f(t) dt$ . Then

$$F'(x_0) = f(x_0) = \frac{F(b) - F(a)}{b - a}$$

as desired.

2. Theorem (Integration by parts): Let  $F, G : [a, b] \rightarrow \mathbb{R}$  be differentiable with  $F' = f$ ,  $G' = g$  and with  $f$  and  $g$  both integrable. Then

$$\int_a^b Fg = F(b)G(b) - F(a)G(a) - \int_a^b fG$$

- Just use the product rule plus the FTC to prove.
- We have

$$\begin{aligned} \int_a^b (FG)' &= \int_a^b fG + \int_a^b Fg \\ F(b)G(b) - F(a)G(a) &= \int_a^b fG + \int_a^b Fg \\ \int_a^b Fg &= F(b)G(b) - F(a)G(a) - \int_a^b fG \end{aligned}$$

3. Theorem ( $u$ -substitution).

- Follows similarly from the chain rule and FTC.

- Integration of vector-valued functions.

- If  $f : [a, b] \rightarrow \mathbb{R}^k$ , we define  $\int_a^b f$  by

$$\int_a^b f = \left( \int_a^b f_1, \dots, \int_a^b f_k \right)$$

- Alternatively, you can define  $\int_a^b f$  using  $P$ ,  $U(f, P)$ ,  $L(f, P)$ , etc. and then prove that the integral exists iff all  $f_i$  are integrable and in this case the above definition holds.
- Rectifiable curves: Let  $\gamma : [a, b] \rightarrow \mathbb{R}^k$  be a continuous function.
- Plan: Define the length of  $\gamma$  and show that we can compute it with an integral.
  - Idea: For polygonal paths, we know how to define length. So let's approximate  $\gamma$  by polygons and take a limit.
  - Ref: Given a partition  $P$ , then define the length of  $\gamma$  with respect to  $P$  as  $\Lambda(\gamma, P)$ . Let the length of  $\gamma$  be  $\Lambda(\gamma) = \sup_P \Lambda(\gamma, P)$  if this limit exists in this case, we call  $\gamma$  **rectifiable**.
- Fractals are not rectifiable — their length diverges.
- Theorem: Suppose  $\gamma$  is continuously differentiable (i.e.,  $\gamma$  is differentiable and  $\gamma'$  is continuous). Then  $\gamma$  is rectifiable and

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)| dt$$

- Notice: If  $P \leq P'$ , then  $\Lambda(\gamma, P) \leq \Lambda(\gamma, P')$ . (Prove with triangle inequality.)
- WTS: For all partitions  $P$ ,  $\Lambda(\gamma, P) \leq \int_a^b |\gamma'(t)| dt$  and thus  $\Lambda(\gamma) \leq \int_a^b |\gamma'(t)| dt$ .
- We have that

$$\begin{aligned} \Lambda(\gamma, P) &= \sum_{i=1}^n |\gamma(x_i) - \gamma(x_{i-1})| \\ &= \sum_{i=1}^n \left| \int_{x_{i-1}}^{x_i} \gamma'(t) dt \right| \\ &\leq \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |\gamma'(t)| dt \\ &= \int_a^b |\gamma'(t)| dt \end{aligned}$$

- Catch up.
  - I should make up PSets 1-2.
  - Exams have less than Rudin-strength problems.
  - Exams are mostly true/false (and of that, mostly false, provide a counterexample).

## 6.2 Exam 1 Additional Topics

- A continuous function that is not always differentiable.

$$f(x) = |x|$$

- A differentiable function with a discontinuous derivative.

$$f(x) = x^2 \sin \frac{1}{x}$$

- A vector-valued function that doesn't satisfy the MVT.

$$\mathbf{f}(x) = e^{ix}$$

- Between 0 and  $2\pi$ .

- A pair of vector-valued functions that don't satisfy L'Hôpital's rule.

$$f(x) = x \qquad g(x) = x + x^2 e^{i/x^2}$$