Chapter 1

The Algebra and Topology of $\mathbb{R}^n$

1.1 Notes

1/10:

- Syllabus.
  - In his mind, homework is the main setting where learning takes place.
- We're going to be studying analysis, or calculus, on **manifolds** this quarter.
- **Manifold**: A "space" that looks like Euclidean space $\mathbb{R}^n$ locally.
  - The surfaces of a sphere and torus are common examples of 2-dimensional manifolds.
  - With regard to the above definition, think about how people in ancient times didn't think the Earth was a sphere because it looked like a plane locally.
- This class will look much like a calculus course, in that we first talk about limits, then differentiation, then integration, and culminating in the fundamental theory of calculus.
- Last quarter, we primarily developed linear algebra and basic topology on metric spaces.
  - Chapter 1 of Munkres (1991) is a review of what's needed from last quarter.
  - This is all basically continuity.
- Thus, we can start right up with differentiation.

1.2 The Algebra and Topology of $\mathbb{R}^n$

From Munkres (1991).

1/17:

- "In the first part of this book, $\mathbb{R}^n$ and its subspaces are the only vector spaces with which we shall be concerned. In later chapters, we shall deal with more general vector spaces" (Munkres, 1991, p. 2).
- **Inner product**: *Denoted by* $\langle \mathbf{x}, \mathbf{y} \rangle$.
- **Euclidean norm**: The following norm. *Denoted by* $\|\mathbf{x}\|$. *Given by*

$$\|\mathbf{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

- **Sup norm** (of n-tuples): The following norm. *Denoted by* $|\mathbf{x}|$. *Given by*

$$|\mathbf{x}| = \max\{|x_1|, \dots, |x_n|\}$$

- Note that the Euclidean norm and sup norm satisfy the inequalities

$$|\mathbf{x}| \leq \|\mathbf{x}\| \leq \sqrt{n}|\mathbf{x}|$$

for all $\mathbf{x} \in \mathbb{R}^n$.

- **Sup norm** (of matrices): The following norm. *Denoted by* $\boldsymbol{|A|}$. *Given by*

$$|A| = \max\{|a_{ij}| : i \in [n],\ j \in [m]\}$$

- Theorem 1.3: If A has size n by m and B has size m by p, then

$$|A \cdot B| \leq m|A|\,|B|$$

- **Echelon form**: *Also known as* **stairstep form**.
- **Transpose** (of A): *Denoted by* $\boldsymbol{A^{\mathrm{tr}}}$.
- Theorem 2.1: Let A be an n-by-m matrix. Any elementary row operation on A may be carried out by premultiplying A by the corresponding elementary matrix.
  - "We will use this result later on when we prove the change of variables theorem for a multiple integral" (Munkres, 1991, p. 12).
- **Determinant** (of A): *Denoted by* $\mathbf{det}\,\boldsymbol{A}$. *Not denoted by* $|A|$.
- **Determinant function**: A function that assigns to each n-by-n matrix A a real number denoted $\det A$ and satisfies the following axioms.
  1. If B is the matrix obtained by exchanging any two rows of A, then $\det B = -\det A$.
  2. Given i, the function $\det A$ is linear as a function of the i^{th} row alone.
  3. $\det I_n = 1$.
- Corollary 2.9: The determinant function is uniquely characterized by its three axioms.
- $\boldsymbol{\epsilon}$**-neighborhood** (of x_0): The following set, where X is a metric space with metric d, $x_0 \in X$, and $\epsilon > 0$. *Also known as* $\boldsymbol{\epsilon}$**-neighborhood centered at** $\boldsymbol{x_0}$. *Denoted by* $\boldsymbol{U(x_0;\epsilon)}$. *Given by*

$$U(x_0;\epsilon) = \{x \mid d(x, x_0) < \epsilon\}$$

- **Topological property** (of X): A property of a metric space X that depends only on the collection of open sets of X, rather than on the specific metric involved.
  - Examples include limits, continuity, and compactness.
- **Interior** (of $A \subset \mathbb{R}^n$): The union of all open sets of $\mathbb{R}^n$ that are contained in A. *Denoted by* $\mathbf{Int}\,\boldsymbol{A}$.
- **Exterior** (of $A \subset \mathbb{R}^n$): The union of all open sets of $\mathbb{R}^n$ that are disjoint from A. *Denoted by* $\mathbf{Ext}\,\boldsymbol{A}$.
- **Boundary** (of $A \subset \mathbb{R}^n$): The set of all points of $\mathbb{R}^n$ that are contained in neither $\operatorname{Int} A$ nor $\operatorname{Ext} A$. *Denoted by* $\mathbf{Bd}\,\boldsymbol{A}$.
  - $\mathbf{x} \in \operatorname{Bd} A$ iff every open set containing $\mathbf{x}$ intersects both A and $\mathbb{R}^n \setminus A$.