Chapter 9

Functions of Several Variables

9.1 Notes

2/14: • Plan:
1. Warm-up with matrices.
2. The total derivatives of $f : \mathbb{R}^n \to \mathbb{R}^m$ ($n = m = 2$, i.e., $f : \mathbb{C} \to \mathbb{C}$).
3. Basic properties: Chain rule, relation with partial derivatives, implicit function theorem.

- Let V, W be finite-dimensional vector spaces over $\mathbb{R}$. We let $L(V, W)$ be the vector space of all linear transformations $\phi : V \to W$.
- If we pick bases $N_1, \dots, N_n$ of V and $w_1, \dots, w_m$ of W, then $V \cong \mathbb{R}^n$ and $W \cong \mathbb{R}^m$. It follows that $L(V, W) \cong \mathbb{R}^{mn}$.
- $L(V, W) \times L(W, U) \xrightarrow{\text{compose}} L(V, U)$, i.e., $\mathbb{R}^{mn} \times \mathbb{R}^{nl} \xrightarrow[\text{mult.}]{\text{matrix}} \mathbb{R}^{ml}$.
- Sup norm: If A is an $m \times n$ real matrix, then $\|A\| = \sup_{\substack{\mathbf{x} \in \mathbb{R}^n \\ |\mathbf{x}| = 1}} |A\mathbf{x}|$.
  - Basic properties:
    1. $|A\mathbf{x}| \leq \|A\||x|$.
    2. $\|A\| < \infty$ and all $A : \mathbb{R}^n \to \mathbb{R}^m$ are uniformly continuous.
    3. $\|A\| = 0 \iff A = 0$.
    4. $\|cA\| = |c|\|A\|$.
    5. $\|A + B\| \leq \|A\| + \|B\|$.
    6. $\|AB\| \leq \|A\|\|B\|$.
  - Note that we get a metric space structure on $L(V, W)$ by defining $d(A, B) = \|A - B\|$.
- Proves that 1 and 2 imply the uniform continuity of all A (via Lipschitz continuity).
- **Differentiable** (multivariate function f at $\mathbf{x}_0$): A function $f : U \to \mathbb{R}^m$ ($U \subset \mathbb{R}^n$) such that to $\mathbf{x}_0 \in U$ there corresponds some linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{|f(\mathbf{x}_0 - \mathbf{h}) - f(\mathbf{x}_0) - A\mathbf{h}}{|\mathbf{h}|} = 0$$

- **Total derivative** (of f multivariate at $\mathbf{x}_0$): The linear transformation A in the above definition. *Denoted by* $\boldsymbol{f'(\mathbf{x}_0)}$.
- "An proof and progress in mathematics" - Thurston.

  - Relating to the old one dimensional derivative.
  - A paper we'd find rather impressionistic right now.
- Propositions ahead of us.
  - Proposition: Suppose that f is differentiable at $\mathbf{x}_0 \in U$ and A, B are both derivatives of f at $\mathbf{x}_0$. Then $A = B$.
  - Proposition: Differentiable implies continuous.
  - Proposition: Sum rule, product rule, quotient rule.