1 Differentiation

From Rudin (1976).

Chapter 5

1. Let f be defined for all real x, and suppose that

$$|f(y) - f(x)| \leq (y - x)^2$$

for all real x and y. Prove that f is constant.

Proof. To prove that f is constant, Theorem 5.11b tells us that it will suffice to show that f is differentiable on $\mathbb{R}$ with derivative $f' = 0$. Let $x \in \mathbb{R}$ be arbitrary. We want to show that for all $\epsilon > 0$, there exists a δ such that if $y \in \mathbb{R}$ and $0 < |y - x| < \delta$, then $|[f(y) - f(x)]/(y - x) - 0| < \epsilon$. Let ϵ be arbitrary. Choose $\delta = \epsilon$. Then we have that

$$\begin{aligned}\left|\frac{f(y) - f(x)}{y - x} - 0\right| &= \frac{|f(y) - f(x)|}{|y - x|} \\ &\leq \frac{(y - x)^2}{|y - x|} \\ &\leq |y - x| \\ &< \epsilon\end{aligned}$$

as desired. $\square$

2. Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b) and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)}$$

for $a < x < b$.

Proof. To prove that f is strictly increasing on (a, b), it will suffice to show that $x < y$ implies $f(x) < f(y)$ for all $x, y \in (a, b)$. Let $x, y \in (a, b)$ satisfy $x < y$. Since f is differentiable on (a, b), it is differentiable on $(x, y) \subset (a, b)$ and (by Theorem 5.2) continuous on $[x, y] \subset (a, b)$. Thus, by the MVT, there exists $c \in (x, y)$ such that

$$f(y) - f(x) = (y - x)f'(c)$$

But since $x < y$, $y - x > 0$. This combined with the fact that $f'(c) > 0$ by definition implies that $(y - x)f'(c) > 0$. Consequently,

$$f(x) < f(x) + (y - x)f'(c) = f(y)$$

as desired.

Since f is strictly increasing (and hence 1-1) on (a, b), we may construct a well-defined inverse function $g : f[(a, b)] \to (a, b)$ for it by

$$g(f(x)) = x$$

for all $f(x) \in f[(a, b)]$. It follows by the fact that $f'(x) > 0$ for all $x \in (a, b)$, the definitions of $f'(x)$ and $g'(f(x))$, and Theorem 3.3d that

$$\frac{1}{f'(x)} = \frac{1}{\lim_{y \to x} \frac{f(y) - f(x)}{y - x}}$$

$$\begin{aligned}
&= \lim_{y \to x} \frac{1}{\frac{f(y)-f(x)}{y-x}} \\
&= \lim_{y \to x} \frac{y-x}{f(y)-f(x)} \\
&= \lim_{y \to x} \frac{g(f(y))-g(f(x))}{f(y)-f(x)} \\
&= g'(f(x))
\end{aligned}$$

as desired. □

3. Suppose g is a real function on $\mathbb{R}^1$, with bounded derivative (say $|g'| \leq M$). Fix $\epsilon > 0$ and define $f(x) = x + \epsilon g(x)$. Prove that f is one-to-one if ϵ is small enough. (A set of admissable values of ϵ can be determined which depends only on M.)

Proof. Neglecting the trivial case where $M = 0$, take $\epsilon = 1/2M$. It follows that

$$\begin{aligned}
0 &< 1 - \frac{1}{2} \\
&= 1 + \frac{1}{2M} \cdot -M \\
&\leq 1 + \epsilon g'(x) \\
&= \frac{\mathrm{d}}{\mathrm{d}x}(x) + \frac{\mathrm{d}}{\mathrm{d}x}(\epsilon g) \\
&= f'(x)
\end{aligned}$$

Therefore, by Problem 5.2, f is strictly increasing and, hence, one-to-one. □

4. If

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$$

where $C_0, \dots, C_n$ are real constants, prove that the equation

$$C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 and 1.

Proof. Consider the polynomial

$$f(x) = C_0 x + \frac{C_1}{2}x^2 + \cdots + \frac{C_n}{n+1}x^{n+1}$$

We have that $f(0) = 0$ (by direct substitution) and $f(1) = 0$ (by the constraint on the coefficients). Thus, since f is continuous on $[0,1]$ and differentiable on $(0,1)$ (as a polynomial), we have by the MVT that there exists $x \in (0,1)$ such that

$$\begin{aligned}
f(1) - f(0) &= (1-0)f'(x) \\
f'(x) &= 0 \\
C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n &= 0
\end{aligned}$$

as desired. □

5. Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

Proof. To prove that $\lim_{x\to\infty} g(x) = 0$, it will suffice to show that for every $\epsilon > 0$, there exists $N > 0$ such that if $x > N$, then $|g(x) - 0| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{x\to\infty} f'(x) = 0$ by hypothesis, we know that there exists $N > 0$ such that if $x > N$, then $|f'(x)| < \epsilon$. Choose this N to be our N. Let $x > N$ be arbitrary. Applying the MVT to f on the interval $[x, x+1]$ proves the existence of a c within that closed interval such that

$$f(x+1) - f(x) = f'(c)(x + 1 - x) = f'(c)$$

Additionally, since $c > x > N$, we have that $|f'(c)| < \epsilon$. Therefore, we have that

$$\begin{aligned} |g(x)| &= |f(x+1) - f(x)| \\ &= |f'(c)| \\ &< \epsilon \end{aligned}$$

as desired. $\square$