4 Sequences and Series of Functions

From Rudin (1976).

Chapter 7

2/9: **1.** Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

2. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.

3. Construct sequences $\{f_n\}, \{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converge on E).

4. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?

7. For $n = 1, 2, 3, \dots$ and x real, put

$$f_n(x) = \frac{x}{1 + nx^2}$$

Show that $\{f_n\}$ converges uniformly to a function f and that the equation

$$f'(x) = \lim_{n \to \infty} f_n'(x)$$

is correct if $x \neq 0$ but false if $x = 0$.