

MATH 20410 (Analysis in \mathbb{R}^n II – Accelerated) Problem Sets

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1 Differentiation

From Rudin (1976).

Chapter 5

1. Let f be defined for all real x , and suppose that

$$|f(y) - f(x)| \leq (y - x)^2$$

for all real x and y . Prove that f is constant.

Proof. To prove that f is constant, Theorem 5.11b tells us that it will suffice to show that f is differentiable on \mathbb{R} with derivative $f' = 0$. Let $x \in \mathbb{R}$ be arbitrary. We want to show that for all $\epsilon > 0$, there exists a δ such that if $y \in \mathbb{R}$ and $0 < |y - x| < \delta$, then $|(f(y) - f(x))/(y - x) - 0| < \epsilon$. Let ϵ be arbitrary. Choose $\delta = \epsilon$. Then we have that

$$\begin{aligned} \left| \frac{f(y) - f(x)}{y - x} - 0 \right| &= \frac{|f(y) - f(x)|}{|y - x|} \\ &\leq \frac{(y - x)^2}{|y - x|} \\ &\leq |y - x| \\ &< \epsilon \end{aligned}$$

as desired. □

2. Suppose $f'(x) > 0$ in (a, b) . Prove that f is strictly increasing in (a, b) and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)}$$

for $a < x < b$.

Proof. To prove that f is strictly increasing on (a, b) , it will suffice to show that $x < y$ implies $f(x) < f(y)$ for all $x, y \in (a, b)$. Let $x, y \in (a, b)$ satisfy $x < y$. Since f is differentiable on (a, b) , it is differentiable on $(x, y) \subset (a, b)$ and (by Theorem 5.2) continuous on $[x, y] \subset (a, b)$. Thus, by the MVT, there exists $c \in (x, y)$ such that

$$f(y) - f(x) = (y - x)f'(c)$$

But since $x < y$, $y - x > 0$. This combined with the fact that $f'(c) > 0$ by definition implies that $(y - x)f'(c) > 0$. Consequently,

$$f(y) < f(x) + (y - x)f'(c) = f(y)$$

as desired.

Since f is strictly increasing (and hence 1-1) on (a, b) , we may construct a well-defined inverse function $g : f[(a, b)] \rightarrow (a, b)$ for it by

$$g(f(x)) = x$$

for all $f(x) \in f[(a, b)]$. It follows by the fact that $f'(x) > 0$ for all $x \in (a, b)$, the definitions of $f'(x)$ and $g'(f(x))$, and Theorem 3.3d that

$$\frac{1}{f'(x)} = \frac{1}{\lim_{y \rightarrow x} \frac{f(y) - f(x)}{y - x}}$$

$$\begin{aligned}
&= \lim_{y \rightarrow x} \frac{1}{\frac{f(y)-f(x)}{y-x}} \\
&= \lim_{y \rightarrow x} \frac{y-x}{f(y)-f(x)} \\
&= \lim_{y \rightarrow x} \frac{g(f(y))-g(f(x))}{f(y)-f(x)} \\
&= g'(f(x))
\end{aligned}$$

as desired. \square

3. Suppose g is a real function on \mathbb{R}^1 , with bounded derivative (say $|g'| \leq M$). Fix $\epsilon > 0$ and define $f(x) = x + \epsilon g(x)$. Prove that f is one-to-one if ϵ is small enough. (A set of admissible values of ϵ can be determined which depends only on M .)

Proof. Neglecting the trivial case where $M = 0$, take $\epsilon = 1/2M$. It follows that

$$\begin{aligned}
0 &< 1 - \frac{1}{2} \\
&= 1 + \frac{1}{2M} \cdot -M \\
&\leq 1 + \epsilon g'(x) \\
&= \frac{d}{dx}(x) + \frac{d}{dx}(\epsilon g) \\
&= f'(x)
\end{aligned}$$

Therefore, by Problem 5.2, f is strictly increasing and, hence, one-to-one. \square

4. If

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$$

where C_0, \dots, C_n are real constants, prove that the equation

$$C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n = 0$$

has at least one real root between 0 and 1.

Proof. Consider the polynomial

$$f(x) = C_0x + \frac{C_1}{2}x^2 + \cdots + \frac{C_n}{n+1}x^{n+1}$$

We have that $f(0) = 0$ (by direct substitution) and $f(1) = 0$ (by the constraint on the coefficients). Thus, since f is continuous on $[0, 1]$ and differentiable on $(0, 1)$ (as a polynomial), we have by the MVT that there exists $x \in (0, 1)$ such that

$$\begin{aligned}
f(1) - f(0) &= (1 - 0)f'(x) \\
f'(x) &= 0 \\
C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n &= 0
\end{aligned}$$

as desired. \square

5. Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \rightarrow 0$ as $x \rightarrow +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \rightarrow 0$ as $x \rightarrow +\infty$.

Proof. To prove that $\lim_{x \rightarrow \infty} g(x) = 0$, it will suffice to show that for every $\epsilon > 0$, there exists $N > 0$ such that if $x > N$, then $|g(x) - 0| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{x \rightarrow \infty} f'(x) = 0$ by hypothesis, we know that there exists $N > 0$ such that if $x > N$, then $|f'(x)| < \epsilon$. Choose this N to be our N . Let $x > N$ be arbitrary. Applying the MVT to f on the interval $[x, x+1]$ proves the existence of a c within that closed interval such that

$$f(x+1) - f(x) = f'(c)(x+1-x) = f'(c)$$

Additionally, since $c > x > N$, we have that $|f'(c)| < \epsilon$. Therefore, we have that

$$\begin{aligned} |g(x)| &= |f(x+1) - f(x)| \\ &= |f'(c)| \\ &< \epsilon \end{aligned}$$

as desired. □

2 Differentiation II / Integration

From Rudin (1976).

Chapter 5

8. Suppose f' is continuous on $[a, b]$ and $\epsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \epsilon$$

whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$. (This could be expressed by saying that f is **uniformly differentiable** on $[a, b]$ if f' is continuous on $[a, b]$.) Does this hold for vector-valued functions, too?

Proof. By Theorem 2.40, $[a, b]$ is compact. This combined with the fact that f' is continuous implies by Theorem 4.19 that f' is uniformly continuous. Thus, there exists $\delta > 0$ such that if $x, y \in [a, b]$ and $|y - x| < \delta$, then $|f'(y) - f'(x)| < \epsilon$. Choose this δ to be our δ . Let $x, t \in [a, b]$ be such that $0 < |t - x| < \delta$. Then since f is continuous on $[t, x] \subset [a, b]$ and differentiable on $(t, x) \subset [a, b]$, we have by the MVT that there exists $c \in (t, x)$ such that

$$\begin{aligned} f(t) - f(x) &= (t - x)f'(c) \\ \frac{f(t) - f(x)}{t - x} &= f'(c) \end{aligned}$$

Additionally, since $t < c < x$ and $|t - x| < \delta$, we must have $|c - x| < \delta$, meaning that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| = |f'(c) - f'(x)| < \epsilon$$

as desired.

And yes, this does hold for vector-valued functions, which we can treat component-wise. \square

17. Suppose f is a real, three times differentiable function on $[-1, 1]$ such that

$$f(-1) = 0 \qquad f(0) = 0 \qquad f(1) = 1 \qquad f'(0) = 0$$

Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1, 1)$. Note that equality holds for $\frac{1}{2}(x^3 + x^2)$. (Hint: Use Theorem 5.15 with $\alpha = 0$ and $\beta = \pm 1$ to show that there exist $s \in (0, 1)$ and $t \in (-1, 0)$ such that $f^{(3)}(s) + f^{(3)}(t) = 6$.)

Proof. Since f is three times differentiable on $[-1, 1]$, we know that f'' is differentiable on $[-1, 1]$. It follows by Theorem 5.2 that f'' is continuous on $[-1, 1]$. Thus, since f is defined on $[-1, 1]$, $3 \in \mathbb{N}$, f'' is continuous on $[-1, 1]$, $f^{(3)}$ is defined on $(-1, 1)$, $0, 1 \in [-1, 1]$ such that $0 \neq 1$, and we can define

$$P(t) = \sum_{k=0}^2 \frac{f^{(k)}(0)}{k!} (t - 0)^k$$

we have by Taylor's theorem that there exists $s \in (0, 1)$ such that

$$\begin{aligned} f(1) &= P(1) + \frac{f^{(3)}(s)}{3!} (1 - 0)^3 \\ 1 - \left[\frac{f(0)}{0!} (1 - 0)^0 + \frac{f'(0)}{1!} (1 - 0)^1 + \frac{f''(0)}{2!} (1 - 0)^2 \right] &= \frac{f^{(3)}(s)}{3!} \\ 1 - \left[\frac{f''(0)}{2} \right] &= \frac{f^{(3)}(s)}{6} \\ 6 - 3f''(0) &= f^{(3)}(s) \end{aligned}$$

Similarly, we have that there exists $t \in (-1, 0)$ such that

$$\begin{aligned} f(-1) &= P(-1) + \frac{f^{(3)}(t)}{3!}(-1-0)^3 \\ 0 - \left[\frac{f(0)}{0!}(-1-0)^0 + \frac{f'(0)}{1!}(-1-0)^1 + \frac{f''(0)}{2!}(-1-0)^2 \right] &= -\frac{f^{(3)}(t)}{3!} \\ - \left[\frac{f''(0)}{2} \right] &= -\frac{f^{(3)}(t)}{6} \\ 3f''(0) &= f^{(3)}(s) \end{aligned}$$

Thus,

$$f^{(3)}(s) + f^{(3)}(t) = 3f''(0) + 6 - 3f''(0) = 6$$

Now suppose for the sake of contradiction that for all $x \in (-1, 1)$, we have $f^{(3)}(x) < 3$. Then $f^{(3)}(s) < 3$ and $f^{(3)}(t) < 3$. It follows that $f^{(3)}(s) + f^{(3)}(t) < 6$, a contradiction. \square

25. Suppose f is twice differentiable on $[a, b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geq \delta > 0$, and $0 \leq f''(x) \leq M$ for all $x \in [a, b]$. Let ξ be the unique point in (a, b) at which $f(\xi) = 0$. Complete the details in the following outline of **Newton's method** for computing ξ .

- (a) Choose $x_1 \in (\xi, b)$ and define $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Interpret this geometrically, in terms of a tangent to the graph of f .

Answer. Since we can rearrange the above to $0 - f(x_n) = f'(x_n)(x_{n+1} - x_n)$, we know that x_{n+1} is the point at which the tangent to f at x_n crosses the x -axis. In other words, the zero of the tangent line

$$y - f(x_n) = f'(x_n)(x - x_n)$$

is $(x_{n+1}, 0)$. \square

- (b) Prove that $x_{n+1} < x_n$ and that

$$\lim_{n \rightarrow \infty} x_n = \xi$$

Proof. To prove that $x_{n+1} < x_n$, it will suffice to show that $f(x_n), f'(x_n) > 0$. Since $f'(x) > 0$ for all $x \in [a, b]$ by hypothesis, we know that $f'(x_n) > 0$. As to $f(x_n)$, suppose for the sake of contradiction that $f(x_n) \leq 0$. We know that $f(\xi) = 0$, $f(b) > 0$, and $\xi < x_n < b$. Since ξ is the *unique* point at which $f(\xi) = 0$ by hypothesis and $x_n \neq \xi$, we know that $f(x_n) \neq 0$. And if $f(x_n) < 0$, we have by the intermediate value theorem for f continuous that there exists $c \in (x_n, b)$ such that $f(c) = 0$. But since $\xi < x_n < c < b$, $c \neq \xi$, and thus we have a contradiction here, too.

Having established that $\{x_n\}$ is a monotonically decreasing sequence, Theorem 3.14 tells us that to show that it converges, it will suffice to show that it is bounded. Clearly, $\{x_n\}$ is bounded above by x_1 . And on the bottom side, $\{x_n\}$ is bounded by ξ : If there were $x_n < \xi$, this would imply that $f(x_n) < 0$ by a symmetric argument to the above, meaning that $f(x_n)/f'(x_n) < 0$ and implying that $x_{n+1} > x_n$, a contradiction. Furthermore, we know that the limit (call it x) equals ξ since

$$\begin{aligned} x &= x - \frac{f(x)}{f'(x)} \\ f(x) &= 0 \end{aligned}$$

so $x = \xi$ by the uniqueness of ξ . \square

(c) Use Taylor's theorem to show that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

for some $t_n \in (\xi, x_n)$.

Proof. Since f is defined on $[a, b]$, $2 \in \mathbb{N}$, f' is continuous on $[a, b]$, f'' is defined on (a, b) , $\xi, x_n \in [a, b]$ with $\xi \neq x_n$, and

$$P(t) = \sum_{k=0}^1 \frac{f^{(k)}(x_n)}{k!}(t - x_n)^k$$

we have by Taylor's theorem that there exists $t_n \in (\xi, x_n)$ such that

$$\begin{aligned} f(\xi) &= \left[\frac{f(x_n)}{0!}(\xi - x_n)^0 + \frac{f'(x_n)}{1!}(\xi - x_n)^1 \right] + \frac{f''(t_n)}{2!}(\xi - x_n)^2 \\ 0 &= f(x_n) - f'(x_n)(x_n - \xi) + \frac{f''(t_n)}{2}(x_n - \xi)^2 \\ x_n - \frac{f(x_n)}{f'(x_n)} - \xi &= \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2 \\ x_{n+1} - \xi &= \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2 \end{aligned}$$

as desired. □

(d) If $A = M/2\delta$, deduce that

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A}[A(x_1 - \xi)]^{2n}$$

(Compare with Chapter 3, Exercises 16 and 18.)

Proof. We have from part (b) that $x_i > \xi$ for all $i \in \mathbb{N}$, so naturally $0 \leq x_{n+1} - \xi$. As to the other part of the question, we induct on n . For the base case $n = 1$, we have that

$$\begin{aligned} x_2 - \xi &= \frac{f''(t_1)}{2f'(x_1)}(x_1 - \xi)^2 \\ &\leq \frac{M}{2\delta}(x_1 - \xi)^2 \\ &= \frac{2\delta}{M} \left[\frac{M}{2\delta}(x_1 - \xi) \right]^2 \\ &= \frac{1}{A}[A(x_1 - \xi)]^{2 \cdot 1} \end{aligned}$$

Now suppose inductively that we have proven the claim for $n - 1$; we now seek to prove it for n . Indeed, we have that

$$\begin{aligned} x_{n+1} - \xi &= \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2 \\ &\leq \frac{M}{2\delta}(x_n - \xi)^2 \\ &\leq A \left(\frac{1}{A}[A(x_1 - \xi)]^{2(n-1)} \right)^2 \\ &= \frac{1}{A}[A(x_1 - \xi)]^{2n} \end{aligned}$$

as desired. □

- (e) Show that Newton's method amounts to finding a fixed point of the function g defined by

$$g(x) = x - \frac{f(x)}{f'(x)}$$

How does $g'(x)$ behave for x near ξ ?

Proof. A fixed point of the function g is a point x such that

$$\begin{aligned} g(x) &= x \\ x - \frac{f(x)}{f'(x)} &= x \\ f(x) &= 0 \end{aligned}$$

Thus, if we want to find a point x where $f(x) = 0$, it is equivalent to find a point x such that $g(x) = x$.

As to the other part of the question, we have by the rules of derivatives that

$$\begin{aligned} g'(x) &= 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{f'(x)^2} \\ &= \frac{f(x)f''(x)}{f'(x)^2} \\ &\leq \frac{M}{\delta^2} f(x) \end{aligned}$$

Thus, since $f(x) \rightarrow 0$ as $x \rightarrow \xi$, $g'(x) \rightarrow 0$ as $x \rightarrow \xi$. □

- (f) Put $f(x) = \sqrt[3]{x}$ on $(-\infty, \infty)$ and try Newton's method. What happens?

Answer. We have by the power rule that

$$f'(x) = \frac{1}{3x^{2/3}}$$

Choose $x_1 = 1$. Then

$$\begin{aligned} x_2 &= 1 - \frac{f(1)}{f'(1)} = -2 \\ x_3 &= 1 - \frac{f(-2)}{f'(-2)} = 7 \\ x_4 &= 1 - \frac{f(7)}{f'(7)} = -20 \\ &\vdots \end{aligned}$$

It appears that the series is diverging to ∞ while alternating from positive to negative. In fact, since $x_3 > x_2$, contrary to part (b), we know that something must be wrong (i.e., one of our hypotheses must not be met). Upon further investigation, we can determine that on $[-1, 1]$, we have $f''(1) = -2/9 < 0$; thus, our last hypothesis is the issue with this function. □

Chapter 6

1. Suppose α increases on $[a, b]$, $a \leq x_0 \leq b$, α is continuous at x_0 , $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathcal{R}(\alpha)$ and that $\int f d\alpha = 0$.

Proof. Since f is bounded on $[a, b]$ with only one discontinuity on $[a, b]$ and α is continuous at the point at which f is discontinuous, Theorem 6.10 implies that $f \in \mathcal{R}(\alpha)$, as desired. It follows that $\inf U(P, f, \alpha) = \sup L(P, f, \alpha) = \int f d\alpha$. But since $L(P, f, \alpha) = 0$ for all P (there is no infinite interval $[x_i, x_{i+1}] \subset [a, b]$ that does not contain 0, and f is bounded below by 0), we know that

$$\int f d\alpha = \sup L(P, f, \alpha) = 0$$

as desired. □

2. Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x) dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$. (Compare this with Exercise 1.)

Proof. Suppose for the sake of contradiction that $f(x) \neq 0$ for some x . By the definition of f , this must mean that $f(x) > 0$. It follows since f is continuous that there exists some $N_r(x)$ such that $f(y) > 0$ for all $y \in N_r(x)$. Now consider the partition

$$P = \{a, x - r/2, x + r/2, b\}$$

of $[a, b]$. But since $m_2 > 0$, we have that

$$\begin{aligned} 0 &< m_1[(x - r/2) - a] + m_2[(x + r/2) - (x - r/2)] + m_3[b - (x + r/2)] \\ &= L(P, f) \\ &\leq \int_a^b f(x) dx \end{aligned} \quad \text{Theorem 6.4}$$

a contradiction. □

4. If $f(x) = 0$ for all irrational x and $f(x) = 1$ for all rational x , prove that $f \notin \mathcal{R}$ on $[a, b]$ for any $a < b$.

Proof. Let P be an arbitrary partition of $[a, b]$. Since the rationals and irrationals are dense in the reals, we know that for any $[x_i, x_{i+1}]$, $f(x) = 0$ for some $x \in [x_i, x_{i+1}]$ and $f(x) = 1$ for some $x \in [x_i, x_{i+1}]$. Thus, we have that $L(P, f) = 0$ and $U(P, f) = b - a$. It follows that if $a < b$,

$$\sup L(P, f) = 0 \neq b - a = \inf U(P, f)$$

so $f \notin \mathcal{R}$, as desired. □

3 Integration II

From Rudin (1976).

Chapter 6

2/2: 3. Define three functions $\beta_1, \beta_2, \beta_3$ as follows:

$$\beta_1 = \begin{cases} 0 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases} \quad \beta_2 = \begin{cases} 0 & x < 0 \\ 1 & x = 0 \\ 1 & x > 0 \end{cases} \quad \beta_3 = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases}$$

Let f be a bounded function on $[-1, 1]$.

(a) Prove that $f \in \mathcal{R}(\beta_1)$ if and only if $f(0+) = f(0)$ and that then

$$\int f d\beta_1 = f(0)$$

Proof. Suppose first that $f \in \mathcal{R}(\beta_1)$ with $\int f d\beta_1 = f(0)$. To prove that $f(0+) = f(0)$, it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in [-1, 1]$ and $0 \leq x < \delta$, then $|f(x) - f(0)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $f \in \mathcal{R}(\beta_1)$ by hypothesis, we have by Theorem 6.6 that there exists a partition P of $[-1, 1]$ such that $U(P, f, \beta_1) - L(P, f, \beta_1) < \epsilon$. Now let $x_i = \min\{x \in P : x > 0\}$; we know that such an object exists since there exist elements of P greater than zero (namely 1) and P is finite. It follows by the definition of β_1 that $\Delta x_i = 1$ and $\Delta x_j = 0$ for $j \neq i$. Thus, $U(P, f, \beta_1) = M_i$ and $L(P, f, \beta_1) = m_i$ (which exist because f is bounded on $[-1, 1]$). At this point, we are ready to choose δ , which we take to be $\delta = x_i$. Now to confirm that this δ works: Let $0 \leq x < \delta$. By the definition of x_i, x_{i-1} , $m_i \leq f(x) \leq M_i$ and $m_i \leq f(0) \leq M_i$. But since $M_i - m_i < \epsilon$ as per the above, we have that $|f(x) - f(0)| < \epsilon$, as desired.

Now suppose that $f(0+) = f(0)$. To prove that $f \in \mathcal{R}(\beta_1)$, Theorem 6.6 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a P such that $U(P, f, \beta_1) - L(P, f, \beta_1) < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $f(0+) = f(0)$, we know that there exists a $\delta' > 0$ such that if $x \in [-1, 1]$ and $0 \leq x < \delta'$, then $|f(x) - f(0)| < \epsilon/3$. Let $\delta = \min(\delta'/2, 1)$. Thus, we may define $P = \{-1, 0, \delta, 1\}$. We have

$$\begin{aligned} U(P, f, \beta_1) &= \sum_{i=1}^3 M_i \Delta \beta_{1_i} & L(P, f, \beta_1) &= \sum_{i=1}^3 m_i \Delta \beta_{1_i} \\ &= M_2 & &= m_2 \end{aligned}$$

(which exist because f is bounded on $[-1, 1]$). Consequently, $M_2 \leq f(0) + \epsilon/3$. $m_2 \geq f(0) - \epsilon/3$. Therefore,

$$\begin{aligned} U(P, f, \beta_1) - L(P, f, \beta_1) &= M_2 - m_2 \\ &\leq [f(0) + \frac{\epsilon}{3}] - [f(0) - \frac{\epsilon}{3}] \\ &= \frac{2\epsilon}{3} \\ &< \epsilon \end{aligned}$$

as desired.

As to proving that $\int f d\beta_1$, we know that $M_2 \leq f(0) + \epsilon/3$ for arbitrarily small ϵ implies $M_2 \leq f(0)$. Similarly, $m_2 \geq f(0)$. Thus,

$$\inf U(P, f, \beta_1) \leq U(P, f, \beta_1) = M_2 \leq f(0) \leq m_2 = L(P, f, \beta_1) \leq \sup L(P, f, \beta_1)$$

But by Theorem 6.5, $\sup L(P, f, \beta_1) \leq \inf U(P, f, \beta_1)$. Therefore,

$$\int_{-1}^1 f d\beta_1 = \sup L(P, f, \beta_1) = \inf U(P, f, \beta_1) = f(0)$$

as desired. □

- (b) State and prove a similar result for β_2 .

Proof. The result will be $f \in \mathcal{R}(\beta_2)$ if and only if $f(0-) = f(0)$ and that then

$$\int f d\beta = f(0)$$

The proof of this result is entirely symmetric to the proof of the previous result. □

- (c) Prove that $f \in \mathcal{R}(\beta_3)$ if and only if f is continuous at 0.

Proof. Suppose first that $f \in \mathcal{R}(\beta_3)$. To prove that f is continuous at 0, it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in [-1, 1]$ and $|x| < \delta$, then $|f(x) - f(0)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $f \in \mathcal{R}(\beta_3)$ by hypothesis, we have by Theorem 6.6 that there exists a partition P of $[-1, 1]$ such that $U(P, f, \beta_3) - L(P, f, \beta_3) < \epsilon/2$. Now let $x_i = \max\{x \in P : x < 0\}$ and let $x_j = \min\{x \in P : x > 0\}$. Choose $\delta = \min\{|x_i|, |x_j|\}$. Let $P^* = P \cup \{-\delta, 0, \delta\}$ be a refinement of P . It follows by the definition of β_3 and a reenumeration of P^* that $U(P^*, f, \beta_3) = (M_{i-1} + M_i)/2$ and $L(P^*, f, \beta_3) = (m_{i-1} + m_i)/2$. Now let $|x| < \delta$. We divide into two cases ($x \geq 0$ and $x < 0$). If $x \geq 0$, then $m_i \leq f(x) \leq M_i$ and $m_i \leq f(0) \leq M_i$. But then we have that

$$\begin{aligned} |f(x) - f(0)| &\leq M_i - m_i \\ &\leq (M_{i-1} - m_{i-1}) + (M_i - m_i) \\ &= 2 \left[\frac{M_{i-1} + M_i}{2} - \frac{m_{i-1} + m_i}{2} \right] \\ &= 2[U(P^*, f, \beta_3) - L(P^*, f, \beta_3)] \\ &< \epsilon \end{aligned}$$

as desired. The proof is symmetric in the other case.

Now suppose that f is continuous at 0. To prove that $f \in \mathcal{R}(\beta_3)$, Theorem 6.6 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a P such that $U(P, f, \beta_3) - L(P, f, \beta_3) < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f is continuous at 0, we know that there exists a $\delta' > 0$ such that if $x \in [-1, 1]$ and $|x| < \delta'$, then $|f(x) - f(0)| < \epsilon/3$. Choose $\delta = \min(\delta'/2, 1)$. Consider $P = \{-1, -\delta/2, \delta/2, 1\}$. It follows as before that $U(P, f, \beta_3) = M_2$ and $L(P, f, \beta_3) = m_2$. Consequently, $M_2 \leq f(0) + \epsilon/3$ and $m_2 \geq f(0) - \epsilon/3$. Therefore,

$$\begin{aligned} U(P, f, \beta_3) - L(P, f, \beta_3) &= M_2 - m_2 \\ &\leq [f(0) + \frac{\epsilon}{3}] - [f(0) - \frac{\epsilon}{3}] \\ &= \frac{2\epsilon}{3} \\ &< \epsilon \end{aligned}$$

as desired. □

- (d) If f is continuous at 0, prove that

$$\int f d\beta_1 = \int f d\beta_2 = \int f d\beta_3 = f(0)$$

Proof. If f is continuous at 0, then $f(0+) = f(0) = f(0-)$. It follows that

$$f(0) = \int f \, d\beta_1 \quad \text{Part (a)}$$

$$= \int f \, d\beta_2 \quad \text{Part (b)}$$

$$= \int f \, d\beta_3 \quad \text{Part (c)}$$

Note that calculating the exact value of $\int f \, d\beta_3$ is symmetric to the proof in part (a). \square

5. Suppose f is a bounded real function on $[a, b]$, and $f^2 \in \mathcal{R}$ on $[a, b]$. Does it follow that $f \in \mathcal{R}$? Does the answer change if we assume that $f^3 \in \mathcal{R}$?

Proof. $f^2 \in \mathcal{R} \nRightarrow f \in \mathcal{R}$: Consider the bounded real function $f : [a, b] \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & x \notin \mathbb{Q} \\ -1 & x \in \mathbb{Q} \end{cases}$$

Since $f^2(x) = 1$ for all $x \in [a, b]$, $f^2 \in \mathcal{R}$ as a constant function. However, by Exercise 6.4 and a clever application of Theorem 6.12 (to relate it to the function explicitly considered in Exercise 6.4), we know that $f \notin \mathcal{R}$.

$f^3 \in \mathcal{R} \Rightarrow f \in \mathcal{R}$: Let $f : [a, b] \rightarrow \mathbb{R}$ be any bounded real function such that $f^3 \in \mathcal{R}$. To prove that $f \in \mathcal{R}$, Theorem 6.11 tells us that it will suffice to show that there exist $m, M \in \mathbb{R}$ such that $m \leq f \leq M$ and that there exists a continuous function $\phi : [m, M] \rightarrow \mathbb{R}$ such that $f = \phi \circ f^3$. Since f is bounded by hypothesis, we can pick $m, M \in \mathbb{R}$ such that $m \leq f \leq M$. Now let $\phi : [m, M] \rightarrow \mathbb{R}$ be defined by

$$\phi(x) = \sqrt[3]{x}$$

for all $x \in [m, M]$. It is obvious that ϕ is continuous and that $\phi \circ f^3 = f$, as desired. \square

7. Suppose f is a real function on $(0, 1]$ and $f \in \mathcal{R}$ on $[c, 1]$ for every $c > 0$. Define

$$\int_0^1 f(x) \, dx = \lim_{c \rightarrow 0} \int_c^1 f(x) \, dx$$

if this limit exists (and is finite).

- (a) If $f \in \mathcal{R}$ on $[0, 1]$, show that this definition of the integral agrees with the old one.

Proof. To prove that $\int_0^1 f = \lim_{c \rightarrow 0} \int_c^1 f$, it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $c \in (0, 1]$ and $c < \delta$, then

$$\left| \int_0^c f \right| = \left| \int_c^1 f - \int_0^1 f \right| < \epsilon$$

Let $\epsilon > 0$ be arbitrary. Since f is integrable, f is bounded, i.e., there exists $M \in \mathbb{R}$ such that $|f(x)| < M$ for all $x \in [0, 1]$. Choose $\delta = \epsilon/M$. Let $c \in (0, 1]$ be such that $c < \delta$. Then by Theorem 6.12d,

$$\left| \int_0^c f \right| \leq M(c - 0) < \epsilon$$

as desired. \square

- (b) Construct a function f such that the above limit exists, although it fails to exist with $|f|$ in place of f .

Proof. Let $f : (0, 1] \rightarrow \mathbb{R}$ be defined by

$$f(x) = (-1)^n n$$

for $1/n < x \leq 1/(n-1)$ ($n = 2, 3, \dots$). It follows since f is a constant function save one terminal discontinuity on each $[1/n, 1/(n-1)]$ that

$$\begin{aligned} \int_{1/n}^{1/(n-1)} f &= (-1)^n n \cdot \left(\frac{1}{n-1} - \frac{1}{n} \right) \\ &= \frac{(-1)^n n}{n(n-1)} \\ &= \frac{(-1)^n}{n-1} \end{aligned}$$

for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned} \int_{1/N}^1 f &= \sum_{n=2}^N \int_{1/n}^{1/(n-1)} f \\ &= \sum_{n=2}^N \frac{(-1)^n}{n-1} \end{aligned}$$

Thus,

$$\lim_{c \rightarrow 0} \int_c^1 f = \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1}$$

which converges by Theorem 3.43. However, the limit fails to exist if f is replaced by $|f|$, because in that case, the integral is equal to the harmonic series, which diverges to infinity. \square

8. Suppose $f \in \mathcal{R}$ on $[a, b]$ for every $b > a$ where a is fixed. Define

$$\int_a^{\infty} f(x) \, dx = \lim_{b \rightarrow \infty} \int_a^b f(x) \, dx$$

if this limit exists (and is finite). In that case, we say that the integral on the left **converges**. If it also converges after f has been replaced by $|f|$, it is said to converge **absolutely**.

Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1, \infty)$. Prove that $\int_1^{\infty} f(x) \, dx$ converges if and only if $\sum_{n=1}^{\infty} f(n)$ converges. (This is the so-called “integral test” for convergence of series.)

Proof. To prove the claim, we will show that

$$\sum_{n=2}^N f(n) \leq \int_1^N f \leq \sum_{n=1}^{N-1} f(n) \leq f(1) + \int_1^{N-1} f(x) \, dx$$

It will follow since both the sum and the integral limit are monotonically increasing as $N \rightarrow \infty$ ($f \geq 0$) and both are bounded below and above by (a function of) the other, both converge or diverge together. Let's begin.

Since f is monotonically decreasing on $[1, \infty)$, we know that $f(n) \leq f(x)$ for all $1 \leq x \leq n$ ($n \in \mathbb{N}$). Thus, by Theorem 6.12b,

$$\int_{n-1}^n f(n) \, dx \leq \int_{n-1}^n f(x) \, dx$$

Therefore,

$$\sum_{n=2}^N f(n) = \sum_{n=2}^N \int_{n-1}^n f(n) \, dx \quad \text{Theorem 6.12d}$$

$$\leq \sum_{n=2}^N \int_{n-1}^n f(x) \, dx$$

$$= \int_1^N f(x) \, dx \quad \text{Theorem 6.12c}$$

for all $N = 2, 3, 4, \dots$, thereby establishing the left inequality above.

Since f is monotonically decreasing on $[1, \infty)$, we know that $f(x) \leq f(n)$ for all $x \geq n$ ($n \in \mathbb{N}$). Thus, by Theorem 6.12b,

$$\int_n^{n+1} f(x) \, dx \leq \int_n^{n+1} f(n) \, dx$$

Therefore,

$$\int_1^N f(x) \, dx = \sum_{n=1}^{N-1} \left(\int_n^{n+1} f(x) \, dx \right) \quad \text{Theorem 6.12c}$$

$$\leq \sum_{n=1}^{N-1} \left(\int_n^{n+1} f(n) \, dx \right)$$

$$= \sum_{n=1}^{N-1} f(n) \quad \text{Theorem 6.12d}$$

for all $N = 2, 3, 4, \dots$, thereby establishing the middle inequality above.

From our statement about $f(n)$ and $f(x)$ from the left inequality, we have by Theorem 6.12b that

$$\int_{n-1}^n f(n) \, dx \leq \int_{n-1}^n f(x) \, dx$$

Therefore,

$$\sum_{n=1}^{N-1} f(n) = f(1) + \sum_{n=2}^{N-1} \int_{n-1}^n f(n) \, dx \quad \text{Theorem 6.12d}$$

$$\leq f(1) + \sum_{n=2}^{N-1} \int_{n-1}^n f(x) \, dx$$

$$= f(1) + \int_1^{N-1} f(x) \, dx \quad \text{Theorem 6.12c}$$

for all $N = 2, 3, 4, \dots$, thereby establishing the right inequality above. \square

10. Let p, q be positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

Prove the following statements.

(a) If $u, v \geq 0$, then

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}$$

Equality holds if and only if $u^p = v^q$.

Discussion. To prove the desired inequality, it will suffice to show that

$$0 \leq \frac{u^p}{p} + \frac{v^q}{q} - uv$$

i.e., that for all $u, v \geq 0$, the expression on the right above is nonnegative. To consider all such values at once, we can consider applying our analysis toolbox to $f : [0, \infty)^2 \rightarrow \mathbb{R}$ defined by

$$f(u, v) = \frac{u^p}{p} + \frac{v^q}{q} - uv$$

with the goal of proving that it is nonnegative everywhere on its domain. However, since we do not yet know multivariable calculus, it will suffice to fix $u \geq 0$ and analyze $f : [0, \infty) \rightarrow \mathbb{R}$ defined by

$$f(v) = \frac{u^p}{p} + \frac{v^q}{q} - uv$$

Let's begin. □

Proof. Fix $u \geq 0$. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be defined by

$$f(v) = \frac{u^p}{p} + \frac{v^q}{q} - uv$$

It follows from the definition of f that to prove the desired inequality, it will suffice to show that f is nonnegative everywhere on its domain. Let's begin.

Since f is a polynomial in v , f is differentiable. Thus, we may consider

$$f'(v) = v^{q-1} - u$$

As a function of a positive power ($q/(q-1) = p > 0$ and $q > 0$ imply $q-1 > 0$) of its variable (minus a constant), f' is strictly increasing. Additionally, we have that

$$\begin{aligned} 0 &= f'(v) \\ u &= v^{q-1} \\ &= v^{q/p} \\ v &= u^{p/q} \end{aligned}$$

Thus, we know that $f' < 0$ on $(0, u^{p/q})$ and $f' > 0$ on $(u^{p/q}, \infty)$. It follows by the strict version of Theorem 5.11 that f is strictly decreasing on $(0, u^{p/q})$ and strictly increasing on $(u^{p/q}, \infty)$. Furthermore, since f is differentiable (hence continuous by Theorem 5.2), we know that $f(0) \geq f(u^{p/q})$. Combining the last several results, we have that $f(u^{p/q})$ is the minimum of f over $[0, \infty)$, and hence equal to the minimum value of f over $[0, \infty)$. But since

$$\begin{aligned} f(u^{p/q}) &= \frac{u^p}{p} + \frac{(u^{p/q})^q}{q} - uu^{p/q} \\ &= \frac{u^p}{p} + \frac{u^p}{q} - u^{p/q+1} \\ &= u^p \left(\frac{1}{p} + \frac{1}{q} \right) - u^p \\ &= 0 \end{aligned}$$

we know that $f(v) \geq 0$ on its domain, as desired.

Additionally, since f is strictly decreasing on $(0, u^{p/q})$ and strictly increasing on $(u^{p/q}, \infty)$, we know that $f(v) = 0$ iff $v = u^{p/q}$, i.e., iff $v^q = u^p$, as desired. □

(b) If $f, g \in \mathcal{R}(\alpha)$, $f, g \geq 0$, and

$$\int_a^b f^p d\alpha = 1 = \int_a^b g^q d\alpha$$

then

$$\int_a^b fg d\alpha \leq 1$$

Proof. By Theorem 6.13a, the hypothesis $f, g \in \mathcal{R}(\alpha)$ implies that $fg \in \mathcal{R}(\alpha)$. Thus, we have that

$$\int_a^b fg d\alpha \leq \int_a^b \left(\frac{f^p}{p} + \frac{g^q}{q} \right) d\alpha \quad \text{Theorem 6.12b}$$

$$= \frac{1}{p} \int_a^b f^p d\alpha + \frac{1}{q} \int_a^b g^q d\alpha \quad \text{Theorem 6.12a}$$

$$= \frac{1}{p} + \frac{1}{q}$$

$$= 1$$

as desired. □

(c) If f, g are complex functions in $\mathcal{R}(\alpha)$, then

$$\left| \int_a^b fg d\alpha \right| \leq \left(\int_a^b |f|^p d\alpha \right)^{1/p} \left(\int_a^b |g|^q d\alpha \right)^{1/q}$$

This is **Hölder's inequality**. When $p = q = 2$, it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)

Proof. By Theorem 6.11 with $\phi(y) = |y|^p$ (resp. $\phi(y) = |y|^q$), the hypothesis $f, g \in \mathcal{R}(\alpha)$ implies that $|f|^p, |g|^q \in \mathcal{R}(\alpha)$. Thus, we may let

$$I_f = \left(\int_a^b |f|^p d\alpha \right)^{1/p} \quad I_g = \left(\int_a^b |g|^q d\alpha \right)^{1/q}$$

We divide into two cases ($I_f = 0$ or $I_g = 0$, and $I_f, I_g \neq 0$). In the first case, WLOG let $I_f = 0$. Then since $0 \leq |f|^p$, it follows that $f = 0$ on $[a, b]$. Thus

$$\left| \int_a^b fg d\alpha \right| = 0 \leq 0 = I_f I_g = \left(\int_a^b |f|^p d\alpha \right)^{1/p} \left(\int_a^b |g|^q d\alpha \right)^{1/q}$$

as desired. In the other case, it follows that

$$\begin{aligned} I_f^p &= \int_a^b |f|^p d\alpha & I_g^q &= \int_a^b |g|^q d\alpha \\ 1 &= \int_a^b \left| \frac{f}{I_f} \right|^p d\alpha & 1 &= \int_a^b \left| \frac{g}{I_g} \right|^q d\alpha \end{aligned} \quad \text{Theorem 6.12a}$$

Thus, since $|f/I_f|, |g/I_g| \in \mathcal{R}(\alpha)$ by Theorems 6.12 and 6.13 and $|f/I_f|, |g/I_g| \geq 0$ by the defini-

tion of the absolute value, we have that

$$\begin{aligned}
 \left| \int_a^b fg \, d\alpha \right| &\leq \int_a^b |fg| \, d\alpha && \text{Theorem 6.13b} \\
 &= I_f I_g \int_a^b \left| \frac{f}{I_f} \right| \left| \frac{g}{I_g} \right| d\alpha \\
 &\leq I_f I_g \cdot 1 && \text{Part (b)} \\
 &= \left(\int_a^b |f|^p \, d\alpha \right)^{1/p} \left(\int_a^b |g|^q \, d\alpha \right)^{1/q}
 \end{aligned}$$

as desired. \square

11. Let α be a fixed increasing function on $[a, b]$. For $u \in \mathcal{R}(\alpha)$, define

$$\|u\|_2 = \left(\int_a^b |u|^2 \, d\alpha \right)^{1/2}$$

Suppose $f, g, h \in \mathcal{R}(\alpha)$, and prove the triangle inequality

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2$$

as a consequence of the Schwarz inequality, as in the proof of Theorem 1.37.

Proof. By Theorems 6.12a and 6.13b, the hypothesis that $f, g, h \in \mathcal{R}(\alpha)$ implies that $|f - g|, |g - h| \in \mathcal{R}(\alpha)$. Thus, we have that

$$\begin{aligned}
 \|f - h\|_2^2 &= \int_a^b |f - h|^2 \, d\alpha \\
 &= \int_a^b |(f - g) + (g - h)|^2 \, d\alpha \\
 &= \int_a^b |f - g|^2 \, d\alpha + 2 \int_a^b |f - g| \cdot |g - h| \, d\alpha + \int_a^b |g - h|^2 \, d\alpha \\
 &\leq \int_a^b |f - g|^2 \, d\alpha + 2 \left(\int_a^b |f - g|^2 \, d\alpha \right)^{1/2} \left(\int_a^b |g - h|^2 \, d\alpha \right)^{1/2} + \int_a^b |g - h|^2 \, d\alpha \\
 &= \|f - g\|_2^2 + 2\|f - g\|_2 \|g - h\|_2 + \|g - h\|_2^2 \\
 &= (\|f - g\|_2 + \|g - h\|_2)^2
 \end{aligned}$$

Taking square roots of both sides of the inequality yields the desired result. \square

4 Sequences and Series of Functions

From Rudin (1976).

Chapter 7

- 2/9: 1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.
2. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E , prove that $\{f_n + g_n\}$ converges uniformly on E . If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E .
3. Construct sequences $\{f_n\}, \{g_n\}$ which converge uniformly on some set E , but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converge on E).
4. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?

7. For $n = 1, 2, 3, \dots$ and x real, put

$$f_n(x) = \frac{x}{1 + nx^2}$$

Show that $\{f_n\}$ converges uniformly to a function f and that the equation

$$f'(x) = \lim_{n \rightarrow \infty} f'_n(x)$$

is correct if $x \neq 0$ but false if $x = 0$.

References

Rudin, W. (1976). *Principles of mathematical analysis* (A. A. Arthur & S. L. Langman, Eds.; Third). McGraw-Hill.