Chapter 7

Sequences and Series of Functions

7.1 Notes

1/31:

- Midterm on differentiation and integration, and a bit of stuff from this week.
- Plan:
  - Talk about sequences of functions, all with the same domain and range, converging.
  - Address what properties of f_n remain in the limit (e.g., continuity, differentiability, integrability).
    - The answer depends on what we mean by "convergence."
    - $f_n \to f$ pointwise implies basically nothing.
    - $f_n \to f$ uniformly implies that basically everything works out nicely.
- We'll restrict ourselves to real functions because those have all the properties (integrability, differentiability, etc.) that we care about.
- **Pointwise** (convergent sequence $\{f_n\}$ to f): A sequence of functions $\{f_n\}$ such that for all $x \in X$, the sequence $\{f_n(x)\}$ converges to $f(x)$, where $f_n : X \to \mathbb{R}$ for all $n \in \mathbb{N}$ and $f : X \to \mathbb{R}$. *Denoted by* $f_n \to f$.
- Bad functions.
  - Consider $f_n : [0,1] \to \mathbb{R}$ defined by $x \mapsto x^n$. Each f_n is continuous, but f is not (zero everywhere except $f(1) = 1$)[1].
  - Consider $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = x^2/(1+x^2)^n$, and $f(x) = \sum_{n=0}^{\infty} f_n(x)$. As a geometric series, $f(x) = 1 + x^2$ when $x \neq 0$ but $f(0) = 0$. Thus, the limit exists but is not continuous once again.
  - Consider $f_m : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto \lim_{n\to\infty} \cos^{2n}(m!\pi x)$. Each f_m is integrable, but the limit f is the function that's 1 for rationals and zero for irrationals. In particular, f is not integrable.
    - We take even powers of the cosine to make it always positive.
    - We use $\cos^2(x)$ just because its always between $[0,1]$, and we know when it is equal to 1.
    - In particular, $\cos^2(\pi x)$ is equal to 1 at every integer, $\cos^2(2\pi x)$ is equal to 1 at every half integer. $\cos^2(6\pi x)$ is equal to 1 at every one-sixth of an integer.
    - Then raising it to the n^{th} power just makes it spiky.
- Aside: Interchanging limits.
  - If all f_n are continuous, then $\lim_{x\to x_0} f_n(x) = f_n(x_0)$.

[1] Questions that require counterexamples like this could show up on the midterm!

- The question "is f continuous" is equivalent to being able to interchange limits:

$$\lim_{x\to x_0}\lim_{n\to\infty} f_n(x) = f(x_0) = \lim_{n\to\infty}\lim_{x\to x_0} f_n(x)$$

- Sequence example showing we need to be careful interchanging limits: $s_{n,m} = m/(m+n)$.

- All of this pathology goes away with the right definition, though.
- **Uniformly** (convergent sequence $\{f_n\}$ to f): A sequence of functions $\{f_n\}$ such that for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$ for all $x \in X$, where $f_n : X \to \mathbb{R}$ for all $n \in \mathbb{N}$ and $f : X \to \mathbb{R}$.
- Proposition (Cauchy criterion for uniform convergence): $f_n \to f$ uniformly iff for all $\epsilon > 0$, there exists N such that for all $m, n \geq N$ and for all $x \in X$, $|f_n(x) - f_m(x)| < \epsilon$.
  - Forward direction: Let $\epsilon > 0$. Suppose $f_n \to f$ uniformly. Choose N such that the functions are within $\epsilon/2$. Then

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

2/2:

- Office hours tomorrow 4-5 PM.
- Plan:
  1. More on uniform convergence.
     - Limit of continuous functions is continuous.
     - Limit of the integral of functions is the integral of the limit.
  2. $\mathcal{C}(X)$ perspectives on uniform convergence.
- Corollary (Weierstraß M-test): If there exist constants $M_n \in \mathbb{R}$ such that $|f_n(x)| \leq M_n$ for all x and $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly.
- Theorem: $f_n : X \to \mathbb{R}$, f_n continuous at $x_0 \in X$ for all n, and $f_n \to f$ uniformly imply f continuous at x_0.
  - Idea:
    - "$\epsilon/3$ trick": Find δ such that if $|x - x_0| < \delta$, then

$$|f(x) - f(x_0)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

  - Proof:
    - $f_n \to f$ uniformly implies there exists $N \in \mathbb{N}$ such that $|f_N(x) - f(x)| < \epsilon/3$ for all $x \in X$.
    - f_N continuous at x_0: There exists δ such that if $d(x, x_0) < \delta$, then $|f_N(x) - f_N(x_0)| < \epsilon/3$.
    - Thus, by the $\epsilon/3$ trick, we have the continuity of f.
- Theorem: $f_n : [a, b] \to \mathbb{R}$ integrable and $f_n \to f$ uniformly implies f is integrable and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$$

  - Plan:
    1. Show f is integrable.
    2. Show $\int f = \lim \int f_n$.
  - Proof:

- Fix some partition P and write

$$U(f_n, P) = \sum_{i=1}^{k} M_i^n \Delta x_i \qquad\qquad L(f_n, P) = \sum_{i=1}^{k} m_i^n \Delta x_i$$
$$U(f, P) = \sum_{i=1}^{k} M_i \Delta x_i \qquad\qquad L(f, P) = \sum_{i=1}^{k} m_i \Delta x_i$$

- WTS: For large enough n, we can make all $|M_i^n - M_i| < \epsilon$ and $|m_i^n - m_i| < \epsilon$.
- We can do the above by uniform convergence.
- We end up with

$$\begin{aligned} |U(f_n, P) - U(f, P)| &\leq \sum_{i=1}^{n} |M_i^n - M_i| \Delta x_i \\ &< \epsilon(b-a) \end{aligned}$$

- Klug had assumed that the functions achieve their maxima/minima on the same interval and forgets how to finish the proof.

- Defining a norm on $\mathcal{C}(X)$.

$$\|f\| = \sup_{x \in X} |f(x)|$$

  - This makes $\mathcal{C}(X)$ into a vector space.
  - We can now define our metric $d(f,g)$ by $d(f,g) = \|f - g\|$.

- $f_n \to f \iff f$ is bounded.
  - $f_n \to f$ uniformly $\iff \lim_{n\to\infty} \sup |f_n(x) - f(x)| = 0 \iff f_n \to f$ is $\mathcal{C}(X)$.

- Corollary to the Weierstraß M-test: $\mathcal{C}(X)$ is complete (i.e., all uniformly Cauchy sequences converge).
  - Assume $\{f_n\}$ is Cauchy. Then by the Cauchy criterion for uniform convergence, f_n converges uniformly to some f. But this f must be continuous, too, meaning $f \in \mathcal{C}(X)$.