MATH 20800 (Honors Analysis in $\mathbb{R}^n$ II) Notes

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Contents

Chapter 1

The Algebra and Topology of $\mathbb{R}^n$

1.1 Notes

- Syllabus.

    - In his mind, homework is the main setting where learning takes place.

- We're going to be studying analysis, or calculus, on **manifolds** this quarter.

- **Manifold**: A "space" that looks like Euclidean space $\mathbb{R}^n$ locally.

    - The surfaces of a sphere and torus are common examples of 2-dimensional manifolds.
    - With regard to the above definition, think about how people in ancient times didn't think the Earth was a sphere because it looked like a plane locally.

- This class will look much like a calculus course, in that we first talk about limits, then differentiation, then integration, and culminating in the fundamental theory of calculus.

- Last quarter, we primarily developed linear algebra and basic topology on metric spaces.

    - Chapter 1 of Munkres (1991) is a review of what's needed from last quarter.
    - This is all basically continuity.

- Thus, we can start right up with differentiation.

Chapter 2

Differentiation

2.1 Notes

• Since manifolds look like Euclidean spaces locally, we basically only need to study differentiation on Euclidean spaces.

  • Set up: Let $U \subset \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}^n$ be a function.

  • Idea: The derivative of f at some point $\mathbf{a} \in U$ is "the best linear approximation" to f at $\mathbf{a}$.

  • **Differentiable** (function f at $\mathbf{a}$): A function f for which there exists a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$ such that
  $$\lim_{\mathbf{h} \to \mathbf{0}} \frac{f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - A\mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0}$$

  • **Total derivative** (of f at $\mathbf{a}$): The linear transformation A corresponding to a differentiable function f. *Denoted by* $\boldsymbol{Df}(\mathbf{a})$.

  • Questions to ask:

    1. When does the total derivative exist?
    2. When it does exist, can there be multiple?
    3. When it exists and is unique, how do I calculate it?

  • Proposition: If A, B are linear transformations that both satisfy the definition, then $A = B$.

    – We have

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - A\mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0} \qquad\qquad \lim_{\mathbf{h} \to \mathbf{0}} \frac{f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - B\mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0}$$

    – It follows by subtracting the right equation above from the left one that

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{A\mathbf{h} - B\mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0}$$

    – Apply linearity: For $\mathbf{v} \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $t > 0$, we have

$$\frac{A(t\mathbf{v}) - B(t\mathbf{v})}{t} = A\mathbf{v} - B\mathbf{v}$$

– Therefore, since $t\mathbf{v} \to 0$ as $t \to 0$, we have by the above that

$$\mathbf{0} = \lim_{t \to 0} \frac{A(t\mathbf{v}) - B(t\mathbf{v})}{\|t\mathbf{v}\|}$$

$$= \lim_{t \to 0} \frac{A\mathbf{v} - B\mathbf{v}}{\|\mathbf{v}\|}$$

$$\mathbf{0} \cdot \|\mathbf{v}\| = \lim_{t \to 0}(A\mathbf{v} - B\mathbf{v})$$

$$\mathbf{0} = A\mathbf{v} - B\mathbf{v}$$

$$B\mathbf{v} = A\mathbf{v}$$

- Example: Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be linear, i.e., $f(\mathbf{v}) = A\mathbf{v}$ for some linear transformation A. Then for all $\mathbf{a} \in \mathbb{R}^n$, $Df(\mathbf{a}) = A$ is constant.

    – We have from the definition that

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - A\mathbf{h}}{\|\mathbf{h}\|} = \lim_{\mathbf{h} \to \mathbf{0}} \frac{f(\mathbf{a}) + f(\mathbf{h}) - f(\mathbf{a}) - f(\mathbf{h})}{\|\mathbf{h}\|}$$

$$= \lim_{\mathbf{h} \to \mathbf{0}} \frac{\mathbf{0}}{\|\mathbf{h}\|}$$

$$= \mathbf{0}$$

- Theorem: If f is differentiable at $\mathbf{a}$, then f is continuous at $\mathbf{a}$.

    – By definition, there exists a linear transformation A such that

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - A\mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0}$$

    – Additionally, we have that

$$f(\mathbf{a} + \mathbf{h}) = f(\mathbf{a}) + A\mathbf{h} + \|\mathbf{h}\| \left(\frac{f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - A\mathbf{h}}{\|\mathbf{h}\|} \right)$$

    – As $\mathbf{h} \to \mathbf{0}$, the right-hand side of the above equation goes to $f(\mathbf{a})$.
        ■ As a linear transformation, $A\mathbf{h} \to \mathbf{0}$ as $\mathbf{h} \to \mathbf{0}$.
        ■ Clearly $\|\mathbf{h}\| \to \mathbf{0}$ as $\mathbf{h} \to \mathbf{0}$.
        ■ And we have by definition that the last term goes to $\mathbf{0}$ as $\mathbf{h} \to \mathbf{0}$.
    – Therefore, f is continuous at $\mathbf{a}$.

- Observation: A function $f : U \to \mathbb{R}^m$ is given by an m-tuple of functions $f_1 : U \to \mathbb{R}$ known as components. $f = (f_1, \ldots, f_m)$.

- Proposition: f is differentiable at $\mathbf{a} \in U$ iff each component function f_i is differentiable at $\mathbf{a}$. In this case,

$$Df(\mathbf{a}) = (Df_1(\mathbf{a}), \ldots, Df_m(\mathbf{a}))$$

    – We know that

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - A\mathbf{h}}{\|\mathbf{h}\|} \in \mathbb{R}^m$$

    – Thus, the limit is zero iff the limit of each component is zero.
    – We have that the ith component of the vector on the left below is equal to the number on the right; we call the common value $L_i(\mathbf{h})$.

$$\left(\frac{f(\mathbf{a} + \mathbf{h}) - f(\mathbf{a}) - A\mathbf{h}}{\|\mathbf{h}\|} \right)_i = \frac{f_i(\mathbf{a} + \mathbf{h}) - f_i(\mathbf{a}) - (A\mathbf{h})_i}{\|\mathbf{h}\|}$$

    – The upshot is that f is differentiable at $\mathbf{a}$ iff $\lim_{\mathbf{h} \to \mathbf{0}} L_i(\mathbf{h}) = \mathbf{0}$ iff the linear transformation $\mathbf{h} \mapsto (A\mathbf{h})_i : \mathbb{R}^m \to \mathbb{R}$ is the total derivative of f_i.

- Now, each f_i is a function of n variables, i.e., $f_i(x_1, \ldots, x_n)$ where $x_1, \ldots, x_n$ are coordinates on $\mathbb{R}^n$.

References

Munkres, J. R. (1991). *Analysis on manifolds.* Addison-Wesley.