3 Integration II

From Rudin (1976).

Chapter 6

2/2: **3.** Define three functions $\beta_1, \beta_2, \beta_3$ as follows:

$$\beta_1 = \begin{cases} 0 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases} \qquad \beta_2 = \begin{cases} 0 & x < 0 \\ 1 & x = 0 \\ 1 & x > 0 \end{cases} \qquad \beta_3 = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases}$$

Let f be a bounded function on $[-1, 1]$.

(a) Prove that $f \in \mathscr{R}(\beta_1)$ if and only if $f(0+) = f(0)$ and that then

$$\int f \, \mathrm{d}\beta_1 = f(0)$$

(b) State and prove a similar result for β_2.

(c) Prove that $f \in \mathscr{R}(\beta_3)$ if and only if f is continuous at 0.

(d) If f is continuous at 0, prove that

$$\int f \, \mathrm{d}\beta_1 = \int f \, \mathrm{d}\beta_2 = \int f \, \mathrm{d}\beta_3 = f(0)$$

5. Suppose f is a bounded real function on $[a, b]$, and $f^2 \in \mathscr{R}$ on $[a, b]$. Does it follow that $f \in \mathscr{R}$? Does the answer change if we assume that $f^3 \in \mathscr{R}$?

7. Suppose f is a real function on $(0, 1]$ and $f \in \mathscr{R}$ on $[c, 1]$ for every $c > 0$. Define

$$\int_0^1 f(x) \, \mathrm{d}x = \lim_{c \to 0} \int_c^1 f(x) \, \mathrm{d}x$$

if this limit exists (and is finite).

(a) If $f \in \mathscr{R}$ on $[0, 1]$, show that this definition of the integral agrees with the old one.

(b) Construct a function f such that the above limit exists, although it fails to exist with $|f|$ in place of f.

8. Suppose $f \in \mathscr{R}$ on $[a, b]$ for every $b > a$ where a is fixed. Define

$$\int_a^\infty f(x) \, \mathrm{d}x = \lim_{b \to \infty} \int_a^b f(x) \, \mathrm{d}x$$

if this limit exists (and is finite). In that case, we say that the integral on the left **converges**. If it also converges after f has been replaced by $|f|$, it is said to converge **absolutely**.

Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1, \infty)$. Prove that $\int_1^\infty f(x) \, \mathrm{d}x$ converges if and only if $\sum_{n=1}^\infty f(n)$ converges. (This is the so-called "integral test" for convergence of series.)

10. Let p, q be positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

Prove the following statements.

(a) If $u, v \geq 0$, then

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}$$

Equality holds if and only if $u^p = v^q$.

(b) If $f, g \in \mathscr{R}(\alpha)$, $f, g \geq 0$, and

$$\int_a^b f^p \, \mathrm{d}\alpha = 1 = \int_a^b g^q \, \mathrm{d}\alpha$$

then

$$\int_a^b fg \, \mathrm{d}\alpha \leq 1$$

(c) If f, g are complex functions in $\mathscr{R}(\alpha)$, then

$$\left| \int_a^b fg \, \mathrm{d}\alpha \right| \leq \left(\int_a^b |f|^p \, \mathrm{d}\alpha \right)^{1/p} \left(\int_a^b |g|^q \, \mathrm{d}\alpha \right)^{1/q}$$

This is **Hölder's inequality**. When $p = q = 2$, it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)

11. Let α be a fixed increasing function on $[a, b]$. For $u \in \mathscr{R}(\alpha)$, define

$$\|u\|_2 = \left(\int_a^b |u|^2 \, \mathrm{d}\alpha \right)^{1/2}$$

Suppose $f, g, h \in \mathscr{R}(\alpha)$, and prove the triangle inequality

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2$$

as a consequence of the Schwarz inequality, as in the proof of Theorem 1.37.