Chapter 7

Sequences and Series of Functions

7.1 Notes

1/31:

- Midterm on differentiation and integration, and a bit of stuff from this week.
- Plan:
  - Talk about sequences of functions, all with the same domain and range, converging.
  - Address what properties of f_n remain in the limit (e.g., continuity, differentiability, integrability).
    - The answer depends on what we mean by "convergence."
    - $f_n \to f$ pointwise implies basically nothing.
    - $f_n \to f$ uniformly implies that basically everything works out nicely.
- We'll restrict ourselves to real functions because those have all the properties (integrability, differentiability, etc.) that we care about.
- **Pointwise** (convergent sequence $\{f_n\}$ to f): A sequence of functions $\{f_n\}$ such that for all $x \in X$, the sequence $\{f_n(x)\}$ converges to $f(x)$, where $f_n : X \to \mathbb{R}$ for all $n \in \mathbb{N}$ and $f : X \to \mathbb{R}$. *Denoted by* $f_n \to f$.
- Bad functions.
  - Consider $f_n : [0, 1] \to \mathbb{R}$ defined by $x \mapsto x^n$. Each f_n is continuous, but f is not (zero everywhere except $f(1) = 1$)[1].
  - Consider $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = x^2/(1 + x^2)^n$, and $f(x) = \sum_{n=0}^{\infty} f_n(x)$. As a geometric series, $f(x) = 1 + x^2$ when $x \neq 0$ but $f(0) = 0$. Thus, the limit exists but is not continuous once again.
  - Consider $f_m : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto \lim_{n\to\infty} \cos^{2n}(m!\pi x)$. Each f_m is integrable, but the limit f is the function that's 1 for rationals and zero for irrationals. In particular, f is not integrable.
    - We take even powers of the cosine to make it always positive.
    - We use $\cos^2(x)$ just because its always between $[0, 1]$, and we know when it is equal to 1.
    - In particular, $\cos^2(\pi x)$ is equal to 1 at every integer, $\cos^2(2\pi x)$ is equal to 1 at every half integer. $\cos^2(6\pi x)$ is equal to 1 at every one-sixth of an integer.
    - Then raising it to the n^{th} power just makes it spiky.
- Aside: Interchanging limits.
  - If all f_n are continuous, then $\lim_{x\to x_0} f_n(x) = f_n(x_0)$.

[1] Questions that require counterexamples like this could show up on the midterm!

  - The question "is f continuous" is equivalent to being able to interchange limits:

$$\lim_{x\to x_0}\lim_{n\to\infty} f_n(x) = f(x_0) = \lim_{n\to\infty}\lim_{x\to x_0} f_n(x)$$

  - Sequence example showing we need to be careful interchanging limits: $s_{n,m} = m/(m+n)$.

- All of this pathology goes away with the right definition, though.
- **Uniformly** (convergent sequence $\{f_n\}$ to f): A sequence of functions $\{f_n\}$ such that for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$ for all $x \in X$, where $f_n : X \to \mathbb{R}$ for all $n \in \mathbb{N}$ and $f : X \to \mathbb{R}$.
- Proposition (Cauchy criterion for uniform convergence): $f_n \to f$ uniformly iff for all $\epsilon > 0$, there exists N such that for all $m, n \geq N$ and for all $x \in X$, $|f_n(x) - f_m(x)| < \epsilon$.
  - Forward direction: Let $\epsilon > 0$. Suppose $f_n \to f$ uniformly. Choose N such that the functions are within $\epsilon/2$. Then

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$