Chapter 8

Some Special Functions

8.1 Notes

3/7:
- Plan:
  1. Go over some of the hits in chapter 8.
  2. Define sine.
  3. Power series.
  4. Exponential functions (log, sin, cos).
- Proposition (power series properties): If $\sum_{n=0}^{\infty} a_n x^n$ converges for all $|x| < R$, and $f : B_R(0) \to \mathbb{R}$ is defined by

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

  then:

  (a) f is continuous.
  - From the root test, $\sum_{n=0}^{\infty} a_n x^n$ is in fact absolutely convergent on $(-R, R)$. Therefore, on any interval $[-R+\epsilon, R-\epsilon]$ $(0 < \epsilon < R)$, we have

$$|a_n x^n| \leq |a_n||R+\epsilon|^n$$

    so by the M-test, $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-R+\epsilon, R-\epsilon]$. Then since $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-R+\epsilon, R-\epsilon]$, we have (a) since all $\sum_{n=0}^{N} a_n x^n$ are continuous.

  (b) f is differentiable with $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.
  - (b) follows similarly to (a) by uniform convergence.
  - Note that $\limsup \sqrt[n]{|na_n|} = \limsup \sqrt[n]{|a_n|}$ (since $\lim_{n\to\infty} \sqrt[n]{n} = 1$).
  - Therefore, $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges on $(-R, R)$.

  (c) More generally, f is infinitely differentiable with

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k}$$

  - Now (c) follows as in the proof of (b).

  (d) We have the identity

$$a_k = \frac{f^{(k)}(0)}{k!}$$

  - (d) follows from (c) by plugging in zero.

- Note that historically, the analysis of power series motivated the development of all of the Chapter 7 theorems; we simply learned those first without motivation to present the proofs in an ordered manner.
- Aside: Consider the exponential function x^y for $x, y \in \mathbb{R}$ with $x \geq 0$.
  - We define it for natural numbers and integers fairly easily, then rationals, and then for reals as the supremum of exponentials of the entries in the Dedekind cut below $x \in \mathbb{R}$.
  - Under this definition, we can confirm our normal exponential rules and then that x^y is continuous.
- Recall that
$$\mathrm{e} = \sum_{n=0}^{\infty} \frac{1}{n!}$$
- So now we are going to construct $E(x)$, $L(x)$, $C(x)$, and $S(x)$ (which are just e^x, $\ln(x)$, $\cos(x)$, and $\sin(x)$).
- Define $E : \mathbb{C} \to \mathbb{C}$ by
$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
  - By the proposition, it converges and is continuous for all $z \in \mathbb{C}$.
  - For the real numbers, E is differentiable. (E is also complex-differentiable, but we won't go into that).
- Proposition: $E(z)E(w) = E(z+w)$ for all $z, w \in \mathbb{C}$.
  - We have by the Cauchy product (Mertens' theorem) that
$$\begin{aligned} E(z)E(w) &= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{w^n}{n!}\right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^k w^{n-k}}{k!(n-k)!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}(z+w)^n \\ &= E(z+w) \end{aligned}$$
- Corollary: $E(z)E(-z) = E(0) = 1$ for all $z \in \mathbb{C}$.
- $E(x) > 0$ for $x \geq 0$.
  - It follows since $E(z+w) = E(z)E(w)$ that $E(x) > 0$ for all $x \in \mathbb{R}$.
- $\mathrm{d}E/\mathrm{d}x = E$; E is the unique, normalized ($E(0) = 1$) function such that this is true.
  - We can prove this from the power series definition.
- $E(x) \to \infty$ as $x \to \infty$ and $E(x) \to 0$ as $x \to -\infty$. (Also from the power series definition.)
- $0 \leq x_1 < x_2$ implies that $E(x_1) < E(x_2)$.
  - Either from $\mathrm{d}E/\mathrm{d}x = E > 0$ or from the power series definition.
  - It follows from $E(z+w) = E(z)E(w)$ that $x_1 < x_2$ implies $E(x_1) < E(x_2)$.

3/9:
- Plan:
  1. Keep going with E, L, C, and S.
  2. Prove the fundamental theorem of algebra.
- Define
$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
  - Recall that $E(z+w) = E(z)E(w)$.
- Theorem: $E(x) = \mathrm{e}^x$ for all $x \in \mathbb{R}$.
  - $E(1) = \mathrm{e}^1$ (by definition).
  - $E(n) = \mathrm{e}^n$ (by $E(z+w) = E(z)E(w)$).
  - $[E(p/q)]^q = E(p) = \mathrm{e}^p$ (by $E(z+w) = E(z)E(w)$).
  - $E(p/q) = \mathrm{e}^{p/q}$ for all $p/q \in \mathbb{Q}$.
  - $E(x) = \mathrm{e}^x$ for all $x \in \mathbb{R}$ (since both LHS and RHS are continuous functions that agree on $\mathbb{Q}$).
- Briefly: $E : \mathbb{R} \to \mathbb{R}^+$ is a strictly increasing surjective function. Thus, we have an inverse function $L : \mathbb{R}^+ \to \mathbb{R}$.
- Theorem: L is differentiable (and therefore continuous).
  - Since $E' = E > 0$ everywhere, we may apply the inverse function theorem at every point.
- Now by the chain rule, $E(L(x)) = x$ for all $x \in \mathbb{R}^+$, so taking derivatives yields
$$\begin{aligned} E'(L(x))L'(x) &= 1 \\ E(L(x))L'(x) &= 1 \\ xL'(x) &= 1 \\ L'(x) &= \frac{1}{x} \end{aligned}$$
- Proposition:
  1. $L(uw) = L(u) + L(w)$.
  2. $L(x) = \int_1^x t^{-1}\,\mathrm{d}t$.
- Trig functions:
$$C(x) = \frac{1}{2}[E(ix) + E(-ix)] \qquad\qquad S(x) = \frac{1}{2i}[E(ix) - E(-ix)]$$
  - You can use these definitions to prove trig identities, having derived them geometrically.
- Proposition: If $x \in \mathbb{R}$, then $C(x), S(x) \in \mathbb{R}$.
  - Key observation: $E(\bar{z}) = \overline{E(z)}$.
  - We have
$$\begin{aligned} \overline{C(x)} &= \frac{1}{2}[\overline{E(ix)} + \overline{E(-ix)}] \\ &= \frac{1}{2}[E(-ix) + E(ix)] \\ &= C(x) \end{aligned}$$

  - Symmetric for $S(x)$.
- Note that we could equally well define C,S by
$$C(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k} \qquad\qquad S(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1}$$
- Proposition: $E(ix)=C(x)+iS(x)$.
- Proposition: C,S are differentiable with
$$C'(x)=-S(x) \qquad\qquad S'(x)=C(x)$$
- Proposition: For all $x\in\mathbb{R}$, $|E(ix)|=1$.
  - We have that
$$|E(ix)|^2=E(ix)\overline{E(ix)}=E(ix)E(-ix)=E(0)=1$$
  - Taking square roots of both sides yields the desired result.
- The above result proves that the imaginary axis maps onto the unit circle in the complex plane.
- We now define π and all that.
  - Goal: Show that for all $z\in\mathbb{C}$ with $|z|=1$, there exists a unique $\theta\in[0,2\pi)$ such that $\mathrm{e}^{i\theta}=z$. Further, $E(ix)$ has period 2π.
- Proposition: $C(x)^2+S(x)^2=1$.
  - Use $E(ix)=C(x)+iS(x)$ and $|E(ix)|=1$.
- Proposition: There exists some positive number x such that $C(x)=0$.
  - Suppose (contradiction): $C(x)>0$ for all $x>0$ (since $C(0)=1$).
  - Thus, $S'(x)>0$ for all $x>0$.
  - Consequently, given $0<x<y$,
$$S(x)(y-x)<\int_x^y S(t)\,\mathrm{d}t=C(x)-C(y)\leq 2$$
  - But we can choose y large enough to make $S(x)(y-x)>2$, a contradiction.
- $\boldsymbol{\pi}$: The real number such that $\pi/2$ is the unique smallest positive real number with $C(\pi/2)=0$.
  - We know that a unique smallest number exists because since $C(0)=1$ and C is continuous, there exists a neighborhood around 0 where C is nonzero.
- Proposition: $S(\pi/2)=1$.
  - We have
$$\begin{aligned}C(\pi/2)^2+S(\pi/2)^2&=1\\ S(\pi/2)&=\pm 1\end{aligned}$$
  - Furthermore, since $S(0)=0$ and $S'(x)=C(x)$ is positive on $[0,\pi/2)$, we know that S is increasing and thus $S(\pi/2)=+1$.

8.2 Chapter 8: Some Special Functions

3/10:

- **Analytic function**: A function that can be represented by a power series.
- Theorem 8.1: If
$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$
converges for $|x| < R$, then...
  1. f converges uniformly on $[-R+\epsilon, R-\epsilon]$ for all $\epsilon > 0$;
  2. f is continuous and differentiable on $(-R, R)$;
  3. We have the identity
  $$f'(x) = \sum_{n=1}^{\infty} nc_n x^{n-1}$$
  for all $|x| < R$.
- Corollary: If f satisfies the hypotheses of Theorem 8.1, then f has derivatives of all orders in $(-R, R)$ given by
$$f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} c_n x^{n-k}$$
In particular,
$$f^{(k)}(0) = k!c_k$$
for all $k \in \mathbb{N}_0$.
- Note that there exist functions f that have derivatives of all orders at a point but cannot be expanded in a power series at that point (see Exercise 8.1).
- Theorem 8.2: If $\sum c_n$ converges and
$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$
for $|x| < 1$, then
$$\lim_{x \to 1} f(x) = \sum_{n=0}^{\infty} c_n$$
- Theorem 8.3: If $\{a_{ij}\}$ $(i, j \in \mathbb{N})$ is a double sequence, $\{b_i\}$ is defined by
$$b_i = \sum_{j=1}^{\infty} |a_{ij}|$$
for all $i \in \mathbb{N}$, and $\sum b_i$ converges, then
$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$$
- Theorem 8.4: If
$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$
converges for $|x| < R$ and $a \in (-R, R)$, then f can be expanded in a power series about $x = a$ which converges in $|x - a| < R - |a|$ and
$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$
for all $|x - a| < R - |a|$.

  - "This is an extension of Theorem 5.15 and is als known as Taylor's theorem" (Rudin, 1976, p. 176).

- Theorem 8.5: If $\sum a_n x^n, \sum b_n x^n$ converge on $S = (-R, R)$, E is the set of all $x \in S$ at which
$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$$
and E has a limit point in S, then $a_n = b_n$ for $n \in \mathbb{N}_0$. Hence, the above equation holds for all $x \in S$.
- $\boldsymbol{E}$: The function defined as follows for all $z \in \mathbb{C}$. *Given by*
$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
- We have that $E(z+w) = E(z)E(w)$ and thus $E(z)E(-z) = E(z-z) = E(0) = 1$ for all $z, w \in \mathbb{C}$.
- Thus, $E(x) = 1/E(-x) > 0$ for all $x \in \mathbb{R}$.
- It follows since $E(x) \to \infty$ as $x \to \infty$ that $E(x) \to 0$ as $x \to -\infty$.
- $0 < x < y$ implies $E(x) < E(y)$.
- We have that
$$E'(z) = \lim_{h \to 0} \frac{E(z+h) - E(z)}{h} = E(z) \lim_{h \to 0} \frac{E(h) - 1}{h} = E(z)$$
- Rudin (1976) proves that $E(x) = \mathrm{e}^x$ for all $x \in \mathbb{R}$ as in class.
- Theorem 8.6: Let e^x be defined on $\mathbb{R}$ as above. Then
  (a) e^x is continuous and differentiable for all x.
  (b) $(\mathrm{e}^x)' = \mathrm{e}^x$.
  (c) e^x is a strictly increasing function of x, and $\mathrm{e}^x > 0$.
  (d) $\mathrm{e}^{x+y} = \mathrm{e}^x \mathrm{e}^y$.
  (e) $\mathrm{e}^x \to \infty$ as $x \to \infty$ and $\mathrm{e}^x \to 0$ as $x \to -\infty$.
  (f) $\lim_{x \to \infty} x^n \mathrm{e}^{-x} = 0$ for all n.
- Theorem 8.6f shows that e^x tends to infinity faster than any power of x.
- $\boldsymbol{L}$: The inverse of E, implied to exist by the IVT since E is strictly increasing and differentiable on $\mathbb{R}$.
- Differentiating $L(E(x)) = x$ with the chain rule reveals that $L'(y) = 1/y$.
- Since $L(1) = L(E(0)) = 0$, the FTC implies that $L(y) = \int_1^y \mathrm{d}x\,/x$.
- If $E(x) = u$ and $E(y) = v$, then
$$\begin{aligned} L(uv) &= L(E(x)E(y)) \\ &= L(E(x+y)) \\ &= x + y \\ &= L(u) + L(v) \end{aligned}$$
- We define $x^n = E(nL(x))$ for all $x > 0$ and $n \in \mathbb{N}$, which we can extend analogously to before to x^y for any $x > 0$ and $y \in \mathbb{R}$.
- In the same vein, we have that
$$(x^\alpha)' = E(\alpha L(x)) \cdot \frac{\alpha}{x} = \alpha x^{\alpha - 1}$$

- We also have $\lim_{x\to\infty} x^{-\alpha}\log x = 0$, i.e., that $\log x \to \infty$ slower than any positive power of x.
- We define

$$C(x) = \frac{1}{2}[E(ix) + E(-ix)] \qquad\qquad S(x) = \frac{1}{2i}[E(ix) - E(-ix)]$$

- We know that $E(\bar{z}) = \overline{E(z)}$, so $C(x), S(x)$ are real for real x.
- Also, $E(ix) = C(x) + iS(x)$.
- We have $|E(ix)| = 1$ for all $x \in \mathbb{R}$.
- We have $C(0) = 1$ and $S(0) = 0$.
- We have

$$C'(x) = -S(x) \qquad\qquad S'(x) = C(x)$$

- Rudin (1976) proves, as in class, that there exist positive numbers x for which $C(x) = 0$.
- A smallest positive number such that $C(x) = 0$ exists since $f^{-1}(\{0\})$ is closed as the preimage of a closed set under a continuous function.
- We can prove as in class that $C(\pi/2) = 0$ and $S(\pi/2) = 1$. It follows that

$$E(i\frac{\pi}{2}) = i$$

  so that, by the addition formula, $E(2\pi i) = 1$, and hence $E(z + 2\pi i) = E(z)$ by the addition formula for all $z \in \mathbb{C}$.
- Theorem 8.7:
  (a) E is periodic with period $2\pi i$.
  (b) C, S are periodic with period 2π.
  (c) $0 < t < 2\pi$ implies that $E(it) \neq 1$.
  (d) $z \in \mathbb{C}$ with $|z| = 1$ implies there is a unique $t \in [0, 2\pi)$ with $E(it) = z$.
- Calculating the circumference of a circle.
  - Consider the curve $\gamma : [0, 2\pi] \to \mathbb{C}$ defined by

$$\gamma(t) = E(it)$$

  - This is a simple closed curve in the plane whose range is exactly the unit circle in the plane.
  - Thus, since $\gamma'(t) = iE(it)$, the length of γ (i.e., the circumference of the unit circle) is

$$\int_0^{2\pi} |\gamma'(t)| \, \mathrm{d}t = 2\pi$$

  - This shows that π has the same geometric significance in analysis with which it was originally defined in geometry.
- Similarly, we can consider the triangle with vertices at $z_1 = 0$, $z_2 = \gamma(t_0)$, and $z_3 = C(t_0)$ to recover the original geometric definition of $C(t)$.
  - We can do the same with S.