1 Norms and Differentiation

1/21: **1.** Let V be a vector space over $\mathbb{R}$. Recall that a norm on V is a function

$$\| \ \| : V \to \mathbb{R}$$

such that

- $\|\lambda \mathbf{v}\| = |\lambda| \cdot \|\mathbf{v}\|$ for all $\lambda \in \mathbb{R}$, $\mathbf{v} \in V$;
- $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$ for all $\mathbf{v}, \mathbf{w} \in V$;
- $\|\mathbf{v}\| = 0$ iff $\mathbf{v} = \mathbf{0}$.

A norm defines a metric on V given by $d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|$. We will say that two norms $\| \ \|_1$ and $\| \ \|_2$ are equivalent if there exist constants $C_1, C_2 \in \mathbb{R}$ with $0 < C_1 \leq C_2$ such that for all $\mathbf{v} \in V$,

$$C_1 \|\mathbf{v}\|_2 \leq \|\mathbf{v}\|_1 \leq C_2 \|\mathbf{v}\|_2$$

In this problem, you will show that any two norms on a finite dimensional vector space are equivalent.

(a) Show that if $\| \ \|_1$ and $\| \ \|_2$ are two equivalent norms on V, then a subset $U \subset V$ is open with respect to $\| \ \|_1$ iff it is open with respect to $\| \ \|_2$. (Recall that a subset $U \subset V$ is open with respect to a norm $\| \ \|$ if for every $\mathbf{v} \in U$, there exists an $\epsilon > 0$ such that for every $\mathbf{w} \in V$ satisfying $\|\mathbf{v} - \mathbf{w}\| < \epsilon$, $\mathbf{w} \in U$.)

(b) Let V be a finite dimensional vector space over $\mathbb{R}$ with basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Let

$$\| \ \|_1 : V \to \mathbb{R}$$

denote the function given by

$$\|a_1 \mathbf{e}_1 + \cdots + a_n \mathbf{e}_n\|_1 = \sum_{i=1}^{n} |a_i|$$

Show that $\| \ \|_1$ defines a norm on V.

(c) Let V and $\| \ \|_1$ be as in the previous part, and let

$$\| \ \| : V \to \mathbb{R}$$

be another norm on V. Show that the function $\| \ \| : V \to \mathbb{R}$ is continuous with respect to the metric defined by $\| \ \|_1$. Deduce that there exist constants $C_1, C_2 \in \mathbb{R}$ such that for all $\mathbf{v} \in V$ with $\|\mathbf{v}\|_1 = 1$,

$$C_1 \leq \|\mathbf{v}\| \leq C_2$$

(Hint: Use the fact that the unit sphere with respect to $\| \ \|_1$ is compact.)

(d) Prove that any two norms on a finite dimensional vector space are equivalent.

2. Let $U \subset \mathbb{R}^n$ be an open subset, and suppose that a function

$$f : U \to \mathbb{R}^m$$

is differentiable at a point $\mathbf{x}_0 \in U$. For a real number $\lambda > 0$, let g_λ denote the function

$$g_\lambda(\mathbf{x}) = \frac{f(\mathbf{x}_0 + \lambda(\mathbf{x} - \mathbf{x}_0)) - f(\mathbf{x}_0)}{\lambda}$$

Prove that g_λ converges to the linear function $g(\mathbf{x}) = Df(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$ as $\lambda \to 0$, where the limit is taken in the topology of uniform convergence on compact sets. In other words, for every compact subset $K \subset \mathbb{R}^n$, prove that the restriction $g_\lambda|_K$ converges uniformly to $g|_K$. This is a precise formulation of the idea that a differentiable function looks linear when "zoomed in" at a point.

3. (a) Let

$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Prove that $D_{\mathbf{v}}f(0)$ exists for all vectors $\mathbf{v} \in \mathbb{R}^2$ but that f is not continuous at $(0,0)$ (and in particular, not differentiable there).

(b) Let

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Prove that f is differentiable at zero but both partial derivatives are not continuous at zero.

4. Let $U \subset \mathbb{R}^n$ be an open subset, and $f : U \to \mathbb{R}$ a function. Suppose that for $\mathbf{a} \in U$, the partial derivatives $\partial f/\partial x_i$ $(i = 1, \ldots, n)$ exist and are bounded in a neighborhood of $\mathbf{a}$. Prove that f is continuous at $\mathbf{a}$.

5. Let

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(a) Show that f is of class C^1 on $\mathbb{R}^2$.

(b) Show that both $\partial^2 f/\partial x \partial y$ and $\partial^2 f/\partial y \partial x$ exist at $(0,0)$, but that

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) \neq \frac{\partial^2 f}{\partial y \partial x}(0,0)$$

6. Let M_n denote the space of n-by-n matrices (which can be identified with $\mathbb{R}^{n^2}$), and let

$$\mathrm{GL}_n \subset M_n$$

denote the subset of invertible matrices. In this problem, you will show that the operation of matrix inverse $\mathrm{inv} : \mathrm{GL}_n \to \mathrm{GL}_n$ defined by

$$\mathrm{inv}(A) = A^{-1}$$

is smooth and compute its derivative.

(a) Let $H \in M_n$ be a matrix such that $\|H\| < 1$ (where $\| \ \|$ denotes the operator norm). Show that $I + H$ is invertible (where $I \in M_n$ is the identity matrix). Use this to show that $\mathrm{GL}_n \subset M_n$ is open. In particular, for a matrix $A \in \mathrm{GL}_n$, if inv is differentiable at A, then the total derivative can be regarded as a linear function

$$D\,\mathrm{inv}(A) : M_n \to M_n$$

(b) Show directly that inv is differentiable at the identity I with derivative

$$D\,\mathrm{inv}(I)(X) = -X$$

for $X \in M_n$. (Hint: Show that for $\|H\| < 1$, $(I+H)^{-1} - I + H = H^2(I+H)^{-1}$.)

(c) Let $\mathrm{mult} : M_n \times M_n \to M_n$ be the function defined by matrix multiplication, i.e.,

$$\mathrm{mult}(A_1, A_2) = A_1 A_2$$

Show that mult is smooth and the derivative at $(A_1, A_2) \in M_n \times M_n$ is given by

$$D\,\mathrm{mult}((A_1, A_2))(H_1, H_2) = A_1 H_2 + H_1 A_2$$

for $(H_1, H_2) \in M_n \times M_n$.

(d) Use the chain rule to show that inv is differentiable at every $A \in \mathrm{GL}_n$ and that

$$D\,\mathrm{inv}(A)(X) = -A^{-1} X A^{-1}$$

Deduce that inv is smooth.