MATH 20410 (Analysis in $\mathbb{R}^n$ II – Accelerated) Notes

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Contents

Chapter 6

The Riemann-Stieltjes Integral

6.1 Notes

1/28:
- Plan:
  1. Finish up Fundamental Theorem of Calculus proof.
  2. Basic consequences.
  3. Rectifiable curves.
- Recall that we're given $f:[a,b]\to\mathbb{R}$ continuous, $f:[a,b]\to\mathbb{R}$, and $x\mapsto\int_a^x f(t)\,\mathrm{d}t$.
- Goal: Show $F'(x_0)=f(x_0)$.
  - WTS: Find δ such that $|x-x_0|<\delta$ implies

$$
\begin{aligned}
\left|\frac{1}{x-x_0}\int_{x_0}^x f(t)\,\mathrm{d}t-f(x_0)\right| &= \left|\frac{1}{x-x_0}\int_{x_0}^x f(t)\,\mathrm{d}t-\frac{1}{x-x_0}\int_{x_0}^x f(x_0)\,\mathrm{d}t\right|\\
&= \frac{1}{|x-x_0|}\left|\int_{x_0}^x (f(t)-f(x_0))\,\mathrm{d}t\right|\\
&\leq \frac{1}{|x-x_0|}\int_{x_0}^x |f(t)-f(x_0)|\,\mathrm{d}t\\
&< \epsilon
\end{aligned}
$$

  - Since f is continuous, there exists δ such that if $|x-x_0|<\delta$, then $|f(x)-f(x_0)|<\epsilon$.
  - Now

$$
\begin{aligned}
\frac{1}{|x-x_0|}\int_{x_0}^x |f(t)-f(x_0)|\,\mathrm{d}t &< \frac{1}{|x-x_0|}\int_{x_0}^x \epsilon\,\mathrm{d}t\\
&= \epsilon
\end{aligned}
$$

- Applications:
  1. Theorem (MVT for integration): $f:[a,b]\to\mathbb{R}$ continuous, then there exists $x_0\in[a,b]$ such that

$$f(x_0)=\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

     - Apply MVT to $F(x)=\int_a^x f(t)\,\mathrm{d}t$. Then

$$F'(x_0)=f(x_0)=\frac{F(b)-F(a)}{b-a}$$

       as desired.

2. Theorem (Integration by parts): Let $F, G : [a, b] \to \mathbb{R}$ be differentiable with $F' = f$, $G' = g$ and with f and g both integrable. Then

$$\int_a^b Fg = F(b)G(b) - F(a)G(a) - \int_a^b fG$$

   - Just use the product rule plus the FTC to prove.
   - We have

$$\begin{aligned} \int_a^b (FG)' &= \int_a^b fG + \int_a^b Fg \\ F(b)G(b) - F(a)G(a) &= \int_a^b fG + \int_a^b Fg \\ \int_a^b Fg &= F(b)G(b) - F(a)G(a) - \int_a^b fG \end{aligned}$$

3. Theorem (u-substitution).
   - Follows similarly from the chain rule and FTC.

- Integration of vector-valued functions.
- If $f : [a, b] \to \mathbb{R}^k$, we define $\int_a^b f$ by

$$\int_a^b f = \left(\int_a^b f_1, \dots, \int_a^b f_k \right)$$

- Alternatively, you can define $\int_a^b f$ using P, $U(f, P)$, $L(f, P)$, etc. and then prove that the integral exists iff all f_i are integrable and in this case the above definition holds.
- Rectifiable curves: Let $\gamma : [a, b] \to \mathbb{R}^k$ be a continuous function.
- Plan: Define the length of γ and show that we can compute it with an integral.
  - Idea: For polygonal paths, we know how to define length. So let's approximate γ by polygons and take a limit.
  - Ref: Given a partition P, then define the length of γ with respect to P as $\Lambda(\gamma, P)$. Let the length of γ be $\Lambda(\gamma) = \sup_P \Lambda(\gamma, P)$ if this limit exists in this case, we call γ **rectifiable**.
- Fractals are not rectifiable — their length diverges.
- Theorem: Suppose γ is continuously differentiable (i.e., γ is differentiable and γ' is continuous). Then γ si rectifiable and

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)| \, \mathrm{d}t$$

  - Notice: If $P \leq P'$, then $\Lambda(\gamma, P) \leq \Lambda(\gamma, P')$. (Prove with triangle inequality.)
  - WTS: For all partitions P, $\Lambda(\gamma, P) \leq \int_a^b |\gamma'(t)| \, \mathrm{d}t$ and thus $\Lambda(\gamma) \leq \int_a^b |\gamma'(t)| \, \mathrm{d}t$.
  - We have that

$$\begin{aligned} \Lambda(\gamma, P) &= \sum_{i=1}^n |\gamma(x_i) - \gamma(x_{i-1}) \\ &= \sum_{i=1}^n \left| \int_{x_{i-1}}^{x_i} \gamma'(t) \, \mathrm{d}t \right| \\ &\leq \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |\gamma'(t)| \, \mathrm{d}t \\ &= \int_a^b |\gamma'(t)| \, \mathrm{d}t \end{aligned}$$

- Catch up.
  - I should make up PSets 1-2.
  - Exams have less than Rudin-strength problems.
  - Exams are mostly true/false (and of that, mostly false, provide a counterexample).

6.2 Exam 1 Additional Topics

- A continuous function that is not always differentiable.

$$f(x) = |x|$$

- A differentiable function with a discontinuous derivative.

$$f(x) = x^2 \sin \frac{1}{x}$$

- A vector-valued function that doesn't satisfy the MVT.

$$\mathbf{f}(x) = \mathrm{e}^{ix}$$

  - Between 0 and 2π.

- A pair of vector-valued functions that don't satisfy L'Hôpital's rule.

$$f(x) = x \qquad\qquad g(x) = x + x^2 \mathrm{e}^{i/x^2}$$

Chapter 7

Sequences and Series of Functions

7.1 Notes

1/31:

- Midterm on differentiation and integration, and a bit of stuff from this week.
- Plan:
  - Talk about sequences of functions, all with the same domain and range, converging.
  - Address what properties of f_n remain in the limit (e.g., continuity, differentiability, integrability).
    - The answer depends on what we mean by "convergence."
    - $f_n \to f$ pointwise implies basically nothing.
    - $f_n \to f$ uniformly implies that basically everything works out nicely.
- We'll restrict ourselves to real functions because those have all the properties (integrability, differentiability, etc.) that we care about.
- **Pointwise** (convergent sequence $\{f_n\}$ to f): A sequence of functions $\{f_n\}$ such that for all $x \in X$, the sequence $\{f_n(x)\}$ converges to $f(x)$, where $f_n : X \to \mathbb{R}$ for all $n \in \mathbb{N}$ and $f : X \to \mathbb{R}$. *Denoted by* $f_n \to f$.
- Bad functions.
  - Consider $f_n : [0,1] \to \mathbb{R}$ defined by $x \mapsto x^n$. Each f_n is continuous, but f is not (zero everywhere except $f(1) = 1$)[1].
  - Consider $f_n : \mathbb{R} \to \mathbb{R}$ defined by $f_n(x) = x^2/(1+x^2)^n$, and $f(x) = \sum_{n=0}^{\infty} f_n(x)$. As a geometric series, $f(x) = 1 + x^2$ when $x \neq 0$ but $f(0) = 0$. Thus, the limit exists but is not continuous once again.
  - Consider $f_m : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto \lim_{n\to\infty} \cos^{2n}(m!\pi x)$. Each f_m is integrable, but the limit f is the function that's 1 for rationals and zero for irrationals. In particular, f is not integrable.
    - We take even powers of the cosine to make it always positive.
    - We use $\cos^2(x)$ just because its always between $[0,1]$, and we know when it is equal to 1.
    - In particular, $\cos^2(\pi x)$ is equal to 1 at every integer, $\cos^2(2\pi x)$ is equal to 1 at every half integer. $\cos^2(6\pi x)$ is equal to 1 at every one-sixth of an integer.
    - Then raising it to the n^{th} power just makes it spiky.
- Aside: Interchanging limits.
  - If all f_n are continuous, then $\lim_{x\to x_0} f_n(x) = f_n(x_0)$.

[1] Questions that require counterexamples like this could show up on the midterm!

  - The question "is f continuous" is equivalent to being able to interchange limits:

$$\lim_{x \to x_0} \lim_{n \to \infty} f_n(x) = f(x_0) = \lim_{n \to \infty} \lim_{x \to x_0} f_n(x)$$

  - Sequence example showing we need to be careful interchanging limits: $s_{n,m} = m/(m+n)$.

- All of this pathology goes away with the right definition, though.
- **Uniformly** (convergent sequence $\{f_n\}$ to f): A sequence of functions $\{f_n\}$ such that for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$ for all $x \in X$, where $f_n : X \to \mathbb{R}$ for all $n \in \mathbb{N}$ and $f : X \to \mathbb{R}$.
- Proposition (Cauchy criterion for uniform convergence): $f_n \to f$ uniformly iff for all $\epsilon > 0$, there exists N such that for all $m, n \geq N$ and for all $x \in X$, $|f_n(x) - f_m(x)| < \epsilon$.
  - Forward direction: Let $\epsilon > 0$. Suppose $f_n \to f$ uniformly. Choose N such that the functions are within $\epsilon/2$. Then

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

2/2:
- Office hours tomorrow 4-5 PM.
- Plan:
  1. More on uniform convergence.
     - Limit of continuous functions is continuous.
     - Limit of the integral of functions is the integral of the limit.
  2. $\mathcal{C}(X)$ perspectives on uniform convergence.
- Corollary (Weierstraß M-test): If there exist constants $M_n \in \mathbb{R}$ such that $|f_n(x)| \leq M_n$ for all x and $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly.
- Theorem: $f_n : X \to \mathbb{R}$, f_n continuous at $x_0 \in X$ for all n, and $f_n \to f$ uniformly imply f continuous at x_0.
  - Idea:
    - "$\epsilon/3$ trick": Find δ such that if $|x - x_0| < \delta$, then

$$|f(x) - f(x_0)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

  - Proof:
    - $f_n \to f$ uniformly implies there exists $N \in \mathbb{N}$ such that $|f_N(x) - f(x)| < \epsilon/3$ for all $x \in X$.
    - f_N continuous at x_0: There exists δ such that if $d(x, x_0) < \delta$, then $|f_N(x) - f_N(x_0)| < \epsilon/3$.
    - Thus, by the $\epsilon/3$ trick, we have the continuity of f.
- Defining a norm on $\mathcal{C}(X)$.

$$\|f\| = \sup_{x \in X} |f(x)|$$

  - This makes $\mathcal{C}(X)$ into a vector space.
  - We can now define our metric $d(f,g)$ by $d(f,g) = \|f - g\|$.
- $f_n \to f \iff f$ is bounded.
  - $f_n \to f$ uniformly $\iff \lim_{n \to \infty} \sup |f_n(x) - f(x)| = 0 \iff f_n \to f$ is $\mathcal{C}(X)$.
- Corollary to the Weierstraß M-test: $\mathcal{C}(X)$ is complete (i.e., all uniformly Cauchy sequences converge).

- Assume $\{f_n\}$ is Cauchy. Then by the Cauchy criterion for uniform convergence, f_n converges uniformly to some f. But this f must be continuous, too, meaning $f \in \mathcal{C}(X)$.

2/4:

- Plan.
  1. $\int \lim f_n = \lim \int f_n$.
  2. $\mathrm{d}x \lim f_n = \lim \mathrm{d}x\, f_n$.
  3. Definitions: Pointwise/uniform boundedness, equicontinuity.
- Theorem: $f_n : [a,b] \to \mathbb{R}$ integrable and $f_n \to f$ uniformly implies f is integrable and

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$$

  - Plan:
    1. Show f is integrable.
    2. Show $\int f = \lim \int f_n$.
  - Proof:
    - Let $\epsilon_n = \sup_{x\in[a,b]} |f(x) - f_n(x)|$.
    - Since $f_n \to f$ uniformly, $\epsilon_n \to 0$ as $n \to \infty$.
    - By definition, $f_n - \epsilon_n \leq f \leq f_n + \epsilon_n$.
    - Thus, by Theorems 6.4 and 6.5,

$$\int_a^b (f_n - \epsilon_n) = \underline{\int} (f_n - \epsilon_n) \leq \underline{\int} f \leq \bar{f} \leq \int_a^b (f_n + \epsilon_n)$$

    - It follows since

$$0 \leq \overline{\int} f - \underline{\int} f \leq \int_a^b (f_n + \epsilon_n) - \int_a^b (f_n - \epsilon_n) = (b-a)...$$

      that f is integrable.
    - Hence,

$$\begin{aligned} \int_a^b (f_n - \epsilon_n) \leq \int_a^b f \leq \int_a^b (f_n - \epsilon_n) \\ \left| \int_a^b f_n - \int_a^b f \right| \leq \epsilon_n \\ \lim_{n\to\infty} \int_a^b f_n = \int_a^b f \end{aligned}$$

- Theorem: $f_n : [a,b] \to \mathbb{R}$, each f_n differentiable, $f_n \to f$ pointwise, and $(f_n)' \to g$ uniformly implies that f is differentiable and $f' = g$.
  - Note that you can do better: Substituting $f_n(x_0)$ converging for some $x_0 \in [a,b]$ for $f_n \to f$ pointwise still implies the desired result.
  - Idea: We use the $\epsilon/3$ trick; 2/3 will be easy and 1/3 will be tricky.
  - Goal: We want

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - g(x_0) \right| < \epsilon$$

    for some δ with $0 < |x - x_0| < \delta$. We will show that

$$\begin{aligned} &\left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{f_N(x) - f_N(x_0)}{x - x_0} + \frac{f_N(x) - f_N(x_0)}{x - x_0} - f_N'(x_0) + f_N'(x_0) - g(x_0) \right| \\ \leq &\left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{f_N(x) - f_N(x_0)}{x - x_0} \right| + \left| \frac{f_N(x) - f_N(x_0)}{x - x_0} - f_N'(x_0) \right| + |f_N'(x_0) - g(x_0)| \end{aligned}$$

- For the middle inequality, use Chapter 5, Exercise 8.
- For the right inequality, use the uniform convergence condition.
- For the left inequality, it will suffice to show the Cauchy condition

$$\left|\frac{f_n(x)-f_n(x_0)}{x-x_0}-\frac{f_m(x)-f_m(x_0)}{x-x_0}\right|<\frac{\epsilon}{3}$$

so, noting that the left term above is equal to

$$\left|\frac{[f_n(x)-f_m(x)]-[f_n(x_0)-f_m(x_0)]}{x-x_0}\right|$$

which is equal to $|f_n'(c)-f_m'(c)|$ by the MVT, from which we can apply the Cauchy form of the uniform convergence of $(f_n)'$ condition.

- **Pointwise bounded** ($\{f_n\}$): A sequence of real functions $\{f_n\}$ such that for all $x\in X$, there exists $M_x\in\mathbb{R}$ such that $|f_n(x)|\leq M_x$ for all $n\in\mathbb{N}$.

- **Uniformly bounded** ($\{f_n\}$): A sequence of real functions $\{f_n\}$ for which there exists $M\in\mathbb{R}$ such that for all $x\in X$ and $n\in\mathbb{N}$, $|f_n(x)|\leq M$.

- Proposition: $f_n:E\to\mathbb{R}$, $\{f_n\}$ is pointwise bounded, and E is countable implies there is a subsequence $\{f_{n_k}\}$ that converges pointwise.
  - Enumerate $E=\{x_1,x_2,\dots\}$.
  - Then since $\{f_n(x_m)\}$ is bounded for all m by hypothesis, it always has a convergent subsequence.
  - The claim is if you look at the sequence of diagonal functions, it is such a subsequence, i.e., if $f_1(x_1)$ is the first term for x_1, $f_3(x_2)$ is the second term for x_2, $f_{11}(x_3)$ is the third term for x_3, and so on, $f_1,f_3,f_{11},\dots$ is such a subsequence.

2/9:

- Build up to the Arzelà-Ascoli theorem.

2/11:

- The Arzelà-Ascoli theorem.

Chapter 9

Functions of Several Variables

9.1 Notes

2/14: - Plan:
  1. Warm-up with matrices.
  2. The total derivatives of $f : \mathbb{R}^n \to \mathbb{R}^m$ ($n = m = 2$, i.e., $f : \mathbb{C} \to \mathbb{C}$).
  3. Basic properties: Chain rule, relation with partial derivatives, implicit function theorem.
- Let V, W be finite-dimensional vector spaces over $\mathbb{R}$. We let $\mathcal{L}(V, W)$ be the vector space of all linear transformations $\phi : V \to W$.
- If we pick bases $N_1, \dots, N_n$ of V and $w_1, \dots, w_m$ of W, then $V \cong \mathbb{R}^n$ and $W \cong \mathbb{R}^m$. It follows that $\mathcal{L}(V, W) \cong \mathbb{R}^{mn}$.
- $\mathcal{L}(V, W) \times \mathcal{L}(W, U) \xrightarrow{\text{compose}} \mathcal{L}(V, U)$, i.e., $\mathbb{R}^{mn} \times \mathbb{R}^{nl} \xrightarrow[\text{mult.}]{\text{matrix}} \mathbb{R}^{ml}$.
- Sup norm: If A is an $m \times n$ real matrix, then $\|A\| = \sup_{\substack{\mathbf{x} \in \mathbb{R}^n \\ |\mathbf{x}|=1}} |A\mathbf{x}|$.
  - Basic properties:
    1. $|A\mathbf{x}| \leq \|A\||x|$.
    2. $\|A\| < \infty$ and all $A : \mathbb{R}^n \to \mathbb{R}^m$ are uniformly continuous.
    3. $\|A\| = 0 \iff A = 0$.
    4. $\|cA\| = |c|\|A\|$.
    5. $\|A + B\| \leq \|A\| + \|B\|$.
    6. $\|AB\| \leq \|A\|\|B\|$.
  - Note that we get a metric space structure on $\mathcal{L}(V, W)$ by defining $d(A, B) = \|A - B\|$.
- Proves that 1 and 2 imply the uniform continuity of all A (via Lipschitz continuity).
- **Differentiable** (multivariate function f at $\mathbf{x}_0$): A function $f : U \to \mathbb{R}^m$ ($U \subset \mathbb{R}^n$) such that to $\mathbf{x}_0 \in U$ there corresponds some linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{|f(\mathbf{x}_0 - \mathbf{h}) - f(\mathbf{x}_0) - A\mathbf{h}}{|\mathbf{h}|} = 0$$

- **Total derivative** (of f multivariate at $\mathbf{x}_0$): The linear transformation A in the above definition. *Denoted by* $\boldsymbol{f'(\mathbf{x}_0)}$.
- "An proof and progress in mathematics" - Thurston.

  - Relating to the old one dimensional derivative.
  - A paper we'd find rather impressionistic right now.
- Propositions ahead of us.
  - Proposition: Suppose that f is differentiable at $\mathbf{x}_0 \in U$ and A, B are both derivatives of f at $\mathbf{x}_0$. Then $A = B$.
  - Proposition: Differentiable implies continuous.
  - Proposition: Sum rule, product rule, quotient rule.