

MATH 20410 (Analysis in \mathbb{R}^n II – Accelerated) Problem Sets

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1 Differentiation

From Rudin (1976).

Chapter 5

1. Let f be defined for all real x , and suppose that

$$|f(y) - f(x)| \leq (y - x)^2$$

for all real x and y . Prove that f is constant.

Proof. To prove that f is constant, Theorem 5.11b tells us that it will suffice to show that f is differentiable on \mathbb{R} with derivative $f' = 0$. Let $x \in \mathbb{R}$ be arbitrary. We want to show that for all $\epsilon > 0$, there exists a δ such that if $y \in \mathbb{R}$ and $0 < |y - x| < \delta$, then $|(f(y) - f(x))/(y - x) - 0| < \epsilon$. Let ϵ be arbitrary. Choose $\delta = \epsilon$. Then we have that

$$\begin{aligned} \left| \frac{f(y) - f(x)}{y - x} - 0 \right| &= \frac{|f(y) - f(x)|}{|y - x|} \\ &\leq \frac{(y - x)^2}{|y - x|} \\ &\leq |y - x| \\ &< \epsilon \end{aligned}$$

as desired. □

2. Suppose $f'(x) > 0$ in (a, b) . Prove that f is strictly increasing in (a, b) and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)}$$

for $a < x < b$.

Proof. To prove that f is strictly increasing on (a, b) , it will suffice to show that $x < y$ implies $f(x) < f(y)$ for all $x, y \in (a, b)$. Let $x, y \in (a, b)$ satisfy $x < y$. Since f is differentiable on (a, b) , it is differentiable on $(x, y) \subset (a, b)$ and (by Theorem 5.2) continuous on $[x, y] \subset (a, b)$. Thus, by the MVT, there exists $c \in (x, y)$ such that

$$f(y) - f(x) = (y - x)f'(c)$$

But since $x < y$, $y - x > 0$. This combined with the fact that $f'(c) > 0$ by definition implies that $(y - x)f'(c) > 0$. Consequently,

$$f(y) < f(x) + (y - x)f'(c) = f(y)$$

as desired.

Since f is strictly increasing (and hence 1-1) on (a, b) , we may construct a well-defined inverse function $g : f[(a, b)] \rightarrow (a, b)$ for it by

$$g(f(x)) = x$$

for all $f(x) \in f[(a, b)]$. It follows by the fact that $f'(x) > 0$ for all $x \in (a, b)$, the definitions of $f'(x)$ and $g'(f(x))$, and Theorem 3.3d that

$$\frac{1}{f'(x)} = \frac{1}{\lim_{y \rightarrow x} \frac{f(y) - f(x)}{y - x}}$$

$$\begin{aligned}
&= \lim_{y \rightarrow x} \frac{1}{\frac{f(y)-f(x)}{y-x}} \\
&= \lim_{y \rightarrow x} \frac{y-x}{f(y)-f(x)} \\
&= \lim_{y \rightarrow x} \frac{g(f(y))-g(f(x))}{f(y)-f(x)} \\
&= g'(f(x))
\end{aligned}$$

as desired. \square

3. Suppose g is a real function on \mathbb{R}^1 , with bounded derivative (say $|g'| \leq M$). Fix $\epsilon > 0$ and define $f(x) = x + \epsilon g(x)$. Prove that f is one-to-one if ϵ is small enough. (A set of admissible values of ϵ can be determined which depends only on M .)

Proof. Neglecting the trivial case where $M = 0$, take $\epsilon = 1/2M$. It follows that

$$\begin{aligned}
0 &< 1 - \frac{1}{2} \\
&= 1 + \frac{1}{2M} \cdot -M \\
&\leq 1 + \epsilon g'(x) \\
&= \frac{d}{dx}(x) + \frac{d}{dx}(\epsilon g) \\
&= f'(x)
\end{aligned}$$

Therefore, by Problem 5.2, f is strictly increasing and, hence, one-to-one. \square

4. If

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$$

where C_0, \dots, C_n are real constants, prove that the equation

$$C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n = 0$$

has at least one real root between 0 and 1.

Proof. Consider the polynomial

$$f(x) = C_0x + \frac{C_1}{2}x^2 + \cdots + \frac{C_n}{n+1}x^{n+1}$$

We have that $f(0) = 0$ (by direct substitution) and $f(1) = 0$ (by the constraint on the coefficients). Thus, since f is continuous on $[0, 1]$ and differentiable on $(0, 1)$ (as a polynomial), we have by the MVT that there exists $x \in (0, 1)$ such that

$$\begin{aligned}
f(1) - f(0) &= (1 - 0)f'(x) \\
f'(x) &= 0 \\
C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n &= 0
\end{aligned}$$

as desired. \square

5. Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \rightarrow 0$ as $x \rightarrow +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \rightarrow 0$ as $x \rightarrow +\infty$.

Proof. To prove that $\lim_{x \rightarrow \infty} g(x) = 0$, it will suffice to show that for every $\epsilon > 0$, there exists $N > 0$ such that if $x > N$, then $|g(x) - 0| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{x \rightarrow \infty} f'(x) = 0$ by hypothesis, we know that there exists $N > 0$ such that if $x > N$, then $|f'(x)| < \epsilon$. Choose this N to be our N . Let $x > N$ be arbitrary. Applying the MVT to f on the interval $[x, x+1]$ proves the existence of a c within that closed interval such that

$$f(x+1) - f(x) = f'(c)(x+1-x) = f'(c)$$

Additionally, since $c > x > N$, we have that $|f'(c)| < \epsilon$. Therefore, we have that

$$\begin{aligned} |g(x)| &= |f(x+1) - f(x)| \\ &= |f'(c)| \\ &< \epsilon \end{aligned}$$

as desired. □

2 Differentiation II / Integration

From Rudin (1976).

Chapter 5

8. Suppose f' is continuous on $[a, b]$ and $\epsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \epsilon$$

whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$. (This could be expressed by saying that f is **uniformly differentiable** on $[a, b]$ if f' is continuous on $[a, b]$.) Does this hold for vector-valued functions, too?

Proof. By Theorem 2.40, $[a, b]$ is compact. This combined with the fact that f' is continuous implies by Theorem 4.19 that f' is uniformly continuous. Thus, there exists $\delta > 0$ such that if $x, y \in [a, b]$ and $|y - x| < \delta$, then $|f'(y) - f'(x)| < \epsilon$. Choose this δ to be our δ . Let $x, t \in [a, b]$ be such that $0 < |t - x| < \delta$. Then since f is continuous on $[t, x] \subset [a, b]$ and differentiable on $(t, x) \subset [a, b]$, we have by the MVT that there exists $c \in (t, x)$ such that

$$\begin{aligned} f(t) - f(x) &= (t - x)f'(c) \\ \frac{f(t) - f(x)}{t - x} &= f'(c) \end{aligned}$$

Additionally, since $t < c < x$ and $|t - x| < \delta$, we must have $|c - x| < \delta$, meaning that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| = |f'(c) - f'(x)| < \epsilon$$

as desired.

And yes, this does hold for vector-valued functions, which we can treat component-wise. \square

17. Suppose f is a real, three times differentiable function on $[-1, 1]$ such that

$$f(-1) = 0 \qquad f(0) = 0 \qquad f(1) = 1 \qquad f'(0) = 0$$

Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1, 1)$. Note that equality holds for $\frac{1}{2}(x^3 + x^2)$. (Hint: Use Theorem 5.15 with $\alpha = 0$ and $\beta = \pm 1$ to show that there exist $s \in (0, 1)$ and $t \in (-1, 0)$ such that $f^{(3)}(s) + f^{(3)}(t) = 6$.)

Proof. Since f is three times differentiable on $[-1, 1]$, we know that f'' is differentiable on $[-1, 1]$. It follows by Theorem 5.2 that f'' is continuous on $[-1, 1]$. Thus, since f is defined on $[-1, 1]$, $3 \in \mathbb{N}$, f'' is continuous on $[-1, 1]$, $f^{(3)}$ is defined on $(-1, 1)$, $0, 1 \in [-1, 1]$ such that $0 \neq 1$, and we can define

$$P(t) = \sum_{k=0}^2 \frac{f^{(k)}(0)}{k!} (t - 0)^k$$

we have by Taylor's theorem that there exists $s \in (0, 1)$ such that

$$\begin{aligned} f(1) &= P(1) + \frac{f^{(3)}(s)}{3!} (1 - 0)^3 \\ 1 - \left[\frac{f(0)}{0!} (1 - 0)^0 + \frac{f'(0)}{1!} (1 - 0)^1 + \frac{f''(0)}{2!} (1 - 0)^2 \right] &= \frac{f^{(3)}(s)}{3!} \\ 1 - \left[\frac{f''(0)}{2} \right] &= \frac{f^{(3)}(s)}{6} \\ 6 - 3f''(0) &= f^{(3)}(s) \end{aligned}$$

Similarly, we have that there exists $t \in (-1, 0)$ such that

$$\begin{aligned} f(-1) &= P(-1) + \frac{f^{(3)}(t)}{3!}(-1-0)^3 \\ 0 - \left[\frac{f(0)}{0!}(-1-0)^0 + \frac{f'(0)}{1!}(-1-0)^1 + \frac{f''(0)}{2!}(-1-0)^2 \right] &= -\frac{f^{(3)}(t)}{3!} \\ - \left[\frac{f''(0)}{2} \right] &= -\frac{f^{(3)}(t)}{6} \\ 3f''(0) &= f^{(3)}(s) \end{aligned}$$

Thus,

$$f^{(3)}(s) + f^{(3)}(t) = 3f''(0) + 6 - 3f''(0) = 6$$

Now suppose for the sake of contradiction that for all $x \in (-1, 1)$, we have $f^{(3)}(x) < 3$. Then $f^{(3)}(s) < 3$ and $f^{(3)}(t) < 3$. It follows that $f^{(3)}(s) + f^{(3)}(t) < 6$, a contradiction. \square

25. Suppose f is twice differentiable on $[a, b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geq \delta > 0$, and $0 \leq f''(x) \leq M$ for all $x \in [a, b]$. Let ξ be the unique point in (a, b) at which $f(\xi) = 0$. Complete the details in the following outline of **Newton's method** for computing ξ .

- (a) Choose $x_1 \in (\xi, b)$ and define $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Interpret this geometrically, in terms of a tangent to the graph of f .

Answer. Since we can rearrange the above to $0 - f(x_n) = f'(x_n)(x_{n+1} - x_n)$, we know that x_{n+1} is the point at which the tangent to f at x_n crosses the x -axis. In other words, the zero of the tangent line

$$y - f(x_n) = f'(x_n)(x - x_n)$$

is $(x_{n+1}, 0)$. \square

- (b) Prove that $x_{n+1} < x_n$ and that

$$\lim_{n \rightarrow \infty} x_n = \xi$$

Proof. To prove that $x_{n+1} < x_n$, it will suffice to show that $f(x_n), f'(x_n) > 0$. Since $f'(x) > 0$ for all $x \in [a, b]$ by hypothesis, we know that $f'(x_n) > 0$. As to $f(x_n)$, suppose for the sake of contradiction that $f(x_n) \leq 0$. We know that $f(\xi) = 0$, $f(b) > 0$, and $\xi < x_n < b$. Since ξ is the *unique* point at which $f(\xi) = 0$ by hypothesis and $x_n \neq \xi$, we know that $f(x_n) \neq 0$. And if $f(x_n) < 0$, we have by the intermediate value theorem for f continuous that there exists $c \in (x_n, b)$ such that $f(c) = 0$. But since $\xi < x_n < c < b$, $c \neq \xi$, and thus we have a contradiction here, too.

Having established that $\{x_n\}$ is a monotonically decreasing sequence, Theorem 3.14 tells us that to show that it converges, it will suffice to show that it is bounded. Clearly, $\{x_n\}$ is bounded above by x_1 . And on the bottom side, $\{x_n\}$ is bounded by ξ : If there were $x_n < \xi$, this would imply that $f(x_n) < 0$ by a symmetric argument to the above, meaning that $f(x_n)/f'(x_n) < 0$ and implying that $x_{n+1} > x_n$, a contradiction. Furthermore, we know that the limit (call it x) equals ξ since

$$\begin{aligned} x &= x - \frac{f(x)}{f'(x)} \\ f(x) &= 0 \end{aligned}$$

so $x = \xi$ by the uniqueness of ξ . \square

(c) Use Taylor's theorem to show that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

for some $t_n \in (\xi, x_n)$.

Proof. Since f is defined on $[a, b]$, $2 \in \mathbb{N}$, f' is continuous on $[a, b]$, f'' is defined on (a, b) , $\xi, x_n \in [a, b]$ with $\xi \neq x_n$, and

$$P(t) = \sum_{k=0}^1 \frac{f^{(k)}(x_n)}{k!}(t - x_n)^k$$

we have by Taylor's theorem that there exists $t_n \in (\xi, x_n)$ such that

$$\begin{aligned} f(\xi) &= \left[\frac{f(x_n)}{0!}(\xi - x_n)^0 + \frac{f'(x_n)}{1!}(\xi - x_n)^1 \right] + \frac{f''(t_n)}{2!}(\xi - x_n)^2 \\ 0 &= f(x_n) - f'(x_n)(x_n - \xi) + \frac{f''(t_n)}{2}(x_n - \xi)^2 \\ x_n - \frac{f(x_n)}{f'(x_n)} - \xi &= \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2 \\ x_{n+1} - \xi &= \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2 \end{aligned}$$

as desired. □

(d) If $A = M/2\delta$, deduce that

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A}[A(x_1 - \xi)]^{2n}$$

(Compare with Chapter 3, Exercises 16 and 18.)

Proof. We have from part (b) that $x_i > \xi$ for all $i \in \mathbb{N}$, so naturally $0 \leq x_{n+1} - \xi$. As to the other part of the question, we induct on n . For the base case $n = 1$, we have that

$$\begin{aligned} x_2 - \xi &= \frac{f''(t_1)}{2f'(x_1)}(x_1 - \xi)^2 \\ &\leq \frac{M}{2\delta}(x_1 - \xi)^2 \\ &= \frac{2\delta}{M} \left[\frac{M}{2\delta}(x_1 - \xi) \right]^2 \\ &= \frac{1}{A}[A(x_1 - \xi)]^{2 \cdot 1} \end{aligned}$$

Now suppose inductively that we have proven the claim for $n - 1$; we now seek to prove it for n . Indeed, we have that

$$\begin{aligned} x_{n+1} - \xi &= \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2 \\ &\leq \frac{M}{2\delta}(x_n - \xi)^2 \\ &\leq A \left(\frac{1}{A}[A(x_1 - \xi)]^{2(n-1)} \right)^2 \\ &= \frac{1}{A}[A(x_1 - \xi)]^{2n} \end{aligned}$$

as desired. □

- (e) Show that Newton's method amounts to finding a fixed point of the function g defined by

$$g(x) = x - \frac{f(x)}{f'(x)}$$

How does $g'(x)$ behave for x near ξ ?

Proof. A fixed point of the function g is a point x such that

$$\begin{aligned} g(x) &= x \\ x - \frac{f(x)}{f'(x)} &= x \\ f(x) &= 0 \end{aligned}$$

Thus, if we want to find a point x where $f(x) = 0$, it is equivalent to find a point x such that $g(x) = x$.

As to the other part of the question, we have by the rules of derivatives that

$$\begin{aligned} g'(x) &= 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{f'(x)^2} \\ &= \frac{f(x)f''(x)}{f'(x)^2} \\ &\leq \frac{M}{\delta^2} f(x) \end{aligned}$$

Thus, since $f(x) \rightarrow 0$ as $x \rightarrow \xi$, $g'(x) \rightarrow 0$ as $x \rightarrow \xi$. □

- (f) Put $f(x) = \sqrt[3]{x}$ on $(-\infty, \infty)$ and try Newton's method. What happens?

Answer. We have by the power rule that

$$f'(x) = \frac{1}{3x^{2/3}}$$

Choose $x_1 = 1$. Then

$$\begin{aligned} x_2 &= 1 - \frac{f(1)}{f'(1)} = -2 \\ x_3 &= 1 - \frac{f(-2)}{f'(-2)} = 7 \\ x_4 &= 1 - \frac{f(7)}{f'(7)} = -20 \\ &\vdots \end{aligned}$$

It appears that the series is diverging to ∞ while alternating from positive to negative. In fact, since $x_3 > x_2$, contrary to part (b), we know that something must be wrong (i.e., one of our hypotheses must not be met). Upon further investigation, we can determine that on $[-1, 1]$, we have $f''(1) = -2/9 < 0$; thus, our last hypothesis is the issue with this function. □

Chapter 6

1. Suppose α increases on $[a, b]$, $a \leq x_0 \leq b$, α is continuous at x_0 , $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathcal{R}(\alpha)$ and that $\int f d\alpha = 0$.

Proof. Since f is bounded on $[a, b]$ with only one discontinuity on $[a, b]$ and α is continuous at the point at which f is discontinuous, Theorem 6.10 implies that $f \in \mathcal{R}(\alpha)$, as desired. It follows that $\inf U(P, f, \alpha) = \sup L(P, f, \alpha) = \int f d\alpha$. But since $L(P, f, \alpha) = 0$ for all P (there is no infinite interval $[x_i, x_{i+1}] \subset [a, b]$ that does not contain 0, and f is bounded below by 0), we know that

$$\int f d\alpha = \sup L(P, f, \alpha) = 0$$

as desired. \square

2. Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x) dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$. (Compare this with Exercise 1.)

Proof. Suppose for the sake of contradiction that $f(x) \neq 0$ for some x . By the definition of f , this must mean that $f(x) > 0$. It follows since f is continuous that there exists some $N_r(x)$ such that $f(y) > 0$ for all $y \in N_r(x)$. Now consider the partition

$$P = \{a, x - r/2, x + r/2, b\}$$

of $[a, b]$. But since $m_2 > 0$, we have that

$$\begin{aligned} 0 &< m_1[(x - r/2) - a] + m_2[(x + r/2) - (x - r/2)] + m_3[b - (x + r/2)] \\ &= L(P, f) \\ &\leq \int_a^b f(x) dx \end{aligned} \quad \text{Theorem 6.4}$$

a contradiction. \square

4. If $f(x) = 0$ for all irrational x and $f(x) = 1$ for all rational x , prove that $f \notin \mathcal{R}$ on $[a, b]$ for any $a < b$.

Proof. Let P be an arbitrary partition of $[a, b]$. Since the rationals and irrationals are dense in the reals, we know that for any $[x_i, x_{i+1}]$, $f(x) = 0$ for some $x \in [x_i, x_{i+1}]$ and $f(x) = 1$ for some $x \in [x_i, x_{i+1}]$. Thus, we have that $L(P, f) = 0$ and $U(P, f) = b - a$. It follows that if $a < b$,

$$\sup L(P, f) = 0 \neq b - a = \inf U(P, f)$$

so $f \notin \mathcal{R}$, as desired. \square

3 Integration II

From Rudin (1976).

Chapter 6

2/2: 3. Define three functions $\beta_1, \beta_2, \beta_3$ as follows:

$$\beta_1 = \begin{cases} 0 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases} \quad \beta_2 = \begin{cases} 0 & x < 0 \\ 1 & x = 0 \\ 1 & x > 0 \end{cases} \quad \beta_3 = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases}$$

Let f be a bounded function on $[-1, 1]$.

(a) Prove that $f \in \mathcal{R}(\beta_1)$ if and only if $f(0+) = f(0)$ and that then

$$\int f d\beta_1 = f(0)$$

Proof. Suppose first that $f \in \mathcal{R}(\beta_1)$ with $\int f d\beta_1 = f(0)$. To prove that $f(0+) = f(0)$, it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in [-1, 1]$ and $0 \leq x < \delta$, then $|f(x) - f(0)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $f \in \mathcal{R}(\beta_1)$ by hypothesis, we have by Theorem 6.6 that there exists a partition P of $[-1, 1]$ such that $U(P, f, \beta_1) - L(P, f, \beta_1) < \epsilon$. Now let $x_i = \min\{x \in P : x > 0\}$; we know that such an object exists since there exist elements of P greater than zero (namely 1) and P is finite. It follows by the definition of β_1 that $\Delta x_i = 1$ and $\Delta x_j = 0$ for $j \neq i$. Thus, $U(P, f, \beta_1) = M_i$ and $L(P, f, \beta_1) = m_i$ (which exist because f is bounded on $[-1, 1]$). At this point, we are ready to choose δ , which we take to be $\delta = x_i$. Now to confirm that this δ works: Let $0 \leq x < \delta$. By the definition of x_i, x_{i-1} , $m_i \leq f(x) \leq M_i$ and $m_i \leq f(0) \leq M_i$. But since $M_i - m_i < \epsilon$ as per the above, we have that $|f(x) - f(0)| < \epsilon$, as desired.

Now suppose that $f(0+) = f(0)$. To prove that $f \in \mathcal{R}(\beta_1)$, Theorem 6.6 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a P such that $U(P, f, \beta_1) - L(P, f, \beta_1) < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $f(0+) = f(0)$, we know that there exists a $\delta' > 0$ such that if $x \in [-1, 1]$ and $0 \leq x < \delta'$, then $|f(x) - f(0)| < \epsilon/3$. Let $\delta = \min(\delta'/2, 1)$. Thus, we may define $P = \{-1, 0, \delta, 1\}$. We have

$$\begin{aligned} U(P, f, \beta_1) &= \sum_{i=1}^3 M_i \Delta \beta_{1_i} & L(P, f, \beta_1) &= \sum_{i=1}^3 m_i \Delta \beta_{1_i} \\ &= M_2 & &= m_2 \end{aligned}$$

(which exist because f is bounded on $[-1, 1]$). Consequently, $M_2 \leq f(0) + \epsilon/3$. $m_2 \geq f(0) - \epsilon/3$. Therefore,

$$\begin{aligned} U(P, f, \beta_1) - L(P, f, \beta_1) &= M_2 - m_2 \\ &\leq [f(0) + \frac{\epsilon}{3}] - [f(0) - \frac{\epsilon}{3}] \\ &= \frac{2\epsilon}{3} \\ &< \epsilon \end{aligned}$$

as desired.

As to proving that $\int f d\beta_1$, we know that $M_2 \leq f(0) + \epsilon/3$ for arbitrarily small ϵ implies $M_2 \leq f(0)$. Similarly, $m_2 \geq f(0)$. Thus,

$$\inf U(P, f, \beta_1) \leq U(P, f, \beta_1) = M_2 \leq f(0) \leq m_2 = L(P, f, \beta_1) \leq \sup L(P, f, \beta_1)$$

But by Theorem 6.5, $\sup L(P, f, \beta_1) \leq \inf U(P, f, \beta_1)$. Therefore,

$$\int_{-1}^1 f d\beta_1 = \sup L(P, f, \beta_1) = \inf U(P, f, \beta_1) = f(0)$$

as desired. □

- (b) State and prove a similar result for β_2 .

Proof. The result will be $f \in \mathcal{R}(\beta_2)$ if and only if $f(0-) = f(0)$ and that then

$$\int f d\beta = f(0)$$

The proof of this result is entirely symmetric to the proof of the previous result. □

- (c) Prove that $f \in \mathcal{R}(\beta_3)$ if and only if f is continuous at 0.

Proof. Suppose first that $f \in \mathcal{R}(\beta_3)$. To prove that f is continuous at 0, it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in [-1, 1]$ and $|x| < \delta$, then $|f(x) - f(0)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $f \in \mathcal{R}(\beta_3)$ by hypothesis, we have by Theorem 6.6 that there exists a partition P of $[-1, 1]$ such that $U(P, f, \beta_3) - L(P, f, \beta_3) < \epsilon/2$. Now let $x_i = \max\{x \in P : x < 0\}$ and let $x_j = \min\{x \in P : x > 0\}$. Choose $\delta = \min\{|x_i|, |x_j|\}$. Let $P^* = P \cup \{-\delta, 0, \delta\}$ be a refinement of P . It follows by the definition of β_3 and a reenumeration of P^* that $U(P^*, f, \beta_3) = (M_{i-1} + M_i)/2$ and $L(P^*, f, \beta_3) = (m_{i-1} + m_i)/2$. Now let $|x| < \delta$. We divide into two cases ($x \geq 0$ and $x < 0$). If $x \geq 0$, then $m_i \leq f(x) \leq M_i$ and $m_i \leq f(0) \leq M_i$. But then we have that

$$\begin{aligned} |f(x) - f(0)| &\leq M_i - m_i \\ &\leq (M_{i-1} - m_{i-1}) + (M_i - m_i) \\ &= 2 \left[\frac{M_{i-1} + M_i}{2} - \frac{m_{i-1} + m_i}{2} \right] \\ &= 2[U(P^*, f, \beta_3) - L(P^*, f, \beta_3)] \\ &< \epsilon \end{aligned}$$

as desired. The proof is symmetric in the other case.

Now suppose that f is continuous at 0. To prove that $f \in \mathcal{R}(\beta_3)$, Theorem 6.6 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a P such that $U(P, f, \beta_3) - L(P, f, \beta_3) < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f is continuous at 0, we know that there exists a $\delta' > 0$ such that if $x \in [-1, 1]$ and $|x| < \delta'$, then $|f(x) - f(0)| < \epsilon/3$. Choose $\delta = \min(\delta'/2, 1)$. Consider $P = \{-1, -\delta/2, \delta/2, 1\}$. It follows as before that $U(P, f, \beta_3) = M_2$ and $L(P, f, \beta_3) = m_2$. Consequently, $M_2 \leq f(0) + \epsilon/3$ and $m_2 \geq f(0) - \epsilon/3$. Therefore,

$$\begin{aligned} U(P, f, \beta_3) - L(P, f, \beta_3) &= M_2 - m_2 \\ &\leq [f(0) + \frac{\epsilon}{3}] - [f(0) - \frac{\epsilon}{3}] \\ &= \frac{2\epsilon}{3} \\ &< \epsilon \end{aligned}$$

as desired. □

- (d) If f is continuous at 0, prove that

$$\int f d\beta_1 = \int f d\beta_2 = \int f d\beta_3 = f(0)$$

Proof. If f is continuous at 0, then $f(0+) = f(0) = f(0-)$. It follows that

$$f(0) = \int f \, d\beta_1 \quad \text{Part (a)}$$

$$= \int f \, d\beta_2 \quad \text{Part (b)}$$

$$= \int f \, d\beta_3 \quad \text{Part (c)}$$

Note that calculating the exact value of $\int f \, d\beta_3$ is symmetric to the proof in part (a). \square

5. Suppose f is a bounded real function on $[a, b]$, and $f^2 \in \mathcal{R}$ on $[a, b]$. Does it follow that $f \in \mathcal{R}$? Does the answer change if we assume that $f^3 \in \mathcal{R}$?

Proof. $f^2 \in \mathcal{R} \nRightarrow f \in \mathcal{R}$: Consider the bounded real function $f : [a, b] \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & x \notin \mathbb{Q} \\ -1 & x \in \mathbb{Q} \end{cases}$$

Since $f^2(x) = 1$ for all $x \in [a, b]$, $f^2 \in \mathcal{R}$ as a constant function. However, by Exercise 6.4 and a clever application of Theorem 6.12 (to relate it to the function explicitly considered in Exercise 6.4), we know that $f \notin \mathcal{R}$.

$f^3 \in \mathcal{R} \Rightarrow f \in \mathcal{R}$: Let $f : [a, b] \rightarrow \mathbb{R}$ be any bounded real function such that $f^3 \in \mathcal{R}$. To prove that $f \in \mathcal{R}$, Theorem 6.11 tells us that it will suffice to show that there exist $m, M \in \mathbb{R}$ such that $m \leq f \leq M$ and that there exists a continuous function $\phi : [m, M] \rightarrow \mathbb{R}$ such that $f = \phi \circ f^3$. Since f is bounded by hypothesis, we can pick $m, M \in \mathbb{R}$ such that $m \leq f \leq M$. Now let $\phi : [m, M] \rightarrow \mathbb{R}$ be defined by

$$\phi(x) = \sqrt[3]{x}$$

for all $x \in [m, M]$. It is obvious that ϕ is continuous and that $\phi \circ f^3 = f$, as desired. \square

7. Suppose f is a real function on $(0, 1]$ and $f \in \mathcal{R}$ on $[c, 1]$ for every $c > 0$. Define

$$\int_0^1 f(x) \, dx = \lim_{c \rightarrow 0} \int_c^1 f(x) \, dx$$

if this limit exists (and is finite).

- (a) If $f \in \mathcal{R}$ on $[0, 1]$, show that this definition of the integral agrees with the old one.

Proof. To prove that $\int_0^1 f = \lim_{c \rightarrow 0} \int_c^1 f$, it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $c \in (0, 1]$ and $c < \delta$, then

$$\left| \int_0^c f \right| = \left| \int_c^1 f - \int_0^1 f \right| < \epsilon$$

Let $\epsilon > 0$ be arbitrary. Since f is integrable, f is bounded, i.e., there exists $M \in \mathbb{R}$ such that $|f(x)| < M$ for all $x \in [0, 1]$. Choose $\delta = \epsilon/M$. Let $c \in (0, 1]$ be such that $c < \delta$. Then by Theorem 6.12d,

$$\left| \int_0^c f \right| \leq M(c - 0) < \epsilon$$

as desired. \square

- (b) Construct a function f such that the above limit exists, although it fails to exist with $|f|$ in place of f .

Proof. Let $f : (0, 1] \rightarrow \mathbb{R}$ be defined by

$$f(x) = (-1)^n n$$

for $1/n < x \leq 1/(n-1)$ ($n = 2, 3, \dots$). It follows since f is a constant function save one terminal discontinuity on each $[1/n, 1/(n-1)]$ that

$$\begin{aligned} \int_{1/n}^{1/(n-1)} f &= (-1)^n n \cdot \left(\frac{1}{n-1} - \frac{1}{n} \right) \\ &= \frac{(-1)^n n}{n(n-1)} \\ &= \frac{(-1)^n}{n-1} \end{aligned}$$

for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned} \int_{1/N}^1 f &= \sum_{n=2}^N \int_{1/n}^{1/(n-1)} f \\ &= \sum_{n=2}^N \frac{(-1)^n}{n-1} \end{aligned}$$

Thus,

$$\lim_{c \rightarrow 0} \int_c^1 f = \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1}$$

which converges by Theorem 3.43. However, the limit fails to exist if f is replaced by $|f|$, because in that case, the integral is equal to the harmonic series, which diverges to infinity. \square

8. Suppose $f \in \mathcal{R}$ on $[a, b]$ for every $b > a$ where a is fixed. Define

$$\int_a^{\infty} f(x) \, dx = \lim_{b \rightarrow \infty} \int_a^b f(x) \, dx$$

if this limit exists (and is finite). In that case, we say that the integral on the left **converges**. If it also converges after f has been replaced by $|f|$, it is said to converge **absolutely**.

Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1, \infty)$. Prove that $\int_1^{\infty} f(x) \, dx$ converges if and only if $\sum_{n=1}^{\infty} f(n)$ converges. (This is the so-called “integral test” for convergence of series.)

Proof. To prove the claim, we will show that

$$\sum_{n=2}^N f(n) \leq \int_1^N f \leq \sum_{n=1}^{N-1} f(n) \leq f(1) + \int_1^{N-1} f(x) \, dx$$

It will follow since both the sum and the integral limit are monotonically increasing as $N \rightarrow \infty$ ($f \geq 0$) and both are bounded below and above by (a function of) the other, both converge or diverge together. Let's begin.

Since f is monotonically decreasing on $[1, \infty)$, we know that $f(n) \leq f(x)$ for all $1 \leq x \leq n$ ($n \in \mathbb{N}$). Thus, by Theorem 6.12b,

$$\int_{n-1}^n f(n) \, dx \leq \int_{n-1}^n f(x) \, dx$$

Therefore,

$$\sum_{n=2}^N f(n) = \sum_{n=2}^N \int_{n-1}^n f(n) \, dx \quad \text{Theorem 6.12d}$$

$$\leq \sum_{n=2}^N \int_{n-1}^n f(x) \, dx$$

$$= \int_1^N f(x) \, dx \quad \text{Theorem 6.12c}$$

for all $N = 2, 3, 4, \dots$, thereby establishing the left inequality above.

Since f is monotonically decreasing on $[1, \infty)$, we know that $f(x) \leq f(n)$ for all $x \geq n$ ($n \in \mathbb{N}$). Thus, by Theorem 6.12b,

$$\int_n^{n+1} f(x) \, dx \leq \int_n^{n+1} f(n) \, dx$$

Therefore,

$$\int_1^N f(x) \, dx = \sum_{n=1}^{N-1} \left(\int_n^{n+1} f(x) \, dx \right) \quad \text{Theorem 6.12c}$$

$$\leq \sum_{n=1}^{N-1} \left(\int_n^{n+1} f(n) \, dx \right)$$

$$= \sum_{n=1}^{N-1} f(n) \quad \text{Theorem 6.12d}$$

for all $N = 2, 3, 4, \dots$, thereby establishing the middle inequality above.

From our statement about $f(n)$ and $f(x)$ from the left inequality, we have by Theorem 6.12b that

$$\int_{n-1}^n f(n) \, dx \leq \int_{n-1}^n f(x) \, dx$$

Therefore,

$$\sum_{n=1}^{N-1} f(n) = f(1) + \sum_{n=2}^{N-1} \int_{n-1}^n f(n) \, dx \quad \text{Theorem 6.12d}$$

$$\leq f(1) + \sum_{n=2}^{N-1} \int_{n-1}^n f(x) \, dx$$

$$= f(1) + \int_1^{N-1} f(x) \, dx \quad \text{Theorem 6.12c}$$

for all $N = 2, 3, 4, \dots$, thereby establishing the right inequality above. \square

10. Let p, q be positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

Prove the following statements.

(a) If $u, v \geq 0$, then

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}$$

Equality holds if and only if $u^p = v^q$.

Discussion. To prove the desired inequality, it will suffice to show that

$$0 \leq \frac{u^p}{p} + \frac{v^q}{q} - uv$$

i.e., that for all $u, v \geq 0$, the expression on the right above is nonnegative. To consider all such values at once, we can consider applying our analysis toolbox to $f : [0, \infty)^2 \rightarrow \mathbb{R}$ defined by

$$f(u, v) = \frac{u^p}{p} + \frac{v^q}{q} - uv$$

with the goal of proving that it is nonnegative everywhere on its domain. However, since we do not yet know multivariable calculus, it will suffice to fix $u \geq 0$ and analyze $f : [0, \infty) \rightarrow \mathbb{R}$ defined by

$$f(v) = \frac{u^p}{p} + \frac{v^q}{q} - uv$$

Let's begin. □

Proof. Fix $u \geq 0$. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be defined by

$$f(v) = \frac{u^p}{p} + \frac{v^q}{q} - uv$$

It follows from the definition of f that to prove the desired inequality, it will suffice to show that f is nonnegative everywhere on its domain. Let's begin.

Since f is a polynomial in v , f is differentiable. Thus, we may consider

$$f'(v) = v^{q-1} - u$$

As a function of a positive power ($q/(q-1) = p > 0$ and $q > 0$ imply $q-1 > 0$) of its variable (minus a constant), f' is strictly increasing. Additionally, we have that

$$\begin{aligned} 0 &= f'(v) \\ u &= v^{q-1} \\ &= v^{q/p} \\ v &= u^{p/q} \end{aligned}$$

Thus, we know that $f' < 0$ on $(0, u^{p/q})$ and $f' > 0$ on $(u^{p/q}, \infty)$. It follows by the strict version of Theorem 5.11 that f is strictly decreasing on $(0, u^{p/q})$ and strictly increasing on $(u^{p/q}, \infty)$. Furthermore, since f is differentiable (hence continuous by Theorem 5.2), we know that $f(0) \geq f(u^{p/q})$. Combining the last several results, we have that $f(u^{p/q})$ is the minimum of f over $[0, \infty)$, and hence equal to the minimum value of f over $[0, \infty)$. But since

$$\begin{aligned} f(u^{p/q}) &= \frac{u^p}{p} + \frac{(u^{p/q})^q}{q} - uu^{p/q} \\ &= \frac{u^p}{p} + \frac{u^p}{q} - u^{p/q+1} \\ &= u^p \left(\frac{1}{p} + \frac{1}{q} \right) - u^p \\ &= 0 \end{aligned}$$

we know that $f(v) \geq 0$ on its domain, as desired.

Additionally, since f is strictly decreasing on $(0, u^{p/q})$ and strictly increasing on $(u^{p/q}, \infty)$, we know that $f(v) = 0$ iff $v = u^{p/q}$, i.e., iff $v^q = u^p$, as desired. □

(b) If $f, g \in \mathcal{R}(\alpha)$, $f, g \geq 0$, and

$$\int_a^b f^p d\alpha = 1 = \int_a^b g^q d\alpha$$

then

$$\int_a^b fg d\alpha \leq 1$$

Proof. By Theorem 6.13a, the hypothesis $f, g \in \mathcal{R}(\alpha)$ implies that $fg \in \mathcal{R}(\alpha)$. Thus, we have that

$$\int_a^b fg d\alpha \leq \int_a^b \left(\frac{f^p}{p} + \frac{g^q}{q} \right) d\alpha \quad \text{Theorem 6.12b}$$

$$= \frac{1}{p} \int_a^b f^p d\alpha + \frac{1}{q} \int_a^b g^q d\alpha \quad \text{Theorem 6.12a}$$

$$= \frac{1}{p} + \frac{1}{q}$$

$$= 1$$

as desired. □

(c) If f, g are complex functions in $\mathcal{R}(\alpha)$, then

$$\left| \int_a^b fg d\alpha \right| \leq \left(\int_a^b |f|^p d\alpha \right)^{1/p} \left(\int_a^b |g|^q d\alpha \right)^{1/q}$$

This is **Hölder's inequality**. When $p = q = 2$, it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)

Proof. By Theorem 6.11 with $\phi(y) = |y|^p$ (resp. $\phi(y) = |y|^q$), the hypothesis $f, g \in \mathcal{R}(\alpha)$ implies that $|f|^p, |g|^q \in \mathcal{R}(\alpha)$. Thus, we may let

$$I_f = \left(\int_a^b |f|^p d\alpha \right)^{1/p} \quad I_g = \left(\int_a^b |g|^q d\alpha \right)^{1/q}$$

We divide into two cases ($I_f = 0$ or $I_g = 0$, and $I_f, I_g \neq 0$). In the first case, WLOG let $I_f = 0$. Then since $0 \leq |f|^p$, it follows that $f = 0$ on $[a, b]$. Thus

$$\left| \int_a^b fg d\alpha \right| = 0 \leq 0 = I_f I_g = \left(\int_a^b |f|^p d\alpha \right)^{1/p} \left(\int_a^b |g|^q d\alpha \right)^{1/q}$$

as desired. In the other case, it follows that

$$\begin{aligned} I_f^p &= \int_a^b |f|^p d\alpha & I_g^q &= \int_a^b |g|^q d\alpha \\ 1 &= \int_a^b \left| \frac{f}{I_f} \right|^p d\alpha & 1 &= \int_a^b \left| \frac{g}{I_g} \right|^q d\alpha \end{aligned} \quad \text{Theorem 6.12a}$$

Thus, since $|f/I_f|, |g/I_g| \in \mathcal{R}(\alpha)$ by Theorems 6.12 and 6.13 and $|f/I_f|, |g/I_g| \geq 0$ by the defini-

tion of the absolute value, we have that

$$\begin{aligned}
 \left| \int_a^b fg \, d\alpha \right| &\leq \int_a^b |fg| \, d\alpha && \text{Theorem 6.13b} \\
 &= I_f I_g \int_a^b \left| \frac{f}{I_f} \right| \left| \frac{g}{I_g} \right| d\alpha \\
 &\leq I_f I_g \cdot 1 && \text{Part (b)} \\
 &= \left(\int_a^b |f|^p \, d\alpha \right)^{1/p} \left(\int_a^b |g|^q \, d\alpha \right)^{1/q}
 \end{aligned}$$

as desired. \square

11. Let α be a fixed increasing function on $[a, b]$. For $u \in \mathcal{R}(\alpha)$, define

$$\|u\|_2 = \left(\int_a^b |u|^2 \, d\alpha \right)^{1/2}$$

Suppose $f, g, h \in \mathcal{R}(\alpha)$, and prove the triangle inequality

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2$$

as a consequence of the Schwarz inequality, as in the proof of Theorem 1.37.

Proof. By Theorems 6.12a and 6.13b, the hypothesis that $f, g, h \in \mathcal{R}(\alpha)$ implies that $|f - g|, |g - h| \in \mathcal{R}(\alpha)$. Thus, we have that

$$\begin{aligned}
 \|f - h\|_2^2 &= \int_a^b |f - h|^2 \, d\alpha \\
 &= \int_a^b |(f - g) + (g - h)|^2 \, d\alpha \\
 &= \int_a^b |f - g|^2 \, d\alpha + 2 \int_a^b |f - g| \cdot |g - h| \, d\alpha + \int_a^b |g - h|^2 \, d\alpha \\
 &\leq \int_a^b |f - g|^2 \, d\alpha + 2 \left(\int_a^b |f - g|^2 \, d\alpha \right)^{1/2} \left(\int_a^b |g - h|^2 \, d\alpha \right)^{1/2} + \int_a^b |g - h|^2 \, d\alpha \\
 &= \|f - g\|_2^2 + 2\|f - g\|_2 \|g - h\|_2 + \|g - h\|_2^2 \\
 &= (\|f - g\|_2 + \|g - h\|_2)^2
 \end{aligned}$$

Taking square roots of both sides of the inequality yields the desired result. \square

4 Sequences and Series of Functions

From Rudin (1976).

Chapter 7

- 2/9: 1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

Proof. Let $\{f_n\}$ be an arbitrary uniformly convergent sequence of bounded functions. To prove that it is uniformly bounded, it will suffice to find a number M such that $|f_n(x)| < M$ for all $x \in E$ and $n \in \mathbb{N}$. Let f be the function such that $f_n \Rightarrow f$, and let $M_n = \sup_{x \in E} |f_n(x) - f(x)|$ for each $n \in \mathbb{N}$ (the boundedness of each f_n implies that such an M_n always exists). Thus, based on the last two definitions, we can invoke Theorem 7.9 to learn that $M_n \rightarrow 0$ as $n \rightarrow \infty$. But since $\{M_n\}$ converges, Theorem 3.2c implies that $\{M_n\}$ is bounded, say by $M/2$. Taking M to be our M yields that for an arbitrary $x \in E$ and $n \in \mathbb{N}$,

$$|f_n(x)| \leq M_n \leq \frac{M}{2} < M$$

as desired. \square

2. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E , prove that $\{f_n + g_n\}$ converges uniformly on E . If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E .

Proof. To prove that $\{f_n + g_n\}$ converges uniformly on E to $f + g$, it will suffice to show that for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|(f_n + g_n)(x) - (f + g)(x)| < \epsilon$ for all $x \in E$. Let $\epsilon > 0$ be arbitrary. Since $f_n \rightarrow f$ uniformly on E , there exists N_1 such that if $n \geq N_1$, then $|f_n(x) - f(x)| < \epsilon/2$ for all $x \in E$. Similarly, there exists N_2 such that if $n \geq N_2$, then $|g_n(x) - g(x)| < \epsilon/2$ for all $x \in E$. Choose $N = \max(N_1, N_2)$. Now suppose $n \geq N$, and let $x \in E$ be arbitrary. It follows from the first condition that $n \geq N \geq N_1$ and $n \geq N \geq N_2$, so

$$\begin{aligned} |(f_n + g_n)(x) - (f + g)(x)| &= |f_n(x) - f(x) + g_n(x) - g(x)| \\ &\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

as desired.

To prove that $\{f_n g_n\}$ converges uniformly on E to fg , it will suffice to show that for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|(f_n g_n)(x) - (fg)(x)| < \epsilon$ for all $x \in E$. Let $\epsilon > 0$ be arbitrary. Since f_n, g_n are uniformly convergent sequences of bounded functions, Exercise 1 implies that they are uniformly bounded, i.e., there exists $M^f, M^g \in \mathbb{R}$ such that $|f_n| < M^f$ and $|g_n| < M^g$ for all $n \in \mathbb{N}$. If we take $M = \max(M^f, M^g)$, then we have $|f_n| < M$ and $|g_n| < M$ for all $n \in \mathbb{N}$. Note that the same inequality holds for f and g . Now, as before, we may pick N such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon/2M$ and $|g_n(x) - g(x)| < \epsilon/2M$ for all $x \in E$. It follows that for any $n \geq N$ and $x \in E$,

$$\begin{aligned} |(f_n g_n)(x) - (fg)(x)| &= |f_n(x) \cdot (g_n(x) - g(x)) + g(x) \cdot (f_n(x) - f(x))| \\ &= |f_n(x)| \cdot |g_n(x) - g(x)| + |g(x)| \cdot |f_n(x) - f(x)| \\ &< M \cdot \frac{\epsilon}{2M} + M \cdot \frac{\epsilon}{2M} \\ &= \epsilon \end{aligned}$$

as desired. \square

3. Construct sequences $\{f_n\}, \{g_n\}$ which converge uniformly on some set E , but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converge on E).

Proof. Let

$$f_n(x) = x \qquad g_n(x) = \frac{1}{n}$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then $\{f_n\}$ converges uniformly to $f(x) = x$ (by choosing $N = 1$ for any ϵ) and $\{g_n\}$ converges uniformly to $g(x) = 0$ (by choosing $1/N < \epsilon$ with the Archimedean principle). However, while $\{f_n g_n\}$ converges pointwise to $(fg)(x) = 0$ by Theorem 3.3c, it does not converge uniformly since for any n , choosing $x = n$ yields $(f_n g_n)(x) = 1$. \square

4. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?

Proof. Absolute convergence values: The series converges absolutely for any

$$x \in (-\infty, -1) \cup \left(\bigcup_{k=1}^{\infty} \left(-\frac{1}{k^2}, -\frac{1}{(k+1)^2} \right) \right) \cup (0, \infty)$$

We prove this via casework as follows.

Let $x \in (0, \infty)$. Then we have

$$\sum_{n=1}^{\infty} \left| \frac{1}{1+n^2x} \right| = \sum_{n=1}^{\infty} \frac{1}{1+n^2x} \leq \sum_{n=1}^{\infty} \frac{1}{n^2x} = \frac{1}{x} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{c}{x}$$

where $c \in \mathbb{R}$ is finite by Theorem 3.28. Therefore, since the sum is monotonically increasing and bounded, Theorem 3.14 implies that the sum overall converges, as desired.

Let $x \in (-\infty, -1)$. Then we have

$$n^2x + 1 < n^2x + n^2 = n^2(x+1)$$

Since $x < -1$,

$$n^2x + 1 < 0 \qquad n^2(x+1) < 0$$

for all $n \in \mathbb{N}$. Thus,

$$\begin{aligned} n^2x + 1 &< n^2(x+1) \\ \frac{n^2x + 1}{n^2(x+1)} &> 1 \\ \frac{1}{n^2(x+1)} &< \frac{1}{n^2x + 1} \\ \left| \frac{1}{n^2x + 1} \right| &< \left| \frac{1}{n^2(x+1)} \right| \end{aligned}$$

for all $n \in \mathbb{N}$. It follows that

$$\sum_{n=1}^{\infty} \left| \frac{1}{n^2x + 1} \right| < \sum_{n=1}^{\infty} \left| \frac{1}{n^2(x+1)} \right| = \frac{1}{x+1} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{c}{x+1}$$

where $c \in \mathbb{R}$ is finite by Theorem 3.28. Therefore, since the sum is monotonically increasing and bounded, Theorem 3.14 implies that the sum overall converges, as desired.

Let $x \in (-1/k^2, -1/(k+1)^2)$. For right now, we consider only the sum for $n \geq \sqrt{2}(k+1)$, leaving finitely many terms out of the sum. Let $\delta = 1/(k+1)^2$. It follows that

$$\begin{aligned} n &\geq \sqrt{2}(k+1) & x &< -\frac{1}{(k+1)^2} \\ n &\geq \sqrt{\frac{2}{1/(k+1)^2}} & -x &> \frac{1}{(k+1)^2} \\ n^2 &\geq \frac{2}{\delta} \\ \frac{\delta}{2} &\geq \frac{1}{n^2} \end{aligned}$$

Additionally, since $n \geq \sqrt{2}(k+1) > k$ (hence $n^2 \geq (k+1)^2$) and $x < -1/(k+1)^2$, we have that

$$\begin{aligned} n^2 x &< (k+1)^2 \cdot -\frac{1}{(k+1)^2} \\ n^2 x &< -1 \\ n^2 x + 1 &< 0 \end{aligned}$$

Thus, for $n \geq \sqrt{2}(k+1)$, we have that

$$\left| \frac{1}{1+n^2 x} \right| = \frac{1}{n^2(-x) - 1} < \frac{1}{n^2 \delta - 1} = \frac{1}{n^2} \cdot \frac{1}{\delta - 1/n^2} \leq \frac{1}{n^2} \cdot \frac{1}{\delta - \delta/2} = \frac{2}{\delta n^2}$$

Therefore, since $|f_n(x)| \leq M_n = 2/\delta n^2$ and $\sum M_n$ converges by Theorem 3.28, the comparison test implies that $\sum |f_n(x)|$ converges, as desired. Adding on the finitely many terms we left out of the summation will not change this fact.

Note that the series diverges for $x = 0$ since each term becomes 1 in this case. Additionally, the series fails to exist for $x = -1/k^2$ ($k \in \mathbb{N}$) since the k^{th} term is undefined in this case.

Uniform convergence intervals: The series converges uniformly on any

$$[a, b] \subset (-\infty, -1) \cup \left(\bigcup_{k=1}^{\infty} \left(-\frac{1}{k^2}, -\frac{1}{(k+1)^2} \right) \right) \cup (0, \infty)$$

This is because any such interval will be a subset of either $(-\infty, -1)$, $(0, \infty)$, or a set of the form $(-1/k^2, -1/(k+1)^2)$ ($k \in \mathbb{N}$). Thus, we may take as $\sum M_n$ the supremum on $[a, b]$ of the appropriate bound derived above (either c/x , $c/(x+1)$, or $2c/\delta$, respectively; all supremums of which will exist by the definition of $[a, b]$) and apply Theorem 7.10.

Non-uniform convergence intervals: Any interval containing one or more of the points in the set $\{0\} \cup \{-1/n^2\}_{n=1}^{\infty}$, by the above.

Points of continuity: The series is continuous at all points at which it converges.

Let x be a point at which f converges. Then by the first part of the proof, x is an element of an open set G . Thus, let $N_{2r}(x) \subset G$, and consider $[x-r, x+r]$. By the above, f converges uniformly on this interval. Additionally, each f_n is continuous on this interval by definition. Thus, by Theorem 7.12, f is continuous at x , as desired.

Boundedness: f is not bounded.

If we suppose for the sake of contradiction that f is bounded by m , we nevertheless find that

$$f\left(\frac{1}{4m^2}\right) > \sum_{n=1}^{2m} \frac{1}{1 + \frac{n^2}{4m^2}} = \sum_{n=1}^{2m} \frac{(2m)^2}{(2m)^2 + n^2} \geq \sum_{n=1}^{2m} \frac{(2m)^2}{(2m)^2 + (2m)^2} = \sum_{n=1}^{2m} \frac{1}{2} = m$$

□

7. For $n = 1, 2, 3, \dots$ and x real, put

$$f_n(x) = \frac{x}{1 + nx^2}$$

Show that $\{f_n\}$ converges uniformly to a function f and that the equation

$$f'(x) = \lim_{n \rightarrow \infty} f'_n(x)$$

is correct if $x \neq 0$ but false if $x = 0$.

Proof. To prove that $\{f_n\}$ converges uniformly to f defined by $f(x) = 0$ ($x \in \mathbb{R}$), Theorem 7.9 tells us that it will suffice to show that $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$ and that the sequence $\{M_n\}$ defined by $M_n = \sup_{x \in \mathbb{R}} |f_n(x)|$ tends to zero as $n \rightarrow \infty$. Since

$$f_n(x) = \frac{x}{1 + nx^2} < \frac{x}{nx^2} = \frac{1}{x} \cdot \frac{1}{n} \rightarrow 0$$

as $n \rightarrow \infty$ for all $x \neq 0$ and $f_n(0) = 0$ for all n , $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$, as desired. Additionally, by the Schwarz inequality, if a_1, a_2, b_1, b_2 are real numbers, then

$$|a_1 b_1 + a_2 b_2|^2 \leq (|a_1|^2 + |a_2|^2)(|b_1|^2 + |b_2|^2)$$

It follows that

$$\begin{aligned} |2\sqrt{n}x|^2 &= \left| \underbrace{1}_{a_1} \cdot \underbrace{\sqrt{n}x}_{b_1} + \underbrace{\sqrt{n}x}_{a_2} \cdot \underbrace{1}_{b_2} \right|^2 \leq (|1|^2 + |\sqrt{n}x|^2)(|\sqrt{n}x|^2 + |1|^2) = (1 + nx^2)^2 \\ |2\sqrt{n}x| &\leq |1 + nx^2| \\ \frac{1}{|1 + nx^2|} &\leq \frac{1}{2\sqrt{n}|x|} \\ \frac{|x|}{|1 + nx^2|} &\leq \frac{1}{2\sqrt{n}} \\ \left| \frac{x}{1 + nx^2} \right| &\leq \frac{1}{2\sqrt{n}} \end{aligned}$$

for all $x \neq 0$, $n \in \mathbb{N}$. This combined with the facts that $f_n(0) = 0 < \frac{1}{2\sqrt{n}}$ for all $n \in \mathbb{N}$ and $f_n(1/\sqrt{n}) = 1/2\sqrt{n}$ for all $n \in \mathbb{N}$ implies that $M_n = 1/2\sqrt{n}$. Thus, $M_n \rightarrow 0$ as $n \rightarrow \infty$, as desired.

$f'(x) = 0$ for all $x \in \mathbb{R}$. Additionally,

$$f'_n(x) = \frac{1 - nx^2}{(1 + nx^2)^2} \leq \frac{1 - nx^2}{(nx^2)^2} = \frac{1}{x^4} \cdot \frac{1}{n^2} - \frac{1}{x^2} \cdot \frac{1}{n} \rightarrow 0$$

as $n \rightarrow \infty$ for all $x \neq 0$, as desired. However, $f'_n(0) = 1$ for all $n \in \mathbb{N}$, as desired. □

5 Sequences and Series of Functions II / Functions of Several Variables

From Rudin (1976).

Chapter 7

2/16: 5. Let

$$f_n(x) = \begin{cases} 0 & x < \frac{1}{n+1} \\ \sin^2 \frac{\pi}{x} & \frac{1}{n+1} \leq x \leq \frac{1}{n} \\ 0 & \frac{1}{n} < x \end{cases}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that absolute convergence, even for all x , does not imply uniform convergence.

Proof. To prove that $\{f_n\}$ converges pointwise to the continuous function f defined by $f(x) = 0$ for all $x \in \mathbb{R}$, it will suffice to show that for every $\epsilon > 0$ and for every $x \in \mathbb{R}$, there exists an integer N such that if $n \geq N$, then $|f_n(x)| < \epsilon$. Let $\epsilon > 0$ and $x \in \mathbb{R}$ be arbitrary. We divide into three cases ($x \in \{1/n\}_{n=1}^\infty$, $x \in [0, 1] \setminus \{1/n\}_{n=1}^\infty$, and $x \notin [0, 1]$).

If $x \in \{1/n\}_{n=1}^\infty$, let $x = 1/k$. Then by the definition of $f_n(x)$, we have that

$$f_i(x) = \begin{cases} 0 & i < k - 1 \\ \sin^2 \frac{\pi}{1/k} = \sin^2 k\pi = 0 & i = k - 1, k \\ 0 & i > k \end{cases}$$

Thus, choose $N = 1$. It follows that if $n \geq N$, then

$$|f_n(x)| = 0 < \epsilon$$

as desired.

If $x \in [0, 1] \setminus \{1/n\}_{n=1}^\infty$, let $x \in (1/[(N-1)+1], 1/(N-1))$ where $N \in \mathbb{N}$. Choose this N to be our N . It follows that if $n \geq N$, then

$$\frac{1}{n} \leq \frac{1}{N} = \frac{1}{(N-1)+1} < x$$

so by definition,

$$|f_n(x)| = 0 < \epsilon$$

as desired.

If $x \notin [0, 1]$, then either $x < 1/(n+1)$ for all $n \in \mathbb{N}$ or $x > 1/n$ for all $n \in \mathbb{N}$. Either way, we choosing $N = 1$ yields that if $n \geq N$, then

$$|f_n(x)| = 0 < \epsilon$$

as desired.

To prove that $\{f_n\}$ does not converge uniformly to f , Theorem 7.9 tells us that it will suffice to show that if $M_n = \sup_{x \in \mathbb{R}} |f_n(x) - f(x)|$, then $M_n \not\rightarrow 0$ as $n \rightarrow \infty$. Let $n \in \mathbb{N}$ be arbitrary. Since $n < n + 1/2 < n + 1$ and hence $1/(n+1) \leq 2/(2n+1) \leq 1/n$, we have by the properties of the sine function that

$$f_n\left(\frac{2}{2n+1}\right) = \sin^2 \left[\frac{\pi}{2/(2n+1)} \right] = \sin^2 \left[\frac{2n+1}{2} \pi \right] = \sin^2 \left[\left(n + \frac{1}{2} \right) \pi \right] = 1$$

and that $f_n(x) \leq 1$ everywhere else. Thus, $M_n = 1$ for all $n \in \mathbb{N}$. But then $M_n \not\rightarrow 0$ as $n \rightarrow \infty$, as desired.

It follows by an argument symmetric to the above that while $\sum f_n$ converges absolutely to

$$f(x) = \begin{cases} 0 & x \leq 0 \\ \sin^2 \frac{\pi}{x} & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$$

$M_n = 1$ for all $n \in \mathbb{N}$. □

6. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x .

Proof. Let $[a, b]$ be an arbitrary bounded interval, and let $f_n(x) = (-1)^n \frac{x^2 + n}{n^2}$. To prove that the series converges uniformly on $[a, b]$, Theorem 7.8 tells us that it will suffice to show that for every $\epsilon > 0$, there exists an N such that if $n, m \geq N$ (WLOG let $n \leq m$) and $x \in [a, b]$, then

$$\left| \sum_{i=n}^m f_i(x) \right| < \epsilon$$

Let $\epsilon > 0$ be arbitrary. Define $m = \max(|a|, |b|)$ (note that since $a \neq b$ by definition, $m > 0$). By consecutive applications of Theorem 3.43, we know that both $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converge. Thus, by consecutive applications of Theorem 3.22, there exist integers N_1, N_2 such that $m \geq n \geq N_1$ implies the left result below and $m \geq n \geq N_2$ implies the right result below.

$$\left| \sum_{k=n}^m (-1)^k \frac{1}{k^2} \right| < \frac{\epsilon}{2m^2} \qquad \left| \sum_{k=n}^m (-1)^k \frac{1}{k} \right| < \frac{\epsilon}{2}$$

Choose $N = \max(N_1, N_2)$. Now let $n, m \geq N$ with WLOG $n \leq m$, and let $x \in [a, b]$. It follows that

$$\begin{aligned} \left| \sum_{k=n}^m f_k(x) \right| &= \left| \sum_{k=n}^m (-1)^k \frac{x^2 + k}{k^2} \right| \\ &= \left| x^2 \sum_{k=n}^m (-1)^k \frac{1}{k^2} + \sum_{k=n}^m (-1)^k \frac{1}{k} \right| \\ &\leq |x^2| \cdot \left| \sum_{k=n}^m (-1)^k \frac{1}{k^2} \right| + \left| \sum_{k=n}^m (-1)^k \frac{1}{k} \right| \\ &\leq m^2 \cdot \left| \sum_{k=n}^m (-1)^k \frac{1}{k^2} \right| + \left| \sum_{k=n}^m (-1)^k \frac{1}{k} \right| \\ &< m^2 \cdot \frac{\epsilon}{2m^2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

as desired.

To prove that the series does not converge absolutely for any value of x , let $x \in \mathbb{R}$ be arbitrary. Then

$$\begin{aligned} \sum_{n=1}^{\infty} \left| (-1)^n \frac{x^2 + n}{n^2} \right| &= x^2 \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n} \\ &\geq \sum_{n=1}^{\infty} \frac{1}{n} \end{aligned}$$

where the latter series diverges by Theorem 3.28, yielding the desired result. □

8. If

$$I(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

if $\{x_n\}$ is a sequence of distinct points of (a, b) , and if $\sum |c_n|$ converges, prove that the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n)$$

converges uniformly on $[a, b]$, and that f is continuous for every $x \neq x_n$.

Proof. Let $f_n(x) = c_n I(x - x_n)$ for all $n \in \mathbb{N}$. To prove that f converges uniformly on $[a, b]$, Theorem 7.10 tells us that it will suffice to show that $|f_n(x)| \leq M_n$ for all $x \in [a, b]$ and $\sum M_n$ converges. Let $M_n = c_n$ for all $n \in \mathbb{N}$. Then for any $x \in [a, b]$,

$$|f_n(x)| = c_n I(x - x_n) \leq c_n = M_n$$

as desired. Additionally, $\sum M_n = \sum c_n$ converges, as desired. This completes the proof.

For the second part of the proof, let $x \notin \{x_n\}$. Then every f_n is continuous at x by definition. Thus, f is a uniformly convergent sequence of functions continuous at x , so by Theorem 7.12, f is continuous at x . \square

9. Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E . Prove that

$$\lim_{n \rightarrow \infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \rightarrow x$ and $x \in E$. Is the converse of this true?

Proof. Let $\{x_n\} \subset E$ be an arbitrary sequence of points that converges to some $x \in E$. To prove that $\lim_{n \rightarrow \infty} f_n(x_n) = f(x)$, it will suffice to show that for every $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|f_n(x_n) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\{f_n\}$ is a uniformly convergent sequence of continuous functions, Theorem 7.12 implies that f is a continuous function. Thus, there exists a $\delta > 0$ such that if $y \in E$ and $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon/2$. Additionally, since $x_n \rightarrow x$, there exists an N_1 such that if $n \geq N_1$, $|x_n - x| < \delta$. Furthermore, since f_n converges uniformly to f , there exists N_2 such that if $n \geq N_2$, then $|f_n(y) - f(y)| < \epsilon/2$ for all $y \in E$. In particular, $|f_n(x_n) - f(x_n)| < \epsilon/2$. Choose $N = \max(N_1, N_2)$. Let $n \geq N$ be arbitrary. Then

$$\begin{aligned} |f_n(x_n) - f(x)| &\leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

as desired.

No, it is not true in general that if $\{f_n\}$ is a sequence of continuous functions for which $\lim_{n \rightarrow \infty} f_n(x_n) = f(x)$ for every sequence of points $x_n \in E$ such that $x_n \rightarrow x$ and $x \in E$, then f_n converges uniformly. Consider the sequence of functions from Exercise 7.5. This is a sequence of continuous functions for which $\lim_{n \rightarrow \infty} f_n(x_n) = f(x)$ for any sequence $\{x_n\}$ of the desired type since we can always choose N large enough so that the moving “hump” and neighborhood of x containing all remaining x_n are separated forever more. Moreover, by Exercise 7.5, $\{f_n\}$ does not converge uniformly, as desired. \square

Chapter 9

1. If S is a nonempty subset of a vector space X , prove (as asserted in Section 9.1) that the span of S is a vector space.

Proof. Let $S = \{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ (the proof is symmetric if S is infinite).

To prove that $\text{span}(S)$ is a vector space, it will suffice to show that $\text{span}(S)$ is nonempty and that for all $\mathbf{x}, \mathbf{y} \in \text{span}(S)$ and $c \in \mathbb{C}$, $(\mathbf{x} + \mathbf{y}) \in \text{span}(S)$ and $c\mathbf{x} \in \text{span}(S)$. Since S is nonempty, there exists $\mathbf{x} \in S$; thus, $1\mathbf{x} \in \text{span}(S)$, so $\text{span}(S)$ is nonempty, as desired. Let $\mathbf{x}, \mathbf{y} \in \text{span}(S)$ and $c \in \mathbb{C}$. There exist $a_1, \dots, a_n, b_1, \dots, b_n$ such that

$$\mathbf{x} = a_1\mathbf{u}_1 + \dots + a_n\mathbf{u}_n \qquad \mathbf{y} = b_1\mathbf{u}_1 + \dots + b_n\mathbf{u}_n$$

It follows by the definition of $\text{span}(S)$ that

$$\begin{aligned} (a_1 + b_1)\mathbf{u}_1 + \dots + (a_n + b_n)\mathbf{u}_n &= \mathbf{x} + \mathbf{y} \in \text{span}(S) \\ ca_1\mathbf{u}_1 + \dots + ca_n\mathbf{u}_n &= c\mathbf{x} \in \text{span}(S) \end{aligned}$$

as desired. □

2. Prove (as asserted in Section 9.6) that BA is linear if A and B are linear transformations. Prove also that A^{-1} is linear and invertible.

Proof. Let $A \in L(X, Y)$ and $B \in L(Y, Z)$. Then for all $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2 \in X$ and $c \in \mathbb{C}$,

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 \qquad A(c\mathbf{x}) = cA\mathbf{x}$$

and for all $\mathbf{y}, \mathbf{y}_1, \mathbf{y}_2 \in Y$ and $c \in \mathbb{C}$,

$$B(\mathbf{y}_1 + \mathbf{y}_2) = B\mathbf{y}_1 + B\mathbf{y}_2 \qquad B(c\mathbf{y}) = cB\mathbf{y}$$

It follows that for any $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2 \in X$ and $c \in \mathbb{C}$, we have that

$$\begin{aligned} BA(\mathbf{x}_1 + \mathbf{x}_2) &= B(A\mathbf{x}_1 + A\mathbf{x}_2) & BA(c\mathbf{x}) &= B(cA\mathbf{x}) \\ &= BA\mathbf{x}_1 + BA\mathbf{x}_2 & &= cBA\mathbf{x} \end{aligned}$$

so BA is a linear transformation, as desired.

Let $A \in L(X, Y)$ be invertible. Since A is a linear transformation, the same equalities from above still apply. Thus,

$$\begin{aligned} \mathbf{x}_1 + \mathbf{x}_2 &= \mathbf{x}_1 + \mathbf{x}_2 & c\mathbf{x} &= c\mathbf{x} \\ I(\mathbf{x}_1 + \mathbf{x}_2) &= I\mathbf{x}_1 + I\mathbf{x}_2 & I(c\mathbf{x}) &= cI\mathbf{x} \\ AA^{-1}(\mathbf{x}_1 + \mathbf{x}_2) &= AA^{-1}\mathbf{x}_1 + AA^{-1}\mathbf{x}_2 & AA^{-1}(c\mathbf{x}) &= cAA^{-1}\mathbf{x} \\ A(A^{-1}(\mathbf{x}_1 + \mathbf{x}_2)) &= A(A^{-1}\mathbf{x}_1 + A^{-1}\mathbf{x}_2) & A(A^{-1}(c\mathbf{x})) &= A(cA^{-1}\mathbf{x}) \\ A^{-1}(\mathbf{x}_1 + \mathbf{x}_2) &= A^{-1}\mathbf{x}_1 + A^{-1}\mathbf{x}_2 & A^{-1}(c\mathbf{x}) &= cA^{-1}\mathbf{x} \end{aligned}$$

where we use the fact that A is one-to-one for the last equality in both cases. To prove that A^{-1} is invertible, it will suffice to show that it is one-to-one and onto. Suppose $A^{-1}\mathbf{x} = A^{-1}\mathbf{y}$. Then

$$\begin{aligned} AA^{-1}\mathbf{x} &= AA^{-1}\mathbf{y} \\ I\mathbf{x} &= I\mathbf{y} \\ \mathbf{x} &= \mathbf{y} \end{aligned}$$

proving that A^{-1} is one-to-one, as desired. Now suppose $\mathbf{y} \in X$. Then $A\mathbf{y} = \mathbf{x}$ for some $\mathbf{x} \in X$. It follows that

$$A^{-1}\mathbf{x} = A^{-1}A\mathbf{y} = I\mathbf{y} = \mathbf{y}$$

proving that A^{-1} is onto, as desired. □

3. Assume $A \in L(X, Y)$ and $A\mathbf{x} = \mathbf{0}$ only when $\mathbf{x} = \mathbf{0}$. Prove that A is then 1-1.

Proof. If we suppose that $A\mathbf{x} = A\mathbf{y}$, then by linearity,

$$\begin{aligned}\mathbf{0} &= A\mathbf{x} - A\mathbf{y} \\ &= A(\mathbf{x} - \mathbf{y})\end{aligned}$$

It follows by hypothesis that $\mathbf{x} - \mathbf{y} = \mathbf{0}$, hence $\mathbf{x} = \mathbf{y}$, proving that A is 1-1, as desired. \square

4. Prove (as asserted in Section 9.30) that null spaces and ranges of linear transformations are vector spaces.

Proof. Let $A \in L(X, Y)$.

Suppose $\mathbf{x}_1, \mathbf{x}_2 \in \text{null } A$. Then $A\mathbf{x}_1 = \mathbf{0}$ and $A\mathbf{x}_2 = \mathbf{0}$. It follows that

$$\begin{aligned}\mathbf{0} &= A\mathbf{x}_1 + A\mathbf{x}_2 \\ &= A(\mathbf{x}_1 + \mathbf{x}_2)\end{aligned}$$

so $(\mathbf{x}_1 + \mathbf{x}_2) \in \text{null } A$, as desired.

Suppose $\mathbf{x} \in \text{null } A$ and $c \in \mathbb{C}$. Then $A\mathbf{x} = \mathbf{0}$. It follows that

$$\begin{aligned}\mathbf{0} &= c \cdot \mathbf{0} \\ &= cA\mathbf{x} \\ &= A(c\mathbf{x})\end{aligned}$$

so $c\mathbf{x} \in \text{null } A$, as desired.

Suppose $\mathbf{y}_1, \mathbf{y}_2 \in \text{range } A$. Then there exist $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that $A\mathbf{x}_1 = \mathbf{y}_1$ and $A\mathbf{x}_2 = \mathbf{y}_2$. It follows that

$$\begin{aligned}A(\mathbf{x}_1 + \mathbf{x}_2) &= A\mathbf{x}_1 + A\mathbf{x}_2 \\ &= \mathbf{y}_1 + \mathbf{y}_2\end{aligned}$$

so $(\mathbf{y}_1 + \mathbf{y}_2) \in \text{range } A$, as desired.

Suppose $\mathbf{y} \in \text{range } A$ and $c \in \mathbb{C}$. Then there exists $\mathbf{x} \in X$ such that $A\mathbf{x} = \mathbf{y}$. It follows that

$$\begin{aligned}A(c\mathbf{x}) &= cA\mathbf{x} \\ &= c\mathbf{y}\end{aligned}$$

so $c\mathbf{y} \in \text{range } A$, as desired. \square

6 Functions of Several Variables II

From Rudin (1976).

Chapter 9

- 2/22: 5. Prove that to every $A \in L(\mathbb{R}^n, \mathbb{R}^1)$ corresponds a unique $\mathbf{y} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$. Prove also that $\|A\| = |\mathbf{y}|$. (Hint: Under certain conditions, equality holds in the Schwarz inequality.)

Proof. Let \mathbf{y} be defined as follows.

$$\mathbf{y} = \begin{bmatrix} A\mathbf{e}_1 \\ \vdots \\ A\mathbf{e}_n \end{bmatrix}$$

Then if $\mathbf{x} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$, we have that

$$A\mathbf{x} = a_1 \cdot A\mathbf{e}_1 + \cdots + a_n \cdot A\mathbf{e}_n = \mathbf{x} \cdot \mathbf{y}$$

as desired.

Now suppose that \mathbf{z} satisfies $A\mathbf{x} = \mathbf{x} \cdot \mathbf{z}$ for all $\mathbf{x} \in \mathbb{R}^n$. Then

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} &= \mathbf{x} \cdot \mathbf{z} \\ \mathbf{x} \cdot (\mathbf{y} - \mathbf{z}) &= 0 \end{aligned}$$

for all $\mathbf{x} \in \mathbb{R}^n$. In particular, if $\mathbf{x} = \mathbf{y} - \mathbf{z}$, then

$$\begin{aligned} 0 &= (\mathbf{y} - \mathbf{z}) \cdot (\mathbf{y} - \mathbf{z}) \\ &= \|\mathbf{y} - \mathbf{z}\|^2 \\ \mathbf{0} &= \mathbf{y} - \mathbf{z} \\ \mathbf{z} &= \mathbf{y} \end{aligned}$$

as desired.

We also have by the Cauchy-Schwarz inequality that

$$\|A\| = \sup_{\substack{\mathbf{x} \in \mathbb{R}^n \\ |\mathbf{x}|=1}} |A\mathbf{x}| = \sup_{\substack{\mathbf{x} \in \mathbb{R}^n \\ |\mathbf{x}|=1}} |\mathbf{x} \cdot \mathbf{y}| \leq \sup_{\substack{\mathbf{x} \in \mathbb{R}^n \\ |\mathbf{x}|=1}} |\mathbf{x}| \cdot |\mathbf{y}| = |\mathbf{y}|$$

Moreover, equality holds by noting that if $\mathbf{x} = \mathbf{y}/|\mathbf{y}|$, then

$$|\mathbf{x} \cdot \mathbf{y}| = \frac{|\mathbf{y}|^2}{|\mathbf{y}|} = |\mathbf{y}|$$

so the leftmost supremum is also at least $|\mathbf{y}|$, as desired. \square

6. If

$$f(x, y) = \begin{cases} 0 & (x, y) = (0, 0) \\ \frac{xy}{x^2 + y^2} & (x, y) \neq (0, 0) \end{cases}$$

prove that $(D_1f)(x, y)$ and $(D_2f)(x, y)$ exist at every point of \mathbb{R}^2 , although f is not continuous at $(0, 0)$.

Proof. Let $(x, y) \in \mathbb{R}^2$ be arbitrary. We divide into two cases ($(x, y) \neq (0, 0)$ and $(x, y) = (0, 0)$). If $(x, y) \neq (0, 0)$, then as the sum, product, and quotient of linear (hence differentiable) functions, f is

differentiable at (x, y) . Therefore, by Theorem 9.17, $(D_1f)(x, y)$ and $(D_2f)(x, y)$ exist. On the other hand, if $(x, y) = (0, 0)$, then

$$\begin{aligned}(D_jf)(0, 0) &= \lim_{t \rightarrow 0} \frac{f(t\mathbf{e}_j) - f(0, 0)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\frac{t \cdot 0}{t^2 + 0^2} - 0}{t} \\ &= \lim_{t \rightarrow 0} 0 \\ &= 0\end{aligned}$$

However, since

$$f(t, t) = \frac{t^2}{t^2 + t^2} = \frac{1}{2}$$

for all t (notably arbitrarily small t), we find points $(x, y) \in \mathbb{R}^2$ arbitrarily close to $(0, 0)$ for which $f(x, y) = 1/2$. Therefore, f is not continuous at $(0, 0)$. \square

7. Suppose that f is a real-valued function defined in an open set $E \subset \mathbb{R}^n$, and that the partial derivatives D_1f, \dots, D_nf are bounded on E . Prove that f is continuous in E . (Hint: Proceed as in the proof of Theorem 9.21.)

Proof. To prove that f is continuous at an arbitrary $\mathbf{x} \in E$, it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $\mathbf{x} + \mathbf{h} \in E$ and $|\mathbf{h}| < \delta$, then $|f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since each D_jf is bounded, we know that there exist M_1, \dots, M_n such that $|D_jf| \leq M_j$ ($j = 1, \dots, n$). Define $M = \max M_j$. Choose $\delta = \epsilon/nM$. Let \mathbf{h} be such that $\mathbf{x} + \mathbf{h} \in E$ and $|\mathbf{h}| < \delta$. Supposing $\mathbf{h} = \sum h_j \mathbf{e}_j$, put $\mathbf{v}_0 = \mathbf{0}$ and $\mathbf{v}_k = h_1 \mathbf{e}_1 + \dots + h_k \mathbf{e}_k$ ($k = 1, \dots, n$). Then, letting $\theta_j \in (0, 1)$ for all j , we have that

$$\begin{aligned}|f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})| &= \left| \sum_{j=1}^n [f(\mathbf{x} + \mathbf{v}_j) - f(\mathbf{x} + \mathbf{v}_{j-1})] \right| \\ &\leq \sum_{j=1}^n |f(\mathbf{x} + \mathbf{v}_j) - f(\mathbf{x} + \mathbf{v}_{j-1})| \\ &= \sum_{j=1}^n [h_j (D_jf)(\mathbf{x} + \mathbf{v}_{j-1} + \theta_j h_j \mathbf{e}_j)] \quad \text{MVT} \\ &\leq M \sum_{j=1}^n h_j \\ &\leq nM |\mathbf{h}| \\ &< \epsilon\end{aligned}$$

as desired. \square

8. Suppose that f is a differentiable real function in an open set $E \subset \mathbb{R}^n$, and that f has a local maximum at a point $\mathbf{x} \in E$. Prove that $f'(\mathbf{x}) = 0$.

Proof. To prove that $f'(\mathbf{x}) = 0$, Theorem 9.17 tells us that it will suffice to show that $(D_jf)(\mathbf{x}) = 0$ for all $1 \leq j \leq n$. Let j be an arbitrary natural number between 1 and n , inclusive. Define $U_j = (\mathbf{x} + \text{span}(\mathbf{e}_j)) \cap E$ and let $\mathbf{x} = (x_1, \dots, x_n)$. Then by definition, $(D_jf)(\mathbf{x}) = (f|_{U_j})'(x_j)$ and $f|_{U_j}(x_j)$ is a local maximum of $f|_{U_j}$. It follows by Theorem 5.8 that

$$(D_jf)(\mathbf{x}) = (f|_{U_j})'(x_j) = 0$$

as desired. \square

10. If f is a real function defined in a convex open set $E \subset \mathbb{R}^n$, such that $(D_1 f)(\mathbf{x}) = 0$ for every $\mathbf{x} \in E$, prove that $f(\mathbf{x})$ depends only on x_2, \dots, x_n . Show that the convexity of E can be replaced by a weaker condition, but that some condition is required. For example, if $n = 2$ and E is shaped like a horseshoe, the statement may be false.

Proof. To prove that $f(\mathbf{x})$ depends only on x_2, \dots, x_n , it will suffice to show that if $\mathbf{x}, \mathbf{y} \in E$ are such that $\mathbf{x} = (x, x_2, \dots, x_n)$ and $\mathbf{y} = (y, x_2, \dots, x_n)$, then $f(\mathbf{x}) = f(\mathbf{y})$. Let $\mathbf{x}, \mathbf{y} \in E$ be arbitrary points that satisfy the previous condition. Since E is convex, $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in E$ for all $\lambda \in (0, 1)$. Define $U = \{\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} : 0 \leq \lambda \leq 1\}$. Then since $(f|_U)'(t_1) = (D_1 f)(\mathbf{t}) = 0$ on U , Theorem 5.17b implies that $f|_U$ is constant. In particular, since $\mathbf{x}, \mathbf{y} \in U$, $f(\mathbf{x}) = f(\mathbf{y})$, as desired.

The following weaker condition will suffice: E open is a set such that $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in E$ for all $0 < \lambda < 1$ and $\mathbf{x}, \mathbf{y} \in E$ that satisfy $\mathbf{x} - \mathbf{y} \in \text{span}(\mathbf{e}_1)$. Note that the above proof still works with this condition in place of convexity because the arbitrary \mathbf{x}, \mathbf{y} to which we applied the definition of convexity satisfy $\mathbf{x} - \mathbf{y} \in \text{span}(\mathbf{e}_1)$.

If we do not assert any condition, we may consider as a counterexample the subset of \mathbb{R}^2

$$E = ((-2, -1) \times (1, 2)) \cup ((1, 2) \times (1, 2))$$

and the function $f : E \rightarrow \mathbb{R}$ defined by

$$f(x, y) = \text{sgn}(x) + y$$

Since $f(x, y) = -1 + y$ for $x \in (-2, -1) \times (1, 2)$ and $f(x, y) = 1 + y$ for $x \in (1, 2) \times (1, 2)$, clearly $(D_1 f)(\mathbf{x}) = 0$ for all $\mathbf{x} \in E$. Yet $f(-1.5, 0) = -1$ and $f(1.5, 0) = 1$. \square

11. If f and g are differentiable real functions in \mathbb{R}^n , prove that

$$\nabla(fg) = f\nabla g + g\nabla f$$

and that

$$\nabla\left(\frac{1}{f}\right) = -\frac{\nabla f}{f^2}$$

wherever $f \neq 0$.

Proof. Since, by definition, partial derivatives only take limits over linear subsets of \mathbb{R}^n , i.e., ones that are isomorphic to \mathbb{R} , they behave like the one-dimensional derivatives of f restricted to such linear subsets. More precisely, $D_j f$ behaves like the derivative of $(f|_{N_r(\mathbf{x}) \cap \text{span}(\mathbf{e}_j)})'$ where \mathbf{x} is the point at which we're taking the derivative and r is taken to be small enough so that $N_r(\mathbf{x})$ is a subset of the open domain of f . In particular, this means that partial derivatives obey the usual sum, product, and quotient rules, as well as other rules carried over from the analysis of functions of a single variable. Thus, we have that for any $\mathbf{x} \in \mathbb{R}^n$,

$$\begin{aligned} (\nabla(fg))(\mathbf{x}) &= \sum_{j=1}^n (D_j(fg))(\mathbf{x})\mathbf{e}_j \\ &= \sum_{j=1}^n (fD_jg + D_jfg)(\mathbf{x})\mathbf{e}_j \\ &= f(\mathbf{x}) \sum_{j=1}^n (D_jg)(\mathbf{x})\mathbf{e}_j + g(\mathbf{x}) \sum_{j=1}^n (D_jf)(\mathbf{x})\mathbf{e}_j \\ &= f(\mathbf{x})(\nabla g)(\mathbf{x}) + (\nabla f)(\mathbf{x})g(\mathbf{x}) \\ &= (f\nabla g + g\nabla f)(\mathbf{x}) \end{aligned}$$

and symmetrically for $\nabla(1/f)$. \square

17. Let $\mathbf{f} = (f_1, f_2)$ be the mapping of \mathbb{R}^2 into \mathbb{R}^2 given by

$$f_1(x, y) = e^x \cos y$$

$$f_2(x, y) = e^x \sin y$$

(a) What is the range of f ?

Proof. The range of f is $\mathbb{R}^2 \setminus \{\mathbf{0}\}$.

Both trigonometric functions have range $[-1, 1]$ and e^x has range $(0, \infty)$, so we can make e^x arbitrarily large and then take y such that the trigonometric function equals 1 or -1 . However, since $\cos y = \pm 1$ when $\sin y = 0$ and vice versa, we can never achieve $(0, 0)$. \square

(b) Show that the Jacobian of f is not zero at any point of \mathbb{R}^2 . Thus, every point of \mathbb{R}^2 has a neighborhood in which f is one-to-one. Nevertheless, f is not one-to-one on \mathbb{R}^2 .

Proof. Let $(x, y) \in \mathbb{R}^2$ be arbitrary. We divide into two cases ($y = n\pi$ and $y \neq n\pi$ [$n \in \mathbb{Z}$]). If $y = n\pi$, then since $(\sin y)' = \cos y$, $D_2 f_2 = e^x \cos y$, $e^x \neq 0$ for all x , and $\cos n\pi = \pm 1$ ($n \in \mathbb{Z}$), we have that $(D_2 f_2)(x, y) \neq 0$. If $y \neq n\pi$, then since $(e^x)' = e^x$ (Theorem 8.6b), $D_1 f_2 = e^x \sin y$, $e^x \neq 0$ for all x , and $\sin y \neq 0$ for all $y \neq n\pi$, we have that $(D_1 f_2)(x, y) \neq 0$.

It follows by Theorem 9.24 that every point of \mathbb{R}^2 has a neighborhood in which f is 1-1.

Since the trigonometric functions are periodic with period 2π , $f(0, 0) = f(0, 2\pi)$, for instance. \square

(c) Put $\mathbf{a} = (0, \pi/3)$, $\mathbf{b} = f(\mathbf{a})$, and let \mathbf{g} be the continuous inverse of \mathbf{f} , defined in a neighborhood of \mathbf{b} , such that $\mathbf{g}(\mathbf{b}) = \mathbf{a}$. Find an explicit formula for \mathbf{g} , compute $\mathbf{f}'(\mathbf{a})$ and $\mathbf{g}'(\mathbf{b})$, and verify that

$$\mathbf{g}'(\mathbf{b}) = [\mathbf{f}'(\mathbf{g}(\mathbf{b}))]^{-1}$$

Proof. Let $U = \mathbb{R} \times (0, \pi/2)$. Then $V = f(U) = (0, \infty)^2$. Thus, we may define $\mathbf{g} : V \rightarrow U$ by

$$g_1(x, y) = \frac{1}{2} \ln(x^2 + y^2) \qquad g_2(x, y) = \tan^{-1} \left(\frac{y}{x} \right)$$

Under this definition, we can easily see that if $(x, y) \in U$, then

$$\begin{aligned} g(f(x, y)) &= g(e^x \cos y, e^x \sin y) \\ &= \left(\frac{1}{2} \ln((e^x \cos y)^2 + (e^x \sin y)^2), \tan^{-1} \left(\frac{e^x \sin y}{e^x \cos y} \right) \right) \\ &= \left(\frac{1}{2} \ln(e^{2x}(\cos^2 y + \sin^2 y)), \tan^{-1}(\tan y) \right) \\ &= (x, y) \end{aligned}$$

as desired. We can also compute that

$$\begin{aligned} \mathbf{f}'(\mathbf{a}) &= \begin{bmatrix} (D_1 f_1)(\mathbf{a}) & (D_2 f_1)(\mathbf{a}) \\ (D_1 f_2)(\mathbf{a}) & (D_2 f_2)(\mathbf{a}) \end{bmatrix} & \mathbf{g}'(\mathbf{b}) &= \begin{bmatrix} (D_1 f_1)(\mathbf{a}) & (D_2 f_1)(\mathbf{a}) \\ (D_1 f_2)(\mathbf{a}) & (D_2 f_2)(\mathbf{a}) \end{bmatrix} \\ &= \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix} & &= \begin{bmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} & &= \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \end{aligned}$$

It follows from the definition of matrix multiplication that

$$\mathbf{g}'(\mathbf{b})\mathbf{f}'(\mathbf{g}(\mathbf{b})) = \mathbf{g}'(\mathbf{b})\mathbf{f}'(\mathbf{a}) = I \qquad \mathbf{f}'(\mathbf{g}(\mathbf{b}))\mathbf{g}'(\mathbf{b}) = \mathbf{f}'(\mathbf{a})\mathbf{g}'(\mathbf{b}) = I$$

as desired. \square

(d) What are the images under \mathbf{f} of lines parallel to the coordinate axes?

Proof. Let $U \subset \mathbb{R}^2$ denote a line parallel to the x -axis and a distance t away, and let $V \subset \mathbb{R}^2$ denote a line parallel to the y -axis and a distance t away. Formally,

$$U = \{(t, y) : y \in \mathbb{R}\} \qquad V = \{(x, t) : x \in \mathbb{R}\}$$

where $t \in \mathbb{R}$.

We first describe $\mathbf{f}(U)$. Since $x = t$ for all $(x, y) \in U$, $e^x = e^t$. Since y ranges over \mathbb{R} as (x, y) ranges over U , the image of U under $\cos y$ is $[-1, 1]$. Similarly, the image of U under $\sin y$ is $[-1, 1]$. Thus, $\mathbf{f}(U) = (f_1(U), f_2(U))$ where the two components are given by

$$f_1(U) = [-e^t, e^t] \qquad f_2(U) = [-e^t, e^t]$$

We now describe $\mathbf{f}(V)$. Since x ranges over \mathbb{R} as (x, y) ranges over V , the image of V under e^x is $(0, \infty)$. Since $y = t$ for all $(x, y) \in V$, $\cos y = 0$ if $t = \frac{\pi}{2} + n\pi$, $\cos y > 0$ if $t \in (-\frac{\pi}{2} + 2\pi n, \frac{\pi}{2} + 2\pi n)$, and $\cos y < 0$ if $t \in (\frac{\pi}{2} + 2\pi n, \frac{3\pi}{2} + 2\pi n)$ ($n \in \mathbb{Z}$). Similarly, $\sin y = 0$ if $t = n\pi$, $\sin y > 0$ if $t \in (2n\pi, (2n+1)\pi)$, and $\sin y < 0$ if $t \in ((2n-1)\pi, 2n\pi)$ ($n \in \mathbb{Z}$). Thus, $\mathbf{f}(V) = (f_1(V), f_2(V))$ where the two components are given by

$$f_1(V) = \begin{cases} \{0\} & t = \frac{\pi}{2} + n\pi \\ (0, \infty) & -\frac{\pi}{2} + 2\pi n < t < \frac{\pi}{2} + 2\pi n \\ (0, -\infty) & \frac{\pi}{2} + 2\pi n < t < \frac{3\pi}{2} + 2\pi n \end{cases} \quad f_2(V) = \begin{cases} \{0\} & t = n\pi \\ (0, \infty) & 2n\pi < t < (2n+1)\pi \\ (0, -\infty) & (2n-1)\pi < t < 2n\pi \end{cases}$$

where $n \in \mathbb{Z}$ in every occurrence. □

7 Functions of Several Variables III

From Rudin (1976).

Chapter 9

3/2: 9. If \mathbf{f} is a differentiable mapping of a *connected* open set $E \subset \mathbb{R}^n$ into \mathbb{R}^m , and if $\mathbf{f}'(\mathbf{x}) = 0$ for every $\mathbf{x} \in E$, prove that \mathbf{f} is constant in E .

13. Suppose \mathbf{f} is a differentiable mapping of \mathbb{R}^1 into \mathbb{R}^3 such that $|\mathbf{f}(t)| = 1$ for every t . Prove that $\mathbf{f}'(t) \cdot \mathbf{f}(t) = 0$. Interpret this result geometrically.

14. Define

$$f(x, y) = \begin{cases} 0 & (x, y) = (0, 0) \\ \frac{x^3}{x^2 + y^2} & (x, y) \neq (0, 0) \end{cases}$$

- (a) Prove that $D_1 f$ and $D_2 f$ are bounded functions in \mathbb{R}^2 . (Hence f is continuous.)
 - (b) Let \mathbf{u} be any unit vector in \mathbb{R}^2 . Show that the directional derivative $(D_{\mathbf{u}} f)(0, 0)$ exists, and that its absolute value is at most 1.
 - (c) Let γ be a differentiable mapping of \mathbb{R}^1 into \mathbb{R}^2 (in other words, γ is a differentiable curve in \mathbb{R}^2), with $\gamma(0) = (0, 0)$ and $|\gamma'(0)| > 0$. Put $g(t) = f(\gamma(t))$ and prove that g is differentiable for every $t \in \mathbb{R}^1$. If $\gamma \in C^1$, prove that $g \in C^1$.
 - (d) In spite of this, prove that f is not differentiable at $(0, 0)$. (Hint: The formula $(D_{\mathbf{u}} f)(\mathbf{x}) = \sum_{i=1}^n (D_i f)(\mathbf{x}) u_i$ fails.)
16. Show that the continuity of \mathbf{f}' at the point \mathbf{a} is needed in the inverse function theorem, even in the case $n = 1$: If

$$f(t) = \begin{cases} t + 2t^2 \sin \frac{1}{t} & t \neq 0 \\ 0 & t = 0 \end{cases}$$

then $f'(0) = 1$ and f' is bounded in $(-1, 1)$, but f is not one-to-one in any neighborhood of 0.

References

Rudin, W. (1976). *Principles of mathematical analysis* (A. A. Arthur & S. L. Langman, Eds.; Third). McGraw-Hill.