3 Integration II

From Rudin (1976).

Chapter 6

2/2: **3.** Define three functions $\beta_1, \beta_2, \beta_3$ as follows:

$$\beta_1 = \begin{cases} 0 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases} \qquad \beta_2 = \begin{cases} 0 & x < 0 \\ 1 & x = 0 \\ 1 & x > 0 \end{cases} \qquad \beta_3 = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases}$$

Let f be a bounded function on $[-1, 1]$.

(a) Prove that $f \in \mathscr{R}(\beta_1)$ if and only if $f(0+) = f(0)$ and that then

$$\int f \, \mathrm{d}\beta_1 = f(0)$$

Proof. Suppose first that $f \in \mathscr{R}(\beta_1)$ with $\int f \, \mathrm{d}\beta_1 = f(0)$. To prove that $f(0+) = f(0)$, it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in [-1, 1]$ and $0 \leq x < \delta$, then $|f(x) - f(0)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $f \in \mathscr{R}(\beta_1)$ by hypothesis, we have by Theorem 6.6 that there exists a partition P of $[-1, 1]$ such that $U(P, f, \beta_1) - L(P, f, \beta_1) < \epsilon$. Now let $x_i = \min\{x \in P : x > 0\}$; we know that such an object exists since there exist elements of P greater than zero (namely 1) and P is finite. It follows by the definition of β_1 that $\Delta x_i = 1$ and $\Delta x_j = 0$ for $j \neq i$. Thus, $U(P, f, \beta_1) = M_i$ and $L(P, f, \beta_1) = m_i$ (which exist because f is bounded on $[-1, 1]$). At this point, we are ready to choose δ, whcih we take to be $\delta = x_i$. Now to confirm that this δ works: Let $0 \leq x < \delta$. By the definition of x_i, x_{i-1}, $m_i \leq f(x) \leq M_i$ and $m_i \leq f(0) \leq M_i$. But since $M_i - m_i < \epsilon$ as per the above, we have that $|f(x) - f(0)| < \epsilon$, as desired.

Now suppose that $f(0+) = f(0)$. To prove that $f \in \mathscr{R}(\beta_1)$, Theorem 6.6 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a P such that $U(P, f, \beta_1) - L(P, f, \beta_2) < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $f(0+) = f(0)$, we know that there exists a $\delta' > 0$ such that if $x \in [-1, 1]$ and $0 \leq x < \delta'$, then $|f(x) - f(0)| < \epsilon/3$. Let $\delta = \min(\delta'/2, 1)$. Thus, we may define $P = \{-1, 0, \delta, 1\}$. We have

$$U(P, f, \beta_1) = \sum_{i=1}^{3} M_i \Delta\beta_{1_i} = M_2 \qquad L(P, f, \beta_1) = \sum_{i=1}^{3} m_i \Delta\beta_{1_i} = m_2$$

(which exist because f is bounded on $[-1, 1]$). Consequently, $M_2 \leq f(0) + \epsilon/3$. $m_2 \geq f(0) - \epsilon/3$. Therefore,

$$\begin{aligned} U(P, f, \beta_1) - L(P, f, \beta_1) &= M_2 - m_2 \\ &\leq [f(0) + \tfrac{\epsilon}{3}] - [f(0) - \tfrac{\epsilon}{3}] \\ &= \frac{2\epsilon}{3} \\ &< \epsilon \end{aligned}$$

as desired.

As to proving that $\int f \, \mathrm{d}\beta_1$, we know that $M_2 \leq f(0) + \epsilon/3$ for arbitrarily small ϵ implies $M_2 \leq f(0)$. Similarly, $m_2 \geq f(0)$. Thus,

$$\inf U(P, f, \beta_1) \leq U(P, f, \beta_1) = M_2 \leq f(0) \leq m_2 = L(P, f, \beta_1) \leq \sup L(P, f, \beta_1)$$

But by Theorem 6.5, $\sup L(P,f,\beta_1) \leq \inf U(P,f,\beta_1)$. Therefore,

$$\int_{-1}^{1} f \, \mathrm{d}\beta_1 = \sup L(P,f,\beta_1) = \inf U(P,f,\beta_1) = f(0)$$

as desired. $\square$

(b) State and prove a similar result for β_2.

Proof. The result will be $f \in \mathscr{R}(\beta_2)$ if and only if $f(0-) = f(0)$ and that then

$$\int f \, \mathrm{d}\beta = f(0)$$

The proof of this result is entirely symmetric to the proof of the previous result. $\square$

(c) Prove that $f \in \mathscr{R}(\beta_3)$ if and only if f is continuous at 0.

Proof. Suppose first that $f \in \mathscr{R}(\beta_3)$. To prove that f is continuous at 0, it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in [-1,1]$ and $|x| < \delta$, then $|f(x) - f(0)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $f \in \mathscr{R}(\beta_3)$ by hypothesis, we have by Theorem 6.6 that there exists a partition P of $[-1,1]$ such that $U(P,f,\beta_3) - L(P,f,\beta_3) < \epsilon/2$. Now let $x_i = \max\{x \in P : x < 0\}$ and let $x_j = \min\{x \in P : x > 0\}$. Choose $\delta = \min\{|x_i|, |x_j|\}$. Let $P^* = P \cup \{-\delta, 0, \delta\}$ be a refinement of P. It follows by the definition of β_3 and a reenumeration of P^* that $U(P^*,f,\beta_3) = (M_{i-1} + M_i)/2$ and $L(P^*,f,\beta_3) = (m_{i-1} + m_i)/2$. Now let $|x| < \delta$. We divide into two cases ($x \geq 0$ and $x < 0$). If $x \geq 0$, then $m_i \leq f(x) \leq M_i$ and $m_i \leq f(0) \leq M_i$. But then we have that

$$\begin{aligned}
|f(x) - f(0)| &\leq M_i - m_i \\
&\leq (M_{i-1} - m_{i-1}) + (M_i - m_i) \\
&= 2\left[\frac{M_{i-1} + M_i}{2} - \frac{m_{i-1} + m_i}{2}\right] \\
&= 2[U(P^*,f,\beta_3) - L(P^*,f,\beta_3)] \\
&< \epsilon
\end{aligned}$$

as desired. The proof is symmetric in the other case.

Now suppose that f is continuous at 0. To prove that $f \in \mathscr{R}(\beta_3)$, Theorem 6.6 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a P such that $U(P,f,\beta_3) - L(P,f,\beta_3) < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f is continuous at 0, we know that there exists a $\delta' > 0$ such that if $x \in [-1,1]$ and $|x| < \delta'$, then $|f(x) - f(0)| < \epsilon/3$. Choose $\delta = \min(\delta'/2, 1)$. Consider $P = \{-1, -\delta/2, \delta/2, 1\}$. It follows as before that $U(P,f,\beta_3) = M_2$ and $L(P,f,\beta_3) = m_2$. Consequently, $M_2 \leq f(0) + \epsilon/3$ and $m_2 \geq f(0) - \epsilon/3$. Therefore,

$$\begin{aligned}
U(P,f,\beta_3) - L(P,f,\beta_3) &= M_2 - m_2 \\
&\leq [f(0) + \tfrac{\epsilon}{3}] - [f(0) - \tfrac{\epsilon}{3}] \\
&= \frac{2\epsilon}{3} \\
&< \epsilon
\end{aligned}$$

as desired. $\square$

(d) If f is continuous at 0, prove that

$$\int f \, \mathrm{d}\beta_1 = \int f \, \mathrm{d}\beta_2 = \int f \, \mathrm{d}\beta_3 = f(0)$$

Proof. If f is continuous at 0, then $f(0+) = f(0) = f(0-)$. It follows that

$$\begin{aligned} f(0) &= \int f \,\mathrm{d}\beta_1 && \text{Part (a)} \\ &= \int f \,\mathrm{d}\beta_2 && \text{Part (b)} \\ &= \int f \,\mathrm{d}\beta_3 && \text{Part (c)} \end{aligned}$$

Note that calculating the exact value of $\int f \,\mathrm{d}\beta_3$ is symmetric to the proof in part (a). ☐

5. Suppose f is a bounded real function on $[a,b]$, and $f^2 \in \mathscr{R}$ on $[a,b]$. Does it follow that $f \in \mathscr{R}$? Does the answer change if we assume that $f^3 \in \mathscr{R}$?

Proof. $\underline{f^2 \in \mathscr{R} \not\Rightarrow f \in \mathscr{R}}$: Consider the bounded real function $f : [a,b] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & x \notin \mathbb{Q} \\ -1 & x \in \mathbb{Q} \end{cases}$$

Since $f^2(x) = 1$ for all $x \in [a,b]$, $f^2 \in \mathscr{R}$ as a constant function. However, by Exercise 6.4 and a clever application of Theorem 6.12 (to relate it to the function explicitly considered in Exercise 6.4), we know that $f \notin \mathscr{R}$.

$\underline{f^3 \in \mathscr{R} \Rightarrow f \in \mathscr{R}}$: Let $f : [a,b] \to \mathbb{R}$ be any bounded real function such that $f^3 \in \mathscr{R}$. To prove that $f \in \mathscr{R}$, Theorem 6.11 tells us that it will suffice to show that there exist $m, M \in \mathbb{R}$ such that $m \leq f \leq M$ and that there exists a continuous function $\phi : [m,M] \to \mathbb{R}$ such that $f = \phi \circ f^3$. Since f is bounded by hypothesis, we can pick $m, M \in \mathbb{R}$ such that $m \leq f \leq M$. Now let $\phi : [m,M] \to \mathbb{R}$ be defined by

$$\phi(x) = \sqrt[3]{x}$$

for all $x \in [m,M]$. It is obvious that ϕ is continuous and that $\phi \circ f^3 = f$, as desired. ☐

7. Suppose f is a real function on $(0,1]$ and $f \in \mathscr{R}$ on $[c,1]$ for every $c > 0$. Define

$$\int_0^1 f(x) \,\mathrm{d}x = \lim_{c \to 0} \int_c^1 f(x) \,\mathrm{d}x$$

if this limit exists (and is finite).

(a) If $f \in \mathscr{R}$ on $[0,1]$, show that this definition of the integral agrees with the old one.

Proof. To prove that $\int_0^1 f = \lim_{c \to 0} \int_c^1 f$, it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $c \in (0,1]$ and $c < \delta$, then

$$\left| \int_0^c f \right| = \left| \int_c^1 f - \int_0^1 f \right| < \epsilon$$

Let $\epsilon > 0$ be arbitrary. Since f is integrable, f is bounded, i.e., there exists $M \in \mathbb{R}$ such that $|f(x)| < M$ for all $x \in [0,1]$. Choose $\delta = \epsilon / M$. Let $c \in (0,1]$ be such that $c < \delta$. Then by Theorem 6.12d,

$$\begin{aligned} \left| \int_0^c f \right| &\leq M(c - 0) \\ &< \epsilon \end{aligned}$$

as desired. ☐

(b) Construct a function f such that the above limit exists, although it fails to exist with $|f|$ in place of f.

Proof. Let $f : (0,1] \to \mathbb{R}$ be defined by

$$f(x) = (-1)^n n$$

for $1/n < x \leq 1/(n-1)$ $(n = 2, 3, \dots)$. It follows since f is a constant function save one terminal discontinuity on each $[1/n, 1/(n-1)]$ that

$$\begin{aligned}\int_{1/n}^{1/(n-1)} f &= (-1)^n n \cdot \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= \frac{(-1)^n n}{n(n-1)} \\ &= \frac{(-1)^n}{n-1}\end{aligned}$$

for all $n \in \mathbb{N}$. It follows that

$$\begin{aligned}\int_{1/N}^{1} f &= \sum_{n=2}^{N} \int_{1/n}^{1/(n-1)} f \\ &= \sum_{n=2}^{N} \frac{(-1)^n}{n-1}\end{aligned}$$

Thus,

$$\lim_{c \to 0} \int_c^1 f = \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1}$$

which converges by Theorem 3.43. However, the limit fails to exist if f is replaced by $|f|$, because in that case, the integral is equal to the harmonic series, which diverges to infinity. □

8. Suppose $f \in \mathscr{R}$ on $[a, b]$ for every $b > a$ where a is fixed. Define

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{b \to \infty} \int_a^b f(x)\,\mathrm{d}x$$

if this limit exists (and is finite). In that case, we say that the integral on the left **converges**. If it also converges after f has been replaced by $|f|$, it is said to converge **absolutely**.

Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1, \infty)$. Prove that $\int_1^\infty f(x)\,\mathrm{d}x$ converges if and only if $\sum_{n=1}^\infty f(n)$ converges. (This is the so-called "integral test" for convergence of series.)

Proof. To prove the claim, we will show that

$$\sum_{n=2}^{N} f(n) \leq \int_1^N f \leq \sum_{n=1}^{N-1} f(n) \leq f(1) + \int_1^{N-1} f(x)\,\mathrm{d}x$$

It will follow since both the sum and the integral limit are monotonically increasing as $N \to \infty$ $(f \geq 0)$ and both are bounded below and above by (a function of) the other, both converge or diverge together. Let's begin.

Since f is monotonically decreasing on $[1, \infty)$, we know that $f(n) \leq f(x)$ for all $1 \leq x \leq n$ $(n \in \mathbb{N})$. Thus, by Theorem 6.12b,

$$\int_{n-1}^{n} f(n)\,\mathrm{d}x \leq \int_{n-1}^{n} f(x)\,\mathrm{d}x$$

Therefore,

$$\begin{aligned}\sum_{n=2}^{N} f(n) &= \sum_{n=2}^{N} \int_{n-1}^{n} f(n)\,\mathrm{d}x && \text{Theorem 6.12d}\\ &\leq \sum_{n=2}^{N} \int_{n-1}^{n} f(x)\,\mathrm{d}x \\ &= \int_{1}^{N} f(x)\,\mathrm{d}x && \text{Theorem 6.12c}\end{aligned}$$

for all $N = 2, 3, 4, \dots$, thereby establishing the left inequality above.

Since f is monotonically decreasing on $[1, \infty)$, we know that $f(x) \leq f(n)$ for all $x \geq n$ ($n \in \mathbb{N}$). Thus, by Theorem 6.12b,

$$\int_{n}^{n+1} f(x)\,\mathrm{d}x \leq \int_{n}^{n+1} f(n)\,\mathrm{d}x$$

Therefore,

$$\begin{aligned}\int_{1}^{N} f(x)\,\mathrm{d}x &= \sum_{n=1}^{N-1} \left(\int_{n}^{n+1} f(x)\,\mathrm{d}x\right) && \text{Theorem 6.12c}\\ &\leq \sum_{n=1}^{N-1} \left(\int_{n}^{n+1} f(n)\,\mathrm{d}x\right) \\ &= \sum_{n=1}^{N-1} f(n) && \text{Theorem 6.12d}\end{aligned}$$

for all $N = 2, 3, 4, \dots$, thereby establishing the middle inequality above.

From our statement about $f(n)$ and $f(x)$ from the left inequality, we have by Theorem 6.12b that

$$\int_{n-1}^{n} f(n)\,\mathrm{d}x \leq \int_{n-1}^{n} f(x)\,\mathrm{d}x$$

Therefore,

$$\begin{aligned}\sum_{n=1}^{N-1} f(n) &= f(1) + \sum_{n=2}^{N-1} \int_{n-1}^{n} f(n)\,\mathrm{d}x && \text{Theorem 6.12d}\\ &\leq f(1) + \sum_{n=2}^{N-1} \int_{n-1}^{n} f(x)\,\mathrm{d}x \\ &= f(1) + \int_{1}^{N-1} f(x)\,\mathrm{d}x && \text{Theorem 6.12c}\end{aligned}$$

for all $N = 2, 3, 4, \dots$, thereby establishing the right inequality above. □

10. Let p, q be positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

Prove the following statements.

(a) If $u, v \geq 0$, then

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}$$

Equality holds if and only if $u^p = v^q$.

Discussion. To prove the desired inequality, it will suffice to show that

$$0 \leq \frac{u^p}{p} + \frac{v^q}{q} - uv$$

i.e., that for all $u, v \geq 0$, the expression on the right above is nonnegative. To consider all such values at once, we can consider applying our analysis toolbox to $f : [0, \infty)^2 \to \mathbb{R}$ defined by

$$f(u, v) = \frac{u^p}{p} + \frac{v^q}{q} - uv$$

with the goal of proving that it is nonnegative everywhere on its domain. However, since we do not yet know multivariable calculus, it will suffice to fix $u \geq 0$ and analyze $f : [0, \infty) \to \mathbb{R}$ defined by

$$f(v) = \frac{u^p}{p} + \frac{v^q}{q} - uv$$

Let's begin. $\square$

Proof. Fix $u \geq 0$. Let $f : [0, \infty) \to \mathbb{R}$ be defined by

$$f(v) = \frac{u^p}{p} + \frac{v^q}{q} - uv$$

It follows from the definition of f that to prove the desired inequality, it will suffice to show that f is nonnegative everywhere on its domain. Let's begin.

Since f is a polynomial in v, f is differentiable. Thus, we may consider

$$f'(v) = v^{q-1} - u$$

As a function of a positive power ($q/(q-1) = p > 0$ and $q > 0$ imply $q - 1 > 0$) of its variable (minus a constant), f' is strictly increasing. Additionally, we have that

$$\begin{aligned} 0 &= f'(v) \\ u &= v^{q-1} \\ &= v^{q/p} \\ v &= u^{p/q} \end{aligned}$$

Thus, we know that $f' < 0$ on $(0, u^{p/q})$ and $f' > 0$ on $(u^{p/q}, \infty)$. It follows by the strict version of Theorem 5.11 that f is strictly decreasing on $(0, u^{p/q})$ and strictly increasing on $(u^{p/q}, \infty)$. Furthermore, since f is differentiable (hence continuous by Theorem 5.2), we know that $f(0) \geq f(u^{p/q})$. Combining the last several results, we have that $f(u^{p/q})$ is the minimum of f over $[0, \infty)$, and hence equal to the minimum value of f over $[0, \infty)$. But since

$$\begin{aligned} f(u^{p/q}) &= \frac{u^p}{p} + \frac{(u^{p/q})^q}{q} - uu^{p/q} \\ &= \frac{u^p}{p} + \frac{u^p}{q} - u^{p/q+1} \\ &= u^p \left(\frac{1}{p} + \frac{1}{q} \right) - u^p \\ &= 0 \end{aligned}$$

we know that $f(v) \geq 0$ on its domain, as desired.

Additionally, since f is strictly decreasing on $(0, u^{p/q})$ and strictly increasing on $(u^{p/q}, \infty)$, we know that $f(v) = 0$ iff $v = u^{p/q}$, i.e., iff $v^q = u^p$, as desired. $\square$

(b) If $f,g\in\mathscr{R}(\alpha)$, $f,g\geq 0$, and

$$\int_a^b f^p\,\mathrm{d}\alpha = 1 = \int_a^b g^q\,\mathrm{d}\alpha$$

then

$$\int_a^b fg\,\mathrm{d}\alpha \leq 1$$

Proof. By Theorem 6.13a, the hypothesis $f,g\in\mathscr{R}(\alpha)$ implies that $fg\in\mathscr{R}(\alpha)$. Thus, we have that

$$\begin{aligned}
\int_a^b fg\,\mathrm{d}\alpha &\leq \int_a^b \left(\frac{f^p}{p}+\frac{g^q}{q}\right)\mathrm{d}\alpha && \text{Theorem 6.12b}\\
&= \frac{1}{p}\int_a^b f^p\,\mathrm{d}\alpha + \frac{1}{q}\int_a^b g^q\,\mathrm{d}\alpha && \text{Theorem 6.12a}\\
&= \frac{1}{p}+\frac{1}{q}\\
&= 1
\end{aligned}$$

as desired. □

(c) If f,g are complex functions in $\mathscr{R}(\alpha)$, then

$$\left|\int_a^b fg\,\mathrm{d}\alpha\right| \leq \left(\int_a^b |f|^p\,\mathrm{d}\alpha\right)^{1/p}\left(\int_a^b |g|^q\,\mathrm{d}\alpha\right)^{1/q}$$

This is **Hölder's inequality**. When $p=q=2$, it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)

Proof. By Theorem 6.11 with $\phi(y)=|y|^p$ (resp. $\phi(y)=|y|^q$), the hypothesis $f,g\in\mathscr{R}(\alpha)$ implies that $|f|^p,|g|^q\in\mathscr{R}(\alpha)$. Thus, we may let

$$I_f = \left(\int_a^b |f|^p\,\mathrm{d}\alpha\right)^{1/p} \qquad\qquad I_g = \left(\int_a^b |g|^q\,\mathrm{d}\alpha\right)^{1/q}$$

We divide into two cases ($I_f=0$ or $I_g=0$, and $I_f,I_g\neq 0$). In the first case, WLOG let $I_f=0$. Then since $0\leq|f|^p$, it follows that $f=0$ on $[a,b]$. Thus

$$\left|\int_a^b fg\,\mathrm{d}\alpha\right| = 0 \leq 0 = I_fI_g = \left(\int_a^b |f|^p\,\mathrm{d}\alpha\right)^{1/p}\left(\int_a^b |g|^q\,\mathrm{d}\alpha\right)^{1/q}$$

as desired. In the other case, it follows that

$$\begin{aligned}
I_f^p &= \int_a^b |f|^p\,\mathrm{d}\alpha & I_g^q &= \int_a^b |g|^q\,\mathrm{d}\alpha\\
1 &= \int_a^b \left|\frac{f}{I_f}\right|^p\mathrm{d}\alpha & 1 &= \int_a^b \left|\frac{g}{I_g}\right|^q\mathrm{d}\alpha && \text{Theorem 6.12a}
\end{aligned}$$

Thus, since $|f/I_f|,|g/I_g|\in\mathscr{R}(\alpha)$ by Theorems 6.12 and 6.13 and $|f/I_f|,|g/I_g|\geq 0$ by the definition of the absolute value, we have that

$$\begin{aligned}
\left|\int_a^b fg\,\mathrm{d}\alpha\right| &\leq \int_a^b |fg|\,\mathrm{d}\alpha && \text{Theorem 6.13b}\\
&= I_f I_g \int_a^b \left|\frac{f}{I_f}\right|\left|\frac{g}{I_g}\right|\mathrm{d}\alpha\\
&\leq I_f I_g \cdot 1 && \text{Part (b)}\\
&= \left(\int_a^b |f|^p\,\mathrm{d}\alpha\right)^{1/p}\left(\int_a^b |g|^q\,\mathrm{d}\alpha\right)^{1/q}
\end{aligned}$$

as desired. $\square$

11. Let α be a fixed increasing function on $[a,b]$. For $u \in \mathscr{R}(\alpha)$, define

$$\|u\|_2 = \left(\int_a^b |u|^2\,\mathrm{d}\alpha\right)^{1/2}$$

Suppose $f,g,h \in \mathscr{R}(\alpha)$, and prove the triangle inequality

$$\|f-h\|_2 \leq \|f-g\|_2 + \|g-h\|_2$$

as a consequence of the Schwarz inequality, as in the proof of Theorem 1.37.

Proof. By Theorems 6.12a and 6.13b, the hypothesis that $f,g,h \in \mathscr{R}(\alpha)$ implies that $|f-g|, |g-h| \in \mathscr{R}(\alpha)$. Thus, we have that

$$\begin{aligned}
\|f-h\|_2^2 &= \int_a^b |f-h|^2\,\mathrm{d}\alpha\\
&= \int_a^b |(f-g)+(g-h)|^2\,\mathrm{d}\alpha\\
&= \int_a^b |f-g|^2\,\mathrm{d}\alpha + 2\int_a^b |f-g|\cdot|g-h|\,\mathrm{d}\alpha + \int_a^b |g-h|^2\,\mathrm{d}\alpha\\
&\leq \int_a^b |f-g|^2\,\mathrm{d}\alpha + 2\left(\int_a^b |f-g|^2\,\mathrm{d}\alpha\right)^{1/2}\left(\int_a^b |g-h|^2\,\mathrm{d}\alpha\right)^{1/2} + \int_a^b |g-h|^2\,\mathrm{d}\alpha\\
&= \|f-g\|_2^2 + 2\|f-g\|_2\|g-h\|_2 + \|g-h\|_2^2\\
&= (\|f-g\|_2 + \|g-h\|_2)^2
\end{aligned}$$

Taking square roots of both sides of the inequality yields the desired result. $\square$