4 Sequences and Series of Functions

From Rudin (1976).

Chapter 7

2/9: **1.** Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

Proof. Let $\{f_n\}$ be an arbitrary uniformly convergent sequence of bounded functions. To prove that it is uniformly bounded, it will suffice to find a number M such that $|f_n(x)| < M$ for all $x \in E$ and $n \in \mathbb{N}$. Let f be the function such that $f_n \rightrightarrows f$, and let $M_n = \sup_{x \in E} |f_n(x) - f(x)|$ for each $n \in \mathbb{N}$ (the boundedness of each f_n implies that such an M_n always exists). Thus, based on the last two definitions, we can invoke Theorem 7.9 to learn that $M_n \to 0$ as $n \to \infty$. But since $\{M_n\}$ converges, Theorem 3.2c implies that $\{M_n\}$ is bounded, say by $M/2$. Taking M to be our M yields that for an arbitrary $x \in E$ and $n \in \mathbb{N}$,

$$|f_n(x)| \leq M_n \leq \frac{M}{2} < M$$

as desired. ☐

2. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.

Proof. To prove that $\{f_n + g_n\}$ converges uniformly on E to $f + g$, it will suffice to show that for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|(f_n + g_n)(x) - (f + g)(x)| < \epsilon$ for all $x \in E$. Let $\epsilon > 0$ be arbitrary. Since $f_n \to f$ uniformly on E, there exists N_1 such that if $n \geq N_1$, then $|f_n(x) - f(x)| < \epsilon/2$ for all $x \in E$. Similarly, there exists N_2 such that if $n \geq N_2$, then $|g_n(x) - g(x)| < \epsilon/2$ for all $x \in E$. Choose $N = \max(N_1, N_2)$. Now suppose $n \geq N$, and let $x \in E$ be arbitrary. It follows from the first condition that $n \geq N \geq N_1$ and $n \geq N \geq N_2$, so

$$\begin{aligned}
|(f_n + g_n)(x) - (f + g)(x)| &= |f_n(x) - f(x) + g_n(x) - g(x)| \\
&\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

as desired.

To prove that $\{f_n g_n\}$ converges uniformly on E to fg, it will suffice to show that for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|(f_n g_n)(x) - (fg)(x)| < \epsilon$ for all $x \in E$. Let $\epsilon > 0$ be arbitrary. Since f_n, g_n are uniformly convergent sequences of bounded functions, Exercise 1 implies that they are uniformly bounded, i.e., there exists $M^f, M^g \in \mathbb{R}$ such that $|f_n| < M^f$ and $|g_n| < M^g$ for all $n \in \mathbb{N}$. If we take $M = \max(M^f, M^g)$, then we have $|f_n| < M$ and $|g_n| < M$ for all $n \in \mathbb{N}$. Note that the same inequality holds for f and g. Now, as before, we may pick N such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon/2M$ and $|g_n(x) - g(x)| < \epsilon/2M$ for all $x \in E$. It follows that for any $n \geq N$ and $x \in E$,

$$\begin{aligned}
|(f_n g_n)(x) - (fg)(x)| &= |f_n(x) \cdot (g_n(x) - g(x)) + g(x) \cdot (f_n(x) - f(x))| \\
&= |f_n(x)| \cdot |g_n(x) - g(x)| + |g(x)| \cdot |f_n(x) - f(x)| \\
&< M \cdot \frac{\epsilon}{2M} + M \cdot \frac{\epsilon}{2M} \\
&= \epsilon
\end{aligned}$$

as desired. ☐

3. Construct sequences $\{f_n\}, \{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converge on E).

Proof. Let

$$f_n(x) = x \qquad\qquad g_n(x) = \frac{1}{n}$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then $\{f_n\}$ converges uniformly to $f(x) = x$ (by choosing $N = 1$ for any ϵ) and $\{g_n\}$ converges uniformly to $g(x) = 0$ (by choosing $1/N < \epsilon$ with the Archimedean principle). However, while $\{f_n g_n\}$ converges pointwise to $(fg)(x) = 0$ by Theorem 3.3c, it does not converge uniformly since for any n, choosing $x = n$ yields $(f_n g_n)(x) = 1$. □

4. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?

Proof. Absolute convergence values: The series converges absolutely for any

$$x \in (-\infty, -1) \cup \left(\bigcup_{k=1}^{\infty} \left(-\frac{1}{k^2}, -\frac{1}{(k+1)^2} \right) \right) \cup (0, \infty)$$

We prove this via casework as follows.

Let $x \in (0, \infty)$. Then we have

$$\sum_{n=1}^{\infty} \left| \frac{1}{1 + n^2 x} \right| = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x} \leq \sum_{n=1}^{\infty} \frac{1}{n^2 x} = \frac{1}{x} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{c}{x}$$

where $c \in \mathbb{R}$ is finite by Theorem 3.28. Therefore, since the sum is monotonically increasing and bounded, Theorem 3.14 implies that the sum overall converges, as desired.

Let $x \in (-\infty, -1)$. Then we have

$$n^2 x + 1 < n^2 x + n^2 = n^2(x + 1)$$

Since $x < -1$,

$$n^2 x + 1 < 0 \qquad\qquad n^2(x + 1) < 0$$

for all $n \in \mathbb{N}$. Thus,

$$\begin{aligned} n^2 x + 1 &< n^2(x + 1) \\ \frac{n^2 x + 1}{n^2(x + 1)} &> 1 \\ \frac{1}{n^2(x + 1)} &< \frac{1}{n^2 x + 1} \\ \left| \frac{1}{n^2 x + 1} \right| &< \left| \frac{1}{n^2(x + 1)} \right| \end{aligned}$$

for all $n \in \mathbb{N}$. It follows that

$$\sum_{n=1}^{\infty} \left| \frac{1}{n^2 x + 1} \right| < \sum_{n=1}^{\infty} \left| \frac{1}{n^2(x + 1)} \right| = \frac{1}{x + 1} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{c}{x + 1}$$

where $c \in \mathbb{R}$ is finite by Theorem 3.28. Therefore, since the sum is monotonically increasing and bounded, Theorem 3.14 implies that the sum overall converges, as desired.

Let $x \in (-1/k^2, -1/(k+1)^2)$. For right now, we consider only the sum for $n \geq \sqrt{2}(k+1)$, leaving finitely many terms out of the sum. Let $\delta = 1/(k+1)^2$. It follows that

$$\begin{aligned} n &\geq \sqrt{2}(k+1) & x &< -\frac{1}{(k+1)^2} \\ n &\geq \sqrt{\frac{2}{1/(k+1)^2}} & -x &> \frac{1}{(k+1)^2} \\ n^2 &\geq \frac{2}{\delta} \\ \frac{\delta}{2} &\geq \frac{1}{n^2} \end{aligned}$$

Additionally, since $n \geq \sqrt{2}(k+1) > k$ (hence $n^2 \geq (k+1)^2$) and $x < -1/(k+1)^2$, we have that

$$\begin{aligned} n^2x &< (k+1)^2 \cdot -\frac{1}{(k+1)^2} \\ n^2x &< -1 \\ n^2x + 1 &< 0 \end{aligned}$$

Thus, for $n \geq \sqrt{2}(k+1)$, we have that

$$\left|\frac{1}{1+n^2x}\right| = \frac{1}{n^2(-x)-1} < \frac{1}{n^2\delta - 1} = \frac{1}{n^2} \cdot \frac{1}{\delta - 1/n^2} \leq \frac{1}{n^2} \cdot \frac{1}{\delta - \delta/2} = \frac{2}{\delta n^2}$$

Therefore, since $|f_n(x)| \leq M_n = 2/\delta n^2$ and $\sum M_n$ converges by Theorem 3.28, the comparison test implies that $\sum |f_n(x)|$ converges, as desired. Adding on the finitely many terms we left out of the summation will not change this fact.

Note that the series diverges for $x = 0$ since each term becomes 1 in this case. Additionally, the series fails to exist for $x = -1/k^2$ ($k \in \mathbb{N}$) since the k^{th} term is undefined in this case.

Uniform convergence intervals: The series converges uniformly on any

$$[a, b] \subset (-\infty, -1) \cup \left(\bigcup_{k=1}^{\infty} \left(-\frac{1}{k^2}, -\frac{1}{(k+1)^2}\right)\right) \cup (0, \infty)$$

This is because any such interval will be a subset of either $(-\infty, -1)$, $(0, \infty)$, or a set of the form $(-1/k^2, -1/(k+1)^2)$ ($k \in \mathbb{N}$). Thus, we may take as $\sum M_n$ the supremum on $[a, b]$ of the appropriate bound derived above (either c/x, $c/(x+1)$, or $2c/\delta$, respectively; all supremums of which will exist by the definition of $[a, b]$) and apply Theorem 7.10.

Non-uniform convergence intervals: Any interval containing one or more of the points in the set $\{0\} \cup \{-1/n^2\}_{n=1}^{\infty}$, by the above.

Points of continuity: The series is continuous at all points at which it converges.

Let x be a point at which f converges. Then by the first part of the proof, x is an element of an open set G. Thus, let $N_{2r}(x) \subset G$, and consider $[x - r, x + r]$. By the above, f converges uniformly on this interval. Additionally, each f_n is continuous on this interval by definition. Thus, by Theorem 7.12, f is continuous at x, as desired.

Boundedness: f is not bounded.

If we suppose for the sake of contradiction that f is bounded by m, we nevertheless find that

$$f(\tfrac{1}{4m^2}) > \sum_{n=1}^{2m} \frac{1}{1+\frac{n^2}{4m^2}} = \sum_{n=1}^{2m} \frac{(2m)^2}{(2m)^2+n^2} \geq \sum_{n=1}^{2m} \frac{(2m)^2}{(2m)^2+(2m)^2} = \sum_{n=1}^{2m} \frac{1}{2} = m$$

□

7. For $n = 1, 2, 3, \dots$ and x real, put

$$f_n(x) = \frac{x}{1+nx^2}$$

Show that $\{f_n\}$ converges uniformly to a function f and that the equation

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

is correct if $x \neq 0$ but false if $x = 0$.

Proof. To prove that $\{f_n\}$ converges uniformly to f defined by $f(x) = 0$ ($x \in \mathbb{R}$), Theorem 7.9 tells us that it will suffice to show that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$ and that the sequence $\{M_n\}$ defined by $M_n = \sup_{x\in\mathbb{R}} |f_n(x)|$ tends to zero as $n \to \infty$. Since

$$f_n(x) = \frac{x}{1+nx^2} < \frac{x}{nx^2} = \frac{1}{x} \cdot \frac{1}{n} \to 0$$

as $n \to \infty$ for all $x \neq 0$ and $f_n(0) = 0$ for all n, $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$, as desired. Additionally, by the Schwarz inequality, if a_1, a_2, b_1, b_2 are real numbers, then

$$|a_1b_1 + a_2b_2|^2 \leq (|a_1|^2 + |a_2|^2)(|b_1|^2 + |b_2|^2)$$

It follows that

$$\begin{gathered}
|2\sqrt{n}x|^2 = |\underbrace{1}_{a_1} \cdot \underbrace{\sqrt{n}x}_{b_1} + \underbrace{\sqrt{n}x}_{a_2} \cdot \underbrace{1}_{b_2}|^2 \leq (|1|^2 + |\sqrt{n}x|^2)(|\sqrt{n}x|^2 + |1|^2) = (1+nx^2)^2 \\
|2\sqrt{n}x| \leq |1+nx^2| \\
\frac{1}{|1+nx^2|} \leq \frac{1}{2\sqrt{n}|x|} \\
\frac{|x|}{|1+nx^2|} \leq \frac{1}{2\sqrt{n}} \\
\left|\frac{x}{1+nx^2}\right| \leq \frac{1}{2\sqrt{n}}
\end{gathered}$$

for all $x \neq 0$, $n \in \mathbb{N}$. This combined with the facts that $f_n(0) = 0 < \frac{1}{2\sqrt{n}}$ for all $n \in \mathbb{N}$ and $f_n(1/\sqrt{n}) = 1/2\sqrt{n}$ for all $n \in \mathbb{N}$ implies that $M_n = 1/2\sqrt{n}$. Thus, $M_n \to 0$ as $n \to \infty$, as desired.

$f'(x) = 0$ for all $x \in \mathbb{R}$. Additionally,

$$f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2} \leq \frac{1-nx^2}{(nx^2)^2} = \frac{1}{x^4} \cdot \frac{1}{n^2} - \frac{1}{x^2} \cdot \frac{1}{n} \to 0$$

as $n \to \infty$ for all $x \neq 0$, as desired. However, $f_n'(0) = 1$ for all $n \in \mathbb{N}$, as desired. □