

Chapter 7

Sequences and Series of Functions

7.1 Notes

- 1/31:
- Midterm on differentiation and integration, and a bit of stuff from this week.
 - Plan:
 - Talk about sequences of functions, all with the same domain and range, converging.
 - Address what properties of f_n remain in the limit (e.g., continuity, differentiability, integrability).
 - The answer depends on what we mean by “convergence.”
 - $f_n \rightarrow f$ pointwise implies basically nothing.
 - $f_n \rightarrow f$ uniformly implies that basically everything works out nicely.
 - We’ll restrict ourselves to real functions because those have all the properties (integrability, differentiability, etc.) that we care about.
 - **Pointwise** (convergent sequence $\{f_n\}$ to f): A sequence of functions $\{f_n\}$ such that for all $x \in X$, the sequence $\{f_n(x)\}$ converges to $f(x)$, where $f_n : X \rightarrow \mathbb{R}$ for all $n \in \mathbb{N}$ and $f : X \rightarrow \mathbb{R}$. Denoted by $f_n \rightarrow f$.
 - Bad functions.
 - Consider $f_n : [0, 1] \rightarrow \mathbb{R}$ defined by $x \mapsto x^n$. Each f_n is continuous, but f is not (zero everywhere except $f(1) = 1$)^[1].
 - Consider $f_n : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f_n(x) = x^2/(1 + x^2)^n$, and $f(x) = \sum_{n=0}^{\infty} f_n(x)$. As a geometric series, $f(x) = 1 + x^2$ when $x \neq 0$ but $f(0) = 0$. Thus, the limit exists but is not continuous once again.
 - Consider $f_m : \mathbb{R} \rightarrow \mathbb{R}$ defined by $x \mapsto \lim_{n \rightarrow \infty} \cos^{2n}(m!\pi x)$. Each f_m is integrable, but the limit f is the function that’s 1 for rationals and zero for irrationals. In particular, f is not integrable.
 - We take even powers of the cosine to make it always positive.
 - We use $\cos^2(x)$ just because it’s always between $[0, 1]$, and we know when it is equal to 1.
 - In particular, $\cos^2(\pi x)$ is equal to 1 at every integer, $\cos^2(2\pi x)$ is equal to 1 at every half integer. $\cos^2(6\pi x)$ is equal to 1 at every one-sixth of an integer.
 - Then raising it to the n^{th} power just makes it spiky.
 - Aside: Interchanging limits.
 - If all f_n are continuous, then $\lim_{x \rightarrow x_0} f_n(x) = f_n(x_0)$.

¹Questions that require counterexamples like this could show up on the midterm!

- The question “is f continuous” is equivalent to being able to interchange limits:

$$\lim_{x \rightarrow x_0} \lim_{n \rightarrow \infty} f_n(x) = f(x_0) = \lim_{n \rightarrow \infty} \lim_{x \rightarrow x_0} f_n(x)$$

- Sequence example showing we need to be careful interchanging limits: $s_{n,m} = m/(m+n)$.
- All of this pathology goes away with the right definition, though.
- **Uniformly** (convergent sequence $\{f_n\}$ to f): A sequence of functions $\{f_n\}$ such that for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$ for all $x \in X$, where $f_n : X \rightarrow \mathbb{R}$ for all $n \in \mathbb{N}$ and $f : X \rightarrow \mathbb{R}$.
- Proposition (Cauchy criterion for uniform convergence): $f_n \rightarrow f$ uniformly iff for all $\epsilon > 0$, there exists N such that for all $m, n \geq N$ and for all $x \in X$, $|f_n(x) - f_m(x)| < \epsilon$.
 - Forward direction: Let $\epsilon > 0$. Suppose $f_n \rightarrow f$ uniformly. Choose N such that the functions are within $\epsilon/2$. Then

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

2/2:

- Office hours tomorrow 4-5 PM.
- Plan:
 1. More on uniform convergence.
 - Limit of continuous functions is continuous.
 - Limit of the integral of functions is the integral of the limit.
 2. $\mathcal{C}(X)$ perspectives on uniform convergence.
- Corollary (Weierstraß M-test): If there exist constants $M_n \in \mathbb{R}$ such that $|f_n(x)| \leq M_n$ for all x and $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly.
- Theorem: $f_n : X \rightarrow \mathbb{R}$, f_n continuous at $x_0 \in X$ for all n , and $f_n \rightarrow f$ uniformly imply f continuous at x_0 .
 - Idea:
 - “ $\epsilon/3$ trick”: Find δ such that if $|x - x_0| < \delta$, then

$$|f(x) - f(x_0)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

- Proof:

- $f_n \rightarrow f$ uniformly implies there exists $N \in \mathbb{N}$ such that $|f_N(x) - f(x)| < \epsilon/3$ for all $x \in X$.
 - f_N continuous at x_0 : There exists δ such that if $d(x, x_0) < \delta$, then $|f_N(x) - f_N(x_0)| < \epsilon/3$.
 - Thus, by the $\epsilon/3$ trick, we have the continuity of f .
- Defining a norm on $\mathcal{C}(X)$.

$$\|f\| = \sup_{x \in X} |f(x)|$$

- This makes $\mathcal{C}(X)$ into a vector space.
- We can now define our metric $d(f, g)$ by $d(f, g) = \|f - g\|$.
- $f_n \rightarrow f \iff f$ is bounded.
 - $f_n \rightarrow f$ uniformly $\iff \lim_{n \rightarrow \infty} \sup |f_n(x) - f(x)| = 0 \iff f_n \rightarrow f$ is $\mathcal{C}(X)$.
- Corollary to the Weierstraß M-test: $\mathcal{C}(X)$ is complete (i.e., all uniformly Cauchy sequences converge).

- Assume $\{f_n\}$ is Cauchy. Then by the Cauchy criterion for uniform convergence, f_n converges uniformly to some f . But this f must be continuous, too, meaning $f \in \mathcal{C}(X)$.

2/4:

- Plan.

1. $\int \lim f_n = \lim \int f_n$.
2. $\mathrm{d}x \lim f_n = \lim \mathrm{d}x f_n$.
3. Definitions: Pointwise/uniform boundedness, equicontinuity.

- Theorem: $f_n : [a, b] \rightarrow \mathbb{R}$ integrable and $f_n \rightarrow f$ uniformly implies f is integrable and

$$\int_a^b f = \lim_{n \rightarrow \infty} \int_a^b f_n$$

- Plan:

1. Show f is integrable.
2. Show $\int f = \lim \int f_n$.

- Proof:

- Let $\epsilon_n = \sup_{x \in [a, b]} |f(x) - f_n(x)|$.
- Since $f_n \rightarrow f$ uniformly, $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$.
- By definition, $f_n - \epsilon_n \leq f \leq f_n + \epsilon_n$.
- Thus, by Theorems 6.4 and 6.5,

$$\int_a^b (f_n - \epsilon_n) = \int_a^b (f_n - \epsilon_n) \leq \int_a^b f \leq \int_a^b (f_n + \epsilon_n)$$

- It follows since

$$0 \leq \int_a^b f - \int_a^b f_n \leq \int_a^b (f_n + \epsilon_n) - \int_a^b (f_n - \epsilon_n) = (b - a) \dots$$

that f is integrable.

- Hence,

$$\begin{aligned} \int_a^b (f_n - \epsilon_n) &\leq \int_a^b f \leq \int_a^b (f_n + \epsilon_n) \\ \left| \int_a^b f_n - \int_a^b f \right| &\leq \epsilon_n \\ \lim_{n \rightarrow \infty} \int_a^b f_n &= \int_a^b f \end{aligned}$$

- Theorem: $f_n : [a, b] \rightarrow \mathbb{R}$, each f_n differentiable, $f_n \rightarrow f$ pointwise, and $(f_n)' \rightarrow g$ uniformly implies that f is differentiable and $f' = g$.

- Note that you can do better: Substituting $f_n(x_0)$ converging for some $x_0 \in [a, b]$ for $f_n \rightarrow f$ pointwise still implies the desired result.
- Idea: We use the $\epsilon/3$ trick; $2/3$ will be easy and $1/3$ will be tricky.
- Goal: We want

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - g(x_0) \right| < \epsilon$$

for some δ with $0 < |x - x_0| < \delta$. We will show that

$$\begin{aligned} &\left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{f_N(x) - f_N(x_0)}{x - x_0} + \frac{f_N(x) - f_N(x_0)}{x - x_0} - f'_N(x_0) + f'_N(x_0) - g(x_0) \right| \\ &\leq \left| \frac{f(x) - f(x_0)}{x - x_0} - \frac{f_N(x) - f_N(x_0)}{x - x_0} \right| + \left| \frac{f_N(x) - f_N(x_0)}{x - x_0} - f'_N(x_0) \right| + |f'_N(x_0) - g(x_0)| \end{aligned}$$

- For the middle inequality, use Chapter 5, Exercise 8.
- For the right inequality, use the uniform convergence condition.
- For the left inequality, it will suffice to show the Cauchy condition

$$\left| \frac{f_n(x) - f_n(x_0)}{x - x_0} - \frac{f_m(x) - f_m(x_0)}{x - x_0} \right| < \frac{\epsilon}{3}$$

so, noting that the left term above is equal to

$$\left| \frac{[f_n(x) - f_m(x)] - [f_n(x_0) - f_m(x_0)]}{x - x_0} \right|$$

which is equal to $|f'_n(c) - f'_m(c)|$ by the MVT, from which we can apply the Cauchy form of the uniform convergence of $(f_n)'$ condition.

- **Pointwise bounded** ($\{f_n\}$): A sequence of real functions $\{f_n\}$ such that for all $x \in X$, there exists $M_x \in \mathbb{R}$ such that $|f_n(x)| \leq M_x$ for all $n \in \mathbb{N}$.
- **Uniformly bounded** ($\{f_n\}$): A sequence of real functions $\{f_n\}$ for which there exists $M \in \mathbb{R}$ such that for all $x \in X$ and $n \in \mathbb{N}$, $|f_n(x)| \leq M$.
- Proposition: $f_n : E \rightarrow \mathbb{R}$, $\{f_n\}$ is pointwise bounded, and E is countable implies there is a subsequence $\{f_{n_k}\}$ that converges pointwise.
 - Enumerate $E = \{x_1, x_2, \dots\}$.
 - Then since $\{f_n(x_m)\}$ is bounded for all m by hypothesis, it always has a convergent subsequence.
 - The claim is if you look at the sequence of diagonal functions, it is such a subsequence, i.e., if $f_1(x_1)$ is the first term for x_1 , $f_3(x_2)$ is the second term for x_2 , $f_{11}(x_3)$ is the third term for x_3 , and so on, f_1, f_3, f_{11}, \dots is such a subsequence.

2/9: • Build up to the Arzelà-Ascoli theorem.

2/11: • The Arzelà-Ascoli theorem.