MATH 20410 (Analysis in $\mathbb{R}^n$ II – Accelerated) Problem Sets

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Contents

1 Differentiation

From Rudin (1976).

Chapter 5

1. Let f be defined for all real x, and suppose that

$$|f(y) - f(x)| \le (y - x)^2$$

for all real x and y. Prove that f is constant.

Proof. To prove that f is constant, Theorem 5.11b tells us that it will suffice to show that f is differentiable on $\mathbb{R}$ with derivative $f' = 0$. Let $x \in \mathbb{R}$ be arbitrary. We want to show that for all $\epsilon > 0$, there exists a δ such that if $y \in \mathbb{R}$ and $0 < |y - x| < \delta$, then $|[f(y) - f(x)]/(y - x) - 0| < \epsilon$. Let ϵ be arbitrary. Choose $\delta = \epsilon$. Then we have that

$$
\begin{aligned}
\left| \frac{f(y) - f(x)}{y - x} - 0 \right| &= \frac{|f(y) - f(x)|}{|y - x|} \\
&\le \frac{(y - x)^2}{|y - x|} \\
&\le |y - x| \\
&< \epsilon
\end{aligned}
$$

as desired. $\square$

2. Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b) and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)}$$

for $a < x < b$.

Proof. To prove that f is strictly increasing on (a, b), it will suffice to show that $x < y$ implies $f(x) < f(y)$ for all $x, y \in (a, b)$. Let $x, y \in (a, b)$ satisfy $x < y$. Since f is differentiable on (a, b), it is differentiable on $(x, y) \subset (a, b)$ and (by Theorem 5.2) continuous on $[x, y] \subset (a, b)$. Thus, by the MVT, there exists $c \in (x, y)$ such that

$$f(y) - f(x) = (y - x)f'(c)$$

But since $x < y$, $y - x > 0$. This combined with the fact that $f'(c) > 0$ by definition implies that $(y - x)f'(c) > 0$. Consequently,

$$f(x) < f(x) + (y - x)f'(c) = f(y)$$

as desired.

Since f is strictly increasing (and hence 1-1) on (a, b), we may construct a well-defined inverse function $g : f[(a, b)] \to (a, b)$ for it by

$$g(f(x)) = x$$

for all $f(x) \in f[(a, b)]$. It follows by the fact that $f'(x) > 0$ for all $x \in (a, b)$, the definitions of $f'(x)$ and $g'(f(x))$, and Theorem 3.3d that

$$\frac{1}{f'(x)} = \frac{1}{\lim_{y \to x} \frac{f(y) - f(x)}{y - x}}$$

$$= \lim_{y \to x} \frac{1}{\frac{f(y)-f(x)}{y-x}}$$

$$= \lim_{y \to x} \frac{y-x}{f(y)-f(x)}$$

$$= \lim_{y \to x} \frac{g(f(y))-g(f(x))}{f(y)-f(x)}$$

$$= g'(f(x))$$

as desired.   □

3. Suppose g is a real function on $\mathbb{R}^1$, with bounded derivative (say $|g'| \leq M$). Fix $\epsilon > 0$ and define $f(x) = x + \epsilon g(x)$. Prove that f is one-to-one if ϵ is small enough. (A set of admissable values of ϵ can be determined which depends only on M.)

Proof. Neglecting the trivial case where $M = 0$, take $\epsilon = 1/2M$. It follows that

$$0 < 1 - \frac{1}{2}$$

$$= 1 + \frac{1}{2M} \cdot -M$$

$$\leq 1 + \epsilon g'(x)$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}(x) + \frac{\mathrm{d}}{\mathrm{d}x}(\epsilon g)$$

$$= f'(x)$$

Therefore, by Problem 5.2, f is strictly increasing and, hence, one-to-one.   □

4. If

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$$

where $C_0, \ldots, C_n$ are real constants, prove that the equation

$$C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 and 1.

Proof. Consider the polynomial

$$f(x) = C_0 x + \frac{C_1}{2}x^2 + \cdots + \frac{C_n}{n+1}x^{n+1}$$

We have that $f(0) = 0$ (by direct substitution) and $f(1) = 0$ (by the constraint on the coefficients). Thus, since f is continuous on $[0, 1]$ and differentiable on $(0, 1)$ (as a polynomial), we have by the MVT that there exists $x \in (0, 1)$ such that

$$f(1) - f(0) = (1 - 0)f'(x)$$

$$f'(x) = 0$$

$$C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n = 0$$

as desired.   □

5. Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

Proof. To prove that $\lim_{x \to \infty} g(x) = 0$, it will suffice to show that for every $\epsilon > 0$, there exists $N > 0$ such that if $x > N$, then $|g(x) - 0| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\lim_{x \to \infty} f'(x) = 0$ by hypothesis, we know that there exists $N > 0$ such that if $x > N$, then $|f'(x)| < \epsilon$. Choose this N to be our N. Let $x > N$ be arbitrary. Applying the MVT to f on the interval $[x, x + 1]$ proves the existence of a c within that closed interval such that

$$f(x + 1) - f(x) = f'(c)(x + 1 - x) = f'(c)$$

Additionally, since $c > x > N$, we have that $|f'(c)| < \epsilon$. Therefore, we have that

$$\begin{aligned} |g(x)| &= |f(x + 1) - f(x)| \\ &= |f'(c)| \\ &< \epsilon \end{aligned}$$

as desired. $\square$

2   Differentiation II / Integration

From Rudin (1976).

Chapter 5

8. Suppose f' is continuous on $[a, b]$ and $\epsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \epsilon$$

whenever $0 < |t - x| < \delta$, $a \le x \le b$, $a \le t \le b$. (This could be expressed by saying that f is **uniformly differentiable** on $[a, b]$ if f' is continuous on $[a, b]$.) Does this hold for vector-valued functions, too?

Proof. By Theorem 2.40, $[a, b]$ is compact. This combined with the fact that f' is continuous implies by Theorem 4.19 that f' is uniformly continuous. Thus, there exists $\delta > 0$ such that if $x, y \in [a, b]$ and $|y - x| < \delta$, then $|f'(y) - f'(x)| < \epsilon$. Choose this δ to be our δ. Let $x, t \in [a, b]$ be such that $0 < |t - x| < \delta$. Then since f is continuous on $[t, x] \subset [a, b]$ and differentiable on $(t, x) \subset [a, b]$, we have by the MVT that there exists $c \in (t, x)$ such that

$$f(t) - f(x) = (t - x)f'(c)$$
$$\frac{f(t) - f(x)}{t - x} = f'(c)$$

Additionally, since $t < c < x$ and $|t - x| < \delta$, we must have $|c - x| < \delta$, meaning that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| = |f'(c) - f'(x)| < \epsilon$$

as desired.

And yes, this does hold for vector-valued functions, which we can treat component-wise. $\qquad \square$

17. Suppose f is a real, three times differentiable function on $[-1, 1]$ such that

$$f(-1) = 0 \qquad\qquad f(0) = 0 \qquad\qquad f(1) = 1 \qquad\qquad f'(0) = 0$$

Prove that $f^{(3)}(x) \ge 3$ for some $x \in (-1, 1)$. Note that equality holds for $\frac{1}{2}(x^3 + x^2)$. (Hint: Use Theorem 5.15 with $\alpha = 0$ and $\beta = \pm 1$ to show that there exist $s \in (0, 1)$ and $t \in (-1, 0)$ such that $f^{(3)}(s) + f^{(3)}(t) = 6$.)

Proof. Since f is three times differentiable on $[-1, 1]$, we know that f'' is differentiable on $[-1, 1]$. It follows by Theorem 5.2 that f'' is continuous on $[-1, 1]$. Thus, since f is defined on $[-1, 1]$, $3 \in \mathbb{N}$, f'' is continuous on $[-1, 1]$, $f^{(3)}$ is defined on $(-1, 1)$, $0, 1 \in [-1, 1]$ such that $0 \ne 1$, and we can define

$$P(t) = \sum_{k=0}^{2} \frac{f^{(k)}(0)}{k!}(t - 0)^k$$

we have by Taylor's theorem that there exists $s \in (0, 1)$ such that

$$f(1) = P(1) + \frac{f^{(3)}(s)}{3!}(1 - 0)^3$$

$$1 - \left[\frac{f(0)}{0!}(1 - 0)^0 + \frac{f'(0)}{1!}(1 - 0)^1 + \frac{f''(0)}{2!}(1 - 0)^2 \right] = \frac{f^{(3)}(s)}{3!}$$

$$1 - \left[\frac{f''(0)}{2} \right] = \frac{f^{(3)}(s)}{6}$$

$$6 - 3f''(0) = f^{(3)}(s)$$

Similarly, we have that there exists $t \in (-1, 0)$ such that

$$f(-1) = P(-1) + \frac{f^{(3)}(t)}{3!}(-1-0)^3$$

$$0 - \left[\frac{f(0)}{0!}(-1-0)^0 + \frac{f'(0)}{1!}(-1-0)^1 + \frac{f''(0)}{2!}(-1-0)^2\right] = -\frac{f^{(3)}(t)}{3!}$$

$$-\left[\frac{f''(0)}{2}\right] = -\frac{f^{(3)}(t)}{6}$$

$$3f''(0) = f^{(3)}(s)$$

Thus,

$$f^{(3)}(s) + f^{(3)}(t) = 3f''(0) + 6 - 3f''(0) = 6$$

Now suppose for the sake of contradiction that for all $x \in (-1, 1)$, we have $f^{(3)}(x) < 3$. Then $f^{(3)}(s) < 3$ and $f^{(3)}(t) < 3$. It follows that $f^{(3)}(s) + f^{(3)}(t) < 6$, a contradiction. $\square$

25. Suppose f is twice differentiable on $[a, b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geq \delta > 0$, and $0 \leq f''(x) \leq M$ for all $x \in [a, b]$. Let ξ be the unique point in (a, b) at which $f(\xi) = 0$. Complete the details in the following outline of **Newton's method** for computing ξ.

(a) Choose $x_1 \in (\xi, b)$ and define $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Interpret this geometrically, in terms of a tangent to the graph of f.

Answer. Since we can rearrange the above to $0 - f(x_n) = f'(x_n)(x_{n+1} - x_n)$, we know that x_{n+1} is the point at which the tangent to f at x_n crosses the x-axis. In other words, the zero of the tangent line

$$y - f(x_n) = f'(x_n)(x - x_n)$$

is $(x_{n+1}, 0)$. $\square$

(b) Prove that $x_{n+1} < x_n$ and that

$$\lim_{n \to \infty} x_n = \xi$$

Proof. To prove that $x_{n+1} < x_n$, it will suffice to show that $f(x_n), f'(x_n) > 0$. Since $f'(x) > 0$ for all $x \in [a, b]$ by hypothesis, we know that $f'(x_n) > 0$. As to $f(x_n)$, suppose for the sake of contradiction that $f(x_n) \leq 0$. We know that $f(\xi) = 0$, $f(b) > 0$, and $\xi < x_n < b$. Since ξ is the *unique* point at which $f(\xi) = 0$ by hypothesis and $x_n \neq \xi$, we know that $f(x_n) \neq 0$. And if $f(x_n) < 0$, we have by the intermediate value theorem for f continuous that there exists $c \in (x_n, b)$ such that $f(c) = 0$. But since $\xi < x_n < c < b$, $c \neq \xi$, and thus we have a contradiction here, too.

Having established that $\{x_n\}$ is a monotonically decreasing sequence, Theorem 3.14 tells us that to show that it converges, it will suffice to show that it is bounded. Clearly, $\{x_n\}$ is bounded above by x_1. And on the bottom side, $\{x_n\}$ is bounded by ξ: If there were $x_n < \xi$, this would imply that $f(x_n) < 0$ by a symmetric argument to the above, meaning that $f(x_n)/f'(x_n) < 0$ and implying that $x_{n+1} > x_n$, a contradiction. Furthermore, we know that the limit (call it x) equals ξ since

$$x = x - \frac{f(x)}{f'(x)}$$

$$f(x) = 0$$

so $x = \xi$ by the uniqueness of ξ. $\square$

(c) Use Taylor's theorem to show that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

for some $t_n \in (\xi, x_n)$.

Proof. Since f is defined on $[a, b]$, $2 \in \mathbb{N}$, f' is continuous on $[a, b]$, f'' is defined on (a, b), $\xi, x_n \in [a, b]$ with $\xi \neq x_n$, and

$$P(t) = \sum_{k=0}^{1} \frac{f^{(k)}(x_n)}{k!}(t - x_n)^k$$

we have by Taylor's theorem that there exists $t_n \in (\xi, x_n)$ such that

$$f(\xi) = \left[\frac{f(x_n)}{0!}(\xi - x_n)^0 + \frac{f'(x_n)}{1!}(\xi - x_n)^1\right] + \frac{f''(t_n)}{2!}(\xi - x_n)^2$$

$$0 = f(x_n) - f'(x_n)(x_n - \xi) + \frac{f''(t_n)}{2}(x_n - \xi)^2$$

$$x_n - \frac{f(x_n)}{f'(x_n)} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

as desired.   □

(d) If $A = M/2\delta$, deduce that

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A}[A(x_1 - \xi)]^{2n}$$

(Compare with Chapter 3, Exercises 16 and 18.)

Proof. We have from part (b) that $x_i > \xi$ for all $i \in \mathbb{N}$, so naturally $0 \leq x_{n+1} - \xi$. As to the other part of the question, we induct on n. For the base case $n = 1$, we have that

$$\begin{aligned}
x_2 - \xi &= \frac{f''(t_1)}{2f'(x_1)}(x_1 - \xi)^2 \\
&\leq \frac{M}{2\delta}(x_1 - \xi)^2 \\
&= \frac{2\delta}{M}\left[\frac{M}{2\delta}(x_1 - \xi)\right]^2 \\
&= \frac{1}{A}[A(x_1 - \xi)]^{2 \cdot 1}
\end{aligned}$$

Now suppose inductively that we have proven the claim for $n - 1$; we now seek to prove it for n. Indeed, we have that

$$\begin{aligned}
x_{n+1} - \xi &= \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2 \\
&\leq \frac{M}{2\delta}(x_n - \xi)^2 \\
&\leq A\left(\frac{1}{A}[A(x_1 - \xi)]^{2(n-1)}\right)^2 \\
&= \frac{1}{A}[A(x_1 - \xi)]^{2n}
\end{aligned}$$

as desired.   □

(e) Show that Newton's method amounts to finding a fixed point of the function g defined by

$$g(x) = x - \frac{f(x)}{f'(x)}$$

How does $g'(x)$ behave for x near ξ?

Proof. A fixed point of the function g is a point x such that

$$g(x) = x$$
$$x - \frac{f(x)}{f'(x)} = x$$
$$f(x) = 0$$

Thus, if we want to find a point x where $f(x) = 0$, it is equivalent to find a point x such that $g(x) = x$.

As to the other part of the question, we have by the rules of derivatives that

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{f'(x)^2}$$
$$= \frac{f(x)f''(x)}{f'(x)^2}$$
$$\leq \frac{M}{\delta^2} f(x)$$

Thus, since $f(x) \to 0$ as $x \to \xi$, $g'(x) \to 0$ as $x \to \xi$.   □

(f) Put $f(x) = \sqrt[3]{x}$ on $(-\infty, \infty)$ and try Newton's method. What happens?

Answer. We have by the power rule that

$$f'(x) = \frac{1}{3x^{2/3}}$$

Choose $x_1 = 1$. Then

$$x_2 = 1 - \frac{f(1)}{f'(1)} = -2$$
$$x_3 = 1 - \frac{f(-2)}{f'(-2)} = 7$$
$$x_4 = 1 - \frac{f(7)}{f'(7)} = -20$$
$$\vdots$$

It appears that the series is diverging to ∞ while alternating from positive to negative. In fact, since $x_3 > x_2$, contrary to part (b), we know that something must be wrong (i.e., one of our hypotheses must not be met). Upon further investigation, we can determine that on $[-1, 1]$, we have $f''(1) = -2/9 < 0$; thus, our last hypothesis is the issue with this function.   □

Chapter 6

1. Suppose α increases on $[a, b]$, $a \leq x_0 \leq b$, α is continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathscr{R}(\alpha)$ and that $\int f \, d\alpha = 0$.

Proof. Since f is bounded on $[a, b]$ with only one discontinuity on $[a, b]$ and α is continuous at the point at which f is discontinuous, Theorem 6.10 implies that $f \in \mathscr{R}(\alpha)$, as desired. It follows that $\inf U(P, f, \alpha) = \sup L(P, f, \alpha) = \int f \, d\alpha$. But since $L(P, f, \alpha) = 0$ for all P (there is no infinite interval $[x_i, x_{i+1}] \subset [a, b]$ that does not contain 0, and f is bounded below by 0), we know that

$$\int f \, d\alpha = \sup L(P, f, \alpha) = 0$$

as desired.   □

2. Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x) \, dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$. (Compare this with Exercise 1.)

Proof. Suppose for the sake of contradiction that $f(x) \neq 0$ for some x. By the definition of f, this must mean that $f(x) > 0$. It follows since f is continuous that there exists some $N_r(x)$ such that $f(y) > 0$ for all $y \in N_r(x)$. Now consider the partition

$$P = \{a, x - r/2, x + r/2, b\}$$

of $[a, b]$. But since $m_2 > 0$, we have that

$$
\begin{aligned}
0 &< m_1[(x - r/2) - a] + m_2[(x + r/2) - (x - r/2)] + m_3[b - (x + r/2)] \\
&= L(P, f) \\
&\leq \int_a^b f(x) \, dx \hspace{4cm} \text{Theorem 6.4}
\end{aligned}
$$

a contradiction.   □

4. If $f(x) = 0$ for all irrational x and $f(x) = 1$ for all rational x, prove that $f \notin \mathscr{R}$ on $[a, b]$ for any $a < b$.

Proof. Let P be an arbitrary partition of $[a, b]$. Since the rationals and irrationals are dense in the reals, we know that for any $[x_i, x_{i+1}]$, $f(x) = 0$ for some $x \in [x_i, x_{i+1}]$ and $f(x) = 1$ for some $x \in [x_i, x_{i+1}]$. Thus, we have that $L(P, f) = 0$ and $U(P, f) = b - a$. It follows that if $a < b$,

$$\sup L(P, f) = 0 \neq b - a = \inf U(P, f)$$

so $f \notin \mathscr{R}$, as desired.   □

3   Integration II

From Rudin (1976).

Chapter 6

2/2:   **3.** Define three functions $\beta_1, \beta_2, \beta_3$ as follows:

$$\beta_1 = \begin{cases} 0 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases} \qquad \beta_2 = \begin{cases} 0 & x < 0 \\ 1 & x = 0 \\ 1 & x > 0 \end{cases} \qquad \beta_3 = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases}$$

Let f be a bounded function on $[-1, 1]$.

(a) Prove that $f \in \mathscr{R}(\beta_1)$ if and only if $f(0+) = f(0)$ and that then

$$\int f \, \mathrm{d}\beta_1 = f(0)$$

Proof. Suppose first that $f \in \mathscr{R}(\beta_1)$ with $\int f \, \mathrm{d}\beta_1 = f(0)$. To prove that $f(0+) = f(0)$, it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in [-1, 1]$ and $0 \leq x < \delta$, then $|f(x) - f(0)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $f \in \mathscr{R}(\beta_1)$ by hypothesis, we have by Theorem 6.6 that there exists a partition P of $[-1, 1]$ such that $U(P, f, \beta_1) - L(P, f, \beta_1) < \epsilon$. Now let $x_i = \min\{x \in P : x > 0\}$; we know that such an object exists since there exist elements of P greater than zero (namely 1) and P is finite. It follows by the definition of β_1 that $\Delta x_i = 1$ and $\Delta x_j = 0$ for $j \neq i$. Thus, $U(P, f, \beta_1) = M_i$ and $L(P, f, \beta_1) = m_i$ (which exist because f is bounded on $[-1, 1]$). At this point, we are ready to choose δ, whcih we take to be $\delta = x_i$. Now to confirm that this δ works: Let $0 \leq x < \delta$. By the definition of x_i, x_{i-1}, $m_i \leq f(x) \leq M_i$ and $m_i \leq f(0) \leq M_i$. But since $M_i - m_i < \epsilon$ as per the above, we have that $|f(x) - f(0)| < \epsilon$, as desired.

Now suppose that $f(0+) = f(0)$. To prove that $f \in \mathscr{R}(\beta_1)$, Theorem 6.6 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a P such that $U(P, f, \beta_1) - L(P, f, \beta_2) < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $f(0+) = f(0)$, we know that there exists a $\delta' > 0$ such that if $x \in [-1, 1]$ and $0 \leq x < \delta'$, then $|f(x) - f(0)| < \epsilon/3$. Let $\delta = \min(\delta'/2, 1)$. Thus, we may define $P = \{-1, 0, \delta, 1\}$. We have

$$U(P, f, \beta_1) = \sum_{i=1}^{3} M_i \Delta\beta_{1_i} \qquad\qquad L(P, f, \beta_1) = \sum_{i=1}^{3} m_i \Delta\beta_{1_i}$$
$$= M_2 \qquad\qquad\qquad\qquad\qquad\qquad = m_2$$

(which exist because f is bounded on $[-1, 1]$). Consequently, $M_2 \leq f(0) + \epsilon/3$. $m_2 \geq f(0) - \epsilon/3$. Therefore,

$$\begin{aligned} U(P, f, \beta_1) - L(P, f, \beta_1) &= M_2 - m_2 \\ &\leq [f(0) + \tfrac{\epsilon}{3}] - [f(0) - \tfrac{\epsilon}{3}] \\ &= \frac{2\epsilon}{3} \\ &< \epsilon \end{aligned}$$

as desired.

As to proving that $\int f \, \mathrm{d}\beta_1$, we know that $M_2 \leq f(0) + \epsilon/3$ for arbitrarily small ϵ implies $M_2 \leq f(0)$. Similarly, $m_2 \geq f(0)$. Thus,

$$\inf U(P, f, \beta_1) \leq U(P, f, \beta_1) = M_2 \leq f(0) \leq m_2 = L(P, f, \beta_1) \leq \sup L(P, f, \beta_1)$$

But by Theorem 6.5, $\sup L(P, f, \beta_1) \leq \inf U(P, f, \beta_1)$. Therefore,

$$\int_{-1}^{1} f \, \mathrm{d}\beta_1 = \sup L(P, f, \beta_1) = \inf U(P, f, \beta_1) = f(0)$$

as desired.   □

(b) State and prove a similar result for β_2.

Proof. The result will be $f \in \mathscr{R}(\beta_2)$ if and only if $f(0-) = f(0)$ and that then

$$\int f \, \mathrm{d}\beta = f(0)$$

The proof of this result is entirely symmetric to the proof of the previous result.   □

(c) Prove that $f \in \mathscr{R}(\beta_3)$ if and only if f is continuous at 0.

Proof. Suppose first that $f \in \mathscr{R}(\beta_3)$. To prove that f is continuous at 0, it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in [-1, 1]$ and $|x| < \delta$, then $|f(x) - f(0)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $f \in \mathscr{R}(\beta_3)$ by hypothesis, we have by Theorem 6.6 that there exists a partition P of $[-1, 1]$ such that $U(P, f, \beta_3) - L(P, f, \beta_3) < \epsilon/2$. Now let $x_i = \max\{x \in P : x < 0\}$ and let $x_j = \min\{x \in P : x > 0\}$. Choose $\delta = \min\{|x_i|, |x_j|\}$. Let $P^* = P \cup \{-\delta, 0, \delta\}$ be a refinement of P. It follows by the definition of β_3 and a reenumeration of P^* that $U(P^*, f, \beta_3) = (M_{i-1} + M_i)/2$ and $L(P^*, f, \beta_3) = (m_{i-1} + m_i)/2$. Now let $|x| < \delta$. We divide into two cases ($x \geq 0$ and $x < 0$). If $x \geq 0$, then $m_i \leq f(x) \leq M_i$ and $m_i \leq f(0) \leq M_i$. But then we have that

$$\begin{aligned}
|f(x) - f(0)| &\leq M_i - m_i \\
&\leq (M_{i-1} - m_{i-1}) + (M_i - m_i) \\
&= 2\left[\frac{M_{i-1} + M_i}{2} - \frac{m_{i-1} + m_i}{2}\right] \\
&= 2[U(P^*, f, \beta_3) - L(P^*, f, \beta_3)] \\
&< \epsilon
\end{aligned}$$

as desired. The proof is symmetric in the other case.

Now suppose that f is continuous at 0. To prove that $f \in \mathscr{R}(\beta_3)$, Theorem 6.6 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a P such that $U(P, f, \beta_3) - L(P, f, \beta_3) < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f is continuous at 0, we know that there exists a $\delta' > 0$ such that if $x \in [-1, 1]$ and $|x| < \delta'$, then $|f(x) - f(0)| < \epsilon/3$. Choose $\delta = \min(\delta'/2, 1)$. Consider $P = \{-1, -\delta/2, \delta/2, 1\}$. It follows as before that $U(P, f, \beta_3) = M_2$ and $L(P, f, \beta_3) = m_2$. Consequently, $M_2 \leq f(0) + \epsilon/3$ and $m_2 \geq f(0) - \epsilon/3$. Therefore,

$$\begin{aligned}
U(P, f, \beta_3) - L(P, f, \beta_3) &= M_2 - m_2 \\
&\leq [f(0) + \tfrac{\epsilon}{3}] - [f(0) - \tfrac{\epsilon}{3}] \\
&= \frac{2\epsilon}{3} \\
&< \epsilon
\end{aligned}$$

as desired.   □

(d) If f is continuous at 0, prove that

$$\int f \, \mathrm{d}\beta_1 = \int f \, \mathrm{d}\beta_2 = \int f \, \mathrm{d}\beta_3 = f(0)$$

Proof. If f is continuous at 0, then $f(0+) = f(0) = f(0-)$. It follows that

$$f(0) = \int f \, \mathrm{d}\beta_1 \hspace{4cm} \text{Part (a)}$$

$$= \int f \, \mathrm{d}\beta_2 \hspace{4cm} \text{Part (b)}$$

$$= \int f \, \mathrm{d}\beta_3 \hspace{4cm} \text{Part (c)}$$

Note that calculating the exact value of $\int f \, \mathrm{d}\beta_3$ is symmetric to the proof in part (a). $\qquad\square$

5. Suppose f is a bounded real function on $[a, b]$, and $f^2 \in \mathscr{R}$ on $[a, b]$. Does it follow that $f \in \mathscr{R}$? Does the answer change if we assume that $f^3 \in \mathscr{R}$?

Proof. $\underline{f^2 \in \mathscr{R} \nRightarrow f \in \mathscr{R}}$: Consider the bounded real function $f : [a, b] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & x \notin \mathbb{Q} \\ -1 & x \in \mathbb{Q} \end{cases}$$

Since $f^2(x) = 1$ for all $x \in [a, b]$, $f^2 \in \mathscr{R}$ as a constant function. However, by Exercise 6.4 and a clever application of Theorem 6.12 (to relate it to the function explicitly considered in Exercise 6.4), we know that $f \notin \mathscr{R}$.

$\underline{f^3 \in \mathscr{R} \Rightarrow f \in \mathscr{R}}$: Let $f : [a, b] \to \mathbb{R}$ be any bounded real function such that $f^3 \in \mathscr{R}$. To prove that $f \in \mathscr{R}$, Theorem 6.11 tells us that it will suffice to show that there exist $m, M \in \mathbb{R}$ such that $m \leq f \leq M$ and that there exists a continuous function $\phi : [m, M] \to \mathbb{R}$ such that $f = \phi \circ f^3$. Since f is bounded by hypothesis, we can pick $m, M \in \mathbb{R}$ such that $m \leq f \leq M$. Now let $\phi : [m, M] \to \mathbb{R}$ be defined by

$$\phi(x) = \sqrt[3]{x}$$

for all $x \in [m, M]$. It is obvious that ϕ is continuous and that $\phi \circ f^3 = f$, as desired. $\qquad\square$

7. Suppose f is a real function on $(0, 1]$ and $f \in \mathscr{R}$ on $[c, 1]$ for every $c > 0$. Define

$$\int_0^1 f(x) \, \mathrm{d}x = \lim_{c \to 0} \int_c^1 f(x) \, \mathrm{d}x$$

if this limit exists (and is finite).

(a) If $f \in \mathscr{R}$ on $[0, 1]$, show that this definition of the integral agrees with the old one.

Proof. To prove that $\int_0^1 f = \lim_{c \to 0} \int_c^1 f$, it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $c \in (0, 1]$ and $c < \delta$, then

$$\left| \int_0^c f \right| = \left| \int_c^1 f - \int_0^1 f \right| < \epsilon$$

Let $\epsilon > 0$ be arbitrary. Since f is integrable, f is bounded, i.e., there exists $M \in \mathbb{R}$ such that $|f(x)| < M$ for all $x \in [0, 1]$. Choose $\delta = \epsilon/M$. Let $c \in (0, 1]$ be such that $c < \delta$. Then by Theorem 6.12d,

$$\left| \int_0^c f \right| \leq M(c - 0)$$

$$< \epsilon$$

as desired. $\qquad\square$

(b) Construct a function f such that the above limit exists, although it fails to exist with $|f|$ in place of f.

Proof. Let $f : (0, 1] \to \mathbb{R}$ be defined by

$$f(x) = (-1)^n n$$

for $1/n < x \leq 1/(n-1)$ $(n = 2, 3, \dots)$. It follows since f is a constant function save one terminal discontinuity on each $[1/n, 1/(n-1)]$ that

$$\int_{1/n}^{1/(n-1)} f = (-1)^n n \cdot \left(\frac{1}{n-1} - \frac{1}{n} \right)$$
$$= \frac{(-1)^n n}{n(n-1)}$$
$$= \frac{(-1)^n}{n-1}$$

for all $n \in \mathbb{N}$. It follows that

$$\int_{1/N}^{1} f = \sum_{n=2}^{N} \int_{1/n}^{1/(n-1)} f$$
$$= \sum_{n=2}^{N} \frac{(-1)^n}{n-1}$$

Thus,

$$\lim_{c \to 0} \int_c^1 f = \sum_{n=2}^{\infty} \frac{(-1)^n}{n-1}$$

which converges by Theorem 3.43. However, the limit fails to exist if f is replaced by $|f|$, because in that case, the integral is equal to the harmonic series, which diverges to infinity. $\qquad\square$

8. Suppose $f \in \mathscr{R}$ on $[a, b]$ for every $b > a$ where a is fixed. Define

$$\int_a^{\infty} f(x)\,\mathrm{d}x = \lim_{b \to \infty} \int_a^b f(x)\,\mathrm{d}x$$

if this limit exists (and is finite). In that case, we say that the integral on the left **converges**. If it also converges after f has been replaced by $|f|$, it is said to converge **absolutely**.

Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1, \infty)$. Prove that $\int_1^{\infty} f(x)\,\mathrm{d}x$ converges if and only if $\sum_{n=1}^{\infty} f(n)$ converges. (This is the so-called "integral test" for convergence of series.)

Proof. To prove the claim, we will show that

$$\sum_{n=2}^{N} f(n) \leq \int_1^N f \leq \sum_{n=1}^{N-1} f(n) \leq f(1) + \int_1^{N-1} f(x)\,\mathrm{d}x$$

It will follow since both the sum and the integral limit are monotonically increasing as $N \to \infty$ $(f \geq 0)$ and both are bounded below and above by (a function of) the other, both converge or diverge together. Let's begin.

Since f is monotonically decreasing on $[1, \infty)$, we know that $f(n) \leq f(x)$ for all $1 \leq x \leq n$ $(n \in \mathbb{N})$. Thus, by Theorem 6.12b,

$$\int_{n-1}^n f(n)\,\mathrm{d}x \leq \int_{n-1}^n f(x)\,\mathrm{d}x$$

Therefore,

$$\sum_{n=2}^{N} f(n) = \sum_{n=2}^{N} \int_{n-1}^{n} f(n)\,\mathrm{d}x \qquad\qquad \text{Theorem 6.12d}$$

$$\leq \sum_{n=2}^{N} \int_{n-1}^{n} f(x)\,\mathrm{d}x$$

$$= \int_{1}^{N} f(x)\,\mathrm{d}x \qquad\qquad \text{Theorem 6.12c}$$

for all $N = 2, 3, 4, \ldots$, thereby establishing the left inequality above.

Since f is monotonically decreasing on $[1, \infty)$, we know that $f(x) \leq f(n)$ for all $x \geq n$ ($n \in \mathbb{N}$). Thus, by Theorem 6.12b,

$$\int_{n}^{n+1} f(x)\,\mathrm{d}x \leq \int_{n}^{n+1} f(n)\,\mathrm{d}x$$

Therefore,

$$\int_{1}^{N} f(x)\,\mathrm{d}x = \sum_{n=1}^{N-1} \left(\int_{n}^{n+1} f(x)\,\mathrm{d}x \right) \qquad\qquad \text{Theorem 6.12c}$$

$$\leq \sum_{n=1}^{N-1} \left(\int_{n}^{n+1} f(n)\,\mathrm{d}x \right)$$

$$= \sum_{n=1}^{N-1} f(n) \qquad\qquad \text{Theorem 6.12d}$$

for all $N = 2, 3, 4, \ldots$, thereby establishing the middle inequality above.

From our statement about $f(n)$ and $f(x)$ from the left inequality, we have by Theorem 6.12b that

$$\int_{n-1}^{n} f(n)\,\mathrm{d}x \leq \int_{n-1}^{n} f(x)\,\mathrm{d}x$$

Therefore,

$$\sum_{n=1}^{N-1} f(n) = f(1) + \sum_{n=2}^{N-1} \int_{n-1}^{n} f(n)\,\mathrm{d}x \qquad\qquad \text{Theorem 6.12d}$$

$$\leq f(1) + \sum_{n=2}^{N-1} \int_{n-1}^{n} f(x)\,\mathrm{d}x$$

$$= f(1) + \int_{1}^{N-1} f(x)\,\mathrm{d}x \qquad\qquad \text{Theorem 6.12c}$$

for all $N = 2, 3, 4, \ldots$, thereby establishing the right inequality above. $\qquad\square$

10. Let p, q be positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

Prove the following statements.

(a) If $u, v \geq 0$, then

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}$$

Equality holds if and only if $u^p = v^q$.

Discussion. To prove the desired inequality, it will suffice to show that

$$0 \leq \frac{u^p}{p} + \frac{v^q}{q} - uv$$

i.e., that for all $u, v \geq 0$, the expression on the right above is nonnegative. To consider all such values at once, we can consider applying our analysis toolbox to $f : [0, \infty)^2 \to \mathbb{R}$ defined by

$$f(u, v) = \frac{u^p}{p} + \frac{v^q}{q} - uv$$

with the goal of proving that it is nonnegative everywhere on its domain. However, since we do not yet know multivariable calculus, it will suffice to fix $u \geq 0$ and analyze $f : [0, \infty) \to \mathbb{R}$ defined by

$$f(v) = \frac{u^p}{p} + \frac{v^q}{q} - uv$$

Let's begin. $\qquad\square$

Proof. Fix $u \geq 0$. Let $f : [0, \infty) \to \mathbb{R}$ be defined by

$$f(v) = \frac{u^p}{p} + \frac{v^q}{q} - uv$$

It follows from the definition of f that to prove the desired inequality, it will suffice to show that f is nonnegative everywhere on its domain. Let's begin.

Since f is a polynomial in v, f is differentiable. Thus, we may consider

$$f'(v) = v^{q-1} - u$$

As a function of a positive power ($q/(q-1) = p > 0$ and $q > 0$ imply $q - 1 > 0$) of its variable (minus a constant), f' is strictly increasing. Additionally, we have that

$$0 = f'(v)$$
$$u = v^{q-1}$$
$$= v^{q/p}$$
$$v = u^{p/q}$$

Thus, we know that $f' < 0$ on $(0, u^{p/q})$ and $f' > 0$ on $(u^{p/q}, \infty)$. It follows by the strict version of Theorem 5.11 that f is strictly decreasing on $(0, u^{p/q})$ and strictly increasing on $(u^{p/q}, \infty)$. Furthermore, since f is differentiable (hence continuous by Theorem 5.2), we know that $f(0) \geq f(u^{p/q})$. Combining the last several results, we have that $f(u^{p/q})$ is the minimum of f over $[0, \infty)$, and hence equal to the minimum value of f over $[0, \infty)$. But since

$$f(u^{p/q}) = \frac{u^p}{p} + \frac{(u^{p/q})^q}{q} - uu^{p/q}$$
$$= \frac{u^p}{p} + \frac{u^p}{q} - u^{p/q+1}$$
$$= u^p \left(\frac{1}{p} + \frac{1}{q} \right) - u^p$$
$$= 0$$

we know that $f(v) \geq 0$ on its domain, as desired.

Additionally, since f is strictly decreasing on $(0, u^{p/q})$ and strictly increasing on $(u^{p/q}, \infty)$, we know that $f(v) = 0$ iff $v = u^{p/q}$, i.e., iff $v^q = u^p$, as desired. $\qquad\square$

(b) If $f, g \in \mathcal{R}(\alpha)$, $f, g \geq 0$, and

$$\int_a^b f^p \, \mathrm{d}\alpha = 1 = \int_a^b g^q \, \mathrm{d}\alpha$$

then

$$\int_a^b fg \, \mathrm{d}\alpha \leq 1$$

Proof. By Theorem 6.13a, the hypothesis $f, g \in \mathcal{R}(\alpha)$ implies that $fg \in \mathcal{R}(\alpha)$. Thus, we have that

$$\int_a^b fg \, \mathrm{d}\alpha \leq \int_a^b \left(\frac{f^p}{p} + \frac{g^q}{q} \right) \mathrm{d}\alpha \qquad\qquad \text{Theorem 6.12b}$$

$$= \frac{1}{p} \int_a^b f^p \, \mathrm{d}\alpha + \frac{1}{q} \int_a^b g^q \, \mathrm{d}\alpha \qquad\qquad \text{Theorem 6.12a}$$

$$= \frac{1}{p} + \frac{1}{q}$$

$$= 1$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

(c) If f, g are complex functions in $\mathcal{R}(\alpha)$, then

$$\left| \int_a^b fg \, \mathrm{d}\alpha \right| \leq \left(\int_a^b |f|^p \, \mathrm{d}\alpha \right)^{1/p} \left(\int_a^b |g|^q \, \mathrm{d}\alpha \right)^{1/q}$$

This is **Hölder's inequality**. When $p = q = 2$, it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)

Proof. By Theorem 6.11 with $\phi(y) = |y|^p$ (resp. $\phi(y) = |y|^q$), the hypothesis $f, g \in \mathcal{R}(\alpha)$ implies that $|f|^p, |g|^q \in \mathcal{R}(\alpha)$. Thus, we may let

$$I_f = \left(\int_a^b |f|^p \, \mathrm{d}\alpha \right)^{1/p} \qquad\qquad\qquad I_g = \left(\int_a^b |g|^q \, \mathrm{d}\alpha \right)^{1/q}$$

We divide into two cases ($I_f = 0$ or $I_g = 0$, and $I_f, I_g \neq 0$). In the first case, WLOG let $I_f = 0$. Then since $0 \leq |f|^p$, it follows that $f = 0$ on $[a, b]$. Thus

$$\left| \int_a^b fg \, \mathrm{d}\alpha \right| = 0 \leq 0 = I_f I_g = \left(\int_a^b |f|^p \, \mathrm{d}\alpha \right)^{1/p} \left(\int_a^b |g|^q \, \mathrm{d}\alpha \right)^{1/q}$$

as desired. In the other case, it follows that

$$I_f^p = \int_a^b |f|^p \, \mathrm{d}\alpha \qquad\qquad\qquad I_g^q = \int_a^b |g|^q \, \mathrm{d}\alpha$$

$$1 = \int_a^b \left| \frac{f}{I_f} \right|^p \, \mathrm{d}\alpha \qquad\qquad\qquad 1 = \int_a^b \left| \frac{g}{I_g} \right|^q \, \mathrm{d}\alpha \qquad\qquad \text{Theorem 6.12a}$$

Thus, since $|f/I_f|, |g/I_g| \in \mathcal{R}(\alpha)$ by Theorems 6.12 and 6.13 and $|f/I_f|, |g/I_g| \geq 0$ by the defini-

tion of the absolute value, we have that

$$\left| \int_a^b fg \, \mathrm{d}\alpha \right| \leq \int_a^b |fg| \, \mathrm{d}\alpha \qquad\qquad\qquad \text{Theorem 6.13b}$$

$$= I_f I_g \int_a^b \left| \frac{f}{I_f} \right| \left| \frac{g}{I_g} \right| \mathrm{d}\alpha$$

$$\leq I_f I_g \cdot 1 \qquad\qquad\qquad\qquad \text{Part (b)}$$

$$= \left(\int_a^b |f|^p \, \mathrm{d}\alpha \right)^{1/p} \left(\int_a^b |g|^q \, \mathrm{d}\alpha \right)^{1/q}$$

as desired. □

11. Let α be a fixed increasing function on $[a,b]$. For $u \in \mathscr{R}(\alpha)$, define

$$\|u\|_2 = \left(\int_a^b |u|^2 \, \mathrm{d}\alpha \right)^{1/2}$$

Suppose $f, g, h \in \mathscr{R}(\alpha)$, and prove the triangle inequality

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2$$

as a consequence of the Schwarz inequality, as in the proof of Theorem 1.37.

Proof. By Theorems 6.12a and 6.13b, the hypothesis that $f, g, h \in \mathscr{R}(\alpha)$ implies that $|f - g|, |g - h| \in \mathscr{R}(\alpha)$. Thus, we have that

$$\|f - h\|_2^2 = \int_a^b |f - h|^2 \, \mathrm{d}\alpha$$

$$= \int_a^b |(f - g) + (g - h)|^2 \, \mathrm{d}\alpha$$

$$= \int_a^b |f - g|^2 \, \mathrm{d}\alpha + 2 \int_a^b |f - g| \cdot |g - h| \, \mathrm{d}\alpha + \int_a^b |g - h|^2 \, \mathrm{d}\alpha$$

$$\leq \int_a^b |f - g|^2 \, \mathrm{d}\alpha + 2 \left(\int_a^b |f - g|^2 \, \mathrm{d}\alpha \right)^{1/2} \left(\int_a^b |g - h|^2 \, \mathrm{d}\alpha \right)^{1/2} + \int_a^b |g - h|^2 \, \mathrm{d}\alpha$$

$$= \|f - g\|_2^2 + 2\|f - g\|_2 \|g - h\|_2 + \|g - h\|_2^2$$

$$= (\|f - g\|_2 + \|g - h\|_2)^2$$

Taking square roots of both sides of the inequality yields the desired result. □

4   Sequences and Series of Functions

From Rudin (1976).

Chapter 7

2/9:   **1.** Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

Proof. Let $\{f_n\}$ be an arbitrary uniformly convergent sequence of bounded functions. To prove that it is uniformly bounded, it will suffice to find a number M such that $|f_n(x)| < M$ for all $x \in E$ and $n \in \mathbb{N}$. Let f be the function such that $f_n \rightrightarrows f$, and let $M_n = \sup_{x \in E} |f_n(x) - f(x)|$ for each $n \in \mathbb{N}$ (the boundedness of each f_n implies that such an M_n always exists). Thus, based on the last two definitions, we can invoke Theorem 7.9 to learn that $M_n \to 0$ as $n \to \infty$. But since $\{M_n\}$ converges, Theorem 3.2c implies that $\{M_n\}$ is bounded, say by $M/2$. Taking M to be our M yields that for an arbitrary $x \in E$ and $n \in \mathbb{N}$,

$$|f_n(x)| \leq M_n \leq \frac{M}{2} < M$$

as desired. $\qquad\qquad\square$

2. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.

Proof. To prove that $\{f_n + g_n\}$ converges uniformly on E to $f + g$, it will suffice to show that for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|(f_n + g_n)(x) - (f + g)(x)| < \epsilon$ for all $x \in E$. Let $\epsilon > 0$ be arbitrary. Since $f_n \to f$ uniformly on E, there exists N_1 such that if $n \geq N_1$, then $|f_n(x) - f(x)| < \epsilon/2$ for all $x \in E$. Similarly, there exists N_2 such that if $n \geq N_2$, then $|g_n(x) - g(x)| < \epsilon/2$ for all $x \in E$. Choose $N = \max(N_1, N_2)$. Now suppose $n \geq N$, and let $x \in E$ be arbitrary. It follows from the first condition that $n \geq N \geq N_1$ and $n \geq N \geq N_2$, so

$$\begin{aligned}
|(f_n + g_n)(x) - (f + g)(x)| &= |f_n(x) - f(x) + g_n(x) - g(x)| \\
&\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

as desired.

To prove that $\{f_n g_n\}$ converges uniformly on E to fg, it will suffice to show that for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|(f_n g_n)(x) - (fg)(x)| < \epsilon$ for all $x \in E$. Let $\epsilon > 0$ be arbitrary. Since f_n, g_n are uniformly convergent sequences of bounded functions, Exercise 1 implies that they are uniformly bounded, i.e., there exists $M^f, M^g \in \mathbb{R}$ such that $|f_n| < M^f$ and $|g_n| < M^g$ for all $n \in \mathbb{N}$. If we take $M = \max(M^f, M^g)$, then we have $|f_n| < M$ and $|g_n| < M$ for all $n \in \mathbb{N}$. Note that the same inequality holds for f and g. Now, as before, we may pick N such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon/2M$ and $|g_n(x) - g(x)| < \epsilon/2M$ for all $x \in E$. It follows that for any $n \geq N$ and $x \in E$,

$$\begin{aligned}
|(f_n g_n)(x) - (fg)(x)| &= |f_n(x) \cdot (g_n(x) - g(x)) + g(x) \cdot (f_n(x) - f(x))| \\
&= |f_n(x)| \cdot |g_n(x) - g(x)| + |g(x)| \cdot |f_n(x) - f(x)| \\
&< M \cdot \frac{\epsilon}{2M} + M \cdot \frac{\epsilon}{2M} \\
&= \epsilon
\end{aligned}$$

as desired. $\qquad\qquad\square$

3. Construct sequences $\{f_n\}, \{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converge on E).

Proof. Let

$$f_n(x) = x \qquad\qquad\qquad g_n(x) = \frac{1}{n}$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then $\{f_n\}$ converges uniformly to $f(x) = x$ (by choosing $N = 1$ for any ϵ) and $\{g_n\}$ converges uniformly to $g(x) = 0$ (by choosing $1/N < \epsilon$ with the Archimedean principle). However, while $\{f_n g_n\}$ converges pointwise to $(fg)(x) = 0$ by Theorem 3.3c, it does not converge uniformly since for any n, choosing $x = n$ yields $(f_n g_n)(x) = 1$. $\qquad\square$

4. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?

Proof. <u>Absolute convergence values:</u> The series converges absolutely for any

$$x \in (-\infty, -1) \cup \left(\bigcup_{k=1}^{\infty} \left(-\frac{1}{k^2}, -\frac{1}{(k+1)^2} \right) \right) \cup (0, \infty)$$

We prove this via casework as follows.

Let $x \in (0, \infty)$. Then we have

$$\sum_{n=1}^{\infty} \left| \frac{1}{1 + n^2 x} \right| = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x} \leq \sum_{n=1}^{\infty} \frac{1}{n^2 x} = \frac{1}{x} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{c}{x}$$

where $c \in \mathbb{R}$ is finite by Theorem 3.28. Therefore, since the sum is monotonically increasing and bounded, Theorem 3.14 implies that the sum overall converges, as desired.

Let $x \in (-\infty, -1)$. Then we have

$$n^2 x + 1 < n^2 x + n^2 = n^2 (x + 1)$$

Since $x < -1$,

$$n^2 x + 1 < 0 \qquad\qquad\qquad n^2 (x + 1) < 0$$

for all $n \in \mathbb{N}$. Thus,

$$n^2 x + 1 < n^2 (x + 1)$$
$$\frac{n^2 x + 1}{n^2 (x + 1)} > 1$$
$$\frac{1}{n^2 (x + 1)} < \frac{1}{n^2 x + 1}$$
$$\left| \frac{1}{n^2 x + 1} \right| < \left| \frac{1}{n^2 (x + 1)} \right|$$

for all $n \in \mathbb{N}$. It follows that

$$\sum_{n=1}^{\infty} \left| \frac{1}{n^2 x + 1} \right| < \sum_{n=1}^{\infty} \left| \frac{1}{n^2 (x + 1)} \right| = \frac{1}{x + 1} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{c}{x + 1}$$

where $c \in \mathbb{R}$ is finite by Theorem 3.28. Therefore, since the sum is monotonically increasing and bounded, Theorem 3.14 implies that the sum overall converges, as desired.

Let $x \in (-1/k^2, -1/(k+1)^2)$. For right now, we consider only the sum for $n \geq \sqrt{2}(k+1)$, leaving finitely many terms out of the sum. Let $\delta = 1/(k+1)^2$. It follows that

$$n \geq \sqrt{2}(k+1) \qquad\qquad\qquad x < -\frac{1}{(k+1)^2}$$

$$n \geq \sqrt{\frac{2}{1/(k+1)^2}} \qquad\qquad -x > \frac{1}{(k+1)^2}$$

$$n^2 \geq \frac{2}{\delta}$$

$$\frac{\delta}{2} \geq \frac{1}{n^2}$$

Additionally, since $n \geq \sqrt{2}(k+1) > k$ (hence $n^2 \geq (k+1)^2$) and $x < -1/(k+1)^2$, we have that

$$n^2 x < (k+1)^2 \cdot -\frac{1}{(k+1)^2}$$

$$n^2 x < -1$$

$$n^2 x + 1 < 0$$

Thus, for $n \geq \sqrt{2}(k+1)$, we have that

$$\left| \frac{1}{1+n^2 x} \right| = \frac{1}{n^2(-x)-1} < \frac{1}{n^2\delta - 1} = \frac{1}{n^2} \cdot \frac{1}{\delta - 1/n^2} \leq \frac{1}{n^2} \cdot \frac{1}{\delta - \delta/2} = \frac{2}{\delta n^2}$$

Therefore, since $|f_n(x)| \leq M_n = 2/\delta n^2$ and $\sum M_n$ converges by Theorem 3.28, the comparison test implies that $\sum |f_n(x)|$ converges, as desired. Adding on the finitely many terms we left out of the summation will not change this fact.

Note that the series diverges for $x = 0$ since each term becomes 1 in this case. Additionally, the series fails to exist for $x = -1/k^2$ ($k \in \mathbb{N}$) since the kth term is undefined in this case.

Uniform convergence intervals: The series converges uniformly on any

$$[a,b] \subset (-\infty, -1) \cup \left(\bigcup_{k=1}^{\infty} \left(-\frac{1}{k^2}, -\frac{1}{(k+1)^2} \right) \right) \cup (0, \infty)$$

This is because any such interval will be a subset of either $(-\infty, -1)$, $(0, \infty)$, or a set of the form $(-1/k^2, -1/(k+1)^2)$ ($k \in \mathbb{N}$). Thus, we may take as $\sum M_n$ the supremum on $[a,b]$ of the appropriate bound derived above (either c/x, $c/(x+1)$, or $2c/\delta$, respectively; all supremums of which will exist by the definition of $[a,b]$) and apply Theorem 7.10.

Non-uniform convergence intervals: Any interval containing one or more of the points in the set $\{0\} \cup \{-1/n^2\}_{n=1}^{\infty}$, by the above.

Points of continuity: The series is continuous at all points at which it converges.

Let x be a point at which f converges. Then by the first part of the proof, x is an element of an open set G. Thus, let $N_{2r}(x) \subset G$, and consider $[x-r, x+r]$. By the above, f converges uniformly on this interval. Additionally, each f_n is continuous on this interval by definition. Thus, by Theorem 7.12, f is continuous at x, as desired.

Boundedness: f is not bounded.

If we suppose for the sake of contradiction that f is bounded by m, we nevertheless find that

$$f(\tfrac{1}{4m^2}) > \sum_{n=1}^{2m} \frac{1}{1+\frac{n^2}{4m^2}} = \sum_{n=1}^{2m} \frac{(2m)^2}{(2m)^2+n^2} \geq \sum_{n=1}^{2m} \frac{(2m)^2}{(2m)^2+(2m)^2} = \sum_{n=1}^{2m} \frac{1}{2} = m$$

$\square$

7. For $n = 1, 2, 3, \ldots$ and x real, put

$$f_n(x) = \frac{x}{1+nx^2}$$

Show that $\{f_n\}$ converges uniformly to a function f and that the equation

$$f'(x) = \lim_{n\to\infty} f'_n(x)$$

is correct if $x \neq 0$ but false if $x = 0$.

Proof. To prove that $\{f_n\}$ converges uniformly to f defined by $f(x) = 0$ $(x \in \mathbb{R})$, Theorem 7.9 tells us that it will suffice to show that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$ and that the sequence $\{M_n\}$ defined by $M_n = \sup_{x\in\mathbb{R}} |f_n(x)|$ tends to zero as $n \to \infty$. Since

$$f_n(x) = \frac{x}{1+nx^2} < \frac{x}{nx^2} = \frac{1}{x} \cdot \frac{1}{n} \to 0$$

as $n \to \infty$ for all $x \neq 0$ and $f_n(0) = 0$ for all n, $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$, as desired. Additionally, by the Schwarz inequality, if a_1, a_2, b_1, b_2 are real numbers, then

$$|a_1 b_1 + a_2 b_2|^2 \leq (|a_1|^2 + |a_2|^2)(|b_1|^2 + |b_2|^2)$$

It follows that

$$|2\sqrt{n}x|^2 = |\underbrace{1}_{a_1} \cdot \underbrace{\sqrt{n}x}_{b_1} + \underbrace{\sqrt{n}x}_{a_2} \cdot \underbrace{1}_{b_2}|^2 \leq (|1|^2 + |\sqrt{n}x|^2)(|\sqrt{n}x|^2 + |1|^2) = (1+nx^2)^2$$

$$|2\sqrt{n}x| \leq |1+nx^2|$$

$$\frac{1}{|1+nx^2|} \leq \frac{1}{2\sqrt{n}|x|}$$

$$\frac{|x|}{|1+nx^2|} \leq \frac{1}{2\sqrt{n}}$$

$$\left|\frac{x}{1+nx^2}\right| \leq \frac{1}{2\sqrt{n}}$$

for all $x \neq 0$, $n \in \mathbb{N}$. This combined with the facts that $f_n(0) = 0 < \frac{1}{2\sqrt{n}}$ for all $n \in \mathbb{N}$ and $f_n(1/\sqrt{n}) = 1/2\sqrt{n}$ for all $n \in \mathbb{N}$ implies that $M_n = 1/2\sqrt{n}$. Thus, $M_n \to 0$ as $n \to \infty$, as desired. $f'(x) = 0$ for all $x \in \mathbb{R}$. Additionally,

$$f'_n(x) = \frac{1-nx^2}{(1+nx^2)^2} \leq \frac{1-nx^2}{(nx^2)^2} = \frac{1}{x^4} \cdot \frac{1}{n^2} - \frac{1}{x^2} \cdot \frac{1}{n} \to 0$$

as $n \to \infty$ for all $x \neq 0$, as desired. However, $f'_n(0) = 1$ for all $n \in \mathbb{N}$, as desired. $\square$

5   Sequences and Series of Functions II / Functions of Several Variables

From Rudin (1976).

Chapter 7

2/16:   **5.** Let

$$f_n(x) = \begin{cases} 0 & x < \frac{1}{n+1} \\ \sin^2 \frac{\pi}{x} & \frac{1}{n+1} \le x \le \frac{1}{n} \\ 0 & \frac{1}{n} < x \end{cases}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that absolute convergence, even for all x, does not imply uniform convergence.

Proof. To prove that $\{f_n\}$ converges pointwise to the continuous function f defined by $f(x) = 0$ for all $x \in \mathbb{R}$, it will suffice to show that for every $\epsilon > 0$ and for every $x \in \mathbb{R}$, there exists an integer N such that if $n \ge N$, then $|f_n(x)| < \epsilon$. Let $\epsilon > 0$ and $x \in \mathbb{R}$ be arbitrary. We divide into three cases $(x \in \{1/n\}_{n=1}^\infty, x \in [0,1] \setminus \{1/n\}_{n=1}^\infty,$ and $x \notin [0,1])$.

If $x \in \{1/n\}_{n=1}^\infty$, let $x = 1/k$. Then by the definition of $f_n(x)$, we have that

$$f_i(x) = \begin{cases} 0 & i < k-1 \\ \sin^2 \frac{\pi}{1/k} = \sin^2 k\pi = 0 & i = k-1, k \\ 0 & i > k \end{cases}$$

Thus, choose $N = 1$. It follows that if $n \ge N$, then

$$|f_n(x)| = 0 < \epsilon$$

as desired.

If $x \in [0,1] \setminus \{1/n\}_{n=1}^\infty$, let $x \in (1/[(N-1)+1], 1/(N-1))$ where $N \in \mathbb{N}$. Choose this N to be our N. It follows that if $n \ge N$, then

$$\frac{1}{n} \le \frac{1}{N} = \frac{1}{(N-1)+1} < x$$

so by definition,

$$|f_n(x)| = 0 < \epsilon$$

as desired.

If $x \notin [0,1]$, then either $x < 1/(n+1)$ for all $n \in \mathbb{N}$ or $x > 1/n$ for all $n \in \mathbb{N}$. Either way, we choosing $N = 1$ yields that if $n \ge N$, then

$$|f_n(x)| = 0 < \epsilon$$

as desired.

To prove that $\{f_n\}$ does not converge uniformly to f, Theorem 7.9 tells us that it will suffice to show that if $M_n = \sup_{x \in \mathbb{R}} |f_n(x) - f(x)|$, then $M_n \nrightarrow 0$ as $n \to \infty$. Let $n \in \mathbb{N}$ be arbitrary. Since $n < n + 1/2 < n + 1$ and hence $1/(n+1) \le 2/(2n+1) \le 1/n$, we have by the properties of the sine function that

$$f_n(\tfrac{2}{2n+1}) = \sin^2 \left[\frac{\pi}{2/(2n+1)} \right] = \sin^2 \left[\frac{2n+1}{2} \pi \right] = \sin^2 \left[\left(n + \frac{1}{2} \right) \pi \right] = 1$$

and that $f_n(x) \le 1$ everywhere else. Thus, $M_n = 1$ for all $n \in \mathbb{N}$. But then $M_n \nrightarrow 0$ as $n \to \infty$, as desired.

It follows by an argument symmetric to the above that while $\sum f_n$ converges absolutely to

$$f(x) = \begin{cases} 0 & x \leq 0 \\ \sin^2 \frac{\pi}{x} & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$$

$M_n = 1$ for all $n \in \mathbb{N}$.   □

6. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

Proof. Let $[a, b]$ be an arbitrary bounded interval, and let $f_n(x) = (-1)^n \frac{x^2 + n}{n^2}$. To prove that the series converges uniformly on $[a, b]$, Theorem 7.8 tells us that it will suffice to show that for every $\epsilon > 0$, there exists an N such that if $n, m \geq N$ (WLOG let $n \leq m$) and $x \in [a, b]$, then

$$\left| \sum_{i=n}^{m} f_i(x) \right| < \epsilon$$

Let $\epsilon > 0$ be arbitrary. Define $m = \max(|a|, |b|)$ (note that since $a \neq b$ by definition, $m > 0$). By consecutive applications of Theorem 3.43, we know that both $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converge. Thus, by consecutive applications of Theorem 3.22, there exist integers N_1, N_2 such that $m \geq n \geq N_1$ implies the left result below and $m \geq n \geq N_2$ implies the right result below.

$$\left| \sum_{k=n}^{m} (-1)^k \frac{1}{k^2} \right| < \frac{\epsilon}{2m^2} \qquad\qquad \left| \sum_{k=n}^{m} (-1)^k \frac{1}{k} \right| < \frac{\epsilon}{2}$$

Choose $N = \max(N_1, N_2)$. Now let $n, m \geq N$ with WLOG $n \leq m$, and let $x \in [a, b]$. It follows that

$$\begin{aligned} \left| \sum_{k=n}^{m} f_k(x) \right| &= \left| \sum_{k=n}^{m} (-1)^k \frac{x^2 + k}{k^2} \right| \\ &= \left| x^2 \sum_{k=n}^{m} (-1)^k \frac{1}{k^2} + \sum_{k=n}^{m} (-1)^k \frac{1}{k} \right| \\ &\leq |x^2| \cdot \left| \sum_{k=n}^{m} (-1)^k \frac{1}{k^2} \right| + \left| \sum_{k=n}^{m} (-1)^k \frac{1}{k} \right| \\ &\leq m^2 \cdot \left| \sum_{k=n}^{m} (-1)^k \frac{1}{k^2} \right| + \left| \sum_{k=n}^{m} (-1)^k \frac{1}{k} \right| \\ &< m^2 \cdot \frac{\epsilon}{2m^2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

as desired.

To prove that the series does not converge absolutely for any value of x, let $x \in \mathbb{R}$ be arbitrary. Then

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{x^2 + n}{n^2} \right| = x^2 \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n}$$

$$\geq \sum_{n=1}^{\infty} \frac{1}{n}$$

where the latter series diverges by Theorem 3.28, yielding the desired result.   □

8. If
$$I(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$$

if $\{x_n\}$ is a sequence of distinct points of (a, b), and if $\sum |c_n|$ converges, prove that the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n)$$

converges uniformly on $[a, b]$, and that f is continuous for every $x \ne x_n$.

Proof. Let $f_n(x) = c_n I(x - x_n)$ for all $n \in \mathbb{N}$. To prove that f converges uniformly on $[a, b]$, Theorem 7.10 tells us that it will suffice to show that $|f_n(x)| \le M_n$ for all $x \in [a, b]$ and $\sum M_n$ converges. Let $M_n = c_n$ for all $n \in \mathbb{N}$. Then for any $x \in [a, b]$,

$$|f_n(x)| = c_n I(x - x_n) \le c_n = M_n$$

as desired. Additionally, $\sum M_n = \sum c_n$ converges, as desired. This completes the proof.

For the second part of the proof, let $x \notin \{x_n\}$. Then every f_n is continuous at x by definition. Thus, f is a uniformly convergent sequence of functions continuous at x, so by Theorem 7.12, f is continuous at x. □

9. Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that
$$\lim_{n \to \infty} f_n(x_n) = f(x)$$
for every sequence of points $x_n \in E$ such that $x_n \to x$ and $x \in E$. Is the converse of this true?

Proof. Let $\{x_n\} \subset E$ be an arbitrary sequence of points that converges to some $x \in E$. To prove that $\lim_{n \to \infty} f_n(x_n) = f(x)$, it will suffice to show that for every $\epsilon > 0$, there exists an N such that if $n \ge N$, then $|f_n(x_n) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\{f_n\}$ is a uniformly convergent sequence of continuous functions, Theorem 7.12 implies that f is a continuous function. Thus, there exists a $\delta > 0$ such that if $y \in E$ and $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon/2$. Additionally, since $x_n \to x$, there exists an N_1 such that if $n \ge N_1$, $|x_n - x| < \delta$. Furthermore, since f_n converges uniformly to f, there exists N_2 such that if $n \ge N_2$, then $|f_n(y) - f(y)| < \epsilon/2$ for all $y \in E$. In particular, $|f_n(x_n) - f(x_n)| < \epsilon/2$. Choose $N = \max(N_1, N_2)$. Let $n \ge N$ be arbitrary. Then

$$|f_n(x_n) - f(x)| \le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$
$$= \epsilon$$

as desired.

No, it is not true in general that if $\{f_n\}$ is a sequence of continuous functions for which $\lim_{n \to \infty} f_n(x_n) = f(x)$ for every sequence of points $x_n \in E$ such that $x_n \to x$ and $x \in E$, then f_n converges uniformly. Consider the sequence of functions from Exercise 7.5. This is a sequence of continuous functions for which $\lim_{n \to \infty} f_n(x_n) = f(x)$ for any sequence $\{x_n\}$ of the desired type since we can always choose N large enough so that the moving "hump" and neighborhood of x containing all remaining x_n are separated forever more. Moreover, by Exercise 7.5, $\{f_n\}$ does not converge uniformly, as desired. □

Chapter 9

1. If S is a nonempty subset of a vector space X, prove (as asserted in Section 9.1) that the span of S is a vector space.

Proof. Let $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ (the proof is symmetric if S is infinite).

To prove that span(S) is a vector space, it will suffice to show that span(S) is nonempty and that for all $\mathbf{x}, \mathbf{y} \in$ span(S) and $c \in \mathbb{C}$, $(\mathbf{x} + \mathbf{y}) \in$ span(S) and $c\mathbf{x} \in$ span(S). Since S is nonempty, there exists $\mathbf{x} \in S$; thus, $1\mathbf{x} \in$ span(S), so span(S) is nonempty, as desired. Let $\mathbf{x}, \mathbf{y} \in$ span(S) and $c \in \mathbb{C}$. There exist $a_1, \ldots, a_n, b_1, \ldots, b_n$ such that

$$\mathbf{x} = a_1\mathbf{u}_1 + \cdots + a_n\mathbf{u}_n \qquad\qquad \mathbf{y} = b_1\mathbf{u}_1 + \cdots + b_n\mathbf{u}_n$$

It follows by the definition of span(S) that

$$(a_1 + b_1)\mathbf{u}_1 + \cdots + (a_n + b_n)\mathbf{u}_n = \mathbf{x} + \mathbf{y} \in \text{span}(S)$$
$$ca_1\mathbf{u}_1 + \cdots + ca_n\mathbf{u}_n = c\mathbf{x} \in \text{span}(S)$$

as desired. $\qquad\square$

2. Prove (as asserted in Section 9.6) that BA is linear if A and B are linear transformations. Prove also that A^{-1} is linear and invertible.

Proof. Let $A \in L(X, Y)$ and $B \in L(Y, Z)$. Then for all $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2 \in X$ and $c \in \mathbb{C}$,

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 \qquad\qquad A(c\mathbf{x}) = cA\mathbf{x}$$

and for all $\mathbf{y}, \mathbf{y}_1, \mathbf{y}_2 \in Y$ and $c \in \mathbb{C}$,

$$B(\mathbf{y}_1 + \mathbf{y}_2) = B\mathbf{y}_1 + B\mathbf{y}_2 \qquad\qquad B(c\mathbf{y}) = cB\mathbf{y}$$

It follows that for any $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2 \in X$ and $c \in \mathbb{C}$, we have that

$$\begin{aligned} BA(\mathbf{x}_1 + \mathbf{x}_2) &= B(A\mathbf{x}_1 + A\mathbf{x}_2) & BA(c\mathbf{x}) &= B(cA\mathbf{x}) \\ &= BA\mathbf{x}_1 + BA\mathbf{x}_2 & &= cBA\mathbf{x} \end{aligned}$$

so BA is a linear transformation, as desired.

Let $A \in L(X, Y)$ be invertible. Since A is a linear transformation, the same equalities from above still apply. Thus,

$$\begin{aligned} \mathbf{x}_1 + \mathbf{x}_2 &= \mathbf{x}_1 + \mathbf{x}_2 & c\mathbf{x} &= c\mathbf{x} \\ I(\mathbf{x}_1 + \mathbf{x}_2) &= I\mathbf{x}_1 + I\mathbf{x}_2 & I(c\mathbf{x}) &= cI\mathbf{x} \\ AA^{-1}(\mathbf{x}_1 + \mathbf{x}_2) &= AA^{-1}\mathbf{x}_1 + AA^{-1}\mathbf{x}_2 & AA^{-1}(c\mathbf{x}) &= cAA^{-1}\mathbf{x} \\ A(A^{-1}(\mathbf{x}_1 + \mathbf{x}_2)) &= A(A^{-1}\mathbf{x}_1 + A^{-1}\mathbf{x}_2) & A(A^{-1}(c\mathbf{x})) &= A(cA^{-1}\mathbf{x}) \\ A^{-1}(\mathbf{x}_1 + \mathbf{x}_2) &= A^{-1}\mathbf{x}_1 + A^{-1}\mathbf{x}_2 & A^{-1}(c\mathbf{x}) &= cA^{-1}\mathbf{x} \end{aligned}$$

where we use the fact that A is one-to-one for the last equality in both cases. To prove that A^{-1} is invertible, it will suffice to show that it is one-to-one and onto. Suppose $A^{-1}\mathbf{x} = A^{-1}\mathbf{y}$. Then

$$AA^{-1}\mathbf{x} = AA^{-1}\mathbf{y}$$
$$I\mathbf{x} = I\mathbf{y}$$
$$\mathbf{x} = \mathbf{y}$$

proving that A^{-1} is one-to-one, as desired. Now suppose $\mathbf{y} \in X$. Then $A\mathbf{y} = \mathbf{x}$ for some $\mathbf{x} \in X$. It follows that
$$A^{-1}\mathbf{x} = A^{-1}A\mathbf{y} = I\mathbf{y} = \mathbf{y}$$

proving that A^{-1} is onto, as desired. $\qquad\square$

3. Assume $A \in L(X, Y)$ and $A\mathbf{x} = \mathbf{0}$ only when $\mathbf{x} = \mathbf{0}$. Prove that A is then 1-1.

Proof. If we suppose that $A\mathbf{x} = A\mathbf{y}$, then by linearity,

$$\mathbf{0} = A\mathbf{x} - A\mathbf{y}$$
$$= A(\mathbf{x} - \mathbf{y})$$

It follows by hypothesis that $\mathbf{x} - \mathbf{y} = \mathbf{0}$, hence $\mathbf{x} = \mathbf{y}$, proving that A is 1-1, as desired. $\qquad\square$

4. Prove (as asserted in Section 9.30) that null spaces and ranges of linear transformations are vector spaces.

Proof. Let $A \in L(X, Y)$.

Suppose $\mathbf{x}_1, \mathbf{x}_2 \in \text{null } A$. Then $A\mathbf{x}_1 = \mathbf{0}$ and $A\mathbf{x}_2 = \mathbf{0}$. It follows that

$$\mathbf{0} = A\mathbf{x}_1 + A\mathbf{x}_2$$
$$= A(\mathbf{x}_1 + \mathbf{x}_2)$$

so $(\mathbf{x}_1 + \mathbf{x}_2) \in \text{null } A$, as desired.

Suppose $\mathbf{x} \in \text{null } A$ and $c \in \mathbb{C}$. Then $A\mathbf{x} = \mathbf{0}$. It follows that

$$\mathbf{0} = c \cdot \mathbf{0}$$
$$= cA\mathbf{x}$$
$$= A(c\mathbf{x})$$

so $c\mathbf{x} \in \text{null } A$, as desired.

Suppose $\mathbf{y}_1, \mathbf{y}_2 \in \text{range } A$. Then there exist $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that $A\mathbf{x}_1 = \mathbf{y}_1$ and $A\mathbf{x}_2 = \mathbf{y}_2$. It follows that

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2$$
$$= \mathbf{y}_1 + \mathbf{y}_2$$

so $(\mathbf{y}_1 + \mathbf{y}_2) \in \text{range } A$, as desired.

Suppose $\mathbf{y} \in \text{range } A$ and $c \in \mathbb{C}$. Then there exists $\mathbf{x} \in X$ such that $A\mathbf{x} = \mathbf{y}$. It follows that

$$A(c\mathbf{x}) = cA\mathbf{x}$$
$$= c\mathbf{y}$$

so $c\mathbf{y} \in \text{range } A$, as desired. $\qquad\square$

6   Functions of Several Variables II

From Rudin (1976).

Chapter 9

2/22:   **5.** Prove that to every $A \in L(\mathbb{R}^n, \mathbb{R}^1)$ corresponds a unique $\mathbf{y} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$. Prove also that $\|A\| = |\mathbf{y}|$. (Hint: Under certain conditions, equality holds in the Schwarz inequality.)

Proof. Let $\mathbf{y}$ be defined as follows.

$$\mathbf{y} = \begin{bmatrix} A\mathbf{e}_1 \\ \vdots \\ A\mathbf{e}_n \end{bmatrix}$$

Then if $\mathbf{x} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$, we have that

$$A\mathbf{x} = a_1 \cdot A\mathbf{e}_1 + \cdots + a_n \cdot A\mathbf{e}_n = \mathbf{x} \cdot \mathbf{y}$$

as desired.

Now suppose that $\mathbf{z}$ satisfies $A\mathbf{x} = \mathbf{x} \cdot \mathbf{z}$ for all $\mathbf{x} \in \mathbb{R}^n$. Then

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot \mathbf{z}$$
$$\mathbf{x} \cdot (\mathbf{y} - \mathbf{z}) = 0$$

for all $\mathbf{x} \in \mathbb{R}^n$. In particular, if $\mathbf{x} = \mathbf{y} - \mathbf{z}$, then

$$\begin{aligned} 0 &= (\mathbf{y} - \mathbf{z}) \cdot (\mathbf{y} - \mathbf{z}) \\ &= \|\mathbf{y} - \mathbf{z}\|^2 \\ \mathbf{0} &= \mathbf{y} - \mathbf{z} \\ \mathbf{z} &= \mathbf{y} \end{aligned}$$

as desired.

We also have by the Cauchy-Schwarz inequality that

$$\|A\| = \sup_{\substack{\mathbf{x} \in \mathbb{R}^n \\ |\mathbf{x}|=1}} |A\mathbf{x}| = \sup_{\substack{\mathbf{x} \in \mathbb{R}^n \\ |\mathbf{x}|=1}} |\mathbf{x} \cdot \mathbf{y}| \leq \sup_{\substack{\mathbf{x} \in \mathbb{R}^n \\ |\mathbf{x}|=1}} |\mathbf{x}| \cdot |\mathbf{y}| = |\mathbf{y}|$$

Moreover, equality holds by noting that if $\mathbf{x} = \mathbf{y}/|\mathbf{y}|$, then

$$|\mathbf{x} \cdot \mathbf{y}| = \frac{|\mathbf{y}|^2}{|\mathbf{y}|} = |\mathbf{y}|$$

so the leftmost supremum is also at least $|\mathbf{y}|$, as desired. $\square$

6. If

$$f(x,y) = \begin{cases} 0 & (x,y) = (0,0) \\ \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \end{cases}$$

prove that $(D_1 f)(x,y)$ and $(D_2 f)(x,y)$ exist at every point of $\mathbb{R}^2$, although f is not continuous at $(0,0)$.

Proof. Let $(x,y) \in \mathbb{R}^2$ be arbitrary. We divide into two cases ($(x,y) \neq (0,0)$ and $(x,y) = (0,0)$). If $(x,y) \neq (0,0)$, then as the sum, product, and quotient of linear (hence differentiable) functions, f is

differentiable at (x, y). Therefore, by Theorem 9.17, $(D_1 f)(x, y)$ and $(D_2 f)(x, y)$ exist. On the other hand, if $(x, y) = (0, 0)$, then

$$
\begin{aligned}
(D_j f)(0, 0) &= \lim_{t \to 0} \frac{f(t \mathbf{e}_j) - f(0, 0)}{t} \\
&= \lim_{t \to 0} \frac{\frac{t \cdot 0}{t^2 + 0^2} - 0}{t} \\
&= \lim_{t \to 0} 0 \\
&= 0
\end{aligned}
$$

However, since

$$
f(t, t) = \frac{t^2}{t^2 + t^2} = \frac{1}{2}
$$

for all t (notably arbitrarily small t), we find points $(x, y) \in \mathbb{R}^2$ arbitrarily close to $(0, 0)$ for which $f(x, y) = 1/2$. Therefore, f is not continuous at $(0, 0)$. $\square$

7. Suppose that f is a real-valued function defined in an open set $E \subset \mathbb{R}^n$, and that the partial derivatives $D_1 f, \ldots, D_n f$ are bounded on E. Prove that f is continuous in E. (Hint: Proceed as in the proof of Theorem 9.21.)

Proof. To prove that f is continuous at an arbitrary $\mathbf{x} \in E$, it will suffice to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $\mathbf{x} + \mathbf{h} \in E$ and $|\mathbf{h}| < \delta$, then $|f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since each $D_j f$ is bounded, we know that there exist $M_1, \ldots, M_n$ such that $|D_j f| \leq M_j$ $(j = 1, \ldots, n)$. Define $M = \max M_j$. Choose $\delta = \epsilon/nM$. Let $\mathbf{h}$ be such that $\mathbf{x} + \mathbf{h} \in E$ and $|\mathbf{h}| < \delta$. Supposing $\mathbf{h} = \sum h_j \mathbf{e}_j$, put $\mathbf{v}_0 = \mathbf{0}$ and $\mathbf{v}_k = h_1 \mathbf{e}_1 + \cdots + h_k \mathbf{e}_k$ $(k = 1, \ldots, n)$. Then, letting $\theta_j \in (0, 1)$ for all j, we have that

$$
\begin{aligned}
|f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})| &= \left| \sum_{j=1}^n [f(\mathbf{x} + \mathbf{v}_j) - f(\mathbf{x} + \mathbf{v}_{j-1})] \right| \\
&\leq \sum_{j=1}^n |f(\mathbf{x} + \mathbf{v}_j) - f(\mathbf{x} + \mathbf{v}_{j-1})| \\
&= \sum_{j=1}^n [h_j (D_j f)(\mathbf{x} + \mathbf{v}_{j-1} + \theta_j h_j \mathbf{e}_j)] \qquad \text{MVT} \\
&\leq M \sum_{j=1}^n h_j \\
&\leq nM |\mathbf{h}| \\
&< \epsilon
\end{aligned}
$$

as desired. $\square$

8. Suppose that f is a differentiable real function in an open set $E \subset \mathbb{R}^n$, and that f has a local maximum at a point $\mathbf{x} \in E$. Prove that $f'(\mathbf{x}) = 0$.

Proof. To prove that $f'(\mathbf{x}) = 0$, Theorem 9.17 tells us that it will suffice to show that $(D_j f)(\mathbf{x}) = 0$ for all $1 \leq j \leq n$. Let j be an arbitrary natural number between 1 and n, inclusive. Define $U_j = (\mathbf{x} + \mathrm{span}(\mathbf{e}_j)) \cap E$ and let $\mathbf{x} = (x_1, \ldots, x_n)$. Then by definition, $(D_j f)(\mathbf{x}) = (f|_{U_j})'(x_j)$ and $f|_{U_j}(x_j)$ is a local maximum of $f|_{U_j}$. It follows by Theorem 5.8 that

$$
(D_j f)(\mathbf{x}) = (f|_{U_j})'(x_j) = 0
$$

as desired. $\square$

10. If f is a real function defined in a convex open set $E \subset \mathbb{R}^n$, such that $(D_1 f)(\mathbf{x}) = 0$ for every $\mathbf{x} \in E$, prove that $f(\mathbf{x})$ depends only on $x_2, \dots, x_n$. Show that the convexity of E can be replaced by a weaker condition, but that some condition is required. For example, if $n = 2$ and E is shaped like a horseshoe, the statement may be false.

Proof. To prove that $f(\mathbf{x})$ depends only on $x_2, \dots, x_n$, it will suffice to show that if $\mathbf{x}, \mathbf{y} \in E$ are such that $\mathbf{x} = (x, x_2, \dots, x_n)$ and $\mathbf{y} = (y, x_2, \dots, x_n)$, then $f(\mathbf{x}) = f(\mathbf{y})$. Let $\mathbf{x}, \mathbf{y} \in E$ be arbitrary points that satisfy the previous condition. Since E is convex, $\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in E$ for all $\lambda \in (0, 1)$. Define $U = \{\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} : 0 \le \lambda \le 1\}$. Then since $(f|_U)'(t_1) = (D_1 f)(\mathbf{t}) = 0$ on U, Theorem 5.17b implies that $f|_U$ is constant. In particular, since $\mathbf{x}, \mathbf{y} \in U$, $f(\mathbf{x}) = f(\mathbf{y})$, as desired.

The following weaker condition will suffice: E open is a set such that $\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in E$ for all $0 < \lambda < 1$ and $\mathbf{x}, \mathbf{y} \in E$ *that satisfy* $\mathbf{x} - \mathbf{y} \in \mathrm{span}(\mathbf{e}_1)$. Note that the above proof still works with this condition in place of convexity because the arbitrary $\mathbf{x}, \mathbf{y}$ to which we applied the definition of convexity satisfy $\mathbf{x} - \mathbf{y} \in \mathrm{span}(\mathbf{e}_1)$.

If we do not assert any condition, we may consider as a counterexample the subset of $\mathbb{R}^2$

$$E = ((-2, -1) \times (1, 2)) \cup ((1, 2) \times (1, 2))$$

and the function $f : E \to \mathbb{R}$ defined by

$$f(x, y) = \mathrm{sgn}(x) + y$$

Since $f(x, y) = -1 + y$ for $x \in (-2, -1) \times (1, 2)$ and $f(x, y) = 1 + y$ for $x \in (1, 2) \times (1, 2)$, clearly $(D_1 f)(\mathbf{x}) = 0$ for all $\mathbf{x} \in E$. Yet $f(-1.5, 0) = -1$ and $f(1.5, 0) = 1$. $\qquad \square$

11. If f and g are differentiable real functions in $\mathbb{R}^n$, prove that

$$\nabla(fg) = f\nabla g + g\nabla f$$

and that

$$\nabla \left(\frac{1}{f} \right) = -\frac{\nabla f}{f^2}$$

wherever $f \neq 0$.

Proof. Since, by definition, partial derivatives only take limits over linear subsets of $\mathbb{R}^n$, i.e., ones that are isomorphic to $\mathbb{R}$, they behave like the one-dimensional derivatives of f restricted to such linear subsets. More precisely, $D_j f$ behaves like the derivative of $(f|_{N_r(\mathbf{x}) \cap \mathrm{span}(\mathbf{e}_j)})'$ where $\mathbf{x}$ is the point at which we're taking the derivative and r is taken to be small enough so that $N_r(\mathbf{x})$ is a subset of the open domain of f. In particular, this means that partial derivatives obey the usual sum, product, and quotient rules, as well as other rules carried over from the analysis of functions of a single variable. Thus, we have that for any $\mathbf{x} \in \mathbb{R}^n$,

$$
\begin{aligned}
(\nabla(fg))(\mathbf{x}) &= \sum_{j=1}^{n} (D_j(fg))(\mathbf{x})\mathbf{e}_j \\
&= \sum_{j=1}^{n} (fD_j g + D_j fg)(\mathbf{x})\mathbf{e}_j \\
&= f(\mathbf{x}) \sum_{j=1}^{n} (D_j g)(\mathbf{x})\mathbf{e}_j + g(\mathbf{x}) \sum_{j=1}^{n} (D_j f)(\mathbf{x})\mathbf{e}_j \\
&= f(\mathbf{x})(\nabla g)(\mathbf{x}) + (\nabla f)(\mathbf{x})g(\mathbf{x}) \\
&= (f\nabla g + g\nabla f)(\mathbf{x})
\end{aligned}
$$

and symmetrically for $\nabla(1/f)$. $\qquad \square$

17. Let $\mathbf{f} = (f_1, f_2)$ be the mapping of $\mathbb{R}^2$ into $\mathbb{R}^2$ given by

$$f_1(x, y) = \mathrm{e}^x \cos y \qquad\qquad f_2(x, y) = \mathrm{e}^x \sin y$$

(a) What is the range of f?

Proof. The range of f is $\mathbb{R}^2 \setminus \{\mathbf{0}\}$.

Both trigonometric functions have range $[-1, 1]$ and e^x has range $(0, \infty)$, so we can make e^x arbitrarily large and then take y such that such that the trigonometric function equals 1 or -1. However, since $\cos y = \pm 1$ when $\sin y = 0$ and vice versa, we can never achieve $(0, 0)$. $\qquad\square$

(b) Show that the Jacobian of f is not zero at any point of $\mathbb{R}^2$. Thus, every point of $\mathbb{R}^2$ has a neighborhood in which f is one-to-one. Nevertheless, f is not one-to-one on $\mathbb{R}^2$.

Proof. Let $(x, y) \in \mathbb{R}^2$ be arbitrary. We divide into two cases ($y = n\pi$ and $y \neq n\pi$ $[n \in \mathbb{Z}]$). If $y = n\pi$, then since $(\sin y)' = \cos y$, $D_2 f_2 = \mathrm{e}^x \cos y$, $\mathrm{e}^x \neq 0$ for all x, and $\cos n\pi = \pm 1$ $(n \in \mathbb{Z})$, we have that $(D_2 f_2)(x, y) \neq 0$. If $y \neq n\pi$, then since $(\mathrm{e}^x)' = \mathrm{e}^x$ (Theorem 8.6b), $D_1 f_2 = \mathrm{e}^x \sin y$, $\mathrm{e}^x \neq 0$ for all x, and $\sin y \neq 0$ for all $y \neq n\pi$, we have that $(D_1 f_2)(x, y) \neq 0$.

It follows by Theorem 9.24 that every point of $\mathbb{R}^2$ has a neighborhood in which f is 1-1.

Since the trigonometric functions are periodic with period 2π, $f(0, 0) = f(0, 2\pi)$, for instance. $\quad\square$

(c) Put $\mathbf{a} = (0, \pi/3)$, $\mathbf{b} = f(\mathbf{a})$, and let $\mathbf{g}$ be the continuous inverse of $\mathbf{f}$, defined in a neighborhood of $\mathbf{b}$, such that $\mathbf{g}(\mathbf{b}) = \mathbf{a}$. Find an explicit formula for $\mathbf{g}$, compute $\mathbf{f}'(\mathbf{a})$ and $\mathbf{g}'(\mathbf{b})$, and verify that

$$\mathbf{g}'(\mathbf{b}) = [\mathbf{f}'(\mathbf{g}(\mathbf{b}))]^{-1}$$

Proof. Let $U = \mathbb{R} \times (0, \pi/2)$. Then $V = f(U) = (0, \infty)^2$. Thus, we may define $\mathbf{g} : V \to U$ by

$$g_1(x, y) = \frac{1}{2} \ln\left(x^2 + y^2\right) \qquad\qquad g_2(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$$

Under this definition, we can easily see that if $(x, y) \in U$, then

$$g(f(x, y)) = g(\mathrm{e}^x \cos y, \mathrm{e}^x \sin y)$$

$$= \left(\frac{1}{2} \ln\left((\mathrm{e}^x \cos y)^2 + (\mathrm{e}^x \sin y)^2\right), \tan^{-1}\left(\frac{\mathrm{e}^x \sin y}{\mathrm{e}^x \cos y}\right)\right)$$

$$= \left(\frac{1}{2} \ln\left(\mathrm{e}^{2x}(\cos^2 y + \sin^2 y)\right), \tan^{-1}(\tan y)\right)$$

$$= (x, y)$$

as desired. We can also compute that

$$\mathbf{f}'(\mathbf{a}) = \begin{bmatrix} (D_1 f_1)(\mathbf{a}) & (D_2 f_1)(\mathbf{a}) \\ (D_1 f_2)(\mathbf{a}) & (D_2 f_2)(\mathbf{a}) \end{bmatrix} \qquad\qquad \mathbf{g}'(\mathbf{b}) = \begin{bmatrix} (D_1 f_1)(\mathbf{a}) & (D_2 f_1)(\mathbf{a}) \\ (D_1 f_2)(\mathbf{a}) & (D_2 f_2)(\mathbf{a}) \end{bmatrix}$$

$$= \begin{bmatrix} \mathrm{e}^x \cos y & -\mathrm{e}^x \sin y \\ \mathrm{e}^x \sin y & \mathrm{e}^x \cos y \end{bmatrix} \qquad\qquad = \begin{bmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \qquad\qquad\qquad = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

It follows from the definition of matrix multiplication that

$$\mathbf{g}'(\mathbf{b})\mathbf{f}'(\mathbf{g}(\mathbf{b})) = \mathbf{g}'(\mathbf{b})\mathbf{f}'(\mathbf{a}) = I \qquad\qquad \mathbf{f}'(\mathbf{g}(\mathbf{b}))\mathbf{g}'(\mathbf{b}) = \mathbf{f}'(\mathbf{a})\mathbf{g}'(\mathbf{b}) = I$$

as desired. $\qquad\square$

(d) What are the images under $\mathbf{f}$ of lines parallel to the coordinate axes?

Proof. Let $U \subset \mathbb{R}^2$ denote a line parallel to the x-axis and a distance t away, and let $V \subset \mathbb{R}^2$ denote a line parallel to the y-axis and a distance t away. Formally,

$$U = \{(t, y) : y \in \mathbb{R}\} \qquad\qquad V = \{(x, t) : x \in \mathbb{R}\}$$

where $t \in \mathbb{R}$.

We first describe $\mathbf{f}(U)$. Since $x = t$ for all $(x, y) \in U$, $\mathrm{e}^x = \mathrm{e}^t$. Since y ranges over $\mathbb{R}$ as (x, y) ranges over U, the image of U under $\cos y$ is $[-1, 1]$. Similarly, the image of U under $\sin y$ is $[-1, 1]$. Thus, $\mathbf{f}(U) = (f_1(U), f_2(U))$ where the two components are given by

$$f_1(U) = [-\mathrm{e}^t, \mathrm{e}^t] \qquad\qquad f_2(U) = [-\mathrm{e}^t, \mathrm{e}^t]$$

We now describe $\mathbf{f}(V)$. Since x ranges over $\mathbb{R}$ as (x, y) ranges over V, the image of V under e^x is $(0, \infty)$. Since $y = t$ for all $(x, y) \in U$, $\cos y = 0$ if $t = \frac{\pi}{2} + n\pi$, $\cos y > 0$ if $t \in (-\frac{\pi}{2} + 2\pi n, \frac{\pi}{2} + 2\pi n)$, and $\cos y < 0$ if $t \in (\frac{\pi}{2} + 2\pi n, \frac{3\pi}{2} + 2\pi n)$ $(n \in \mathbb{Z})$. Similarly, $\sin y = 0$ if $t = n\pi$, $\sin y > 0$ if $t \in (2n\pi, (2n+1)\pi)$, and $\sin y < 0$ if $t \in ((2n-1)\pi, 2n\pi)$ $(n \in \mathbb{Z})$. Thus, $\mathbf{f}(V) = (f_1(V), f_2(V))$ where the two components are given by

$$f_1(V) = \begin{cases} \{0\} & t = \frac{\pi}{2} + n\pi \\ (0, \infty) & -\frac{\pi}{2} + 2\pi n < t < \frac{\pi}{2} + 2\pi n \\ (0, -\infty) & \frac{\pi}{2} + 2\pi n < t < \frac{3\pi}{2} + 2\pi n \end{cases} \qquad f_2(V) = \begin{cases} \{0\} & t = n\pi \\ (0, \infty) & 2n\pi < t < (2n+1)\pi \\ (0, -\infty) & (2n-1)\pi < t < 2n\pi \end{cases}$$

where $n \in \mathbb{Z}$ in every occurrence. $\square$

7   Functions of Several Variables III

From Rudin (1976).

Chapter 9

3/2:   **9.** If $\mathbf{f}$ is a differentiable mapping of a *connected* open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, and if $\mathbf{f}'(\mathbf{x}) = 0$ for every $\mathbf{x} \in E$, prove that $\mathbf{f}$ is constant in E.

Proof. Suppose for the sake of contradiction that $\mathbf{f}$ is not constant on E. Then there exist $\mathbf{a}, \mathbf{b} \in E$ such that $\mathbf{f}(\mathbf{a}) \neq \mathbf{f}(\mathbf{b})$. Let

$$A = \mathbf{f}^{-1}(\{\mathbf{f}(\mathbf{a})\}) \qquad\qquad B = E \setminus A$$

Since $\mathbf{a} \in A$ and $\mathbf{b} \in B$, A, B are nonempty. Additionally, we clearly have that $E = A \cup B$ and $A \cup B = \emptyset$. Furthermore, since A is closed by Theorem 4.8, Theorem 2.27b implies that $\bar{A} = A$ and hence

$$\bar{A} \cap B = A \cap B = \emptyset$$

These results combined with the fact that E is connected by hypothesis implies that $A \cap \bar{B} \neq \emptyset$. Let $\mathbf{x} \in A \cap \bar{B}$. It follows by the definition of the closure of B and since $A \cap B = \emptyset$ that $\mathbf{x} \in A \cap LP(B)$, where $LP(B)$ denotes the set of all limit points of B. Now since E is open, there exists $N_r(\mathbf{x}) \subset E$. Thus, since $\mathbf{x} \in LP(B)$, there exists $\mathbf{c} \in N_r(\mathbf{x})$ such that $\mathbf{c} \in B$. With $\mathbf{x} \in A$ and $\mathbf{c} \in B$, we have that $\mathbf{f}(\mathbf{c}) \neq \mathbf{f}(\mathbf{a})$. But $N_r(\mathbf{x})$ is convex, so since $\mathbf{f}'(\mathbf{x}) = 0$ for all points in $N_r(\mathbf{x}) \subset E$, we have by the Corollary to Theorem 9.19 that $\mathbf{f}$ is constant on $N_r(\mathbf{x})$, a contradiction. $\qquad\square$

13. Suppose $\mathbf{f}$ is a differentiable mapping of $\mathbb{R}^1$ into $\mathbb{R}^3$ such that $|\mathbf{f}(t)| = 1$ for every t. Prove that $\mathbf{f}'(t) \cdot \mathbf{f}(t) = 0$. Interpret this result geometrically.

Proof. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(t) = |\mathbf{f}(t)|^2 - 1$$

for all $t \in \mathbb{R}$. Since g is a constant function by definition, $g' = 0$. Additionally, since

$$1 = |\mathbf{f}(t)| = |\mathbf{f}(t)|^2 = \sum_{i=1}^{3} [f_i(t)]^2$$

we have by the sum and the chain rule that

$$g'(t) = \sum_{i=1}^{3} 2 f_i(t) f_i'(t) - 0 = 2\mathbf{f}(t) \cdot \mathbf{f}'(t)$$

for all $t \in \mathbb{R}$. But this implies by the transitive property that

$$\mathbf{f}(t) \cdot \mathbf{f}'(t) = 0$$

as desired, where we have incorporated the 2 into the zero.

Geometrically, this means that if a path is drawn along the surface of the unit sphere, the tangent to the path (which will be tangent to the unit sphere) is perpendicular to the normal (i.e., the radial position vector). $\qquad\square$

14. Define

$$f(x, y) = \begin{cases} 0 & (x, y) = (0, 0) \\ \frac{x^3}{x^2 + y^2} & (x, y) \neq (0, 0) \end{cases}$$

(a) Prove that $D_1 f$ and $D_2 f$ are bounded functions in $\mathbb{R}^2$. (Hence f is continuous.)

Proof. We have by the rules of derivatives that

$$D_1 f(x,y) = \frac{(x^2+y^2)\cdot 3x^2 - x^3 \cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{x^4 + 3x^2 y^2}{(x^2+y^2)^2}$$

$$D_2 f(x,y) = \frac{(x^2+y^2)\cdot 0 + x^3 \cdot 2y}{(x^2+y^2)}$$

$$= \frac{2x^3 y}{(x^2+y^2)^2}$$

or, in polar coordinates,

$$= \frac{r^4 \cos^4 \theta + 3r^2 \cos^2 \theta r^2 \sin^2 \theta}{r^4}$$

$$= \cos^4 \theta + 3\cos^2 \theta \sin^2 \theta$$

$$= \frac{2r^3 \cos^3 \theta r \sin \theta}{r^4}$$

$$= 2\cos^3 \theta \sin \theta$$

Since cosine and sine are bounded, their products above are bounded. □

(b) Let $\mathbf{u}$ be any unit vector in $\mathbb{R}^2$. Show that the directional derivative $(D_{\mathbf{u}}f)(0,0)$ exists, and that its absolute value is at most 1.

Proof. Let $\mathbf{u} = (r,\theta)$ in polar coordinates. Then $D_{\mathbf{u}}f$ becomes

$$\frac{\mathrm{d}f}{\mathrm{d}r} = \cos^3 \theta$$

which is indeed bounded by $[-1,1]$. □

(c) Let γ be a differentiable mapping of $\mathbb{R}^1$ into $\mathbb{R}^2$ (in other words, γ is a differentiable curve in $\mathbb{R}^2$), with $\gamma(0) = (0,0)$ and $|\gamma'(0)| > 0$. Put $g(t) = f(\gamma(t))$ and prove that g is differentiable for every $t \in \mathbb{R}^1$. If $\gamma \in C^1$, prove that $g \in C^1$.

Proof. We divide into two cases ($t \neq 0$ and $t = 0$). If $t \neq 0$, then $\gamma(t) \neq \mathbf{0}$, so g is differentiable by Theorem 9.15. If $t = 0$, then let $\mathbf{u} = \gamma'(0)$. It follows by part (b) that $D_{\mathbf{u}}f(0,0)$ exists, i.e., $g'(0)$ exists. □

(d) In spite of this, prove that f is not differentiable at $(0,0)$. (Hint: The formula $(D_{\mathbf{u}}f)(\mathbf{x}) = \sum_{i=1}^{n}(D_i f)(\mathbf{x})u_i$ fails.)

Proof. Let $\mathbf{u} = (u_1, u_2)$ in Cartesian coordinates. Then

$$\mathbf{u} = \left(\sqrt{u_1^2 + u_2^2}, \arctan(u_2/u_1) \right)$$

in polar coordinates. It follows that

$$(D_{\mathbf{u}}f)(0,0) = \cos(\arctan(u_2/u_1))$$

$$= \frac{u_1}{\sqrt{u_1^2 + u_2^2}}$$

However, $\sum_{i=1}^{2}(D_i f)(0,0)u_i$ is of indeterminate form, so $\mathbf{f}$ is not differentiable by the contrapositive of Theorem 9.17. □

16. Show that the continuity of $\mathbf{f}'$ at the point $\mathbf{a}$ is needed in the inverse function theorem, even in the case $n = 1$: If

$$f(t) = \begin{cases} t + 2t^2 \sin \frac{1}{t} & t \neq 0 \\ 0 & t = 0 \end{cases}$$

then $f'(0) = 1$ and f' is bounded in $(-1,1)$, but f is not one-to-one in any neighborhood of 0.

Proof. We have previously established that the derivative of $t^2 \sin(1/t)$ is zero at $t = 0$. Thus, by the rules of derivatives, $f'(0) = 1 + 2 \cdot 0 = 1$ and for $t \neq 0$,

$$f'(t) = 1 + 4t \sin \frac{1}{t} + 2t^2 \cos \frac{1}{t} \cdot -\frac{1}{t^2}$$
$$= 1 + 4t \sin \frac{1}{t} - 2 \cos \frac{1}{t}$$

which is clearly bounded on any bounded interval as the sum of a constant function, the product of a constantly increasing function and a function bounded on the real line, and a function bounded on the real line. However, consider an arbitrary neighborhood $N_r(0)$. Choose $n \in \mathbb{N}$ such that $1/n < r$. Then $1/n\pi < r$. It follows that

$$f'(1/n\pi) = 1 + \frac{4}{n\pi} \sin n\pi - 2 \cos n\pi = 1 + 2 \cdot \pm 1$$

Thus, either $f'(1/n\pi) = -1$ and $f'(1/(n+1)\pi) = 3$ or vice versa. Either way, for some x on that interval, $f'(x) = 0$, meaning that there is a local maximum at x, meaning that f is not 1-1 in the vicinity of this maximum, i.e., on $N_r(0)$. $\qquad\square$

References

Rudin, W. (1976). *Principles of mathematical analysis* (A. A. Arthur & S. L. Langman, Eds.; Third). McGraw-Hill.