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February 9, 2022

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1 Differentiation

From Rudin (1976).

Chapter 5

1. Let f be defined for all real x, and suppose that

$$|f(y) - f(x)| \le (y - x)^2$$

for all real x and y. Prove that f is constant.

Proof. To prove that f is constant, Theorem 5.11b tells us that it will suffice to show that f is differentiable on \mathbb{R} with derivative f'=0. Let $x\in\mathbb{R}$ be arbitrary. We want to show that for all $\epsilon>0$, there exists a δ such that if $y\in\mathbb{R}$ and $0<|y-x|<\delta$, then $|[f(y)-f(x)]/(y-x)-0|<\epsilon$. Let ϵ be arbitrary. Choose $\delta=\epsilon$. Then we have that

$$\left| \frac{f(y) - f(x)}{y - x} - 0 \right| = \frac{|f(y) - f(x)|}{|y - x|}$$

$$\leq \frac{(y - x)^2}{|y - x|}$$

$$\leq |y - x|$$

$$< \epsilon$$

as desired. \Box

2. Suppose f'(x) > 0 in (a, b). Prove that f is strictly increasing in (a, b) and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)}$$

for a < x < b.

Proof. To prove that f is strictly increasing on (a,b), it will suffice to show that x < y implies f(x) < f(y) for all $x, y \in (a,b)$. Let $x,y \in (a,b)$ satisfy x < y. Since f is differentiable on (a,b), it is differentiable on $(x,y) \subset (a,b)$ and (by Theorem 5.2) continuous on $[x,y] \subset (a,b)$. Thus, by the MVT, there exists $c \in (x,y)$ such that

$$f(y) - f(x) = (y - x)f'(c)$$

But since x < y, y - x > 0. This combined with the fact that f'(c) > 0 by definition implies that (y - x)f'(c) > 0. Consequently,

$$f(x) < f(x) + (y - x)f'(c) = f(y)$$

as desired.

Since f is strictly increasing (and hence 1-1) on (a, b), we may construct a well-defined inverse function $g: f[(a, b)] \to (a, b)$ for it by

$$g(f(x)) = x$$

for all $f(x) \in f[(a,b)]$. It follows by the fact that f'(x) > 0 for all $x \in (a,b)$, the definitions of f'(x) and g'(f(x)), and Theorem 3.3d that

$$\frac{1}{f'(x)} = \frac{1}{\lim_{y \to x} \frac{f(y) - f(x)}{y - x}}$$

$$= \lim_{y \to x} \frac{1}{\frac{f(y) - f(x)}{y - x}}$$

$$= \lim_{y \to x} \frac{y - x}{f(y) - f(x)}$$

$$= \lim_{y \to x} \frac{g(f(y)) - g(f(x))}{f(y) - f(x)}$$

$$= g'(f(x))$$

as desired.

3. Suppose g is a real function on \mathbb{R}^1 , with bounded derivative (say $|g'| \leq M$). Fix $\epsilon > 0$ and define $f(x) = x + \epsilon g(x)$. Prove that f is one-to-one if ϵ is small enough. (A set of admissable values of ϵ can be determined which depends only on M.)

Proof. Neglecting the trivial case where M=0, take $\epsilon=1/2M$. It follows that

$$0 < 1 - \frac{1}{2}$$

$$= 1 + \frac{1}{2M} \cdot -M$$

$$\leq 1 + \epsilon g'(x)$$

$$= \frac{d}{dx}(x) + \frac{d}{dx}(\epsilon g)$$

$$= f'(x)$$

Therefore, by Problem 5.2, f is strictly increasing and, hence, one-to-one.

4. If

$$C_0 + \frac{C_1}{2} + \dots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0$$

where C_0, \ldots, C_n are real constants, prove that the equation

$$C_0 + C_1 x + \dots + C_{n-1} x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 and 1.

Proof. Consider the polynomial

$$f(x) = C_0 x + \frac{C_1}{2} x^2 + \dots + \frac{C_n}{n+1} x^{n+1}$$

We have that f(0) = 0 (by direct substitution) and f(1) = 0 (by the constraint on the coefficients). Thus, since f is continuous on [0,1] and differentiable on (0,1) (as a polynomial), we have by the MVT that there exists $x \in (0,1)$ such that

$$f(1) - f(0) = (1 - 0)f'(x)$$
$$f'(x) = 0$$
$$C_0 + C_1 x + \dots + C_{n-1} x^{n-1} + C_n x^n = 0$$

as desired. \Box

5. Suppose f is defined and differentiable for every x > 0, and $f'(x) \to 0$ as $x \to +\infty$. Put g(x) = f(x+1) - f(x). Prove that $g(x) \to 0$ as $x \to +\infty$.

Proof. To prove that $\lim_{x\to\infty}g(x)=0$, it will suffice to show that for every $\epsilon>0$, there exists N>0 such that if x>N, then $|g(x)-0|<\epsilon$. Let $\epsilon>0$ be arbitrary. Since $\lim_{x\to\infty}f'(x)=0$ by hypothesis, we know that there exists N>0 such that if x>N, then $|f'(x)|<\epsilon$. Choose this N to be our N. Let x>N be arbitrary. Applying the MVT to f on the interval [x,x+1] proves the existence of a c within that closed interval such that

$$f(x+1) - f(x) = f'(c)(x+1-x) = f'(c)$$

Additionally, since c > x > N, we have that $|f'(c)| < \epsilon$. Therefore, we have that

$$|g(x)| = |f(x+1) - f(x)|$$
$$= |f'(c)|$$
$$< \epsilon$$

as desired. \Box

2 Differentiation II / Integration

From Rudin (1976).

Chapter 5

8. Suppose f' is continuous on [a,b] and $\epsilon>0$. Prove that there exists $\delta>0$ such that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \epsilon$$

whenever $0 < |t-x| < \delta$, $a \le x \le b$, $a \le t \le b$. (This could be expressed by saying that f is **uniformly** differentiable on [a, b] if f' is continuous on [a, b].) Does this hold for vector-valued functions, too?

Proof. By Theorem 2.40, [a,b] is compact. This combined with the fact that f' is continuous implies by Theorem 4.19 that f' is uniformly continuous. Thus, there exists $\delta > 0$ such that if $x,y \in [a,b]$ and $|y-x| < \delta$, then $|f'(y) - f'(x)| < \epsilon$. Choose this δ to be our δ . Let $x,t \in [a,b]$ be such that $0 < |t-x| < \delta$. Then since f is continuous on $[t,x] \subset [a,b]$ and differentiable on $(t,x) \subset [a,b]$, we have by the MVT that there exists $c \in (t,x)$ such that

$$f(t) - f(x) = (t - x)f'(c)$$
$$\frac{f(t) - f(x)}{t - x} = f'(c)$$

Additionally, since t < c < x and $|t - x| < \delta$, we must have $|c - x| < \delta$, meaning that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| = |f'(c) - f'(x)| < \epsilon$$

as desired.

And yes, this does hold for vector-valued functions, which we can treat component-wise.

17. Suppose f is a real, three times differentiable function on [-1,1] such that

$$f(-1) = 0$$
 $f(0) = 0$ $f(1) = 1$ $f'(0) = 0$

Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1,1)$. Note that equality holds for $\frac{1}{2}(x^3 + x^2)$. (Hint: Use Theorem 5.15 with $\alpha = 0$ and $\beta = \pm 1$ to show that there exist $s \in (0,1)$ and $t \in (-1,0)$ such that $f^{(3)}(s) + f^{(3)}(t) = 6$.)

Proof. Since f is three times differentiable on [-1,1], we know that f'' is differentiable on [-1,1]. It follows by Theorem 5.2 that f'' is continuous on [-1,1]. Thus, since f is defined on [-1,1], $f \in \mathbb{N}$, f'' is continuous on [-1,1], $f^{(3)}$ is defined on (-1,1), $0,1 \in [-1,1]$ such that $0 \neq 1$, and we can define

$$P(t) = \sum_{k=0}^{2} \frac{f^{(k)}(0)}{k!} (t-0)^{k}$$

we have by Taylor's theorem that there exists $s \in (0,1)$ such that

$$f(1) = P(1) + \frac{f^{(3)}(s)}{3!} (1 - 0)^3$$

$$1 - \left[\frac{f(0)}{0!} (1 - 0)^0 + \frac{f'(0)}{1!} (1 - 0)^1 + \frac{f''(0)}{2!} (1 - 0)^2 \right] = \frac{f^{(3)}(s)}{3!}$$

$$1 - \left[\frac{f''(0)}{2} \right] = \frac{f^{(3)}(s)}{6}$$

$$6 - 3f''(0) = f^{(3)}(s)$$

Similarly, we have that there exists $t \in (-1,0)$ such that

$$f(-1) = P(-1) + \frac{f^{(3)}(t)}{3!}(-1 - 0)^3$$

$$0 - \left[\frac{f(0)}{0!}(-1 - 0)^0 + \frac{f'(0)}{1!}(-1 - 0)^1 + \frac{f''(0)}{2!}(-1 - 0)^2\right] = -\frac{f^{(3)}(t)}{3!}$$

$$-\left[\frac{f''(0)}{2}\right] = -\frac{f^{(3)}(t)}{6}$$

$$3f''(0) = f^{(3)}(s)$$

Thus,

$$f^{(3)}(s) + f^{(3)}(t) = 3f''(0) + 6 - 3f''(0) = 6$$

Now suppose for the sake of contradiction that for all $x \in (-1,1)$, we have $f^{(3)}(x) < 3$. Then $f^{(3)}(s) < 3$ and $f^{(3)}(t) < 3$. It follows that $f^{(3)}(s) + f^{(3)}(t) < 6$, a contradiction.

- **25.** Suppose f is twice differentiable on [a, b], f(a) < 0, f(b) > 0, $f'(x) \ge \delta > 0$, and $0 \le f''(x) \le M$ for all $x \in [a, b]$. Let ξ be the unique point in (a, b) at which $f(\xi) = 0$. Complete the details in the following outline of **Newton's method** for computing ξ .
 - (a) Choose $x_1 \in (\xi, b)$ and define $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Interpret this geometrically, in terms of a tangent to the graph of f.

Answer. Since we can rearrange the above to $0 - f(x_n) = f'(x_n)(x_{n+1} - x_n)$, we know that x_{n+1} is the point at which the tangent to f at x_n crosses the x-axis. In other words, the zero of the tangent line

$$y - f(x_n) = f'(x_n)(x - x_n)$$

(b) Prove that $x_{n+1} < x_n$ and that

is $(x_{n+1}, 0)$.

$$\lim_{n\to\infty} x_n = \xi$$

Proof. To prove that $x_{n+1} < x_n$, it will suffice to show that $f(x_n), f'(x_n) > 0$. Since f'(x) > 0 for all $x \in [a,b]$ by hypothesis, we know that $f'(x_n) > 0$. As to $f(x_n)$, suppose for the sake of contradiction that $f(x_n) \le 0$. We know that $f(\xi) = 0$, f(b) > 0, and $\xi < x_n < b$. Since ξ is the *unique* point at which $f(\xi) = 0$ by hypothesis and $x_n \ne \xi$, we know that $f(x_n) \ne 0$. And if $f(x_n) < 0$, we have by the intermediate value theorem for f continuous that there exists $c \in (x_n, b)$ such that f(c) = 0. But since $\xi < x_n < c < b$, $c \ne \xi$, and thus we have a contradiction here, too.

Having established that $\{x_n\}$ is a monotonically decreasing sequence, Theorem 3.14 tells us that to show that it converges, it will suffice to show that it is bounded. Clearly, $\{x_n\}$ is bounded above by x_1 . And on the bottom side, $\{x_n\}$ is bounded by ξ : If there were $x_n < \xi$, this would imply that $f(x_n) < 0$ by a symmetric argument to the above, meaning that $f(x_n)/f'(x_n) < 0$ and implying that $x_{n+1} > x_n$, a contradiction. Furthermore, we know that the limit (call it x) equals ξ since

$$x = x - \frac{f(x)}{f'(x)}$$
$$f(x) = 0$$

so $x = \xi$ by the uniqueness of ξ .

(c) Use Taylor's theorem to show that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

for some $t_n \in (\xi, x_n)$.

Proof. Since f is defined on [a,b], $2 \in \mathbb{N}$, f' is continuous on [a,b], f'' is defined on (a,b), $\xi, x_n \in [a,b]$ with $\xi \neq x_n$, and

$$P(t) = \sum_{k=0}^{1} \frac{f^{(k)}(x_n)}{k!} (t - x_n)^k$$

we have by Taylor's theorem that there exists $t_n \in (\xi, x_n)$ such that

$$f(\xi) = \left[\frac{f(x_n)}{0!} (\xi - x_n)^0 + \frac{f'(x_n)}{1!} (\xi - x_n)^1 \right] + \frac{f''(t_n)}{2!} (\xi - x_n)^2$$

$$0 = f(x_n) - f'(x_n)(x_n - \xi) + \frac{f''(t_n)}{2} (x_n - \xi)^2$$

$$x_n - \frac{f(x_n)}{f'(x_n)} - \xi = \frac{f''(t_n)}{2f'(x_n)} (x_n - \xi)^2$$

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)} (x_n - \xi)^2$$

as desired.

(d) If $A = M/2\delta$, deduce that

$$0 \le x_{n+1} - \xi \le \frac{1}{A} [A(x_1 - \xi)]^{2n}$$

(Compare with Chapter 3, Exercises 16 and 18.)

Proof. We have from part (b) that $x_i > \xi$ for all $i \in \mathbb{N}$, so naturally $0 \le x_{n+1} - \xi$. As to the other part of the question, we induct on n. For the base case n = 1, we have that

$$x_{2} - \xi = \frac{f''(t_{1})}{2f'(x_{1})}(x_{1} - \xi)^{2}$$

$$\leq \frac{M}{2\delta}(x_{1} - \xi)^{2}$$

$$= \frac{2\delta}{M} \left[\frac{M}{2\delta}(x_{1} - \xi) \right]^{2}$$

$$= \frac{1}{4} [A(x_{1} - \xi)]^{2 \cdot 1}$$

Now suppose inductively that we have proven the claim for n-1; we now seek to prove it for n. Indeed, we have that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)} (x_n - \xi)^2$$

$$\leq \frac{M}{2\delta} (x_n - \xi)^2$$

$$\leq A \left(\frac{1}{A} [A(x_1 - \xi)]^{2(n-1)} \right)^2$$

$$= \frac{1}{A} [A(x_1 - \xi)]^{2n}$$

as desired. \Box

(e) Show that Newton's method amounts to finding a fixed point of the function g defined by

$$g(x) = x - \frac{f(x)}{f'(x)}$$

How does g'(x) behave for x near ξ ?

Proof. A fixed point of the function g is a point x such that

$$g(x) = x$$
$$x - \frac{f(x)}{f'(x)} = x$$
$$f(x) = 0$$

Thus, if we want to find a point x where f(x) = 0, it is equivalent to find a point x such that g(x) = x.

As to the other part of the question, we have by the rules of derivatives that

$$g'(x) = 1 - \frac{f'(x)f'(x) - f(x)f''(x)}{f'(x)^2}$$
$$= \frac{f(x)f''(x)}{f'(x)^2}$$
$$\leq \frac{M}{\delta^2}f(x)$$

Thus, since $f(x) \to 0$ as $x \to \xi$, $g'(x) \to 0$ as $x \to \xi$.

(f) Put $f(x) = \sqrt[3]{x}$ on $(-\infty, \infty)$ and try Newton's method. What happens?

Answer. We have by the power rule that

$$f'(x) = \frac{1}{3x^{2/3}}$$

Choose $x_1 = 1$. Then

$$x_2 = 1 - \frac{f(1)}{f'(1)} = -2$$

$$x_3 = 1 - \frac{f(-2)}{f'(-2)} = 7$$

$$x_4 = 1 - \frac{f(7)}{f'(7)} = -20$$

$$\vdots$$

It appears that the series is diverging to ∞ while alternating from positive to negative. In fact, since $x_3 > x_2$, contrary to part (b), we know that something must be wrong (i.e., one of our hypotheses must not be met). Upon further investigation, we can determine that on [-1,1], we have f''(1) = -2/9 < 0; thus, our last hypothesis is the issue with this function.

Chapter 6

1. Suppose α increases on [a,b], $a \leq x_0 \leq b$, α is continuous at x_0 , $f(x_0) = 1$, and f(x) = 0 if $x \neq x_0$. Prove that $f \in \mathcal{R}(\alpha)$ and that $\int f \, d\alpha = 0$.

Proof. Since f is bounded on [a,b] with only one discontinuity on [a,b] and α is continuous at the point at which f is discontinuous, Theorem 6.10 implies that $f \in \mathcal{R}(\alpha)$, as desired. It follows that inf $U(P,f,\alpha) = \sup L(P,f,\alpha) = \int f \, d\alpha$. But since $L(P,f,\alpha) = 0$ for all P (there is no infinite interval $[x_i,x_{i+1}] \subset [a,b]$ that does not contain 0, and f is bounded below by 0), we know that

$$\int f \, \mathrm{d}\alpha = \sup L(P, f, \alpha) = 0$$

as desired. \Box

2. Suppose $f \ge 0$, f is continuous on [a, b], and $\int_a^b f(x) dx = 0$. Prove that f(x) = 0 for all $x \in [a, b]$. (Compare this with Exercise 1.)

Proof. Suppose for the sake of contradiction that $f(x) \neq 0$ for some x. By the definition of f, this must mean that f(x) > 0. It follows since f is continuous that there exists some $N_r(x)$ such that f(y) > 0 for all $y \in N_r(x)$. Now consider the partition

$$P = \{a, x - r/2, x + r/2, b\}$$

of [a, b]. But since $m_2 > 0$, we have that

$$0 < m_1[(x-r/2)-a] + m_2[(x+r/2)-(x-r/2)] + m_3[b-(x+r/2)]$$

$$= L(P,f)$$

$$\leq \int_a^b f(x) dx$$
 Theorem 6.4

a contradiction. \Box

4. If f(x) = 0 for all irrational x and f(x) = 1 for all rational x, prove that $f \notin \mathcal{R}$ on [a, b] for any a < b.

Proof. Let P be an arbitrary partition of [a, b]. Since the rationals and irrationals are dense in the reals, we know that for any $[x_i, x_{i+1}]$, f(x) = 0 for some $x \in [x_i, x_{i+1}]$ and f(x) = 1 for some $x \in [x_i, x_{i+1}]$. Thus, we have that L(P, f) = 0 and U(P, f) = b - a. It follows that if a < b,

$$\sup L(P, f) = 0 \neq b - a = \inf U(P, f)$$

so $f \notin \mathcal{R}$, as desired.

3 Integration II

From Rudin (1976).

Chapter 6

2/2: **3.** Define three functions $\beta_1, \beta_2, \beta_3$ as follows:

$$\beta_1 = \begin{cases} 0 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases} \qquad \beta_2 = \begin{cases} 0 & x < 0 \\ 1 & x = 0 \\ 1 & x > 0 \end{cases} \qquad \beta_3 = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases}$$

Let f be a bounded function on [-1,1].

(a) Prove that $f \in \mathcal{R}(\beta_1)$ if and only if f(0+) = f(0) and that then

$$\int f \, \mathrm{d}\beta_1 = f(0)$$

Proof. Suppose first that $f \in \mathcal{R}(\beta_1)$ with $\int f \, \mathrm{d}\beta_1 = f(0)$. To prove that f(0+) = f(0), it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in [-1,1]$ and $0 \le x < \delta$, then $|f(x) - f(0)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $f \in \mathcal{R}(\beta_1)$ by hypothesis, we have by Theorem 6.6 that there exists a partition P of [-1,1] such that $U(P,f,\beta_1) - L(P,f,\beta_1) < \epsilon$. Now let $x_i = \min\{x \in P : x > 0\}$; we know that such an object exists since there exist elements of P greater than zero (namely 1) and P is finite. It follows by the definition of β_1 that $\Delta x_i = 1$ and $\Delta x_j = 0$ for $j \ne i$. Thus, $U(P,f,\beta_1) = M_i$ and $L(P,f,\beta_1) = m_i$ (which exist because f is bounded on [-1,1]). At this point, we are ready to choose δ , which we take to be $\delta = x_i$. Now to confirm that this δ works: Let $0 \le x < \delta$. By the definition of $x_i, x_{i-1}, m_i \le f(x) \le M_i$ and $m_i \le f(0) \le M_i$. But since $M_i - m_i < \epsilon$ as per the above, we have that $|f(x) - f(0)| < \epsilon$, as desired.

Now suppose that f(0+)=f(0). To prove that $f\in \mathcal{R}(\beta_1)$, Theorem 6.6 tells us that it will suffice to show that for every $\epsilon>0$, there exists a P such that $U(P,f,\beta_1)-L(P,f,\beta_2)<\epsilon$. Let $\epsilon>0$ be arbitrary. Since f(0+)=f(0), we know that there exists a $\delta'>0$ such that if $x\in[-1,1]$ and $0\leq x<\delta'$, then $|f(x)-f(0)|<\epsilon/3$. Let $\delta=\min(\delta'/2,1)$. Thus, we may define $P=\{-1,0,\delta,1\}$. We have

$$U(P, f, \beta_1) = \sum_{i=1}^{3} M_i \Delta \beta_{1_i}$$

$$= M_2$$

$$L(P, f, \beta_1) = \sum_{i=1}^{3} m_i \Delta \beta_{1_i}$$

$$= m_2$$

(which exist because f is bounded on [-1,1]). Consequently, $M_2 \le f(0) + \epsilon/3$. $m_2 \ge f(0) - \epsilon/3$. Therefore,

$$U(P, f, \beta_1) - L(P, f, \beta_1) = M_2 - m_2$$

$$\leq [f(0) + \frac{\epsilon}{3}] - [f(0) - \frac{\epsilon}{3}]$$

$$= \frac{2\epsilon}{3}$$

$$< \epsilon$$

as desired.

As to proving that $\int f d\beta_1$, we know that $M_2 \leq f(0) + \epsilon/3$ for arbitrarily small ϵ implies $M_2 \leq f(0)$. Similarly, $m_2 \geq f(0)$. Thus,

$$\inf U(P, f, \beta_1) \le U(P, f, \beta_1) = M_2 \le f(0) \le m_2 = L(P, f, \beta_1) \le \sup L(P, f, \beta_1)$$

But by Theorem 6.5, $\sup L(P, f, \beta_1) \leq \inf U(P, f, \beta_1)$. Therefore,

$$\int_{-1}^{1} f \, d\beta_1 = \sup L(P, f, \beta_1) = \inf U(P, f, \beta_1) = f(0)$$

as desired. \Box

(b) State and prove a similar result for β_2 .

Proof. The result will be $f \in \mathcal{R}(\beta_2)$ if and only if f(0-) = f(0) and that then

$$\int f \, \mathrm{d}\beta = f(0)$$

The proof of this result is entirely symmetric to the proof of the previous result.

(c) Prove that $f \in \mathcal{R}(\beta_3)$ if and only if f is continuous at 0.

Proof. Suppose first that $f \in \mathcal{R}(\beta_3)$. To prove that f is continuous at 0, it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $x \in [-1,1]$ and $|x| < \delta$, then $|f(x) - f(0)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $f \in \mathcal{R}(\beta_3)$ by hypothesis, we have by Theorem 6.6 that there exists a partition P of [-1,1] such that $U(P,f,\beta_3) - L(P,f,\beta_3) < \epsilon/2$. Now let $x_i = \max\{x \in P : x < 0\}$ and let $x_j = \min\{x \in P : x > 0\}$. Choose $\delta = \min\{|x_i|, |x_j|\}$. Let $P^* = P \cup \{-\delta, 0, \delta\}$ be a refinement of P. It follows by the definition of β_3 and a reenumeration of P^* that $U(P^*, f, \beta_3) = (M_{i-1} + M_i)/2$ and $L(P^*, f, \beta_3) = (m_{i-1} + m_i)/2$. Now let $|x| < \delta$. We divide into two cases $(x \ge 0)$ and $(x \le 0)$. If $(x \ge 0)$, then $(x \ge 0)$ and $(x \le 0)$ and $(x \ge 0)$ and (x

$$|f(x) - f(0)| \le M_i - m_i$$

$$\le (M_{i-1} - m_{i-1}) + (M_i - m_i)$$

$$= 2 \left[\frac{M_{i-1} + M_i}{2} - \frac{m_{i-1} + m_i}{2} \right]$$

$$= 2[U(P^*, f, \beta_3) - L(P^*, f, \beta_3)]$$

$$< \epsilon$$

as desired. The proof is symmetric in the other case.

Now suppose that f is continuous at 0. To prove that $f \in \mathcal{R}(\beta_3)$, Theorem 6.6 tells us that it will suffice to show that for every $\epsilon > 0$, there exists a P such that $U(P, f, \beta_3) - L(P, f, \beta_3) < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since f is continuous at 0, we know that there exists a $\delta' > 0$ such that if $x \in [-1, 1]$ and $|x| < \delta'$, then $|f(x) - f(0)| < \epsilon/3$. Choose $\delta = \min(\delta'/2, 1)$. Consider $P = \{-1, -\delta/2, \delta/2, 1\}$. It follows as before that $U(P, f, \beta_3) = M_2$ and $L(P, f, \beta_3) = m_2$. Consequently, $M_2 \le f(0) + \epsilon/3$ and $m_2 \ge f(0) - \epsilon/3$. Therefore,

$$U(P, f, \beta_3) - L(P, f, \beta_3) = M_2 - m_2$$

$$\leq [f(0) + \frac{\epsilon}{3}] - [f(0) - \frac{\epsilon}{3}]$$

$$= \frac{2\epsilon}{3}$$

$$< \epsilon$$

as desired. \Box

(d) If f is continuous at 0, prove that

$$\int f \, \mathrm{d}\beta_1 = \int f \, \mathrm{d}\beta_2 = \int f \, \mathrm{d}\beta_3 = f(0)$$

Proof. If f is continuous at 0, then f(0+) = f(0) = f(0-). It follows that

$$f(0) = \int f \, \mathrm{d}\beta_1$$
 Part (a)

$$= \int f \, \mathrm{d}\beta_2$$
 Part (b)

$$= \int f \, \mathrm{d}\beta_3$$
 Part (c)

Note that calculating the exact value of $\int f d\beta_3$ is symmetric to the proof in part (a).

5. Suppose f is a bounded real function on [a,b], and $f^2 \in \mathcal{R}$ on [a,b]. Does it follow that $f \in \mathcal{R}$? Does the answer change if we assume that $f^3 \in \mathcal{R}$?

Proof. $f^2 \in \mathcal{R} \Rightarrow f \in \mathcal{R}$: Consider the bounded real function $f:[a,b] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & x \notin \mathbb{Q} \\ -1 & x \in \mathbb{Q} \end{cases}$$

Since $f^2(x) = 1$ for all $x \in [a, b]$, $f^2 \in \mathcal{R}$ as a constant function. However, by Exercise 6.4 and a clever application of Theorem 6.12 (to relate it to the function explicitly considered in Exercise 6.4), we know that $f \notin \mathcal{R}$.

 $\underline{f^3} \in \mathscr{R} \Rightarrow f \in \mathscr{R}$: Let $f:[a,b] \to \mathbb{R}$ be any bounded real function such that $f^3 \in \mathscr{R}$. To prove that $f \in \mathscr{R}$, Theorem 6.11 tells us that it will suffice to show that there exist $m, M \in \mathbb{R}$ such that $m \leq f \leq M$ and that there exists a continuous function $\phi:[m,M] \to \mathbb{R}$ such that $f = \phi \circ f^3$. Since f is bounded by hypothesis, we can pick $m,M \in \mathbb{R}$ such that $m \leq f \leq M$. Now let $\phi:[m,M] \to \mathbb{R}$ be defined by

$$\phi(x) = \sqrt[3]{x}$$

for all $x \in [m, M]$. It is obvious that ϕ is continuous and that $\phi \circ f^3 = f$, as desired.

7. Suppose f is a real function on (0,1] and $f \in \mathcal{R}$ on [c,1] for every c>0. Define

$$\int_0^1 f(x) \, \mathrm{d}x = \lim_{c \to 0} \int_c^1 f(x) \, \mathrm{d}x$$

if this limit exists (and is finite).

(a) If $f \in \mathcal{R}$ on [0,1], show that this definition of the integral agrees with the old one.

Proof. To prove that $\int_0^1 f = \lim_{c\to 0} \int_c^1 f$, it will suffice to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $c \in (0,1]$ and $c < \delta$, then

$$\left| \int_0^c f \right| = \left| \int_c^1 f - \int_0^1 f \right| < \epsilon$$

Let $\epsilon > 0$ be arbitrary. Since f is integrable, f is bounded, i.e., there exists $M \in \mathbb{R}$ such that |f(x)| < M for all $x \in [0,1]$. Choose $\delta = \epsilon/M$. Let $c \in (0,1]$ be such that $c < \delta$. Then by Theorem 6.12d,

$$\left| \int_0^c f \right| \le M(c - 0)$$

$$< \epsilon$$

as desired.

(b) Construct a function f such that the above limit exists, although it fails to exist with |f| in place of f.

Proof. Let $f:(0,1]\to\mathbb{R}$ be defined by

$$f(x) = (-1)^n n$$

for $1/n < x \le 1/(n-1)$ (n=2,3,...). It follows since f is a constant function save one terminal discontinuity on each [1/n,1/(n-1)] that

$$\int_{1/n}^{1/(n-1)} f = (-1)^n n \cdot \left(\frac{1}{n-1} - \frac{1}{n}\right)$$
$$= \frac{(-1)^n n}{n(n-1)}$$
$$= \frac{(-1)^n}{n-1}$$

for all $n \in \mathbb{N}$. It follows that

$$\int_{1/N}^{1} f = \sum_{n=2}^{N} \int_{1/n}^{1/(n-1)} f$$
$$= \sum_{n=2}^{N} \frac{(-1)^n}{n-1}$$

Thus,

$$\lim_{c \to 0} \int_{c}^{1} f = \sum_{n=2}^{\infty} \frac{(-1)^{n}}{n-1}$$

which converges by Theorem 3.43. However, the limit fails to exist if f is replaced by |f|, because in that case, the integral is equal to the harmonic series, which diverges to infinity.

8. Suppose $f \in \mathcal{R}$ on [a, b] for every b > a where a is fixed. Define

$$\int_{a}^{\infty} f(x) \, \mathrm{d}x = \lim_{b \to \infty} \int_{a}^{b} f(x) \, \mathrm{d}x$$

if this limit exists (and is finite). In that case, we say that the integral on the left **converges**. If it also converges after f has been replaced by |f|, it is said to converge **absolutely**.

Assume that $f(x) \ge 0$ and that f decreases monotonically on $[1, \infty)$. Prove that $\int_1^\infty f(x) \, \mathrm{d}x$ converges if and only if $\sum_{n=1}^\infty f(n)$ converges. (This is the so-called "integral test" for convergence of series.)

Proof. To prove the claim, we will show that

$$\sum_{n=2}^{N} f(n) \le \int_{1}^{N} f \le \sum_{n=1}^{N-1} f(n) \le f(1) + \int_{1}^{N-1} f(x) \, \mathrm{d}x$$

It will follow since both the sum and the integral limit are monotonically increasing as $N \to \infty$ ($f \ge 0$) and both are bounded below and above by (a function of) the other, both converge or diverge together. Let's begin.

Since f is monotonically decreasing on $[1, \infty)$, we know that $f(n) \leq f(x)$ for all $1 \leq x \leq n$ $(n \in \mathbb{N})$. Thus, by Theorem 6.12b,

$$\int_{n-1}^{n} f(n) \, \mathrm{d}x \le \int_{n-1}^{n} f(x) \, \mathrm{d}x$$

Therefore,

$$\sum_{n=2}^{N} f(n) = \sum_{n=2}^{N} \int_{n-1}^{n} f(n) dx$$
 Theorem 6.12d
$$\leq \sum_{n=2}^{N} \int_{n-1}^{n} f(x) dx$$

$$= \int_{1}^{N} f(x) dx$$
 Theorem 6.12c

for all $N = 2, 3, 4, \ldots$, thereby establishing the left inequality above.

Since f is monotonically decreasing on $[1, \infty)$, we know that $f(x) \leq f(n)$ for all $x \geq n$ $(n \in \mathbb{N})$. Thus, by Theorem 6.12b,

$$\int_{n}^{n+1} f(x) \, \mathrm{d}x \le \int_{n}^{n+1} f(n) \, \mathrm{d}x$$

Therefore,

$$\int_{1}^{N} f(x) dx = \sum_{n=1}^{N-1} \left(\int_{n}^{n+1} f(x) dx \right)$$
 Theorem 6.12c

$$\leq \sum_{n=1}^{N-1} \left(\int_{n}^{n+1} f(n) dx \right)$$

$$= \sum_{n=1}^{N-1} f(n)$$
 Theorem 6.12d

for all $N = 2, 3, 4, \ldots$, thereby establishing the middle inequality above.

From our statement about f(n) and f(x) from the left inequality, we have by Theorem 6.12b that

$$\int_{n-1}^{n} f(n) \, \mathrm{d}x \le \int_{n-1}^{n} f(x) \, \mathrm{d}x$$

Therefore,

$$\sum_{n=1}^{N-1} f(n) = f(1) + \sum_{n=2}^{N-1} \int_{n-1}^{n} f(n) dx$$
 Theorem 6.12d

$$\leq f(1) + \sum_{n=2}^{N-1} \int_{n-1}^{n} f(x) dx$$

$$= f(1) + \int_{1}^{N-1} f(x) dx$$
 Theorem 6.12c

for all $N = 2, 3, 4, \ldots$, thereby establishing the right inequality above.

10. Let p, q be positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

Prove the following statements.

(a) If $u, v \geq 0$, then

$$uv \le \frac{u^p}{p} + \frac{v^q}{q}$$

Equality holds if and only if $u^p = v^q$.

Discussion. To prove the desired inequality, it will suffice to show that

$$0 \le \frac{u^p}{p} + \frac{v^q}{q} - uv$$

i.e., that for all $u, v \ge 0$, the expression on the right above is nonnegative. To consider all such values at once, we can consider applying our analysis toolbox to $f: [0, \infty)^2 \to \mathbb{R}$ defined by

$$f(u,v) = \frac{u^p}{p} + \frac{v^q}{q} - uv$$

with the goal of proving that it is nonnegative everywhere on its domain. However, since we do not yet know multivariable calculus, it will suffice to fix $u \geq 0$ and analyze $f : [0, \infty) \to \mathbb{R}$ defined by

$$f(v) = \frac{u^p}{p} + \frac{v^q}{q} - uv$$

Let's begin.

Proof. Fix $u \geq 0$. Let $f: [0, \infty) \to \mathbb{R}$ be defined by

$$f(v) = \frac{u^p}{p} + \frac{v^q}{q} - uv$$

It follows from the definition of f that to prove the desired inequality, it will suffice to show that f is nonnegative everywhere on its domain. Let's begin.

Since f is a polynomial in v, f is differentiable. Thus, we may consider

$$f'(v) = v^{q-1} - u$$

As a function of a positive power (q/(q-1) = p > 0 and q > 0 imply q-1 > 0) of its variable (minus a constant), f' is strictly increasing. Additionally, we have that

$$0 = f'(v)$$

$$u = v^{q-1}$$

$$= v^{q/p}$$

$$v = u^{p/q}$$

Thus, we know that f' < 0 on $(0, u^{p/q})$ and f' > 0 on $(u^{p/q}, \infty)$. It follows by the strict version of Theorem 5.11 that f is strictly decreasing on $(0, u^{p/q})$ and strictly increasing on $(u^{p/q}, \infty)$. Furthermore, since f is differentiable (hence continuous by Theorem 5.2), we know that $f(0) \ge f(u^{p/q})$. Combining the last several results, we have that $f(u^{p/q})$ is the minimum of f over $[0, \infty)$, and hence equal to the minimum value of f over $[0, \infty)$. But since

$$f(u^{p/q}) = \frac{u^p}{p} + \frac{(u^{p/q})^q}{q} - uu^{p/q}$$
$$= \frac{u^p}{p} + \frac{u^p}{q} - u^{p/q+1}$$
$$= u^p \left(\frac{1}{p} + \frac{1}{q}\right) - u^p$$

we know that $f(v) \geq 0$ on its domain, as desired.

Additionally, since f is strictly decreasing on $(0, u^{p/q})$ and strictly increasing on $(u^{p/q}, \infty)$, we know that f(v) = 0 iff $v = u^{p/q}$, i.e., iff $v^q = u^p$, as desired.

(b) If $f, g \in \mathcal{R}(\alpha)$, $f, g \geq 0$, and

$$\int_{a}^{b} f^{p} d\alpha = 1 = \int_{a}^{b} g^{q} d\alpha$$

then

$$\int_{a}^{b} fg \, \mathrm{d}\alpha \le 1$$

Proof. By Theorem 6.13a, the hypothesis $f, g \in \mathcal{R}(\alpha)$ implies that $fg \in \mathcal{R}(\alpha)$. Thus, we have that

$$\int_{a}^{b} f g \, d\alpha \le \int_{a}^{b} \left(\frac{f^{p}}{p} + \frac{g^{q}}{q} \right) d\alpha$$
 Theorem 6.12b
$$= \frac{1}{p} \int_{a}^{b} f^{p} \, d\alpha + \frac{1}{q} \int_{a}^{b} g^{q} \, d\alpha$$
 Theorem 6.12a
$$= \frac{1}{p} + \frac{1}{q}$$

$$= 1$$

as desired. \Box

(c) If f, g are complex functions in $\mathcal{R}(\alpha)$, then

$$\left| \int_{a}^{b} f g \, d\alpha \right| \leq \left(\int_{a}^{b} |f|^{p} \, d\alpha \right)^{1/p} \left(\int_{a}^{b} |g|^{q} \, d\alpha \right)^{1/q}$$

This is **Hölder's inequality**. When p = q = 2, it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)

Proof. By Theorem 6.11 with $\phi(y) = |y|^p$ (resp. $\phi(y) = |y|^q$), the hypothesis $f, g \in \mathcal{R}(\alpha)$ implies that $|f|^p, |g|^q \in \mathcal{R}(\alpha)$. Thus, we may let

$$I_f = \left(\int_a^b |f|^p \,\mathrm{d}\alpha\right)^{1/p} \qquad \qquad I_g = \left(\int_a^b |g|^q \,\mathrm{d}\alpha\right)^{1/q}$$

We divide into two cases ($I_f = 0$ or $I_g = 0$, and $I_f, I_g \neq 0$). In the first case, WLOG let $I_f = 0$. Then since $0 \leq |f|^p$, it follows that f = 0 on [a, b]. Thus

$$\left| \int_a^b f g \, d\alpha \right| = 0 \le 0 = I_f I_g = \left(\int_a^b |f|^p \, d\alpha \right)^{1/p} \left(\int_a^b |g|^q \, d\alpha \right)^{1/q}$$

as desired. In the other case, it follows that

$$I_f^p = \int_a^b |f|^p d\alpha \qquad \qquad I_g^q = \int_a^b |g|^q d\alpha$$

$$1 = \int_a^b \left| \frac{f}{I_f} \right|^p d\alpha \qquad \qquad 1 = \int_a^b \left| \frac{g}{I_g} \right|^q d\alpha \qquad \qquad \text{Theorem 6.12a}$$

Thus, since $|f/I_f|, |g/I_g| \in \mathcal{R}(\alpha)$ by Theorems 6.12 and 6.13 and $|f/I_f|, |g/I_g| \ge 0$ by the defini-

tion of the absolute value, we have that

$$\left| \int_{a}^{b} f g \, d\alpha \right| \leq \int_{a}^{b} |fg| \, d\alpha \qquad \text{Theorem 6.13b}$$

$$= I_{f} I_{g} \int_{a}^{b} \left| \frac{f}{I_{f}} \right| \left| \frac{g}{I_{g}} \right| \, d\alpha$$

$$\leq I_{f} I_{g} \cdot 1 \qquad \text{Part (b)}$$

$$= \left(\int_{a}^{b} |f|^{p} \, d\alpha \right)^{1/p} \left(\int_{a}^{b} |g|^{q} \, d\alpha \right)^{1/q}$$

as desired.

11. Let α be a fixed increasing function on [a,b]. For $u \in \mathcal{R}(\alpha)$, define

$$||u||_2 = \left(\int_a^b |u|^2 \, \mathrm{d}\alpha\right)^{1/2}$$

Suppose $f, g, h \in \mathcal{R}(\alpha)$, and prove the triangle inequality

$$||f - h||_2 \le ||f - g||_2 + ||g - h||_2$$

as a consequence of the Schwarz inequality, as in the proof of Theorem 1.37.

Proof. By Theorems 6.12a and 6.13b, the hypothesis that $f, g, h \in \mathcal{R}(\alpha)$ implies that $|f - g|, |g - h| \in \mathcal{R}(\alpha)$. Thus, we have that

$$\begin{split} \|f-h\|_2^2 &= \int_a^b |f-h|^2 \,\mathrm{d}\alpha \\ &= \int_a^b |(f-g) + (g-h)|^2 \,\mathrm{d}\alpha \\ &= \int_a^b |f-g|^2 \,\mathrm{d}\alpha + 2 \int_a^b |f-g| \cdot |g-h| \,\mathrm{d}\alpha + \int_a^b |g-h|^2 \,\mathrm{d}\alpha \\ &\leq \int_a^b |f-g|^2 \,\mathrm{d}\alpha + 2 \left(\int_a^b |f-g|^2 \,\mathrm{d}\alpha\right)^{1/2} \left(\int_a^b |g-h|^2 \,\mathrm{d}\alpha\right)^{1/2} + \int_a^b |g-h|^2 \,\mathrm{d}\alpha \\ &= \|f-g\|_2^2 + 2\|f-g\|_2 \|g-h\|_2 + \|g-h\|_2^2 \\ &= (\|f-g\|_2 + \|g-h\|_2)^2 \end{split}$$

Taking square roots of both sides of the inequality yields the desired result.

4 Sequences and Series of Functions

From Rudin (1976).

Chapter 7

2/9:

1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

Proof. Let $\{f_n\}$ be an arbitrary uniformly convergent sequence of bounded functions. To prove that it is uniformly bounded, it will suffice to find a number M such that $|f_n(x)| < M$ for all $x \in E$ and $n \in \mathbb{N}$. Let f be the function such that $f_n \rightrightarrows f$, and let $M_n = \sup_{x \in E} |f_n(x) - f(x)|$ for each $n \in \mathbb{N}$ (the boundedness of each f_n implies that such an M_n always exists). Thus, based on the last two definitions, we can invoke Theorem 7.9 to learn that $M_n \to 0$ as $n \to \infty$. But since $\{M_n\}$ converges, Theorem 3.2c implies that $\{M_n\}$ is bounded, say by M/2. Taking M to be our M yields that for an arbitrary $x \in E$ and $n \in \mathbb{N}$,

 $|f_n(x)| \le M_n \le \frac{M}{2} < M$

as desired.

2. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_ng_n\}$ converges uniformly on E.

Proof. To prove that $\{f_n+g_n\}$ converges uniformly on E to f+g, it will suffice to show that for all $\epsilon>0$, there exists an N such that if $n\geq N$, then $|(f_n+g_n)(x)-(f+g)(x)|<\epsilon$ for all $x\in E$. Let $\epsilon>0$ be arbitrary. Since $f_n\to f$ uniformly on E, there exists N_1 such that if $n\geq N_1$, then $|f_n(x)-f(x)|<\epsilon/2$ for all $x\in E$. Similarly, there exists N_2 such that if $n\geq N_2$, then $|g_n(x)-g(x)|<\epsilon/2$ for all $x\in E$. Choose $N=\max(N_1,N_2)$. Now suppose $n\geq N$, and let $x\in E$ be arbitrary. It follows from the first condition that $n\geq N\geq N_1$ and $n\geq N\geq N_2$, so

$$|(f_n + g_n)(x) - (f + g)(x)| = |f_n(x) - f(x) + g_n(x) - g(x)|$$

$$\leq |f_n(x) - f(x)| + |g_n(x) - g(x)|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$= \epsilon$$

as desired.

To prove that $\{f_ng_n\}$ converges uniformly on E to fg, it will suffice to show that for all $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|(f_ng_n)(x) - (fg)(x)| < \epsilon$ for all $x \in E$. Let $\epsilon > 0$ be arbitrary. Since f_n, g_n are uniformly convergent sequences of bounded functions, Exercise 1 implies that they are uniformly bounded, i.e., there exists $M^f, M^g \in \mathbb{R}$ such that $|f_n| < M^f$ and $|g_n| < M^g$ for all $n \in \mathbb{N}$. If we take $M = \max(M^f, M^g)$, then we have $|f_n| < M$ and $|g_n| < M$ for all $n \in \mathbb{N}$. Note that the same inequality holds for f and g. Now, as before, we may pick N such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon/2M$ and $|g_n(x) - g(x)| < \epsilon/2M$ for all $x \in E$. It follows that for any $n \geq N$ and $x \in E$,

$$|(f_n g_n)(x) - (fg)(x)| = |f_n(x) \cdot (g_n(x) - g(x)) + g(x) \cdot (f_n(x) - f(x))|$$

$$= |f_n(x)| \cdot |g_n(x) - g(x)| + |g(x)| \cdot |f_n(x) - f(x)|$$

$$< M \cdot \frac{\epsilon}{2M} + M \cdot \frac{\epsilon}{2M}$$

$$= \epsilon$$

as desired. \Box

3. Construct sequences $\{f_n\}, \{g_n\}$ which converge uniformly on some set E, but such that $\{f_ng_n\}$ does not converge uniformly on E (of course, $\{f_ng_n\}$ must converge on E).

Proof. Let

$$f_n(x) = x g_n(x) = \frac{1}{n}$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then $\{f_n\}$ converges uniformly to f(x) = x (by choosing N = 1 for any ϵ) and $\{g_n\}$ converges uniformly to g(x) = 0 (by choosing $1/N < \epsilon$ with the Archimedean principle). However, while $\{f_ng_n\}$ converges pointwise to (fg)(x) = 0 by Theorem 3.3c, it does not converge uniformly since for any n, choosing x = n yields $(f_ng_n)(x) = 1$.

4. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?

Proof. Absolute convergence values: The series converges absolutely for any

$$x \in (-\infty, -1) \cup \left(\bigcup_{k=1}^{\infty} \left(-\frac{1}{k^2}, -\frac{1}{(k+1)^2}\right)\right) \cup (0, \infty)$$

We prove this via casework as follows.

Let $x \in (0, \infty)$. Then we have

$$\sum_{n=1}^{\infty} \left| \frac{1}{1+n^2 x} \right| = \sum_{n=1}^{\infty} \frac{1}{1+n^2 x} \le \sum_{n=1}^{\infty} \frac{1}{n^2 x} = \frac{1}{x} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{c}{x}$$

where $c \in \mathbb{R}$ is finite by Theorem 3.28. Therefore, since the sum is monotonically increasing and bounded, Theorem 3.14 implies that the sum overall converges, as desired.

Let $x \in (-\infty, -1)$. Then we have

$$n^2x + 1 < n^2x + n^2 = n^2(x+1)$$

Since x < -1,

$$n^2x + 1 < 0 n^2(x+1) < 0$$

for all $n \in \mathbb{N}$. Thus,

$$n^{2}x + 1 < n^{2}(x+1)$$

$$\frac{n^{2}x + 1}{n^{2}(x+1)} > 1$$

$$\frac{1}{n^{2}(x+1)} < \frac{1}{n^{2}x + 1}$$

$$\left| \frac{1}{n^{2}x + 1} \right| < \left| \frac{1}{n^{2}(x+1)} \right|$$

for all $n \in \mathbb{N}$. It follows that

$$\sum_{n=1}^{\infty} \left| \frac{1}{n^2 x + 1} \right| < \sum_{n=1}^{\infty} \left| \frac{1}{n^2 (x+1)} \right| = \frac{1}{x+1} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{c}{x+1}$$

where $c \in \mathbb{R}$ is finite by Theorem 3.28. Therefore, since the sum is monotonically increasing and bounded, Theorem 3.14 implies that the sum overall converges, as desired.

Let $x \in (-1/k^2, -1/(k+1)^2)$. For right now, we consider only the sum for $n \ge \sqrt{2}(k+1)$, leaving finitely many terms out of the sum. Let $\delta = 1/(k+1)^2$. It follows that

$$n \ge \sqrt{2(k+1)}$$

$$x < -\frac{1}{(k+1)^2}$$

$$n \ge \sqrt{\frac{2}{1/(k+1)^2}}$$

$$-x > \frac{1}{(k+1)^2}$$

$$n^2 \ge \frac{2}{\delta}$$

$$\frac{\delta}{2} \ge \frac{1}{n^2}$$

Additionally, since $n \ge \sqrt{2}(k+1) > k$ (hence $n^2 \ge (k+1)^2$) and $x < -1/(k+1)^2$, we have that

$$n^{2}x < (k+1)^{2} \cdot -\frac{1}{(k+1)^{2}}$$

$$n^{2}x < -1$$

$$n^{2}x + 1 < 0$$

Thus, for $n \ge \sqrt{2}(k+1)$, we have that

$$\left| \frac{1}{1+n^2x} \right| = \frac{1}{n^2(-x)-1} < \frac{1}{n^2\delta-1} = \frac{1}{n^2} \cdot \frac{1}{\delta-1/n^2} \le \frac{1}{n^2} \cdot \frac{1}{\delta-\delta/2} = \frac{2}{\delta n^2}$$

Therefore, since $|f_n(x)| \leq M_n = 2/\delta n^2$ and $\sum M_n$ converges by Theorem 3.28, the comparison test implies that $\sum |f_n(x)|$ converges, as desired. Adding on the finitely many terms we left out of the summation will not change this fact.

Note that the series diverges for x=0 since each term becomes 1 in this case. Additionally, the series fails to exist for $x=-1/k^2$ ($k \in \mathbb{N}$) since the k^{th} term is undefined in this case.

Uniform convergence intervals: The series converges uniformly on any

$$[a,b]\subset (-\infty,-1)\cup \left(\bigcup_{k=1}^{\infty}\left(-\frac{1}{k^2},-\frac{1}{(k+1)^2}\right)\right)\cup (0,\infty)$$

This is because any such interval will be a subset of either $(-\infty, -1)$, $(0, \infty)$, or a set of the form $(-1/k^2, -1/(k+1)^2)$ $(k \in \mathbb{N})$. Thus, we may take as $\sum M_n$ the supremum on [a, b] of the appropriate bound derived above (either c/x, c/(x+1), or $2c/\delta$, respectively; all supremums of which will exist by the definition of [a, b]) and apply Theorem 7.10.

Non-uniform convergence intervals: Any interval containing one or more of the points in the set $\{0\} \cup \{-1/n^2\}_{n=1}^{\infty}$, by the above.

Points of continuity: The series is continuous at all points at which it converges.

Let x be a point at which f converges. Then by the first part of the proof, x is an element of an open set G. Thus, let $N_{2r}(x) \subset G$, and consider [x-r,x+r]. By the above, f converges uniformly on this interval. Additionally, each f_n is continuous on this interval by definition. Thus, by Theorem 7.12, f is continuous at x, as desired.

Boundedness: f is not bounded.

If we suppose for the sake of contradiction that f is bounded by m, we nevertheless find that

$$f(\frac{1}{4m^2}) > \sum_{n=1}^{2m} \frac{1}{1 + \frac{n^2}{4m^2}} = \sum_{n=1}^{2m} \frac{(2m)^2}{(2m)^2 + n^2} \ge \sum_{n=1}^{2m} \frac{(2m)^2}{(2m)^2 + (2m)^2} = \sum_{n=1}^{2m} \frac{1}{2} = m$$

7. For n = 1, 2, 3, ... and x real, put

$$f_n(x) = \frac{x}{1 + nx^2}$$

Show that $\{f_n\}$ converges uniformly to a function f and that the equation

$$f'(x) = \lim_{n \to \infty} f'_n(x)$$

is correct if $x \neq 0$ but false if x = 0.

Proof. To prove that $\{f_n\}$ converges uniformly to f defined by f(x) = 0 ($x \in \mathbb{R}$), Theorem 7.9 tells us that it will suffice to show that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$ and that the sequence $\{M_n\}$ defined by $M_n = \sup_{x \in \mathbb{R}} |f_n(x)|$ tends to zero as $n \to \infty$. Since

$$f_n(x) = \frac{x}{1 + nx^2} < \frac{x}{nx^2} = \frac{1}{x} \cdot \frac{1}{n} \to 0$$

as $n \to \infty$ for all $x \neq 0$ and $f_n(0) = 0$ for all n, $\lim_{n \to \infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$, as desired. Additionally, by the Schwarz inequality, if a_1, a_2, b_1, b_2 are real numbers, then

$$|a_1b_1 + a_2b_2|^2 \le (|a_1|^2 + |a_2|^2)(|b_1|^2 + |b_2|^2)$$

It follows that

$$|2\sqrt{n}x|^2 = |\underbrace{1}_{a_1} \cdot \underbrace{\sqrt{n}x}_{b_1} + \underbrace{\sqrt{n}x}_{a_2} \cdot \underbrace{1}_{b_2}|^2 \le (|1|^2 + |\sqrt{n}x|^2)(|\sqrt{n}x|^2 + |1|^2) = (1 + nx^2)^2$$

$$|2\sqrt{n}x| \le |1 + nx^2|$$

$$\frac{1}{|1 + nx^2|} \le \frac{1}{2\sqrt{n}|x|}$$

$$\frac{|x|}{|1 + nx^2|} \le \frac{1}{2\sqrt{n}}$$

$$\left|\frac{x}{1 + nx^2}\right| \le \frac{1}{2\sqrt{n}}$$

for all $x \neq 0$, $n \in \mathbb{N}$. This combined with the facts that $f_n(0) = 0 < \frac{1}{2\sqrt{n}}$ for all $n \in \mathbb{N}$ and $f_n(1/\sqrt{n}) = 1/2\sqrt{n}$ for all $n \in \mathbb{N}$ implies that $M_n = 1/2\sqrt{n}$. Thus, $M_n \to 0$ as $n \to \infty$, as desired. f'(x) = 0 for all $x \in \mathbb{R}$. Additionally,

$$f'_n(x) = \frac{1 - nx^2}{(1 + nx^2)^2} \le \frac{1 - nx^2}{(nx^2)^2} = \frac{1}{x^4} \cdot \frac{1}{n^2} - \frac{1}{x^2} \cdot \frac{1}{n} \to 0$$

as $n \to \infty$ for all $x \neq 0$, as desired. However, $f'_n(0) = 1$ for all $n \in \mathbb{N}$, as desired.

References MATH 20410

References

Rudin, W. (1976). Principles of mathematical analysis (A. A. Arthur & S. L. Langman, Eds.; Third). McGraw-Hill.