8 Functions of Several Variables IV / Special Functions

From Rudin (1976).

Chapter 8

3/11: **1.** Define

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Prove that f has derivatives of all orders at $x = 0$, and that $f^{(n)}(0) = 0$ for $n = 1, 2, \ldots$.

Proof. We induct on n. For the base case $n = 1$, we have that

$$\begin{aligned} f^{(1)}(0) &= \lim_{h \to 0} \frac{f^{(0)}(h) - f^{(0)}(0)}{h} \\ &= \lim_{h \to 0} \frac{e^{-1/h^2}}{h} \\ &= \lim_{h \to 0} h^{-1} e^{-1/h^2} \end{aligned}$$

Let $x = h^{-2}$. Then $h = x^{-1/2}$. Additionally, as $h \to 0$, $x \to \infty$. Thus, the above limit is equal to

$$\lim_{x \to \infty} x^{1/2} e^{-x}$$

which equals zero by Theorem 8.6e. Furthermore, we can calculate by the rules of derivatives that if $x \neq 0$, then

$$f^{(1)}(x) = 2x^{-3} e^{-1/x^2}$$

Thus, $f^{(1)}(x)$ is of the form $\sum_{i=1}^{m} a_i x^{-b_i} e^{-1/x^2}$ where $a_i, b_i \in \mathbb{N}_0$ for all $i = 1, \ldots, m$[1]. Now suppose inductively that

$$f^{(n-1)}(x) = \begin{cases} 0 & x = 0 \\ \sum_{i=1}^{m} a_i x^{-b_i} e^{-1/x^2} & x \neq 0 \end{cases}$$

Then

$$\begin{aligned} f^{(n)}(0) &= \lim_{h \to 0} \frac{f^{(n-1)}(h) - f^{(n-1)}(0)}{h} \\ &= \lim_{h \to 0} \frac{\sum_{i=1}^{m} a_i h^{-b_i} e^{-1/h^2}}{h} \\ &= \sum_{i=1}^{m} a_i \lim_{h \to 0} h^{-b_i - 1} e^{-1/h^2} \\ &= \sum_{i=1}^{m} a_i \cdot 0 && \text{Theorem 8.6e} \\ &= 0 \end{aligned}$$

Furthermore, we can calculate by the rules of derivatives that if $x \neq 0$, then

$$f^{(n)}(x) = \sum_{i=1}^{m} a_i \left[-b_i x^{-b_i - 1} e^{-1/x^2} + 2x^{-b_i - 3} e^{-1/x^2} \right]$$

which is of the form $\sum_{i=1}^{m} a_i x^{-b_i} e^{-1/x^2}$, as desired. $\square$

[1] Although this expression may look a bit esoteric, one can readily confirm that $f = f^{(0)}$ satisfies it with $m = 1$, $a_1 = 1$, $b_1 = 0$ and $f' = f^{(1)}$ satisfies it with $m = 1$, $a_1 = 2$, $b_1 = 3$.

6. Suppose $f(x)f(y) = f(x+y)$ for all real x and y.

(a) Assuming that f is differentiable and not zero, prove that

$$f(x) = \mathrm{e}^{cx}$$

where c is a constant.

Proof. By Theorem 5.2, if f is differentiable, then f is continuous. Thus, $f(x) = \mathrm{e}^{cx}$ for some c by part (b). □

(b) Prove the same thing, assuming only that f is continuous.

Proof. $f(0)^2 = f(0)$, so $f(0) = 0, 1$. If $f(0) = 0$, then $f(x) = f(x)f(0) = 0$ for all x. But f is nonzero by hypothesis, so $f(0) = 1$.
$f(x)f(-x) = f(0) = 1$, so $f(x) \neq 0$ for any x. Continuity/IVT implies that f is strictly positive.
f can be strictly increasing, constant, or strictly decreasing. □

10. Prove that $\sum_{p \text{ prime}} 1/p$ diverges. (This shows that the primes form a fairly substantial subset of the positive integers.) (Hint: Given N, let $p_1, \dots, p_k$ be those primes that divide at least one integer less than or equal to N. Then

$$\begin{aligned}\sum_{n=1}^{N} \frac{1}{n} &\leq \prod_{j=1}^{k} \left(1 + \frac{1}{p_j} + \frac{1}{p_j^2} + \cdots\right) \\ &= \prod_{j=1}^{k} \left(1 - \frac{1}{p_j}\right)^{-1} \\ &\leq \exp\left(\sum_{j=1}^{k} \frac{2}{p_j}\right)\end{aligned}$$

The last inequality holds because

$$(1-x)^{-1} \leq \mathrm{e}^{2x}$$

if $0 \leq x \leq 1/2$.)

Proof. Since the harmonic series diverges, we can make $\sum_{n=1}^{N} 1/n$ as big as necessary. This combined with the fact that log is strictly increasing proves that if we suppose the $\sum_{p \text{ prime}} 1/p$ is bounded above by some M, then there exists N such that $\log\left(\sum_{n=1}^{N} 1/n\right) > 2M$. For this N, then, the inequality proves that $2\sum_{j=1}^{k} 1/p_k$ is necessarily greater, as desired. □

Chapter 9

20. Take $n = m = 1$ in the implicit function theorem, and interpret the theorem (as well as its proof) graphically.

Proof. If $n = m = 1$, then $f(x,y) = 0$ describes a set of points in the plane (namely $f^{-1}(\{0\})$). Since f is continuous, this set could contain one or more connected lines and/or one or more entire connected regions. However, we are only interested in places where $f^{-1}(\{0\})$ is a connected line. We identify such places with a condition on the partial derivative of f, namely that it not equal to zero with respect to one coordinate or the other. WLOG, let this coordinate be y. Indeed, if (a,b) makes $f(a,b) = 0$ and $D_2 f(a,b) \neq 0$, then visualizing the graph of f as a surface in 3-space shows that the tangent line to f at (a,b) parallel to the y axis has nonzero slope, and thus, $f(a,y) \neq 0$ for all y sufficiently close to b. Thus, since $f \in \mathscr{C}^1$, and hence $D_2 f \neq 0$ in some neighborhood (U) of (a,b), there is a one-to-one correspondence between points x, y such that $f(x,y) = 0$ in this neighborhood.

As for the proof, the linear map $\mathbf{F}'(a,b)$ is invertible since its Jacobian is given by

$$\begin{bmatrix} D_1 f(a,b) & D_2 f(a,b) \\ D_1\pi_1(a,b) & D_2\pi_1(a,b) \end{bmatrix} = \begin{bmatrix} D_1 f(a,b) & D_2 f(a,b) \\ 1 & 0 \end{bmatrix}$$

where π_1 denotes the map $(x,y) \mapsto x$ and $D_2 f(a,b) \neq 0$ by definition. Thus, it also implies a 1-1 region for F near (a,b). Consequently, we can find a set of points (x,y) near $(0,b)$ in the codomain such that $\mathbf{F}$ is 1-1 on this range. Considering only the points for which $x = 0$ maintains one-to-oneness and allows us to identify our unique (x,y) near (a,b). $\square$