Chapter 2

Differentiation

2.1 Notes

1/10:

- Since manifolds look like Euclidean spaces locally, we basically only need to study differentiation on Euclidean spaces.
- Set up: Let $U \subset \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}^n$ be a function.
- Idea: The derivative of f at some point $\mathbf{a} \in U$ is "the best linear approximation" to f at $\mathbf{a}$.
- **Differentiable** (function f at $\mathbf{a}$): A function f for which there exists a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$ such that
$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - A\mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0}$$
- **Total derivative** (of f at $\mathbf{a}$): The linear transformation A corresponding to a differentiable function f. *Denoted by* $\boldsymbol{Df}(\mathbf{a})$.
- Questions to ask:
  1. When does the total derivative exist?
  2. When it does exist, can there be multiple?
  3. When it exists and is unique, how do I calculate it?
- Proposition: If A, B are linear transformations that both satisfy the definition, then $A = B$.
  - We have
$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - A\mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0} \qquad \lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - B\mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0}$$
  - It follows by subtracting the right equation above from the left one that
$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{A\mathbf{h} - B\mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0}$$
  - Apply linearity: For $\mathbf{v} \in \mathbb{R}^n$ and $t \in \mathbb{R}$, $t > 0$, we have
$$\frac{A(t\mathbf{v}) - B(t\mathbf{v})}{t} = A\mathbf{v} - B\mathbf{v}$$

- Therefore, since $t\mathbf{v} \to 0$ as $t \to 0$, we have by the above that

$$\begin{aligned}
\mathbf{0} &= \lim_{t\to 0} \frac{A(t\mathbf{v}) - B(t\mathbf{v})}{\|t\mathbf{v}\|} \\
&= \lim_{t\to 0} \frac{A\mathbf{v} - B\mathbf{v}}{\|\mathbf{v}\|} \\
\mathbf{0} \cdot \|\mathbf{v}\| &= \lim_{t\to 0}(A\mathbf{v} - B\mathbf{v}) \\
\mathbf{0} &= A\mathbf{v} - B\mathbf{v} \\
B\mathbf{v} &= A\mathbf{v}
\end{aligned}$$

- Example: Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be linear, i.e., $f(\mathbf{v}) = A\mathbf{v}$ for some linear transformation A. Then for all $\mathbf{a} \in \mathbb{R}^n$, $Df(\mathbf{a}) = A$ is constant.
  - We have from the definition that

$$\begin{aligned}
\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - A\mathbf{h}}{\|\mathbf{h}\|} &= \lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{a}) + f(\mathbf{h}) - f(\mathbf{a}) - f(\mathbf{h})}{\|\mathbf{h}\|} \\
&= \lim_{\mathbf{h}\to\mathbf{0}} \frac{\mathbf{0}}{\|\mathbf{h}\|} \\
&= \mathbf{0}
\end{aligned}$$

- Theorem: If f is differentiable at $\mathbf{a}$, then f is continuous at $\mathbf{a}$.
  - By definition, there exists a linear transformation A such that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - A\mathbf{h}}{\|\mathbf{h}\|} = \mathbf{0}$$

  - Additionally, we have that

$$f(\mathbf{a}+\mathbf{h}) = f(\mathbf{a}) + A\mathbf{h} + \|\mathbf{h}\| \left(\frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - A\mathbf{h}}{\|\mathbf{h}\|} \right)$$

  - As $\mathbf{h} \to \mathbf{0}$, the right-hand side of the above equation goes to $f(\mathbf{a})$.
    - As a linear transformation, $A\mathbf{h} \to \mathbf{0}$ as $\mathbf{h} \to \mathbf{0}$.
    - Clearly $\|\mathbf{h}\| \to \mathbf{0}$ as $\mathbf{h} \to \mathbf{0}$.
    - And we have by definition that the last term goes to $\mathbf{0}$ as $\mathbf{h} \to \mathbf{0}$.
  - Therefore, f is continuous at $\mathbf{a}$.
- Observation: A function $f : U \to \mathbb{R}^m$ is given by an m-tuple of functions $f_1 : U \to \mathbb{R}$ known as components. $f = (f_1, \dots, f_m)$.
- Proposition: f is differentiable at $\mathbf{a} \in U$ iff each component function f_i is differentiable at $\mathbf{a}$. In this case,

$$Df(\mathbf{a}) = (Df_1(\mathbf{a}), \dots, Df_m(\mathbf{a}))$$

  - We know that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - A\mathbf{h}}{\|\mathbf{h}\|} \in \mathbb{R}^m$$

  - Thus, the limit is zero iff the limit of each component is zero.
  - We have that the i^{th} component of the vector on the left below is equal to the number on the right; we call the common value $L_i(\mathbf{h})$.

$$\left(\frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - A\mathbf{h}}{\|\mathbf{h}\|} \right)_i = \frac{f_i(\mathbf{a}+\mathbf{h}) - f_i(\mathbf{a}) - (A\mathbf{h})_i}{\|\mathbf{h}\|}$$

  - The upshot is that f is differentiable at $\mathbf{a}$ iff $\lim_{\mathbf{h}\to\mathbf{0}} L_i(\mathbf{h}) = \mathbf{0}$ iff the linear transformation $\mathbf{h} \mapsto (A\mathbf{h})_i : \mathbb{R}^m \to \mathbb{R}$ is the total derivative of f_i.
- Now, each f_i is a function of n variables, i.e., $f_i(x_1, \dots, x_n)$ where $x_1, \dots, x_n$ are coordinates on $\mathbb{R}^n$.