

Chapter 1

Multilinear Algebra

1.1 Notes

3/28: • Motivation for the course and an overview of Guillemin and Haine (2018).

3/30: • Plan:

– More (multi)linear algebra.

• Dual spaces.

• Let V be an n -dimensional real vector space.

• **Hom** (V, \mathbb{R}): The set of all homomorphisms (i.e., linear maps) from V to \mathbb{R} . *Also known as V^* .*

• **Dual basis** (for V^*): The set of linear transformations from V to \mathbb{R} defined by

$$e_j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

where e_1, \dots, e_n is a basis of V . *Denoted by e_1^*, \dots, e_n^* .*

• Check: e_1^*, \dots, e_n^* are a basis for V^* .

– Are they linearly independent? Let $c_1 e_1^* + \dots + c_n e_n^* = 0 \in \text{Hom}(V, \mathbb{R})$. Then

$$c_i = (c_1 e_1^* + \dots + c_n e_n^*)(e_i) = 0 \in \mathbb{R}$$

as desired.

– Span? Let $\varphi \in \text{Hom}(V, \mathbb{R})$. Then we can verify that

$$\varphi(e_1) e_1^* + \dots + \varphi(e_n) e_n^* = \varphi$$

■ We prove this by verifying the previous statement on the basis of V (if two linear transformations have the same action on the basis of a vector space, they are equal).

• With a choice of basis for V , we obtain an isomorphism $\varepsilon : V \rightarrow V^*$ with the mapping $e_i \mapsto e_i^*$ for all i .

• The dual space is known as such because $(V^*)^* \cong V$, where \cong is **canonical** (no choice of basis is needed).

• One more property of dual spaces: **functoriality**.

- Given a linear transformation $A : V \rightarrow W$, we know that $A^* : W^* \rightarrow V^*$ where A^* is the transpose of A . In particular, if $\varphi \in W^*$, then $\varphi \circ A : V \rightarrow \mathbb{R}$.
- Claim: A^* is linear.
- **Functoriality:** If $A : V \rightarrow W$ and $B : W \rightarrow U$, then $B^* : U^* \rightarrow W^*$ and $A^* : W^* \rightarrow V^*$. The functoriality statement is that $(B \circ A)^* = A^* \circ B^*$.
- A^* is the **pullback** (or transpose) of A .
- Let v_1, \dots, v_n be a basis for V and w_1, \dots, w_m be a basis for W . Then $[A]_{v_1, \dots, v_n}^{w_1, \dots, w_m} = A$ is the matrix of the linear transformation A with respect to these bases. Then if v_1^*, \dots, v_n^* and w_1^*, \dots, w_m^* are the corresponding dual bases, then $[A^*]_{v_1^*, \dots, v_n^*}^{w_1^*, \dots, w_m^*} = A^T$. We can and should verify this for ourselves.
- This is over the real numbers, so A^* is just the transpose because there are no complex numbers of which to take the conjugate!
- A generalization: Tensors.
- **k -tensor:** A **multilinear** map

$$T : \underbrace{V \times \cdots \times V}_{k \text{ times}} \rightarrow \mathbb{R}$$

- **Multilinear** (map T): A function T such that

$$\begin{aligned} T(v_1, \dots, v_i^1 + v_i^2, \dots, v_k) &= T(v_1, \dots, v_i^1, \dots, v_k) + T(v_1, \dots, v_i^2, \dots, v_k) \\ T(v_1, \dots, \lambda v_i, \dots, v_k) &= \lambda T(v_1, \dots, v_i, \dots, v_k) \end{aligned}$$

for all $(v_1, \dots, v_k) \in V^k$.

- The determinant is an n -tensor!
- 1-tensors are just covectors.
- $\mathcal{L}^k(V)$: The vector space of all k -tensors on V .
- Calculating $\dim \mathcal{L}^k(V)$. (Answer not given in this class.)
- Let $A : V \rightarrow W$. Then $A^* : \mathcal{L}^k(W) \rightarrow \mathcal{L}^k(V)$.
 - Check $(A \circ B)^* = B^* \circ A^*$.
- **Multi-index of n of length k :** A k -tuple (i_1, \dots, i_k) where each $i_j \in \mathbb{N}$ satisfies $1 \leq i_j \leq n$ ($j = 1, \dots, k$). Denoted by \mathbf{I} .
- Let e_1, \dots, e_n be a basis for V .
- **Tensor product** (of $T_1 \in \mathcal{L}^k(V)$, $T_2 \in \mathcal{L}^l(V)$): The function from V^{k+l} to \mathbb{R} defined by

$$(v_1, \dots, v_{k+l}) \mapsto T_1(v_1, \dots, v_k) T_2(v_{k+1}, \dots, v_{k+l})$$

Denoted by $T_1 \otimes T_2$.

- Claims:
 1. $T_1 \otimes T_2 \in \mathcal{L}^{k+l}(V)$.
 2. $A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$.
- $e_{\mathbf{I}}^*$: The function $e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*$, where $\mathbf{I} = (i_1, \dots, i_k)$ is a multi-index of n of length k .
- Claim: Letting \mathbf{I} range over all n^k multi-indices of n of length k , the $e_{\mathbf{I}}^*$ are a basis for $\mathcal{L}^k(V)$.

- If $V = \mathbb{R}$, then $V = \mathbb{R}e_1$. If $V = \mathbb{R}^2$, then $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$.
- We know that $L^1(V) = V^* = \mathbb{R}e_1^*$. Thus, $e_1^* \otimes e_2^* : V \times V \rightarrow \mathbb{R}$. Thus, for example,

$$(e_1^* \otimes e_2^*)((1, 2), (3, 4)) = e_1^*(1, 2) \cdot e_2^*(3, 4) = 1 \cdot 4 = 4$$

4/1:

- Plan: More multilinear algebra.
 - Properties of the tensor product.
 - Sign of a permutation.
 - Alternating tensors (lead into differential forms down the road).
- Recall: V is an n -dimensional vector space over \mathbb{R} with basis e_1, \dots, e_n . $\mathcal{L}^k(V)$ is the vector space of k -tensors on V . $\{e_I^* \mid I \text{ a multiindex of } n \text{ of length } k\}$ is a basis for $\mathcal{L}^k(V)$.

- For example, if $V = \mathbb{R}^2$ and $T \in \mathcal{L}^2(V)$, then

$$T(a_1e_1 + a_2e_2, b_1e_1 + b_2e_2) = a_1b_1T(e_1, e_1) + a_1b_2T(e_1, e_2) + a_2b_1T(e_2, e_1) + a_2b_2T(e_2, e_2)$$

- A basis of $\mathcal{L}^2(V)$ is

$$\{e_1^* \otimes e_1^*, e_1^* \otimes e_2^*, e_2^* \otimes e_1^*, e_2^* \otimes e_2^*\}$$

- Recall that some basic properties are

$$e_1^* \otimes e_2^*((1, 2), (3, 4)) = 1 \cdot 4 = 4 \qquad e_2^* \otimes e_1^*((1, 2), (3, 4)) = 2 \cdot 3 = 6$$

- It follows by the initial decomposition of T that

$$T = a_1b_1e_1^* \otimes e_1^* + a_1b_2e_1^* \otimes e_2^* + a_2b_1e_2^* \otimes e_1^* + a_2b_2e_2^* \otimes e_2^*$$

- Important consequence: To know the action of T on an arbitrary pair of vectors, you need only know its action on the basis; a higher-dimensional generalization of the earlier property.
- Note that

$$e_I^*(e_J) = \delta_{IJ} = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$$

- Basic properties of the tensor product.

1. *Right-distributive*: If $T_1 \in \mathcal{L}^k(V)$ and $T_2, T_3 \in \mathcal{L}^\ell(V)$, then

$$T_1 \otimes (T_2 + T_3) = T_1 \otimes T_2 + T_1 \otimes T_3$$

2. *Left-distributive*: If $T_1, T_2 \in \mathcal{L}^k(V)$ and $T_3 \in \mathcal{L}^\ell(V)$, then

$$(T_1 + T_2) \otimes T_3 = T_1 \otimes T_3 + T_2 \otimes T_3$$

3. *Associative*: If $T_1 \in \mathcal{L}^k(V)$, $T_2 \in \mathcal{L}^\ell(V)$, and $T_3 \in \mathcal{L}^m(V)$, then

$$T_1 \otimes (T_2 \otimes T_3) = (T_1 \otimes T_2) \otimes T_3 = T_1 \otimes T_2 \otimes T_3$$

4. *Scalar multiplication*: If $T_1 \in \mathcal{L}^k(V)$, $T_2 \in \mathcal{L}^\ell(V)$, and $\lambda \in \mathbb{R}$, then

$$(\lambda T_1) \otimes T_2 = \lambda(T_1 \otimes T_2) = T_1 \otimes (\lambda T_2)$$

- Note that the tensor product is not commutative.
- Aside: Defining the sign of a permutation.

- S_A : The set of all automorphisms of A (bijections from A to A), where A is a set.
- S_n : The set $S_{[n]}$.
- Given $\sigma_1, \sigma_2 \in S_n$, $\sigma_1 \circ \sigma_2 \in S_n$.
 - Thus, S_n is a **group**.
- **Transposition**: A function $\tau \in S_n$ such that

$$\tau(k) = \begin{cases} j & k = i \\ i & k = j \\ k & k \neq i, j \end{cases}$$

for some $i, j \in [n]$. Denoted by $\tau_{i,j}$.

- Theorem: An element of S_n can be written as the product of transpositions (i.e., for all $\sigma \in S_n$, there exist $\tau_1, \dots, \tau_m \in S_n$ such that $\sigma = \tau_1 \circ \dots \circ \tau_m$).
- **Sign** (of $\sigma \in S_n$): The number (mod 2) of transpositions whose product equals σ . Denoted by $(-1)^\sigma$, $\text{sign}(\sigma)$.
- Theorem: The sign of σ is well-defined. Additionally,

$$(-1)^{\sigma_1 \sigma_2} = (-1)^{\sigma_1} \cdot (-1)^{\sigma_2}$$

- Example: Consider the identity permutation. $(-1)^\sigma = +1$. We can think of this as the product of zero transpositions or, for instance, as the product of the two transpositions $\tau_{1,2} \circ \tau_{1,2}$. Another example would be $\tau_{2,3} \circ \tau_{1,2} \circ \tau_{1,2} \circ \tau_{2,3}$.
- Theorem: Let X_i be a rational or polynomial function for each $i \in \mathbb{N}$. Then

$$(-1)^\sigma = \prod_{i < j} \frac{X_{\sigma(i)} - X_{\sigma(j)}}{X_i - X_j}$$

- Example: For the permutation $\sigma = (1, 2, 3)$, we have

$$\begin{aligned} (-1)^\sigma &= \frac{X_{\sigma(1)} - X_{\sigma(2)}}{X_1 - X_2} \cdot \frac{X_{\sigma(1)} - X_{\sigma(3)}}{X_1 - X_3} \cdot \frac{X_{\sigma(2)} - X_{\sigma(3)}}{X_2 - X_3} \\ &= \frac{X_2 - X_3}{X_1 - X_2} \cdot \frac{X_2 - X_1}{X_1 - X_3} \cdot \frac{X_3 - X_1}{X_2 - X_3} \\ &= \frac{-(X_1 - X_2)}{X_1 - X_2} \cdot \frac{-(X_1 - X_3)}{X_1 - X_3} \cdot \frac{X_2 - X_3}{X_2 - X_3} \\ &= -1 \cdot -1 \cdot 1 \\ &= +1 \end{aligned}$$

which squares with the fact that $\sigma = \tau_{1,2} \circ \tau_{2,3}$.

- Claims to verify with the above formula:
 1. $\text{sign}(\sigma) \in \{\pm 1\}$.
 2. $\text{sign}(\tau_{i,j}) = -1$.
 3. $\text{sign}(\sigma_1 \sigma_2) = \text{sign}(\sigma_1) \text{sign}(\sigma_2)$.

4/4:

- Plan:
 - More multilinear algebra.

- Alternating k -tensors — 2 views:
 1. As a subspace of $\mathcal{L}^k(V)$.
 2. As a quotient of $\mathcal{L}^k(V)$.
- Next time: Operators as alternating tensors.
 - Wedge products.
 - Interior products.
 - Pullbacks.
- Recall: $\dim V = n$, e_1, \dots, e_n a basis, $\mathcal{L}^k(V)$ the space of k -tensors, $\sigma \in S_k$ implies $(-1)^\sigma \in \{\pm 1\}$, key property: $(-1)^{\sigma_1 \sigma_2} = (-1)^{\sigma_1} (-1)^{\sigma_2}$.
- T^σ : The k -tensor over V defined by

$$T^\sigma(v_1, \dots, v_k) = T(v_{\bar{\sigma}(1)}, \dots, v_{\bar{\sigma}(k)})$$

where $T \in \mathcal{L}^k(V)$, $\sigma \in S_k$, and $\bar{\sigma}$ denotes the inverse of σ .

- Example: $n = 2$, $k = 2$. Let $T = e_1^* \otimes e_2^* \in \mathcal{L}^2(V)$. Let $\sigma = \tau_{1,2}$. Then $T^\sigma = e_2^* \otimes e_1^*$.
- Another property is $e_I^\sigma = e_{\sigma(I)}^*$.
- Properties:
 1. $T^{\sigma_1 \sigma_2} = (T^{\sigma_1})^{\sigma_2}$.
 2. $(T_1 + T_2)^\sigma = T_1^\sigma + T_2^\sigma$.
 3. $(cT)^\sigma = cT^\sigma$.
- Thus, you can view $\sigma : \mathcal{L}^k(V) \rightarrow \mathcal{L}^k(V)$ as a linear map!
- **Alternating k -tensor**: A tensor $T \in \mathcal{L}^k(V)$ such that $T^\sigma = (-1)^\sigma T$ for all $\sigma \in S_k$.
 - Equivalently, $T^\tau = -T$ for all $\tau \in S_k$.
- An example of an alternating 2-tensor when $\dim V = 2$ is $T = e_1^* \otimes e_2^* - e_2^* \otimes e_1^*$.
 - Naturally, $T_{1,2}^\tau = -T$, and $\tau_{1,2}$ is the unique transposition in S_2 .
- $e_1^* \otimes e_2^*$ is *not* an alternating 2-tensor since $(e_1^* \otimes e_2^*)^\tau = e_2^* \otimes e_1^* \neq (-1)^\tau (e_1^* \otimes e_2^*)$.
- We can look at $n = 2$, $k = 1$ for ourselves.
- Note: If T_1, T_2 are both alternating k -tensors, then $T_1 + T_2$ is also alternating, as is cT_1 for all $c \in \mathbb{R}$.
- $\mathcal{A}^k(V)$: The vector space of alternating k -tensors.
- **Alt** (T): The function $\text{Alt} : \mathcal{L}^k(V) \rightarrow \mathcal{L}^k(V)$ defined by

$$\text{Alt}(T) = \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma$$

- Properties:
 1. $\text{im}(\text{Alt}) = \mathcal{A}^k(V)$.
 2. $\mathcal{L}^k(V) / \ker(\text{Alt}) = \Lambda^k(V^*)$ is isomorphic to $\mathcal{A}^k(V)$.
 3. $\text{Alt}(T)^\sigma = (-1)^\sigma \text{Alt}(T)$.

– Proof:

$$\begin{aligned}
 \text{Alt}(T)^{\sigma'} &= \left(\sum_{\sigma \in S_k} (-1)^\sigma T^\sigma \right)^{\sigma'} \\
 &= \sum_{\sigma \in S_k} (-1)^\sigma T^{\sigma\sigma'} \\
 &= (-1)^{\sigma'} \sum_{\sigma \in S_k} (-1)^{\sigma'} (-1)^\sigma T^{\sigma\sigma'} \\
 &= (-1)^{\sigma'} \sum_{\sigma \in S_k} (-1)^{\sigma\sigma'} T^{\sigma\sigma'} \\
 &= (-1)^{\sigma'} \text{Alt}(T)
 \end{aligned}$$

– The last equality holds because summing over all σ is the same as summing over all $\sigma' \circ \sigma$.

– This implies $\text{im}(\text{Alt}) \leq \mathcal{A}^k(V)$.

4. If $T \in \mathcal{A}^k(V)$, $\text{Alt}(T) = k!T$.

– We have

$$\begin{aligned}
 \text{Alt}(T) &= \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma \\
 &= \sum_{\sigma \in S_k} (-1)^\sigma (-1)^\sigma T \\
 &= \sum_{\sigma \in S_k} T \\
 &= k!T
 \end{aligned}$$

where $T^\sigma = (-1)^\sigma T$ since $T \in \mathcal{A}^k(V)$ by definition.

– This implies that $\text{im}(\text{Alt}) = \mathcal{A}^k(V)$: $\text{Alt}(\frac{1}{k!}T) = T \in \mathcal{A}^k(V)$.

5. $\text{Alt}(T^\sigma) = \text{Alt}(T)^\sigma$.

6. $\text{Alt} : \mathcal{L}^k(V) \rightarrow \mathcal{L}^k(V)$ is linear.

• Warning: Some people take $\text{Alt}(T) = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma$ ^[1].

• Example: $n = k = 2$. We have

$$\text{Alt}(e_1^* \otimes e_2^*) = e_1^* \otimes e_2^* - e_2^* \otimes e_1^*$$

• **Non-repeating** (multi-index I): A multi-index I such that $i_{j_1} \neq i_{j_2}$ for all $j_1 \neq j_2$.

• **Increasing** (multi-index I): A multi-index I such that $i_1 < \dots < i_k$.

• Claim: $\{\text{Alt}(e_I^*)\}$ where I is non-repeating and increasing is a basis for $\mathcal{A}^k(V)$. There are $\binom{n}{k}$ of these; thus, $\dim \mathcal{A}^k(V) = \binom{n}{k}$.

4/6:

• Klug will be in Texas on Monday and thus is cancelling class on Monday. Homework is now due next Friday. We'll have weekly homeworks going forward after that.

• Plan:

– $\text{Alt} : \mathcal{L}^k(V) \twoheadrightarrow \mathcal{A}^k(V)$ ^[2].

– Goal: Identify $\ker(\text{Alt}) = \mathcal{I}^k(V)$, where $\mathcal{I}^k(V)$ is the space of **redundant** k -tensors^[3].

¹Klug prefers this convention, but the text takes the other one.

²The two-headed right arrow denotes a surjective map.

³The \mathcal{I} in $\mathcal{I}^k(V)$ stands for “ideal.”

- Then: Operations on alternating tensors, e.g.,
 - Wedge product.
 - Interior product.
 - Orientations.
- Claim: $\{\text{Alt}(e_I^*) \mid I \text{ non-repeating, increasing multi-index}\}$ is a basis for $\mathcal{A}^k(V)$.
 - Left as an exercise to us.
- **Redundant** (k -tensor): A k -tensor of the form

$$\ell_1 \otimes \cdots \otimes \ell_i \otimes \ell_i \otimes \ell_{i+2} \otimes \cdots \otimes \ell_k$$

where $\ell_1, \dots, \ell_k \in V^*$.

- $\mathcal{I}^k(V)$: The span of all redundant k -tensors.
 - Note that not every k -tensor in $\mathcal{I}^k(V)$ is a redundant.
- **Decomposable** (k -tensor): A k -tensor of the form $\ell_1 \otimes \cdots \otimes \ell_k$ for $\ell_i \in \mathcal{L}^1(V)$.
 - It often suffices to prove things for decomposable tensors.
- Properties.
 1. If $T \in \mathcal{I}^k(V)$, then $\text{Alt}(T) = 0$, i.e., $\mathcal{I}^k(V) \leq \ker(\text{Alt})$.
 - “Proof by example”: If $T = \ell_1 \otimes \ell_1 \otimes \ell_2 \otimes \ell_3$, then $T^{\tau_{1,2}} = T$. It follows from the properties of Alt that

$$\begin{aligned}\text{Alt}(T) &= \text{Alt}(T^{\tau_{1,2}}) = (-1)^{\tau_{1,2}} \text{Alt}(T) = -\text{Alt}(T) \\ 2 \text{Alt}(T) &= 0 \\ \text{Alt}(T) &= 0\end{aligned}$$

2. If $T \in \mathcal{I}^r(V)$ and $T' \in \mathcal{L}^s(V)$, then

$$T \otimes T' \in \mathcal{I}^{r+s}(V)$$

Similarly, if $T \in \mathcal{L}^r(V)$ and $T' \in \mathcal{I}^s(V)$, then

$$T \otimes T' \in \mathcal{I}^{r+s}(V)$$

- Proof: It suffices to assume that T is redundant. Obviously adding more tensors to the direct product will not change the redundancy of the initial tensor. Example: $\ell_1 \otimes \ell_1 \otimes \ell_2$ is just as redundant as $\ell_1 \otimes \ell_1 \otimes \ell_2 \otimes T$.
3. If $T \in \mathcal{L}^k(V)$ and $\sigma \in S_k$, then

$$T^\sigma = (-1)^\sigma T + S$$

for some $S \in \mathcal{I}^k(V)$.

- Proof by example: It suffices to check this for decomposable tensors (a tensor is just a sum of decomposable tensors). Take $k = 2$. Let $T = \ell_1 \otimes \ell_2$. Let $\sigma = \tau_{1,2}$. Then

$$T^\sigma - (-1)^\sigma T = \ell_2 \otimes \ell_1 + \ell_1 \otimes \ell_2 = (\ell_1 + \ell_2) \otimes (\ell_1 + \ell_2) - \ell_1 \otimes \ell_1 - \ell_2 \otimes \ell_2$$

- Actual proof: It suffices to assume T is decomposable. We induct on the number of transpositions needed to write σ as a product of **adjacent** transpositions.

- Base case: $\sigma = \tau_{i,i+1}$. Then

$$\begin{aligned} T^{\tau_{i,i+1}} + T &= \ell_1 \otimes \cdots \otimes (\ell_i + \ell_{i+1}) \otimes (\ell_i + \ell_{i+1}) \otimes \cdots \otimes \ell_k \\ &\quad - \ell_1 \otimes \cdots \otimes \ell_i \otimes \ell_i \otimes \cdots \otimes \ell_k \\ &\quad - \ell_1 \otimes \cdots \otimes \ell_{i+1} \otimes \ell_{i+1} \otimes \cdots \otimes \ell_k \end{aligned}$$

- Inductive step: If $\sigma = \beta\tau$, then

$$\begin{aligned} T^\sigma &= T^{\beta\tau} \\ &= (-1)^\tau T^\beta + \text{stuff in } \mathcal{I}^k(V) \\ &= (-1)^\tau [(-1)^\beta T + \text{stuff in } \mathcal{I}^k(V)] + \text{stuff in } \mathcal{I}^k(V) \end{aligned}$$

- If $T \in \mathcal{L}^k(V)$, then

$$\text{Alt}(T) = k!T + W$$

for some $W \in \mathcal{I}^k(V)$.

- We have that

$$\begin{aligned} \text{Alt}(T) &= \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma \\ &= \sum_{\sigma \in S_k} (-1)^\sigma [(-1)^\sigma T + S_\sigma] \\ &= \sum_{\sigma \in S_k} T + \sum_{\sigma \in S_k} (-1)^\sigma S_\sigma \\ &= k!T + W \end{aligned}$$

- $\mathcal{I}^k(V) = \ker(\text{Alt})$.

- We have that $\mathcal{I}^k(V) \leq \ker(\text{Alt})$ by property 1.
- Now suppose $T \in \ker(\text{Alt})$. Then $\text{Alt}(T) = 0$. Then by property 4,

$$\begin{aligned} \text{Alt}(T) &= k!T + W \\ 0 &= k!T + W \\ T &= -\frac{1}{k!}W \in \mathcal{I}^k(V) \end{aligned}$$

- Warning: If $T \in \mathcal{A}^r(V)$ and $T' \in \mathcal{A}^s(V)$, then we do not necessarily have $T \otimes T' \in \mathcal{A}^{r+s}(V)$.

- Example: $e_1^*, e_2^* \in \mathcal{A}^1(V)$ have $e_1^* \otimes e_2^* \notin \mathcal{A}^2(V)$.

- **Adjacent** (transposition): A transposition of the form $\tau_{i,i+1}$.

4/8:

- Recall that $\mathcal{A}^k(V) \hookrightarrow \mathcal{L}^k(V)^{[4]}$

- Functoriality: $(A \circ B)^* = B^* \circ A^*$.

- A^* takes $\mathcal{L}^k(W) \rightarrow \mathcal{L}^k(V)$ and $\mathcal{A}^k(W) \rightarrow \mathcal{A}^k(V)$.

- $\dim(\Lambda^k(V)) = \binom{n}{k}$.

- Special case $k = n$: $\dim \Lambda^n(V) = 1$.

- If $A : V \rightarrow V$ induces a map $\Lambda^n(V^*) \rightarrow \Lambda^n(V^*)$ defined by the determinant.

- Aside: $\Lambda^k(V)$ is “exterior powers.”

⁴The hooked right arrow denotes an injective map.

- Plan: Wedge products + basis for $\Lambda^k(V)$.
- **Wedge product:** A function $\wedge : \Lambda^k(V^*) \times \Lambda^\ell(V^*) \rightarrow \Lambda^{k+\ell}(V)$.
 - We denote elements of $\Lambda^k(V^*)$ by ω_1, ω_2 , etc.
- If $\pi : \mathcal{L}^k(V) \rightarrow \Lambda^k(V^*)$ sends $T \mapsto \omega$, $\omega_1 = \pi(T_1)$, and $\omega_2 = \pi(T_2)$, then $\omega_1 \wedge \omega_2 = \pi(T_1 \otimes T_2)$.
 - Note that $\ker(\pi) = \mathcal{I}^k(V)$.
- Properties.

1. This is well defined, i.e., this does not depend on the choice of T_1, T_2 .

- Consider $T_1 + W_1, T_2 + W_2$ with $W_1, W_2 \in \mathcal{I}^k(V)$.
- We check that $\pi[(T_1 + W_1) \otimes (T_2 + W_2)] = \pi(T_1 \otimes T_2)$.
- Since $W_1 \otimes T_2, T_1 \otimes W_2, W_1 \otimes W_2 \in \mathcal{I}^{k+\ell}(V)$, we have that

$$\begin{aligned} \pi[(T_1 + W_1) \otimes (T_2 + W_2)] &= \pi(T_1 \otimes T_2 + W_1 \otimes T_2 + T_1 \otimes W_2 + W_1 \otimes W_2) \\ &= \pi(T_1 \otimes T_2) + \pi(W_1 \otimes T_2) + \pi(T_1 \otimes W_2) + \pi(W_1 \otimes W_2) \\ &= \pi(T_1 \otimes T_2) \end{aligned}$$

2. Associative: We have that

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge \omega_2 \wedge \omega_3$$

- Follows from the definition of \wedge in terms of π and properties of the tensor product.

3. Distributive: We have that

$$(\omega_1 + \omega_2) \wedge \omega_3 = \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_3 \quad \omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$$

- Follows from the definition of \wedge in terms of π and properties of the tensor product.

4. Linear: We have that

$$(c\omega_1) \wedge \omega_2 = c(\omega_1 \wedge \omega_2) = \omega_1 \wedge (c\omega_2)$$

- Follows from the definition of \wedge in terms of π and properties of the tensor product.

5. Anticommutative: We have that

$$\omega_1 \wedge \omega_2 = (-1)^{k\ell} \omega_2 \wedge \omega_1$$

- It suffices to assume that $w_1 = \ell_1 \wedge \cdots \wedge \ell_k, w_2 = \ell'_1 \wedge \cdots \wedge \ell'_\ell$.

■ We have

$$(\ell_1 \wedge \cdots \wedge \ell_k) \wedge (\ell'_1 \wedge \cdots \wedge \ell'_\ell) = (-1)^k (\ell'_1 \wedge \cdots \wedge \ell'_\ell) \wedge (\ell_1 \wedge \cdots \wedge \ell_k)$$

- Let $\ell_1, \dots, \ell_k \in \Lambda^1(V^*) = V^* = \mathcal{L}^1(V)$.
- Recall that $\mathcal{I}^1(V) = \{0\}$.
- Claim: $\ell_{\sigma(1)} \wedge \cdots \wedge \ell_{\sigma(k)} = (-1)^\sigma \ell_1 \wedge \cdots \wedge \ell_k$ for all $\sigma \in S_k$.

■ Recall that $T^\sigma = (-1)^\sigma T + W$ for some $W \in \mathcal{I}^k(V)$.

■ Let $T = \ell_1 \otimes \cdots \otimes \ell_k$.

■ Then

$$\begin{aligned} (\ell_1 \otimes \cdots \otimes \ell_k)^\sigma &= \ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)} \\ &= (-1)^\sigma \ell_1 \otimes \cdots \otimes \ell_k + W \end{aligned}$$

■ Then hit both sides by π , noting that $\pi(W) = 0$.

- Example:

1. $n = 2, k = \ell = 1$. Consider $e_1^*, e_2^* \in \mathcal{L}^1(V) = V^* = \mathcal{A}^1(V) = \Lambda^1(V^*)$. Then

$$e_1^* \wedge e_2^* = (-1)e_2^* \wedge e_1^* \qquad e_1^* \wedge e_1^* = 0 = e_2^* \wedge e_2^*$$

2. $n = 4$. We have $e_1^* \wedge (3e_1^* + 2e_2^* + 3e_3^*) = 3(e_1^* \wedge e_1^*) + 2(e_1^* \wedge e_2^*) + 3(e_1^* \wedge e_3^*)$. We also have $(e_1^* \wedge e_2^*) \wedge (e_1^* \wedge e_2^*) = 0$.

4/13:

- Plan:
 - Finish multilinear algebra.
 - Basis for $\Lambda^k(V^*)$.
 - Talk a bit about pullbacks and the determinant.
 - **Orientations** of vector spaces.
 - The **interior product**.
- Basis for $\Lambda^k(V^*)$.
 - Recall that $\{\text{Alt}(e_I^*) \mid I \text{ is a nonrepeating, increasing partition of } n \text{ into } k \text{ parts}\}$ is a basis for $\mathcal{A}^k(V)$.
- Alt is an isomorphism from $\Lambda^k(V^*)$ to $\mathcal{A}^k(V)$.
- If we have an injective map from $\mathcal{A}^k(V)$ to $\mathcal{L}^k(V)$ and π a projection map from $\mathcal{L}^k(V)$ to the quotient space $\mathcal{A}^k(V^*)$ gives rise to $\pi|_{\mathcal{A}^k(V)}$.
- Claim:
 1. $\pi|_{\mathcal{A}^k(V)}$ is an isomorphism.
 2. $\pi(\text{Alt}(e_I^*)) = k!\pi(e_I^*)$.
- (2) implies that $\{\pi(e_I^*) = e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*, I \text{ non-repeating and increasing}\}$ is a basis for $\Lambda^k(V^*)$.
- Examples:
 1. $n = 2 = \dim V, V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$.
 - $\Lambda^0(V^*) = \mathbb{R}$ since $\binom{n}{0} = 1$.
 - $\Lambda^1(V^*) = \mathbb{R}e_1^* \oplus \mathbb{R}e_2^*$ since $\binom{n}{1} = 2$.
 - $\Lambda^2(V^*) = \mathbb{R}e_1^* \wedge e_2^*$ since $\binom{n}{2} = 1$.
 - For the second to last one, note that $e_1^* \wedge e_2^* = -e_2^* \wedge e_1^*$.
 - $\Lambda^3(V^*) = 0$ since $\binom{2}{3} = 0$.
 - For the last one, note that all $e_1^* \wedge e_1^* \wedge e_2^* = 0$.
 2. $n = 3, V = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3$.
 - $\binom{n}{0} = 1$: $\Lambda^0(V^*) = \mathbb{R}$.
 - $\binom{n}{1} = 3$: $\Lambda^1(V^*) = \mathbb{R}e_1^* \oplus \mathbb{R}e_2^* \oplus \mathbb{R}e_3^*$.
 - $\binom{n}{2} = 3$: $\Lambda^2(V^*) = \mathbb{R}e_1^* \wedge e_2^* \oplus \mathbb{R}e_2^* \wedge e_3^* \oplus \mathbb{R}e_1^* \wedge e_3^*$.
 - $\binom{n}{3} = 1$: $\Lambda^3(V^*) = \mathbb{R}e_1^* \wedge e_2^* \wedge e_3^*$.
 - $\binom{n}{m} = 0$ ($m > n$): $\Lambda^m(V^*) = \Lambda^4(V^*) = 0$.
- If $A : V \rightarrow W, \omega_1 \in \Lambda^k(W^*), \omega_2 \in \Lambda^\ell(W^*)$, then

$$A^*(\omega_1 \wedge \omega_2) = A^*\omega_1 \wedge A^*\omega_2$$

- **Determinant:** Let $\dim V = n$. Let $A : V \rightarrow V$ be a linear transformation. This induces a pullback $A^* : \Lambda^n(V^*) \rightarrow \Lambda^n(V^*)$. The top exterior power $k = n$ implies $\binom{n}{n} = 1$. We define $\det(A)$ to be the unique real number such that $A^*(v) = \det(A)v$.

- This determinant is the one we know.

– A^* sends $e_1^* \wedge \cdots \wedge e_n^*$ to $A^*e_1^* \wedge \cdots \wedge A^*e_n^*$ which equals $A^*(e_1^* \wedge \cdots \wedge e_n^*)$ or $\det(A)$

- Sanity check.

1. $\det(\text{id}) = 1$.

– $\text{id}(e_1^* \wedge \cdots \wedge e_n^*) = \text{id}e_1^* \wedge \cdots \wedge \text{id}e_n^* = 1 \cdot e_1^* \wedge \cdots \wedge e_n^*$.

2. If A is not an isomorphism, then $\det(A) = 0$.

– If A is not an isomorphism, then there exists $v_1 \in \ker A$ with $v_1 \neq 0$. Let v_1^*, \dots, v_n^* be a basis of V^* . So the pullback of this wedge is the wedge of the pullbacks, but $A^*v_1^* = 0$, so

$$A^*(v_1^* \wedge \cdots \wedge v_n^*) = (A^*v_1^*) \wedge \cdots \wedge (A^*v_n^*) = 0 \wedge \cdots \wedge (A^*v_n^*) = 0 = 0 \cdot v_1^* \wedge \cdots \wedge v_n^*$$

3. $\det(AB) = \det(A)\det(B)$.

– Let $A : V \rightarrow V$ and $B : V \rightarrow V$.

– We have $(AB)^* = B^*A^*$; in particular, $n = k$, $V = W = U = V$.

- Recall: If we pick a basis for V , e_1, \dots, e_n .

– Implies $[a_{ij}] = [A]_{e_1, \dots, e_n}^{e_1, \dots, e_n}$.

- Does $\det(A) = \det([a_{ij}]) = \sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$?

– If $A : V \rightarrow V$, we know that $A^* : \Lambda^n(V^*) \rightarrow \Lambda^n(V^*)$ takes $e_1^* \wedge \cdots \wedge e_n^* \mapsto A^*(e_1^* \wedge \cdots \wedge e_n^*)$. We WTS

$$A^*(e_1^* \wedge \cdots \wedge e_n^*) = \left[\sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} \right] e_1^* \wedge \cdots \wedge e_n^*$$

– We have that

$$\begin{aligned} A^*(e_1^* \wedge \cdots \wedge e_n^*) &= A^*e_1^* \wedge \cdots \wedge A^*e_n^* \\ &= \left(\sum_{i_1=1}^n a_{i_1,1} e_{i_1}^* \right) \wedge \cdots \wedge \left(\sum_{i_n=1}^n a_{i_n,n} e_{i_n}^* \right) \\ &= \sum_{i_1, \dots, i_n} a_{i_1,1} \cdots a_{i_n,n} e_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \\ &= \left[\sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} \right] e_1^* \wedge \cdots \wedge e_n^* \end{aligned}$$

where the sign arises from the need to reorder $e_{i_1}^* \wedge \cdots \wedge e_{i_n}^*$ and the antisymmetry of the wedge product.

1.2 Chapter 1: Multilinear Algebra

From Guillemin and Haine (2018).

3/31:

- Guillemin and Haine (2018) defines real vector spaces, the operations on them, their basic properties, and the zero vector.
- **Linearly independent** (vectors v_1, \dots, v_k): A finite set of vectors $v_1, \dots, v_k \in V$ such that the map from \mathbb{R}^k to V defined by $(c_1, \dots, c_k) \mapsto c_1v_1 + \cdots + c_kv_k$ is injective.
- **Spanning** (vectors v_1, \dots, v_k): We require that the above map is surjective.

- Guillemin and Haine (2018) defines basis, finite-dimensional vector space, dimension, subspace, linear map, and kernel.
- **Image** (of $A : V \rightarrow W$): The range space of A , a subspace of W . *Also known as $\mathbf{im}(A)$.*
- Guillemin and Haine (2018) defines the matrix of a linear map.
- **Inner product** (on V): A map $B : V \times V \rightarrow \mathbb{R}$ with the following three properties.

– *Bilinearity*: For vectors $v, v_1, v_2, w \in V$ and $\lambda \in \mathbb{R}$, we have

$$B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w)$$

and

$$B(\lambda v, w) = \lambda B(v, w)$$

– *Symmetry*: For vectors $v, w \in V$, we have $B(v, w) = B(w, v)$.

– *Positivity*: For every vector $v \in V$, we have $B(v, v) \geq 0$. Moreover, if $v \neq 0$, then $B(v, v) > 0$.

- **W-coset**: A set of the form $\{v + w \mid w \in W\}$, where W is a subspace V and $v \in V$. *Denoted by $v + W$.*
 - If $v_1 - v_2 \in W$, then $v_1 + W = v_2 + W$.
 - It follows that the distinct W -cosets decompose V into a disjoint collection of subsets of V .

- **Quotient space** (of V by W): The set of distinct W -cosets in V , along with the following definitions of vector addition and scalar multiplication.

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W \qquad \lambda(v + W) = (\lambda v) + W$$

Denoted by V/W .

- **Quotient map**: The linear map $\pi : V \rightarrow V/W$ defined by

$$\pi(v) = v + W$$

– π is surjective.

– Note that $\ker(\pi) = W$ since for all $w \in W$, $\pi(w) = w + W = 0 + W$, which is the zero vector in V/W .

- If V, W are finite dimensional, then

$$\dim(V/W) = \dim(V) - \dim(W)$$

- Proposition 1.2.9: Let $A : V \rightarrow U$ be a linear map. If $W \subset \ker(A)$, then there exists a unique linear map $A^\sharp : V/W \rightarrow U$ with the property that $A = A^\sharp \circ \pi$, where $\pi : V \rightarrow V/W$ is the quotient map.

– This proposition rephrases in terms of quotient spaces the fact that if $w \in W$, then $A(v+w) = Av$.

- **Dual space** (of V): The set of all linear functions $\ell : V \rightarrow \mathbb{R}$, along with the following definitions of vector addition and scalar multiplication.

$$(\ell_1 + \ell_2)(v) = \ell_1(v) + \ell_2(v) \qquad (\lambda\ell)(v) = \lambda \cdot \ell(v)$$

Denoted by V^ .*

- **Dual basis** (of e_1, \dots, e_n a basis of V): The basis of V^* consisting of the n functions that take every $v = c_1 e_1 + \dots + c_n e_n$ to one of the c_i . *Denoted by e_1^*, \dots, e_n^* . Given by*

$$e_i^*(v) = c_i$$

for all $v \in V$.

- Claim 1.2.12: If V is an n -dimensional vector space with basis e_1, \dots, e_n , then e_1^*, \dots, e_n^* is a basis of V^* .

Proof. We will first prove that e_1^*, \dots, e_n^* spans V^* . Let $\ell \in V^*$ be arbitrary. Set $\lambda_i = \ell(e_i)$ for all $i \in [n]$. Define $\ell' = \sum_{i=1}^n \lambda_i e_i^*$. Then

$$\ell'(e_j) = \sum_{i=1}^n \lambda_i e_i^*(e_j) = \lambda_j \cdot 1 = \ell(e_j)$$

for all $j \in [n]$. Therefore, since ℓ, ℓ' take identical values on the basis of V , $\ell = \ell'$, as desired.

We now prove that e_1^*, \dots, e_n^* is linearly independent. Let $\sum_{i=1}^n \lambda_i e_i^* = 0$. Then for all $j \in [n]$,

$$\lambda_j = \left(\sum_{i=1}^n \lambda_i e_i^* \right) (e_j) = 0$$

as desired. □

- **Transpose** (of A): The map from W^* to V^* defined by $\ell \mapsto \ell \circ A$ for all $\ell \in W^*$. Denoted by A^* .
- Claim 1.2.15: If e_1, \dots, e_n is a basis of V , f_1, \dots, f_m is a basis of W , e_1^*, \dots, e_n^* and f_1^*, \dots, f_m^* are the corresponding dual bases, and $[a_{i,j}]$ is the $m \times n$ matrix of A with respect to $\{e_j\}, \{f_i\}$, then the linear map A^* is defined in terms of $\{f_i^*\}, \{e_j^*\}$ by the transpose matrix $(a_{j,i})$.

Proof. Let $[c_{j,i}]$ be the $n \times m$ matrix of A^* with respect to $\{f_i^*\}, \{e_j^*\}$. We seek to prove that $a_{i,j} = c_{j,i}$ ($1 \leq i \leq m, 1 \leq j \leq n$).

By the definition of $[a_{i,j}]$ and $[c_{j,i}]$, we have that

$$A^* f_i^* = \sum_{k=1}^n c_{k,i} e_k^* \qquad A e_j = \sum_{k=1}^m a_{k,j} f_k$$

It follows that

$$[A^* f_i^*](e_j) = \left[\sum_{k=1}^n c_{k,i} e_k^* \right] (e_j) = c_{j,i}$$

and

$$[A^* f_i^*](e_j) = f_i^*(A e_j) = f_i^* \left(\sum_{k=1}^m a_{k,j} f_k \right) = a_{i,j}$$

so transitivity implies the desired result. □

4/4:

- V^k : The set of all k -tuples (v_1, \dots, v_k) where $v_1, \dots, v_k \in V$ a vector space.
– Note that

$$V^k = \underbrace{V \oplus \dots \oplus V}_{k \text{ times}}$$

where “ \oplus ” denotes the direct sum.

- **Linear** (function in its i^{th} variable): A function $T : V^k \rightarrow \mathbb{R}$ such that the map from V to \mathbb{R} defined by $v \mapsto T(v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_k)$ is linear, where all v_j save v_i are fixed.
- **k -linear** (function T): A function $T : V^k \rightarrow \mathbb{R}$ that is linear in its i^{th} variable for $i = 1, \dots, k$. Also known as **k -tensor**.
- $\mathcal{L}^k(V)$: The set of all k -tensors in V .

- Since the sum $T_1 + T_2$ of two k -linear functions $T_1, T_2 : V^k \rightarrow \mathbb{R}$ is just another k -linear function, and λT_1 is k -linear for all $\lambda \in \mathbb{R}$, we have that $\mathcal{L}^k(V)$ is a vector space.

- Convention: 0-tensors are just the real numbers. Mathematically, we define

$$\mathcal{L}^0(V) = \mathbb{R}$$

- Note that $\mathcal{L}^1(V) = V^*$.
- Defines multi-indices of n of length k .
- Lemma 1.3.5: If $n, k \in \mathbb{N}$, then there are exactly n^k multi-indices of n of length k .
- T_I : The real number $T(e_{i_1}, \dots, e_{i_k})$, where $T \in \mathcal{L}^k(V)$, e_1, \dots, e_n is a basis of V , and I is a multi-index of n of length k .
- Proposition 1.3.7: The real numbers T_I determine T , i.e., if T, T' are k -tensors and $T_I = T'_I$ for all I , then $T = T'$.

Proof. We induct on n . For the base case $n = 1$, $T \in (\mathbb{R}^k)^*$ and we have already proven this result. Now suppose inductively that the assertion is true for $n - 1$. For each e_i , let T_i be the $(k - 1)$ -tensor defined by

$$(v_1, \dots, v_{n-1}) \mapsto T(v_1, \dots, v_{n-1}, e_i)$$

Then for an arbitrary $v = c_1 e_1 + \dots + c_n e_n$,

$$T(v_1, \dots, v_{n-1}, v) = \sum_{i=1}^n c_i T_i(v_1, \dots, v_{n-1})$$

so the T_i 's determine T . Applying the inductive hypothesis completes the proof. \square

- **Tensor product:** The function $\otimes : \mathcal{L}^k(V) \times \mathcal{L}^\ell(V) \rightarrow \mathcal{L}^{k+\ell}(V)$ defined by

$$(T_1 \otimes T_2)(v_1, \dots, v_{k+\ell}) = T_1(v_1, \dots, v_k) T_2(v_{k+1}, \dots, v_{k+\ell})$$

for all $T_1 \in \mathcal{L}^k(V)$ and $T_2 \in \mathcal{L}^\ell(V)$.

- Note that by the definition of 0-tensors as real numbers, if $a \in \mathbb{R}$ and $T \in \mathcal{L}^k(V)$, then

$$a \otimes T = T \otimes a = aT$$

- Proposition 1.3.9: Associativity, distributivity of scalar multiplication, and left and right distributive laws for the tensor product.
- **Decomposable** (k -tensor): A k -tensor T for which there exist $\ell_1, \dots, \ell_k \in V^*$ such that

$$T = \ell_1 \otimes \dots \otimes \ell_k$$

- Defines e_I^* .
- Theorem 1.3.13: V a vector space with basis e_1, \dots, e_n and $0 \leq k \leq n$ implies the k -tensors e_I^* form a basis of $\mathcal{L}^k(V)$.

Proof. Spanning: Let $T \in \mathcal{L}^k(V)$ be arbitrary. Define

$$T' = \sum_I T_I e_I^*$$

Since

$$T'_J = T'(e_{j_1}, \dots, e_{j_k}) = \sum_I T_I e_I^*(e_{j_1}, \dots, e_{j_k}) = T_J e_J^*(e_{j_1}, \dots, e_{j_k}) = T_J$$

for all J , Proposition 1.3.7 asserts that $T = T'$. Therefore, since every $T_I \in \mathbb{R}$, $T = T' \in \text{span}(e_I^*)$.

Linear independence: Suppose

$$T = \sum_I c_I e_I^* = 0$$

for some set of constants $c_I \in \mathbb{R}$. Then

$$0 = T(e_{j_1}, \dots, e_{j_k}) = \sum_I c_I e_I^*(e_{j_1}, \dots, e_{j_k}) = c_J$$

for all J , as desired. □

- Corollary 1.3.15: If $\dim V = n$, then $\dim(\mathcal{L}^k(V)) = n^k$.

Proof. Follows immediately from Lemma 1.3.5. □

- **Pullback** (of T by the map A): The k -tensor $A^*T : V^k \rightarrow \mathbb{R}$ defined by

$$(A^*T)(v_1, \dots, v_k) = T(Av_1, \dots, Av_k)$$

where V, W are finite-dimensional vector spaces, $A : V \rightarrow W$ is linear, and $T \in \mathcal{L}^k(W)$.

- Proposition 1.3.18: The map $A^* : \mathcal{L}^k(W) \rightarrow \mathcal{L}^k(V)$ defined by $T \mapsto A^*T$ is linear.

- Identities:

– If $T_1 \in \mathcal{L}^k(W)$ and $T_2 \in \mathcal{L}^m(W)$, then

$$A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$$

– If U is a vector space, $B : U \rightarrow V$ is linear, and $T \in \mathcal{L}^k(W)$, then $(AB)^*T = B^*(A^*T)$. Hence,

$$(AB)^* = B^*A^*$$

- 4/13: • **Σ_k** : The set containing the natural numbers 1 through k . *Given by*

$$\Sigma_k = \{1, 2, \dots, k\}$$

- **Permutation of order k** : A bijection on Σ_k . *Denoted by σ .*
- **Product** (of σ_1, σ_2): The composition $\sigma_1 \circ \sigma_2$, i.e., the map

$$i \mapsto \sigma_1(\sigma_2(i))$$

Denoted by $\sigma_1\sigma_2$.

- **Inverse** (of σ): The permutation of order k which is the inverse bijection of σ . *Denoted by σ^{-1} .*
- **Permutation group** (of Σ_k): The set of all permutations of order k . *Also known as **symmetric group on k letters**. Denoted by S_k .*
- Lemma 1.4.2: The group S_k has $k!$ elements.

- **Transposition:** A permutation of order k defined by

$$\ell \mapsto \begin{cases} j & \ell = i \\ i & \ell = j \\ \ell & \ell \neq i, j \end{cases}$$

for all $\ell \in \Sigma_k$, where $i, j \in \Sigma_k$. Denoted by $\tau_{i,j}$.

- **Elementary transposition:** A transposition of the form $\pi_{i,i+1}$.
- Theorem 1.4.4: Every $\sigma \in S_k$ can be written as a product of (a finite number of) transpositions.

Proof. We induct on k .

For the base case $k = 2$, the identity permutation of S_2 is the “product” of zero transpositions, and the only other permutation is a transposition (the “product” of one transposition, namely itself).

Now suppose inductively that we have proven the claim for $k - 1$. Let $\sigma \in S_k$ be arbitrary. Suppose $\sigma(k) = i$. Then $\tau_{i,k}\sigma(k) = k$. Since $(\tau_{i,k}\sigma)|_{\Sigma_{k-1}} \in S_{k-1}$, we have by the inductive hypothesis that $(\tau_{i,k}\sigma)|_{\Sigma_{k-1}} = \tau_1 \cdots \tau_m$ for some set of permutations $\tau_1, \dots, \tau_m \in S_{k-1}$. For each τ_j ($1 \leq j \leq m$), define $\tau'_j \in S_k$

$$\tau'_j(\ell) = \begin{cases} \tau_j(\ell) & \ell < k \\ \ell & \ell = k \end{cases}$$

It follows that

$$\begin{aligned} \tau_{i,k}\sigma &= \tau'_1 \cdots \tau'_m \\ \sigma &= \tau_{i,k}\tau'_1 \cdots \tau'_m \end{aligned}$$

as desired. □

- Theorem 1.4.5: Every transposition can be written as a product of elementary transpositions.

Proof. Let $\tau_{i,j} \in S_k$, and let $i < j$ WLOG. Then we have that

$$\tau_{i,j} = \prod_{\ell=i}^{j-1} \tau_{\ell,\ell+1}$$

as desired. □

- Corollary 1.4.6: Every permutation can be written as a product of elementary transpositions.
- **Sign** (of σ): The number ± 1 assigned to σ by the expression

$$\prod_{i < j} \frac{x_{\sigma(i)} - x_{\sigma(j)}}{x_i - x_j}$$

where x_1, \dots, x_k are coordinate functions on \mathbb{R}^k . Denoted by $(-1)^\sigma$.

- Claim 1.4.9: The sign defines a group homomorphism $S_k \rightarrow \{\pm 1\}$. That is, for $\sigma_1, \sigma_2 \in S_k$, we have

$$(-1)^{\sigma_1\sigma_2} = (-1)^{\sigma_1}(-1)^{\sigma_2}$$

Proof. For all $i < j$, define p, q such that p is the lesser of $\sigma_2(i), \sigma_2(j)$ and q is the greater of $\sigma_2(i), \sigma_2(j)$. Formally,

$$p = \begin{cases} \sigma_2(i) & \sigma_2(i) < \sigma_2(j) \\ \sigma_2(j) & \sigma_2(j) < \sigma_2(i) \end{cases} \quad q = \begin{cases} \sigma_2(j) & \sigma_2(i) < \sigma_2(j) \\ \sigma_2(i) & \sigma_2(j) < \sigma_2(i) \end{cases}$$

It follows that if $\sigma_2(i) < \sigma_2(j)$, then

$$\frac{x_{\sigma_1\sigma_2(i)} - x_{\sigma_1\sigma_2(j)}}{x_{\sigma_2(i)} - x_{\sigma_2(j)}} = \frac{x_{\sigma_1(p)} - x_{\sigma_1(q)}}{x_p - x_q}$$

and if $\sigma_2(j) < \sigma_2(i)$, then

$$\frac{x_{\sigma_1\sigma_2(i)} - x_{\sigma_1\sigma_2(j)}}{x_{\sigma_2(i)} - x_{\sigma_2(j)}} = \frac{x_{\sigma_1(q)} - x_{\sigma_1(p)}}{x_q - x_p} = \frac{x_{\sigma_1(p)} - x_{\sigma_1(q)}}{x_p - x_q}$$

Therefore,

$$\begin{aligned} (-1)^{\sigma_1\sigma_2} &= \prod_{i < j} \frac{x_{\sigma_1\sigma_2(i)} - x_{\sigma_1\sigma_2(j)}}{x_i - x_j} \\ &= \prod_{i < j} \frac{x_{\sigma_1\sigma_2(i)} - x_{\sigma_1\sigma_2(j)}}{x_{\sigma_2(i)} - x_{\sigma_2(j)}} \cdot \frac{x_{\sigma_2(i)} - x_{\sigma_2(j)}}{x_i - x_j} \\ &= \prod_{i < j} \frac{x_{\sigma_1(p)} - x_{\sigma_1(q)}}{x_p - x_q} \cdot \prod_{i < j} \frac{x_{\sigma_2(i)} - x_{\sigma_2(j)}}{x_i - x_j} \\ &= (-1)^{\sigma_1}(-1)^{\sigma_2} \end{aligned}$$

as desired. \square

- Proposition 1.4.11: If σ is the product of an odd number of transpositions, then $(-1)^\sigma = -1$, and if σ is the product of an even number of transpositions, then $(-1)^\sigma = +1$.

Proof. Follows from the fact that $(-1)^\sigma = -1$ (see Exercise 1.4.ii). \square

- T^σ : The k -tensor defined by

$$T^\sigma(v_1, \dots, v_k) = T(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)})$$

where $T \in \mathcal{L}^k(V)$, V is an n -dimensional vector space, and $\sigma \in S_k$.

- Proposition 1.4.14:

1. If $T = \ell_1 \otimes \dots \otimes \ell_k$ ($\ell_i \in V^*$), then $T^\sigma = \ell_{\sigma(1)} \otimes \dots \otimes \ell_{\sigma(k)}$.

Proof. If $v_1, \dots, v_k \in V$, then

$$\begin{aligned} T^\sigma(v_1, \dots, v_k) &= T(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)}) \\ &= [\ell_1 \otimes \dots \otimes \ell_k](v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)}) \\ &= \ell_1(v_{\sigma^{-1}(1)}) \dots \ell_k(v_{\sigma^{-1}(k)}) \\ &= \ell_{\sigma(1)}(v_1) \dots \ell_{\sigma(k)}(v_k) \\ &= [\ell_{\sigma(1)} \otimes \dots \otimes \ell_{\sigma(k)}](v_1, \dots, v_k) \end{aligned}$$

as desired. Note that we can justify the fourth equality by noting that if $\sigma^{-1}(i) = q$, then the i^{th} term in the product is $\ell_{\sigma(q)}(v_q)$, so since σ is a bijection, the product can be arranged to the form on the right-hand side of equality four. \square

2. The assignment $T \mapsto T^\sigma$ is a linear map from $\mathcal{L}^k(V) \rightarrow \mathcal{L}^k(V)$.

Proof. See Exercise 1.4.iii. □

3. If $\sigma_1, \sigma_2 \in S_k$, we have $T^{\sigma_1\sigma_2} = (T^{\sigma_1})^{\sigma_2}$.

Proof. Let $T = \ell_1 \otimes \cdots \otimes \ell_k^{[5]}$. Then

$$T^{\sigma_1} = \ell_{\sigma_1(1)} \otimes \cdots \otimes \ell_{\sigma_1(k)} = \ell'_1 \otimes \cdots \otimes \ell'_k$$

and thus

$$(T^{\sigma_1})^{\sigma_2} = \ell'_{\sigma_2(1)} \otimes \cdots \otimes \ell'_{\sigma_2(k)}$$

Let $\sigma_2(i) = j$. Then since $\ell'_p = \ell_{\sigma_1(p)}$ by definition, we have that $\ell'_{\sigma_2(j)} = \ell_{\sigma_1(\sigma_2(j))}$. Therefore,

$$\begin{aligned} (T^{\sigma_1})^{\sigma_2} &= \ell'_{\sigma_2(1)} \otimes \cdots \otimes \ell'_{\sigma_2(k)} \\ &= \ell_{\sigma_1(\sigma_2(1))} \otimes \cdots \otimes \ell_{\sigma_1(\sigma_2(k))} \\ &= \ell_{\sigma_1\sigma_2(1)} \otimes \cdots \otimes \ell_{\sigma_1\sigma_2(k)} \\ &= T^{\sigma_1\sigma_2} \end{aligned}$$

as desired. □

- **Alternating** (k -tensor): A k -tensor $T \in \mathcal{L}^k(V)$ such that $T^\sigma = (-1)^\sigma T$ for all $\sigma \in S_k$.
- $\mathcal{A}^k(V)$: The set of all alternating k -tensors in $\mathcal{L}^k(V)$.
 - Proposition 1.4.14(2) implies that $(T_1 + T_2)^\sigma = T_1^\sigma + T_2^\sigma$ and $(\lambda T_1)^\sigma = \lambda T_1^\sigma$; it follows that $\mathcal{A}^k(V)$ is a vector space.
- **Alternation operation**: The function from $\mathcal{L}^k(V) \rightarrow \mathcal{L}^k(V)$ defined by

$$T \mapsto \sum_{\tau \in S_k} (-1)^\tau T^\tau$$

Denoted by **Alt**.

- Proposition 1.4.17: For $T \in \mathcal{L}^k(V)$ and $\sigma \in S_k$, we have that

1. $\text{Alt}(T)^\sigma = (-1)^\sigma \text{Alt } T$.

Proof. We have that

$$\begin{aligned} \text{Alt}(T)^\sigma &= \left(\sum_{\tau \in S_k} (-1)^\tau T^\tau \right)^\sigma \\ &= \sum_{\tau \in S_k} (-1)^\tau (T^\tau)^\sigma && \text{Proposition 1.4.14(2)} \\ &= \sum_{\tau \in S_k} (-1)^\tau T^{\tau\sigma} && \text{Proposition 1.4.14(3)} \\ &= (-1)^\sigma \sum_{\tau \in S_k} (-1)^{\tau\sigma} T^{\tau\sigma} \\ &= (-1)^\sigma \sum_{\tau\sigma \in S_k} (-1)^{\tau\sigma} T^{\tau\sigma} \\ &= (-1)^\sigma \text{Alt } T \end{aligned}$$

as desired. □

⁵What gives us the right to assume T is decomposable?

2. If $T \in \mathcal{A}^k(V)$, then $\text{Alt } T = k!T$.

Proof. Since $T \in \mathcal{A}^k(V)$, we know that $T^\sigma = (-1)^\sigma T$. Therefore,

$$\text{Alt } T = \sum_{\tau \in S_k} (-1)^\tau T^\tau = \sum_{\tau \in S_k} (-1)^\tau (-1)^\tau T = \sum_{\tau \in S_k} T = k!T$$

where the last equality holds because the cardinality of S_k is $k!$. □

3. $\text{Alt}(T^\sigma) = \text{Alt}(T)^\sigma$.

Proof. We have that

$$\text{Alt}(T^\sigma) = \sum_{\tau \in S_k} (-1)^\tau T^{\tau\sigma} = (-1)^\sigma \sum_{\tau \in S_k} (-1)^{\tau\sigma} T^{\tau\sigma} = (-1)^\sigma \text{Alt}(T) = \text{Alt}(T)^\sigma$$

as desired. □

4. The alternation operation is linear.

Proof. Follows by Proposition 1.4.14. □

- **Repeating** (multi-index I): A multi-index I of length k such that $i_r = i_s$ for some $r \neq s$.
- **Strictly increasing** (multi-index I): A multi-index I of length k such that $i_1 < i_2 < \dots < i_k$.
- I^σ : The multi-index of length k defined by

$$I^\sigma = (i_{\sigma(1)}, \dots, i_{\sigma(k)})$$

- If I is non-repeating, there is a unique $\sigma \in S_k$ such that I^σ is strictly increasing.
- ψ_I : The following k -tensor. *Given by*

$$\psi_I = \text{Alt}(e_I^*)$$

- Proposition 1.4.20:

1. $\psi_{I^\sigma} = (-1)^\sigma \psi_I$.

Proof. We have that

$$\psi_{I^\sigma} = \text{Alt}(e_{I^\sigma}^*) = \text{Alt}[(e_I^*)^\sigma] = \text{Alt}(e_I^*)^\sigma = (-1)^\sigma \text{Alt}(e_I^*) = (-1)^\sigma \psi_I$$

as desired. □

2. If I is repeating, then $\psi_I = 0$.

Proof. Suppose $I = (i_1, \dots, i_k)$ is such that $i_r = i_s$ for some distinct $r, s \in \Sigma_k$. Then $e_I^* = e_{I^{\tau_{i_r, i_s}}}^*$, so

$$\psi_I = \psi_{I^{\tau_{i_r, i_s}}} = (-1)^{\tau_{i_r, i_s}} \psi_I = -\psi_I$$

Therefore, we must have $\psi_I = 0$, as desired. □

3. If I and J are strictly increasing, then

$$\psi_I(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$$

Proof. We have by definition that

$$\psi_I(e_{j_1}, \dots, e_{j_k}) = \sum_{\tau} (-1)^{\tau} e_{I^{\tau}}^*(e_{j_1}, \dots, e_{j_k})$$

This combined with the facts that

$$e_{I^{\tau}}^*(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1 & I^{\tau} = J \\ 0 & I^{\tau} \neq J \end{cases}$$

I^{τ} is strictly increasing iff $I^{\tau} = I$, and the above equation is nonzero iff $I^{\tau} = I = J$ implies the desired result. \square

- Conclusion 1.4.22: If $T \in \mathcal{A}^k(V)$, then we can write T as a sum

$$T = \sum_I c_I \psi_I$$

with I 's strictly increasing.

Proof. Let $T \in \mathcal{A}^k(V)$ be arbitrary. By Theorem 1.3.13,

$$T = \sum_J a_J e_J^*$$

for some set of $a_J \in \mathbb{R}$. It follows since $\text{Alt}(T) = k!T$ that

$$T = \frac{1}{k!} \sum a_J \text{Alt}(e_J^*) = \sum b_J \psi_J$$

We can disregard all repeating terms in the sum since they are zero by Proposition 1.4.20(2); for every non-repeating term J , we can write $J = I^{\sigma}$, where I is strictly increasing and hence $\psi_J = (-1)^{\sigma} \psi_I$. \square

- Claim 1.4.24: The c_I 's of Conclusion 1.4.22 are unique.

Proof. For J strictly increasing, we have

$$T_J = T(e_{j_1}, \dots, e_{j_k}) = \sum_I c_I \psi_I(e_{j_1}, \dots, e_{j_k}) = c_J$$

\square

- Proposition 1.4.26: The alternating tensors ψ_I with I strictly increasing are a basis for $\mathcal{A}^k(V)$.

Proof. Spanning: See Conclusion 1.4.22.

Linear independence: See Claim 1.4.24. \square

- We have that

$$\dim \mathcal{A}^k(V) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

- Hint in proving this claim: “Show that every strictly increasing multi-index of length k determines a k -element subset of $\{1, \dots, n\}$ and vice versa.” (Guillemin & Haine, 2018, p. 16).
- Note also that if $k > n$, every multi-index has a repeat somewhere, meaning that $\dim \mathcal{A}^k(V) = \binom{n}{k} = 0$.