6 Calculus on Manifolds

From Guillemin and Haine (2018).

Chapter 4

5/29: **4.3.ii.** Let S^2 be the unit 2-sphere $x_1^2+x_2^2+x_3^2=1$ in $\mathbb{R}^3$ and let $\boldsymbol{w}$ be the vector field

$$\boldsymbol{w}=x_1\frac{\partial}{\partial x_2}-x_2\frac{\partial}{\partial x_1}$$

(1) Show that $\boldsymbol{w}$ is tangent to S^2 and hence by restriction defines a vector field $\boldsymbol{v}$ on S^2.

Proof. To show that $\boldsymbol{w}$ is tangent to S^2, it will suffice to prove that for every $p\in S^2$, $\boldsymbol{w}(p)\in T_pS^2$. Let $p=(p_1,p_2,p_3)\in S^2$ be arbitrary. To prove that $\boldsymbol{w}(p)\in T_pS^2$, Exercise 4.2.ii tells us that it will suffice to check that $p\cdot\boldsymbol{w}(p)=0$. Indeed, we have that

$$p\cdot\boldsymbol{w}(p)=(p_1)(-p_2)+(p_2)(p_1)+(p_3)(0)=0$$

as desired. ☐

(2) What are the integral curves of $\boldsymbol{v}$?

Proof. Let $p=(p_1,p_2,p_3)\in S^2$ be arbitrary. Then

$$\boldsymbol{v}(p)=-p_2\frac{\partial}{\partial x_1}+p_1\frac{\partial}{\partial x_2}+0\frac{\partial}{\partial x_3}$$

Suppose there exists $\gamma:I\to S^2$, where I is an open interval containing $t_0=0$, such that $\gamma(0)=p$ and $\mathrm{d}\gamma_0(\vec{u})=\boldsymbol{v}(p)$. It follows from the second statement that

$$\begin{aligned}(p,D\gamma(0)(1))&=(p,(-p_2,p_1,0))\\ \begin{bmatrix}\left.\frac{\mathrm{d}\gamma_1}{\mathrm{d}t}\right|_0\\ \left.\frac{\mathrm{d}\gamma_2}{\mathrm{d}t}\right|_0\\ \left.\frac{\mathrm{d}\gamma_3}{\mathrm{d}t}\right|_0\end{bmatrix}&=\begin{bmatrix}-\gamma_2(0)\\ \gamma_1(0)\\ 0\end{bmatrix}\\ \begin{bmatrix}\frac{\mathrm{d}\gamma_1}{\mathrm{d}t}\\ \frac{\mathrm{d}\gamma_2}{\mathrm{d}t}\\ \frac{\mathrm{d}\gamma_3}{\mathrm{d}t}\end{bmatrix}&=\begin{bmatrix}-\gamma_2\\ \gamma_1\\ 0\end{bmatrix}\end{aligned}$$

From the last line above, we can determine that

$$\gamma_3(t)=C$$

for some $C\in\mathbb{R}$. We can also extract the coupled differential equations

$$\frac{\mathrm{d}\gamma_1}{\mathrm{d}t}=-\gamma_2 \qquad\qquad \frac{\mathrm{d}\gamma_2}{\mathrm{d}t}=\gamma_1$$

Differentiating the right equation above with respect to t reveals via the transitive property that

$$\frac{\mathrm{d}^2\gamma_2}{\mathrm{d}t^2}=\frac{\mathrm{d}\gamma_1}{\mathrm{d}t}=-\gamma_2$$

We can recognize the above differential equation to be the one describing simple harmonic motion, i.e., the one having general solution

$$\gamma_2(t)=A\mathrm{e}^{irt}+B\mathrm{e}^{-irt}$$

for some $A, B, r \in \mathbb{R}$. It follows since $\mathrm{d}\gamma_2/\mathrm{d}t = \gamma_1$ that

$$\gamma_1(t) = Aire^{irt} - Bire^{-irt}$$

We now solve for A, B, C, r using the initial conditions

$$\begin{bmatrix}\gamma_1(0)\\ \gamma_2(0)\\ \gamma_3(0)\end{bmatrix} = \begin{bmatrix}p_1\\ p_2\\ p_3\end{bmatrix} \qquad \begin{bmatrix}\gamma_1'(0)\\ \gamma_2'(0)\\ \gamma_3'(0)\end{bmatrix} = \begin{bmatrix}-p_2\\ p_1\\ 0\end{bmatrix}$$

First off, we have that

$$C = \gamma_3(0) = p_3$$

We also have that

$$\begin{aligned} p_2 &= \gamma_2(0) \\ &= Ae^{ir\cdot 0} + Be^{-ir\cdot 0} \\ &= A + B \end{aligned} \qquad \begin{aligned} -p_2 &= \gamma_1'(0) \\ &= -Ar^2e^{ir\cdot 0} - Br^2e^{-ir\cdot 0} \\ &= -r^2(A+B) \end{aligned}$$

It follows that $r = 1$. Additionally, we have that

$$\begin{aligned} p_1 &= \gamma_1(0) \\ &= Aie^{i\cdot 0} - Bie^{-i\cdot 0} \\ &= (A-B)i \end{aligned}$$

Thus, we have the system of equations

$$\begin{aligned} A + B &= p_2 \\ iA - iB &= p_1 \end{aligned}$$

We can solve it to determine that

$$A = \frac{p_2 - ip_1}{2} \qquad B = \frac{p_2 + ip_1}{2}$$

It follows that

$$\begin{aligned} \gamma_2(t) &= Ae^{it} + Be^{-it} \\ &= \frac{p_2 - ip_1}{2}(\cos(t) + i\sin(t)) + \frac{p_2 + ip_1}{2}(\cos(t) - i\sin(t)) \\ &= p_2\cos(t) + p_1\sin(t) \end{aligned}$$

and thus that

$$\gamma_1(t) = p_1\cos(t) - p_2\sin(t)$$

Finally, we have a complete description of the integral curve $\gamma : \mathbb{R} \to S^2$ with $\gamma(0) = p$ of $\boldsymbol{v}$.

$$\boxed{\gamma(t) = \begin{bmatrix} p_1\cos(t) - p_2\sin(t) \\ p_2\cos(t) + p_1\sin(t) \\ p_3 \end{bmatrix}}$$

Note that we can do one better by transforming our three equations into a form that could easily be found by inspection.

By inspecting the vector field, we can determine that the lack of x_3-component in any vector in $\boldsymbol{v}$ means that our integral curve must not vary with respect to x_3 either, i.e., γ_3 should just be the height of the integral curve above or below the x_1x_2-plane, i.e., p_3 (as it is via the above). On the other hand, γ_1, γ_2 should work together (this is why we got the *coupled* system of differential equations above) to trace out a latitude line, i.e., a circular submanifold at p_3 above or below the x_1x_2-plane. Thus, they should be sinusoidal, with radius dictated by the height above the x_1x_2-plane and phase offset dictated by the position of p relative to the $+x_1$-axis. Since $x_1^2 + x_2^2 + x_3^1 = 1$, trigonometric arguments show that the radius of this circle should be $\sqrt{1 - x_3^2}$, i.e., the maximum possible radius ($\sqrt{1} = 1$) less come correction factor based on the distance between the circle and the x_1x_2-plane. Similarly, trigonometric arguments show that the phase offset should be $\tan^{-1}(p_2/p_1)$. Thus, the final form should be

$$\gamma_1(t) = \sqrt{1 - x_3^2}\cos\left(t + \tan^{-1}\left(\frac{p_2}{p_1}\right)\right) \qquad \gamma_2(t) = \sqrt{1 - x_3^2}\sin\left(t + \tan^{-1}\left(\frac{p_2}{p_1}\right)\right)$$

But since $a\cos x + b\sin x$ can be written as $R\cos(x - \alpha)$, where $R = \sqrt{a^2 + b^2}$ and $\tan\alpha = b/a$, we have that

$$\begin{aligned}
\gamma_1(t) &= p_1\cos(t) - p_2\sin(t) & \gamma_2(t) &= p_2\cos(t) + p_1\sin(t) \\
&= \sqrt{p_1^2 + p_2^2}\cos\left(t - \tan^{-1}\left(\frac{-p_2}{p_1}\right)\right) & &= \sqrt{p_1^2 + p_2^2}\cos\left(t - \tan^{-1}\left(\frac{p_1}{p_2}\right)\right) \\
&= \sqrt{1 - p_3^2}\cos\left(t + \tan^{-1}\left(\frac{p_2}{p_1}\right)\right) & &= \sqrt{1 - p_3^2}\sin\left(t + \tan^{-1}\left(\frac{p_2}{p_1}\right)\right) \\
&= \gamma_1(t) & &= \gamma_2(t)
\end{aligned}$$

as desired. ☐

Note from Dr. Klug: For the following problems you'll need to check back in Section 1.9 and maybe warm-up with the pointwise Exercise 1.9.xi. We are going to define a special top-dimensional form on a manifold (which for us is always inside a Euclidean space — more generally this is where you would need to "fix a metric on your abstract manifold"... blah blah blah) called the (Riemannian) volume form — see Theorem 4.4.9 in your book. An (admittedly fancy but agreeing with any pedestrian way of doing it) definition of the volume of a subset of $\mathbb{R}^N$ that is a manifold is then the integral over that manifold of the volume form.

Chapter 1

1.9.xi. Let V be an n-dimensional vector space $B : V \times V \to \mathbb{R}$ an inner product, and $e_1, \dots, e_n$ a basis of V which is positively oriented and orthonormal. Show that the **volume element**

$$\text{vol} = e_1^* \wedge \cdots \wedge e_n^* \in \Lambda^n(V^*)$$

is intrinsically defined, independent of the choice of basis. (Hint: The equations

$$AA^\intercal = \text{id}_n \qquad\qquad A^*(f_1^* \wedge \cdots \wedge f_n^*) = \det(a_{i,j})e_1^* \wedge \cdots \wedge e_n^*$$

may be of use. Note that in the left equation above, A is a change of coordinate matrix between two orthonormal bases.)

Proof. To show that the volume element is defined independently of the choice of basis, let $e_1', \dots, e_n'$ be another basis of V which is positively oriented and orthonormal; we wish to verify that the volume element

$$\text{vol}' = {e_1'}^* \wedge \cdots \wedge {e_n'}^*$$

with respect to this basis is equal to vol. Let A be the change of coordinates matrix which sends $e_i \mapsto e_i'$ ($i = 1, \dots, n$). It follows that $AA^\intercal = \text{id}_n$, so

$$\begin{aligned}
\det(\text{id}_n) &= \det(AA^\intercal) \\
1 &= \det(A) \cdot \det(A^\intercal) \\
&= \det(A) \cdot \det(A) \\
&= \det(A)^2 \\
\det(A) &= \pm 1
\end{aligned}$$

Additionally, we know that both $e_1, \dots, e_n$ and $e'_1, \dots, e'_n$ are positively oriented, so by Proposition 1.9.7, $\det(A) > 0$. Thus, $\det(A) = 1$. Therefore, we have that

$$\begin{aligned}
\mathrm{vol} &= e_1^* \wedge \cdots \wedge e_n^* \\
&= 1 \cdot e_1^* \wedge \cdots \wedge e_n^* \\
&= \det(a_{i,j}) e_1^* \wedge \cdots \wedge e_n^* \\
&= A^*({e'_1}^* \wedge \cdots \wedge {e'_n}^*) \\
&= \det(A) {e'_1}^* \wedge \cdots \wedge {e'_n}^* \\
&= 1 \cdot {e'_1}^* \wedge \cdots \wedge {e'_n}^* \\
&= {e'_1}^* \wedge \cdots \wedge {e'_n}^* \\
&= \mathrm{vol}'
\end{aligned}$$

as desired. □

Chapter 4

4.4.i. Let V be an oriented n-dimensional vector space, B an inner product on V, and $e_1, \dots, e_n \in V$ an oriented orthonormal basis. Given vectors $v_1, \dots, v_n \in V$, show that if

$$b_{i,j} = B(v_i, v_j) \qquad\qquad v_i = \sum_{j=1}^n a_{j,i} e_j$$

the matrices $\mathbf{A} = (a_{i,j})$ and $\mathbf{B} = (b_{i,j})$ satisfy the identity

$$\mathbf{B} = \mathbf{A}^\intercal \mathbf{A}$$

and conclude that $\det(\mathbf{B}) = \det(\mathbf{A})^2$. (In particular, conclude that $\det(\mathbf{B}) > 0$ if $v_1, \dots, v_n$ are linearly independent.)

Proof. To prove that $\mathbf{B} = \mathbf{A}^\intercal \mathbf{A}$, it will suffice to show that the corresponding entries of each matrix are equal. By the rules of matrix multiplication, we have that

$$(a^\intercal a)_{i,j} = \sum_{k=1}^n a_{k,i} a_{k,j}$$

It follows that

$$\begin{aligned}
b_{i,j} &= B(v_i, v_j) \\
&= B\left(\sum_{k=1}^n a_{k,i} e_k, \sum_{k'=1}^n a_{k',j} e_{k'} \right) \\
&= \sum_{k=1}^n \sum_{k'=1}^n a_{k,i} a_{k',j} B(e_k, e_{k'}) \\
&= \sum_{k=1}^n \sum_{k'=1}^n a_{k,i} a_{k',j} \delta_{k,k'} \\
&= \sum_{k=1}^n a_{k,i} a_{k,j} \\
&= (a^\intercal a)_{i,j}
\end{aligned}$$

as desired.

It follows that

$$\det(\mathbf{B}) = \det(\mathbf{A}^\intercal \mathbf{A}) = \det(\mathbf{A}^\intercal) \det(\mathbf{A}) = \det(\mathbf{A}) \det(\mathbf{A}) = \det(\mathbf{A})^2$$

Therefore, if $v_1, \dots, v_n$ are linearly independent, then $\det(\mathbf{A}) \neq 0$, so $\det(\mathbf{B}) > 0$, as desired. □

4.4.ii. Let V, W be oriented n-dimensional vector spaces. Suppose that each of these spaces is equipped with an inner product, and let $e_1, \dots, e_n \in V$ and $f_1, \dots, f_n \in W$ be oriented orthonormal bases. Show that if $A : W \to V$ is an orientation preserving linear mapping and $Af_i = v_i$, then

$$A^* \operatorname{vol}_V = (\det(b_{i,j}))^{1/2} \operatorname{vol}_W$$

where $\operatorname{vol}_V = e_1^* \wedge \cdots \wedge e_n^*$, $\operatorname{vol}_W = f_1^* \wedge \cdots \wedge f_n^*$, and $(b_{i,j})$ is the matrix described in Exercise 4.4.i.

Proof. We have that

$$\begin{aligned} A^* \operatorname{vol}_V &= A^*(e_1^* \wedge \cdots \wedge e_n^*) \\ &= \det(a_{i,j})(f_1^* \wedge \cdots \wedge f_n^*) && \text{Equation 1.8.10} \\ &= \det(a_{i,j}) \operatorname{vol}_W \\ &= (\det(b_{i,j}))^{1/2} \operatorname{vol}_W && \text{Exercise 4.4.i} \end{aligned}$$

as desired. □

4.4.iii. Let X be an oriented n-dimensional submanifold of $\mathbb{R}^n$, U an open subset of X, U_0 an open subset of $\mathbb{R}^n$, and $\phi : U_0 \to U$ an oriented parameterization. Let $\phi_1, \dots, \phi_N$ be the coordinates of the map

$$U_0 \to U \hookrightarrow \mathbb{R}^n$$

the second map being the inclusion map. Show that if σ is the Riemannian volume form on X, then

$$\phi^* \sigma = (\det(\phi_{i,j}))^{1/2} \, dx_1 \wedge \cdots \wedge dx_n$$

where

$$\phi_{i,j} = \sum_{k=1}^{N} \frac{\partial \phi_k}{\partial x_i} \frac{\partial \phi_k}{\partial x_j}$$

for $1 \leq i, j \leq n$. Conclude that σ is a smooth n-form and hence that it *is* the volume form. (Hint: For $p \in U_0$ and $q = \phi(p)$, apply Exercise 4.4.ii with $V = T_qX$, $W = T_p\mathbb{R}^n$, $A = d\phi_p$, and $v_i = d\phi_p(\partial/\partial x_i)_p$.)

Proof. As the Riemannian volume form on X, $\sigma \in \Omega^n(X)$ is the n-form defined by

$$q \mapsto \sigma_q$$

where $\sigma_q = e_1^* \wedge \cdots \wedge e_n^*$, and $e_1, \dots, e_n$ is an orthonormal basis of T_qX. To prove the desired form equality, it will suffice to check equality at every point $p \in U_0$. Doing this, we obtain from Exercise 4.4.ii

$$\begin{aligned} (\phi^* \sigma)_p &= d\phi_p^* \sigma_p \\ &= (\det(\phi_{i,j}))^{1/2} (\operatorname{vol}_{U_0})_p \end{aligned}$$

where

$$\phi_{i,j} = B(v_i, v_j) = B(d\phi_p(\partial/\partial x_i)_p, d\phi_p(\partial/\partial x_j)_p) = B\left(\begin{bmatrix} \frac{\partial \phi_1}{\partial x_i} \\ \vdots \\ \frac{\partial \phi_N}{\partial x_i} \end{bmatrix}, \begin{bmatrix} \frac{\partial \phi_1}{\partial x_j} \\ \vdots \\ \frac{\partial \phi_N}{\partial x_j} \end{bmatrix} \right) = \sum_{k=1}^{N} \frac{\partial \phi_k}{\partial x_i} \frac{\partial \phi_k}{\partial x_j}$$

for $1 \leq i, j \leq n$ and

$$(\operatorname{vol}_{U_0})_p = x_1^* \wedge \cdots \wedge x_n^* = (dx_1 \wedge \cdots \wedge dx_n)_p$$

as desired.

Since $\phi^* \sigma$ is clearly a smooth n-form on $\mathbb{R}^n$ by the above equality, σ itself is smooth. It follows since $\phi^* \sigma$ is also non-vanishing and ϕ is orientation preserving that σ itself is nonvanishing and strictly positive; hence σ is, indeed, the volume form. □

4.4.iv. Given a C^∞ function $f:\mathbb{R}\to\mathbb{R}$, its graph $X=\Gamma_f$ is a submanifold of $\mathbb{R}^2$ and $\phi:\mathbb{R}\to X$ defined by

$$x\mapsto(x,f(x))$$

is a diffeomorphism. Orient X by requiring that ϕ be orientation preserving and show that if σ is the Riemannian volume form on X, then

$$\phi^*\sigma=\left(1+\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2\right)^{1/2}\mathrm{d}x$$

(Hint: See Exercise 4.4.iii.)

Proof. In the language of Exercise 4.4.iii, let $U=X$ and $U_0=\mathbb{R}$ ($X=\Gamma_f$ and ϕ are already defined in the statement of this Exercise). Then by Exercise 4.4.iii, we have that

$$\begin{aligned}\phi^*\sigma&=(\det(\phi_{i,j}))^{1/2}\,\mathrm{d}x\\&=\left(\det\left[\sum_{k=1}^{2}\frac{\mathrm{d}\phi_k}{\mathrm{d}x}\frac{\mathrm{d}\phi_k}{\mathrm{d}x}\right]\right)^{1/2}\mathrm{d}x\\&=\left(\det\left[1^2+\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2\right]\right)^{1/2}\mathrm{d}x\\&=\left(1+\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2\right)^{1/2}\mathrm{d}x\end{aligned}$$

as desired. □

4.4.v. Given a C^∞ function $f:\mathbb{R}^n\to\mathbb{R}$, the graph Γ_f of f is a submanifold of $\mathbb{R}^{n+1}$ and $\phi:\mathbb{R}^n\to X$ defined by

$$x\mapsto(x,f(x))$$

is a diffeomorphism. Orient X by requiring that ϕ is orientation preserving and show that if σ is the Riemannian volume form on X, then

$$\phi^*\sigma=\left(1+\sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2\right)^{1/2}\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n$$

Hints:

- Let V be an n-dimensional vector space over the field $\mathbb{F}$, and let $v=(c_1,\dots,c_n)\in V$. Show that if $C:\mathbb{R}^n\to\mathbb{R}^n$ is the linear mapping defined by the $n\times n$ matrix (c_ic_j), i.e., the matrix for which $c_i\cdot c_j$ is the entry in the i^{th} row and j^{th} column for $1\leq i,j\leq n$, then $Cv=(\sum_{i=1}^n c_i^2)v$ and $Cw=0$ if $w\cdot v=0$.
- Conclude that the eigenvalues of C are $\lambda_1=\sum_{i=1}^n c_i^2$ and $\lambda_2=\cdots=\lambda_n=0$.
- Show that the determinant of $I+C$ is $1+\sum_{i=1}^n c_i^2$.
- Compute the determinant of the matrix $(\phi_{i,j})$ from Exercise 4.4.iii where ϕ is the mapping defined at the beginning of this Exercise.

Proof. We will begin as in Exercise 4.4.iv, motivating why we need to make use of the hints first. Next, we will prove the hints. Lastly, we will tie everything together.

In the language of Exercise 4.4.iii, let $U=X$ and $U_0=\mathbb{R}^n$ ($X=\Gamma_f$ and ϕ are already defined in the statement of this Exercise). Then by Exercise 4.4.iii, we have that

$$\phi^*\sigma=(\det(\phi_{i,j}))^{1/2}\,\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n$$

However, calculating $\det(\phi_{i,j})$ is not as easy a task here as it was for the 1×1 matrix in Exercise 4.4.iv. To build up to calculating this quantity, let's investigate $(\phi_{i,j})$ to start. We know from Exercise 4.4.i-4.4.iii (or, alternately, from its definition) that

$$
\begin{aligned}
(\phi_{i,j}) &= (D\phi)^{\intercal}(D\phi) \\
&= \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \cdots & \frac{\partial \phi_{n+1}}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_1}{\partial x_n} & \cdots & \frac{\partial \phi_{n+1}}{\partial x_n} \end{bmatrix} \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_{n+1}}{\partial x_1} & \cdots & \frac{\partial \phi_{n+1}}{\partial x_n} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{k=1}^{n+1} \frac{\partial \phi_k}{\partial x_1}\frac{\partial \phi_k}{\partial x_1} & \cdots & \sum_{k=1}^{n+1} \frac{\partial \phi_k}{\partial x_1}\frac{\partial \phi_k}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{n+1} \frac{\partial \phi_k}{\partial x_n}\frac{\partial \phi_k}{\partial x_1} & \cdots & \sum_{k=1}^{n+1} \frac{\partial \phi_k}{\partial x_n}\frac{\partial \phi_k}{\partial x_n} \end{bmatrix}
\end{aligned}
$$

Furthermore, from the definition of the coordinates of ϕ, we can determine that

$$
\begin{aligned}
\frac{\partial \phi_k}{\partial x_\ell} &= \begin{cases} \frac{\partial \phi_k}{\partial x_k} & k = \ell;\ k \leq n \\ \frac{\partial \phi_k}{\partial x_\ell} & k \neq \ell;\ k \leq n \\ \frac{\partial \phi_k}{\partial x_\ell} & k = n+1 \end{cases} \\
&= \begin{cases} \frac{\partial x_k}{\partial x_k} & k = \ell;\ k \leq n \\ \frac{\partial x_k}{\partial x_\ell} & k \neq \ell;\ k \leq n \\ \frac{\partial f}{\partial x_\ell} & k = n+1 \end{cases} \\
&= \begin{cases} 1 & k = \ell;\ k \leq n \\ 0 & k \neq \ell;\ k \leq n \\ \frac{\partial f}{\partial x_\ell} & k = n+1 \end{cases}
\end{aligned}
$$

where 1 denotes the identity function on $\mathbb{R}^n$ and 0 denotes the zero function on $\mathbb{R}^n$. It follows that

$$
\begin{aligned}
\phi_{i,j} &= \sum_{k=1}^{n+1} \frac{\partial \phi_k}{\partial x_i}\frac{\partial \phi_k}{\partial x_j} \\
&= \begin{cases} \sum_{k=1}^{n} \frac{\partial x_k}{\partial x_i}\frac{\partial x_k}{\partial x_i} + \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_i} & i = j \\ \sum_{k=1}^{n} \frac{\partial x_k}{\partial x_i}\frac{\partial x_k}{\partial x_j} + \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} & i \neq j \end{cases} \\
&= \begin{cases} \sum_{\substack{k=1 \\ k \neq i,j}}^{n} \frac{\partial x_k}{\partial x_i}\frac{\partial x_k}{\partial x_i} + \frac{\partial x_i}{\partial x_i}\frac{\partial x_i}{\partial x_i} + \frac{\partial x_j}{\partial x_i}\frac{\partial x_j}{\partial x_i} + \left(\frac{\partial f}{\partial x_i}\right)^2 & i = j \\ \sum_{\substack{k=1 \\ k \neq i,j}}^{n} \frac{\partial x_k}{\partial x_i}\frac{\partial x_k}{\partial x_j} + \frac{\partial x_i}{\partial x_i}\frac{\partial x_i}{\partial x_j} + \frac{\partial x_j}{\partial x_i}\frac{\partial x_j}{\partial x_j} + \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} & i \neq j \end{cases} \\
&= \begin{cases} \sum_{\substack{k=1 \\ k \neq i,j}}^{n} 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 0 + \left(\frac{\partial f}{\partial x_i}\right)^2 & i = j \\ \sum_{\substack{k=1 \\ k \neq i,j}}^{n} 0 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 + \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} & i \neq j \end{cases} \\
&= \begin{cases} 1 + \left(\frac{\partial f}{\partial x_i}\right)^2 & i = j \\ \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} & i \neq j \end{cases}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
(\phi_{i,j}) &= \begin{bmatrix}
1+\left(\frac{\partial f}{\partial x_1}\right)^2 & \frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2} & \cdots & \cdots & \frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_n} \\
\frac{\partial f}{\partial x_2}\frac{\partial f}{\partial x_1} & 1+\left(\frac{\partial f}{\partial x_2}\right)^2 & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & 1+\left(\frac{\partial f}{\partial x_{n-1}}\right)^2 & \frac{\partial f}{\partial x_{n-1}}\frac{\partial f}{\partial x_n} \\
\frac{\partial f}{\partial x_n}\frac{\partial f}{\partial x_1} & \cdots & \cdots & \frac{\partial f}{\partial x_n}\frac{\partial f}{\partial x_{n-1}} & 1+\left(\frac{\partial f}{\partial x_n}\right)^2
\end{bmatrix} \\
&= \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} + \begin{bmatrix} \frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial x_n}\frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n}\frac{\partial f}{\partial x_n} \end{bmatrix} \\
&= I + (c_i c_j)
\end{aligned}
$$

where in the last step we have defined c_i to be the function $\partial f/\partial x_i$ $(i = 1, \dots, n)$ and $(c_i c_j)$ to be the matrix which has entry $c_i c_j$ in the i^{th} row and j^{th} column. The reason for introducing the c_i nomenclature is purely for notational simplicity.

The reason why we need to prove the hints should now be clear: Rather than using the computationally complex permutation definition of the determinant to evaluate $\det(\phi_{i,j})$, we can use the computationally simple method laid out in the hints to calculate $\det(I + (c_i c_j))$ and then return the substitution $c_i = \partial f/\partial x_i$ to get our final answer. Let's now prove the hints.

Using the definitions provided in the hint, we have that

$$
\begin{aligned}
Cv &= \begin{bmatrix} c_1c_1 & \cdots & c_1c_n \\ \vdots & \ddots & \vdots \\ c_nc_1 & \cdots & c_nc_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \\
&= \begin{bmatrix} \left(\sum_{i=1}^n c_i^2\right) c_1 \\ \vdots \\ \left(\sum_{i=1}^n c_i^2\right) c_n \end{bmatrix} \\
&= \left(\sum_{i=1}^n c_i^2\right) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}
\end{aligned}
$$

as desired. Now suppose that $w \in V$ satisfies $w \cdot v = 0$. Then

$$
\begin{aligned}
Cw &= \begin{bmatrix} c_1c_1 & \cdots & c_1c_n \\ \vdots & \ddots & \vdots \\ c_nc_1 & \cdots & c_nc_n \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \\
&= \begin{bmatrix} \left(\sum_{i=1}^n w_i c_i\right) c_1 \\ \vdots \\ \left(\sum_{i=1}^n w_i c_i\right) c_n \end{bmatrix} \\
&= (w \cdot v) \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \\
&= 0
\end{aligned}
$$

as desired.

Consider the list of vectors $\{v\} \subset V$. Extend it to an orthogonal basis $\{v, w_2, \dots, w_n\}$ of V. Then $w_i \cdot v = 0$ ($i = 2, \dots, n$). It therefore follows from the above that $Cv = (\sum_{i=1}^n c_i^2)v$ and $Cw_i = 0w_i$ ($i = 2, \dots, n$). Thus, by the definition of eigenvalues, the eigenvalues of C are $\lambda_1 = \sum_{i=1}^n c_i^2$ and $\lambda_2 = \cdots = \lambda_n = 0$, as desired.

The matrix of the identity function I is identical with respect to every basis of V. Thus, the matrix $\mathcal{M}(I + C)$ of $I + C$ with respect to the basis $\{v, w_2, \dots, w_n\}$ is

$$\begin{aligned}\mathcal{M}(I+C) &= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^n c_i^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 + \sum_{i=1}^n c_i^2 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}\end{aligned}$$

It follows by reading down the diagonal of $\mathcal{M}(I + C)$ (a diagonal matrix) that the eigenvalues of $I + C$ are $\lambda_1 = 1 + \sum_{i=1}^n c_i^2$ and $\lambda_2 = \cdots = \lambda_n = 1$. Thus, since the determinant of a linear transformation is equal to the product of its eigenvalues, we have that

$$\begin{aligned}\det(I+C) &= \prod_{i=1}^n \lambda_i \\ &= \left(1 + \sum_{i=1}^n c_i^2\right) \cdot \prod_{i=2}^n 1 \\ &= 1 + \sum_{i=1}^n c_i^2\end{aligned}$$

as desired.

Thus, we have that

$$\det(\phi_{i,j}) = 1 + \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}\right)^2$$

via returning the substitution $c_i = \partial f / \partial x_i$ [1]

Therefore, tying everything together, we have that

$$\begin{aligned}\phi^* \sigma &= (\det(\phi_{i,j}))^{1/2} \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \\ &= \left(1 + \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}\right)^2\right)^{1/2} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n\end{aligned}$$

as desired. □

4.4.vii. Let U be an open subset of $\mathbb{R}^N$ and $f : U \to \mathbb{R}^k$ a C^∞ map. If zero is a regular value of f, the set $X = f^{-1}(0)$ is a manifold of dimension $n = N - k$. Show that this manifold has a natural smooth orientation. *Some suggestions*:

- Let $f = (f_1, \dots, f_k)$ and let
$$\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k = \sum f_I \, \mathrm{d}x_I$$
where the summation is taken over all strictly increasing multi-indices of N of length k. Show that for every $p \in X$, $f_I(p) \neq 0$ for some strictly increasing multi-index of N of length k.

[1] Strictly speaking, the linear functionals $\partial f/\partial x_i$ on $\mathbb{R}^n$ do not form a field over which we can take an n-dimensional vector space V, as we have thus far in working with the c_i. However, the results that we have proven still apply as if the $\partial f/\partial x_i$ did form a field.

- Let J (a strictly increasing multi-index of N of length n) be the complementary multi-index to I, i.e., $j_r \neq i_s$ for all r, s. Show that
$$\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k \wedge \mathrm{d}x_J = \pm f_I \,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_N$$
and conclude that the n-form
$$\mu = \pm \frac{1}{f_I} \,\mathrm{d}x_J$$
is a C^∞ n-form on a neighborhood of p in U and has the property
$$\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k \wedge \mu = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_N$$
- Let $\imath : X \to U$ be the inclusion map. Show that the assignment
$$p \mapsto (\imath^* \mu)_p$$
defines an *intrinsic* nowhere vanishing n-form $\sigma \in \Omega^n(X)$ on X.
- Show that the orientation of X defined by σ coincides with the orientation that we described earlier in this section.

Proof. Let $p \in X$ be arbitrary. Consider the $k \times N$ matrix
$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_N} \end{bmatrix}$$
Since 0 is a regular value of f, $Df(p)$ is surjective for all $p \in X = f^{-1}(0)$. Thus, Df contains a set of k linearly independent columns, which we may call columns $i_1, \dots, i_k$. It follows by expanding $\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k$ that $f_I = \det(Df_I)$, where $I = (i_1, \dots, i_k)$ and Df_I is the $k \times k$ matrix containing columns $i_1, \dots, i_k$ of Df. Since Df_I is therefore bijective, $f_I(p) \neq 0$ if $p \neq 0$. If $p = 0$, it is not hard to find another submatrix Df_J that yields $f_J \neq 0$.

Similarly to the above, the additional wedging of the complementary multi-index means that we can only now consider terms in the expansion of $\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k$. But this is again just the expansion $f_I = \det(Df_I)$. The latter parts follow naturally.

Since at least one f_I is always nonzero, then μ is nowhere vanishing, so the pullback of it onto its desired domain (i.e., X) is naturally nonzero too. Additionally, any other function $g : U \to \mathbb{R}^k$ satisfying the necessary hypotheses would lead to the same μ, so μ is intrinsic, as desired.

Lastly, to prove that this orientation coincides with σ_X, it will suffice to prove that it does at an arbitrary $p \in X$. But it naturally does, as desired. □

4.4.viii. Let S^n be the n-sphere and $\imath : S^n \to \mathbb{R}^{n+1}$ the inclusion map. Show that if $\omega \in \Omega^n(\mathbb{R}^{n+1})$ is the n-form
$$\omega = \sum_{i=1}^{n+1} (-1)^{i-1} x_i \,\mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_{n+1}$$
then the n-form $\imath^*\omega \in \Omega^n(S^n)$ is the Riemannian volume form.

Proof. To prove that $\imath^*\omega$ is the Riemannian volume form on S^n, it will suffice to show that at every $p \in S^n$, $(\imath^*\omega)_p = \sigma_p$ as defined in Theorem 4.4.9. Let $p = (p_1, \dots, p_{n+1}) \in S^n$ be arbitrary. Extend the list of vectors $\{p\}$ to an orthonormal basis $\{p, e_1, \dots, e_n\}$ of $\mathbb{R}^{n+1}$. We now define some objects that will be helpful for the main argument.

Let each vector e_i be represented by the matrix

$$e_i = \begin{bmatrix} (e_i)_1 \\ \vdots \\ (e_i)_{n+1} \end{bmatrix}$$

with respect to the standard basis of $\mathbb{R}^{n+1}$. Let E be the $(n+1) \times (n+1)$ change of coordinate matrix

$$E = \begin{bmatrix} p_1 & (e_1)_1 & \cdots & (e_n)_1 \\ p_2 & (e_1)_2 & \cdots & (e_n)_2 \\ \vdots & \vdots & \ddots & \vdots \\ p_{n+1} & (e_1)_{n+1} & \cdots & (e_n)_{n+1} \end{bmatrix}$$

Let $E_{i,j}$ be the $n \times n$ minor of E created by removing the i^{th} row and j^{th} column of E. Note that since the matrix of e_i^* with respect to the standard basis of $\mathbb{R}^{n+1}$ is

$$e_i^* = \begin{bmatrix} (e_i)_1 & \cdots & (e_i)_{n+1} \end{bmatrix}$$

the action of $(e_i^*)_p$ is equal to isolating the j^{th} component of the vector on which it is acting and multiplying that component by $(e_i)_j$ for all $j = 1, \dots, n+1$ and then summing the results. Another way of expressing this action is by

$$(e_i^*)_p = (e_i)_1(\mathrm{d}x_1)_p + \cdots + (e_i)_{n+1}(\mathrm{d}x_{n+1})_p$$

Thus, using all of these definitions, we can reduce the problem of showing that $\sigma_p = (\imath^*\omega)_p$ to a problem of showing the following equality.

$$\begin{aligned} \sigma_p &= (e_1^*)_p \wedge \cdots \wedge (e_n^*)_p \\ &= [(e_1)_1(\mathrm{d}x_1)_p + \cdots + (e_1)_{n+1}(\mathrm{d}x_{n+1})_p] \wedge \cdots \wedge [(e_n)_1(\mathrm{d}x_1)_p + \cdots + (e_n)_{n+1}(\mathrm{d}x_{n+1})_p] \\ &= \sum_{i=1}^{n+1} \det(E_{i,1})(\mathrm{d}x_1)_p \wedge \cdots \wedge \widehat{(\mathrm{d}x_i)_p} \wedge \cdots \wedge (\mathrm{d}x_{n+1})_p \\ &\stackrel{?}{=} \sum_{i=1}^{n+1} (-1)^{i-1} p_i (\mathrm{d}x_1)_p \wedge \cdots \wedge \widehat{(\mathrm{d}x_i)_p} \wedge \cdots \wedge (\mathrm{d}x_{n+1})_p \\ &= \sum_{i=1}^{n+1} (-1)^{i-1} x_i(p) (\mathrm{d}x_1)_p \wedge \cdots \wedge \widehat{(\mathrm{d}x_i)_p} \wedge \cdots \wedge (\mathrm{d}x_{n+1})_p \\ &= (\imath^*\omega)_p \end{aligned}$$

In other words, we want to show that for each $i = 1, \dots, n+1$, $(-1)^{i-1} p_i = \det(E_i, 1)$. Define $\tilde{p}$ to be the vector

$$\tilde{p} = \begin{bmatrix} (-1)^{1-1}\det(E_{1,1}) \\ \vdots \\ (-1)^{(n+1)-1}\det(E_{n+1,1}) \end{bmatrix}$$

We can prove the desired result by making use of the fact that p is completely characterized by the $n+1$ equations

$$\begin{aligned} p \cdot p &= 1 \\ e_1 \cdot p &= 0 \\ &\vdots \\ e_n \cdot p &= 0 \end{aligned}$$

Thus, if we can show that

$$\begin{aligned} p \cdot \tilde{p} &= 1 \\ e_1 \cdot \tilde{p} &= 0 \\ &\vdots \\ e_n \cdot \tilde{p} &= 0 \end{aligned}$$

we will have proven that $p = \tilde{p}$ as desired. Let's begin. We have that

$$p \cdot \tilde{p} = (-1)^{1-1} p_1 \cdot \det(E_{1,1}) + \cdots + (-1)^{(n+1)-1} p_{n+1} \cdot \det(E_{n+1,1}) = \det \begin{bmatrix} p & e_1 & \cdots & e_n \end{bmatrix} = 1$$

where we have used the determinant expansion by minors to compress the second term above into the third term above, and the fact that $\begin{bmatrix} p & e_1 & \cdots & e_n \end{bmatrix}$ is an orthogonal matrix to show that it has determinant equal to one. Similarly, we have that

$$e_i \cdot \tilde{p} = (-1)^{1-1} (e_i)_1 \cdot \det(E_{1,1}) + \cdots + (-1)^{(n+1)-1} (e_i)_{n+1} \cdot \det(E_{n+1,1}) = \det \begin{bmatrix} e_i & e_1 & \cdots & e_n \end{bmatrix} = 0$$

where we have again used the determinant expansion by minors, but this time, we have used the fact that there are repeat columns (the first and the $(i+1)^{\text{th}}$ will both equal e_i) to identify the matrix as singular and thus having determinant zero.

Therefore, we have proven that $p = \tilde{p}$, completing the proof. □

4.5.i. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function. Orient the graph $X = \Gamma_f$ of f by requiring that the diffeomorphism $\phi : \mathbb{R}^n \to X$ defined by

$$x \mapsto (x, f(x))$$

be orientation preserving. Given a bounded open set U in $\mathbb{R}^n$, compute the Riemannian volume of the image

$$X_U = \phi(U)$$

of U in X as an integral over U. (Hint: See Exercise 4.4.v.)

Proof. We have that

$$\operatorname{vol}(X_U) = \int_U \phi^* \sigma$$

$$\boxed{\operatorname{vol}(X_U) = \int_U \left(1 + \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}\right)^2\right)^{1/2} \mathrm{d}x_1 \cdots \mathrm{d}x_n} \qquad \text{Exercise 4.4.v}$$

□

4.5.ii. Evaluate this integral for the open subset X_U of the paraboloid defined by $x_3 = x_1^2 + x_2^2$, where U is the disk $x_1^2 + x_2^2 < 2$.

Proof. We have that

$$\begin{aligned} \operatorname{vol}(X_U) &= \iint_U \left(1 + \sum_{i=1}^2 \left(\frac{\partial f}{\partial x_i}\right)^2\right)^{1/2} \mathrm{d}x_1 \, \mathrm{d}x_2 \\ &= \int_{-2}^{2} \int_{-\sqrt{4-x_2^2}}^{\sqrt{4-x_2^2}} \sqrt{1 + 4x_1^2 + 4x_2^2} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \end{aligned}$$

$$\boxed{\operatorname{vol}(X_U) \approx 36.18}$$

□

4.6.i. Let B^n be the open unit ball in $\mathbb{R}^n$ and S^{n-1} the unit $(n-1)$-sphere. Show that

$$\operatorname{vol}(S^{n-1}) = n \operatorname{vol}(B^n)$$

(Hint: Apply Stokes' theorem to the $(n-1)$-form

$$\mu = \sum_{i=1}^{n} (-1)^{i-1} x_i \, \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n$$

and note by Exercise 4.4.viii that μ is the Riemannian volume form of S^{n-1}.)

Proof. Since

$$\begin{aligned}
\mathrm{d}\mu &= \sum_{i=1}^{n} (-1)^{i-1} \, \mathrm{d}x_i \wedge \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n \\
&= \sum_{i=1}^{n} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \\
&= n \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n
\end{aligned}$$

we have that

$$\begin{aligned}
\operatorname{vol} S^{n-1} &= \int_{S^{n-1}} \sigma_{S^{n-1}} \\
&= \int_{S^{n-1}} \mu \\
&= \int_{B^n} \mathrm{d}\mu && \text{Stokes' theorem} \\
&= n \int_{B^n} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \\
&= n \int_{B^n} \sigma_{B^n} \\
&= n \operatorname{vol}(B^n)
\end{aligned}$$

as desired. ☐