# 1 Multilinear Algebra

*From Guillemin and Haine (2018).*

## Chapter 1

**1.2.iv.** Let $U$, $V$, and $W$ be vector spaces and let $A : V \to W$ and $B : U \to V$ be linear mappings. Show that $(AB)^* = B^*A^*$.

**1.2.v.** Let $V = \mathbb{R}^2$ and let $W$ be the $x_1$-axis, i.e., the one-dimensional subspace

$$\{(x_1, 0) \mid x_1 \in \mathbb{R}\}$$

of $\mathbb{R}^2$.

(1) Show that the $W$-cosets are the lines $x_2 = a$ parallel to the $x_1$-axis.

(2) Show that the sum of the cosets $x_2 = a$ and $x_2 = b$ is the coset $x_2 = a + b$.

(3) Show that the scalar multiple of the coset $x_2 = c$ by the number $\lambda$ is the coset $x_2 = \lambda c$.

**1.2.vi.** (1) Let $(V^*)^*$ be the dual of the vector space $V^*$. For every $v \in V$, let $\mathrm{ev}_v : V^* \to \mathbb{R}$ be the **evaluation function** $\mathrm{ev}_v(\ell) = \ell(v)$. Show that the $\mathrm{ev}_v$ is a linear function on $V^*$, i.e., an element of $(V^*)^*$, and show that the map $\mathrm{ev} = \mathrm{ev}_{(-)} : V \to (V^*)^*$ defined by $v \mapsto \mathrm{ev}_v$ is a linear map of $V$ into $(V^*)^*$.

(2) If $V$ is finite dimensional, show that the map ev is bijective. Conclude that there is a natural identification of $V$ with $(V^*)^*$, i.e., that $V$ and $(V^*)^*$ are two descriptions of the same object. (Hint: $\dim(V^*)^* = \dim V^* = \dim V$, so since $\dim(V) = \dim(\ker(A)) + \dim(\mathrm{im}(A))$, it suffices to show that ev is injective.)

**1.2.xi.** Let $V$ be a vector space.

(1) Let $B : V \times V \to \mathbb{R}$ be an inner product on $V$. For all $v \in V$, let $\ell_v : V \to \mathbb{R}$ be the function $\ell_v(w) = B(v, w)$. Show that $\ell_v$ is linear, and show that the map $L : V \to V^*$ defined by $v \mapsto \ell_v$ is a linear mapping.

(2) If $V$ is finite dimensional, prove that $L$ is bijective. Conclude that if $V$ has an inner product, one gets from it a natural identification of $V$ with $V^*$. (Hint: Since $\dim V = \dim V^*$ and $\dim(V) = \dim(\ker(A)) + \dim(\mathrm{im}(A))$, it suffices to show that $\ker(L) = 0$. Now note that if $v \neq \mathbf{0}$, then $\ell_v(v) = B(v, v)$ is a positive number.)

**1.3.i.** Verify that there are exactly $n^k$ multi-indices of length $k$.

**1.3.ii.** Prove that the map $A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(W)$ defined by $T \mapsto A^*T$ is linear.

**1.3.iii.** Verify that

$$A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$$

**1.3.iv.** Verify that

$$(AB)^*T = B^*(A^*T)$$

**1.3.vii.** Let $T$ be a $k$-tensor and $v$ be a vector. Define $T_v : V^{k-1} \to \mathbb{R}$ by

$$T_v(v_1, \ldots, v_{k-1}) = T(v, v_1, \ldots, v_{k-1})$$

Show that $T_v$ is a $(k-1)$-tensor.

**1.3.viii.** Show that if $T_1$ is an $r$-tensor and $T_2$ is an $s$-tensor, then if $r > 0$,

$$(T_1 \otimes T_2)_v = (T_1)_v \otimes T_2$$

**1.3.ix.** Let $A : V \to W$ be a linear map, let $v \in V$, and let $w = Av$. Show that for all $T \in \mathcal{L}^k(W)$,

$$A^*(T_w) = (A^*T)_v$$

**1.4.i.** Show that there are exactly $k!$ permutations of order $k$. (Hint: Induction on $k$: Let $\sigma \in S_k$, and let $\sigma(k) = i$ $(1 \leq i \leq k)$. Show that $\tau_{i,k}\sigma$ leaves $k$ fixed and hence is, in effect, a permutation of $\Sigma_{k-1}$.)

**1.4.ii.** Prove that if $\tau \in S_k$ is a transposition, $(-1)^\tau = -1$. Deduce from this that if $\sigma$ is the product of an odd number of transpositions, then $(-1)^\sigma = -1$, and if $\sigma$ is the product of an even number of transpositions, then $(-1)^\sigma = +1$.

**1.4.iii.** Prove that the assignment $T \mapsto T^\sigma$ is a linear map $\mathcal{L}^k(V) \to \mathcal{L}^k(V)$.

**1.4.vi.** Show that every one of the six elements of $S_3$ is either a transposition or can be written as a product of two transpositions.

**1.4.ix.** Let $A : V \to W$ be a linear mapping. Show that if $T \in \mathcal{A}^k(W)$, then $A^*T \in \mathcal{A}^k(V)$.

**1.5.i.** A $k$-tensor $T \in \mathcal{L}^k(V)$ is **symmetric** if $T^\sigma = T$ for all $\sigma \in S_k$. Show that the set $\mathcal{S}^k(V)$ of symmetric $k$-tensors is a vector subspace of $\mathcal{L}^k(V)$.

**1.6.i.** Verify the following three equations, where $\lambda \in \mathbb{R}$.

(1) $\lambda(\omega_1 \wedge \omega_2) = (\lambda\omega_1) \wedge \omega_2 = \omega_1 \wedge (\lambda\omega_2)$.

(2) $(\omega_1 + \omega_2) \wedge \omega_3 = \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_3$.

(3) $\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$.

**1.6.ii.** Verify the following multiplicative law for the wedge product.

$$\omega_1 \wedge \omega_2 = (-1)^{rs}\omega_2 \wedge \omega_1$$

**1.6.iv.** If $\omega, \mu \in \Lambda^r(V^*)$, prove that

$$(\omega + \mu)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} \omega^\ell \wedge \mu^{k-\ell}$$

(Hint: As in freshman calculus, prove this binomial theorem by induction using the identity $\binom{k}{\ell} = \binom{k-1}{\ell-1} + \binom{k-1}{\ell}$.)

**1.7.i.** Prove that if $T$ is the decomposable $k$-tensor $\ell_1 \otimes \cdots \otimes \ell_k$, then

$$\imath_v T = \sum_{r=1}^{k} (-1)^{r-1} \ell_r(v) \ell_1 \otimes \cdots \otimes \hat{\ell}_r \otimes \cdots \otimes \ell_k$$

where the hat over $\ell_r$ means that $\ell_r$ is deleted from the tensor product.

**1.7.ii.** Prove that if $T_1 \in \mathcal{L}^p(V)$ and $T_2 \in \mathcal{L}^q(V)$, then

$$\imath_v(T_1 \otimes T_2) = \imath_v T_1 \otimes T_2 + (-1)^p T_1 \otimes \imath_v T_2$$

**1.7.iii.** Show that if $T \in \mathcal{A}^k(V)$, then $\imath_v T = kT_v$, where $T_v$ is defined as in Exercise 1.3.vii. In particular, conclude that $\imath_v T \in \mathcal{A}^{k-1}(V)$. (See Exercise 1.4.viii, which asserts that $T \in \mathcal{A}^k(V)$ implies $T_v \in \mathcal{A}^{k-1}(V)$.)

**1.8.i.** Verify the following assertions.

(1) The map $A^* : \Lambda^k(W^*) \to \Lambda^k(V^*)$ sending $\omega \mapsto A^*\omega$ is linear.

(2) If $\omega_i \in \Lambda^{k_i}(W^*)$ $(i = 1, 2)$, then

$$A^*(\omega_1 \wedge \omega_2) = A^*(\omega_1) \wedge A^*(\omega_2)$$

(3) If $U$ is a vector space and $B : U \to V$ is a linear map, then for $\omega \in \Lambda^k(W^*)$,

$$B^* A^* \omega = (AB)^* \omega$$

**1.8.ii.** Deduce from the fact "$A : V \to V$ not surjective implies $\det(A) = 0$" a well-known fact about determinants of $n \times n$ matrices: If two columns are equal, the determinant is zero.

**1.8.iv.** Deduce from Exercise 1.8.i another well-known fact about determinants of $n \times n$ matrices: If $(b_{i,j})$ is the inverse of $[a_{i,j}]$, its determinant is the inverse of the determinant of $[a_{i,j}]$.

**1.8.v.** Extract from the formula $\det([a_{i,j}]) = \sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$ the following well-known formula for determinants of $2 \times 2$ matrices.

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

**1.9.i.** Prove that if $e_1, \ldots, e_n$ is a positively oriented basis of $V$, then the basis $e_1, \ldots, e_{i-1}, -e_i, e_{i+1}, \ldots, e_n$ is negatively oriented.

**1.9.ii.** Show that the argument in the proof of Theorem 1.9.9 can be modified to prove that if $V$ and $W$ are oriented, then these orientations induce a natural orientation on $V/W$.