Week 4

Differential Forms

4.1 Overview of Differential Forms

4/18:

- Office Hours on Wednesday, 4:00-5:00 PM.
- Plan:
  - An impressionistic overview of what (differential) forms do/are.
  - Tangent spaces.
  - Vector fields/integral curves.
  - 1-forms; a warm-up to k-forms.
- Impressionistic overview of the rest of Guillemin and Haine (2018).
  - An open subset $U \subset \mathbb{R}^n$; $n = 2$ and $n = 3$ are nice.
  - Sometimes, we'll have some functions $F : U \to V$; this is where pullbacks come into play.
  - At every point $p \in U$, we'll define a vector space (the tangent space $T_p\mathbb{R}^n$). Associated to that vector space you get our whole slew of associated spaces (the dual space $T_p^*\mathbb{R}^n$, and all of the higher exterior powers $\Lambda^k(T_p^*\mathbb{R}^n)$).
  - We let $\omega \in \Omega^k(U)$ be a k-form in the space of k-forms.
  - ω assigns (smoothly) to every point $p \in U$ an element of $\Lambda^k(T_p^*\mathbb{R}^n)$.
  - Question: What really is a k-form?
    - Answer: Something that can be integrated on k-dimensional subsets.
    - If $k = 1$, i.e., $\omega \in \Omega^1(U)$, then U can be integrated over curves.
  - If we take $k = 0$, then $\Omega^0(U) = C^\infty(U)$, i.e., the set of all smooth functions $f : U \to \mathbb{R}$.
    - Guillemin and Haine (2018) doesn't, but Klug will and we should distinguish between functions $F : U \to V$ and $f : U \to \mathbb{R}$.
  - We will soon construct a map $d : \Omega^0(U) \to \Omega^1(U)$ (the **exterior derivative**) that is rather like the gradient but not quite.
    - d is linear.
    - Maps from vector spaces are heretofore assumed to be linear unless stated otherwise.
  - The 1-forms in $\operatorname{im}(d)$ are special: $\int_\gamma \mathrm{d}f = f(\gamma(b)) - f(\gamma(a))$ only depends on the endpoints of $\gamma : [a, b] \to U$! The integral is *path-independent*.
  - A generalization of this fact is that instead of integrating along the surface M, we can integrate along the boundary curve:
    $$\int_M \mathrm{d}\omega = \int_{\partial M} \omega$$
    This is **Stokes' theorem**.

    - M is a k-dimensional subset of $U \subset \mathbb{R}^n$.
  - Note that we have all manner of functions d that we could differentiate between (because they are functions) but nobody does.

$$0 \to \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(U) \xrightarrow{d} 0$$

  - Theorem: $d^2 = d \circ d = 0$.
    - Corollary: $\operatorname{im}(d^{n-1}) \subset \ker(d^n)$.
  - We'll define $H^k_{dR}(U) = \ker(d)/\operatorname{im}(d)$.
    - These will be finite dimensional, even though all the individual vector spaces will be infinite dimensional.
    - These will tell us about the shape of U; basically, if all of these equal zero, U is simply connected. If some are nonzero, U has some holes.
  - For small values of n and k, this d will have some nice geometric interpretations (div, grad, curl, n'at).
  - We'll have additional operations on forms such as the wedge product.

- **Tangent space** (of p): The following set. *Denoted by* $\boldsymbol{T_p\mathbb{R}^n}$. *Given by*

$$T_p\mathbb{R}^n = \{(p, v) : v \in \mathbb{R}^n\}$$

  - This is naturally a vector space with addition and scalar multiplication defined as follows.

$$(p, v_1) + (p, v_2) = (p, v_1 + v_2) \qquad\qquad \lambda(p, v) = (p, \lambda v)$$

  - The point is that

$$T_p\mathbb{R}^n \neq T_q\mathbb{R}^n$$

    for $p \neq q$ even though the spaces are isomorphic.
  - Aside: $F : U \to V$ differentiable and $p \in U$ induce a map $\mathrm{d}F_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ called the "derivative at p."
    - We will see that the matrix of this map is the Jacobian.
  - Chain rule: If $U \xrightarrow{F} V \xrightarrow{G} W$, then

$$\mathrm{d}(G \circ F)_p = \mathrm{d}G_{F(p)} \circ \mathrm{d}F_p$$

- This is round 1 of our discussion on tangent spaces.
- Round 2, later on, will be submanifolds such as T_pM: The tangent space to a point p of a manifold M.
- **Vector field** (on U): A function that assigns to each $p \in U$ an element of $T_p\mathbb{R}^n$.
  - A constant vector field would be $p \mapsto (p, v)$, visualized as a field of vectors at every p all pointing the same direction. For example, we could take $v = (1, 1)$. *picture*
  - Special case: $v = e_1, e_2, \dots, e_n$. Here we use the notation $e_i = \mathrm{d}/\mathrm{d}x_i$.
  - Example: $n = 2$, $U = \mathbb{R}^2 \setminus \{(0, 0)\}$. We could take a vector field that spins us around in circles.
  - Notice that for all p, $\mathrm{d}/\mathrm{d}x_1 \mid_p, \dots, \mathrm{d}/\mathrm{d}x_n \mid_p \in T_p\mathbb{R}^n$ are a basis.
    - Thus, any vector field v on U can be written uniquely as

$$v = f_1 \frac{\mathrm{d}}{\mathrm{d}x_1} + \cdots + f_n \frac{\mathrm{d}}{\mathrm{d}x_n}$$

      where the $f_1, \dots, f_n$ are functions $f_i : U \to \mathbb{R}$.

4.2 The Lie Derivative and 1-Forms

4/20:

- Plan:
  - Vector fields and their integral curves.
  - Lie derivatives.
  - 1-forms and k-forms.
  - $\Omega^0(U) \xrightarrow{d} \Omega^1(U)$.
- Notation.
  - $U \subset \mathbb{R}^n$.
  - v denotes a vector field on U.
    - Note that the set of all vector fields on U constitute the vector space ??.
  - $v_p \in T_p\mathbb{R}^n$.
  - $\omega_p \in \Lambda^k(T_p^*\mathbb{R}^n)$.
  - $\mathrm{d}/\mathrm{d}x_i \mid_p = (p, e_i) \in T_p\mathbb{R}^n$.
- Recall that any vector field v on U can be written uniquely as

$$v = g_1 \frac{\mathrm{d}}{\mathrm{d}x_1} + \cdots + g_n \frac{\mathrm{d}}{\mathrm{d}x_n}$$

  where the $g_i : U \to \mathbb{R}$.
- **Smooth** (vector field): A vector field v for which all g_i are smooth.
- From now on, we assume unless stated otherwise that all vector fields are smooth.
- **Lie derivative** (of f wrt. v): The function $L_v f : U \to \mathbb{R}$ defined by $p \mapsto D_{v_p}(f)(p)$, where v is a vector field on U and $f : U \to \mathbb{R}$ (always smooth).
  - Recall that $D_{v_p}(f)(p)$ denotes the directional derivative of f in the direction v_p at p.
  - As some examples, we have

$$L_{\mathrm{d}/\mathrm{d}x_i} f = \frac{\mathrm{d}f}{\mathrm{d}x_i} \qquad\qquad L_{(g_1 \frac{\mathrm{d}}{\mathrm{d}x_1} + \cdots + g_n \frac{\mathrm{d}}{\mathrm{d}x_n})} f = g_1 \frac{\mathrm{d}f}{\mathrm{d}x_1} + \cdots + g_n \frac{\mathrm{d}f}{\mathrm{d}x_n}$$

- Property.
  1. Product rule: $L_v(f_1 f_2) = (L_v f_1) f_2 + f_1 (L_v f_2)$.
- Later: Geometric meaning to the expression $L_v f = 0$.
  - Satisfied iff f is constant on the integral curves of v. As if f "flows along" the vector field.
- We define $T_p^*\mathbb{R}^n = (T_p\mathbb{R}^n)^*$.
- 1-forms:
  - A (differential) 1-form on $U \subset \mathbb{R}^n$ is a function $\omega : p \mapsto \omega_p \in T_p^*\mathbb{R}^n$.
  - A "co-vector field"
- Notation: $\mathrm{d}x_i$ is the 1-form that at p is $(p, e_i^*) \in T_p^*\mathbb{R}^n$.
- For example, if $U = \mathbb{R}^2$ and $\omega = \mathrm{d}x_1$, then we have the vector field of "unit vectors pointing to the right at each point."

- Note: Given any 1-form ω on U, we can write ω uniquely as
$$\omega = g_1 \, \mathrm{d}x_1 + \cdots + g_n \, \mathrm{d}x_n$$
for some set of smooth $g_i : U \to \mathbb{R}$.
- Notation:
  - $\Omega^1(U)$ is the set of all smooth 1-forms.
  - Notice that $\Omega^1(U)$ is a vector space.
- Given $\omega \in \Omega^1(U)$ and a vector field v on U, we can define $\omega(v) : U \to \mathbb{R}$ by $p \mapsto \omega_p(v_p)$.
- If $U = \mathbb{R}^2$, we have that
$$\mathrm{d}x \left(\frac{\mathrm{d}}{\mathrm{d}x} \right) = 1 \qquad\qquad \mathrm{d}x \left(\frac{\mathrm{d}}{\mathrm{d}y} \right) = 0$$
- Note that $\mathrm{d}x \, , \mathrm{d}y$ are not a basis for $\Omega^1(U)$ since the latter is infinite dimensional.
- Exterior derivative for 0/1 forms.
  - Let $d : \Omega^0(U) \to \Omega^1(U)$ take $f : U \to \mathbb{R}$ to $\frac{\partial f}{\partial x_1} \, \mathrm{d}x_1 + \cdots + \frac{\partial f}{\partial x_n} \, \mathrm{d}x_n$.
  - This represents the gradient as a 1-form.
- Check:
  1. Linear.
  2. $\mathrm{d}x_i = \mathrm{d}(x_i)$, where $x_i : \mathbb{R}^n \to \mathbb{R}$ is the i^{th} coordinate function.

4.3 Integral Curves

4/22:
- Plan:
  - Clear up a bit of notational confusion.
  - Discuss integral curves of vectors fields.
  - k-forms.
  - Exterior derivatives $d : \Omega^k(U) \to \Omega^{k+1}(U)$ (definition and properties).
- Notation:
  - $F : \mathbb{R}^n \to \mathbb{R}^m$ smooth.
  - We are used to denoting derivatives by big D: $DF_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$ where bases of the two spaces are $e_1, \dots, e_n$ and $e_1, \dots, e_m$ has matrix equal to the Jacobian:
$$[DF_p] = \left[\tfrac{\mathrm{d}F_i}{\mathrm{d}x_j}(p) \right]$$
  - The book often uses small d: $f : U \to \mathbb{R}$ has $df_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}$, where the latter set is isomorphic to $\mathbb{R}$.
  - $df : p \to df_p \in T_p^*\mathbb{R}^n$.
  - Klug said
$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \, \mathrm{d}x_i$$
  - Homework 1 defined $\mathrm{d}f = df$?

- Sometimes three perspectives help you keep this all straight:
  1. Abstract nonsense: The definition of the derivative.
  2. How do I compute it: Apply the formula.
  3. What is it: E.g., magnitude of the directional derivative in the direction of steepest ascent.

- For the homework,
  - Let ω be a 1-form in $\Omega^1(U)$.
  - Let $\gamma : [a, b] \to U$ be a curve in U.
  - Then $\mathrm{d}\gamma_p = \gamma'_p : T_p\mathbb{R} \to T_{\gamma(p)}\mathbb{R}^n$ is a function that takes in points of the curve and spits out tangent vectors.
  - Integrating swallows 1-forms and spits out numbers.
  $$\int_\gamma \omega = \int_a^b \omega(\gamma'(t))\,\mathrm{d}t$$
  - Problem: If $\omega = \mathrm{d}f$, then
  $$\int_\gamma \omega = f(\gamma(b)) - f(\gamma(a))$$
  regardless of the path.
  - Question: Given a 1-form ω, is $\omega = \mathrm{d}f$ for some f?
  - Homework: Explicit U, ω, closed γ such that $\int_\gamma \omega \neq 0$ implies that $\omega \neq \mathrm{d}f$. This motivates and leads into the de Rham cohomology.

- Aside: It won't hurt (for now) to think of 1-forms as vector fields.

- Integral curves: Let $U \subset \mathbb{R}^n$, v be a (smooth) vector field on U. A curve $\gamma : (a, b) \to U$ is an **integral curve** for v if $\gamma'(t) = v_{\gamma(t)}$.

- Examples:
  - If $U = \mathbb{R}^2$ and $\gamma = \mathrm{d}/\mathrm{d}x$, then the integral curve is the line from left to right traveling at unit speed. The curve has to always have as it's tangent vector the unit vector pointing right (which is the vector at every point in the vector field).
  - Vector fields flow everything around. An integral curve is the trajectory of a particle subjected to the vector field as a force field.

- Main points:
  1. These integral curves always exist (locally) and often exist globally (cases in which they do are called **complete vector fields**).
  2. They are unique given a starting point $p \in U$.

- An incomplete vector field is one such as the "all roads lead to Rome" vector field where everything always points inward. This is because integral curves cannot be defined for all "time" (real numbers, positive and negative).

- The proofs are in the book; they require an existence/uniqueness result for ODEs and the implicit function theorem.

- Aside: $f : U \to \mathbb{R}$, v a vector field, implies that $L_v f = 0$ means that f is constant along all the integral curves of v. This also means that f is integral for v.

- **Pullback** (of 1-forms): If $F : U \to V$, $d : \Omega^0(U) \to \Omega^1(U)$, and $d : \Omega^0(V) \to \Omega^1(V)$, then we get an induced map $F^* : \Omega^0(V) \to \Omega^0(U)$. If $f : V \to \mathbb{R}$, then $f \circ F$ is involved.
  - We're basically saying that if we have $\mathrm{Hom}(A, X)$ (the set of all functions from A to X) and $\mathrm{Hom}(B, X)$, then if we have $F : A \to B$, we get an induced map $F^* : \mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X)$ that is precomposed with F.

4.4 Chapter 2: Differential Forms

From Guillemin and Haine (2018).

5/5:
- Goals for this chapter.
  - Generalize to n dimensions the basic operations of 3D vector calculus (**divergence**, **gradient**, and **curl**).
    - div and grad are pretty straightforward, but curl is more subtle.
  - Substitute **differential forms** for **vector fields** to discover to a natural generalization of the operations, in particular, where all three operations are special cases of **exterior differentiation**.
- Introducing vector fields and their dual objects (**one-forms**).
- **Tangent space** (to $\mathbb{R}^n$ at p): The set of pairs (p, v) for all $v \in \mathbb{R}^n$. *Denoted by* $\boldsymbol{T_p\mathbb{R}^n}$. *Given by*

$$T_p\mathbb{R}^n = \{(p, v) \mid v \in \mathbb{R}^n\}$$

- Operations on the tangent space.
  - Directly, we identify $T_p\mathbb{R}^n \cong \mathbb{R}^n$ by $(p, v) \mapsto v$ to make $T_p\mathbb{R}^n$ a vector space.
  - Explicitly, we define

$$(p, v_1) + (p, v_2) = (p, v_1 + v_2) \qquad \lambda(p, v) = (p, \lambda v)$$

  for all $v, v_1, v_2 \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.
- **Derivative** (of f at p): The linear map from $\mathbb{R}^n \to \mathbb{R}^m$ defined by the following $m \times n$ matrix, where $U \subset \mathbb{R}^n$ is open and $f : U \to \mathbb{R}^m$ is a C^1-mapping. *Denoted by* $\boldsymbol{Df(p)}$. *Given by*

$$Df(p) = \left[\frac{\partial f_i}{\partial x_j}(p)\right]$$

- $\mathbf{d}\boldsymbol{f_p}$: The linear map from $T_p\mathbb{R}^n \to T_q\mathbb{R}^m$ defined as follows, where $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^m$ is a C^1-mapping, and $q = f(p)$. *Given by*

$$\mathrm{d}f_p(p, v) = (q, Df(p)v)$$

  - Guillemin and Haine (2018) also refer to this as the "base-pointed" version of the derivative of f at p.
- The chain rule for the base-pointed version, where $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^n$ is a C^1-mapping, $\mathrm{im}(f) \subset V$ open, and $g : V \to \mathbb{R}^k$ is a C^1-mapping.

$$\mathrm{d}g_q \circ \mathrm{d}f_p = \mathrm{d}(f \circ g)_p$$

- **Vector field** (on $\mathbb{R}^3$): A function which attaches to each point $p \in \mathbb{R}^3$ a base-pointed arrow $(p, v) \in T_p\mathbb{R}^3$.
  - These vector fields are the typical subject of vector calculus.
- **Vector field** (on U): A function which assigns to each point $p \in U$ a vector in $T_p\mathbb{R}^n$, where $U \subset \mathbb{R}^n$ is open. *Denoted by* $\boldsymbol{v}$.
  - We denote the value of $\boldsymbol{v}$ at p by either $\boldsymbol{v}(p)$ or $\boldsymbol{v}_p$.
- **Constant** (vector field): A vector field of the form $p \mapsto (p, v)$, where $v \in \mathbb{R}^n$ is fixed.
- $\boldsymbol{\partial/\partial x_i}$: The constant vector field having $v = e_i$.

- $\boldsymbol{fv}$: The vector field defined on U as follows, where $f : U \to \mathbb{R}$. *Given by*

$$p \mapsto f(p)\boldsymbol{v}(p)$$

  - Note that we are invoking our definition of scalar multiplication on $T_p\mathbb{R}^n$ here.

- **Sum** (of $\boldsymbol{v}_1, \boldsymbol{v}_2$): The vector field on U defined as follows. *Denoted by* $\boldsymbol{v_1 + v_2}$. *Given by*

$$p \mapsto \boldsymbol{v}_1(p) + \boldsymbol{v}_2(p)$$

  - Note that we are invoking our definition of addition on $T_p\mathbb{R}^n$ here.

- The list of vectors $(\partial/\partial x_1)_p, \dots, (\partial/\partial x_n)_p$ constitutes a basis of $T_p\mathbb{R}^n$.
  - Recall that $(\partial/\partial x_i)_p = (p, e_i)$.
  - Thus, if $\boldsymbol{v}$ is a vector field on U, it has a unique decomposition

$$\boldsymbol{v} = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i}$$

    where each $g_i : U \to \mathbb{R}$.

- $\boldsymbol{C^\infty}$ (vector field): A vector field such that $g_i \in C^\infty(U)$ for all g_i's in its unique decomposition.

- **Lie derivative** (of f with respect to $\boldsymbol{v}$): The function from $U \to \mathbb{R}$ defined as follows, where $U \subset \mathbb{R}^n$, $f : U \to \mathbb{R}$ is a C^1-mapping, and $\boldsymbol{v}(p) = (p, v)$. *Denoted by* $\boldsymbol{L_v f}$. *Given by*

$$L_{\boldsymbol{v}} f(p) = Df(p)v$$

  - A more explicit formula for the Lie derivative is

$$L_{\boldsymbol{v}} f = \sum_{i=1}^{n} g_i \frac{\partial f}{\partial x_i}$$

  - The vector field decides the direction in which we take the derivative at each point. Instead of having to take a derivative everywhere in one direction at a time, we can now take a derivative in a different direction at every point!

- Lemma 2.1.11: Let U be an open subset of $\mathbb{R}^n$, $\boldsymbol{v}$ a vector field on U, and $f_1, f_2 \in C^1(U)$. Then

$$L_{\boldsymbol{v}}(f_1 \cdot f_2) = L_{\boldsymbol{v}}(f_1) \cdot f_2 + f_1 \cdot L_{\boldsymbol{v}}(f_2)$$

  Proof. See Exercise 2.1.ii. □

- **Cotangent space** (to $\mathbb{R}^n$ at p): The dual vector space to $T_p\mathbb{R}^n$. *Denoted by* $\boldsymbol{T_p^*\mathbb{R}^n}$. *Given by*

$$T_p^*\mathbb{R}^n = (T_p\mathbb{R}^n)^*$$

- **Cotangent vector** (to $\mathbb{R}^n$ at p): An element of $T_p^*\mathbb{R}^n$.

- **Differential one-form** (on U): A function which assigns to each point $p \in U$ a cotangent vector. *Also known as* **one-form** (on U). *Denoted by* $\boldsymbol{\omega}$. *Given by*

$$p \mapsto \omega_p$$

- Note that by identifying $T_p\mathbb{R} \cong \mathbb{R}$, we have that $\mathrm{d}f_p \in T_p^*\mathbb{R}^n$, assuming that $f : U \to \mathbb{R}$.

- Geometric example: Consider $f:\mathbb{R}^2\to\mathbb{R}$ such that $f\in C^1$. By the latter condition, we know that the graph of f is a "smooth" surface in $\mathbb{R}^3$, i.e., one without any abrupt changes in derivative (consider the graph of the piecewise function defined by $-x^2$ for $x<0$ and x^2 for $x\geq 0$, for example). What $\mathrm{d}f_p$ does is take a point (p_1,p_2,q), where $q=f(p)$, on the surface and a vector v with tail at (p,q), and give us a number representing the magnitude of the instantaneous change of f at p in the direction v. Thus, $\mathrm{d}f_p$ contains, in a sense, all of the information concerning the rate of change of f at p.

- $\mathbf{d}\boldsymbol{f}$: The one-form on U defined as follows. *Given by*

$$p\mapsto \mathrm{d}f_p$$

  - Continuing with the geometric example: What $\mathrm{d}f$ does is take every point p across the surface and return all of the information concerning the rate of change of f at p (packaged neatly by $\mathrm{d}f_p$).

- **Pointwise product** (of ϕ with ω): The one-form on U defined as follows, where $\phi:U\to\mathbb{R}$ and ω is a one-form. *Denoted by* $\boldsymbol{\phi\omega}$. *Given by*

$$(\phi\omega)_p=\phi(p)\omega_p$$

- **Pointwise sum** (of ω_1,ω_2): The one-form on U defined as follows. *Denoted by* $\boldsymbol{\omega_1+\omega_2}$. *Given by*

$$(\omega_1+\omega_2)_p=(\omega_1)_p+(\omega_2)_p$$

- $\boldsymbol{x_i}$: The function from $U\to\mathbb{R}$ defined as follows. *Given by*

$$x_i(u_1,\dots,u_n)=u_i$$

  - x_i is constantly increasing in the x_i-direction, and constant in every other direction.

- $(\mathbf{d}\boldsymbol{x_i})_{\boldsymbol{p}}$: The linear map from $T_p\mathbb{R}^n\to\mathbb{R}$ (i.e., the cotangent vector in $T_p^*\mathbb{R}^n$) defined as follows. *Given by*

$$(\mathrm{d}x_i)_p(p,a_1x_1+\cdots+a_nx_n)=a_1$$

  - Naturally, the instantaneous change in x_i at any point p in the direction $\boldsymbol{v}(p)$ will just be the magnitude of $\boldsymbol{v}(p)$ in the x_i-direction.
  - It follows immediately that

$$(\mathrm{d}x_i)_p\left(\frac{\partial}{\partial x_j}\right)_p=\delta_{ij}$$

- Consequently, the list of cotangent vectors $(\mathrm{d}x_1)_p,\dots,(\mathrm{d}x_n)_p$ constitutes a basis of $T_p^*\mathbb{R}^n$ that is **dual** to the basis $(\partial/\partial x_1)_p,\dots,(\partial/\partial x_n)_p$ of $T_p\mathbb{R}^n$.

- $\mathbf{d}\boldsymbol{x_i}$: The one-form on U defined as follows. *Given by*

$$p\mapsto(\mathrm{d}x_i)_p$$

  - Thus, if $\omega_p:T_p\mathbb{R}^n\to\mathbb{R}$, it has a unique decomposition

$$\omega_p=\sum_{i=1}^n f_i(p)(\mathrm{d}x_i)_p$$

    where every $f_i:U\to\mathbb{R}$.
  - Similarly, $\omega:U\to T_p^*\mathbb{R}^n$ has a unique decomposition

$$\omega=\sum_{i=1}^n f_i\mathrm{d}x_i$$

- **Smooth** (one-form): A one form for which the associated functions $f_1, \dots, f_n \in C^\infty$. *Also known as* $\boldsymbol{C^\infty}$ (one-form).
- Lemma 2.1.18: Let U be an open subset of $\mathbb{R}^n$. If $f : U \to \mathbb{R}$ is a C^∞ function, then

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \, \mathrm{d}x_i$$

- **Interior product** (of $\boldsymbol{v}$ with ω): The function which combines a point $p \in U$, the vector $\boldsymbol{v}(p) \in T_p\mathbb{R}^n$, and the functional $\omega_p \in T_p^*\mathbb{R}^n$ to yield a real number. *Denoted by* $\boldsymbol{\iota_{v(p)}\omega_p}$.
- Examples.
  - If

$$\boldsymbol{v} = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i} \qquad\qquad \omega = \sum_{i=1}^{n} f_i \mathrm{d}x_i$$

    then

$$\iota_{\boldsymbol{v}}\omega = \sum_{i=1}^{n} f_i g_i$$

    - We use multiplication and the fact that $(\mathrm{d}x_i)_p(\partial/\partial x_j)_p = \delta_{ij}$ to obtain this result.
    - If $\boldsymbol{v}, \omega \in C^\infty$, so is $\iota_{\boldsymbol{v}}\omega$, where C^∞ refers to three different sets of smooth objects (vector fields, one-forms, and functions, respectively[1]).
  - As with f, if $\phi \in C^\infty(U)$, then

$$\mathrm{d}\phi = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} \mathrm{d}x_i$$

  - It follows if $\boldsymbol{v}$ is defined as in the first example that

$$\iota_{\boldsymbol{v}}\mathrm{d}\phi = \sum_{i=1}^{n} g_i \frac{\partial \phi}{\partial x_i} = L_{\boldsymbol{v}}\phi$$

- **Integral curve** (of $\boldsymbol{v}$): A C^1 curve $\gamma : (a,b) \to U$ such that for all $t \in (a,b)$,

$$\boldsymbol{v}(\gamma(t)) = (\gamma(t), \tfrac{\mathrm{d}\gamma}{\mathrm{d}t}(t))$$

  where $U \subset \mathbb{R}^n$ is open and $\boldsymbol{v}$ is a vector field on U.
  - An equivalent condition if $\boldsymbol{v} = \sum_{i=1}^{n} g_i \, \mathrm{d}/\mathrm{d}x_i$ and $g : U \to \mathbb{R}^n$ is defined by $(g_1, \dots, g_n)$ is that γ satisfies the system of differential equations

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = g(\gamma(t))$$

- Theorem 2.2.4 (existence of integral curves): Let $U \subset \mathbb{R}^n$ open, $\boldsymbol{v}$ a vector field on U. If $p_0 \in U$ and $a \in \mathbb{R}$, then there exist $I = (a - T, a + T)$ for some $T \in \mathbb{R}$, $U_0 = N_r(p_0) \subset U$, and $\gamma_p : I \to U$ such that $\gamma_p(a) = p$ for all $p \in U_0$.
- Theorem 2.2.5 (uniqueness of integral curves): Let $U \subset \mathbb{R}^n$ open, $\boldsymbol{v}$ a vector field on U, and $\gamma_1 : I_1 \to U$ and $\gamma_2 : I_2 \to U$ integral curves for $\boldsymbol{v}$. If $a \in I_1 \cap I_2$ and $\gamma_1(a) = \gamma_2(a)$, then

$$\gamma_1|_{I_1 \cap I_2} = \gamma_2|_{I_1 \cap I_2}$$

  and the curve $\gamma : I_1 \cup I_2 \to U$ defined by

$$\gamma(t) = \begin{cases} \gamma_1(t) & t \in I_1 \\ \gamma_2(t) & t \in I_2 \end{cases}$$

  is an integral curve for $\boldsymbol{v}$.

[1] Technically, these objects are all types of functions, though, so it is fair to call them all smooth.

- Theorem 2.2.6 (smooth dependence on initial data): Let $V \subset U \subset \mathbb{R}^n$ open, $\boldsymbol{v}$ a C^∞-vector field on V, $I \subset \mathbb{R}$ an open interval, and $a \in I$. Let $h : V \times I \to U$ have the following properties.
  1. $h(p, a) = p$.
  2. For all $p \in V$, the curve $\gamma_p : I \to U$ defined by $\gamma_p(t) = h(p, t)$ is an integral curve of $\boldsymbol{v}$.

  Then $h \in C^\infty$.
- **Autonomous** (system of ODEs): A system of ODEs that does not explicitly depend on the independent variable.
- $\mathrm{d}\gamma/\mathrm{d}t = g(\gamma(t))$ is autonomous since g does not depend on t.
- Theorem 2.2.7: Let $I = (a, b)$. For all $c \in \mathbb{R}$, define $I_c = (a-c, b-c)$. If $\gamma : I \to U$ is an integral curve, then the reparameterized curve $\gamma_c : I_c \to U$ defined by
  $$\gamma_c(t) = \gamma(t + c)$$
  is an integral curve.
  - Note that this is truly just a reparameterization; we still have, for instance,
    $$\gamma_c(a - c) = \gamma(a - c + c) = \gamma(a) \qquad \gamma_c(b - c) = \gamma(b - c + c) = \gamma(b)$$
- **Integral** (of the system $\mathrm{d}\gamma/\mathrm{d}t = g(\gamma(t))$): A C^1-function $\phi : U \to \mathbb{R}$ such that for every integral curve $\gamma(t)$, the function $t \mapsto \phi(\gamma(t))$ is constant.
  - An alternate condition is that for all t,
    $$0 = \frac{\mathrm{d}}{\mathrm{d}t}\phi(\gamma(t)) = (D\phi)_{\gamma(t)}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right) = (D\phi)_{\gamma(t)}(v) = L_v\phi(p)$$
    where $\boldsymbol{v}(p) = (p, v)$.
- Theorem 2.2.9: Let $U \subset \mathbb{R}^n$ open, $\phi \in C^1(u)$. Then ϕ is an integral of the system $\mathrm{d}\gamma/\mathrm{d}t = g(\gamma(t))$ iff $L_{\boldsymbol{v}}\phi = 0$.
- **Complete** (vector field): A vector field $\boldsymbol{v}$ on U such that for every $p \in U$, there exists an integral curve $\gamma : \mathbb{R} \to U$ with $\gamma(0) = p$.
  - Alternatively, for every p, there exists an integral curve that starts at p and exists for all time.
- **Maximal** (integral curve): An integral curve $\gamma : [0, b) \to U$ with $\gamma(0) = p$ such that it cannot be extended to an interval $[0, b')$ with $b' > b$.
- For a maximal curve, either...
  1. $b = +\infty$;
  2. $|\gamma(t)| \to +\infty$ as $t \to b$;
  3. The limit set of $\{\gamma(t) \mid 0 \leq t < b\}$ contains points on the boundary of U.
- Eliminating 2 and 3, as can be done with the following lemma, provides a means of proving that γ exists for all positive time.
- Lemma 2.2.11: The scenarios 2 and 3 above cannot happen if there exists a proper C^1-function $\phi : U \to \mathbb{R}$ with $L_{\boldsymbol{v}}\phi = 0$.

  Proof. Suppose there exists $\phi \in C^1$ such that $L_{\boldsymbol{v}}\phi = 0$. Then ϕ is constant on $\gamma(t)$ (say with value $c \in \mathbb{R}$) by definition. But then since $\{c\} \subset \mathbb{R}$ is compact and $\phi \in C^1$, $\phi^{-1}(c) \subset U$ is compact and, importantly, contains $\mathrm{im}(\gamma)$. The compactness of this set implies that γ can neither "run off to infinity" as in scenario 2 or "run off the boundary" as in scenario 3. □

- Theorem 2.2.12: If there exists a proper C^1-function $\phi : U \to \mathbb{R}$ with the property $L_{\boldsymbol{v}}\phi = 0$, then the vector field $\boldsymbol{v}$ is complete.

  Proof. Apply a similar argument to the interval $(-b, 0]$ and join the two results. □

- Example: Let $U = \mathbb{R}^2$ and let $\boldsymbol{v}$ be the vector field
$$\boldsymbol{v} = x^3 \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$$
  Then $\phi(x, y) = 2y^2 + x^4$ is a proper function with the above property.
  - Note that indeed, as per Theorem 2.2.12, we have that
$$\begin{aligned} L_{\boldsymbol{v}}\phi &= x^3 \frac{\partial \phi}{\partial y} - y \frac{\partial \phi}{\partial x} \\ &= x^3 \cdot 4y - y \cdot 4x^3 \\ &= 0 \end{aligned}$$

- We now build up to an alternate completeness condition (Theorem 2.2.15).
- **Support** (of $\boldsymbol{v}$): The following set. *Denoted by* **supp** $\boldsymbol{(v)}$. *Given by*
$$\operatorname{supp}(\boldsymbol{v}) = \overline{\{q \in U \mid \boldsymbol{v}(q) \neq 0\}}$$
- **Compactly supported** (vector field $\boldsymbol{v}$): A vector field $\boldsymbol{v}$ for which $\operatorname{supp}(\boldsymbol{v})$ is compact.
- Theorem 2.2.15: If $\boldsymbol{v}$ is compactly supported, then $\boldsymbol{v}$ is complete.

  Proof. Let $p \in U$ be such that $\boldsymbol{v}(p) = 0$. Define $\gamma_0 : (-\infty, \infty) \to U$ by $\gamma_0(t) = p$ for all $t \in (-\infty, \infty)$. Since
$$\frac{\mathrm{d}\gamma_0}{\mathrm{d}t} = 0 = \boldsymbol{v}(p) = \boldsymbol{v}(\gamma(t))$$
  we know that γ_0 is an integral curve of $\boldsymbol{v}$.

  Now consider an arbitrary integral curve $\gamma : (-a, b) \to U$ having the property $\gamma(t_0) = p$ for some $t_0 \in (-a, b)$. It follows by Theorem 2.2.5 that γ and γ_0 coincide on the interval $(-a, a)$.

  By hypothesis, $\operatorname{supp}(\boldsymbol{v})$ is compact. Basic set theory tells us that for γ arbitrary, either $\gamma(t) \in \operatorname{supp}(\boldsymbol{v})$ for all t or there exists t_0 such that $\gamma(t_0) \in U \setminus \operatorname{supp}(\boldsymbol{v})$. But then by the definition of $\operatorname{supp}(\boldsymbol{v})$, $\boldsymbol{v}(\gamma(t_0)) = 0$. Thus, letting $p = \gamma(t_0)$, we have an associated γ_0 that γ "runs along" while outside the support. It follows that in either case, γ cannot go off to ∞ or go off the boundary of U as $t \to b$. □

- **Bump function**: A function $f : \mathbb{R}^n \to \mathbb{R}$ which is both smooth and compactly supported.
- $\boldsymbol{C_0^\infty(\mathbb{R}^n)}$: The vector space of all bump functions with domain $\mathbb{R}^n$.
- An application of Theorem 2.2.15.
  - Suppose $\boldsymbol{v}$ is a vector field on U and we want to inspect the integral curves of $\boldsymbol{v}$ on some $A \subset U$ compact. Let $\rho \in C_0^\infty(U)$ be such that $\rho(p) = 1$ for all $p \in N_r(A)$, where $N_r(A)$ is some neighborhood of the set A. Then the vector field $\boldsymbol{w} = \rho\boldsymbol{v}$ is compactly supported and hence complete. However, it is also identical to $\boldsymbol{v}$ on A, so its integral curves on A coincide with those of $\boldsymbol{v}$ on A.

- $\boldsymbol{f_t}$: The map from $U \to U$ defined as follows, where $\boldsymbol{v}$ is complete. *Given by*
$$f_t(p) = \gamma_p(t)$$

- Note that it is the fact that $\boldsymbol{v}$ is complete that justifies the existence of an integral curve $\gamma_p : \mathbb{R} \to U$ with $\gamma_p(0) = p$ for all $p \in U$.

- Properties of f_t.

1. $\boldsymbol{v} \in C^\infty$ implies $f_t \in C^\infty$.

   Proof. By Theorem 2.2.6. ☐

2. $f_0 = \mathrm{id}_U$.

   Proof. We have
$$f_0(p) = \gamma_p(0) = p = \mathrm{id}_U(p)$$
   as desired. ☐

3. $f_t \circ f_a = f_{t+a}$.

   Proof. Let $q = f_a(p)$. Since $\boldsymbol{v}$ is complete and $q \in U$, there exists γ_q such that $\gamma_q(0) = q$. It follows that $\gamma_p(a) = f_a(p) = q = \gamma_q(0)$. Thus, by Theorem 2.2.7, $\gamma_q(t)$ and $\gamma_p(t+a)$ are both integral curves of $\boldsymbol{v}$ with the same initial point. Therefore,
$$(f_t \circ f_a)(p) = f_t(q) = \gamma_q(t) = \gamma_p(t+a) = f_{t+a}(p)$$
   for all t, as desired. ☐

4. $f_t \circ f_{-t} = \mathrm{id}_U$.

   Proof. See properties 2 and 3. ☐

5. $f_{-t} = f_t^{-1}$.

   Proof. See property 4. ☐

- Thus, f_t is a C^∞ **diffeomorphism**.
  - "Hence, if $\boldsymbol{v}$ is complete, it generates a **one-parameter group** f_t $(-\infty < t < \infty)$ of C^∞-diffeomorphisms of U" (Guillemin & Haine, 2018, p. 40).
- **Diffeomorphism**: An isomorphism of smooth manifolds. In particular, it is an invertible function that maps one differentiable manifold to another such that both the function and its inverse are differentiable.
- **One-parameter group**: A continuous group homomorphism $\varphi : \mathbb{R} \to G$ from the real line $\mathbb{R}$ (as an additive group) to some other topological group G.
- If $\boldsymbol{v}$ is not complete, there is an analogous result, but it is trickier to formulate.
- ***f*-related** (vector fields $\boldsymbol{v}, \boldsymbol{w}$): Two vector fields $\boldsymbol{v}, \boldsymbol{w}$ such that
$$\mathrm{d}f_p(\boldsymbol{v}(p)) = \boldsymbol{w}(f(p))$$
  for all $p \in U$, where $\boldsymbol{v}$ is a C^∞-vector field on $U \subset \mathbb{R}^n$ open, $\boldsymbol{w}$ is a C^∞-vector field on $W \subset \mathbb{R}^m$ open, and $f : U \to W$ is a C^∞ map.
  - An alternate formulation is that in terms of coordinates,
$$w_i(q) = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} v_j(p)$$
    where
$$\boldsymbol{v} = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i} \boldsymbol{w} \qquad = \sum_{j=1}^m w_j \frac{\partial}{\partial y_i}$$
    for $v_i \in C^k(U)$ and $w_j \in C^k(W)$.

- If $m = n$ and f is a C^∞ diffeomorphism, then $\boldsymbol{w}$ is the vector field defined by the equation
$$w_i = \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j} v_j \right) \circ f^{-1}$$
- Theorem 2.2.18: If $f : U \to W$ is a C^∞ diffeomorphism and $\boldsymbol{v}$ is a C^∞ vector field on U, then there exists a unique C^∞ vector field $\boldsymbol{w}$ on W having the property that $\boldsymbol{v}$ and $\boldsymbol{w}$ are f-related.

  Proof. See the above. □
- **Pushforward** (of $\boldsymbol{v}$ by f): The vector field $\boldsymbol{w}$ shown to exist by Theorem 2.2.18. *Denoted by* $\boldsymbol{f_* v}$.
- Theorem 2.2.20: Let $U_1, U_2 \subset \mathbb{R}^n$ open, $\boldsymbol{v}_1, \boldsymbol{v}_2$ vector fields on U_1, U_2, and $f : U_1 \to U_2$ a C^∞ map. If $\boldsymbol{v}_1, \boldsymbol{v}_2$ are f-related, every integral curve $\gamma : I \to U_1$ of $\boldsymbol{v}_1$ gets mapped by f onto an integral curve $f \circ \gamma : I \to U_2$ of $\boldsymbol{v}_2$.

  Proof. We want to show that
$$\boldsymbol{v}_2((f \circ \gamma)(t)) = \left((f \circ \gamma)(t),\, \tfrac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)\big|_t\right)$$
  We are given that
$$\boldsymbol{v}_1(\gamma(t)) = \left(\gamma(t),\, \tfrac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_t\right) \qquad \mathrm{d}f_p(\boldsymbol{v}_1(p)) = \boldsymbol{v}_2(f(p))$$
  Let $p = \gamma(t)$ and $q = f(p)$. Then
$$\begin{aligned} \boldsymbol{v}_2((f \circ \gamma)(t)) &= \boldsymbol{v}_2(f(p)) \\ &= \mathrm{d}f_p(\boldsymbol{v}_1(p)) \\ &= \mathrm{d}f_p(\boldsymbol{v}_1(\gamma(t))) \\ &= \mathrm{d}f_p\left(\gamma(t),\, \tfrac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_t\right) \\ &= \mathrm{d}f_p\left(p,\, \tfrac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_t\right) \\ &= \left(q, Df(p)\left(\tfrac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_t\right)\right) \\ &= \left((f \circ \gamma)(t),\, \tfrac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)\big|_t\right) \end{aligned}$$
  as desired. □
- Corollary 2.2.21: In the setting of Theorem 2.2.20, suppose $\boldsymbol{v}_1, \boldsymbol{v}_2$ are complete. Let $(f_{i,t})_{t \in \mathbb{R}} : U_i \to U_i$ be the one-parameter group of diffeomorphisms generated by $\boldsymbol{v}_i$. Then
$$f \circ f_{1,t} = f_{2,t} \circ f$$
  Proof. We have that
$$(f \circ f_{1,t})(p) = (f \circ \gamma_p)(t)$$
  By Theorem 2.2.20, the above is an integral curve of $\boldsymbol{v}_2$. Let $f(p) = q$. Then
$$\begin{aligned} (f_{2,t} \circ f)(p) &= f_{2,t}(q) \\ &= \gamma_q(t) \end{aligned}$$
  ...

  Guillemin and Haine (2018) proves that if $\phi \in C^\infty(U_2)$ and $f^*\phi = \phi \circ f$, then
$$f^* L_{\boldsymbol{v}_2} \phi = L_{\boldsymbol{v}_1} f^* \phi$$
  by virtue of the observations that if $f(p) = q$, then at the point p, the right-hand side above is $(\mathrm{d}\phi)_q \circ \mathrm{d}f_p(\boldsymbol{v}_1(p))$ by the chain rule and by definition the left hand side is $\mathrm{d}\phi_q(\boldsymbol{v}_2(q))$. Moreover, by definition, $\boldsymbol{v}_2(q) = \mathrm{d}f_p(\boldsymbol{v}_1(p))$ so the two sides are the same. □

- Theorem 2.2.22: For $i = 1, 2, 3$, let $U_i \subset \mathbb{R}^{n_i}$ open and $\boldsymbol{v}_i$ a vector field on U_i. For $i = 1, 2$, let $f_i : U_i \to U_{i+1}$ be a C^∞ map. If $\boldsymbol{v}_1, \boldsymbol{v}_2$ are f_1-related and $\boldsymbol{v}_2, \boldsymbol{v}_3$ are f_2-related, then $\boldsymbol{v}_1, \boldsymbol{v}_3$ are $(f_2 \circ f_1)$-related. In particular, if f_1, f_2 are diffeomorphisms, we have

$$(f_2)_*(f_1)_*\boldsymbol{v}_1 = (f_2 \circ f_1)_*\boldsymbol{v}_1$$

- **Pullback** (of μ on U): The function from $U \to T_p^*\mathbb{R}^n$ defined as follows, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open, $f : U \to V$ is a C^∞ map, and μ is a one-form on V. *Denoted by* $\boldsymbol{f^*\mu}$. *Given by*

$$p \mapsto \mu_{f(p)} \circ \mathrm{d}f_p$$

- If $\phi : V \to \mathbb{R}$ is a C^∞ map and $\mu = \mathrm{d}\phi$, then

$$\mu_q \circ \mathrm{d}f_p = \mathrm{d}\phi_q \circ \mathrm{d}f_p = \mathrm{d}(\phi \circ f)_p$$

  - In other words,
$$f^*\mu = \mathrm{d}\phi \circ f$$

- Theorem 2.2.24: If μ is a C^∞ one-form on V, its pullback $f^*\mu$ is C^∞.

  Proof. See Exercise 2.2.ii. □