5 Manifolds

From Guillemin and Haine (2018).

Chapter 4

5/22: **4.1.i.** Show that the set of solutions to the system of equations

$$\begin{aligned} x_1^2 + \cdots + x_n^2 &= 1 \\ x_1 + \cdots + x_n &= 0 \end{aligned}$$

is an $(n-2)$-dimensional submanifold of $\mathbb{R}^n$.

Proof. Let $f : \mathbb{R}^n \to \mathbb{R}^2$ be defined by

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mapsto \begin{bmatrix} x_1^2 + \cdots + x_n^2 - 1 \\ x_1 + \cdots + x_n \end{bmatrix}$$

Then the set of solutions to the given system of equations is equal to $f^{-1}(0)$, where $0 \in \mathbb{R}^2$.

The task now becomes a problem of proving that that $f^{-1}(0)$ is an $(n-2)$-dimensional submanifold of $\mathbb{R}^n$. To do so, Theorem 4.1.7 tells us that it will suffice to show that 0 is a regular value of f. Suppose for the sake of contradiction that 0 is not a regular value of f. Then there exists $p \in f^{-1}(0)$ such that f is not a submersion at p. It follows that $Df(p) : \mathbb{R}^n \to \mathbb{R}^2$ is not surjective. Thus, the rank of the matrix of $Df(p)$ must be less than two. Consequently, all columns in the matrix

$$\mathcal{M}(Df(p)) = \begin{bmatrix} 2p_1 & \cdots & 2p_n \\ 1 & \cdots & 1 \end{bmatrix}$$

where $p = (p_1, \dots, p_n)$ must be equal. It follows that $p_1 = \cdots = p_n$. This combined with the fact that $p_1 + \cdots + p_n = 0$ means that $p_i = 0$ for all $i = 1, \dots, n$. But then $p_1^2 + \cdots + p_n^2 - 1 = -1 \neq 0$, a contradiction. $\square$

4.1.ii. Let $S^{n-1} \subset \mathbb{R}^n$ be the $(n-1)$-sphere and let

$$X_a = \{x \in S^{n-1} \mid x_1 + \cdots + x_n = a\}$$

For what values of a is X_a an $(n-2)$-dimensional submanifold of S^{n-1}?

Proof. We first determine which values of a yield a nonempty X_a. Then, we determine which of these X_a describe $(n-2)$-dimensional submanifolds of S^{n-1}.

For the first part, suppose $x \in S^{n-1}$. Then $x_1^2 + \cdots + x_n^2 = 1$. It follows by the Cauchy-Schwarz inequality that

$$\begin{aligned} |a| &= |x_1 + \cdots + x_n| \\ &= |x_1 \cdot 1 + \cdots + x_n \cdot 1| \\ &\leq \sqrt{x_1^2 + \cdots + x_n^2} \cdot \sqrt{1^2 + \cdots + 1^2} \\ &= \sqrt{1} \cdot \sqrt{n} \\ &= \sqrt{n} \end{aligned}$$

Now for the second part. Let $f_a : \mathbb{R}^n \to \mathbb{R}^2$ be defined by

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mapsto \begin{bmatrix} x_1^2 + \cdots + x_n^2 - 1 \\ x_1 + \cdots + x_n - a \end{bmatrix}$$

Then $X_a = f_a^{-1}(0)$. Thus, we want to find the set of all a such that 0 is a regular value of f. Suppose a is not in this set. Then 0 is not a regular value of f_a. It follows by a similar argument to that used in Exercise 4.1.i that $x_1 = \cdots = x_n$. This combined with the fact that $a = x_1 + \cdots + x_n = nx_i$ implies that $x_i = a/n$ for $i = 1, \dots, n$. And this result combined with the fact that $x_1^2 + \cdots + x_n^2 = 1$ implies that

$$\begin{aligned} 1 &= x_1^2 + \cdots + x_n^2 \\ &= nx_i^2 \\ &= a^2/n \\ a &= \pm\sqrt{n} \end{aligned}$$

Therefore, if $|a| \leq \sqrt{n}$ and $|a| \neq \sqrt{n}$, we know that

$$\boxed{|a| < \sqrt{n}}$$

□

4.1.iii. Show that if X_i is an n_i-dimensional submanifold of $\mathbb{R}^{N_i}$ for $i = 1, 2$, then

$$X_1 \times X_2 \subset \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$$

is an $(n_1 + n_2)$-dimensional submanifold of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$.

Proof. Taking the hint from Guillemin and Haine (2018, p. 98), we approach this problem from the perspective of the definition of an n-manifold, as opposed that of Theorem 4.1.7. Additionally, note that any time "i" appears for the remainder of this proof, it is a stand-in for $1, 2$.

To prove that $X_1 \times X_2 \subset \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ is an $(n_1 + n_2)$-manifold, it will suffice to show that for every $p \in X_1 \times X_2$, there exists a neighborhood $V \subset \mathbb{R}^{N_1+N_2}$ of p, an open subset $U \subset \mathbb{R}^{n_1+n_2}$, and a diffeomorphism $\phi : U \to (X_1 \times X_2) \cap V$. Let $p \in X_1 \times X_2$ be arbitrary. Suppose $p = (p_1, p_2)$, where p_i is an n_i-tuple. It follows that $p_i \in X_i$. Therefore, since X_i is an n_i-manifold, there exists a neighborhood $V_i \subset \mathbb{R}^{N_i}$ of p_i, an open subset $U_i \subset \mathbb{R}^{n_i}$, and a diffeomorphism $\phi_i : U_i \to X_i \cap V_i$. Let $V = V_1 \times V_2$, $U = U_1 \times U_2$, and $\phi(x_1, x_2) = (\phi_1(x_1), \phi_2(x_2))$. Naturally, $V \subset \mathbb{R}^{N_1+N_2}$ and $U \subset \mathbb{R}^{n_1+n_2}$. Additionally, endowing $\mathbb{R}^{N_1+N_2} = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ with the product topology ensures that V is a neighborhood of p and endowing $\mathbb{R}^{n_1+n_2} = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with the product topology ensures that U is open. Lastly, defining ϕ as the "product" of two diffeomorphisms guarantees that ϕ, itself, is also a diffeomorphism. □

4.1.v. Let $g : \mathbb{R}^n \to \mathbb{R}^k$ be a C^∞ map and let $X = \Gamma_g$ be the graph of g. Prove directly that X is an n-manifold by proving that the map $\gamma_g : \mathbb{R}^n \to X$ defined by

$$x \mapsto (x, g(x))$$

is a diffeomorphism.

Proof. It's clear that γ_g is a C^∞ map since each of its components are C^∞. It is a diffeomorphism since it's inverse is the map $\pi : \gamma_g \to \mathbb{R}^n$ given by $\pi(x, g(x)) = x$, which is also clearly C^∞. □

4.1.vi. Prove that the orthogonal group $O(n)$ is an $n(n-1)/2$-manifold. *Hints*:

- Let $f : \mathcal{M}_n \to \mathcal{S}_n$ be the map
$$f(A) = A^\intercal A - \mathrm{id}_n$$
show that $O(n) = f^{-1}(0)$.
- Show that
$$f(A + \varepsilon B) = A^\intercal A + \varepsilon(A^\intercal B + B^\intercal A) + \varepsilon^2 B^\intercal B - \mathrm{id}_n$$

- Conclude that the derivative of f at A is the map given by

$$B \mapsto A^\intercal B + B^\intercal A$$

- Let $A \in O(n)$. Show that if $C \in \mathcal{S}_n$ and $B = AC/2$, then $Df(A)(B) = C$.
- Conclude that the derivative of f is surjective at A.
- Conclude that 0 is a regular value of the mapping f.

Proof. As per Guillemin and Haine (2018, p. 100), the set $\mathcal{M}_n$ of all $n \times n$ matrices is isomorphic to $\mathbb{R}^{n^2}$ (one degree of freedom for each matrix element), and the set $\mathcal{S}_n$ of all symmetric $n \times n$ matrices is isomorphic to $\mathbb{R}^{n(n+1)/2}$ (one degree of freedom for each matrix element in the upper triangle). Additionally,

$$\begin{aligned} n^2 - \frac{n(n+1)}{2} &= n^2 - \frac{1}{2}n^2 - \frac{1}{2}n \\ &= \frac{1}{2}n^2 - \frac{1}{2}n \\ &= \frac{n(n-1)}{2} \end{aligned}$$

To prove that $O(n)$ is an $n(n-1)/2$-manifold, Theorem 4.1.7 tells us that it will suffice to find a function $f : \mathcal{M}_n \to \mathcal{S}_n$ with regular value 0 such that $O(n) = f^{-1}(0)$.

We first define a function f that we will prove fits all of the above requirements. Let f be described by the relation

$$A \mapsto A^\intercal A - \mathrm{id}_n$$

By the properties of matrix multiplication, $A^\intercal A \in \mathcal{S}_n$ regardless of whether or not A is. Since $\mathcal{S}_n$ is a vector space, subtracting $\mathrm{id}_n \in \mathcal{S}_n$ will not take the difference out of $\mathcal{S}_n$. Thus, f does map arbitrary $n \times n$ matrices to symmetric $n \times n$ matrices, as desired. Moreover, if $A \in O(n)$, then $A^\intercal A = \mathrm{id}_n$. It follows that

$$\begin{aligned} f(A) &= A^\intercal A - \mathrm{id}_n \\ &= \mathrm{id}_n - \mathrm{id}_n \\ &= 0 \end{aligned}$$

$A \notin O(n)$ implies a similar result. Therefore, $O(n) = f^{-1}(0)$.

We now build up to proving that 0 is a regular value of f. To prove this, we will need to check that f is a submersion at all $A \in O(n) = f^{-1}(0)$, i.e., that $Df(A)$ is surjective for all such A. To confirm this, we will calculate $Df(A)$ for an arbitrary $A \in O(n)$ and show directly that for all $C \in \mathcal{S}_n$, there exists $B \in \mathcal{M}_n$ such that $Df(A)(B) = C$. Let's begin.

We have from first principles that

$$0 = \lim_{H \to 0} \frac{|f(A+H) - f(A) - Df(A)(H)|}{|H|}$$

where we take $|\cdot|$ to be any matrix norm (e.g., the operator norm or the Frobenius norm). If we take $H = \varepsilon B$, where $\varepsilon \in \mathbb{R}_{>0}$, then we can work with the limit definition of the derivative more easily. First off, we can determine that

$$\begin{aligned} f(A + \varepsilon B) &= (A + \varepsilon B)^\intercal (A + \varepsilon B) - \mathrm{id}_n \\ &= A^\intercal A + A^\intercal (\varepsilon B) + (\varepsilon B)^\intercal A + (\varepsilon B)^\intercal (\varepsilon B) - \mathrm{id}_n \\ &= A^\intercal A + \varepsilon (A^\intercal B + B^\intercal A) + \varepsilon^2 B^\intercal B - \mathrm{id}_n \end{aligned}$$

Plugging this back into the limit definition, we have that

$$\begin{aligned}
0 &= \lim_{H\to 0}\frac{|f(A+H)-f(A)-Df(A)(H)|}{|H|}\\
&= \lim_{\varepsilon\to 0}\frac{|[A^\intercal A+\varepsilon(A^\intercal B+B^\intercal A)+\varepsilon^2B^\intercal B-\mathrm{id}_n]-[A^\intercal A-\mathrm{id}_n]-Df(A)(\varepsilon B)|}{|\varepsilon B|}\\
&= \lim_{\varepsilon\to 0}\frac{|[\varepsilon(A^\intercal B+B^\intercal A)+\varepsilon^2B^\intercal B]-Df(A)(\varepsilon B)|}{|\varepsilon B|}\\
&= \lim_{\varepsilon\to 0}\frac{\varepsilon|(A^\intercal B+B^\intercal A)+\varepsilon B^\intercal B-Df(A)(B)|}{\varepsilon|B|}\\
&= \lim_{\varepsilon\to 0}\frac{|(A^\intercal B+B^\intercal A)+\varepsilon B^\intercal B-Df(A)(B)|}{|B|}
\end{aligned}$$

From here, it is easy to see that if we let $Df(A)$ send

$$B\mapsto A^\intercal B+B^\intercal A$$

then the above limit evaluates to 0, as desired.

Let $A\in O(n)$ be arbitrary, and let $C\in\mathcal{S}_n$ be arbitrary. We want to find $B\in\mathcal{M}_n$ such that $Df(A)(B)=C$. Choose $B=AC/2$. Then

$$\begin{aligned}
Df(A)(B) &= A^\intercal B+B^\intercal A\\
&= \frac{1}{2}[A^\intercal AC+(AC)^\intercal A]\\
&= \frac{1}{2}[A^\intercal AC+C^\intercal A^\intercal A]\\
&= \frac{1}{2}[\mathrm{id}_n C+C\,\mathrm{id}_n]\\
&= C
\end{aligned}$$

as desired. □

4.2.i. What is the tangent space to the quadric

$$Q=\{(x_1,\dots,x_n)\in\mathbb{R}^n\mid x_n=x_1^2+\cdots+x_{n-1}^2\}$$

at the point $(1,0,\dots,0,1)$?

Proof. Let $f:\mathbb{R}^{n-1}\to\mathbb{R}$ be defined by

$$(x_1,\dots,x_{n-1})\mapsto x_1^2+\cdots+x_{n-1}^2$$

From here, elementary set theory can demonstrate that $Q=\Gamma_f$. It follows by Example 4.1.4(1) that Q is an $(n-1)$-manifold in $\mathbb{R}^n$, and $\phi:\mathbb{R}^{n-1}\to\mathbb{R}^n$ defined by $x\mapsto(x,f(x))$ is a parametrization of Q at p for all $p\in Q$.

We can calculate that

$$D\phi(x)=\begin{bmatrix}
\frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_1}{\partial x_2} & \cdots & \frac{\partial\phi_1}{\partial x_{n-1}}\\
\frac{\partial\phi_2}{\partial x_1} & \frac{\partial\phi_2}{\partial x_2} & \cdots & \frac{\partial\phi_2}{\partial x_{n-1}}\\
\vdots & \vdots & \ddots & \vdots\\
\frac{\partial\phi_{n-1}}{\partial x_1} & \frac{\partial\phi_{n-1}}{\partial x_2} & \cdots & \frac{\partial\phi_{n-1}}{\partial x_{n-1}}\\
\frac{\partial\phi_n}{\partial x_1} & \frac{\partial\phi_n}{\partial x_2} & \cdots & \frac{\partial\phi_n}{\partial x_{n-1}}
\end{bmatrix}=\begin{bmatrix}
1 & 0 & \cdots & 0\\
0 & 1 & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots\\
0 & 0 & \cdots & 1\\
2x_1 & 2x_2 & \cdots & 2x_{n-1}
\end{bmatrix}$$

Now let $p = (1, 0, \dots, 0, 1)$, and let $q = \phi^{-1}(p) = (1, 0, \dots, 0)$. Then

$$D\phi(q) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 2 & 0 & \cdots & 0 \end{bmatrix}$$

so that if $v = (v_1, \dots, v_{n-1}) \in \mathbb{R}^{n-1}$ is arbitrary, then

$$D\phi(q)(v) = \begin{bmatrix} v_1 \\ \vdots \\ v_{n-1} \\ 2v_1 \end{bmatrix}$$

This combined with the fact that $\mathrm{d}\phi_q : T_q\mathbb{R}^{n-1} \to T_p\mathbb{R}^n$ is defined by $(q, v) \mapsto (p, Df(q)(v))$ shows that

$$T_pQ = \operatorname{im}(\mathrm{d}\phi_q)$$

$$\boxed{T_pQ = \operatorname{span}\left\{\left(p, \begin{bmatrix} v_1 \\ \vdots \\ v_{n-1} \\ 2v_1 \end{bmatrix}\right)\right\}}$$

over all $(v_1, \dots, v_{n-1}) \in \mathbb{R}^{n-1}$. This should also make intuitive sense. At $(1, 0, \dots, 0)$, the quadric is changing, but only in the x_1-direction, and its slope there in that direction should be $2q_1 = 2$. The slope is not changing in any of the other directions, so those components of the tangent vector should be mapped by the identity function, as they are. $\square$

4.2.ii. Show that the tangent space to the $(n-1)$-sphere S^{n-1} at p is the space of vectors $(p, v) \in T_p\mathbb{R}^n$ satisfying $p \cdot v = 0$.

Proof. Let $p = (p_1, \dots, p_n) \in S^{n-1}$ be arbitrary. We first define the requisite diffeomorphism.

Adapting Example 4.1.4(6) from Guillemin and Haine (2018, p. 98), we know that we can easily define a diffeomorphism ϕ (see below for details) from a subset of $\mathbb{R}^{n-1}$ to the portion of S^{n-1} lying in the positive half-space above the hyperplane $x_n = 0$. But what if p lies outside this positive half-space? Well, we are helped by the fact that if $p \in S^{n-1}$, some p_i is nonzero. Thus, we can take p to lie in the region of S^{n-1} either above or below the hyperplane $x_i = 0$, and a simple isomorphism of $\mathbb{R}^n$ that, in particular, sends this region of S^{n-1} to the region of S^{n-1} above the hyperplane $x_n = 0$ is, if p lies above $x_i = 0$, the coordinate exchange function $f_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$(x_1, \dots, x_n) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

where $\sigma = \tau_{i,n}$ and, if p lies below $x_i = 0$, the coordinate exchange function $-f_\sigma$. Thus, for p arbitrary, our complete diffeomorphism is $\pm f_\sigma \circ \phi$.

We now define ϕ. Let U be the open unit ball centered at the origin in $\mathbb{R}^{n-1}$. Let V be the half-space of $\mathbb{R}^n$ above the hyperplane $x_n = 0$ (i.e., all points $(x_1, \dots, x_n) \in \mathbb{R}^n$ such that $x_n > 0$). Then, as described above, $S^{n-1} \cap V$ is the portion of S^{n-1} lying above the hyperplane $x_n = 0$. The diffeomorphism $\phi : U \to S^{n-1} \cap V$ which projects each point in U "up" onto the surface of the hypersphere is given by

$$(x_1, \dots, x_{n-1}) \mapsto \left(x_1, \dots, x_{n-1}, \sqrt{1 - \left(x_1^2 + \cdots + x_{n-1}^2\right)}\right)$$

We now divide into two cases (the needed diffeomorphism is $f_\sigma \circ \phi$, and the needed diffeomorphism is $-f_\sigma \circ \phi$). Note that the proof of the second case is entirely symmetric to that of the first case, and thus will not be discussed further.

Let $r = f_\sigma^{-1}(p)$ and let $q = (f_\sigma \circ \phi)^{-1}(p)$. We now define $\mathrm{d}(f_\sigma \circ \phi)_q$. First off, by the chain rule,

$$\mathrm{d}(f_\sigma \circ \phi)_q = \mathrm{d}(f_\sigma)_r \circ \mathrm{d}\phi_q$$

Additionally, we know that in general,

$$Df_\sigma(x) = \begin{bmatrix} \frac{\partial (f_\sigma)_1}{\partial x_1} & \cdots & \frac{\partial (f_\sigma)_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial (f_\sigma)_n}{\partial x_1} & \cdots & \frac{\partial (f_\sigma)_n}{\partial x_n} \end{bmatrix} \qquad D\phi(x) = \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \cdots & \frac{\partial \phi_1}{\partial x_{n-1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_{n-1}}{\partial x_1} & \cdots & \frac{\partial \phi_{n-1}}{\partial x_{n-1}} \\ \frac{\partial \phi_n}{\partial x_1} & \cdots & \frac{\partial \phi_n}{\partial x_{n-1}} \end{bmatrix}$$

$$= P_\sigma \qquad\qquad = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ \frac{-x_1}{\sqrt{1-\left(x_1^2+\cdots+x_{n-1}^2\right)}} & \cdots & \frac{-x_{n-1}}{\sqrt{1-\left(x_1^2+\cdots+x_{n-1}^2\right)}} \end{bmatrix}$$

where P_σ is the permutation matrix which differs from the identity in that its i^{th} and n^{th} columns are interchanged. It follows that

$$T_pS^{n-1} = \operatorname{span}\left\{\left(p, P_\sigma \begin{bmatrix} w_1 \\ \vdots \\ w_{n-1} \\ \frac{-q_1w_1-\cdots-q_{n-1}w_{n-1}}{\sqrt{1-\left(q_1^2+\cdots+q_{n-1}^2\right)}} \end{bmatrix}\right)\right\}$$

for $(w_1, \dots, w_{n-1}) \in U$.

We now use a bidirectional inclusion argument to complete the proof.

Let $(p, v) \in T_pS^{n-1}$ be arbitrary. Then some $p_i \neq 0$. It follows that

$$\begin{aligned}
(f_\sigma \circ \phi)(q_1, \dots, q_{n-1}) &= f_\sigma(\phi(q_1, \dots, q_{n-1})) \\
&= f_\sigma\left(q_1, \dots, q_{n-1}, \sqrt{1 - \left(q_1^2 + \cdots + q_{n-1}^2\right)}\right) \\
&= \left(q_1, \dots, q_{i-1}, \sqrt{1 - \left(q_1^2 + \cdots + q_{n-1}^2\right)}, q_{i+1}, \dots, q_{n-1}, q_i\right) \\
&= (p_1, \dots, p_{i-1}, p_i, p_{i+1}, \dots, p_{n-1}, p_n) \\
&= p
\end{aligned}$$

Thus, we have that

$$\begin{aligned}
p \cdot v &= p_1v_1 + \cdots + p_nv_n \\
&= q_1w_1 + \cdots + q_{i-1}w_{i-1} + \sqrt{1 - \left(q_1^2 + \cdots + q_{n-1}^2\right)} \cdot \frac{-q_1w_1 - \cdots - q_{n-1}w_{n-1}}{\sqrt{1 - \left(q_1^2 + \cdots + q_{n-1}^2\right)}} \\
&\quad + q_{i+1}w_{i+1} + \cdots + q_{n-1}w_{n-1} + q_iw_i \\
&= 0
\end{aligned}$$

as desired.

Now suppose that $(p, v) \in T_p\mathbb{R}^n$ is such that $p \cdot v = 0$. Then

$$\begin{aligned}
0 &= p \cdot v \\
&= p_1v_1 + \cdots + p_nv_n \\
&= q_1v_1 + \cdots + q_{i-1}v_{i-1} + \sqrt{1 - \left(q_1^2 + \cdots + q_{n-1}^2\right)} \cdot v_i + q_{i+1}v_{i+1} + \cdots + q_nv_n \\
\sqrt{1 - \left(q_1^2 + \cdots + q_{n-1}^2\right)} \cdot v_i &= -q_1v_1 - \cdots - q_{i-1}v_{i-1} - q_{i+1}v_{i+1} + \cdots + q_nv_n \\
v_i &= \frac{-q_1v_1 - \cdots - q_{i-1}v_{i-1} - q_{i+1}v_{i+1} + \cdots + q_nv_n}{\sqrt{1 - \left(q_1^2 + \cdots + q_{n-1}^2\right)}}
\end{aligned}$$

so with some reindexing, v fits the form of the vector in the span defining T_pS^{n-1}, as desired. □

4.2.iii. Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be a C^∞ map and let $X = \Gamma_f$. What is the tangent space to X at $(a, f(a))$?

Proof. As per Example 4.1.4(1), $\phi : \mathbb{R}^n \to \mathbb{R}^{n+k}$ defined by

$$x \mapsto (x, f(x))$$

is a suitable diffeomorphism for all $p \in X$. It follows that $D\phi(x)$ is an $(n + k) \times n$ matrix where the top $n \times n$ matrix is id_n and the bottom $k \times n$ matrix is $Df(x)$. Let $p = (a, f(a))$. Then

$$\boxed{T_pX = \operatorname{span}\left\{\left(p, \begin{bmatrix} v_1 \\ \vdots \\ v_n \\ \sum_{i=1}^n \left.\frac{\partial f_1}{\partial x_i}\right|_a v_i \\ \vdots \\ \sum_{i=1}^n \left.\frac{\partial f_k}{\partial x_i}\right|_a v_i \end{bmatrix}\right)\right\}}$$

for $(v_1, \dots, v_n) \in \mathbb{R}^n$. □

4.2.iv. Let $\sigma : S^{n-1} \to S^{n-1}$ be the antipodal map $\sigma(x) = -x$. What is the derivative of σ at $p \in S^{n-1}$?

Proof. Let $\tilde{\sigma} : \mathbb{R}^n \to \mathbb{R}^n$ be the extension of the antipodal map to $\mathbb{R}^n$. Then $D\tilde{\sigma}(x) = -\mathrm{id}_n$. It follows that the derivative of σ at any $p \in S^{n-1}$ is the map $\mathrm{d}\sigma_p : T_pS^{n-1} \to T_{-p}S^{n-1}$ defined by

$$\boxed{\mathrm{d}\sigma_p(p, v) = (-p, -v)}$$

□

4.2.v. Let $X_i \subset \mathbb{R}^{N_i}$ $(i = 1, 2)$ be an n_i-manifold and let $p_i \in X_i$. Define X to be the Cartesian product

$$X_1 \times X_2 \subset \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$$

and let $p = (p_1, p_2)$. Show that $T_pX \cong T_{p_1}X_1 \oplus T_{p_2}X_2$.

Proof. Let $f : T_pX \to T_{p_1}X_1 \oplus T_{p_2}X_2$ be defined by

$$(p, v) \mapsto ((p_1, v_1), (p_2, v_2))$$

We can check componentwise that f is bijective, as desired. □