Week 5

Differentiation

5.1 k-Forms

4/25: • Definitions and examples of k-forms.

5.2 Vector Calculus Operations

4/27: • Announcements.
    - No class this Friday, next Monday.
    - Midterm next Friday.
        - Up through Chapter 2.
        - The exam will likely be computationally heavy.
        - Compute d, pullbacks, interior products, Lie derivatives, etc.
        - Emphasis on Chapter 2 as opposed to Chapter 1 even though it all builds on itself.
        - He'll probably cook up a few problems too.
    - There is a recorded lecture for us.
        - On Chapter 3 content.
        - We'll cover Chapter 3 in kind of an impressionistic way as it is.
    - There are also some notes on the physics stuff.

- Vector calculus operations.
    - In one dimension, you have functions, and you take derivatives.
        - The derivative operation does essentially map $\Omega^0 \to \Omega^1$ or $C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$.
    - In two dimensions, ...
        - $\mathrm{d}^2 = 0$ reflects the fact that gradient vector fields are curl-free.
    - If you want to understand the 2D-curl business...
        - $\mathrm{curl}(v) : \mathbb{R}^2 \to \mathbb{R}$ is intuitively about balls spinning around in a vector field.
        - There's also a nice formula to compute it.
        - And then there's a connection with $\mathrm{d} : \Omega^1 \to \Omega^2$.
    - In 3D, you can take top-dimensional forms (which are just functions) and bottom-dimensional forms (which are by definition functions) and you can work out an identification between them.
    - Note that $\mathrm{curl} : \mathfrak{X}(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$, where $\mathfrak{X}(\mathbb{R}^2)$ is the space of vector fields.

- The musical operator $\sharp$ identifies forms with vector fields, i.e., $\sharp : \Omega^1 \to \mathfrak{X}(\mathbb{R}^2)$.

- Properties of exterior derivatives $\mathrm{d} : \Omega^k(U) \to \Omega^{k+1}(U)$.

  1. $\mathrm{d}(\omega_1 + \omega_2) = \mathrm{d}\omega_1 + \mathrm{d}\omega_2$ and $\mathrm{d}(\lambda\omega) = \lambda\,\mathrm{d}\omega$.
  2. Product rule $\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge \mathrm{d}\omega_2$.
     - Special case $k = \ell = 0$. Then
       $$\mathrm{d}(fg) = g\,\mathrm{d}f + f\,\mathrm{d}g$$
       which is the usual product rule for gradient.
     - Claim:
       $$\mathrm{d}\left(\sum_I f_I\,\mathrm{d}x_I\right) = \sum_I \mathrm{d}f_I \wedge \mathrm{d}x_I$$
       - Let $\omega_1 \in \Omega^k$ and $\omega_2 \in \Omega^\ell$ be defined by
         $$\omega_1 = \sum_I f_I\,\mathrm{d}x_I \qquad \omega_2 = \sum_J g_J\,\mathrm{d}x_J$$
         where we're summing over all I such that $|I| = k$ and all J such that $|J| = \ell$. Then
         $$\begin{aligned}\omega_1 \wedge \omega_2 &= \sum_{I,J} f_I g_J\,\mathrm{d}x_I \wedge \mathrm{d}x_J\\ \mathrm{d}(\omega_1 \wedge \omega_2) &= \sum_{I,J} \mathrm{d}(f_I g_J) \wedge \mathrm{d}x_I \wedge \mathrm{d}x_J\end{aligned}$$
       - Note that
         $$\mathrm{d}(f_I g_J) = g_J\,\mathrm{d}f_I + f_I\,\mathrm{d}g_J$$
         and
         $$\mathrm{d}g_J \wedge \mathrm{d}x_I = (-1)^k\,\mathrm{d}x_I \wedge \mathrm{d}g_J$$
       - These identities allow us to take the previous equation to
         $$\begin{aligned}\mathrm{d}(\omega_1 \wedge \omega_2) &= \sum_{I,J} g_J\,\mathrm{d}f_I \wedge \mathrm{d}x_I \wedge \mathrm{d}x_J + (-1)^k f_I\,\mathrm{d}x_I \wedge \mathrm{d}g_J \wedge \mathrm{d}x_J\\ &= \sum_{I,J} (\mathrm{d}f_I \wedge \mathrm{d}x_I) \wedge (g_J\,\mathrm{d}x_J) + \sum_{I,J} (f_I\,\mathrm{d}x_I) \wedge (\mathrm{d}g_J \wedge \mathrm{d}x_J)\\ &= \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^k \omega_1\,\mathrm{d}\omega_2\end{aligned}$$
  3. $\mathrm{d}^2 = 0$.
     - Let $\omega = \sum_I f_I\,\mathrm{d}x_I$.
     - Then
       $$\begin{aligned}\mathrm{d}^2(\omega) &= \mathrm{d}(\mathrm{d}\omega)\\ &= \mathrm{d}\left(\sum_I \mathrm{d}f_I \wedge \mathrm{d}x_I\right)\\ &= \sum_I \mathrm{d}(\mathrm{d}f_I \wedge \mathrm{d}x_I) && \text{Property 1}\\ &= \sum_I \mathrm{d}(\mathrm{d}f_I) \wedge \mathrm{d}x_I && \text{Property 2}\end{aligned}$$
       so it suffices to just show that $\mathrm{d}^2 f = 0$ for all $f \in \Omega^0$.

  - We know that $\mathrm{d}f = \sum_{i=1}^n \partial f/\partial x_i \, \mathrm{d}x_i$. Thus,

$$\begin{aligned}
\mathrm{d}(\mathrm{d}f) &= \sum_i \mathrm{d}\left(\frac{\partial f}{\partial x_i}\right) \wedge \mathrm{d}x_i \\
&= \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i} \, \mathrm{d}x_j \wedge \mathrm{d}x_i \\
&= 0
\end{aligned}$$

  - The last equality holds because of commuting partial derivatives for smooth f, and the fact that changing order introduces a negative sign by some property.

- In fact, if we fix $\mathrm{d}^0 : \Omega^0(U) \to \Omega^1(U)$ to be the "gradient," then these properties characterize the function d on its domain and codomain. In particular, d is the unique function on its domain and codomain that satisfies these properties.
  - We define it by
$$\mathrm{d}\left(\sum_I f_I \, \mathrm{d}x_I\right) = \sum_I \mathrm{d}f_I \wedge \mathrm{d}x_I$$
  - The above properties characterize it axiomatically.
  - We can prove this uniqueness theorem.
- **Closed** (form): A form $\omega \in \Omega^k(U)$ such that $\mathrm{d}\omega = 0$.
- **Exact** (form): A form $\omega \in \Omega^k(U)$ such that $\omega = \mathrm{d}\eta$ for some $\eta \in \Omega^{k-1}(U)$.
- $\mathrm{d}^2 = 0$ implies closed and exact implies closed.
- **Poincaré lemma**: Locally closed forms are exact.

5.3 Chapter 2: Differential Forms

From Guillemin and Haine (2018).

5/5: - As we formed the k^{th} exterior powers $\Lambda^k(V^*)$, we can form the k^{th} exterior powers $\Lambda^k(T_p^*\mathbb{R}^n)$.
  - In particular, we can take the vector space $\mathcal{L}^k(T_p\mathbb{R}^n)$ (of all k-tensors on the tangent space of p) and the span $\mathcal{I}^k(T_p\mathbb{R}^n)$ (of all redundant k-tensors on the tangent space of p) and form their quotient space $\Lambda^k(T_p^*\mathbb{R}^n)$.
  - This quotient space will be isomorphic to the set $\mathcal{A}^k(T_p\mathbb{R}^n)$ of all alternating tensors on the tangent space of p. As such, elements of $\Lambda^k(T_p^*\mathbb{R}^n)$ can be thought of as k-linear alternating tensors.
- Since $\Lambda^1(T_p^*\mathbb{R}^n) = T_p^*\mathbb{R}^n$, we can think of a one-form as a function which takes as its value at p an element of the space $\Lambda^1(T_p^*\mathbb{R}^n)$.
- ***k*-form** (on U): A function which assigns to each point $p \in U$ an element $\omega_p \in \Lambda^k(T_p^*\mathbb{R}^n)$, where $U \subset \mathbb{R}^n$ is open.
- We can use the wedge product to construct k-forms.
  - Let $\omega_1, \dots, \omega_k$ be one-forms. Then $\omega_1 \wedge \cdots \wedge \omega_k$ is the k-form whose value at p is the wedge product
$$(\omega_1 \wedge \cdots \wedge \omega_k)_p = (\omega_1)_p \wedge \cdots \wedge (\omega_k)_p$$

  - Let $f_1,\dots,f_k$ be real-valued functions in $C^\infty(U)$. Suppose $\omega_i = \mathrm{d}f_i$. Then we may obtain the k-form whose value at p is

$$(\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_k)_p = (\mathrm{d}f_1)_p\wedge\cdots\wedge(\mathrm{d}f_k)_p$$

- Since $(\mathrm{d}x_1)_p,\dots,(\mathrm{d}x_n)_p$ are a basis of $T_p^*\mathbb{R}^n$, the wedge products

$$(\mathrm{d}x_I)_p = (\mathrm{d}x_{i_1})_p\wedge\cdots\wedge(\mathrm{d}x_{i_k})_p$$

  where $I=(i_1,\dots,i_k)$ is a strictly increasing multi-index of n of length k form a basis of $\Lambda^k(T_p^*\mathbb{R}^n)$.
- Thus, every $\omega_p\in\Lambda^k(T_p^*\mathbb{R}^n)$ has a unique decomposition

$$\omega_p = \sum_I c_I(\mathrm{d}x_I)_p$$

  where every $c_I\in\mathbb{R}$.
- Similarly, every k-form ω on U has a unique decomposition

$$\omega = \sum_I f_I\,\mathrm{d}x_I$$

  where every $f_I : U\to\mathbb{R}$.
- **Class C^r** (k-form): A k-form ω for which every f_I in its decomposition is in $C^r(U)$.
- From here on out, we assume unless otherwise stated that all k-forms we consider are of class C^∞.
- **$\Omega^k(U)$**: The set of k-forms of class C^∞ on U.
- **$f\omega$**: The k-form defined as follows, where $f\in C^\infty(U)$ and $\omega\in\Omega^k(U)$. *Given by*

$$p\mapsto f(p)\omega_p$$

- **Sum** (of ω_1,ω_2): The k-form defined as follows, where $\omega_1,\omega_2\in\Omega^k(U)$. *Denoted by* $\boldsymbol{\omega_1+\omega_2}$. *Given by*

$$p\mapsto(\omega_1)_p+(\omega_2)_p$$

- **Wedge product** (of ω_1,ω_2): The (k_1+k_2)-form defined as follows, where $\omega_1\in\Omega^{k_1}(U)$ and $\omega_2\in\Omega^{k_2}(U)$. *Denoted by* $\boldsymbol{\omega_1\wedge\omega_2}$. *Given by*

$$p\mapsto(\omega_1)_p\wedge(\omega_2)_p$$

- **Zero-form**: A function which assigns to each $p\in U$ an element of $\Lambda^0(T_p^*\mathbb{R}^n)=\mathbb{R}$. *Also known as* **real-valued function**.
- It follows from the definition of zero-forms that

$$\Omega^0(U) = C^\infty(U)$$

- **Exterior differentiation operation**: The operator from $\Omega^0(U)\to\Omega^1(U)$ which associates to a function $f\in C^\infty(U)$ the 1-form $\mathrm{d}f$. *Denoted by* $\mathbf{d}$.
- We now seek to define a generalized version of the exterior differentiation operation; in particular, we would like to define an analogous function $\mathrm{d}:\Omega^k(U)\to\Omega^{k+1}(U)$.
- Desired properties of exterior differentiation.

1. If $\omega_1, \omega_2 \in \Omega^k(U)$, then
$$\mathrm{d}(\omega_1 + \omega_2) = \mathrm{d}\omega_1 + \mathrm{d}\omega_2$$
2. If $\omega_1 \in \Omega^k(U)$ and $\omega_2 \in \Omega^\ell(U)$, then
$$\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge \mathrm{d}\omega_2$$
3. If $\omega \in \Omega^k(U)$, then
$$\mathrm{d}(\mathrm{d}\omega) = 0$$

- Consequences of these properties.
- Lemma 2.4.5: Let $U \subset \mathbb{R}^n$ open. If $f_1, \dots, f_k \in C^\infty(U)$, then
$$\mathrm{d}(\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k) = 0$$

  Proof. We induct on k. For the base case $k = 1$, we have that $\mathrm{d}(\mathrm{d}f_1) = 0$ by Property 3. Now suppose inductively that we have proven the claim for $k-1$ functions; we now seek to prove it for k functions. Let $\mu = \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_{k-1}$. Then by the induction hypothesis, $\mathrm{d}\mu = 0$. Therefore,
$$\begin{aligned}
\mathrm{d}(\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k) &= \mathrm{d}(\mu \wedge \mathrm{d}f_k) \\
&= \mathrm{d}\mu \wedge \mathrm{d}f_k + (-1)^{k-1}\mu \wedge \mathrm{d}(\mathrm{d}f_k) && \text{Property 2} \\
&= 0
\end{aligned}$$
  as desired. □

- A special case of Lemma 2.4.5 is that
$$\mathrm{d}(\mathrm{d}x_I) = 0$$
- Now since every k-form $\omega \in \Omega^k(U)$ has a unique decomposition in terms of the $\mathrm{d}x_I$, Property 2 and the above equation reveal that
$$\mathrm{d}\omega = \sum_I \mathrm{d}f_I \wedge \mathrm{d}x_I$$
- Therefore, if there exists an operator d satisfying Properties 1-3, then d necessarily has the above form. All that's left is to show that the operator defined above has these properties.
- Proposition 2.4.10: Let $U \subset \mathbb{R}^n$ be open. There is a unique operator $\mathrm{d} : \Omega^*(U) \to \Omega^{*+1}(U)$ satisfying Properties 1-3.

  Proof. ... □

- **Closed** (k-form): A k-form $\omega \in \Omega^k(U)$ for which $\mathrm{d}\omega = 0$.
- **Exact** (k-form): A k-form $\omega \in \Omega^k(U)$ such that $\omega = \mathrm{d}\mu$ for some $\mu \in \Omega^{k-1}(U)$.
- Property 3 implies that every exact k-form is closed.
  - The converse is not true even for 1-forms (see Exercise 2.1.iii).
  - "It is a very interesting (and hard) question to determine if an open set U has the following property: For $k > 0$, every closed k-form is exact" (Guillemin & Haine, 2018, p. 49).
    - Note that we do not consider zero-forms since there are no (-1)-forms for which to define exactness.
- If $f \in C^\infty(U)$ and $\mathrm{d}f = 0$, then f is constant on connected components of U (see Exercise 2.2.iii).
- Lemma 2.4.16 (Poincaré lemma): If ω is a closed form on U of degree $k > 0$, then for every point $p \in U$, there exists a neighborhood of p on which ω is exact.

  Proof. See Exercises 2.4.v and 2.4.vi. □