MATH 20510 (Analysis in $\mathbb{R}^n$ III – Accelerated) Notes

Steven Labalme

May 14, 2022

Weeks

Week 1

Tensors

1.1 Course Motivation

3/28: • Motivation for the course and an overview of Guillemin and Haine (2018).

1.2 Defining Tensors and Their Operations

3/30: • Plan:
  - More (multi)linear algebra.
- Dual spaces.
- Let V be an n-dimensional real vector space.
- **Hom** $(\boldsymbol{V}, \mathbb{R})$: The set of all homomorphisms (i.e., linear maps) from V to $\mathbb{R}$. *Also known as* $\boldsymbol{V^*}$.
- **Dual basis** (for V^*): The set of linear transformations from V to $\mathbb{R}$ defined by

$$e_j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

  where $e_1, \dots, e_n$ is a basis of V. *Denoted by* $e_1^*, \dots, e_n^*$.
- Check: $e_1^*, \dots, e_n^*$ are a basis for V^*.
  - Are they linearly independent? Let $c_1e_1^* + \cdots + c_ne_n^* = 0 \in \operatorname{Hom}(V, \mathbb{R})$. Then

$$c_i = (c_1e_1^* + \cdots + c_ne_n^*)(e_i) = 0 \in \mathbb{R}$$

    as desired.
  - Span? Let $\varphi \in \operatorname{Hom}(V, \mathbb{R})$. Then we can verify that

$$\varphi(e_1)e_1^* + \cdots + \varphi(e_n)e_n^* = \varphi$$

    - We prove this by verifying the previous statement on the basis of V (if two linear transformations have the same action on the basis of a vector space, they are equal).
- With a choice of basis for V, we obtain an isomorphism $\varepsilon : V \to V^*$ with the mapping $e_i \mapsto e_i^*$ for all i.
- The dual space is known as such because $(V^*)^* \cong V$, where $\cong$ is **canonical** (no choice of basis is needed).

- One more property of dual spaces: **functoriality**.
  - Given a linear transformation $A : V \to W$, we know that $A^* : W^* \to V^*$ where A^* is the transpose of A. In particular, if $\varphi \in W^*$, then $\varphi \circ A : V \to \mathbb{R}$.
  - Claim: A^* is linear.
- **Functoriality**: If $A : V \to W$ and $B : W \to U$, then $B^* : U^* \to W^*$ and $A^* : W^* \to V^*$. The functoriality statement is that $(B \circ A)^* = A^* \circ B^*$.
- A^* is the **pullback** (or transpose) of A.
- Let $v_1, \dots, v_n$ be a basis for V and $w_1, \dots, w_m$ be a basis for W. Then $[A]_{v_1,\dots,v_n}^{w_1,\dots,w_m} = A$ is the matrix of the linear transformation A with respect to these bases. Then if $v_1^*, \dots, v_n^*$ and $w_1^*, \dots, w_m^*$ are the corresponding dual bases, then $[A^*]_{v_1^*,\dots,v_n^*}^{w_1^*,\dots,w_m^*} = A^T$. We can and should verify this for ourselves.
- This is over the real numbers, so A^* is just the transpose because there are no complex numbers of which to take the conjugate!
- A generalization: Tensors.
- ***k*-tensor**: A **multilinear** map
$$T : \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}$$
- **Multilinear** (map T): A function T such that
$$\begin{aligned} T(v_1, \dots, v_i^1 + v_i^2, \dots, v_k) &= T(v_1, \dots, v_i^1, \dots, v_k) + T(v_1, \dots, v_i^2, \dots, v_k) \\ T(v_1, \dots, \lambda v_i, \dots, v_k) &= \lambda T(v_1, \dots, v_i, \dots, v_k) \end{aligned}$$
  for all $(v_1, \dots, v_k) \in V^k$.
- The determinant is an n-tensor!
- 1-tensors are just covectors.
- $\boldsymbol{\mathcal{L}^k(V)}$: The vector space of all k-tensors on V.
- Calculating $\dim \mathcal{L}^k(V)$. (Answer not given in this class.)
- Let $A : V \to W$. Then $A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$.
  - Check $(A \circ B)^* = B^* \circ A^*$.
- **Multi-index of *n* of length *k***: A k-tuple $(i_1, \dots, i_k)$ where each $i_j \in \mathbb{N}$ satisfies $1 \leq i_j \leq n$ $(j = 1, \dots, k)$. *Denoted by* $\boldsymbol{I}$.
- Let $e_1, \dots, e_n$ be a basis for V.
- **Tensor product** (of $T_1 \in \mathcal{L}^k(V)$, $T_2 \in L^l(V)$): The function from V^{k+l} to $\mathbb{R}$ defined by
$$(v_1, \dots, v_{k+l}) \mapsto T_1(v_1, \dots, v_k) T_2(v_{k+1}, \dots, v_{k+l})$$
  Denoted by $\boldsymbol{T_1 \otimes T_2}$.
- Claims:
  1. $T_1 \otimes T_2 \in L^{k+l}(V)$.
  2. $A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$.
- $\boldsymbol{e_I^*}$: The function $e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*$, where $I = (i_1, \dots, i_k)$ is a multi-index of n of length k.

- Claim: Letting I range over all n^k multi-indices of n of length k, the e_I^* are a basis for $\mathcal{L}^k(V)$.
- If $V=\mathbb{R}$, then $V=\mathbb{R}e_1$. If $V=\mathbb{R}^2$, then $V=\mathbb{R}e_1\oplus\mathbb{R}e_2$.
- We know that $L^1(V)=V^*=Re_1^*$. Thus, $e_1^*\otimes e_2^*:V\times V\to\mathbb{R}$. Thus, for example,

$$(e_1^*\otimes e_2^*)((1,2),(3,4))=e_1^*(1,2)\cdot e_2^*(3,4)=1\cdot 4=4$$

1.3 The Tensor Product and Permutations

4/1:

- Plan: More multilinear algebra.
  - Properties of the tensor product.
  - Sign of a permutation.
  - Alternating tensors (lead into differential forms down the road).
- Recall: V is an n-dimensional vector space over $\mathbb{R}$ with basis $e_1,\dots,e_n$. $\mathcal{L}^k(V)$ is the vector space of k-tensors on V. $\{e_I^*\mid I \text{ a multiindex of } n \text{ of length } k\}$ is a basis for $\mathcal{L}^k(V)$.
- For example, if $V=\mathbb{R}^2$ and $T\in\mathcal{L}^2(V)$, then

$$T(a_1e_1+a_2e_2,b_1e_1+b_2e_2)=a_1b_1T(e_1,e_1)+a_1b_2T(e_1,e_2)+a_2b_1T(e_2,e_1)+a_2b_2T(e_2,e_2)$$

  - A basis of $\mathcal{L}^2(V)$ is

$$\{e_1^*\otimes e_1^*,e_1^*\otimes e_2^*,e_2^*\otimes e_1^*,e_2^*\otimes e_2^*\}$$

  - Recall that some basic properties are

$$e_1^*\otimes e_2^*((1,2),(3,4))=1\cdot 4=4 \qquad e_2^*\otimes e_1^*((1,2),(3,4))=2\cdot 3=6$$

  - It follows by the initial decomposition of T that

$$T=a_1b_1e_1^*\otimes e_1^*+a_1b_2e_1^*\otimes e_2^*+a_2b_1e_2^*\otimes e_1^*+a_2b_2e_2^*\otimes e_2^*$$

- Important consequence: To know the action of T on an arbitrary pair of vectors, you need only know its action on the basis; a higher-dimensional generalization of the earlier property.
- Note that

$$e_I^*(e_J)=\delta_{IJ}=\begin{cases}1 & I=J\\ 0 & I\neq J\end{cases}$$

- Basic properties of the tensor product.
  1. *Right-distributive*: If $T_1\in\mathcal{L}^k(V)$ and $T_2,T_3\in\mathcal{L}^\ell(V)$, then

$$T_1\otimes(T_2+T_3)=T_1\otimes T_2+T_1\otimes T_3$$

  2. *Left-distributive*: If $T_1,T_2\in\mathcal{L}^k(V)$ and $T_3\in\mathcal{L}^\ell(V)$, then

$$(T_1+T_2)\otimes T_3=T_1\otimes T_3+T_2\otimes T_3$$

  3. *Associative*: If $T_1\in\mathcal{L}^k(V)$, $T_2\in\mathcal{L}^\ell(V)$, and $T_3\in\mathcal{L}^m(V)$, then

$$T_1\otimes(T_2\otimes T_3)=(T_1\otimes T_2)\otimes T_2=T_1\otimes T_2\otimes T_3$$

  4. *Scalar multiplication*: If $T_1\in\mathcal{L}^k(V)$, $T_2\in\mathcal{L}^\ell(V)$, and $\lambda\in\mathbb{R}$, then

$$(\lambda T_1)\otimes T_2=\lambda(T_1\otimes T_2)=T_1\otimes(\lambda T_2)$$

- Note that the tensor product is not commutative.
- Aside: Defining the sign of a permutation.
- $\boldsymbol{S_A}$: The set of all automorphisms of A (bijections from A to A), where A is a set.
- $\boldsymbol{S_n}$: The set $S_{[n]}$.
- Given $\sigma_1, \sigma_2 \in S_n$, $\sigma_1 \circ \sigma_2 \in S_n$.
  - Thus, S_n is a **group**.
- **Transposition**: A function $\tau \in S_n$ such that

$$\tau(k) = \begin{cases} j & k = i \\ i & k = j \\ k & k \neq i, j \end{cases}$$

  for some $i, j \in [n]$. *Denoted by* $\boldsymbol{\tau_{i,j}}$.
- Theorem: An element of S_n can be written as the product of transpositions (i.e., for all $\sigma \in S_n$, there exist $\tau_1, \dots, \tau_m \in S_n$ such that $\sigma = \tau_1 \circ \cdots \circ \tau_m$).
- **Sign** (of $\sigma \in S_n$): The number (mod 2) of transpositions whose product equals σ. *Denoted by* $\boldsymbol{(-1)^\sigma}$, $\textbf{sign}\,\boldsymbol{(\sigma)}$.
- Theorem: The sign of σ is well-defined. Additionally,

$$(-1)^{\sigma_1\sigma_2} = (-1)^{\sigma_1} \cdot (-1)^{\sigma_2}$$

- Example: Consider the identity permutation. $(-1)^\sigma = +1$. We can think of this as the product of zero transpositions or, for instance, as the product of the two transpositions $\tau_{1,2} \circ \tau_{1,2}$. Another example would be $\tau_{2,3} \circ \tau_{1,2} \circ \tau_{1,2} \circ \tau_{2,3}$.
- Theorem: Let X_i be a rational or polynomial function for each $i \in \mathbb{N}$. Then

$$(-1)^\sigma = \prod_{i<j} \frac{X_{\sigma(i)} - X_{\sigma(j)}}{X_i - X_j}$$

- Example: For the permutation $\sigma = (1, 2, 3)$, we have

$$\begin{aligned}
(-1)^\sigma &= \frac{X_{\sigma(1)} - X_{\sigma(2)}}{X_1 - X_2} \cdot \frac{X_{\sigma(1)} - X_{\sigma(3)}}{X_1 - X_3} \cdot \frac{X_{\sigma(2)} - X_{\sigma(3)}}{X_2 - X_3} \\
&= \frac{X_2 - X_3}{X_1 - X_2} \cdot \frac{X_2 - X_1}{X_1 - X_3} \cdot \frac{X_3 - X_1}{X_2 - X_3} \\
&= \frac{-(X_1 - X_2)}{X_1 - X_2} \cdot \frac{-(X_1 - X_3)}{X_1 - X_3} \cdot \frac{X_2 - X_3}{X_2 - X_3} \\
&= -1 \cdot -1 \cdot 1 \\
&= +1
\end{aligned}$$

  which squares with the fact that $\sigma = \tau_{1,2} \circ \tau_{2,3}$.
- Claims to verify with the above formula:
  1. $\operatorname{sign}(\sigma) \in \{\pm 1\}$.
  2. $\operatorname{sign}(\tau_{i,j}) = -1$.
  3. $\operatorname{sign}(\sigma_1\sigma_2) = \operatorname{sign}(\sigma_1)\operatorname{sign}(\sigma_2)$.

1.4 Chapter 1: Multilinear Algebra

From Guillemin and Haine (2018).

3/31:

- Guillemin and Haine (2018) defines real vector spaces, the operations on them, their basic properties, and the zero vector.
- **Linearly independent** (vectors $v_1, \dots, v_k$): A finite set of vectors $v_1, \dots, v_k \in V$ such that the map from $\mathbb{R}^k$ to V defined by $(c_1, \dots, c_k) \mapsto c_1 v_1 + \cdots + c_k v_k$ is injective.
- **Spanning** (vectors $v_1, \dots, v_k$): We require that the above map is surjective.
- Guillemin and Haine (2018) defines basis, finite-dimensional vector space, dimension, subspace, linear map, and kernel.
- **Image** (of $A : V \to W$): The range space of A, a subspace of W. *Also known as* $\mathbf{im}\,(\boldsymbol{A})$.
- Guillemin and Haine (2018) defines the matrix of a linear map.
- **Inner product** (on V): A map $B : V \times V \to \mathbb{R}$ with the following three properties.
  - *Bilinearity*: For vectors $v, v_1, v_2, w \in V$ and $\lambda \in \mathbb{R}$, we have

    $$B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w)$$

    and

    $$B(\lambda v, w) = \lambda B(v, w)$$
  - *Symmetry*: For vectors $v, w \in V$, we have $B(v, w) = B(w, v)$.
  - *Positivity*: For every vector $v \in V$, we have $B(v, v) \geq 0$. Moreover, if $v \neq 0$, then $B(v, v) > 0$.
- $\boldsymbol{W}$**-coset**: A set of the form $\{v + w \mid w \in W\}$, where W is a subspace V and $v \in V$. *Denoted by* $\boldsymbol{v + W}$.
  - If $v_1 - v_2 \in W$, then $v_1 + W = v_2 + W$.
  - It follows that the distinct W-cosets decompose V into a disjoint collection of subsets of V.
- **Quotient space** (of V by W): The set of distinct W-cosets in V, along with the following definitions of vector addition and scalar multiplication.

  $$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W \qquad \lambda(v + W) = (\lambda v) + W$$

  Denoted by $\boldsymbol{V/W}$.
- **Quotient map**: The linear map $\pi : V \to V/W$ defined by

  $$\pi(v) = v + W$$
  - π is surjective.
  - Note that $\ker(\pi) = W$ since for all $w \in W$, $\pi(w) = w + W = 0 + W$, which is the zero vector in V/W.
- If V, W are finite dimensional, then

  $$\dim(V/W) = \dim(V) - \dim(W)$$
- Proposition 1.2.9: Let $A : V \to U$ be a linear map. If $W \subset \ker(A)$, then there exists a unique linear map $A^\sharp : V/W \to U$ with the property that $A = A^\sharp \circ \pi$, where $\pi : V \to V/W$ is the quotient map.
  - This proposition rephrases in terms of quotient spaces the fact that if $w \in W$, then $A(v+w) = Av$.

- **Dual space** (of V): The set of all linear functions $\ell : V \to \mathbb{R}$, along with the following definitions of vector addition and scalar multiplication.

$$(\ell_1 + \ell_2)(v) = \ell_1(v) + \ell_2(v) \qquad\qquad (\lambda\ell)(v) = \lambda \cdot \ell(v)$$

Denoted by V^*.

- **Dual basis** (of $e_1, \dots, e_n$ a basis of V): The basis of V^* consisting of the n functions that take every $v = c_1e_1 + \cdots + c_ne_n$ to one of the c_i. *Denoted by* $\boldsymbol{e_1^*, \dots, e_n^*}$. *Given by*

$$e_i^*(v) = c_i$$

for all $v \in V$.

- Claim 1.2.12: If V is an n-dimensional vector space with basis $e_1, \dots, e_n$, then $e_1^*, \dots, e_n^*$ is a basis of V^*.

Proof. We will first prove that $e_1^*, \dots, e_n^*$ spans V^*. Let $\ell \in V^*$ be arbitrary. Set $\lambda_i = \ell(e_i)$ for all $i \in [n]$. Define $\ell' = \sum_{i=1}^n \lambda_i e_i^*$. Then

$$\ell'(e_j) = \sum_{i=1}^n \lambda_i e_i^*(e_j) = \lambda_j \cdot 1 = \ell(e_j)$$

for all $j \in [n]$. Therefore, since ℓ, ℓ' take identical values on the basis of V, $\ell = \ell'$, as desired.

We now prove that $e_1^*, \dots, e_n^*$ is linearly independent. Let $\sum_{i=1}^n \lambda_i e_i^* = 0$. Then for all $j \in [n]$,

$$\lambda_j = \left(\sum_{i=1}^n \lambda_i e_i^* \right)(e_j) = 0$$

as desired. $\square$

- **Transpose** (of A): The map from W^* to V^* defined by $\ell \mapsto \ell \circ A$ for all $\ell \in W^*$. *Denoted by* $\boldsymbol{A^*}$.
- Claim 1.2.15: If $e_1, \dots, e_n$ is a basis of V, $f_1, \dots, f_m$ is a basis of W, $e_1^*, \dots, e_n^*$ and $f_1^*, \dots, f_m^*$ are the corresponding dual bases, and $[a_{i,j}]$ is the $m \times n$ matrix of A with respect to $\{e_j\}, \{f_i\}$, then the linear map A^* is defined in terms of $\{f_i^*\}, \{e_j^*\}$ by the transpose matrix $(a_{j,i})$.

Proof. Let $[c_{j,i}]$ be the $n \times m$ matrix of A^* with respect to $\{f_i^*\}, \{e_j^*\}$. We seek to prove that $a_{i,j} = c_{j,i}$ ($1 \leq i \leq m$, $1 \leq j \leq n$).

By the definition of $[a_{i,j}]$ and $[c_{j,i}]$, we have that

$$A^* f_i^* = \sum_{k=1}^n c_{k,i} e_k^* \qquad\qquad Ae_j = \sum_{k=1}^m a_{k,j} f_k$$

It follows that

$$[A^* f_i^*](e_j) = \left[\sum_{k=1}^n c_{k,i} e_k^* \right](e_j) = c_{j,i}$$

and

$$[A^* f_i^*](e_j) = f_i^*(Ae_j) = f_i^* \left(\sum_{k=1}^m a_{k,j} f_k \right) = a_{i,j}$$

so transitivity implies the desired result. $\square$

4/4: - $\boldsymbol{V^k}$: The set of all k-tuples $(v_1, \dots, v_k)$ where $v_1, \dots, v_k \in V$ a vector space.

  - Note that
$$V^k = \underbrace{V \times \cdots \times V}_{k \text{ times}}$$
where "$\times$" denotes the Cartesian product.

- **Linear** (function in its i^{th} variable): A function $T : V^k \to \mathbb{R}$ such that the map from V to $\mathbb{R}$ defined by $v \mapsto T(v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_k)$ is linear, where all v_j save v_i are fixed.
- ***k*-linear** (function T): A function $T : V^k \to \mathbb{R}$ that is linear in its i^{th} variable for $i = 1, \dots, k$. *Also known as* ***k*-tensor**.
- $\boldsymbol{\mathcal{L}^k(V)}$: The set of all k-tensors in V.
  - Since the sum $T_1 + T_2$ of two k-linear functions $T_1, T_2 : V^k \to \mathbb{R}$ is just another k-linear function, and λT_1 is k-linear for all $\lambda \in \mathbb{R}$, we have that $\mathcal{L}^k(V)$ is a vector space.
- Convention: 0-tensors are just the real numbers. Mathematically, we define
$$\mathcal{L}^0(V) = \mathbb{R}$$
- Note that $\mathcal{L}^1(V) = V^*$.
- Defines multi-indices of n of length k.
- Lemma 1.3.5: If $n, k \in \mathbb{N}$, then there are exactly n^k multi-indices of n of length k.
- $\boldsymbol{T_I}$: The real number $T(e_{i_1}, \dots, e_{i_k})$, where $T \in \mathcal{L}^k(V)$, $e_1, \dots, e_n$ is a basis of V, and I is a multi-index of n of length k.
- Proposition 1.3.7: The real numbers T_I determine T, i.e., if T, T' are k-tensors and $T_I = T'_I$ for all I, then $T = T'$.

  Proof. We induct on n. For the base case $n = 1$, $T \in (\mathbb{R}^k)^*$ and we have already proven this result. Now suppose inductively that the assertion is true for $n - 1$. For each e_i, let T_i be the $(k-1)$-tensor defined by
$$(v_1, \dots, v_{n-1}) \mapsto T(v_1, \dots, v_{n-1}, e_i)$$
Then for an arbitrary $v = c_1 e_1 + \cdots + c_n e_n$,
$$T(v_1, \dots, v_{n-1}, v) = \sum_{i=1}^{n} c_i T_i(v_1, \dots, v_{n-1})$$
so the T_i's determine T. Applying the inductive hypothesis completes the proof. $\square$

- **Tensor product**: The function $\otimes : \mathcal{L}^k(V) \times \mathcal{L}^\ell(V) \to \mathcal{L}^{k+\ell}(V)$ defined by
$$(T_1 \otimes T_2)(v_1, \dots, v_{k+\ell}) = T_1(v_1, \dots, v_k) T_2(v_{k+1}, \dots, v_{k+\ell})$$
for all $T_1 \in \mathcal{L}^k(V)$ and $T_2 \in \mathcal{L}^\ell(V)$.
- Note that by the definition of 0-tensors as real numbers, if $a \in \mathbb{R}$ and $T \in \mathcal{L}^k(V)$, then
$$a \otimes T = T \otimes a = aT$$
- Proposition 1.3.9: Associativity, distributivity of scalar multiplication, and left and right distributive laws for the tensor product.
- **Decomposable** (k-tensor): A k-tensor T for which there exist $\ell_1, \dots, \ell_k \in V^*$ such that
$$T = \ell_1 \otimes \cdots \otimes \ell_k$$

- Defines e_I^*.
- Theorem 1.3.13: V a vector space with basis $e_1, \dots, e_n$ and $0 \leq k \leq n$ implies the k-tensors e_I^* form a basis of $\mathcal{L}^k(V)$.

  Proof. Spanning: Let $T \in \mathcal{L}^k(V)$ be arbitrary. Define

  $$T' = \sum_I T_I e_I^*$$

  Since

  $$T'_J = T'(e_{j_1}, \dots, e_{j_k}) = \sum_I T_I e_I^*(e_{j_1}, \dots, e_{j_k}) = T_J e_J^*(e_{j_1}, \dots, e_{j_k}) = T_J$$

  for all J, Proposition 1.3.7 asserts that $T = T'$. Therefore, since every $T_I \in \mathbb{R}$, $T = T' \in \operatorname{span}(e_I^*)$.

  Linear independence: Suppose

  $$T = \sum_I c_I e_I^* = 0$$

  for some set of constants $c_I \in \mathbb{R}$. Then

  $$0 = T(e_{j_1}, \dots, e_{j_k}) = \sum_I c_I e_I^*(e_{j_1}, \dots, e_{j_k}) = c_J$$

  for all J, as desired. □

- Corollary 1.3.15: If $\dim V = n$, then $\dim(\mathcal{L}^k(V)) = n^k$.

  Proof. Follows immediately from Lemma 1.3.5. □

- **Pullback** (of T by the map A): The k-tensor $A^*T : V^k \to \mathbb{R}$ defined by

  $$(A^*T)(v_1, \dots, v_k) = T(Av_1, \dots, Av_k)$$

  where V, W are finite-dimensional vector spaces, $A : V \to W$ is linear, and $T \in \mathcal{L}^k(W)$.

- Proposition 1.3.18: The map $A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$ defined by $T \mapsto A^*T$ is linear.
- Identities:
  - If $T_1 \in \mathcal{L}^k(W)$ and $T_2 \in \mathcal{L}^m(W)$, then

    $$A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$$

  - If U is a vector space, $B : U \to V$ is linear, and $T \in \mathcal{L}^k(W)$, then $(AB)^*T = B^*(A^*T)$. Hence,

    $$(AB)^* = B^*A^*$$

4/13:

- $\boldsymbol{\Sigma_k}$: The set containing the natural numbers 1 through k. *Given by*

  $$\Sigma_k = \{1, 2, \dots, k\}$$

- **Permutation of order $\boldsymbol{k}$**: A bijection on Σ_k. *Denoted by* $\boldsymbol{\sigma}$.
- **Product** (of σ_1, σ_2): The composition $\sigma_1 \circ \sigma_2$, i.e., the map

  $$i \mapsto \sigma_1(\sigma_2(i))$$

  Denoted by $\boldsymbol{\sigma_1 \sigma_2}$.

- **Inverse** (of σ): The permutation of order k which is the inverse bijection of σ. *Denoted by* $\boldsymbol{\sigma^{-1}}$.
- **Permutation group** (of Σ_k): The set of all permutations of order k. *Also known as* **symmetric group on *k* letters**. *Denoted by* $\boldsymbol{S_k}$.
- Lemma 1.4.2: The group S_k has $k!$ elements.
- **Transposition**: A permutation of order k defined by

$$\ell \mapsto \begin{cases} j & \ell = i \\ i & \ell = j \\ \ell & \ell \neq i, j \end{cases}$$

  for all $\ell \in \Sigma_k$, where $i, j \in \Sigma_k$. *Denoted by* $\boldsymbol{\tau_{i,j}}$.
- **Elementary transposition**: A transposition of the form $\pi_{i,i+1}$.
- Theorem 1.4.4: Every $\sigma \in S_k$ can be written as a product of (a finite number of) transpositions.

  Proof. We induct on k.

  For the base case $k = 2$, the identity permutation of S_2 is the "product" of zero transpositions, and the only other permutation is a transposition (the "product" of one transposition, namely itself).

  Now suppose inductively that we have proven the claim for $k - 1$. Let $\sigma \in S_k$ be arbitrary. Suppose $\sigma(k) = i$. Then $\tau_{i,k}\sigma(k) = k$. Since $(\tau_{i,k}\sigma)|_{\Sigma_{k-1}} \in S_{k-1}$, we have by the inductive hypothesis that $(\tau_{i,k}\sigma)|_{\Sigma_{k-1}} = \tau_1 \cdots \tau_m$ for some set of permutations $\tau_1, \dots, \tau_m \in S_{k-1}$. For each τ_j $(1 \leq j \leq m)$, define $\tau_j' \in S_k$

$$\tau_j'(\ell) = \begin{cases} \tau_j(\ell) & \ell < k \\ \ell & \ell = k \end{cases}$$

  It follows that

$$\begin{aligned} \tau_{i,k}\sigma &= \tau_1' \cdots \tau_m' \\ \sigma &= \tau_{i,k}\tau_1' \cdots \tau_m' \end{aligned}$$

  as desired. □

- Theorem 1.4.5: Every transposition can be written as a product of elementary transpositions.

  Proof. Let $\tau_{i,j} \in S_k$, and let $i < j$ WLOG. Then we have that

$$\tau_{i,j} = \prod_{\ell=i}^{i-1} \tau_{\ell,\ell+1}$$

  as desired. □

- Corollary 1.4.6: Every permutation can be written as a product of elementary transpositions.
- **Sign** (of σ): The number ± 1 assigned to σ by the expression

$$\prod_{i<j} \frac{x_{\sigma(i)} - x_{\sigma(j)}}{x_i - x_j}$$

  where $x_1, \dots, x_k$ are coordinate functions on $\mathbb{R}^k$. *Denoted by* $\boldsymbol{(-1)^\sigma}$.
- Claim 1.4.9: The sign defines a group homomorphism $S_k \to \{\pm 1\}$. That is, for $\sigma_1, \sigma_2 \in S_k$, we have

$$(-1)^{\sigma_1\sigma_2} = (-1)^{\sigma_1}(-1)^{\sigma_2}$$

Proof. For all $i < j$, define p, q such that p is the lesser of $\sigma_2(i), \sigma_2(j)$ and q is the greater of $\sigma_2(i), \sigma_2(j)$. Formally,

$$p = \begin{cases} \sigma_2(i) & \sigma_2(i) < \sigma_2(j) \\ \sigma_2(j) & \sigma_2(j) < \sigma_2(i) \end{cases} \qquad q = \begin{cases} \sigma_2(j) & \sigma_2(i) < \sigma_2(j) \\ \sigma_2(i) & \sigma_2(j) < \sigma_2(i) \end{cases}$$

It follows that if $\sigma_2(i) < \sigma_2(j)$, then

$$\frac{x_{\sigma_1\sigma_2(i)} - x_{\sigma_1\sigma_2(j)}}{x_{\sigma_2(i)} - x_{\sigma_2(j)}} = \frac{x_{\sigma_1(p)} - x_{\sigma_1(q)}}{x_p - x_q}$$

and if $\sigma_2(j) < \sigma_2(i)$, then

$$\frac{x_{\sigma_1\sigma_2(i)} - x_{\sigma_1\sigma_2(j)}}{x_{\sigma_2(i)} - x_{\sigma_2(j)}} = \frac{x_{\sigma_1(q)} - x_{\sigma_1(p)}}{x_q - x_p} = \frac{x_{\sigma_1(p)} - x_{\sigma_1(q)}}{x_p - x_q}$$

Therefore,

$$\begin{aligned}
(-1)^{\sigma_1\sigma_2} &= \prod_{i<j} \frac{x_{\sigma_1\sigma_2(i)} - x_{\sigma_1\sigma_2(j)}}{x_i - x_j} \\
&= \prod_{i<j} \frac{x_{\sigma_1\sigma_2(i)} - x_{\sigma_1\sigma_2(j)}}{x_{\sigma_2(i)} - x_{\sigma_2(j)}} \cdot \frac{x_{\sigma_2(i)} - x_{\sigma_2(j)}}{x_i - x_j} \\
&= \prod_{i<j} \frac{x_{\sigma_1(p)} - x_{\sigma_1(q)}}{x_p - x_q} \cdot \prod_{i<j} \frac{x_{\sigma_2(i)} - x_{\sigma_2(j)}}{x_i - x_j} \\
&= (-1)^{\sigma_1}(-1)^{\sigma_2}
\end{aligned}$$

as desired. ☐

- Proposition 1.4.11: If σ is the product of an odd number of transpositions, then $(-1)^\sigma = -1$, and if σ is the product of an even number of transpositions, then $(-1)^\sigma = +1$.

  Proof. Follows from the fact that $(-1)^\sigma = -1$ (see Exercise 1.4.ii). ☐

- $\boldsymbol{T^\sigma}$: The k-tensor defined by

  $$T^\sigma(v_1, \dots, v_k) = T(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)})$$

  where $T \in \mathcal{L}^k(V)$, V is an n-dimensional vector space, and $\sigma \in S_k$.

- Proposition 1.4.14:

  1. If $T = \ell_1 \otimes \cdots \otimes \ell_k$ ($\ell_i \in V^*$), then $T^\sigma = \ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}$.

     Proof. If $v_1, \dots, v_k \in V$, then

     $$\begin{aligned}
     T^\sigma(v_1, \dots, v_k) &= T(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)}) \\
     &= [\ell_1 \otimes \cdots \otimes \ell_k](v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)}) \\
     &= \ell_1(v_{\sigma^{-1}(1)}) \cdots \ell_k(v_{\sigma^{-1}(k)}) \\
     &= \ell_{\sigma(1)}(v_1) \cdots \ell_{\sigma(k)}(v_2) \\
     &= [\ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}](v_1, \dots, v_k)
     \end{aligned}$$

     as desired. Note that we can justify the fourth equality by nothing that if $\sigma^{-1}(i) = q$, then the i^{th} term in the product is $\ell_{\sigma(q)}(v_q)$, so since σ is a bijection, the product can be arranged to the form on the right-hand side of equality four. ☐

2. The assignment $T \mapsto T^\sigma$ is a linear map from $\mathcal{L}^k(V) \to \mathcal{L}^k(V)$.

   Proof. See Exercise 1.4.iii. □

3. If $\sigma_1, \sigma_2 \in S_k$, we have $T^{\sigma_1\sigma_2} = (T^{\sigma_1})^{\sigma_2}$.

   Proof. Let $T = \ell_1 \otimes \cdots \otimes \ell_k$[1]. Then

$$T^{\sigma_1} = \ell_{\sigma_1(1)} \otimes \cdots \otimes \ell_{\sigma_1(k)} = \ell'_1 \otimes \cdots \otimes \ell'_k$$

   and thus

$$(T^{\sigma_1})^{\sigma_2} = \ell'_{\sigma_2(1)} \otimes \cdots \otimes \ell'_{\sigma_2(k)}$$

   Let $\sigma_2(i) = j$. Then since $\ell'_p = \ell_{\sigma_1(p)}$ by definition, we have that $\ell'_{\sigma_2(j)} = \ell_{\sigma_1(\sigma_2(j))}$. Therefore,

$$\begin{aligned}(T^{\sigma_1})^{\sigma_2} &= \ell'_{\sigma_2(1)} \otimes \cdots \otimes \ell'_{\sigma_2(k)} \\ &= \ell_{\sigma_1(\sigma_2(1))} \otimes \cdots \otimes \ell_{\sigma_1(\sigma_2(k))} \\ &= \ell_{\sigma_1\sigma_2(1)} \otimes \cdots \otimes \ell_{\sigma_1\sigma_2(k)} \\ &= T^{\sigma_1\sigma_2}\end{aligned}$$

   as desired. □

- **Alternating** (k-tensor): A k-tensor $T \in \mathcal{L}^k(V)$ such that $T^\sigma = (-1)^\sigma T$ for all $\sigma \in S_k$.
- $\boldsymbol{\mathcal{A}^k(V)}$: The set of all alternating k-tensors in $\mathcal{L}^k(V)$.
  - Proposition 1.4.14(2) implies that $(T_1 + T_2)^\sigma = T_1^\sigma + T_2^\sigma$ and $(\lambda T)^\sigma = \lambda T^\sigma$; it follows that $\mathcal{A}^k(V)$ is a vector space.
- **Alternation operation**: The function from $\mathcal{L}^k(V) \to \mathcal{L}^k(V)$ defined by

$$T \mapsto \sum_{\tau \in S_k} (-1)^\tau T^\tau$$

  Denoted by **Alt**.
- Proposition 1.4.17: For $T \in \mathcal{L}^k(V)$ and $\sigma \in S_k$, we have that

1. $\operatorname{Alt}(T)^\sigma = (-1)^\sigma \operatorname{Alt} T$.

   Proof. We have that

$$\begin{aligned}\operatorname{Alt}(T)^\sigma &= \left(\sum_{\tau \in S_k} (-1)^\tau T^\tau\right)^\sigma \\ &= \sum_{\tau \in S_k} (-1)^\tau (T^\tau)^\sigma && \text{Proposition 1.4.14(2)} \\ &= \sum_{\tau \in S_k} (-1)^\tau T^{\tau\sigma} && \text{Proposition 1.4.14(3)} \\ &= (-1)^\sigma \sum_{\tau \in S_k} (-1)^{\tau\sigma} T^{\tau\sigma} \\ &= (-1)^\sigma \sum_{\tau\sigma \in S_k} (-1)^{\tau\sigma} T^{\tau\sigma} \\ &= (-1)^\sigma \operatorname{Alt} T\end{aligned}$$

   as desired. □

[1] What gives us the right to assume T is decomposable?

2. If $T \in \mathcal{A}^k(V)$, then $\operatorname{Alt} T = k!T$.

   Proof. Since $T \in \mathcal{A}^k(V)$, we know that $T^\sigma = (-1)^\sigma T$. Therefore,

   $$\operatorname{Alt} T = \sum_{\tau \in S_k} (-1)^\tau T^\tau = \sum_{\tau \in S_k} (-1)^\tau (-1)^\tau T = \sum_{\tau \in S_k} T = k!T$$

   where the last equality holds because the cardinality of S_k is $k!$. ☐

3. $\operatorname{Alt}(T^\sigma) = \operatorname{Alt}(T)^\sigma$.

   Proof. We have that

   $$\operatorname{Alt}(T^\sigma) = \sum_{\tau \in S_k} (-1)^\tau T^{\tau\sigma} = (-1)^\sigma \sum_{\tau \in S_k} (-1)^{\tau\sigma} T^{\tau\sigma} = (-1)^\sigma \operatorname{Alt}(T) = \operatorname{Alt}(T)^\sigma$$

   as desired. ☐

4. The alternation operation is linear.

   Proof. Follows by Proposition 1.4.14. ☐

- **Repeating** (multi-index I): A multi-index I of length k such that $i_r = i_s$ for some $r \neq s$.
- **Strictly increasing** (multi-index I): A multi-index I of length k such that $i_1 < i_2 < \cdots < i_r$.
- $\boldsymbol{I^\sigma}$: The multi-index of length k defined by

  $$I^\sigma = (i_{\sigma(1)}, \dots, i_{\sigma(k)})$$

- If I is non-repeating, there is a unique $\sigma \in S_k$ such that I^σ is strictly increasing.
- $\boldsymbol{\psi_I}$: The following k-tensor. *Given by*

  $$\psi_I = \operatorname{Alt}(e_I^*)$$

- Proposition 1.4.20:

1. $\psi_{I^\sigma} = (-1)^\sigma \psi_I$.

   Proof. We have that

   $$\psi_{I^\sigma} = \operatorname{Alt}(e_{I^\sigma}^*) = \operatorname{Alt}[(e_I^*)^\sigma] = \operatorname{Alt}(e_I^*)^\sigma = (-1)^\sigma \operatorname{Alt}(e_I^*) = (-1)^\sigma \psi_I$$

   as desired. ☐

2. If I is repeating, then $\psi_I = 0$.

   Proof. Suppose $I = (i_1, \dots, i_k)$ is such that $i_r = i_s$ for some distinct $r, s \in \Sigma_k$. Then $e_I^* = e_{I^{\tau_{i_r,i_s}}}^*$, so

   $$\psi_I = \psi_{I^{\tau_{i_r,i_s}}} = (-1)^{\tau_{i_r,i_s}} \psi_I = -\psi_I$$

   Therefore, we must have $\psi_I = 0$, as desired. ☐

3. If I and J are strictly increasing, then

   $$\psi_I(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$$

Proof. We have by definition that

$$\psi_I(e_{j_1}, \dots, e_{j_k}) = \sum_\tau (-1)^\tau e^*_{I^\tau}(e_{j_1}, \dots, e_{j_k})$$

This combined with the facts that

$$e^*_{I^\tau}(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1 & I^\tau = J \\ 0 & I^\tau \neq J \end{cases}$$

I^τ is strictly increasing iff $I^\tau = I$, and the above equation is nonzero iff $I^\tau = I = J$ implies the desired result. ☐

- Conclusion 1.4.22: If $T \in \mathcal{A}^k(V)$, then we can write T as a sum

$$T = \sum_I c_I \psi_I$$

with I's strictly increasing.

Proof. Let $T \in \mathcal{A}^k(V)$ be arbitrary. By Theorem 1.3.13,

$$T = \sum_J a_J e^*_J$$

for some set of $a_J \in \mathbb{R}$. It follows since $\operatorname{Alt}(T) = k!T$ that

$$T = \frac{1}{k!} \sum a_J \operatorname{Alt}(e^*_J) = \sum b_J \psi_J$$

We can disregard all repeating terms in the sum since they are zero by Proposition 1.4.20(2); for every non-repeating term J, we can write $J = I^\sigma$, where I is strictly increasing and hence $\psi_J = (-1)^\sigma \psi_I$. ☐

- Claim 1.4.24: The c_I's of Conclusion 1.4.22 are unique.

Proof. For J strictly increasing, we have

$$T_J = T(e_{j_1}, \dots, e_{j_k}) = \sum_I c_I \psi_I(e_{j_1}, \dots, e_{j_k}) = c_J$$

☐

- Proposition 1.4.26: The alternating tensors ψ_I with I strictly increasing are a basis for $\mathcal{A}^k(V)$.

Proof. Spanning: See Conclusion 1.4.22.

Linear independence: See Claim 1.4.24. ☐

- We have that

$$\dim \mathcal{A}^k(V) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

  - Hint in proving this claim: "Show that every strictly increasing multi-index of length k determines a k-element subset of $\{1, \dots, n\}$ and vice versa." (Guillemin & Haine, 2018, p. 16).
  - Note also that if $k > n$, every multi-index has a repeat somewhere, meaning that $\dim \mathcal{A}^k(V) = \binom{n}{k} = 0$.

Week 2

Tensor Classifications

2.1 Alternating Tensors

4/4:

- Plan:
  - More multilinear algebra.
  - Alternating k-tensors — 2 views:
    1. As a subspace of $\mathcal{L}^k(V)$.
    2. As a quotient of $\mathcal{L}^k(V)$.
  - Next time: Operators as alternating tensors.
    - Wedge products.
    - Interior products.
    - Pullbacks.
- Recall: $\dim V = n$, $e_1, \dots, e_n$ a basis, $\mathcal{L}^k(V)$ the space of k-tensors, $\sigma \in S_k$ implies $(-1)^\sigma \in \{\pm 1\}$, key property: $(-1)^{\sigma_1\sigma_2} = (-1)^{\sigma_1}(-1)^{\sigma_2}$.
- $\boldsymbol{T^\sigma}$: The k-tensor over V defined by

$$T^\sigma(v_1, \dots, v_k) = T(v_{\bar{\sigma}(1)}, \dots, v_{\bar{\sigma}(k)})$$

  where $T \in \mathcal{L}^k(V)$, $\sigma \in S_k$, and $\bar{\sigma}$ denotes the inverse of σ.
- Example: $n = 2$, $k = 2$. Let $T = e_1^* \otimes e_2^* \in \mathcal{L}^2(V)$. Let $\sigma = \tau_{1,2}$. Then $T^\sigma = e_2^* \otimes e_1^*$.
- Another property is $e_I^\sigma = e_{\sigma(I)}^*$.
- Properties:
  1. $T^{\sigma_1\sigma_2} = (T^{\sigma_1})^{\sigma_2}$.
  2. $(T_1 + T_2)^\sigma = T_1^\sigma + T_2^\sigma$.
  3. $(cT)^\sigma = cT^\sigma$.
- Thus, you can view $\sigma : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ as a linear map!
- **Alternating $\boldsymbol{k}$-tensor**: A tensor $T \in \mathcal{L}^k(V)$ such that $T^\sigma = (-1)^\sigma T$ for all $\sigma \in S_k$.
  - Equivalently, $T^\tau = -T$ for all $\tau \in S_k$.
- An example of an alternating 2-tensor when $\dim V = 2$ is $T = e_1^* \otimes e_2^* - e_2^* \otimes e_1^*$.
  - Naturally, $T^{\tau_{1,2}} = -T$, and $\tau_{1,2}$ is the unique transposition in S_2.

- $e_1^* \otimes e_2^*$ is *not* an alternating 2-tensor since $(e_1^* \otimes e_2^*)^\tau = e_2^* \otimes e_1^* \neq (-1)^\tau (e_1^* \otimes e_2^*)$.
- We can look at $n = 2$, $k = 1$ for ourselves.
- Note: If T_1, T_2 are both alternating k-tensors, then $T_1 + T_2$ is also alternating, as is cT_1 for all $c \in \mathbb{R}$.
- $\boldsymbol{\mathcal{A}^k(V)}$: The vector space of alternating k-tensors.
- **Alt (T)**: The function $\text{Alt} : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ defined by

$$\text{Alt}(T) = \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma$$

- Properties:
  1. $\text{im}(\text{Alt}) = \mathcal{A}^k(V)$.
  2. $\mathcal{L}^k(V)/\ker(\text{Alt}) = \Lambda^k(V^*)$ is isomorphic to $\mathcal{A}^k(V)$.
  3. $\text{Alt}(T)^\sigma = (-1)^\sigma \text{Alt}(T)$.
     - Proof:

$$\begin{aligned}
\text{Alt}(T)^{\sigma'} &= \left(\sum_{\sigma \in S_k} (-1)^\sigma T^\sigma\right)^{\sigma'} \\
&= \sum_{\sigma \in S_k} (-1)^\sigma T^{\sigma\sigma'} \\
&= (-1)^{\sigma'} \sum_{\sigma \in S_k} (-1)^{\sigma'} (-1)^\sigma T^{\sigma\sigma'} \\
&= (-1)^{\sigma'} \sum_{\sigma \in S_k} (-1)^{\sigma\sigma'} T^{\sigma\sigma'} \\
&= (-1)^{\sigma'} \text{Alt}(T)
\end{aligned}$$

     - The last equality holds because summing over all σ is the same as summing over all $\sigma' \circ \sigma$.
     - This implies $\text{im}(\text{Alt}) \leq \mathcal{A}^k(V)$.
  4. If $T \in \mathcal{A}^k(T)$, $\text{Alt}(T) = k!T$.
     - We have

$$\begin{aligned}
\text{Alt}(T) &= \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma \\
&= \sum_{\sigma \in S_k} (-1)^\sigma (-1)^\sigma T \\
&= \sum_{\sigma \in S_k} T \\
&= k!T
\end{aligned}$$

       where $T^\sigma = (-1)^\sigma T$ since $T \in \mathcal{A}^k(V)$ by definition.
     - This implies that $\text{im}(\text{Alt}) = \mathcal{A}^k(V)$: $\text{Alt}(\frac{1}{k!}T) = T \in \mathcal{A}^k(V)$.
  5. $\text{Alt}(T^\sigma) = \text{Alt}(T)^\sigma$.
  6. $\text{Alt} : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ is linear.
- Warning: Some people take $\text{Alt}(T) = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma$[1].

[1]Klug prefers this convention, but the text takes the other one.

- Example: $n = k = 2$. We have

$$\operatorname{Alt}(e_1^* \otimes e_2^*) = e_1^* \otimes e_2^* - e_2^* \otimes e_1^*$$

- **Non-repeating** (multi-index I): A multi-index I such that $i_{j_1} \neq i_{j_2}$ for all $j_1 \neq j_2$.
- **Increasing** (multi-index I): A multi-index I such that $i_1 < \cdots < i_k$.
- Claim: $\{\operatorname{Alt}(e_I^*)\}$ where I is non-repeating and increasing is a basis for $\mathcal{A}^k(V)$. There are $\binom{n}{k}$ of these; thus, $\dim \mathcal{A}^k(V) = \binom{n}{k}$.

2.2 Redundant Tensors and Alternatization

4/6: - Klug will be in Texas on Monday and thus is cancelling class on Monday. Homework is now due next Friday. We'll have weekly homeworks going forward after that.
- Plan:
  - $\operatorname{Alt} : \mathcal{L}^k(V) \twoheadrightarrow \mathcal{A}^k(V)$[2].
  - Goal: Identify $\ker(\operatorname{Alt}) = \mathcal{I}^k(V)$, where $\mathcal{I}^k(V)$ is the space of **redundant** k-tensors[3].
  - Then: Operations on alternating tensors, e.g.,
    - Wedge product.
    - Interior product.
    - Orientations.
- Claim: $\{\operatorname{Alt}(e_I^*) \mid I \text{ non-repeating, increasing multi-index}\}$ is a basis for $\mathcal{A}^k(V)$.
  - Left as an exercise to us.
- **Redundant** (k-tensor): A k-tensor of the form

$$\ell_1 \otimes \cdots \otimes \ell_i \otimes \ell_i \otimes \ell_{i+2} \otimes \cdots \otimes \ell_k$$

  where $\ell_1, \dots, \ell_k \in V^*$.
- $\boldsymbol{\mathcal{I}^k(V)}$: The span of all redundant k-tensors.
  - Note that not every k-tensor in $\mathcal{I}^k(V)$ is a redundant.
- **Decomposable** (k-tensor): A k-tensor of the form $\ell_1 \otimes \cdots \otimes \ell_k$ for $\ell_i \in \mathcal{L}^1(V)$.
  - It often suffices to prove things for decomposable tensors.
- Properties.
  1. If $T \in \mathcal{I}^k(V)$, then $\operatorname{Alt}(T) = 0$, i.e., $\mathcal{I}^k(V) \leq \ker(\operatorname{Alt})$.
     - "Proof by example": If $T = \ell_1 \otimes \ell_1 \otimes \ell_2 \otimes \ell_3$, then $T^{\tau_{1,2}} = T$. It follows from the properties of Alt that

$$\begin{aligned}
\operatorname{Alt}(T) = \operatorname{Alt}(T^{\tau_{1,2}}) &= (-1)^{\tau_{1,2}} \operatorname{Alt}(T) = -\operatorname{Alt}(T) \\
2\operatorname{Alt}(T) &= 0 \\
\operatorname{Alt}(T) &= 0
\end{aligned}$$

[2]The two-headed right arrow denotes a surjective map.
[3]The $\mathcal{I}$ in $\mathcal{I}^k(V)$ stands for "ideal."

2. If $T \in \mathcal{I}^r(V)$ and $T' \in \mathcal{L}^s(V)$, then
$$T \otimes T' \in \mathcal{I}^{r+s}(V)$$
Similarly, if $T \in \mathcal{L}^r(V)$ and $T \in \mathcal{I}^s(V)$, then
$$T \otimes T' \in \mathcal{I}^{r+s}(V)$$
   – Proof: It suffices to assume that T is redundant. Obviously adding more tensors to the direct product will not change the redundancy of the initial tensor. Example: $\ell_1 \otimes \ell_1 \otimes \ell_2$ is just as redundant as $\ell_1 \otimes \ell_1 \otimes \ell_2 \otimes T$.
3. If $T \in \mathcal{L}^k(V)$ and $\sigma \in S_k$, then
$$T^\sigma = (-1)^\sigma T + S$$
for some $S \in \mathcal{I}^k(V)$.
   – Proof by example: It suffices to check this for decomposable tensors (a tensor is just a sum of decomposable tensors). Take $k = 2$. Let $T = \ell_1 \otimes \ell_2$. Let $\sigma = \tau_{1,2}$. Then
$$T^\sigma - (-1)^\sigma T = \ell_2 \otimes \ell_1 + \ell_1 \otimes \ell_2 = (\ell_1 + \ell_2) \otimes (\ell_1 + \ell_2) - \ell_1 \otimes \ell_1 - \ell_2 \otimes \ell_2$$
   – Actual proof: It suffices to assume T is decomposable. We induct on the number of transpositions needed to write σ as a product of **adjacent** transpositions.
   – Base case: $\sigma = \tau_{i,i+1}$. Then
$$\begin{aligned} T^{\tau_{i,i+1}} + T &= \ell_1 \otimes \cdots \otimes (\ell_i + \ell_{i+1}) \otimes (\ell_i + \ell_{i+1}) \otimes \cdots \otimes \ell_k \\ &\quad - \ell_1 \otimes \cdots \otimes \ell_i \otimes \ell_i \otimes \cdots \otimes \ell_k \\ &\quad - \ell_1 \otimes \cdots \otimes \ell_{i+1} \otimes \ell_{i+1} \otimes \cdots \otimes \ell_k \end{aligned}$$
   – Inductive step: If $\sigma = \beta\tau$, then
$$\begin{aligned} T^\sigma &= T^{\beta\tau} \\ &= (-1)^\tau T^\beta + \text{stuff in } \mathcal{I}^k(V) \\ &= (-1)^\tau[(-1)^\beta T + \text{stuff in } \mathcal{I}^k(V)] + \text{stuff in } \mathcal{I}^k(V) \end{aligned}$$
4. If $T \in \mathcal{L}^k(V)$, then
$$\text{Alt}(T) = k!T + W$$
for some $W \in \mathcal{I}^k(V)$.
   – We have that
$$\begin{aligned} \text{Alt}(T) &= \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma \\ &= \sum_{\sigma \in S_k} (-1)^\sigma [(-1)^\sigma T + S_\sigma] \\ &= \sum_{\sigma \in S_k} T + \sum_{\sigma \in S_k} (-1)^\sigma S_\sigma \\ &= k!T + W \end{aligned}$$
5. $\mathcal{I}^k(V) = \ker(\text{Alt})$.
   – We have that $\mathcal{I}^k(V) \leq \ker(\text{Alt})$ by property 1.
   – Now suppose $T \in \ker(\text{Alt})$. Then $\text{Alt}(T) = 0$. Then by property 4,
$$\begin{aligned} \text{Alt}(T) &= k!T + W \\ 0 &= k!T + W \\ T &= -\frac{1}{k!} W \in \mathcal{I}^k(V) \end{aligned}$$

- Warning: If $T \in \mathcal{A}^r(V)$ and $T' \in \mathcal{A}^s(V)$, then we do not necessarily have $T \otimes T' \in \mathcal{A}^{r+s}(V)$.
  - Example: $e_1^*, e_2^* \in \mathcal{A}^1(V)$ have $e_1^* \otimes e_2^* \notin \mathcal{A}^2(V)$.
- **Adjacent** (transposition): A transposition of the form $\tau_{i,i+1}$.

2.3 The Wedge Product

4/8:

- Recall that $\mathcal{A}^k(V) \hookrightarrow \mathcal{L}^k(V)$[4]
- Functoriality: $(A \circ B)^* = B^* \circ A^*$.
  - A^* takes $\mathcal{L}^k(W) \to \mathcal{L}^k(V)$ and $\mathcal{A}^k(W) \to \mathcal{A}^k(V)$.
- $\dim(\Lambda^k(V)) = \binom{n}{k}$.
  - Special case $k = n$: $\dim \Lambda^n(V) = 1$.
  - If $A : V \to V$ induces a map $\Lambda^n(V^*) \to \Lambda^n(V^*)$ defined by the determinant.
- Aside: $\Lambda^k(V)$ is "exterior powers."
- Plan: Wedge products + basis for $\Lambda^k(V)$.
- **Wedge product**: A function $\wedge : \Lambda^k(V^*) \times \Lambda^\ell(V^*) \to \Lambda^{k+\ell}(V)$.
  - We denote elements of $\Lambda^k(V^*)$ by ω_1, ω_2, etc.
- If $\pi : \mathcal{L}^k(V) \to \Lambda^k(V^*)$ sends $T \mapsto \omega$, $\omega_1 = \pi(T_1)$, and $\omega_2 = \pi(T_2)$, then $\omega_1 \wedge \omega_2 = \pi(T_1 \otimes T_2)$.
  - Note that $\ker(\pi) = \mathcal{I}^k(V)$.
- Properties.
  1. This is well defined, i.e., this does not depend on the choice of T_1, T_2.
     - Consider $T_1 + W_1, T_2 + W_2$ with $W_1, W_2 \in \mathcal{I}^k(V)$.
     - We check that $\pi[(T_1 + W_1) \otimes (T_2 + W_2)] = \pi(T_1 \otimes T_2)$.
     - Since $W_1 \otimes T_2, T_1 \otimes W_2, W_1 \otimes W_2 \in \mathcal{I}^{k+\ell}(V)$, we have that

$$\begin{aligned}\pi[(T_1 + W_1) \otimes (T_2 + W_2)] &= \pi(T_1 \otimes T_2 + W_1 \otimes T_2 + T_1 \otimes W_2 + W_1 \otimes W_2)\\ &= \pi(T_1 \otimes T_2) + \pi(W_1 \otimes T_2) + \pi(T_1 \otimes W_2) + \pi(W_1 \otimes W_2)\\ &= \pi(T_1 \otimes T_2)\end{aligned}$$

  2. Associative: We have that

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge \omega_2 \wedge \omega_3$$

     - Follows from the definition of $\wedge$ in terms of π and properties of the tensor product.
  3. Distributive: We have that

$$(\omega_1 + \omega_2) \wedge \omega_3 = \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_3 \qquad \omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$$

     - Follows from the definition of $\wedge$ in terms of π and properties of the tensor product.
  4. Linear: We have that

$$(c\omega_1) \wedge \omega_2 = c(\omega_1 \wedge \omega_2) = \omega_1 \wedge (c\omega_2)$$

     - Follows from the definition of $\wedge$ in terms of π and properties of the tensor product.

[4] The hooked right arrow denotes an injective map.

5. Anticommutative: We have that

$$\omega_1 \wedge \omega_2 = (-1)^{k\ell} \omega_2 \wedge \omega_1$$

- It suffices to assume that $w_1 = \ell_1 \wedge \cdots \wedge \ell_k, w_2 = \ell'_1 \wedge \cdots \wedge \ell'_\ell$.
  - We have

$$(\ell_1 \wedge \cdots \wedge \ell_k) \wedge (\ell'_1 \wedge \cdots \wedge \ell'_\ell) = (-1)^k (\ell'_1 \wedge \cdots \wedge \ell'_\ell) \wedge (\ell_1 \wedge \cdots \wedge \ell_k)$$

- Let $\ell_1, \dots, \ell_k \in \Lambda^1(V^*) = V^* = \mathcal{L}^1(V)$.
- Recall that $\mathcal{I}^1(V) = \{0\}$.
- Claim: $\ell_{\sigma(1)} \wedge \cdots \wedge \ell_{\sigma(k)} = (-1)^\sigma \ell_1 \wedge \cdots \wedge \ell_k$ for all $\sigma \in S_k$.
  - Recall that $T^\sigma = (-1)^\sigma T + W$ for some $W \in \mathcal{I}^k(V)$.
  - Let $T = \ell_1 \otimes \cdots \otimes \ell_k$.
  - Then

$$\begin{aligned}(\ell_1 \otimes \cdots \otimes \ell_k)^\sigma &= \ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)} \\ &= (-1)^\sigma \ell_1 \otimes \cdots \otimes \ell_k + W\end{aligned}$$

  - Then hit both sides by π, noting that $\pi(W) = 0$.

- Example:

1. $n = 2$, $k = \ell = 1$. Consider $e_1^*, e_2^* \in \mathcal{L}^1(V) = V^* = \mathcal{A}^1(V) = \Lambda^1(V^*)$. Then

$$e_1^* \wedge e_2^* = (-1) e_2^* \wedge e_1^* \qquad\qquad e_1^* \wedge e_1^* = 0 = e_2^* \wedge e_2^*$$

2. $n = 4$. We have $e_1^* \wedge (3e_1^* + 2e_2^* + 3e_2^*) = 3(e_1^* \wedge e_1^*) + 2(e_1^* \wedge e_2^*) + 3(e_1^* \wedge e_3^*)$. We also have $(e_1^* \wedge e_2^*) \wedge (e_1^* \wedge e_2^*) = 0$.

2.4 Chapter 1: Multilinear Algebra

From Guillemin and Haine (2018).

4/14:
- Having discussed $\operatorname{im}(\operatorname{Alt}) = \mathcal{A}^k(V)$ in some detail now, we move onto $\ker(\operatorname{Alt})$.
- **Redundant** (decomposable k-tensor): A decomposable k-tensor $\ell_1 \otimes \cdots \otimes \ell_k$ such that for some $i \in [k-1]$, $\ell_i = \ell_{i+1}$.
- $\boldsymbol{\mathcal{I}^k(V)}$: The linear span of the set of redundant k-tensors.
- Convention: There are no redundant 1-tensors. Mathematically, we define

$$\mathcal{I}^1(V) = 0$$

- Proposition 1.5.2: $T \in \mathcal{I}^k(V)$ implies $\operatorname{Alt}(T) = 0$.

Proof. Let $T = \ell_1 \otimes \cdots \otimes \ell_k$ with $\ell_i = \ell_{i+1}$. Then if $\sigma = \tau_{i,i+1}$, we have that $T^\sigma = T$ and $(-1)^\sigma = -1$. Therefore,

$$\begin{aligned}\operatorname{Alt}(T) &= \operatorname{Alt}(T^\sigma) \\ &= \operatorname{Alt}(T)^\sigma && \text{Proposition 1.4.17(3)} \\ &= (-1)^\sigma \operatorname{Alt}(T) && \text{Proposition 1.4.17(1)} \\ &= -\operatorname{Alt}(T)\end{aligned}$$

so we must have that $\operatorname{Alt}(T) = 0$, as desired. ∎

- Proposition 1.5.3: $T \in \mathcal{I}^r(V)$ and $T' \in \mathcal{L}^s(V)$ imply $T \otimes T', T' \otimes T \in \mathcal{I}^{r+s}(V)$.

  Proof. We first justify why we need only prove this claim for T' decomposable. As an element of $\mathcal{L}^s(V)$, we know that $T' = \sum a_I e_I^*$ for some set of $a_I \in \mathbb{R}$. Since each e_I^* is decomposable, this means that T' is a linear combination of decomposable tensors. This combined with the fact that the tensor product is linear means that

  $$T \otimes T' = T \otimes \sum a_I e_I^* = \sum a_I (T \otimes e_I^*)$$

  and similarly for $T' \otimes T$. Thus, if we can prove that each $T \otimes e_I^* \in \mathcal{I}^{r+s}(V)$, it will follow since $\mathcal{I}^k(V)$ is a vector space that $\sum a_I(T \otimes e_I^*) = T \otimes T' \in \mathcal{I}^{r+s}(V)$. In other words, we need only prove that $T \otimes T' \in \mathcal{I}^{r+s}(V)$ for T' decomposable, as desired.

  Let $T = \ell_1 \otimes \cdots \otimes \ell_r$ with $\ell_i = \ell_{i+1}$, and let $T' = \ell_1' \otimes \cdots \otimes \ell_s'$. It follows that

  $$T \otimes T' = (\ell_1 \otimes \cdots \otimes \ell_i \otimes \ell_{i+1} \otimes \cdots \otimes \ell_r) \otimes (\ell_1' \otimes \cdots \otimes \ell_s')$$

  is redundant and hence in $\mathcal{I}^{r+s}(V)$, as desired. The argument is symmetric for $T' \otimes T$. $\square$

- Proposition 1.5.4: $T \in \mathcal{L}^k(V)$ and $\sigma \in S_k$ imply

  $$T^\sigma = (-1)^\sigma T + S$$

  where $S \in \mathcal{I}^k(V)$.

  Proof. As with Proposition 1.5.3, the linearity of $\sigma : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ allows us to assume that T is decomposable.

  By Theorem 1.4.5, σ can be written as a product of m elementary transpositions. To prove the claim, we induct on m.

  For the base case $m = 1$, let $\sigma = \tau_{i,i+1}$. If $T_1 = \ell_1 \otimes \cdots \otimes \ell_{i-1}$ and $T_2 = \ell_{i+2} \otimes \cdots \otimes \ell_k$, then

  $$\begin{aligned} T^\sigma - (-1)^\sigma T &= T_1 \otimes (\ell_{i+1} \otimes \ell_i \pm \ell_i \otimes \ell_{i+1}) \otimes T_2 \\ &= T_1 \otimes [(\ell_i + \ell_{i+1}) \otimes (\ell_i + \ell_{i+1}) \mp \ell_i \otimes \ell_i \mp \ell_{i+1} \otimes \ell_{i+1}] \otimes T_2 \end{aligned}$$

  i.e., $T^\sigma - (-1)^\sigma T$ is the sum of three redundant k-tensors, and thus a redundant k-tensor in and of itself, as desired. Note that even though only the middle portion is explicitly redundant, Proposition 1.5.3 allows us to call the whole tensor product redundant.

  Now suppose inductively that we have proven the claim for $m - 1$. Let $\sigma = \tau\beta$ where β is the product of $m - 1$ elementary transpositions and τ is an elementary transposition. Then

  $$\begin{aligned} T^\sigma &= (T^\beta)^\tau && \text{Proposition 1.4.14(3)} \\ &= (-1)^\tau T^\beta + \cdots && \text{Base case} \\ &= (-1)^\tau (-1)^\beta T + \cdots && \text{Inductive hypothesis} \\ &= (-1)^\sigma T + \cdots && \text{Claim 1.4.9} \end{aligned}$$

  where the dots are elements of $\mathcal{I}^k(V)$. $\square$

- Corollary 1.5.6: $T \in \mathcal{L}^k(V)$ implies

  $$\text{Alt}(T) = k!T + W$$

  where $W \in \mathcal{I}^k(V)$.

Proof. By definition,

$$\operatorname{Alt}(T) = \sum_{\sigma\in S_k} (-1)^\sigma T^\sigma$$

By Proposition 1.5.4,

$$T^\sigma = (-1)^\sigma T + W_\sigma$$

for all $\sigma \in S_k$ with each $W_\sigma \in \mathcal{I}^k(V)$. It follows by combining the above two results that

$$\operatorname{Alt}(T) = \sum_{\sigma\in S_k} (-1)^\sigma[(-1)^\sigma T + W_\sigma] = \sum_{\sigma\in S_k} T + \sum_{\sigma\in S_k} (-1)^\sigma W_\sigma = k!T + W$$

where $W = \sum_{\sigma\in S_k} (-1)^\sigma W_\sigma$ is an element of $\mathcal{I}^k(V)$ as a linear combination of elements of $\mathcal{I}^k(V)$. □

- Corollary 1.5.8: Let V be a vector space and $k \geq 1$. Then

$$\mathcal{I}^k(V) = \ker(\operatorname{Alt} : \mathcal{L}^k(V) \to \mathcal{A}^k(V))$$

Proof. Suppose first that $T \in \mathcal{I}^k(V)$. Then by Proposition 1.5.2, $\operatorname{Alt}(T) = 0$, so $T \in \ker(\operatorname{Alt})$, as desired.

Now suppose that $T \in \ker(\operatorname{Alt})$. Then $\operatorname{Alt}(T) = 0$, so by Corollary 1.5.6,

$$\begin{aligned} 0 &= k!T + W \\ T &= -\frac{1}{k!}W \end{aligned}$$

Therefore, as a scalar multiple of an element of $\mathcal{I}^k(V)$, $T \in \mathcal{I}^k(V)$. □

- Theorem 1.5.9: Every $T \in \mathcal{L}^k(V)$ has a unique decomposition $T = T_1 + T_2$ where $T_1 \in \mathcal{A}^k(V)$ and $T_2 \in \mathcal{I}^k(V)$.

Proof. By Corollary 1.5.6, we have that

$$\begin{aligned} \operatorname{Alt}(T) &= k!T + W \\ T &= \underbrace{\left(\frac{1}{k!}\operatorname{Alt}(T)\right)}_{T_1} + \underbrace{\left(-\frac{1}{k!}W\right)}_{T_2} \end{aligned}$$

As to uniqueness, suppose $0 = T_1 + T_2$ where $T_1 \in \mathcal{A}^k(V)$ and $T_2 \in \mathcal{I}^k(V)$. Then

$$\begin{aligned} 0 &= \operatorname{Alt}(0) = \operatorname{Alt}(T_1 + T_2) = \operatorname{Alt}(T_1) + \operatorname{Alt}(T_2) = k!T_1 + 0 = k!T_1 \\ T_1 &= 0 \end{aligned}$$

so $T_2 = 0$, too. □

- $\boldsymbol{\Lambda^k(V^*)}$: The quotient of the vector space $\mathcal{L}^k(V)$ by the subspace $\mathcal{I}^k(V)$. *Given by*

$$\Lambda^k(V^*) = \mathcal{L}^k(V)/\mathcal{I}^k(V)$$

- The quotient map $\pi : \mathcal{L}^k(V) \to \Lambda^k(V^*)$ defined by $T \mapsto T + \mathcal{I}^k(V)$ is onto and has $\ker(\pi) = \mathcal{I}^k(V)$.
- Theorem 1.5.13: $\pi : \mathcal{L}^k(V) \to \Lambda^k(V^*)$ maps $\mathcal{A}^k(V)$ bijectively onto $\Lambda^k(V^*)$.

Proof. Theorem 1.5.9 implies that every $T + \mathcal{I}^k(V)$ contains a unique $T_1 \in \mathcal{A}^k(V)$. Thus, for every element of $\Lambda^k(V^*)$, there is a unique element of $\mathcal{A}^k(V)$ which gets mapped onto it by π. □

- Note that since $\mathcal{A}^k(V)$ and $\Lambda^k(V)$ are in bijective correspondence, many texts do not distinguish between them. There are some advantages to making the distinction, though.
- The tensor product and pullback operatios give rise to similar operations on the spaces $\Lambda^k(V^*)$.
- **Wedge product**: The function $\wedge : \Lambda^{k_1}(V^*) \times \Lambda^{k_2}(V^*) \to \Lambda^{k_1+k_2}(V^*)$ defined by

$$\omega_1 \wedge \omega_2 = \pi(T_1 \otimes T_2)$$

  where for $i = 1, 2$, $\omega_i \in \Lambda^{k_i}(V^*)$, and $\omega_i = \pi(T_i)$ for some $T_i \in \mathcal{L}^{k_i}(V)$.
  - Note that it is Theorem 1.5.13 that allows us to find T_i such that $\omega_i = \pi(T_i)$.
- Claim 1.6.3: The wedge product is well-defined, i.e., it does not depend on our choices of T_i.

  Proof. We prove WLOG that $\wedge$ is well defined with respect to T_1. Suppose $\omega_1 = \pi(T_1) = \pi(T_1')$. Then by the definition of the quotient map, $T_1' = T_1 + W_1$ for some $W_1 \in \mathcal{I}^{k_1}(V)$. But this means that

$$T_1' \otimes T_2 = (T_1 + W_1) \otimes T_2 = T_1 \otimes T_2 + W_1 \otimes T_2$$

  where $W_1 \otimes T_2 \in \mathcal{I}^{k_1+k_2}(V)$ by Proposition 1.5.3. It follows that

$$\pi(T_1' \otimes T_2) = \pi(T_1 \otimes T_2)$$

  $\square$

- The wedge product also generalizes to higher orders, obeying associativity, scalar multiplication, and distributivity.
- **Decomposable element** (of $\Lambda^k(V^*)$): An element of $\Lambda^k(V^*)$ of the form $\ell_1 \wedge \cdots \wedge \ell_k$ where $\ell_1, \dots, \ell_k \in V^*$.
- Claim 1.6.8: The following wedge product identity holds for decomposable elements of $\Lambda^k(V^*)$.

$$\ell_{\sigma(1)} \wedge \cdots \wedge \ell_{\sigma(k)} = (-1)^\sigma \ell_1 \wedge \cdots \wedge \ell_k$$

  Proof. Let $T = \ell_1 \otimes \cdots \otimes \ell_k$. It follows by Proposition 1.4.14(1) that $T^\sigma = \ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}$. Therefore, we have that

$$\begin{aligned}
\ell_{\sigma(1)} \wedge \cdots \wedge \ell_{\sigma(k)} &= \pi(\ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}) \\
&= \pi(T^\sigma) \\
&= \pi[(-1)^\sigma T + W] \\
&= (-1)^\sigma \pi(T) \\
&= (-1)^\sigma \pi(\ell_1 \otimes \cdots \otimes \ell_k) \\
&= (-1)^\sigma \ell_1 \wedge \cdots \wedge \ell_k
\end{aligned}$$

  as desired. $\square$

- An important consequence of Claim 1.6.8 is that

$$\ell_1 \wedge \ell_2 = -\ell_2 \wedge \ell_1$$

- Theorem 1.6.10: If $\omega_1 \in \Lambda^r(V^*)$ and $\omega_2 \in \Lambda^s(V^*)$, then

$$\omega_1 \wedge \omega_2 = (-1)^{rs} \omega_2 \wedge \omega_1$$

  - This can be deduced from Claim 1.6.8.

  - Hint: It suffices to prove this for decomposable elements, i.e., for $\omega_1 = \ell_1 \wedge \cdots \wedge \ell_r$ and $\omega_2 = \ell_1' \wedge \cdots \wedge \ell_s'$.

- Theorem 1.6.13: The elements

$$e_{i_1}^* \wedge \cdots \wedge e_{i_k}^* = \pi(e_I^*) = \pi(e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*)$$

  with I strictly increasing are basis vectors of $\Lambda^k(V^*)$.

  Proof. Follows from the facts that the ψ_I for I strictly increasing constitute a basis of $\mathcal{A}^k(V)$ by Proposition 1.4.26 and π is an isomorphism $\mathcal{A}^k(V) \to \Lambda^k(V^*)$. $\square$

Week 3

Multilinear Spaces, Operations, and Conventions

3.1 Exterior Powers Basis and the Determinant

4/13:
- Plan:
  - Finish multilinear algebra.
  - Basis for $\Lambda^k(V^*)$.
  - Talk a bit about pullbacks and the determinant.
  - **Orientations** of vector spaces.
  - The **interior product**.
- Basis for $\Lambda^k(V^*)$.
  - Recall that $\{\text{Alt}(e_I^*) \mid I \text{ is a nonrepeating, increasing partition of } n \text{ into } k \text{ parts}\}$ is a basis for $\mathcal{A}^k(V)$.
- Alt is an isomorphism from $\Lambda^k(V^*)$ to $\mathcal{A}^k(V)$.
- If we have an injective map from $\mathcal{A}^k(V)$ to $\mathcal{L}^k(V)$ and π a projection map from $\mathcal{L}^k(V)$ to the quotient space $\mathcal{A}^k(V^*)$ gives rise to $\pi|_{\mathcal{A}^k(V)}$.
- Claim:
  1. $\pi|_{\mathcal{A}^k(V)}$ is an isomorphism.
  2. $\pi(\text{Alt}(e_I^*)) = k!\pi(e_I^*)$.

  (2) implies that $\{\pi(e_I^*) = e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*,\ I \text{ non-repeating and increasing}\}$ is a basis for $\Lambda^k(V^*)$.
- Examples:
  1. $n = 2 = \dim V$, $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$.
     - $\Lambda^0(V^*) = \mathbb{R}$ since $\binom{n}{0} = 1$.
     - $\Lambda^1(V^*) = \mathbb{R}e_1^* \oplus \mathbb{R}e_2^*$ since $\binom{n}{1} = 2$.
     - $\Lambda^2(V^*) = \mathbb{R}e_1^* \wedge e_2^*$ since $\binom{n}{2} = 1$.
       - For the second to last one, note that $e_1^* \wedge e_2^* = -e_2^* \wedge e_1^*$.
     - $\Lambda^3(V^*) = 0$ since $\binom{2}{3} = 0$.
       - For the last one, note that all $e_1^* \wedge e_1^* \wedge e_2^* = 0$.
  2. $n = 3$, $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3$.

  - $\binom{n}{0} = 1$: $\Lambda^0(V^*) = \mathbb{R}$.
  - $\binom{n}{1} = 3$: $\Lambda^1(V^*) = \mathbb{R}e_1^* \oplus \mathbb{R}e_2^* \oplus \mathbb{R}e_3^*$.
  - $\binom{n}{2} = 3$: $\Lambda^2(V^*) = \mathbb{R}e_1^* \wedge e_2^* \oplus \mathbb{R}e_2^* \wedge e_3^* \oplus \mathbb{R}e_1^* \wedge e_3^*$.
  - $\binom{n}{3} = 1$: $\Lambda^3(V^*) = \mathbb{R}e_1^* \wedge e_2^* \wedge e_3^*$.
  - $\binom{n}{m} = 0$ $(m > n)$: $\Lambda^m(V^*) = \Lambda^4(V^*) = 0$.
- If $A : V \to W$, $\omega_1 \in \Lambda^k(W^*)$, $\omega_2 \in \Lambda^\ell(W^*)$, then

$$A^*(\omega_1 \wedge \omega_2) = A^*\omega_1 \wedge A^*\omega_2$$

- **Determinant**: Let $\dim V = n$. Let $A : V \to V$ be a linear transformation. This induces a pullback $A^* : \Lambda^n(V^*) \to \Lambda^n(V^*)$. The top exterior power $k = n$ implies $\binom{k}{n} = 1$. We define $\det(A)$ to be the unique real number such that $A^*(v) = \det(A)v$.
- This determinant is the one we know.
  - A^* sends $e_1^* \wedge \cdots \wedge e_n^*$ to $A^*e_1^* \wedge \cdots \wedge A^*e_n^*$ which equals $A^*(e_1^* \wedge \cdots \wedge e_n^*)$ or $\det(A)$
- Sanity check.
  1. $\det(\mathrm{id}) = 1$.
     - $\mathrm{id}(e_1^* \wedge \cdots \wedge e_n^*) = \mathrm{id}\, e_1^* \wedge \cdots \wedge \mathrm{id}\, e_n^* = 1 \cdot e_1^* \wedge \cdots \wedge e_n^*$.
  2. If A is not an isomorphism, then $\det(A) = 0$.
     - If A is not an isomorphism, then there exists $v_1 \in \ker A$ with $v_1 \neq 0$. Let $v_1^*, \dots, v_n^*$ be a basis of V^*. So the pullback of this wedge is the wedge of the pullbacks, but $A^*v_1^* = 0$, so

$$A^*(v_1^* \wedge \cdots \wedge v_n^*) = (A^*v_1^*) \wedge \cdots \wedge (A^*v_n^*) = 0 \wedge \cdots \wedge (A^*v_n^*) = 0 = 0 \cdot v_1^* \wedge \cdots \wedge v_n^*$$

  3. $\det(AB) = \det(A)\det(B)$.
     - Let $A : V \to V$ and $B : V \to V$.
     - We have $(AB)^* = B^*A^*$; in particular, $n = k$, $V = W = U = V$.
- Recall: If we pick a basis for V, $e_1, \dots, e_n$.
  - Implies $[a_{ij}] = [A]_{e_1,\dots,e_n}^{e_1,\dots,e_n}$.
- Does $\det(A) = \det([a_{ij}]) = \sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$?
  - If $A : V \to V$, we know that $A^* : \Lambda^n(V^*) \to \Lambda^n(V^*)$ takes $e_1^* \wedge \cdots \wedge e_n^* \mapsto A^*(e_1^* \wedge \cdots \wedge e_n^*)$. We WTS

$$A^*(e_1^* \wedge \cdots \wedge e_n^*) = \left[\sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}\right] e_1^* \wedge \cdots \wedge e_n^*$$

  - We have that

$$\begin{aligned}
A^*(e_1^* \wedge \cdots \wedge e_n^*) &= A^*e_1^* \wedge \cdots \wedge A^*e_n^* \\
&= \left(\sum_{i_1=1}^n a_{i_1,1} e_{i_1}^*\right) \wedge \cdots \wedge \left(\sum_{i_n=1}^n a_{i_n,n} e_{i_n}^*\right) \\
&= \sum_{i_1,\dots,i_n} a_{i_1,1} \cdots a_{i_n,n} e_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \\
&= \left[\sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}\right] e_1^* \wedge \cdots \wedge e_n^*
\end{aligned}$$

    where the sign arises from the need to reorder $e_{i_1}^* \wedge \cdots \wedge e_{i_n}^*$ and the antisymmetry of the wedge product.

3.2 The Interior Product and Orientations

4/15:

- Plan:
  - Orientations.
  - Interior product.
- **Interior product**: We know that $\Lambda^k(V^*) \cong \mathcal{A}^k(V)$. Fix $v \in V$. Define $\iota_v : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$.
  - Wrong way: We take $\iota_v : \mathcal{L}^k(V) \to \mathcal{L}^{k-1}(V)$.

$$T \mapsto \sum_{r=1}^{k} (-1)^{r-1} T(v_1, \dots, v_r, \dots, v_{k-1})$$

  - Right way: First define $\varphi_v : \mathcal{A}^k(V) \to \mathcal{A}^{k-1}(V)$ by

$$T \mapsto T_v(v_1, \dots, v_{k-1}) = T(v, v_1, \dots, v_{k-1})$$

    - Check: $T_{v_1+v_2} = T_{v_1} + T_{v_2}$. $T_{\lambda v} = \lambda T_v$. $\varphi_v^{k-1} \circ \varphi_v^k = 0$ implies $\varphi_v \circ \varphi_w = -\varphi_w \circ \varphi_v$.
- Properties:

0. $\iota_v T \in \mathcal{L}^{k-1}(V)$.
1. ι_v is a linear map. This is all happening in the set $\text{Hom}(\mathcal{L}^k(V), \mathcal{L}^{k-1}(V))$.
2. $\iota_{v_1+v_2} = \iota_{v_1} + \iota_{v_2}$; $\iota_{\lambda v} = \lambda \iota_v$.
3. "Product rule": If $T_1 \in \mathcal{L}^p(V)$ and $T_1 \in \mathcal{L}^q(V)$, then $\iota_v(T_1 \otimes T_2) = \iota_v T_1 \otimes T_2 + (-1)^p T_1 \otimes \iota_v T_2$.
4. We have

$$\iota_v(\ell_1 \otimes \cdots \otimes \ell_k) = \sum_{r=1}^{k} (-1)^{r-1} \ell_r(v) \ell_1 \otimes \cdots \otimes \hat{\ell}_r \otimes \cdots \otimes \ell_k$$

5. $\iota_v \circ \iota_v = 0 \in \text{Hom}(\mathcal{L}^k(V), \mathcal{L}^{k-2}(V))$.
   - Note that this is related to $d^2 = 0$ from the first day of class (alongside $\int_m \mathrm{d}w = \int_{\partial m} w$).
   - Proof: We induct on k. It suffices to prove the result for T decomposable.
   - Trivial base case for $k = 1$.
   - We have that

$$\begin{aligned}
(\iota_v \circ \iota_v)(\ell_1 \otimes \cdots \otimes \ell_{k-1} \otimes \ell) &= \iota_v(\iota_v T \otimes \ell + (-1)^{k-1} \ell(v) T) \\
&= \iota_v(\iota_v T \otimes \ell) + (-1)^{k-1} \ell(v) \iota_v T \\
&= (-1)^{k-2} \ell(v) \iota_v T + (-1)^{k-1} \ell(v) \iota_v T \\
&= (-1)^{k-2} \ell(v) \iota_v T - (-1)^{k-2} \ell(v) \iota_v T \\
&= 0
\end{aligned}$$

6. If $T \in \mathcal{I}^k(V)$, then $\iota_v T \in \mathcal{I}^{k-1}(V)$.
   - Thus, ι_v induces a map $\iota_v : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$.
   - Proof: It suffices to check this for decomposables.
7. $\iota_{v_1} \circ \iota_{v_2} = -\iota_{v_2} \circ \iota_{v_1}$.

- Orientations:
  - A vector space V should have two orientations.
  - Two bases $e_1, \dots, e_n$ and $f_1, \dots, f_n$ are **orientation equivalent** if $T : V \to V$ an isomorphism has positive determinant. Otherwise, they are **orientation-inequivalent**.

  - An orientation on V is a choice of equivalence classes of bases under the equivalence relation on bases.
  - $T : V \to W$ given orientations, T preserves or reverses orientations.

- Fancy orientations.
  - An orientation on a 1D vector space L is a division into two halves.
  - Def: An orientation of V is an orientation of $\Lambda^n(V^*)$.

- We can prove that they're both the same.
  - If W and V are both oriented, then V/W gets a canonical orientation.

3.3 Chapter 1: Multilinear Algebra

From Guillemin and Haine (2018).

4/14:
- $\boldsymbol{\iota_v T}$: The $(k-1)$-tensor defined by

$$(\iota_v T)(v_1, \dots, v_{k-1}) = \sum_{r=1}^{k} (-1)^{r-1} T(v_1, \dots, v_{r-1}, v, v_r, \dots, v_{k-1})$$

  where $T \in \mathcal{L}^k(V)$, $k \in \mathbb{N}_0$, V is a vector space, and $v \in V$.

- If $v = v_1 + v_2$, then

$$\iota_v T = \iota_{v_1} T + \iota_{v_2} T$$

- If $T = T_1 + T_2$, then

$$\iota_v T = \iota_v T_1 + \iota_v T_2$$

- Lemma 1.7.4: If $T = \ell_1 \otimes \cdots \otimes \ell_k$, then

$$\iota_v T = \sum_{r=1}^{k} (-1)^{r-1} \ell_r(v) \ell_1 \otimes \cdots \otimes \hat{\ell}_r \otimes \cdots \otimes \ell_k$$

  where the hat over ℓ_r means that ℓ_r is deleted from the tensor product.

- Lemma 1.7.6: $T_1 \in \mathcal{L}^p(V)$ and $T_2 \in \mathcal{L}^q(V)$ imply

$$\iota_v(T_1 \otimes T_2) = \iota_v T_1 \otimes T_2 + (-1)^p T_1 \otimes \iota_v T_2$$

- Lemma 1.7.8: $T \in \mathcal{L}^k(V)$ implies that for all $v \in V$, we have

$$\iota_v(\iota_v T) = 0$$

  Proof. It suffices by linearity to prove this for decomposable tensors. We induct on k. For the base case $k = 1$, the claim is trivially true. Now suppose inductively that we have proven the claim for $k-1$. Consider $\ell_1 \otimes \cdots \otimes \ell_k$. Taking $T = \ell_1 \otimes \cdots \otimes \ell_{k-1}$ and $\ell = \ell_k$, we obtain

$$\iota_v(\iota_v(T \otimes \ell)) = \iota_v(\iota_v T) \otimes \ell + (-1)^{k-2} \ell(v) \iota_v T + (-1)^{k-1} \ell(v) \iota_v T$$

  The first term is zero by the inductive hypothesis, and the second two cancel each other out, as desired. ☐

- Claim 1.7.10: For all $v_1, v_2 \in V$, we have that

$$\iota_{v_1} \iota_{v_2} = -\iota_{v_2} \iota_{v_1}$$

Proof. Let $v = v_1 + v_2$. Then $\iota_v = \iota_{v_1} + \iota_{v_2}$. Therefore,

$$\begin{aligned}
0 &= \iota_v \iota_v && \text{Lemma 1.7.8}\\
&= (\iota_{v_1} + \iota_{v_2})(\iota_{v_1} + \iota_{v_2})\\
&= \iota_{v_1}\iota_{v_1} + \iota_{v_1}\iota_{v_2} + \iota_{v_2}\iota_{v_1} + \iota_{v_2}\iota_{v_2}\\
&= \iota_{v_1}\iota_{v_2} + \iota_{v_2}\iota_{v_1} && \text{Lemma 1.7.8}
\end{aligned}$$

yielding the desired result. □

- Lemma 1.7.11: If $T \in \mathcal{L}^k(V)$ is redundant, then so is $\iota_v T$.

  Proof. Let $T = T_1 \otimes \ell \otimes \ell \otimes T_2$ where $\ell \in V^*$, $T_1 \in \mathcal{L}^p(V)$, and $T_2 \in \mathcal{L}^q(V)$. By Lemma 1.7.6, we have that
  $$\iota_v T = \iota_v T_1 \otimes \ell \otimes \ell \otimes T_2 + (-1)^p T_1 \otimes \iota_v(\ell \otimes \ell) \otimes T_2 + (-1)^{p+2} T_1 \otimes \ell \otimes \ell \otimes \iota_v T_2$$
  Thus, since the first and third terms above are redundant and $\iota_v(\ell \otimes \ell) = \ell(v)\ell - \ell(v)\ell = 0$ by Lemma 1.7.4, we have the desired result. □

- $\boldsymbol{\iota_v \omega}$: The $\mathcal{I}^k(V)$-coset $\pi(\iota_v T)$, where $\omega = \pi(T)$.
- Proves that $\iota_v \omega$ does not depend on the choice of T.
- **Inner product operation**: The linear map $\iota_v : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$.
- The inner product has the following important identities.

$$\begin{aligned}
\iota_{(v_1+v_2)}\omega &= \iota_{v_1}\omega + \iota_{v_2}\omega\\
\iota_v(\omega_1 \wedge \omega_2) &= \iota_v \omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge \omega_2\\
\iota_v(\iota_v \omega) &= 0\\
\iota_{v_1}\iota_{v_2}\omega &= -\iota_{v_2}\iota_{v_1}\omega
\end{aligned}$$

4/18:

- As we developed the pullback $A^*T \in \mathcal{L}^k(V)$, we now look to develop a pullback on $\Lambda^k(V^*)$.
- Lemma 1.8.1: If $T \in \mathcal{I}^k(W)$, then $A^*T \in \mathcal{I}^k(V)$.

  Proof. It suffices to prove this for redundant k-tensors. Let $T = \ell_1 \otimes \cdots \otimes \ell_k$ be such that $\ell_i = \ell_{i+1}$. Then we have that
  $$\begin{aligned}
  A^*T &= A^*(\ell_1 \otimes \cdots \otimes \ell_k)\\
  &= A^*\ell_1 \otimes \cdots \otimes A^*\ell_k && \text{Exercise 1.3.iii}
  \end{aligned}$$
  where $A^*\ell_i = A^*\ell_{i+1}$ so that $A^*T \in \mathcal{I}^k(V)$, as desired. □

- $\boldsymbol{A^*\omega}$: The $\mathcal{I}^k(W)$-coset $\pi(A^*T)$, where $\omega = \pi(T)$.
- Claim 1.8.3: $A^*\omega$ is well-defined.

  Proof. Suppose $\omega = \pi(T) = \pi(T')$. Then $T = T' + S$ where $S \in \mathcal{I}^k(W)$. It follows that $A^*T = A^*T' + A^*S$, but since $A^*S \in \mathcal{I}^k(V)$ (Lemma 1.8.1), we have that
  $$\pi(A^*T) = \pi(A^*T')$$
  as desired. □

- Proposition 1.8.4. The map $A^* : \Lambda^k(W^*) \to \Lambda^k(V^*)$ sending $\omega \mapsto A^*\omega$ is linear. Moreover,

1. If $\omega_i \in \Lambda^{k_i}(W^*)$ $(i = 1, 2)$, then

$$A^*(\omega_1 \wedge \omega_2) = A^*(\omega_1) \wedge A^*(\omega_2)$$

2. If U is a vector space and $B : U \to V$ is a linear map, then for $\omega \in \Lambda^k(W^*)$,

$$B^* A^* \omega = (AB)^* \omega$$

(Hint: This proposition follows immediately from Exercises 1.3.iii-1.3.iv.)

- **Determinant** (of A): The number a such that $A^*\omega = a\omega$, where $A^* : \Lambda^n(V^*) \to \Lambda^n(V^*)$. *Denoted by* $\boldsymbol{\det(A)}$.
- Proposition 1.8.7: If A and B are linear mappings of V into V, then

$$\det(AB) = \det(A)\det(B)$$

Proof. Proposition 1.8.4(2) implies that

$$\begin{aligned}\det(AB)\omega &= (AB)^*\omega \\ &= B^*(A^*\omega) \\ &= \det(B)A^*\omega \\ &= \det(B)\det(A)\omega\end{aligned}$$

as desired. ☐

- $\mathbf{id}_{\boldsymbol{V}}$: The identity map on V.
- Proposition 1.8.8: $\det(\mathrm{id}_V) = 1$.
  - Hint: id_V^* is the identity map on $\Lambda^n(V^*)$.
- Proposition 1.8.9: If $A : V \to V$ is not surjective, then $\det(A) = 0$.

Proof. Let $W = \mathrm{im}(A)$. If A is not onto, $\dim W < n$, implying that $\Lambda^n(W^*) = 0$. Now let $A = i_W B$ where i_W is the inclusion map of W into V and B is the mapping A regarded as a mapping from V to W. It follows by Proposition 1.8.4(1) that if $\omega \in \Lambda^n(V^*)$, then

$$A^*\omega = B^* i_W^* \omega$$

where $i_W^*\omega = 0$ as an element of $\Lambda^n(W^*)$. ☐

- Deriving the typical formula for the determinant.
  - Let V, W be n-dimensional vector spaces with respective bases $e_1, \dots, e_n$ and $f_1, \dots, f_n$.
  - Denote the corresponding dual bases by $e_1^*, \dots, e_n^*$ and $f_1^*, \dots, f_n^*$.
  - Let $A : V \to W$. Recall that if the matrix of A is $[a_{i,j}]$, then the matrix of $A^* : W^* \to V^*$ is $(a_{j,i})$, i.e., if

$$Ae_j = \sum_{i=1}^{n} a_{i,j} f_i$$

then

$$A^* f_j^* = \sum_{i=1}^{n} a_{j,i} e_i^*$$

- It follows that

$$\begin{aligned}A^*(f_1^* \wedge \cdots \wedge f_n^*) &= A^* f_1^* \wedge \cdots \wedge A^* f_n^* \\ &= \sum_{1 \leq k_1, \dots, k_n \leq n} (a_{1,k_1} e_{k_1}^*) \wedge \cdots \wedge (a_{n,k_n} e_{k_n}^*) \\ &= \sum_{1 \leq k_1, \dots, k_n \leq n} a_{1,k_1} \cdots a_{n,k_n} e_{k_1}^* \wedge \cdots \wedge e_{k_n}^*\end{aligned}$$

- At this point, we are summing over all possible lists of length n containing the numbers between 1 and n at each index.
  - However, any list in which a number repeats will lead to a wedge product of a linear functional with itself, making that term equal to zero.
  - Thus, it is only necessary to sum over those terms that are non-repeating.
  - But the terms that are non repeating are exactly the permutations $\sigma \in S_n$.
- Thus,

$$\begin{aligned}A^*(f_1^* \wedge \cdots \wedge f_n^*) &= \sum_{\sigma \in S_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} (e_1^* \wedge \cdots \wedge e_n^*)^\sigma \\ &= \sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} e_1^* \wedge \cdots \wedge e_n^* \\ &= \det([a_{i,j}]) e_1^* \wedge \cdots \wedge e_n^*\end{aligned}$$

- If $V = W$ and $e_i = f_i$ $(i = 1, \dots, n)$, then we may define $\omega = e_1^* \wedge \cdots \wedge e_n^* = f_1^* \wedge \cdots \wedge f_n^* \in \Lambda^k(V^*)$ to obtain

$$A^*\omega = \det([a_{i,j}])\omega$$

  which proves that

$$\det(A) = \det([a_{i,j}])$$

  as desired.

- **Orientation** (of ℓ): A choice of one of the disconnected components of $\ell \setminus \{0\}$, where $\ell \subset \mathbb{R}^2$ is a straight line through the origin.
- **Orientation** (of L): A choice of one of the connected components of $L \setminus \{0\}$, where L is a one-dimensional vector space.
- **Positive component** (of $L \setminus \{0\}$): The component chosen in the orientation of L. *Denoted by* $\boldsymbol{L}_+$.
- **Negative component** (of $L \setminus \{0\}$): The component chosen in the orientation of L. *Denoted by* $\boldsymbol{L}_-$.
- **Positively oriented** ($v \in L$): A vector $v \in L$ such that $v \in L_+$.
- **Orientation** (of V) An orientation of the one-dimensional vector space $\Lambda^n(V^*)$, where V is an n-dimensional vector space.
- "One important way of assigning an orientation to V is to choose a basis $e_1, \dots, e_n$ of V. Then if $e_1^*, \dots, e_n^*$ is the dual basis, we can orient $\Lambda^n(V^*)$ by requiring that $e_1^* \wedge \cdots \wedge e_n^*$ be in the positive component of $\Lambda^n(V^*)$" (Guillemin & Haine, 2018, p. 29).
- **Positively oriented** (ordered basis $e_1, \dots, e_n$ of V): An ordered basis $e_1, \dots, e_n \in V$ such that $e_1^* \wedge \cdots \wedge e_n^* \in \Lambda^n(V^*)_+$.
- Proposition 1.9.7: If $e_1, \dots, e_n$ is positively oriented, then $f_1, \dots, f_n$ is positively oriented iff $\det[a_{i,j}] > 0$ where $e_j = \sum_{i=1}^n a_{i,j} f_i$.

Proof. We have that

$$f_1^* \wedge \cdots \wedge f_n^* = \det[a_{i,j}] e_1^* \wedge \cdots \wedge e_n^*$$

□

- Corollary 1.9.8: If $e_1, \dots, e_n$ is a positively oriented basis of V, then the basis

$$e_1, \dots, e_{i-1}, -e_i, e_{i+1}, \dots, e_n$$

is negatively oriented.

- Theorem 1.9.9: Given orientations on V and V/W (where $\dim V = n > 1$, $W \leq V$, and $\dim W = k < n$), one gets from these orientations a natural orientation on W.

Proof. The orientations on V and V/W come prepackaged with a basis. We first apply an orientation to W based on these bases, and then show that any choice of basis for $V, V/W$ induces a basis with the same orientation on W. Let's begin.

Let $r = n - k$, and let $\pi : V \to V/W$. By Exercises 1.2.i and 1.2.ii, we may choose a basis $e_1, \dots, e_n$ of V such that $e_{r+1}, \dots, e_n$ is a basis of W. It follows that $\pi(e_1), \dots, \pi(e_r)$ is a basis of V/W. WLOG[1], take $\pi(e_1), \dots, \pi(e_r)$ and $e_1, \dots, e_n$ to be positively oriented on V/W and V, respectively. Assign to W the orientation associated with $e_{r+1}, \dots, e_n$.

Now suppose $f_1, \dots, f_n$ is another basis of V such that $f_{r+1}, \dots, f_n$ is a basis of W. Let $A = [a_{i,j}]$ express $e_1, \dots, e_n$ as linear combinations of $f_1, \dots, f_n$, i.e., let

$$e_j = \sum_{i=1}^{n} a_{i,j} f_i$$

for all $j = 1, \dots, n$. Now as will be explained below, A must have the form

$$A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

where B is the $r \times r$ matrix expressing $\pi(e_1), \dots, \pi(e_r)$ as linear combinations of $\pi(f_1), \dots, \pi(f_r)$, and D is the $k \times k$ matrix expressing the basis vectors $e_{r+1}, \dots, e_n$ as linear combinations of $f_{r+1}, \dots, f_n$. We have just explained B and D. We don't particularly care about C or have a good way of defining its structure. We can, however, take the block labeled zero to be the $k \times r$ zero matrix by Proposition 1.2.9; in particular, since these components of these vectors will be fed into π and fall within W, they can moved around wherever without altering the identities of the W-cosets to which they pertain. Having justified this structure for A, we see that we can take

$$\det(A) = \det(B) \det(D)$$

It follows by Proposition 1.9.7 as well as the positivity of $\det(A)$ and $\det(B)$ that $\det(D)$ is positive, and hence the orientation of $e_{r+1}, \dots, e_n$ and $f_{r+1}, \dots, f_n$ are one and the same. □

- **Orientation-preserving** (map A): A bijective linear map $A : V_1 \to V_2$, where V_1, V_2 are oriented n-dimensional vector spaces, such that for all $\omega \in \Lambda^n(V_2^*)_+$, we have that $A^*\omega \in \Lambda^n(V_1^*)_+$.
- If $V_1 = V_2$, A is orientation-preserving iff $\det(A) > 0$.
- Proposition 1.9.14: Let V_1, V_2, V_3 be oriented n-dimensional vector spaces, and let $A_1 : V_1 \to V_2$ and $A_2 : V_2 \to V_3$ be bijective linear maps. Then if A_1, A_2 are orientation preserving, so is $A_2 \circ A_1$.

[1] If the first basis is negatively oriented, we may substitute $-e_1$ for e_1. If the second basis is negatively oriented, we may substitute $-e_n$ for e_n.

Week 4

Differential Forms

4.1 Overview of Differential Forms

4/18:

- Office Hours on Wednesday, 4:00-5:00 PM.
- Plan:
  - An impressionistic overview of what (differential) forms do/are.
  - Tangent spaces.
  - Vector fields/integral curves.
  - 1-forms; a warm-up to k-forms.
- Impressionistic overview of the rest of Guillemin and Haine (2018).
  - An open subset $U \subset \mathbb{R}^n$; $n = 2$ and $n = 3$ are nice.
  - Sometimes, we'll have some functions $F : U \to V$; this is where pullbacks come into play.
  - At every point $p \in U$, we'll define a vector space (the tangent space $T_p\mathbb{R}^n$). Associated to that vector space you get our whole slew of associated spaces (the dual space $T_p^*\mathbb{R}^n$, and all of the higher exterior powers $\Lambda^k(T_p^*\mathbb{R}^n)$).
  - We let $\omega \in \Omega^k(U)$ be a k-form in the space of k-forms.
  - ω assigns (smoothly) to every point $p \in U$ an element of $\Lambda^k(T_p^*\mathbb{R}^n)$.
  - Question: What really is a k-form?
    - Answer: Something that can be integrated on k-dimensional subsets.
    - If $k = 1$, i.e., $\omega \in \Omega^1(U)$, then U can be integrated over curves.
  - If we take $k = 0$, then $\Omega^0(U) = C^\infty(U)$, i.e., the set of all smooth functions $f : U \to \mathbb{R}$.
    - Guillemin and Haine (2018) doesn't, but Klug will and we should distinguish between functions $F : U \to V$ and $f : U \to \mathbb{R}$.
  - We will soon construct a map $d : \Omega^0(U) \to \Omega^1(U)$ (the **exterior derivative**) that is rather like the gradient but not quite.
    - d is linear.
    - Maps from vector spaces are heretofore assumed to be linear unless stated otherwise.
  - The 1-forms in $\operatorname{im}(d)$ are special: $\int_\gamma \mathrm{d}f = f(\gamma(b)) - f(\gamma(a))$ only depends on the endpoints of $\gamma : [a, b] \to U$! The integral is *path-independent*.
  - A generalization of this fact is that instead of integrating along the surface M, we can integrate along the boundary curve:
    $$\int_M \mathrm{d}\omega = \int_{\partial M} \omega$$
    This is **Stokes' theorem**.

        - M is a k-dimensional subset of $U \subset \mathbb{R}^n$.
    - Note that we have all manner of functions d that we could differentiate between (because they are functions) but nobody does.

$$0 \to \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(U) \xrightarrow{d} 0$$

    - Theorem: $d^2 = d \circ d = 0$.
        - Corollary: $\operatorname{im}(d^{n-1}) \subset \ker(d^n)$.
    - We'll define $H^k_{dR}(U) = \ker(d)/\operatorname{im}(d)$.
        - These will be finite dimensional, even though all the individual vector spaces will be infinite dimensional.
        - These will tell us about the shape of U; basically, if all of these equal zero, U is simply connected. If some are nonzero, U has some holes.
    - For small values of n and k, this d will have some nice geometric interpretations (div, grad, curl, n'at).
    - We'll have additional operations on forms such as the wedge product.

- **Tangent space** (of p): The following set. *Denoted by* $\boldsymbol{T_p\mathbb{R}^n}$. *Given by*

$$T_p\mathbb{R}^n = \{(p, v) : v \in \mathbb{R}^n\}$$

    - This is naturally a vector space with addition and scalar multiplication defined as follows.

$$(p, v_1) + (p, v_2) = (p, v_1 + v_2) \qquad\qquad \lambda(p, v) = (p, \lambda v)$$

    - The point is that

$$T_p\mathbb{R}^n \neq T_q\mathbb{R}^n$$

    for $p \neq q$ even though the spaces are isomorphic.
    - Aside: $F : U \to V$ differentiable and $p \in U$ induce a map $\mathrm{d}F_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ called the "derivative at p."
        - We will see that the matrix of this map is the Jacobian.
    - Chain rule: If $U \xrightarrow{F} V \xrightarrow{G} W$, then

$$\mathrm{d}(G \circ F)_p = \mathrm{d}G_{F(p)} \circ \mathrm{d}F_p$$

- This is round 1 of our discussion on tangent spaces.
- Round 2, later on, will be submanifolds such as T_pM: The tangent space to a point p of a manifold M.
- **Vector field** (on U): A function that assigns to each $p \in U$ an element of $T_p\mathbb{R}^n$.
    - A constant vector field would be $p \mapsto (p, v)$, visualized as a field of vectors at every p all pointing the same direction. For example, we could take $v = (1, 1)$. *picture*
    - Special case: $v = e_1, e_2, \dots, e_n$. Here we use the notation $e_i = \mathrm{d}/\mathrm{d}x_i$.
    - Example: $n = 2$, $U = \mathbb{R}^2 \setminus \{(0, 0)\}$. We could take a vector field that spins us around in circles.
    - Notice that for all p, $\mathrm{d}/\mathrm{d}x_1 \,|_p, \dots, \mathrm{d}/\mathrm{d}x_n \,|_p \in T_p\mathbb{R}^n$ are a basis.
        - Thus, any vector field v on U can be written uniquely as

$$v = f_1 \frac{\mathrm{d}}{\mathrm{d}x_1} + \cdots + f_n \frac{\mathrm{d}}{\mathrm{d}x_n}$$

        where the $f_1, \dots, f_n$ are functions $f_i : U \to \mathbb{R}$.

4.2 The Lie Derivative and 1-Forms

4/20:
- Plan:
  - Vector fields and their integral curves.
  - Lie derivatives.
  - 1-forms and k-forms.
  - $\Omega^0(U) \xrightarrow{d} \Omega^1(U)$.
- Notation.
  - $U \subset \mathbb{R}^n$.
  - v denotes a vector field on U.
    - Note that the set of all vector fields on U constitute the vector space ??.
  - $v_p \in T_p\mathbb{R}^n$.
  - $\omega_p \in \Lambda^k(T_p^*\mathbb{R}^n)$.
  - $\mathrm{d}/\mathrm{d}x_i \mid_p = (p, e_i) \in T_p\mathbb{R}^n$.
- Recall that any vector field v on U can be written uniquely as

$$v = g_1 \frac{\mathrm{d}}{\mathrm{d}x_1} + \cdots + g_n \frac{\mathrm{d}}{\mathrm{d}x_n}$$

  where the $g_i : U \to \mathbb{R}$.
- **Smooth** (vector field): A vector field v for which all g_i are smooth.
- From now on, we assume unless stated otherwise that all vector fields are smooth.
- **Lie derivative** (of f wrt. v): The function $L_v f : U \to \mathbb{R}$ defined by $p \mapsto D_{v_p}(f)(p)$, where v is a vector field on U and $f : U \to \mathbb{R}$ (always smooth).
  - Recall that $D_{v_p}(f)(p)$ denotes the directional derivative of f in the direction v_p at p.
  - As some examples, we have

$$L_{\mathrm{d}/\mathrm{d}x_i} f = \frac{\mathrm{d}f}{\mathrm{d}x_i} \qquad\qquad L_{(g_1 \frac{\mathrm{d}}{\mathrm{d}x_1} + \cdots + g_n \frac{\mathrm{d}}{\mathrm{d}x_n})} f = g_1 \frac{\mathrm{d}f}{\mathrm{d}x_1} + \cdots + g_n \frac{\mathrm{d}f}{\mathrm{d}x_n}$$

- Property.
  1. Product rule: $L_v(f_1 f_2) = (L_v f_1) f_2 + f_1 (L_v f_2)$.
- Later: Geometric meaning to the expression $L_v f = 0$.
  - Satisfied iff f is constant on the integral curves of v. As if f "flows along" the vector field.
- We define $T_p^*\mathbb{R}^n = (T_p\mathbb{R}^n)^*$.
- 1-forms:
  - A (differential) 1-form on $U \subset \mathbb{R}^n$ is a function $\omega : p \mapsto \omega_p \in T_p^*\mathbb{R}^n$.
  - A "co-vector field"
- Notation: $\mathrm{d}x_i$ is the 1-form that at p is $(p, e_i^*) \in T_p^*\mathbb{R}^n$.
- For example, if $U = \mathbb{R}^2$ and $\omega = \mathrm{d}x_1$, then we have the vector field of "unit vectors pointing to the right at each point."

- Note: Given any 1-form ω on U, we can write ω uniquely as

$$\omega = g_1 \, \mathrm{d}x_1 + \cdots + g_n \, \mathrm{d}x_n$$

for some set of smooth $g_i : U \to \mathbb{R}$.

- Notation:
  - $\Omega^1(U)$ is the set of all smooth 1-forms.
  - Notice that $\Omega^1(U)$ is a vector space.
- Given $\omega \in \Omega^1(U)$ and a vector field v on U, we can define $\omega(v) : U \to \mathbb{R}$ by $p \mapsto \omega_p(v_p)$.
- If $U = \mathbb{R}^2$, we have that

$$\mathrm{d}x\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) = 1 \qquad \mathrm{d}x\left(\frac{\mathrm{d}}{\mathrm{d}y}\right) = 0$$

- Note that $\mathrm{d}x \, , \mathrm{d}y$ are not a basis for $\Omega^1(U)$ since the latter is infinite dimensional.
- Exterior derivative for 0/1 forms.
  - Let $d : \Omega^0(U) \to \Omega^1(U)$ take $f : U \to \mathbb{R}$ to $\frac{\partial f}{\partial x_1} \, \mathrm{d}x_1 + \cdots + \frac{\partial f}{\partial x_n} \, \mathrm{d}x_n$.
  - This represents the gradient as a 1-form.
- Check:
  1. Linear.
  2. $\mathrm{d}x_i = \mathrm{d}(x_i)$, where $x_i : \mathbb{R}^n \to \mathbb{R}$ is the i^{th} coordinate function.

4.3 Integral Curves

4/22:

- Plan:
  - Clear up a bit of notational confusion.
  - Discuss integral curves of vectors fields.
  - k-forms.
  - Exterior derivatives $d : \Omega^k(U) \to \Omega^{k+1}(U)$ (definition and properties).
- Notation:
  - $F : \mathbb{R}^n \to \mathbb{R}^m$ smooth.
  - We are used to denoting derivatives by big D: $DF_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$ where bases of the two spaces are $e_1, \dots, e_n$ and $e_1, \dots, e_m$ has matrix equal to the Jacobian:

$$[DF_p] = \left[\frac{\mathrm{d}F_i}{\mathrm{d}x_j}(p)\right]$$

  - The book often uses small d: $f : U \to \mathbb{R}$ has $df_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}$, where the latter set is isomorphic to $\mathbb{R}$.
  - $df : p \to df_p \in T_p^*\mathbb{R}^n$.
  - Klug said

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \, \mathrm{d}x_i$$

  - Homework 1 defined $\mathrm{d}f = df$?

  - Sometimes three perspectives help you keep this all straight:
    1. Abstract nonsense: The definition of the derivative.
    2. How do I compute it: Apply the formula.
    3. What is it: E.g., magnitude of the directional derivative in the direction of steepest ascent.
- For the homework,
  - Let ω be a 1-form in $\Omega^1(U)$.
  - Let $\gamma : [a,b] \to U$ be a curve in U.
  - Then $\mathrm{d}\gamma_p = \gamma'_p : T_p\mathbb{R} \to T_{\gamma(p)}\mathbb{R}^n$ is a function that takes in points of the curve and spits out tangent vectors.
  - Integrating swallows 1-forms and spits out numbers.
$$\int_\gamma \omega = \int_a^b \omega(\gamma'(t))\,\mathrm{d}t$$
  - Problem: If $\omega = \mathrm{d}f$, then
$$\int_\gamma \omega = f(\gamma(b)) - f(\gamma(a))$$
    regardless of the path.
  - Question: Given a 1-form ω, is $\omega = \mathrm{d}f$ for some f?
  - Homework: Explicit U, ω, closed γ such that $\int_\gamma \omega \neq 0$ implies that $\omega \neq \mathrm{d}f$. This motivates and leads into the de Rham cohomology.
- Aside: It won't hurt (for now) to think of 1-forms as vector fields.
- Integral curves: Let $U \subset \mathbb{R}^n$, v be a (smooth) vector field on U. A curve $\gamma : (a,b) \to U$ is an **integral curve** for v if $\gamma'(t) = v_{\gamma(t)}$.
- Examples:
  - If $U = \mathbb{R}^2$ and $\gamma = \mathrm{d}/\mathrm{d}x$, then the integral curve is the line from left to right traveling at unit speed. The curve has to always have as it's tangent vector the unit vector pointing right (which is the vector at every point in the vector field).
  - Vector fields flow everything around. An integral curve is the trajectory of a particle subjected to the vector field as a force field.
- Main points:
  1. These integral curves always exist (locally) and often exist globally (cases in which they do are called **complete vector fields**).
  2. They are unique given a starting point $p \in U$.
- An incomplete vector field is one such as the "all roads lead to Rome" vector field where everything always points inward. This is because integral curves cannot be defined for all "time" (real numbers, positive and negative).
- The proofs are in the book; they require an existence/uniqueness result for ODEs and the implicit function theorem.
- Aside: $f : U \to \mathbb{R}$, v a vector field, implies that $L_v f = 0$ means that f is constant along all the integral curves of v. This also means that f is integral for v.
- **Pullback** (of 1-forms): If $F : U \to V$, $d : \Omega^0(U) \to \Omega^1(U)$, and $d : \Omega^0(V) \to \Omega^1(V)$, then we get an induced map $F^* : \Omega^0(V) \to \Omega^0(U)$. If $f : V \to \mathbb{R}$, then $f \circ F$ is involved.
  - We're basically saying that if we have $\mathrm{Hom}(A, X)$ (the set of all functions from A to X) and $\mathrm{Hom}(B, X)$, then if we have $F : A \to B$, we get an induced map $F^* : \mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X)$ that is precomposed with F.

4.4 Chapter 2: Differential Forms

From Guillemin and Haine (2018).

5/5:
- Goals for this chapter.
  - Generalize to n dimensions the basic operations of 3D vector calculus (**divergence**, **gradient**, and **curl**).
    - div and grad are pretty straightforward, but curl is more subtle.
  - Substitute **differential forms** for **vector fields** to discover to a natural generalization of the operations, in particular, where all three operations are special cases of **exterior differentiation**.
- Introducing vector fields and their dual objects (**one-forms**).
- **Tangent space** (to $\mathbb{R}^n$ at p): The set of pairs (p,v) for all $v\in\mathbb{R}^n$. *Denoted by* $\boldsymbol{T_p\mathbb{R}^n}$. *Given by*

$$T_p\mathbb{R}^n = \{(p,v) \mid v\in\mathbb{R}^n\}$$

- Operations on the tangent space.
  - Directly, we identify $T_p\mathbb{R}^n\cong\mathbb{R}^n$ by $(p,v)\mapsto v$ to make $T_p\mathbb{R}^n$ a vector space.
  - Explicitly, we define

$$(p,v_1)+(p,v_2)=(p,v_1+v_2) \qquad\qquad \lambda(p,v)=(p,\lambda v)$$

    for all $v,v_1,v_2\in\mathbb{R}^n$ and $\lambda\in\mathbb{R}$.
- **Derivative** (of f at p): The linear map from $\mathbb{R}^n\to\mathbb{R}^m$ defined by the following $m\times n$ matrix, where $U\subset\mathbb{R}^n$ is open and $f:U\to\mathbb{R}^m$ is a C^1-mapping. *Denoted by* $\boldsymbol{Df(p)}$. *Given by*

$$Df(p)=\left[\frac{\partial f_i}{\partial x_j}(p)\right]$$

- $\mathbf{d}\boldsymbol{f_p}$: The linear map from $T_p\mathbb{R}^n\to T_q\mathbb{R}^m$ defined as follows, where $U\subset\mathbb{R}^n$ open, $f:U\to\mathbb{R}^m$ is a C^1-mapping, and $q=f(p)$. *Given by*

$$\mathrm{d}f_p(p,v)=(q,Df(p)v)$$

  - Guillemin and Haine (2018) also refer to this as the "base-pointed" version of the derivative of f at p.
- The chain rule for the base-pointed version, where $U\subset\mathbb{R}^n$ open, $f:U\to\mathbb{R}^n$ is a C^1-mapping, $\operatorname{im}(f)\subset V$ open, and $g:V\to\mathbb{R}^k$ is a C^1-mapping.

$$\mathrm{d}g_q\circ\mathrm{d}f_p=\mathrm{d}(f\circ g)_p$$

- **Vector field** (on $\mathbb{R}^3$): A function which attaches to each point $p\in\mathbb{R}^3$ a base-pointed arrow $(p,v)\in T_p\mathbb{R}^3$.
  - These vector fields are the typical subject of vector calculus.
- **Vector field** (on U): A function which assigns to each point $p\in U$ a vector in $T_p\mathbb{R}^n$, where $U\subset\mathbb{R}^n$ is open. *Denoted by* $\boldsymbol{v}$.
  - We denote the value of $\boldsymbol{v}$ at p by either $\boldsymbol{v}(p)$ or $\boldsymbol{v}_p$.
- **Constant** (vector field): A vector field of the form $p\mapsto(p,v)$, where $v\in\mathbb{R}^n$ is fixed.
- $\boldsymbol{\partial/\partial x_i}$: The constant vector field having $v=e_i$.

- $\boldsymbol{fv}$: The vector field defined on U as follows, where $f : U \to \mathbb{R}$. *Given by*

$$p \mapsto f(p)\boldsymbol{v}(p)$$

  - Note that we are invoking our definition of scalar multiplication on $T_p\mathbb{R}^n$ here.

- **Sum** (of $\boldsymbol{v}_1, \boldsymbol{v}_2$): The vector field on U defined as follows. *Denoted by* $\boldsymbol{v_1 + v_2}$. *Given by*

$$p \mapsto \boldsymbol{v}_1(p) + \boldsymbol{v}_2(p)$$

  - Note that we are invoking our definition of addition on $T_p\mathbb{R}^n$ here.

- The list of vectors $(\partial/\partial x_1)_p, \dots, (\partial/\partial x_n)_p$ constitutes a basis of $T_p\mathbb{R}^n$.
  - Recall that $(\partial/\partial x_i)_p = (p, e_i)$.
  - Thus, if $\boldsymbol{v}$ is a vector field on U, it has a unique decomposition

$$\boldsymbol{v} = \sum_{i=1}^n g_i \frac{\partial}{\partial x_i}$$

    where each $g_i : U \to \mathbb{R}$.

- $\boldsymbol{C^\infty}$ (vector field): A vector field such that $g_i \in C^\infty(U)$ for all g_i's in its unique decomposition.

- **Lie derivative** (of f with respect to $\boldsymbol{v}$): The function from $U \to \mathbb{R}$ defined as follows, where $U \subset \mathbb{R}^n$, $f : U \to \mathbb{R}$ is a C^1-mapping, and $\boldsymbol{v}(p) = (p, v)$. *Denoted by* $\boldsymbol{L_v f}$. *Given by*

$$L_{\boldsymbol{v}} f(p) = Df(p)v$$

  - A more explicit formula for the Lie derivative is

$$L_{\boldsymbol{v}} f = \sum_{i=1}^n g_i \frac{\partial f}{\partial x_i}$$

  - The vector field decides the direction in which we take the derivative at each point. Instead of having to take a derivative everywhere in one direction at a time, we can now take a derivative in a different direction at every point!

- Lemma 2.1.11: Let U be an open subset of $\mathbb{R}^n$, $\boldsymbol{v}$ a vector field on U, and $f_1, f_2 \in C^1(U)$. Then

$$L_{\boldsymbol{v}}(f_1 \cdot f_2) = L_{\boldsymbol{v}}(f_1) \cdot f_2 + f_1 \cdot L_{\boldsymbol{v}}(f_2)$$

  Proof. See Exercise 2.1.ii. □

- **Cotangent space** (to $\mathbb{R}^n$ at p): The dual vector space to $T_p\mathbb{R}^n$. *Denoted by* $\boldsymbol{T_p^*\mathbb{R}^n}$. *Given by*

$$T_p^*\mathbb{R}^n = (T_p\mathbb{R}^n)^*$$

- **Cotangent vector** (to $\mathbb{R}^n$ at p): An element of $T_p^*\mathbb{R}^n$.

- **Differential one-form** (on U): A function which assigns to each point $p \in U$ a cotangent vector. *Also known as* **one-form** (on U). *Denoted by* $\boldsymbol{\omega}$. *Given by*

$$p \mapsto \omega_p$$

- Note that by identifying $T_p\mathbb{R} \cong \mathbb{R}$, we have that $\mathrm{d}f_p \in T_p^*\mathbb{R}^n$, assuming that $f : U \to \mathbb{R}$.

- Geometric example: Consider $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f \in C^1$. By the latter condition, we know that the graph of f is a "smooth" surface in $\mathbb{R}^3$, i.e., one without any abrupt changes in derivative (consider the graph of the piecewise function defined by $-x^2$ for $x < 0$ and x^2 for $x \geq 0$, for example). What $\mathrm{d}f_p$ does is take a point (p_1, p_2, q), where $q = f(p)$, on the surface and a vector v with tail at (p, q), and give us a number representing the magnitude of the instantaneous change of f at p in the direction v. Thus, $\mathrm{d}f_p$ contains, in a sense, all of the information concerning the rate of change of f at p.

- $\boldsymbol{\mathrm{d}f}$: The one-form on U defined as follows. *Given by*

$$p \mapsto \mathrm{d}f_p$$

  - Continuing with the geometric example: What $\mathrm{d}f$ does is take every point p across the surface and return all of the information concerning the rate of change of f at p (packaged neatly by $\mathrm{d}f_p$).

- **Pointwise product** (of ϕ with ω): The one-form on U defined as follows, where $\phi : U \to \mathbb{R}$ and ω is a one-form. *Denoted by* $\boldsymbol{\phi\omega}$. *Given by*

$$(\phi\omega)_p = \phi(p)\omega_p$$

- **Pointwise sum** (of ω_1, ω_2): The one-form on U defined as follows. *Denoted by* $\boldsymbol{\omega_1 + \omega_2}$. *Given by*

$$(\omega_1 + \omega_2)_p = (\omega_1)_p + (\omega_2)_p$$

- $\boldsymbol{x_i}$: The function from $U \to \mathbb{R}$ defined as follows. *Given by*

$$x_i(u_1, \dots, u_n) = u_i$$

  - x_i is constantly increasing in the x_i-direction, and constant in every other direction.

- $\boldsymbol{(\mathrm{d}x_i)_p}$: The linear map from $T_p\mathbb{R}^n \to \mathbb{R}$ (i.e., the cotangent vector in $T_p^*\mathbb{R}^n$) defined as follows. *Given by*

$$(\mathrm{d}x_i)_p(p, a_1x_1 + \cdots + a_nx_n) = a_1$$

  - Naturally, the instantaneous change in x_i at any point p in the direction $\boldsymbol{v}(p)$ will just be the magnitude of $\boldsymbol{v}(p)$ in the x_i-direction.
  - It follows immediately that

$$(\mathrm{d}x_i)_p \left(\frac{\partial}{\partial x_j} \right)_p = \delta_{ij}$$

- Consequently, the list of cotangent vectors $(\mathrm{d}x_1)_p, \dots, (\mathrm{d}x_n)_p$ constitutes a basis of $T_p^*\mathbb{R}^n$ that is **dual** to the basis $(\partial/\partial x_1)_p, \dots, (\partial/\partial x_n)_p$ of $T_p\mathbb{R}^n$.

- $\boldsymbol{\mathrm{d}x_i}$: The one-form on U defined as follows. *Given by*

$$p \mapsto (\mathrm{d}x_i)_p$$

  - Thus, if $\omega_p : T_p\mathbb{R}^n \to \mathbb{R}$, it has a unique decomposition

$$\omega_p = \sum_{i=1}^{n} f_i(p)(\mathrm{d}x_i)_p$$

    where every $f_i : U \to \mathbb{R}$.
  - Similarly, $\omega : U \to T_p^*\mathbb{R}^n$ has a unique decomposition

$$\omega = \sum_{i=1}^{n} f_i \mathrm{d}x_i$$

- **Smooth** (one-form): A one form for which the associated functions $f_1, \dots, f_n \in C^\infty$. *Also known as* $\boldsymbol{C^\infty}$ (one-form).
- Lemma 2.1.18: Let U be an open subset of $\mathbb{R}^n$. If $f : U \to \mathbb{R}$ is a C^∞ function, then

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\,\mathrm{d}x_i$$

- **Interior product** (of $\boldsymbol{v}$ with ω): The function which combines a point $p \in U$, the vector $\boldsymbol{v}(p) \in T_p\mathbb{R}^n$, and the functional $\omega_p \in T_p^*\mathbb{R}^n$ to yield a real number. *Denoted by* $\boldsymbol{\iota_{v(p)}\omega_p}$.
- Examples.
  - If

$$\boldsymbol{v} = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i} \qquad\qquad \omega = \sum_{i=1}^{n} f_i \mathrm{d}x_i$$

    then

$$\iota_{\boldsymbol{v}}\omega = \sum_{i=1}^{n} f_i g_i$$

    - We use multiplication and the fact that $(\mathrm{d}x_i)_p(\,\partial/\partial x_j\,)_p = \delta_{ij}$ to obtain this result.
    - If $\boldsymbol{v}, \omega \in C^\infty$, so is $\iota_{\boldsymbol{v}}\omega$, where C^∞ refers to three different sets of smooth objects (vector fields, one-forms, and functions, respectively[1]).
  - As with f, if $\phi \in C^\infty(U)$, then

$$\mathrm{d}\phi = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}\mathrm{d}x_i$$

  - It follows if $\boldsymbol{v}$ is defined as in the first example that

$$\iota_{\boldsymbol{v}}\mathrm{d}\phi = \sum_{i=1}^{n} g_i \frac{\partial \phi}{\partial x_i} = L_{\boldsymbol{v}}\phi$$

- **Integral curve** (of $\boldsymbol{v}$): A C^1 curve $\gamma : (a, b) \to U$ such that for all $t \in (a, b)$,

$$\boldsymbol{v}(\gamma(t)) = (\gamma(t), \tfrac{\mathrm{d}\gamma}{\mathrm{d}t}(t))$$

  where $U \subset \mathbb{R}^n$ is open and $\boldsymbol{v}$ is a vector field on U.
  - An equivalent condition if $\boldsymbol{v} = \sum_{i=1}^{n} g_i \; \mathrm{d}/\mathrm{d}x_i$ and $g : U \to \mathbb{R}^n$ is defined by $(g_1, \dots, g_n)$ is that γ satisfies the system of differential equations

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = g(\gamma(t))$$

- Theorem 2.2.4 (existence of integral curves): Let $U \subset \mathbb{R}^n$ open, $\boldsymbol{v}$ a vector field on U. If $p_0 \in U$ and $a \in \mathbb{R}$, then there exist $I = (a - T, a + T)$ for some $T \in \mathbb{R}$, $U_0 = N_r(p_0) \subset U$, and $\gamma_p : I \to U$ such that $\gamma_p(a) = p$ for all $p \in U_0$.
- Theorem 2.2.5 (uniqueness of integral curves): Let $U \subset \mathbb{R}^n$ open, $\boldsymbol{v}$ a vector field on U, and $\gamma_1 : I_1 \to U$ and $\gamma_2 : I_2 \to U$ integral curves for $\boldsymbol{v}$. If $a \in I_1 \cap I_2$ and $\gamma_1(a) = \gamma_2(a)$, then

$$\gamma_1|_{I_1 \cap I_2} = \gamma_2|_{I_1 \cap I_2}$$

  and the curve $\gamma : I_1 \cup I_2 \to U$ defined by

$$\gamma(t) = \begin{cases} \gamma_1(t) & t \in I_1 \\ \gamma_2(t) & t \in I_2 \end{cases}$$

  is an integral curve for $\boldsymbol{v}$.

[1] Technically, these objects are all types of functions, though, so it is fair to call them all smooth.

- Theorem 2.2.6 (smooth dependence on initial data): Let $V \subset U \subset \mathbb{R}^n$ open, $\boldsymbol{v}$ a C^∞-vector field on V, $I \subset \mathbb{R}$ an open interval, and $a \in I$. Let $h : V \times I \to U$ have the following properties.
  1. $h(p, a) = p$.
  2. For all $p \in V$, the curve $\gamma_p : I \to U$ defined by $\gamma_p(t) = h(p, t)$ is an integral curve of $\boldsymbol{v}$.

  Then $h \in C^\infty$.
- **Autonomous** (system of ODEs): A system of ODEs that does not explicitly depend on the independent variable.
- $\mathrm{d}\gamma/\mathrm{d}t = g(\gamma(t))$ is autonomous since g does not depend on t.
- Theorem 2.2.7: Let $I = (a, b)$. For all $c \in \mathbb{R}$, define $I_c = (a - c, b - c)$. If $\gamma : I \to U$ is an integral curve, then the reparameterized curve $\gamma_c : I_c \to U$ defined by
$$\gamma_c(t) = \gamma(t + c)$$
  is an integral curve.
  - Note that this is truly just a reparameterization; we still have, for instance,
$$\gamma_c(a - c) = \gamma(a - c + c) = \gamma(a) \qquad\qquad \gamma_c(b - c) = \gamma(b - c + c) = \gamma(b)$$
- **Integral** (of the system $\mathrm{d}\gamma/\mathrm{d}t = g(\gamma(t))$): A C^1-function $\phi : U \to \mathbb{R}$ such that for every integral curve $\gamma(t)$, the function $t \mapsto \phi(\gamma(t))$ is constant.
  - An alternate condition is that for all t,
$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\phi(\gamma(t)) = (D\phi)_{\gamma(t)}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right) = (D\phi)_{\gamma(t)}(v) = L_v\phi(p)$$
    where $\boldsymbol{v}(p) = (p, v)$.
- Theorem 2.2.9: Let $U \subset \mathbb{R}^n$ open, $\phi \in C^1(u)$. Then ϕ is an integral of the system $\mathrm{d}\gamma/\mathrm{d}t = g(\gamma(t))$ iff $L_{\boldsymbol{v}}\phi = 0$.
- **Complete** (vector field): A vector field $\boldsymbol{v}$ on U such that for every $p \in U$, there exists an integral curve $\gamma : \mathbb{R} \to U$ with $\gamma(0) = p$.
  - Alternatively, for every p, there exists an integral curve that starts at p and exists for all time.
- **Maximal** (integral curve): An integral curve $\gamma : [0, b) \to U$ with $\gamma(0) = p$ such that it cannot be extended to an interval $[0, b')$ with $b' > b$.
- For a maximal curve, either...
  1. $b = +\infty$;
  2. $|\gamma(t)| \to +\infty$ as $t \to b$;
  3. The limit set of $\{\gamma(t) \mid 0 \leq t < b\}$ contains points on the boundary of U.
- Eliminating 2 and 3, as can be done with the following lemma, provides a means of proving that γ exists for all positive time.
- Lemma 2.2.11: The scenarios 2 and 3 above cannot happen if there exists a proper C^1-function $\phi : U \to \mathbb{R}$ with $L_{\boldsymbol{v}}\phi = 0$.

  Proof. Suppose there exists $\phi \in C^1$ such that $L_{\boldsymbol{v}}\phi = 0$. Then ϕ is constant on $\gamma(t)$ (say with value $c \in \mathbb{R}$) by definition. But then since $\{c\} \subset \mathbb{R}$ is compact and $\phi \in C^1$, $\phi^{-1}(c) \subset U$ is compact and, importantly, contains $\operatorname{im}(\gamma)$. The compactness of this set implies that γ can neither "run off to infinity" as in scenario 2 or "run off the boundary" as in scenario 3. $\square$

- Theorem 2.2.12: If there exists a proper C^1-function $\phi : U \to \mathbb{R}$ with the property $L_{\boldsymbol{v}}\phi = 0$, then the vector field $\boldsymbol{v}$ is complete.

  Proof. Apply a similar argument to the interval $(-b, 0]$ and join the two results. □

- Example: Let $U = \mathbb{R}^2$ and let $\boldsymbol{v}$ be the vector field
$$\boldsymbol{v} = x^3 \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$$
  Then $\phi(x, y) = 2y^2 + x^4$ is a proper function with the above property.
  - Note that indeed, as per Theorem 2.2.12, we have that
$$\begin{aligned} L_{\boldsymbol{v}}\phi &= x^3 \frac{\partial \phi}{\partial y} - y \frac{\partial \phi}{\partial x} \\ &= x^3 \cdot 4y - y \cdot 4x^3 \\ &= 0 \end{aligned}$$
- We now build up to an alternate completeness condition (Theorem 2.2.15).
- **Support** (of $\boldsymbol{v}$): The following set. *Denoted by* **supp ($\boldsymbol{v}$)**. *Given by*
$$\operatorname{supp}(\boldsymbol{v}) = \overline{\{q \in U \mid \boldsymbol{v}(q) \neq 0\}}$$
- **Compactly supported** (vector field $\boldsymbol{v}$): A vector field $\boldsymbol{v}$ for which $\operatorname{supp}(\boldsymbol{v})$ is compact.
- Theorem 2.2.15: If $\boldsymbol{v}$ is compactly supported, then $\boldsymbol{v}$ is complete.

  Proof. Let $p \in U$ be such that $\boldsymbol{v}(p) = 0$. Define $\gamma_0 : (-\infty, \infty) \to U$ by $\gamma_0(t) = p$ for all $t \in (-\infty, \infty)$. Since
$$\frac{\mathrm{d}\gamma_0}{\mathrm{d}t} = 0 = \boldsymbol{v}(p) = \boldsymbol{v}(\gamma(t))$$
  we know that γ_0 is an integral curve of $\boldsymbol{v}$.

  Now consider an arbitrary integral curve $\gamma : (-a, b) \to U$ having the property $\gamma(t_0) = p$ for some $t_0 \in (-a, b)$. It follows by Theorem 2.2.5 that γ and γ_0 coincide on the interval $(-a, a)$.

  By hypothesis, $\operatorname{supp}(\boldsymbol{v})$ is compact. Basic set theory tells us that for γ arbitrary, either $\gamma(t) \in \operatorname{supp}(\boldsymbol{v})$ for all t or there exists t_0 such that $\gamma(t_0) \in U \setminus \operatorname{supp}(\boldsymbol{v})$. But then by the definition of $\operatorname{supp}(\boldsymbol{v})$, $\boldsymbol{v}(\gamma(t_0)) = 0$. Thus, letting $p = \gamma(t_0)$, we have an associated γ_0 that γ "runs along" while outside the support. It follows that in either case, γ cannot go off to ∞ or go off the boundary of U as $t \to b$. □

- **Bump function**: A function $f : \mathbb{R}^n \to \mathbb{R}$ which is both smooth and compactly supported.
- $\boldsymbol{C_0^\infty(\mathbb{R}^n)}$: The vector space of all bump functions with domain $\mathbb{R}^n$.
- An application of Theorem 2.2.15.
  - Suppose $\boldsymbol{v}$ is a vector field on U and we want to inspect the integral curves of $\boldsymbol{v}$ on some $A \subset U$ compact. Let $\rho \in C_0^\infty(U)$ be such that $\rho(p) = 1$ for all $p \in N_r(A)$, where $N_r(A)$ is some neighborhood of the set A. Then the vector field $\boldsymbol{w} = \rho\boldsymbol{v}$ is compactly supported and hence complete. However, it is also identical to $\boldsymbol{v}$ on A, so its integral curves on A coincide with those of $\boldsymbol{v}$ on A.
- $\boldsymbol{f_t}$: The map from $U \to U$ defined as follows, where $\boldsymbol{v}$ is complete. *Given by*
$$f_t(p) = \gamma_p(t)$$

- Note that it is the fact that $\boldsymbol{v}$ is complete that justifies the existence of an integral curve $\gamma_p : \mathbb{R} \to U$ with $\gamma_p(0) = p$ for all $p \in U$.

- Properties of f_t.
  1. $\boldsymbol{v} \in C^\infty$ implies $f_t \in C^\infty$.

     Proof. By Theorem 2.2.6. □
  2. $f_0 = \mathrm{id}_U$.

     Proof. We have
     $$f_0(p) = \gamma_p(0) = p = \mathrm{id}_U(p)$$
     as desired. □
  3. $f_t \circ f_a = f_{t+a}$.

     Proof. Let $q = f_a(p)$. Since $\boldsymbol{v}$ is complete and $q \in U$, there exists γ_q such that $\gamma_q(0) = q$. It follows that $\gamma_p(a) = f_a(p) = q = \gamma_q(0)$. Thus, by Theorem 2.2.7, $\gamma_q(t)$ and $\gamma_p(t+a)$ are both integral curves of $\boldsymbol{v}$ with the same initial point. Therefore,
     $$(f_t \circ f_a)(p) = f_t(q) = \gamma_q(t) = \gamma_p(t+a) = f_{t+a}(p)$$
     for all t, as desired. □
  4. $f_t \circ f_{-t} = \mathrm{id}_U$.

     Proof. See properties 2 and 3. □
  5. $f_{-t} = f_t^{-1}$.

     Proof. See property 4. □

- Thus, f_t is a C^∞ **diffeomorphism**.
  - "Hence, if $\boldsymbol{v}$ is complete, it generates a **one-parameter group** f_t ($-\infty < t < \infty$) of C^∞-diffeomorphisms of U" (Guillemin & Haine, 2018, p. 40).

- **Diffeomorphism**: An isomorphism of smooth manifolds. In particular, it is an invertible function that maps one differentiable manifold to another such that both the function and its inverse are differentiable.

- **One-parameter group**: A continuous group homomorphism $\varphi : \mathbb{R} \to G$ from the real line $\mathbb{R}$ (as an additive group) to some other topological group G.

- If $\boldsymbol{v}$ is not complete, there is an analogous result, but it is trickier to formulate.

- ***f*-related** (vector fields $\boldsymbol{v}, \boldsymbol{w}$): Two vector fields $\boldsymbol{v}, \boldsymbol{w}$ such that
  $$\mathrm{d}f_p(\boldsymbol{v}(p)) = \boldsymbol{w}(f(p))$$
  for all $p \in U$, where $\boldsymbol{v}$ is a C^∞-vector field on $U \subset \mathbb{R}^n$ open, $\boldsymbol{w}$ is a C^∞-vector field on $W \subset \mathbb{R}^m$ open, and $f : U \to W$ is a C^∞ map.
  - An alternate formulation is that in terms of coordinates,
    $$w_i(q) = \sum_{j=1}^n \frac{\partial f_i}{\partial x_j} v_j(p)$$
    where
    $$\boldsymbol{v} = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i} \boldsymbol{w} \qquad = \sum_{j=1}^m w_j \frac{\partial}{\partial y_i}$$
    for $v_i \in C^k(U)$ and $w_j \in C^k(W)$.

- If $m = n$ and f is a C^∞ diffeomorphism, then $\boldsymbol{w}$ is the vector field defined by the equation

$$w_i = \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j} v_j \right) \circ f^{-1}$$

- Theorem 2.2.18: If $f : U \to W$ is a C^∞ diffeomorphism and $\boldsymbol{v}$ is a C^∞ vector field on U, then there exists a unique C^∞ vector field $\boldsymbol{w}$ on W having the property that $\boldsymbol{v}$ and $\boldsymbol{w}$ are f-related.

  Proof. See the above. $\square$

- **Pushforward** (of $\boldsymbol{v}$ by f): The vector field $\boldsymbol{w}$ shown to exist by Theorem 2.2.18. *Denoted by* $\boldsymbol{f_* v}$.
- Theorem 2.2.20: Let $U_1, U_2 \subset \mathbb{R}^n$ open, $\boldsymbol{v}_1, \boldsymbol{v}_2$ vector fields on U_1, U_2, and $f : U_1 \to U_2$ a C^∞ map. If $\boldsymbol{v}_1, \boldsymbol{v}_2$ are f-related, every integral curve $\gamma : I \to U_1$ of $\boldsymbol{v}_1$ gets mapped by f onto an integral curve $f \circ \gamma : I \to U_2$ of $\boldsymbol{v}_2$.

  Proof. We want to show that

$$\boldsymbol{v}_2((f \circ \gamma)(t)) = \left((f \circ \gamma)(t),\, \tfrac{\mathrm{d}}{\mathrm{d}t} (f \circ \gamma) \big|_t \right)$$

  We are given that

$$\boldsymbol{v}_1(\gamma(t)) = \left(\gamma(t),\, \tfrac{\mathrm{d}\gamma}{\mathrm{d}t} \Big|_t \right) \qquad\qquad \mathrm{d}f_p(\boldsymbol{v}_1(p)) = \boldsymbol{v}_2(f(p))$$

  Let $p = \gamma(t)$ and $q = f(p)$. Then

$$\begin{aligned}
\boldsymbol{v}_2((f \circ \gamma)(t)) &= \boldsymbol{v}_2(f(p)) \\
&= \mathrm{d}f_p(\boldsymbol{v}_1(p)) \\
&= \mathrm{d}f_p(\boldsymbol{v}_1(\gamma(t))) \\
&= \mathrm{d}f_p \left(\gamma(t),\, \tfrac{\mathrm{d}\gamma}{\mathrm{d}t} \Big|_t \right) \\
&= \mathrm{d}f_p \left(p,\, \tfrac{\mathrm{d}\gamma}{\mathrm{d}t} \Big|_t \right) \\
&= \left(q, Df(p) \left(\tfrac{\mathrm{d}\gamma}{\mathrm{d}t} \Big|_t \right) \right) \\
&= \left((f \circ \gamma)(t),\, \tfrac{\mathrm{d}}{\mathrm{d}t} (f \circ \gamma) \big|_t \right)
\end{aligned}$$

  as desired. $\square$

- Corollary 2.2.21: In the setting of Theorem 2.2.20, suppose $\boldsymbol{v}_1, \boldsymbol{v}_2$ are complete. Let $(f_{i,t})_{t \in \mathbb{R}} : U_i \to U_i$ be the one-parameter group of diffeomorphisms generated by $\boldsymbol{v}_i$. Then

$$f \circ f_{1,t} = f_{2,t} \circ f$$

  Proof. We have that

$$(f \circ f_{1,t})(p) = (f \circ \gamma_p)(t)$$

  By Theorem 2.2.20, the above is an integral curve of $\boldsymbol{v}_2$. Let $f(p) = q$. Then

$$\begin{aligned}
(f_{2,t} \circ f)(p) &= f_{2,t}(q) \\
&= \gamma_q(t)
\end{aligned}$$

  ...

  Guillemin and Haine (2018) proves that if $\phi \in C^\infty(U_2)$ and $f^* \phi = \phi \circ f$, then

$$f^* L_{\boldsymbol{v}_2} \phi = L_{\boldsymbol{v}_1} f^* \phi$$

  by virtue of the observations that if $f(p) = q$, then at the point p, the right-hand side above is $(\mathrm{d}\phi)_q \circ \mathrm{d}f_p(\boldsymbol{v}_1(p))$ by the chain rule and by definition the left hand side is $\mathrm{d}\phi_q(\boldsymbol{v}_2(q))$. Moreover, by definition, $\boldsymbol{v}_2(q) = \mathrm{d}f_p(\boldsymbol{v}_1(p))$ so the two sides are the same. $\square$

- Theorem 2.2.22: For $i = 1, 2, 3$, let $U_i \subset \mathbb{R}^{n_i}$ open and $\boldsymbol{v}_i$ a vector field on U_i. For $i = 1, 2$, let $f_i : U_i \to U_{i+1}$ be a C^∞ map. If $\boldsymbol{v}_1, \boldsymbol{v}_2$ are f_1-related and $\boldsymbol{v}_2, \boldsymbol{v}_3$ are f_2-related, then $\boldsymbol{v}_1, \boldsymbol{v}_3$ are $(f_2 \circ f_1)$-related. In particular, if f_1, f_2 are diffeomorphisms, we have
$$(f_2)_*(f_1)_*\boldsymbol{v}_1 = (f_2 \circ f_1)_*\boldsymbol{v}_1$$
- **Pullback** (of μ on U): The function from $U \to T_p^*\mathbb{R}^n$ defined as follows, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open, $f : U \to V$ is a C^∞ map, and μ is a one-form on V. *Denoted by* $\boldsymbol{f^*\mu}$. *Given by*
$$p \mapsto \mu_{f(p)} \circ \mathrm{d}f_p$$
- If $\phi : V \to \mathbb{R}$ is a C^∞ map and $\mu = \mathrm{d}\phi$, then
$$\mu_q \circ \mathrm{d}f_p = \mathrm{d}\phi_q \circ \mathrm{d}f_p = \mathrm{d}(\phi \circ f)_p$$
  - In other words,
$$f^*\mu = \mathrm{d}\phi \circ f$$
- Theorem 2.2.24: If μ is a C^∞ one-form on V, its pullback $f^*\mu$ is C^∞.

  Proof. See Exercise 2.2.ii. □

Week 5

Differentiation

5.1 Vector Calculus Operations

4/27: • Announcements.

  - No class this Friday, next Monday.
  - Midterm next Friday.
    - Up through Chapter 2.
    - The exam will likely be computationally heavy.
    - Compute d, pullbacks, interior products, Lie derivatives, etc.
    - Emphasis on Chapter 2 as opposed to Chapter 1 even though it all builds on itself.
    - He'll probably cook up a few problems too.
  - There is a recorded lecture for us.
    - On Chapter 3 content.
    - We'll cover Chapter 3 in kind of an impressionistic way as it is.
  - There are also some notes on the physics stuff.

- Vector calculus operations.
  - In one dimension, you have functions, and you take derivatives.
    - The derivative operation does essentially map $\Omega^0 \to \Omega^1$ or $C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$.
  - In two dimensions, ...
    - $\mathrm{d}^2 = 0$ reflects the fact that gradient vector fields are curl-free.
  - If you want to understand the 2D-curl business...
    - $\mathrm{curl}(v) : \mathbb{R}^2 \to \mathbb{R}$ is intuitively about balls spinning around in a vector field.
    - There's also a nice formula to compute it.
    - And then there's a connection with $\mathrm{d} : \Omega^1 \to \Omega^2$.
  - In 3D, you can take top-dimensional forms (which are just functions) and bottom-dimensional forms (which are by definition functions) and you can work out an identification between them.
  - Note that $\mathrm{curl} : \mathfrak{X}(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$, where $\mathfrak{X}(\mathbb{R}^2)$ is the space of vector fields.
- The musical operator $\sharp$ identifies forms with vector fields, i.e., $\sharp : \Omega^1 \to \mathfrak{X}(\mathbb{R}^2)$.
- Properties of exterior derivatives $\mathrm{d} : \Omega^k(U) \to \Omega^{k+1}(U)$.
  1. $\mathrm{d}(\omega_1 + \omega_2) = \mathrm{d}\omega_1 + \mathrm{d}\omega_2$ and $\mathrm{d}(\lambda\omega) = \lambda\,\mathrm{d}\omega$.
  2. Product rule $\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge \mathrm{d}\omega_2$.

- Special case $k = \ell = 0$. Then
$$\mathrm{d}(fg) = g\,\mathrm{d}f + f\,\mathrm{d}g$$
which is the usual product rule for gradient.
- Claim:
$$\mathrm{d}\left(\sum_I f_I\,\mathrm{d}x_I\right) = \sum_I \mathrm{d}f_I \wedge \mathrm{d}x_I$$
  - Let $\omega_1 \in \Omega^k$ and $\omega_2 \in \Omega^\ell$ be defined by
$$\omega_1 = \sum_I f_I\,\mathrm{d}x_I \qquad\qquad \omega_2 = \sum_J g_J\,\mathrm{d}x_J$$
where we're summing over all I such that $|I| = k$ and all J such that $|J| = \ell$. Then
$$\omega_1 \wedge \omega_2 = \sum_{I,J} f_I g_J\,\mathrm{d}x_I \wedge \mathrm{d}x_J\,\mathrm{d}(\omega_1 \wedge \omega_2) \qquad = \sum_{I,J} \mathrm{d}(f_I g_J) \wedge \mathrm{d}x_I \wedge \mathrm{d}x_J$$
  - Note that
$$\mathrm{d}(f_I g_J) = g_J\,\mathrm{d}f_I + f_I\,\mathrm{d}g_J$$
and
$$\mathrm{d}g_J \wedge \mathrm{d}x_I = (-1)^k\,\mathrm{d}x_I \wedge \mathrm{d}g_J$$
  - These identities allow us to take the previous equation to
$$\begin{aligned}\mathrm{d}(\omega_1 \wedge \omega_2) &= \sum_{I,J} g_J\,\mathrm{d}f_I \wedge \mathrm{d}x_I \wedge \mathrm{d}x_J + (-1)^k f_I\,\mathrm{d}x_I \wedge \mathrm{d}g_J \wedge \mathrm{d}x_J \\ &= \sum_{I,J} (\mathrm{d}f_I \wedge \mathrm{d}x_I) \wedge (g_J\,\mathrm{d}x_J) + \sum_{I,J} (f_I\,\mathrm{d}x_I) \wedge (ddg_J \wedge \mathrm{d}x_J) \\ &= \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^k \omega_1\,\mathrm{d}\omega_2\end{aligned}$$

3. $\mathrm{d}^2 = 0$.
   - Let $\omega = \sum_I f_I\,\mathrm{d}x_I$.
   - Then
$$\begin{aligned}\mathrm{d}^2(\omega) &= \mathrm{d}(\mathrm{d}\omega) \\ &= \mathrm{d}\left(\sum_I \mathrm{d}f_I \wedge \mathrm{d}x_I\right) \\ &= \sum_I \mathrm{d}(\mathrm{d}f_I \wedge \mathrm{d}x_I) && \text{Property 1} \\ &= \sum_I \mathrm{d}(\mathrm{d}f_I) \wedge \mathrm{d}x_I && \text{Property 2}\end{aligned}$$
so it suffices to just show that $\mathrm{d}^2 f = 0$ for all $f \in \Omega^0$.
   - We know that $\mathrm{d}f = \sum_{i=1}^n \partial f/\partial x_i\,\mathrm{d}x_i$. Thus,
$$\begin{aligned}\mathrm{d}(\mathrm{d}f) &= \sum_i \mathrm{d}\left(\frac{\partial f}{\partial x_i}\right) \wedge \mathrm{d}x_i \\ &= \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i}\,\mathrm{d}x_j \wedge \mathrm{d}x_i \\ &= 0\end{aligned}$$

  - The last equality holds because of commuting partial derivatives for smooth f, and the fact that changing order introduces a negative sign by some property.

- In fact, if we fix $\mathrm{d}^0 : \Omega^0(U) \to \Omega^1(U)$ to be the "gradient," then these properties characterize the function d on its domain and codomain. In particular, d is the unique function on its domain and codomain that satisfies these properties.
  - We define it by
$$\mathrm{d}\left(\sum_I f_I\,\mathrm{d}x_I\right) = \sum_I \mathrm{d}f_I \wedge \mathrm{d}x_I$$
  - The above properties characterize it axiomatically.
  - We can prove this uniqueness theorem.
- **Closed** (form): A form $\omega \in \Omega^k(U)$ such that $\mathrm{d}\omega = 0$.
- **Exact** (form): A form $\omega \in \Omega^k(U)$ such that $\omega = \mathrm{d}\eta$ for some $\eta \in \Omega^{k-1}(U)$.
- $\mathrm{d}^2 = 0$ implies closed and exact implies closed.
- **Poincaré lemma**: Locally closed forms are exact.

5.2 Chapter 2: Differential Forms

From Guillemin and Haine (2018).

5/5:
- As we formed the k^{th} exterior powers $\Lambda^k(V^*)$, we can form the k^{th} exterior powers $\Lambda^k(T_p^*\mathbb{R}^n)$.
  - Since $\Lambda^1(T_p^*\mathbb{R}^n) = T_p^*\mathbb{R}^n$, we can think of a one-form as a function which takes its value at p in the space $\Lambda^1(T_p^*\mathbb{R}^n)$.
- ***k*-form** (on U): A function which assigns to each point $p \in U$ an element $\omega_p \in \Lambda^k(T_p^*\mathbb{R}^n)$, where $U \subset \mathbb{R}^n$ is open.
- We can use the wedge product to construct k-forms.
  - Let $\omega_1, \dots, \omega_k$ be one-forms. Then $\omega_1 \wedge \cdots \wedge \omega_k$ is the k-form whose value at p is the wedge product
$$(\omega_1 \wedge \cdots \wedge \omega_k)_p = (\omega_1)_p \wedge \cdots \wedge (\omega_k)_p$$
  - Let $f_1, \dots, f_k$ be real-valued functions in $C^\infty(U)$. Suppose $\omega_i = \mathrm{d}f_i$. Then we may obtain the k-form whose value at p is
$$(\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k)_p = (\mathrm{d}f_1)_p \wedge \cdots \wedge (\mathrm{d}f_k)_p$$
- Since $(\mathrm{d}x_1)_p, \dots, (\mathrm{d}x_n)_p$ are a basis of $T_p^*\mathbb{R}^n$, the wedge products
$$(\mathrm{d}x_I)_p = (\mathrm{d}x_{i_1})_p \wedge \cdots \wedge (\mathrm{d}x_{i_k})_p$$
where $I = (i_1, \dots, i_k)$ is a strictly increasing multi-index of n of length k form a basis of $\Lambda^k(T_p^*\mathbb{R}^n)$.
- Thus, every $\omega_p \in \Lambda^k(T_p^*\mathbb{R}^n)$ has a unique decomposition
$$\omega_p = \sum_I c_I(\mathrm{d}x_I)_p$$
where every $c_I \in \mathbb{R}$.

- Similarly, every k-form ω on U has a unique decomposition

$$\omega = \sum_I f_I \, \mathrm{d}x_I$$

  where every $f_I : U \to \mathbb{R}$.

- **Class C^r** (k-form): A k-form ω for which every f_I in its decomposition is in $C^r(U)$.
- From here on out, we assume unless otherwise stated that all k-forms we consider are of class C^∞.
- **$\Omega^k(U)$**: The set of k-forms of class C^∞ on U.
- **$f\omega$**: The k-form defined as follows, where $f \in C^\infty(U)$ and $\omega \in \Omega^k(U)$. *Given by*

$$p \mapsto f(p)\omega_p$$

- **Sum** (of ω_1, ω_2): The k-form defined as follows, where $\omega_1, \omega_2 \in \Omega^k(U)$. *Denoted by* $\boldsymbol{\omega_1 + \omega_2}$. *Given by*

$$p \mapsto (\omega_1)_p + (\omega_2)_p$$

- **Wedge product** (of ω_1, ω_2): The $(k_1 + k_2)$-form defined as follows, where $\omega_1 \in \Omega^{k_1}(U)$ and $\omega_2 \in \Omega^{k_2}(U)$. *Denoted by* $\boldsymbol{\omega_1 \wedge \omega_2}$. *Given by*

$$p \mapsto (\omega_1)_p \wedge (\omega_2)_p$$

- **Zero-form**: A function which assigns to each $p \in U$ an element of $\Lambda^0(T_p^*\mathbb{R}^n) = \mathbb{R}$. *Also known as* **real-valued function**.
- It follows from the definition of zero-forms that

$$\Omega^0(U) = C^\infty(U)$$

- **Exterior differentiation operation**: The operator from $\Omega^0(U) \to \Omega^1(U)$ which associates to a function $f \in C^\infty(U)$ the 1-form $\mathrm{d}f$. *Denoted by* $\mathbf{d}$.
- We now seek to define a generalized version of the exterior differentiation operation; in particular, we would like to define an analogous function $\mathrm{d} : \Omega^k(U) \to \Omega^{k+1}(U)$.
- Desired properties of exterior differentiation.
  1. If $\omega_1, \omega_2 \in \Omega^k(U)$, then
  $$\mathrm{d}(\omega_1 + \omega_2) = \mathrm{d}\omega_1 + \mathrm{d}\omega_2$$
  2. If $\omega_1 \in \Omega^k(U)$ and $\omega_2 \in \Omega^\ell(U)$, then
  $$\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge \mathrm{d}\omega_2$$
  3. If $\omega \in \Omega^k(U)$, then
  $$\mathrm{d}(\mathrm{d}\omega) = 0$$
- Consequences of these properties.
- Lemma 2.4.5: Let $U \subset \mathbb{R}^n$ open. If $f_1, \dots, f_k \in C^\infty(U)$, then

$$\mathrm{d}(\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k) = 0$$

Proof. We induct on k. For the base case $k = 1$, we have that $\mathrm{d}(\mathrm{d}f_1) = 0$ by Property 3. Now suppose inductively that we have proven the claim for $k-1$ functions; we now seek to prove it for k functions. Let $\mu = \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_{k-1}$. Then by the induction hypothesis, $\mathrm{d}\mu = 0$. Therefore,

$$\begin{aligned}
\mathrm{d}(\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k) &= \mathrm{d}(\mu \wedge \mathrm{d}f_k) \\
&= \mathrm{d}\mu \wedge \mathrm{d}f_k + (-1)^{k-1}\mu \wedge \mathrm{d}(\mathrm{d}f_k) && \text{Property 2} \\
&= 0
\end{aligned}$$

as desired. ☐

- A special case of Lemma 2.4.5 is that
$$\mathrm{d}(\mathrm{d}x_I) = 0$$
- Now since every k-form $\omega \in \Omega^k(U)$ has a unique decomposition in terms of the $\mathrm{d}x_I$, Property 2 and the above equation reveal that
$$\mathrm{d}\omega = \sum_I \mathrm{d}f_I \wedge \mathrm{d}x_I$$
- Therefore, if there exists an operator d satisfying Properties 1-3, then d necessarily has the above form. All that's left is to show that the operator defined above has these properties.
- Proposition 2.4.10: Let $U \subset \mathbb{R}^n$ be open. There is a unique operator $\mathrm{d} : \Omega^*(U) \to \Omega^{*+1}(U)$ satisfying Properties 1-3.

Proof. ... ☐

- **Closed** (k-form): A k-form $\omega \in \Omega^k(U)$ for which $\mathrm{d}\omega = 0$.
- **Exact** (k-form): A k-form $\omega \in \Omega^k(U)$ such that $\omega = \mathrm{d}\mu$ for some $\mu \in \Omega^{k-1}(U)$.
- Property 3 implies that every exact k-form is closed.
  - The converse is not true even for 1-forms (see Exercise 2.1.iii).
  - "It is a very interesting (and hard) question to determine if an open set U has the following property: For $k > 0$, every closed k-form is exact" (Guillemin & Haine, 2018, p. 49).
    - Note that we do not consider zero-forms since there are no (-1)-forms for which to define exactness.
- If $f \in C^\infty(U)$ and $\mathrm{d}f = 0$, then f is constant on connected components of U (see Exercise 2.2.iii).
- Lemma 2.4.16 (Poincaré lemma): If ω is a closed form on U of degree $k > 0$, then for every point $p \in U$, there exists a neighborhood of p on which ω is exact.

Proof. See Exercises 2.4.v and 2.4.vi. ☐

Week 6

Operations on Forms

6.1 The Pullback

5/4:

- Klug got his flight to his wedding paid for by giving a talk at a nearby institution!
- Homework 3 now due Monday (the stuff will be on the exam though).
- Office hours today from 5:00-6:00.
- Exam Friday.
- Next week will be Chapter 3.
  - Integration of top-dimensional forms, i.e., if we're in 2D space, we'll integrate 2D forms; in 3D space, we'll integrate 3D forms, etc.
- Pullbacks of k-forms.
  - Let $F : U \to V$ be smooth where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$.
  - This induces $F^* : \Omega^k(V) \to \Omega^k(U)$.
  - We have $\mathrm{d}F_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$, which also induces $\mathrm{d}F_p^* : \Lambda^k(T_{F(p)}^*\mathbb{R}^m) \to \Lambda^k(T_p^*\mathbb{R}^n)$.
  - Note that F^* maps $\omega \mapsto F^*\omega$ where $F^*\omega_p = \mathrm{d}F_p^*\omega_{F(p)}$.
- In formulas...

$$\omega = \sum_I \varphi_I \,\mathrm{d}x_I \qquad\qquad F^*\omega = \sum_I F^*\varphi_I \,\mathrm{d}F_I$$

  - φ_I is just a function.
  - Recall that $F^*\varphi_I = \varphi_I \circ F : U \to \mathbb{R}$.
  - If $I = (i_1, \dots, i_k)$, then $\mathrm{d}F_I = \mathrm{d}F_{i_1} \wedge \cdots \wedge \mathrm{d}F_{i_k}$.
  - Recall that $F_{i_j} : U \to \mathbb{R}$ sends $x \mapsto x_{i_j}$ (the component of F).
  - There is a derivation that gets you from the above abstract definition of the pullback to the below concrete form.
- Note that $\mathrm{d}F_p$ is the kind of thing we worked on last quarter?
- Properties of the pullback (let $U \xrightarrow{F} V \xrightarrow{G} W$).
  1. F^* is linear.
  2. $F^*(\omega_1 \wedge \omega_2) = F^*\omega_1 \wedge F^*\omega_2$.

3. $(F \circ G)^* = G^* \circ F^*$.
4. $\mathrm{d} \circ F^* = F^* \circ \mathrm{d}$. *picture; Commutative diagram*

- Properties 1-3 follow from our Chapter 1 pointwise properties.
  - They also yield the explicit formula for $F^*\omega$ given above.
- Property 4:
  - First: Recall that the following diagram holds. *picture*
  - Check: $\mathrm{d}F_I = F^* \,\mathrm{d}x_I$ where $\mathrm{d}F_{i_1} \wedge \cdots \wedge \mathrm{d}F_{i_k}$ where $I = (i_1, \dots, i_k)$.
  - Now we prove the property by taking

$$\begin{aligned}
\mathrm{d}F_I &= F^*(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}) \\
&= F^* \,\mathrm{d}x_{i_1} \wedge \cdots \wedge F^* \,\mathrm{d}x_{i_k} && \text{Property 2} \\
&= \mathrm{d}(F^* x_{i_1}) \wedge \cdots \wedge \mathrm{d}(F^* x_{i_k}) \\
&= \mathrm{d}F_{i_1} \wedge \cdots \wedge \mathrm{d}F_{i_k}
\end{aligned}$$

  - Now we have that if $\omega = \sum_I \varphi_I \,\mathrm{d}x_I$, then

$$\begin{aligned}
\mathrm{d}(F^*\omega) &= \mathrm{d}\left(\sum_I F^*\varphi_I \,\mathrm{d}F_I\right) \\
&= \sum_I \mathrm{d}(F^*\varphi_I \wedge \mathrm{d}F_I) \\
&= \sum_I \mathrm{d}(F^*\varphi_I) \wedge \mathrm{d}F_I \\
&= \sum_I F^* \,\mathrm{d}\varphi_I \wedge F^* \,\mathrm{d}x_I \\
&= \sum_I F^*(\mathrm{d}\varphi_I \wedge \mathrm{d}x_I) \\
&= F^* \left(\sum_I \mathrm{d}\varphi_I \wedge \mathrm{d}x_I\right) \\
&= F^* \,\mathrm{d}\omega
\end{aligned}$$

    where the second equality holds by the linearity of d and we insert the wedge because multiplication is the same as wedging a zero-form, the third equality holds by the product rule $\mathrm{d}^2 = 0$, the fourth equality holds because d and F^* commute for 0-forms, and the fifth equality holds by Property 2.

- $\mathrm{d}^2 = 0$ generalizes curl and all of those identities.
- Two other operations.
- **Interior product**: Given v a vector field on U, we have $\iota_v : \Omega^k(U) \to \Omega^{k-1}(U)$ that sends $\omega \mapsto \iota_v\omega$.
  - Its properties follow from the properties of the pointwise stuff.
    1. $\iota_v(\omega_1 + \omega_2) = \iota_v\omega_1 + \iota_v\omega_2$.
    2. $\iota_v(\omega_1 \wedge \omega_2) = \cdots$.
    3. $\iota_v \circ \iota_w = -\iota_w \circ \iota_v$.
- **Lie derivative**: If v is a vector field on U, then $L_v : \Omega^k(U) \to \Omega^k(U)$ sends $\omega \mapsto \mathrm{d}\iota_v\omega + \iota_v \,\mathrm{d}\omega$.
  - Note that we use ι to drop the index and d to raise it back up, and then vice versa.
- Check: Agrees with previous definition for Ω^0.

- Cartan's magic formula is what we're taking to be the definition of the Lie derivative.
- Properties.
  1. $L_v \circ \mathrm{d} = \mathrm{d} \circ L_v$.
  2. $L_v(\omega \wedge \eta) = L_v\omega \wedge \eta + \omega \wedge L_v\eta$.
     - Proof:

$$\begin{aligned}\mathrm{d}(\iota_v \mathrm{d} + \mathrm{d}\iota_v) &= \mathrm{d}\iota_v \mathrm{d} \\ &= \iota_v(\iota_v \mathrm{d} + \mathrm{d}\iota_v)\end{aligned}$$

- We should find an explicit formula for the Lie derivative.
  - Your vector field will be of the form

$$v = \sum f_i \, \partial/\partial x_i$$

  - Your form will be of the form

$$\omega = \sum \varphi_I \, \mathrm{d}x_I$$

6.2 Chapter 2: Differential Forms

From Guillemin and Haine (2018).

5/5: • **Interior product** (of $\boldsymbol{v}$ with ω): The $(k-1)$-form on U defined as follows, where $U \subset \mathbb{R}^n$ open, $\boldsymbol{v}$ a vector field on U, and $\omega \in \Omega^k(U)$. *Denoted by* $\boldsymbol{\iota_v \omega}$. *Given by*

$$p \mapsto \iota_{\boldsymbol{v}(p)}\omega_p$$

- By definition, $\iota_{\boldsymbol{v}(p)}\omega_p \in \Lambda^{k-1}(T_p^*\mathbb{R}^n)$.

References

Guillemin, V., & Haine, P. J. (2018). *Differential forms* [https://math.mit.edu/classes/18.952/2018SP/files/18.952_book.pdf].