MATH 20510 (Analysis in $\mathbb{R}^n$ III – Accelerated) Notes

Steven Labalme

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Contents

Chapter 1

Multilinear Algebra

1.1 Notes

3/28: • Motivation for the course and an overview of Guillemin and Haine (2018).

3/30: • Plan:

  - More (multi)linear algebra.

- Dual spaces.
- Let V be an n-dimensional real vector space.
- **Hom $(V, \mathbb{R})$**: The set of all homomorphisms (i.e., linear maps) from V to $\mathbb{R}$. *Also known as* $\boldsymbol{V^*}$.
- **Dual basis** (for V^*): The set of linear transformations from V to $\mathbb{R}$ defined by

$$e_j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

  where $e_1, \dots, e_n$ is a basis of V. *Denoted by* $e_1^*, \dots, e_n^*$.
- Check: $e_1^*, \dots, e_n^*$ are a basis for V^*.
  - Are they linearly independent? Let $c_1 e_1^* + \cdots + c_n e_n^* = 0 \in \text{Hom}(V, \mathbb{R})$. Then

$$c_i = (c_1 e_1^* + \cdots + c_n e_n^*)(e_i) = 0 \in \mathbb{R}$$

    as desired.
  - Span? Let $\varphi \in \text{Hom}(V, \mathbb{R})$. Then we can verify that

$$\varphi(e_1)e_1^* + \cdots + \varphi(e_n)e_n^* = \varphi$$

    - We prove this by verifying the previous statement on the basis of V (if two linear transformations have the same action on the basis of a vector space, they are equal).
- With a choice of basis for V, we obtain an isomorphism $\varepsilon : V \to V^*$ with the mapping $e_i \mapsto e_i^*$ for all i.
- The dual space is known as such because $(V^*)^* \cong V$, where $\cong$ is **canonical** (no choice of basis is needed).
- One more property of dual spaces: **functoriality**.

- Given a linear transformation $A : V \to W$, we know that $A^* : W^* \to V^*$ where A^* is the transpose of A. In particular, if $\varphi \in W^*$, then $\varphi \circ A : V \to \mathbb{R}$.
- Claim: A^* is linear.

- **Functoriality**: If $A : V \to W$ and $B : W \to U$, then $B^* : U^* \to W^*$ and $A^* : W^* \to V^*$. The functoriality statement is that $(B \circ A)^* = A^* \circ B^*$.
- A^* is the **pullback** (or transpose) of A.
- Let $v_1, \dots, v_n$ be a basis for V and $w_1, \dots, w_m$ be a basis for W. Then $[A]_{v_1,\dots,v_n}^{w_1,\dots,w_m} = A$ is the matrix of the linear transformation A with respect to these bases. Then if $v_1^*, \dots, v_n^*$ and $w_1^*, \dots, w_m^*$ are the corresponding dual bases, then $[A^*]_{v_1^*,\dots,v_n^*}^{w_1^*,\dots,w_m^*} = A^T$. We can and should verify this for ourselves.
- This is over the real numbers, so A^* is just the transpose because there are no complex numbers of which to take the conjugate!
- A generalization: Tensors.
- ***k*-tensor**: A **multilinear** map
$$T : \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}$$
- **Multilinear** (map T): A function T such that
$$\begin{aligned} T(v_1, \dots, v_i^1 + v_i^2, \dots, v_k) &= T(v_1, \dots, v_i^1, \dots, v_k) + T(v_1, \dots, v_i^2, \dots, v_k) \\ T(v_1, \dots, \lambda v_i, \dots, v_k) &= \lambda T(v_1, \dots, v_i, \dots, v_k) \end{aligned}$$
for all $(v_1, \dots, v_k) \in V^k$.
- The determinant is an n-tensor!
- 1-tensors are just covectors.
- $\boldsymbol{\mathcal{L}^k(V)}$: The vector space of all k-tensors on V.
- Calculating $\dim \mathcal{L}^k(V)$. (Answer not given in this class.)
- Let $A : V \to W$. Then $A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$.
  - Check $(A \circ B)^* = B^* \circ A^*$.
- **Multi-index of $\boldsymbol{n}$ of length $\boldsymbol{k}$**: A k-tuple $(i_1, \dots, i_k)$ where each $i_j \in \mathbb{N}$ satisfies $1 \leq i_j \leq n$ $(j = 1, \dots, k)$. *Denoted by* $\boldsymbol{I}$.
- Let $e_1, \dots, e_n$ be a basis for V.
- **Tensor product** (of $T_1 \in \mathcal{L}^k(V)$, $T_2 \in L^l(V)$): The function from V^{k+l} to $\mathbb{R}$ defined by
$$(v_1, \dots, v_{k+l}) \mapsto T_1(v_1, \dots, v_k) T_2(v_{k+1}, \dots, v_{k+l})$$
Denoted by $\boldsymbol{T_1 \otimes T_2}$.
- Claims:
  1. $T_1 \otimes T_2 \in L^{k+l}(V)$.
  2. $A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$.
- $\boldsymbol{e_I^*}$: The function $e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*$, where $I = (i_1, \dots, i_k)$ is a multi-index of n of length k.
- Claim: Letting I range over all n^k multi-indices of n of length k, the e_I^* are a basis for $\mathcal{L}^k(V)$.

- If $V=\mathbb{R}$, then $V=\mathbb{R}e_1$. If $V=\mathbb{R}^2$, then $V=\mathbb{R}e_1\oplus\mathbb{R}e_2$.
- We know that $L^1(V)=V^*=Re_1^*$. Thus, $e_1^*\otimes e_2^*:V\times V\to\mathbb{R}$. Thus, for example,

$$(e_1^*\otimes e_2^*)((1,2),(3,4))=e_1^*(1,2)\cdot e_2^*(3,4)=1\cdot 4=4$$

4/1:

- Plan: More multilinear algebra.
  - Properties of the tensor product.
  - Sign of a permutation.
  - Alternating tensors (lead into differential forms down the road).
- Recall: V is an n-dimensional vector space over $\mathbb{R}$ with basis $e_1,\dots,e_n$. $\mathcal{L}^k(V)$ is the vector space of k-tensors on V. $\{e_I^*\mid I\text{ a multiindex of }n\text{ of length }k\}$ is a basis for $\mathcal{L}^k(V)$.
- For example, if $V=\mathbb{R}^2$ and $T\in\mathcal{L}^2(V)$, then

$$T(a_1e_1+a_2e_2,b_1e_1+b_2e_2)=a_1b_1T(e_1,e_1)+a_1b_2T(e_1,e_2)+a_2b_1T(e_2,e_1)+a_2b_2T(e_2,e_2)$$

  - A basis of $\mathcal{L}^2(V)$ is

$$\{e_1^*\otimes e_1^*,e_1^*\otimes e_2^*,e_2^*\otimes e_1^*,e_2^*\otimes e_2^*\}$$

  - Recall that some basic properties are

$$e_1^*\otimes e_2^*((1,2),(3,4))=1\cdot 4=4 \qquad e_2^*\otimes e_1^*((1,2),(3,4))=2\cdot 3=6$$

  - It follows by the initial decomposition of T that

$$T=a_1b_1e_1^*\otimes e_1^*+a_1b_2e_1^*\otimes e_2^*+a_2b_1e_2^*\otimes e_1^*+a_2b_2e_2^*\otimes e_2^*$$

- Important consequence: To know the action of T on an arbitrary pair of vectors, you need only know its action on the basis; a higher-dimensional generalization of the earlier property.
- Note that

$$e_I^*(e_J)=\delta_{IJ}=\begin{cases}1 & I=J\\ 0 & I\neq J\end{cases}$$

- Basic properties of the tensor product.
  1. *Right-distributive*: If $T_1\in\mathcal{L}^k(V)$ and $T_2,T_3\in\mathcal{L}^\ell(V)$, then

$$T_1\otimes(T_2+T_3)=T_1\otimes T_2+T_1\otimes T_3$$

  2. *Left-distributive*: If $T_1,T_2\in\mathcal{L}^k(V)$ and $T_3\in\mathcal{L}^\ell(V)$, then

$$(T_1+T_2)\otimes T_3=T_1\otimes T_3+T_2\otimes T_3$$

  3. *Associative*: If $T_1\in\mathcal{L}^k(V)$, $T_2\in\mathcal{L}^\ell(V)$, and $T_3\in\mathcal{L}^m(V)$, then

$$T_1\otimes(T_2\otimes T_3)=(T_1\otimes T_2)\otimes T_2=T_1\otimes T_2\otimes T_3$$

  4. *Scalar multiplication*: If $T_1\in\mathcal{L}^k(V)$, $T_2\in\mathcal{L}^\ell(V)$, and $\lambda\in\mathbb{R}$, then

$$(\lambda T_1)\otimes T_2=\lambda(T_1\otimes T_2)=T_1\otimes(\lambda T_2)$$

- Note that the tensor product is not commutative.
- Aside: Defining the sign of a permutation.

- $\boldsymbol{S_A}$: The set of all automorphisms of A (bijections from A to A), where A is a set.
- $\boldsymbol{S_n}$: The set $S_{[n]}$.
- Given $\sigma_1, \sigma_2 \in S_n$, $\sigma_1 \circ \sigma_2 \in S_n$.
  - Thus, S_n is a **group**.
- **Transposition**: A function $\tau \in S_n$ such that

$$\tau(k) = \begin{cases} j & k = i \\ i & k = j \\ k & k \neq i, j \end{cases}$$

  for some $i, j \in [n]$. *Denoted by* $\boldsymbol{\tau_{i,j}}$.
- Theorem: An element of S_n can be written as the product of transpositions (i.e., for all $\sigma \in S_n$, there exist $\tau_1, \dots, \tau_m \in S_n$ such that $\sigma = \tau_1 \circ \cdots \circ \tau_m$).
- **Sign** (of $\sigma \in S_n$): The number (mod 2) of transpositions whose product equals σ. *Denoted by* $\boldsymbol{(-1)^\sigma}$, $\mathbf{sign}\,\boldsymbol{(\sigma)}$.
- Theorem: The sign of σ is well-defined. Additionally,

$$(-1)^{\sigma_1\sigma_2} = (-1)^{\sigma_1} \cdot (-1)^{\sigma_2}$$

- Example: Consider the identity permutation. $(-1)^\sigma = +1$. We can think of this as the product of zero transpositions or, for instance, as the product of the two transpositions $\tau_{1,2} \circ \tau_{1,2}$. Another example would be $\tau_{2,3} \circ \tau_{1,2} \circ \tau_{1,2} \circ \tau_{2,3}$.
- Theorem: Let X_i be a rational or polynomial function for each $i \in \mathbb{N}$. Then

$$(-1)^\sigma = \prod_{i<j} \frac{X_{\sigma(i)} - X_{\sigma(j)}}{X_i - X_j}$$

- Example: For the permutation $\sigma = (1, 2, 3)$, we have

$$\begin{aligned} (-1)^\sigma &= \frac{X_{\sigma(1)} - X_{\sigma(2)}}{X_1 - X_2} \cdot \frac{X_{\sigma(1)} - X_{\sigma(3)}}{X_1 - X_3} \cdot \frac{X_{\sigma(2)} - X_{\sigma(3)}}{X_2 - X_3} \\ &= \frac{X_2 - X_3}{X_1 - X_2} \cdot \frac{X_2 - X_1}{X_1 - X_3} \cdot \frac{X_3 - X_1}{X_2 - X_3} \\ &= \frac{-(X_1 - X_2)}{X_1 - X_2} \cdot \frac{-(X_1 - X_3)}{X_1 - X_3} \cdot \frac{X_2 - X_3}{X_2 - X_3} \\ &= -1 \cdot -1 \cdot 1 \\ &= +1 \end{aligned}$$

  which squares with the fact that $\sigma = \tau_{1,2} \circ \tau_{2,3}$.
- Claims to verify with the above formula:
  1. $\operatorname{sign}(\sigma) \in \{\pm 1\}$.
  2. $\operatorname{sign}(\tau_{i,j}) = -1$.
  3. $\operatorname{sign}(\sigma_1\sigma_2) = \operatorname{sign}(\sigma_1)\operatorname{sign}(\sigma_2)$.

4/4:
- Plan:
  - More multilinear algebra.

- Alternating k-tensors — 2 views:
  1. As a subspace of $\mathcal{L}^k(V)$.
  2. As a quotient of $\mathcal{L}^k(V)$.
- Next time: Operators as alternating tensors.
  - Wedge products.
  - Interior products.
  - Pullbacks.

- Recall: $\dim V = n$, $e_1, \dots, e_n$ a basis, $\mathcal{L}^k(V)$ the space of k-tensors, $\sigma \in S_k$ implies $(-1)^\sigma \in \{\pm 1\}$, key property: $(-1)^{\sigma_1\sigma_2} = (-1)^{\sigma_1}(-1)^{\sigma_2}$.
- $\boldsymbol{T^\sigma}$: The k-tensor over V defined by

$$T^\sigma(v_1, \dots, v_k) = T(v_{\bar{\sigma}(1)}, \dots, v_{\bar{\sigma}(k)})$$

  where $T \in \mathcal{L}^k(V)$, $\sigma \in S_k$, and $\bar{\sigma}$ denotes the inverse of σ.
- Example: $n = 2$, $k = 2$. Let $T = e_1^* \otimes e_2^* \in \mathcal{L}^2(V)$. Let $\sigma = \tau_{1,2}$. Then $T^\sigma = e_2^* \otimes e_1^*$.
- Another property is $e_I^\sigma = e_{\sigma(I)}^*$.
- Properties:
  1. $T^{\sigma_1\sigma_2} = (T^{\sigma_1})^{\sigma_2}$.
  2. $(T_1 + T_2)^\sigma = T_1^\sigma + T_2^\sigma$.
  3. $(cT)^\sigma = cT^\sigma$.
- Thus, you can view $\sigma : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ as a linear map!
- **Alternating $\boldsymbol{k}$-tensor**: A tensor $T \in \mathcal{L}^k(V)$ such that $T^\sigma = (-1)^\sigma T$ for all $\sigma \in S_k$.
  - Equivalently, $T^\tau = -T$ for all $\tau \in S_k$.
- An example of an alternating 2-tensor when $\dim V = 2$ is $T = e_1^* \otimes e_2^* - e_2^* \otimes e_1^*$.
  - Naturally, $T_{1,2}^\tau = -T$, and $\tau_{1,2}$ is the unique transposition in S_2.
- $e_1^* \otimes e_2^*$ is *not* an alternating 2-tensor since $(e_1^* \otimes e_2^*)^\tau = e_2^* \otimes e_1^* \neq (-1)^\tau(e_1^* \otimes e_2^*)$.
- We can look at $n = 2$, $k = 1$ for ourselves.
- Note: If $T - 1, T_2$ are both alternating k-tensors, then $T_1 + T_2$ is also alternating, as is cT_1 for all $c \in \mathbb{R}$.
- $\boldsymbol{\mathcal{A}^k(V)}$: The vector space of alternating k-tensors.
- **Alt $\boldsymbol{(T)}$**: The function $\text{Alt} : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ defined by

$$\text{Alt}(T) = \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma$$

- Properties:
  1. $\text{im}(\text{Alt}) = \mathcal{A}^k(V)$.
  2. $\mathcal{L}^k(V)/\ker(\text{Alt}) = \Lambda^k(V^*)$ is isomorphic to $\mathcal{A}^k(V)$.
  3. $\text{Alt}(T)^\sigma = (-1)^\sigma \text{Alt}(T)$.

- Proof:

$$\begin{aligned}
\mathrm{Alt}(T)^{\sigma'} &= \left(\sum_{\sigma\in S_k}(-1)^\sigma T^\sigma\right)^{\sigma'} \\
&= \sum_{\sigma\in S_k}(-1)^\sigma T^{\sigma\sigma'} \\
&= (-1)^{\sigma'}\sum_{\sigma\in S_k}(-1)^{\sigma'}(-1)^\sigma T^{\sigma\sigma'} \\
&= (-1)^{\sigma'}\sum_{\sigma\in S_k}(-1)^{\sigma\sigma'} T^{\sigma\sigma'} \\
&= (-1)^{\sigma'}\,\mathrm{Alt}(T)
\end{aligned}$$

- The last equality holds because summing over all σ is the same as summing over all $\sigma'\circ\sigma$.
- This implies $\mathrm{im}(\mathrm{Alt}) \leq \mathcal{A}^k(V)$.

4. If $T\in\mathcal{A}^k(T)$, $\mathrm{Alt}(T) = k!T$.
   - We have

$$\begin{aligned}
\mathrm{Alt}(T) &= \sum_{\sigma\in S_k}(-1)^\sigma T^\sigma \\
&= \sum_{\sigma\in S_k}(-1)^\sigma(-1)^\sigma T \\
&= \sum_{\sigma\in S_k} T \\
&= k!T
\end{aligned}$$

   where $T^\sigma = (-1)^\sigma T$ since $T\in\mathcal{A}^k(V)$ by definition.
   - This implies that $\mathrm{im}(\mathrm{Alt}) = \mathcal{A}^k(V)$: $\mathrm{Alt}(\frac{1}{k!}T) = T\in\mathcal{A}^k(V)$.
5. $\mathrm{Alt}(T^\sigma) = \mathrm{Alt}(T)^\sigma$.
6. $\mathrm{Alt} : \mathcal{L}^k(V)\to\mathcal{L}^k(V)$ is linear.

- Warning: Some people take $\mathrm{Alt}(T) = \frac{1}{k!}\sum_{\sigma\in S_k}(-1)^\sigma T^\sigma$[1].
- Example: $n=k=2$. We have

$$\mathrm{Alt}(e_1^*\otimes e_2^*) = e_1^*\otimes e_2^* - e_2^*\otimes e_1^*$$

- **Non-repeating** (multi-index I): A multi-index I such that $i_{j_1}\neq i_{j_2}$ for all $j_1\neq j_2$.
- **Increasing** (multi-index I): A multi-index I such that $i_1<\cdots<i_k$.
- Claim: $\{\mathrm{Alt}(e_I^*)\}$ where I is non-repeating and increasing is a basis for $\mathcal{A}^k(V)$. There are $\binom{n}{k}$ of these; thus, $\dim\mathcal{A}^k(V) = \binom{n}{k}$.

4/6:

- Klug will be in Texas on Monday and thus is cancelling class on Monday. Homework is now due next Friday. We'll have weekly homeworks going forward after that.
- Plan:
  - $\mathrm{Alt} : \mathcal{L}^k(V)\twoheadrightarrow\mathcal{A}^k(V)$[2].
  - Goal: Identify $\ker(\mathrm{Alt}) = \mathcal{I}^k(V)$, where $\mathcal{I}^k(V)$ is the space of **redundant** k-tensors[3].

[1]Klug prefers this convention, but the text takes the other one.
[2]The two-headed right arrow denotes a surjective map.
[3]The $\mathcal{I}$ in $\mathcal{I}^k(V)$ stands for "ideal."

  - Then: Operations on alternating tensors, e.g.,
    - Wedge product.
    - Interior product.
    - Orientations.
- Claim: $\{\text{Alt}(e_I^*) \mid I \text{ non-repeating, increasing multi-index}\}$ is a basis for $\mathcal{A}^k(V)$.
  - Left as an exercise to us.
- **Redundant** (k-tensor): A k-tensor of the form

$$\ell_1 \otimes \cdots \otimes \ell_i \otimes \ell_i \otimes \ell_{i+2} \otimes \cdots \otimes \ell_k$$

where $\ell_1, \dots, \ell_k \in V^*$.
- $\boldsymbol{\mathcal{I}^k(V)}$: The span of all redundant k-tensors.
  - Note that not every k-tensor in $\mathcal{I}^k(V)$ is a redundant.
- **Decomposable** (k-tensor): A k-tensor of the form $\ell_1 \otimes \cdots \otimes \ell_k$ for $\ell_i \in \mathcal{L}^1(V)$.
  - It often suffices to prove things for decomposable tensors.
- Properties.
  1. If $T \in \mathcal{I}^k(V)$, then $\text{Alt}(T) = 0$, i.e., $\mathcal{I}^k(V) \leq \ker(\text{Alt})$.
     - "Proof by example": If $T = \ell_1 \otimes \ell_1 \otimes \ell_2 \otimes \ell_3$, then $T^{\tau_{1,2}} = T$. It follows from the properties of Alt that

$$\begin{aligned}
\text{Alt}(T) &= \text{Alt}(T^{\tau_{1,2}}) = (-1)^{\tau_{1,2}} \text{Alt}(T) = -\text{Alt}(T) \\
2\,\text{Alt}(T) &= 0 \\
\text{Alt}(T) &= 0
\end{aligned}$$

  2. If $T \in \mathcal{I}^r(V)$ and $T' \in \mathcal{L}^s(V)$, then

$$T \otimes T' \in \mathcal{I}^{r+s}(V)$$

     Similarly, if $T \in \mathcal{L}^r(V)$ and $T \in \mathcal{I}^s(V)$, then

$$T \otimes T' \in \mathcal{I}^{r+s}(V)$$

     - Proof: It suffices to assume that T is redundant. Obviously adding more tensors to the direct product will not change the redundancy of the initial tensor. Example: $\ell_1 \otimes \ell_1 \otimes \ell_2$ is just as redundant as $\ell_1 \otimes \ell_1 \otimes \ell_2 \otimes T$.
  3. If $T \in \mathcal{L}^k(V)$ and $\sigma \in S_k$, then

$$T^\sigma = (-1)^\sigma T + S$$

     for some $S \in \mathcal{I}^k(V)$.
     - Proof by example: It suffices to check this for decomposable tensors (a tensor is just a sum of decomposable tensors). Take $k = 2$. Let $T = \ell_1 \otimes \ell_2$. Let $\sigma = \tau_{1,2}$. Then

$$T^\sigma - (-1)^\sigma T = \ell_2 \otimes \ell_1 + \ell_1 \otimes \ell_2 = (\ell_1 + \ell_2) \otimes (\ell_1 + \ell_2) - \ell_1 \otimes \ell_1 - \ell_2 \otimes \ell_2$$

     - Actual proof: It suffices to assume T is decomposable. We induct on the number of transpositions needed to write σ as a product of **adjacent** transpositions.

- Base case: $\sigma = \tau_{i,i+1}$. Then

$$\begin{aligned} T^{\tau_{i,i+1}} + T = \ell_1 \otimes \cdots \otimes (\ell_i + \ell_{i+1}) \otimes (\ell_i + \ell_{i+1}) \otimes \cdots \otimes \ell_k \\ - \ell_1 \otimes \cdots \otimes \ell_i \otimes \ell_i \otimes \cdots \otimes \ell_k \\ - \ell_1 \otimes \cdots \otimes \ell_{i+1} \otimes \ell_{i+1} \otimes \cdots \otimes \ell_k \end{aligned}$$

- Inductive step: If $\sigma = \beta\tau$, then

$$\begin{aligned} T^\sigma &= T^{\beta\tau} \\ &= (-1)^\tau T^\beta + \text{stuff in } \mathcal{I}^k(V) \\ &= (-1)^\tau[(-1)^\beta T + \text{stuff in } \mathcal{I}^k(V)] + \text{stuff in } \mathcal{I}^k(V) \end{aligned}$$

4. If $T \in \mathcal{L}^k(V)$, then

$$\text{Alt}(T) = k!T + W$$

for some $W \in \mathcal{I}^k(V)$.
   - We have that

$$\begin{aligned} \text{Alt}(T) &= \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma \\ &= \sum_{\sigma \in S_k} (-1)^\sigma [(-1)^\sigma T + S_\sigma] \\ &= \sum_{\sigma \in S_k} T + \sum_{\sigma \in S_k} (-1)^\sigma S_\sigma \\ &= k!T + W \end{aligned}$$

5. $\mathcal{I}^k(V) = \ker(\text{Alt})$.
   - We have that $\mathcal{I}^k(V) \leq \ker(\text{Alt})$ by property 1.
   - Now suppose $T \in \ker(\text{Alt})$. Then $\text{Alt}(T) = 0$. Then by property 4,

$$\begin{aligned} \text{Alt}(T) &= k!T + W \\ 0 &= k!T + W \\ T &= -\frac{1}{k!}W \in \mathcal{I}^k(V) \end{aligned}$$

- Warning: If $T \in \mathcal{A}^r(V)$ and $T' \in \mathcal{A}^s(V)$, then we do not necessarily have $T \otimes T' \in \mathcal{A}^{r+s}(V)$.
  - Example: $e_1^*, e_2^* \in \mathcal{A}^1(V)$ have $e_1^* \otimes e_2^* \notin \mathcal{A}^2(V)$.
- **Adjacent** (transposition): A transposition of the form $\tau_{i,i+1}$.

4/8:

- Recall that $\mathcal{A}^k(V) \hookrightarrow \mathcal{L}^k(V)$[4]
- Functoriality: $(A \circ B)^* = B^* \circ A^*$.
  - A^* takes $\mathcal{L}^k(W) \to \mathcal{L}^k(V)$ and $\mathcal{A}^k(W) \to \mathcal{A}^k(V)$.
- $\dim(\Lambda^k(V)) = \binom{n}{k}$.
  - Special case $k = n$: $\dim \Lambda^n(V) = 1$.
  - If $A : V \to V$ induces a map $\Lambda^n(V^*) \to \Lambda^n(V^*)$ defined by the determinant.
- Aside: $\Lambda^k(V)$ is "exterior powers."

[4] The hooked right arrow denotes an injective map.

- Plan: Wedge products + basis for $\Lambda^k(V)$.
- **Wedge product**: A function $\wedge : \Lambda^k(V^*) \times \Lambda^\ell(V^*) \to \Lambda^{k+\ell}(V)$.
  - We denote elements of $\Lambda^k(V^*)$ by ω_1, ω_2, etc.
- If $\pi : \mathcal{L}^k(V) \to \Lambda^k(V^*)$ sends $T \mapsto \omega$, $\omega_1 = \pi(T_1)$, and $\omega_2 = \pi(T_2)$, then $\omega_1 \wedge \omega_2 = \pi(T_1 \otimes T_2)$.
  - Note that $\ker(\pi) = \mathcal{I}^k(V)$.
- Properties.
  1. This is well defined, i.e., this does not depend on the choice of T_1, T_2.
     - Consider $T_1 + W_1, T_2 + W_2$ with $W_1, W_2 \in \mathcal{I}^k(V)$.
     - We check that $\pi[(T_1 + W_1) \otimes (T_2 + W_2)] = \pi(T_1 \otimes T_2)$.
     - Since $W_1 \otimes T_2, T_1 \otimes W_2, W_1 \otimes W_2 \in \mathcal{I}^{k+\ell}(V)$, we have that

$$\begin{aligned}\pi[(T_1 + W_1) \otimes (T_2 + W_2)] &= \pi(T_1 \otimes T_2 + W_1 \otimes T_2 + T_1 \otimes W_2 + W_1 \otimes W_2)\\ &= \pi(T_1 \otimes T_2) + \pi(W_1 \otimes T_2) + \pi(T_1 \otimes W_2) + \pi(W_1 \otimes W_2)\\ &= \pi(T_1 \otimes T_2)\end{aligned}$$

  2. Associative: We have that

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge \omega_2 \wedge \omega_3$$

     - Follows from the definition of $\wedge$ in terms of π and properties of the tensor product.
  3. Distributive: We have that

$$(\omega_1 + \omega_2) \wedge \omega_3 = \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_3 \qquad \omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$$

     - Follows from the definition of $\wedge$ in terms of π and properties of the tensor product.
  4. Linear: We have that

$$(c\omega_1) \wedge \omega_2 = c(\omega_1 \wedge \omega_2) = \omega_1 \wedge (c\omega_2)$$

     - Follows from the definition of $\wedge$ in terms of π and properties of the tensor product.
  5. Anticommutative: We have that

$$\omega_1 \wedge \omega_2 = (-1)^{k\ell} \omega_2 \wedge \omega_1$$

     - It suffices to assume that $w_1 = \ell_1 \wedge \cdots \wedge \ell_k, w_2 = \ell'_1 \wedge \cdots \wedge \ell'_\ell$.
       - We have

$$(\ell_1 \wedge \cdots \wedge \ell_k) \wedge (\ell'_1 \wedge \cdots \wedge \ell'_\ell) = (-1)^k (\ell'_1 \wedge \cdots \wedge \ell'_\ell) \wedge (\ell_1 \wedge \cdots \wedge \ell_k)$$

     - Let $\ell_1, \ldots, \ell_k \in \Lambda^1(V^*) = V^* = \mathcal{L}^1(V)$.
     - Recall that $\mathcal{I}^1(V) = \{0\}$.
     - Claim: $\ell_{\sigma(1)} \wedge \cdots \wedge \ell_{\sigma(k)} = (-1)^\sigma \ell_1 \wedge \cdots \wedge \ell_k$ for all $\sigma \in S_k$.
       - Recall that $T^\sigma = (-1)^\sigma T + W$ for some $W \in \mathcal{I}^k(V)$.
       - Let $T = \ell_1 \otimes \cdots \otimes \ell_k$.
       - Then

$$\begin{aligned}(\ell_1 \otimes \cdots \otimes \ell_k)^\sigma &= \ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}\\ &= (-1)^\sigma \ell_1 \otimes \cdots \otimes \ell_k + W\end{aligned}$$

       - Then hit both sides by π, noting that $\pi(W) = 0$.
- Example:

1. $n=2$, $k=\ell=1$. Consider $e_1^*, e_2^* \in \mathcal{L}^1(V) = V^* = \mathcal{A}^1(V) = \Lambda^1(V^*)$. Then

$$e_1^* \wedge e_2^* = (-1)e_2^* \wedge e_1^* \qquad\qquad e_1^* \wedge e_1^* = 0 = e_2^* \wedge e_2^*$$

2. $n=4$. We have $e_1^* \wedge (3e_1^* + 2e_2^* + 3e_2^*) = 3(e_1^* \wedge e_1^*) + 2(e_1^* \wedge e_2^*) + 3(e_1^* \wedge e_3^*)$. We also have $(e_1^* \wedge e_2^*) \wedge (e_1^* \wedge e_2^*) = 0$.

4/13:

- Plan:
  - Finish multilinear algebra.
  - Basis for $\Lambda^k(V^*)$.
  - Talk a bit about pullbacks and the determinant.
  - **Orientations** of vector spaces.
  - The **interior product**.
- Basis for $\Lambda^k(V^*)$.
  - Recall that $\{\text{Alt}(e_I^*) \mid I \text{ is a nonrepeating, increasing partition of } n \text{ into } k \text{ parts}\}$ is a basis for $\mathcal{A}^k(V)$.
- Alt is an isomorphism from $\Lambda^k(V^*)$ to $\mathcal{A}^k(V)$.
- If we have an injective map from $\mathcal{A}^k(V)$ to $\mathcal{L}^k(V)$ and π a projection map from $\mathcal{L}^k(V)$ to the quotient space $\mathcal{A}^k(V^*)$ gives rise to $\pi|_{\mathcal{A}^k(V)}$.
- Claim:
  1. $\pi|_{\mathcal{A}^k(V)}$ is an isomorphism.
  2. $\pi(\text{Alt}(e_I^*)) = k!\pi(e_I^*)$.

  (2) implies that $\{\pi(e_I^*) = e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*,\ I \text{ non-repeating and increasing}\}$ is a basis for $\Lambda^k(V^*)$.
- Examples:
  1. $n = 2 = \dim V$, $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$.
     - $\Lambda^0(V^*) = \mathbb{R}$ since $\binom{n}{0} = 1$.
     - $\Lambda^1(V^*) = \mathbb{R}e_1^* \oplus \mathbb{R}e_2^*$ since $\binom{n}{1} = 2$.
     - $\Lambda^2(V^*) = \mathbb{R}e_1^* \wedge e_2^*$ since $\binom{n}{2} = 1$.
       - For the second to last one, note that $e_1^* \wedge e_2^* = -e_2^* \wedge e_1^*$.
     - $\Lambda^3(V^*) = 0$ since $\binom{2}{3} = 0$.
       - For the last one, note that all $e_1^* \wedge e_1^* \wedge e_2^* = 0$.
  2. $n=3$, $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3$.
     - $\binom{n}{0} = 1$: $\Lambda^0(V^*) = \mathbb{R}$.
     - $\binom{n}{1} = 3$: $\Lambda^1(V^*) = \mathbb{R}e_1^* \oplus \mathbb{R}e_2^* \oplus \mathbb{R}e_3^*$.
     - $\binom{n}{2} = 3$: $\Lambda^2(V^*) = \mathbb{R}e_1^* \wedge e_2^* \oplus \mathbb{R}e_2^* \wedge e_3^* \oplus \mathbb{R}e_1^* \wedge e_3^*$.
     - $\binom{n}{3} = 1$: $\Lambda^3(V^*) = \mathbb{R}e_1^* \wedge e_2^* \wedge e_3^*$.
     - $\binom{n}{m} = 0$ $(m > n)$: $\Lambda^m(V^*) = \Lambda^4(V^*) = 0$.
- If $A : V \to W$, $\omega_1 \in \Lambda^k(W^*)$, $\omega_2 \in \Lambda^\ell(W^*)$, then

$$A^*(\omega_1 \wedge \omega_2) = A^*\omega_1 \wedge A^*\omega_2$$

- **Determinant**: Let $\dim V = n$. Let $A : V \to V$ be a linear transformation. This induces a pullback $A^* : \Lambda^n(V^*) \to \Lambda^n(V^*)$. The top exterior power $k = n$ implies $\binom{k}{n} = 1$. We define $\det(A)$ to be the unique real number such that $A^*(v) = \det(A)v$.

- This determinant is the one we know.
  - A^* sends $e_1^* \wedge \cdots \wedge e_n^*$ to $A^*e_1^* \wedge \cdots \wedge A^*e_n^*$ which equals $A^*(e_1^* \wedge \cdots \wedge e_n^*)$ or $\det(A)$
- Sanity check.
  1. $\det(\mathrm{id}) = 1$.
     - $\mathrm{id}(e_1^* \wedge \cdots \wedge e_n^*) = \mathrm{id}\, e_1^* \wedge \cdots \wedge \mathrm{id}\, e_n^* = 1 \cdot e_1^* \wedge \cdots \wedge e_n^*$.
  2. If A is not an isomorphism, then $\det(A) = 0$.
     - If A is not an isomorphism, then there exists $v_1 \in \ker A$ with $v_1 \neq 0$. Let $v_1^*, \dots, v_n^*$ be a basis of V^*. So the pullback of this wedge is the wedge of the pullbacks, but $A^*v_1^* = 0$, so

$$A^*(v_1^* \wedge \cdots \wedge v_n^*) = (A^*v_1^*) \wedge \cdots \wedge (A^*v_n^*) = 0 \wedge \cdots \wedge (A^*v_n^*) = 0 = 0 \cdot v_1^* \wedge \cdots \wedge v_n^*$$

  3. $\det(AB) = \det(A)\det(B)$.
     - Let $A : V \to V$ and $B : V \to V$.
     - We have $(AB)^* = B^*A^*$; in particular, $n = k$, $V = W = U = V$.
- Recall: If we pick a basis for V, $e_1, \dots, e_n$.
  - Implies $[a_{ij}] = [A]_{e_1,\dots,e_n}^{e_1,\dots,e_n}$.
- Does $\det(A) = \det([a_{ij}]) = \sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$?
  - If $A : V \to V$, we know that $A^* : \Lambda^n(V^*) \to \Lambda^n(V^*)$ takes $e_1^* \wedge \cdots \wedge e_n^* \mapsto A^*(e_1^* \wedge \cdots \wedge e_n^*)$. We WTS

$$A^*(e_1^* \wedge \cdots \wedge e_n^*) = \left[\sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}\right] e_1^* \wedge \cdots \wedge e_n^*$$

  - We have that

$$\begin{aligned}
A^*(e_1^* \wedge \cdots \wedge e_n^*) &= A^*e_1^* \wedge \cdots \wedge A^*e_n^* \\
&= \left(\sum_{i_1=1}^n a_{i_1,1} e_{i_1}^*\right) \wedge \cdots \wedge \left(\sum_{i_n=1}^n a_{i_n,n} e_{i_n}^*\right) \\
&= \sum_{i_1,\dots,i_n} a_{i_1,1} \cdots a_{i_n,n} e_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \\
&= \left[\sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}\right] e_1^* \wedge \cdots \wedge e_n^*
\end{aligned}$$

    where the sign arises from the need to reorder $e_{i_1}^* \wedge \cdots \wedge e_{i_n}^*$ and the antisymmetry of the wedge product.

1.2 Chapter 1: Multilinear Algebar

From Guillemin and Haine (2018).

3/31:

- Guillemin and Haine (2018) defines real vector spaces, the operations on them, their basic properties, and the zero vector.
- **Linearly independent** (vectors $v_1, \dots, v_k$): A finite set of vectors $v_1, \dots, v_k \in V$ such that the map from $\mathbb{R}^k$ to V defined by $(c_1, \dots, c_k) \mapsto c_1v_1 + \cdots + c_kv_k$ is injective.
- **Spanning** (vectors $v_1, \dots, v_k$): We require that the above map is surjective.

- Guillemin and Haine (2018) defines basis, finite-dimensional vector space, dimension, subspace, linear map, and kernel.
- **Image** (of $A : V \to W$): The range space of A, a subspace of W. *Also known as* $\mathbf{im}\,(\boldsymbol{A})$.
- Guillemin and Haine (2018) defines the matrix of a linear map.
- **Inner product** (on V): A map $B : V \times V \to \mathbb{R}$ with the following three properties.
  - *Bilinearity*: For vectors $v, v_1, v_2, w \in V$ and $\lambda \in \mathbb{R}$, we have
    $$B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w)$$
    and
    $$B(\lambda v, w) = \lambda B(v, w)$$
  - *Symmetry*: For vectors $v, w \in V$, we have $B(v, w) = B(w, v)$.
  - *Positivity*: For every vector $v \in V$, we have $B(v, v) \geq 0$. Moreover, if $v \neq 0$, then $B(v, v) > 0$.
- **$\boldsymbol{W}$-coset**: A set of the form $\{v + w \mid w \in W\}$, where W is a subspace V and $v \in V$. *Denoted by* $\boldsymbol{v + W}$.
  - If $v_1 - v_2 \in W$, then $v_1 + W = v_2 + W$.
  - It follows that the distinct W-cosets decompose V into a disjoint collection of subsets of V.
- **Quotient space** (of V by W): The set of distinct W-cosets in V, along with the following definitions of vector addition and scalar multiplication.
  $$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W \qquad \lambda(v + W) = (\lambda v) + W$$
  Denoted by $\boldsymbol{V/W}$.
- **Quotient map**: The linear map $\pi : V \to V/W$ defined by
  $$\pi(v) = v + W$$
  - π is surjective.
  - Note that $\ker(\pi) = W$ since for all $w \in W$, $\pi(w) = w + W = 0 + W$, which is the zero vector in V/W.
- If V, W are finite dimensional, then
  $$\dim(V/W) = \dim(V) - \dim(W)$$
- Proposition 1.2.9: Let $A : V \to U$ be a linear map. If $W \subset \ker(A)$, then there exists a unique linear map $A^\sharp : V/W \to U$ with the property that $A = A^\sharp \circ \pi$, where $\pi : V \to V/W$ is the quotient map.
  - This proposition rephrases in terms of quotient spaces the fact that if $w \in W$, then $A(v+w) = Av$.
- **Dual space** (of V): The set of all linear functions $\ell : V \to \mathbb{R}$, along with the following definitions of vector addition and scalar multiplication.
  $$(\ell_1 + \ell_2)(v) = \ell_1(v) + \ell_2(v) \qquad (\lambda\ell)(v) = \lambda \cdot \ell(v)$$
  Denoted by V^*.
- **Dual basis** (of $e_1, \dots, e_n$ a basis of V): The basis of V^* consisting of the n functions that take every $v = c_1e_1 + \cdots + c_ne_n$ to one of the c_i. *Denoted by* $\boldsymbol{e_1^*, \dots, e_n^*}$. *Given by*
  $$e_i^*(v) = c_i$$
  for all $v \in V$.

- Claim 1.2.12: If V is an n-dimensional vector space with basis $e_1, \dots, e_n$, then $e_1^*, \dots, e_n^*$ is a basis of V^*.

  Proof. We will first prove that $e_1^*, \dots, e_n^*$ spans V^*. Let $\ell \in V^*$ be arbitrary. Set $\lambda_i = \ell(e_i)$ for all $i \in [n]$. Define $\ell' = \sum_{i=1}^n \lambda_i e_i^*$. Then

  $$\ell'(e_j) = \sum_{i=1}^n \lambda_i e_i^*(e_j) = \lambda_j \cdot 1 = \ell(e_j)$$

  for all $j \in [n]$. Therefore, since ℓ, ℓ' take identical values on the basis of V, $\ell = \ell'$, as desired.

  We now prove that $e_1^*, \dots, e_n^*$ is linearly independent. Let $\sum_{i=1}^n \lambda_i e_i^* = 0$. Then for all $j \in [n]$,

  $$\lambda_j = \left(\sum_{i=1}^n \lambda_i e_i^* \right) (e_j) = 0$$

  as desired. □

- **Transpose** (of A): The map from W^* to V^* defined by $\ell \mapsto \ell \circ A$ for all $\ell \in W^*$. *Denoted by* $\boldsymbol{A^*}$.

- Claim 1.2.15: If $e_1, \dots, e_n$ is a basis of V, $f_1, \dots, f_m$ is a basis of W, $e_1^*, \dots, e_n^*$ and $f_1^*, \dots, f_m^*$ are the corresponding dual bases, and $[a_{i,j}]$ is the $m \times n$ matrix of A with respect to $\{e_j\}, \{f_i\}$, then the linear map A^* is defined in terms of $\{f_i^*\}, \{e_j^*\}$ by the transpose matrix $(a_{j,i})$.

  Proof. Let $[c_{j,i}]$ be the $n \times m$ matrix of A^* with respect to $\{f_i^*\}, \{e_j^*\}$. We seek to prove that $a_{i,j} = c_{j,i}$ ($1 \leq i \leq m$, $1 \leq j \leq n$).

  By the definition of $[a_{i,j}]$ and $[c_{j,i}]$, we have that

  $$A^* f_i^* = \sum_{k=1}^n c_{k,i} e_k^* \qquad\qquad A e_j = \sum_{k=1}^m a_{k,j} f_k$$

  It follows that

  $$[A^* f_i^*](e_j) = \left[\sum_{k=1}^n c_{k,i} e_k^* \right] (e_j) = c_{j,i}$$

  and

  $$[A^* f_i^*](e_j) = f_i^*(A e_j) = f_i^* \left(\sum_{k=1}^m a_{k,j} f_k \right) = a_{i,j}$$

  so transitivity implies the desired result. □

4/4:

- $\boldsymbol{V^k}$: The set of all k-tuples $(v_1, \dots, v_k)$ where $v_1, \dots, v_k \in V$ a vector space.
  - Note that

    $$V^k = \underbrace{V \oplus \cdots \oplus V}_{k \text{ times}}$$

    where "$\oplus$" denotes the direct sum.

- **Linear** (function in its i^{th} variable): A function $T : V^k \to \mathbb{R}$ such that the map from V to $\mathbb{R}$ defined by $v \mapsto T(v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_k)$ is linear, where all v_j save v_i are fixed.

- $\boldsymbol{k}$**-linear** (function T): A function $T : V^k \to \mathbb{R}$ that is linear in its i^{th} variable for $i = 1, \dots, k$. *Also known as* $\boldsymbol{k}$**-tensor**.

- $\boldsymbol{\mathcal{L}^k(V)}$: The set of all k-tensors in V.

- Since the sum $T_1 + T_2$ of two k-linear functions $T_1, T_2 : V^k \to \mathbb{R}$ is just another k-linear function, and λT_1 is k-linear for all $\lambda \in \mathbb{R}$, we have that $\mathcal{L}^k(V)$ is a vector space.

- Convention: 0-tensors are just the real numbers. Mathematically, we define

$$\mathcal{L}^0(V) = \mathbb{R}$$

- Note that $\mathcal{L}^1(V) = V^*$.
- Defines multi-indices of n of length k.
- Lemma 1.3.5: If $n, k \in \mathbb{N}$, then there are exactly n^k multi-indices of n of length k.
- $\boldsymbol{T_I}$: The real number $T(e_{i_1}, \dots, e_{i_k})$, where $T \in \mathcal{L}^k(V)$, $e_1, \dots, e_n$ is a basis of V, and I is a multi-index of n of length k.
- Proposition 1.3.7: The real numbers T_I determine T, i.e., if T, T' are k-tensors and $T_I = T'_I$ for all I, then $T = T'$.

Proof. We induct on n. For the base case $n = 1$, $T \in (\mathbb{R}^k)^*$ and we have already proven this result. Now suppose inductively that the assertion is true for $n - 1$. For each e_i, let T_i be the $(k-1)$-tensor defined by

$$(v_1, \dots, v_{n-1}) \mapsto T(v_1, \dots, v_{n-1}, e_i)$$

Then for an arbitrary $v = c_1 e_1 + \cdots + c_n e_n$,

$$T(v_1, \dots, v_{n-1}, v) = \sum_{i=1}^{n} c_i T_i(v_1, \dots, v_{n-1})$$

so the T_i's determine T. Applying the inductive hypothesis completes the proof. ☐

- **Tensor product**: The function $\otimes : \mathcal{L}^k(V) \times \mathcal{L}^\ell(V) \to \mathcal{L}^{k+\ell}(V)$ defined by

$$(T_1 \otimes T_2)(v_1, \dots, v_{k+\ell}) = T_1(v_1, \dots, v_k) T_2(v_{k+1}, \dots, v_{k+\ell})$$

for all $T_1 \in \mathcal{L}^k(V)$ and $T_2 \in \mathcal{L}^\ell(V)$.

- Note that by the definition of 0-tensors as real numbers, if $a \in \mathbb{R}$ and $T \in \mathcal{L}^k(V)$, then

$$a \otimes T = T \otimes a = aT$$

- Proposition 1.3.9: Associativity, distributivity of scalar multiplication, and left and right distributive laws for the tensor product.
- **Decomposable** (k-tensor): A k-tensor T for which there exist $\ell_1, \dots, \ell_k \in V^*$ such that

$$T = \ell_1 \otimes \cdots \otimes \ell_k$$

- Defines e_I^*.
- Theorem 1.3.13: V a vector space with basis $e_1, \dots, e_n$ and $0 \leq k \leq n$ implies the k-tensors e_I^* form a basis of $\mathcal{L}^k(V)$.

Proof. Spanning: Let $T \in \mathcal{L}^k(V)$ be arbitrary. Define

$$T' = \sum_I T_I e_I^*$$

Since

$$T'_J = T'(e_{j_1}, \dots, e_{j_k}) = \sum_I T_I e_I^*(e_{j_1}, \dots, e_{j_k}) = T_J e_J^*(e_{j_1}, \dots, e_{j_k}) = T_J$$

for all J, Proposition 1.3.7 asserts that $T = T'$. Therefore, since every $T_I \in \mathbb{R}$, $T = T' \in \operatorname{span}(e_I^*)$.

Linear independence: Suppose

$$T = \sum_I c_I e_I^* = 0$$

for some set of constants $c_I \in \mathbb{R}$. Then

$$0 = T(e_{j_1}, \dots, e_{j_k}) = \sum_I c_I e_I^*(e_{j_1}, \dots, e_{j_k}) = c_J$$

for all J, as desired. $\square$

- Corollary 1.3.15: If $\dim V = n$, then $\dim(\mathcal{L}^k(V)) = n^k$.

  Proof. Follows immediately from Lemma 1.3.5. $\square$

- **Pullback** (of T by the map A): The k-tensor $A^*T : V^k \to \mathbb{R}$ defined by

  $$(A^*T)(v_1, \dots, v_k) = T(Av_1, \dots, Av_k)$$

  where V, W are finite-dimensional vector spaces, $A : V \to W$ is linear, and $T \in \mathcal{L}^k(W)$.
- Proposition 1.3.18: The map $A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$ defined by $T \mapsto A^*T$ is linear.
- Identities:
  - If $T_1 \in \mathcal{L}^k(W)$ and $T_2 \in \mathcal{L}^m(W)$, then

    $$A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$$

  - If U is a vector space, $B : U \to V$ is linear, and $T \in \mathcal{L}^k(W)$, then $(AB)^*T = B^*(A^*T)$. Hence,

    $$(AB)^* = B^*A^*$$

4/13:

- $\boldsymbol{\Sigma_k}$: The set containing the natural numbers 1 through k. *Given by*

  $$\Sigma_k = \{1, 2, \dots, k\}$$

- **Permutation of order $\boldsymbol{k}$**: A bijection on Σ_k. *Denoted by* $\boldsymbol{\sigma}$.
- **Product** (of σ_1, σ_2): The composition $\sigma_1 \circ \sigma_2$, i.e., the map

  $$i \mapsto \sigma_1(\sigma_2(i))$$

  Denoted by $\boldsymbol{\sigma_1\sigma_2}$.
- **Inverse** (of σ): The permutation of order k which is the inverse bijection of σ. *Denoted by* $\boldsymbol{\sigma^{-1}}$.
- **Permutation group** (of Σ_k): The set of all permutations of order k. *Also known as* **symmetric group on $\boldsymbol{k}$ letters**. *Denoted by* $\boldsymbol{S_k}$.
- Lemma 1.4.2: The group S_k has $k!$ elements.

- **Transposition**: A permutation of order k defined by

$$\ell \mapsto \begin{cases} j & \ell = i \\ i & \ell = j \\ \ell & \ell \neq i,j \end{cases}$$

for all $\ell \in \Sigma_k$, where $i,j \in \Sigma_k$. *Denoted by* $\boldsymbol{\tau_{i,j}}$.

- **Elementary transposition**: A transposition of the form $\pi_{i,i+1}$.
- Theorem 1.4.4: Every $\sigma \in S_k$ can be written as a product of (a finite number of) transpositions.

Proof. We induct on k.

For the base case $k = 2$, the identity permutation of S_2 is the "product" of zero transpositions, and the only other permutation is a transposition (the "product" of one transposition, namely itself).

Now suppose inductively that we have proven the claim for $k-1$. Let $\sigma \in S_k$ be arbitrary. Suppose $\sigma(k) = i$. Then $\tau_{i,k}\sigma(k) = k$. Since $(\tau_{i,k}\sigma)|_{\Sigma_{k-1}} \in S_{k-1}$, we have by the inductive hypothesis that $(\tau_{i,k}\sigma)|_{\Sigma_{k-1}} = \tau_1 \cdots \tau_m$ for some set of permutations $\tau_1, \dots, \tau_m \in S_{k-1}$. For each τ_j $(1 \leq j \leq m)$, define $\tau_j' \in S_k$

$$\tau_j'(\ell) = \begin{cases} \tau_j(\ell) & \ell < k \\ \ell & \ell = k \end{cases}$$

It follows that

$$\begin{aligned} \tau_{i,k}\sigma &= \tau_1' \cdots \tau_m' \\ \sigma &= \tau_{i,k}\tau_1' \cdots \tau_m' \end{aligned}$$

as desired. □

- Theorem 1.4.5: Every transposition can be written as a product of elementary transpositions.

Proof. Let $\tau_{i,j} \in S_k$, and let $i < j$ WLOG. Then we have that

$$\tau_{i,j} = \prod_{\ell=i}^{i-1} \tau_{\ell,\ell+1}$$

as desired. □

- Corollary 1.4.6: Every permutation can be written as a product of elementary transpositions.
- **Sign** (of σ): The number ± 1 assigned to σ by the expression

$$\prod_{i<j} \frac{x_{\sigma(i)} - x_{\sigma(j)}}{x_i - x_j}$$

where $x_1, \dots, x_k$ are coordinate functions on $\mathbb{R}^k$. *Denoted by* $\boldsymbol{(-1)^\sigma}$.

- Claim 1.4.9: The sign defines a group homomorphism $S_k \to \{\pm 1\}$. That is, for $\sigma_1, \sigma_2 \in S_k$, we have

$$(-1)^{\sigma_1\sigma_2} = (-1)^{\sigma_1}(-1)^{\sigma_2}$$

Proof. For all $i < j$, define p, q such that p is the lesser of $\sigma_2(i), \sigma_2(j)$ and q is the greater of $\sigma_2(i), \sigma_2(j)$. Formally,

$$p = \begin{cases} \sigma_2(i) & \sigma_2(i) < \sigma_2(j) \\ \sigma_2(j) & \sigma_2(j) < \sigma_2(i) \end{cases} \qquad q = \begin{cases} \sigma_2(j) & \sigma_2(i) < \sigma_2(j) \\ \sigma_2(i) & \sigma_2(j) < \sigma_2(i) \end{cases}$$

It follows that if $\sigma_2(i) < \sigma_2(j)$, then

$$\frac{x_{\sigma_1\sigma_2(i)} - x_{\sigma_1\sigma_2(j)}}{x_{\sigma_2(i)} - x_{\sigma_2(j)}} = \frac{x_{\sigma_1(p)} - x_{\sigma_1(q)}}{x_p - x_q}$$

and if $\sigma_2(j) < \sigma_2(i)$, then

$$\frac{x_{\sigma_1\sigma_2(i)} - x_{\sigma_1\sigma_2(j)}}{x_{\sigma_2(i)} - x_{\sigma_2(j)}} = \frac{x_{\sigma_1(q)} - x_{\sigma_1(p)}}{x_q - x_p} = \frac{x_{\sigma_1(p)} - x_{\sigma_1(q)}}{x_p - x_q}$$

Therefore,

$$\begin{aligned}
(-1)^{\sigma_1\sigma_2} &= \prod_{i<j} \frac{x_{\sigma_1\sigma_2(i)} - x_{\sigma_1\sigma_2(j)}}{x_i - x_j} \\
&= \prod_{i<j} \frac{x_{\sigma_1\sigma_2(i)} - x_{\sigma_1\sigma_2(j)}}{x_{\sigma_2(i)} - x_{\sigma_2(j)}} \cdot \frac{x_{\sigma_2(i)} - x_{\sigma_2(j)}}{x_i - x_j} \\
&= \prod_{i<j} \frac{x_{\sigma_1(p)} - x_{\sigma_1(q)}}{x_p - x_q} \cdot \prod_{i<j} \frac{x_{\sigma_2(i)} - x_{\sigma_2(j)}}{x_i - x_j} \\
&= (-1)^{\sigma_1}(-1)^{\sigma_2}
\end{aligned}$$

as desired. $\square$

- Proposition 1.4.11: If σ is the product of an odd number of transpositions, then $(-1)^\sigma = -1$, and if σ is the product of an even number of transpositions, then $(-1)^\sigma = +1$.

  Proof. Follows from the fact that $(-1)^\sigma = -1$ (see Exercise 1.4.ii). $\square$

- $\boldsymbol{T^\sigma}$: The k-tensor defined by

  $$T^\sigma(v_1, \dots, v_k) = T(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)})$$

  where $T \in \mathcal{L}^k(V)$, V is an n-dimensional vector space, and $\sigma \in S_k$.

- Proposition 1.4.14:

  1. If $T = \ell_1 \otimes \cdots \otimes \ell_k$ ($\ell_i \in V^*$), then $T^\sigma = \ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}$.

     Proof. If $v_1, \dots, v_k \in V$, then

     $$\begin{aligned}
     T^\sigma(v_1, \dots, v_k) &= T(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)}) \\
     &= [\ell_1 \otimes \cdots \otimes \ell_k](v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)}) \\
     &= \ell_1(v_{\sigma^{-1}(1)}) \cdots \ell_k(v_{\sigma^{-1}(k)}) \\
     &= \ell_{\sigma(1)}(v_1) \cdots \ell_{\sigma(k)}(v_2) \\
     &= [\ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}](v_1, \dots, v_k)
     \end{aligned}$$

     as desired. Note that we can justify the fourth equality by nothing that if $\sigma^{-1}(i) = q$, then the i^{th} term in the product is $\ell_{\sigma(q)}(v_q)$, so since σ is a bijection, the product can be arranged to the form on the right-hand side of equality four. $\square$

2. The assignment $T \mapsto T^\sigma$ is a linear map from $\mathcal{L}^k(V) \to \mathcal{L}^k(V)$.

   Proof. See Exercise 1.4.iii. $\square$

3. If $\sigma_1, \sigma_2 \in S_k$, we have $T^{\sigma_1\sigma_2} = (T^{\sigma_1})^{\sigma_2}$.

   Proof. Let $T = \ell_1 \otimes \cdots \otimes \ell_k$[5]. Then

$$T^{\sigma_1} = \ell_{\sigma_1(1)} \otimes \cdots \otimes \ell_{\sigma_1(k)} = \ell_1' \otimes \cdots \otimes \ell_k'$$

   and thus

$$(T^{\sigma_1})^{\sigma_2} = \ell'_{\sigma_2(1)} \otimes \cdots \otimes \ell'_{\sigma_2(k)}$$

   Let $\sigma_2(i) = j$. Then since $\ell'_p = \ell_{\sigma_1(p)}$ by definition, we have that $\ell'_{\sigma_2(j)} = \ell_{\sigma_1(\sigma_2(j))}$. Therefore,

$$\begin{aligned}
(T^{\sigma_1})^{\sigma_2} &= \ell'_{\sigma_2(1)} \otimes \cdots \otimes \ell'_{\sigma_2(k)}\\
&= \ell_{\sigma_1(\sigma_2(1))} \otimes \cdots \otimes \ell_{\sigma_1(\sigma_2(k))}\\
&= \ell_{\sigma_1\sigma_2(1)} \otimes \cdots \otimes \ell_{\sigma_1\sigma_2(k)}\\
&= T^{\sigma_1\sigma_2}
\end{aligned}$$

   as desired. $\square$

- **Alternating** (k-tensor): A k-tensor $T \in \mathcal{L}^k(V)$ such that $T^\sigma = (-1)^\sigma T$ for all $\sigma \in S_k$.
- $\boldsymbol{\mathcal{A}^k(V)}$: The set of all alternating k-tensors in $\mathcal{L}^k(V)$.
  - Proposition 1.4.14(2) implies that $(T_1 + T_2)^\sigma = T_1^\sigma + T_2^\sigma$ and $(\lambda T_1)^\sigma = \lambda T_1^\sigma$; it follows that $\mathcal{A}^k(V)$ is a vector space.
- **Alternation operation**: The function from $\mathcal{L}^k(V) \to \mathcal{L}^k(V)$ defined by

$$T \mapsto \sum_{\tau \in S_k} (-1)^\tau T^\tau$$

  Denoted by **Alt**.
- Proposition 1.4.17: For $T \in \mathcal{L}^k(V)$ and $\sigma \in S_k$, we have that

1. $\operatorname{Alt}(T)^\sigma = (-1)^\sigma \operatorname{Alt} T$.

   Proof. We have that

$$\begin{aligned}
\operatorname{Alt}(T)^\sigma &= \left(\sum_{\tau \in S_k} (-1)^\tau T^\tau\right)^\sigma\\
&= \sum_{\tau \in S_k} (-1)^\tau (T^\tau)^\sigma && \text{Proposition 1.4.14(2)}\\
&= \sum_{\tau \in S_k} (-1)^\tau T^{\tau\sigma} && \text{Proposition 1.4.14(3)}\\
&= (-1)^\sigma \sum_{\tau \in S_k} (-1)^{\tau\sigma} T^{\tau\sigma}\\
&= (-1)^\sigma \sum_{\tau\sigma \in S_k} (-1)^{\tau\sigma} T^{\tau\sigma}\\
&= (-1)^\sigma \operatorname{Alt} T
\end{aligned}$$

   as desired. $\square$

[5] What gives us the right to assume T is decomposable?

2. If $T \in \mathcal{A}^k(V)$, then $\operatorname{Alt} T = k!T$.

   Proof. Since $T \in \mathcal{A}^k(V)$, we know that $T^\sigma = (-1)^\sigma T$. Therefore,

   $$\operatorname{Alt} T = \sum_{\tau \in S_k} (-1)^\tau T^\tau = \sum_{\tau \in S_k} (-1)^\tau (-1)^\tau T = \sum_{\tau \in S_k} T = k!T$$

   where the last equality holds because the cardinality of S_k is $k!$. ☐

3. $\operatorname{Alt}(T^\sigma) = \operatorname{Alt}(T)^\sigma$.

   Proof. We have that

   $$\operatorname{Alt}(T^\sigma) = \sum_{\tau \in S_k} (-1)^\tau T^{\tau\sigma} = (-1)^\sigma \sum_{\tau \in S_k} (-1)^{\tau\sigma} T^{\tau\sigma} = (-1)^\sigma \operatorname{Alt}(T) = \operatorname{Alt}(T)^\sigma$$

   as desired. ☐

4. The alternation operation is linear.

   Proof. Follows by Proposition 1.4.14. ☐

- **Repeating** (multi-index I): A multi-index I of length k such that $i_r = i_s$ for some $r \neq s$.
- **Strictly increasing** (multi-index I): A multi-index I of length k such that $i_1 < i_2 < \cdots < i_r$.
- $\boldsymbol{I^\sigma}$: The multi-index of length k defined by

  $$I^\sigma = (i_{\sigma(1)}, \dots, i_{\sigma(k)})$$

- If I is non-repeating, there is a unique $\sigma \in S_k$ such that I^σ is strictly increasing.
- $\boldsymbol{\psi_I}$: The following k-tensor. *Given by*

  $$\psi_I = \operatorname{Alt}(e_I^*)$$

- Proposition 1.4.20:

1. $\psi_{I^\sigma} = (-1)^\sigma \psi_I$.

   Proof. We have that

   $$\psi_{I^\sigma} = \operatorname{Alt}(e_{I^\sigma}^*) = \operatorname{Alt}[(e_I^*)^\sigma] = \operatorname{Alt}(e_I^*)^\sigma = (-1)^\sigma \operatorname{Alt}(e_I^*) = (-1)^\sigma \psi_I$$

   as desired. ☐

2. If I is repeating, then $\psi_I = 0$.

   Proof. Suppose $I = (i_1, \dots, i_k)$ is such that $i_r = i_s$ for some distinct $r, s \in \Sigma_k$. Then $e_I^* = e_{I^{\tau_{i_r,i_s}}}^*$, so

   $$\psi_I = \psi_{I^{\tau_{i_r,i_s}}} = (-1)^{\tau_{i_r,i_s}} \psi_I = -\psi_I$$

   Therefore, we must have $\psi_I = 0$, as desired. ☐

3. If I and J are strictly increasing, then

   $$\psi_I(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$$

Proof. We have by definition that

$$\psi_I(e_{j_1},\dots,e_{j_k}) = \sum_\tau (-1)^\tau e_{I^\tau}^*(e_{j_1},\dots,e_{j_k})$$

This combined with the facts that

$$e_{I^\tau}^*(e_{j_1},\dots,e_{j_k}) = \begin{cases} 1 & I^\tau = J \\ 0 & I^\tau \neq J \end{cases}$$

I^τ is strictly increasing iff $I^\tau = I$, and the above equation is nonzero iff $I^\tau = I = J$ implies the desired result. ☐

- Conclusion 1.4.22: If $T \in \mathcal{A}^k(V)$, then we can write T as a sum

$$T = \sum_I c_I \psi_I$$

with I's strictly increasing.

Proof. Let $T \in \mathcal{A}^k(V)$ be arbitrary. By Theorem 1.3.13,

$$T = \sum_J a_J e_J^*$$

for some set of $a_J \in \mathbb{R}$. It follows since $\text{Alt}(T) = k!T$ that

$$T = \frac{1}{k!} \sum a_J \,\text{Alt}(e_J^*) = \sum b_J \psi_J$$

We can disregard all repeating terms in the sum since they are zero by Proposition 1.4.20(2); for every non-repeating term J, we can write $J = I^\sigma$, where I is strictly increasing and hence $\psi_J = (-1)^\sigma \psi_I$. ☐

- Claim 1.4.24: The c_I's of Conclusion 1.4.22 are unique.

Proof. For J strictly increasing, we have

$$T_J = T(e_{j_1},\dots,e_{j_k}) = \sum_I c_I \psi_I(e_{j_1},\dots,e_{j_k}) = c_J$$

☐

- Proposition 1.4.26: The alternating tensors ψ_I with I strictly increasing are a basis for $\mathcal{A}^k(V)$.

Proof. Spanning: See Conclusion 1.4.22.

Linear independence: See Claim 1.4.24. ☐

- We have that

$$\dim \mathcal{A}^k(V) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

  - Hint in proving this claim: "Show that every strictly increasing multi-index of length k determines a k-element subset of $\{1,\dots,n\}$ and vice versa." (Guillemin & Haine, 2018, p. 16).
  - Note also that if $k > n$, every multi-index has a repeat somewhere, meaning that $\dim \mathcal{A}^k(V) = \binom{n}{k} = 0$.

References

Guillemin, V., & Haine, P. J. (2018). *Differential forms* [https://math.mit.edu/classes/18.952/2018SP/files/18.952_book.pdf].