2 Differential Forms

From Guillemin and Haine (2018).

Chapter 2

4/29: **2.1.i.** Let U be an open subset of $\mathbb{R}^n$. If $f : U \to \mathbb{R}$ is a C^∞ function, then

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \, \mathrm{d}x_i$$

2.1.ii. Let U be an open subset of $\mathbb{R}^n$, $\boldsymbol{v}$ a vector field on U, and $f_1, f_2 \in C^1(U)$. Then

$$L_{\boldsymbol{v}}(f_1 \cdot f_2) = L_{\boldsymbol{v}}(f_1) \cdot f_2 + f_1 \cdot L_{\boldsymbol{v}}(f_2)$$

2.1.iii. Let U be an open subset of $\mathbb{R}^n$ and $\boldsymbol{v}_1, \boldsymbol{v}_2$ vector fields on U. Show that there is a unique vector field $\boldsymbol{w}$ on U with the property

$$L_{\boldsymbol{w}}\phi = L_{\boldsymbol{v}_1}(L_{\boldsymbol{v}_2}\phi) - L_{\boldsymbol{v}_2}(L_{\boldsymbol{v}_1}\phi)$$

for all $\phi \in C^\infty(U)$.

2.1.iv. The vector field $\boldsymbol{w}$ in Exercise 2.1.iii is called the **Lie bracket** of the vector fields $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ and is denoted by $[\boldsymbol{v}_1, \boldsymbol{v}_2]$. Verify that the Lie bracket is **skew-symmetric**, i.e.,

$$[\boldsymbol{v}_1, \boldsymbol{v}_2] = -[\boldsymbol{v}_2, \boldsymbol{v}_1]$$

and satisfies the **Jacobi identity**

$$[\boldsymbol{v}_1, [\boldsymbol{v}_2, \boldsymbol{v}_3]] + [\boldsymbol{v}_2, [\boldsymbol{v}_3, \boldsymbol{v}_1]] + [\boldsymbol{v}_3, [\boldsymbol{v}_1, \boldsymbol{v}_2]] = 0$$

Thus, the Lie bracket defines the structure of a **Lie algebra**. (Hint: Prove analogous identities for $L_{\boldsymbol{v}_1}$, $L_{\boldsymbol{v}_2}$, and $L_{\boldsymbol{v}_3}$.)

2.1.vii. Let U be an open subset of $\mathbb{R}^n$, and let $\gamma : [a, b] \to U$, $t \mapsto (\gamma_1(t), \dots, \gamma_n(t))$ be a C^1 curve. Given a C^∞ one-form $\omega = \sum_{i=1}^n f_i \, \mathrm{d}x_i$ on U, define the **line integral** of ω over γ to be the integral

$$\int_\gamma \omega = \sum_{i=1}^{n} \int_a^b f_i(\gamma(t)) \frac{\mathrm{d}\gamma_i}{\mathrm{d}t} \, \mathrm{d}t$$

Show that if $\omega = \mathrm{d}f$ for some $f \in C^\infty(U)$,

$$\int_\gamma \omega = f(\gamma(b)) - f(\gamma(a))$$

In particular, conclude that if γ is a closed curve, i.e., $\gamma(a) = \gamma(b)$, this integral is zero.

2.1.viii. Let ω be the C^∞ one-form on $\mathbb{R}^2 \setminus \{0\}$ defined by

$$\omega = \frac{x_1 \, \mathrm{d}x_2 - x_2 \, \mathrm{d}x_1}{x_1^2 + x_2^2}$$

and let $\gamma : [0, 2\pi] \to \mathbb{R}^2 \setminus \{0\}$ be the closed curve $t \mapsto (\cos t, \sin t)$. Compute the line integral $\int_\gamma \omega$ and note that $\int_\gamma \omega \neq 0$. Conclude that ω is not of the form $\mathrm{d}f$ for $f \in C^\infty(\mathbb{R}^2 \setminus \{0\})$.

2.2.i. For $i = 1, 2$, let U_i be an open subset of $\mathbb{R}^{n_i}$, $\boldsymbol{v}_i$ a vector field on U_i, and $f : U_1 \to U_2$ a C^∞-map. If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are f-related, every integral curve $\gamma : I \to U_1$ of $\boldsymbol{v}_1$ gets mapped by f onto an integral curve $f \circ \gamma : I \to U_2$ of $\boldsymbol{v}_2$.

2.2.ii. Let U, V be open subsets of $\mathbb{R}^n$ and $f : U \to V$ an C^k map.

(1) Show that for $\phi \in C^\infty(V)$, the pullback can be rewritten

$$f^* \,\mathrm{d}\phi = \mathrm{d}f^*\phi$$

(2) Let μ be the one-form

$$\mu = \sum_{i=1}^{m} \phi_i \,\mathrm{d}x_i$$

on V for all $\phi_i \in C^\infty(V)$. Show that if $f = (f_1, \ldots, f_m)$, then

$$f^*\mu = \sum_{i=1}^{m} f^*\phi_i \,\mathrm{d}f_i$$

(3) Show that if μ is C^∞ and f is C^∞, $f^*\mu$ is C^∞.

2.2.iv. (1) Let $U = \mathbb{R}^2$ and let $\boldsymbol{v}$ be the vector field $x_1 \,\partial/\partial x_2 - x_2 \,\partial/\partial x_1$. Show that the curve

$$t \mapsto (r\cos(t+\theta), r\sin(t+\theta))$$

for $t \in \mathbb{R}$ is the unique integral curve of $\boldsymbol{v}$ passing through the point $(r\cos\theta, r\sin\theta)$ at $t = 0$.

(2) Let $U = \mathbb{R}^n$ and let $\boldsymbol{v}$ be the constant vector field $\sum_{i=1}^{n} c_i \,\partial/\partial x_i$. Show that the curve

$$t \mapsto a + t(c_1, \ldots, c_n)$$

for $t \in \mathbb{R}$ is the unique integral curve of $\boldsymbol{v}$ passing through $a \in \mathbb{R}^n$ at $t = 0$.

(3) Let $U = \mathbb{R}^n$ and let $\boldsymbol{v}$ be the vector field $\sum_{i=1}^{n} x_i \,\partial/\partial x_i$. Show that the curve

$$t \mapsto \mathrm{e}^t(a_1, \ldots, a_n)$$

for $t \in \mathbb{R}$ is the unique integral curve of $\boldsymbol{v}$ passing through a at $t = 0$.

2.2.viii. Let $\boldsymbol{v}$ be the vector field on $\mathbb{R}$ given by $x^2 \,\mathrm{d}/\mathrm{d}x$. Show that the curve

$$x(t) = \frac{a}{a - at}$$

is an integral curve of $\boldsymbol{v}$ with initial point $x(0) = a$. Conclude that for $a > 0$, the curve

$$x(t) = \frac{a}{1 - at}$$

on $0 < t < 1/a$ is a maximal integral curve. (In particular, conclude that $\boldsymbol{v}$ is not complete.)

2.3.i. Let $\omega \in \Omega^2(\mathbb{R}^4)$ be the 2-form $\mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_3 \wedge \mathrm{d}x_4$. Compute $\omega \wedge \omega$.

2.3.ii. Let $\omega_1, \omega_2, \omega_3 \in \Omega^1(\mathbb{R}^3)$ be the 1-forms

$$\begin{aligned}
\omega_1 &= x_2 \,\mathrm{d}x_3 - x_3 \,\mathrm{d}x_2 \\
\omega_2 &= x_3 \,\mathrm{d}x_1 - x_1 \,\mathrm{d}x_3 \\
\omega_3 &= x_1 \,\mathrm{d}x_2 - x_2 \,\mathrm{d}x_1
\end{aligned}$$

Compute the following.

(1) $\omega_1 \wedge \omega_2$.

(2) $\omega_2 \wedge \omega_3$.

(3) $\omega_3 \wedge \omega_1$.

(4) $\omega_1 \wedge \omega_2 \wedge \omega_3$.

2.3.iii. Let U be an open subset of $\mathbb{R}^n$ and $f_1, \dots, f_n \in C^\infty(U)$. Show that

$$\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n = \det \left[\frac{\partial f_i}{\partial x_j} \right] \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

2.3.iv. Let U be an open subset of $\mathbb{R}^n$. Show that every $(n-1)$-form $\omega \in \Omega^{n-1}(U)$ can be written uniquely as a sum

$$\sum_{i=1}^{n} f_i \, \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n$$

where $f_i \in C^\infty(U)$ and $\widehat{\mathrm{d}x_i}$ indicates that $\mathrm{d}x_i$ is to be omitted from the wedge product $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$.

2.3.v. Let $\mu = \sum_{i=1}^{n} x_i \, \mathrm{d}x_i$. Show that there exists an $(n-1)$-form $\omega \in \Omega^{n-1}(\mathbb{R}^n \setminus \{0\})$ with the property

$$\mu \wedge \omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

2.3.vi. Let J be the multi-index $(j_1, \dots, j_k)$ and let $\mathrm{d}x_J = \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_k}$. Show that $\mathrm{d}x_J = 0$ if $j_r = j_s$ for some $r \neq s$ and show that if the numbers $j_1, \dots, j_k$ are all distinct, then

$$\mathrm{d}x_J = (-1)^\sigma \, \mathrm{d}x_I$$

where $I = (i_1, \dots, i_k)$ is the strictly increasing rearrangement of $(j_1, \dots, j_k)$ and σ is the permutation

$$(j_1, \dots, j_k) \mapsto (i_1, \dots, i_k)$$