Week 4

Differential Forms

4.1 Overview of Differential Forms

4/18:

- Office Hours on Wednesday, 4:00-5:00 PM.
- Plan:
  - An impressionistic overview of what (differential) forms do/are.
  - Tangent spaces.
  - Vector fields/integral curves.
  - 1-forms; a warm-up to k-forms.
- Impressionistic overview of the rest of Guillemin and Haine (2018).
  - An open subset $U \subset \mathbb{R}^n$; $n = 2$ and $n = 3$ are nice.
  - Sometimes, we'll have some functions $F : U \to V$; this is where pullbacks come into play.
  - At every point $p \in U$, we'll define a vector space (the tangent space $T_p\mathbb{R}^n$). Associated to that vector space you get our whole slew of associated spaces (the dual space $T_p^*\mathbb{R}^n$, and all of the higher exterior powers $\Lambda^k(T_p^*\mathbb{R}^n)$).
  - We let $\omega \in \Omega^k(U)$ be a k-form in the space of k-forms.
  - ω assigns (smoothly) to every point $p \in U$ an element of $\Lambda^k(T_p^*\mathbb{R}^n)$.
  - Question: What really is a k-form?
    - Answer: Something that can be integrated on k-dimensional subsets.
    - If $k = 1$, i.e., $\omega \in \Omega^1(U)$, then U can be integrated over curves.
  - If we take $k = 0$, then $\Omega^0(U) = C^\infty(U)$, i.e., the set of all smooth functions $f : U \to \mathbb{R}$.
    - Guillemin and Haine (2018) doesn't, but Klug will and we should distinguish between functions $F : U \to V$ and $f : U \to \mathbb{R}$.
  - We will soon construct a map $\mathrm{d} : \Omega^0(U) \to \Omega^1(U)$ (the **exterior derivative**) that is rather like the gradient but not quite.
    - d is linear.
    - Maps from vector spaces are heretofore assumed to be linear unless stated otherwise.
  - The 1-forms in im(d) are special: $\int_\gamma \mathrm{d}f = f(\gamma(b)) - f(\gamma(a))$ only depends on the endpoints of $\gamma : [a, b] \to U$! The integral is *path-independent*.
  - A generalization of this fact is that instead of integrating along the surface M, we can integrate along the boundary curve:
    $$\int_M \mathrm{d}\omega = \int_{\partial M} \omega$$
    This is **Stokes' theorem**.

  - M is a k-dimensional subset of $U \subset \mathbb{R}^n$.
- Note that we have all manner of functions d that we could differentiate between (because they are functions) but nobody does.

$$0 \to \Omega^0(U) \xrightarrow{\mathrm{d}} \Omega^1(U) \xrightarrow{\mathrm{d}} \Omega^2(U) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n(U) \xrightarrow{\mathrm{d}} 0$$

- Theorem: $\mathrm{d}^2 = \mathrm{d} \circ \mathrm{d} = 0$.
  - Corollary: $\operatorname{im}(\mathrm{d}^{n-1}) \subset \ker(\mathrm{d}^n)$.
- We'll define $H^k_{dR}(U) = \ker(\mathrm{d}) / \operatorname{im}(\mathrm{d})$.
  - These will be finite dimensional, even though all the individual vector spaces will be infinite dimensional.
  - These will tell us about the shape of U; basically, if all of these equal zero, U is simply connected. If some are nonzero, U has some holes.
- For small values of n and k, this d will have some nice geometric interpretations (div, grad, curl, n'at).
- We'll have additional operations on forms such as the wedge product.

- **Tangent space** (of p): The following set. *Denoted by* $\boldsymbol{T_p\mathbb{R}^n}$. *Given by*

$$T_p\mathbb{R}^n = \{(p, v) : v \in \mathbb{R}^n\}$$

  - This is naturally a vector space with addition and scalar multiplication defined as follows.

$$(p, v_1) + (p, v_2) = (p, v_1 + v_2) \qquad\qquad \lambda(p, v) = (p, \lambda v)$$

  - The point is that

$$T_p\mathbb{R}^n \neq T_q\mathbb{R}^n$$

    for $p \neq q$ even though the spaces are isomorphic.
    - When in $\mathbb{R}^n$ alone, it may seem silly to define what is essentially just $\mathbb{R}^n$ again. After all, in $\mathbb{R}^n$, $(p, v) \in T_p\mathbb{R}^n$ and $(q, v) \in T_q(\mathbb{R}^n)$ both point in the same direction and are basically identical.
    - However, when we get to manifolds (see Figure 8.4), isomorphic tangent spaces may not have vectors that point in the same direction in the space *containing* the manifold!
  - Aside: $F : U \to V$ differentiable and $p \in U$ induce a map $\mathrm{d}F_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ called the "derivative at p."
    - We will see that the matrix of this map is the Jacobian.
  - Chain rule: If $U \xrightarrow{F} V \xrightarrow{G} W$, then

$$\mathrm{d}(G \circ F)_p = \mathrm{d}G_{F(p)} \circ \mathrm{d}F_p$$

- This is round 1 of our discussion on tangent spaces.
- Round 2, later on, will be submanifolds such as T_pM: The tangent space to a point p of a manifold M.
- **Vector field** (on U): A function that assigns to each $p \in U$ an element of $T_p\mathbb{R}^n$.
  - A constant vector field would be $p \mapsto (p, v)$, visualized as a field of vectors at every p all pointing the same direction. For example, we could take $v = (1, 1)$.
  - Special case: $v = e_1, e_2, \dots, e_n$. Here we use the notation $\partial/\partial x_i$ to denote the vector field with $v = e_i$.
  - Example: $n = 2$, $U = \mathbb{R}^2 \setminus \{(0, 0)\}$. We could take a vector field that spins us around in circles.

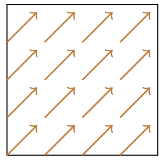


Figure 4.1: The constant vector field $v = (1,1)$.

- Notice that for all p, $\partial/\partial x_1 \mid_p, \dots, \partial/\partial x_n \mid_p \in T_p\mathbb{R}^n$ are a basis of $T_p\mathbb{R}^n$.
  - Thus, any vector field $\boldsymbol{v}$ on U can be written uniquely as

$$\boldsymbol{v} = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}$$

where the $f_1, \dots, f_n$ are functions $f_i : U \to \mathbb{R}$.

4.2 The Lie Derivative and 1-Forms

4/20: • Plan:
  - Vector fields and their integral curves.
  - Lie derivatives.
  - 1-forms and k-forms.
  - $\Omega^0(U) \xrightarrow{\mathrm{d}} \Omega^1(U)$.
- Notation.
  - $U \subset \mathbb{R}^n$.
  - $\boldsymbol{v}$ denotes a vector field on U.
    - Note that the set of all vector fields on U constitute the vector space $\mathfrak{X}(U)$.
  - $\boldsymbol{v}_p \in T_p\mathbb{R}^n$.
  - $\omega_p \in \Lambda^k(T_p^*\mathbb{R}^n)$.
  - $\partial/\partial x_i \mid_p = (p, e_i) \in T_p\mathbb{R}^n$.
- Recall that any vector field $\boldsymbol{v}$ on U can be written uniquely as

$$\boldsymbol{v} = g_1 \frac{\partial}{\partial x_1} + \cdots + g_n \frac{\partial}{\partial x_n}$$

where the $g_i : U \to \mathbb{R}$.
- **Smooth** (vector field): A vector field $\boldsymbol{v}$ for which all g_i are smooth.
- From now on, we assume unless stated otherwise that all vector fields are smooth.
- **Lie derivative** (of f wrt. $\boldsymbol{v}$): The function $L_{\boldsymbol{v}} f : U \to \mathbb{R}$ defined by $p \mapsto D_{\boldsymbol{v}_p}(f)(p)$, where $\boldsymbol{v}$ is a vector field on U and $f : U \to \mathbb{R}$ (always smooth).
  - Recall that $D_{\boldsymbol{v}_p}(f)(p)$ denotes the directional derivative of f in the direction $\boldsymbol{v}_p$[1] at p.
  - As some examples, we have

$$L_{\partial/\partial x_i} f = \frac{\partial f}{\partial x_i} \qquad L_{(g_1 \frac{\partial}{\partial x_1} + \cdots + g_n \frac{\partial}{\partial x_n})} f = g_1 \frac{\partial f}{\partial x_1} + \cdots + g_n \frac{\partial f}{\partial x_n}$$

[1]Note that by "in the direction $\boldsymbol{v}_p$," we mean in the direction v where $\boldsymbol{v}_p = (p, v)$.

- Property.
  1. Product rule: $L_{\boldsymbol{v}}(f_1 f_2) = (L_{\boldsymbol{v}} f_1) f_2 + f_1 (L_{\boldsymbol{v}} f_2)$.
- Later: Geometric meaning to the expression $L_{\boldsymbol{v}} f = 0$.
  - Satisfied iff f is constant on the integral curves of $\boldsymbol{v}$. As if f "flows along" the vector field.
- We define $T_p^*\mathbb{R}^n = (T_p\mathbb{R}^n)^*$.
- 1-forms:
  - A (differential) 1-form on $U \subset \mathbb{R}^n$ is a function $\omega : p \mapsto \omega_p \in T_p^*\mathbb{R}^n$.
  - A "co-vector field."
- Notation: $\mathrm{d}x_i$ is the 1-form that at p is the functional defined by

$$(p, v) \mapsto e_i^*(v)$$

  - For example, if $U = \mathbb{R}^2$ and $\omega = \mathrm{d}x_1$, then ω_p returns (as a scalar) the x_1-component of any vector v fed to it as a $(p, v) \in T_pU$ pair.
- Note: Given any 1-form ω on U, we can write ω uniquely as

$$\omega = g_1 \, \mathrm{d}x_1 + \cdots + g_n \, \mathrm{d}x_n$$

  for some set of smooth $g_i : U \to \mathbb{R}$.
- Notation:
  - $\Omega^1(U)$ is the set of all smooth 1-forms.
  - Notice that $\Omega^1(U)$ is a vector space.
- Given $\omega \in \Omega^1(U)$ and a vector field $\boldsymbol{v}$ on U, we can define $\omega(\boldsymbol{v}) : U \to \mathbb{R}$ by $p \mapsto \omega_p(\boldsymbol{v}_p)$.
- If $U = \mathbb{R}^2$, we have that

$$\mathrm{d}x\left(\frac{\partial}{\partial x}\right) = 1 \qquad\qquad \mathrm{d}x\left(\frac{\partial}{\partial y}\right) = 0$$

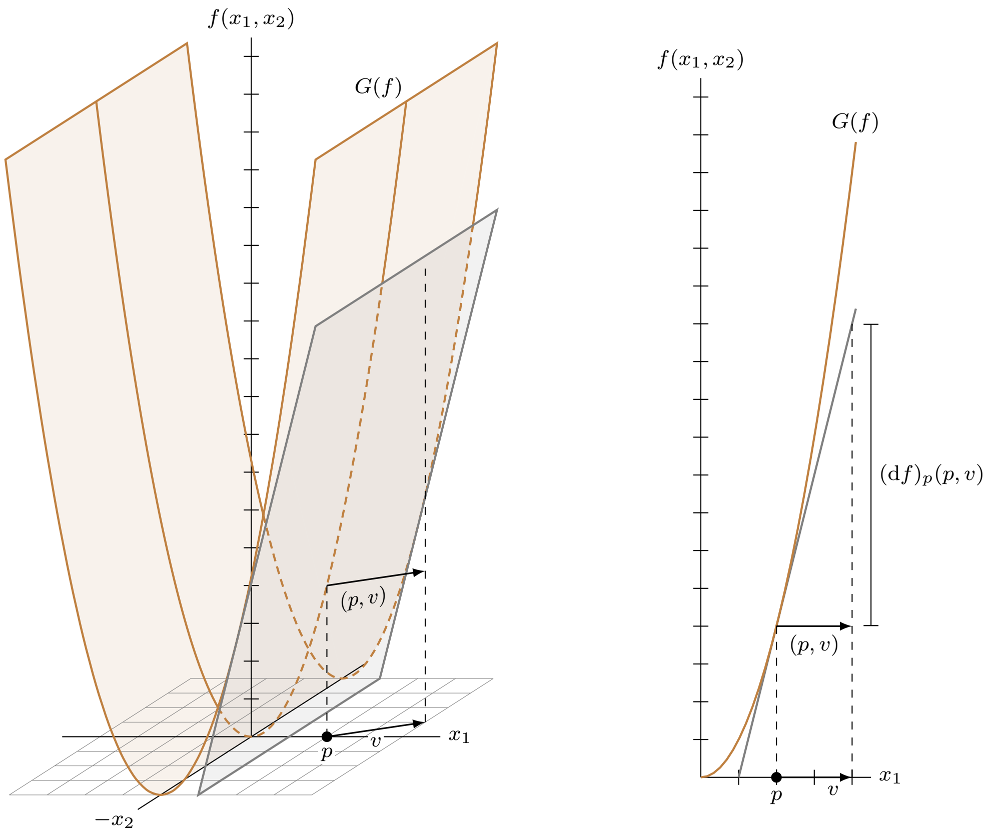
  - Note that in the above equation, 1 represents the identity function on U and 0 represents the zero function on $\mathbb{R}^2$.
- $\mathrm{d}x_1, \dots, \mathrm{d}x_n$ are not a basis for $\Omega^1(U)$ since the latter is infinite dimensional.
  - In fact, at each point $p \in U$, we add n dimensions to $\Omega^1(U)$, one for each basis vector of the basis $\partial/\partial x_1 \mid_p, \dots, \partial/\partial x_n \mid_p$ of $T_p\mathbb{R}^n$.
  - Do not confuse our ability to decompose a one-form to $\sum_{i=1}^n g_i \, \mathrm{d}x_i$ with the $\mathrm{d}x_i$ being a basis for $\Omega^1(U)$. The difference is that the g_i are functions, not constants; if the $\mathrm{d}x_i$ were a basis of $\Omega^1(U)$, then any $\omega \in \Omega^1(U)$ would be able to be decomposed into $\omega = \sum_{i=1}^n c_i \, \mathrm{d}x_i$ for $c_i \in \mathbb{R}$.
- **Exterior derivative** (for 0/1 forms): The function from $\Omega^0(U) \to \Omega^1(U)$ (recall that $\Omega^0(U) \cong C^\infty(U)$) defined as follows. *Denoted by* $\mathbf{d}$. *Given by*

$$f \mapsto \frac{\partial f}{\partial x_1} \, \mathrm{d}x_1 + \cdots + \frac{\partial f}{\partial x_n} \, \mathrm{d}x_n$$

  - This represents the gradient as a 1-form.
- As the notation would suggest, the exterior derivative is nothing but a formalization of the familiar, intuitive concept of the differential.

  - From computational calculus, we may intuitively rearrange the Leibniz derivative notation to give a quantity called the differential. For example, if $\mathrm{d}f/\mathrm{d}x = f'$, then the differential is $\mathrm{d}f = f'\,\mathrm{d}x$.
  - Note, however, that the exterior derivative generalizes much more nicely than the differential, permitting many later results.

- Example: The exterior derivative d^0 for a function $f : \mathbb{R}^2 \to \mathbb{R}$.



(a) Describing a parabolic cylinder.

(b) A 2D cross section.

Figure 4.2: The exterior derivative d^0 for a function $f : \mathbb{R}^2 \to \mathbb{R}$.

  - Consider the coordinate function $x_1 : \mathbb{R}^2 \to \mathbb{R}$. The graph $G(f)$ of the related function $f = x_1^2$ is a parabolic cylinder in $\mathbb{R}^3$. This graph is depicted by the brown surface in Figure 4.2a.
  - Let's think about how we would interpret the differential $\mathrm{d}f$ from computational calculus. We would think of it as the infinitesimal change in f as a function of the infinitesimal change in $\mathrm{d}x$ and $\mathrm{d}y$. In this case, we would have $\mathrm{d}f = 2x\,\mathrm{d}x$, which would intuitively tell us something about how much $\mathrm{d}f$ changes (e.g., linearly, just a little bit, but more extremely at higher values of x). We might even picture a few arrows and a right triangle tangent to the graph, but these arrows are infinitely small and we can't describe them any more than via a vague picture. Indeed, this whole idea of "infinitesimals" is not well defined, and the differential is limited in its applications beyond being a small aid to our intuition and a clever bit of notation.
  - Now let's build a picture of $\mathrm{d}f$ in the context of differential forms. As a one-form, $\mathrm{d}f$ should take a point $p \in \mathbb{R}^2$ and a vector $v \in \mathbb{R}^2$ and return the instantaneous rate of change of f at p in the direction v (scaled by $|v|$)[2].

[2] Technically, $\mathrm{d}f$ should take a point $p \in \mathbb{R}^2$ to a cotangent vector $(\mathrm{d}f)_p \in T_p^*\mathbb{R}^2$, which in turn takes an object $(p, v) \in T_p\mathbb{R}^2$, isolates the vector component $v \in \mathbb{R}^2$, and return the instantaneous rate of change of f at p in the direction v (scaled by $|v|$).

    - More tangibly, consider the case where $p = (2,0)$ and $v = (2,1)$. Then geometrically, $(\mathrm{d}f)_p(p,v)$ takes us to $(2,0)$ in the x_1x_2-plane, projects us up onto the surface $G(f)$ (i.e., to the point $(2,0,4) \in \mathbb{R}^3$), extends from that point on the surface a vector with x_1-component 2 and x_2-component 1 (this vector lives in $T_p\mathbb{R}^2$), and measures the distance from the tip of this vector to the tangent plane to $G(f)$ at $(2,0,4)$; this distance is 8 units long. Therefore, $(\mathrm{d}f)_p(p,v) = 8$ for p, v as defined.
- Now we know what $\mathrm{d}f$ does. But say we want to express $\mathrm{d}f$ in terms of the basis of $\Omega^1(\mathbb{R}^2)$, i.e., in terms of $\mathrm{d}x_1, \mathrm{d}x_2$, as we would want to to further work with it algebraically.
    - Applying the three properties defining the exterior derivative (see Section 2.4 of Guillemin and Haine (2018)), we can determine that

$$\begin{aligned}\mathrm{d}\big(x_1^2\big) &= \mathrm{d}(x_1 \cdot x_1)\\ &= x_1\,\mathrm{d}x_1 + (-1)^0 x_1\,\mathrm{d}x_1\\ &= 2x_1\,\mathrm{d}x_1\end{aligned}$$

        - ➢ As a sanity check, note that

$$(2x_1\,\mathrm{d}x_1)_p(p,v) = 2 \cdot x_1(p) \cdot (\mathrm{d}x_1)_p(p,v) = 2 \cdot 2 \cdot 2 = 8$$

        as expected.
    - Moreover, this should make intuitive sense. $2x_1$ is the "derivative" of x_1^2 — this means that it tells us the instantaneous rate of change of x_1^2, specifically that of it in the x_1 direction. The only thing left is scaling it appropriately to our direction vector, but $(\mathrm{d}x_1)_p$ takes care of that by isolating the x_1-component of v.
        - ➢ Notice the extreme conceptual similarity (but slight increase in rigor) between this concept and the naïve understanding of the differential.
    - This notion generalizes to functions that have nonzero rates of change in more than one direction at p via the properties of vector addition and the definition of the exterior derivative as the "gradient" expression. Think, for instance, about the paraboloid $f = x_1^2 + x_2^2$.
- Note that since $(\mathrm{d}f)_p$ is a linear transformation, the gray plane in Figure 4.2a is a decent intuitive visualization of the graph of $(\mathrm{d}f)_p$.
    - Delving further into the relationship between $(\mathrm{d}f)_p$ and the total derivative $Df(p)$ of f at p, we know that $Df(p)$ is given by the Jacobian

$$\begin{aligned}Df(p) &= \begin{bmatrix} \left.\frac{\partial f}{\partial x_1}\right|_p & \left.\frac{\partial f}{\partial x_2}\right|_p \end{bmatrix}\\ &= \begin{bmatrix} 4 & 0 \end{bmatrix}\end{aligned}$$

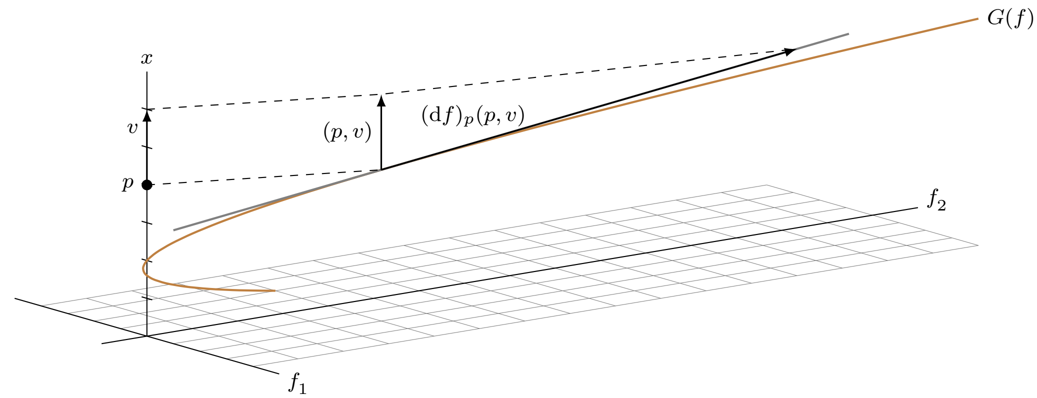
    - This matrix is very closely related to the grey plane in Figure 4.2a. In fact, if we view $Df(p)$ as a function from $\mathbb{R}^2 \to \mathbb{R}$, we realize that the grey plane is just the graph $G(Df(p))$ of $Df(p)$ translated 2 units along the x_1-axis and 4 units along the $f(x_1, x_2)$-axis so as to be tangent to $(p, f(p)) \in \mathbb{R}^3$.
    - Furthermore, this matrix is the function which relates v to $(\mathrm{d}f)_p(p,v)$. Indeed,

$$(\mathrm{d}f)_p(p,v) = \begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

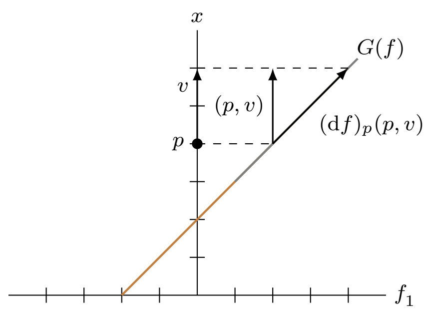
    where v_1 and v_2 are the x_1- and x_2-components of v, respectively. This is exemplified by our specific example since

$$(\mathrm{d}f)_p(p,v) = \begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4 \cdot 2 + 0 \cdot 1 = 8$$

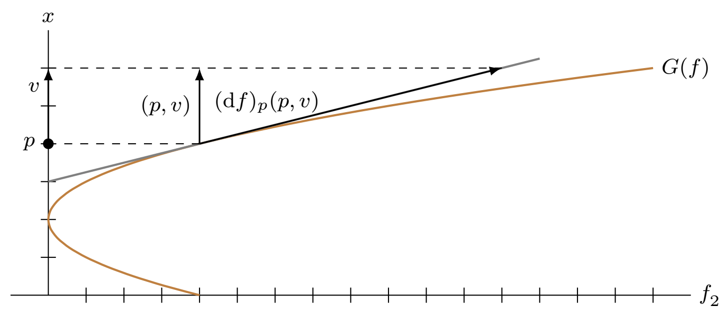
    in agreement with the above.



(a) General picture.



(b) View in the $+f_2$ direction.



(c) View in the $-f_1$ direction.

Figure 4.3: The exterior derivative d^0 for a function $f : \mathbb{R} \to \mathbb{R}^2$.

- In fact, from this perspective, we see that the one-form $\mathrm{d}f$ is entirely analogous to the "unpointed" total derivative Df, which should help justify the similarity in notation.

- Example: The exterior derivative d^0 for a function $f : \mathbb{R} \to \mathbb{R}^2$.
  - Consider the parametric function $f : \mathbb{R} \to \mathbb{R}^2$ described by the relation

$$f(t) = \big(t-2, (t-2)^2\big)$$

  - One way to visualize the graph $G(f)$ is as a parabola "being drawn" from left to right across $\mathbb{R}^2$. In this mental movie, the third dimension of the graph is time. However, we can equally well use a spatial third dimension, as in Figure 4.3. The 3D and 2D-plus-time pictures are related as follows: Every point of the parabola drawn up until time t in the 2D plus time picture is just every point of $G(f)$ beneath the horizontal plane $z = t$.
  - Now let's move back into differential forms and describe $\mathrm{d}f$. At every point p, the graph of $(\mathrm{d}f)_p$ can be thought of geometrically as a tangent line to $G(f)$. The set of all these tangent lines (and the relation between them and the points p) is contained in $\mathrm{d}f$.
  - As a specific example, consider $p = 4$ and $v = 2$ as in Figure 4.3. Geometrically, we can see that this will lead to

$$(\mathrm{d}f)_p(p,v) = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

- In terms of linear transformations, we have that

$$\begin{aligned} Df(p) &= \begin{bmatrix} \left.\frac{\partial f_1}{\partial t}\right|_p \\ \left.\frac{\partial f_2}{\partial t}\right|_p \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 4 \end{bmatrix} \end{aligned}$$

so that

$$(\mathrm{d}f)_p(p, v) = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

- Thus, we see that one-forms describe the slope of continuous functions that are multivariate in both domain and/or codomain[3].
- Check:
  1. d is linear.
  2. $\mathrm{d}x_i = \mathrm{d}(x_i)$, where $x_i : \mathbb{R}^n \to \mathbb{R}$ is the i^{th} coordinate function.

4.3 Integral Curves

4/22:

- Plan:
  - Clear up a bit of notational confusion.
  - Discuss integral curves of vector fields.
  - k-forms.
  - Exterior derivatives $\mathrm{d} : \Omega^k(U) \to \Omega^{k+1}(U)$ (definition and properties).
- Notation:
  - $F : \mathbb{R}^n \to \mathbb{R}^m$ smooth.
  - We are used to denoting derivatives by big D: $DF_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$ where bases of the two spaces are $e_1, \dots, e_n$ and $e_1, \dots, e_m$ has matrix equal to the Jacobian:

$$[DF_p] = \left[\frac{\partial F_i}{\partial x_j}(p)\right]$$

  - The book often uses small d: $f : U \to \mathbb{R}$ has $\mathrm{d}f_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}$, where the latter set is isomorphic to $\mathbb{R}$.
  - $\mathrm{d}f$ sends $p \mapsto \mathrm{d}f_p \in T_p^*\mathbb{R}^n$.
  - Klug said

$$\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \, \mathrm{d}x_i$$

  - Homework 1 defined $\mathrm{d}f = \mathrm{d}f$?
  - Sometimes three perspectives help you keep this all straight:
    1. Abstract nonsense: The definition of the derivative.
    2. How do I compute it: Apply the formula.
    3. What is it: E.g., magnitude of the directional derivative in the direction of steepest ascent.
- For the homework,

[3] Then what are k-forms for?

  - Let ω be a 1-form in $\Omega^1(U)$.
  - Let $\gamma : [a,b] \to U$ be a curve in U.
  - Then $\mathrm{d}\gamma_p = \gamma'_p : T_p\mathbb{R} \to T_{\gamma(p)}\mathbb{R}^n$ is a function that takes in points of the curve and spits out tangent vectors.
  - Integrating swallows 1-forms and spits out numbers.

$$\int_\gamma \omega = \int_a^b \omega(\gamma'(t))\,\mathrm{d}t$$

  - Problem: If $\omega = \mathrm{d}f$, then

$$\int_\gamma \omega = f(\gamma(b)) - f(\gamma(a))$$

    regardless of the path.
  - Question: Given a 1-form ω, is $\omega = \mathrm{d}f$ for some f?
  - Homework: Explicit U, ω, closed γ such that $\int_\gamma \omega \neq 0$ implies that $\omega \neq \mathrm{d}f$. This motivates and leads into the de Rham cohomology.

- Aside: It won't hurt (for now) to think of 1-forms as vector fields.
- **Integral curve** (for $\boldsymbol{v}$): A curve $\gamma : (a,b) \to U$ such that

$$\gamma'(t) = \boldsymbol{v}_{\gamma(t)}$$

  where $U \subset \mathbb{R}^n$ and $\boldsymbol{v}$ is a (smooth) vector field on U.

- Examples:
  - If $U = \mathbb{R}^2$ and $\boldsymbol{v} = \partial/\partial x$, then the integral curve is the line from left to right traveling at unit speed. The curve has to always have as its tangent vector the unit vector pointing right (which is the vector at every point in the vector field).
  - Vector fields flow everything around. An integral curve is the trajectory of a particle subjected to the vector field as a *velocity* field (the vector field is not a force field or acceleration field).
- Main points:
  1. These integral curves always exist (locally) and often exist globally (cases in which they do are called **complete vector fields**).
  2. They are unique given a starting point $p \in U$.
- An incomplete vector field is one such as the "all roads lead to Rome" vector field where everything always points inward. This is because integral curves cannot be defined for all "time" (real numbers, positive and negative).
- The proofs are in the book; they require an existence/uniqueness result for ODEs and the implicit function theorem.
- Aside: $f : U \to \mathbb{R}$, $\boldsymbol{v}$ a vector field, implies that $L_{\boldsymbol{v}} f = 0$ means that f is constant along all the integral curves of $\boldsymbol{v}$. This also means that f is **integral** for v.
- **Pullback** (of 1-forms): If $F : U \to V$, $\mathrm{d} : \Omega^0(U) \to \Omega^1(U)$, and $\mathrm{d} : \Omega^0(V) \to \Omega^1(V)$, then we get an induced map $F^* : \Omega^0(V) \to \Omega^0(U)$. If $f : V \to \mathbb{R}$, then $f \circ F$ is involved.
  - We're basically saying that if we have $\mathrm{Hom}(A, X)$ (the set of all functions from A to X) and $\mathrm{Hom}(B, X)$, then if we have $F : A \to B$, we get an induced map $F^* : \mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X)$ that is precomposed with F.

4.4 Chapter 2: Differential Forms

From Guillemin and Haine (2018).

5/5: - Goals for this chapter.
  - Generalize to n dimensions the basic operations of 3D vector calculus (**divergence**, **gradient**, and **curl**).
    - div and grad are pretty straightforward, but curl is more subtle.
  - Substitute **differential forms** for **vector fields** to discover a natural generalization of the operations, in particular, where all three operations are special cases of **exterior differentiation**.
- Introducing vector fields and their dual objects (**one-forms**).
- **Tangent space** (to $\mathbb{R}^n$ at p): The set of pairs (p,v) for all $v\in\mathbb{R}^n$. *Denoted by* $\boldsymbol{T_p\mathbb{R}^n}$. *Given by*

$$T_p\mathbb{R}^n=\{(p,v)\mid v\in\mathbb{R}^n\}$$

- Operations on the tangent space.
  - Directly, we identify $T_p\mathbb{R}^n\cong\mathbb{R}^n$ by $(p,v)\mapsto v$ to make $T_p\mathbb{R}^n$ a vector space.
  - Explicitly, we define

$$(p,v_1)+(p,v_2)=(p,v_1+v_2) \qquad\qquad \lambda(p,v)=(p,\lambda v)$$

  for all $v,v_1,v_2\in\mathbb{R}^n$ and $\lambda\in\mathbb{R}$.
- **Derivative** (of f at p): The linear map from $\mathbb{R}^n\to\mathbb{R}^m$ defined by the following $m\times n$ matrix, where $U\subset\mathbb{R}^n$ is open and $f:U\to\mathbb{R}^m$ is a C^1-mapping. *Denoted by* $\boldsymbol{Df(p)}$. *Given by*

$$Df(p)=\left[\frac{\partial f_i}{\partial x_j}(p)\right]$$

- $\mathbf{d}\boldsymbol{f_p}$: The linear map from $T_p\mathbb{R}^n\to T_q\mathbb{R}^m$ defined as follows, where $U\subset\mathbb{R}^n$ open, $f:U\to\mathbb{R}^m$ is a C^1-mapping, and $q=f(p)$. *Given by*

$$\mathrm{d}f_p(p,v)=(q,Df(p)v)$$

  - Guillemin and Haine (2018) also refer to this as the "base-pointed" version of the derivative of f at p.
- The chain rule for the base-pointed version, where $U\subset\mathbb{R}^n$ open, $f:U\to\mathbb{R}^n$ is a C^1-mapping, $\operatorname{im}(f)\subset V$ open, and $g:V\to\mathbb{R}^k$ is a C^1-mapping.

$$\mathrm{d}g_q\circ\mathrm{d}f_p=\mathrm{d}(g\circ f)_p$$

- Example: The chain rule for single-variable f,g.

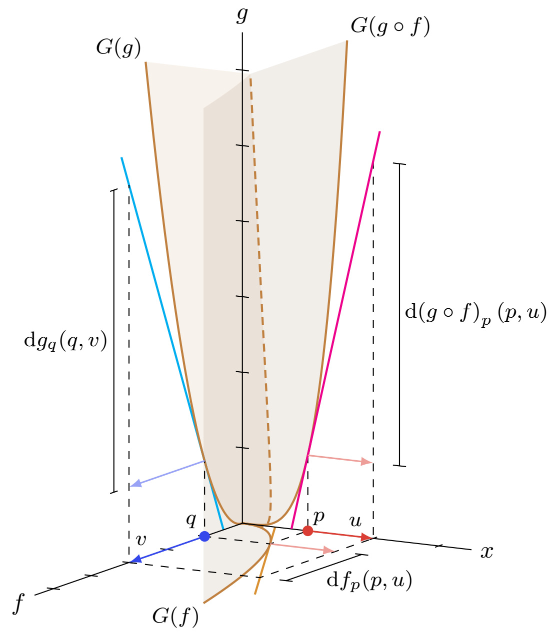


Figure 4.4: The chain rule for single-variable f, g.

- Let $U, V, W = \mathbb{R}$. Consider the functions $f : U \to V$ and $g : V \to W$, both described by the relation

$$x \mapsto x^2$$

- Since the 3 spaces U, V, W are all one-dimensional, a complete geometric representation of the actions of f, g and their composition can be realized in $\mathbb{R}^3$. This is what is depicted in Figure 4.4. Let's now state what the elements of it are.
  - The red dot on the x-axis labeled p depicts a point in U.
  - The red arrow on the x-axis labeled u depicts a vector in U.
    - ➢ Together, p and u depict the tangent vector $(p, u) \in T_pU$.
  - The blue dot on the f-axis labeled q depicts the point $f(p) \in V$.
  - The blue arrow on the f-axis labeled v depicts the vector $Df(p)(u) \in V$.
    - ➢ Together, q and v depict the tangent vector $(q, v) \in T_qV$.
  - The solid brown line in the fx-plane labeled $G(f)$ depicts the graph of f. As we would expect, it is a parabola in x, and a subset of the space $U \times V = \mathbb{R}^2 \subset \mathbb{R}^3 = U \times V \times W$.
  - The solid brown line in the gf-plane labeled $G(g)$ depicts the graph of g. Note that g does not depend on x, but rather takes f as its independent variable. Thus, $G(g)$ is a parabola *in* f, and a subset of the space $V \times W = \mathbb{R}^2 \subset \mathbb{R}^3 = U \times V \times W$. Indeed, it is the depiction of g as a function of f that facilitates composition.
  - This brings us to the solid brown line in the gx-plane labeled $G(g \circ f)$. This line depicts the graph of $g \circ f$. As we can determine from middle-school algebra, $g \circ f$ is a quartic function, and a subset of the space $U \times W = \mathbb{R}^2 \subset \mathbb{R}^3 = U \times V \times W$.
  - The orange line in the fx-plane is tangent to $G(f)$ at (p, q). It will be used to illustrate the relationship between (p, u) and $\mathrm{d}f_p(p, u)$.
  - The cyan line in the gf-plane is tangent to $G(g)$ at $(q, g(q))$. It will be used to illustrate the relationship between (q, v) and $\mathrm{d}g_q(q, v)$.
  - The magenta line in the gx-plane is tangent to $G(g \circ f)$ at $(p, g(q))$. It will be used to illustrate the relationship between (p, u) and $\mathrm{d}(g \circ f)_p\,(p, u)$.
  - The only elements left at this point are the translucent surfaces and their dashed line of intersection. Although it may not be strictly necessary to include these in the diagram,

I believe they more fully illustrate the relationship between all of the parts of this setup. Indeed, this curve in $\mathbb{R}^3$ contains all of the information conveyed by f (via its projection into the fx-plane), by g (via its projection into the gf-plane), and by $g \circ f$ (via its projection into the gx-plane). On the contrary, any one of f, g, or $g \circ f$ is missing some of the information contained in the other two. For example, given the equation $g \circ f = x^4$, there are infinitely many possible functions f, g that satisfy this equation, and one would need to specify either f or g in order to obtain the other uniquely. However, this curve from $U \to U \times V \times W$ says it all.

- The gist of this diagram is that if we want to find the slope of $g \circ f$ at p, we can go about this two ways.
  - Directly, we may plug (p, u) into the covector $\mathrm{d}(g \circ f)_p$.
    - Graphically, this is equivalent to moving (p, u) upwards parallel to the g-axis until p touches $G(g \circ f)$, and then measuring the distance from the tip of the translated vector (shown in light red) to the magenta tangent line.
  - Alternatively, we may rely solely on information about the slopes of f and g independently. Indeed, we may plug (p, u) into $\mathrm{d}f_p$, yielding (q, v), and then plug this result into $\mathrm{d}g_q$.
    - Graphically, this is equivalent to moving (p, u) outwards parallel to the f-axis until p touches $G(f)$ and then measuring both the distance from the base of the translated vector to p, yielding q, and the tip of the translated vector (also shown in light red) to the orange tangent line, yielding v. Having obtained q and v, we could project them onto the f-axis, obtaining a workable input for the next step. This next step is much the same as the first: We move (q, v) upwards parallel to the g-axis until q touches $G(g)$ and then measure the distance from the tip of the translated vector (shown in light blue) to the cyan tangent line.
  - In higher dimensions, the "measuring" described above would have to be done for every relevant component.
- In the specific example drawn, where $p = 1$, $q = f(p) = 1$, $u = 1$, and $v = Df(p)(u) = 2$, we can confirm by inspection that both $\mathrm{d}(g \circ f)_p\,(p, u)$ and $\mathrm{d}g_q(q, v)$ are 4 units long.
- From a computational point of view, we have that

$$\begin{aligned} Df &= \begin{bmatrix} \frac{\partial f}{\partial x} \end{bmatrix} \\ &= \begin{bmatrix} 2x \end{bmatrix} \end{aligned} \qquad \begin{aligned} Dg &= \begin{bmatrix} \frac{\partial g}{\partial f} \end{bmatrix} \\ &= \begin{bmatrix} 2f \end{bmatrix} \end{aligned} \qquad \begin{aligned} D(g \circ f) &= \begin{bmatrix} \frac{\partial g}{\partial x} \end{bmatrix} \\ &= \begin{bmatrix} 4x^3 \end{bmatrix} \end{aligned}$$

  so that

$$\begin{aligned} Df(p) &= \begin{bmatrix} \left.\frac{\partial f}{\partial x}\right|_p \end{bmatrix} \\ &= \begin{bmatrix} 2 \end{bmatrix} \end{aligned} \qquad \begin{aligned} Dg(q) &= \begin{bmatrix} \left.\frac{\partial g}{\partial f}\right|_p \end{bmatrix} \\ &= \begin{bmatrix} 2 \end{bmatrix} \end{aligned} \qquad \begin{aligned} D(g \circ f)(p) &= \begin{bmatrix} \left.\frac{\partial g}{\partial x}\right|_p \end{bmatrix} \\ &= \begin{bmatrix} 4 \end{bmatrix} \end{aligned}$$

  and hence

$$D(g \circ f)(p) = \begin{bmatrix} 4 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} = Dg(p) \circ Df(p)$$

- **Vector field** (on $\mathbb{R}^3$): A function which attaches to each point $p \in \mathbb{R}^3$ a base-pointed arrow $(p, v) \in T_p\mathbb{R}^3$.
  - These vector fields are the typical subject of vector calculus.
- **Vector field** (on U): A function which assigns to each point $p \in U$ a vector in $T_p\mathbb{R}^n$, where $U \subset \mathbb{R}^n$ is open. *Denoted by* $\boldsymbol{v}$.
  - We denote the value of $\boldsymbol{v}$ at p by either $\boldsymbol{v}(p)$ or $\boldsymbol{v}_p$.
- **Constant** (vector field): A vector field of the form $p \mapsto (p, v)$, where $v \in \mathbb{R}^n$ is fixed.
- $\boldsymbol{\partial / \partial x_i}$: The constant vector field having $v = e_i$.

- $\boldsymbol{fv}$: The vector field defined on U as follows, where $f : U \to \mathbb{R}$. *Given by*

$$p \mapsto f(p)\boldsymbol{v}(p)$$

  - Note that we are invoking our definition of scalar multiplication on $T_p\mathbb{R}^n$ here.

- **Sum** (of $\boldsymbol{v}_1, \boldsymbol{v}_2$): The vector field on U defined as follows. *Denoted by* $\boldsymbol{v_1 + v_2}$. *Given by*

$$p \mapsto \boldsymbol{v}_1(p) + \boldsymbol{v}_2(p)$$

  - Note that we are invoking our definition of addition on $T_p\mathbb{R}^n$ here.

- The list of vectors $(\partial/\partial x_1)_p, \dots, (\partial/\partial x_n)_p$ constitutes a basis of $T_p\mathbb{R}^n$.
  - Recall that $(\partial/\partial x_i)_p = (p, e_i)$.
  - Thus, if $\boldsymbol{v}$ is a vector field on U, it has a unique decomposition

$$\boldsymbol{v} = \sum_{i=1}^n g_i \frac{\partial}{\partial x_i}$$

    where each $g_i : U \to \mathbb{R}$.

- $\boldsymbol{C^\infty}$ (vector field): A vector field such that $g_i \in C^\infty(U)$ for all g_i's in its unique decomposition.

- **Lie derivative** (of f with respect to $\boldsymbol{v}$): The function from $U \to \mathbb{R}$ defined as follows, where $U \subset \mathbb{R}^n$, $f : U \to \mathbb{R}$ is a C^1-mapping, and $\boldsymbol{v}(p) = (p, v)$. *Denoted by* $\boldsymbol{L_v f}$. *Given by*

$$L_{\boldsymbol{v}} f(p) = Df(p)v$$

  - A more explicit formula for the Lie derivative is

$$L_{\boldsymbol{v}} f = \sum_{i=1}^n g_i \frac{\partial f}{\partial x_i}$$

  - The vector field decides the direction in which we take the derivative at each point. Instead of having to take a derivative everywhere in one direction at a time, we can now take a derivative in a different direction at every point!

- Lemma 2.1.11: Let U be an open subset of $\mathbb{R}^n$, $\boldsymbol{v}$ a vector field on U, and $f_1, f_2 \in C^1(U)$. Then

$$L_{\boldsymbol{v}}(f_1 \cdot f_2) = L_{\boldsymbol{v}}(f_1) \cdot f_2 + f_1 \cdot L_{\boldsymbol{v}}(f_2)$$

  Proof. See Exercise 2.1.ii. □

- **Cotangent space** (to $\mathbb{R}^n$ at p): The dual vector space to $T_p\mathbb{R}^n$. *Denoted by* $\boldsymbol{T_p^*\mathbb{R}^n}$. *Given by*

$$T_p^*\mathbb{R}^n = (T_p\mathbb{R}^n)^*$$

- **Cotangent vector** (to $\mathbb{R}^n$ at p): An element of $T_p^*\mathbb{R}^n$.

- **Differential one-form** (on U): A function which assigns to each point $p \in U$ a cotangent vector. *Also known as* **one-form** (on U). *Denoted by* $\boldsymbol{\omega}$. *Given by*

$$p \mapsto \omega_p$$

- Note that by identifying $T_q\mathbb{R} \cong \mathbb{R}$, we have that $\mathrm{d}f_p \in T_p^*\mathbb{R}^n$, assuming that $f : U \to \mathbb{R}$.

  - Geometric example: Consider $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f \in C^1$. By the latter condition, we know that the graph of f is a "smooth" surface in $\mathbb{R}^3$, i.e., one without any abrupt changes in derivative (consider the graph of the piecewise function defined by $-x^2$ for $x < 0$ and x^2 for $x \geq 0$, for example). What $\mathrm{d}f_p$ does is take a point (p_1, p_2, q), where $q = f(p)$, on the surface and a vector v with tail at (p, q), and give us a number representing the magnitude of the instantaneous change of f at p in the direction v. Thus, $\mathrm{d}f_p$ contains, in a sense, all of the information concerning the rate of change of f at p.

- **$\mathbf{d}\boldsymbol{f}$**: The one-form on U defined as follows. *Given by*

$$p \mapsto \mathrm{d}f_p$$

  - Continuing with the geometric example: What $\mathrm{d}f$ does is take every point p across the surface and return all of the information concerning the rate of change of f at p (packaged neatly by $\mathrm{d}f_p$).

- **Pointwise product** (of ϕ with ω): The one-form on U defined as follows, where $\phi : U \to \mathbb{R}$ and ω is a one-form. *Denoted by* $\boldsymbol{\phi\omega}$. *Given by*

$$(\phi\omega)_p = \phi(p)\omega_p$$

- **Pointwise sum** (of ω_1, ω_2): The one-form on U defined as follows. *Denoted by* $\boldsymbol{\omega_1 + \omega_2}$. *Given by*

$$(\omega_1 + \omega_2)_p = (\omega_1)_p + (\omega_2)_p$$

- $\boldsymbol{x_i}$: The function from $U \to \mathbb{R}$ defined as follows. *Given by*

$$x_i(u_1, \dots, u_n) = u_i$$

  - x_i is constantly increasing in the x_i-direction, and constant in every other direction.

- $\boldsymbol{(\mathrm{d}x_i)_p}$: The linear map from $T_p\mathbb{R}^n \to \mathbb{R}$ (i.e., the cotangent vector in $T_p^*\mathbb{R}^n$) defined as follows. *Given by*

$$(\mathrm{d}x_i)_p(p, a_1x_1 + \cdots + a_nx_n) = a_i$$

  - Naturally, the instantaneous change in x_i at any point p in the direction $\boldsymbol{v}(p)$ will just be the magnitude of $\boldsymbol{v}(p)$ in the x_i-direction.
  - Note that as per the discussion associated with Figure 4.2, we can also think of $(\mathrm{d}x_i)_p$ as returning the product of the derivative of x_i at p (which will always be 1, regardless of where p is or which integer i is) and the magnitude of v in the x_i-direction. This notion can be summed up by the statement

$$\mathrm{d}x_i = \mathrm{d}(x_i) = 1\,\mathrm{d}x_i$$

  - It follows immediately that

$$(\mathrm{d}x_i)_p\left(\frac{\partial}{\partial x_j}\right)_p = \delta_{ij}$$

- Consequently, the list of cotangent vectors $(\mathrm{d}x_1)_p, \dots, (\mathrm{d}x_n)_p$ constitutes a basis of $T_p^*\mathbb{R}^n$ that is **dual** to the basis $(\partial/\partial x_1)_p, \dots, (\partial/\partial x_n)_p$ of $T_p\mathbb{R}^n$.

- $\mathbf{d}\boldsymbol{x_i}$: The one-form on U defined as follows. *Given by*

$$p \mapsto (\mathrm{d}x_i)_p$$

  - Thus, if $\omega_p \in T_p^*\mathbb{R}^n$, it has a unique decomposition

$$\omega_p = \sum_{i=1}^n f_i(p)(\mathrm{d}x_i)_p$$

    where every $f_i : U \to \mathbb{R}$.

- Similarly, $\omega \in \Omega^1(U)$ has a unique decomposition

$$\omega = \sum_{i=1}^{n} f_i \mathrm{d}x_i$$

- **Smooth** (one-form): A one-form for which the associated functions $f_1, \dots, f_n \in C^\infty$. *Also known as* $\boldsymbol{C^\infty}$ (one-form).
- Lemma 2.1.18: Let U be an open subset of $\mathbb{R}^n$. If $f : U \to \mathbb{R}$ is a C^∞ function, then

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \, \mathrm{d}x_i$$

- **Interior product** (of $\boldsymbol{v}$ with ω): The function on U defined as follows, where $\boldsymbol{v}$ is a vector field over U and ω is a one-form on U. *Denoted by* $\boldsymbol{\iota_v \omega}$. *Given by*

$$p \mapsto \iota_{\boldsymbol{v}(p)} \omega_p$$

  - Note that $\iota_{\boldsymbol{v}(p)} \omega_p$ denotes the interior product of the vector $\boldsymbol{v}(p)$ and the one-tensor ω_p.
  - What we are doing with this definition:
    - We first learned to take the interior product of a vector and a tensor. In particular, for every vector $v \in V$, we defined a function ι_v which took the k-tensor in question to a specifically defined $(k-1)$-tensor.
    - What we are now doing is taking a vector field (a collection of vectors indexed by the points $p \in U$) and a one-form (a collection of 1-tensors indexed by the points $p \in U$) and defining the inner product of a vector field and a one-form as the function which, at each point $p \in U$, evaluates to the inner product of the vector $\boldsymbol{v}(p)$ and the 1-tensor ω_p.
    - This is very much analogous to the step up from cotangent vectors to one-forms (which describe a set of cotangent vectors indexed by the points of a vector space).
- Examples.
  - If

$$\boldsymbol{v} = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i} \qquad\qquad \omega = \sum_{i=1}^{n} f_i \mathrm{d}x_i$$

  then

$$\iota_{\boldsymbol{v}} \omega = \sum_{i=1}^{n} f_i g_i$$

    - Proof: By definition, we know that

$$\boldsymbol{v}(p) = \sum_{i=1}^{n} g_i(p) \left. \frac{\partial}{\partial x_i} \right|_p \qquad\qquad \omega_p = \sum_{i=1}^{n} f_i(p) (\mathrm{d}x_i)_p$$

    It follows by the definition of the interior product of a vector and a tensor that

$$\begin{aligned}
\iota_{\boldsymbol{v}(p)} \omega_p &= \sum_{i=1}^{1} \omega_p(\boldsymbol{v}(p)) \\
&= \omega_p(\boldsymbol{v}(p)) \\
&= \left[\sum_{i=1}^{n} f_i(p) (\mathrm{d}x_i)_p \right] \left(\sum_{j=1}^{n} g_j(p) \left. \frac{\partial}{\partial x_j} \right|_p \right)
\end{aligned}$$

We now invoke linearity.

$$= \sum_{i,j=1}^{n} f_i(p)g_j(p) \cdot (\mathrm{d}x_i)_p \left(\frac{\partial}{\partial x_j}\right)_p$$
$$= \sum_{i,j=1}^{n} f_i(p)g_j(p) \cdot \delta_{ij}$$

All terms where $i \neq j$ are equal to zero, so only the n terms where $i = j$ remain.

$$= \sum_{i=1}^{n} f_i(p)g_i(p)$$

Thus, using the definition of $\iota_{\boldsymbol{v}}\omega$, we have by transitivity that

$$\iota_{\boldsymbol{v}}\omega = \sum_{i=1}^{n} f_i g_i$$

  - Notice how the interior product is finally starting to look like a form of multiplication: In particular, we can view the inner product through a naïve lens as "taking the componentwise product of $\boldsymbol{v}$ and ω and using the fact that $(\mathrm{d}x_i)_p(\partial/\partial x_j)_p = \delta_{ij}$ to obtain this result."
  - If $\boldsymbol{v}, \omega \in C^\infty$, so is $\iota_{\boldsymbol{v}}\omega$, where C^∞ refers to three different sets of smooth objects (vector fields, one-forms, and functions, respectively[4]).

- As with f, if $\phi \in C^\infty(U)$, then

$$\mathrm{d}\phi = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} \mathrm{d}x_i$$

- It follows if $\boldsymbol{v}$ is defined as in the first example that

$$\iota_{\boldsymbol{v}}\mathrm{d}\phi = \sum_{i=1}^{n} g_i \frac{\partial \phi}{\partial x_i} = L_{\boldsymbol{v}}\phi$$

- **Integral curve** (of $\boldsymbol{v}$): A C^1 curve $\gamma : (a, b) \to U$ such that for all $t \in (a, b)$,

$$\boldsymbol{v}(\gamma(t)) = (\gamma(t), \gamma'(t))$$

  where $U \subset \mathbb{R}^n$ is open and $\boldsymbol{v}$ is a vector field on U.

  - An equivalent condition if $\boldsymbol{v} = \sum_{i=1}^{n} g_i \, \partial/\partial x_i$ and $g : U \to \mathbb{R}^n$ is defined by $(g_1, \dots, g_n)$ is that γ satisfies the system of differential equations

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = g(\gamma(t))$$

    - Verbally, we must have "for all $1 \leq i \leq n$ that the change in γ with respect to t in the x_i-direction is equal to the x_i-component of $\boldsymbol{v}$ at every point in $\gamma((a, b)) \subset U$."

- Theorem 2.2.4 (existence of integral curves): Let $U \subset \mathbb{R}^n$ open, $\boldsymbol{v}$ a vector field on U. If $p_0 \in U$ and $a \in \mathbb{R}$, then there exist $I = (a - T, a + T)$ for some $T \in \mathbb{R}$, $U_0 = N_r(p_0) \subset U$, and, for all $p \in U_0$, an integral curve $\gamma_p : I \to U$ such that $\gamma_p(a) = p$.

- Theorem 2.2.5 (uniqueness of integral curves): Let $U \subset \mathbb{R}^n$ open, $\boldsymbol{v}$ a vector field on U, and $\gamma_1 : I_1 \to U$ and $\gamma_2 : I_2 \to U$ integral curves for $\boldsymbol{v}$. If $a \in I_1 \cap I_2$ and $\gamma_1(a) = \gamma_2(a)$, then

$$\gamma_1|_{I_1 \cap I_2} = \gamma_2|_{I_1 \cap I_2}$$

[4]Technically, these objects are all types of functions, though, so it is fair to call them all smooth.

and the curve $\gamma : I_1 \cup I_2 \to U$ defined by

$$\gamma(t) = \begin{cases} \gamma_1(t) & t \in I_1 \\ \gamma_2(t) & t \in I_2 \end{cases}$$

is an integral curve for $\boldsymbol{v}$.

- Theorem 2.2.6 (smooth dependence on initial data): Let $V \subset U \subset \mathbb{R}^n$ open, $\boldsymbol{v}$ a C^∞-vector field on V, $I \subset \mathbb{R}$ an open interval, and $a \in I$. Let $h : V \times I \to U$ have the following properties.
  1. $h(p, a) = p$.
  2. For all $p \in V$, the curve $\gamma_p : I \to U$ defined by $\gamma_p(t) = h(p, t)$ is an integral curve of $\boldsymbol{v}$.

  Then $h \in C^\infty$.
- **Autonomous** (system of ODEs): A system of ODEs that does not explicitly depend on the independent variable.
- $\mathrm{d}\gamma/\mathrm{d}t = g(\gamma(t))$ is autonomous since g does not depend on t.
- Theorem 2.2.7: Let $I = (a, b)$. For all $c \in \mathbb{R}$, define $I_c = (a - c, b - c)$. If $\gamma : I \to U$ is an integral curve, then the reparameterized curve $\gamma_c : I_c \to U$ defined by

$$\gamma_c(t) = \gamma(t + c)$$

  is an integral curve.
  - Note that this is truly just a reparameterization; we still have, for instance,

$$\gamma_c(a - c) = \gamma(a - c + c) = \gamma(a) \qquad \gamma_c(b - c) = \gamma(b - c + c) = \gamma(b)$$

- **Integral** (of the system $\mathrm{d}\gamma/\mathrm{d}t = g(\gamma(t))$): A C^1-function $\phi : U \to \mathbb{R}$ such that for every integral curve $\gamma(t)$, the function $t \mapsto \phi(\gamma(t))$ is constant.
  - To visualize this definition, consider the case where $\phi : \mathbb{R}^2 \to \mathbb{R}$.
    - Here, the graph $G(\phi)$ of ϕ is a C^1 (continuously differentiable, so continuous and with no abrupt changes in slope) surface in $\mathbb{R}^3$.
    - In particular, what this definition is saying is that if ϕ is an integral of $\boldsymbol{v}$, then projecting an integral curve in $\mathbb{R}^2 \cong \mathbb{R}^2 \times \{0\}$ up onto the surface $G(\phi)$ gives a contour line, i.e., a path along which all points are the same height above the xy-plane.
  - An alternate condition is that if $p = \gamma(t)$ and $v = \boldsymbol{v}(p) = \gamma'(t) = D\gamma(t)$, then for all t,

$$\begin{aligned} 0 &= D(\phi \circ \gamma)(t) = D\phi(\gamma(t)) \cdot D\gamma(t) = D\phi(p)v = L_{\boldsymbol{v}}\phi(p) \\ 0 &= L_{\boldsymbol{v}}\phi(\gamma(t)) \end{aligned}$$

    - To visualize this alternate condition, consider again the case where $\phi : \mathbb{R}^2 \to \mathbb{R}$.
    - Imagine you are standing on the surface $G(\phi)$ and want to walk along it. However, as you walk, you want to stay at the same height above the xy-plane. In other words, you want to walk in the direction such that your change in elevation will be zero. Naturally, at every point along $G(\phi)$ (that is not a local maximum or minimum), there will be such a direction you can walk in. The vectors indicating these directions compose $\boldsymbol{v}$. And naturally, the directional derivative/change in height of ϕ in these directions will be zero. But this directional derivative is just the Lie derivative by definition.
  - Note that taking the gradient (from vector calc) of the integral will not actually return the original vector field; rather, all the vectors in $\nabla\phi$ will be perpendicular to those in $\boldsymbol{v}$.
    - Would curl return the original vector field?

- Theorem 2.2.9: Let $U \subset \mathbb{R}^n$ open, $\phi \in C^1(U)$. Then ϕ is an integral of the system $\mathrm{d}\gamma/\mathrm{d}t = g(\gamma(t))$ iff $L_{\boldsymbol{v}}\phi = 0$.
- **Complete** (vector field): A vector field $\boldsymbol{v}$ on U such that for every $p \in U$, there exists an integral curve $\gamma : \mathbb{R} \to U$ with $\gamma(0) = p$.
  - Alternatively, for every p, there exists an integral curve that starts at p and exists for all time.
- **Maximal** (integral curve): An integral curve $\gamma : [0, b) \to U$ with $\gamma(0) = p$ such that it cannot be extended to an interval $[0, b')$ with $b' > b$.
- For a maximal curve, either...
  1. $b = +\infty$;
  2. $|\gamma(t)| \to +\infty$ as $t \to b$;
  3. The limit points of $\{\gamma(t) \mid 0 \leq t < b\}$ contain elements of the boundary of U.
- Eliminating 2 and 3, as can be done with the following lemma, provides a means of proving that γ exists for all positive time.
- Lemma 2.2.11: The scenarios 2 and 3 above cannot happen if there exists a proper C^1-function $\phi : U \to \mathbb{R}$ with $L_{\boldsymbol{v}}\phi = 0$.

  Proof. Suppose there exists $\phi \in C^1$ such that $L_{\boldsymbol{v}}\phi = 0$. Then ϕ is constant on $\gamma(t)$ (say with value $c \in \mathbb{R}$) by definition. But then since $\{c\} \subset \mathbb{R}$ is compact and $\phi \in C^1$, $\phi^{-1}(c) \subset U$ is compact and, importantly, contains $\mathrm{im}(\gamma)$. The compactness of this set implies that γ can neither "run off to infinity" as in scenario 2 or "run off the boundary" as in scenario 3. □

- Theorem 2.2.12: If there exists a proper C^1-function $\phi : U \to \mathbb{R}$ with the property $L_{\boldsymbol{v}}\phi = 0$, then the vector field $\boldsymbol{v}$ is complete.

  Proof. Apply a similar argument to the interval $(-b, 0]$ and join the two results. □

- Example: Let $U = \mathbb{R}^2$ and let $\boldsymbol{v}$ be the vector field
$$\boldsymbol{v} = x^3 \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$$
  Then $\phi(x, y) = 2y^2 + x^4$ is a proper function with the above property.
  - Note that indeed, as per Theorem 2.2.12, we have that
$$\begin{aligned} L_{\boldsymbol{v}}\phi &= x^3 \frac{\partial \phi}{\partial y} - y \frac{\partial \phi}{\partial x} \\ &= x^3 \cdot 4y - y \cdot 4x^3 \\ &= 0 \end{aligned}$$
- We now build up to an alternate completeness condition (Theorem 2.2.15).
- **Support** (of $\boldsymbol{v}$): The following set. *Denoted by* **supp($\boldsymbol{v}$)**. *Given by*
$$\mathrm{supp}(\boldsymbol{v}) = \overline{\{q \in U \mid \boldsymbol{v}(q) \neq 0\}}$$
- **Compactly supported** (vector field $\boldsymbol{v}$): A vector field $\boldsymbol{v}$ for which $\mathrm{supp}(\boldsymbol{v})$ is compact.
- Theorem 2.2.15: If $\boldsymbol{v}$ is compactly supported, then $\boldsymbol{v}$ is complete.

Proof. Let $p \in U$ be such that $\boldsymbol{v}(p) = 0$. Define $\gamma_0 : (-\infty, \infty) \to U$ by $\gamma_0(t) = p$ for all $t \in (-\infty, \infty)$. Since

$$\frac{\mathrm{d}\gamma_0}{\mathrm{d}t} = 0 = \boldsymbol{v}(p) = \boldsymbol{v}(\gamma(t))$$

we know that γ_0 is an integral curve of $\boldsymbol{v}$.

Now consider an arbitrary integral curve $\gamma : (-a, b) \to U$ having the property $\gamma(t_0) = p$ for some $t_0 \in (-a, b)$. It follows by Theorem 2.2.5 that γ and γ_0 coincide on the interval $(-a, a)$.

By hypothesis, $\operatorname{supp}(\boldsymbol{v})$ is compact. Basic set theory tells us that for γ arbitrary, either $\gamma(t) \in \operatorname{supp}(\boldsymbol{v})$ for all t or there exists t_0 such that $\gamma(t_0) \in U \setminus \operatorname{supp}(\boldsymbol{v})$. But then by the definition of $\operatorname{supp}(\boldsymbol{v})$, $\boldsymbol{v}(\gamma(t_0)) = 0$. Thus, letting $p = \gamma(t_0)$, we have an associated γ_0 that γ "runs along" while outside the support. It follows that in either case, γ cannot go off to ∞ or go off the boundary of U as $t \to b$. ☐

- Intuition for Theorem 2.2.15.
  - We seek to prove that $\boldsymbol{v}$ is complete. One way we could prove that $\boldsymbol{v}$ is *not* complete is to find an integral curve that "runs off" of U at some point in positive or negative time.
  - With the introduction of the support, we can break down integral curves into two types: Those that remain in $\operatorname{supp}(\boldsymbol{v})$ for all time, necessarily always moving with some nonzero speed; and those that leave $\operatorname{supp}(\boldsymbol{v})$ eventually and "get stuck," i.e., sooner or later find themselves at a fixed point from which they cannot move for the rest of time.
  - It follows since $\operatorname{supp}(\boldsymbol{v})$ is a *compact* subset of an open set U that between $\operatorname{supp}(\boldsymbol{v})$ $\mathbb{R}^n \setminus U$, there is a buffer zone[5] of points $q \in U$ with $\boldsymbol{v}(q) = 0$. Thus, integral curves of the first kind meander forever, never leaving U, and curves of the second kind get stuck before they can. Either way, the curve can be defined for all time t.
- **Bump function**: A function $f : \mathbb{R}^n \to \mathbb{R}$ which is both smooth and compactly supported.
- Example:
  - $\Psi : \mathbb{R} \to \mathbb{R}$ defined by

$$\Psi(x) = \begin{cases} \exp\left(-\frac{1}{1-x^2}\right) & x \in (-1, 1) \\ 0 & \text{otherwise} \end{cases}$$

- $\boldsymbol{C_0^\infty(\mathbb{R}^n)}$: The vector space of all bump functions with domain $\mathbb{R}^n$.
- An application of Theorem 2.2.15.
  - Suppose $\boldsymbol{v}$ is a vector field on U and we want to inspect the integral curves of $\boldsymbol{v}$ on some $A \subset U$ compact. Let $\rho \in C_0^\infty(U)$ be such that $\rho(p) = 1$ for all $p \in N_r(A)$, where $N_r(A)$ is some neighborhood of the set A. Then the vector field $\boldsymbol{w} = \rho\boldsymbol{v}$ is compactly supported and hence complete. However, it is also identical to $\boldsymbol{v}$ on A, so its integral curves on A coincide with those of $\boldsymbol{v}$ on A.
- $\boldsymbol{f_t}$: The map from $U \to U$ defined as follows, where $\boldsymbol{v}$ is complete. *Given by*

$$f_t(p) = \gamma_p(t)$$

  where $\gamma_p : \mathbb{R} \to U$ satisfies $\gamma_p(0) = p$.
  - Note that it is the fact that $\boldsymbol{v}$ is complete that justifies the existence of an integral curve γ_p for all $p \in U$.
  - What f_t does: f_t takes every point in the set/"manifold" U and moves them, along their integral curves as defined by the vector field $\boldsymbol{v}$, to a new point at time t. There are definite parallels to a homotopy herein.

[5] Is there? Consider all roads lead to Rome over the open unit circle in $\mathbb{R}^2$. Curves can just run right off here, right, even though the support is compact?

- Properties of f_t.
  1. $\boldsymbol{v} \in C^\infty$ implies $f_t \in C^\infty$.

     Proof. By Theorem 2.2.6. □
  2. $f_0 = \mathrm{id}_U$.

     Proof. We have
     $$f_0(p) = \gamma_p(0) = p = \mathrm{id}_U(p)$$
     as desired. □
  3. $f_t \circ f_a = f_{t+a}$.

     Proof. Let $q = f_a(p)$. Since $\boldsymbol{v}$ is complete and $q \in U$, there exists γ_q such that $\gamma_q(0) = q$. It follows that $\gamma_p(a) = f_a(p) = q = \gamma_q(0)$. Thus, by Theorem 2.2.7, $\gamma_q(t)$ and $\gamma_p(t+a)$ are both integral curves of $\boldsymbol{v}$ with the same initial point. Therefore,
     $$(f_t \circ f_a)(p) = f_t(q) = \gamma_q(t) = \gamma_p(t+a) = f_{t+a}(p)$$
     for all t, as desired. □
  4. $f_t \circ f_{-t} = \mathrm{id}_U$.

     Proof. See properties 2 and 3. □
  5. $f_{-t} = f_t^{-1}$.

     Proof. See property 4. □
- Thus, f_t is a C^∞ **diffeomorphism**.
  - "Hence, if $\boldsymbol{v}$ is complete, it generates a **one-parameter group** f_t ($-\infty < t < \infty$) of C^∞-diffeomorphisms of U" (Guillemin & Haine, 2018, p. 40).
- **Diffeomorphism**: An isomorphism of smooth manifolds. In particular, it is an invertible function that maps one differentiable manifold to another such that both the function and its inverse are differentiable.
- **One-parameter group**: A continuous group homomorphism $\varphi : \mathbb{R} \to G$ from the real line $\mathbb{R}$ (as an additive group) to some other topological group G.
- If $\boldsymbol{v}$ is not complete, there is an analogous result, but it is trickier to formulate.
- $\boldsymbol{f}$**-related** (vector fields $\boldsymbol{v}, \boldsymbol{w}$): Two vector fields $\boldsymbol{v}, \boldsymbol{w}$ such that
  $$\mathrm{d}f_p(\boldsymbol{v}(p)) = \boldsymbol{w}(f(p))$$
  for all $p \in U$, where $\boldsymbol{v}$ is a C^∞-vector field on $U \subset \mathbb{R}^n$ open, $\boldsymbol{w}$ is a C^∞-vector field on $W \subset \mathbb{R}^m$ open, and $f : U \to W$ is a C^∞ map.
- Example: f-related $\boldsymbol{v}, \boldsymbol{w}$ for $f : \mathbb{R} \to \mathbb{R}$.
  - Let $U, W = \mathbb{R}_{\geq 0}$. Consider the function $f : U \to W$ described by the relation
    $$x \mapsto x^2$$
  - Figure 4.5 makes clear that in the same way that p and $f(p)$ are "related" by f, the vectors $\boldsymbol{v}(p)$ and $\boldsymbol{w}(f(p))$ are "related" by $\mathrm{d}f_p$. Indeed, the idea of "f-relatedness" simply implies that every vector in $\boldsymbol{v}(U)$ is so paired with a vector in $\boldsymbol{w}(W)$.
  - Let's now think about what we gain by introducing f-relatedness.

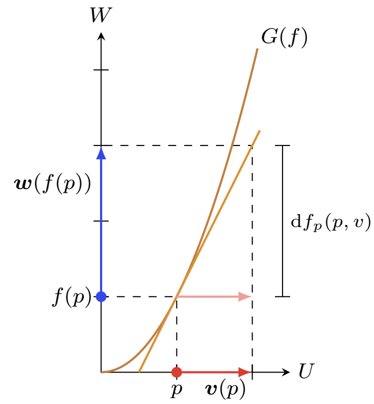


Figure 4.5: f-related $\boldsymbol{v}, \boldsymbol{w}$ for $f : \mathbb{R} \to \mathbb{R}$.

- $\mathrm{d}f_p$, as a linear transformation, takes in any vector and spits out another. $\mathrm{d}f$ attaches such a covector to each point of the domain (into which we can *later* feed whatever vector we want).
- f-relatedness allows us to *preselect* the vectors we want to feed into each $\mathrm{d}f_p$, run them through, and get the results, as indexed by $f(p)$.

– Let's now consider a specific example $\boldsymbol{v}$ and $\boldsymbol{w}$ and check that they are f-related.

- Let

$$\begin{aligned} \boldsymbol{v} : U &\to TU \\ p &\mapsto (p, e_1) \end{aligned} \qquad\qquad \begin{aligned} \boldsymbol{w} : W &\to TW \\ f(p) &\mapsto (f(p), 2p \cdot e_1) \end{aligned}$$

where TU, TW are the **tangent bundles** of U, W, respectively. Note that $\boldsymbol{v} = \partial/\partial x$.

- Let $p \in U$ be arbitrary. We know that

$$\begin{aligned} Df(p) &= \begin{bmatrix} \left.\frac{\partial f}{\partial x}\right|_p \end{bmatrix} \\ &= \begin{bmatrix} 2p \end{bmatrix} \end{aligned} \qquad\qquad \boldsymbol{v}(p) = \begin{bmatrix} 1 \end{bmatrix} \qquad\qquad \boldsymbol{w}(f(p)) = \begin{bmatrix} 2p \end{bmatrix}$$

from which it follows that

$$\mathrm{d}f_p(\boldsymbol{v}(p)) = \begin{bmatrix} 2p \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} 2p \end{bmatrix} = \boldsymbol{w}(f(p))$$

as desired.

- **Tangent bundle** (of a set U): The disjoint union of the tangent spaces of U. *Denoted by* $\boldsymbol{TU}$. *Given by*

$$TU = \bigcup_{p \in U} T_p U$$

- An alternate formulation of f-relatedness.
  – In terms of coordinates,

$$w_i(q) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} v_j(p)$$

where

$$\boldsymbol{v} = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i} \qquad\qquad \boldsymbol{w} = \sum_{j=1}^{m} w_j \frac{\partial}{\partial y_i}$$

for $v_i \in C^k(U)$ and $w_j \in C^k(W)$, and $f(p) = q$.

- Derivation: If $f(p) = q$, $\boldsymbol{v}(p) = (p, v)$, $v = (v_1, \dots, v_n)$, $\boldsymbol{w}(q) = (q, w)$, and $w = (w_1, \dots, w_m)$, then

$$\begin{aligned}
\boldsymbol{w}(q) &= \mathrm{d}f_p(\boldsymbol{v}(p)) \\
&= Df(p)v \\
&= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \\
\begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} &= \begin{bmatrix} \sum_{j=1}^n \frac{\partial f_1}{\partial x_j} v_j \\ \vdots \\ \sum_{j=1}^n \frac{\partial f_m}{\partial x_j} v_j \end{bmatrix}
\end{aligned}$$

so comparing i^{th} indices gives the above formula.

- If $m = n$ and f is a C^∞ diffeomorphism, then

$$\boldsymbol{w} = \sum_{i=1}^m w_i \frac{\partial}{\partial y_i}$$

where

$$w_i = \sum_{j=1}^n \left(\frac{\partial f_i}{\partial x_j} v_j \right) \circ f^{-1}$$

- Theorem 2.2.18: If $f : U \to W$ is a C^∞ diffeomorphism and $\boldsymbol{v}$ is a C^∞ vector field on U, then there exists a unique C^∞ vector field $\boldsymbol{w}$ on W having the property that $\boldsymbol{v}$ and $\boldsymbol{w}$ are f-related.

Proof. See the above. $\square$

- **Pushforward** (of $\boldsymbol{v}$ by f): The vector field $\boldsymbol{w}$ shown to exist by Theorem 2.2.18. *Denoted by* $\boldsymbol{f_* v}$.
- Theorem 2.2.20: Let $U_1, U_2 \subset \mathbb{R}^n$ open, $\boldsymbol{v}_1, \boldsymbol{v}_2$ vector fields on U_1, U_2, and $f : U_1 \to U_2$ a C^∞ map. If $\boldsymbol{v}_1, \boldsymbol{v}_2$ are f-related, every integral curve $\gamma : I \to U_1$ of $\boldsymbol{v}_1$ gets mapped by f onto an integral curve $f \circ \gamma : I \to U_2$ of $\boldsymbol{v}_2$.

Proof. We want to show that

$$\boldsymbol{v}_2((f \circ \gamma)(t)) = \left((f \circ \gamma)(t), \tfrac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)\big|_t \right)$$

We are given that

$$\boldsymbol{v}_1(\gamma(t)) = \left(\gamma(t), \tfrac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_t \right) \qquad \mathrm{d}f_p(\boldsymbol{v}_1(p)) = \boldsymbol{v}_2(f(p))$$

Let $p = \gamma(t)$ and $q = f(p)$. Then

$$\begin{aligned}
\boldsymbol{v}_2((f \circ \gamma)(t)) &= \boldsymbol{v}_2(f(p)) \\
&= \mathrm{d}f_p(\boldsymbol{v}_1(p)) \\
&= \mathrm{d}f_p(\boldsymbol{v}_1(\gamma(t))) \\
&= \mathrm{d}f_p \left(\gamma(t), \tfrac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_t \right) \\
&= \mathrm{d}f_p \left(p, \tfrac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_t \right) \\
&= \left(q, Df(p) \left(\tfrac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_t \right) \right) \\
&= \left((f \circ \gamma)(t), \tfrac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)\big|_t \right)
\end{aligned}$$

as desired. $\square$

- Corollary 2.2.21: In the setting of Theorem 2.2.20, suppose $\boldsymbol{v}_1, \boldsymbol{v}_2$ are complete. Let $(f_{i,t})_{t\in\mathbb{R}} : U_i \to U_i$ be the one-parameter group of diffeomorphisms generated by $\boldsymbol{v}_i$. Then

$$f \circ f_{1,t} = f_{2,t} \circ f$$

Proof. We have that

$$(f \circ f_{1,t})(p) = (f \circ \gamma_p)(t)$$

By Theorem 2.2.20, the above is an integral curve of $\boldsymbol{v}_2$. Let $f(p) = q$. Then

$$\begin{aligned}(f_{2,t} \circ f)(p) &= f_{2,t}(q)\\ &= \gamma_q(t)\end{aligned}$$

...

Guillemin and Haine (2018) proves that if $\phi \in C^\infty(U_2)$ and $f^*\phi = \phi \circ f$, then

$$f^* L_{\boldsymbol{v}_2}\phi = L_{\boldsymbol{v}_1} f^*\phi$$

by virtue of the observations that if $f(p) = q$, then at the point p, the right-hand side above is $(\mathrm{d}\phi)_q \circ \mathrm{d}f_p(\boldsymbol{v}_1(p))$ by the chain rule and by definition the left hand side is $\mathrm{d}\phi_q(\boldsymbol{v}_2(q))$. Moreover, by definition, $\boldsymbol{v}_2(q) = \mathrm{d}f_p(\boldsymbol{v}_1(p))$ so the two sides are the same. □

- Theorem 2.2.22: For $i = 1, 2, 3$, let $U_i \subset \mathbb{R}^{n_i}$ open and $\boldsymbol{v}_i$ a vector field on U_i. For $i = 1, 2$, let $f_i : U_i \to U_{i+1}$ be a C^∞ map. If $\boldsymbol{v}_1, \boldsymbol{v}_2$ are f_1-related and $\boldsymbol{v}_2, \boldsymbol{v}_3$ are f_2-related, then $\boldsymbol{v}_1, \boldsymbol{v}_3$ are $(f_2 \circ f_1)$-related. In particular, if f_1, f_2 are diffeomorphisms, we have

$$(f_2)_*(f_1)_*\boldsymbol{v}_1 = (f_2 \circ f_1)_*\boldsymbol{v}_1$$

- **Pullback** (of μ along f): The one-form on U defined as follows, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open, $f : U \to V$ is a C^∞ map, and μ is a one-form on V. *Denoted by* $\boldsymbol{f^*\mu}$. *Given by*

$$p \mapsto \mu_{f(p)} \circ \mathrm{d}f_p$$

  - This definition does stick with the theme of pullbacks being precompositions; it is just a bit more complicated because a one-form takes two inputs instead of one: a point and a tangent vector.
  - To feed a point $p \in U$ and a vector $v \in U$ into a one-form on V, we use f to send $p \mapsto f(p)$ and $\mathrm{d}f_p$ to send $v \mapsto \mathrm{d}f_p(p, v)$. Hence the above definition.

- If $\phi : V \to \mathbb{R}$ is a C^∞ map and $\mu = \mathrm{d}\phi$, then

$$\mu_q \circ \mathrm{d}f_p = \mathrm{d}\phi_q \circ \mathrm{d}f_p = \mathrm{d}(\phi \circ f)_p$$

  - In other words,

$$f^*\mu = \mathrm{d}\phi \circ f$$

- Theorem 2.2.24: If μ is a C^∞ one-form on V, its pullback $f^*\mu$ is C^∞.

Proof. See Exercise 2.2.ii. □