MATH 20510 (Analysis in $\mathbb{R}^n$ III – Accelerated) Problem Sets

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Contents

1 Multilinear Algebra

From Guillemin and Haine (2018).

Chapter 1

1.2.iv. Let U, V, and W be vector spaces and let $A : V \to W$ and $B : U \to V$ be linear mappings. Show that $(AB)^* = B^*A^*$.

Proof. Clearly, both $(AB)^*$ and B^*A^* send W^* to U^*. Thus, we need only verify that both maps have the same action on every element of W^*.

Let $\ell \in W^*$ be arbitrary. Then

$$(AB)^*\ell = \ell \circ AB = (\ell \circ A) \circ B = A^*\ell \circ B$$

where $A^*\ell \in V^*$. It follows in a similar fashion that

$$A^*\ell \circ B = B^*(A^*\ell) = (B^*A^*)\ell$$

where we have the last equality above by the associativity of the composition operation. Transitivity between the first and second equations above finishes the proof. □

1.2.v. Let $V = \mathbb{R}^2$ and let W be the x_1-axis, i.e., the one-dimensional subspace

$$\{(x_1, 0) \mid x_1 \in \mathbb{R}\}$$

of $\mathbb{R}^2$.

(1) Show that the W-cosets are the lines $x_2 = a$ parallel to the x_1-axis.

Proof. Let $v + W \in V/W$ be arbitrary. Let $v = (v_1, v_2)$. Then

$$\begin{aligned}
v + W &= \{v + w \mid w \in \{(x_1, 0) \mid x_1 \in \mathbb{R}\}\} \\
&= \{v + (x_1, 0) \mid x_1 \in \mathbb{R}\} \\
&= \{(v_1 + x_1, v_2) \mid x_1 \in \mathbb{R}\} \\
&= \{(x_1, v_2) \mid x_1 \in \mathbb{R}\}
\end{aligned}$$

Since every line $x_2 = a$ is a set of the form $\{(x_1, a) \mid x_1 \in \mathbb{R}\}$, we have that $v + W$ is equal to the line $x_2 = v_2$, as desired. □

(2) Show that the sum of the cosets $x_2 = a$ and $x_2 = b$ is the coset $x_2 = a + b$.

Proof. By part (1), every line $x_2 = a$ is a set of the form $(0, a) + W$. Therefore, by the definition of addition on V/W,

$$\begin{aligned}
[(0, a) + W] + [(0, b) + W] &= [(0, a) + (0, b)] + W \\
&= (0, a + b) + W
\end{aligned}$$

as desired. □

(3) Show that the scalar multiple of the coset $x_2 = c$ by the number λ is the coset $x_2 = \lambda c$.

Proof. Proceeding in a similar manner to part (2), we have that

$$\begin{aligned}
\lambda[(0, c) + W] &= [\lambda(0, c)] + W \\
&= (0, \lambda c) + W
\end{aligned}$$

as desired. □

1.2.vi. (1) Let $(V^*)^*$ be the dual of the vector space V^*. For every $v \in V$, let $\mathrm{ev}_v : V^* \to \mathbb{R}$ be the **evaluation function** $\mathrm{ev}_v(\ell) = \ell(v)$. Show that the ev_v is a linear function on V^*, i.e., an element of $(V^*)^*$, and show that the map $\mathrm{ev} = \mathrm{ev}_{(-)} : V \to (V^*)^*$ defined by $v \mapsto \mathrm{ev}_v$ is a linear map of V into $(V^*)^*$.

Proof. Let $v \in V$, $\ell_1, \ell_2, \ell \in V^*$, and $\lambda \in \mathbb{R}$ be arbitrary. Then

$$\begin{aligned} \mathrm{ev}_v(\ell_1 + \ell_2) &= (\ell_1 + \ell_2)(v) \\ &= \ell_1(v) + \ell_2(v) \\ &= \mathrm{ev}_v(\ell_1) + \mathrm{ev}_v(\ell_2) \end{aligned} \qquad \begin{aligned} \mathrm{ev}_v(\lambda\ell) &= (\lambda\ell)(v) \\ &= \lambda\ell(v) \\ &= \lambda\,\mathrm{ev}_v(\ell) \end{aligned}$$

so ev_v is linear, as desired.

Let $v_1, v_2, v \in V$, $\ell \in V^*$, and $\lambda \in \mathbb{R}$ be arbitrary. Then

$$\begin{aligned} \mathrm{ev}(v_1 + v_2)(\ell) &= \mathrm{ev}_{v_1+v_2}(\ell) \\ &= \ell(v_1 + v_2) \\ &= \ell(v_1) + \ell(v_2) \\ &= \mathrm{ev}_{v_1}(\ell) + \mathrm{ev}_{v_2}(\ell) \\ &= \mathrm{ev}(v_1)(\ell) + \mathrm{ev}(v_2)(\ell) \\ &= [\mathrm{ev}(v_1) + \mathrm{ev}(v_2)](\ell) \end{aligned} \qquad \begin{aligned} \mathrm{ev}(\lambda v)(\ell) &= \mathrm{ev}_{\lambda v}(\ell) \\ &= \ell(\lambda v) \\ &= \lambda\ell(v) \\ &= \lambda\,\mathrm{ev}_v(\ell) \\ &= \lambda\,\mathrm{ev}(v)(\ell) \\ &= [\lambda\,\mathrm{ev}(v)](\ell) \end{aligned}$$

Thus, $\mathrm{ev}(v_1 + v_2)$ and $\mathrm{ev}(v_1) + \mathrm{ev}(v_2)$, and $\mathrm{ev}(\lambda v)$ and $\lambda\,\mathrm{ev}(v)$ have the same action pairwise on every $\ell \in V^*$. Consequently, the two pairs of functions in V^* are both equal pairwise. Therefore, ev itself is linear. □

(2) If V is finite dimensional, show that the map ev is bijective. Conclude that there is a natural identification of V with $(V^*)^*$, i.e., that V and $(V^*)^*$ are two descriptions of the same object. (Hint: $\dim(V^*)^* = \dim V^* = \dim V$, so since $\dim(V) = \dim(\ker(A)) + \dim(\mathrm{im}(A))$, it suffices to show that ev is injective.)

Proof. Taking the hint, we seek to show that ev is injective. Suppose $v_1 \neq v_2$. WLOG let $v_2 \neq 0$. Let $\ell : V \to \mathbb{R}$ be defined by

$$\ell(v) = \begin{cases} \|v\| & v = \lambda v_2 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned} \ell(v_1) &\neq \ell(v_2) \\ \mathrm{ev}_{v_1}(\ell) &\neq \mathrm{ev}_{v_2}(\ell) \\ \mathrm{ev}(v_1)(\ell) &\neq \mathrm{ev}(v_2)(\ell) \end{aligned}$$

as desired. □

1.2.xi. Let V be a vector space.

(1) Let $B : V \times V \to \mathbb{R}$ be an inner product on V. For all $v \in V$, let $\ell_v : V \to \mathbb{R}$ be the function $\ell_v(w) = B(v, w)$. Show that ℓ_v is linear, and show that the map $L : V \to V^*$ defined by $v \mapsto \ell_v$ is a linear mapping.

Proof. Since

$$\begin{aligned} \ell_v(w_1 + w_2) &= B(v, w_1 + w_2) \\ &= B(w_1 + w_2, v) \\ &= B(w_1, v) + B(w_2, v) \\ &= B(v, w_1) + B(v, w_2) \\ &= \ell_v(w_1) + \ell_v(w_2) \end{aligned} \qquad \begin{aligned} \ell_v(\lambda w) &= B(v, \lambda w) \\ &= B(\lambda w, v) \\ &= \lambda B(w, v) \\ &= \lambda B(v, w) \\ &= \lambda\ell_v(w) \end{aligned}$$

we have that ℓ_v is linear, as desired. Note that each step follows either from the definition of ℓ_v or one of the three inner product properties (bilinearity, symmetry, and positivity).

Since

$$\begin{aligned} [L(v_1+v_2)](w) &= \ell_{v_1+v_2}(w) \\ &= B(v_1+v_2,w) \\ &= B(v_1,w)+B(v_2,w) \\ &= \ell_{v_1}(w)+\ell_{v_2}(w) \\ &= L(v_1)(w)+L(v_2)(w) \\ &= [L(v_1)+L(v_2)](w) \end{aligned} \qquad \begin{aligned} [L(\lambda v)](w) &= \ell_{\lambda v}(w) \\ &= B(\lambda v,w) \\ &= \lambda B(v,w) \\ &= \lambda \ell_v(w) \\ &= \lambda L(v)(w) \\ &= [\lambda L(v)](w) \end{aligned}$$

we know that the functions $L(v_1+v_2)$ and $L(v_1)+L(v_2)$ have the same action on every $w \in V$. Thus they are equal. A symmetric statement holds for $L(\lambda v)$ and $\lambda L(v)$. ☐

(2) If V is finite dimensional, prove that L is bijective. Conclude that if V has an inner product, one gets from it a natural identification of V with V^*. (Hint: Since $\dim V = \dim V^*$ and $\dim(V) = \dim(\ker(A)) + \dim(\operatorname{im}(A))$, it suffices to show that $\ker(L) = 0$. Now note that if $v \neq \mathbf{0}$, then $\ell_v(v) = B(v,v)$ is a positive number.)

Proof. Taking the hint, suppose $L(v) = 0 \in V^*$ for some $v \in V$. Thus, for all $w \in V$ (and, in particular, for v), we have that

$$0 = L(v)(v) = \ell_v(v) = B(v,v)$$

But then by the positivity of the inner product, $v = 0$, as desired. ☐

1.3.i. Verify that there are exactly n^k multi-indices of length k.

Proof. Let $(i_1,\dots,i_k)$ be a multi-index of n of length k. We independently pick each i_j to be any one of the n numbers between 1 and n, inclusive. Thus, for each of the n values of i_1, there are n possible values of i_2. For each of the n^2 values of (i_1,i_2), there are n possible values of i_3. Continuing on in this fashion inductively confirms that there are always exactly n^k multi-indices of length k. ☐

1.3.ii. Prove that the map $A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$ defined by $T \mapsto A^*T$ is linear.

Proof. We have that

$$\begin{aligned} [A^*(T_1+T_2)](v_1,\dots,v_k) &= (T_1+T_2)(Av_1,\dots,Av_k) \\ &= T_1(Av_1,\dots,Av_k)+T_2(Av_1,\dots,Av_k) \\ &= A^*T_1(v_1,\dots,v_k)+A^*T_2(v_1,\dots,v_k) \\ &= [A^*T_1+A^*T_2](v_1,\dots,v_k) \end{aligned}$$

and

$$\begin{aligned} [A^*(\lambda T)](v_1,\dots,v_k) &= (\lambda T)(Av_1,\dots,Av_k) \\ &= \lambda T(Av_1,\dots,Av_k) \\ &= \lambda (A^*T)(v_1,\dots,v_k) \\ &= [\lambda (A^*T)](v_1,\dots,v_k) \end{aligned}$$

as desired. ☐

1.3.iii. Verify that

$$A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$$

Proof. Let $T_1 \in \mathcal{L}^k(W)$ and $T_2 \in \mathcal{L}^\ell(W)$. Then

$$\begin{aligned}
[A^*(T_1 \otimes T_2)](v_1, \dots, v_{k+\ell}) &= (T_1 \otimes T_2)(Av_1, \dots, Av_{k+\ell}) \\
&= T_1(Av_1, \dots, Av_k)T_2(Av_{k+1}, \dots, Av_{k+\ell}) \\
&= (A^*T_1)(v_1, \dots, v_k)(A^*T_2)(v_{k+1}, \dots, v_{k+\ell}) \\
&= [A^*(T_1) \otimes A^*(T_2)](v_1, \dots, v_{k+\ell})
\end{aligned}$$

as desired. □

1.3.iv. Verify that

$$(AB)^*T = B^*(A^*T)$$

Proof. Let U, V, W be vector spaces, $A : V \to W$, $B : U \to V$, and $T \in \mathcal{L}^k(W)$. Then

$$\begin{aligned}
[(AB)^*T](v_1, \dots, v_k) &= T(ABv_1, \dots, ABv_k) \\
&= A^*T(Bv_1, \dots, Bv_k) \\
&= [B^*(A^*T)](v_1, \dots, v_k)
\end{aligned}$$

as desired. □

1.3.vii. Let T be a k-tensor and v be a vector. Define $T_v : V^{k-1} \to \mathbb{R}$ by

$$T_v(v_1, \dots, v_{k-1}) = T(v, v_1, \dots, v_{k-1})$$

Show that T_v is a $(k-1)$-tensor.

Proof. For the sake of space and ease of notation, I will show only that T_v is linear in its 1^{st} variable. However, a symmetric argument would work in the generalized i^{th} case. This being established, it will follow that T_v is $(k-1)$-linear and thus a $(k-1)$-tensor, as desired. Let's begin.

We have that

$$\begin{aligned}
T_v(v_1 + v_1', \dots, v_{k-1}) &= T(v, v_1 + v_1', \dots, v_{k-1}) \\
&= T(v, v_1, \dots, v_{k-1}) + T(v, v_1', \dots, v_{k-1}) \\
&= T_v(v_1, \dots, v_{k-1}) + T_v(v_1', \dots, v_{k-1})
\end{aligned}$$

and

$$\begin{aligned}
T_v(\lambda v_1, \dots, v_{k-1}) &= T(v, \lambda v_1, \dots, v_{k-1}) \\
&= \lambda T(v, v_1, \dots, v_{k-1}) \\
&= \lambda T_v(v_1, \dots, v_{k-1})
\end{aligned}$$

as desired. □

1.3.viii. Show that if T_1 is an r-tensor and T_2 is an s-tensor, then if $r > 0$,

$$(T_1 \otimes T_2)_v = (T_1)_v \otimes T_2$$

Proof. We have that

$$\begin{aligned}
[(T_1 \otimes T_2)_v](v_1, \dots, v_{r+s-1}) &= (T_1 \otimes T_2)(v, v_1, \dots, v_{r+s-1}) \\
&= T_1(v, v_1, \dots, v_{r-1})T_2(v_r, \dots, v_{r+s-1}) \\
&= (T_1)_v(v_1, \dots, v_{r-1})T_2(v_r, \dots, v_{r+s-1}) \\
&= [(T_1)_v \otimes T_2](v_1, \dots, v_{r+s-1})
\end{aligned}$$

as desired. □

1.3.ix. Let $A : V \to W$ be a linear map, let $v \in V$, and let $w = Av$. Show that for all $T \in \mathcal{L}^k(W)$,

$$A^*(T_w) = (A^*T)_v$$

Proof. We have that

$$\begin{aligned}
[A^*(T_w)](v_1, \dots, v_{k-1}) &= T_w(Av_1, \dots, Av_{k-1}) \\
&= T(w, Av_1, \dots, Av_{k-1}) \\
&= T(Av, Av_1, \dots, Av_{k-1}) \\
&= (A^*T)(v, v_1, \dots, v_k) \\
&= [(A^*T)_v](v_1, \dots, v_k)
\end{aligned}$$

as desired. □

1.4.i. Show that there are exactly $k!$ permutations of order k. (Hint: Induction on k: Let $\sigma \in S_k$, and let $\sigma(k) = i$ ($1 \leq i \leq k$). Show that $\tau_{i,k}\sigma$ leaves k fixed and hence is, in effect, a permutation of Σ_{k-1}.)

Proof. We induct on k. For the base case $k = 1$, there is clearly only $1! = 1$ possible bijection from a singleton set to itself. Now suppose inductively that we have proven the claim for $k - 1$. Let $\sigma \in S_k$ be arbitrary. Suppose $\sigma(k) = i$. It follows that $(\tau_{i,k}\sigma)(k) = \tau_{i,k}(i) = k$. Thus, since $\tau_{i,k}\sigma$ is a bijection on Σ_k, $(\tau_{i,k}\sigma)|_{\Sigma_{k-1}} \in S_{k-1}$. Consequently, by the inductive hypothesis, there are $(k-1)!$ possible permutations $(\tau_{i,k}\sigma)|_{\Sigma_{k-1}}$. Furthermore, to each of these permutations, there correspond k distinct permutations in S_k (i.e., those obtained by iterating i from 1 through k). Thus, there are $k \cdot (k-1)! = k!$ permutations of order k, as desired. □

1.4.ii. Prove that if $\tau \in S_k$ is a transposition, $(-1)^\tau = -1$. Deduce from this that if σ is the product of an odd number of transpositions, then $(-1)^\sigma = -1$, and if σ is the product of an even number of transpositions, then $(-1)^\sigma = +1$.

Proof. We induct on k.

For the base case $k = 2$, the only possible transposition is $\tau_{1,2}$. For this transposition, we have

$$(-1)^{\tau_{1,2}} = \prod_{i<j} \frac{x_{\tau_{1,2}(i)} - x_{\tau_{1,2}(j)}}{x_i - x_j} = \frac{x_{\tau_{1,2}(1)} - x_{\tau_{1,2}(2)}}{x_1 - x_2} = \frac{x_2 - x_1}{x_1 - x_2} = -1$$

as desired.

Now suppose inductively that we have proven the claim for $k-1$. Let $\tau_{p,q} \in S_k$ with $p < q$ WLOG. We divide into two cases ($q \neq k$ and $q = k$).

If $q \neq k$, then as in Exercise 1.4.i, we can identify $\tau_{p,q}$ with an element $\tau'_{p,q} \in S_{k-1}$. By the inductive hypothesis,

$$-1 = (-1)^{\tau'_{p,q}} = \prod_{\substack{i<j \\ j \neq k}} \frac{x_{\tau_{p,q}(i)} - x_{\tau_{p,q}(j)}}{x_i - x_j}$$

It follows that

$$(-1)^{\tau_{p,q}} = \prod_{i<j} \frac{x_{\tau_{p,q}(i)} - x_{\tau_{p,q}(j)}}{x_i - x_j} = \prod_{\substack{i<j \\ j \neq k}} \frac{x_{\tau_{p,q}(i)} - x_{\tau_{p,q}(j)}}{x_i - x_j} \cdot \prod_{i=1}^{k-1} \frac{x_{\tau_{p,q}(i)} - x_{\tau_{p,q}(k)}}{x_i - x_k} = -1 \cdot 1 = -1$$

where we evaluate

$$\begin{aligned}\prod_{i=1}^{k-1}\frac{x_{\tau_{p,q}(i)}-x_{\tau_{p,q}(k)}}{x_i-x_k} &= \prod_{i=1}^{k-1}\frac{x_{\tau_{p,q}(i)}-x_k}{x_i-x_k}\\ &= \prod_{\substack{i=1\\ i\neq p,q}}^{k-1}\frac{x_{\tau_{p,q}(i)}-x_k}{x_i-x_k}\cdot\frac{x_{\tau_{p,q}(p)}-x_k}{x_p-x_k}\cdot\frac{x_{\tau_{p,q}(q)}-x_k}{x_q-x_k}\\ &= \prod_{\substack{i=1\\ i\neq p,q}}^{k-1}\frac{x_i-x_k}{x_i-x_k}\cdot\frac{x_q-x_k}{x_p-x_k}\cdot\frac{x_p-x_k}{x_q-x_k}\\ &= 1\end{aligned}$$

If $q=k$, then we divide into two subcases ($p=k-1$ and $p\neq k-1$). If $p=k-1$, then $\tau_{p,q}=\tau_{k-1,k}$. Therefore,

$$\begin{aligned}&(-1)^{\tau_{p,q}}\\ &= \prod_{i<j}\frac{x_{\tau_{k-1,k}(i)}-x_{\tau_{k-1,k}(j)}}{x_i-x_j}\\ &= \prod_{\substack{i<j\\ j<k-1}}\frac{x_{\tau_{k-1,k}(i)}-x_{\tau_{k-1,k}(j)}}{x_i-x_j}\cdot\prod_{i=1}^{k-2}\frac{x_{\tau_{k-1,k}(i)}-x_{\tau_{k-1,k}(k-1)}}{x_i-x_{k-1}}\cdot\prod_{i=1}^{k-2}\frac{x_{\tau_{k-1,k}(i)}-x_{\tau_{k-1,k}(k)}}{x_i-x_k}\cdot\frac{x_{\tau_{k-1,k}(k-1)}-x_{\tau_{k-1,k}(k)}}{x_{k-1}-x_k}\\ &= \prod_{\substack{i<j\\ j<k-1}}\frac{x_i-x_j}{x_i-x_j}\cdot\prod_{i=1}^{k-2}\frac{x_i-x_k}{x_i-x_{k-1}}\cdot\prod_{i=1}^{k-2}\frac{x_i-x_{k-1}}{x_i-x_k}\cdot\frac{x_k-x_{k-1}}{x_{k-1}-x_k}\\ &= \prod_{\substack{i<j\\ j<k-1}}\frac{x_i-x_j}{x_i-x_j}\cdot\prod_{i=1}^{k-2}\left(\frac{x_i-x_{k-1}}{x_i-x_{k-1}}\frac{x_i-x_k}{x_i-x_k}\right)\cdot\frac{x_k-x_{k-1}}{x_{k-1}-x_k}\\ &= 1\cdot 1\cdot -1\\ &= -1\end{aligned}$$

If $p\neq k-1$, then $\tau_{p,q}=\tau_{p,k}=\tau_{k-1,k}\tau_{p,k-1}\tau_{k-1,k}$. By our argument for the case $q\neq k$, we know that $(-1)^{\tau_{p,k-q}}=-1$, and by our argument for the case $q=k$ and $p=k-1$, we know that $(-1)^{\tau_{k-1,k}}=-1$. Therefore, by Claim 1.4.9,

$$(-1)^{\tau_{p,q}}=(-1)^{\tau_{k-1,k}\tau_{p,k-1}\tau_{k-1,k}}=(-1)^{\tau_{k-1,k}}(-1)^{\tau_{p,k-1}}(-1)^{\tau_{k-1,k}}=-1\cdot -1\cdot -1=-1$$

as desired.

It follows by Claim 1.4.9 that if $\sigma\in S_k$ can be decomposed into $\sigma=\tau_1\cdots\tau_n$ where $n|2=1$, then

$$(-1)^\sigma=(-1)^{\tau_1\cdots\tau_n}=(-1)^{\tau_1}\cdots(-1)^{\tau_n}=\underbrace{(-1)\cdots(-1)}_{n\text{ times}}=-1$$

as desired.

The proof is symmetric for even permutations. □

1.4.iii. Prove that the assignment $T\mapsto T^\sigma$ is a linear map $\mathcal{L}^k(V)\to\mathcal{L}^k(V)$.

Proof. We have that

$$\begin{aligned}(T_1+T_2)^\sigma(v_1,\dots,v_k) &= (T_1+T_2)(v_{\sigma^{-1}(1)},\dots,v_{\sigma^{-1}(k)})\\ &= T_1(v_{\sigma^{-1}(1)},\dots,v_{\sigma^{-1}(k)})+T_2(v_{\sigma^{-1}(1)},\dots,v_{\sigma^{-1}(k)})\\ &= T_1^\sigma(v_1,\dots,v_k)+T_2^\sigma(v_1,\dots,v_k)\end{aligned}$$

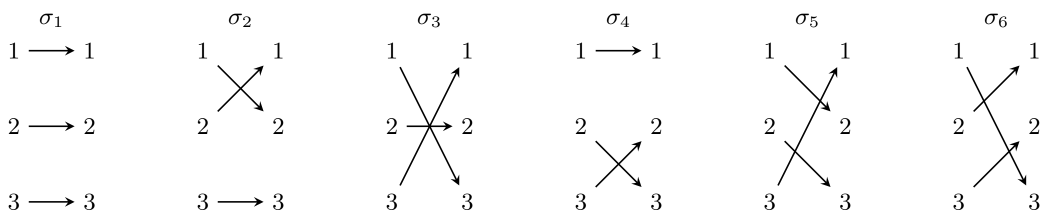
and

$$
\begin{aligned}
(\lambda T)^\sigma(v_1,\dots,v_k) &= (\lambda T)(v_{\sigma^{-1}(1)},\dots,v_{\sigma^{-1}(k)}) \\
&= \lambda T(v_{\sigma^{-1}(1)},\dots,v_{\sigma^{-1}(k)}) \\
&= \lambda T^\sigma(v_1,\dots,v_k)
\end{aligned}
$$

as desired. □

1.4.vi. Show that every one of the six elements of S_3 is either a transposition or can be written as a product of two transpositions.

Proof. The six elements $\sigma_1,\dots,\sigma_6 \in S_3$ are the permutations



It follows that we may write

$$
\sigma_1 = \tau_{1,2}\tau_{1,2} \qquad \sigma_2 = \tau_{1,2} \qquad \sigma_3 = \tau_{1,3} \qquad \sigma_4 = \tau_{2,3} \qquad \sigma_5 = \tau_{1,2}\tau_{2,3} \qquad \sigma_6 = \tau_{1,2}\tau_{1,3}
$$

□

1.4.ix. Let $A : V \to W$ be a linear mapping. Show that if $T \in \mathcal{A}^k(W)$, then $A^*T \in \mathcal{A}^k(V)$.

Proof. Since $T \in \mathcal{A}^k(W)$, we know that $T^\sigma = (-1)^\sigma T$ for all $\sigma \in S_k$. It follows that

$$
\begin{aligned}
(A^*T)^\sigma(v_1,\dots,v_k) &= (A^*T)(v_{\sigma^{-1}(1)},\dots,v_{\sigma^{-1}(k)}) \\
&= T(Av_{\sigma^{-1}(1)},\dots,Av_{\sigma^{-1}(k)}) \\
&= T^\sigma(Av_1,\dots,Av_k) \\
&= (-1)^\sigma T(Av_1,\dots,Av_k) \\
&= (-1)^\sigma A^*T(v_1,\dots,v_k)
\end{aligned}
$$

as desired. □

1.5.i. A k-tensor $T \in \mathcal{L}^k(V)$ is **symmetric** if $T^\sigma = T$ for all $\sigma \in S_k$. Show that the set $\mathcal{S}^k(V)$ of symmetric k-tensors is a vector subspace of $\mathcal{L}^k(V)$.

Proof. To prove that $\mathcal{S}^k(V) \leq \mathcal{L}^k(V)$, it will suffice to show that it contains the additive identity of $\mathcal{L}^k(V)$ (i.e., the zero tensor), and that it is closed under addition and scalar multiplication. Since we clearly have

$$
0^\sigma(v_1,\dots,v_k) = 0(v_{\sigma^{-1}(1)},\dots,v_{\sigma^{-1}(k)}) = 0(v_1,\dots,v_k)
$$

we know that $\mathcal{S}^k(V)$ contains the additive identity. Now suppose $T_1, T_2 \in \mathcal{S}^k(V)$. Then since

$$
(T_1+T_2)^\sigma = T_1^\sigma + T_2^\sigma = T_1 + T_2
$$

where the first equality holds because of the linearity of $\sigma : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ and the second equality holds since $T_1, T_2 \in \mathcal{S}^k(V)$, $\mathcal{S}^k(V)$ is closed under addition. Similarly, the fact that

$$
(\lambda T)^\sigma = \lambda T^\sigma = \lambda T
$$

confirms that $\mathcal{S}^k(V)$ is closed under scalar multiplication. □

1.6.i. Verify the following three equations, where $\lambda \in \mathbb{R}$.

(1) $\lambda(\omega_1 \wedge \omega_2) = (\lambda\omega_1) \wedge \omega_2 = \omega_1 \wedge (\lambda\omega_2)$.

Proof. We have that

$$\begin{aligned}\lambda(\omega_1 \wedge \omega_2) &= \lambda\pi(T_1 \otimes T_2)\\ &= \pi[(\lambda T_1) \otimes T_2]\\ &= (\lambda\omega_1) \wedge \omega_2\end{aligned}$$

It follows by a symmetric argument that $\lambda(\omega_1 \wedge \omega_2) = \omega_1 \wedge (\lambda\omega_2)$. □

(2) $(\omega_1 + \omega_2) \wedge \omega_3 = \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_3$.

Proof. We have that

$$\begin{aligned}(\omega_1 + \omega_2) \wedge \omega_3 &= \pi[(T_1 + T_2) \otimes T_3]\\ &= \pi[T_1 \otimes T_3 + T_2 \otimes T_3]\\ &= \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_3\end{aligned}$$

as desired. □

(3) $\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$.

Proof. We have that

$$\begin{aligned}\omega_1 \wedge (\omega_2 + \omega_3) &= \pi[T_1 \otimes (T_2 + T_3)]\\ &= \pi[T_1 \otimes T_2 + T_1 \otimes T_3]\\ &= \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3\end{aligned}$$

as desired. □

1.6.ii. Verify the following multiplicative law for the wedge product.

$$\omega_1 \wedge \omega_2 = (-1)^{rs}\omega_2 \wedge \omega_1$$

Proof. As per Guillemin and Haine (2018), it suffices to prove this claim for decomposable elements. As such, let $\omega_1 = \ell_1 \wedge \cdots \wedge \ell_r$ and let $\omega_2 = \ell'_1 \wedge \cdots \wedge \ell'_s$. Let $\sigma \in S_{r+s}$ be the permutation

$$\sigma(x) = \begin{cases} x + s & x \leq r \\ x - r & x > r \end{cases}$$

We can write σ as a product of elementary transpositions in a systematic manner as follows.

$$\sigma = \prod_{j=s-1}^{0} \prod_{i=1}^{r} \tau_{i+j,\, i+j+1}$$

Clearly, there are rs of these transpositions, so $(-1)^\sigma = (-1)^{rs}$. Therefore, we have that

$$\begin{aligned}\omega_1 \wedge \omega_2 &= (\ell_1 \wedge \cdots \wedge \ell_r) \wedge (\ell'_1 \wedge \cdots \wedge \ell'_s)\\ &= (-1)^\sigma(\ell'_1 \wedge \cdots \wedge \ell'_s) \wedge (\ell_1 \wedge \cdots \wedge \ell_r)\\ &= (-1)^{rs}\omega_2 \wedge \omega_1\end{aligned}$$

□

1.6.iv. If $\omega, \mu \in \Lambda^r(V^*)$, prove that

$$(\omega+\mu)^k = \sum_{\ell=0}^{k}\binom{k}{\ell}\omega^\ell \wedge \mu^{k-\ell}$$

(Hint: As in freshman calculus, prove this binomial theorem by induction using the identity $\binom{k}{\ell} = \binom{k-1}{\ell-1} + \binom{k-1}{\ell}$.)

Proof. We induct on k.

For the base case $k=1$, we have that

$$\begin{aligned}\sum_{\ell=0}^{1}\binom{1}{\ell}\omega^\ell \wedge \mu^{1-\ell} &= \binom{1}{0}\omega^0 \wedge \mu^{1-0} + \binom{1}{1}\omega^1 \wedge \mu^{1-1}\\ &= \mu + \omega\\ &= (\omega+\mu)^1\end{aligned}$$

as desired.

Now suppose inductively that we have proven the claim for $k-1$. Then

$$\begin{aligned}(\omega+\mu)^k &= (\omega+\mu)^1(\omega+\mu)^{k-1}\\ &= (\omega+\mu)\sum_{\ell=0}^{k-1}\binom{k-1}{\ell}\omega^\ell \wedge \mu^{(k-1)-\ell}\\ &= \sum_{\ell=0}^{k-1}\binom{k-1}{\ell}\omega^{\ell+1} \wedge \mu^{(k-1)-\ell} + \sum_{\ell=0}^{k-1}\binom{k-1}{\ell}\omega^\ell \wedge \mu^{k-\ell}\\ &= \sum_{\ell=1}^{k}\binom{k-1}{\ell-1}\omega^\ell \wedge \mu^{k-\ell} + \sum_{\ell=0}^{k-1}\binom{k-1}{\ell}\omega^\ell \wedge \mu^{k-\ell}\\ &= \binom{k-1}{k-1}\omega^{k-1} \wedge \mu^1 + \sum_{\ell=1}^{k-1}\left[\binom{k-1}{\ell-1} + \binom{k-1}{\ell}\right]\omega^\ell \wedge \mu^{k-\ell} + \binom{k-1}{0}\omega^0 \wedge \mu^k\\ &= \binom{k}{k}\omega^{k-1} \wedge \mu^1 + \sum_{\ell=1}^{k-1}\binom{k}{\ell}\omega^\ell \wedge \mu^{k-\ell} + \binom{k}{0}\omega^0 \wedge \mu^k\\ &= \sum_{\ell=0}^{k}\binom{k}{\ell}\omega^\ell \wedge \mu^{k-\ell}\end{aligned}$$

as desired. □

1.7.i. Prove that if T is the decomposable k-tensor $\ell_1 \otimes \cdots \otimes \ell_k$, then

$$\iota_v T = \sum_{r=1}^{k}(-1)^{r-1}\ell_r(v)\ell_1 \otimes \cdots \otimes \hat{\ell}_r \otimes \cdots \otimes \ell_k$$

where the hat over ℓ_r means that ℓ_r is deleted from the tensor product.

Proof. We have that

$$\begin{aligned}
(\iota_v T)(v_1,\dots,v_{k-1}) &= \sum_{r=1}^{k}(-1)^{r-1}T(v_1,\dots,v_{r-1},v,v_r,\dots,v_{k-1})\\
&= \sum_{r=1}^{k}(-1)^{r-1}[\ell_1\otimes\cdots\otimes\ell_{r-1}\otimes\ell_r\otimes\ell_{r+1}\otimes\cdots\otimes\ell_k](v_1,\dots,v_{r-1},v,v_r,\dots,v_{k-1})\\
&= \sum_{r=1}^{k}(-1)^{r-1}\ell_1(v_1)\cdots\ell_{r-1}(v_{r-1})\ell_r(v)\ell_{r+1}(v_r)\cdots\ell_k(v_{k-1})\\
&= \sum_{r=1}^{k}(-1)^{r-1}\ell_r(v)\ell_1(v_1)\cdots\ell_{r-1}(v_{r-1})\ell_{r+1}(v_r)\cdots\ell_k(v_{k-1})\\
&= \sum_{r=1}^{k}(-1)^{r-1}\ell_r(v)[\ell_1\otimes\cdots\otimes\hat{\ell}_r\otimes\cdots\otimes\ell_k](v_1,\dots,v_{k-1})
\end{aligned}$$

as desired. ☐

1.7.ii. Prove that if $T_1\in\mathcal{L}^p(V)$ and $T_2\in\mathcal{L}^q(V)$, then

$$\iota_v(T_1\otimes T_2)=\iota_vT_1\otimes T_2+(-1)^pT_1\otimes\iota_vT_2$$

Proof. We have that

$$\begin{aligned}
[\iota_v(T_1\otimes T_2)](v_1,\dots,v_{p+q-1}) &= \sum_{r=1}^{p+q}(-1)^{r-1}(T_1\otimes T_2)(v_1,\dots,v_{r-1},v,v_r,\dots,v_{p+q-1})\\
&= \sum_{r=1}^{p}(-1)^{r-1}(T_1\otimes T_2)(v_1,\dots,v_{r-1},v,v_r,\dots,v_{p+q-1})\\
&\quad+\sum_{r=p+1}^{p+q}(-1)^{r-1}(T_1\otimes T_2)(v_1,\dots,v_{r-1},v,v_r,\dots,v_{p+q-1})\\
&= \sum_{r=1}^{p}(-1)^{r-1}T_1(v_1,\dots,v_{r-1},v,v_r,\dots,v_{p-1})T_2(v_p,\dots,v_{p+q-1})\\
&\quad+\sum_{r=p+1}^{p+q}(-1)^{r-1}T_1(v_1,\dots,v_p)T_2(v_{p+1},\dots,v_{r-1},v,v_r,\dots,v_{p+q-1})\\
&= \left[\sum_{r=1}^{p}(-1)^{r-1}T_1(v_1,\dots,v_{r-1},v,v_r,\dots,v_{p-1})\right]\cdot T_2(v_p,\dots,v_{p+q-1})\\
&\quad+T_1(v_1,\dots,v_p)\cdot\sum_{r=p+1}^{p+q}(-1)^{r-1}T_2(v_{p+1},\dots,v_{r-1},v,v_r,\dots,v_{p+q-1})\\
&= \left[\sum_{r=1}^{p}(-1)^{r-1}T_1(v_1,\dots,v_{r-1},v,v_r,\dots,v_{p-1})\right]\cdot T_2(v_p,\dots,v_{p+q-1})\\
&\quad+T_1(v_1,\dots,v_p)\cdot(-1)^p\sum_{r=1}^{q}(-1)^{r-1}T_2(v_{p+1},\dots,v_{p+r-1},v,v_{p+r},\dots,v_{p+q-1})\\
&= (\iota_vT_1)(v_1,\dots,v_{p-1})\cdot T_2(v_p,\dots,v_{p+q-1})\\
&\quad+(-1)^pT_1(v_1,\dots,v_p)\cdot(\iota_vT_2)(v_{p+1},\dots,v_{p+q-1})\\
&= (\iota_vT_1\otimes T_2)(v_1,\dots,v_{p+q-1})+(-1)^p(T_1\otimes\iota_vT_2)(v_1,\dots,v_{p+q-1})\\
&= [\iota_vT_1\otimes T_2+(-1)^pT_1\otimes\iota_vT_2](v_1,\dots,v_{p+q-1})
\end{aligned}$$

as desired. ☐

1.7.iii. Show that if $T \in \mathcal{A}^k(V)$, then $\iota_v T = kT_v$, where T_v is defined as in Exercise 1.3.vii. In particular, conclude that $\iota_v T \in \mathcal{A}^{k-1}(V)$. (See Exercise 1.4.viii, which asserts that $T \in \mathcal{A}^k(V)$ implies $T_v \in \mathcal{A}^{k-1}(V)$.)

Proof. Suppose $T \in \mathcal{A}^k(V)$. Let $\sigma \in S_k$ be the permutation that moves the r^{th} index to the first place and shifts all $r-1$ indices to its left up one. For example, if $r = 4$ and $\sigma \in S_6$, $\sigma(1,2,3,4,5,6) = (4,1,2,3,5,6)$. More relevant to our situation would be the ability of σ to do the following.

$$\sigma(v_1, v_2, v_3, v, v_4, v_5) = \sigma(v, v_1, v_2, v_3, v_4, v_5)$$

Going back to the general case, since we have

$$\sigma = \prod_{i=1}^{r-1} \tau_{i,i+1}$$

we can determine that

$$(-1)^\sigma = (-1)^{r-1}$$

Therefore, by the above and since $T^\sigma = (-1)^\sigma T$ as an alternating k-tensor,

$$\begin{aligned}
(\iota_v T)(v_1, \dots, v_{k-1}) &= \sum_{r=1}^{k} (-1)^{r-1} T(v_1, \dots, v_{r-1}, v, v_r, \dots, v_{k-1}) \\
&= \sum_{r=1}^{k} (-1)^{\sigma} T(v_1, \dots, v_{r-1}, v, v_r, \dots, v_{k-1}) \\
&= \sum_{r=1}^{k} T^{\sigma}(v_1, \dots, v_{r-1}, v, v_r, \dots, v_{k-1}) \\
&= \sum_{r=1}^{k} T(v, v_1, \dots, v_{k-1}) \\
&= \sum_{r=1}^{k} T_v(v_1, \dots, v_{k-1}) \\
&= kT_v(v_1, \dots, v_{k-1})
\end{aligned}$$

as desired.

As stated in the question, we may invoke Exercise 1.4.vii to determine that $\iota_v T = kT_v \in \mathcal{A}^{k-1}(V)$. ☐

1.8.i. Verify the following assertions.

(1) The map $A^* : \Lambda^k(W^*) \to \Lambda^k(V^*)$ sending $\omega \mapsto A^*\omega$ is linear.

Proof. We have that

$$\begin{aligned}
A^*(\omega_1 + \omega_2) &= \pi(A^*(T_1 + T_2)) \\
&= \pi(A^*T_1 + A^*T_2) \\
&= \pi(A^*T_1) + \pi(A^*T_2) \\
&= A^*\omega_1 + A^*\omega_2
\end{aligned}
\qquad
\begin{aligned}
A^*(\lambda\omega) &= \pi(A^*(\lambda T)) \\
&= \pi(\lambda A^*T) \\
&= \lambda\pi(A^*T) \\
&= \lambda A^*\omega
\end{aligned}$$

as desired. ☐

(2) If $\omega_i \in \Lambda^{k_i}(W^*)$ $(i = 1, 2)$, then

$$A^*(\omega_1 \wedge \omega_2) = A^*(\omega_1) \wedge A^*(\omega_2)$$

Proof. We have that

$$\begin{aligned} A^*(\omega_1 \wedge \omega_2) &= A^*(\pi(T_1 \otimes T_2)) \\ &= \pi(A^*(T_1 \otimes T_2)) \\ &= \pi(A^*T_1 \otimes A^*T_2) \\ &= \pi(A^*T_1) \wedge \pi(A^*T_2) \\ &= A^*(\omega_1) \wedge A^*(\omega_2) \end{aligned}$$

as desired. □

(3) If U is a vector space and $B : U \to V$ is a linear map, then for $\omega \in \Lambda^k(W^*)$,

$$B^*A^*\omega = (AB)^*\omega$$

Proof. We have that

$$\begin{aligned} B^*A^*\omega &= B^*(\pi(A^*T)) \\ &= \pi(B^*A^*T) \\ &= \pi((AB)^*T) \\ &= (AB^*)\omega \end{aligned}$$

as desired. □

1.8.ii. Deduce from the fact "$A : V \to V$ not surjective implies $\det(A) = 0$" a well-known fact about determinants of $n \times n$ matrices: If two columns are equal, the determinant is zero.

Proof. If an $n \times n$ matrix has two identical columns, then the dimension of its range space is at most $n - 1$. Thus, A is not surjective, and hence has $\det(A) = 0$. □

1.8.iv. Deduce from Exercise 1.8.i another well-known fact about determinants of $n \times n$ matrices: If $(b_{i,j})$ is the inverse of $[a_{i,j}]$, its determinant is the inverse of the determinant of $[a_{i,j}]$.

Proof. Let $(b_{i,j}) = [a_{i,j}]^{-1}$. Then

$$(b_{i,j})[a_{i,j}] = \mathrm{id}_V$$

It follows from Propositions 1.8.7 and 1.8.8 (which in turn follow from Exercise 1.8.i) that

$$\begin{aligned} \det(b_{i,j}) \det[a_{i,j}] &= \det(\mathrm{id}_V) = 1 \\ \det(b_{i,j}) &= \frac{1}{\det[a_{i,j}]} \end{aligned}$$

as desired. □

1.8.v. Extract from the formula $\det([a_{i,j}]) = \sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$ the following well-known formula for determinants of 2×2 matrices.

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Proof. The two elements of S_2 are the identity permutation (which we will refer to as σ_1) and $\tau_{1,2}$ (which we will refer to as σ_2). It follows that for the $n = 2$ case,

$$\begin{aligned}
\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &= \sum_{\sigma \in S_2} (-1)^\sigma a_{1,\sigma(1)} a_{2,\sigma(2)} \\
&= (-1)^{\sigma_1} a_{1,\sigma_1(1)} a_{2,\sigma_1(2)} + (-1)^{\sigma_2} a_{1,\sigma_2(1)} a_{2,\sigma_2(2)} \\
&= (1) a_{1,1} a_{2,2} + (-1) a_{1,2} a_{2,1} \\
&= a_{1,1} a_{2,2} - a_{1,2} a_{2,1}
\end{aligned}$$

as desired. $\square$

1.9.i. Prove that if $e_1, \dots, e_n$ is a positively oriented basis of V, then the basis $e_1, \dots, e_{i-1}, -e_i, e_{i+1}, \dots, e_n$ is negatively oriented.

Proof. Since $e_1, \dots, e_n$ is a positively oriented basis of V, we know that $e_1^* \wedge \cdots \wedge e_n^* \in \Lambda^n(V^*)_+$. This combined with the fact that

$$e_1 \wedge \cdots \wedge e_{i-1}, -e_i, e_{i+1} \wedge \cdots \wedge e_n = -e_1^* \wedge \cdots \wedge e_n^* \notin \Lambda^n(V^*)$$

implies that the given basis is negatively oriented, as desired. $\square$

1.9.ii. Show that the argument in the proof of Theorem 1.9.9 can be modified to prove that if V and W are oriented, then these orientations induce a natural orientation on V/W.

Proof. Let $W \leq V$, $\dim V = n > 1$, $\dim W = k < n$, and $r = n - k$. WLOG choose $e_1, \dots, e_n$ a positively oriented basis of V such that $e_{r+1}, \dots, e_n$ is a positively oriented basis of W. It follows that $\pi(e_1), \dots, \pi(e_r)$ for a basis of V/W. Assign to V/W the orientation associated with this basis. Now suppose $\pi(f_1), \dots, \pi(f_r)$ is another basis of V/W. $\square$

2 Differential Forms

From Guillemin and Haine (2018).

Chapter 2

4/29: **2.1.i.** Let U be an open subset of $\mathbb{R}^n$. If $f: U \to \mathbb{R}$ is a C^∞ function, then

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \, \mathrm{d}x_i$$

Proof. The object on the left side of the above equality is a one-form. The object on the right side of the equality is the pointwise sum of n pointwise products of the functions $\frac{\partial f}{\partial x_i}: U \to \mathbb{R}$ with the one-forms $\mathrm{d}x_i$; thus, it is a one-form, too.

We want to prove that these two one-forms are equal. But under which definition of equality are we working? Each one-form is technically just a function from $U \to T_p^*\mathbb{R}^n$. Thus, we need only verify that both one-forms have the same action on every $p \in U$.

Let $p \in U$ be arbitrary. We now seek to verify that

$$\mathrm{d}f_p = \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \, \mathrm{d}x_i \right)_p$$

But once again, both sides are functions; specifically, they are both cotangent vectors to $\mathbb{R}^n$ at p. Thus, we need to verify that both cotangent vectors have the same action on every $(p, v) \in T_p\mathbb{R}^n$.

Let $(p, v) \in T_p\mathbb{R}^n$ be arbitrary. Additionally, let $v = (v_1, \dots, v_n)$. Then

$$\begin{aligned}
\mathrm{d}f_p(p, v) &= Df(p)v \\
&= \sum_{i=1}^{n} \left. \frac{\partial f}{\partial x_i} \right|_p v_i \\
&= \sum_{i=1}^{n} \left. \frac{\partial f}{\partial x_i} \right|_p (\mathrm{d}x_i)_p(p, v) \\
&= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \, \mathrm{d}x_i \right)_p (p, v) \\
&= \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \, \mathrm{d}x_i \right)_p (p, v)
\end{aligned}$$

as desired. □

2.1.ii. Let U be an open subset of $\mathbb{R}^n$, $\boldsymbol{v}$ a vector field on U, and $f_1, f_2 \in C^1(U)$. Then

$$L_{\boldsymbol{v}}(f_1 \cdot f_2) = L_{\boldsymbol{v}}(f_1) \cdot f_2 + f_1 \cdot L_{\boldsymbol{v}}(f_2)$$

Proof. Let

$$\boldsymbol{v} = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i}$$

By the definition of the Lie derivative, we have that

$$L_{\boldsymbol{v}}(f_1 \cdot f_2) = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i}(f_1 \cdot f_2)$$

$$= \sum_{i=1}^{n} g_i \left(\frac{\partial f_1}{\partial x_i} \cdot f_2 + f_1 \cdot \frac{\partial f_2}{\partial x_i} \right)$$
$$= f_2 \cdot \sum_{i=1}^{n} g_i \frac{\partial f_1}{\partial x_i} + f_1 \cdot \sum_{i=1}^{n} g_i \frac{\partial f_2}{\partial x_i}$$
$$= L_{\boldsymbol{v}}(f_1) \cdot f_2 + f_1 \cdot L_{\boldsymbol{v}}(f_2)$$

as desired. □

2.1.iii. Let U be an open subset of $\mathbb{R}^n$ and $\boldsymbol{v}_1, \boldsymbol{v}_2$ vector fields on U. Show that there is a unique vector field $\boldsymbol{w}$ on U with the property

$$L_{\boldsymbol{w}}\phi = L_{\boldsymbol{v}_1}(L_{\boldsymbol{v}_2}\phi) - L_{\boldsymbol{v}_2}(L_{\boldsymbol{v}_1}\phi)$$

for all $\phi \in C^\infty(U)$.

Proof. Let $\phi \in C^\infty(U)$ be arbitrary. Additionally, let

$$\boldsymbol{v}_1 = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i} \qquad \boldsymbol{v}_2 = \sum_{i=1}^{n} h_i \frac{\partial}{\partial x_i}$$

Then

$$L_{\boldsymbol{v}_1}\phi = \sum_{i=1}^{n} g_i \frac{\partial \phi}{\partial x_i} \qquad L_{\boldsymbol{v}_2}\phi = \sum_{i=1}^{n} h_i \frac{\partial \phi}{\partial x_i}$$

so that

$$\begin{aligned} L_{\boldsymbol{v}_1}(L_{\boldsymbol{v}_2}\phi) &= L_{\boldsymbol{v}_1}\left(\sum_{i=1}^{n} h_i \frac{\partial \phi}{\partial x_i} \right) \\ &= \sum_{i=1}^{n} L_{\boldsymbol{v}_1}\left(h_i \frac{\partial \phi}{\partial x_i} \right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} g_j \frac{\partial}{\partial x_j}\left(h_i \frac{\partial \phi}{\partial x_i} \right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} g_j \left(\frac{\partial h_i}{\partial x_j}\frac{\partial \phi}{\partial x_i} + h_i \frac{\partial^2 \phi}{\partial x_j \partial x_i} \right) \end{aligned}$$

$$\begin{aligned} L_{\boldsymbol{v}_2}(L_{\boldsymbol{v}_1}\phi) &= L_{\boldsymbol{v}_2}\left(\sum_{i=1}^{n} g_i \frac{\partial \phi}{\partial x_i} \right) \\ &= \sum_{i=1}^{n} L_{\boldsymbol{v}_2}\left(g_i \frac{\partial \phi}{\partial x_i} \right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} h_j \frac{\partial}{\partial x_j}\left(g_i \frac{\partial \phi}{\partial x_i} \right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} h_j \left(\frac{\partial g_i}{\partial x_j}\frac{\partial \phi}{\partial x_i} + g_i \frac{\partial^2 \phi}{\partial x_j \partial x_i} \right) \end{aligned}$$

It follows that

$$\begin{aligned} L_{\boldsymbol{v}_1}(L_{\boldsymbol{v}_2}\phi) - L_{\boldsymbol{v}_2}(L_{\boldsymbol{v}_1}\phi) &= \sum_{i=1}^{n}\sum_{j=1}^{n} g_j \left(\frac{\partial h_i}{\partial x_j}\frac{\partial \phi}{\partial x_i} + h_i \frac{\partial^2 \phi}{\partial x_j \partial x_i} \right) - \sum_{i=1}^{n}\sum_{j=1}^{n} h_j \left(\frac{\partial g_i}{\partial x_j}\frac{\partial \phi}{\partial x_i} + g_i \frac{\partial^2 \phi}{\partial x_j \partial x_i} \right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \left[\left(g_j \frac{\partial h_i}{\partial x_j} - h_j \frac{\partial g_i}{\partial x_j} \right) \frac{\partial \phi}{\partial x_i} + (g_j h_i - h_j g_i) \frac{\partial^2 \phi}{\partial x_j \partial x_i} \right] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \left[\left(\frac{\partial}{\partial x_j}(g_j h_i - h_j g_i) \right) \frac{\partial \phi}{\partial x_i} + (g_j h_i - h_j g_i) \frac{\partial^2 \phi}{\partial x_j \partial x_i} \right] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j}(g_j h_i - h_j g_i) \right) \frac{\partial \phi}{\partial x_i} \end{aligned}$$

and hence that

$$\boldsymbol{w} = \sum_{i=1}^{n} \underbrace{\sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j}(g_j h_i - h_j g_i) \right)}_{\text{functions } U \to \mathbb{R}} \frac{\partial}{\partial x_i}$$

□

2.1.iv. The vector field $\boldsymbol{w}$ in Exercise 2.1.iii is called the **Lie bracket** of the vector fields $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ and is denoted by $[\boldsymbol{v}_1, \boldsymbol{v}_2]$. Verify that the Lie bracket is **skew-symmetric**, i.e.,

$$[\boldsymbol{v}_1, \boldsymbol{v}_2] = -[\boldsymbol{v}_2, \boldsymbol{v}_1]$$

and satisfies the **Jacobi identity**

$$[\boldsymbol{v}_1, [\boldsymbol{v}_2, \boldsymbol{v}_3]] + [\boldsymbol{v}_2, [\boldsymbol{v}_3, \boldsymbol{v}_1]] + [\boldsymbol{v}_3, [\boldsymbol{v}_1, \boldsymbol{v}_2]] = 0$$

Thus, the Lie bracket defines the structure of a **Lie algebra**. (Hint: Prove analogous identities for $L_{\boldsymbol{v}_1}$, $L_{\boldsymbol{v}_2}$, and $L_{\boldsymbol{v}_3}$.)

Proof. Throughout this problem, let

$$\boldsymbol{v}_1 = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i} \qquad \boldsymbol{v}_2 = \sum_{i=1}^n g_i \frac{\partial}{\partial x_i} \qquad \boldsymbol{v}_3 = \sum_{i=1}^n h_i \frac{\partial}{\partial x_i}$$

Then

$$[\boldsymbol{v}_1, \boldsymbol{v}_2] = \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial}{\partial x_j}(f_j g_i - g_j f_i) \right) \frac{\partial}{\partial x_i}$$

It follows that

$$\begin{aligned}
-[\boldsymbol{v}_1, \boldsymbol{v}_2] &= -\sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial}{\partial x_j}(f_j g_i - g_j f_i) \right) \frac{\partial}{\partial x_i} \\
&= \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial}{\partial x_j}(f_i g_j - g_i f_j) \right) \frac{\partial}{\partial x_i} \\
&= \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial}{\partial x_j}(f_j g_i - g_j f_i) \right) \frac{\partial}{\partial x_i} \\
&= [\boldsymbol{v}_1, \boldsymbol{v}_2]
\end{aligned}$$

where the third equality holds by reindexing the symmetric sum.

Additionally, we have that

$$[\boldsymbol{v}_2, \boldsymbol{v}_3] = \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial}{\partial x_j}(g_j h_i - h_j g_i) \right) \frac{\partial}{\partial x_i}$$

and

$$[\boldsymbol{v}_3, \boldsymbol{v}_1] = \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial}{\partial x_j}(h_j f_i - f_j h_i) \right) \frac{\partial}{\partial x_i}$$

It follows that

$$\begin{aligned}
[\boldsymbol{v}_1, [\boldsymbol{v}_2, \boldsymbol{v}_3]] &= \sum_{i=1}^n \sum_{j=1}^n \left[\frac{\partial}{\partial x_j} \left(f_j \sum_{k=1}^n \left[\frac{\partial}{\partial x_k}(g_k h_i - h_k g_i) \right] - f_i \sum_{k=1}^n \left[\frac{\partial}{\partial x_k}(g_k h_j - h_k g_j) \right] \right) \right] \frac{\partial}{\partial x_i} \\
&= 0
\end{aligned}$$

where we note that any i, j term in the double sum and the corresponding j, i term add to zero. We can prove a similar identity for $[\boldsymbol{v}_2, [\boldsymbol{v}_3, \boldsymbol{v}_1]]$ and $[\boldsymbol{v}_3, [\boldsymbol{v}_1, \boldsymbol{v}_2]]$. Thus,

$$[\boldsymbol{v}_1, [\boldsymbol{v}_2, \boldsymbol{v}_3]] + [\boldsymbol{v}_2, [\boldsymbol{v}_3, \boldsymbol{v}_1]] + [\boldsymbol{v}_3, [\boldsymbol{v}_1, \boldsymbol{v}_2]] = 0 + 0 + 0 = 0$$

so the Lie bracket satisfies the Jacobi identity, as desired. $\square$

2.1.vii. Let U be an open subset of $\mathbb{R}^n$, and let $\gamma : [a,b] \to U$, $t \mapsto (\gamma_1(t), \dots, \gamma_n(t))$ be a C^1 curve. Given a C^∞ one-form $\omega = \sum_{i=1}^n f_i \, \mathrm{d}x_i$ on U, define the **line integral** of ω over γ to be the integral

$$\int_\gamma \omega = \sum_{i=1}^n \int_a^b f_i(\gamma(t)) \frac{\mathrm{d}\gamma_i}{\mathrm{d}t} \, \mathrm{d}t$$

Show that if $\omega = \mathrm{d}f$ for some $f \in C^\infty(U)$,

$$\int_\gamma \omega = f(\gamma(b)) - f(\gamma(a))$$

In particular, conclude that if γ is a closed curve, i.e., $\gamma(a) = \gamma(b)$, this integral is zero.

Proof. Since $\gamma : [a,b] \to U$ (where U is open), we know that there exist $N_{r_1}(\gamma(a)) \subset U$ and $N_{r_2}(\gamma(b)) \subset U$. Thus, we may extend γ to some open superset $(a,b)^+ \supset [a,b]$ in a C^1 fashion, i.e., along the tangent vectors to $\gamma(a)$ and $\gamma(b)$ at a and b, respectively. From now on, when we refer to γ, we will be discussing $\gamma : (a,b)^+ \to U$. With this adjustment, we can show that $f \circ \gamma$ satisfies the hypotheses for the multivariable chain rule at $t \in [a,b]$ arbitrary.

$(a,b)^+$ is open in $\mathbb{R}$ by definition. Since $\gamma \in C^1(\mathbb{R})$, $\gamma : (a,b)^+ \to U$ is differentiable at $t \in [a,b] \subset \mathbb{R}^n$. $U \supset \gamma((a,b)^+)$ is an open set in $\mathbb{R}^n$ by hypothesis. Since $f \in C^\infty(U)$, $f : U \to \mathbb{R}$ is differentiable at $\gamma(t)$. Therefore, we have by Theorem 9.15 of Rudin (1976) that

$$\begin{aligned}
(f \circ \gamma)'(t) &= D(f \circ \gamma)(t) \\
&= Df(\gamma(t)) \circ D\gamma(t) \\
&= \begin{bmatrix} \left.\frac{\partial f}{\partial x_1}\right|_{\gamma(t)} & \cdots & \left.\frac{\partial f}{\partial x_n}\right|_{\gamma(t)} \end{bmatrix} \begin{bmatrix} \left.\frac{\partial \gamma_1}{\partial t}\right|_t \\ \vdots \\ \left.\frac{\partial \gamma_n}{\partial t}\right|_t \end{bmatrix} \\
&= \sum_{i=1}^n \left.\frac{\partial f}{\partial x_i}\right|_{\gamma(t)} \left.\frac{\partial \gamma_i}{\partial t}\right|_t \\
&= \sum_{i=1}^n \left.\frac{\partial f}{\partial x_i}\right|_{\gamma(t)} \frac{\partial \gamma_i}{\partial t}
\end{aligned}$$

Now suppose $\omega = \mathrm{d}f$. Then by Lemma 2.1.18, each $f_i = \partial f / \partial x_i$. It follows that

$$\begin{aligned}
\int_\gamma \omega &= \sum_{i=1}^n \int_a^b \left.\frac{\partial f}{\partial x_i}\right|_{\gamma(t)} \frac{\mathrm{d}\gamma_i}{\mathrm{d}t} \, \mathrm{d}t \\
&= \int_a^b \sum_{i=1}^n \left.\frac{\partial f}{\partial x_i}\right|_{\gamma(t)} \frac{\mathrm{d}\gamma_i}{\mathrm{d}t} \, \mathrm{d}t \\
&= \int_a^b (f \circ \gamma)'(t) \, \mathrm{d}t \\
&= f(\gamma(b)) - f(\gamma(a))
\end{aligned}$$

as desired.

Now suppose that γ is a closed curve. Then

$$\begin{aligned}
\int_\gamma \omega &= f(\gamma(b)) - f(\gamma(a)) \\
&= f(\gamma(a)) - f(\gamma(a)) \\
&= 0
\end{aligned}$$

as desired. $\square$

2.1.viii. Let ω be the C^∞ one-form on $\mathbb{R}^2 \setminus \{0\}$ defined by

$$\omega = \frac{x_1\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_1}{x_1^2 + x_2^2}$$

and let $\gamma : [0, 2\pi] \to \mathbb{R}^2 \setminus \{0\}$ be the closed curve $t \mapsto (\cos t, \sin t)$. Compute the line integral $\int_\gamma \omega$ and note that $\int_\gamma \omega \neq 0$. Conclude that ω is not of the form $\mathrm{d}f$ for $f \in C^\infty(\mathbb{R}^2 \setminus \{0\})$.

Proof. From the given definition of ω, we can determine that

$$f_1(x_1, x_2) = -\frac{x_2}{x_1^2 + x_2^2} \qquad\qquad f_2(x_1, x_2) = \frac{x_1}{x_1^2 + x_2^2}$$

We also have that

$$\gamma_1(t) = \cos t \qquad\qquad \gamma_2(t) = \sin t$$

Thus, we know that

$$\begin{aligned}
\int_\gamma \omega &= \sum_{i=1}^{2} \int_0^{2\pi} f_i(\gamma(t)) \frac{\mathrm{d}\gamma_i}{\mathrm{d}t}\,\mathrm{d}t \\
&= \int_0^{2\pi} f_1(\cos t, \sin t) \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\cos t)\,\mathrm{d}t + \int_0^{2\pi} f_2(\cos t, \sin t) \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\sin t)\,\mathrm{d}t \\
&= \int_0^{2\pi} -\sin t \cdot -\sin t\,\mathrm{d}t + \int_0^{2\pi} \cos t \cdot \cos t\,\mathrm{d}t \\
&= \int_0^{2\pi} \mathrm{d}t \\
\boxed{\int_\gamma \omega = 2\pi}
\end{aligned}$$

Since $\int_\gamma \omega \neq 0$ and $\gamma(0) = \gamma(2\pi) = (1, 0)$, we have by Exercise 2.1.vii that $\omega \neq \mathrm{d}f$. $\square$

2.2.i. For $i = 1, 2$, let U_i be an open subset of $\mathbb{R}^{n_i}$, $\boldsymbol{v}_i$ a vector field on U_i, and $f : U_1 \to U_2$ a C^∞-map. If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are f-related, every integral curve $\gamma : I \to U_1$ of $\boldsymbol{v}_1$ gets mapped by f onto an integral curve $f \circ \gamma : I \to U_2$ of $\boldsymbol{v}_2$.

Proof. We want to show that

$$\boldsymbol{v}_2((f \circ \gamma)(t)) = \left((f \circ \gamma)(t),\, \tfrac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)\big|_t\right)$$

We are given that

$$\boldsymbol{v}_1(\gamma(t)) = \left(\gamma(t),\, \tfrac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_t\right) \qquad\qquad \mathrm{d}f_p(\boldsymbol{v}_1(p)) = \boldsymbol{v}_2(f(p))$$

Let $p = \gamma(t)$ and $q = f(p)$. Then

$$\begin{aligned}
\boldsymbol{v}_2((f \circ \gamma)(t)) &= \boldsymbol{v}_2(f(p)) \\
&= \mathrm{d}f_p(\boldsymbol{v}_1(p)) \\
&= \mathrm{d}f_p(\boldsymbol{v}_1(\gamma(t))) \\
&= \mathrm{d}f_p\left(\gamma(t),\, \tfrac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_t\right) \\
&= \mathrm{d}f_p\left(p,\, \tfrac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_t\right) \\
&= \left(q, Df(p)\left(\tfrac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_t\right)\right) \\
&= \left((f \circ \gamma)(t),\, \tfrac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)\big|_t\right)
\end{aligned}$$

as desired. $\square$

2.2.ii. Let U, V be open subsets of $\mathbb{R}^n$ and $f : U \to V$ an C^k map.

(1) Show that for $\phi \in C^\infty(V)$, the pullback can be rewritten

$$f^* \,\mathrm{d}\phi = \mathrm{d}f^*\phi$$

Proof. We have that

$$\begin{aligned}(f^*\mathrm{d}\phi)(p) &= \mathrm{d}\phi_{f(p)} \circ \mathrm{d}f_p \\ &= \mathrm{d}(\phi \circ f)_p \\ &= \mathrm{d}f^*\phi\end{aligned}$$

where $f^*\phi = \phi \circ f$ is another variation of the pullback. $\square$

(2) Let μ be the one-form

$$\mu = \sum_{i=1}^{n} \phi_i \,\mathrm{d}x_i$$

on V for all $\phi_i \in C^\infty(V)$. Show that if $f = (f_1, \dots, f_n)$, then

$$f^*\mu = \sum_{i=1}^{n} f^*\phi_i \,\mathrm{d}f_i$$

Proof. We have that

$$\begin{aligned}(f^*\mu)(p) &= \mu_{f(p)} \circ \mathrm{d}f_p \\ &= \sum_{i=1}^{n} \phi_i(f(p))(\mathrm{d}x_i)_p \circ \mathrm{d}f_p \\ &= \sum_{i=1}^{n} (\phi_i \circ f)(p)(\mathrm{d}f_i)_p \\ &= \sum_{i=1}^{n} f^*\phi_i(p)(\mathrm{d}f_i)_p\end{aligned}$$

where we have $(\mathrm{d}x_i)_p \circ \mathrm{d}f_p = \mathrm{d}f_i$ since

$$\begin{aligned}[(\mathrm{d}x_i)_p \circ \mathrm{d}f_p](p, v) &= (\mathrm{d}x_i)_p(q, Df(p)v) \\ &= \left(q_i, \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} v_j\right) \\ &= (q_i, Df_i(p)v) \\ &= (\mathrm{d}f_i)_p(p, v)\end{aligned}$$

$\square$

(3) Show that if μ is C^∞ and f is C^∞, $f^*\mu$ is C^∞.

Proof. To prove that $f^*\mu \in C^\infty$, it will suffice to show by (2) that every $f^*\phi_i \in C^\infty$. But this is obvious since $f^*\phi_i = \phi_i \circ f$ where the latter two composed functions are both C^∞. $\square$

2.2.iv. (1) Let $U = \mathbb{R}^2$ and let $\boldsymbol{v}$ be the vector field $x_1 \,\partial/\partial x_2 - x_2 \,\partial/\partial x_1$. Show that the curve

$$t \mapsto (r\cos(t + \theta), r\sin(t + \theta))$$

for $t \in \mathbb{R}$ is the unique integral curve of $\boldsymbol{v}$ passing through the point $(r\cos\theta, r\sin\theta)$ at $t = 0$.

Proof. We first will check that the above curve, which we will call $\gamma : \mathbb{R} \to U$, is *an* integral curve of $\boldsymbol{v}$ passing through $(r\cos\theta, r\sin\theta)$ at $t = 0$. To verify the integral curve part, we first note that since that $g_1, g_2 : U \to \mathbb{R}$ are defined by

$$g_1(x_1, x_2) = -x_2 \qquad\qquad g_2(x_1, x_2) = x_1$$

we may define $g : U \to \mathbb{R}^2$ by

$$g(x_1, x_2) = (-x_2, x_1)$$

Thus, we need show that

$$\begin{aligned}
\frac{\mathrm{d}\gamma}{\mathrm{d}t} &\overset{?}{=} g(\gamma(t)) \\
\left(\frac{\mathrm{d}}{\mathrm{d}t}(r\cos(t+\theta)), \frac{\mathrm{d}}{\mathrm{d}t}(r\sin(t+\theta))\right) &\overset{?}{=} g(r\cos(t+\theta), r\sin(t+\theta)) \\
(-r\sin(t+\theta), r\cos(t+\theta)) &\overset{\checkmark}{=} (-r\sin(t+\theta), r\cos(t+\theta))
\end{aligned}$$

To verify the passing through the point at $t = 0$ part, we need only plug in $t = 0$ and observe the equivalence:

$$\gamma(0) = (r\cos(0+\theta), r\sin(0+\theta)) = (r\cos\theta, r\sin\theta)$$

We now check that γ is the *unique* such curve. But if $\tilde{\gamma}$ is an integral curve passing through $(r\cos\theta, r\sin\theta)$ at $t = 0$, we have that $\gamma = \tilde{\gamma}$ by Theorem 2.2.5. □

(2) Let $U = \mathbb{R}^n$ and let $\boldsymbol{v}$ be the constant vector field $\sum_{i=1}^n c_i\, \partial/\partial x_i$. Show that the curve

$$t \mapsto a + t(c_1, \ldots, c_n)$$

for $t \in \mathbb{R}$ is the unique integral curve of $\boldsymbol{v}$ passing through $a \in \mathbb{R}^n$ at $t = 0$.

Proof. Applying the same strategy in part (a), we call the given integral curve γ and define $g : U \to \mathbb{R}^n$ by

$$g(x_1, \ldots, x_n) = (c_1, \ldots, c_n)$$

Then we have the following.

<u>*An* integral curve</u>:

$$\begin{aligned}
\frac{\mathrm{d}\gamma}{\mathrm{d}t} &\overset{?}{=} g(\gamma(t)) \\
\left(\frac{\mathrm{d}}{\mathrm{d}t}(a + tc_1), \ldots, \frac{\mathrm{d}}{\mathrm{d}t}(a + tc_n)\right) &\overset{?}{=} g(a + tc_1, \ldots, a + tc_n) \\
(c_1, \ldots, c_n) &\overset{\checkmark}{=} (c_1, \ldots, c_n)
\end{aligned}$$

<u>$\gamma(0) = a$</u>:

$$\begin{aligned}
\gamma(0) &= a + 0 \cdot (c_1, \ldots, c_n) \\
&= a
\end{aligned}$$

<u>*Unique* integral curve</u>: Apply Theorem 2.2.5. □

(3) Let $U = \mathbb{R}^n$ and let $\boldsymbol{v}$ be the vector field $\sum_{i=1}^n x_i\, \partial/\partial x_i$. Show that the curve

$$t \mapsto \mathrm{e}^t(a_1, \ldots, a_n)$$

for $t \in \mathbb{R}$ is the unique integral curve of $\boldsymbol{v}$ passing through a at $t = 0$.

Proof. Applying the same strategy in parts (a)-(b), we call the given integral curve γ and define $g : U \to \mathbb{R}^n$ by

$$g(x_1, \dots, x_n) = (x_1, \dots, x_n)$$

Then we have the following.

An integral curve:

$$\begin{aligned}\frac{\mathrm{d}\gamma}{\mathrm{d}t} &\overset{?}{=} g(\gamma(t))\\ \left(\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^t a_1), \dots, \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^t a_n)\right) &\overset{?}{=} g(\mathrm{e}^t a_1, \dots, \mathrm{e}^t a_n)\\ (\mathrm{e}^t a_1, \dots, \mathrm{e}^t a_n) &\overset{\checkmark}{=} (\mathrm{e}^t a_1, \dots, \mathrm{e}^t a_n)\end{aligned}$$

$\gamma(0) = a$:

$$\begin{aligned}\gamma(0) &= \mathrm{e}^0(a_1, \dots, a_n)\\ &= a\end{aligned}$$

Unique integral curve: Apply Theorem 2.2.5. □

2.2.viii. Let $\boldsymbol{v}$ be the vector field on $\mathbb{R}$ given by $x^2\, \partial/\partial x$. Show that the curve

$$x(t) = \frac{a}{1 - at}$$

is an integral curve of $\boldsymbol{v}$ with initial point $x(0) = a$. Conclude that for $a > 0$, the curve

$$x(t) = \frac{a}{1 - at}$$

on $0 < t < 1/a$ is a maximal integral curve. (In particular, conclude that $\boldsymbol{v}$ is not complete.)

Proof. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = x^2$$

An integral curve:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &\overset{?}{=} g(x(t))\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a}{1 - at}\right) &\overset{?}{=} g\left(\frac{a}{1 - at}\right)\\ \frac{(1 - at)(0) - a(-a)}{(1 - at)^2} &\overset{?}{=} \left(\frac{a}{1 - at}\right)^2\\ \frac{a^2}{(1 - at)^2} &\overset{\checkmark}{=} \frac{a^2}{(1 - at)^2}\end{aligned}$$

$x(0) = a$:

$$\begin{aligned}x(0) &= \frac{a}{1 - a \cdot 0}\\ &= a\end{aligned}$$

x on $0 < t < 1/a$ is a maximal integral curve: Suppose for the sake of contradiction that there exists an $a > 0$ to which there corresponds a number $b > 1/a$ such that $x(t) = a/(1 - at)$ on $(0, b)$ is an integral curve. It can be proven with an ϵ, δ argument that x is continuous on $(0, 1/a)$ and on $(1/a, b)$, but that there is a discontinuity at $1/a$. But since x is an integral curve, we have by definition that x is C^1 (hence continuous) on $(0, b)$, a contradiction. Therefore, x is a maximal integral curve on the specified integral.

Choose $a = 1 > 0$. By the above, no integral curve $\gamma : \mathbb{R} \to \mathbb{R}$ exists with $\gamma(0) = 1$, so $\boldsymbol{v}$ cannot be complete, as desired. □

2.3.i. Let $\omega \in \Omega^2(\mathbb{R}^4)$ be the 2-form $\mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_3 \wedge \mathrm{d}x_4$. Compute $\omega \wedge \omega$.

Proof. By the definition of the wedge product for k-forms, all properties proven for the wedge product of tensors carry over. This result will not be stated again, though it will be used again.

By the distributive law, we have that

$$\begin{aligned}\omega \wedge \omega &= [(\mathrm{d}x_1 \wedge \mathrm{d}x_2) + (\mathrm{d}x_3 \wedge \mathrm{d}x_4)] \wedge [(\mathrm{d}x_1 \wedge \mathrm{d}x_2) + (\mathrm{d}x_3 \wedge \mathrm{d}x_4)] \\ &= (\mathrm{d}x_1 \wedge \mathrm{d}x_2) \wedge (\mathrm{d}x_1 \wedge \mathrm{d}x_2) + 2(\mathrm{d}x_1 \wedge \mathrm{d}x_2) \wedge (\mathrm{d}x_3 \wedge \mathrm{d}x_4) + (\mathrm{d}x_3 \wedge \mathrm{d}x_4) \wedge (\mathrm{d}x_3 \wedge \mathrm{d}x_4)\end{aligned}$$

By the anticommutative law, a decomposable element wedged with itself is zero.

$$= 2\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4$$

$\square$

2.3.ii. Let $\omega_1, \omega_2, \omega_3 \in \Omega^1(\mathbb{R}^3)$ be the 1-forms

$$\begin{aligned}\omega_1 &= x_2\,\mathrm{d}x_3 - x_3\,\mathrm{d}x_2 \\ \omega_2 &= x_3\,\mathrm{d}x_1 - x_1\,\mathrm{d}x_3 \\ \omega_3 &= x_1\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_1\end{aligned}$$

Compute the following.

(1) $\omega_1 \wedge \omega_2$.

Proof. We have that

$$\begin{aligned}\omega_1 \wedge \omega_2 &= (x_2\mathrm{d}x_3 - x_3\mathrm{d}x_2) \wedge (x_3\mathrm{d}x_1 - x_1\mathrm{d}x_3) \\ &= x_2x_3\mathrm{d}x_3 \wedge \mathrm{d}x_1 - x_2x_1\mathrm{d}x_3 \wedge \mathrm{d}x_3 - x_3^2\mathrm{d}x_2 \wedge \mathrm{d}x_1 + x_1x_3\mathrm{d}x_2 \wedge \mathrm{d}x_3 \\ &= x_3^2\mathrm{d}x_1 \wedge \mathrm{d}x_2 - x_2x_3\mathrm{d}x_1 \wedge \mathrm{d}x_3 + x_1x_3\mathrm{d}x_2 \wedge \mathrm{d}x_3\end{aligned}$$

$\square$

(2) $\omega_2 \wedge \omega_3$.

Proof. We have that

$$\begin{aligned}\omega_2 \wedge \omega_3 &= (x_3\,\mathrm{d}x_1 - x_1\,\mathrm{d}x_3) \wedge (x_1\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_1) \\ &= x_3x_1\mathrm{d}x_1 \wedge \mathrm{d}x_2 - x_3x_2\mathrm{d}x_1 \wedge \mathrm{d}x_1 - x_1^2\mathrm{d}x_3 \wedge \mathrm{d}x_2 + x_1x_2\mathrm{d}x_3 \wedge \mathrm{d}x_1 \\ &= x_1x_3\mathrm{d}x_1 \wedge \mathrm{d}x_2 - x_1x_2\mathrm{d}x_1 \wedge \mathrm{d}x_3 + x_1^2\mathrm{d}x_2 \wedge \mathrm{d}x_3\end{aligned}$$

$\square$

(3) $\omega_3 \wedge \omega_1$.

Proof. We have that

$$\begin{aligned}\omega_3 \wedge \omega_1 &= (x_1\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_1) \wedge (x_2\,\mathrm{d}x_3 - x_3\,\mathrm{d}x_2) \\ &= x_1x_2\mathrm{d}x_2 \wedge \mathrm{d}x_3 - x_1x_3\mathrm{d}x_2 \wedge \mathrm{d}x_2 - x_2^2\mathrm{d}x_1 \wedge \mathrm{d}x_3 + x_2x_3\mathrm{d}x_1 \wedge \mathrm{d}x_2 \\ &= x_2x_3\mathrm{d}x_1 \wedge \mathrm{d}x_2 - x_2^2\mathrm{d}x_1 \wedge \mathrm{d}x_3 + x_1x_2\mathrm{d}x_2 \wedge \mathrm{d}x_3\end{aligned}$$

$\square$

(4) $\omega_1 \wedge \omega_2 \wedge \omega_3$.

Proof. We have that

$$\begin{aligned}
\omega_1 \wedge \omega_2 \wedge \omega_3 &= (\omega_1 \wedge \omega_2) \wedge \omega_3 \\
&= (x_3^2 \mathrm{d}x_1 \wedge \mathrm{d}x_2 - x_2x_3\mathrm{d}x_1 \wedge \mathrm{d}x_3 + x_1x_3\mathrm{d}x_2 \wedge \mathrm{d}x_3) \wedge (x_1\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_1) \\
&= x_3^2x_1\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_2 - x_3^2x_2\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_1 - x_2x_3x_1\mathrm{d}x_1 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_2 \\
&\quad + x_2x_3x_2\mathrm{d}x_1 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_1 + x_1x_3x_1\mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_2 - x_1x_3x_2\mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_1 \\
&= x_1x_2x_3\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 - x_1x_2x_3\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \\
&= 0
\end{aligned}$$

∎

2.3.iii. Let U be an open subset of $\mathbb{R}^n$ and $f_1, \dots, f_n \in C^\infty(U)$. Show that

$$\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n = \det\left[\frac{\partial f_i}{\partial x_j}\right] \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

Proof. By Lemma 2.1.18,

$$\mathrm{d}f_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}\mathrm{d}x_j$$

for all $i = 1, \dots, n$. It follows that

$$\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n = \sum_{j=1}^{n} \frac{\partial f_1}{\partial x_j}\mathrm{d}x_j \wedge \cdots \wedge \sum_{j=1}^{n} \frac{\partial f_n}{\partial x_j}\mathrm{d}x_j$$

If we apply the distributive law for the wedge product, there will be n^n terms in the resulting sum. Every term contains the product of n partial derivatives as a scalar multiple in front of the wedge product of n one-forms. For the partial derivatives, each of the n functions f_i will be represented exactly once. However, for the one-forms (and corresponding variables of differentiation), any number from 1 through n can be represented up to n times. Thus, we need to sum terms of the form

$$\frac{\partial f_1}{\partial x_{i_1}} \cdots \frac{\partial f_n}{\partial x_{i_n}}\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_n}$$

over the multi-indices of n of length n. Consequently,

$$\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n = \sum_{I} \frac{\partial f_1}{\partial x_{i_1}} \cdots \frac{\partial f_n}{\partial x_{i_n}}\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_n}$$

We now consider which terms in the sum are equal to zero. By the anticommutative property of the wedge product, any repeating multi-index will lead to a term whose wedge product evaluates to zero. Thus, we can restrict our sum to the *non-repeating* multi-indices of n of length n.

Every non-repeating multi-index of n of length n is equal to the n-tuple $(\sigma(1), \dots, \sigma(n))$ for some $\sigma \in S_n$. Thus, instead of summing over the multi-indices of n of length n, we can sum over the permutations in S_n:

$$\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n = \sum_{\sigma \in S_n} \frac{\partial f_1}{\partial x_{\sigma(1)}} \cdots \frac{\partial f_n}{\partial x_{\sigma(n)}}\mathrm{d}x_{\sigma(1)} \wedge \cdots \wedge \mathrm{d}x_{\sigma(n)}$$

But by an extension of Claim 1.6.8,

$$\mathrm{d}x_{\sigma(1)} \wedge \cdots \wedge \mathrm{d}x_{\sigma(n)} = (-1)^\sigma \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

Therefore, we can factor out the one-form from the sum and equate the sum with the determinant, as desired.

$$\begin{aligned} \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n &= \left[\sum_{\sigma \in S_n} (-1)^\sigma \frac{\partial f_1}{\partial x_{\sigma(1)}} \cdots \frac{\partial f_n}{\partial x_{\sigma(n)}}\right] \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \\ &= \det\left[\frac{\partial f_i}{\partial x_j}\right] \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \end{aligned}$$

□

2.3.iv. Let U be an open subset of $\mathbb{R}^n$. Show that every $(n-1)$-form $\omega \in \Omega^{n-1}(U)$ can be written uniquely as a sum

$$\sum_{i=1}^{n} f_i \,\mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n$$

where $f_i \in C^\infty(U)$ and $\widehat{\mathrm{d}x_i}$ indicates that $\mathrm{d}x_i$ is to be omitted from the wedge product $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$.

Proof. Let $\omega \in \Omega^{n-1}(U)$ be arbitrary. Then ω has a decomposition

$$\omega = \sum_I f_I \,\mathrm{d}x_I$$

where we sum over the multi-indices of n of length $n-1$. However, since any wedge product with a repeat evaluates to zero (anticommutative property), we need only sum over the non-repeating multi-indices of n of length $n-1$. Moreover, all of these can be reordered so that they are strictly increasing by some $\sigma \in S_{n-1}$. The resulting sign $(-1)^\sigma$ and multiple functions f_I, if applicable, can be combined into one new function f_i and reindexed. □

2.3.v. Let $\mu = \sum_{i=1}^n x_i \,\mathrm{d}x_i$. Show that there exists an $(n-1)$-form $\omega \in \Omega^{n-1}(\mathbb{R}^n \setminus \{0\})$ with the property

$$\mu \wedge \omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

Proof. Define $1/0 = \pm\infty$ and $0 \cdot \pm\infty = 1$. Let $\omega = (1/x_1)\,\mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n$. Then

$$\begin{aligned} \mu \wedge \omega &= \left(\sum_{i=1}^n x_i \,\mathrm{d}x_i\right) \wedge \left(\frac{1}{x_1} \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n\right) \\ &= \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \end{aligned}$$

where all terms save the first cancel by the anticommutative property. □

2.3.vi. Let J be the multi-index $(j_1, \dots, j_k)$ and let $\mathrm{d}x_J = \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_k}$. Show that $\mathrm{d}x_J = 0$ if $j_r = j_s$ for some $r \neq s$ and show that if the numbers $j_1, \dots, j_k$ are all distinct, then

$$\mathrm{d}x_J = (-1)^\sigma \,\mathrm{d}x_I$$

where $I = (i_1, \dots, i_k)$ is the strictly increasing rearrangement of $(j_1, \dots, j_k)$ and σ is the permutation

$$(j_1, \dots, j_k) \mapsto (i_1, \dots, i_k)$$

Proof. Suppose first that $j_r = j_s$ for some $r \neq s$. We wish to prove that $\mathrm{d}x_J = 0$, where "0" denotes the zero element of $\Omega^k(\mathbb{R}^n)$. To do this, we need to show that $\mathrm{d}x_J$ sends every $p \in \mathbb{R}^n$ to the zero k-tensor in $\Lambda^k(T_p^*\mathbb{R}^n) \cong \mathcal{A}^k(T_p\mathbb{R}^n)$.

Let $p \in \mathbb{R}^n$ be arbitrary. Then

$$\begin{aligned}
\mathrm{d}x_J\,(p) &= (\mathrm{d}x_{j_1})_p \wedge \cdots \wedge (\mathrm{d}x_{j_k})_p \\
&= (\mathrm{d}x_{\tau_{r,s}(j_1)})_p \wedge \cdots \wedge (\mathrm{d}x_{\tau_{r,s}(j_k)})_p \\
&= (-1)^{\tau_{r,s}}(\mathrm{d}x_{j_1})_p \wedge \cdots \wedge (\mathrm{d}x_{j_k})_p && \text{Claim 1.6.8} \\
&= -(\mathrm{d}x_{j_1})_p \wedge \cdots \wedge (\mathrm{d}x_{j_k})_p \\
&= -\,\mathrm{d}x_J\,(p) \\
2\,\mathrm{d}x_J\,(p) &= 0 \\
\mathrm{d}x_J\,(p) &= 0
\end{aligned}$$

as desired.

Now suppose that the numbers $j_1, \dots, j_k$ are all distinct. Then like before, we need to show that $\mathrm{d}x_J$ and $(-1)^\sigma\,\mathrm{d}x_I$ send every $p \in \mathbb{R}^n$ to the same k-tensor in $\Lambda^k(T_p^*\mathbb{R}^n)$.

Let $p \in \mathbb{R}^n$ be arbitrary. Then

$$\begin{aligned}
\mathrm{d}x_J\,(p) &= (\mathrm{d}x_{j_1})_p \wedge \cdots \wedge (\mathrm{d}x_{j_k})_p \\
&= (\mathrm{d}x_{\sigma(j_1)})_p \wedge \cdots \wedge (\mathrm{d}x_{\sigma(j_k)})_p \\
&= (-1)^\sigma(\mathrm{d}x_{i_1})_p \wedge \cdots \wedge (\mathrm{d}x_{i_k})_p && \text{Claim 1.6.8} \\
&= (-1)^\sigma\,\mathrm{d}x_I\,(p)
\end{aligned}$$

as desired. $\square$

3 Operations on Forms

From Guillemin and Haine (2018).

Chapter 2

5/9: **2.4.i.** Compute the exterior derivatives of the following differential forms.

(1) $x_1\,\mathrm{d}x_2\wedge\mathrm{d}x_3$.

Proof. We have that

$$\begin{aligned}\mathrm{d}(x_1\,\mathrm{d}x_2\wedge\mathrm{d}x_3) &= \mathrm{d}(x_1\,\mathrm{d}x_2)\wedge\mathrm{d}x_3+(-1)^1x_1\,\mathrm{d}x_2\wedge\mathrm{d}(\mathrm{d}x_3)\\ &= \mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3 - x_1\,\mathrm{d}x_2\wedge 0\\ \boxed{\mathrm{d}(x_1\,\mathrm{d}x_2\wedge\mathrm{d}x_3) = \mathrm{d}x_1\wedge\mathrm{d}x_2\wedge\mathrm{d}x_3}\end{aligned}$$

□

(2) $x_1\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_1$.

Proof. We have that

$$\begin{aligned}\mathrm{d}(x_1\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_1) &= \mathrm{d}(x_1\,\mathrm{d}x_2) - \mathrm{d}(x_2\,\mathrm{d}x_1)\\ &= \mathrm{d}x_1\wedge\mathrm{d}x_2 - \mathrm{d}x_2\wedge\mathrm{d}x_1\\ &= \mathrm{d}x_1\wedge\mathrm{d}x_2 + \mathrm{d}x_1\wedge\mathrm{d}x_2\\ \boxed{\mathrm{d}(x_1\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_1) = 2\,\mathrm{d}x_1\wedge\mathrm{d}x_2}\end{aligned}$$

□

(3) $\mathrm{e}^{-f}\,\mathrm{d}f$ where $f=\sum_{i=1}^n x_i^2$.

Proof. We state as a lemma first that

$$\begin{aligned}\mathrm{d}f &= \sum_{i=1}^n \mathrm{d}\left(x_i^2\right)\\ &= 2\sum_{i=1}^n x_i\,\mathrm{d}x_i\end{aligned}$$

It follows that

$$\begin{aligned}\mathrm{d}\left(\mathrm{e}^{-f}\,\mathrm{d}f\right) &= \mathrm{d}\left(\mathrm{e}^{-f}\right)\wedge\mathrm{d}f\\ &= \mathrm{e}^{-f}\,\mathrm{d}(-f)\wedge\mathrm{d}f\\ &= -4\mathrm{e}^{-f}\sum_{i=1}^n x_i\,\mathrm{d}x_i\wedge\sum_{i=1}^n x_i\,\mathrm{d}x_i\\ &= -4\mathrm{e}^{-f}\sum_I x_{i_1}x_{i_2}\,\mathrm{d}x_{i_1}\wedge\mathrm{d}x_{i_2}\end{aligned}$$

where we sum over the multi-indices I of n of length 2. However, since all repeating multi-indices equal zero, we can eliminate those terms from the sum. Additionally, we can pair up all strictly increasing and strictly decreasing terms with the same two numbers (e.g., $(1,2)\sim(2,1)$, $(1,3)\sim(3,1)$, etc.). Invoking the anticommutative property, we can rewrite the sum such that we only sum over the non-repeating, strictly increasing multi-indices of n of length 2.

$$\boxed{\mathrm{d}\left(\mathrm{e}^{-f}\,\mathrm{d}f\right) = -4\mathrm{e}^{-f}\sum_{1\leq i_1<i_2\leq n}(x_{i_1}x_{i_2}-x_{i_2}x_{i_1})\,\mathrm{d}x_{i_1}\wedge\mathrm{d}x_{i_2}}$$

□

(4) $\sum_{i=1}^n x_i \, \mathrm{d}x_i$.

Proof. We have that

$$\mathrm{d}\left(\sum_{i=1}^n x_i \, \mathrm{d}x_i\right) = \sum_{i=1}^n \mathrm{d}(x_i \, \mathrm{d}x_i)$$

$$\boxed{\mathrm{d}\left(\sum_{i=1}^n x_i \, \mathrm{d}x_i\right) = \sum_{i=1}^n \mathrm{d}x_i \wedge \mathrm{d}x_i}$$

□

(5) $\sum_{i=1}^n (-1)^i x_i \, \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n$.

Proof. We have that

$$\begin{aligned}
\mathrm{d}\left(\sum_{i=1}^n (-1)^i x_i \, \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n\right) &= \sum_{i=1}^n (-1)^i \, \mathrm{d}\Big(x_i \, \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n\Big) \\
&= \sum_{i=1}^n (-1)^i \, \mathrm{d}x_i \wedge \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n
\end{aligned}$$

$$\boxed{\mathrm{d}\left(\sum_{i=1}^n (-1)^i x_i \, \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n\right) = -n \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n}$$

Note that we get from the second to the third line as follows: The $i = 1$ term is

$$(-1)^1 \, \mathrm{d}x_1 \wedge \widehat{\mathrm{d}x_1} \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n = -\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

The $i = 2$ term is

$$\begin{aligned}
(-1)^2 \, \mathrm{d}x_2 \wedge \mathrm{d}x_1 \wedge \widehat{\mathrm{d}x_2} \wedge \mathrm{d}x_3 \wedge \cdots \wedge \mathrm{d}x_n &= (1) \cdot (-1)^{\tau}_{1,2} \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \\
&= -\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n
\end{aligned}$$

where we have used Claim 1.6.8[1] and the odd permutation $\tau_{1,2}$ to rearrange the term. The $i = 3$ term is

$$\begin{aligned}
(-1)^3 \, \mathrm{d}x_3 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \widehat{\mathrm{d}x_3} \wedge \mathrm{d}x_4 \wedge \cdots \wedge \mathrm{d}x_n &= (-1) \cdot (-1)^{\tau_{1,2}\tau_{2,3}} \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \\
&= -\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n
\end{aligned}$$

From here, we should be able to see that all n terms will evaluate to $-\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$, so we can simply add them up and stick n out front as a coefficient. This intuitive justification can be formalized with an induction argument. □

2.4.ii. Solve the equation $\mathrm{d}\mu = \omega$ for $\mu \in \Omega^1(\mathbb{R}^3)$, where ω is the 2-form...

General Solution. The following is a derivation of the general solution to the equation $\mathrm{d}\mu = \omega$, where ω is a two-form with the structure

$$\omega = f \, \mathrm{d}x_i \wedge \mathrm{d}x_j$$

Let $\omega \in \Omega^2(U)$ be a two-form on U with the above structure. Notice that if we take $\mu = g \, \mathrm{d}x_j$ by inspection, then $\mathrm{d}\mu = \mathrm{d}g \wedge \mathrm{d}x_j$. By comparing this equation with the definition of ω, we can determine that

$$\begin{aligned}
\mathrm{d}g &= f \, \mathrm{d}x_i \\
g &= \int f \, \mathrm{d}x_i
\end{aligned}$$

[1]Technically, we use the natural extension of Claim 1.6.8 to the wedge product of one-forms.

Therefore, the solution to $\mathrm{d}\mu = \omega$ is the one-form

$$\mu = \left(\int f \,\mathrm{d}x_i\right) \mathrm{d}x_j$$

□

(1) $\mathrm{d}x_2 \wedge \mathrm{d}x_3$.

Proof. Using the above formula, we have

$$\mu = \left(\int \mathrm{d}x_2\right) \mathrm{d}x_3$$
$$\boxed{\mu = x_2 \,\mathrm{d}x_3}$$

□

(2) $x_2 \,\mathrm{d}x_2 \wedge \mathrm{d}x_3$.

Proof. Using the above formula, we have

$$\mu = \left(\int x_2 \,\mathrm{d}x_2\right) \mathrm{d}x_3$$
$$\boxed{\mu = \frac{1}{2}x_2^2 \,\mathrm{d}x_3}$$

□

(3) $(x_1^2 + x_2^2) \,\mathrm{d}x_1 \wedge \mathrm{d}x_2$.

Proof. Using the above formula, we have

$$\mu = \left(\int (x_1^2 + x_2^2) \,\mathrm{d}x_1\right) \mathrm{d}x_2$$
$$\boxed{\mu = \left(\frac{1}{3}x_1^3 + x_1 x_2^2\right) \mathrm{d}x_2}$$

□

(4) $\cos(x_1) \,\mathrm{d}x_1 \wedge \mathrm{d}x_3$.

Proof. Using the above formula, we have

$$\mu = \left(\int \cos(x_1) \,\mathrm{d}x_1\right) \mathrm{d}x_3$$
$$\boxed{\mu = \sin(x_1) \,\mathrm{d}x_3}$$

□

2.4.iii. Let U be an open subset of $\mathbb{R}^n$.

(1) Show that if $\mu \in \Omega^k(U)$ is exact and $\omega \in \Omega^\ell(U)$ is closed then $\mu \wedge \omega$ is exact. *Hint*: See the second desired property of the exterior derivative.

Proof. To prove that $\mu\wedge\omega$ is exact, it will suffice to show that there exists some $\eta \in \Omega^{k+\ell-1}(U)$ such that $\mathrm{d}\eta = \mu \wedge \omega$. Since μ is exact, we know that there exists $\tilde{\mu} \in \Omega^{k-1}(U)$ such that $\mathrm{d}\tilde{\mu} = \mu$. Since ω is closed, we know that $\mathrm{d}\omega = 0$. Working off of a principle similar to the general proof in Exercise 2.4.ii, we can discover by inspection that taking

$$\eta = \tilde{\mu} \wedge \omega$$

makes it so that

$$\begin{aligned}
\mathrm{d}\eta &= \mathrm{d}(\tilde{\mu} \wedge \omega) \\
&= \mathrm{d}\tilde{\mu} \wedge \omega + (-1)^{k-1}\tilde{\mu} \wedge \mathrm{d}\omega \\
&= \mu \wedge \omega + (-1)^{k-1}\tilde{\mu} \wedge 0 \\
&= \mu \wedge \omega
\end{aligned}$$

as desired. □

(2) In particular, $\mathrm{d}x_1$ is exact, so if $\omega \in \Omega^\ell(U)$ is closed, then $\mathrm{d}x_1 \wedge \omega = \mathrm{d}\mu$. What is μ?

Proof. Since $\mathrm{d}x_1 = \mathrm{d}(x_1)$, we have by part (1) that

$$\boxed{\mu = x_1\omega}$$

□

2.4.iv. Let Q be the rectangle $(a_1, b_1) \times \cdots \times (a_n, b_n)$. Show that if ω is in $\Omega^n(Q)$, then ω is exact. *Hint*: Let $\omega = f\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ with $f \in C^\infty(Q)$ and let g be the function defined by

$$g(x_1, \dots, x_n) = \int_{a_1}^{x_1} f(t, x_2, \dots, x_n)\,\mathrm{d}t$$

Show that $\omega = \mathrm{d}(g\,\mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n)$.

Proof. Let $\omega \in \Omega^n(Q)$ be arbitrary. Then $\omega = f\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ for some $f \in C^\infty(Q)$. To prove that ω is exact, it will suffice to show that $\omega = \mathrm{d}\mu$ for some $\mu \in \Omega^{n-1}(Q)$.

Let g be defined as in the hint. If we take $\mu = g\,\mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n$, then

$$\begin{aligned}
\mathrm{d}\mu &= \mathrm{d}(g\,\mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n) \\
&= \mathrm{d}g \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n \\
&= f(x_1, x_2, \dots, x_n)\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n \\
&= f\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \\
&= \omega
\end{aligned}$$

as desired, where we get from the second to the third line above with the Fundamental Theorem of Calculus. □

2.5.i. Verify the following properties of the interior product, where $U \subset \mathbb{R}^n$ open, $\boldsymbol{v}, \boldsymbol{w}$ are vector fields on U, $\omega_1, \omega_2, \omega \in \Omega^k(U)$, and $\mu \in \Omega^\ell(U)$.

(1) *Linearity in the form*: We have

$$\iota_{\boldsymbol{v}}(\omega_1 + \omega_2) = \iota_{\boldsymbol{v}}\omega_1 + \iota_{\boldsymbol{v}}\omega_2$$

Proof. To prove that the above two forms are equal, it will suffice to show that they evaluate to identical elements of $\Lambda^{k-1}(V^*)$ for all $p \in U$. Let $p \in U$ be arbitrary. Then

$$\begin{aligned}
[\iota_{\boldsymbol{v}}(\omega_1 + \omega_2)]_p &= \iota_{\boldsymbol{v}(p)}[(\omega_1 + \omega_2)_p] \\
&= \iota_{\boldsymbol{v}(p)}[(\omega_1)_p + (\omega_2)_p] \\
&= \iota_{\boldsymbol{v}(p)}(\omega_1)_p + \iota_{\boldsymbol{v}(p)}(\omega_2)_p \\
&= (\iota_{\boldsymbol{v}}\omega_1 + \iota_{\boldsymbol{v}}\omega_2)_p
\end{aligned}$$

where we get from the first to the second line and the third to the fourth line using the definition of the interior product, and from the second to the third line using the linearity of the interior product operation. ☐

(2) *Linearity in the vector field*: We have

$$\iota_{\boldsymbol{v}+\boldsymbol{w}}\omega = \iota_{\boldsymbol{v}}\omega + \iota_{\boldsymbol{w}}\omega$$

Proof. As in the previous part, we have that

$$\begin{aligned}
[\iota_{\boldsymbol{v}+\boldsymbol{w}}\omega]_p &= \iota_{(\boldsymbol{v}+\boldsymbol{w})(p)}\omega_p \\
&= \iota_{\boldsymbol{v}(p)+\boldsymbol{w}(p)}\omega_p \\
&= \iota_{\boldsymbol{v}(p)}\omega_p + \iota_{\boldsymbol{w}(p)}\omega_p \\
&= [\iota_{\boldsymbol{v}}\omega]_p + [\iota_{\boldsymbol{w}}\omega]_p \\
&= [\iota_{\boldsymbol{v}}\omega + \iota_{\boldsymbol{w}}\omega]_p
\end{aligned}$$

☐

(3) *Derivation property*: We have

$$\iota_{\boldsymbol{v}}(\omega \wedge \mu) = \iota_{\boldsymbol{v}}\omega \wedge \mu + (-1)^k \omega \wedge \iota_{\boldsymbol{v}}\mu$$

Proof. As in the previous parts, we have that

$$\begin{aligned}
[\iota_{\boldsymbol{v}}(\omega \wedge \mu)]_p &= \iota_{\boldsymbol{v}(p)}(\omega \wedge \mu)_p \\
&= \iota_{\boldsymbol{v}(p)}(\omega_p \wedge \mu_p) \\
&= \iota_{\boldsymbol{v}(p)}\omega_p \wedge \mu_p + (-1)^k \omega_p \wedge \iota_{\boldsymbol{v}(p)}\mu_p \\
&= (\iota_{\boldsymbol{v}}\omega)_p \wedge \mu_p + (-1)^k \omega_p \wedge (\iota_{\boldsymbol{v}}\mu)_p \\
&= (\iota_{\boldsymbol{v}}\omega \wedge \mu)_p + (-1)^k (\omega_p \wedge \iota_{\boldsymbol{v}}\mu)_p \\
&= [\iota_{\boldsymbol{v}}\omega \wedge \mu + (-1)^k \omega_p \wedge \iota_{\boldsymbol{v}}\mu]_p
\end{aligned}$$

☐

(4) The identity

$$\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{w}}\omega) = -\iota_{\boldsymbol{w}}(\iota_{\boldsymbol{v}}\omega)$$

Proof. As in the previous parts, we have that

$$\begin{aligned}
[\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{w}}\omega)]_p &= \iota_{\boldsymbol{v}(p)}(\iota_{\boldsymbol{w}}\omega)_p \\
&= \iota_{\boldsymbol{v}(p)}(\iota_{\boldsymbol{w}(p)}\omega_p) \\
&= -\iota_{\boldsymbol{v}(p)}(\iota_{\boldsymbol{w}(p)}\omega_p) \\
&= -\iota_{\boldsymbol{v}(p)}(\iota_{\boldsymbol{w}}\omega)_p \\
&= -[\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{w}}\omega)]_p \\
&= [-\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{w}}\omega)]_p
\end{aligned}$$

☐

(5) The identity, as a special case of (4),

$$\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}\omega) = 0$$

Proof. We have that

$$\begin{aligned}
\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}\omega) &= -\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}\omega) \\
2\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}\omega) &= 0 \\
\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}\omega) &= 0
\end{aligned}$$

□

(6) If $\omega = \mu_1 \wedge \cdots \wedge \mu_k$ (i.e., if ω is **decomposable**), then

$$\iota_{\boldsymbol{v}}\omega = \sum_{r=1}^{k} (-1)^{r-1} \iota_{\boldsymbol{v}}(\mu_r) \mu_1 \wedge \cdots \wedge \widehat{\mu_r} \wedge \cdots \wedge \mu_k$$

Proof. To prove that the above two forms are equal, it will suffice to show that they evaluate to identical elements of $\Lambda^{k-1}(V^*)$ for all $p \in U$. Let $p \in U$ be arbitrary. Also let $(\mu_i)_p = \pi(\ell_i)$ for $i = 1, \dots, k$ and, thus, $\omega_p = \pi(\ell_1 \otimes \cdots \otimes \ell_k)$. Then

$$\begin{aligned}
[\iota_{\boldsymbol{v}}\omega]_p &= \iota_{\boldsymbol{v}(p)}\omega_p \\
&= \pi\left[\iota_{\boldsymbol{v}(p)}(\ell_1 \otimes \cdots \otimes \ell_k)\right] \\
&= \pi\left[\sum_{r=1}^{k} (-1)^{r-1} \ell_r(\boldsymbol{v}(p)) \ell_1 \otimes \cdots \otimes \hat{\ell}_r \otimes \cdots \otimes \ell_k\right] && \text{Lemma 1.7.4} \\
&= \sum_{r=1}^{k} (-1)^{r-1} \pi\left[\iota_{\boldsymbol{v(p)}}(\ell_r) \ell_1 \otimes \cdots \otimes \hat{\ell}_r \otimes \cdots \otimes \ell_k\right] \\
&= \sum_{r=1}^{k} (-1)^{r-1} \iota_{\boldsymbol{v(p)}}(\mu_r)_p (\mu_1)_p \wedge \cdots \wedge \widehat{(\mu_r)_p} \wedge \cdots \wedge (\mu_k)_p \\
&= \left[\sum_{r=1}^{k} (-1)^{r-1} \iota_{\boldsymbol{v}}(\mu_r) \mu_1 \wedge \cdots \wedge \widehat{\mu_r} \wedge \cdots \wedge \mu_k\right]_p
\end{aligned}$$

as desired. □

2.5.ii. Show that if ω is the k-form $\mathrm{d}x_I$ and $\boldsymbol{v}$ is the vector field $\partial/\partial x_r$, then $\iota_{\boldsymbol{v}}\omega$ is given by

$$\iota_{\boldsymbol{v}}\omega = \sum_{j=1}^{k} (-1)^{j-1} \delta_{r,i_j} \,\mathrm{d}x_{I_j}$$

where

$$\delta_{r,i_j} = \begin{cases} 1 & r = i_j \\ 0 & r \neq i_j \end{cases} \qquad\qquad I_j = (i_1, \dots, \widehat{i_j}, \dots, i_k)$$

In the above, 1 represents the identity function on U, and 0 represents the zero function on U.

Proof. We have that $\omega = \mathrm{d}x_I = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$. Therefore, by Properties 2.5.3(6),

$$\begin{aligned}
\iota_{\boldsymbol{v}}\omega &= \sum_{j=1}^{k} (-1)^{j-1} \iota_{\boldsymbol{v}}(\mathrm{d}x_{i_j}) \,\mathrm{d}x_{i_1} \wedge \cdots \wedge \widehat{\mathrm{d}x_{i_j}} \wedge \cdots \wedge \mathrm{d}x_{i_k} \\
&= \sum_{j=1}^{k} (-1)^{j-1} \,\mathrm{d}x_{i_j}\,(\partial/\partial x_r)\,\mathrm{d}x_{i_1} \wedge \cdots \wedge \widehat{\mathrm{d}x_{i_j}} \wedge \cdots \wedge \mathrm{d}x_{i_k} \\
&= \sum_{j=1}^{k} (-1)^{j-1} \delta_{r,i_j} \,\mathrm{d}x_{I_j}
\end{aligned}$$

as desired. □

2.5.iii. Show that if ω is the n-form $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ and $\boldsymbol{v}$ is the vector field $\sum_{i=1}^n f_i \, \partial/\partial x_i$, then $\iota_{\boldsymbol{v}}\omega$ is given by

$$\iota_{\boldsymbol{v}}\omega = \sum_{r=1}^n (-1)^{r-1} f_r \, \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_r} \wedge \cdots \wedge \mathrm{d}x_n$$

Proof. Let $I_j = (1, \dots, \hat{j}, \dots, n)$. Then we have that

$$\begin{aligned}
\iota_{\boldsymbol{v}}\omega &= \sum_{r=1}^n f_r \iota_{\partial/\partial x_r} \omega && \text{Properties 2.5.3(2)} \\
&= \sum_{r=1}^n f_r \left(\sum_{j=1}^n (-1)^{j-1} \delta_{r,j} \, \mathrm{d}x_{I_j} \right) && \text{Exercise 2.5.ii} \\
&= \sum_{r=1}^n f_r \left((-1)^{r-1} \delta_{r,r} \, \mathrm{d}x_{I_r} \right) \\
&= \sum_{r=1}^n (-1)^{r-1} f_r \, \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_r} \wedge \cdots \wedge \mathrm{d}x_n
\end{aligned}$$

as desired. □

2.5.iv. Let $U \subset \mathbb{R}^n$ open and $\boldsymbol{v}$ a C^∞ vector field on U. Show that for $\omega \in \Omega^k(U)$,

$$\mathrm{d}(L_{\boldsymbol{v}}\omega) = L_{\boldsymbol{v}}(\mathrm{d}\omega) \qquad \iota_{\boldsymbol{v}}(L_{\boldsymbol{v}}\omega) = L_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}\omega)$$

Hint: Deduce the first of these identities using the identity $\mathrm{d}(\mathrm{d}\mu) = 0$ and the second using the identity $\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}\mu) = 0$.

Proof. Let $\omega \in \Omega^k(U)$ be arbitrary. Then

$$\begin{aligned}
\mathrm{d}(L_{\boldsymbol{v}}\omega) &= \mathrm{d}(\iota_{\boldsymbol{v}}(\mathrm{d}\omega) + \mathrm{d}(\iota_{\boldsymbol{v}}\omega)) \\
&= \mathrm{d}(\iota_{\boldsymbol{v}}(\mathrm{d}\omega)) + \mathrm{d}(\mathrm{d}(\iota_{\boldsymbol{v}}\omega)) \\
&= \mathrm{d}(\iota_{\boldsymbol{v}}(\mathrm{d}\omega)) + 0 \\
&= 0 + \mathrm{d}(\iota_{\boldsymbol{v}}(\mathrm{d}\omega)) \\
&= \iota_{\boldsymbol{v}}(0) + \mathrm{d}(\iota_{\boldsymbol{v}}(\mathrm{d}\omega)) \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}(\mathrm{d}\omega)) + \mathrm{d}(\iota_{\boldsymbol{v}}(\mathrm{d}\omega)) \\
&= L_{\boldsymbol{v}}(\mathrm{d}\omega)
\end{aligned}
\qquad
\begin{aligned}
\iota_{\boldsymbol{v}}(L_{\boldsymbol{v}}\omega) &= \iota_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}(\mathrm{d}\omega) + \mathrm{d}(\iota_{\boldsymbol{v}}\omega)) \\
&= \iota_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}(\mathrm{d}\omega)) + \iota_{\boldsymbol{v}}(\mathrm{d}(\iota_{\boldsymbol{v}}\omega)) \\
&= 0 + \iota_{\boldsymbol{v}}(\mathrm{d}(\iota_{\boldsymbol{v}}\omega)) \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}(\iota_{\boldsymbol{v}}\omega)) + 0 \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}(\iota_{\boldsymbol{v}}\omega)) + \mathrm{d}(0) \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}(\iota_{\boldsymbol{v}}\omega)) + \mathrm{d}(\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}\omega)) \\
&= L_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}\omega)
\end{aligned}$$

as desired. □

2.5.v. Given $\omega_i \in \Omega^{k_i}(U)$ for $i = 1, 2$, show that

$$L_{\boldsymbol{v}}(\omega_1 \wedge \omega_2) = L_{\boldsymbol{v}}\omega_1 \wedge \omega_2 + \omega_1 \wedge L_{\boldsymbol{v}}\omega_2$$

Hint: Plug $\omega = \omega_1 \wedge \omega_2$ into the definition of the Lie derivative and use the second desired property of exterior differentiation along with the derivation property of the interior product to evaluate the resulting expression.

Proof. Let $\omega = \omega_1 \wedge \omega_2$. Then

$$\begin{aligned}
L_{\boldsymbol{v}}(\omega_1 \wedge \omega_2) &= L_{\boldsymbol{v}}\omega \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}\omega) + \mathrm{d}(\iota_{\boldsymbol{v}}\omega) \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}(\omega_1 \wedge \omega_2)) + \mathrm{d}(\iota_{\boldsymbol{v}}(\omega_1 \wedge \omega_2)) \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}\omega_1 \wedge \omega_2 + (-1)^{k_1}\omega_1 \wedge \mathrm{d}\omega_2) + \mathrm{d}(\iota_{\boldsymbol{v}}(\omega_1 \wedge \omega_2)) \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}\omega_1 \wedge \omega_2 + (-1)^{k_1}\omega_1 \wedge \mathrm{d}\omega_2) + \mathrm{d}\big(\iota_{\boldsymbol{v}}\omega_1 \wedge \omega_2 + (-1)^{k_1}\omega_1 \wedge \iota_{\boldsymbol{v}}\omega_2\big) \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}\omega_1 \wedge \omega_2) \\
&\quad + (-1)^{k_1}\iota_{\boldsymbol{v}}(\omega_1 \wedge \mathrm{d}\omega_2) \\
&\quad + \mathrm{d}(\iota_{\boldsymbol{v}}\omega_1 \wedge \omega_2) \\
&\quad + (-1)^{k_1}\,\mathrm{d}(\omega_1 \wedge \iota_{\boldsymbol{v}}\omega_2) \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}\omega_1) \wedge \omega_2 + (-1)^{k_1+1}\mathrm{d}\omega_1 \wedge \iota_{\boldsymbol{v}}\omega_2 \\
&\quad + (-1)^{k_1}(\iota_{\boldsymbol{v}}\omega_1 \wedge \mathrm{d}\omega_2 + (-1)^{k_1}\omega_1 \wedge \iota_{\boldsymbol{v}}(\mathrm{d}\omega_2)) \\
&\quad + \mathrm{d}(\iota_{\boldsymbol{v}}\omega_1) \wedge \omega_2 + (-1)^{k_1-1}\iota_{\boldsymbol{v}}\omega_1 \wedge \mathrm{d}\omega_2 \\
&\quad + (-1)^{k_1}(\mathrm{d}\omega_1 \wedge \iota_{\boldsymbol{v}}\omega_2 + (-1)^{k_1}\omega_1 \wedge \mathrm{d}(\iota_{\boldsymbol{v}}\omega_2)) \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}\omega_1) \wedge \omega_2 + (-1)^{k_1+1}\mathrm{d}\omega_1 \wedge \iota_{\boldsymbol{v}}\omega_2 \\
&\quad + (-1)^{k_1}\iota_{\boldsymbol{v}}\omega_1 \wedge \mathrm{d}\omega_2 + (-1)^{2k_1}\omega_1 \wedge \iota_{\boldsymbol{v}}(\mathrm{d}\omega_2) \\
&\quad + \mathrm{d}(\iota_{\boldsymbol{v}}\omega_1) \wedge \omega_2 + (-1)^{k_1-1}\iota_{\boldsymbol{v}}\omega_1 \wedge \mathrm{d}\omega_2 \\
&\quad + (-1)^{k_1}\mathrm{d}\omega_1 \wedge \iota_{\boldsymbol{v}}\omega_2 + (-1)^{2k_1}\omega_1 \wedge \mathrm{d}(\iota_{\boldsymbol{v}}\omega_2) \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}\omega_1) \wedge \omega_2 - (-1)^{k_1}\mathrm{d}\omega_1 \wedge \iota_{\boldsymbol{v}}\omega_2 \\
&\quad + (-1)^{k_1}\iota_{\boldsymbol{v}}\omega_1 \wedge \mathrm{d}\omega_2 + \omega_1 \wedge \iota_{\boldsymbol{v}}(\mathrm{d}\omega_2) \\
&\quad + \mathrm{d}(\iota_{\boldsymbol{v}}\omega_1) \wedge \omega_2 - (-1)^{k_1}\iota_{\boldsymbol{v}}\omega_1 \wedge \mathrm{d}\omega_2 \\
&\quad + (-1)^{k_1}\mathrm{d}\omega_1 \wedge \iota_{\boldsymbol{v}}\omega_2 + \omega_1 \wedge \mathrm{d}(\iota_{\boldsymbol{v}}\omega_2) \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}\omega_1) \wedge \omega_2 \\
&\quad + \omega_1 \wedge \iota_{\boldsymbol{v}}(\mathrm{d}\omega_2) \\
&\quad + \mathrm{d}(\iota_{\boldsymbol{v}}\omega_1) \wedge \omega_2 \\
&\quad + \omega_1 \wedge \mathrm{d}(\iota_{\boldsymbol{v}}\omega_2) \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}\omega_1) \wedge \omega_2 + \mathrm{d}(\iota_{\boldsymbol{v}}\omega_1) \wedge \omega_2 + \omega_1 \wedge \iota_{\boldsymbol{v}}(\mathrm{d}\omega_2) + \omega_1 \wedge \mathrm{d}(\iota_{\boldsymbol{v}}\omega_2) \\
&= (\iota_{\boldsymbol{v}}(\mathrm{d}\omega_1) + \mathrm{d}(\iota_{\boldsymbol{v}}\omega_1)) \wedge \omega_2 + \omega_1 \wedge (\iota_{\boldsymbol{v}}(\mathrm{d}\omega_2) + \mathrm{d}(\iota_{\boldsymbol{v}}\omega_2)) \\
&= L_{\boldsymbol{v}}\omega_1 \wedge \omega_2 + \omega_1 \wedge L_{\boldsymbol{v}}\omega_2
\end{aligned}$$

as desired. □

2.6.i. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the map

$$f(x_1, x_2, x_3) = (x_1x_2, x_2x_3^2, x_3^3)$$

Compute the pullback $f^*\omega$ for the following forms.

(1) $\omega = x_2\,\mathrm{d}x_3$.

Proof. We have that

$$\begin{aligned}
f^*\omega &= f^*x_2 \cdot \mathrm{d}f_3 \\
&= x_2(x_1x_2, x_2x_3^2, x_3^3) \cdot 3x_3^2\,\mathrm{d}x_3 \\
\end{aligned}$$
$$\boxed{f^*\omega = 3x_2x_3^4\,\mathrm{d}x_3}$$

□

(2) $\omega = x_1\,\mathrm{d}x_1 \wedge \mathrm{d}x_3$.

Proof. We have that

$$\begin{aligned}
f^*\omega &= f^*x_1 \cdot \mathrm{d}f_1 \wedge \mathrm{d}f_3\\
&= x_1(x_1x_2, x_2x_3^2, x_3^3) \cdot (x_1\,\mathrm{d}x_2 + x_2\,\mathrm{d}x_1) \wedge 3x_3^2\,\mathrm{d}x_3\\
\Aboxed{f^*\omega &= 3x_1^2x_2x_3^2\,\mathrm{d}x_2 \wedge \mathrm{d}x_3 + 3x_1x_2^2x_3^2\,\mathrm{d}x_1 \wedge \mathrm{d}x_3}
\end{aligned}$$

□

(3) $\omega = x_1\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$.

Proof. We have that

$$\begin{aligned}
f^*\omega &= f^*x_1 \cdot \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \mathrm{d}f_3\\
&= x_1(x_1x_2, x_2x_3^2, x_3^3) \cdot (x_1\,\mathrm{d}x_2 + x_2\,\mathrm{d}x_1) \wedge (2x_2x_3\,\mathrm{d}x_3 + x_3^2\,\mathrm{d}x_2) \wedge 3x_3^2\,\mathrm{d}x_3\\
\Aboxed{f^*\omega &= 3x_1x_2^2x_3^4\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3}
\end{aligned}$$

where, from the second to the third line, we cancel all wedge products with repeats. □

2.6.ii. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be the map

$$f(x_1, x_2) = (x_1^2, x_2^2, x_1x_2)$$

Compute the pullback $f^*\omega$ for the following forms.

(1) $\omega = x_2\,\mathrm{d}x_2 + x_3\,\mathrm{d}x_3$.

Proof. We have that

$$\begin{aligned}
f^*\omega &= f^*x_2 \cdot \mathrm{d}f_2 + f^*x_3 \cdot \mathrm{d}f_3\\
&= x_2(x_1^2, x_2^2, x_1x_2) \cdot 2x_2\,\mathrm{d}x_2 + x_3(x_1^2, x_2^2, x_1x_2) \cdot (x_1\,\mathrm{d}x_2 + x_2\,\mathrm{d}x_1)\\
\Aboxed{f^*\omega &= x_1x_2^2\,\mathrm{d}x_1 + (2x_2^3 + x_1^2x_2)\,\mathrm{d}x_2}
\end{aligned}$$

□

(2) $\omega = x_1\,\mathrm{d}x_2 \wedge \mathrm{d}x_3$.

Proof. We have that

$$\begin{aligned}
f^*\omega &= f^*x_1 \cdot \mathrm{d}f_2 \wedge \mathrm{d}f_3\\
&= x_1(x_1^2, x_2^2, x_1x_2) \cdot 2x_2\,\mathrm{d}x_2 \wedge (x_1\,\mathrm{d}x_2 + x_2\,\mathrm{d}x_1)\\
&= 2x_1^2x_2^2\,\mathrm{d}x_2 \wedge \mathrm{d}x_1\\
\Aboxed{f^*\omega &= -2x_1^2x_2^2\,\mathrm{d}x_1 \wedge \mathrm{d}x_2}
\end{aligned}$$

□

(3) $\omega = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$.

Proof. We have that

$$\begin{aligned}
f^*\omega &= \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \mathrm{d}f_3\\
&= 2x_1\,\mathrm{d}x_1 \wedge 2x_2\,\mathrm{d}x_2 \wedge (x_1\,\mathrm{d}x_2 + x_2\,\mathrm{d}x_1)\\
\Aboxed{f^*\omega &= 0}
\end{aligned}$$

where 0 denotes the zero element of $\Omega^3(\mathbb{R}^2)$. □

2.6.iii. Let $U \subset \mathbb{R}^n$ open, $V \subset \mathbb{R}^m$ open, $f : U \to V$ a C^∞ map, and $\gamma : [a,b] \to U$ a C^∞ curve. Show that for $\omega \in \Omega^1(V)$,

$$\int_\gamma f^*\omega = \int_\eta \omega$$

where $\eta : [a,b] \to V$ is the curve $\eta(t) = f(\gamma(t))$. (See Exercise 2.1.vii.)

Proof. Since $\omega \in \Omega^1(V)$, we know that

$$\omega = \sum_{j=1}^m g_j \, \mathrm{d}x_j$$

for some $g_i \in C^\infty(V)$. It follows that

$$\begin{aligned} f^*\omega &= \sum_{j=1}^m f^* g_j \, \mathrm{d}f_j \\ &= \sum_{j=1}^m f^* g_j \left(\sum_{i=1}^n \frac{\partial f_j}{\partial x_i} \, \mathrm{d}x_i \right) \\ &= \sum_{i=1}^n \left(\sum_{j=1}^m f^* g_j \frac{\partial f_j}{\partial x_i} \right) \mathrm{d}x_i \end{aligned}$$

Additionally, let $\gamma_1, \dots, \gamma_n$ be the coordinate functions of γ, let $\eta_1, \dots, \eta_m$ be the coordinate functions of η, and let $f_1, \dots, f_m$ be the coordinate functions of f. It follows that

$$\begin{aligned} \int_\gamma f^*\omega &= \sum_{i=1}^n \int_a^b \left[\sum_{j=1}^m f^* g_j \frac{\partial f_j}{\partial x_i} \right] (\gamma(t)) \frac{\mathrm{d}\gamma_i}{\mathrm{d}t} \, \mathrm{d}t \\ &= \sum_{i=1}^n \sum_{j=1}^m \int_a^b [f^* g_j](\gamma(t)) \frac{\partial f_j}{\partial x_i} \frac{\mathrm{d}\gamma_i}{\mathrm{d}t} \, \mathrm{d}t \\ &= \sum_{j=1}^m \int_a^b [g_j \circ f](\gamma(t)) \left(\sum_{i=1}^n \frac{\partial f_j}{\partial x_i} \frac{\mathrm{d}\gamma_i}{\mathrm{d}t} \right) \mathrm{d}t \\ &= \sum_{j=1}^m \int_a^b g_j(f(\gamma(t))) \frac{\mathrm{d}(f_j \circ \gamma)}{\mathrm{d}t} \, \mathrm{d}t \\ &= \sum_{j=1}^m \int_a^b g_j(\eta(t)) \frac{\mathrm{d}\eta_j}{\mathrm{d}t} \, \mathrm{d}t \\ &= \int_\eta \omega \end{aligned}$$

□

4 Integration of Forms

From Guillemin and Haine (2018).

Chapter 3

5/17: **3.2.i.** Let $f : \mathbb{R} \to \mathbb{R}$ be a compactly supported function of class C^r with support on the interval (a, b). Show that the following are equivalent.

(1) $\int_a^b f(x)\,\mathrm{d}x = 0$.

(2) There exists a function $g : \mathbb{R} \to \mathbb{R}$ of class C^{r+1} with support on (a, b) with $\mathrm{d}g/\mathrm{d}x = f$.

(Hint: Show that the function $g(x) = \int_a^x f(s)\,\mathrm{d}s$ is compactly supported.)

Proof. Suppose first that $\int_a^b f(x)\,\mathrm{d}x = 0$. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$x \mapsto \int_a^x f(s)\,\mathrm{d}s$$

By the FTC, $\mathrm{d}g/\mathrm{d}x = f$ and, hence, $g \in C^{r+1}(\mathbb{R})$. Moreover, since f is supported on (a, b), we know that $f(x) = 0$ for all $x \leq a$ and $x \geq b$. It follows that

$$g(x) = \int_a^x f(x)\,\mathrm{d}x = \int_a^x 0\,\mathrm{d}x = 0$$

for all $x \leq a$ and that

$$g(x) = \int_a^x f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^x f(x)\,\mathrm{d}x = 0 + \int_b^x 0\,\mathrm{d}x = 0$$

for all $x \geq b$. Thus, g is supported on (a, b). Moreover, since $\mathrm{supp}(g) \subset \mathbb{R}$ is closed by definition and bounded (as a subset of (a, b)), the Heine-Borel theorem proves that g is compactly supported.

Now suppose that there exists a function $g : \mathbb{R} \to \mathbb{R}$ of class C^{r+1} with support on (a, b) and with $\mathrm{d}g/\mathrm{d}x = f$. Then by the FTC,

$$\int_a^b f(x)\,\mathrm{d}x = g(b) - g(a) = 0 - 0 = 0$$

as desired. ☐

3.6.iii. Show that the Brouwer fixed point theorem isn't true if one replaces the closed unit ball by the open unit ball. (Hint: Let U be the open unit ball (i.e., the interior of B^n). Show that the map $h : U \to \mathbb{R}^n$ defined by

$$h(x) = \frac{x}{1 - \|x\|^2}$$

is a diffeomorphism of U onto $\mathbb{R}^n$, and show that there are lots of mappings of $\mathbb{R}^n$ onto $\mathbb{R}^n$ which do not have fixed points.)

Proof. It appears that taking the hint will not suffice to prove the claim. After all, proving that there exist continuous mappings $h : U \to \mathbb{R}^n$ with no fixed point will not negate the modified Brouwer fixed point theorem; we would need to find a continuous mapping $f : U \to U$ with no fixed points. Fortunately, this is not hard to do — let $x = (1, 0, \dots, 0) \in \mathbb{R}^n$ and choose $f : U \to U$ defined by the rule "take every $p \in U$ to the midpoint of the line $\overline{px}$." This is clearly a continuous mapping of $U \to U$ with no fixed points. ☐

3.6.iv. Show that the fixed point in the Brouwer theorem doesn't have to be an interior point of B^n, i.e., show that it can lie on the boundary.

Proof. Take the mapping f from the proof of Exercise 3.6.iii. There, the fixed point is x. $\square$

3.6.v. If we identify $\mathbb{C}$ with $\mathbb{R}^2$ via the mapping $(x, y) \mapsto x + iy$, we can think of a $\mathbb{C}$-linear mapping of $\mathbb{C}$ into itself, i.e., a mapping of the form $z \mapsto cz$ for a fixed $c \in \mathbb{C}$ as an $\mathbb{R}$-linear mapping of $\mathbb{R}^2$ into itself. Show that the determinant of this mapping is $|c|^2$.

Proof. Let $c = a + ib$. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the real form of the described complex mapping, i.e.,

$$f(x, y) = (\operatorname{Re}(c \cdot (x + iy)), \operatorname{Im}(c \cdot (x + iy)))$$

Then since

$$(a + ib)(x + iy) = ax + aiy + ibx - by = (ax - by) + i(bx + ay)$$

we have that

$$f(x, y) = (ax - by, bx + ay)$$

It follows that the matrix of f is

$$\mathcal{M}(f) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

The determinant of $\mathcal{M}(f)$ is hence

$$\det[\mathcal{M}(f)] = (a)(a) - (-b)(b) = a^2 + b^2 = \left(\sqrt{a^2 + b^2}\right)^2 = |c|^2$$

as desired. $\square$

3.6.vi. (1) Let $f : \mathbb{C} \to \mathbb{C}$ be the mapping $f(z) = z^n$. Show that $Df(z)$ is the linear map

$$Df(z) = nz^{n-1}$$

given by multiplication by nz^{n-1}. (Hint: Argue from first principles. Show that for $h \in \mathbb{C} = \mathbb{R}^2$,

$$\frac{(z + h)^n - z^n - nz^{n-1}h}{|h|}$$

tends to zero as $|h| \to 0$.)

Proof. We have that

$$\begin{aligned}
0 &\stackrel{?}{=} \lim_{|h| \to 0} \frac{(z + h)^n - z^n - nz^{n-1}h}{|h|} \\
0 &\stackrel{?}{=} \lim_{|h| \to 0} \frac{\sum_{k=0}^{n} \binom{n}{k} z^{n-k} h^k - z^n - nz^{n-1}h}{|h|} \\
0 &\stackrel{?}{=} \lim_{|h| \to 0} \frac{z^n + nz^{n-1}h + \sum_{k=2}^{n} \binom{n}{k} z^{n-k} h^k - z^n - nz^{n-1}h}{|h|} \\
0 &\stackrel{?}{=} \lim_{|h| \to 0} \frac{\sum_{k=2}^{n} \binom{n}{k} z^{n-k} h^k}{|h|} \\
0 &\stackrel{?}{=} \lim_{|h| \to 0} \sum_{k=2}^{n} \binom{n}{k} z^{n-k} h^{k-1} \\
0 &\stackrel{?}{=} \sum_{k=2}^{n} \binom{n}{k} z^{n-k} 0^{k-1} \\
0 &\stackrel{\checkmark}{=} 0
\end{aligned}$$

as desired. $\square$

(2) Conclude from Exercise 3.6.v that

$$\det(Df(z)) = n^2|z|^{2n-2}$$

Proof. By calling "nz^{n-1}" a linear map, we mean the linear map $x \mapsto nz^{n-1} \cdot x$ for $x \in \mathbb{C}$ and $\cdot : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ the multiplication operation on $\mathbb{C}$. Thus, in the context of Exercise 3.6.v, $c = nz^{n-1}$. It follows that

$$\det(Df(z)) = |nz^{n-1}|^2 = n^2|z^{n-1}|^2 = n^2|z^{2n-2}| = n^2|z|^{2n-2}$$

as desired. □

(3) Show that at every point $z \in \mathbb{C} \setminus \{0\}$, f is orientation preserving.

Proof. Let $z \in \mathbb{C} \setminus \{0\}$ be arbitrary. To prove that f is orientation preserving at z, it will suffice to show that $\det[Df(z)] > 0$. But since $n > 0$ and $|z| > 0$ for $z \neq 0$, we have by part (2) that

$$\det[Df(z)] = n^2|z|^{2n-2} > 0$$

as desired. □

(4) Show that every point $w \in \mathbb{C} \setminus \{0\}$ is a regular value of f and that

$$f^{-1}(w) = \{z_1, \dots, z_n\}$$

with $\sigma_{z_i} = +1$.

Proof. By part (3), $\det[Df(z)] > 0$ for all $z \in \mathbb{C} \setminus \{0\}$. Thus, no $z \in \mathbb{C} \setminus \{0\}$ is a critical point of f. Additionally,

$$\det[Df(0)] = n^2|0|^{2n-2} = 0$$

so 0 is the lone critical value of f and element of C_f. Moreover, since $f(0) = 0$, $f(C_f) = \{0\}$, so the set of regular values of f is

$$f(\mathbb{C}) \setminus f(C_f) = \mathbb{C} \setminus \{0\}$$

as desired.

Additionally, by DeMoivre's Theorem, there are exactly n roots $z_1, \dots, z_n$ of the function z^n for all z. Lastly, by part (3), f is orientation preserving at all z, including $z_1, \dots, z_n$; therefore, $\sigma_{z_i} = +1$ for all $i = 1, \dots, n$. □

(5) Conclude that the degree of f is n.

Proof. By part (4) and Theorem 3.6.4,

$$\deg(f) = \sum_{i=1}^{n} \sigma_{z_i} = \sum_{i=1}^{n} +1 = n$$

as desired. □

3.7.i. What are the set of critical points and the image of the set of critical points for the following maps from $\mathbb{R} \to \mathbb{R}$?

(1) The map $f_1(x) = (x^2 - 1)^2$.

Answer.

Critical points: $-1, 0, 1$
Critical values: $0, 1$

□

(2) The map $f_2(x) = \sin(x) + x$.

Answer.

$$\boxed{\begin{array}{l}\text{Critical points: } \pi + 2\pi z,\ \ z \in \mathbb{Z} \\ \text{Critical values: } \pi + 2\pi z,\ \ z \in \mathbb{Z}\end{array}}$$

□

(3) The map

$$f_3(x) = \begin{cases} 0 & x \leq 0 \\ \mathrm{e}^{-1/x} & x > 0 \end{cases}$$

Answer.

$$\boxed{\begin{array}{l}\text{Critical point: } 0 \\ \text{Critical value: } 0\end{array}}$$

□

3.7.ii. (Sard's theorem for affine maps) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an **affine map**, i.e., a map of the form $f(x) = A(x) + x_0$ where $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map and $x_0 \in \mathbb{R}^n$. Prove Sard's theorem for f.

Proof. We have that

$$Df(x) = A$$

for all $x \in \mathbb{R}^n$. We divide into two cases ($\det A = 0$ and $\det A \neq 0$). If $\det A = 0$, $f(\mathbb{R}^n) \setminus f(C_f) = \emptyset$ which is open and dense in $\mathbb{R}^n$. If $\det A \neq 0$, $f(\mathbb{R}^n) \setminus f(C_f) = \mathbb{R}^n$ which is open and dense in $\mathbb{R}^n$. □

5 Manifolds

From Guillemin and Haine (2018).

Chapter 4

5/22: **4.1.i.** Show that the set of solutions to the system of equations

$$\begin{aligned}x_1^2 + \cdots + x_n^2 &= 1\\ x_1 + \cdots + x_n &= 0\end{aligned}$$

is an $(n-2)$-dimensional submanifold of $\mathbb{R}^n$.

Proof. Let $f : \mathbb{R}^n \to \mathbb{R}^2$ be defined by

$$\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} \mapsto \begin{bmatrix}x_1^2 + \cdots + x_n^2 - 1\\ x_1 + \cdots + x_n\end{bmatrix}$$

Then the set of solutions to the given system of equations is equal to $f^{-1}(0)$, where $0 \in \mathbb{R}^2$.

The task now becomes a problem of proving that that $f^{-1}(0)$ is an $(n-2)$-dimensional submanifold of $\mathbb{R}^n$. To do so, Theorem 4.1.7 tells us that it will suffice to show that 0 is a regular value of f. Suppose for the sake of contradiction that 0 is not a regular value of f. Then there exists $p \in f^{-1}(0)$ such that f is not a submersion at p. It follows that $Df(p) : \mathbb{R}^n \to \mathbb{R}^2$ is not surjective. Thus, the rank of the matrix of $Df(p)$ must be less than two. Consequently, all columns in the matrix

$$\mathcal{M}(Df(p)) = \begin{bmatrix}2p_1 & \cdots & 2p_n\\ 1 & \cdots & 1\end{bmatrix}$$

where $p = (p_1, \dots, p_n)$ must be equal. It follows that $p_1 = \cdots = p_n$. This combined with the fact that $p_1 + \cdots + p_n = 0$ means that $p_i = 0$ for all $i = 1, \dots, n$. But then $p_1^2 + \cdots + p_n^2 - 1 = -1 \neq 0$, a contradiction. $\square$

4.1.ii. Let $S^{n-1} \subset \mathbb{R}^n$ be the $(n-1)$-sphere and let

$$X_a = \{x \in S^{n-1} \mid x_1 + \cdots + x_n = a\}$$

For what values of a is X_a an $(n-2)$-dimensional submanifold of S^{n-1}?

Proof. We first determine which values of a yield a nonempty X_a. Then, we determine which of these X_a describe $(n-2)$-dimensional submanifolds of S^{n-1}.

For the first part, suppose $x \in S^{n-1}$. Then $x_1^2 + \cdots + x_n^2 = 1$. It follows by the Cauchy-Schwarz inequality that

$$\begin{aligned}|a| &= |x_1 + \cdots + x_n|\\ &= |x_1 \cdot 1 + \cdots + x_n \cdot 1|\\ &\leq \sqrt{x_1^2 + \cdots + x_n^2} \cdot \sqrt{1^2 + \cdots + 1^2}\\ &= \sqrt{1} \cdot \sqrt{n}\\ &= \sqrt{n}\end{aligned}$$

Now for the second part. Let $f_a : \mathbb{R}^n \to \mathbb{R}^2$ be defined by

$$\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix} \mapsto \begin{bmatrix}x_1^2 + \cdots + x_n^2 - 1\\ x_1 + \cdots + x_n - a\end{bmatrix}$$

Then $X_a = f_a^{-1}(0)$. Thus, we want to find the set of all a such that 0 is a regular value of f. Suppose a is not in this set. Then 0 is not a regular value of f_a. It follows by a similar argument to that used in Exercise 4.1.i that $x_1 = \cdots = x_n$. This combined with the fact that $a = x_1 + \cdots + x_n = nx_i$ implies that $x_i = a/n$ for $i = 1, \dots, n$. And this result combined with the fact that $x_1^2 + \cdots + x_n^2 = 1$ implies that

$$\begin{aligned} 1 &= x_1^2 + \cdots + x_n^2 \\ &= nx_i^2 \\ &= a^2/n \\ a &= \pm\sqrt{n} \end{aligned}$$

Therefore, if $|a| \leq \sqrt{n}$ and $|a| \neq \sqrt{n}$, we know that

$$\boxed{|a| < \sqrt{n}}$$

□

4.1.iii. Show that if X_i is an n_i-dimensional submanifold of $\mathbb{R}^{N_i}$ for $i = 1, 2$, then

$$X_1 \times X_2 \subset \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$$

is an $(n_1 + n_2)$-dimensional submanifold of $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$.

Proof. Taking the hint from Guillemin and Haine (2018, p. 98), we approach this problem from the perspective of the definition of an n-manifold, as opposed that of Theorem 4.1.7. Additionally, note that any time "i" appears for the remainder of this proof, it is a stand-in for $1, 2$.

To prove that $X_1 \times X_2 \subset \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ is an $(n_1 + n_2)$-manifold, it will suffice to show that for every $p \in X_1 \times X_2$, there exists a neighborhood $V \subset \mathbb{R}^{N_1+N_2}$ of p, an open subset $U \subset \mathbb{R}^{n_1+n_2}$, and a diffeomorphism $\phi : U \to (X_1 \times X_2) \cap V$. Let $p \in X_1 \times X_2$ be arbitrary. Suppose $p = (p_1, p_2)$, where p_i is an n_i-tuple. It follows that $p_i \in X_i$. Therefore, since X_i is an n_i-manifold, there exists a neighborhood $V_i \subset \mathbb{R}^{N_i}$ of p_i, an open subset $U_i \subset \mathbb{R}^{n_i}$, and a diffeomorphism $\phi_i : U_i \to X_i \cap V_i$. Let $V = V_1 \times V_2$, $U = U_1 \times U_2$, and $\phi(x_1, x_2) = (\phi_1(x_1), \phi_2(x_2))$. Naturally, $V \subset \mathbb{R}^{N_1+N_2}$ and $U \subset \mathbb{R}^{n_1+n_2}$. Additionally, endowing $\mathbb{R}^{N_1+N_2} = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ with the product topology ensures that V is a neighborhood of p and endowing $\mathbb{R}^{n_1+n_2} = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ with the product topology ensures that U is open. Lastly, defining ϕ as the "product" of two diffeomorphisms guarantees that ϕ, itself, is also a diffeomorphism. □

4.1.v. Let $g : \mathbb{R}^n \to \mathbb{R}^k$ be a C^∞ map and let $X = \Gamma_g$ be the graph of g. Prove directly that X is an n-manifold by proving that the map $\gamma_g : \mathbb{R}^n \to X$ defined by

$$x \mapsto (x, g(x))$$

is a diffeomorphism.

Proof. It's clear that γ_g is a C^∞ map since each of its components are C^∞. It is a diffeomorphism since it's inverse is the map $\pi : \gamma_g \to \mathbb{R}^n$ given by $\pi(x, g(x)) = x$, which is also clearly C^∞. □

4.1.vi. Prove that the orthogonal group $O(n)$ is an $n(n-1)/2$-manifold. *Hints*:

- Let $f : \mathcal{M}_n \to \mathcal{S}_n$ be the map
$$f(A) = A^\intercal A - \mathrm{id}_n$$
show that $O(n) = f^{-1}(0)$.
- Show that
$$f(A + \varepsilon B) = A^\intercal A + \varepsilon(A^\intercal B + B^\intercal A) + \varepsilon^2 B^\intercal B - \mathrm{id}_n$$

- Conclude that the derivative of f at A is the map given by

$$B \mapsto A^\intercal B + B^\intercal A$$

- Let $A \in O(n)$. Show that if $C \in \mathcal{S}_n$ and $B = AC/2$, then $Df(A)(B) = C$.
- Conclude that the derivative of f is surjective at A.
- Conclude that 0 is a regular value of the mapping f.

Proof. As per Guillemin and Haine (2018, p. 100), the set $\mathcal{M}_n$ of all $n \times n$ matrices is isomorphic to $\mathbb{R}^{n^2}$ (one degree of freedom for each matrix element), and the set $\mathcal{S}_n$ of all symmetric $n \times n$ matrices is isomorphic to $\mathbb{R}^{n(n+1)/2}$ (one degree of freedom for each matrix element in the upper triangle). Additionally,

$$\begin{aligned} n^2 - \frac{n(n+1)}{2} &= n^2 - \frac{1}{2}n^2 - \frac{1}{2}n \\ &= \frac{1}{2}n^2 - \frac{1}{2}n \\ &= \frac{n(n-1)}{2} \end{aligned}$$

To prove that $O(n)$ is an $n(n-1)/2$-manifold, Theorem 4.1.7 tells us that it will suffice to find a function $f : \mathcal{M}_n \to \mathcal{S}_n$ with regular value 0 such that $O(n) = f^{-1}(0)$.

We first define a function f that we will prove fits all of the above requirements. Let f be described by the relation

$$A \mapsto A^\intercal A - \mathrm{id}_n$$

By the properties of matrix multiplication, $A^\intercal A \in \mathcal{S}_n$ regardless of whether or not A is. Since $\mathcal{S}_n$ is a vector space, subtracting $\mathrm{id}_n \in \mathcal{S}_n$ will not take the difference out of $\mathcal{S}_n$. Thus, f does map arbitrary $n \times n$ matrices to symmetric $n \times n$ matrices, as desired. Moreover, if $A \in O(n)$, then $A^\intercal A = \mathrm{id}_n$. It follows that

$$\begin{aligned} f(A) &= A^\intercal A - \mathrm{id}_n \\ &= \mathrm{id}_n - \mathrm{id}_n \\ &= 0 \end{aligned}$$

$A \notin O(n)$ implies a similar result. Therefore, $O(n) = f^{-1}(0)$.

We now build up to proving that 0 is a regular value of f. To prove this, we will need to check that f is a submersion at all $A \in O(n) = f^{-1}(0)$, i.e., that $Df(A)$ is surjective for all such A. To confirm this, we will calculate $Df(A)$ for an arbitrary $A \in O(n)$ and show directly that for all $C \in \mathcal{S}_n$, there exists $B \in \mathcal{M}_n$ such that $Df(A)(B) = C$. Let's begin.

We have from first principles that

$$0 = \lim_{H \to 0} \frac{|f(A+H) - f(A) - Df(A)(H)|}{|H|}$$

where we take $|\cdot|$ to be any matrix norm (e.g., the operator norm or the Frobenius norm). If we take $H = \varepsilon B$, where $\varepsilon \in \mathbb{R}_{>0}$, then we can work with the limit definition of the derivative more easily. First off, we can determine that

$$\begin{aligned} f(A + \varepsilon B) &= (A + \varepsilon B)^\intercal (A + \varepsilon B) - \mathrm{id}_n \\ &= A^\intercal A + A^\intercal (\varepsilon B) + (\varepsilon B)^\intercal A + (\varepsilon B)^\intercal (\varepsilon B) - \mathrm{id}_n \\ &= A^\intercal A + \varepsilon (A^\intercal B + B^\intercal A) + \varepsilon^2 B^\intercal B - \mathrm{id}_n \end{aligned}$$

Plugging this back into the limit definition, we have that

$$\begin{aligned}
0 &= \lim_{H\to 0} \frac{|f(A+H)-f(A)-Df(A)(H)|}{|H|} \\
&= \lim_{\varepsilon\to 0} \frac{|[A^\intercal A+\varepsilon(A^\intercal B+B^\intercal A)+\varepsilon^2 B^\intercal B-\mathrm{id}_n]-[A^\intercal A-\mathrm{id}_n]-Df(A)(\varepsilon B)|}{|\varepsilon B|} \\
&= \lim_{\varepsilon\to 0} \frac{|[\varepsilon(A^\intercal B+B^\intercal A)+\varepsilon^2 B^\intercal B]-Df(A)(\varepsilon B)|}{|\varepsilon B|} \\
&= \lim_{\varepsilon\to 0} \frac{\varepsilon|(A^\intercal B+B^\intercal A)+\varepsilon B^\intercal B-Df(A)(B)|}{\varepsilon|B|} \\
&= \lim_{\varepsilon\to 0} \frac{|(A^\intercal B+B^\intercal A)+\varepsilon B^\intercal B-Df(A)(B)|}{|B|}
\end{aligned}$$

From here, it is easy to see that if we let $Df(A)$ send

$$B \mapsto A^\intercal B + B^\intercal A$$

then the above limit evaluates to 0, as desired.

Let $A \in O(n)$ be arbitrary, and let $C \in \mathcal{S}_n$ be arbitrary. We want to find $B \in \mathcal{M}_n$ such that $Df(A)(B) = C$. Choose $B = AC/2$. Then

$$\begin{aligned}
Df(A)(B) &= A^\intercal B + B^\intercal A \\
&= \frac{1}{2}[A^\intercal AC + (AC)^\intercal A] \\
&= \frac{1}{2}[A^\intercal AC + C^\intercal A^\intercal A] \\
&= \frac{1}{2}[\mathrm{id}_n C + C\,\mathrm{id}_n] \\
&= C
\end{aligned}$$

as desired. ☐

4.2.i. What is the tangent space to the quadric

$$Q = \{(x_1,\dots,x_n)\in\mathbb{R}^n \mid x_n = x_1^2+\cdots+x_{n-1}^2\}$$

at the point $(1,0,\dots,0,1)$?

Proof. Let $f:\mathbb{R}^{n-1}\to\mathbb{R}$ be defined by

$$(x_1,\dots,x_{n-1}) \mapsto x_1^2+\cdots+x_{n-1}^2$$

From here, elementary set theory can demonstrate that $Q=\Gamma_f$. It follows by Example 4.1.4(1) that Q is an $(n-1)$-manifold in $\mathbb{R}^n$, and $\phi:\mathbb{R}^{n-1}\to\mathbb{R}^n$ defined by $x\mapsto(x,f(x))$ is a parametrization of Q at p for all $p\in Q$.

We can calculate that

$$D\phi(x) = \begin{bmatrix}
\frac{\partial\phi_1}{\partial x_1} & \frac{\partial\phi_1}{\partial x_2} & \cdots & \frac{\partial\phi_1}{\partial x_{n-1}} \\
\frac{\partial\phi_2}{\partial x_1} & \frac{\partial\phi_2}{\partial x_2} & \cdots & \frac{\partial\phi_2}{\partial x_{n-1}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial\phi_{n-1}}{\partial x_1} & \frac{\partial\phi_{n-1}}{\partial x_2} & \cdots & \frac{\partial\phi_{n-1}}{\partial x_{n-1}} \\
\frac{\partial\phi_n}{\partial x_1} & \frac{\partial\phi_n}{\partial x_2} & \cdots & \frac{\partial\phi_n}{\partial x_{n-1}}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
2x_1 & 2x_2 & \cdots & 2x_{n-1}
\end{bmatrix}$$

Now let $p = (1, 0, \dots, 0, 1)$, and let $q = \phi^{-1}(p) = (1, 0, \dots, 0)$. Then

$$D\phi(q) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 2 & 0 & \cdots & 0 \end{bmatrix}$$

so that if $v = (v_1, \dots, v_{n-1}) \in \mathbb{R}^{n-1}$ is arbitrary, then

$$D\phi(q)(v) = \begin{bmatrix} v_1 \\ \vdots \\ v_{n-1} \\ 2v_1 \end{bmatrix}$$

This combined with the fact that $\mathrm{d}\phi_q : T_q\mathbb{R}^{n-1} \to T_p\mathbb{R}^n$ is defined by $(q, v) \mapsto (p, Df(q)(v))$ shows that

$$T_pQ = \operatorname{im}(\mathrm{d}\phi_q)$$

$$\boxed{T_pQ = \operatorname{span}\left\{\left(p, \begin{bmatrix} v_1 \\ \vdots \\ v_{n-1} \\ 2v_1 \end{bmatrix}\right)\right\}}$$

over all $(v_1, \dots, v_{n-1}) \in \mathbb{R}^{n-1}$. This should also make intuitive sense. At $(1, 0, \dots, 0)$, the quadric is changing, but only in the x_1-direction, and its slope there in that direction should be $2q_1 = 2$. The slope is not changing in any of the other directions, so those components of the tangent vector should be mapped by the identity function, as they are. □

4.2.ii. Show that the tangent space to the $(n-1)$-sphere S^{n-1} at p is the space of vectors $(p, v) \in T_p\mathbb{R}^n$ satisfying $p \cdot v = 0$.

Proof. Let $p = (p_1, \dots, p_n) \in S^{n-1}$ be arbitrary. We first define the requisite diffeomorphism.

Adapting Example 4.1.4(6) from Guillemin and Haine (2018, p. 98), we know that we can easily define a diffeomorphism ϕ (see below for details) from a subset of $\mathbb{R}^{n-1}$ to the portion of S^{n-1} lying in the positive half-space above the hyperplane $x_n = 0$. But what if p lies outside this positive half-space? Well, we are helped by the fact that if $p \in S^{n-1}$, some p_i is nonzero. Thus, we can take p to lie in the region of S^{n-1} either above or below the hyperplane $x_i = 0$, and a simple isomorphism of $\mathbb{R}^n$ that, in particular, sends this region of S^{n-1} to the region of S^{n-1} above the hyperplane $x_n = 0$ is, if p lies above $x_i = 0$, the coordinate exchange function $f_\sigma : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$(x_1, \dots, x_n) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

where $\sigma = \tau_{i,n}$ and, if p lies below $x_i = 0$, the coordinate exchange function $-f_\sigma$. Thus, for p arbitrary, our complete diffeomorphism is $\pm f_\sigma \circ \phi$.

We now define ϕ. Let U be the open unit ball centered at the origin in $\mathbb{R}^{n-1}$. Let V be the half-space of $\mathbb{R}^n$ above the hyperplane $x_n = 0$ (i.e., all points $(x_1, \dots, x_n) \in \mathbb{R}^n$ such that $x_n > 0$). Then, as described above, $S^{n-1} \cap V$ is the portion of S^{n-1} lying above the hyperplane $x_n = 0$. The diffeomorphism $\phi : U \to S^{n-1} \cap V$ which projects each point in U "up" onto the surface of the hypersphere is given by

$$(x_1, \dots, x_{n-1}) \mapsto \left(x_1, \dots, x_{n-1}, \sqrt{1 - \left(x_1^2 + \cdots + x_{n-1}^2\right)}\right)$$

We now divide into two cases (the needed diffeomorphism is $f_\sigma \circ \phi$, and the needed diffeomorphism is $-f_\sigma \circ \phi$). Note that the proof of the second case is entirely symmetric to that of the first case, and thus will not be discussed further.

Let $r = f_\sigma^{-1}(p)$ and let $q = (f_\sigma \circ \phi)^{-1}(p)$. We now define $\mathrm{d}(f_\sigma \circ \phi)_q$. First off, by the chain rule,

$$\mathrm{d}(f_\sigma \circ \phi)_q = \mathrm{d}(f_\sigma)_r \circ \mathrm{d}\phi_q$$

Additionally, we know that in general,

$$Df_\sigma(x) = \begin{bmatrix} \frac{\partial (f_\sigma)_1}{\partial x_1} & \cdots & \frac{\partial (f_\sigma)_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial (f_\sigma)_n}{\partial x_1} & \cdots & \frac{\partial (f_\sigma)_n}{\partial x_n} \end{bmatrix} \qquad D\phi(x) = \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \cdots & \frac{\partial \phi_1}{\partial x_{n-1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_{n-1}}{\partial x_1} & \cdots & \frac{\partial \phi_{n-1}}{\partial x_{n-1}} \\ \frac{\partial \phi_n}{\partial x_1} & \cdots & \frac{\partial \phi_n}{\partial x_{n-1}} \end{bmatrix}$$

$$= P_\sigma \qquad\qquad = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \\ \frac{-x_1}{\sqrt{1-(x_1^2+\cdots+x_{n-1}^2)}} & \cdots & \frac{-x_{n-1}}{\sqrt{1-(x_1^2+\cdots+x_{n-1}^2)}} \end{bmatrix}$$

where P_σ is the permutation matrix which differs from the identity in that its i^{th} and n^{th} columns are interchanged. It follows that

$$T_pS^{n-1} = \operatorname{span}\left\{ \left(p, P_\sigma \begin{bmatrix} w_1 \\ \vdots \\ w_{n-1} \\ \frac{-q_1 w_1 - \cdots - q_{n-1} w_{n-1}}{\sqrt{1-(q_1^2+\cdots+q_{n-1}^2)}} \end{bmatrix} \right) \right\}$$

for $(w_1, \dots, w_{n-1}) \in U$.

We now use a bidirectional inclusion argument to complete the proof.

Let $(p, v) \in T_pS^{n-1}$ be arbitrary. Then some $p_i \neq 0$. It follows that

$$\begin{aligned}
(f_\sigma \circ \phi)(q_1, \dots, q_{n-1}) &= f_\sigma(\phi(q_1, \dots, q_{n-1})) \\
&= f_\sigma\left(q_1, \dots, q_{n-1}, \sqrt{1 - (q_1^2 + \cdots + q_{n-1}^2)}\right) \\
&= \left(q_1, \dots, q_{i-1}, \sqrt{1 - (q_1^2 + \cdots + q_{n-1}^2)}, q_{i+1}, \dots, q_{n-1}, q_i\right) \\
&= (p_1, \dots, p_{i-1}, p_i, p_{i+1}, \dots, p_{n-1}, p_n) \\
&= p
\end{aligned}$$

Thus, we have that

$$\begin{aligned}
p \cdot v &= p_1 v_1 + \cdots + p_n v_n \\
&= q_1 w_1 + \cdots + q_{i-1} w_{i-1} + \sqrt{1 - (q_1^2 + \cdots + q_{n-1}^2)} \cdot \frac{-q_1 w_1 - \cdots - q_{n-1} w_{n-1}}{\sqrt{1 - (q_1^2 + \cdots + q_{n-1}^2)}} \\
&\quad + q_{i+1} w_{i+1} + \cdots + q_{n-1} w_{n-1} + q_i w_i \\
&= 0
\end{aligned}$$

as desired.

Now suppose that $(p,v) \in T_p\mathbb{R}^n$ is such that $p \cdot v = 0$. Then

$$\begin{aligned}
0 &= p \cdot v \\
&= p_1v_1 + \cdots + p_nv_n \\
&= q_1v_1 + \cdots + q_{i-1}v_{i-1} + \sqrt{1-\left(q_1^2+\cdots+q_{n-1}^2\right)} \cdot v_i + q_{i+1}v_{i+1} + \cdots + q_nv_n \\
\sqrt{1-\left(q_1^2+\cdots+q_{n-1}^2\right)} \cdot v_i &= -q_1v_1 - \cdots - q_{i-1}v_{i-1} - q_{i+1}v_{i+1} + \cdots + q_nv_n \\
v_i &= \frac{-q_1v_1 - \cdots - q_{i-1}v_{i-1} - q_{i+1}v_{i+1} + \cdots + q_nv_n}{\sqrt{1-\left(q_1^2+\cdots+q_{n-1}^2\right)}}
\end{aligned}$$

so with some reindexing, v fits the form of the vector in the span defining T_pS^{n-1}, as desired. □

4.2.iii. Let $f : \mathbb{R}^n \to \mathbb{R}^k$ be a C^∞ map and let $X = \Gamma_f$. What is the tangent space to X at $(a, f(a))$?

Proof. As per Example 4.1.4(1), $\phi : \mathbb{R}^n \to \mathbb{R}^{n+k}$ defined by

$$x \mapsto (x, f(x))$$

is a suitable diffeomorphism for all $p \in X$. It follows that $D\phi(x)$ is an $(n+k) \times n$ matrix where the top $n \times n$ matrix is id_n and the bottom $k \times n$ matrix is $Df(x)$. Let $p = (a, f(a))$. Then

$$\boxed{T_pX = \operatorname{span}\left\{\left(p, \begin{bmatrix} v_1 \\ \vdots \\ v_n \\ \sum_{i=1}^n \left.\frac{\partial f_1}{\partial x_i}\right|_a v_i \\ \vdots \\ \sum_{i=1}^n \left.\frac{\partial f_k}{\partial x_i}\right|_a v_i \end{bmatrix}\right)\right\}}$$

for $(v_1, \dots, v_n) \in \mathbb{R}^n$. □

4.2.iv. Let $\sigma : S^{n-1} \to S^{n-1}$ be the antipodal map $\sigma(x) = -x$. What is the derivative of σ at $p \in S^{n-1}$?

Proof. Let $\tilde{\sigma} : \mathbb{R}^n \to \mathbb{R}^n$ be the extension of the antipodal map to $\mathbb{R}^n$. Then $D\tilde{\sigma}(x) = -\mathrm{id}_n$. It follows that the derivative of σ at any $p \in S^{n-1}$ is the map $\mathrm{d}\sigma_p : T_pS^{n-1} \to T_{-p}S^{n-1}$ defined by

$$\boxed{\mathrm{d}\sigma_p(p,v) = (-p,-v)}$$

□

4.2.v. Let $X_i \subset \mathbb{R}^{N_i}$ $(i = 1, 2)$ be an n_i-manifold and let $p_i \in X_i$. Define X to be the Cartesian product

$$X_1 \times X_2 \subset \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$$

and let $p = (p_1, p_2)$. Show that $T_pX \cong T_{p_1}X_1 \oplus T_{p_2}X_2$.

Proof. Let $f : T_pX \to T_{p_1}X_1 \oplus T_{p_2}X_2$ be defined by

$$(p, v) \mapsto ((p_1, v_1), (p_2, v_2))$$

We can check componentwise that f is bijective, as desired. □

6 Calculus on Manifolds

From Guillemin and Haine (2018).

Chapter 4

5/29: **4.3.ii.** Let S^2 be the unit 2-sphere $x_1^2 + x_2^2 + x_3^2 = 1$ in $\mathbb{R}^3$ and let $\boldsymbol{w}$ be the vector field

$$\boldsymbol{w} = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}$$

(1) Show that $\boldsymbol{w}$ is tangent to S^2 and hence by restriction defines a vector field $\boldsymbol{v}$ on S^2.

Proof. To show that $\boldsymbol{w}$ is tangent to S^2, it will suffice to prove that for every $p \in S^2$, $\boldsymbol{w}(p) \in T_pS^2$. Let $p = (p_1, p_2, p_3) \in S^2$ be arbitrary. To prove that $\boldsymbol{w}(p) \in T_pS^2$, Exercise 4.2.ii tells us that it will suffice to check that $p \cdot \boldsymbol{w}(p) = 0$. Indeed, we have that

$$p \cdot \boldsymbol{w}(p) = (p_1)(-p_2) + (p_2)(p_1) + (p_3)(0) = 0$$

as desired. ☐

(2) What are the integral curves of $\boldsymbol{v}$?

Proof. Let $p = (p_1, p_2, p_3) \in S^2$ be arbitrary. Then

$$\boldsymbol{v}(p) = -p_2 \frac{\partial}{\partial x_1} + p_1 \frac{\partial}{\partial x_2} + 0 \frac{\partial}{\partial x_3}$$

Suppose there exists $\gamma : I \to S^2$, where I is an open interval containing $t_0 = 0$, such that $\gamma(0) = p$ and $\mathrm{d}\gamma_0(\vec{u}) = \boldsymbol{v}(p)$. It follows from the second statement that

$$
\begin{aligned}
(p, D\gamma(0)(1)) &= (p, (-p_2, p_1, 0)) \\
\begin{bmatrix} \left.\frac{\mathrm{d}\gamma_1}{\mathrm{d}t}\right|_0 \\ \left.\frac{\mathrm{d}\gamma_2}{\mathrm{d}t}\right|_0 \\ \left.\frac{\mathrm{d}\gamma_3}{\mathrm{d}t}\right|_0 \end{bmatrix} &= \begin{bmatrix} -\gamma_2(0) \\ \gamma_1(0) \\ 0 \end{bmatrix} \\
\begin{bmatrix} \frac{\mathrm{d}\gamma_1}{\mathrm{d}t} \\ \frac{\mathrm{d}\gamma_2}{\mathrm{d}t} \\ \frac{\mathrm{d}\gamma_3}{\mathrm{d}t} \end{bmatrix} &= \begin{bmatrix} -\gamma_2 \\ \gamma_1 \\ 0 \end{bmatrix}
\end{aligned}
$$

From the last line above, we can determine that

$$\gamma_3(t) = C$$

for some $C \in \mathbb{R}$. We can also extract the coupled differential equations

$$\frac{\mathrm{d}\gamma_1}{\mathrm{d}t} = -\gamma_2 \qquad\qquad \frac{\mathrm{d}\gamma_2}{\mathrm{d}t} = \gamma_1$$

Differentiating the right equation above with respect to t reveals via the transitive property that

$$\frac{\mathrm{d}^2\gamma_2}{\mathrm{d}t^2} = \frac{\mathrm{d}\gamma_1}{\mathrm{d}t} = -\gamma_2$$

We can recognize the above differential equation to be the one describing simple harmonic motion, i.e., the one having general solution

$$\gamma_2(t) = A\mathrm{e}^{irt} + B\mathrm{e}^{-irt}$$

for some $A, B, r \in \mathbb{R}$. It follows since $\mathrm{d}\gamma_2/\mathrm{d}t = \gamma_1$ that

$$\gamma_1(t) = Aire^{irt} - Bire^{-irt}$$

We now solve for A, B, C, r using the initial conditions

$$\begin{bmatrix}\gamma_1(0)\\\gamma_2(0)\\\gamma_3(0)\end{bmatrix} = \begin{bmatrix}p_1\\p_2\\p_3\end{bmatrix} \qquad \begin{bmatrix}\gamma_1'(0)\\\gamma_2'(0)\\\gamma_3'(0)\end{bmatrix} = \begin{bmatrix}-p_2\\p_1\\0\end{bmatrix}$$

First off, we have that

$$C = \gamma_3(0) = p_3$$

We also have that

$$\begin{aligned} p_2 &= \gamma_2(0) \\ &= Ae^{ir\cdot 0} + Be^{-ir\cdot 0} \\ &= A + B \end{aligned} \qquad \begin{aligned} -p_2 &= \gamma_1'(0) \\ &= -Ar^2e^{ir\cdot 0} - Br^2e^{-ir\cdot 0} \\ &= -r^2(A+B) \end{aligned}$$

It follows that $r = 1$. Additionally, we have that

$$\begin{aligned} p_1 &= \gamma_1(0) \\ &= Aie^{i\cdot 0} - Bie^{-i\cdot 0} \\ &= (A-B)i \end{aligned}$$

Thus, we have the system of equations

$$\begin{aligned} A + B &= p_2 \\ iA - iB &= p_1 \end{aligned}$$

We can solve it to determine that

$$A = \frac{p_2 - ip_1}{2} \qquad B = \frac{p_2 + ip_1}{2}$$

It follows that

$$\begin{aligned} \gamma_2(t) &= Ae^{it} + Be^{-it} \\ &= \frac{p_2 - ip_1}{2}(\cos(t) + i\sin(t)) + \frac{p_2 + ip_1}{2}(\cos(t) - i\sin(t)) \\ &= p_2\cos(t) + p_1\sin(t) \end{aligned}$$

and thus that

$$\gamma_1(t) = p_1\cos(t) - p_2\sin(t)$$

Finally, we have a complete description of the integral curve $\gamma : \mathbb{R} \to S^2$ with $\gamma(0) = p$ of $\boldsymbol{v}$.

$$\boxed{\gamma(t) = \begin{bmatrix} p_1\cos(t) - p_2\sin(t) \\ p_2\cos(t) + p_1\sin(t) \\ p_3 \end{bmatrix}}$$

Note that we can do one better by transforming our three equations into a form that could easily be found by inspection.

By inspecting the vector field, we can determine that the lack of x_3-component in any vector in $\boldsymbol{v}$ means that our integral curve must not vary with respect to x_3 either, i.e., γ_3 should just be the height of the integral curve above or below the x_1x_2-plane, i.e., p_3 (as it is via the above). On the other hand, γ_1, γ_2 should work together (this is why we got the *coupled* system of differential equations above) to trace out a latitude line, i.e., a circular submanifold at p_3 above or below the x_1x_2-plane. Thus, they should be sinusoidal, with radius dictated by the height above the x_1x_2-plane and phase offset dictated by the position of p relative to the $+x_1$-axis. Since $x_1^2 + x_2^2 + x_3^1 = 1$, trigonometric arguments show that the radius of this circle should be $\sqrt{1 - x_3^2}$, i.e., the maximum possible radius ($\sqrt{1} = 1$) less come correction factor based on the distance between the circle and the x_1x_2-plane. Similarly, trigonometric arguments show that the phase offset should be $\tan^{-1}(p_2/p_1)$. Thus, the final form should be

$$\gamma_1(t) = \sqrt{1 - x_3^2}\cos\left(t + \tan^{-1}\left(\frac{p_2}{p_1}\right)\right) \qquad \gamma_2(t) = \sqrt{1 - x_3^2}\sin\left(t + \tan^{-1}\left(\frac{p_2}{p_1}\right)\right)$$

But since $a\cos x + b\sin x$ can be written as $R\cos(x - \alpha)$, where $R = \sqrt{a^2 + b^2}$ and $\tan\alpha = b/a$, we have that

$$\begin{aligned}
\gamma_1(t) &= p_1\cos(t) - p_2\sin(t) & \gamma_2(t) &= p_2\cos(t) + p_1\sin(t) \\
&= \sqrt{p_1^2 + p_2^2}\cos\left(t - \tan^{-1}\left(\frac{-p_2}{p_1}\right)\right) & &= \sqrt{p_1^2 + p_2^2}\cos\left(t - \tan^{-1}\left(\frac{p_1}{p_2}\right)\right) \\
&= \sqrt{1 - p_3^2}\cos\left(t + \tan^{-1}\left(\frac{p_2}{p_1}\right)\right) & &= \sqrt{1 - p_3^2}\sin\left(t + \tan^{-1}\left(\frac{p_2}{p_1}\right)\right) \\
&= \gamma_1(t) & &= \gamma_2(t)
\end{aligned}$$

as desired. $\square$

Note from Dr. Klug: For the following problems you'll need to check back in Section 1.9 and maybe warm-up with the pointwise Exercise 1.9.xi. We are going to define a special top-dimensional form on a manifold (which for us is always inside a Euclidean space — more generally this is where you would need to "fix a metric on your abstract manifold"... blah blah blah) called the (Riemannian) volume form — see Theorem 4.4.9 in your book. An (admittedly fancy but agreeing with any pedestrian way of doing it) definition of the volume of a subset of $\mathbb{R}^N$ that is a manifold is then the integral over that manifold of the volume form.

Chapter 1

1.9.xi. Let V be an n-dimensional vector space $B : V \times V \to \mathbb{R}$ an inner product, and $e_1, \dots, e_n$ a basis of V which is positively oriented and orthonormal. Show that the **volume element**

$$\text{vol} = e_1^* \wedge \cdots \wedge e_n^* \in \Lambda^n(V^*)$$

is intrinsically defined, independent of the choice of basis. (Hint: The equations

$$AA^\intercal = \text{id}_n \qquad\qquad A^*(f_1^* \wedge \cdots \wedge f_n^*) = \det(a_{i,j})e_1^* \wedge \cdots \wedge e_n^*$$

may be of use. Note that in the left equation above, A is a change of coordinate matrix between two orthonormal bases.)

Proof. To show that the volume element is defined independently of the choice of basis, let $e_1', \dots, e_n'$ be another basis of V which is positively oriented and orthonormal; we wish to verify that the volume element

$$\text{vol}' = {e_1'}^* \wedge \cdots \wedge {e_n'}^*$$

with respect to this basis is equal to vol. Let A be the change of coordinates matrix which sends $e_i \mapsto e_i'$ ($i = 1, \dots, n$). It follows that $AA^\intercal = \text{id}_n$, so

$$\begin{aligned}
\det(\text{id}_n) &= \det(AA^\intercal) \\
1 &= \det(A) \cdot \det(A^\intercal) \\
&= \det(A) \cdot \det(A) \\
&= \det(A)^2 \\
\det(A) &= \pm 1
\end{aligned}$$

Additionally, we know that both $e_1, \ldots, e_n$ and $e'_1, \ldots, e'_n$ are positively oriented, so by Proposition 1.9.7, $\det(A) > 0$. Thus, $\det(A) = 1$. Therefore, we have that

$$\begin{aligned}
\mathrm{vol} &= e_1^* \wedge \cdots \wedge e_n^* \\
&= 1 \cdot e_1^* \wedge \cdots \wedge e_n^* \\
&= \det(a_{i,j}) e_1^* \wedge \cdots \wedge e_n^* \\
&= A^*({e'_1}^* \wedge \cdots \wedge {e'_n}^*) \\
&= \det(A) {e'_1}^* \wedge \cdots \wedge {e'_n}^* \\
&= 1 \cdot {e'_1}^* \wedge \cdots \wedge {e'_n}^* \\
&= {e'_1}^* \wedge \cdots \wedge {e'_n}^* \\
&= \mathrm{vol}'
\end{aligned}$$

as desired. □

Chapter 4

4.4.i. Let V be an oriented n-dimensional vector space, B an inner product on V, and $e_1, \ldots, e_n \in V$ an oriented orthonormal basis. Given vectors $v_1, \ldots, v_n \in V$, show that if

$$b_{i,j} = B(v_i, v_j) \qquad\qquad v_i = \sum_{j=1}^{n} a_{j,i} e_j$$

the matrices $\mathbf{A} = (a_{i,j})$ and $\mathbf{B} = (b_{i,j})$ satisfy the identity

$$\mathbf{B} = \mathbf{A}^\intercal \mathbf{A}$$

and conclude that $\det(\mathbf{B}) = \det(\mathbf{A})^2$. (In particular, conclude that $\det(\mathbf{B}) > 0$ if $v_1, \ldots, v_n$ are linearly independent.)

Proof. To prove that $\mathbf{B} = \mathbf{A}^\intercal \mathbf{A}$, it will suffice to show that the corresponding entries of each matrix are equal. By the rules of matrix multiplication, we have that

$$(a^\intercal a)_{i,j} = \sum_{k=1}^{n} a_{k,i} a_{k,j}$$

It follows that

$$\begin{aligned}
b_{i,j} &= B(v_i, v_j) \\
&= B\left(\sum_{k=1}^{n} a_{k,i} e_k, \sum_{k'=1}^{n} a_{k',j} e_{k'} \right) \\
&= \sum_{k=1}^{n} \sum_{k'=1}^{n} a_{k,i} a_{k',j} B(e_k, e_{k'}) \\
&= \sum_{k=1}^{n} \sum_{k'=1}^{n} a_{k,i} a_{k',j} \delta_{k,k'} \\
&= \sum_{k=1}^{n} a_{k,i} a_{k,j} \\
&= (a^\intercal a)_{i,j}
\end{aligned}$$

as desired.

It follows that

$$\det(\mathbf{B}) = \det(\mathbf{A}^\intercal \mathbf{A}) = \det(\mathbf{A}^\intercal) \det(\mathbf{A}) = \det(\mathbf{A}) \det(\mathbf{A}) = \det(\mathbf{A})^2$$

Therefore, if $v_1, \ldots, v_n$ are linearly independent, then $\det(\mathbf{A}) \neq 0$, so $\det(\mathbf{B}) > 0$, as desired. □

4.4.ii. Let V, W be oriented n-dimensional vector spaces. Suppose that each of these spaces is equipped with an inner product, and let $e_1, \dots, e_n \in V$ and $f_1, \dots, f_n \in W$ be oriented orthonormal bases. Show that if $A : W \to V$ is an orientation preserving linear mapping and $Af_i = v_i$, then

$$A^* \operatorname{vol}_V = (\det(b_{i,j}))^{1/2} \operatorname{vol}_W$$

where $\operatorname{vol}_V = e_1^* \wedge \cdots \wedge e_n^*$, $\operatorname{vol}_W = f_1^* \wedge \cdots \wedge f_n^*$, and $(b_{i,j})$ is the matrix described in Exercise 4.4.i.

Proof. We have that

$$\begin{aligned} A^* \operatorname{vol}_V &= A^*(e_1^* \wedge \cdots \wedge e_n^*) \\ &= \det(a_{i,j})(f_1^* \wedge \cdots \wedge f_n^*) && \text{Equation 1.8.10} \\ &= \det(a_{i,j}) \operatorname{vol}_W \\ &= (\det(b_{i,j}))^{1/2} \operatorname{vol}_W && \text{Exercise 4.4.i} \end{aligned}$$

as desired. □

4.4.iii. Let X be an oriented n-dimensional submanifold of $\mathbb{R}^n$, U an open subset of X, U_0 an open subset of $\mathbb{R}^n$, and $\phi : U_0 \to U$ an oriented parameterization. Let $\phi_1, \dots, \phi_N$ be the coordinates of the map

$$U_0 \to U \hookrightarrow \mathbb{R}^n$$

the second map being the inclusion map. Show that if σ is the Riemannian volume form on X, then

$$\phi^* \sigma = (\det(\phi_{i,j}))^{1/2} \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

where

$$\phi_{i,j} = \sum_{k=1}^{N} \frac{\partial \phi_k}{\partial x_i} \frac{\partial \phi_k}{\partial x_j}$$

for $1 \leq i, j \leq n$. Conclude that σ is a smooth n-form and hence that it *is* the volume form. (Hint: For $p \in U_0$ and $q = \phi(p)$, apply Exercise 4.4.ii with $V = T_q X$, $W = T_p \mathbb{R}^n$, $A = \mathrm{d}\phi_p$, and $v_i = \mathrm{d}\phi_p(\partial/\partial x_i)_p$.)

Proof. As the Riemannian volume form on X, $\sigma \in \Omega^n(X)$ is the n-form defined by

$$q \mapsto \sigma_q$$

where $\sigma_q = e_1^* \wedge \cdots \wedge e_n^*$, and $e_1, \dots, e_n$ is an orthonormal basis of $T_q X$. To prove the desired form equality, it will suffice to check equality at every point $p \in U_0$. Doing this, we obtain from Exercise 4.4.ii

$$\begin{aligned} (\phi^* \sigma)_p &= \mathrm{d}\phi_p^* \sigma_p \\ &= (\det(\phi_{i,j}))^{1/2} (\operatorname{vol}_{U_0})_p \end{aligned}$$

where

$$\phi_{i,j} = B(v_i, v_j) = B(\mathrm{d}\phi_p(\partial/\partial x_i)_p, \mathrm{d}\phi_p(\partial/\partial x_j)_p) = B\left(\begin{bmatrix} \frac{\partial \phi_1}{\partial x_i} \\ \vdots \\ \frac{\partial \phi_N}{\partial x_i} \end{bmatrix}, \begin{bmatrix} \frac{\partial \phi_1}{\partial x_j} \\ \vdots \\ \frac{\partial \phi_N}{\partial x_j} \end{bmatrix} \right) = \sum_{k=1}^{N} \frac{\partial \phi_k}{\partial x_i} \frac{\partial \phi_k}{\partial x_j}$$

for $1 \leq i, j \leq n$ and

$$(\operatorname{vol}_{U_0})_p = x_1^* \wedge \cdots \wedge x_n^* = (\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n)_p$$

as desired.

Since $\phi^* \sigma$ is clearly a smooth n-form on $\mathbb{R}^n$ by the above equality, σ itself is smooth. It follows since $\phi^* \sigma$ is also non-vanishing and ϕ is orientation preserving that σ itself is nonvanishing and strictly positive; hence σ is, indeed, the volume form. □

4.4.iv. Given a C^∞ function $f : \mathbb{R} \to \mathbb{R}$, its graph $X = \Gamma_f$ is a submanifold of $\mathbb{R}^2$ and $\phi : \mathbb{R} \to X$ defined by

$$x \mapsto (x, f(x))$$

is a diffeomorphism. Orient X by requiring that ϕ be orientation preserving and show that if σ is the Riemannian volume form on X, then

$$\phi^*\sigma = \left(1 + \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2\right)^{1/2} \mathrm{d}x$$

(Hint: See Exercise 4.4.iii.)

Proof. In the language of Exercise 4.4.iii, let $U = X$ and $U_0 = \mathbb{R}$ ($X = \Gamma_f$ and ϕ are already defined in the statement of this Exercise). Then by Exercise 4.4.iii, we have that

$$\begin{aligned}
\phi^*\sigma &= (\det(\phi_{i,j}))^{1/2}\, \mathrm{d}x \\
&= \left(\det\left[\sum_{k=1}^{2} \frac{\mathrm{d}\phi_k}{\mathrm{d}x}\frac{\mathrm{d}\phi_k}{\mathrm{d}x}\right]\right)^{1/2} \mathrm{d}x \\
&= \left(\det\left[1^2 + \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2\right]\right)^{1/2} \mathrm{d}x \\
&= \left(1 + \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^2\right)^{1/2} \mathrm{d}x
\end{aligned}$$

as desired. □

4.4.v. Given a C^∞ function $f : \mathbb{R}^n \to \mathbb{R}$, the graph Γ_f of f is a submanifold of $\mathbb{R}^{n+1}$ and $\phi : \mathbb{R}^n \to X$ defined by

$$x \mapsto (x, f(x))$$

is a diffeomorphism. Orient X by requiring that ϕ is orientation preserving and show that if σ is the Riemannian volume form on X, then

$$\phi^*\sigma = \left(1 + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\right)^2\right)^{1/2} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

Hints:

- Let V be an n-dimensional vector space over the field $\mathbb{F}$, and let $v = (c_1, \dots, c_n) \in V$. Show that if $C : \mathbb{R}^n \to \mathbb{R}^n$ is the linear mapping defined by the $n \times n$ matrix $(c_i c_j)$, i.e., the matrix for which $c_i \cdot c_j$ is the entry in the i^{th} row and j^{th} column for $1 \leq i, j \leq n$, then $Cv = (\sum_{i=1}^{n} c_i^2)v$ and $Cw = 0$ if $w \cdot v = 0$.
- Conclude that the eigenvalues of C are $\lambda_1 = \sum_{i=1}^{n} c_i^2$ and $\lambda_2 = \cdots = \lambda_n = 0$.
- Show that the determinant of $I + C$ is $1 + \sum_{i=1}^{n} c_i^2$.
- Compute the determinant of the matrix $(\phi_{i,j})$ from Exercise 4.4.iii where ϕ is the mapping defined at the beginning of this Exercise.

Proof. We will begin as in Exercise 4.4.iv, motivating why we need to make use of the hints first. Next, we will prove the hints. Lastly, we will tie everything together.

In the language of Exercise 4.4.iii, let $U = X$ and $U_0 = \mathbb{R}^n$ ($X = \Gamma_f$ and ϕ are already defined in the statement of this Exercise). Then by Exercise 4.4.iii, we have that

$$\phi^*\sigma = (\det(\phi_{i,j}))^{1/2}\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

However, calculating $\det(\phi_{i,j})$ is not as easy a task here as it was for the 1×1 matrix in Exercise 4.4.iv. To build up to calculating this quantity, let's investigate $(\phi_{i,j})$ to start. We know from Exercise 4.4.i-4.4.iii (or, alternately, from its definition) that

$$\begin{aligned}
(\phi_{i,j}) &= (D\phi)^{\intercal}(D\phi) \\
&= \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \cdots & \frac{\partial \phi_{n+1}}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_1}{\partial x_n} & \cdots & \frac{\partial \phi_{n+1}}{\partial x_n} \end{bmatrix} \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \cdots & \frac{\partial \phi_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \phi_{n+1}}{\partial x_1} & \cdots & \frac{\partial \phi_{n+1}}{\partial x_n} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{k=1}^{n+1} \frac{\partial \phi_k}{\partial x_1}\frac{\partial \phi_k}{\partial x_1} & \cdots & \sum_{k=1}^{n+1} \frac{\partial \phi_k}{\partial x_1}\frac{\partial \phi_k}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \sum_{k=1}^{n+1} \frac{\partial \phi_k}{\partial x_n}\frac{\partial \phi_k}{\partial x_1} & \cdots & \sum_{k=1}^{n+1} \frac{\partial \phi_k}{\partial x_n}\frac{\partial \phi_k}{\partial x_n} \end{bmatrix}
\end{aligned}$$

Furthermore, from the definition of the coordinates of ϕ, we can determine that

$$\begin{aligned}
\frac{\partial \phi_k}{\partial x_\ell} &= \begin{cases} \frac{\partial \phi_k}{\partial x_k} & k = \ell;\ k \leq n \\ \frac{\partial \phi_k}{\partial x_\ell} & k \neq \ell;\ k \leq n \\ \frac{\partial \phi_k}{\partial x_\ell} & k = n+1 \end{cases} \\
&= \begin{cases} \frac{\partial x_k}{\partial x_k} & k = \ell;\ k \leq n \\ \frac{\partial x_k}{\partial x_\ell} & k \neq \ell;\ k \leq n \\ \frac{\partial f}{\partial x_\ell} & k = n+1 \end{cases} \\
&= \begin{cases} 1 & k = \ell;\ k \leq n \\ 0 & k \neq \ell;\ k \leq n \\ \frac{\partial f}{\partial x_\ell} & k = n+1 \end{cases}
\end{aligned}$$

where 1 denotes the identity function on $\mathbb{R}^n$ and 0 denotes the zero function on $\mathbb{R}^n$. It follows that

$$\begin{aligned}
\phi_{i,j} &= \sum_{k=1}^{n+1} \frac{\partial \phi_k}{\partial x_i}\frac{\partial \phi_k}{\partial x_j} \\
&= \begin{cases} \displaystyle\sum_{k=1}^{n} \frac{\partial x_k}{\partial x_i}\frac{\partial x_k}{\partial x_i} + \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_i} & i = j \\ \displaystyle\sum_{k=1}^{n} \frac{\partial x_k}{\partial x_i}\frac{\partial x_k}{\partial x_j} + \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} & i \neq j \end{cases} \\
&= \begin{cases} \displaystyle\sum_{\substack{k=1 \\ k\neq i,j}}^{n} \frac{\partial x_k}{\partial x_i}\frac{\partial x_k}{\partial x_i} + \frac{\partial x_i}{\partial x_i}\frac{\partial x_i}{\partial x_i} + \frac{\partial x_j}{\partial x_i}\frac{\partial x_j}{\partial x_i} + \left(\frac{\partial f}{\partial x_i}\right)^2 & i = j \\ \displaystyle\sum_{\substack{k=1 \\ k\neq i,j}}^{n} \frac{\partial x_k}{\partial x_i}\frac{\partial x_k}{\partial x_j} + \frac{\partial x_i}{\partial x_i}\frac{\partial x_i}{\partial x_j} + \frac{\partial x_j}{\partial x_i}\frac{\partial x_j}{\partial x_j} + \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} & i \neq j \end{cases} \\
&= \begin{cases} \displaystyle\sum_{\substack{k=1 \\ k\neq i,j}}^{n} 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 0 + \left(\frac{\partial f}{\partial x_i}\right)^2 & i = j \\ \displaystyle\sum_{\substack{k=1 \\ k\neq i,j}}^{n} 0 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 + \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} & i \neq j \end{cases} \\
&= \begin{cases} 1 + \left(\frac{\partial f}{\partial x_i}\right)^2 & i = j \\ \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} & i \neq j \end{cases}
\end{aligned}$$

Thus,

$$
\begin{aligned}
(\phi_{i,j}) &= \begin{bmatrix}
1+\left(\frac{\partial f}{\partial x_1}\right)^2 & \frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_2} & \cdots & \cdots & \frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_n} \\
\frac{\partial f}{\partial x_2}\frac{\partial f}{\partial x_1} & 1+\left(\frac{\partial f}{\partial x_2}\right)^2 & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & 1+\left(\frac{\partial f}{\partial x_{n-1}}\right)^2 & \frac{\partial f}{\partial x_{n-1}}\frac{\partial f}{\partial x_n} \\
\frac{\partial f}{\partial x_n}\frac{\partial f}{\partial x_1} & \cdots & \cdots & \frac{\partial f}{\partial x_n}\frac{\partial f}{\partial x_{n-1}} & 1+\left(\frac{\partial f}{\partial x_n}\right)^2
\end{bmatrix} \\
&= \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} + \begin{bmatrix} \frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_1}\frac{\partial f}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f}{\partial x_n}\frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n}\frac{\partial f}{\partial x_n} \end{bmatrix} \\
&= I + (c_i c_j)
\end{aligned}
$$

where in the last step we have defined c_i to be the function $\partial f/\partial x_i$ $(i = 1, \ldots, n)$ and $(c_i c_j)$ to be the matrix which has entry $c_i c_j$ in the i^{th} row and j^{th} column. The reason for introducing the c_i nomenclature is purely for notational simplicity.

The reason why we need to prove the hints should now be clear: Rather than using the computationally complex permutation definition of the determinant to evaluate $\det(\phi_{i,j})$, we can use the computationally simple method laid out in the hints to calculate $\det(I + (c_i c_j))$ and then return the substitution $c_i = \partial f/\partial x_i$ to get our final answer. Let's now prove the hints.

Using the definitions provided in the hint, we have that

$$
\begin{aligned}
Cv &= \begin{bmatrix} c_1c_1 & \cdots & c_1c_n \\ \vdots & \ddots & \vdots \\ c_nc_1 & \cdots & c_nc_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \\
&= \begin{bmatrix} \left(\sum_{i=1}^n c_i^2\right) c_1 \\ \vdots \\ \left(\sum_{i=1}^n c_i^2\right) c_n \end{bmatrix} \\
&= \left(\sum_{i=1}^n c_i^2\right) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}
\end{aligned}
$$

as desired. Now suppose that $w \in V$ satisfies $w \cdot v = 0$. Then

$$
\begin{aligned}
Cw &= \begin{bmatrix} c_1c_1 & \cdots & c_1c_n \\ \vdots & \ddots & \vdots \\ c_nc_1 & \cdots & c_nc_n \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \\
&= \begin{bmatrix} \left(\sum_{i=1}^n w_i c_i\right) c_1 \\ \vdots \\ \left(\sum_{i=1}^n w_i c_i\right) c_n \end{bmatrix} \\
&= (w \cdot v) \cdot \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \\
&= 0
\end{aligned}
$$

as desired.

Consider the list of vectors $\{v\} \subset V$. Extend it to an orthogonal basis $\{v, w_2, \dots, w_n\}$ of V. Then $w_i \cdot v = 0$ $(i = 2, \dots, n)$. It therefore follows from the above that $Cv = (\sum_{i=1}^n c_i^2)v$ and $Cw_i = 0w_i$ $(i = 2, \dots, n)$. Thus, by the definition of eigenvalues, the eigenvalues of C are $\lambda_1 = \sum_{i=1}^n c_i^2$ and $\lambda_2 = \cdots = \lambda_n = 0$, as desired.

The matrix of the identity function I is identical with respect to every basis of V. Thus, the matrix $\mathcal{M}(I+C)$ of $I+C$ with respect to the basis $\{v, w_2, \dots, w_n\}$ is

$$\begin{aligned} \mathcal{M}(I+C) &= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^n c_i^2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 + \sum_{i=1}^n c_i^2 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \end{aligned}$$

It follows by reading down the diagonal of $\mathcal{M}(I+C)$ (a diagonal matrix) that the eigenvalues of $I+C$ are $\lambda_1 = 1 + \sum_{i=1}^n c_i^2$ and $\lambda_2 = \cdots = \lambda_n = 1$. Thus, since the determinant of a linear transformation is equal to the product of its eigenvalues, we have that

$$\begin{aligned} \det(I+C) &= \prod_{i=1}^n \lambda_i \\ &= \left(1 + \sum_{i=1}^n c_i^2\right) \cdot \prod_{i=2}^n 1 \\ &= 1 + \sum_{i=1}^n c_i^2 \end{aligned}$$

as desired.

Thus, we have that

$$\det(\phi_{i,j}) = 1 + \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}\right)^2$$

via returning the substitution $c_i = \partial f/\partial x_i$ [2]

Therefore, tying everything together, we have that

$$\begin{aligned} \phi^*\sigma &= (\det(\phi_{i,j}))^{1/2}\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \\ &= \left(1 + \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i}\right)^2\right)^{1/2} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \end{aligned}$$

as desired. □

4.4.vii. Let U be an open subset of $\mathbb{R}^N$ and $f : U \to \mathbb{R}^k$ a C^∞ map. If zero is a regular value of f, the set $X = f^{-1}(0)$ is a manifold of dimension $n = N - k$. Show that this manifold has a natural smooth orientation. *Some suggestions*:

- Let $f = (f_1, \dots, f_k)$ and let
$$\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k = \sum f_I\, \mathrm{d}x_I$$
where the summation is taken over all strictly increasing multi-indices of N of length k. Show that for every $p \in X$, $f_I(p) \neq 0$ for some strictly increasing multi-index of N of length k.

[2] Strictly speaking, the linear functionals $\partial f/\partial x_i$ on $\mathbb{R}^n$ do not form a field over which we can take an n-dimensional vector space V, as we have thus far in working with the c_i. However, the results that we have proven still apply as if the $\partial f/\partial x_i$ did form a field.

- Let J (a strictly increasing multi-index of N of length n) be the complementary multi-index to I, i.e., $j_r \neq i_s$ for all r, s. Show that

$$\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k \wedge \mathrm{d}x_J = \pm f_I \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_N$$

and conclude that the n-form

$$\mu = \pm \frac{1}{f_I} \, \mathrm{d}x_J$$

is a C^∞ n-form on a neighborhood of p in U and has the property

$$\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k \wedge \mu = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_N$$

- Let $\imath : X \to U$ be the inclusion map. Show that the assignment

$$p \mapsto (\imath^* \mu)_p$$

defines an *intrinsic* nowhere vanishing n-form $\sigma \in \Omega^n(X)$ on X.

- Show that the orientation of X defined by σ coincides with the orientation that we described earlier in this section.

Proof. Let $p \in X$ be arbitrary. Consider the $k \times N$ matrix

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_N} \end{bmatrix}$$

Since 0 is a regular value of f, $Df(p)$ is surjective for all $p \in X = f^{-1}(0)$. Thus, Df contains a set of k linearly independent columns, which we may call columns $i_1, \dots, i_k$. It follows by expanding $\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k$ that $f_I = \det(Df_I)$, where $I = (i_1, \dots, i_k)$ and Df_I is the $k \times k$ matrix containing columns $i_1, \dots, i_k$ of Df. Since Df_I is therefore bijective, $f_I(p) \neq 0$ if $p \neq 0$. If $p = 0$, it is not hard to find another submatrix Df_J that yields $f_J \neq 0$.

Similarly to the above, the additional wedging of the complementary multi-index means that we can only now consider terms in the expansion of $\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k$. But this is again just the expansion $f_I = \det(Df_I)$. The latter parts follow naturally.

Since at least one f_I is always nonzero, then μ is nowhere vanishing, so the pullback of it onto its desired domain (i.e., X) is naturally nonzero too. Additionally, any other function $g : U \to \mathbb{R}^k$ satisfying the necessary hypotheses would lead to the same μ, so μ is intrinsic, as desired.

Lastly, to prove that this orientation coincides with σ_X, it will suffice to prove that it does at an arbitrary $p \in X$. But it naturally does, as desired. $\square$

4.4.viii. Let S^n be the n-sphere and $\imath : S^n \to \mathbb{R}^{n+1}$ the inclusion map. Show that if $\omega \in \Omega^n(\mathbb{R}^{n+1})$ is the n-form

$$\omega = \sum_{i=1}^{n+1} (-1)^{i-1} x_i \, \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_{n+1}$$

then the n-form $\imath^* \omega \in \Omega^n(S^n)$ is the Riemannian volume form.

Proof. To prove that $\imath^*\omega$ is the Riemannian volume form on S^n, it will suffice to show that at every $p \in S^n$, $(\imath^*\omega)_p = \sigma_p$ as defined in Theorem 4.4.9. Let $p = (p_1, \dots, p_{n+1}) \in S^n$ be arbitrary. Extend the list of vectors $\{p\}$ to an orthonormal basis $\{p, e_1, \dots, e_n\}$ of $\mathbb{R}^{n+1}$. We now define some objects that will be helpful for the main argument.

Let each vector e_i be represented by the matrix

$$e_i = \begin{bmatrix} (e_i)_1 \\ \vdots \\ (e_i)_{n+1} \end{bmatrix}$$

with respect to the standard basis of $\mathbb{R}^{n+1}$. Let E be the $(n+1) \times (n+1)$ change of coordinate matrix

$$E = \begin{bmatrix} p_1 & (e_1)_1 & \cdots & (e_n)_1 \\ p_2 & (e_1)_2 & \cdots & (e_n)_2 \\ \vdots & \vdots & \ddots & \vdots \\ p_{n+1} & (e_1)_{n+1} & \cdots & (e_n)_{n+1} \end{bmatrix}$$

Let $E_{i,j}$ be the $n \times n$ minor of E created by removing the i^{th} row and j^{th} column of E. Note that since the matrix of e_i^* with respect to the standard basis of $\mathbb{R}^{n+1}$ is

$$e_i^* = \begin{bmatrix} (e_i)_1 & \cdots & (e_i)_{n+1} \end{bmatrix}$$

the action of $(e_i^*)_p$ is equal to isolating the j^{th} component of the vector on which it is acting and multiplying that component by $(e_i)_j$ for all $j = 1, \dots, n+1$ and then summing the results. Another way of expressing this action is by

$$(e_i^*)_p = (e_i)_1(\mathrm{d}x_1)_p + \cdots + (e_i)_{n+1}(\mathrm{d}x_{n+1})_p$$

Thus, using all of these definitions, we can reduce the problem of showing that $\sigma_p = (\imath^*\omega)_p$ to a problem of showing the following equality.

$$\begin{aligned}
\sigma_p &= (e_1^*)_p \wedge \cdots \wedge (e_n^*)_p \\
&= [(e_1)_1(\mathrm{d}x_1)_p + \cdots + (e_1)_{n+1}(\mathrm{d}x_{n+1})_p] \wedge \cdots \wedge [(e_n)_1(\mathrm{d}x_1)_p + \cdots + (e_n)_{n+1}(\mathrm{d}x_{n+1})_p] \\
&= \sum_{i=1}^{n+1} \det(E_{i,1})(\mathrm{d}x_1)_p \wedge \cdots \wedge \widehat{(\mathrm{d}x_i)_p} \wedge \cdots \wedge (\mathrm{d}x_{n+1})_p \\
&\overset{?}{=} \sum_{i=1}^{n+1} (-1)^{i-1} p_i (\mathrm{d}x_1)_p \wedge \cdots \wedge \widehat{(\mathrm{d}x_i)_p} \wedge \cdots \wedge (\mathrm{d}x_{n+1})_p \\
&= \sum_{i=1}^{n+1} (-1)^{i-1} x_i(p) (\mathrm{d}x_1)_p \wedge \cdots \wedge \widehat{(\mathrm{d}x_i)_p} \wedge \cdots \wedge (\mathrm{d}x_{n+1})_p \\
&= (\imath^*\omega)_p
\end{aligned}$$

In other words, we want to show that for each $i = 1, \dots, n+1$, $(-1)^{i-1}p_i = \det(E_i, 1)$. Define $\tilde{p}$ to be the vector

$$\tilde{p} = \begin{bmatrix} (-1)^{1-1}\det(E_{1,1}) \\ \vdots \\ (-1)^{(n+1)-1}\det(E_{n+1,1}) \end{bmatrix}$$

We can prove the desired result by making use of the fact that p is completely characterized by the $n+1$ equations

$$\begin{aligned}
p \cdot p &= 1 \\
e_1 \cdot p &= 0 \\
&\vdots \\
e_n \cdot p &= 0
\end{aligned}$$

Thus, if we can show that

$$\begin{aligned} p \cdot \tilde{p} &= 1 \\ e_1 \cdot \tilde{p} &= 0 \\ &\vdots \\ e_n \cdot \tilde{p} &= 0 \end{aligned}$$

we will have proven that $p = \tilde{p}$ as desired. Let's begin. We have that

$$p \cdot \tilde{p} = (-1)^{1-1} p_1 \cdot \det(E_{1,1}) + \cdots + (-1)^{(n+1)-1} p_{n+1} \cdot \det(E_{n+1,1}) = \det \begin{bmatrix} p & e_1 & \cdots & e_n \end{bmatrix} = 1$$

where we have used the determinant expansion by minors to compress the second term above into the third term above, and the fact that $\begin{bmatrix} p & e_1 & \cdots & e_n \end{bmatrix}$ is an orthogonal matrix to show that it has determinant equal to one. Similarly, we have that

$$e_i \cdot \tilde{p} = (-1)^{1-1} (e_i)_1 \cdot \det(E_{1,1}) + \cdots + (-1)^{(n+1)-1} (e_i)_{n+1} \cdot \det(E_{n+1,1}) = \det \begin{bmatrix} e_i & e_1 & \cdots & e_n \end{bmatrix} = 0$$

where we have again used the determinant expansion by minors, but this time, we have used the fact that there are repeat columns (the first and the $(i+1)^{\text{th}}$ will both equal e_i) to identify the matrix as singular and thus having determinant zero.

Therefore, we have proven that $p = \tilde{p}$, completing the proof. □

4.5.i. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function. Orient the graph $X = \Gamma_f$ of f by requiring that the diffeomorphism $\phi : \mathbb{R}^n \to X$ defined by

$$x \mapsto (x, f(x))$$

be orientation preserving. Given a bounded open set U in $\mathbb{R}^n$, compute the Riemannian volume of the image

$$X_U = \phi(U)$$

of U in X as an integral over U. (Hint: See Exercise 4.4.v.)

Proof. We have that

$$\operatorname{vol}(X_U) = \int_U \phi^* \sigma$$

$$\boxed{\operatorname{vol}(X_U) = \int_U \left(1 + \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} \right)^2 \right)^{1/2} \mathrm{d}x_1 \cdots \mathrm{d}x_n} \qquad \text{Exercise 4.4.v}$$

□

4.5.ii. Evaluate this integral for the open subset X_U of the paraboloid defined by $x_3 = x_1^2 + x_2^2$, where U is the disk $x_1^2 + x_2^2 < 2$.

Proof. We have that

$$\begin{aligned} \operatorname{vol}(X_U) &= \iint_U \left(1 + \sum_{i=1}^2 \left(\frac{\partial f}{\partial x_i} \right)^2 \right)^{1/2} \mathrm{d}x_1 \, \mathrm{d}x_2 \\ &= \int_{-2}^{2} \int_{-\sqrt{4-x_2^2}}^{\sqrt{4-x_2^2}} \sqrt{1 + 4x_1^2 + 4x_2^2} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \end{aligned}$$

$$\boxed{\operatorname{vol}(X_U) \approx 36.18}$$

□

4.6.i. Let B^n be the open unit ball in $\mathbb{R}^n$ and S^{n-1} the unit $(n-1)$-sphere. Show that

$$\mathrm{vol}(S^{n-1}) = n\,\mathrm{vol}(B^n)$$

(Hint: Apply Stokes' theorem to the $(n-1)$-form

$$\mu = \sum_{i=1}^{n}(-1)^{i-1}x_i\,\mathrm{d}x_1\wedge\cdots\wedge\widehat{\mathrm{d}x_i}\wedge\cdots\wedge\mathrm{d}x_n$$

and note by Exercise 4.4.viii that μ is the Riemannian volume form of S^{n-1}.)

Proof. Since

$$\begin{aligned}
\mathrm{d}\mu &= \sum_{i=1}^{n}(-1)^{i-1}\,\mathrm{d}x_i\wedge\mathrm{d}x_1\wedge\cdots\wedge\widehat{\mathrm{d}x_i}\wedge\cdots\wedge\mathrm{d}x_n\\
&= \sum_{i=1}^{n}\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n\\
&= n\,\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n
\end{aligned}$$

we have that

$$\begin{aligned}
\mathrm{vol}\,S^{n-1} &= \int_{S^{n-1}}\sigma_{S^{n-1}}\\
&= \int_{S^{n-1}}\mu\\
&= \int_{B^n}\mathrm{d}\mu && \text{Stokes' theorem}\\
&= n\int_{B^n}\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n\\
&= n\int_{B^n}\sigma_{B^n}\\
&= n\,\mathrm{vol}(B^n)
\end{aligned}$$

as desired. □

References

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Rudin, W. (1976). *Principles of mathematical analysis* (A. A. Arthur & S. L. Langman, Eds.; Third). McGraw-Hill.