MATH 20510 (Analysis in \mathbb{R}^n III – Accelerated) Notes

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Chapter 1

Multilinear Algebra

1.1 Notes

• Plan:

3/30:

3/28: • Motivation for the course and an overview of Guillemin and Haine (2018).

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- More (multi)linear algebra.

• Dual spaces.

 \bullet Let V be an n-dimensional real vector space.

• Hom (V,\mathbb{R}) : The set of all homomorphisms (i.e., linear maps) from V to \mathbb{R} . Also known as V^* .

• Dual basis (for V^*): The set of linear transformations from V to \mathbb{R} defined by

$$e_j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

where e_1, \ldots, e_n is a basis of V. Denoted by e_1^*, \ldots, e_n^* .

• Check: e_1^*, \ldots, e_n^* are a basis for V^* .

– Are they linearly independent? Let $c_1e_1^* + \cdots + c_ne_n^* = 0 \in \text{Hom}(V, \mathbb{R})$. Then

$$c_i = (c_1 e_1^* + \dots + c_n e_n^*)(e_i) = 0 \in \mathbb{R}$$

as desired.

- Span? Let $\varphi \in \text{Hom}(V, \mathbb{R})$. Then we can verify that

$$\varphi(e_1)e_1^* + \cdots + \varphi(e_n)e_n^* = \varphi$$

- \blacksquare We prove this by verifying the previous statement on the basis of V (if two linear transformations have the same action on the basis of a vector space, they are equal).
- With a choice of basis for V, we obtain an isomorphism $\varepsilon: V \to V^*$ with the mapping $e_i \mapsto e_i^*$ for all i.
- The dual space is known as such because $(V^*)^* \cong V$, where \cong is **canonical** (no choice of basis is needed).
- One more property of dual spaces: functoriality.

- Given a linear transformation $A: V \to W$, we know that $A^*: W^* \to V^*$ where A^* is the transpose of A. In particular, if $\varphi \in W^*$, then $\varphi \circ A: V \to \mathbb{R}$.
- Claim: A^* is linear.
- Functoriality: If $A:V\to W$ and $B:W\to U$, then $B^*:U^*\to W^*$ and $A^*:W^*\to V^*$. The functoriality statement is that $(B\circ A)^*=A^*\circ B^*$.
- A^* is the **pullback** (or transpose) of A.
- Let v_1, \ldots, v_n be a basis for V and w_1, \ldots, w_m be a basis for W. Then $[A]_{v_1, \ldots, v_n}^{w_1, \ldots, w_m} = A$ is the matrix of the linear transformation A with respect to these bases. Then if v_1^*, \ldots, v_n^* and w_1^*, \ldots, w_m^* are the corresponding dual bases, then $[A^*]_{v_1^*, \ldots, v_n^*}^{w_1^*, \ldots, w_n^*} = A^T$. We can and should verify this for ourselves.
- This is over the real numbers, so A^* is just the transpose because there are no complex numbers of which to take the conjugate!
- A generalization: Tensors.
- k-tensor: A multilinear map

$$T: \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}$$

• Multilinear (map T): A function T such that

$$T(v_1, \dots, v_i^1 + v_i^2, \dots, v_k) = T(v_1, \dots, v_i^1, \dots, v_k) + T(v_1, \dots, v_i^2, \dots, v_k)$$
$$T(v_1, \dots, \lambda v_i, \dots, v_k) = \lambda T(v_1, \dots, v_i, \dots, v_k)$$

for all $(v_1, \ldots, v_k) \in V^k$.

- The determinant is an *n*-tensor!
- 1-tensors are just covectors.
- $L^k(V)$: The vector space of all k-tensors on V.
- Calculating dim $L^k(V)$. (Answer not given in this class.)
- Let $A: V \to W$. Then $A^*: L^k(W) \to L^k(V)$.
 - Check $(A \circ B)^* = B^* \circ A^*$.
- Multi-index of n of length k: A k-tuple (i_1, \ldots, i_k) where each $i_j \in \mathbb{N}$ satisfies $1 \leq i_j \leq n$ $(j = 1, \ldots, k)$. Denoted by I.
- Let e_1, \ldots, e_n be a basis for V.
- **Tensor product** (of $T_1 \in L^k(V)$, $T_2 \in L^l(V)$): The function from V^{k+l} to \mathbb{R} defined by

$$(v_1, \ldots, v_{k+l}) \mapsto T_1(v_1, \ldots, v_k) T_2(v_{k+1}, \ldots, v_{k+l})$$

Denoted by $T_1 \otimes T_2$.

- Claims:
 - 1. $T_1 \otimes T_2 \in L^{k+l}(V)$.
 - 2. $A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$.
- e_I^* : The function $e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*$, where $I = (i_1, \dots, i_k)$ is a multi-index of n of length k.
- Claim: Letting I range over all n^k multi-indices of n of length k, the e_I^* are a basis for $L^k(V)$.

- If $V = \mathbb{R}$, then $V = \mathbb{R}e_1$. If $V = \mathbb{R}^2$, then $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$.
- We know that $L^1(V) = V^* = Re_1^*$. Thus, $e_1^* \otimes e_2^* : V \times V \to \mathbb{R}$. Thus, for example,

$$(e_1^* \otimes e_2^*)((1,2),(3,4)) = e_1^*(1,2) \cdot e_2^*(3,4) = 1 \cdot 4 = 4$$

- 4/1: Plan: More multilinear algebra.
 - Properties of the tensor product.
 - Sign of a permutation.
 - Alternating tensors (lead into differential forms down the road).
 - Recall: V is an n-dimensional vector space over \mathbb{R} with basis e_1, \ldots, e_n . $\mathcal{L}^k(V)$ is the vector space of k-tensors on V. $\{e_I^* \mid I \text{ a multiindex of } n \text{ of length } k\}$ is a basis for $\mathcal{L}^k(V)$.
 - For example, if $V = \mathbb{R}^2$ and $T \in \mathcal{L}^2(V)$, then

$$T(a_1e_1 + a_2e_2, b_1e_1 + b_2e_2) = a_1b_1T(e_1, e_1) + a_1b_2T(e_1, e_2) + a_2b_1T(e_2, e_1) + a_2b_2T(e_2, e_2)$$

- A basis of $\mathcal{L}^2(V)$ is

$$\{e_1^* \otimes e_1^*, e_1^* \otimes e_2^*, e_2^* \otimes e_1^*, e_2^* \otimes e_2^*\}$$

- Recall that some basic properties are

$$e_1^* \otimes e_2^*((1,2),(3,4)) = 1 \cdot 4 = 4$$
 $e_2^* \otimes e_1^*((1,2),(3,4)) = 2 \cdot 3 = 6$

- It follows by the initial decomposition of T that

$$T = a_1 b_1 e_1^* \otimes e_1^* + a_1 b_2 e_1^* \otimes e_2^* + a_2 b_1 e_2^* \otimes e_1^* + a_2 b_2 e_2^* \otimes e_2^*$$

- Important consequence: To know the action of T on an arbitrary pair of vectors, you need only know its action on the basis; a higher-dimensional generalization of the earlier property.
- Note that

$$e_I^*(e_J) = \delta_{IJ} = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$$

- Basic properties of the tensor product.
 - 1. Right-distributive: If $T_1 \in \mathcal{L}^k(V)$ and $T_2, T_3 \in \mathcal{L}^{\ell}(V)$, then

$$T_1 \otimes (T_2 + T_3) = T_1 \otimes T_2 + T_1 \otimes T_3$$

2. Left-distributive: If $T_1, T_2 \in \mathcal{L}^k(V)$ and $T_3 \in \mathcal{L}^{\ell}(V)$, then

$$(T_1 + T_2) \otimes T_3 = T_1 \otimes T_3 + T_2 \otimes T_3$$

3. Associative: If $T_1 \in \mathcal{L}^k(V)$, $T_2 \in \mathcal{L}^\ell(V)$, and $T_3 \in \mathcal{L}^m(V)$, then

$$T_1 \otimes (T_2 \otimes T_3) = (T_1 \otimes T_2) \otimes T_2 = T_1 \otimes T_2 \otimes T_3$$

4. Scalar multiplication: If $T_1 \in \mathcal{L}^k(V)$, $T_2 \in \mathcal{L}^{\ell}(V)$, and $\lambda \in \mathbb{R}$, then

$$(\lambda T_1) \otimes T_2 = \lambda (T_1 \otimes T_2) = T_1 \otimes (\lambda T_2)$$

- Note that the tensor product is not commutative.
- Aside: Defining the sign of a permutation.

- S_A : The set of all automorphisms of A (bijections from A to A), where A is a set.
- S_n : The set $S_{[n]}$.
- Given $\sigma_1, \sigma_2 \in S_n, \sigma_1 \circ \sigma_2 \in S_n$.
 - Thus, S_n is a **group**.
- Transposition: A function $\tau \in S_n$ such that

$$\tau(k) = \begin{cases} j & k = i \\ i & k = j \\ k & k \neq i, j \end{cases}$$

for some $i, j \in [n]$. Denoted by $\tau_{i,j}$.

- Theorem: An element of S_n can be written as the product of transpositions (i.e., for all $\sigma \in S_n$, there exist $\tau_1, \ldots, \tau_m \in S_n$ such that $\sigma = \tau_1 \circ \cdots \circ \tau_m$).
- Sign (of $\sigma \in S_n$): The number (mod 2) of transpositions whose product equals σ . Denoted by $(-1)^{\sigma}$, sign (σ) .
- Theorem: The sign of σ is well-defined. Additionally,

$$(-1)^{\sigma_1 \sigma_2} = (-1)^{\sigma_1} \cdot (-1)^{\sigma_2}$$

- Example: Consider the identity permutation. $(-1)^{\sigma} = +1$. We can think of this as the product of zero transpositions or, for instance, as the product of the two transpositions $\tau_{1,2} \circ \tau_{1,2}$. Another example would be $\tau_{2,3} \circ \tau_{1,2} \circ \tau_{1,2} \circ \tau_{2,3}$.
- Theorem: Let X_i be a rational or polynomial function for each $i \in \mathbb{N}$. Then

$$(-1)^{\sigma} = \prod_{i < j} \frac{X_{\sigma(i)} - X_{\sigma(j)}}{X_i - X_j}$$

• Example: For the permutation $\sigma = (1, 2, 3)$, we have

$$\begin{split} (-1)^{\sigma} &= \frac{X_{\sigma(1)} - X_{\sigma(2)}}{X_1 - X_2} \cdot \frac{X_{\sigma(1)} - X_{\sigma(3)}}{X_1 - X_3} \cdot \frac{X_{\sigma(2)} - X_{\sigma(3)}}{X_2 - X_3} \\ &= \frac{X_2 - X_3}{X_1 - X_2} \cdot \frac{X_2 - X_1}{X_1 - X_3} \cdot \frac{X_3 - X_1}{X_2 - X_3} \\ &= \frac{-(X_1 - X_2)}{X_1 - X_2} \cdot \frac{-(X_1 - X_3)}{X_1 - X_3} \cdot \frac{X_2 - X_3}{X_2 - X_3} \\ &= -1 \cdot -1 \cdot 1 \\ &= +1 \end{split}$$

which squares with the fact that $\sigma = \tau_{1,2} \circ \tau_{2,3}$.

- Claims to verify with the above formula:
 - 1. $sign(\sigma) \in \{\pm 1\}.$
 - 2. $sign(\tau_{i,i}) = -1$.
 - 3. $\operatorname{sign}(\sigma_1 \sigma_2) = \operatorname{sign}(\sigma_1) \operatorname{sign}(\sigma_2)$.
- 4/4: Plan:
 - More multilinear algebra.

- Alternating k-tensors 2 views:
 - 1. As a subspace of $\mathcal{L}^k(V)$.
 - 2. As a quotient of $\mathcal{L}^k(V)$.
- Next time: Operators as alternating tensors.
 - Wedge products.
 - Interior products.
 - Pullbacks.
- Recall: dim $V = n, e_1, \ldots, e_n$ a basis, $\mathcal{L}^k(V)$ the space of k-tensors, $\sigma \in S_k$ implies $(-1)^{\sigma} \in \{\pm 1\}$, key property: $(-1)^{\sigma_1 \sigma_2} = (-1)^{\sigma_1} (-1)^{\sigma_2}$.
- T^{σ} : The k-tensor over V defined by

$$T^{\sigma}(v_1,\ldots,v_k) = T(v_{\bar{\sigma}(1)},\ldots,v_{\bar{\sigma}(k)})$$

where $T \in \mathcal{L}^k(V)$, $\sigma \in S_k$, and $\bar{\sigma}$ denotes the inverse of σ .

- Example: n=2, k=2. Let $T=e_1^*\otimes e_2^*\in \mathcal{L}^2(V)$. Let $\sigma=\tau_{1,2}$. Then $T^{\sigma}=e_2^*\otimes e_1^*$.
- Another property is $e_I^{\sigma} = e_{\sigma(I)}^*$.
- Properties:
 - 1. $T^{\sigma_1 \sigma_2} = (T^{\sigma_1})^{\sigma_2}$.
 - 2. $(T_1 + T_2)^{\sigma} = T_1^{\sigma} + T_2^{\sigma}$.
 - 3. $(cT)^{\sigma} = cT^{\sigma}$.
- Thus, you can view $\sigma: \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ as a linear map!
- Alternating k-tensor: A tensor $T \in \mathcal{L}^k(V)$ such that $T^{\sigma} = (-1)^{\sigma}T$ for all $\sigma \in S_k$.
 - Equivalently, $T^{\tau} = -T$ for all $\tau \in S_k$.
- An example of an alternating 2-tensor when dim V=2 is $T=e_1^*\otimes e_2^*-e_2^*\otimes e_1^*$.
 - Naturally, $T_{1,2}^{\tau} = -T$, and $\tau_{1,2}$ is the unique transposition in S_2 .
- $e_1^* \otimes e_2^*$ is not an alternating 2-tensor since $(e_1^* \otimes e_2^*)^{\tau} = e_2^* \otimes e_1^* \neq (-1)^{\tau} (e_1^* \otimes e_2^*)$.
- We can look at n=2, k=1 for ourselves.
- Note: If $T-1, T_2$ are both alternating k-tensors, then T_1+T_2 is also alternating, as is cT_1 for all $c \in \mathbb{R}$.
- $\mathcal{A}^k(V)$: The vector space of alternating k-tensors.
- Alt (T): The function Alt : $\mathcal{L}^k(V) \to \mathcal{L}^k(V)$ defined by

$$Alt(T) = \sum_{\sigma \in S_k} (-1)^{\sigma} T^{\sigma}$$

- Properties:
 - 1. $\operatorname{im}(\operatorname{Alt}) = \mathcal{A}^k(V)$.
 - 2. $\mathcal{L}^k(V)/\ker(Alt) = \Lambda^k(V^*)$ is isomorphic to $\mathcal{A}^k(V)$.
 - 3. $Alt(T)^{\sigma} = (-1)^{\sigma} Alt(T)$.

- Proof:

$$\operatorname{Alt}(T)^{\sigma'} = \left(\sum_{\sigma \in S_k} (-1)^{\sigma} T^{\sigma}\right)^{\sigma'}$$

$$= \sum_{\sigma \in S_k} (-1)^{\sigma} T^{\sigma \sigma'}$$

$$= (-1)^{\sigma'} \sum_{\sigma \in S_k} (-1)^{\sigma'} (-1)^{\sigma} T^{\sigma \sigma'}$$

$$= (-1)^{\sigma'} \sum_{\sigma \in S_k} (-1)^{\sigma \sigma'} T^{\sigma \sigma'}$$

$$= (-1)^{\sigma'} \operatorname{Alt}(T)$$

- The last equality holds because summing over all σ is the same as summing over all $\sigma' \circ \sigma$.
- This implies $\operatorname{im}(\operatorname{Alt}) \leq \mathcal{A}^k(V)$.
- 4. If $T \in \mathcal{A}^k(T)$, Alt(T) = k!T.
 - We have

$$\operatorname{Alt}(T) = \sum_{\sigma \in S_k} (-1)^{\sigma} T^{\sigma}$$
$$= \sum_{\sigma \in S_k} (-1)^{\sigma} (-1)^{\sigma} T$$
$$= \sum_{\sigma \in S_k} T$$
$$= k!T$$

where $T^{\sigma} = (-1)^{\sigma}T$ since $T \in \mathcal{A}^k(V)$ by definition.

- This implies that $\operatorname{im}(\operatorname{Alt}) = \mathcal{A}^k(V)$: $\operatorname{Alt}(\frac{1}{k!}T) = T \in \mathcal{A}^k(V)$.
- 5. $Alt(T^{\sigma}) = Alt(T)^{\sigma}$.
- 6. Alt : $\mathcal{L}^k(V) \to \mathcal{L}^k(V)$ is linear.
- Warning: Some people take $Alt(T) = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^{\sigma} T^{\sigma[1]}$.
- Example: n = k = 2. We have

$$Alt(e_1^* \otimes e_2^*) = e_1^* \otimes e_2^* - e_2^* \otimes e_1^*$$

- Non-repeating (multi-index I): A multi-index I such that $i_{j_1} \neq i_{j_2}$ for all $j_1 \neq j_2$.
- Increasing (multi-index I): A multi-index I such that $i_1 < \cdots < i_k$.
- Claim: $\{Alt(e_I^*)\}$ where I is non-repeating and increasing is a basis for $\mathcal{A}^k(V)$. There are $\binom{n}{k}$ of these; thus, $\dim \mathcal{A}^k(V) = \binom{n}{k}$.
- Klug will be in Texas on Monday and thus is cancelling class on Monday. Homework is now due next Friday. We'll have weekly homeworks going forward after that.
 - Plan:

4/6:

- Alt: $\mathcal{L}^k(V) \to \mathcal{A}^k(V)^{[2]}$.
- Goal: Identify $\ker(Alt) = \mathcal{I}^k(V)$, where $\mathcal{I}^k(V)$ is the space of **redundant** k-tensors^[3].

¹Klug prefers this convention, but the text takes the other one.

²The two-headed right arrow denotes a surjective map.

³The \mathcal{I} in $\mathcal{I}^k(V)$ stands for "ideal."

- Then: Operations on alternating tensors, e.g.,
 - Wedge product.
 - Interior product.
 - Orientations.
- Claim: $\{Alt(e_I^*) \mid I \text{ non-repeating, increasing multi-index}\}\$ is a basis for $\mathcal{A}^k(V)$.
 - Left as an exercise to us.
- **Redundant** (k-tensor): A k-tensor of the form

$$\ell_1 \otimes \cdots \otimes \ell_i \otimes \ell_i \otimes \ell_{i+2} \otimes \cdots \otimes \ell_k$$

where $\ell_1, \ldots, \ell_k \in V^*$.

- $\mathcal{I}^k(V)$: The span of all redundant k-tensors.
 - Note that not every k-tensor in $\mathcal{I}^k(V)$ is a redundant.
- **Decomposable** (k-tensor): A k-tensor of the form $\ell_1 \otimes \cdots \otimes \ell_k$ for $\ell_i \in \mathcal{L}^1(V)$.
 - It often suffices to prove things for decomposable tensors.
- Properties.
 - 1. If $T \in \mathcal{I}^k(V)$, then Alt(T) = 0, i.e., $\mathcal{I}^k(V) \leq \ker(Alt)$.
 - "Proof by example": If $T = \ell_1 \otimes \ell_1 \otimes \ell_2 \otimes \ell_3$, then $T^{\tau_{1,2}} = T$. It follows from the properties of Alt that

$$\begin{aligned} \operatorname{Alt}(T) &= \operatorname{Alt}(T^{\tau_{1,2}}) = (-1)^{\tau_{1,2}} \operatorname{Alt}(T) = -\operatorname{Alt}(T) \\ 2 \operatorname{Alt}(T) &= 0 \\ \operatorname{Alt}(T) &= 0 \end{aligned}$$

2. If $T \in \mathcal{I}^r(V)$ and $T' \in \mathcal{L}^s(V)$, then

$$T \otimes T' \in \mathcal{I}^{r+s}(V)$$

Similarly, if $T \in \mathcal{L}^r(V)$ and $T \in \mathcal{I}^s(V)$, then

$$T \otimes T' \in \mathcal{I}^{r+s}(V)$$

- Proof: It suffices to assume that T is redundant. Obviously adding more tensors to the direct product will not change the redundancy of the initial tensor. Example: $\ell_1 \otimes \ell_1 \otimes \ell_2$ is just as redundant as $\ell_1 \otimes \ell_1 \otimes \ell_2 \otimes T$.
- 3. If $T \in \mathcal{L}^k(V)$ and $\sigma \in S_k$, then

$$T^{\sigma} = (-1)^{\sigma}T + S$$

for some $S \in \mathcal{I}^k(V)$.

– Proof by example: It suffices to check this for decomposable tensors (a tensor is just a sum of decomposable tensors). Take k=2. Let $T=\ell_1\otimes\ell_2$. Let $\sigma=\tau_{1,2}$. Then

$$T^{\sigma} - (-1)^{\sigma}T = \ell_2 \otimes \ell_1 + \ell_1 \otimes \ell_2 = (\ell_1 + \ell_2) \otimes (\ell_1 + \ell_2) - \ell_1 \otimes \ell_1 - \ell_2 \otimes \ell_2$$

– Actual proof: It suffices to assume T is decomposable. We induct on the number of transpositions needed to write σ as a product of **adjacent** transpositions.

- Base case: $\sigma = \tau_{i,i+1}$. Then

$$T^{\tau_{i,i+1}} + T = \ell_1 \otimes \cdots \otimes (\ell_i + \ell_{i+1}) \otimes (\ell_i + \ell_{i+1}) \otimes \cdots \otimes \ell_k$$
$$-\ell_1 \otimes \cdots \otimes \ell_i \otimes \ell_i \otimes \cdots \otimes \ell_k$$
$$-\ell_1 \otimes \cdots \otimes \ell_{i+1} \otimes \ell_{i+1} \otimes \cdots \otimes \ell_k$$

- Inductive step: If $\sigma = \beta \tau$, then

$$\begin{split} T^{\sigma} &= T^{\beta\tau} \\ &= (-1)^{\tau} T^{\beta} + \text{stuff in } \mathcal{I}^k(V) \\ &= (-1)^{\tau} [(-1)^{\beta} T + \text{stuff in } \mathcal{I}^k(V)] + \text{stuff in } \mathcal{I}^k(V) \end{split}$$

4. If $T \in \mathcal{L}^k(V)$, then

$$Alt(T) = k!T + W$$

for some $W \in \mathcal{I}^k(V)$.

- We have that

$$\operatorname{Alt}(T) = \sum_{\sigma \in S_k} (-1)^{\sigma} T^{\sigma}$$

$$= \sum_{\sigma \in S_k} (-1)^{\sigma} [(-1)^{\sigma} T + S_{\sigma}]$$

$$= \sum_{\sigma \in S_k} T + \sum_{\sigma \in S_k} (-1)^{\sigma} S_{\sigma}$$

$$= k! T + W$$

- 5. $\mathcal{I}^k(V) = \ker(Alt)$.
 - We have that $\mathcal{I}^k(V) \leq \ker(\mathrm{Alt})$ by property 1.
 - Now suppose $T \in \ker(Alt)$. Then Alt(T) = 0. Then by property 4,

$$Alt(T) = k!T + W$$
$$0 = k!T + W$$
$$T = -\frac{1}{k!}W \in \mathcal{I}^{k}(V)$$

- Warning: If $T \in \mathcal{A}^r(V)$ and $T' \in \mathcal{A}^s(V)$, then we do not necessarily have $T \otimes T' \in \mathcal{A}^{r+s}(V)$.
 - Example: $e_1^*, e_2^* \in \mathcal{A}^1(V)$ have $e_1^* \otimes e_2^* \notin \mathcal{A}^2(V)$.
- Adjacent (transposition): A transposition of the form $\tau_{i,i+1}$.

1.2 Chapter 1: Multilinear Algebar

From Guillemin and Haine (2018).

3/31:

- Guillemin and Haine (2018) defines real vector spaces, the operations on them, their basic properties, and the zero vector.
- Linearly independent (vectors v_1, \ldots, v_k): A finite set of vectors $v_1, \ldots, v_k \in V$ such that the map from \mathbb{R}^k to V defined by $(c_1, \ldots, c_k) \mapsto c_1 v_1 + \cdots + c_k v_k$ is injective.
- Spanning (vectors v_1, \ldots, v_k): We require that the above map is surjective.
- Guillemin and Haine (2018) defines basis, finite-dimensional vector space, dimension, subspace, linear map, and kernel.

- Image (of $A: V \to W$): The range space of A, a subspace of W. Also known as im (A).
- Guillemin and Haine (2018) defines the matrix of a linear map.
- Inner product (on V): A map $B: V \times V \to \mathbb{R}$ with the following three properties.
 - Bilinearity: For vectors $v, v_1, v_2, w \in V$ and $\lambda \in \mathbb{R}$, we have

$$B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w)$$

and

$$B(\lambda v, w) = \lambda B(v, w)$$

- Symmetry: For vectors $v, w \in V$, we have B(v, w) = B(w, v).
- Positivity: For every vector $v \in V$, we have $B(v,v) \geq 0$. Moreover, if $v \neq 0$, then B(v,v) > 0.
- **W-coset**: A set of the form $\{v + w \mid w \in W\}$, where W is a subspace V and $v \in V$. Denoted by v + W.
 - If $v_1 v_2 \in W$, then $v_1 + W = v_2 + W$.
 - It follows that the distinct W-cosets decompose V into a disjoint collection of subsets of V.
- Quotient space (of V by W): The set of distinct W-cosets in V, along with the following definitions of vector addition and scalar multiplication.

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$$
 $\lambda(v + W) = (\lambda v) + W$

Denoted by V/W.

• Quotient map: The linear map $\pi: V \to V/W$ defined by

$$\pi(v) = v + W$$

- $-\pi$ is surjective.
- Note that $\ker(\pi) = W$ since for all $w \in W$, $\pi(w) = w + W = 0 + W$, which is the zero vector in V/W.
- If V, W are finite dimensional, then

$$\dim(V/W) = \dim(V) - \dim(W)$$

- Proposition 1.2.9: Let $A: V \to U$ be a linear map. If $W \subset \ker(A)$, then there exists a unique linear map $A^{\sharp}: V/W \to U$ with the property that $A = A^{\sharp} \circ \pi$, where $\pi: V \to V/W$ is the quotient map.
 - This proposition rephrases in terms of quotient spaces the fact that if $w \in W$, then A(v+w) = Av.
- **Dual space** (of V): The set of all linear functions $\ell: V \to \mathbb{R}$, along with the following definitions of vector addition and scalar multiplication.

$$(\ell_1 + \ell_2)(v) = \ell_1(v) + \ell_2(v) \qquad (\lambda \ell)(v) = \lambda \cdot \ell(v)$$

Denoted by V^* .

• **Dual basis** (of e_1, \ldots, e_n a basis of V): The basis of V^* consisting of the n functions that take every $v = c_1 e_1 + \cdots + c_n e_n$ to one of the c_i . Denoted by e_1^*, \ldots, e_n^* . Given by

$$e_i^*(v) = c_i$$

for all $v \in V$.

• Claim 1.2.12: If V is an n-dimensional vector space with basis e_1, \ldots, e_n , then e_1^*, \ldots, e_n^* is a basis of V^*

Proof. We will first prove that e_1^*, \ldots, e_n^* spans V^* . Let $\ell \in V^*$ be arbitrary. Set $\lambda_i = \ell(e_i)$ for all $i \in [n]$. Define $\ell' = \sum_{i=1}^n \lambda_i e_i^*$. Then

$$\ell'(e_j) = \sum_{i=1}^{n} \lambda_i e_i^*(e_j) = \lambda_j \cdot 1 = \ell(e_j)$$

for all $j \in [n]$. Therefore, since ℓ, ℓ' take identical values on the basis of $V, \ell = \ell'$, as desired. We now prove that e_1^*, \ldots, e_n^* spans V^* . Let $\sum_{i=1}^n \lambda_i e_i^* = 0$. Then for all $j \in [n]$,

$$\lambda_j = \left(\sum_{i=1}^n \lambda_i e_i^*\right)(e_j) = 0$$

as desired.

- Transpose (of A): The map from W^* to V^* defined by $\ell \mapsto A^*\ell = \ell \circ A$ for all $\ell \in W^*$.
- Claim 1.2.15: If e_1, \ldots, e_n is a basis of V, f_1, \ldots, f_m is a basis of W, e_1^*, \ldots, e_n^* and f_1^*, \ldots, f_m^* are the corresponding dual bases, and $[a_{i,j}]$ is the $m \times n$ matrix of A with respect to $\{e_i\}, \{f_i\}$, then the linear map A^* is defined in terms of $\{f_i^*\}, \{e_i^*\}$ by the transpose matrix $(a_{j,i})$.
- 4/4: V^k : The set of all k-tuples (v_1, \ldots, v_k) where $v_1, \ldots, v_k \in V$ a vector space.
 - Note that

$$V^k = \underbrace{V \oplus \cdots \oplus V}_{k \text{ times}}$$

where "

"

denotes the direct sum.

- **Linear** (function in its i^{th} variable): A function $T: V^k \to \mathbb{R}$ such that the map from V to \mathbb{R} defined by $v \mapsto T(v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_k)$ is linear, where all v_i save v_i are fixed.
- **k-linear** (function T): A function $T: V^k \to \mathbb{R}$ that is linear in its i^{th} variable for i = 1, ..., k. Also known as **k-tensor**.
- $\mathcal{L}^k(V)$: The set of all k-tensors in V.
 - Since the sum $T_1 + T_2$ of two k-linear functions $T_1, T_2 : V^k \to \mathbb{R}$ is just another k-linear function, and λT_1 is k-linear for all $\lambda \in \mathbb{R}$, we have that $\mathcal{L}^k(V)$ is a vector space.
- Convention: 0-tensors are just the real numbers. Mathematically, we define

$$\mathcal{L}^0(V) = \mathbb{R}$$

- Note that $\mathcal{L}^1(V) = V^*$.
- Defines multi-indices of n of length k.
- Lemma 1.3.5: If $n, k \in \mathbb{N}$, then there are exactly n^k multi-indices of n of length k.
- T_I : The real number $T(e_{i_1}, \ldots, e_{i_k})$, where $T \in \mathcal{L}^k(V)$, e_1, \ldots, e_n is a basis of V, and I is a multi-index of n of length k.
- Proposition 1.3.7: The real numbers T_I determine T, i.e., if T, T' are k-tensors and $T_I = T'_I$ for all I, then T = T'.

Proof. We induct on n. For the base case n = 1, $T \in (\mathbb{R}^k)^*$ and we have already proven this result. Now suppose inductively that the assertion is true for n - 1. For each e_i , let T_i be the (k - 1)-tensor defined by

$$(v_1, \ldots, v_{n-1}) \mapsto T(v_1, \ldots, v_{n-1}, e_i)$$

Then for an arbitrary $v = c_1 e_1 + \cdots + c_n e_n$,

$$T(v_1, \dots, v_{n-1}, v) = \sum_{i=1}^n c_i T_i(v_1, \dots, v_{n-1})$$

so the T_i 's determine T. Applying the inductive hypothesis completes the proof.

• **Tensor product**: The tensor $T_1 \otimes T_2$ defined by

$$(T_1 \otimes T_2)(v_1, \dots, v_{k+\ell}) = T_1(v_1, \dots, v_k)T_2(v_{k+1}, \dots, v_{k+\ell})$$

where $T_1 \in \mathcal{L}^k(V)$ and $T_2 \in \mathcal{L}^\ell(V)$.

• Note that by the definition of 0-tensors as real numbers, if $a \in \mathbb{R}$ and $T \in \mathcal{L}^k(V)$, then

$$a \otimes T = T \otimes a = aT$$

- Proposition 1.3.9: Associativity, distributivity of scalar multiplication, and left and right distributive laws for the tensor product.
- Decomposable (k-tensor): A k-tensor T for which there exist $\ell_1, \ldots, \ell_k \in V^*$ such that

$$T = \ell_1 \otimes \cdots \otimes \ell_k$$

- Defines e_I^* .
- Theorem 1.3.13: V a vector space with basis e_1, \ldots, e_n and $0 \le k \le n$ implies the k-tensors e_I^* form a basis of $\mathcal{L}^k(V)$.

Proof. Spanning: Let $T \in \mathcal{L}^k(V)$ be arbitrary. Define

$$T' = \sum_{I} T_{I} e_{I}^{*}$$

Since

$$T'_J = T'(e_{j_1}, \dots, e_{j_k}) = \sum_I T_I e_I^*(e_{j_1}, \dots, e_{j_k}) = T_J e_J^*(e_{j_1}, \dots, e_{j_k}) = T_J$$

for all J, Proposition 1.3.7 asserts that T = T'. Therefore, since every $T_I \in \mathbb{R}$, $T = T' \in \text{span}(e_I^*)$.

Linear independence: Suppose

$$T = \sum_{I} c_I e_I^* = 0$$

for some set of constants $c_I \in \mathbb{R}$. Then

$$0 = T(e_{j_1}, \dots, e_{j_k}) = \sum_{I} c_I e_I^*(e_{j_1}, \dots, e_{j_k}) = c_J$$

for all J, as desired.

• Corollary 1.3.15: If dim V = n, then dim $(\mathcal{L}^k(V)) = n^k$.

Proof. Follows immediately from Lemma 1.3.5.

• **Pullback** (of T by the map A): The k-tensor $A^*T: V^k \to \mathbb{R}$ defined by

$$(A^*T)(v_1,\ldots,v_k) = T(Av_1,\ldots,Av_k)$$

where V, W are finite-dimensional vector spaces, $A: V \to W$ is linear, and $T \in \mathcal{L}^k(W)$.

- Proposition 1.3.18: The map $A^*: \mathcal{L}^k(W) \to \mathcal{L}^k(V)$ defined by $T \mapsto A^*T$ is linear.
- Identities:

– If
$$T_1 \in \mathcal{L}^k(W)$$
 and $T_2 \in \mathcal{L}^m(W)$, then

$$A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$$

– If U is a vector space, $B: U \to V$ is linear, and $T \in \mathcal{L}^k(W)$, then

$$(AB)^*T = B^*(A^*T)$$

References

Guillemin, V., & Haine, P. J. (2018). Differential forms [https://math.mit.edu/classes/18.952/2018SP/files/18.952_book.pdf].