Week 3

Multilinear Spaces, Operations, and Conventions

3.1 Exterior Powers Basis and the Determinant

4/13:

- Plan:
  - Finish multilinear algebra.
  - Basis for $\Lambda^k(V^*)$.
  - Talk a bit about pullbacks and the determinant.
  - **Orientations** of vector spaces.
  - The **interior product**.
- Basis for $\Lambda^k(V^*)$.
  - Recall that $\{\text{Alt}(e_I^*) \mid I \text{ is a nonrepeating, increasing partition of } n \text{ into } k \text{ parts}\}$ is a basis for $\mathcal{A}^k(V)$.
- Alt is an isomorphism from $\Lambda^k(V^*)$ to $\mathcal{A}^k(V)$.
- If we have an injective map from $\mathcal{A}^k(V)$ to $\mathcal{L}^k(V)$ and π a projection map from $\mathcal{L}^k(V)$ to the quotient space $\mathcal{A}^k(V^*)$ gives rise to $\pi|_{\mathcal{A}^k(V)}$.
- Claim:
  1. $\pi|_{\mathcal{A}^k(V)}$ is an isomorphism.
  2. $\pi(\text{Alt}(e_I^*)) = k!\pi(e_I^*)$.

  (2) implies that $\{\pi(e_I^*) = e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*,\ I \text{ non-repeating and increasing}\}$ is a basis for $\Lambda^k(V^*)$.
- Examples:
  1. $n = 2 = \dim V$, $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$.
     - $\Lambda^0(V^*) = \mathbb{R}$ since $\binom{n}{0} = 1$.
     - $\Lambda^1(V^*) = \mathbb{R}e_1^* \oplus \mathbb{R}e_2^*$ since $\binom{n}{1} = 2$.
     - $\Lambda^2(V^*) = \mathbb{R}e_1^* \wedge e_2^*$ since $\binom{n}{2} = 1$.
       - For the second to last one, note that $e_1^* \wedge e_2^* = -e_2^* \wedge e_1^*$.
     - $\Lambda^3(V^*) = 0$ since $\binom{2}{3} = 0$.
       - For the last one, note that all $e_1^* \wedge e_1^* \wedge e_2^* = 0$.
  2. $n = 3$, $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3$.

- $\binom{n}{0} = 1$: $\Lambda^0(V^*) = \mathbb{R}$.
- $\binom{n}{1} = 3$: $\Lambda^1(V^*) = \mathbb{R}e_1^* \oplus \mathbb{R}e_2^* \oplus \mathbb{R}e_3^*$.
- $\binom{n}{2} = 3$: $\Lambda^2(V^*) = \mathbb{R}e_1^* \wedge e_2^* \oplus \mathbb{R}e_2^* \wedge e_3^* \oplus \mathbb{R}e_1^* \wedge e_3^*$.
- $\binom{n}{3} = 1$: $\Lambda^3(V^*) = \mathbb{R}e_1^* \wedge e_2^* \wedge e_3^*$.
- $\binom{n}{m} = 0$ $(m > n)$: $\Lambda^m(V^*) = \Lambda^4(V^*) = 0$.

- If $A : V \to W$, $\omega_1 \in \Lambda^k(W^*)$, $\omega_2 \in \Lambda^\ell(W^*)$, then

$$A^*(\omega_1 \wedge \omega_2) = A^*\omega_1 \wedge A^*\omega_2$$

- **Determinant**: Let $\dim V = n$. Let $A : V \to V$ be a linear transformation. This induces a pullback $A^* : \Lambda^n(V^*) \to \Lambda^n(V^*)$. The top exterior power $k = n$ implies $\binom{k}{n} = 1$. We define $\det(A)$ to be the unique real number such that $A^*(v) = \det(A)v$.

- This determinant is the one we know.
  - A^* sends $e_1^* \wedge \cdots \wedge e_n^*$ to $A^*e_1^* \wedge \cdots \wedge A^*e_n^*$ which equals $A^*(e_1^* \wedge \cdots \wedge e_n^*)$ or $\det(A)$

- Sanity check.
  1. $\det(\mathrm{id}) = 1$.
     - $\mathrm{id}(e_1^* \wedge \cdots \wedge e_n^*) = \mathrm{id}\, e_1^* \wedge \cdots \wedge \mathrm{id}\, e_n^* = 1 \cdot e_1^* \wedge \cdots \wedge e_n^*$.
  2. If A is not an isomorphism, then $\det(A) = 0$.
     - If A is not an isomorphism, then there exists $v_1 \in \ker A$ with $v_1 \neq 0$. Let $v_1^*, \ldots, v_n^*$ be a basis of V^*. So the pullback of this wedge is the wedge of the pullbacks, but $A^*v_1^* = 0$, so

$$A^*(v_1^* \wedge \cdots \wedge v_n^*) = (A^*v_1^*) \wedge \cdots \wedge (A^*v_n^*) = 0 \wedge \cdots \wedge (A^*v_n^*) = 0 = 0 \cdot v_1^* \wedge \cdots \wedge v_n^*$$

  3. $\det(AB) = \det(A)\det(B)$.
     - Let $A : V \to V$ and $B : V \to V$.
     - We have $(AB)^* = B^*A^*$; in particular, $n = k$, $V = W = U = V$.

- Recall: If we pick a basis for V, $e_1, \ldots, e_n$.
  - Implies $[a_{ij}] = [A]_{e_1,\ldots,e_n}^{e_1,\ldots,e_n}$.

- Does $\det(A) = \det([a_{ij}]) = \sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$?
  - If $A : V \to V$, we know that $A^* : \Lambda^n(V^*) \to \Lambda^n(V^*)$ takes $e_1^* \wedge \cdots \wedge e_n^* \mapsto A^*(e_1^* \wedge \cdots \wedge e_n^*)$. We WTS

$$A^*(e_1^* \wedge \cdots \wedge e_n^*) = \left[\sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}\right] e_1^* \wedge \cdots \wedge e_n^*$$

  - We have that

$$\begin{aligned}
A^*(e_1^* \wedge \cdots \wedge e_n^*) &= A^*e_1^* \wedge \cdots \wedge A^*e_n^* \\
&= \left(\sum_{i_1=1}^n a_{i_1,1} e_{i_1}^*\right) \wedge \cdots \wedge \left(\sum_{i_n=1}^n a_{i_n,n} e_{i_n}^*\right) \\
&= \sum_{i_1,\ldots,i_n} a_{i_1,1} \cdots a_{i_n,n} e_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \\
&= \left[\sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}\right] e_1^* \wedge \cdots \wedge e_n^*
\end{aligned}$$

  where the sign arises from the need to reorder $e_{i_1}^* \wedge \cdots \wedge e_{i_n}^*$ and the antisymmetry of the wedge product.

3.2 The Interior Product and Orientations

4/15:

- Plan:
  - Orientations.
  - Interior product.
- **Interior product**: We know that $\Lambda^k(V^*) \cong \mathcal{A}^k(V)$. Fix $v \in V$. Define $\iota_v : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$.
  - Wrong way: We take $\iota_v : \mathcal{L}^k(V) \to \mathcal{L}^{k-1}(V)$.

$$T \mapsto \sum_{r=1}^{k} (-1)^{r-1} T(v_1, \dots, v_r, \dots, v_{k-1})$$

  - Right way: First define $\varphi_v : \mathcal{A}^k(V) \to \mathcal{A}^{k-1}(V)$ by

$$T \mapsto T_v(v_1, \dots, v_{k-1}) = T(v, v_1, \dots, v_{k-1})$$

    - Check: $T_{v_1+v_2} = T_{v_1} + T_{v_2}$. $T_{\lambda v} = \lambda T_v$. $\varphi_v^{k-1} \circ \varphi_v^k = 0$ implies $\varphi_v \circ \varphi_w = -\varphi_w \circ \varphi_v$.
- Properties:

0. $\iota_v T \in \mathcal{L}^{k-1}(V)$.
1. ι_v is a linear map. This is all happening in the set $\operatorname{Hom}(\mathcal{L}^k(V), \mathcal{L}^{k-1}(V))$.
2. $\iota_{v_1+v_2} = \iota_{v_1} + \iota_{v_2}$; $\iota_{\lambda v} = \lambda \iota_v$.
3. "Product rule": If $T_1 \in \mathcal{L}^p(V)$ and $T_1 \in \mathcal{L}^q(V)$, then $\iota_v(T_1 \otimes T_2) = \iota_v T_1 \otimes T_2 + (-1)^p T_1 \otimes \iota_v T_2$.
4. We have

$$\iota_v(\ell_1 \otimes \cdots \otimes \ell_k) = \sum_{r=1}^{k} (-1)^{r-1} \ell_r(v) \ell_1 \otimes \cdots \otimes \hat{\ell}_r \otimes \cdots \otimes \ell_k$$

5. $\iota_v \circ \iota_v = 0 \in \operatorname{Hom}(\mathcal{L}^k(V), \mathcal{L}^{k-2}(V))$.
   - Note that this is related to $d^2 = 0$ from the first day of class (alongside $\int_m \mathrm{d}w = \int_{\partial m} w$).
   - Proof: We induct on k. It suffices to prove the result for T decomposable.
   - Trivial base case for $k = 1$.
   - We have that

$$\begin{aligned}
(\iota_v \circ \iota_v)(\ell_1 \otimes \cdots \otimes \ell_{k-1} \otimes \ell) &= \iota_v(\iota_v T \otimes \ell + (-1)^{k-1} \ell(v) T) \\
&= \iota_v(\iota_v T \otimes \ell) + (-1)^{k-1} \ell(v) \iota_v T \\
&= (-1)^{k-2} \ell(v) \iota_v T + (-1)^{k-1} \ell(v) \iota_v T \\
&= (-1)^{k-2} \ell(v) \iota_v T - (-1)^{k-2} \ell(v) \iota_v T \\
&= 0
\end{aligned}$$

6. If $T \in \mathcal{I}^k(V)$, then $\iota_v T \in \mathcal{I}^{k-1}(V)$.
   - Thus, ι_v induces a map $\iota_v : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$.
   - Proof: It suffices to check this for decomposables.
7. $\iota_{v_1} \circ \iota_{v_2} = -\iota_{v_2} \circ \iota_{v_1}$.

- Orientations:
  - A vector space V should have two orientations.
  - Two bases $e_1, \dots, e_n$ and $f_1, \dots, f_n$ are **orientation equivalent** if $T : V \to V$ an isomorphism has positive determinant. Otherwise, they are **orientation-inequivalent**.

  - An orientation on V is a choice of equivalence classes of bases under the equivalence relation on bases.
  - $T : V \to W$ given orientations, T preserves or reverses orientations.
- Fancy orientations.
  - An orientation on a 1D vector space L is a division into two halves.
  - Def: An orientation of V is an orientation of $\Lambda^n(V^*)$.
- We can prove that they're both the same.
  - If W and V are both oriented, then V/W gets a canonical orientation.

3.3 Chapter 1: Multilinear Algebra

From Guillemin and Haine (2018).

4/14:
- $\boldsymbol{\iota_v T}$: The $(k-1)$-tensor defined by

$$(\iota_v T)(v_1, \dots, v_{k-1}) = \sum_{r=1}^{k} (-1)^{r-1} T(v_1, \dots, v_{r-1}, v, v_r, \dots, v_{k-1})$$

  where $T \in \mathcal{L}^k(V)$, $k \in \mathbb{N}_0$, V is a vector space, and $v \in V$.
- If $v = v_1 + v_2$, then

$$\iota_v T = \iota_{v_1} T + \iota_{v_2} T$$

- If $T = T_1 + T_2$, then

$$\iota_v T = \iota_v T_1 + \iota_v T_2$$

- Lemma 1.7.4: If $T = \ell_1 \otimes \cdots \otimes \ell_k$, then

$$\iota_v T = \sum_{r=1}^{k} (-1)^{r-1} \ell_r(v) \ell_1 \otimes \cdots \otimes \hat{\ell}_r \otimes \cdots \otimes \ell_k$$

  where the hat over ℓ_r means that ℓ_r is deleted from the tensor product.
- Lemma 1.7.6: $T_1 \in \mathcal{L}^p(V)$ and $T_2 \in \mathcal{L}^q(V)$ imply

$$\iota_v(T_1 \otimes T_2) = \iota_v T_1 \otimes T_2 + (-1)^p T_1 \otimes \iota_v T_2$$

- Lemma 1.7.8: $T \in \mathcal{L}^k(V)$ implies that for all $v \in V$, we have

$$\iota_v(\iota_v T) = 0$$

  Proof. It suffices by linearity to prove this for decomposable tensors. We induct on k. For the base case $k = 1$, the claim is trivially true. Now suppose inductively that we have proven the claim for $k - 1$. Consider $\ell_1 \otimes \cdots \otimes \ell_k$. Taking $T = \ell_1 \otimes \cdots \otimes \ell_{k-1}$ and $\ell = \ell_k$, we obtain

$$\iota_v(\iota_v(T \otimes \ell)) = \iota_v(\iota_v T) \otimes \ell + (-1)^{k-2} \ell(v) \iota_v T + (-1)^{k-1} \ell(v) \iota_v T$$

  The first term is zero by the inductive hypothesis, and the second two cancel each other out, as desired. ☐

- Claim 1.7.10: For all $v_1, v_2 \in V$, we have that

$$\iota_{v_1} \iota_{v_2} = -\iota_{v_2} \iota_{v_1}$$

Proof. Let $v = v_1 + v_2$. Then $\iota_v = \iota_{v_1} + \iota_{v_2}$. Therefore,

$$\begin{aligned}
0 &= \iota_v \iota_v && \text{Lemma 1.7.8}\\
&= (\iota_{v_1} + \iota_{v_2})(\iota_{v_1} + \iota_{v_2})\\
&= \iota_{v_1}\iota_{v_1} + \iota_{v_1}\iota_{v_2} + \iota_{v_2}\iota_{v_1} + \iota_{v_2}\iota_{v_2}\\
&= \iota_{v_1}\iota_{v_2} + \iota_{v_2}\iota_{v_1} && \text{Lemma 1.7.8}
\end{aligned}$$

yielding the desired result. ☐

- Lemma 1.7.11: If $T \in \mathcal{L}^k(V)$ is redundant, then so is $\iota_v T$.

  Proof. Let $T = T_1 \otimes \ell \otimes \ell \otimes T_2$ where $\ell \in V^*$, $T_1 \in \mathcal{L}^p(V)$, and $T_2 \in \mathcal{L}^q(V)$. By Lemma 1.7.6, we have that

  $$\iota_v T = \iota_v T_1 \otimes \ell \otimes \ell \otimes T_2 + (-1)^p T_1 \otimes \iota_v(\ell \otimes \ell) \otimes T_2 + (-1)^{p+2} T_1 \otimes \ell \otimes \ell \otimes \iota_v T_2$$

  Thus, since the first and third terms above are redundant and $\iota_v(\ell \otimes \ell) = \ell(v)\ell - \ell(v)\ell = 0$ by Lemma 1.7.4, we have the desired result. ☐

- $\boldsymbol{\iota_v \omega}$: The $\mathcal{I}^k(V)$-coset $\pi(\iota_v T)$, where $\omega = \pi(T)$.
- Proves that $\iota_v \omega$ does not depend on the choice of T.
- **Inner product operation**: The linear map $\iota_v : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$.
- The inner product has the following important identities.

$$\begin{aligned}
\iota_{(v_1+v_2)}\omega &= \iota_{v_1}\omega + \iota_{v_2}\omega\\
\iota_v(\omega_1 \wedge \omega_2) &= \iota_v\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge \omega_2\\
\iota_v(\iota_v \omega) &= 0\\
\iota_{v_1}\iota_{v_2}\omega &= -\iota_{v_2}\iota_{v_1}\omega
\end{aligned}$$

4/18:

- As we developed the pullback $A^*T \in \mathcal{L}^k(V)$, we now look to develop a pullback on $\Lambda^k(V^*)$.
- Lemma 1.8.1: If $T \in \mathcal{I}^k(W)$, then $A^*T \in \mathcal{I}^k(V)$.

  Proof. It suffices to prove this for redundant k-tensors. Let $T = \ell_1 \otimes \cdots \otimes \ell_k$ be such that $\ell_i = \ell_{i+1}$. Then we have that

  $$\begin{aligned}
  A^*T &= A^*(\ell_1 \otimes \cdots \otimes \ell_k)\\
  &= A^*\ell_1 \otimes \cdots \otimes A^*\ell_k && \text{Exercise 1.3.iii}
  \end{aligned}$$

  where $A^*\ell_i = A^*\ell_{i+1}$ so that $A^*T \in \mathcal{I}^k(V)$, as desired. ☐

- $\boldsymbol{A^*\omega}$: The $\mathcal{I}^k(W)$-coset $\pi(A^*T)$, where $\omega = \pi(T)$.
- Claim 1.8.3: $A^*\omega$ is well-defined.

  Proof. Suppose $\omega = \pi(T) = \pi(T')$. Then $T = T' + S$ where $S \in \mathcal{I}^k(W)$. It follows that $A^*T = A^*T' + A^*S$, but since $A^*S \in \mathcal{I}^k(V)$ (Lemma 1.8.1), we have that

  $$\pi(A^*T) = \pi(A^*T')$$

  as desired. ☐

- Proposition 1.8.4. The map $A^* : \Lambda^k(W^*) \to \Lambda^k(V^*)$ sending $\omega \mapsto A^*\omega$ is linear. Moreover,

1. If $\omega_i \in \Lambda^{k_i}(W^*)$ $(i = 1, 2)$, then
$$A^*(\omega_1 \wedge \omega_2) = A^*(\omega_1) \wedge A^*(\omega_2)$$
2. If U is a vector space and $B : U \to V$ is a linear map, then for $\omega \in \Lambda^k(W^*)$,
$$B^* A^* \omega = (AB)^* \omega$$

(Hint: This proposition follows immediately from Exercises 1.3.iii-1.3.iv.)

- **Determinant** (of A): The number a such that $A^*\omega = a\omega$, where $A^* : \Lambda^n(V^*) \to \Lambda^n(V^*)$. *Denoted by* $\mathbf{det\,(}\boldsymbol{A}\mathbf{)}$.
- Proposition 1.8.7: If A and B are linear mappings of V into V, then
$$\det(AB) = \det(A)\det(B)$$

  Proof. Proposition 1.8.4(2) implies that
$$\begin{aligned}\det(AB)\omega &= (AB)^*\omega \\ &= B^*(A^*\omega) \\ &= \det(B)A^*\omega \\ &= \det(B)\det(A)\omega\end{aligned}$$
  as desired. □

- $\mathbf{id}_{\boldsymbol{V}}$: The identity map on V.
- Proposition 1.8.8: $\det(\mathrm{id}_V) = 1$.
  - Hint: id_V^* is the identity map on $\Lambda^n(V^*)$.
- Proposition 1.8.9: If $A : V \to V$ is not surjective, then $\det(A) = 0$.

  Proof. Let $W = \mathrm{im}(A)$. If A is not onto, $\dim W < n$, implying that $\Lambda^n(W^*) = 0$. Now let $A = i_W B$ where i_W is the inclusion map of W into V and B is the mapping A regarded as a mapping from V to W. It follows by Proposition 1.8.4(1) that if $\omega \in \Lambda^n(V^*)$, then
$$A^*\omega = B^* i_W^* \omega$$
  where $i_W^*\omega = 0$ as an element of $\Lambda^n(W^*)$. □

- Deriving the typical formula for the determinant.
  - Let V, W be n-dimensional vector spaces with respective bases $e_1, \dots, e_n$ and $f_1, \dots, f_n$.
  - Denote the corresponding dual bases by $e_1^*, \dots, e_n^*$ and $f_1^*, \dots, f_n^*$.
  - Let $A : V \to W$. Recall that if the matrix of A is $[a_{i,j}]$, then the matrix of $A^* : W^* \to V^*$ is $(a_{j,i})$, i.e., if
$$Ae_j = \sum_{i=1}^n a_{i,j} f_i$$
    then
$$A^* f_j^* = \sum_{i=1}^n a_{j,i} e_i^*$$

- It follows that

$$\begin{aligned} A^*(f_1^* \wedge \cdots \wedge f_n^*) &= A^* f_1^* \wedge \cdots \wedge A^* f_n^* \\ &= \sum_{1 \leq k_1, \dots, k_n \leq n} (a_{1,k_1} e_{k_1}^*) \wedge \cdots \wedge (a_{n,k_n} e_{k_n}^*) \\ &= \sum_{1 \leq k_1, \dots, k_n \leq n} a_{1,k_1} \cdots a_{n,k_n} e_{k_1}^* \wedge \cdots \wedge e_{k_n}^* \end{aligned}$$

- At this point, we are summing over all possible lists of length n containing the numbers between 1 and n at each index.
  - However, any list in which a number repeats will lead to a wedge product of a linear functional with itself, making that term equal to zero.
  - Thus, it is only necessary to sum over those terms that are non-repeating.
  - But the terms that are non repeating are exactly the permutations $\sigma \in S_n$.
- Thus,

$$\begin{aligned} A^*(f_1^* \wedge \cdots \wedge f_n^*) &= \sum_{\sigma \in S_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} (e_1^* \wedge \cdots \wedge e_n^*)^\sigma \\ &= \sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} e_1^* \wedge \cdots \wedge e_n^* \\ &= \det([a_{i,j}]) e_1^* \wedge \cdots \wedge e_n^* \end{aligned}$$

- If $V = W$ and $e_i = f_i$ $(i = 1, \dots, n)$, then we may define $\omega = e_1^* \wedge \cdots \wedge e_n^* = f_1^* \wedge \cdots \wedge f_n^* \in \Lambda^k(V^*)$ to obtain

$$A^*\omega = \det([a_{i,j}])\omega$$

  which proves that

$$\det(A) = \det([a_{i,j}])$$

  as desired.

- **Orientation** (of ℓ): A choice of one of the disconnected components of $\ell \setminus \{0\}$, where $\ell \subset \mathbb{R}^2$ is a straight line through the origin.
- **Orientation** (of L): A choice of one of the connected components of $L \setminus \{0\}$, where L is a one-dimensional vector space.
- **Positive component** (of $L \setminus \{0\}$): The component chosen in the orientation of L. *Denoted by* $\boldsymbol{L_+}$.
- **Negative component** (of $L \setminus \{0\}$): The component chosen in the orientation of L. *Denoted by* $\boldsymbol{L_-}$.
- **Positively oriented** ($v \in L$): A vector $v \in L$ such that $v \in L_+$.
- **Orientation** (of V) An orientation of the one-dimensional vector space $\Lambda^n(V^*)$, where V is an n-dimensional vector space.
- "One important way of assigning an orientation to V is to choose a basis $e_1, \dots, e_n$ of V. Then if $e_1^*, \dots, e_n^*$ is the dual basis, we can orient $\Lambda^n(V^*)$ by requiring that $e_1^* \wedge \cdots \wedge e_n^*$ be in the positive component of $\Lambda^n(V^*)$" (Guillemin & Haine, 2018, p. 29).
- **Positively oriented** (ordered basis $e_1, \dots, e_n$ of V): An ordered basis $e_1, \dots, e_n \in V$ such that $e_1^* \wedge \cdots \wedge e_n^* \in \Lambda^n(V^*)_+$.
- Proposition 1.9.7: If $e_1, \dots, e_n$ is positively oriented, then $f_1, \dots, f_n$ is positively oriented iff $\det[a_{i,j}] > 0$ where $e_j = \sum_{i=1}^n a_{i,j} f_i$.

Proof. We have that

$$f_1^* \wedge \cdots \wedge f_n^* = \det[a_{i,j}] e_1^* \wedge \cdots \wedge e_n^*$$

□

- Corollary 1.9.8: If $e_1, \dots, e_n$ is a positively oriented basis of V, then the basis

$$e_1, \dots, e_{i-1}, -e_i, e_{i+1}, \dots, e_n$$

is negatively oriented.

- Theorem 1.9.9: Given orientations on V and V/W (where $\dim V = n > 1$, $W \leq V$, and $\dim W = k < n$), one gets from these orientations a natural orientation on W.

Proof. The orientations on V and V/W come prepackaged with a basis. We first apply an orientation to W based on these bases, and then show that any choice of basis for $V, V/W$ induces a basis with the same orientation on W. Let's begin.

Let $r = n - k$, and let $\pi : V \to V/W$. By Exercises 1.2.i and 1.2.ii, we may choose a basis $e_1, \dots, e_n$ of V such that $e_{r+1}, \dots, e_n$ is a basis of W. It follows that $\pi(e_1), \dots, \pi(e_r)$ is a basis of V/W. WLOG[1], take $\pi(e_1), \dots, \pi(e_r)$ and $e_1, \dots, e_n$ to be positively oriented on V/W and V, respectively. Assign to W the orientation associated with $e_{r+1}, \dots, e_n$.

Now suppose $f_1, \dots, f_n$ is another basis of V such that $f_{r+1}, \dots, f_n$ is a basis of W. Let $A = [a_{i,j}]$ express $e_1, \dots, e_n$ as linear combinations of $f_1, \dots, f_n$, i.e., let

$$e_j = \sum_{i=1}^{n} a_{i,j} f_i$$

for all $j = 1, \dots, n$. Now as will be explained below, A must have the form

$$A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

where B is the $r \times r$ matrix expressing $\pi(e_1), \dots, \pi(e_r)$ as linear combinations of $\pi(f_1), \dots, \pi(f_r)$, and D is the $k \times k$ matrix expressing the basis vectors $e_{r+1}, \dots, e_n$ as linear combinations of $f_{r+1}, \dots, f_n$. We have just explained B and D. We don't particularly care about C or have a good way of defining its structure. We can, however, take the block labeled zero to be the $k \times r$ zero matrix by Proposition 1.2.9; in particular, since these components of these vectors will be fed into π and fall within W, they can moved around wherever without altering the identities of the W-cosets to which they pertain. Having justified this structure for A, we see that we can take

$$\det(A) = \det(B)\det(D)$$

It follows by Proposition 1.9.7 as well as the positivity of $\det(A)$ and $\det(B)$ that $\det(D)$ is positive, and hence the orientation of $e_{r+1}, \dots, e_n$ and $f_{r+1}, \dots, f_n$ are one and the same. □

- **Orientation-preserving** (map A): A bijective linear map $A : V_1 \to V_2$, where V_1, V_2 are oriented n-dimensional vector spaces, such that for all $\omega \in \Lambda^n(V_2^*)_+$, we have that $A^*\omega \in \Lambda^n(V_1^*)_+$.
- If $V_1 = V_2$, A is orientation-preserving iff $\det(A) > 0$.
- Proposition 1.9.14: Let V_1, V_2, V_3 be oriented n-dimensional vector spaces, and let $A_1 : V_1 \to V_2$ and $A_2 : V_2 \to V_3$ be bijective linear maps. Then if A_1, A_2 are orientation preserving, so is $A_2 \circ A_1$.

[1] If the first basis is negatively oriented, we may substitute $-e_1$ for e_1. If the second basis is negatively oriented, we may substitute $-e_n$ for e_n.