Week 5

Differentiation

5.1 Vector Calculus Operations

- Announcements.

    - No class this Friday, next Monday.
    - Midterm next Friday.
        - Up through Chapter 2.
        - The exam will likely be computationally heavy.
        - Compute d, pullbacks, interior products, Lie derivatives, etc.
        - Emphasis on Chapter 2 as opposed to Chapter 1 even though it all builds on itself.
        - He'll probably cook up a few problems too.
    - There is a recorded lecture for us.
        - On Chapter 3 content.
        - We'll cover Chapter 3 in kind of an impressionistic way as it is.
    - There are also some notes on the physics stuff.

- Vector calculus operations.

    - In one dimension, you have functions, and you take derivatives.
        - The derivative operation does essentially map $\Omega^0 \to \Omega^1$ or $C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$.
    - In two dimensions, ...
        - $d^2 = 0$ reflects the fact that gradient vector fields are curl-free.
    - If you want to understand the 2D-curl business. . .
        - $\operatorname{curl}(v) : \mathbb{R}^2 \to \mathbb{R}$ is intuitively about balls spinning around in a vector field.
        - There's also a nice formula to compute it.
        - And then there's a connection with $d : \Omega^1 \to \Omega^2$.
    - In 3D, you can take top-dimensional forms (which are just functions) and bottom-dimensional forms (which are by definition functions) and you can work out an identification between them.
    - Note that $\operatorname{curl} : \mathfrak{X}(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$, where $\mathfrak{X}(\mathbb{R}^2)$ is the space of vector fields.

- The musical operator $\sharp$ identifies forms with vector fields, i.e., $\sharp : \Omega^1 \to \mathfrak{X}(\mathbb{R}^2)$.

- Properties of exterior derivatives $d : \Omega^k(U) \to \Omega^{k+1}(U)$.

    1. $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$ and $d(\lambda\omega) = \lambda \, d\omega$.
    2. Product rule $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$.

– Special case $k = \ell = 0$. Then
$$d(fg) = g\,df + f\,dg$$
which is the usual product rule for gradient.

– Claim:
$$d\left(\sum_I f_I\,dx_I\right) = \sum_I df_I \wedge dx_I$$

■ Let $\omega_1 \in \Omega^k$ and $\omega_2 \in \Omega^\ell$ be defined by
$$\omega_1 = \sum_I f_I\,dx_I \qquad\qquad \omega_2 = \sum_J g_J\,dx_J$$

where we're summing over all I such that $|I| = k$ and all J such that $|J| = \ell$. Then
$$\omega_1 \wedge \omega_2 = \sum_{I,J} f_I g_J\,dx_I \wedge dx_J \,d(\omega_1 \wedge \omega_2) \qquad = \sum_{I,J} d(f_I g_J) \wedge dx_I \wedge dx_J$$

■ Note that
$$d(f_I g_J) = g_J\,df_I + f_I\,dg_J$$
and
$$dg_J \wedge dx_I = (-1)^k\,dx_I \wedge dg_J$$

■ These identities allow us to take the previous equation to
$$d(\omega_1 \wedge \omega_2) = \sum_{I,J} g_J\,df_I \wedge dx_I \wedge dx_J + (-1)^k f_I\,dx_I \wedge dg_J \wedge dx_J$$
$$= \sum_{I,J} (df_I \wedge dx_I) \wedge (g_J\,dx_J) + \sum_{I,J} (f_I\,dx_I) \wedge (ddg_J \wedge dx_J)$$
$$= d\omega_1 \wedge \omega_2 + (-1)^k \omega_1\,d\omega_2$$

3. $d^2 = 0$.

  – Let $\omega = \sum_I f_I\,dx_I$.

  – Then

$$d^2(\omega) = d(d\omega)$$
$$= d\left(\sum_I df_I \wedge dx_I\right)$$
$$= \sum_I d(df_I \wedge dx_I) \qquad\qquad \text{Property 1}$$
$$= \sum_I d(df_I) \wedge dx_I \qquad\qquad \text{Property 2}$$

so it suffices to just show that $d^2 f = 0$ for all $f \in \Omega^0$.

  – We know that $df = \sum_{i=1}^n \partial f / \partial x_i\,dx_i$. Thus,

$$d(df) = \sum_i d\left(\frac{\partial f}{\partial x_i}\right) \wedge dx_i$$
$$= \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i}\,dx_j \wedge dx_i$$
$$= 0$$

- The last equality holds because of commuting partial derivatives for smooth f, and the fact that changing order introduces a negative sign by some property.

- In fact, if we fix $d^0 : \Omega^0(U) \to \Omega^1(U)$ to be the "gradient," then these properties characterize the function d on its domain and codomain. In particular, d is the unique function on its domain and codomain that satisfies these properties.

  - We define it by

  $$d\left(\sum_I f_I \, dx_I\right) = \sum_I df_I \wedge dx_I$$

  - The above properties characterize it axiomatically.
  - We can prove this uniqueness theorem.

- **Closed** (form): A form $\omega \in \Omega^k(U)$ such that $d\omega = 0$.

- **Exact** (form): A form $\omega \in \Omega^k(U)$ such that $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(U)$.

- $d^2 = 0$ implies closed and exact implies closed.

- **Poincaré lemma**: Locally closed forms are exact.

5.2 Chapter 2: Differential Forms

From Guillemin and Haine (2018).

5/5:
- As we formed the kth exterior powers $\Lambda^k(V^*)$, we can form the kth exterior powers $\Lambda^k(T_p^*\mathbb{R}^n)$.

  - Since $\Lambda^1(T_p^*\mathbb{R}^n) = T_p^*\mathbb{R}^n$, we can think of a one-form as a function which takes its value at p in the space $\Lambda^1(T_p^*\mathbb{R}^n)$.

- **k-form** (on U): A function which assigns to each point $p \in U$ an element $\omega_p \in \Lambda^k(T_p^*\mathbb{R}^n)$, where $U \subset \mathbb{R}^n$ is open.

- We can use the wedge product to construct k-forms.

  - Let $\omega_1, \ldots, \omega_k$ be one-forms. Then $\omega_1 \wedge \cdots \wedge \omega_k$ is the k-form whose value at p is the wedge product

  $$(\omega_1 \wedge \cdots \wedge \omega_k)_p = (\omega_1)_p \wedge \cdots \wedge (\omega_k)_p$$

  - Let $f_1, \ldots, f_k$ be real-valued functions in $C^\infty(U)$. Suppose $\omega_i = df_i$. Then we may obtain the k-form whose value at p is

  $$(df_1 \wedge \cdots \wedge df_k)_p = (df_1)_p \wedge \cdots \wedge (df_k)_p$$

- Since $(dx_1)_p, \ldots, (dx_n)_p$ are a basis of $T_p^*\mathbb{R}^n$, the wedge products

  $$(dx_I)_p = (dx_{i_1})_p \wedge \cdots \wedge (dx_{i_k})_p$$

  where $I = (i_1, \ldots, i_k)$ is a strictly increasing multi-index of n of length k form a basis of $\Lambda^k(T_p^*\mathbb{R}^n)$.

- Thus, every $\omega_p \in \Lambda^k(T_p^*\mathbb{R}^n)$ has a unique decomposition

  $$\omega_p = \sum_I c_I (dx_I)_p$$

  where every $c_I \in \mathbb{R}$.

- Similarly, every k-form ω on U has a unique decomposition

$$\omega = \sum_I f_I \, \mathrm{d}x_I$$

  where every $f_I : U \to \mathbb{R}$.

- **Class C^r** (k-form): A k-form ω for which every f_I in its decomposition is in $C^r(U)$.

- From here on out, we assume unless otherwise stated that all k-forms we consider are of class C^∞.

- **$\Omega^k(U)$**: The set of k-forms of class C^∞ on U.

- **$f\omega$**: The k-form defined as follows, where $f \in C^\infty(U)$ and $\omega \in \Omega^k(U)$. *Given by*

$$p \mapsto f(p)\omega_p$$

- **Sum** (of ω_1, ω_2): The k-form defined as follows, where $\omega_1, \omega_2 \in \Omega^k(U)$. *Denoted by $\boldsymbol{\omega_1 + \omega_2}$. Given by*

$$p \mapsto (\omega_1)_p + (\omega_2)_p$$

- **Wedge product** (of ω_1, ω_2): The $(k_1 + k_2)$-form defined as follows, where $\omega_1 \in \Omega^{k_1}(U)$ and $\omega_2 \in \Omega^{k_2}(U)$. *Denoted by $\boldsymbol{\omega_1 \wedge \omega_2}$. Given by*

$$p \mapsto (\omega_1)_p \wedge (\omega_2)_p$$

- **Zero-form**: A function which assigns to each $p \in U$ an element of $\Lambda^0(T_p^*\mathbb{R}^n) = \mathbb{R}$. *Also known as* **real-valued function**.

- It follows from the definition of zero-forms that

$$\Omega^0(U) = C^\infty(U)$$

- **Exterior differentiation operation**: The operator from $\Omega^0(U) \to \Omega^1(U)$ which associates to a function $f \in C^\infty(U)$ the 1-form $\mathrm{d}f$. *Denoted by* $\mathbf{d}$.

- We now seek to define a generalized version of the exterior differentiation operation; in particular, we would like to define an analogous function $\mathrm{d} : \Omega^k(U) \to \Omega^{k+1}(U)$.

- Desired properties of exterior differentiation.

  1. If $\omega_1, \omega_2 \in \Omega^k(U)$, then
  $$\mathrm{d}(\omega_1 + \omega_2) = \mathrm{d}\omega_1 + \mathrm{d}\omega_2$$

  2. If $\omega_1 \in \Omega^k(U)$ and $\omega_2 \in \Omega^\ell(U)$, then
  $$\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge \mathrm{d}\omega_2$$

  3. If $\omega \in \Omega^k(U)$, then
  $$\mathrm{d}(\mathrm{d}\omega) = 0$$

- Consequences of these properties.

- Lemma 2.4.5: Let $U \subset \mathbb{R}^n$ open. If $f_1, \ldots, f_k \in C^\infty(U)$, then

$$\mathrm{d}(\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k) = 0$$

Proof. We induct on k. For the base case $k = 1$, we have that $\mathrm{d}(\mathrm{d}f_1) = 0$ by Property 3. Now suppose inductively that we have proven the claim for $k - 1$ functions; we now seek to prove it for k functions. Let $\mu = \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_{k-1}$. Then by the induction hypothesis, $\mathrm{d}\mu = 0$. Therefore,

$$\begin{aligned}
\mathrm{d}(\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k) &= \mathrm{d}(\mu \wedge \mathrm{d}f_k) \\
&= \mathrm{d}\mu \wedge \mathrm{d}f_k + (-1)^{k-1} \mu \wedge \mathrm{d}(\mathrm{d}f_k) \qquad \text{Property 2} \\
&= 0
\end{aligned}$$

as desired. □

- A special case of Lemma 2.4.5 is that
$$\mathrm{d}(\mathrm{d}x_I) = 0$$

- Now since every k-form $\omega \in \Omega^k(U)$ has a unique decomposition in terms of the $\mathrm{d}x_I$, Property 2 and the above equation reveal that
$$\mathrm{d}\omega = \sum_I \mathrm{d}f_I \wedge \mathrm{d}x_I$$

- Therefore, if there exists an operator d satisfying Properties 1-3, then d necessarily has the above form. All that's left is to show that the operator defined above has these properties.

- Proposition 2.4.10: Let $U \subset \mathbb{R}^n$ be open. There is a unique operator $\mathrm{d} : \Omega^*(U) \to \Omega^{*+1}(U)$ satisfying Properties 1-3.

    Proof. ... □

- **Closed** (k-form): A k-form $\omega \in \Omega^k(U)$ for which $\mathrm{d}\omega = 0$.

- **Exact** (k-form): A k-form $\omega \in \Omega^k(U)$ such that $\omega = \mathrm{d}\mu$ for some $\mu \in \Omega^{k-1}(U)$.

- Property 3 implies that every exact k-form is closed.

    - The converse is not true even for 1-forms (see Exercise 2.1.iii).
    - "It is a very interesting (and hard) question to determine if an open set U has the following property: For $k > 0$, every closed k-form is exact" (Guillemin & Haine, 2018, p. 49).
        - ■ Note that we do not consider zero-forms since there are no (-1)-forms for which to define exactness.

- If $f \in C^\infty(U)$ and $\mathrm{d}f = 0$, then f is constant on connected components of U (see Exercise 2.2.iii).

- Lemma 2.4.16 (Poincaré lemma): If ω is a closed form on U of degree $k > 0$, then for every point $p \in U$, there exists a neighborhood of p on which ω is exact.

    Proof. See Exercises 2.4.v and 2.4.vi. □