Chapter 1

Multilinear Algebra

1.1 Notes

3/28: • Motivation for the course and an overview of Guillemin and Haine (2018).

3/30: • Plan:

  - More (multi)linear algebra.

- Dual spaces.
- Let V be an n-dimensional real vector space.
- **Hom $(\boldsymbol{V},\mathbb{R})$**: The set of all homomorphisms (i.e., linear maps) from V to $\mathbb{R}$. *Also known as* $\boldsymbol{V^*}$.
- **Dual basis** (for V^*): The set of linear transformations from V to $\mathbb{R}$ defined by

$$e_j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

  where $e_1, \dots, e_n$ is a basis of V. *Denoted by* $e_1^*, \dots, e_n^*$.
- Check: $e_1^*, \dots, e_n^*$ are a basis for V^*.
  - Are they linearly independent? Let $c_1 e_1^* + \cdots + c_n e_n^* = 0 \in \operatorname{Hom}(V, \mathbb{R})$. Then

$$c_i = (c_1 e_1^* + \cdots + c_n e_n^*)(e_i) = 0 \in \mathbb{R}$$

    as desired.
  - Span? Let $\varphi \in \operatorname{Hom}(V, \mathbb{R})$. Then we can verify that

$$\varphi(e_1) e_1^* + \cdots + \varphi(e_n) e_n^* = \varphi$$

    - We prove this by verifying the previous statement on the basis of V (if two linear transformations have the same action on the basis of a vector space, they are equal).
- With a choice of basis for V, we obtain an isomorphism $\varepsilon : V \to V^*$ with the mapping $e_i \mapsto e_i^*$ for all i.
- The dual space is known as such because $(V^*)^* \cong V$, where $\cong$ is **canonical** (no choice of basis is needed).
- One more property of dual spaces: **functoriality**.

  - Given a linear transformation $A : V \to W$, we know that $A^* : W^* \to V^*$ where A^* is the transpose of A. In particular, if $\varphi \in W^*$, then $\varphi \circ A : V \to \mathbb{R}$.
  - Claim: A^* is linear.

- **Functoriality**: If $A : V \to W$ and $B : W \to U$, then $B^* : U^* \to W^*$ and $A^* : W^* \to V^*$. The functoriality statement is that $(B \circ A)^* = A^* \circ B^*$.
- A^* is the **pullback** (or transpose) of A.
- Let $v_1, \dots, v_n$ be a basis for V and $w_1, \dots, w_m$ be a basis for W. Then $[A]_{v_1,\dots,v_n}^{w_1,\dots,w_m} = A$ is the matrix of the linear transformation A with respect to these bases. Then if $v_1^*, \dots, v_n^*$ and $w_1^*, \dots, w_m^*$ are the corresponding dual bases, then $[A^*]_{v_1^*,\dots,v_n^*}^{w_1^*,\dots,w_m^*} = A^T$. We can and should verify this for ourselves.
- This is over the real numbers, so A^* is just the transpose because there are no complex numbers of which to take the conjugate!
- A generalization: Tensors.
- ***k*-tensor**: A **multilinear** map

$$T : \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}$$

- **Multilinear** (map T): A function T such that

$$\begin{aligned} T(v_1, \dots, v_i^1 + v_i^2, \dots, v_k) &= T(v_1, \dots, v_i^1, \dots, v_k) + T(v_1, \dots, v_i^2, \dots, v_k) \\ T(v_1, \dots, \lambda v_i, \dots, v_k) &= \lambda T(v_1, \dots, v_i, \dots, v_k) \end{aligned}$$

  for all $(v_1, \dots, v_k) \in V^k$.
- The determinant is an n-tensor!
- 1-tensors are just covectors.
- $\boldsymbol{\mathcal{L}^k(V)}$: The vector space of all k-tensors on V.
- Calculating $\dim \mathcal{L}^k(V)$. (Answer not given in this class.)
- Let $A : V \to W$. Then $A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$.
  - Check $(A \circ B)^* = B^* \circ A^*$.
- **Multi-index of *n* of length *k***: A k-tuple $(i_1, \dots, i_k)$ where each $i_j \in \mathbb{N}$ satisfies $1 \leq i_j \leq n$ $(j = 1, \dots, k)$. *Denoted by* $\boldsymbol{I}$.
- Let $e_1, \dots, e_n$ be a basis for V.
- **Tensor product** (of $T_1 \in \mathcal{L}^k(V)$, $T_2 \in L^l(V)$): The function from V^{k+l} to $\mathbb{R}$ defined by

$$(v_1, \dots, v_{k+l}) \mapsto T_1(v_1, \dots, v_k) T_2(v_{k+1}, \dots, v_{k+l})$$

  Denoted by $\boldsymbol{T_1 \otimes T_2}$.
- Claims:
  1. $T_1 \otimes T_2 \in L^{k+l}(V)$.
  2. $A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$.
- $\boldsymbol{e_I^*}$: The function $e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*$, where $I = (i_1, \dots, i_k)$ is a multi-index of n of length k.
- Claim: Letting I range over all n^k multi-indices of n of length k, the e_I^* are a basis for $\mathcal{L}^k(V)$.

- If $V=\mathbb{R}$, then $V=\mathbb{R}e_1$. If $V=\mathbb{R}^2$, then $V=\mathbb{R}e_1\oplus\mathbb{R}e_2$.
- We know that $L^1(V)=V^*=Re_1^*$. Thus, $e_1^*\otimes e_2^*:V\times V\to\mathbb{R}$. Thus, for example,

$$(e_1^*\otimes e_2^*)((1,2),(3,4))=e_1^*(1,2)\cdot e_2^*(3,4)=1\cdot 4=4$$

4/1:

- Plan: More multilinear algebra.
  - Properties of the tensor product.
  - Sign of a permutation.
  - Alternating tensors (lead into differential forms down the road).
- Recall: V is an n-dimensional vector space over $\mathbb{R}$ with basis $e_1,\dots,e_n$. $\mathcal{L}^k(V)$ is the vector space of k-tensors on V. $\{e_I^*\mid I \text{ a multiindex of } n \text{ of length } k\}$ is a basis for $\mathcal{L}^k(V)$.
- For example, if $V=\mathbb{R}^2$ and $T\in\mathcal{L}^2(V)$, then

$$T(a_1e_1+a_2e_2,b_1e_1+b_2e_2)=a_1b_1T(e_1,e_1)+a_1b_2T(e_1,e_2)+a_2b_1T(e_2,e_1)+a_2b_2T(e_2,e_2)$$

  - A basis of $\mathcal{L}^2(V)$ is
$$\{e_1^*\otimes e_1^*,e_1^*\otimes e_2^*,e_2^*\otimes e_1^*,e_2^*\otimes e_2^*\}$$
  - Recall that some basic properties are
$$e_1^*\otimes e_2^*((1,2),(3,4))=1\cdot 4=4 \qquad\qquad e_2^*\otimes e_1^*((1,2),(3,4))=2\cdot 3=6$$
  - It follows by the initial decomposition of T that
$$T=a_1b_1e_1^*\otimes e_1^*+a_1b_2e_1^*\otimes e_2^*+a_2b_1e_2^*\otimes e_1^*+a_2b_2e_2^*\otimes e_2^*$$
- Important consequence: To know the action of T on an arbitrary pair of vectors, you need only know its action on the basis; a higher-dimensional generalization of the earlier property.
- Note that
$$e_I^*(e_J)=\delta_{IJ}=\begin{cases}1 & I=J\\ 0 & I\neq J\end{cases}$$
- Basic properties of the tensor product.
  1. *Right-distributive*: If $T_1\in\mathcal{L}^k(V)$ and $T_2,T_3\in\mathcal{L}^\ell(V)$, then
$$T_1\otimes(T_2+T_3)=T_1\otimes T_2+T_1\otimes T_3$$
  2. *Left-distributive*: If $T_1,T_2\in\mathcal{L}^k(V)$ and $T_3\in\mathcal{L}^\ell(V)$, then
$$(T_1+T_2)\otimes T_3=T_1\otimes T_3+T_2\otimes T_3$$
  3. *Associative*: If $T_1\in\mathcal{L}^k(V)$, $T_2\in\mathcal{L}^\ell(V)$, and $T_3\in\mathcal{L}^m(V)$, then
$$T_1\otimes(T_2\otimes T_3)=(T_1\otimes T_2)\otimes T_2=T_1\otimes T_2\otimes T_3$$
  4. *Scalar multiplication*: If $T_1\in\mathcal{L}^k(V)$, $T_2\in\mathcal{L}^\ell(V)$, and $\lambda\in\mathbb{R}$, then
$$(\lambda T_1)\otimes T_2=\lambda(T_1\otimes T_2)=T_1\otimes(\lambda T_2)$$
- Note that the tensor product is not commutative.
- Aside: Defining the sign of a permutation.

- $\boldsymbol{S_A}$: The set of all automorphisms of A (bijections from A to A), where A is a set.
- $\boldsymbol{S_n}$: The set $S_{[n]}$.
- Given $\sigma_1, \sigma_2 \in S_n$, $\sigma_1 \circ \sigma_2 \in S_n$.
  - Thus, S_n is a **group**.
- **Transposition**: A function $\tau \in S_n$ such that

$$\tau(k) = \begin{cases} j & k = i \\ i & k = j \\ k & k \neq i, j \end{cases}$$

  for some $i, j \in [n]$. *Denoted by* $\boldsymbol{\tau_{i,j}}$.
- Theorem: An element of S_n can be written as the product of transpositions (i.e., for all $\sigma \in S_n$, there exist $\tau_1, \dots, \tau_m \in S_n$ such that $\sigma = \tau_1 \circ \cdots \circ \tau_m$).
- **Sign** (of $\sigma \in S_n$): The number (mod 2) of transpositions whose product equals σ. *Denoted by* $\boldsymbol{(-1)^\sigma}$, $\textbf{sign}\,\boldsymbol{(\sigma)}$.
- Theorem: The sign of σ is well-defined. Additionally,

$$(-1)^{\sigma_1 \sigma_2} = (-1)^{\sigma_1} \cdot (-1)^{\sigma_2}$$

- Example: Consider the identity permutation. $(-1)^\sigma = +1$. We can think of this as the product of zero transpositions or, for instance, as the product of the two transpositions $\tau_{1,2} \circ \tau_{1,2}$. Another example would be $\tau_{2,3} \circ \tau_{1,2} \circ \tau_{1,2} \circ \tau_{2,3}$.
- Theorem: Let X_i be a rational or polynomial function for each $i \in \mathbb{N}$. Then

$$(-1)^\sigma = \prod_{i<j} \frac{X_{\sigma(i)} - X_{\sigma(j)}}{X_i - X_j}$$

- Example: For the permutation $\sigma = (1, 2, 3)$, we have

$$\begin{aligned}
(-1)^\sigma &= \frac{X_{\sigma(1)} - X_{\sigma(2)}}{X_1 - X_2} \cdot \frac{X_{\sigma(1)} - X_{\sigma(3)}}{X_1 - X_3} \cdot \frac{X_{\sigma(2)} - X_{\sigma(3)}}{X_2 - X_3} \\
&= \frac{X_2 - X_3}{X_1 - X_2} \cdot \frac{X_2 - X_1}{X_1 - X_3} \cdot \frac{X_3 - X_1}{X_2 - X_3} \\
&= \frac{-(X_1 - X_2)}{X_1 - X_2} \cdot \frac{-(X_1 - X_3)}{X_1 - X_3} \cdot \frac{X_2 - X_3}{X_2 - X_3} \\
&= -1 \cdot -1 \cdot 1 \\
&= +1
\end{aligned}$$

  which squares with the fact that $\sigma = \tau_{1,2} \circ \tau_{2,3}$.
- Claims to verify with the above formula:
  1. $\text{sign}(\sigma) \in \{\pm 1\}$.
  2. $\text{sign}(\tau_{i,j}) = -1$.
  3. $\text{sign}(\sigma_1 \sigma_2) = \text{sign}(\sigma_1)\,\text{sign}(\sigma_2)$.

4/4:
- Plan:
  - More multilinear algebra.

  - Alternating k-tensors — 2 views:
    1. As a subspace of $\mathcal{L}^k(V)$.
    2. As a quotient of $\mathcal{L}^k(V)$.
  - Next time: Operators as alternating tensors.
    - Wedge products.
    - Interior products.
    - Pullbacks.
- Recall: $\dim V = n$, $e_1, \dots, e_n$ a basis, $\mathcal{L}^k(V)$ the space of k-tensors, $\sigma \in S_k$ implies $(-1)^\sigma \in \{\pm 1\}$, key property: $(-1)^{\sigma_1\sigma_2} = (-1)^{\sigma_1}(-1)^{\sigma_2}$.
- $\boldsymbol{T^\sigma}$: The k-tensor over V defined by

$$T^\sigma(v_1, \dots, v_k) = T(v_{\bar{\sigma}(1)}, \dots, v_{\bar{\sigma}(k)})$$

  where $T \in \mathcal{L}^k(V)$, $\sigma \in S_k$, and $\bar{\sigma}$ denotes the inverse of σ.
- Example: $n = 2$, $k = 2$. Let $T = e_1^* \otimes e_2^* \in \mathcal{L}^2(V)$. Let $\sigma = \tau_{1,2}$. Then $T^\sigma = e_2^* \otimes e_1^*$.
- Another property is $e_I^\sigma = e_{\sigma(I)}^*$.
- Properties:
  1. $T^{\sigma_1\sigma_2} = (T^{\sigma_1})^{\sigma_2}$.
  2. $(T_1 + T_2)^\sigma = T_1^\sigma + T_2^\sigma$.
  3. $(cT)^\sigma = cT^\sigma$.
- Thus, you can view $\sigma : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ as a linear map!
- **Alternating $\boldsymbol{k}$-tensor**: A tensor $T \in \mathcal{L}^k(V)$ such that $T^\sigma = (-1)^\sigma T$ for all $\sigma \in S_k$.
  - Equivalently, $T^\tau = -T$ for all $\tau \in S_k$.
- An example of an alternating 2-tensor when $\dim V = 2$ is $T = e_1^* \otimes e_2^* - e_2^* \otimes e_1^*$.
  - Naturally, $T_{1,2}^\tau = -T$, and $\tau_{1,2}$ is the unique transposition in S_2.
- $e_1^* \otimes e_2^*$ is *not* an alternating 2-tensor since $(e_1^* \otimes e_2^*)^\tau = e_2^* \otimes e_1^* \neq (-1)^\tau (e_1^* \otimes e_2^*)$.
- We can look at $n = 2$, $k = 1$ for ourselves.
- Note: If T_1, T_2 are both alternating k-tensors, then $T_1 + T_2$ is also alternating, as is cT_1 for all $c \in \mathbb{R}$.
- $\boldsymbol{\mathcal{A}^k(V)}$: The vector space of alternating k-tensors.
- **Alt ($\boldsymbol{T}$)**: The function $\text{Alt} : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ defined by

$$\text{Alt}(T) = \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma$$

- Properties:
  1. $\text{im}(\text{Alt}) = \mathcal{A}^k(V)$.
  2. $\mathcal{L}^k(V) / \ker(\text{Alt}) = \Lambda^k(V^*)$ is isomorphic to $\mathcal{A}^k(V)$.
  3. $\text{Alt}(T)^\sigma = (-1)^\sigma \text{Alt}(T)$.

- Proof:

$$\begin{aligned}\mathrm{Alt}(T)^{\sigma'} &= \left(\sum_{\sigma\in S_k}(-1)^\sigma T^\sigma\right)^{\sigma'}\\ &= \sum_{\sigma\in S_k}(-1)^\sigma T^{\sigma\sigma'}\\ &= (-1)^{\sigma'}\sum_{\sigma\in S_k}(-1)^{\sigma'}(-1)^\sigma T^{\sigma\sigma'}\\ &= (-1)^{\sigma'}\sum_{\sigma\in S_k}(-1)^{\sigma\sigma'}T^{\sigma\sigma'}\\ &= (-1)^{\sigma'}\,\mathrm{Alt}(T)\end{aligned}$$

- The last equality holds because summing over all σ is the same as summing over all $\sigma'\circ\sigma$.
- This implies $\mathrm{im}(\mathrm{Alt})\leq\mathcal{A}^k(V)$.

4. If $T\in\mathcal{A}^k(T)$, $\mathrm{Alt}(T)=k!T$.
   - We have

$$\begin{aligned}\mathrm{Alt}(T) &= \sum_{\sigma\in S_k}(-1)^\sigma T^\sigma\\ &= \sum_{\sigma\in S_k}(-1)^\sigma(-1)^\sigma T\\ &= \sum_{\sigma\in S_k}T\\ &= k!T\end{aligned}$$

   where $T^\sigma=(-1)^\sigma T$ since $T\in\mathcal{A}^k(V)$ by definition.
   - This implies that $\mathrm{im}(\mathrm{Alt})=\mathcal{A}^k(V)$: $\mathrm{Alt}(\frac{1}{k!}T)=T\in\mathcal{A}^k(V)$.
5. $\mathrm{Alt}(T^\sigma)=\mathrm{Alt}(T)^\sigma$.
6. $\mathrm{Alt}:\mathcal{L}^k(V)\to\mathcal{L}^k(V)$ is linear.

- Warning: Some people take $\mathrm{Alt}(T)=\frac{1}{k!}\sum_{\sigma\in S_k}(-1)^\sigma T^\sigma$[1].
- Example: $n=k=2$. We have

$$\mathrm{Alt}(e_1^*\otimes e_2^*)=e_1^*\otimes e_2^*-e_2^*\otimes e_1^*$$

- **Non-repeating** (multi-index I): A multi-index I such that $i_{j_1}\neq i_{j_2}$ for all $j_1\neq j_2$.
- **Increasing** (multi-index I): A multi-index I such that $i_1<\cdots<i_k$.
- Claim: $\{\mathrm{Alt}(e_I^*)\}$ where I is non-repeating and increasing is a basis for $\mathcal{A}^k(V)$. There are $\binom{n}{k}$ of these; thus, $\dim\mathcal{A}^k(V)=\binom{n}{k}$.

4/6:

- Klug will be in Texas on Monday and thus is cancelling class on Monday. Homework is now due next Friday. We'll have weekly homeworks going forward after that.
- Plan:
  - $\mathrm{Alt}:\mathcal{L}^k(V)\twoheadrightarrow\mathcal{A}^k(V)$[2].
  - Goal: Identify $\ker(\mathrm{Alt})=\mathcal{I}^k(V)$, where $\mathcal{I}^k(V)$ is the space of **redundant** k-tensors[3].

[1] Klug prefers this convention, but the text takes the other one.
[2] The two-headed right arrow denotes a surjective map.
[3] The $\mathcal{I}$ in $\mathcal{I}^k(V)$ stands for "ideal."

- Then: Operations on alternating tensors, e.g.,
  - Wedge product.
  - Interior product.
  - Orientations.

- Claim: $\{\text{Alt}(e_I^*) \mid I \text{ non-repeating, increasing multi-index}\}$ is a basis for $\mathcal{A}^k(V)$.
  - Left as an exercise to us.

- **Redundant** (k-tensor): A k-tensor of the form

$$\ell_1 \otimes \cdots \otimes \ell_i \otimes \ell_i \otimes \ell_{i+2} \otimes \cdots \otimes \ell_k$$

where $\ell_1, \dots, \ell_k \in V^*$.

- $\boldsymbol{\mathcal{I}^k(V)}$: The span of all redundant k-tensors.
  - Note that not every k-tensor in $\mathcal{I}^k(V)$ is a redundant.

- **Decomposable** (k-tensor): A k-tensor of the form $\ell_1 \otimes \cdots \otimes \ell_k$ for $\ell_i \in \mathcal{L}^1(V)$.
  - It often suffices to prove things for decomposable tensors.

- Properties.

1. If $T \in \mathcal{I}^k(V)$, then $\text{Alt}(T) = 0$, i.e., $\mathcal{I}^k(V) \leq \ker(\text{Alt})$.
   - "Proof by example": If $T = \ell_1 \otimes \ell_1 \otimes \ell_2 \otimes \ell_3$, then $T^{\tau_{1,2}} = T$. It follows from the properties of Alt that

$$\begin{aligned}
\text{Alt}(T) &= \text{Alt}(T^{\tau_{1,2}}) = (-1)^{\tau_{1,2}} \text{Alt}(T) = -\text{Alt}(T) \\
2\,\text{Alt}(T) &= 0 \\
\text{Alt}(T) &= 0
\end{aligned}$$

2. If $T \in \mathcal{I}^r(V)$ and $T' \in \mathcal{L}^s(V)$, then

$$T \otimes T' \in \mathcal{I}^{r+s}(V)$$

   Similarly, if $T \in \mathcal{L}^r(V)$ and $T \in \mathcal{I}^s(V)$, then

$$T \otimes T' \in \mathcal{I}^{r+s}(V)$$

   - Proof: It suffices to assume that T is redundant. Obviously adding more tensors to the direct product will not change the redundancy of the initial tensor. Example: $\ell_1 \otimes \ell_1 \otimes \ell_2$ is just as redundant as $\ell_1 \otimes \ell_1 \otimes \ell_2 \otimes T$.

3. If $T \in \mathcal{L}^k(V)$ and $\sigma \in S_k$, then

$$T^\sigma = (-1)^\sigma T + S$$

   for some $S \in \mathcal{I}^k(V)$.
   - Proof by example: It suffices to check this for decomposable tensors (a tensor is just a sum of decomposable tensors). Take $k = 2$. Let $T = \ell_1 \otimes \ell_2$. Let $\sigma = \tau_{1,2}$. Then

$$T^\sigma - (-1)^\sigma T = \ell_2 \otimes \ell_1 + \ell_1 \otimes \ell_2 = (\ell_1 + \ell_2) \otimes (\ell_1 + \ell_2) - \ell_1 \otimes \ell_1 - \ell_2 \otimes \ell_2$$

   - Actual proof: It suffices to assume T is decomposable. We induct on the number of transpositions needed to write σ as a product of **adjacent** transpositions.

- Base case: $\sigma = \tau_{i,i+1}$. Then

$$\begin{aligned} T^{\tau_{i,i+1}} + T = \ell_1 \otimes \cdots \otimes (\ell_i + \ell_{i+1}) \otimes (\ell_i + \ell_{i+1}) \otimes \cdots \otimes \ell_k \\ - \ell_1 \otimes \cdots \otimes \ell_i \otimes \ell_i \otimes \cdots \otimes \ell_k \\ - \ell_1 \otimes \cdots \otimes \ell_{i+1} \otimes \ell_{i+1} \otimes \cdots \otimes \ell_k \end{aligned}$$

- Inductive step: If $\sigma = \beta\tau$, then

$$\begin{aligned} T^\sigma &= T^{\beta\tau} \\ &= (-1)^\tau T^\beta + \text{stuff in } \mathcal{I}^k(V) \\ &= (-1)^\tau [(-1)^\beta T + \text{stuff in } \mathcal{I}^k(V)] + \text{stuff in } \mathcal{I}^k(V) \end{aligned}$$

4. If $T \in \mathcal{L}^k(V)$, then

$$\operatorname{Alt}(T) = k!T + W$$

for some $W \in \mathcal{I}^k(V)$.
   - We have that

$$\begin{aligned} \operatorname{Alt}(T) &= \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma \\ &= \sum_{\sigma \in S_k} (-1)^\sigma [(-1)^\sigma T + S_\sigma] \\ &= \sum_{\sigma \in S_k} T + \sum_{\sigma \in S_k} (-1)^\sigma S_\sigma \\ &= k!T + W \end{aligned}$$

5. $\mathcal{I}^k(V) = \ker(\operatorname{Alt})$.
   - We have that $\mathcal{I}^k(V) \leq \ker(\operatorname{Alt})$ by property 1.
   - Now suppose $T \in \ker(\operatorname{Alt})$. Then $\operatorname{Alt}(T) = 0$. Then by property 4,

$$\begin{aligned} \operatorname{Alt}(T) &= k!T + W \\ 0 &= k!T + W \\ T &= -\frac{1}{k!} W \in \mathcal{I}^k(V) \end{aligned}$$

- Warning: If $T \in \mathcal{A}^r(V)$ and $T' \in \mathcal{A}^s(V)$, then we do not necessarily have $T \otimes T' \in \mathcal{A}^{r+s}(V)$.
  - Example: $e_1^*, e_2^* \in \mathcal{A}^1(V)$ have $e_1^* \otimes e_2^* \notin \mathcal{A}^2(V)$.
- **Adjacent** (transposition): A transposition of the form $\tau_{i,i+1}$.

4/8:

- Recall that $\mathcal{A}^k(V) \hookrightarrow \mathcal{L}^k(V)$[4]
- Functoriality: $(A \circ B)^* = B^* \circ A^*$.
  - A^* takes $\mathcal{L}^k(W) \to \mathcal{L}^k(V)$ and $\mathcal{A}^k(W) \to \mathcal{A}^k(V)$.
- $\dim(\Lambda^k(V)) = \binom{n}{k}$.
  - Special case $k = n$: $\dim \Lambda^n(V) = 1$.
  - If $A : V \to V$ induces a map $\Lambda^n(V^*) \to \Lambda^n(V^*)$ defined by the determinant.
- Aside: $\Lambda^k(V)$ is "exterior powers."

[4] The hooked right arrow denotes an injective map.

- Plan: Wedge products + basis for $\Lambda^k(V)$.
- **Wedge product**: A function $\wedge : \Lambda^k(V^*) \times \Lambda^\ell(V^*) \to \Lambda^{k+\ell}(V)$.
  - We denote elements of $\Lambda^k(V^*)$ by ω_1, ω_2, etc.
- If $\pi : \mathcal{L}^k(V) \to \Lambda^k(V^*)$ sends $T \mapsto \omega$, $\omega_1 = \pi(T_1)$, and $\omega_2 = \pi(T_2)$, then $\omega_1 \wedge \omega_2 = \pi(T_1 \otimes T_2)$.
  - Note that $\ker(\pi) = \mathcal{I}^k(V)$.
- Properties.
  1. This is well defined, i.e., this does not depend on the choice of T_1, T_2.
     - Consider $T_1 + W_1, T_2 + W_2$ with $W_1, W_2 \in \mathcal{I}^k(V)$.
     - We check that $\pi[(T_1 + W_1) \otimes (T_2 + W_2)] = \pi(T_1 \otimes T_2)$.
     - Since $W_1 \otimes T_2, T_1 \otimes W_2, W_1 \otimes W_2 \in \mathcal{I}^{k+\ell}(V)$, we have that

$$\begin{aligned}\pi[(T_1 + W_1) \otimes (T_2 + W_2)] &= \pi(T_1 \otimes T_2 + W_1 \otimes T_2 + T_1 \otimes W_2 + W_1 \otimes W_2) \\ &= \pi(T_1 \otimes T_2) + \pi(W_1 \otimes T_2) + \pi(T_1 \otimes W_2) + \pi(W_1 \otimes W_2) \\ &= \pi(T_1 \otimes T_2)\end{aligned}$$

  2. Associative: We have that

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge \omega_2 \wedge \omega_3$$

     - Follows from the definition of $\wedge$ in terms of π and properties of the tensor product.
  3. Distributive: We have that

$$(\omega_1 + \omega_2) \wedge \omega_3 = \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_3 \qquad \omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$$

     - Follows from the definition of $\wedge$ in terms of π and properties of the tensor product.
  4. Linear: We have that

$$(c\omega_1) \wedge \omega_2 = c(\omega_1 \wedge \omega_2) = \omega_1 \wedge (c\omega_2)$$

     - Follows from the definition of $\wedge$ in terms of π and properties of the tensor product.
  5. Anticommutative: We have that

$$\omega_1 \wedge \omega_2 = (-1)^{k\ell} \omega_2 \wedge \omega_1$$

     - It suffices to assume that $w_1 = \ell_1 \wedge \cdots \wedge \ell_k, w_2 = \ell'_1 \wedge \cdots \wedge \ell'_\ell$.
       - We have

$$(\ell_1 \wedge \cdots \wedge \ell_k) \wedge (\ell'_1 \wedge \cdots \wedge \ell'_\ell) = (-1)^k (\ell'_1 \wedge \cdots \wedge \ell'_\ell) \wedge (\ell_1 \wedge \cdots \wedge \ell_k)$$

     - Let $\ell_1, \dots, \ell_k \in \Lambda^1(V^*) = V^* = \mathcal{L}^1(V)$.
     - Recall that $\mathcal{I}^1(V) = \{0\}$.
     - Claim: $\ell_{\sigma(1)} \wedge \cdots \wedge \ell_{\sigma(k)} = (-1)^\sigma \ell_1 \wedge \cdots \wedge \ell_k$ for all $\sigma \in S_k$.
       - Recall that $T^\sigma = (-1)^\sigma T + W$ for some $W \in \mathcal{I}^k(V)$.
       - Let $T = \ell_1 \otimes \cdots \otimes \ell_k$.
       - Then

$$\begin{aligned}(\ell_1 \otimes \cdots \otimes \ell_k)^\sigma &= \ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)} \\ &= (-1)^\sigma \ell_1 \otimes \cdots \otimes \ell_k + W\end{aligned}$$

       - Then hit both sides by π, noting that $\pi(W) = 0$.
- Example:

1. $n=2$, $k=\ell=1$. Consider $e_1^*,e_2^*\in\mathcal{L}^1(V)=V^*=\mathcal{A}^1(V)=\Lambda^1(V^*)$. Then
$$e_1^*\wedge e_2^*=(-1)e_2^*\wedge e_1^* \qquad e_1^*\wedge e_1^*=0=e_2^*\wedge e_2^*$$
2. $n=4$. We have $e_1^*\wedge(3e_1^*+2e_2^*+3e_2^*)=3(e_1^*\wedge e_1^*)+2(e_1^*\wedge e_2^*)+3(e_1^*\wedge e_3^*)$. We also have $(e_1^*\wedge e_2^*)\wedge(e_1^*\wedge e_2^*)=0$.

4/13:

- Plan:
  - Finish multilinear algebra.
  - Basis for $\Lambda^k(V^*)$.
  - Talk a bit about pullbacks and the determinant.
  - **Orientations** of vector spaces.
  - The **interior product**.
- Basis for $\Lambda^k(V^*)$.
  - Recall that $\{\text{Alt}(e_I^*)\mid I \text{ is a nonrepeating, increasing partition of } n \text{ into } k \text{ parts}\}$ is a basis for $\mathcal{A}^k(V)$.
- Alt is an isomorphism from $\Lambda^k(V^*)$ to $\mathcal{A}^k(V)$.
- If we have an injective map from $\mathcal{A}^k(V)$ to $\mathcal{L}^k(V)$ and π a projection map from $\mathcal{L}^k(V)$ to the quotient space $\mathcal{A}^k(V^*)$ gives rise to $\pi|_{\mathcal{A}^k(V)}$.
- Claim:
  1. $\pi|_{\mathcal{A}^k(V)}$ is an isomorphism.
  2. $\pi(\text{Alt}(e_I^*))=k!\pi(e_I^*)$.

  (2) implies that $\{\pi(e_I^*)=e_{i_1}^*\wedge\cdots\wedge e_{i_k}^*,\ I \text{ non-repeating and increasing}\}$ is a basis for $\Lambda^k(V^*)$.
- Examples:
  1. $n=2=\dim V$, $V=\mathbb{R}e_1\oplus\mathbb{R}e_2$.
     - $\Lambda^0(V^*)=\mathbb{R}$ since $\binom{n}{0}=1$.
     - $\Lambda^1(V^*)=\mathbb{R}e_1^*\oplus\mathbb{R}e_2^*$ since $\binom{n}{1}=2$.
     - $\Lambda^2(V^*)=\mathbb{R}e_1^*\wedge e_2^*$ since $\binom{n}{2}=1$.
       - For the second to last one, note that $e_1^*\wedge e_2^*=-e_2^*\wedge e_1^*$.
     - $\Lambda^3(V^*)=0$ since $\binom{2}{3}=0$.
       - For the last one, note that all $e_1^*\wedge e_1^*\wedge e_2^*=0$.
  2. $n=3$, $V=\mathbb{R}e_1\oplus\mathbb{R}e_2\oplus\mathbb{R}e_3$.
     - $\binom{n}{0}=1$: $\Lambda^0(V^*)=\mathbb{R}$.
     - $\binom{n}{1}=3$: $\Lambda^1(V^*)=\mathbb{R}e_1^*\oplus\mathbb{R}e_2^*\oplus\mathbb{R}e_3^*$.
     - $\binom{n}{2}=3$: $\Lambda^2(V^*)=\mathbb{R}e_1^*\wedge e_2^*\oplus\mathbb{R}e_2^*\wedge e_3^*\oplus\mathbb{R}e_1^*\wedge e_3^*$.
     - $\binom{n}{3}=1$: $\Lambda^3(V^*)=\mathbb{R}e_1^*\wedge e_2^*\wedge e_3^*$.
     - $\binom{n}{m}=0$ $(m>n)$: $\Lambda^m(V^*)=\Lambda^4(V^*)=0$.
- If $A:V\to W$, $\omega_1\in\Lambda^k(W^*)$, $\omega_2\in\Lambda^\ell(W^*)$, then
$$A^*(\omega_1\wedge\omega_2)=A^*\omega_1\wedge A^*\omega_2$$
- **Determinant**: Let $\dim V=n$. Let $A:V\to V$ be a linear transformation. This induces a pullback $A^*:\Lambda^n(V^*)\to\Lambda^n(V^*)$. The top exterior power $k=n$ implies $\binom{k}{n}=1$. We define $\det(A)$ to be the unique real number such that $A^*(v)=\det(A)v$.

- This determinant is the one we know.
  - A^* sends $e_1^* \wedge \cdots \wedge e_n^*$ to $A^*e_1^* \wedge \cdots \wedge A^*e_n^*$ which equals $A^*(e_1^* \wedge \cdots \wedge e_n^*)$ or $\det(A)$
- Sanity check.
  1. $\det(\mathrm{id}) = 1$.
     - $\mathrm{id}(e_1^* \wedge \cdots \wedge e_n^*) = \mathrm{id}\, e_1^* \wedge \cdots \wedge \mathrm{id}\, e_n^* = 1 \cdot e_1^* \wedge \cdots \wedge e_n^*$.
  2. If A is not an isomorphism, then $\det(A) = 0$.
     - If A is not an isomorphism, then there exists $v_1 \in \ker A$ with $v_1 \neq 0$. Let $v_1^*, \dots, v_n^*$ be a basis of V^*. So the pullback of this wedge is the wedge of the pullbacks, but $A^*v_1^* = 0$, so
     $$A^*(v_1^* \wedge \cdots \wedge v_n^*) = (A^*v_1^*) \wedge \cdots \wedge (A^*v_n^*) = 0 \wedge \cdots \wedge (A^*v_n^*) = 0 = 0 \cdot v_1^* \wedge \cdots \wedge v_n^*$$
  3. $\det(AB) = \det(A)\det(B)$.
     - Let $A : V \to V$ and $B : V \to V$.
     - We have $(AB)^* = B^*A^*$; in particular, $n = k$, $V = W = U = V$.
- Recall: If we pick a basis for V, $e_1, \dots, e_n$.
  - Implies $[a_{ij}] = [A]_{e_1,\dots,e_n}^{e_1,\dots,e_n}$.
- Does $\det(A) = \det([a_{ij}]) = \sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$?
  - If $A : V \to V$, we know that $A^* : \Lambda^n(V^*) \to \Lambda^n(V^*)$ takes $e_1^* \wedge \cdots \wedge e_n^* \mapsto A^*(e_1^* \wedge \cdots \wedge e_n^*)$. We WTS
  $$A^*(e_1^* \wedge \cdots \wedge e_n^*) = \left[\sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}\right] e_1^* \wedge \cdots \wedge e_n^*$$
  - We have that
  $$\begin{aligned}
  A^*(e_1^* \wedge \cdots \wedge e_n^*) &= A^*e_1^* \wedge \cdots \wedge A^*e_n^* \\
  &= \left(\sum_{i_1=1}^n a_{i_1,1} e_{i_1}^*\right) \wedge \cdots \wedge \left(\sum_{i_n=1}^n a_{i_n,n} e_{i_n}^*\right) \\
  &= \sum_{i_1,\dots,i_n} a_{i_1,1} \cdots a_{i_n,n} e_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \\
  &= \left[\sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}\right] e_1^* \wedge \cdots \wedge e_n^*
  \end{aligned}$$
  where the sign arises from the need to reorder $e_{i_1}^* \wedge \cdots \wedge e_{i_n}^*$ and the antisymmetry of the wedge product.

1.2 Chapter 1: Multilinear Algebar

From Guillemin and Haine (2018).

3/31:

- Guillemin and Haine (2018) defines real vector spaces, the operations on them, their basic properties, and the zero vector.
- **Linearly independent** (vectors $v_1, \dots, v_k$): A finite set of vectors $v_1, \dots, v_k \in V$ such that the map from $\mathbb{R}^k$ to V defined by $(c_1, \dots, c_k) \mapsto c_1v_1 + \cdots + c_kv_k$ is injective.
- **Spanning** (vectors $v_1, \dots, v_k$): We require that the above map is surjective.

- Guillemin and Haine (2018) defines basis, finite-dimensional vector space, dimension, subspace, linear map, and kernel.
- **Image** (of $A : V \to W$): The range space of A, a subspace of W. *Also known as* $\textbf{im}\,\boldsymbol{(A)}$.
- Guillemin and Haine (2018) defines the matrix of a linear map.
- **Inner product** (on V): A map $B : V \times V \to \mathbb{R}$ with the following three properties.
  - *Bilinearity*: For vectors $v, v_1, v_2, w \in V$ and $\lambda \in \mathbb{R}$, we have
    $$B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w)$$
    and
    $$B(\lambda v, w) = \lambda B(v, w)$$
  - *Symmetry*: For vectors $v, w \in V$, we have $B(v, w) = B(w, v)$.
  - *Positivity*: For every vector $v \in V$, we have $B(v, v) \geq 0$. Moreover, if $v \neq 0$, then $B(v, v) > 0$.
- $\boldsymbol{W}$**-coset**: A set of the form $\{v + w \mid w \in W\}$, where W is a subspace V and $v \in V$. *Denoted by* $\boldsymbol{v + W}$.
  - If $v_1 - v_2 \in W$, then $v_1 + W = v_2 + W$.
  - It follows that the distinct W-cosets decompose V into a disjoint collection of subsets of V.
- **Quotient space** (of V by W): The set of distinct W-cosets in V, along with the following definitions of vector addition and scalar multiplication.
  $$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W \qquad \lambda(v + W) = (\lambda v) + W$$
  Denoted by $\boldsymbol{V/W}$.
- **Quotient map**: The linear map $\pi : V \to V/W$ defined by
  $$\pi(v) = v + W$$
  - π is surjective.
  - Note that $\ker(\pi) = W$ since for all $w \in W$, $\pi(w) = w + W = 0 + W$, which is the zero vector in V/W.
- If V, W are finite dimensional, then
  $$\dim(V/W) = \dim(V) - \dim(W)$$
- Proposition 1.2.9: Let $A : V \to U$ be a linear map. If $W \subset \ker(A)$, then there exists a unique linear map $A^\sharp : V/W \to U$ with the property that $A = A^\sharp \circ \pi$, where $\pi : V \to V/W$ is the quotient map.
  - This proposition rephrases in terms of quotient spaces the fact that if $w \in W$, then $A(v+w) = Av$.
- **Dual space** (of V): The set of all linear functions $\ell : V \to \mathbb{R}$, along with the following definitions of vector addition and scalar multiplication.
  $$(\ell_1 + \ell_2)(v) = \ell_1(v) + \ell_2(v) \qquad (\lambda\ell)(v) = \lambda \cdot \ell(v)$$
  Denoted by V^*.
- **Dual basis** (of $e_1, \dots, e_n$ a basis of V): The basis of V^* consisting of the n functions that take every $v = c_1e_1 + \cdots + c_ne_n$ to one of the c_i. *Denoted by* $\boldsymbol{e_1^*, \dots, e_n^*}$. *Given by*
  $$e_i^*(v) = c_i$$
  for all $v \in V$.

- Claim 1.2.12: If V is an n-dimensional vector space with basis $e_1,\dots,e_n$, then $e_1^*,\dots,e_n^*$ is a basis of V^*.

  Proof. We will first prove that $e_1^*,\dots,e_n^*$ spans V^*. Let $\ell \in V^*$ be arbitrary. Set $\lambda_i = \ell(e_i)$ for all $i \in [n]$. Define $\ell' = \sum_{i=1}^n \lambda_i e_i^*$. Then

  $$\ell'(e_j) = \sum_{i=1}^n \lambda_i e_i^*(e_j) = \lambda_j \cdot 1 = \ell(e_j)$$

  for all $j \in [n]$. Therefore, since ℓ, ℓ' take identical values on the basis of V, $\ell = \ell'$, as desired.

  We now prove that $e_1^*,\dots,e_n^*$ is linearly independent. Let $\sum_{i=1}^n \lambda_i e_i^* = 0$. Then for all $j \in [n]$,

  $$\lambda_j = \left(\sum_{i=1}^n \lambda_i e_i^*\right)(e_j) = 0$$

  as desired. □

- **Transpose** (of A): The map from W^* to V^* defined by $\ell \mapsto \ell \circ A$ for all $\ell \in W^*$. *Denoted by* $\boldsymbol{A^*}$.

- Claim 1.2.15: If $e_1,\dots,e_n$ is a basis of V, $f_1,\dots,f_m$ is a basis of W, $e_1^*,\dots,e_n^*$ and $f_1^*,\dots,f_m^*$ are the corresponding dual bases, and $[a_{i,j}]$ is the $m \times n$ matrix of A with respect to $\{e_j\},\{f_i\}$, then the linear map A^* is defined in terms of $\{f_i^*\},\{e_j^*\}$ by the transpose matrix $(a_{j,i})$.

  Proof. Let $[c_{j,i}]$ be the $n \times m$ matrix of A^* with respect to $\{f_i^*\},\{e_j^*\}$. We seek to prove that $a_{i,j} = c_{j,i}$ ($1 \leq i \leq m$, $1 \leq j \leq n$).

  By the definition of $[a_{i,j}]$ and $[c_{j,i}]$, we have that

  $$A^* f_i^* = \sum_{k=1}^n c_{k,i} e_k^* \qquad\qquad Ae_j = \sum_{k=1}^m a_{k,j} f_k$$

  It follows that

  $$[A^* f_i^*](e_j) = \left[\sum_{k=1}^n c_{k,i} e_k^*\right](e_j) = c_{j,i}$$

  and

  $$[A^* f_i^*](e_j) = f_i^*(Ae_j) = f_i^*\left(\sum_{k=1}^m a_{k,j} f_k\right) = a_{i,j}$$

  so transitivity implies the desired result. □

4/4:

- $\boldsymbol{V^k}$: The set of all k-tuples $(v_1,\dots,v_k)$ where $v_1,\dots,v_k \in V$ a vector space.
  - Note that

    $$V^k = \underbrace{V \oplus \cdots \oplus V}_{k \text{ times}}$$

    where "$\oplus$" denotes the direct sum.

- **Linear** (function in its i^{th} variable): A function $T : V^k \to \mathbb{R}$ such that the map from V to $\mathbb{R}$ defined by $v \mapsto T(v_1,\dots,v_{i-1},v,v_{i+1},\dots,v_k)$ is linear, where all v_j save v_i are fixed.

- $\boldsymbol{k}$**-linear** (function T): A function $T : V^k \to \mathbb{R}$ that is linear in its i^{th} variable for $i = 1,\dots,k$. *Also known as* $\boldsymbol{k}$**-tensor**.

- $\boldsymbol{\mathcal{L}^k(V)}$: The set of all k-tensors in V.

  - Since the sum $T_1 + T_2$ of two k-linear functions $T_1, T_2 : V^k \to \mathbb{R}$ is just another k-linear function, and λT_1 is k-linear for all $\lambda \in \mathbb{R}$, we have that $\mathcal{L}^k(V)$ is a vector space.

- Convention: 0-tensors are just the real numbers. Mathematically, we define

$$\mathcal{L}^0(V) = \mathbb{R}$$

- Note that $\mathcal{L}^1(V) = V^*$.
- Defines multi-indices of n of length k.
- Lemma 1.3.5: If $n, k \in \mathbb{N}$, then there are exactly n^k multi-indices of n of length k.
- $\boldsymbol{T_I}$: The real number $T(e_{i_1}, \dots, e_{i_k})$, where $T \in \mathcal{L}^k(V)$, $e_1, \dots, e_n$ is a basis of V, and I is a multi-index of n of length k.
- Proposition 1.3.7: The real numbers T_I determine T, i.e., if T, T' are k-tensors and $T_I = T'_I$ for all I, then $T = T'$.

  Proof. We induct on n. For the base case $n = 1$, $T \in (\mathbb{R}^k)^*$ and we have already proven this result. Now suppose inductively that the assertion is true for $n - 1$. For each e_i, let T_i be the $(k-1)$-tensor defined by

$$(v_1, \dots, v_{n-1}) \mapsto T(v_1, \dots, v_{n-1}, e_i)$$

  Then for an arbitrary $v = c_1 e_1 + \cdots + c_n e_n$,

$$T(v_1, \dots, v_{n-1}, v) = \sum_{i=1}^{n} c_i T_i(v_1, \dots, v_{n-1})$$

  so the T_i's determine T. Applying the inductive hypothesis completes the proof. $\square$

- **Tensor product**: The function $\otimes : \mathcal{L}^k(V) \times \mathcal{L}^\ell(V) \to \mathcal{L}^{k+\ell}(V)$ defined by

$$(T_1 \otimes T_2)(v_1, \dots, v_{k+\ell}) = T_1(v_1, \dots, v_k) T_2(v_{k+1}, \dots, v_{k+\ell})$$

  for all $T_1 \in \mathcal{L}^k(V)$ and $T_2 \in \mathcal{L}^\ell(V)$.

- Note that by the definition of 0-tensors as real numbers, if $a \in \mathbb{R}$ and $T \in \mathcal{L}^k(V)$, then

$$a \otimes T = T \otimes a = aT$$

- Proposition 1.3.9: Associativity, distributivity of scalar multiplication, and left and right distributive laws for the tensor product.
- **Decomposable** (k-tensor): A k-tensor T for which there exist $\ell_1, \dots, \ell_k \in V^*$ such that

$$T = \ell_1 \otimes \cdots \otimes \ell_k$$

- Defines e_I^*.
- Theorem 1.3.13: V a vector space with basis $e_1, \dots, e_n$ and $0 \leq k \leq n$ implies the k-tensors e_I^* form a basis of $\mathcal{L}^k(V)$.

  Proof. Spanning: Let $T \in \mathcal{L}^k(V)$ be arbitrary. Define

$$T' = \sum_I T_I e_I^*$$

Since

$$T'_J = T'(e_{j_1}, \dots, e_{j_k}) = \sum_I T_I e_I^*(e_{j_1}, \dots, e_{j_k}) = T_J e_J^*(e_{j_1}, \dots, e_{j_k}) = T_J$$

for all J, Proposition 1.3.7 asserts that $T = T'$. Therefore, since every $T_I \in \mathbb{R}$, $T = T' \in \operatorname{span}(e_I^*)$.

Linear independence: Suppose

$$T = \sum_I c_I e_I^* = 0$$

for some set of constants $c_I \in \mathbb{R}$. Then

$$0 = T(e_{j_1}, \dots, e_{j_k}) = \sum_I c_I e_I^*(e_{j_1}, \dots, e_{j_k}) = c_J$$

for all J, as desired. $\square$

- Corollary 1.3.15: If $\dim V = n$, then $\dim(\mathcal{L}^k(V)) = n^k$.

  Proof. Follows immediately from Lemma 1.3.5. $\square$

- **Pullback** (of T by the map A): The k-tensor $A^*T : V^k \to \mathbb{R}$ defined by

$$(A^*T)(v_1, \dots, v_k) = T(Av_1, \dots, Av_k)$$

  where V, W are finite-dimensional vector spaces, $A : V \to W$ is linear, and $T \in \mathcal{L}^k(W)$.
- Proposition 1.3.18: The map $A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$ defined by $T \mapsto A^*T$ is linear.
- Identities:
  - If $T_1 \in \mathcal{L}^k(W)$ and $T_2 \in \mathcal{L}^m(W)$, then

$$A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$$

  - If U is a vector space, $B : U \to V$ is linear, and $T \in \mathcal{L}^k(W)$, then $(AB)^*T = B^*(A^*T)$. Hence,

$$(AB)^* = B^*A^*$$

4/13:

- $\boldsymbol{\Sigma_k}$: The set containing the natural numbers 1 through k. *Given by*

$$\Sigma_k = \{1, 2, \dots, k\}$$

- **Permutation of order $\boldsymbol{k}$**: A bijection on Σ_k. *Denoted by* $\boldsymbol{\sigma}$.
- **Product** (of σ_1, σ_2): The composition $\sigma_1 \circ \sigma_2$, i.e., the map

$$i \mapsto \sigma_1(\sigma_2(i))$$

  Denoted by $\boldsymbol{\sigma_1\sigma_2}$.
- **Inverse** (of σ): The permutation of order k which is the inverse bijection of σ. *Denoted by* $\boldsymbol{\sigma^{-1}}$.
- **Permutation group** (of Σ_k): The set of all permutations of order k. *Also known as* **symmetric group on $\boldsymbol{k}$ letters**. *Denoted by* $\boldsymbol{S_k}$.
- Lemma 1.4.2: The group S_k has $k!$ elements.

- **Transposition**: A permutation of order k defined by

$$\ell \mapsto \begin{cases} j & \ell = i \\ i & \ell = j \\ \ell & \ell \neq i,j \end{cases}$$

for all $\ell \in \Sigma_k$, where $i,j \in \Sigma_k$. *Denoted by* $\boldsymbol{\tau_{i,j}}$.

- **Elementary transposition**: A transposition of the form $\pi_{i,i+1}$.
- Theorem 1.4.4: Every $\sigma \in S_k$ can be written as a product of (a finite number of) transpositions.

Proof. We induct on k.

For the base case $k = 2$, the identity permutation of S_2 is the "product" of zero transpositions, and the only other permutation is a transposition (the "product" of one transposition, namely itself).

Now suppose inductively that we have proven the claim for $k-1$. Let $\sigma \in S_k$ be arbitrary. Suppose $\sigma(k) = i$. Then $\tau_{i,k}\sigma(k) = k$. Since $(\tau_{i,k}\sigma)|_{\Sigma_{k-1}} \in S_{k-1}$, we have by the inductive hypothesis that $(\tau_{i,k}\sigma)|_{\Sigma_{k-1}} = \tau_1 \cdots \tau_m$ for some set of permutations $\tau_1, \dots, \tau_m \in S_{k-1}$. For each τ_j $(1 \leq j \leq m)$, define $\tau_j' \in S_k$

$$\tau_j'(\ell) = \begin{cases} \tau_j(\ell) & \ell < k \\ \ell & \ell = k \end{cases}$$

It follows that

$$\begin{aligned} \tau_{i,k}\sigma &= \tau_1' \cdots \tau_m' \\ \sigma &= \tau_{i,k}\tau_1' \cdots \tau_m' \end{aligned}$$

as desired. □

- Theorem 1.4.5: Every transposition can be written as a product of elementary transpositions.

Proof. Let $\tau_{i,j} \in S_k$, and let $i < j$ WLOG. Then we have that

$$\tau_{i,j} = \prod_{\ell=i}^{i-1} \tau_{\ell,\ell+1}$$

as desired. □

- Corollary 1.4.6: Every permutation can be written as a product of elementary transpositions.
- **Sign** (of σ): The number ± 1 assigned to σ by the expression

$$\prod_{i<j} \frac{x_{\sigma(i)} - x_{\sigma(j)}}{x_i - x_j}$$

where $x_1, \dots, x_k$ are coordinate functions on $\mathbb{R}^k$. *Denoted by* $\boldsymbol{(-1)^\sigma}$.

- Claim 1.4.9: The sign defines a group homomorphism $S_k \to \{\pm 1\}$. That is, for $\sigma_1, \sigma_2 \in S_k$, we have

$$(-1)^{\sigma_1\sigma_2} = (-1)^{\sigma_1}(-1)^{\sigma_2}$$

Proof. For all $i < j$, define p, q such that p is the lesser of $\sigma_2(i), \sigma_2(j)$ and q is the greater of $\sigma_2(i), \sigma_2(j)$. Formally,

$$p = \begin{cases} \sigma_2(i) & \sigma_2(i) < \sigma_2(j) \\ \sigma_2(j) & \sigma_2(j) < \sigma_2(i) \end{cases} \qquad q = \begin{cases} \sigma_2(j) & \sigma_2(i) < \sigma_2(j) \\ \sigma_2(i) & \sigma_2(j) < \sigma_2(i) \end{cases}$$

It follows that if $\sigma_2(i) < \sigma_2(j)$, then

$$\frac{x_{\sigma_1\sigma_2(i)} - x_{\sigma_1\sigma_2(j)}}{x_{\sigma_2(i)} - x_{\sigma_2(j)}} = \frac{x_{\sigma_1(p)} - x_{\sigma_1(q)}}{x_p - x_q}$$

and if $\sigma_2(j) < \sigma_2(i)$, then

$$\frac{x_{\sigma_1\sigma_2(i)} - x_{\sigma_1\sigma_2(j)}}{x_{\sigma_2(i)} - x_{\sigma_2(j)}} = \frac{x_{\sigma_1(q)} - x_{\sigma_1(p)}}{x_q - x_p} = \frac{x_{\sigma_1(p)} - x_{\sigma_1(q)}}{x_p - x_q}$$

Therefore,

$$\begin{aligned} (-1)^{\sigma_1\sigma_2} &= \prod_{i<j} \frac{x_{\sigma_1\sigma_2(i)} - x_{\sigma_1\sigma_2(j)}}{x_i - x_j} \\ &= \prod_{i<j} \frac{x_{\sigma_1\sigma_2(i)} - x_{\sigma_1\sigma_2(j)}}{x_{\sigma_2(i)} - x_{\sigma_2(j)}} \cdot \frac{x_{\sigma_2(i)} - x_{\sigma_2(j)}}{x_i - x_j} \\ &= \prod_{i<j} \frac{x_{\sigma_1(p)} - x_{\sigma_1(q)}}{x_p - x_q} \cdot \prod_{i<j} \frac{x_{\sigma_2(i)} - x_{\sigma_2(j)}}{x_i - x_j} \\ &= (-1)^{\sigma_1}(-1)^{\sigma_2} \end{aligned}$$

as desired. □

- Proposition 1.4.11: If σ is the product of an odd number of transpositions, then $(-1)^\sigma = -1$, and if σ is the product of an even number of transpositions, then $(-1)^\sigma = +1$.

  Proof. Follows from the fact that $(-1)^\sigma = -1$ (see Exercise 1.4.ii). □

- $\boldsymbol{T^\sigma}$: The k-tensor defined by

  $$T^\sigma(v_1, \dots, v_k) = T(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)})$$

  where $T \in \mathcal{L}^k(V)$, V is an n-dimensional vector space, and $\sigma \in S_k$.

- Proposition 1.4.14:

  1. If $T = \ell_1 \otimes \cdots \otimes \ell_k$ ($\ell_i \in V^*$), then $T^\sigma = \ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}$.

     Proof. If $v_1, \dots, v_k \in V$, then

     $$\begin{aligned} T^\sigma(v_1, \dots, v_k) &= T(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)}) \\ &= [\ell_1 \otimes \cdots \otimes \ell_k](v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)}) \\ &= \ell_1(v_{\sigma^{-1}(1)}) \cdots \ell_k(v_{\sigma^{-1}(k)}) \\ &= \ell_{\sigma(1)}(v_1) \cdots \ell_{\sigma(k)}(v_2) \\ &= [\ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}](v_1, \dots, v_k) \end{aligned}$$

     as desired. Note that we can justify the fourth equality by nothing that if $\sigma^{-1}(i) = q$, then the i^{th} term in the product is $\ell_{\sigma(q)}(v_q)$, so since σ is a bijection, the product can be arranged to the form on the right-hand side of equality four. □

2. The assignment $T \mapsto T^\sigma$ is a linear map from $\mathcal{L}^k(V) \to \mathcal{L}^k(V)$.

   Proof. See Exercise 1.4.iii. □

3. If $\sigma_1, \sigma_2 \in S_k$, we have $T^{\sigma_1\sigma_2} = (T^{\sigma_1})^{\sigma_2}$.

   Proof. Let $T = \ell_1 \otimes \cdots \otimes \ell_k$[5]. Then

$$T^{\sigma_1} = \ell_{\sigma_1(1)} \otimes \cdots \otimes \ell_{\sigma_1(k)} = \ell'_1 \otimes \cdots \otimes \ell'_k$$

   and thus

$$(T^{\sigma_1})^{\sigma_2} = \ell'_{\sigma_2(1)} \otimes \cdots \otimes \ell'_{\sigma_2(k)}$$

   Let $\sigma_2(i) = j$. Then since $\ell'_p = \ell_{\sigma_1(p)}$ by definition, we have that $\ell'_{\sigma_2(j)} = \ell_{\sigma_1(\sigma_2(j))}$. Therefore,

$$\begin{aligned}
(T^{\sigma_1})^{\sigma_2} &= \ell'_{\sigma_2(1)} \otimes \cdots \otimes \ell'_{\sigma_2(k)} \\
&= \ell_{\sigma_1(\sigma_2(1))} \otimes \cdots \otimes \ell_{\sigma_1(\sigma_2(k))} \\
&= \ell_{\sigma_1\sigma_2(1)} \otimes \cdots \otimes \ell_{\sigma_1\sigma_2(k)} \\
&= T^{\sigma_1\sigma_2}
\end{aligned}$$

   as desired. □

- **Alternating** (k-tensor): A k-tensor $T \in \mathcal{L}^k(V)$ such that $T^\sigma = (-1)^\sigma T$ for all $\sigma \in S_k$.
- $\boldsymbol{\mathcal{A}^k(V)}$: The set of all alternating k-tensors in $\mathcal{L}^k(V)$.
  - Proposition 1.4.14(2) implies that $(T_1 + T_2)^\sigma = T_1^\sigma + T_2^\sigma$ and $(\lambda T)^\sigma = \lambda T^\sigma$; it follows that $\mathcal{A}^k(V)$ is a vector space.
- **Alternation operation**: The function from $\mathcal{L}^k(V) \to \mathcal{L}^k(V)$ defined by

$$T \mapsto \sum_{\tau \in S_k} (-1)^\tau T^\tau$$

  Denoted by **Alt**.
- Proposition 1.4.17: For $T \in \mathcal{L}^k(V)$ and $\sigma \in S_k$, we have that

1. $\operatorname{Alt}(T)^\sigma = (-1)^\sigma \operatorname{Alt} T$.

   Proof. We have that

$$\begin{aligned}
\operatorname{Alt}(T)^\sigma &= \left(\sum_{\tau \in S_k} (-1)^\tau T^\tau\right)^\sigma \\
&= \sum_{\tau \in S_k} (-1)^\tau (T^\tau)^\sigma && \text{Proposition 1.4.14(2)} \\
&= \sum_{\tau \in S_k} (-1)^\tau T^{\tau\sigma} && \text{Proposition 1.4.14(3)} \\
&= (-1)^\sigma \sum_{\tau \in S_k} (-1)^{\tau\sigma} T^{\tau\sigma} \\
&= (-1)^\sigma \sum_{\tau\sigma \in S_k} (-1)^{\tau\sigma} T^{\tau\sigma} \\
&= (-1)^\sigma \operatorname{Alt} T
\end{aligned}$$

   as desired. □

[5] What gives us the right to assume T is decomposable?

2. If $T \in \mathcal{A}^k(V)$, then $\operatorname{Alt} T = k!T$.

   Proof. Since $T \in \mathcal{A}^k(V)$, we know that $T^\sigma = (-1)^\sigma T$. Therefore,
$$\operatorname{Alt} T = \sum_{\tau \in S_k} (-1)^\tau T^\tau = \sum_{\tau \in S_k} (-1)^\tau (-1)^\tau T = \sum_{\tau \in S_k} T = k!T$$
   where the last equality holds because the cardinality of S_k is $k!$. ☐

3. $\operatorname{Alt}(T^\sigma) = \operatorname{Alt}(T)^\sigma$.

   Proof. We have that
$$\operatorname{Alt}(T^\sigma) = \sum_{\tau \in S_k} (-1)^\tau T^{\tau\sigma} = (-1)^\sigma \sum_{\tau \in S_k} (-1)^{\tau\sigma} T^{\tau\sigma} = (-1)^\sigma \operatorname{Alt}(T) = \operatorname{Alt}(T)^\sigma$$
   as desired. ☐

4. The alternation operation is linear.

   Proof. Follows by Proposition 1.4.14. ☐

- **Repeating** (multi-index I): A multi-index I of length k such that $i_r = i_s$ for some $r \neq s$.
- **Strictly increasing** (multi-index I): A multi-index I of length k such that $i_1 < i_2 < \cdots < i_r$.
- $\boldsymbol{I^\sigma}$: The multi-index of length k defined by
$$I^\sigma = (i_{\sigma(1)}, \dots, i_{\sigma(k)})$$
- If I is non-repeating, there is a unique $\sigma \in S_k$ such that I^σ is strictly increasing.
- $\boldsymbol{\psi_I}$: The following k-tensor. *Given by*
$$\psi_I = \operatorname{Alt}(e_I^*)$$
- Proposition 1.4.20:

1. $\psi_{I^\sigma} = (-1)^\sigma \psi_I$.

   Proof. We have that
$$\psi_{I^\sigma} = \operatorname{Alt}(e_{I^\sigma}^*) = \operatorname{Alt}[(e_I^*)^\sigma] = \operatorname{Alt}(e_I^*)^\sigma = (-1)^\sigma \operatorname{Alt}(e_I^*) = (-1)^\sigma \psi_I$$
   as desired. ☐

2. If I is repeating, then $\psi_I = 0$.

   Proof. Suppose $I = (i_1, \dots, i_k)$ is such that $i_r = i_s$ for some distinct $r, s \in \Sigma_k$. Then $e_I^* = e_{I^{\tau_{i_r,i_s}}}^*$, so
$$\psi_I = \psi_{I^{\tau_{i_r,i_s}}} = (-1)^{\tau_{i_r,i_s}} \psi_I = -\psi_I$$
   Therefore, we must have $\psi_I = 0$, as desired. ☐

3. If I and J are strictly increasing, then
$$\psi_I(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$$

Proof. We have by definition that

$$\psi_I(e_{j_1},\dots,e_{j_k}) = \sum_\tau (-1)^\tau e^*_{I^\tau}(e_{j_1},\dots,e_{j_k})$$

This combined with the facts that

$$e^*_{I^\tau}(e_{j_1},\dots,e_{j_k}) = \begin{cases} 1 & I^\tau = J \\ 0 & I^\tau \neq J \end{cases}$$

I^τ is strictly increasing iff $I^\tau = I$, and the above equation is nonzero iff $I^\tau = I = J$ implies the desired result. □

- Conclusion 1.4.22: If $T \in \mathcal{A}^k(V)$, then we can write T as a sum

$$T = \sum_I c_I\psi_I$$

with I's strictly increasing.

Proof. Let $T \in \mathcal{A}^k(V)$ be arbitrary. By Theorem 1.3.13,

$$T = \sum_J a_J e^*_J$$

for some set of $a_J \in \mathbb{R}$. It follows since $\operatorname{Alt}(T) = k!T$ that

$$T = \frac{1}{k!}\sum a_J \operatorname{Alt}(e^*_J) = \sum b_J\psi_J$$

We can disregard all repeating terms in the sum since they are zero by Proposition 1.4.20(2); for every non-repeating term J, we can write $J = I^\sigma$, where I is strictly increasing and hence $\psi_J = (-1)^\sigma\psi_I$. □

- Claim 1.4.24: The c_I's of Conclusion 1.4.22 are unique.

Proof. For J strictly increasing, we have

$$T_J = T(e_{j_1},\dots,e_{j_k}) = \sum_I c_I\psi_I(e_{j_1},\dots,e_{j_k}) = c_J$$

□

- Proposition 1.4.26: The alternating tensors ψ_I with I strictly increasing are a basis for $\mathcal{A}^k(V)$.

Proof. Spanning: See Conclusion 1.4.22.
Linear independence: See Claim 1.4.24. □

- We have that

$$\dim\mathcal{A}^k(V) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

  - Hint in proving this claim: "Show that every strictly increasing multi-index of length k determines a k-element subset of $\{1,\dots,n\}$ and vice versa." (Guillemin & Haine, 2018, p. 16).
  - Note also that if $k > n$, every multi-index has a repeat somewhere, meaning that $\dim\mathcal{A}^k(V) = \binom{n}{k} = 0$.

4/14:
- Having discussed $\operatorname{im}(\operatorname{Alt}) = \mathcal{A}^k(V)$ in some detail now, we move onto $\ker(\operatorname{Alt})$.

- **Redundant** (decomposable k-tensor): A decomposable k-tensor $\ell_1 \otimes \cdots \otimes \ell_k$ such that for some $i \in [k-1]$, $\ell_i = \ell_{i+1}$.
- $\boldsymbol{\mathcal{I}^k(V)}$: The linear span of the set of redundant k-tensors.
- Convention: There are no redundant 1-tensors. Mathematically, we define

$$\mathcal{I}^1(V) = 0$$

- Proposition 1.5.2: $T \in \mathcal{I}^k(V)$ implies $\operatorname{Alt}(T) = 0$.

Proof. Let $T = \ell_1 \otimes \cdots \otimes \ell_k$ with $\ell_i = \ell_{i+1}$. Then if $\sigma = \tau_{i,i+1}$, we have that $T^\sigma = T$ and $(-1)^\sigma = -1$. Therefore,

$$\begin{aligned}
\operatorname{Alt}(T) &= \operatorname{Alt}(T^\sigma) \\
&= \operatorname{Alt}(T)^\sigma && \text{Proposition 1.4.17(3)} \\
&= (-1)^\sigma \operatorname{Alt}(T) && \text{Proposition 1.4.17(1)} \\
&= -\operatorname{Alt}(T)
\end{aligned}$$

so we must have that $\operatorname{Alt}(T) = 0$, as desired. $\square$

- Proposition 1.5.3: $T \in \mathcal{I}^r(V)$ and $T' \in \mathcal{L}^s(V)$ imply $T \otimes T', T' \otimes T \in \mathcal{I}^{r+s}(V)$.

Proof. We first justify why we need only prove this claim for T' decomposable. As an element of $\mathcal{L}^s(V)$, we know that $T' = \sum a_I e_I^*$ for some set of $a_I \in \mathbb{R}$. Since each e_I^* is decomposable, this means that T' is a linear combination of decomposable tensors. This combined with the fact that the tensor product is linear means that

$$T \otimes T' = T \otimes \sum a_I e_I^* = \sum a_I (T \otimes e_I^*)$$

and similarly for $T' \otimes T$. Thus, if we can prove that each $T \otimes e_I^* \in \mathcal{I}^{r+s}(V)$, it will follow since $\mathcal{I}^k(V)$ is a vector space that $\sum a_I(T \otimes e_I^*) = T \otimes T' \in \mathcal{I}^{r+s}(V)$. In other words, we need only prove that $T \otimes T' \in \mathcal{I}^{r+s}(V)$ for T' decomposable, as desired.

Let $T = \ell_1 \otimes \cdots \otimes \ell_r$ with $\ell_i = \ell_{i+1}$, and let $T' = \ell_1' \otimes \cdots \otimes \ell_s'$. It follows that

$$T \otimes T' = (\ell_1 \otimes \cdots \otimes \ell_i \otimes \ell_{i+1} \otimes \cdots \otimes \ell_r) \otimes (\ell_1' \otimes \cdots \otimes \ell_s')$$

is redundant and hence in $\mathcal{I}^{r+s}(V)$, as desired. The argument is symmetric for $T' \otimes T$. $\square$

- Proposition 1.5.4: $T \in \mathcal{L}^k(V)$ and $\sigma \in S_k$ imply

$$T^\sigma = (-1)^\sigma T + S$$

where $S \in \mathcal{I}^k(V)$.

Proof. As with Proposition 1.5.3, the linearity of $\sigma : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ allows us to assume that T is decomposable.

By Theorem 1.4.5, σ can be written as a product of m elementary transpositions. To prove the claim, we induct on m.

For the base case $m = 1$, let $\sigma = \tau_{i,i+1}$. If $T_1 = \ell_1 \otimes \cdots \otimes \ell_{i-1}$ and $T_2 = \ell_{i+2} \otimes \cdots \otimes \ell_k$, then

$$\begin{aligned}
T^\sigma - (-1)^\sigma T &= T_1 \otimes (\ell_{i+1} \otimes \ell_i \pm \ell_i \otimes \ell_{i+1}) \otimes T_2 \\
&= T_1 \otimes [(\ell_i + \ell_{i+1}) \otimes (\ell_i + \ell_{i+1}) \mp \ell_i \otimes \ell_i \mp \ell_{i+1} \otimes \ell_{i+1}] \otimes T_2
\end{aligned}$$

i.e., $T^\sigma - (-1)^\sigma T$ is the sum of three redundant k-tensors, and thus a redundant k-tensor in and of itself, as desired. Note that even though only the middle portion is explicitly redundant, Proposition 1.5.3 allows us to call the whole tensor product redundant.

Now suppose inductively that we have proven the claim for $m-1$. Let $\sigma = \tau\beta$ where β is the product of $m-1$ elementary transpositions and τ is an elementary transposition. Then

$$\begin{aligned}
T^\sigma &= (T^\beta)^\tau && \text{Proposition 1.4.14(3)}\\
&= (-1)^\tau T^\beta + \cdots && \text{Base case}\\
&= (-1)^\tau(-1)^\beta T + \cdots && \text{Inductive hypothesis}\\
&= (-1)^\sigma T + \cdots && \text{Claim 1.4.9}
\end{aligned}$$

where the dots are elements of $\mathcal{I}^k(V)$. ☐

- Corollary 1.5.6: $T \in \mathcal{L}^k(V)$ implies
$$\operatorname{Alt}(T) = k!T + W$$
where $W \in \mathcal{I}^k(V)$.

Proof. By definition,
$$\operatorname{Alt}(T) = \sum_{\sigma\in S_k}(-1)^\sigma T^\sigma$$
By Proposition 1.5.4,
$$T^\sigma = (-1)^\sigma T + W_\sigma$$
for all $\sigma \in S_k$ with each $W_\sigma \in \mathcal{I}^k(V)$. It follows by combining the above two results that
$$\operatorname{Alt}(T) = \sum_{\sigma\in S_k}(-1)^\sigma[(-1)^\sigma T + W_\sigma] = \sum_{\sigma\in S_k} T + \sum_{\sigma\in S_k}(-1)^\sigma W_\sigma = k!T + W$$
where $W = \sum_{\sigma\in S_k}(-1)^\sigma W_\sigma$ is an element of $\mathcal{I}^k(V)$ as a linear combination of elements of $\mathcal{I}^k(V)$. ☐

- Corollary 1.5.8: Let V be a vector space and $k \geq 1$. Then
$$\mathcal{I}^k(V) = \ker(\operatorname{Alt} : \mathcal{L}^k(V) \to \mathcal{A}^k(V))$$

Proof. Suppose first that $T \in \mathcal{I}^k(V)$. Then by Proposition 1.5.2, $\operatorname{Alt}(T) = 0$, so $T \in \ker(\operatorname{Alt})$, as desired.

Now suppose that $T \in \ker(\operatorname{Alt})$. Then $\operatorname{Alt}(T) = 0$, so by Corollary 1.5.6,
$$\begin{aligned}
0 &= k!T + W\\
T &= -\frac{1}{k!}W
\end{aligned}$$
Therefore, as a scalar multiple of an element of $\mathcal{I}^k(V)$, $T \in \mathcal{I}^k(V)$. ☐

- Theorem 1.5.9: Every $T \in \mathcal{L}^k(V)$ has a unique decomposition $T = T_1 + T_2$ where $T_1 \in \mathcal{A}^k(V)$ and $T_2 \in \mathcal{I}^k(V)$.

Proof. By Corollary 1.5.6, we have that
$$\begin{aligned}
\operatorname{Alt}(T) &= k!T + W\\
T &= \underbrace{\left(\frac{1}{k!}\operatorname{Alt}(T)\right)}_{T_1} + \underbrace{\left(-\frac{1}{k!}W\right)}_{T_2}
\end{aligned}$$

As to uniqueness, suppose $0 = T_1 + T_2$ where $T_1 \in \mathcal{A}^k(V)$ and $T_2 \in \mathcal{I}^k(V)$. Then

$$\begin{aligned} 0 = \mathrm{Alt}(0) = \mathrm{Alt}(T_1 + T_2) = \mathrm{Alt}(T_1) + \mathrm{Alt}(T_2) = k!T_1 + 0 = k!T_1 \\ T_1 = 0 \end{aligned}$$

so $T_2 = 0$, too. □

- $\boldsymbol{\Lambda^k(V^*)}$: The quotient of the vector space $\mathcal{L}^k(V)$ by the subspace $\mathcal{I}^k(V)$. *Given by*
$$\Lambda^k(V^*) = \mathcal{L}^k(V)/\mathcal{I}^k(V)$$
- The quotient map $\pi : \mathcal{L}^k(V) \to \Lambda^k(V^*)$ defined by $T \mapsto T + \mathcal{I}^k(V)$ is onto and has $\ker(\pi) = \mathcal{I}^k(V)$.
- Theorem 1.5.13: $\pi : \mathcal{L}^k(V) \to \Lambda^k(V^*)$ maps $\mathcal{A}^k(V)$ bijectively onto $\Lambda^k(V^*)$.

  Proof. Theorem 1.5.9 implies that every $T + \mathcal{I}^k(V)$ contains a unique $T_1 \in \mathcal{A}^k(V)$. Thus, for every element of $\Lambda^k(V^*)$, there is a unique element of $\mathcal{A}^k(V)$ which gets mapped onto it by π. □

- Note that since $\mathcal{A}^k(V)$ and $\Lambda^k(V)$ are in bijective correspondence, many texts do not distinguish between them. There are some advantages to making the distinction, though.
- The tensor product and pullback operatios give rise to similar operations on the spaces $\Lambda^k(V^*)$.
- **Wedge product**: The function $\wedge : \Lambda^{k_1}(V^*) \times \Lambda^{k_2}(V^*) \to \Lambda^{k_1+k_2}(V^*)$ defined by
$$\omega_1 \wedge \omega_2 = \pi(T_1 \otimes T_2)$$
  where for $i = 1, 2$, $\omega_i \in \Lambda^{k_i}(V^*)$, and $\omega_i = \pi(T_i)$ for some $T_i \in \mathcal{L}^{k_i}(V)$.
  - Note that it is Theorem 1.5.13 that allows us to find T_i such that $\omega_i = \pi(T_i)$.
- Claim 1.6.3: The wedge product is well-defined, i.e., it does not depend on our choices of T_i.

  Proof. We prove WLOG that $\wedge$ is well defined with respect to T_1. Suppose $\omega_1 = \pi(T_1) = \pi(T_1')$. Then by the definition of the quotient map, $T_1' = T_1 + W_1$ for some $W_1 \in \mathcal{I}^{k_1}(V)$. But this means that
$$T_1' \otimes T_2 = (T_1 + W_1) \otimes T_2 = T_1 \otimes T_2 + W_1 \otimes T_2$$
  where $W_1 \otimes T_2 \in \mathcal{I}^{k_1+k_2}(V)$ by Proposition 1.5.3. It follows that
$$\pi(T_1' \otimes T_2) = \pi(T_1 \otimes T_2)$$
  □

- The wedge product also generalizes to higher orders, obeying associativity, scalar multiplication, and distributivity.
- **Decomposable element** (of $\Lambda^k(V^*)$): An element of $\Lambda^k(V^*)$ of the form $\ell_1 \wedge \cdots \wedge \ell_k$ where $\ell_1, \dots, \ell_k \in V^*$.
- Claim 1.6.8: The following wedge product identity holds for decomposable elements of $\Lambda^k(V^*)$.
$$\ell_{\sigma(1)} \wedge \cdots \wedge \ell_{\sigma(k)} = (-1)^\sigma \ell_1 \wedge \cdots \wedge \ell_k$$

Proof. Let $T = \ell_1 \otimes \cdots \otimes \ell_k$. It follows by Proposition 1.4.14(1) that $T^\sigma = \ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}$. Therefore, we have that

$$\begin{aligned}
\ell_{\sigma(1)} \wedge \cdots \wedge \ell_{\sigma(k)} &= \pi(\ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}) \\
&= \pi(T^\sigma) \\
&= \pi[(-1)^\sigma T + W] \\
&= (-1)^\sigma \pi(T) \\
&= (-1)^\sigma \pi(\ell_1 \otimes \cdots \otimes \ell_k) \\
&= (-1)^\sigma \ell_1 \wedge \cdots \wedge \ell_k
\end{aligned}$$

as desired. □

- An important consequence of Claim 1.6.8 is that

$$\ell_1 \wedge \ell_2 = -\ell_2 \wedge \ell_1$$

- Theorem 1.6.10: If $\omega_1 \in \Lambda^r(V^*)$ and $\omega_2 \in \Lambda^s(V^*)$, then

$$\omega_1 \wedge \omega_2 = (-1)^{rs} \omega_2 \wedge \omega_1$$

  - This can be deduced from Claim 1.6.8.
  - Hint: It suffices to prove this for decomposable elements, i.e., for $\omega_1 = \ell_1 \wedge \cdots \wedge \ell_r$ and $\omega_2 = \ell_1' \wedge \cdots \wedge \ell_s'$.

- Theorem 1.6.13: The elements

$$e_{i_1}^* \wedge \cdots \wedge e_{i_k}^* = \pi(e_I^*) = \pi(e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*)$$

with I strictly increasing are basis vectors of $\Lambda^k(V^*)$.

Proof. Follows from the facts that the ψ_I for I strictly increasing constitute a basis of $\mathcal{A}^k(V)$ by Proposition 1.4.26 and π is an isomorphism $\mathcal{A}^k(V) \to \Lambda^k(V^*)$. □

- $\boldsymbol{\imath_v T}$: The $(k-1)$-tensor defined by

$$(\imath_v T)(v_1, \dots, v_{k-1}) = \sum_{r=1}^k (-1)^{r-1} T(v_1, \dots, v_{r-1}, v, v_r, \dots, v_{k-1})$$

where $T \in \mathcal{L}^k(V)$, $k \in \mathbb{N}_0$, V is a vector space, and $v \in V$.

- If $v = v_1 + v_2$, then

$$\imath_v T = \imath_{v_1} T + \imath_{v_2} T$$

- If $T = T_1 + T_2$, then

$$\imath_v T = \imath_v T_1 + \imath_v T_2$$

- Lemma 1.7.4: If $T = \ell_1 \otimes \cdots \otimes \ell_k$, then

$$\imath_v T = \sum_{r=1}^k (-1)^{r-1} \ell_r(v) \ell_1 \otimes \cdots \otimes \hat{\ell}_r \otimes \cdots \otimes \ell_k$$

where the hat over ℓ_r means that ℓ_r is deleted from the tensor product.

- Lemma 1.7.6: $T_1 \in \mathcal{L}^p(V)$ and $T_2 \in \mathcal{L}^q(V)$ imply

$$\imath_v(T_1 \otimes T_2) = \imath_v T_1 \otimes T_2 + (-1)^p T_1 \otimes \imath_v T_2$$

- Lemma 1.7.8: $T \in \mathcal{L}^k(V)$ implies that for all $v \in V$, we have

$$\imath_v(\imath_v T) = 0$$

Proof. It suffices by linearity to prove this for decomposable tensors. We induct on k. For the base case $k = 1$, the claim is trivially true. Now suppose inductively that we have proven the claim for $k - 1$. Consider $\ell_1 \otimes \cdots \otimes \ell_k$. Taking $T = \ell_1 \otimes \cdots \otimes \ell_{k-1}$ and $\ell = \ell_k$, we obtain

$$\imath_v(\imath_v(T \otimes \ell)) = \imath_v(\imath_v T) \otimes \ell + (-1)^{k-2}\ell(v)\imath_v T + (-1)^{k-1}\ell(v)\imath_v T$$

The first term is zero by the inductive hypothesis, and the second two cancel each other out, as desired. □

- Claim 1.7.10: For all $v_1, v_2 \in V$, we have that

$$\imath_{v_1}\imath_{v_2} = -\imath_{v_2}\imath_{v_1}$$

Proof. Let $v = v_1 + v_2$. Then $\imath_v = \imath_{v_1} + \imath_{v_2}$. Therefore,

$$\begin{aligned} 0 &= \imath_v \imath_v && \text{Lemma 1.7.8} \\ &= (\imath_{v_1} + \imath_{v_2})(\imath_{v_1} + \imath_{v_2}) \\ &= \imath_{v_1}\imath_{v_1} + \imath_{v_1}\imath_{v_2} + \imath_{v_2}\imath_{v_1} + \imath_{v_2}\imath_{v_2} \\ &= \imath_{v_1}\imath_{v_2} + \imath_{v_2}\imath_{v_1} && \text{Lemma 1.7.8} \end{aligned}$$

yielding the desired result. □

- Lemma 1.7.11: If $T \in \mathcal{L}^k(V)$ is redundant, then so is $\imath_v T$.

Proof. Let $T = T_1 \otimes \ell \otimes \ell \otimes T_2$ where $\ell \in V^*$, $T_1 \in \mathcal{L}^p(V)$, and $T_2 \in \mathcal{L}^q(V)$. By Lemma 1.7.6, we have that

$$\imath_v T = \imath_v T_1 \otimes \ell \otimes \ell \otimes T_2 + (-1)^p T_1 \otimes \imath_v(\ell \otimes \ell) \otimes T_2 + (-1)^{p+2} T_1 \otimes \ell \otimes \ell \otimes \imath_v T_2$$

Thus, since the first and third terms above are redundant and $\imath_v(\ell \otimes \ell) = \ell(v)\ell - \ell(v)\ell = 0$ by Lemma 1.7.4, we have the desired result. □

- $\boldsymbol{\imath_v \omega}$: The $\mathcal{I}^k(V)$-coset $\pi(\imath_v T)$, where $\omega = \pi(T)$.
- Proves that $\imath_v \omega$ does not depend on the choice of T.
- **Inner product operation**: The linear map $\imath_v : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$.
- The inner product has the following important identities.

$$\begin{aligned} \imath_{(v_1+v_2)}\omega &= \imath_{v_1}\omega + \imath_{v_2}\omega \\ \imath_v(\omega_1 \wedge \omega_2) &= \imath_v\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge \omega_2 \\ \imath_v(\imath_v\omega) &= 0 \\ \imath_{v_1}\imath_{v_2}\omega &= -\imath_{v_2}\imath_{v_1}\omega \end{aligned}$$