# 3 Operations on Forms

*From Guillemin and Haine (2018).*

## Chapter 2

5/9: **2.4.i.** Compute the exterior derivatives of the following differential forms.

(1) $x_1\,\mathrm{d}x_2 \wedge \mathrm{d}x_3$.

*Proof.* We have that

$$\begin{aligned}\mathrm{d}(x_1\,\mathrm{d}x_2 \wedge \mathrm{d}x_3) &= \mathrm{d}(x_1\,\mathrm{d}x_2) \wedge \mathrm{d}x_3 + (-1)^1 x_1\,\mathrm{d}x_2 \wedge \mathrm{d}(\mathrm{d}x_3) \\ &= \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 - x_1\,\mathrm{d}x_2 \wedge 0\end{aligned}$$

$$\boxed{\mathrm{d}(x_1\,\mathrm{d}x_2 \wedge \mathrm{d}x_3) = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3}$$

□

(2) $x_1\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_1$.

*Proof.* We have that

$$\begin{aligned}\mathrm{d}(x_1\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_1) &= \mathrm{d}(x_1\,\mathrm{d}x_2) - \mathrm{d}(x_2\,\mathrm{d}x_1) \\ &= \mathrm{d}x_1 \wedge \mathrm{d}x_2 - \mathrm{d}x_2 \wedge \mathrm{d}x_1 \\ &= \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_1 \wedge \mathrm{d}x_2\end{aligned}$$

$$\boxed{\mathrm{d}(x_1\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_1) = 2\,\mathrm{d}x_1 \wedge \mathrm{d}x_2}$$

□

(3) $\mathrm{e}^{-f}\,\mathrm{d}f$ where $f = \sum_{i=1}^n x_i^2$.

*Proof.* We state as a lemma first that

$$\begin{aligned}\mathrm{d}f &= \sum_{i=1}^n \mathrm{d}\big(x_i^2\big) \\ &= 2\sum_{i=1}^n x_i\,\mathrm{d}x_i\end{aligned}$$

It follows that

$$\begin{aligned}\mathrm{d}\big(\mathrm{e}^{-f}\,\mathrm{d}f\big) &= \mathrm{d}\big(\mathrm{e}^{-f}\big) \wedge \mathrm{d}f \\ &= \mathrm{e}^{-f}\,\mathrm{d}(-f) \wedge \mathrm{d}f \\ &= -4\mathrm{e}^{-f}\sum_{i=1}^n x_i\,\mathrm{d}x_i \wedge \sum_{i=1}^n x_i\,\mathrm{d}x_i \\ &= -4\mathrm{e}^{-f}\sum_I x_{i_1}x_{i_2}\,\mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2}\end{aligned}$$

where we sum over the multi-indices $I$ of $n$ of length 2. However, since all repeating multi-indices equal zero, we can eliminate those terms from the sum. Additionally, we can pair up all strictly increasing and strictly decreasing terms with the same two numbers (e.g., $(1,2) \sim (2,1)$, $(1,3) \sim (3,1)$, etc.). Invoking the anticommutative property, we can rewrite the sum such that we only sum over the non-repeating, strictly increasing multi-indices of $n$ of length 2.

$$\boxed{\mathrm{d}\big(\mathrm{e}^{-f}\,\mathrm{d}f\big) = -4\mathrm{e}^{-f}\sum_{1 \leq i_1 < i_2 \leq n}(x_{i_1}x_{i_2} - x_{i_2}x_{i_1})\,\mathrm{d}x_{i_1} \wedge \mathrm{d}x_{i_2}}$$

□

(4) $\sum_{i=1}^{n} x_i \,\mathrm{d}x_i$.

*Proof.* We have that

$$\mathrm{d}\left(\sum_{i=1}^{n} x_i \,\mathrm{d}x_i\right) = \sum_{i=1}^{n} \mathrm{d}(x_i \,\mathrm{d}x_i)$$

$$\boxed{\mathrm{d}\left(\sum_{i=1}^{n} x_i \,\mathrm{d}x_i\right) = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}x_i}$$

□

(5) $\sum_{i=1}^{n} (-1)^i x_i \,\mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n$.

*Proof.* We have that

$$\begin{aligned}
\mathrm{d}\left(\sum_{i=1}^{n} (-1)^i x_i \,\mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n\right) &= \sum_{i=1}^{n} (-1)^i \,\mathrm{d}\Big(x_i \,\mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n\Big) \\
&= \sum_{i=1}^{n} (-1)^i \,\mathrm{d}x_i \wedge \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n
\end{aligned}$$

$$\boxed{\mathrm{d}\left(\sum_{i=1}^{n} (-1)^i x_i \,\mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n\right) = -n \,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n}$$

Note that we get from the second to the third line as follows: The $i = 1$ term is

$$(-1)^1 \,\mathrm{d}x_1 \wedge \widehat{\mathrm{d}x_1} \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n = -\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

The $i = 2$ term is

$$\begin{aligned}
(-1)^2 \,\mathrm{d}x_2 \wedge \mathrm{d}x_1 \wedge \widehat{\mathrm{d}x_2} \wedge \mathrm{d}x_3 \wedge \cdots \wedge \mathrm{d}x_n &= (1) \cdot (-1)^{\tau}_{1,2} \,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \\
&= -\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n
\end{aligned}$$

where we have used Claim 1.6.8[1] and the odd permutation $\tau_{1,2}$ to rearrange the term. The $i = 3$ term is

$$\begin{aligned}
(-1)^3 \,\mathrm{d}x_3 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \widehat{\mathrm{d}x_3} \wedge \mathrm{d}x_4 \wedge \cdots \wedge \mathrm{d}x_n &= (-1) \cdot (-1)^{\tau_{1,2}\tau_{2,3}} \,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \\
&= -\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n
\end{aligned}$$

From here, we should be able to see that all $n$ terms will evaluate to $-\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$, so we can simply add them up and stick $n$ out front as a coefficient. This intuitive justification can be formalized with an induction argument. □

**2.4.ii.** Solve the equation $\mathrm{d}\mu = \omega$ for $\mu \in \Omega^1(\mathbb{R}^3)$, where $\omega$ is the 2-form...

*General Solution.* The following is a derivation of the general solution to the equation $\mathrm{d}\mu = \omega$, where $\omega$ is a two-form with the structure

$$\omega = f \,\mathrm{d}x_i \wedge \mathrm{d}x_j$$

Let $\omega \in \Omega^2(U)$ be a two-form on $U$ with the above structure. Notice that if we take $\mu = g \,\mathrm{d}x_j$ by inspection, then $\mathrm{d}\mu = \mathrm{d}g \wedge \mathrm{d}x_j$. By comparing this equation with the definition of $\omega$, we can determine that

$$\begin{aligned}
\mathrm{d}g &= f \,\mathrm{d}x_i \\
g &= \int f \,\mathrm{d}x_i
\end{aligned}$$

[1] Technically, we use the natural extension of Claim 1.6.8 to the wedge product of one-forms.

Therefore, the solution to $\mathrm{d}\mu = \omega$ is the one-form

$$\mu = \left( \int f \, \mathrm{d}x_i \right) \mathrm{d}x_j$$

□

(1) $\mathrm{d}x_2 \wedge \mathrm{d}x_3$.

*Proof.* Using the above formula, we have

$$\mu = \left( \int \mathrm{d}x_2 \right) \mathrm{d}x_3$$
$$\boxed{\mu = x_2 \, \mathrm{d}x_3}$$

□

(2) $x_2 \, \mathrm{d}x_2 \wedge \mathrm{d}x_3$.

*Proof.* Using the above formula, we have

$$\mu = \left( \int x_2 \, \mathrm{d}x_2 \right) \mathrm{d}x_3$$
$$\boxed{\mu = \frac{1}{2} x_2^2 \, \mathrm{d}x_3}$$

□

(3) $(x_1^2 + x_2^2) \, \mathrm{d}x_1 \wedge \mathrm{d}x_2$.

*Proof.* Using the above formula, we have

$$\mu = \left( \int (x_1^2 + x_2^2) \, \mathrm{d}x_1 \right) \mathrm{d}x_2$$
$$\boxed{\mu = \left( \frac{1}{3} x_1^3 + x_1 x_2^2 \right) \mathrm{d}x_2}$$

□

(4) $\cos(x_1) \, \mathrm{d}x_1 \wedge \mathrm{d}x_3$.

*Proof.* Using the above formula, we have

$$\mu = \left( \int \cos(x_1) \, \mathrm{d}x_1 \right) \mathrm{d}x_3$$
$$\boxed{\mu = \sin(x_1) \, \mathrm{d}x_3}$$

□

**2.4.iii.** Let $U$ be an open subset of $\mathbb{R}^n$.

(1) Show that if $\mu \in \Omega^k(U)$ is exact and $\omega \in \Omega^\ell(U)$ is closed then $\mu \wedge \omega$ is exact. *Hint*: See the second desired property of the exterior derivative.

*Proof.* To prove that $\mu \wedge \omega$ is exact, it will suffice to show that there exists some $\eta \in \Omega^{k+\ell-1}(U)$ such that $\mathrm{d}\eta = \mu \wedge \omega$. Since $\mu$ is exact, we know that there exists $\tilde{\mu} \in \Omega^{k-1}(U)$ such that $\mathrm{d}\tilde{\mu} = \mu$. Since $\omega$ is closed, we know that $\mathrm{d}\omega = 0$. Working off of a principle similar to the general proof in Exercise 2.4.ii, we can discover by inspection that taking

$$\eta = \tilde{\mu} \wedge \omega$$

makes it so that

$$\begin{aligned}
\mathrm{d}\eta &= \mathrm{d}(\tilde{\mu} \wedge \omega) \\
&= \mathrm{d}\tilde{\mu} \wedge \omega + (-1)^{k-1}\tilde{\mu} \wedge \mathrm{d}\omega \\
&= \mu \wedge \omega + (-1)^{k-1}\tilde{\mu} \wedge 0 \\
&= \mu \wedge \omega
\end{aligned}$$

as desired. □

(2) In particular, $\mathrm{d}x_1$ is exact, so if $\omega \in \Omega^\ell(U)$ is closed, then $\mathrm{d}x_1 \wedge \omega = \mathrm{d}\mu$. What is $\mu$?

*Proof.* Since $\mathrm{d}x_1 = \mathrm{d}(x_1)$, we have by part (1) that

$$\boxed{\mu = x_1\omega}$$

□

**2.4.iv.** Let $Q$ be the rectangle $(a_1, b_1) \times \cdots \times (a_n, b_n)$. Show that if $\omega$ is in $\Omega^n(Q)$, then $\omega$ is exact. *Hint*: Let $\omega = f\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ with $f \in C^\infty(Q)$ and let $g$ be the function defined by

$$g(x_1, \dots, x_n) = \int_{a_1}^{x_1} f(t, x_2, \dots, x_n)\,\mathrm{d}t$$

Show that $\omega = \mathrm{d}(g\,\mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n)$.

*Proof.* Let $\omega \in \Omega^n(Q)$ be arbitrary. Then $\omega = f\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ for some $f \in C^\infty(Q)$. To prove that $\omega$ is exact, it will suffice to show that $\omega = \mathrm{d}\mu$ for some $\mu \in \Omega^{n-1}(Q)$.

Let $g$ be defined as in the hint. If we take $\mu = g\,\mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n$, then

$$\begin{aligned}
\mathrm{d}\mu &= \mathrm{d}(g\,\mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n) \\
&= \mathrm{d}g \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n \\
&= f(x_1, x_2, \dots, x_n)\,\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n \\
&= f\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \\
&= \omega
\end{aligned}$$

as desired, where we get from the second to the third line above with the Fundamental Theorem of Calculus. □

**2.5.i.** Verify the following properties of the interior product, where $U \subset \mathbb{R}^n$ open, $\boldsymbol{v}, \boldsymbol{w}$ are vector fields on $U$, $\omega_1, \omega_2, \omega \in \Omega^k(U)$, and $\mu \in \Omega^\ell(U)$.

(1) *Linearity in the form*: We have

$$\iota_{\boldsymbol{v}}(\omega_1 + \omega_2) = \iota_{\boldsymbol{v}}\omega_1 + \iota_{\boldsymbol{v}}\omega_2$$

*Proof.* To prove that the above two forms are equal, it will suffice to show that they evaluate to identical elements of $\Lambda^{k-1}(V^*)$ for all $p \in U$. Let $p \in U$ be arbitrary. Then

$$\begin{aligned}
[\iota_{\boldsymbol{v}}(\omega_1 + \omega_2)]_p &= \iota_{\boldsymbol{v}(p)}[(\omega_1 + \omega_2)_p] \\
&= \iota_{\boldsymbol{v}(p)}[(\omega_1)_p + (\omega_2)_p] \\
&= \iota_{\boldsymbol{v}(p)}(\omega_1)_p + \iota_{\boldsymbol{v}(p)}(\omega_2)_p \\
&= (\iota_{\boldsymbol{v}}\omega_1 + \iota_{\boldsymbol{v}}\omega_2)_p
\end{aligned}$$

where we get from the first to the second line and the third to the fourth line using the definition of the interior product, and from the second to the third line using the linearity of the interior product operation. □

(2) *Linearity in the vector field*: We have

$$\iota_{\boldsymbol{v}+\boldsymbol{w}}\omega = \iota_{\boldsymbol{v}}\omega + \iota_{\boldsymbol{w}}\omega$$

*Proof.* As in the previous part, we have that

$$\begin{aligned}
[\iota_{\boldsymbol{v}+\boldsymbol{w}}\omega]_p &= \iota_{(\boldsymbol{v}+\boldsymbol{w})(p)}\omega_p \\
&= \iota_{\boldsymbol{v}(p)+\boldsymbol{w}(p)}\omega_p \\
&= \iota_{\boldsymbol{v}(p)}\omega_p + \iota_{\boldsymbol{w}(p)}\omega_p \\
&= [\iota_{\boldsymbol{v}}\omega]_p + [\iota_{\boldsymbol{w}}\omega]_p \\
&= [\iota_{\boldsymbol{v}}\omega + \iota_{\boldsymbol{w}}\omega]_p
\end{aligned}$$

□

(3) *Derivation property*: We have

$$\iota_{\boldsymbol{v}}(\omega \wedge \mu) = \iota_{\boldsymbol{v}}\omega \wedge \mu + (-1)^k \omega \wedge \iota_{\boldsymbol{v}}\mu$$

*Proof.* As in the previous parts, we have that

$$\begin{aligned}
[\iota_{\boldsymbol{v}}(\omega \wedge \mu)]_p &= \iota_{\boldsymbol{v}(p)}(\omega \wedge \mu)_p \\
&= \iota_{\boldsymbol{v}(p)}(\omega_p \wedge \mu_p) \\
&= \iota_{\boldsymbol{v}(p)}\omega_p \wedge \mu_p + (-1)^k \omega_p \wedge \iota_{\boldsymbol{v}(p)}\mu_p \\
&= (\iota_{\boldsymbol{v}}\omega)_p \wedge \mu_p + (-1)^k \omega_p \wedge (\iota_{\boldsymbol{v}}\mu)_p \\
&= (\iota_{\boldsymbol{v}}\omega \wedge \mu)_p + (-1)^k (\omega_p \wedge \iota_{\boldsymbol{v}}\mu)_p \\
&= [\iota_{\boldsymbol{v}}\omega \wedge \mu + (-1)^k \omega_p \wedge \iota_{\boldsymbol{v}}\mu]_p
\end{aligned}$$

□

(4) The identity

$$\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{w}}\omega) = -\iota_{\boldsymbol{w}}(\iota_{\boldsymbol{v}}\omega)$$

*Proof.* As in the previous parts, we have that

$$\begin{aligned}
[\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{w}}\omega)]_p &= \iota_{\boldsymbol{v}(p)}(\iota_{\boldsymbol{w}}\omega)_p \\
&= \iota_{\boldsymbol{v}(p)}(\iota_{\boldsymbol{w}(p)}\omega_p) \\
&= -\iota_{\boldsymbol{v}(p)}(\iota_{\boldsymbol{w}(p)}\omega_p) \\
&= -\iota_{\boldsymbol{v}(p)}(\iota_{\boldsymbol{w}}\omega)_p \\
&= -[\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{w}}\omega)]_p \\
&= [-\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{w}}\omega)]_p
\end{aligned}$$

□

(5) The identity, as a special case of (4),

$$\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}\omega) = 0$$

*Proof.* We have that

$$\begin{aligned}
\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}\omega) &= -\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}\omega)\\
2\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}\omega) &= 0\\
\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}\omega) &= 0
\end{aligned}$$

□

(6) If $\omega = \mu_1 \wedge \cdots \wedge \mu_k$ (i.e., if $\omega$ is **decomposable**), then

$$\iota_{\boldsymbol{v}}\omega = \sum_{r=1}^{k} (-1)^{r-1} \iota_{\boldsymbol{v}}(\mu_r)\mu_1 \wedge \cdots \wedge \widehat{\mu_r} \wedge \cdots \wedge \mu_k$$

*Proof.* To prove that the above two forms are equal, it will suffice to show that they evaluate to identical elements of $\Lambda^{k-1}(V^*)$ for all $p \in U$. Let $p \in U$ be arbitrary. Also let $(\mu_i)_p = \pi(\ell_i)$ for $i = 1, \dots, k$ and, thus, $\omega_p = \pi(\ell_1 \otimes \cdots \otimes \ell_k)$. Then

$$\begin{aligned}
[\iota_{\boldsymbol{v}}\omega]_p &= \iota_{\boldsymbol{v}(p)}\omega_p\\
&= \pi\left[\iota_{\boldsymbol{v}(p)}(\ell_1 \otimes \cdots \otimes \ell_k)\right]\\
&= \pi\left[\sum_{r=1}^{k} (-1)^{r-1} \ell_r(\boldsymbol{v}(p))\ell_1 \otimes \cdots \otimes \hat{\ell}_r \otimes \cdots \otimes \ell_k\right] && \text{Lemma 1.7.4}\\
&= \sum_{r=1}^{k} (-1)^{r-1} \pi\left[\iota_{\boldsymbol{v(p)}}(\ell_r)\ell_1 \otimes \cdots \otimes \hat{\ell}_r \otimes \cdots \otimes \ell_k\right]\\
&= \sum_{r=1}^{k} (-1)^{r-1} \iota_{\boldsymbol{v(p)}}(\mu_r)_p(\mu_1)_p \wedge \cdots \wedge \widehat{(\mu_r)_p} \wedge \cdots \wedge (\mu_k)_p\\
&= \left[\sum_{r=1}^{k} (-1)^{r-1} \iota_{\boldsymbol{v}}(\mu_r)\mu_1 \wedge \cdots \wedge \widehat{\mu_r} \wedge \cdots \wedge \mu_k\right]_p
\end{aligned}$$

as desired. □

**2.5.ii.** Show that if $\omega$ is the $k$-form $\mathrm{d}x_I$ and $\boldsymbol{v}$ is the vector field $\partial/\partial x_r$, then $\iota_{\boldsymbol{v}}\omega$ is given by

$$\iota_{\boldsymbol{v}}\omega = \sum_{j=1}^{k} (-1)^{j-1} \delta_{r,i_j}\, \mathrm{d}x_{I_j}$$

where

$$\delta_{r,i_j} = \begin{cases} 1 & r = i_j \\ 0 & r \neq i_j \end{cases} \qquad\qquad I_j = (i_1, \dots, \widehat{i_j}, \dots, i_k)$$

In the above, 1 represents the identity function on $U$, and 0 represents the zero function on $U$.

*Proof.* We have that $\omega = \mathrm{d}x_I = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$. Therefore, by Properties 2.5.3(6),

$$\begin{aligned}
\iota_{\boldsymbol{v}}\omega &= \sum_{j=1}^{k} (-1)^{j-1} \iota_{\boldsymbol{v}}(\mathrm{d}x_{i_j})\, \mathrm{d}x_{i_1} \wedge \cdots \wedge \widehat{\mathrm{d}x_{i_j}} \wedge \cdots \wedge \mathrm{d}x_{i_k}\\
&= \sum_{j=1}^{k} (-1)^{j-1}\, \mathrm{d}x_{i_j}\,(\partial/\partial x_r)\, \mathrm{d}x_{i_1} \wedge \cdots \wedge \widehat{\mathrm{d}x_{i_j}} \wedge \cdots \wedge \mathrm{d}x_{i_k}\\
&= \sum_{j=1}^{k} (-1)^{j-1} \delta_{r,i_j}\, \mathrm{d}x_{I_j}
\end{aligned}$$

as desired. □

**2.5.iii.** Show that if $\omega$ is the $n$-form $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ and $\boldsymbol{v}$ is the vector field $\sum_{i=1}^n f_i \, \partial/\partial x_i$, then $\iota_{\boldsymbol{v}}\omega$ is given by

$$\iota_{\boldsymbol{v}}\omega = \sum_{r=1}^n (-1)^{r-1} f_r \, \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_r} \wedge \cdots \wedge \mathrm{d}x_n$$

*Proof.* Let $I_j = (1, \dots, \hat{j}, \dots, n)$. Then we have that

$$\begin{aligned}
\iota_{\boldsymbol{v}}\omega &= \sum_{r=1}^n f_r \iota_{\partial/\partial x_r} \omega && \text{Properties 2.5.3(2)} \\
&= \sum_{r=1}^n f_r \left( \sum_{j=1}^n (-1)^{j-1} \delta_{r,j} \, \mathrm{d}x_{I_j} \right) && \text{Exercise 2.5.ii} \\
&= \sum_{r=1}^n f_r \left( (-1)^{r-1} \delta_{r,r} \, \mathrm{d}x_{I_r} \right) \\
&= \sum_{r=1}^n (-1)^{r-1} f_r \, \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_r} \wedge \cdots \wedge \mathrm{d}x_n
\end{aligned}$$

as desired. □

**2.5.iv.** Let $U \subset \mathbb{R}^n$ open and $\boldsymbol{v}$ a $C^\infty$ vector field on $U$. Show that for $\omega \in \Omega^k(U)$,

$$\mathrm{d}(L_{\boldsymbol{v}}\omega) = L_{\boldsymbol{v}}(\mathrm{d}\omega) \qquad \iota_{\boldsymbol{v}}(L_{\boldsymbol{v}}\omega) = L_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}\omega)$$

*Hint*: Deduce the first of these identities using the identity $\mathrm{d}(\mathrm{d}\mu) = 0$ and the second using the identity $\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}\mu) = 0$.

*Proof.* Let $\omega \in \Omega^k(U)$ be arbitrary. Then

$$\begin{aligned}
\mathrm{d}(L_{\boldsymbol{v}}\omega) &= \mathrm{d}(\iota_{\boldsymbol{v}}(\mathrm{d}\omega) + \mathrm{d}(\iota_{\boldsymbol{v}}\omega)) \\
&= \mathrm{d}(\iota_{\boldsymbol{v}}(\mathrm{d}\omega)) + \mathrm{d}(\mathrm{d}(\iota_{\boldsymbol{v}}\omega)) \\
&= \mathrm{d}(\iota_{\boldsymbol{v}}(\mathrm{d}\omega)) + 0 \\
&= 0 + \mathrm{d}(\iota_{\boldsymbol{v}}(\mathrm{d}\omega)) \\
&= \iota_{\boldsymbol{v}}(0) + \mathrm{d}(\iota_{\boldsymbol{v}}(\mathrm{d}\omega)) \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}(\mathrm{d}\omega)) + \mathrm{d}(\iota_{\boldsymbol{v}}(\mathrm{d}\omega)) \\
&= L_{\boldsymbol{v}}(\mathrm{d}\omega)
\end{aligned}
\qquad
\begin{aligned}
\iota_{\boldsymbol{v}}(L_{\boldsymbol{v}}\omega) &= \iota_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}(\mathrm{d}\omega) + \mathrm{d}(\iota_{\boldsymbol{v}}\omega)) \\
&= \iota_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}(\mathrm{d}\omega)) + \iota_{\boldsymbol{v}}(\mathrm{d}(\iota_{\boldsymbol{v}}\omega)) \\
&= 0 + \iota_{\boldsymbol{v}}(\mathrm{d}(\iota_{\boldsymbol{v}}\omega)) \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}(\iota_{\boldsymbol{v}}\omega)) + 0 \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}(\iota_{\boldsymbol{v}}\omega)) + \mathrm{d}(0) \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}(\iota_{\boldsymbol{v}}\omega)) + \mathrm{d}(\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}\omega)) \\
&= L_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}\omega)
\end{aligned}$$

as desired. □

**2.5.v.** Given $\omega_i \in \Omega^{k_i}(U)$ for $i = 1, 2$, show that

$$L_{\boldsymbol{v}}(\omega_1 \wedge \omega_2) = L_{\boldsymbol{v}}\omega_1 \wedge \omega_2 + \omega_1 \wedge L_{\boldsymbol{v}}\omega_2$$

*Hint*: Plug $\omega = \omega_1 \wedge \omega_2$ into the definition of the Lie derivative and use the second desired property of exterior differentiation along with the derivation property of the interior product to evaluate the resulting expression.

*Proof.* Let $\omega = \omega_1 \wedge \omega_2$. Then

$$\begin{aligned}
L_{\boldsymbol{v}}(\omega_1 \wedge \omega_2) &= L_{\boldsymbol{v}}\omega \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}\omega) + \mathrm{d}(\iota_{\boldsymbol{v}}\omega) \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}(\omega_1 \wedge \omega_2)) + \mathrm{d}(\iota_{\boldsymbol{v}}(\omega_1 \wedge \omega_2)) \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}\omega_1 \wedge \omega_2 + (-1)^{k_1}\omega_1 \wedge \mathrm{d}\omega_2) + \mathrm{d}(\iota_{\boldsymbol{v}}(\omega_1 \wedge \omega_2)) \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}\omega_1 \wedge \omega_2 + (-1)^{k_1}\omega_1 \wedge \mathrm{d}\omega_2) + \mathrm{d}\big(\iota_{\boldsymbol{v}}\omega_1 \wedge \omega_2 + (-1)^{k_1}\omega_1 \wedge \iota_{\boldsymbol{v}}\omega_2\big) \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}\omega_1 \wedge \omega_2) \\
&\quad + (-1)^{k_1}\iota_{\boldsymbol{v}}(\omega_1 \wedge \mathrm{d}\omega_2) \\
&\quad + \mathrm{d}(\iota_{\boldsymbol{v}}\omega_1 \wedge \omega_2) \\
&\quad + (-1)^{k_1}\,\mathrm{d}(\omega_1 \wedge \iota_{\boldsymbol{v}}\omega_2) \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}\omega_1) \wedge \omega_2 + (-1)^{k_1+1}\mathrm{d}\omega_1 \wedge \iota_{\boldsymbol{v}}\omega_2 \\
&\quad + (-1)^{k_1}(\iota_{\boldsymbol{v}}\omega_1 \wedge \mathrm{d}\omega_2 + (-1)^{k_1}\omega_1 \wedge \iota_{\boldsymbol{v}}(\mathrm{d}\omega_2)) \\
&\quad + \mathrm{d}(\iota_{\boldsymbol{v}}\omega_1) \wedge \omega_2 + (-1)^{k_1-1}\iota_{\boldsymbol{v}}\omega_1 \wedge \mathrm{d}\omega_2 \\
&\quad + (-1)^{k_1}(\mathrm{d}\omega_1 \wedge \iota_{\boldsymbol{v}}\omega_2 + (-1)^{k_1}\omega_1 \wedge \mathrm{d}(\iota_{\boldsymbol{v}}\omega_2)) \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}\omega_1) \wedge \omega_2 + (-1)^{k_1+1}\mathrm{d}\omega_1 \wedge \iota_{\boldsymbol{v}}\omega_2 \\
&\quad + (-1)^{k_1}\iota_{\boldsymbol{v}}\omega_1 \wedge \mathrm{d}\omega_2 + (-1)^{2k_1}\omega_1 \wedge \iota_{\boldsymbol{v}}(\mathrm{d}\omega_2) \\
&\quad + \mathrm{d}(\iota_{\boldsymbol{v}}\omega_1) \wedge \omega_2 + (-1)^{k_1-1}\iota_{\boldsymbol{v}}\omega_1 \wedge \mathrm{d}\omega_2 \\
&\quad + (-1)^{k_1}\mathrm{d}\omega_1 \wedge \iota_{\boldsymbol{v}}\omega_2 + (-1)^{2k_1}\omega_1 \wedge \mathrm{d}(\iota_{\boldsymbol{v}}\omega_2) \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}\omega_1) \wedge \omega_2 - (-1)^{k_1}\mathrm{d}\omega_1 \wedge \iota_{\boldsymbol{v}}\omega_2 \\
&\quad + (-1)^{k_1}\iota_{\boldsymbol{v}}\omega_1 \wedge \mathrm{d}\omega_2 + \omega_1 \wedge \iota_{\boldsymbol{v}}(\mathrm{d}\omega_2) \\
&\quad + \mathrm{d}(\iota_{\boldsymbol{v}}\omega_1) \wedge \omega_2 - (-1)^{k_1}\iota_{\boldsymbol{v}}\omega_1 \wedge \mathrm{d}\omega_2 \\
&\quad + (-1)^{k_1}\mathrm{d}\omega_1 \wedge \iota_{\boldsymbol{v}}\omega_2 + \omega_1 \wedge \mathrm{d}(\iota_{\boldsymbol{v}}\omega_2) \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}\omega_1) \wedge \omega_2 \\
&\quad + \omega_1 \wedge \iota_{\boldsymbol{v}}(\mathrm{d}\omega_2) \\
&\quad + \mathrm{d}(\iota_{\boldsymbol{v}}\omega_1) \wedge \omega_2 \\
&\quad + \omega_1 \wedge \mathrm{d}(\iota_{\boldsymbol{v}}\omega_2) \\
&= \iota_{\boldsymbol{v}}(\mathrm{d}\omega_1) \wedge \omega_2 + \mathrm{d}(\iota_{\boldsymbol{v}}\omega_1) \wedge \omega_2 + \omega_1 \wedge \iota_{\boldsymbol{v}}(\mathrm{d}\omega_2) + \omega_1 \wedge \mathrm{d}(\iota_{\boldsymbol{v}}\omega_2) \\
&= (\iota_{\boldsymbol{v}}(\mathrm{d}\omega_1) + \mathrm{d}(\iota_{\boldsymbol{v}}\omega_1)) \wedge \omega_2 + \omega_1 \wedge (\iota_{\boldsymbol{v}}(\mathrm{d}\omega_2) + \mathrm{d}(\iota_{\boldsymbol{v}}\omega_2)) \\
&= L_{\boldsymbol{v}}\omega_1 \wedge \omega_2 + \omega_1 \wedge L_{\boldsymbol{v}}\omega_2
\end{aligned}$$

as desired. ∎

**2.6.i.** Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be the map

$$f(x_1, x_2, x_3) = (x_1x_2, x_2x_3^2, x_3^3)$$

Compute the pullback $f^*\omega$ for the following forms.

(1) $\omega = x_2\,\mathrm{d}x_3$.

*Proof.* We have that

$$\begin{aligned}
f^*\omega &= f^*x_2 \cdot \mathrm{d}f_3 \\
&= x_2(x_1x_2, x_2x_3^2, x_3^3) \cdot 3x_3^2\,\mathrm{d}x_3
\end{aligned}$$

$$\boxed{f^*\omega = 3x_2x_3^4\,\mathrm{d}x_3}$$

∎

(2) $\omega = x_1 \, \mathrm{d}x_1 \wedge \mathrm{d}x_3$.

*Proof.* We have that

$$\begin{aligned} f^*\omega &= f^*x_1 \cdot \mathrm{d}f_1 \wedge \mathrm{d}f_3 \\ &= x_1(x_1x_2, x_2x_3^2, x_3^3) \cdot (x_1 \, \mathrm{d}x_2 + x_2 \, \mathrm{d}x_1) \wedge 3x_3^2 \, \mathrm{d}x_3 \\ \Aboxed{f^*\omega &= 3x_1^2x_2x_3^2 \, \mathrm{d}x_2 \wedge \mathrm{d}x_3 + 3x_1x_2^2x_3^2 \, \mathrm{d}x_1 \wedge \mathrm{d}x_3} \end{aligned}$$

□

(3) $\omega = x_1 \, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$.

*Proof.* We have that

$$\begin{aligned} f^*\omega &= f^*x_1 \cdot \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \mathrm{d}f_3 \\ &= x_1(x_1x_2, x_2x_3^2, x_3^3) \cdot (x_1 \, \mathrm{d}x_2 + x_2 \, \mathrm{d}x_1) \wedge (2x_2x_3 \, \mathrm{d}x_3 + x_3^2 \, \mathrm{d}x_2) \wedge 3x_3^2 \, \mathrm{d}x_3 \\ \Aboxed{f^*\omega &= 3x_1x_2^2x_3^4 \, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3} \end{aligned}$$

where, from the second to the third line, we cancel all wedge products with repeats. □

**2.6.ii.** Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be the map

$$f(x_1, x_2) = (x_1^2, x_2^2, x_1x_2)$$

Compute the pullback $f^*\omega$ for the following forms.

(1) $\omega = x_2 \, \mathrm{d}x_2 + x_3 \, \mathrm{d}x_3$.

*Proof.* We have that

$$\begin{aligned} f^*\omega &= f^*x_2 \cdot \mathrm{d}f_2 + f^*x_3 \cdot \mathrm{d}f_3 \\ &= x_2(x_1^2, x_2^2, x_1x_2) \cdot 2x_2 \, \mathrm{d}x_2 + x_3(x_1^2, x_2^2, x_1x_2) \cdot (x_1 \, \mathrm{d}x_2 + x_2 \, \mathrm{d}x_1) \\ \Aboxed{f^*\omega &= x_1x_2^2 \, \mathrm{d}x_1 + (2x_2^3 + x_1^2x_2) \, \mathrm{d}x_2} \end{aligned}$$

□

(2) $\omega = x_1 \, \mathrm{d}x_2 \wedge \mathrm{d}x_3$.

*Proof.* We have that

$$\begin{aligned} f^*\omega &= f^*x_1 \cdot \mathrm{d}f_2 \wedge \mathrm{d}f_3 \\ &= x_1(x_1^2, x_2^2, x_1x_2) \cdot 2x_2 \, \mathrm{d}x_2 \wedge (x_1 \, \mathrm{d}x_2 + x_2 \, \mathrm{d}x_1) \\ &= 2x_1^2x_2^2 \, \mathrm{d}x_2 \wedge \mathrm{d}x_1 \\ \Aboxed{f^*\omega &= -2x_1^2x_2^2 \, \mathrm{d}x_1 \wedge \mathrm{d}x_2} \end{aligned}$$

□

(3) $\omega = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$.

*Proof.* We have that

$$\begin{aligned} f^*\omega &= \mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \mathrm{d}f_3 \\ &= 2x_1 \, \mathrm{d}x_1 \wedge 2x_2 \, \mathrm{d}x_2 \wedge (x_1 \, \mathrm{d}x_2 + x_2 \, \mathrm{d}x_1) \\ \Aboxed{f^*\omega &= 0} \end{aligned}$$

where 0 denotes the zero element of $\Omega^3(\mathbb{R}^2)$. □

**2.6.iii.** Let $U \subset \mathbb{R}^n$ open, $V \subset \mathbb{R}^m$ open, $f : U \to V$ a $C^\infty$ map, and $\gamma : [a, b] \to U$ a $C^\infty$ curve. Show that for $\omega \in \Omega^1(V)$,

$$\int_\gamma f^*\omega = \int_\eta \omega$$

where $\eta : [a, b] \to V$ is the curve $\eta(t) = f(\gamma(t))$. (See Exercise 2.1.vii.)

*Proof.* Since $\omega \in \Omega^1(V)$, we know that

$$\omega = \sum_{j=1}^m g_j \, \mathrm{d}x_j$$

for some $g_i \in C^\infty(V)$. It follows that

$$\begin{aligned} f^*\omega &= \sum_{j=1}^m f^*g_j \, \mathrm{d}f_j \\ &= \sum_{j=1}^m f^*g_j \left( \sum_{i=1}^n \frac{\partial f_j}{\partial x_i} \, \mathrm{d}x_i \right) \\ &= \sum_{i=1}^n \left( \sum_{j=1}^m f^*g_j \frac{\partial f_j}{\partial x_i} \right) \mathrm{d}x_i \end{aligned}$$

Additionally, let $\gamma_1, \dots, \gamma_n$ be the coordinate functions of $\gamma$, let $\eta_1, \dots, \eta_m$ be the coordinate functions of $\eta$, and let $f_1, \dots, f_m$ be the coordinate functions of $f$. It follows that

$$\begin{aligned} \int_\gamma f^*\omega &= \sum_{i=1}^n \int_a^b \left[ \sum_{j=1}^m f^*g_j \frac{\partial f_j}{\partial x_i} \right] (\gamma(t)) \frac{\mathrm{d}\gamma_i}{\mathrm{d}t} \, \mathrm{d}t \\ &= \sum_{i=1}^n \sum_{j=1}^m \int_a^b [f^*g_j](\gamma(t)) \frac{\partial f_j}{\partial x_i} \frac{\mathrm{d}\gamma_i}{\mathrm{d}t} \, \mathrm{d}t \\ &= \sum_{j=1}^m \int_a^b [g_j \circ f](\gamma(t)) \left( \sum_{i=1}^n \frac{\partial f_j}{\partial x_i} \frac{\mathrm{d}\gamma_i}{\mathrm{d}t} \right) \mathrm{d}t \\ &= \sum_{j=1}^m \int_a^b g_j(f(\gamma(t))) \frac{\mathrm{d}(f_j \circ \gamma)}{\mathrm{d}t} \, \mathrm{d}t \\ &= \sum_{j=1}^m \int_a^b g_j(\eta(t)) \frac{\mathrm{d}\eta_j}{\mathrm{d}t} \, \mathrm{d}t \\ &= \int_\eta \omega \end{aligned}$$

□