# Week 1

# Tensors

## 1.1 Course Motivation

3/28: • Motivation for the course and an overview of Guillemin and Haine (2018).

## 1.2 Defining Tensors and Their Operations

3/30: • Plan:

  - More (multi)linear algebra.

- Let $V$ be an $n$-dimensional real vector space.
- **Dual space** (of $V$): The set of all homomorphisms from $V$ to $\mathbb{R}$. *Also known as* $\boldsymbol{\mathrm{Hom}(V,\mathbb{R})}$, $\boldsymbol{V^*}$.
  - A homomorphism of vector spaces (such as any $\varphi \in V^*$) is just a linear map or, specifically, a **linear functional**.
- **Linear functional**: A linear map from a vector field to its field of scalars (often $\mathbb{R}$ or $\mathbb{C}$).
- **Dual basis** (for $V^*$): The set of linear transformations from $V$ to $\mathbb{R}$ defined by

$$e_j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

  where $e_1, \dots, e_n$ is a basis of $V$. *Denoted by* $e_1^*, \dots, e_n^*$.
- Check: $e_1^*, \dots, e_n^*$ are a basis for $V^*$.
  - Are they linearly independent? Let $c_1 e_1^* + \cdots + c_n e_n^* = 0 \in \mathrm{Hom}(V,\mathbb{R})$. Then

$$c_i = (c_1 e_1^* + \cdots + c_n e_n^*)(e_i) = 0 \in \mathbb{R}$$

    as desired.
  - Span? Let $\varphi \in \mathrm{Hom}(V,\mathbb{R})$. Then we can verify that

$$\varphi(e_1)e_1^* + \cdots + \varphi(e_n)e_n^* = \varphi$$

    - We prove this by verifying the previous statement on the basis of $V$ (if two linear transformations have the same action on the basis of a vector space, they are equal).
- With a choice of basis for $V$, we obtain an **isomorphism** $\varepsilon : V \to V^*$ defined by $e_i \mapsto e_i^*$ for all $i$.

- **Isomorphism** (between $V, W$): A bijective (equiv., invertible) linear map between two vector spaces $V, W$. *Denoted by* $\boldsymbol{V \cong W}$.
- The dual space is known as such because $(V^*)^* \cong V$, where the isomorphism is **canonical**.
  - The canonical isomorphism $I : V \to (V^*)^*$ is given by $v \mapsto I_v$, where $I_v : V^* \to \mathbb{R}$ is the linear functional defined by
  $$I_v(\varphi) = \varphi(v)$$
  for all $\varphi \in V^*$.
  - Some incomplete thoughts on the proof are commented out in the `.tex` file. See also this link.
- **Canonical** (map): A map between objects that arises naturally from the definition or construction of the objects.
- **Canonical** (map of vector spaces): A map between vector spaces such that no choice of bases is needed to describe it.
  - Equivalently, such a map as described on any choice of bases will be equal to the canonical map.
- **Pullback** (of $A$): The linear transformation from $W^* \to V^*$, where $V, W$ are vector spaces and $A : V \to W$, defined as follows. *Also known as* **transpose**. *Denoted by* $\boldsymbol{A^*}$. *Given by*
  $$A^*(\varphi) = \varphi \circ A$$
  - This object is also known as the transpose because the matrix of $A^*$ is the transpose of the matrix of $A$, provided $V, W$ are real vector spaces.
  - Claim: $A^*$ is linear.
- One more property of dual spaces: **functoriality**.
- **Functoriality**: If $A : V \to W$ and $B : W \to U$, then $B^* : U^* \to W^*$ and $A^* : W^* \to V^*$. The functoriality statement is that $(B \circ A)^* = A^* \circ B^*$.
- Let $v_1, \dots, v_n$ be a basis for $V$ and $w_1, \dots, w_m$ be a basis for $W$. Then $[A]_{v_1,\dots,v_n}^{w_1,\dots,w_m} = A$ is the matrix of the linear transformation $A$ with respect to these bases. Then if $v_1^*, \dots, v_n^*$ and $w_1^*, \dots, w_m^*$ are the corresponding dual bases, then $[A^*]_{v_1^*,\dots,v_n^*}^{w_1^*,\dots,w_m^*} = A^T$. We can and should verify this for ourselves.
- This is over the real numbers, so $A^*$ is just the transpose because there are no complex numbers of which to take the conjugate!
- A generalization: Tensors.
- ***k*-tensor**: A **multilinear** map
  $$T : \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}$$
- **Multilinear** (map $T$): A function $T$ such that
  $$\begin{aligned} T(v_1, \dots, v_i^1 + v_i^2, \dots, v_k) &= T(v_1, \dots, v_i^1, \dots, v_k) + T(v_1, \dots, v_i^2, \dots, v_k) \\ T(v_1, \dots, \lambda v_i, \dots, v_k) &= \lambda T(v_1, \dots, v_i, \dots, v_k) \end{aligned}$$
  for all $(v_1, \dots, v_k) \in V^k$.
- The determinant is an $n$-tensor!
- 1-tensors are just covectors.
- $\boldsymbol{\mathcal{L}^k(V)}$: The vector space of all $k$-tensors on $V$.

- Calculating $\dim \mathcal{L}^k(V)$. (Answer not given in this class.)
- Let $A : V \to W$. Then $A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$.
  - Check $(A \circ B)^* = B^* \circ A^*$.
- **Multi-index of $n$ of length $k$**: A $k$-tuple $(i_1, \dots, i_k)$ where each $i_j \in \mathbb{N}$ satisfies $1 \leq i_j \leq n$ $(j = 1, \dots, k)$. *Denoted by* $\boldsymbol{I}$.
- Let $e_1, \dots, e_n$ be a basis for $V$.
- **Tensor product** (of $T_1 \in \mathcal{L}^k(V)$, $T_2 \in L^l(V)$): The function from $V^{k+l}$ to $\mathbb{R}$ defined by

$$(v_1, \dots, v_{k+l}) \mapsto T_1(v_1, \dots, v_k) T_2(v_{k+1}, \dots, v_{k+l})$$

  *Denoted by* $\boldsymbol{T_1 \otimes T_2}$.
- Claims:
  1. $T_1 \otimes T_2 \in L^{k+l}(V)$.
  2. $A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$.
- $\boldsymbol{e_I^*}$: The function $e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*$, where $I = (i_1, \dots, i_k)$ is a multi-index of $n$ of length $k$.
- Claim: Letting $I$ range over all $n^k$ multi-indices of $n$ of length $k$, the $e_I^*$ are a basis for $\mathcal{L}^k(V)$.
- If $V = \mathbb{R}$, then $V = \mathrm{R}e_1$. If $V = \mathbb{R}^2$, then $V = \mathrm{R}e_1 \oplus \mathrm{R}e_2$.
- We know that $L^1(V) = V^* = Re_1^*$. Thus, $e_1^* \otimes e_2^* : V \times V \to \mathbb{R}$. Thus, for example,

$$(e_1^* \otimes e_2^*)((1,2),(3,4)) = e_1^*(1,2) \cdot e_2^*(3,4) = 1 \cdot 4 = 4$$

## 1.3 The Tensor Product and Permutations

4/1:

- Plan: More multilinear algebra.
  - Properties of the tensor product.
  - Sign of a permutation.
  - Alternating tensors (lead into differential forms down the road).
- Recall: $V$ is an $n$-dimensional vector space over $\mathbb{R}$ with basis $e_1, \dots, e_n$. $\mathcal{L}^k(V)$ is the vector space of $k$-tensors on $V$. $\{e_I^* \mid I \text{ a multiindex of } n \text{ of length } k\}$ is a basis for $\mathcal{L}^k(V)$.
- For example, if $V = \mathbb{R}^2$ and $T \in \mathcal{L}^2(V)$, then

$$T(a_1e_1 + a_2e_2, b_1e_1 + b_2e_2) = a_1b_1T(e_1,e_1) + a_1b_2T(e_1,e_2) + a_2b_1T(e_2,e_1) + a_2b_2T(e_2,e_2)$$

  - A basis of $\mathcal{L}^2(V)$ is

$$\{e_1^* \otimes e_1^*, e_1^* \otimes e_2^*, e_2^* \otimes e_1^*, e_2^* \otimes e_2^*\}$$

  - Recall that some basic properties are

$$e_1^* \otimes e_2^*((1,2),(3,4)) = 1 \cdot 4 = 4 \qquad e_2^* \otimes e_1^*((1,2),(3,4)) = 2 \cdot 3 = 6$$

  - It follows by the initial decomposition of $T$ that

$$T = a_1b_1e_1^* \otimes e_1^* + a_1b_2e_1^* \otimes e_2^* + a_2b_1e_2^* \otimes e_1^* + a_2b_2e_2^* \otimes e_2^*$$

- Important consequence: To know the action of $T$ on an arbitrary pair of vectors, you need only know its action on the basis; a higher-dimensional generalization of the earlier property.

- Note that
$$e_I^*(e_J) = \delta_{IJ} = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$$
- Basic properties of the tensor product.
  1. *Right-distributive*: If $T_1 \in \mathcal{L}^k(V)$ and $T_2, T_3 \in \mathcal{L}^\ell(V)$, then
  $$T_1 \otimes (T_2 + T_3) = T_1 \otimes T_2 + T_1 \otimes T_3$$
  2. *Left-distributive*: If $T_1, T_2 \in \mathcal{L}^k(V)$ and $T_3 \in \mathcal{L}^\ell(V)$, then
  $$(T_1 + T_2) \otimes T_3 = T_1 \otimes T_3 + T_2 \otimes T_3$$
  3. *Associative*: If $T_1 \in \mathcal{L}^k(V)$, $T_2 \in \mathcal{L}^\ell(V)$, and $T_3 \in \mathcal{L}^m(V)$, then
  $$T_1 \otimes (T_2 \otimes T_3) = (T_1 \otimes T_2) \otimes T_2 = T_1 \otimes T_2 \otimes T_3$$
  4. *Scalar multiplication*: If $T_1 \in \mathcal{L}^k(V)$, $T_2 \in \mathcal{L}^\ell(V)$, and $\lambda \in \mathbb{R}$, then
  $$(\lambda T_1) \otimes T_2 = \lambda(T_1 \otimes T_2) = T_1 \otimes (\lambda T_2)$$
- Note that the tensor product is not commutative.
- Aside: Defining the sign of a permutation.
- $\boldsymbol{S_A}$: The set of all automorphisms of $A$ (bijections from $A$ to $A$), where $A$ is a set.
- $\boldsymbol{S_n}$: The set $S_{[n]}$.
- Given $\sigma_1, \sigma_2 \in S_n$, $\sigma_1 \circ \sigma_2 \in S_n$.
  - Thus, $S_n$ is a **group**.
- **Transposition**: A function in $S_n$ such that
$$k \mapsto \begin{cases} j & k = i \\ i & k = j \\ k & k \neq i, j \end{cases}$$
for some $i, j \in [n]$. *Denoted by* $\boldsymbol{\tau_{i,j}}$.
- Theorem: An element of $S_n$ can be written as the product of transpositions (i.e., for all $\sigma \in S_n$, there exist $\tau_1, \dots, \tau_m \in S_n$ such that $\sigma = \tau_1 \circ \cdots \circ \tau_m$).
- **Sign** (of $\sigma \in S_n$): The number (mod 2) of transpositions whose product equals $\sigma$. *Denoted by* $\boldsymbol{(-1)^\sigma}$, $\mathbf{sign}(\boldsymbol{\sigma})$.
- Theorem: The sign of $\sigma$ is well-defined. Additionally,
$$(-1)^{\sigma_1\sigma_2} = (-1)^{\sigma_1} \cdot (-1)^{\sigma_2}$$
- Example: Consider the identity permutation. $(-1)^\sigma = +1$. We can think of this as the product of zero transpositions or, for instance, as the product of the two transpositions $\tau_{1,2} \circ \tau_{1,2}$. Another example would be $\tau_{2,3} \circ \tau_{1,2} \circ \tau_{1,2} \circ \tau_{2,3}$.
- Theorem: Let $X_i$ be a rational or polynomial function for each $i \in \mathbb{N}$. Then
$$(-1)^\sigma = \prod_{i<j} \frac{X_{\sigma(i)} - X_{\sigma(j)}}{X_i - X_j}$$

- Example: For the permutation $\sigma = (1, 2, 3)$, we have

$$\begin{aligned}
(-1)^\sigma &= \frac{X_{\sigma(1)} - X_{\sigma(2)}}{X_1 - X_2} \cdot \frac{X_{\sigma(1)} - X_{\sigma(3)}}{X_1 - X_3} \cdot \frac{X_{\sigma(2)} - X_{\sigma(3)}}{X_2 - X_3} \\
&= \frac{X_2 - X_3}{X_1 - X_2} \cdot \frac{X_2 - X_1}{X_1 - X_3} \cdot \frac{X_3 - X_1}{X_2 - X_3} \\
&= \frac{-(X_1 - X_2)}{X_1 - X_2} \cdot \frac{-(X_1 - X_3)}{X_1 - X_3} \cdot \frac{X_2 - X_3}{X_2 - X_3} \\
&= -1 \cdot -1 \cdot 1 \\
&= +1
\end{aligned}$$

  which checks out with the fact that $\sigma = \tau_{1,2} \circ \tau_{2,3}$.

- Claims to verify with the above formula:
  1. $\operatorname{sign}(\sigma) \in \{\pm 1\}$.
  2. $\operatorname{sign}(\tau_{i,j}) = -1$.
  3. $\operatorname{sign}(\sigma_1\sigma_2) = \operatorname{sign}(\sigma_1)\operatorname{sign}(\sigma_2)$.

## 1.4 Chapter 1: Multilinear Algebra

*From Guillemin and Haine (2018).*

3/31:

- Guillemin and Haine (2018) defines real vector spaces, the operations on them, their basic properties, and the zero vector.
- **Linearly independent** (vectors $v_1, \dots, v_k$): A finite set of vectors $v_1, \dots, v_k \in V$ such that the map from $\mathbb{R}^k$ to $V$ defined by $(c_1, \dots, c_k) \mapsto c_1v_1 + \cdots + c_kv_k$ is injective.
- **Spanning** (vectors $v_1, \dots, v_k$): We require that the above map is surjective.
- Guillemin and Haine (2018) defines basis, finite-dimensional vector space, dimension, subspace, linear map, and kernel.
- **Image** (of $A : V \to W$): The range space of $A$, a subspace of $W$. *Also known as* $\mathbf{im}(\boldsymbol{A})$.
- Guillemin and Haine (2018) defines the matrix of a linear map.
- **Inner product** (on $V$): A map $B : V \times V \to \mathbb{R}$ with the following three properties.
  - *Bilinearity*: For vectors $v, v_1, v_2, w \in V$ and $\lambda \in \mathbb{R}$, we have

$$B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w)$$

    and

$$B(\lambda v, w) = \lambda B(v, w)$$

  - *Symmetry*: For vectors $v, w \in V$, we have $B(v, w) = B(w, v)$.
  - *Positivity*: For every vector $v \in V$, we have $B(v, v) \geq 0$. Moreover, if $v \neq 0$, then $B(v, v) > 0$.
- $\boldsymbol{W}$**-coset**: A set of the form $\{v + w \mid w \in W\}$, where $W$ is a subspace $V$ and $v \in V$. *Denoted by* $\boldsymbol{v + W}$.
  - If $v_1 - v_2 \in W$, then $v_1 + W = v_2 + W$.
  - It follows that the distinct $W$-cosets decompose $V$ into a disjoint collection of subsets of $V$.

- **Quotient space** (of $V$ by $W$): The set of distinct $W$-cosets in $V$, along with the following definitions of vector addition and scalar multiplication.

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W \qquad\qquad \lambda(v + W) = (\lambda v) + W$$

*Denoted by* $\boldsymbol{V/W}$.

- **Quotient map**: The linear map $\pi : V \to V/W$ defined by

$$\pi(v) = v + W$$

  - $\pi$ is surjective.
  - Note that $\ker(\pi) = W$ since for all $w \in W$, $\pi(w) = w + W = 0 + W$, which is the zero vector in $V/W$.

- If $V, W$ are finite dimensional, then

$$\dim(V/W) = \dim(V) - \dim(W)$$

- Proposition 1.2.9: Let $A : V \to U$ be a linear map. If $W \subset \ker(A)$, then there exists a unique linear map $A^\sharp : V/W \to U$ with the property that $A = A^\sharp \circ \pi$, where $\pi : V \to V/W$ is the quotient map.
  - This proposition rephrases in terms of quotient spaces the fact that if $w \in W$, then $A(v+w) = Av$.
  - Another way of thinking about it is that $\pi$ cuts out all of the information that $A$ will lose anyway, and $A^\sharp$ retains all information necessary to reconstruct $A$.

- **Dual space** (of $V$): The set of all linear functions $\ell : V \to \mathbb{R}$, along with the following definitions of vector addition and scalar multiplication.

$$(\ell_1 + \ell_2)(v) = \ell_1(v) + \ell_2(v) \qquad\qquad (\lambda\ell)(v) = \lambda \cdot \ell(v)$$

*Denoted by* $\boldsymbol{V^*}$.

- **Dual basis** (of $e_1, \dots, e_n$ a basis of $V$): The basis of $V^*$ consisting of the $n$ functions that take every $v = c_1e_1 + \cdots + c_ne_n$ to one of the $c_i$. *Denoted by* $\boldsymbol{e_1^*, \dots, e_n^*}$. *Given by*

$$e_i^*(v) = c_i$$

for all $v \in V$.

- Claim 1.2.12: If $V$ is an $n$-dimensional vector space with basis $e_1, \dots, e_n$, then $e_1^*, \dots, e_n^*$ is a basis of $V^*$.

*Proof.* We will first prove that $e_1^*, \dots, e_n^*$ spans $V^*$. Let $\ell \in V^*$ be arbitrary. Set $\lambda_i = \ell(e_i)$ for all $i \in [n]$. Define $\ell' = \sum_{i=1}^n \lambda_i e_i^*$. Then

$$\ell'(e_j) = \sum_{i=1}^n \lambda_i e_i^*(e_j) = \lambda_j \cdot 1 = \ell(e_j)$$

for all $j \in [n]$. Therefore, since $\ell, \ell'$ take identical values on the basis of $V$, $\ell = \ell'$, as desired.

We now prove that $e_1^*, \dots, e_n^*$ is linearly independent. Let $\sum_{i=1}^n \lambda_i e_i^* = 0$. Then for all $j \in [n]$,

$$\lambda_j = \left(\sum_{i=1}^n \lambda_i e_i^*\right)(e_j) = 0$$

as desired. ∎

- **Transpose** (of $A$): The map from $W^*$ to $V^*$ defined by $\ell \mapsto \ell \circ A$ for all $\ell \in W^*$. *Denoted by* $\boldsymbol{A^*}$.
- Claim 1.2.15: If $e_1, \dots, e_n$ is a basis of $V$, $f_1, \dots, f_m$ is a basis of $W$, $e_1^*, \dots, e_n^*$ and $f_1^*, \dots, f_m^*$ are the corresponding dual bases, and $[a_{i,j}]$ is the $m \times n$ matrix of $A$ with respect to $\{e_j\}, \{f_i\}$, then the linear map $A^*$ is defined in terms of $\{f_i^*\}, \{e_j^*\}$ by the transpose matrix $(a_{j,i})$.

  *Proof.* Let $[c_{j,i}]$ be the $n \times m$ matrix of $A^*$ with respect to $\{f_i^*\}, \{e_j^*\}$. We seek to prove that $a_{i,j} = c_{j,i}$ ($1 \leq i \leq m$, $1 \leq j \leq n$).

  By the definition of $[a_{i,j}]$ and $[c_{j,i}]$, we have that

  $$A^* f_i^* = \sum_{k=1}^{n} c_{k,i} e_k^* \qquad\qquad Ae_j = \sum_{k=1}^{m} a_{k,j} f_k$$

  It follows that

  $$[A^* f_i^*](e_j) = \left[\sum_{k=1}^{n} c_{k,i} e_k^*\right](e_j) = c_{j,i}$$

  and

  $$[A^* f_i^*](e_j) = f_i^*(Ae_j) = f_i^*\left(\sum_{k=1}^{m} a_{k,j} f_k\right) = a_{i,j}$$

  so transitivity implies the desired result. □

4/4:
- $\boldsymbol{V^k}$: The set of all $k$-tuples $(v_1, \dots, v_k)$ where $v_1, \dots, v_k \in V$ a vector space.
  - Note that
    $$V^k = \underbrace{V \times \cdots \times V}_{k \text{ times}}$$
    where "$\times$" denotes the Cartesian product.
- **Linear** (function in its $i^{\text{th}}$ variable): A function $T : V^k \to \mathbb{R}$ such that the map from $V$ to $\mathbb{R}$ defined by $v \mapsto T(v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_k)$ is linear, where all $v_j$ save $v_i$ are fixed.
- $\boldsymbol{k}$**-linear** (function $T$): A function $T : V^k \to \mathbb{R}$ that is linear in its $i^{\text{th}}$ variable for $i = 1, \dots, k$. *Also known as* $\boldsymbol{k}$**-tensor**.
- $\boldsymbol{\mathcal{L}^k(V)}$: The set of all $k$-tensors in $V$.
  - Since the sum $T_1 + T_2$ of two $k$-linear functions $T_1, T_2 : V^k \to \mathbb{R}$ is just another $k$-linear function, and $\lambda T_1$ is $k$-linear for all $\lambda \in \mathbb{R}$, we have that $\mathcal{L}^k(V)$ is a vector space.
- Convention[1]: 0-tensors are just the real numbers. Mathematically, we define
  $$\mathcal{L}^0(V) = \mathbb{R}$$
- Note that $\mathcal{L}^1(V) = V^*$.
- Defines multi-indices of $n$ of length $k$.
- Lemma 1.3.5: If $n, k \in \mathbb{N}$, then there are exactly $n^k$ multi-indices of $n$ of length $k$.
- $\boldsymbol{T_I}$: The real number $T(e_{i_1}, \dots, e_{i_k})$, where $T \in \mathcal{L}^k(V)$, $e_1, \dots, e_n$ is a basis of $V$, and $I$ is a multi-index of $n$ of length $k$.
- Proposition 1.3.7: The real numbers $T_I$ determine $T$, i.e., if $T, T'$ are $k$-tensors and $T_I = T'_I$ for all $I$, then $T = T'$.

[1] See the definition of the tensor product and ensuing note for a justification of this convention (just a few lines down).

*Proof.* We induct on $n$. For the base case $n = 1$, $T \in (\mathbb{R}^k)^*$ and we have already proven this result. Now suppose inductively that the assertion is true for $n - 1$. For each $e_i$, let $T_i$ be the $(k-1)$-tensor defined by

$$(v_1, \dots, v_{n-1}) \mapsto T(v_1, \dots, v_{n-1}, e_i)$$

Then for an arbitrary $v = c_1e_1 + \cdots + c_ne_n$,

$$T(v_1, \dots, v_{n-1}, v) = \sum_{i=1}^n c_iT_i(v_1, \dots, v_{n-1})$$

so the $T_i$'s determine $T$. Applying the inductive hypothesis completes the proof. □

- **Tensor product**: The function $\otimes : \mathcal{L}^k(V) \times \mathcal{L}^\ell(V) \to \mathcal{L}^{k+\ell}(V)$ defined by

$$(T_1 \otimes T_2)(v_1, \dots, v_{k+\ell}) = T_1(v_1, \dots, v_k)T_2(v_{k+1}, \dots, v_{k+\ell})$$

for all $T_1 \in \mathcal{L}^k(V)$ and $T_2 \in \mathcal{L}^\ell(V)$.

- Note that by the definition of 0-tensors as real numbers, if $a \in \mathbb{R}$ and $T \in \mathcal{L}^k(V)$, then

$$a \otimes T = T \otimes a = aT$$

- Proposition 1.3.9: Associativity, distributivity of scalar multiplication, and left and right distributive laws for the tensor product.
- **Decomposable** ($k$-tensor): A $k$-tensor $T$ for which there exist $\ell_1, \dots, \ell_k \in V^*$ such that

$$T = \ell_1 \otimes \cdots \otimes \ell_k$$

- Defines $e_I^*$.
- Theorem 1.3.13: $V$ a vector space with basis $e_1, \dots, e_n$ and $0 \leq k \leq n$ implies the $k$-tensors $e_I^*$ form a basis of $\mathcal{L}^k(V)$.

*Proof.* Spanning: Let $T \in \mathcal{L}^k(V)$ be arbitrary. Define

$$T' = \sum_I T_Ie_I^*$$

Since

$$T'_J = T'(e_{j_1}, \dots, e_{j_k}) = \sum_I T_Ie_I^*(e_{j_1}, \dots, e_{j_k}) = T_Je_J^*(e_{j_1}, \dots, e_{j_k}) = T_J$$

for all $J$, Proposition 1.3.7 asserts that $T = T'$. Therefore, since every $T_I \in \mathbb{R}$, $T = T' \in \operatorname{span}(e_I^*)$.

Linear independence: Suppose

$$T = \sum_I c_Ie_I^* = 0$$

for some set of constants $c_I \in \mathbb{R}$. Then

$$0 = T(e_{j_1}, \dots, e_{j_k}) = \sum_I c_Ie_I^*(e_{j_1}, \dots, e_{j_k}) = c_J$$

for all $J$, as desired. □

- Corollary 1.3.15: If $\dim V = n$, then $\dim(\mathcal{L}^k(V)) = n^k$.

*Proof.* Follows immediately from Lemma 1.3.5. □

- **Pullback** (of $T$ by the map $A$): The $k$-tensor $A^*T : V^k \to \mathbb{R}$ defined by
$$(A^*T)(v_1, \dots, v_k) = T(Av_1, \dots, Av_k)$$
where $V, W$ are finite-dimensional vector spaces, $A : V \to W$ is linear, and $T \in \mathcal{L}^k(W)$.
- Proposition 1.3.18: The map $A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$ defined by $T \mapsto A^*T$ is linear.
- Identities:
  - If $T_1 \in \mathcal{L}^k(W)$ and $T_2 \in \mathcal{L}^m(W)$, then
$$A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$$
  - If $U$ is a vector space, $B : U \to V$ is linear, and $T \in \mathcal{L}^k(W)$, then $(AB)^*T = B^*(A^*T)$. Hence,
$$(AB)^* = B^*A^*$$

4/13:

- $\boldsymbol{\Sigma_k}$: The set containing the natural numbers 1 through $k$. *Given by*
$$\Sigma_k = \{1, 2, \dots, k\}$$
- **Permutation of order $\boldsymbol{k}$**: A bijection on $\Sigma_k$. *Denoted by* $\boldsymbol{\sigma}$.
- **Product** (of $\sigma_1, \sigma_2$): The composition $\sigma_1 \circ \sigma_2$, i.e., the map
$$i \mapsto \sigma_1(\sigma_2(i))$$
*Denoted by* $\boldsymbol{\sigma_1\sigma_2}$.
- **Inverse** (of $\sigma$): The permutation of order $k$ which is the inverse bijection of $\sigma$. *Denoted by* $\boldsymbol{\sigma^{-1}}$.
- **Permutation group** (of $\Sigma_k$): The set of all permutations of order $k$. *Also known as* **symmetric group on $\boldsymbol{k}$ letters**. *Denoted by* $\boldsymbol{S_k}$.
- Lemma 1.4.2: The group $S_k$ has $k!$ elements.
- **Transposition**: A permutation of order $k$ defined by
$$\ell \mapsto \begin{cases} j & \ell = i \\ i & \ell = j \\ \ell & \ell \neq i, j \end{cases}$$
for all $\ell \in \Sigma_k$, where $i, j \in \Sigma_k$. *Denoted by* $\boldsymbol{\tau_{i,j}}$.
- **Elementary transposition**: A transposition of the form $\pi_{i,i+1}$.
- Theorem 1.4.4: Every $\sigma \in S_k$ can be written as a product of (a finite number of) transpositions.

  *Proof.* We induct on $k$.

  For the base case $k = 2$, the identity permutation of $S_2$ is the "product" of zero transpositions, and the only other permutation is a transposition (the "product" of one transposition, namely itself).

  Now suppose inductively that we have proven the claim for $k - 1$. Let $\sigma \in S_k$ be arbitrary. Suppose $\sigma(k) = i$. Then $\tau_{i,k}\sigma(k) = k$. Since $(\tau_{i,k}\sigma)|_{\Sigma_{k-1}} \in S_{k-1}$, we have by the inductive hypothesis that $(\tau_{i,k}\sigma)|_{\Sigma_{k-1}} = \tau_1 \cdots \tau_m$ for some set of permutations $\tau_1, \dots, \tau_m \in S_{k-1}$. For each $\tau_j$ $(1 \leq j \leq m)$, define $\tau_j' \in S_k$
$$\tau_j'(\ell) = \begin{cases} \tau_j(\ell) & \ell < k \\ \ell & \ell = k \end{cases}$$

It follows that

$$\begin{aligned}\tau_{i,k}\sigma &= \tau_1' \cdots \tau_m' \\ \sigma &= \tau_{i,k}\tau_1' \cdots \tau_m'\end{aligned}$$

as desired. ☐

- Theorem 1.4.5: Every transposition can be written as a product of elementary transpositions.

  *Proof.* Let $\tau_{i,j} \in S_k$, and let $i < j$ WLOG. Then we have that

  $$\tau_{i,j} = \prod_{\ell=i}^{i-1} \tau_{\ell,\ell+1}$$

  as desired. ☐

- Corollary 1.4.6: Every permutation can be written as a product of elementary transpositions.
- **Sign** (of $\sigma$): The number $\pm 1$ assigned to $\sigma$ by the expression

  $$\prod_{i<j} \frac{x_{\sigma(i)} - x_{\sigma(j)}}{x_i - x_j}$$

  where $x_1, \dots, x_k$ are coordinate functions on $\mathbb{R}^k$. *Denoted by* $\boldsymbol{(-1)^\sigma}$.
- Claim 1.4.9: The sign defines a group homomorphism $S_k \to \{\pm 1\}$. That is, for $\sigma_1, \sigma_2 \in S_k$, we have

  $$(-1)^{\sigma_1\sigma_2} = (-1)^{\sigma_1}(-1)^{\sigma_2}$$

  *Proof.* For all $i < j$, define $p, q$ such that $p$ is the lesser of $\sigma_2(i), \sigma_2(j)$ and $q$ is the greater of $\sigma_2(i), \sigma_2(j)$. Formally,

  $$p = \begin{cases} \sigma_2(i) & \sigma_2(i) < \sigma_2(j) \\ \sigma_2(j) & \sigma_2(j) < \sigma_2(i) \end{cases} \qquad\qquad q = \begin{cases} \sigma_2(j) & \sigma_2(i) < \sigma_2(j) \\ \sigma_2(i) & \sigma_2(j) < \sigma_2(i) \end{cases}$$

  It follows that if $\sigma_2(i) < \sigma_2(j)$, then

  $$\frac{x_{\sigma_1\sigma_2(i)} - x_{\sigma_1\sigma_2(j)}}{x_{\sigma_2(i)} - x_{\sigma_2(j)}} = \frac{x_{\sigma_1(p)} - x_{\sigma_1(q)}}{x_p - x_q}$$

  and if $\sigma_2(j) < \sigma_2(i)$, then

  $$\frac{x_{\sigma_1\sigma_2(i)} - x_{\sigma_1\sigma_2(j)}}{x_{\sigma_2(i)} - x_{\sigma_2(j)}} = \frac{x_{\sigma_1(q)} - x_{\sigma_1(p)}}{x_q - x_p} = \frac{x_{\sigma_1(p)} - x_{\sigma_1(q)}}{x_p - x_q}$$

  Therefore,

  $$\begin{aligned}(-1)^{\sigma_1\sigma_2} &= \prod_{i<j} \frac{x_{\sigma_1\sigma_2(i)} - x_{\sigma_1\sigma_2(j)}}{x_i - x_j} \\ &= \prod_{i<j} \frac{x_{\sigma_1\sigma_2(i)} - x_{\sigma_1\sigma_2(j)}}{x_{\sigma_2(i)} - x_{\sigma_2(j)}} \cdot \frac{x_{\sigma_2(i)} - x_{\sigma_2(j)}}{x_i - x_j} \\ &= \prod_{i<j} \frac{x_{\sigma_1(p)} - x_{\sigma_1(q)}}{x_p - x_q} \cdot \prod_{i<j} \frac{x_{\sigma_2(i)} - x_{\sigma_2(j)}}{x_i - x_j} \\ &= (-1)^{\sigma_1}(-1)^{\sigma_2}\end{aligned}$$

  as desired. ☐

- Proposition 1.4.11: If $\sigma$ is the product of an odd number of transpositions, then $(-1)^\sigma = -1$, and if $\sigma$ is the product of an even number of transpositions, then $(-1)^\sigma = +1$.

  *Proof.* Follows from the fact that $(-1)^\sigma = -1$ (see Exercise 1.4.ii). $\square$

- $\boldsymbol{T^\sigma}$: The $k$-tensor defined by
$$T^\sigma(v_1,\dots,v_k) = T(v_{\sigma^{-1}(1)},\dots,v_{\sigma^{-1}(k)})$$
  where $T \in \mathcal{L}^k(V)$, $V$ is an $n$-dimensional vector space, and $\sigma \in S_k$.

- Proposition 1.4.14:

  1. If $T = \ell_1 \otimes \cdots \otimes \ell_k$ ($\ell_i \in V^*$), then $T^\sigma = \ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}$.

     *Proof.* If $v_1,\dots,v_k \in V$, then
$$\begin{aligned}
T^\sigma(v_1,\dots,v_k) &= T(v_{\sigma^{-1}(1)},\dots,v_{\sigma^{-1}(k)})\\
&= [\ell_1 \otimes \cdots \otimes \ell_k](v_{\sigma^{-1}(1)},\dots,v_{\sigma^{-1}(k)})\\
&= \ell_1(v_{\sigma^{-1}(1)})\cdots\ell_k(v_{\sigma^{-1}(k)})\\
&= \ell_{\sigma(1)}(v_1)\cdots\ell_{\sigma(k)}(v_2)\\
&= [\ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}](v_1,\dots,v_k)
\end{aligned}$$
     as desired. Note that we can justify the fourth equality by nothing that if $\sigma^{-1}(i) = q$, then the $i^{\text{th}}$ term in the product is $\ell_{\sigma(q)}(v_q)$, so since $\sigma$ is a bijection, the product can be arranged to the form on the right-hand side of equality four. $\square$

  2. The assignment $T \mapsto T^\sigma$ is a linear map from $\mathcal{L}^k(V) \to \mathcal{L}^k(V)$.

     *Proof.* See Exercise 1.4.iii. $\square$

  3. If $\sigma_1, \sigma_2 \in S_k$, we have $T^{\sigma_1\sigma_2} = (T^{\sigma_1})^{\sigma_2}$.

     *Proof.* Let $T = \ell_1 \otimes \cdots \otimes \ell_k$[2]. Then
$$T^{\sigma_1} = \ell_{\sigma_1(1)} \otimes \cdots \otimes \ell_{\sigma_1(k)} = \ell'_1 \otimes \cdots \otimes \ell'_k$$
     and thus
$$(T^{\sigma_1})^{\sigma_2} = \ell'_{\sigma_2(1)} \otimes \cdots \otimes \ell'_{\sigma_2(k)}$$
     Let $\sigma_2(i) = j$. Then since $\ell'_p = \ell_{\sigma_1(p)}$ by definition, we have that $\ell'_{\sigma_2(j)} = \ell_{\sigma_1(\sigma_2(j))}$. Therefore,
$$\begin{aligned}
(T^{\sigma_1})^{\sigma_2} &= \ell'_{\sigma_2(1)} \otimes \cdots \otimes \ell'_{\sigma_2(k)}\\
&= \ell_{\sigma_1(\sigma_2(1))} \otimes \cdots \otimes \ell_{\sigma_1(\sigma_2(k))}\\
&= \ell_{\sigma_1\sigma_2(1)} \otimes \cdots \otimes \ell_{\sigma_1\sigma_2(k)}\\
&= T^{\sigma_1\sigma_2}
\end{aligned}$$
     as desired. $\square$

- **Alternating** ($k$-tensor): A $k$-tensor $T \in \mathcal{L}^k(V)$ such that $T^\sigma = (-1)^\sigma T$ for all $\sigma \in S_k$.
- $\boldsymbol{\mathcal{A}^k(V)}$: The set of all alternating $k$-tensors in $\mathcal{L}^k(V)$.
  - Proposition 1.4.14(2) implies that $(T_1 + T_2)^\sigma = T_1^\sigma + T_2^\sigma$ and $(\lambda T)^\sigma = \lambda T^\sigma$; it follows that $\mathcal{A}^k(V)$ is a vector space.

[2]What gives us the right to assume $T$ is decomposable?

- **Alternation operation**: The function from $\mathcal{L}^k(V) \to \mathcal{L}^k(V)$ defined by

$$T \mapsto \sum_{\tau \in S_k} (-1)^\tau T^\tau$$

*Denoted by* **Alt**.

- Proposition 1.4.17: For $T \in \mathcal{L}^k(V)$ and $\sigma \in S_k$, we have that

1. $\operatorname{Alt}(T)^\sigma = (-1)^\sigma \operatorname{Alt} T$.

   *Proof.* We have that

$$\begin{aligned}
\operatorname{Alt}(T)^\sigma &= \left( \sum_{\tau \in S_k} (-1)^\tau T^\tau \right)^\sigma \\
&= \sum_{\tau \in S_k} (-1)^\tau (T^\tau)^\sigma && \text{Proposition 1.4.14(2)} \\
&= \sum_{\tau \in S_k} (-1)^\tau T^{\tau\sigma} && \text{Proposition 1.4.14(3)} \\
&= (-1)^\sigma \sum_{\tau \in S_k} (-1)^{\tau\sigma} T^{\tau\sigma} \\
&= (-1)^\sigma \sum_{\tau\sigma \in S_k} (-1)^{\tau\sigma} T^{\tau\sigma} \\
&= (-1)^\sigma \operatorname{Alt} T
\end{aligned}$$

   as desired. □

2. If $T \in \mathcal{A}^k(V)$, then $\operatorname{Alt} T = k!T$.

   *Proof.* Since $T \in \mathcal{A}^k(V)$, we know that $T^\sigma = (-1)^\sigma T$. Therefore,

$$\operatorname{Alt} T = \sum_{\tau \in S_k} (-1)^\tau T^\tau = \sum_{\tau \in S_k} (-1)^\tau (-1)^\tau T = \sum_{\tau \in S_k} T = k!T$$

   where the last equality holds because the cardinality of $S_k$ is $k!$. □

3. $\operatorname{Alt}(T^\sigma) = \operatorname{Alt}(T)^\sigma$.

   *Proof.* We have that

$$\operatorname{Alt}(T^\sigma) = \sum_{\tau \in S_k} (-1)^\tau T^{\tau\sigma} = (-1)^\sigma \sum_{\tau \in S_k} (-1)^{\tau\sigma} T^{\tau\sigma} = (-1)^\sigma \operatorname{Alt}(T) = \operatorname{Alt}(T)^\sigma$$

   as desired. □

4. The alternation operation is linear.

   *Proof.* Follows by Proposition 1.4.14. □

- **Repeating** (multi-index $I$): A multi-index $I$ of length $k$ such that $i_r = i_s$ for some $r \neq s$.
- **Strictly increasing** (multi-index $I$): A multi-index $I$ of length $k$ such that $i_1 < i_2 < \cdots < i_r$.
- $\boldsymbol{I^\sigma}$: The multi-index of length $k$ defined by

$$I^\sigma = (i_{\sigma(1)}, \dots, i_{\sigma(k)})$$

- If $I$ is non-repeating, there is a unique $\sigma \in S_k$ such that $I^\sigma$ is strictly increasing.

- $\boldsymbol{\psi_I}$: The following $k$-tensor. *Given by*
$$\psi_I = \operatorname{Alt}(e_I^*)$$
- Proposition 1.4.20:
  1. $\psi_{I^\sigma} = (-1)^\sigma \psi_I$.

     *Proof.* We have that
$$\psi_{I^\sigma} = \operatorname{Alt}(e_{I^\sigma}^*) = \operatorname{Alt}[(e_I^*)^\sigma] = \operatorname{Alt}(e_I^*)^\sigma = (-1)^\sigma \operatorname{Alt}(e_I^*) = (-1)^\sigma \psi_I$$
     as desired. □
  2. If $I$ is repeating, then $\psi_I = 0$.

     *Proof.* Suppose $I = (i_1, \dots, i_k)$ is such that $i_r = i_s$ for some distinct $r, s \in \Sigma_k$. Then $e_I^* = e_{I^{\tau_{i_r,i_s}}}^*$, so
$$\psi_I = \psi_{I^{\tau_{i_r,i_s}}} = (-1)^{\tau_{i_r,i_s}} \psi_I = -\psi_I$$
     Therefore, we must have $\psi_I = 0$, as desired. □
  3. If $I$ and $J$ are strictly increasing, then
$$\psi_I(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$$
     *Proof.* We have by definition that
$$\psi_I(e_{j_1}, \dots, e_{j_k}) = \sum_\tau (-1)^\tau e_{I^\tau}^*(e_{j_1}, \dots, e_{j_k})$$
     This combined with the facts that
$$e_{I^\tau}^*(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1 & I^\tau = J \\ 0 & I^\tau \neq J \end{cases}$$
     $I^\tau$ is strictly increasing iff $I^\tau = I$, and the above equation is nonzero iff $I^\tau = I = J$ implies the desired result. □
- Conclusion 1.4.22: If $T \in \mathcal{A}^k(V)$, then we can write $T$ as a sum
$$T = \sum_I c_I \psi_I$$
  with $I$'s strictly increasing.

  *Proof.* Let $T \in \mathcal{A}^k(V)$ be arbitrary. By Theorem 1.3.13,
$$T = \sum_J a_J e_J^*$$
  for some set of $a_J \in \mathbb{R}$. It follows since $\operatorname{Alt}(T) = k!T$ that
$$T = \frac{1}{k!} \sum a_J \operatorname{Alt}(e_J^*) = \sum b_J \psi_J$$
  We can disregard all repeating terms in the sum since they are zero by Proposition 1.4.20(2); for every non-repeating term $J$, we can write $J = I^\sigma$, where $I$ is strictly increasing and hence $\psi_J = (-1)^\sigma \psi_I$. □
- Claim 1.4.24: The $c_I$'s of Conclusion 1.4.22 are unique.

*Proof.* For $J$ strictly increasing, we have

$$T_J = T(e_{j_1}, \dots, e_{j_k}) = \sum_I c_I \psi_I(e_{j_1}, \dots, e_{j_k}) = c_J$$

$\square$

- Proposition 1.4.26: The alternating tensors $\psi_I$ with $I$ strictly increasing are a basis for $\mathcal{A}^k(V)$.

  *Proof.* Spanning: See Conclusion 1.4.22.

  Linear independence: See Claim 1.4.24. $\square$

- We have that
$$\dim \mathcal{A}^k(V) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$
  - Hint in proving this claim: "Show that every strictly increasing multi-index of length $k$ determines a $k$-element subset of $\{1, \dots, n\}$ and vice versa." (Guillemin & Haine, 2018, p. 16).
  - Note also that if $k > n$, every multi-index has a repeat somewhere, meaning that $\dim \mathcal{A}^k(V) = \binom{n}{k} = 0$.