Week 5

Differentiation

5.1 Vector Calculus Operations

4/27:
- Announcements.
  - No class this Friday, next Monday.
  - Midterm next Friday.
    - Up through Chapter 2.
    - The exam will likely be computationally heavy.
    - Compute d, pullbacks, interior products, Lie derivatives, etc.
    - Emphasis on Chapter 2 as opposed to Chapter 1 even though it all builds on itself.
    - He'll probably cook up a few problems too.
  - There is a recorded lecture for us.
    - On Chapter 3 content.
    - We'll cover Chapter 3 in kind of an impressionistic way as it is.
  - There are also some notes on the physics stuff.
- Vector calculus operations.
  - In one dimension, you have functions, and you take derivatives.
    - The derivative operation does essentially map $\Omega^0 \to \Omega^1$ or $C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$.
  - In two dimensions, ...
    - $\mathrm{d}^2 = 0$ reflects the fact that gradient vector fields are curl-free.
  - If you want to understand the 2D-curl business...
    - $\mathrm{curl}(v) : \mathbb{R}^2 \to \mathbb{R}$ is intuitively about balls spinning around in a vector field.
    - There's also a nice formula to compute it.
    - And then there's a connection with $\mathrm{d} : \Omega^1 \to \Omega^2$.
  - In 3D, you can take top-dimensional forms (which are just functions) and bottom-dimensional forms (which are by definition functions) and you can work out an identification between them.
  - Note that $\mathrm{curl} : \mathfrak{X}(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$, where $\mathfrak{X}(\mathbb{R}^2)$ is the space of vector fields.
- The musical operator $\sharp$ identifies forms with vector fields, i.e., $\sharp : \Omega^1 \to \mathfrak{X}(\mathbb{R}^2)$.
- Properties of exterior derivatives $\mathrm{d} : \Omega^k(U) \to \Omega^{k+1}(U)$.
  1. $\mathrm{d}(\omega_1 + \omega_2) = \mathrm{d}\omega_1 + \mathrm{d}\omega_2$ and $\mathrm{d}(\lambda\omega) = \lambda\,\mathrm{d}\omega$.
  2. Product rule $\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge \mathrm{d}\omega_2$.

- Special case $k = \ell = 0$. Then
$$\mathrm{d}(fg) = g\,\mathrm{d}f + f\,\mathrm{d}g$$
which is the usual product rule for gradient.
- Claim:
$$\mathrm{d}\left(\sum_I f_I\,\mathrm{d}x_I\right) = \sum_I \mathrm{d}f_I \wedge \mathrm{d}x_I$$
  - Let $\omega_1 \in \Omega^k$ and $\omega_2 \in \Omega^\ell$ be defined by
$$\omega_1 = \sum_I f_I\,\mathrm{d}x_I \qquad\qquad \omega_2 = \sum_J g_J\,\mathrm{d}x_J$$
where we're summing over all I such that $|I| = k$ and all J such that $|J| = \ell$. Then
$$\omega_1 \wedge \omega_2 = \sum_{I,J} f_I g_J\,\mathrm{d}x_I \wedge \mathrm{d}x_J\,\mathrm{d}(\omega_1 \wedge \omega_2) \qquad = \sum_{I,J} \mathrm{d}(f_I g_J) \wedge \mathrm{d}x_I \wedge \mathrm{d}x_J$$
  - Note that
$$\mathrm{d}(f_I g_J) = g_J\,\mathrm{d}f_I + f_I\,\mathrm{d}g_J$$
and
$$\mathrm{d}g_J \wedge \mathrm{d}x_I = (-1)^k\,\mathrm{d}x_I \wedge \mathrm{d}g_J$$
  - These identities allow us to take the previous equation to
$$\begin{aligned}
\mathrm{d}(\omega_1 \wedge \omega_2) &= \sum_{I,J} g_J\,\mathrm{d}f_I \wedge \mathrm{d}x_I \wedge \mathrm{d}x_J + (-1)^k f_I\,\mathrm{d}x_I \wedge \mathrm{d}g_J \wedge \mathrm{d}x_J\\
&= \sum_{I,J} (\mathrm{d}f_I \wedge \mathrm{d}x_I) \wedge (g_J\,\mathrm{d}x_J) + \sum_{I,J} (f_I\,\mathrm{d}x_I) \wedge (ddg_J \wedge \mathrm{d}x_J)\\
&= \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^k \omega_1\,\mathrm{d}\omega_2
\end{aligned}$$

3. $\mathrm{d}^2 = 0$.
   - Let $\omega = \sum_I f_I\,\mathrm{d}x_I$.
   - Then
$$\begin{aligned}
\mathrm{d}^2(\omega) &= \mathrm{d}(\mathrm{d}\omega)\\
&= \mathrm{d}\left(\sum_I \mathrm{d}f_I \wedge \mathrm{d}x_I\right)\\
&= \sum_I \mathrm{d}(\mathrm{d}f_I \wedge \mathrm{d}x_I) && \text{Property 1}\\
&= \sum_I \mathrm{d}(\mathrm{d}f_I) \wedge \mathrm{d}x_I && \text{Property 2}
\end{aligned}$$
so it suffices to just show that $\mathrm{d}^2 f = 0$ for all $f \in \Omega^0$.
   - We know that $\mathrm{d}f = \sum_{i=1}^n \partial f/\partial x_i\,\mathrm{d}x_i$. Thus,
$$\begin{aligned}
\mathrm{d}(\mathrm{d}f) &= \sum_i \mathrm{d}\left(\frac{\partial f}{\partial x_i}\right) \wedge \mathrm{d}x_i\\
&= \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i}\,\mathrm{d}x_j \wedge \mathrm{d}x_i\\
&= 0
\end{aligned}$$

- The last equality holds because of commuting partial derivatives for smooth f, and the fact that changing order introduces a negative sign by some property.

- In fact, if we fix $\mathrm{d}^0 : \Omega^0(U) \to \Omega^1(U)$ to be the "gradient," then these properties characterize the function d on its domain and codomain. In particular, d is the unique function on its domain and codomain that satisfies these properties.
  - We define it by
$$\mathrm{d}\left(\sum_I f_I \,\mathrm{d}x_I\right) = \sum_I \mathrm{d}f_I \wedge \mathrm{d}x_I$$
  - The above properties characterize it axiomatically.
  - We can prove this uniqueness theorem.
- **Closed** (form): A form $\omega \in \Omega^k(U)$ such that $\mathrm{d}\omega = 0$.
- **Exact** (form): A form $\omega \in \Omega^k(U)$ such that $\omega = \mathrm{d}\eta$ for some $\eta \in \Omega^{k-1}(U)$.
- $\mathrm{d}^2 = 0$ implies closed and exact implies closed.
- **Poincaré lemma**: Locally closed forms are exact.