Week 6

Operations on Forms

6.1 Compact Support and Consequences

5/2:

- Plan:
  - Brouwer's fixed point theorem.
    - The classic fixed point theorem.
    - Several proofs.
  - Compactly supported forms.
  - The Poincaré lemma.
    - Allows us to define the degree of a function $F : U \to V$, where $U, V \subset \mathbb{R}^n$ open.
    - The degree will turn out to be an integer.
    - We will need F to be proper.
    - We'll eventually use the degree to give a proof of the Brouwer's fixed point theorem.
- Theorem (Brouwer's fixed point theorem): Let $B^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ be the closed unit ball in $\mathbb{R}^n$, and let $F : B^n \to B^n$ be continuous. Then there exists $x_0 \in B^n$ such that $F(x_0) = x_0$ (i.e., F has a fixed point).
  - This is a generalized form of what we proved last quarter that a map from $[0,1] \twoheadrightarrow [0,1]$ has a fixed point (IVT and an auxiliary function).
  - Think back to Sharkovsky's theorem last quarter.
  - Another interpretation of Brouwer in $\mathbb{R}^2$: Take a piece of paper, crumple it up, project it down onto where it was, and some point lies exactly above where it was.
- **Support** (of ω): The following set, where $\omega \in \Omega^k(\mathbb{R}^n)$. *Denoted by* **supp($\boldsymbol{\omega}$)**. *Given by*

$$\operatorname{supp}(\omega) = \{p \in \mathbb{R}^n \mid \omega_p \neq 0\}$$

- Example:
  - The support of a bump function on $\mathbb{R}^1$ is the region of the line on which it is not zero.
- **Compactly supported** (form): A form ω for which $\operatorname{supp}(\omega)$ is compact.
- **Compactly supported** (form ω on U): A compactly supported form such that $\operatorname{supp}(\omega) \subset U$.
  - The idea is that we can have some crazy form, but it "dies down" when we get close to the boundary of U.
- $\boldsymbol{\Omega_c^k(U)}$: The vector space of all compactly supported k-forms on U.

  - Thus, the scalar multiple of a compactly supported form on U is still compactly supported, as is the sum of two compactly supported forms on U.
- To get a handle on the degree, we're gonna focus on the top-dimensional space $\Omega_c^n(U)$ of compactly supported forms.
- **Proper** (function): A function $F : U \to V$, where $U, V \in \mathbb{R}^n$ open, for which $F^{-1}(K)$ is compact for any F compact in V.
  - We know that the image of a compact set is compact under a *continuous* function, but we haven't said anything about the inverse image up to this point.
- Example: Sine and cosine are continuous but not proper.
  - Consider $\sin^{-1}(\{0\}) = \{\dots, -\pi, 0, \pi, \dots\}$, which is not bounded and hence not compact (by Heine-Borel).
- The pullback, when restricted to compactly supported forms, maps compactly supported forms to compactly supported forms. Symbolically,

$$F^*[\Omega_c^n(V)] \subset \Omega_c^n(U)$$

- Similarly, $\mathrm{d} : \Omega_c^{n-1}(X) \to \Omega_c^n(X)$.
- $\boldsymbol{n^{\text{th}}}$ **compactly supported de Rahm cohomology group**: The top-dimensional space of forms modulo the image of the $(n-1)$-dimensional space of forms under the exterior derivative. *Denoted by* $\boldsymbol{H_c^n(X)}$. *Given by*

$$H_c^n(X) = \frac{\Omega_c^n(X)}{\mathrm{d}\big(\Omega_c^{n-1}(X)\big)}$$

  - The top is analogous to the kernel of the appropriate d because there's no $n+1$ form so everything just gets mapped to the kernel.
- Since the pullback commutes with the exterior derivative, F will induce a map from $H_c^n(V) \to H_c^n(U)$.
  - Today, we will show that $H_c^n(X) \cong \mathbb{R}$, where the isomorphism is integration.
  - On this function, we're gonna map 1 and that will give us $\deg(F)$.
  - This is something topological: If we move/jiggle F a bit, the degree won't change. The degree is **invariant** under jiggling it around; this is the basis of topology.
  - In fact: For all $\omega \in \Omega_c^n(V)$, we have that

$$\int_U F^*\omega = \deg(F) \int_V \omega$$

  - Another thing that should be familiar from vector calculus is that this is a generalization of a classic change of coordinates integration formula. Specifically, if $F : U \to V$ is a **diffeomorphism** (smooth, bijective, smooth inverse) and $\varphi : V \to \mathbb{R}$, then

$$\int_V \varphi(y)\,\mathrm{d}y = \int_U (\varphi \circ F)(x) |\det DF(x)|\,\mathrm{d}x$$

    - Assume U, V are some bounded open subsets in $\mathbb{R}^n$, though we can get around the boundedness with a more advanced derivation.
    - This formula is just the previous formula in coordinates plus the fact that the degree of a diffeomorphism is ± 1 depending on whether or not it preserves orientation.
    - We'll use this formula over and over again to simplify the domain over which we need to integrate; it's kind of a good old u-substitution type thing.

- **Integral** (of $\omega \in \Omega_c^n(U)$): If $\omega = f\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ is a top-dimensional form, then the integral of ω over U is given as follows. *Denoted by* $\boldsymbol{\int_U \omega}$. *Given by*

$$\int_U \omega = \int_{\mathbb{R}^n} f\,\mathrm{d}x_1 \cdots \mathrm{d}x_n$$

  - Defines integration pictorially as slicing up the plane, taking a point in each region, and multiplying it's value by the area of the region, and then taking finer and finer partitions.

- Theorem (Poincaré lemma — final form): Let $\omega_1, \omega_2 \in \Omega_c^n(U)$. Then $\omega_1 \sim \omega_2$ if $\omega_1 - \omega_2 = \mathrm{d}\mu$ for some $\mu \in \Omega_c^n(U)$ (i.e., $[\omega_1] = [\omega_2]$ in $H_c^n(U)$, where we are representing equivalence classes). Let $\omega_0 \in \Omega_c^n(U)$ with $\int \omega_0 = 1$ (ω_0 is a bump function). Then $\omega \sim c\omega_0$ where c a scalar is given by $c = \int \omega$.
  - We're gonna start small by proving the Poincaré lemma for rectangles.
  - Then we'll have the lemma for general, open, connected subsets of $\mathbb{R}^n$.
  - Then we'll prove the final form above.

- To prove the Poincaré lemma, we need two steps.
  1. Poincaré lemma for rectangles: $\int \omega = 0$ iff $\omega = \mathrm{d}\mu$.
     - The backwards implication is easy.
     - The forwards implication is tricky and requires induction on dimension.
  2. Generalize from rectangles to general regions U.

- Theorem (Poincaré lemma — for rectangles): Let $Q = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$. Take $\omega \in \Omega_c^n(Q)$. Then TFAE.
  1. $\int_Q \omega = 0$.
  2. $\omega = \mathrm{d}\mu$ with $\mu \in \Omega_c^{n-1}(U)$.

  Proof ($2 \Rightarrow 1$). Let $\mu = \sum_{i=1}^n f_i\,\mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n$[1]. Then

$$\begin{aligned}
\mathrm{d}\mu &= \sum_{i=1}^n \mathrm{d}f_i \wedge \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n \\
&= \sum_{i=1}^n \left(\sum_{j=1}^n \frac{\partial f_i}{\partial x_j}\,\mathrm{d}x_j \right) \wedge \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n \\
&= \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}\,\mathrm{d}x_i \wedge \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n \\
&= \sum_{i=1}^n (-1)^{i+1} \frac{\partial f_i}{\partial x_i}\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n
\end{aligned}$$

  Now to show that $\int \mathrm{d}\mu = 0$, it suffices to check that $\int \frac{\partial f_i}{\partial x_i}\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n = 0$ for all i by the distributive property of integration over sums. The conclusion follows from the FTC and the fact that each f_i is supported in Q (i.e., each f_i is zero on the boundary of the rectangle, so the integral will look something like $f_i(b) - f_i(a) = 0 - 0 = 0$). □

  Proof ($1 \Rightarrow 2$). If $1 \Rightarrow 2$ on some $U \subset \mathbb{R}^n$, then $1 \Rightarrow 2$ in $U \times [a_n, b_n] \subset \mathbb{R}^{n+1}$. This inductive step gets us what we need. We'll prove it next time. □

- Motivation/warm up for $1 \Rightarrow 2$.

[1] Note that the carrot to delete something is universal to all fields of math, not just differential geometry.

- Let $n=1$. Then the theorem says $f:\mathbb{R}\to\mathbb{R}$ with $\text{supp}(f)\subset[a,b]$ implies TFAE.
  1. $\int_a^b f = 0$.
  2. $f = \mathrm{d}g/\mathrm{d}x$ for some $g\in\Omega_c^0([a,b])$.
- $2\Rightarrow 1$: We just did this.
- $1\Rightarrow 2$: We let $g(x)=\int_a^x f(t)\,\mathrm{d}t$. We can check that $\mathrm{d}g/\mathrm{d}x = f$, and that $\text{supp}(g)\subset[a,b]$ (since $\int_a^a f(t)\,\mathrm{d}t = 0$ and $\int_a^b f(t)\,\mathrm{d}t = 0$; values smaller and larger are zero by definition).
- ($1\Rightarrow 2$): We know that f starts at zero and ends at zero. We know that the integral (g) of f starts at zero and ends at zero. But then it must be that this integral is a compactly supported function whose derivative is f. Indeed, regardless of how f moves, we know that it must come back to zero, and any positive areas under the curve must be cancelled by negative areas under the curve.
- ($2\Rightarrow 1$): We know that f starts at zero and ends at zero. We know that f is the derivative of a function g that starts at zero and ends at zero. But then the integral of f will just be the ending point of g minus the starting point of g, which are both equally zero, making the integral zero. Indeed, regardless of how g moves, any positive slopes must be cancelled by negative slopes. But these slopes *really are* one and the same as the areas inspected by the integral, as per the FTC!
- An example of two functions that illustrate the point here are $f(x)=\sin(x)$ and $g(x)=1-\cos(x)$ on $[0,2\pi]$.

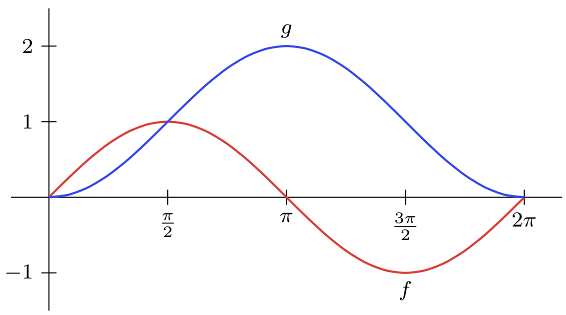

Figure 6.1: Poincaré lemma in one dimension.

6.2 The Pullback

5/4:
- Homework 3 now due Monday (the stuff will be on the exam though).
- Office hours today from 5:00-6:00.
- Exam Friday.
- Next week will be Chapter 3.
  - Integration of top-dimensional forms, i.e., if we're in 2D space, we'll integrate 2-forms; in 3D space, we'll integrate 3-forms; etc.
- Pullbacks of k-forms.
  - Let $U\subset\mathbb{R}^n$ and $V\subset\mathbb{R}^m$.
  - Let $F:U\to V$ be smooth.
  - This induces $F^*:\Omega^k(V)\to\Omega^k(U)$.
  - We have $\mathrm{d}F_p: T_p\mathbb{R}^n\to T_{F(p)}\mathbb{R}^m$, which also induces $\mathrm{d}F_p^*:\Lambda^k(T_{F(p)}^*\mathbb{R}^m)\to\Lambda^k(T_p^*\mathbb{R}^n)$.
  - Note that F^* maps $\omega\mapsto F^*\omega$ where $F^*\omega_p = \mathrm{d}F_p^*\omega_{F(p)}$.

- In formulas, if
$$\omega = \sum_I \varphi_I \,\mathrm{d}x_I$$
then
$$F^*\omega = \sum_I F^*\varphi_I \,\mathrm{d}F_I$$
  - $\varphi_I \in V^*$.
  - Recall that $F^*\varphi_I = \varphi_I \circ F : U \to \mathbb{R}$.
  - If $I = (i_1, \dots, i_k)$, then $\mathrm{d}F_I = \mathrm{d}F_{i_1} \wedge \cdots \wedge \mathrm{d}F_{i_k}$.
  - $F_{i_j} : U \to \mathbb{R}$ sends $p \mapsto x_{i_j} \circ F(p)$, where x_{i_j} (as the ${i_j}^{\text{th}}$ component function) isolates the ${i_j}^{\text{th}}$ component of $F(x)$.
  - There is a derivation that gets you from the above abstract definition of the pullback to the below concrete form.
  - We can prove that $F^*\omega$ has the above form using properties 1-4 below.
- Note that $\mathrm{d}F_p$ is the kind of thing we worked on last quarter?
- Properties of the pullback (let $U \xrightarrow{F} V \xrightarrow{G} W$).
  1. F^* is linear.
  2. $F^*(\omega_1 \wedge \omega_2) = F^*\omega_1 \wedge F^*\omega_2$.
  3. $(F \circ G)^* = G^* \circ F^*$.
  4. $\mathrm{d} \circ F^* = F^* \circ \mathrm{d}$.

$$\begin{tikzcd}
\Omega^0(U) \arrow[d, "\mathrm{d}"'] & \Omega^0(V) \arrow[l, "F^*"'] \arrow[d, "\mathrm{d}"] \arrow[dl, squiggly] \\
\Omega^1(U) & \Omega^1(V) \arrow[l, "F^*"]
\end{tikzcd}$$

Figure 6.2: Commutative diagram.

- Properties 1-3 follow from our Chapter 1 pointwise properties.
  - They also yield the explicit formula for $F^*\omega$ given above.
- Proving property 4.
  - Lemma 1: Figure 6.2 is true, i.e., property 4 holds for zero-forms.
  - Lemma 2: $\mathrm{d}F_I = F^*\,\mathrm{d}x_I$, where $I = (i_1, \dots, i_k)$.

    Proof. We have that
$$\begin{aligned}
\mathrm{d}F_I &= \mathrm{d}F_{i_1} \wedge \cdots \wedge \mathrm{d}F_{i_k} \\
&= \mathrm{d}(x_{i_1} \circ F) \wedge \cdots \wedge \mathrm{d}(x_{i_1} \circ F) \\
&= \mathrm{d}(F^* x_{i_1}) \wedge \cdots \wedge \mathrm{d}(F^* x_{i_k}) \\
&= F^*\,\mathrm{d}(x_{i_1}) \wedge \cdots \wedge F^*\,\mathrm{d}(x_{i_k}) && \text{Lemma 1} \\
&= F^*\,\mathrm{d}x_{i_1} \wedge \cdots \wedge F^*\,\mathrm{d}x_{i_k} \\
&= F^*(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}) && \text{Property 2} \\
&= F^*\,\mathrm{d}x_I
\end{aligned}$$
    as desired. $\square$

- Let $\omega = \sum_I \varphi_I \,\mathrm{d}x_I$. Then

$$\begin{aligned}
\mathrm{d}(F^*\omega) &= \mathrm{d}\left(\sum_I F^*\varphi_I \,\mathrm{d}F_I\right) \\
&= \sum_I \mathrm{d}(F^*\varphi_I \wedge \mathrm{d}F_I) \\
&= \sum_I \mathrm{d}(F^*\varphi_I) \wedge \mathrm{d}F_I \\
&= \sum_I F^*\,\mathrm{d}\varphi_I \wedge F^*\,\mathrm{d}x_I && \text{Lemma 2} \\
&= \sum_I F^*(\mathrm{d}\varphi_I \wedge \mathrm{d}x_I) \\
&= F^*\left(\sum_I \mathrm{d}\varphi_I \wedge \mathrm{d}x_I\right) \\
&= F^*\left(\sum_I \mathrm{d}(\varphi_I \,\mathrm{d}x_I)\right) \\
&= F^*\mathrm{d}\left(\sum_I \varphi_I \,\mathrm{d}x_I\right) \\
&= F^*\,\mathrm{d}\omega
\end{aligned}$$

- $\mathrm{d}^2 = 0$ generalizes curl and all of those identities.
- Two other operations.
- **Interior product**: Given $\boldsymbol{v}$ a vector field on U, we have $\iota_{\boldsymbol{v}} : \Omega^k(U) \to \Omega^{k-1}(U)$ that sends $\omega \mapsto \iota_{\boldsymbol{v}}\omega$.
- Its properties follow from the properties of the pointwise stuff.
  1. $\iota_{\boldsymbol{v}}(\omega_1 + \omega_2) = \iota_{\boldsymbol{v}}\omega_1 + \iota_{\boldsymbol{v}}\omega_2$.
  2. $\iota_{\boldsymbol{v}}(\omega \wedge \mu) = \iota_{\boldsymbol{v}}\omega \wedge \mu + (-1)^k \omega \wedge \iota_{\boldsymbol{v}}\mu$.
  3. $\iota_{\boldsymbol{v}} \circ \iota_{\boldsymbol{w}} = -\iota_{\boldsymbol{w}} \circ \iota_{\boldsymbol{v}}$.
- **Lie derivative**: If $\boldsymbol{v}$ is a vector field on U, then $L_{\boldsymbol{v}} : \Omega^k(U) \to \Omega^k(U)$ sends $\omega \mapsto \mathrm{d}\iota_{\boldsymbol{v}}\omega + \iota_{\boldsymbol{v}}\,\mathrm{d}\omega$.
  - Note that we use ι to drop the index and d to raise it back up, and then vice versa.
- Check: Agrees with previous definition for Ω^0.
- Cartan's magic formula is what we're taking to be the definition of the Lie derivative.
- Properties.
  1. $L_{\boldsymbol{v}} \circ \mathrm{d} = \mathrm{d} \circ L_{\boldsymbol{v}}$.
  2. $L_{\boldsymbol{v}}(\omega \wedge \eta) = L_{\boldsymbol{v}}\omega \wedge \eta + \omega \wedge L_{\boldsymbol{v}}\eta$.
     - Proof:

$$\begin{aligned}
\mathrm{d}(\iota_{\boldsymbol{v}}\mathrm{d} + \mathrm{d}\iota_{\boldsymbol{v}}) &= \mathrm{d}\iota_{\boldsymbol{v}}\mathrm{d} \\
&= \iota_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}\mathrm{d} + \mathrm{d}\iota_{\boldsymbol{v}})
\end{aligned}$$

- We should find an explicit formula for the Lie derivative.
  - Your vector field will be of the form

$$\boldsymbol{v} = \sum f_i \,\partial/\partial x_i$$

  - Your form will be of the form

$$\omega = \sum \varphi_I \,\mathrm{d}x_I$$

6.3 Connections with Vector Calculus

From Klug (2022).

5/26: • 2-dimensional analogues of class content.

  - Let $U \subset \mathbb{R}^2$ and let $\mathfrak{X}(U)$ be the vector space of vector fields on U.
  - 1-forms on U are of the form
$$f\,\mathrm{d}x + g\,\mathrm{d}y$$
  - We have an isomorphism of vector spaces $\sharp : \Omega^1(U) \to \mathfrak{X}(U)$ defined by
$$f\,\mathrm{d}x + g\,\mathrm{d}y \mapsto f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$$
    - The inverse of $\sharp$ is denoted $\flat$.
    - As such, these functions are referred to as the **musical operators**.
  - The exterior derivative of a function on $\mathbb{R}^2$ is
$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y$$
    - This is the **gradient**.
  - The exterior derivative of a one-form on $\mathbb{R}^2$ is
$$\mathrm{d}(f\,\mathrm{d}x + g\,\mathrm{d}y) = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y$$
    - This is related to **Green's theorem**.
    - The expression is called the **2-dimensional curl** (of a vector field), where here we are freely identifying 1-forms and vector fields via $\sharp$.
    - If we (1) make this precise and (2) prove that the intuitive definition of curl agrees with the above formula, we should gain some geometric intuition for d in this particular (co)dimension.
  - The fact that gradient vector fields are curl free, i.e., $\mathrm{curl} \circ \mathrm{grad} = 0$, reflects the fact that $\mathrm{d}^2 = 0$.

- **2-dimensional curl** (of $\boldsymbol{v} \in \mathfrak{X}(U)$): The function from $U \to \mathbb{R}$ describing the way that a ball centered at $p \in U$ would rotate (or "curl") when left in $\boldsymbol{v}$. *Denoted by* $\mathbf{curl}(\boldsymbol{v})$.

- 3-dimensional analogues of class content.
  - Gradient of the zero-form $f : U \to \mathbb{R}$ where $U \subset \mathbb{R}^3$.
$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z$$
    - We have that $\sharp \circ \mathrm{d}^0$ gives the gradient, exactly as in two dimensions.
  - Curl of the one-form $f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}z$.
$$\mathrm{d}(f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}z) = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y + \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right)\mathrm{d}x \wedge \mathrm{d}z$$
    - $\mathrm{curl}(\boldsymbol{v})$ is again a vector field, just with the direction at a point being the axis of rotation of a small ball placed at that point.
    - Once again, we can identify $\Omega^1(U), \Omega^2(U)$ with $\mathfrak{X}(U)$ to learn that d is curl and gradient fields are curl free as a result of $\mathrm{d}^2 = 0$.
  - Divergence of the two-form $f\,\mathrm{d}x \wedge \mathrm{d}y + g\,\mathrm{d}y \wedge \mathrm{d}z + h\,\mathrm{d}x \wedge \mathrm{d}z$.
$$\mathrm{d}(f\,\mathrm{d}x \wedge \mathrm{d}y + g\,\mathrm{d}y \wedge \mathrm{d}z + h\,\mathrm{d}x \wedge \mathrm{d}z) = \left(\frac{\partial f}{\partial z} + \frac{\partial g}{\partial x} - \frac{\partial h}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

- Modulo a sign, this is the **divergence** of a vector field in three dimensions.
- We can identify $\Omega^2(U)$ and $\Omega^3(U)$ with $\mathfrak{X}(U)$ and $\Omega^0(U)$, respectively, to learn that d is div and the fact that $\text{div} \circ \text{curl} = 0$ follows from $\mathrm{d}^2 = 0$.

- **Divergence** (of $\boldsymbol{v} \in \mathfrak{X}(U)$): The function from $U \to \mathbb{R}$ which geometrically represents the compression/stretching of objects placed in the vector field. *Denoted by* $\mathbf{div}(\boldsymbol{v})$.
- Take away: The exterior derivative packages the three operations of vector calculus, and $\mathrm{d}^2 = 0$ generalizes several simple formulas from vector calculus.

6.4 Chapter 2: Differential Forms

From Guillemin and Haine (2018).

5/5:
- **Interior product** (of $\boldsymbol{v}$ with ω): The $(k-1)$-form on U defined as follows, where $U \subset \mathbb{R}^n$ open, $\boldsymbol{v}$ a vector field on U, and $\omega \in \Omega^k(U)$. *Denoted by* $\boldsymbol{\iota_v \omega}$. *Given by*

$$p \mapsto \iota_{\boldsymbol{v}(p)}\omega_p$$

- By definition, $\iota_{\boldsymbol{v}(p)}\omega_p \in \Lambda^{k-1}(T_p^*\mathbb{R}^n)$.

5/26:
- Properties 2.5.3: The following are properties of the interior product defined above, where $U \subset \mathbb{R}^n$ open, $\boldsymbol{v}, \boldsymbol{w}$ are vector fields on U, $\omega_1, \omega_2, \omega \in \Omega^k(U)$, and $\mu \in \Omega^\ell(U)$.
  1. *Linearity in the form*: We have
  $$\iota_{\boldsymbol{v}}(\omega_1 + \omega_2) = \iota_{\boldsymbol{v}}\omega_1 + \iota_{\boldsymbol{v}}\omega_2$$
  2. *Linearity in the vector field*: We have
  $$\iota_{\boldsymbol{v}+\boldsymbol{w}}\omega = \iota_{\boldsymbol{v}}\omega + \iota_{\boldsymbol{w}}\omega$$
  3. *Derivation property*: We have
  $$\iota_{\boldsymbol{v}}(\omega \wedge \mu) = \iota_{\boldsymbol{v}}\omega \wedge \mu + (-1)^k \omega \wedge \iota_{\boldsymbol{v}}\mu$$
  4. The identity
  $$\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{w}}\omega) = -\iota_{\boldsymbol{w}}(\iota_{\boldsymbol{v}}\omega)$$
  5. The identity, as a special case of (4),
  $$\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}\omega) = 0$$
  6. If $\omega = \mu_1 \wedge \cdots \wedge \mu_k$ (i.e., if ω is **decomposable**), then
  $$\iota_{\boldsymbol{v}}\omega = \sum_{r=1}^{k} (-1)^{r-1} \iota_{\boldsymbol{v}}(\mu_r)\mu_1 \wedge \cdots \wedge \widehat{\mu_r} \wedge \cdots \wedge \mu_k$$

- The following are two assertions to prove, both of which are special cases of Property 2.5.3(6).
- Example 2.5.4: If $\boldsymbol{v} = \partial/\partial x_r$ and $\omega = \mathrm{d}x_I$, then

$$\iota_{\boldsymbol{v}}\omega = \sum_{i=1}^{k} (-1)^{i-1} \delta_{i,i_r} \, \mathrm{d}x_{I_r}$$

where

$$\delta_{i,i_r} = \begin{cases} 1 & i = i_r \\ 0 & i \neq i_r \end{cases} \qquad\qquad I_r = (i_1, \dots, \widehat{i_r}, \dots, i_k)$$

- Example 2.5.6: If $\boldsymbol{v} = \sum_{i=1}^n f_i \, \partial/\partial x_i$ and $\omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$, then
$$\iota_{\boldsymbol{v}}\omega = \sum_{r=1}^n (-1)^{r-1} f_r \, \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_r} \wedge \cdots \wedge \mathrm{d}x_n$$
- **Lie derivative** (of ω with respect to $\boldsymbol{v}$): The k-form defined as follows, where $U \subset \mathbb{R}^n$ is open, $\boldsymbol{v}$ is a vector field on U, and $\omega \in \Omega^k(U)$.
$$L_{\boldsymbol{v}}\omega = \iota_{\boldsymbol{v}}(\mathrm{d}\omega) + \mathrm{d}(\iota_{\boldsymbol{v}}\omega)$$
- Properties 2.5.10: The following are properties of the Lie derivative defined above, where $U \subset \mathbb{R}^n$ open, $\boldsymbol{v}$ is a vector field on U, $\omega \in \Omega^k(U)$, and $\mu \in \Omega^\ell(U)$.
  1. *Commutativity with exterior differentiation*: We have
  $$\mathrm{d}(L_{\boldsymbol{v}}\omega) = L_{\boldsymbol{v}}(\mathrm{d}\omega)$$
  2. *Interaction with wedge products*: We have
  $$L_{\boldsymbol{v}}(\omega \wedge \mu) = L_{\boldsymbol{v}}\omega \wedge \mu + \omega \wedge L_{\boldsymbol{v}}\mu$$
- An explicit formula for $L_{\boldsymbol{v}}\omega$.
  - Let $\omega \in \Omega^k(U)$ be defined by $\omega = \sum_I f_I \, \mathrm{d}x_I$ for $f_I \in C^\infty(U)$, and let $\boldsymbol{v} = \sum_{i=1}^n g_i \, \partial/\partial x_i$ for $g_i \in C^\infty(U)$.
  - Then by the above properties,

$$\begin{aligned}
L_{\boldsymbol{v}}\omega &= L_{\boldsymbol{v}}\left(\sum_I f_I \, \mathrm{d}x_I\right) \\
&= \sum_I L_{\boldsymbol{v}}(f_I \, \mathrm{d}x_I) \\
&= \sum_I [(L_{\boldsymbol{v}} f_I) \, \mathrm{d}x_I + f_I(L_{\boldsymbol{v}} \, \mathrm{d}x_I)] \\
&= \sum_I \left[\left(\sum_{i=1}^n g_i \frac{\partial f_I}{\partial x_i}\right) \mathrm{d}x_I + f_I \left(\sum_{r=1}^k \mathrm{d}x_{i_1} \wedge \cdots \wedge L_{\boldsymbol{v}} \, \mathrm{d}x_{i_r} \wedge \cdots \wedge \mathrm{d}x_{i_k}\right)\right] \\
&= \sum_I \left[\left(\sum_{i=1}^n g_i \frac{\partial f_I}{\partial x_i}\right) \mathrm{d}x_I + f_I \left(\sum_{r=1}^k \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}L_{\boldsymbol{v}} x_{i_r} \wedge \cdots \wedge \mathrm{d}x_{i_k}\right)\right] \\
&= \sum_I \left[\left(\sum_{i=1}^n g_i \frac{\partial f_I}{\partial x_i}\right) \mathrm{d}x_I + f_I \left(\sum_{r=1}^k \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}g_{i_r} \wedge \cdots \wedge \mathrm{d}x_{i_k}\right)\right] \\
&= \sum_I \left[\left(\sum_{i=1}^n g_i \frac{\partial f_I}{\partial x_i}\right) \mathrm{d}x_I + f_I \left(\sum_{r=1}^k \mathrm{d}x_{i_1} \wedge \cdots \wedge \left(\sum_{i=1}^n \frac{\partial g_{i_r}}{\partial x_i} \, \mathrm{d}x_i\right) \wedge \cdots \wedge \mathrm{d}x_{i_k}\right)\right] \\
&= \sum_I \left[\left(\sum_{i=1}^n g_i \frac{\partial f_I}{\partial x_i}\right) \mathrm{d}x_I + f_I \left(\sum_{r=1}^k \sum_{i=1}^n \frac{\partial g_{i_r}}{\partial x_i} \, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_{r-1}} \wedge \mathrm{d}x_i \wedge \mathrm{d}x_{i_{r+1}} \wedge \cdots \wedge \mathrm{d}x_{i_k}\right)\right] \\
&= \sum_I \left[\left(\sum_{i=1}^n g_i \frac{\partial f_I}{\partial x_i}\right) \mathrm{d}x_I + f_I \left(\sum_{r=1}^k \sum_{\substack{i=1 \\ i \notin I}}^n \frac{\partial g_{i_r}}{\partial x_i} \, \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_{r-1}} \wedge \mathrm{d}x_i \wedge \mathrm{d}x_{i_{r+1}} \wedge \cdots \wedge \mathrm{d}x_{i_k}\right)\right]
\end{aligned}$$

- Lemma 2.5.13 (the divergence formula): Let $U \subset \mathbb{R}^n$ open, $g_1, \dots, g_n \in C^\infty(U)$, and $\boldsymbol{v} = \sum_{i=1}^n g_i \, \partial/\partial x_i$. Then
$$L_{\boldsymbol{v}}(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n) = \sum_{i=1}^n \left(\frac{\partial g_i}{\partial x_i}\right) \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

- **Pullback** (of ω along f): The k-form on U defined as follows, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open, $f : U \to V$ is a C^∞ map, ω is a k-form on V, $p \in U$, and $q = f(p)$. *Denoted by* $\boldsymbol{f^*\omega}$. *Given by*

$$p \mapsto \mathrm{d}f_p^* \omega_q$$

  - Note that it is because $\mathrm{d}f_p$ is linear that we get an induced pullback $\mathrm{d}f_p^* = (\mathrm{d}f_p)^* : \Lambda^k(T_q^*\mathbb{R}^m) \to \Lambda^k(T_p^*\mathbb{R}^n)$.

- Properties 2.6.4: The following are properties of the pullback defined above, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open and $f : U \to V$ is a C^∞ map.
  1. Let $\phi \in C^\infty(V)$ be a zero-form. Since $\Lambda^0(T_p^*) = \Lambda^0(T_q^*) = \mathbb{R}$, we have that $\mathrm{d}f_p^* = \mathrm{id}_\mathbb{R}$ when $k = 0$. Hence for zero forms,
  $$(f^*\phi)(p) = (\phi \circ f)(p)$$
  for all $p \in U$.
  2. Let $\phi \in \Omega^0(U)$, and let $\mu \in \Omega^1(V)$ be the 1-form $\mu = \mathrm{d}\phi$. By the chain rule,
  $$\mathrm{d}f_p^* \mu_q = (\mathrm{d}f_p)^* \mathrm{d}\phi_q = (\mathrm{d}\phi)_q \circ \mathrm{d}f_p = \mathrm{d}(\phi \circ f)_p$$
  Hence, by property (1),
  $$f^*\mathrm{d}\phi = \mathrm{d}f^*\phi$$
  3. Let $\omega_1, \omega_2 \in \Omega^k(V)$. Then
  $$\mathrm{d}f_p^*(\omega_1 + \omega_2)_q = \mathrm{d}f_p^*(\omega_1)_q + \mathrm{d}f_p^*(\omega_2)_q$$
  so
  $$f^*(\omega_1 + \omega_2) = f^*\omega_1 + f^*\omega_2$$
  4. Since $\mathrm{d}f_p^*$ commutes with the wedge product by Proposition 1.8.4(1), if $\omega_1 \in \Omega^k(V)$ and $\omega_2 \in \Omega^\ell(V)$, then
  $$\mathrm{d}f_p^*[(\omega_1)_q \wedge (\omega_2)_q] = \mathrm{d}f_p^*(\omega_1)_q \wedge \mathrm{d}f_p^*(\omega_2)_q$$
  so
  $$f^*(\omega_1 \wedge \omega_2) = f^*\omega_1 \wedge f^*\omega_2$$
  5. Let $W \subset \mathbb{R}^k$ be open, $g : V \to W$ be a C^∞ map, $p \in U$, $q = f(p)$, and $w = g(q)$. Then $(\mathrm{d}g_q \circ \mathrm{d}f_p)^* : \Lambda^k(T_w^*) \to \Lambda^k(T_p^*)$. But since $(\mathrm{d}g_q) \circ (\mathrm{d}f)_p = \mathrm{d}(g \circ f)_p$ by the chain rule, we have that $\mathrm{d}(g \circ f)_p^* : \Lambda^k(T_w^*) \to \Lambda^k(T_p^*)$. Thus, if $\omega \in \Omega^k(W)$, then
  $$f^*(g^*\omega) = (g \circ f)^*\omega$$

- An explicit formula for $f^*\omega$.
  - Let $\omega \in \Omega^k(V)$ be given by $\omega = \sum_I \phi_I \, \mathrm{d}x_I$, where the $\phi_I \in C^\infty(V)$. Then,
  $$\begin{aligned}
  f^*\omega &= \sum_I f^*\phi_I f^*(\mathrm{d}x_I) && (1)\\
  &= \sum_I (\phi_I \circ f) f^*(\mathrm{d}x_{i_1}) \wedge \cdots \wedge f^*(\mathrm{d}x_{i_k}) && (4)\\
  &= \sum_I (\phi_I \circ f)\, \mathrm{d}f^*x_{i_1} \wedge \cdots \wedge \mathrm{d}f^*x_{i_k} && (2)\\
  &= \sum_I (\phi_I \circ f)\, \mathrm{d}(x_{i_1} \circ f) \wedge \cdots \wedge \mathrm{d}(x_{i_k} \circ f) && (2)\\
  &= \sum_I (\phi_I \circ f) \mathrm{d}f_{i_1} \wedge \cdots \wedge \mathrm{d}f_{i_k}\\
  &= \sum_I f^*\phi_I \mathrm{d}f_I
  \end{aligned}$$
  where the f_{i_j} are the i_j^{th} coordinate functions of the map f.

- Notice that we have showed in the above derivation that

$$f^*(\mathrm{d}x_I) = \mathrm{d}f_I$$

- We now prove that the pullback commutes with exterior differentiation, i.e.,

$$\mathrm{d}(f^*\omega) = f^*\mathrm{d}\omega$$

  - We have that

$$\begin{aligned}
\mathrm{d}(f^*\omega) &= \mathrm{d}\left(\sum_I f^*\phi_I \mathrm{d}f_I\right)\\
&= \sum_I \mathrm{d}(f^*\phi_I \wedge \mathrm{d}f_I)\\
&= \sum_I \left[\mathrm{d}(f^*\phi_I) \wedge \mathrm{d}f_I + (-1)^k f^*\phi_I \wedge \mathrm{d}(\mathrm{d}f_I)\right]\\
&= \sum_I \left[f^*(\mathrm{d}\phi_I) \wedge f^*(\mathrm{d}x_I) + (-1)^k f^*\phi_I \wedge 0\right]\\
&= \sum_I f^*(\mathrm{d}\phi_I) \wedge f^*(\mathrm{d}x_I)\\
&= f^* \sum_I \mathrm{d}\phi_I \wedge \mathrm{d}x_I\\
&= f^*(\mathrm{d}\omega)
\end{aligned}$$

- A special case of $f^*(\mathrm{d}x_I) = \mathrm{d}f_I$:

$$f^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n) = \det\left[\frac{\partial f_i}{\partial x_j}\right] \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

  - Let $U, V \subset \mathbb{R}^n$ open. Then for all $p \in U$,

$$\begin{aligned}
f^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n)_p &= (\mathrm{d}f_1)_p \wedge \cdots \wedge (\mathrm{d}f_n)_p\\
&= \left[\sum_{j=1}^n \left.\frac{\partial f_1}{\partial x_j}\right|_p (\mathrm{d}x_j)_p\right] \wedge \cdots \wedge \left[\sum_{j=1}^n \left.\frac{\partial f_n}{\partial x_j}\right|_p (\mathrm{d}x_j)_p\right]\\
&= \det\left[\left.\frac{\partial f_i}{\partial x_j}\right|_p\right] (\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n)_p
\end{aligned}$$

  - See the argument used in Section 1.8 to derive the typical formula for the determinant for details and context on the above.
- **Homotopy** (between f_0 and f_1): A C^∞ map from $U \times A \to V$ (where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open, $\{0,1\} \subset A \subset \mathbb{R}$ is an open interval, and $f_0, f_1 : U \to V$ are C^∞ maps) such that

$$\begin{aligned}
(x,0) &\mapsto f_0(x)\\
(x,1) &\mapsto f_1(x)
\end{aligned}$$

  Denoted by $\boldsymbol{F}$.
- **Homotopic** (maps): Two maps f_0, f_1 to which there corresponds a homotopy F. *Denoted by* $\boldsymbol{f_0 \simeq f_1}$.
  - "Intuitively, f_0 and f_1 are homotopic if there exists a family of C^∞ maps $f_t : U \to V$ where $f_t(x) = F(x,t)$ which 'smoothly deform f_0 into f_1'" (Guillemin & Haine, 2018, p. 56).

- Theorem 2.6.15: If $U\subset\mathbb{R}^n$ and $V\subset\mathbb{R}^m$ open and $f_0,f_1:U\to V$ homotopic C^∞ maps, then for every closed form $\omega\in\Omega^k(V)$, the form $f_1^*\omega-f_0^*\omega$ is exact.
  - This theorem is closely related to the Poincaré lemma (Lemma 2.4.16) and actually implies a slightly stronger version of it.
- **Contractible** (open subset $U\subset\mathbb{R}^n$): An open subset $U\subset\mathbb{R}^n$ for which there exists a point $p_0\in U$ such that $\mathrm{id}_U:U\to U$ is homotopic to the constant map $f_0:U\to U$ defined by $f_0(p)=p_0$ at p_0.
  - A contractible set is so named because it can be shrunk to a single point continuously.
- Theorem 2.6.15 implies that the Poincaré lemma holds for contractible open subsets of $\mathbb{R}^n$. In particular, if U is contractible, then every closed k-form on U of degree $k>0$ is exact.

  Proof. Let U be contractible, and let $\omega\in\Omega^k(U)$ be closed. Since U is contractible, id_U and f a constant function are homotopic. Thus, by Theorem 2.6.15, $\mathrm{id}_U^*\,\omega-f^*\omega=\omega$ is exact. □

- The three basic operations of 3D vector calculus are gradient, curl, and divergence. These operations are closely related to $\mathrm{d}:\Omega^k(\mathbb{R}^3)\to\Omega^{k+1}(\mathbb{R}^3)$ for $k=0,1,2$, respectively.
  - Gradient and divergence generalize to higher dimensions, with gradient always equal to d^0 and divergence always equal to d^{n-1}.
  - Why we should use differential forms, even in three dimensions: **General covariance**.
    - Translations and rotations of $\mathbb{R}^3$ preserve div and curl, but $\mathrm{d}^0,\mathrm{d}^1,\mathrm{d}^2$ admit all diffeomorphisms of $\mathbb{R}^3$ as symmetries.
- **General covariance**: The desire to formulate the laws of physics in such a way that they admit as large a set of symmetries as possible.
- There are two (natural) ways to convert vector fields into forms.
- Conversion using the *inner* product.
  - Let $B(v,w)=\sum_n v_iw_i$ be the inner product on $\mathbb{R}^n$.
  - By Exercise 1.2.xi, the inner product induces a bijective linear map $L:\mathbb{R}^n\to(\mathbb{R}^n)^*$ such that $L(v)=\ell_v$ iff $\ell_v(w)=B(v,w)$.
  - By identifying $T_p\mathbb{R}^n\cong\mathbb{R}^n$, we may transfer B,L to $T_p\mathbb{R}^n$, providing an inner product B_p on $T_p\mathbb{R}^n$ and a bijective linear map $L_p:T_p\mathbb{R}^n\to T_p^*\mathbb{R}^n$.
    - Note that the only difference between L and L_p (resp. B and B_p) is that L_p eats (p,v) and focuses on v while L eats v directly.
  - The identification $p\mapsto L_p\boldsymbol{v}(p)$ constitutes the 1-form $\boldsymbol{v}^\sharp$.
    - Intuition: $\boldsymbol{v}$ is a vector field. Thus, $v=\boldsymbol{v}(p)$ is the vector in $\boldsymbol{v}$ at point p. What L_p does is take this vector (as part of (p,v)) and return the linear functional $(\ell_v)_p\in T_p^*\mathbb{R}^n$ which sends $(p,w)\mapsto(p,\ell_v(w))$. So essentially, we are identifying with every point p the linear functional that maps every vector w (as part of the ordered pair $(p,w)\in T_p\mathbb{R}^n$) to its inner product with v, $B(v,w)$ (again, as part of the ordered pair $(p,B(v,w))\in T_p\mathbb{R}^n$).
- $\boldsymbol{v^\sharp(p)}$: The cotangent vector
$$\boldsymbol{v}^\sharp(p)=L_p\boldsymbol{v}(p)$$
- Consequences.
  - We have that
$$\boldsymbol{v}=\frac{\partial}{\partial x_i}\quad\Longleftrightarrow\quad\boldsymbol{v}^\sharp=\mathrm{d}x_i$$

- More generally,

$$\boldsymbol{v} = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} \quad \Longleftrightarrow \quad \boldsymbol{v}^\sharp = \sum_{i=1}^{n} f_i \, \mathrm{d}x_i$$

- **Gradient** (of a function f): The following vector field, as determined by $f \in C^\infty(U)$ where $U \subset \mathbb{R}^n$. *Denoted by* $\mathbf{grad}(\boldsymbol{f})$. *Given by*

$$\operatorname{grad}(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}$$

  - This gets converted by $\sharp$ into the 1-form $\sum_{i=1}^{n} \partial f/\partial x_i \, \mathrm{d}x_i = \mathrm{d}f$.
  - Thus, the gradient operation is essentially just the exterior derivative operation d^0.
- Conversion using the *interior* product.
  - Let $\boldsymbol{v} = \sum_{i=1}^{n} f_i \, \partial/\partial x_i$ be a C^∞ vector field on $U \subset \mathbb{R}^n$ open. Let $\Omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$.
  - Then

$$\iota_{\boldsymbol{v}} \Omega = \sum_{r=1}^{n} (-1)^{r-1} f_r \, \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_r} \wedge \cdots \wedge \mathrm{d}x_n$$

  - Since every $(n-1)$-form can be written uniquely as such a sum, the above equation defines a bijective correspondence between vector fields and $(n-1)$-forms.
- The d operation as an operation on vector fields.
  - We may define $\mathrm{d}(\boldsymbol{v})$ by

$$\boldsymbol{v} \mapsto \mathrm{d}\iota_{\boldsymbol{v}} \Omega$$

  - The expression on the right above can related to the **divergence** as follows.

$$\begin{aligned}
\mathrm{d}\iota_{\boldsymbol{v}} \Omega &= \iota_{\boldsymbol{v}}(\mathrm{d}(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n)) + \mathrm{d}(\iota_{\boldsymbol{v}} \Omega) \\
&= L_{\boldsymbol{v}} \Omega \\
&= \operatorname{div}(\boldsymbol{v}) \Omega
\end{aligned}$$

    - The first equality follows by $\mathrm{d}^2 = 0$.
    - The second equality follows by the definition of the Lie derivative of ω with respect to $\boldsymbol{v}$.
    - The third equality follows by Lemma 2.5.13.
- **Divergence** (of a vector field $\boldsymbol{v}$): The following function from $U \to \mathbb{R}$, where $\boldsymbol{v} = \sum_{i=1}^{n} f_i \, \partial/\partial x_i$ is a vector field over U. *Denoted by* $\mathbf{div}(\boldsymbol{v})$. *Given by*

$$\operatorname{div}(\boldsymbol{v}) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$$

  - The above correspondence between $(n-1)$-forms and vector fields converts d into the divergence operation on vector fields.
- **Curl** (of a vector field $\boldsymbol{v}$): The unique vector field $\boldsymbol{w}$ such that $\mathrm{d}(\boldsymbol{v}^\sharp) = \iota_{\boldsymbol{w}} \, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$, where $U \subset \mathbb{R}^3$ open and $\boldsymbol{v}$ is a vector field on U. *Denoted by* $\mathbf{curl}(\boldsymbol{v})$.
- We should confirm that this definition coincides with that from vector calculus. In particular, we should check that if $\boldsymbol{v} = \sum_{i=1}^{3} f_i \, \partial/\partial x_i$, then

$$\operatorname{curl}(\boldsymbol{v}) = \sum_{i=1}^{3} g_i \frac{\partial}{\partial x_i}$$

where

$$\begin{aligned} g_1 &= \frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_2} \\ g_2 &= \frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3} \\ g_3 &= \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1} \end{aligned}$$

- Take aways:
  - The gradient, curl, and divergence operations have differential-form analogues (i.e., $\mathrm{d}^0, \mathrm{d}^1, \mathrm{d}^2$).
  - To define the gradient, we needed the inner product. To define the divergence, we had to equip U with Ω. To define the curl, we needed both.
  - It's these additional structures that explains why diffeomorphisms preserve $\mathrm{d}^0, \mathrm{d}^1, \mathrm{d}^2$, but not $\mathrm{grad}, \mathrm{curl}, \mathrm{div}$.
- Guillemin and Haine (2018) expresses Maxwell's equations in terms of differential forms.
- Guillemin and Haine (2018) introduces symplectic geometry and Hamiltonian mechanics.