Week 2

Tensor Classifications

2.1 Alternating Tensors

4/4:

- Plan:
  - More multilinear algebra.
  - Alternating k-tensors — 2 views:
    1. As a subspace of $\mathcal{L}^k(V)$.
    2. As a quotient of $\mathcal{L}^k(V)$.
  - Next time: Operators as alternating tensors.
    - Wedge products.
    - Interior products.
    - Pullbacks.
- Recall: $\dim V = n$, $e_1, \dots, e_n$ a basis, $\mathcal{L}^k(V)$ the space of k-tensors, $\sigma \in S_k$ implies $(-1)^\sigma \in \{\pm 1\}$, key property: $(-1)^{\sigma_1\sigma_2} = (-1)^{\sigma_1}(-1)^{\sigma_2}$.
- $\boldsymbol{T^\sigma}$: The k-tensor over V defined by

$$T^\sigma(v_1, \dots, v_k) = T(v_{\bar{\sigma}(1)}, \dots, v_{\bar{\sigma}(k)})$$

  where $T \in \mathcal{L}^k(V)$, $\sigma \in S_k$, and $\bar{\sigma}$ denotes the inverse of σ.
- Example: $n = 2$, $k = 2$. Let $T = e_1^* \otimes e_2^* \in \mathcal{L}^2(V)$. Let $\sigma = \tau_{1,2}$. Then $T^\sigma = e_2^* \otimes e_1^*$.
- Another property is $(e_I^*)^\sigma = e_{\sigma(I)}^*$.
- Properties:
  1. $T^{\sigma_1\sigma_2} = (T^{\sigma_1})^{\sigma_2}$.
  2. $(T_1 + T_2)^\sigma = T_1^\sigma + T_2^\sigma$.
  3. $(cT)^\sigma = cT^\sigma$.
- Thus, you can view $\sigma : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ as a linear map!
- **Alternating k-tensor**: A tensor $T \in \mathcal{L}^k(V)$ such that $T^\sigma = (-1)^\sigma T$ for all $\sigma \in S_k$.
  - Equivalently, $T^\tau = -T$ for all $\tau \in S_k$.
- An example of an alternating 2-tensor when $\dim V = 2$ is $T = e_1^* \otimes e_2^* - e_2^* \otimes e_1^*$.
  - Naturally, $T^{\tau_{1,2}} = -T$, and $\tau_{1,2}$ is the unique transposition in S_2.

- $e_1^* \otimes e_2^*$ is *not* an alternating 2-tensor since $(e_1^* \otimes e_2^*)^\tau = e_2^* \otimes e_1^* \neq (-1)^\tau (e_1^* \otimes e_2^*)$.
- We can look at $n = 2$, $k = 1$ for ourselves.
- Note: If T_1, T_2 are both alternating k-tensors, then $T_1 + T_2$ is also alternating, as is cT_1 for all $c \in \mathbb{R}$.
- $\boldsymbol{\mathcal{A}^k(V)}$: The vector space of alternating k-tensors.
- $\mathbf{Alt}(\boldsymbol{T})$: The function $\mathrm{Alt} : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ defined by

$$\mathrm{Alt}(T) = \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma$$

- Properties:
  1. $\mathrm{im}(\mathrm{Alt}) = \mathcal{A}^k(V)$.
  2. $\mathcal{L}^k(V)/\ker(\mathrm{Alt}) = \Lambda^k(V^*)$[1] is isomorphic to $\mathcal{A}^k(V)$.
  3. $\mathrm{Alt}(T)^\sigma = (-1)^\sigma \mathrm{Alt}(T)$.
     - Proof:

$$\begin{aligned}
\mathrm{Alt}(T)^{\sigma'} &= \left(\sum_{\sigma \in S_k} (-1)^\sigma T^\sigma \right)^{\sigma'} \\
&= \sum_{\sigma \in S_k} (-1)^\sigma T^{\sigma\sigma'} \\
&= (-1)^{\sigma'} \sum_{\sigma \in S_k} (-1)^{\sigma'} (-1)^\sigma T^{\sigma\sigma'} \\
&= (-1)^{\sigma'} \sum_{\sigma \in S_k} (-1)^{\sigma\sigma'} T^{\sigma\sigma'} \\
&= (-1)^{\sigma'} \mathrm{Alt}(T)
\end{aligned}$$

     - The last equality holds because summing over all σ is the same as summing over all $\sigma' \circ \sigma$.
     - This implies $\mathrm{im}(\mathrm{Alt}) \leq \mathcal{A}^k(V)$.
  4. If $T \in \mathcal{A}^k(T)$, $\mathrm{Alt}(T) = k!T$.
     - We have

$$\begin{aligned}
\mathrm{Alt}(T) &= \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma \\
&= \sum_{\sigma \in S_k} (-1)^\sigma (-1)^\sigma T \\
&= \sum_{\sigma \in S_k} T \\
&= k!T
\end{aligned}$$

       where $T^\sigma = (-1)^\sigma T$ since $T \in \mathcal{A}^k(V)$ by definition.
     - This implies that $\mathrm{im}(\mathrm{Alt}) = \mathcal{A}^k(V)$: $\mathrm{Alt}(\frac{1}{k!}T) = T \in \mathcal{A}^k(V)$.
  5. $\mathrm{Alt}(T^\sigma) = \mathrm{Alt}(T)^\sigma$.

[1]Note that we use V^* here instead of V because $\Lambda^k(V^*)$ does not indicate some set of functions over the vector space V, but rather the k-dimensional exterior powers of the linear functionals $\ell \in V^*$ that are dual to the vectors $v \in V$. In other words, whereas $\mathcal{A}^k(V)$ denotes the set of alternating k-tensors *acting on* V, $\Lambda^k(V^*)$ denotes the vector space containing all linear combinations of all products of length k of covectors $\ell \in V^*$, where the multiplication operation is the exterior product. Also, $\Lambda^k(V)$ is already otherwise defined as the set of all k-vectors, linear combinations of k-blades, which in turn are exterior products of length k of vectors $v \in V$.

6. $\text{Alt} : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ is linear.

- Warning: Some people take $\text{Alt}(T) = \frac{1}{k!}\sum_{\sigma\in S_k}(-1)^\sigma T^\sigma$[2].
- Example: $n = k = 2$. We have

$$\text{Alt}(e_1^* \otimes e_2^*) = e_1^* \otimes e_2^* - e_2^* \otimes e_1^*$$

- **Non-repeating** (multi-index I): A multi-index I such that $i_{j_1} \neq i_{j_2}$ for all $j_1 \neq j_2$.
- **Increasing** (multi-index I): A multi-index I such that $i_1 < \cdots < i_k$.
- Claim: $\{\text{Alt}(e_I^*)\}$ where I is non-repeating and increasing is a basis for $\mathcal{A}^k(V)$. There are $\binom{n}{k}$ of these; thus, $\dim \mathcal{A}^k(V) = \binom{n}{k}$.
  - Note that we spend so much time on alternating tensors, because we can prove that all of the tensors that we'll eventually be interested in (e.g., those describing the slope of a function of interest at a given point p) *are* alternating.
  - See, for example, Exercise 2.4.i(3). Therein, a two-form is decomposed all the way to a linear combination of alternating two-forms (that become alternating tensors at any point p).
  - It is because of this curious fact that a more robust exploration of the properties of alternating tensors was called for. And this is why we're exploring them now.

2.2 Redundant Tensors and Alternatization

4/6:

- Klug will be in Texas on Monday and thus is cancelling class on Monday. Homework is now due next Friday. We'll have weekly homeworks going forward after that.
- Plan:
  - $\text{Alt} : \mathcal{L}^k(V) \twoheadrightarrow \mathcal{A}^k(V)$[3].
  - Goal: Identify $\ker(\text{Alt}) = \mathcal{I}^k(V)$, where $\mathcal{I}^k(V)$ is the space of **redundant** k-tensors[4].
  - Then: Operations on alternating tensors, e.g.,
    - Wedge product.
    - Interior product.
    - Orientations.
- Claim: $\{\text{Alt}(e_I^*) \mid I \text{ non-repeating, increasing multi-index}\}$ is a basis for $\mathcal{A}^k(V)$.
  - Left as an exercise to us.
- **Redundant** (k-tensor): A k-tensor of the form

$$\ell_1 \otimes \cdots \otimes \ell_i \otimes \ell_i \otimes \ell_{i+2} \otimes \cdots \otimes \ell_k$$

  where $\ell_1, \dots, \ell_k \in V^*$.
- $\boldsymbol{\mathcal{I}^k(V)}$: The span of all redundant k-tensors.
  - Note that not every k-tensor in $\mathcal{I}^k(V)$ is redundant.
- **Decomposable** (k-tensor): A k-tensor of the form $\ell_1 \otimes \cdots \otimes \ell_k$ for some $\ell_i \in \mathcal{L}^1(V)$.
  - It often suffices to prove things for decomposable tensors.

[2]Klug prefers this convention, but the text takes the other one.
[3]The two-headed right arrow denotes a surjective map.
[4]The $\mathcal{I}$ in $\mathcal{I}^k(V)$ stands for "ideal."

- Properties.

  1. If $T \in \mathcal{I}^k(V)$, then $\mathrm{Alt}(T) = 0$, i.e., $\mathcal{I}^k(V) \leq \ker(\mathrm{Alt})$.
     - "Proof by example": If $T = \ell_1 \otimes \ell_1 \otimes \ell_2 \otimes \ell_3$, then $T^{\tau_{1,2}} = T$. It follows from the properties of Alt that

$$\begin{aligned}\mathrm{Alt}(T) &= \mathrm{Alt}(T^{\tau_{1,2}}) = (-1)^{\tau_{1,2}} \mathrm{Alt}(T) = -\mathrm{Alt}(T) \\ 2\,\mathrm{Alt}(T) &= 0 \\ \mathrm{Alt}(T) &= 0\end{aligned}$$

  2. If $T \in \mathcal{I}^r(V)$ and $T' \in \mathcal{L}^s(V)$, then

$$T \otimes T' \in \mathcal{I}^{r+s}(V)$$

     Similarly, if $T \in \mathcal{L}^r(V)$ and $T \in \mathcal{I}^s(V)$, then

$$T \otimes T' \in \mathcal{I}^{r+s}(V)$$

     - Proof: It suffices to assume that T is redundant. Obviously adding more tensors to the direct product will not change the redundancy of the initial tensor. Example: $\ell_1 \otimes \ell_1 \otimes \ell_2$ is just as redundant as $\ell_1 \otimes \ell_1 \otimes \ell_2 \otimes T$.

  3. If $T \in \mathcal{L}^k(V)$ and $\sigma \in S_k$, then

$$T^\sigma = (-1)^\sigma T + S$$

     for some $S \in \mathcal{I}^k(V)$.
     - Proof by example: It suffices to check this for decomposable tensors (a tensor is just a sum of decomposable tensors). Take $k = 2$. Let $T = \ell_1 \otimes \ell_2$. Let $\sigma = \tau_{1,2}$. Then

$$T^\sigma - (-1)^\sigma T = \ell_2 \otimes \ell_1 + \ell_1 \otimes \ell_2 = (\ell_1 + \ell_2) \otimes (\ell_1 + \ell_2) - \ell_1 \otimes \ell_1 - \ell_2 \otimes \ell_2$$

     - Actual proof: It suffices to assume T is decomposable. We induct on the number of transpositions needed to write σ as a product of **adjacent** transpositions.
     - Base case: $\sigma = \tau_{i,i+1}$. Then

$$\begin{aligned}T^{\tau_{i,i+1}} + T = {} & \ell_1 \otimes \cdots \otimes (\ell_i + \ell_{i+1}) \otimes (\ell_i + \ell_{i+1}) \otimes \cdots \otimes \ell_k \\ & - \ell_1 \otimes \cdots \otimes \ell_i \otimes \ell_i \otimes \cdots \otimes \ell_k \\ & - \ell_1 \otimes \cdots \otimes \ell_{i+1} \otimes \ell_{i+1} \otimes \cdots \otimes \ell_k\end{aligned}$$

     - Inductive step: If $\sigma = \beta\tau$, then

$$\begin{aligned}T^\sigma &= T^{\beta\tau} \\ &= (-1)^\tau T^\beta + \text{stuff in } \mathcal{I}^k(V) \\ &= (-1)^\tau [(-1)^\beta T + \text{stuff in } \mathcal{I}^k(V)] + \text{stuff in } \mathcal{I}^k(V)\end{aligned}$$

  4. If $T \in \mathcal{L}^k(V)$, then

$$\mathrm{Alt}(T) = k!T + W$$

     for some $W \in \mathcal{I}^k(V)$.
     - We have that

$$\begin{aligned}\mathrm{Alt}(T) &= \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma \\ &= \sum_{\sigma \in S_k} (-1)^\sigma [(-1)^\sigma T + S_\sigma] \\ &= \sum_{\sigma \in S_k} T + \sum_{\sigma \in S_k} (-1)^\sigma S_\sigma \\ &= k!T + W\end{aligned}$$

5. $\mathcal{I}^k(V) = \ker(\text{Alt})$.
   - We have that $\mathcal{I}^k(V) \leq \ker(\text{Alt})$ by property 1.
   - Now suppose $T \in \ker(\text{Alt})$. Then $\text{Alt}(T) = 0$. Then by property 4,

$$\begin{aligned}\text{Alt}(T) &= k!T + W\\ 0 &= k!T + W\\ T &= -\frac{1}{k!}W \in \mathcal{I}^k(V)\end{aligned}$$

- Warning: If $T \in \mathcal{A}^r(V)$ and $T' \in \mathcal{A}^s(V)$, then we do not necessarily have $T \otimes T' \in \mathcal{A}^{r+s}(V)$.
  - Example: $e_1^*, e_2^* \in \mathcal{A}^1(V)$ have $e_1^* \otimes e_2^* \notin \mathcal{A}^2(V)$.
- **Adjacent** (transposition): A transposition of the form $\tau_{i,i+1}$.

2.3 The Wedge Product

4/8:

- Recall that $\mathcal{A}^k(V) \hookrightarrow \mathcal{L}^k(V)$[5]
- Functoriality: $(A \circ B)^* = B^* \circ A^*$.
  - A^* takes $\mathcal{L}^k(W) \to \mathcal{L}^k(V)$ and $\mathcal{A}^k(W) \to \mathcal{A}^k(V)$.
- $\dim(\Lambda^k(V^*)) = \binom{n}{k}$.
  - Special case $k = n$: $\dim \Lambda^n(V) = 1$.
  - $A : V \to V$ induces a map from $\Lambda^n(V^*) \to \Lambda^n(V^*)$ (the relevant pullback) defined by the determinant.
- Aside: $\Lambda^k(V^*)$ is "exterior powers."
- Plan: Wedge products + basis for $\Lambda^k(V^*)$.
- **Wedge product**: A function $\wedge : \Lambda^k(V^*) \times \Lambda^\ell(V^*) \to \Lambda^{k+\ell}(V)$.
  - We denote elements of $\Lambda^k(V^*)$ by ω_1, ω_2, etc.
- If $\pi : \mathcal{L}^k(V) \to \Lambda^k(V^*)$ sends $T \mapsto \omega$, $\omega_1 = \pi(T_1)$, and $\omega_2 = \pi(T_2)$, then $\omega_1 \wedge \omega_2 = \pi(T_1 \otimes T_2)$.
  - Note that $\ker(\pi) = \mathcal{I}^k(V)$.
- Properties.
  1. This is well defined, i.e., this does not depend on the choice of T_1, T_2.
     - Consider $T_1 + W_1, T_2 + W_2$ with $W_1, W_2 \in \mathcal{I}^k(V)$.
     - We check that $\pi[(T_1 + W_1) \otimes (T_2 + W_2)] = \pi(T_1 \otimes T_2)$.
     - Since $W_1 \otimes T_2, T_1 \otimes W_2, W_1 \otimes W_2 \in \mathcal{I}^{k+\ell}(V)$, we have that

$$\begin{aligned}\pi[(T_1 + W_1) \otimes (T_2 + W_2)] &= \pi(T_1 \otimes T_2 + W_1 \otimes T_2 + T_1 \otimes W_2 + W_1 \otimes W_2)\\ &= \pi(T_1 \otimes T_2) + \pi(W_1 \otimes T_2) + \pi(T_1 \otimes W_2) + \pi(W_1 \otimes W_2)\\ &= \pi(T_1 \otimes T_2)\end{aligned}$$

  2. Associative: We have that

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge \omega_2 \wedge \omega_3$$

[5] The hooked right arrow denotes an injective map.

- Follows from the definition of $\wedge$ in terms of π and properties of the tensor product.

3. Distributive: We have that
$$(\omega_1+\omega_2)\wedge\omega_3=\omega_1\wedge\omega_3+\omega_2\wedge\omega_3 \qquad \omega_1\wedge(\omega_2+\omega_3)=\omega_1\wedge\omega_2+\omega_1\wedge\omega_3$$
   - Follows from the definition of $\wedge$ in terms of π and properties of the tensor product.
4. Linear: We have that
$$(c\omega_1)\wedge\omega_2=c(\omega_1\wedge\omega_2)=\omega_1\wedge(c\omega_2)$$
   - Follows from the definition of $\wedge$ in terms of π and properties of the tensor product.
5. Anticommutative: We have that
$$\omega_1\wedge\omega_2=(-1)^{k\ell}\omega_2\wedge\omega_1$$
   - It suffices to assume that $w_1=\ell_1\wedge\cdots\wedge\ell_k, w_2=\ell_1'\wedge\cdots\wedge\ell_\ell'$.
     - We have
$$(\ell_1\wedge\cdots\wedge\ell_k)\wedge(\ell_1'\wedge\cdots\wedge\ell_\ell')=(-1)^k(\ell_1'\wedge\cdots\wedge\ell_\ell')\wedge(\ell_1\wedge\cdots\wedge\ell_k)$$
   - Let $\ell_1,\dots,\ell_k\in\Lambda^1(V^*)=V^*=\mathcal{L}^1(V)$.
   - Recall that $\mathcal{I}^1(V)=\{0\}$.
   - Claim: $\ell_{\sigma(1)}\wedge\cdots\wedge\ell_{\sigma(k)}=(-1)^\sigma\ell_1\wedge\cdots\wedge\ell_k$ for all $\sigma\in S_k$.
     - Recall that $T^\sigma=(-1)^\sigma T+W$ for some $W\in\mathcal{I}^k(V)$.
     - Let $T=\ell_1\otimes\cdots\otimes\ell_k$.
     - Then
$$\begin{aligned}(\ell_1\otimes\cdots\otimes\ell_k)^\sigma&=\ell_{\sigma(1)}\otimes\cdots\otimes\ell_{\sigma(k)}\\&=(-1)^\sigma\ell_1\otimes\cdots\otimes\ell_k+W\end{aligned}$$
     - Then hit both sides by π, noting that $\pi(W)=0$.

- Example:
  1. $n=2$, $k=\ell=1$. Consider $e_1^*,e_2^*\in\mathcal{L}^1(V)=V^*=\mathcal{A}^1(V)=\Lambda^1(V^*)$. Then
$$e_1^*\wedge e_2^*=(-1)e_2^*\wedge e_1^* \qquad e_1^*\wedge e_1^*=0=e_2^*\wedge e_2^*$$
  2. $n=4$. We have $e_1^*\wedge(3e_1^*+2e_2^*+3e_2^*)=3(e_1^*\wedge e_1^*)+2(e_1^*\wedge e_2^*)+3(e_1^*\wedge e_3^*)$. We also have $(e_1^*\wedge e_2^*)\wedge(e_1^*\wedge e_2^*)=0$.
- Wedging with zero.
  - We wish to further illustrate the definitions behind the on-the-surface simple concept
$$\omega\wedge 0=0$$
  - Let $\omega\in\Lambda^k(V^*)$ be arbitrary. Let 0 denote the zero function of $\Lambda^\ell(V^*)$.
  - Suppose $\omega=\pi(T_1)$ and $0=\pi(T_2)$.
  - Then
$$\begin{aligned}(\omega\wedge 0)(v_1,\dots,v_{k+\ell})&=\pi[(T_1\otimes T_2)(v_1,\dots,v_{k+\ell})]\\&=\pi[T_1(v_1,\dots,v_k)\cdot T_2(v_{k+1},\dots,v_{k+\ell})]\end{aligned}$$
  - Now there are two cases[6]. We could have $T_2(v_{k+1},\dots,v_{k+\ell})=0$, so the argument of π is 0, so everything is zero. But even if we don't have this, $0=\pi(T_2)$ implies that $T_2=0+T_3$ for some $T_3\in\mathcal{I}^\ell(V)$. Thus, under the projection function π, T_2 behaves like the zero element, regardless.
  - Thus, $\omega\wedge 0$ functions as the zero element of $\Lambda^{k+\ell}(V^*)$.
  - Another, perhaps more accurate, way of looking at this is to see that we need to prove that $T_1\otimes T_2\in\mathcal{I}^{k+\ell}(V)$. To do so, it will suffice to show that $T_2\in\mathcal{I}^\ell(V)$. But since $0=\pi(T_2)$, $T_2\in\ker\pi=\mathcal{I}^\ell(V)$, as desired.

[6] Technically, we only need the second case, but for pedagogical purposes, both are presented.

2.4 Chapter 1: Multilinear Algebra

From Guillemin and Haine (2018).

4/14:

- Having discussed $\text{im}(\text{Alt}) = \mathcal{A}^k(V)$ in some detail now, we move onto $\ker(\text{Alt})$.
- **Redundant** (decomposable k-tensor): A decomposable k-tensor $\ell_1 \otimes \cdots \otimes \ell_k$ such that for some $i \in [k-1]$, $\ell_i = \ell_{i+1}$.
- $\boldsymbol{\mathcal{I}^k(V)}$: The linear span of the set of redundant k-tensors.
- Convention: There are no redundant 1-tensors. Hence, we define

$$\mathcal{I}^1(V) = 0$$

- Proposition 1.5.2: $T \in \mathcal{I}^k(V)$ implies $\text{Alt}(T) = 0$.

  Proof. Let $T = \ell_1 \otimes \cdots \otimes \ell_k$ with $\ell_i = \ell_{i+1}$. Then if $\sigma = \tau_{i,i+1}$, we have that $T^\sigma = T$ and $(-1)^\sigma = -1$. Therefore,

$$\begin{aligned}
\text{Alt}(T) &= \text{Alt}(T^\sigma) \\
&= \text{Alt}(T)^\sigma && \text{Proposition 1.4.17(3)} \\
&= (-1)^\sigma \text{Alt}(T) && \text{Proposition 1.4.17(1)} \\
&= -\text{Alt}(T)
\end{aligned}$$

  so we must have that $\text{Alt}(T) = 0$, as desired. □

- Proposition 1.5.3: $T \in \mathcal{I}^r(V)$ and $T' \in \mathcal{L}^s(V)$ imply $T \otimes T', T' \otimes T \in \mathcal{I}^{r+s}(V)$.

  Proof. We first justify why we need only prove this claim for T' decomposable. As an element of $\mathcal{L}^s(V)$, we know that $T' = \sum a_I e_I^*$ for some set of $a_I \in \mathbb{R}$. Since each e_I^* is decomposable, this means that T' is a linear combination of decomposable tensors. This combined with the fact that the tensor product is linear means that

$$T \otimes T' = T \otimes \sum a_I e_I^* = \sum a_I (T \otimes e_I^*)$$

  and similarly for $T' \otimes T$. Thus, if we can prove that each $T \otimes e_I^* \in \mathcal{I}^{r+s}(V)$, it will follow since $\mathcal{I}^k(V)$ is a vector space that $\sum a_I(T \otimes e_I^*) = T \otimes T' \in \mathcal{I}^{r+s}(V)$. In other words, we need only prove that $T \otimes T' \in \mathcal{I}^{r+s}(V)$ for T' decomposable, as desired.

  Let $T = \ell_1 \otimes \cdots \otimes \ell_r$ with $\ell_i = \ell_{i+1}$, and let $T' = \ell_1' \otimes \cdots \otimes \ell_s'$. It follows that

$$T \otimes T' = (\ell_1 \otimes \cdots \otimes \ell_i \otimes \ell_{i+1} \otimes \cdots \otimes \ell_r) \otimes (\ell_1' \otimes \cdots \otimes \ell_s')$$

  is redundant and hence in $\mathcal{I}^{r+s}(V)$, as desired. The argument is symmetric for $T' \otimes T$. □

- Proposition 1.5.4: $T \in \mathcal{L}^k(V)$ and $\sigma \in S_k$ imply

$$T^\sigma = (-1)^\sigma T + S$$

  where $S \in \mathcal{I}^k(V)$.

  Proof. As with Proposition 1.5.3, the linearity of $\sigma : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ allows us to assume that T is decomposable.

  By Theorem 1.4.5, σ can be written as a product of m elementary transpositions. To prove the claim, we induct on m.

For the base case $m = 1$, let $\sigma = \tau_{i,i+1}$. If $T_1 = \ell_1 \otimes \cdots \otimes \ell_{i-1}$ and $T_2 = \ell_{i+2} \otimes \cdots \otimes \ell_k$, then

$$\begin{aligned} T^\sigma - (-1)^\sigma T &= T_1 \otimes (\ell_{i+1} \otimes \ell_i \pm \ell_i \otimes \ell_{i+1}) \otimes T_2 \\ &= T_1 \otimes [(\ell_i + \ell_{i+1}) \otimes (\ell_i + \ell_{i+1}) \mp \ell_i \otimes \ell_i \mp \ell_{i+1} \otimes \ell_{i+1}] \otimes T_2 \end{aligned}$$

i.e., $T^\sigma - (-1)^\sigma T$ is the sum of three redundant k-tensors, and thus a redundant k-tensor in and of itself, as desired. Note that even though only the middle portion is explicitly redundant, Proposition 1.5.3 allows us to call the whole tensor product redundant.

Now suppose inductively that we have proven the claim for $m - 1$. Let $\sigma = \tau\beta$ where β is the product of $m - 1$ elementary transpositions and τ is an elementary transposition. Then

$$\begin{aligned} T^\sigma &= (T^\beta)^\tau && \text{Proposition 1.4.14(3)} \\ &= (-1)^\tau T^\beta + \cdots && \text{Base case} \\ &= (-1)^\tau (-1)^\beta T + \cdots && \text{Inductive hypothesis} \\ &= (-1)^\sigma T + \cdots && \text{Claim 1.4.9} \end{aligned}$$

where the dots are elements of $\mathcal{I}^k(V)$. □

- Corollary 1.5.6: $T \in \mathcal{L}^k(V)$ implies
$$\text{Alt}(T) = k!T + W$$
where $W \in \mathcal{I}^k(V)$.

  Proof. By definition,
$$\text{Alt}(T) = \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma$$
By Proposition 1.5.4,
$$T^\sigma = (-1)^\sigma T + W_\sigma$$
for all $\sigma \in S_k$ with each $W_\sigma \in \mathcal{I}^k(V)$. It follows by combining the above two results that
$$\text{Alt}(T) = \sum_{\sigma \in S_k} (-1)^\sigma [(-1)^\sigma T + W_\sigma] = \sum_{\sigma \in S_k} T + \sum_{\sigma \in S_k} (-1)^\sigma W_\sigma = k!T + W$$
where $W = \sum_{\sigma \in S_k} (-1)^\sigma W_\sigma$ is an element of $\mathcal{I}^k(V)$ as a linear combination of elements of $\mathcal{I}^k(V)$. □

- Corollary 1.5.8: Let V be a vector space and $k \geq 1$. Then
$$\mathcal{I}^k(V) = \ker(\text{Alt} : \mathcal{L}^k(V) \to \mathcal{A}^k(V))$$

  Proof. Suppose first that $T \in \mathcal{I}^k(V)$. Then by Proposition 1.5.2, $\text{Alt}(T) = 0$, so $T \in \ker(\text{Alt})$, as desired.

  Now suppose that $T \in \ker(\text{Alt})$. Then $\text{Alt}(T) = 0$, so by Corollary 1.5.6,
$$\begin{aligned} 0 &= k!T + W \\ T &= -\frac{1}{k!} W \end{aligned}$$
Therefore, as a scalar multiple of an element of $\mathcal{I}^k(V)$, $T \in \mathcal{I}^k(V)$. □

- Theorem 1.5.9: Every $T \in \mathcal{L}^k(V)$ has a unique decomposition $T = T_1 + T_2$ where $T_1 \in \mathcal{A}^k(V)$ and $T_2 \in \mathcal{I}^k(V)$.

Proof. By Corollary 1.5.6, we have that

$$\operatorname{Alt}(T) = k!T + W$$
$$T = \underbrace{\left(\frac{1}{k!}\operatorname{Alt}(T)\right)}_{T_1} + \underbrace{\left(-\frac{1}{k!}W\right)}_{T_2}$$

As to uniqueness, suppose $0 = T_1 + T_2$ where $T_1 \in \mathcal{A}^k(V)$ and $T_2 \in \mathcal{I}^k(V)$. Then

$$0 = \operatorname{Alt}(0) = \operatorname{Alt}(T_1 + T_2) = \operatorname{Alt}(T_1) + \operatorname{Alt}(T_2) = k!T_1 + 0 = k!T_1$$
$$T_1 = 0$$

so $T_2 = 0$, too. □

- $\boldsymbol{\Lambda^k(V^*)}$: The quotient of the vector space $\mathcal{L}^k(V)$ by the subspace $\mathcal{I}^k(V)$. *Given by*

$$\Lambda^k(V^*) = \mathcal{L}^k(V)/\mathcal{I}^k(V)$$

- The quotient map $\pi : \mathcal{L}^k(V) \to \Lambda^k(V^*)$ defined by $T \mapsto T + \mathcal{I}^k(V)$ is onto and has $\ker(\pi) = \mathcal{I}^k(V)$.
- Theorem 1.5.13: $\pi : \mathcal{L}^k(V) \to \Lambda^k(V^*)$ maps $\mathcal{A}^k(V)$ bijectively onto $\Lambda^k(V^*)$.

Proof. Theorem 1.5.9 implies that every $T + \mathcal{I}^k(V)$ contains a unique $T_1 \in \mathcal{A}^k(V)$. Thus, for every element of $\Lambda^k(V^*)$, there is a unique element of $\mathcal{A}^k(V)$ which gets mapped onto it by π. □

- Note that since $\mathcal{A}^k(V)$ and $\Lambda^k(V^*)$ are in bijective correspondence, many texts do not distinguish between them. There are some advantages to making the distinction, though.
  - We can either look at $\mathcal{A}^k(V)$ as the set of all alternating tensors, or as the set of all k-tensors quotient the redundant tensors.
  - The fact that alternating and redundant tensors are orthogonal can probably be related to the fact that a redundant wedge product equals zero and only non-repeating wedges are nonzero.
- The tensor product and pullback operations give rise to similar operations on the spaces $\Lambda^k(V^*)$.
- **Wedge product**: The function $\wedge : \Lambda^{k_1}(V^*) \times \Lambda^{k_2}(V^*) \to \Lambda^{k_1+k_2}(V^*)$ defined by

$$\omega_1 \wedge \omega_2 = \pi(T_1 \otimes T_2)$$

where for $i = 1, 2$, $\omega_i \in \Lambda^{k_i}(V^*)$, and $\omega_i = \pi(T_i)$ for some $T_i \in \mathcal{L}^{k_i}(V)$.

  - Note that it is Theorem 1.5.13 that allows us to find T_i such that $\omega_i = \pi(T_i)$.
- Claim 1.6.3: The wedge product is well-defined, i.e., it does not depend on our choices of T_i.

Proof. We prove WLOG that $\wedge$ is well defined with respect to T_1. Suppose $\omega_1 = \pi(T_1) = \pi(T_1')$. Then by the definition of the quotient map, $T_1' = T_1 + W_1$ for some $W_1 \in \mathcal{I}^{k_1}(V)$. But this means that

$$T_1' \otimes T_2 = (T_1 + W_1) \otimes T_2 = T_1 \otimes T_2 + W_1 \otimes T_2$$

where $W_1 \otimes T_2 \in \mathcal{I}^{k_1+k_2}(V)$ by Proposition 1.5.3. It follows that

$$\pi(T_1' \otimes T_2) = \pi(T_1 \otimes T_2)$$

□

- The wedge product also generalizes to higher orders, obeying associativity, scalar multiplication, and distributivity.

- **Decomposable element** (of $\Lambda^k(V^*)$): An element of $\Lambda^k(V^*)$ of the form $\ell_1\wedge\cdots\wedge\ell_k$ where $\ell_1,\dots,\ell_k\in V^*$.
- Claim 1.6.8: The following wedge product identity holds for decomposable elements of $\Lambda^k(V^*)$.

$$\ell_{\sigma(1)}\wedge\cdots\wedge\ell_{\sigma(k)} = (-1)^\sigma\ell_1\wedge\cdots\wedge\ell_k$$

  Proof. Let $T=\ell_1\otimes\cdots\otimes\ell_k$. It follows by Proposition 1.4.14(1) that $T^\sigma=\ell_{\sigma(1)}\otimes\cdots\otimes\ell_{\sigma(k)}$. Therefore, we have that

$$\begin{aligned}\ell_{\sigma(1)}\wedge\cdots\wedge\ell_{\sigma(k)} &= \pi(\ell_{\sigma(1)}\otimes\cdots\otimes\ell_{\sigma(k)})\\ &= \pi(T^\sigma)\\ &= \pi[(-1)^\sigma T+W]\\ &= (-1)^\sigma\pi(T)\\ &= (-1)^\sigma\pi(\ell_1\otimes\cdots\otimes\ell_k)\\ &= (-1)^\sigma\ell_1\wedge\cdots\wedge\ell_k\end{aligned}$$

  as desired. ☐

- An important consequence of Claim 1.6.8 is that

$$\ell_1\wedge\ell_2 = -\ell_2\wedge\ell_1$$

  - Why the wedge product is anticommutative but not the tensor product: Because every "tensor" that can be wedged is alternating (as an element of $\Lambda^k(V^*)$). Indeed, the tensor product is anticommutative for alternating tensors.
- Theorem 1.6.10: If $\omega_1\in\Lambda^r(V^*)$ and $\omega_2\in\Lambda^s(V^*)$, then

$$\omega_1\wedge\omega_2 = (-1)^{rs}\omega_2\wedge\omega_1$$

  - This can be deduced from Claim 1.6.8.
  - Hint: It suffices to prove this for decomposable elements, i.e., for $\omega_1=\ell_1\wedge\cdots\wedge\ell_r$ and $\omega_2=\ell_1'\wedge\cdots\wedge\ell_s'$.
- Theorem 1.6.13: The elements

$$e_{i_1}^*\wedge\cdots\wedge e_{i_k}^* = \pi(e_I^*) = \pi(e_{i_1}^*\otimes\cdots\otimes e_{i_k}^*)$$

  with I strictly increasing are basis vectors of $\Lambda^k(V^*)$.

  Proof. Follows from the facts that the ψ_I for I strictly increasing constitute a basis of $\mathcal{A}^k(V)$ by Proposition 1.4.26 and π is an isomorphism $\mathcal{A}^k(V)\to\Lambda^k(V^*)$. ☐