2 Differential Forms

From Guillemin and Haine (2018).

Chapter 2

4/29: **2.1.i.** Let U be an open subset of $\mathbb{R}^n$. If $f : U \to \mathbb{R}$ is a C^∞ function, then

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \, \mathrm{d}x_i$$

Proof. The object on the left side of the above equality is a one-form. The object on the right side of the equality is the pointwise sum of n pointwise products of the functions $\frac{\partial f}{\partial x_i} : U \to \mathbb{R}$ with the one-forms $\mathrm{d}x_i$; thus, it is a one-form, too.

We want to prove that these two one-forms are equal. But under which definition of equality are we working? Each one-form is technically just a function from $U \to T_p^*\mathbb{R}^n$. Thus, we need only verify that both one-forms have the same action on every $p \in U$.

Let $p \in U$ be arbitrary. We now seek to verify that

$$\mathrm{d}f_p = \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \, \mathrm{d}x_i \right)_p$$

But once again, both sides are functions; specifically, they are both cotangent vectors to $\mathbb{R}^n$ at p. Thus, we need to verify that both cotangent vectors have the same action on every $(p, v) \in T_p\mathbb{R}^n$.

Let $(p, v) \in T_p\mathbb{R}^n$ be arbitrary. Additionally, let $v = (v_1, \dots, v_n)$. Then

$$\begin{aligned}
\mathrm{d}f_p(p, v) &= Df(p)v \\
&= \sum_{i=1}^{n} \left. \frac{\partial f}{\partial x_i} \right|_p v_i \\
&= \sum_{i=1}^{n} \left. \frac{\partial f}{\partial x_i} \right|_p (\mathrm{d}x_i)_p(p, v) \\
&= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \, \mathrm{d}x_i \right)_p (p, v) \\
&= \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \, \mathrm{d}x_i \right)_p (p, v)
\end{aligned}$$

as desired. ☐

2.1.ii. Let U be an open subset of $\mathbb{R}^n$, $\boldsymbol{v}$ a vector field on U, and $f_1, f_2 \in C^1(U)$. Then

$$L_{\boldsymbol{v}}(f_1 \cdot f_2) = L_{\boldsymbol{v}}(f_1) \cdot f_2 + f_1 \cdot L_{\boldsymbol{v}}(f_2)$$

Proof. Let

$$\boldsymbol{v} = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i}$$

By the definition of the Lie derivative, we have that

$$L_{\boldsymbol{v}}(f_1 \cdot f_2) = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i} (f_1 \cdot f_2)$$

$$= \sum_{i=1}^{n} g_i \left(\frac{\partial f_1}{\partial x_i} \cdot f_2 + f_1 \cdot \frac{\partial f_2}{\partial x_i} \right)$$
$$= f_2 \cdot \sum_{i=1}^{n} g_i \frac{\partial f_1}{\partial x_i} + f_1 \cdot \sum_{i=1}^{n} g_i \frac{\partial f_2}{\partial x_i}$$
$$= L_{\boldsymbol{v}}(f_1) \cdot f_2 + f_1 \cdot L_{\boldsymbol{v}}(f_2)$$

as desired. □

2.1.iii. Let U be an open subset of $\mathbb{R}^n$ and $\boldsymbol{v}_1, \boldsymbol{v}_2$ vector fields on U. Show that there is a unique vector field $\boldsymbol{w}$ on U with the property

$$L_{\boldsymbol{w}}\phi = L_{\boldsymbol{v}_1}(L_{\boldsymbol{v}_2}\phi) - L_{\boldsymbol{v}_2}(L_{\boldsymbol{v}_1}\phi)$$

for all $\phi \in C^\infty(U)$.

Proof. Let $\phi \in C^\infty(U)$ be arbitrary. Additionally, let

$$\boldsymbol{v}_1 = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i} \qquad \boldsymbol{v}_2 = \sum_{i=1}^{n} h_i \frac{\partial}{\partial x_i}$$

Then

$$L_{\boldsymbol{v}_1}\phi = \sum_{i=1}^{n} g_i \frac{\partial \phi}{\partial x_i} \qquad L_{\boldsymbol{v}_2}\phi = \sum_{i=1}^{n} h_i \frac{\partial \phi}{\partial x_i}$$

so that

$$\begin{aligned} L_{\boldsymbol{v}_1}(L_{\boldsymbol{v}_2}\phi) &= L_{\boldsymbol{v}_1}\left(\sum_{i=1}^{n} h_i \frac{\partial \phi}{\partial x_i} \right) \\ &= \sum_{i=1}^{n} L_{\boldsymbol{v}_1}\left(h_i \frac{\partial \phi}{\partial x_i} \right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} g_j \frac{\partial}{\partial x_j}\left(h_i \frac{\partial \phi}{\partial x_i} \right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} g_j \left(\frac{\partial h_i}{\partial x_j}\frac{\partial \phi}{\partial x_i} + h_i \frac{\partial^2 \phi}{\partial x_j \partial x_i} \right) \end{aligned}$$

$$\begin{aligned} L_{\boldsymbol{v}_2}(L_{\boldsymbol{v}_1}\phi) &= L_{\boldsymbol{v}_2}\left(\sum_{i=1}^{n} g_i \frac{\partial \phi}{\partial x_i} \right) \\ &= \sum_{i=1}^{n} L_{\boldsymbol{v}_2}\left(g_i \frac{\partial \phi}{\partial x_i} \right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} h_j \frac{\partial}{\partial x_j}\left(g_i \frac{\partial \phi}{\partial x_i} \right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} h_j \left(\frac{\partial g_i}{\partial x_j}\frac{\partial \phi}{\partial x_i} + g_i \frac{\partial^2 \phi}{\partial x_j \partial x_i} \right) \end{aligned}$$

It follows that

$$\begin{aligned} L_{\boldsymbol{v}_1}(L_{\boldsymbol{v}_2}\phi) - L_{\boldsymbol{v}_2}(L_{\boldsymbol{v}_1}\phi) &= \sum_{i=1}^{n}\sum_{j=1}^{n} g_j \left(\frac{\partial h_i}{\partial x_j}\frac{\partial \phi}{\partial x_i} + h_i \frac{\partial^2 \phi}{\partial x_j \partial x_i} \right) - \sum_{i=1}^{n}\sum_{j=1}^{n} h_j \left(\frac{\partial g_i}{\partial x_j}\frac{\partial \phi}{\partial x_i} + g_i \frac{\partial^2 \phi}{\partial x_j \partial x_i} \right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \left[\left(g_j \frac{\partial h_i}{\partial x_j} - h_j \frac{\partial g_i}{\partial x_j} \right) \frac{\partial \phi}{\partial x_i} + (g_j h_i - h_j g_i) \frac{\partial^2 \phi}{\partial x_j \partial x_i} \right] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \left[\left(\frac{\partial}{\partial x_j}(g_j h_i - h_j g_i) \right) \frac{\partial \phi}{\partial x_i} + (g_j h_i - h_j g_i) \frac{\partial^2 \phi}{\partial x_j \partial x_i} \right] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j}(g_j h_i - h_j g_i) \right) \frac{\partial \phi}{\partial x_i} \end{aligned}$$

and hence that

$$\boldsymbol{w} = \sum_{i=1}^{n} \underbrace{\sum_{j=1}^{n} \left(\frac{\partial}{\partial x_j}(g_j h_i - h_j g_i) \right)}_{\text{functions } U \to \mathbb{R}} \frac{\partial}{\partial x_i}$$

□

2.1.iv. The vector field $\boldsymbol{w}$ in Exercise 2.1.iii is called the **Lie bracket** of the vector fields $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ and is denoted by $[\boldsymbol{v}_1, \boldsymbol{v}_2]$. Verify that the Lie bracket is **skew-symmetric**, i.e.,

$$[\boldsymbol{v}_1, \boldsymbol{v}_2] = -[\boldsymbol{v}_2, \boldsymbol{v}_1]$$

and satisfies the **Jacobi identity**

$$[\boldsymbol{v}_1, [\boldsymbol{v}_2, \boldsymbol{v}_3]] + [\boldsymbol{v}_2, [\boldsymbol{v}_3, \boldsymbol{v}_1]] + [\boldsymbol{v}_3, [\boldsymbol{v}_1, \boldsymbol{v}_2]] = 0$$

Thus, the Lie bracket defines the structure of a **Lie algebra**. (Hint: Prove analogous identities for $L_{\boldsymbol{v}_1}$, $L_{\boldsymbol{v}_2}$, and $L_{\boldsymbol{v}_3}$.)

Proof. Throughout this problem, let

$$\boldsymbol{v}_1 = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i} \qquad \boldsymbol{v}_2 = \sum_{i=1}^n g_i \frac{\partial}{\partial x_i} \qquad \boldsymbol{v}_3 = \sum_{i=1}^n h_i \frac{\partial}{\partial x_i}$$

Then

$$[\boldsymbol{v}_1, \boldsymbol{v}_2] = \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial}{\partial x_j} (f_j g_i - g_j f_i) \right) \frac{\partial}{\partial x_i}$$

It follows that

$$\begin{aligned}
-[\boldsymbol{v}_1, \boldsymbol{v}_2] &= -\sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial}{\partial x_j} (f_j g_i - g_j f_i) \right) \frac{\partial}{\partial x_i} \\
&= \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial}{\partial x_j} (f_i g_j - g_i f_j) \right) \frac{\partial}{\partial x_i} \\
&= \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial}{\partial x_j} (f_j g_i - g_j f_i) \right) \frac{\partial}{\partial x_i} \\
&= [\boldsymbol{v}_1, \boldsymbol{v}_2]
\end{aligned}$$

where the third equality holds by reindexing the symmetric sum.

Additionally, we have that

$$[\boldsymbol{v}_2, \boldsymbol{v}_3] = \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial}{\partial x_j} (g_j h_i - h_j g_i) \right) \frac{\partial}{\partial x_i}$$

and

$$[\boldsymbol{v}_3, \boldsymbol{v}_1] = \sum_{i=1}^n \sum_{j=1}^n \left(\frac{\partial}{\partial x_j} (h_j f_i - f_j h_i) \right) \frac{\partial}{\partial x_i}$$

It follows that

$$\begin{aligned}
[\boldsymbol{v}_1, [\boldsymbol{v}_2, \boldsymbol{v}_3]] &= \sum_{i=1}^n \sum_{j=1}^n \left[\frac{\partial}{\partial x_j} \left(f_j \sum_{k=1}^n \left[\frac{\partial}{\partial x_k} (g_k h_i - h_k g_i) \right] - f_i \sum_{k=1}^n \left[\frac{\partial}{\partial x_k} (g_k h_j - h_k g_j) \right] \right) \right] \frac{\partial}{\partial x_i} \\
&= 0
\end{aligned}$$

where we note that any i, j term in the double sum and the corresponding j, i term add to zero. We can prove a similar identity for $[\boldsymbol{v}_2, [\boldsymbol{v}_3, \boldsymbol{v}_1]]$ and $[\boldsymbol{v}_3, [\boldsymbol{v}_1, \boldsymbol{v}_2]]$. Thus,

$$[\boldsymbol{v}_1, [\boldsymbol{v}_2, \boldsymbol{v}_3]] + [\boldsymbol{v}_2, [\boldsymbol{v}_3, \boldsymbol{v}_1]] + [\boldsymbol{v}_3, [\boldsymbol{v}_1, \boldsymbol{v}_2]] = 0 + 0 + 0 = 0$$

so the Lie bracket satisfies the Jacobi identity, as desired. ∎

2.1.vii. Let U be an open subset of $\mathbb{R}^n$, and let $\gamma : [a,b] \to U$, $t \mapsto (\gamma_1(t), \dots, \gamma_n(t))$ be a C^1 curve. Given a C^∞ one-form $\omega = \sum_{i=1}^n f_i \, \mathrm{d}x_i$ on U, define the **line integral** of ω over γ to be the integral

$$\int_\gamma \omega = \sum_{i=1}^n \int_a^b f_i(\gamma(t)) \frac{\mathrm{d}\gamma_i}{\mathrm{d}t} \, \mathrm{d}t$$

Show that if $\omega = \mathrm{d}f$ for some $f \in C^\infty(U)$,

$$\int_\gamma \omega = f(\gamma(b)) - f(\gamma(a))$$

In particular, conclude that if γ is a closed curve, i.e., $\gamma(a) = \gamma(b)$, this integral is zero.

Proof. Since $\gamma : [a,b] \to U$ (where U is open), we know that there exist $N_{r_1}(\gamma(a)) \subset U$ and $N_{r_2}(\gamma(b)) \subset U$. Thus, we may extend γ to some open superset $(a,b)^+ \supset [a,b]$ in a C^1 fashion, i.e., along the tangent vectors to $\gamma(a)$ and $\gamma(b)$ at a and b, respectively. From now on, when we refer to γ, we will be discussing $\gamma : (a,b)^+ \to U$. With this adjustment, we can show that $f \circ \gamma$ satisfies the hypotheses for the multivariable chain rule at $t \in [a,b]$ arbitrary.

$(a,b)^+$ is open in $\mathbb{R}$ by definition. Since $\gamma \in C^1(\mathbb{R})$, $\gamma : (a,b)^+ \to U$ is differentiable at $t \in [a,b] \subset \mathbb{R}^n$. $U \supset \gamma((a,b)^+)$ is an open set in $\mathbb{R}^n$ by hypothesis. Since $f \in C^\infty(U)$, $f : U \to \mathbb{R}$ is differentiable at $\gamma(t)$. Therefore, we have by Theorem 9.15 of Rudin (1976) that

$$\begin{aligned}
(f \circ \gamma)'(t) &= D(f \circ \gamma)(t) \\
&= Df(\gamma(t)) \circ D\gamma(t) \\
&= \begin{bmatrix} \left.\frac{\partial f}{\partial x_1}\right|_{\gamma(t)} & \cdots & \left.\frac{\partial f}{\partial x_n}\right|_{\gamma(t)} \end{bmatrix} \begin{bmatrix} \left.\frac{\partial \gamma_1}{\partial t}\right|_t \\ \vdots \\ \left.\frac{\partial \gamma_n}{\partial t}\right|_t \end{bmatrix} \\
&= \sum_{i=1}^n \left.\frac{\partial f}{\partial x_i}\right|_{\gamma(t)} \left.\frac{\partial \gamma_i}{\partial t}\right|_t \\
&= \sum_{i=1}^n \left.\frac{\partial f}{\partial x_i}\right|_{\gamma(t)} \frac{\partial \gamma_i}{\partial t}
\end{aligned}$$

Now suppose $\omega = \mathrm{d}f$. Then by Lemma 2.1.18, each $f_i = \partial f / \partial x_i$. It follows that

$$\begin{aligned}
\int_\gamma \omega &= \sum_{i=1}^n \int_a^b \left.\frac{\partial f}{\partial x_i}\right|_{\gamma(t)} \frac{\mathrm{d}\gamma_i}{\mathrm{d}t} \, \mathrm{d}t \\
&= \int_a^b \sum_{i=1}^n \left.\frac{\partial f}{\partial x_i}\right|_{\gamma(t)} \frac{\mathrm{d}\gamma_i}{\mathrm{d}t} \, \mathrm{d}t \\
&= \int_a^b (f \circ \gamma)'(t) \, \mathrm{d}t \\
&= f(\gamma(b)) - f(\gamma(a))
\end{aligned}$$

as desired.

Now suppose that γ is a closed curve. Then

$$\begin{aligned}
\int_\gamma \omega &= f(\gamma(b)) - f(\gamma(a)) \\
&= f(\gamma(a)) - f(\gamma(a)) \\
&= 0
\end{aligned}$$

as desired. ☐

2.1.viii. Let ω be the C^∞ one-form on $\mathbb{R}^2 \setminus \{0\}$ defined by

$$\omega = \frac{x_1 \,\mathrm{d}x_2 - x_2 \,\mathrm{d}x_1}{x_1^2 + x_2^2}$$

and let $\gamma : [0, 2\pi] \to \mathbb{R}^2 \setminus \{0\}$ be the closed curve $t \mapsto (\cos t, \sin t)$. Compute the line integral $\int_\gamma \omega$ and note that $\int_\gamma \omega \neq 0$. Conclude that ω is not of the form $\mathrm{d}f$ for $f \in C^\infty(\mathbb{R}^2 \setminus \{0\})$.

Proof. From the given definition of ω, we can determine that

$$f_1(x_1, x_2) = -\frac{x_2}{x_1^2 + x_2^2} \qquad f_2(x_1, x_2) = \frac{x_1}{x_1^2 + x_2^2}$$

We also have that

$$\gamma_1(t) = \cos t \qquad \gamma_2(t) = \sin t$$

Thus, we know that

$$\begin{aligned}
\int_\gamma \omega &= \sum_{i=1}^{2} \int_0^{2\pi} f_i(\gamma(t)) \frac{\mathrm{d}\gamma_i}{\mathrm{d}t} \,\mathrm{d}t \\
&= \int_0^{2\pi} f_1(\cos t, \sin t) \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\cos t) \,\mathrm{d}t + \int_0^{2\pi} f_2(\cos t, \sin t) \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\sin t) \,\mathrm{d}t \\
&= \int_0^{2\pi} -\sin t \cdot -\sin t \,\mathrm{d}t + \int_0^{2\pi} \cos t \cdot \cos t \,\mathrm{d}t \\
&= \int_0^{2\pi} \mathrm{d}t
\end{aligned}$$

$$\boxed{\int_\gamma \omega = 2\pi}$$

Since $\int_\gamma \omega \neq 0$ and $\gamma(0) = \gamma(2\pi) = (1, 0)$, we have by Exercise 2.1.vii that $\omega \neq \mathrm{d}f$. ☐

2.2.i. For $i = 1, 2$, let U_i be an open subset of $\mathbb{R}^{n_i}$, $\boldsymbol{v}_i$ a vector field on U_i, and $f : U_1 \to U_2$ a C^∞-map. If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are f-related, every integral curve $\gamma : I \to U_1$ of $\boldsymbol{v}_1$ gets mapped by f onto an integral curve $f \circ \gamma : I \to U_2$ of $\boldsymbol{v}_2$.

Proof. We want to show that

$$\boldsymbol{v}_2((f \circ \gamma)(t)) = \left((f \circ \gamma)(t), \tfrac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)\big|_t\right)$$

We are given that

$$\boldsymbol{v}_1(\gamma(t)) = \left(\gamma(t), \tfrac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_t\right) \qquad \mathrm{d}f_p(\boldsymbol{v}_1(p)) = \boldsymbol{v}_2(f(p))$$

Let $p = \gamma(t)$ and $q = f(p)$. Then

$$\begin{aligned}
\boldsymbol{v}_2((f \circ \gamma)(t)) &= \boldsymbol{v}_2(f(p)) \\
&= \mathrm{d}f_p(\boldsymbol{v}_1(p)) \\
&= \mathrm{d}f_p(\boldsymbol{v}_1(\gamma(t))) \\
&= \mathrm{d}f_p\left(\gamma(t), \tfrac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_t\right) \\
&= \mathrm{d}f_p\left(p, \tfrac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_t\right) \\
&= \left(q, Df(p)\left(\tfrac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_t\right)\right) \\
&= \left((f \circ \gamma)(t), \tfrac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)\big|_t\right)
\end{aligned}$$

as desired. ☐

2.2.ii. Let U, V be open subsets of $\mathbb{R}^n$ and $f : U \to V$ an C^k map.

(1) Show that for $\phi \in C^\infty(V)$, the pullback can be rewritten

$$f^* \,\mathrm{d}\phi = \mathrm{d}f^*\phi$$

Proof. We have that

$$\begin{aligned}(f^*\mathrm{d}\phi)(p) &= \mathrm{d}\phi_{f(p)} \circ \mathrm{d}f_p \\ &= \mathrm{d}(\phi \circ f)_p \\ &= \mathrm{d}f^*\phi\end{aligned}$$

where $f^*\phi = \phi \circ f$ is another variation of the pullback. □

(2) Let μ be the one-form

$$\mu = \sum_{i=1}^{n} \phi_i \,\mathrm{d}x_i$$

on V for all $\phi_i \in C^\infty(V)$. Show that if $f = (f_1, \ldots, f_n)$, then

$$f^*\mu = \sum_{i=1}^{n} f^*\phi_i \,\mathrm{d}f_i$$

Proof. We have that

$$\begin{aligned}(f^*\mu)(p) &= \mu_{f(p)} \circ \mathrm{d}f_p \\ &= \sum_{i=1}^{n} \phi_i(f(p))(\mathrm{d}x_i)_p \circ \mathrm{d}f_p \\ &= \sum_{i=1}^{n} (\phi_i \circ f)(p)(\mathrm{d}f_i)_p \\ &= \sum_{i=1}^{n} f^*\phi_i(p)(\mathrm{d}f_i)_p\end{aligned}$$

where we have $(\mathrm{d}x_i)_p \circ \mathrm{d}f_p = \mathrm{d}f_i$ since

$$\begin{aligned}[(\mathrm{d}x_i)_p \circ \mathrm{d}f_p](p, v) &= (\mathrm{d}x_i)_p(q, Df(p)v) \\ &= \left(q_i, \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} v_j\right) \\ &= (q_i, Df_i(p)v) \\ &= (\mathrm{d}f_i)_p(p, v)\end{aligned}$$

□

(3) Show that if μ is C^∞ and f is C^∞, $f^*\mu$ is C^∞.

Proof. To prove that $f^*\mu \in C^\infty$, it will suffice to show by (2) that every $f^*\phi_i \in C^\infty$. But this is obvious since $f^*\phi_i = \phi_i \circ f$ where the latter two composed functions are both C^∞. □

2.2.iv. (1) Let $U = \mathbb{R}^2$ and let $\boldsymbol{v}$ be the vector field $x_1 \,\partial/\partial x_2 - x_2 \,\partial/\partial x_1$. Show that the curve

$$t \mapsto (r\cos(t + \theta), r\sin(t + \theta))$$

for $t \in \mathbb{R}$ is the unique integral curve of $\boldsymbol{v}$ passing through the point $(r\cos\theta, r\sin\theta)$ at $t = 0$.

Proof. We first will check that the above curve, which we will call $\gamma : \mathbb{R} \to U$, is *an* integral curve of $\boldsymbol{v}$ passing through $(r\cos\theta, r\sin\theta)$ at $t = 0$. To verify the integral curve part, we first note that since that $g_1, g_2 : U \to \mathbb{R}$ are defined by

$$g_1(x_1, x_2) = -x_2 \qquad\qquad g_2(x_1, x_2) = x_1$$

we may define $g : U \to \mathbb{R}^2$ by

$$g(x_1, x_2) = (-x_2, x_1)$$

Thus, we need show that

$$\begin{aligned} \frac{\mathrm{d}\gamma}{\mathrm{d}t} &\stackrel{?}{=} g(\gamma(t)) \\ \left(\frac{\mathrm{d}}{\mathrm{d}t}(r\cos(t+\theta)), \frac{\mathrm{d}}{\mathrm{d}t}(r\sin(t+\theta))\right) &\stackrel{?}{=} g(r\cos(t+\theta), r\sin(t+\theta)) \\ (-r\sin(t+\theta), r\cos(t+\theta)) &\stackrel{\checkmark}{=} (-r\sin(t+\theta), r\cos(t+\theta)) \end{aligned}$$

To verify the passing through the point at $t = 0$ part, we need only plug in $t = 0$ and observe the equivalence:

$$\gamma(0) = (r\cos(0+\theta), r\sin(0+\theta)) = (r\cos\theta, r\sin\theta)$$

We now check that γ is the *unique* such curve. But if $\tilde{\gamma}$ is an integral curve passing through $(r\cos\theta, r\sin\theta)$ at $t = 0$, we have that $\gamma = \tilde{\gamma}$ by Theorem 2.2.5. □

(2) Let $U = \mathbb{R}^n$ and let $\boldsymbol{v}$ be the constant vector field $\sum_{i=1}^n c_i\, \partial/\partial x_i$. Show that the curve

$$t \mapsto a + t(c_1, \dots, c_n)$$

for $t \in \mathbb{R}$ is the unique integral curve of $\boldsymbol{v}$ passing through $a \in \mathbb{R}^n$ at $t = 0$.

Proof. Applying the same strategy in part (a), we call the given integral curve γ and define $g : U \to \mathbb{R}^n$ by

$$g(x_1, \dots, x_n) = (c_1, \dots, c_n)$$

Then we have the following.

<u>*An* integral curve</u>:

$$\begin{aligned} \frac{\mathrm{d}\gamma}{\mathrm{d}t} &\stackrel{?}{=} g(\gamma(t)) \\ \left(\frac{\mathrm{d}}{\mathrm{d}t}(a+tc_1), \dots, \frac{\mathrm{d}}{\mathrm{d}t}(a+tc_n)\right) &\stackrel{?}{=} g(a+tc_1, \dots, a+tc_n) \\ (c_1, \dots, c_n) &\stackrel{\checkmark}{=} (c_1, \dots, c_n) \end{aligned}$$

<u>$\gamma(0) = a$</u>:

$$\begin{aligned} \gamma(0) &= a + 0 \cdot (c_1, \dots, c_n) \\ &= a \end{aligned}$$

<u>*Unique* integral curve</u>: Apply Theorem 2.2.5. □

(3) Let $U = \mathbb{R}^n$ and let $\boldsymbol{v}$ be the vector field $\sum_{i=1}^n x_i\, \partial/\partial x_i$. Show that the curve

$$t \mapsto \mathrm{e}^t(a_1, \dots, a_n)$$

for $t \in \mathbb{R}$ is the unique integral curve of $\boldsymbol{v}$ passing through a at $t = 0$.

Proof. Applying the same strategy in parts (a)-(b), we call the given integral curve γ and define $g : U \to \mathbb{R}^n$ by

$$g(x_1, \dots, x_n) = (x_1, \dots, x_n)$$

Then we have the following.
An integral curve:

$$\begin{aligned}\frac{\mathrm{d}\gamma}{\mathrm{d}t} &\stackrel{?}{=} g(\gamma(t)) \\ \left(\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^t a_1), \dots, \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^t a_n)\right) &\stackrel{?}{=} g(\mathrm{e}^t a_1, \dots, \mathrm{e}^t a_n) \\ (\mathrm{e}^t a_1, \dots, \mathrm{e}^t a_n) &\stackrel{\checkmark}{=} (\mathrm{e}^t a_1, \dots, \mathrm{e}^t a_n)\end{aligned}$$

$\gamma(0) = a$:

$$\begin{aligned}\gamma(0) &= \mathrm{e}^0(a_1, \dots, a_n) \\ &= a\end{aligned}$$

Unique integral curve: Apply Theorem 2.2.5. ☐

2.2.viii. Let $\boldsymbol{v}$ be the vector field on $\mathbb{R}$ given by $x^2\, \partial/\partial x$. Show that the curve

$$x(t) = \frac{a}{1 - at}$$

is an integral curve of $\boldsymbol{v}$ with initial point $x(0) = a$. Conclude that for $a > 0$, the curve

$$x(t) = \frac{a}{1 - at}$$

on $0 < t < 1/a$ is a maximal integral curve. (In particular, conclude that $\boldsymbol{v}$ is not complete.)

Proof. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = x^2$$

An integral curve:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &\stackrel{?}{=} g(x(t)) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{a}{1-at}\right) &\stackrel{?}{=} g\left(\frac{a}{1-at}\right) \\ \frac{(1-at)(0) - a(-a)}{(1-at)^2} &\stackrel{?}{=} \left(\frac{a}{1-at}\right)^2 \\ \frac{a^2}{(1-at)^2} &\stackrel{\checkmark}{=} \frac{a^2}{(1-at)^2}\end{aligned}$$

$x(0) = a$:

$$\begin{aligned}x(0) &= \frac{a}{1 - a \cdot 0} \\ &= a\end{aligned}$$

x on $0 < t < 1/a$ is a maximal integral curve: Suppose for the sake of contradiction that there exists an $a > 0$ to which there corresponds a number $b > 1/a$ such that $x(t) = a/(1 - at)$ on $(0, b)$ is an integral curve. It can be proven with an ϵ, δ argument that x is continuous on $(0, 1/a)$ and on $(1/a, b)$, but that there is a discontinuity at $1/a$. But since x is an integral curve, we have by definition that x is C^1 (hence continuous) on $(0, b)$, a contradiction. Therefore, x is a maximal integral curve on the specified integral.

Choose $a = 1 > 0$. By the above, no integral curve $\gamma : \mathbb{R} \to \mathbb{R}$ exists with $\gamma(0) = 1$, so $\boldsymbol{v}$ cannot be complete, as desired. ☐

2.3.i. Let $\omega \in \Omega^2(\mathbb{R}^4)$ be the 2-form $\mathrm{d}x_1 \wedge \mathrm{d}x_2 + \mathrm{d}x_3 \wedge \mathrm{d}x_4$. Compute $\omega \wedge \omega$.

Proof. By the definition of the wedge product for k-forms, all properties proven for the wedge product of tensors carry over. This result will not be stated again, though it will be used again.

By the distributive law, we have that

$$\begin{aligned}\omega \wedge \omega &= [(\mathrm{d}x_1 \wedge \mathrm{d}x_2) + (\mathrm{d}x_3 \wedge \mathrm{d}x_4)] \wedge [(\mathrm{d}x_1 \wedge \mathrm{d}x_2) + (\mathrm{d}x_3 \wedge \mathrm{d}x_4)] \\ &= (\mathrm{d}x_1 \wedge \mathrm{d}x_2) \wedge (\mathrm{d}x_1 \wedge \mathrm{d}x_2) + 2(\mathrm{d}x_1 \wedge \mathrm{d}x_2) \wedge (\mathrm{d}x_3 \wedge \mathrm{d}x_4) + (\mathrm{d}x_3 \wedge \mathrm{d}x_4) \wedge (\mathrm{d}x_3 \wedge \mathrm{d}x_4)\end{aligned}$$

By the anticommutative law, a decomposable element wedged with itself is zero.

$$= 2\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_4$$

□

2.3.ii. Let $\omega_1, \omega_2, \omega_3 \in \Omega^1(\mathbb{R}^3)$ be the 1-forms

$$\begin{aligned}\omega_1 &= x_2\,\mathrm{d}x_3 - x_3\,\mathrm{d}x_2 \\ \omega_2 &= x_3\,\mathrm{d}x_1 - x_1\,\mathrm{d}x_3 \\ \omega_3 &= x_1\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_1\end{aligned}$$

Compute the following.

(1) $\omega_1 \wedge \omega_2$.

Proof. We have that

$$\begin{aligned}\omega_1 \wedge \omega_2 &= (x_2\mathrm{d}x_3 - x_3\mathrm{d}x_2) \wedge (x_3\mathrm{d}x_1 - x_1\mathrm{d}x_3) \\ &= x_2x_3\mathrm{d}x_3 \wedge \mathrm{d}x_1 - x_2x_1\mathrm{d}x_3 \wedge \mathrm{d}x_3 - x_3^2\mathrm{d}x_2 \wedge \mathrm{d}x_1 + x_1x_3\mathrm{d}x_2 \wedge \mathrm{d}x_3 \\ &= x_3^2\mathrm{d}x_1 \wedge \mathrm{d}x_2 - x_2x_3\mathrm{d}x_1 \wedge \mathrm{d}x_3 + x_1x_3\mathrm{d}x_2 \wedge \mathrm{d}x_3\end{aligned}$$

□

(2) $\omega_2 \wedge \omega_3$.

Proof. We have that

$$\begin{aligned}\omega_2 \wedge \omega_3 &= (x_3\,\mathrm{d}x_1 - x_1\,\mathrm{d}x_3) \wedge (x_1\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_1) \\ &= x_3x_1\mathrm{d}x_1 \wedge \mathrm{d}x_2 - x_3x_2\mathrm{d}x_1 \wedge \mathrm{d}x_1 - x_1^2\mathrm{d}x_3 \wedge \mathrm{d}x_2 + x_1x_2\mathrm{d}x_3 \wedge \mathrm{d}x_1 \\ &= x_1x_3\mathrm{d}x_1 \wedge \mathrm{d}x_2 - x_1x_2\mathrm{d}x_1 \wedge \mathrm{d}x_3 + x_1^2\mathrm{d}x_2 \wedge \mathrm{d}x_3\end{aligned}$$

□

(3) $\omega_3 \wedge \omega_1$.

Proof. We have that

$$\begin{aligned}\omega_3 \wedge \omega_1 &= (x_1\,\mathrm{d}x_2 - x_2\,\mathrm{d}x_1) \wedge (x_2\,\mathrm{d}x_3 - x_3\,\mathrm{d}x_2) \\ &= x_1x_2\mathrm{d}x_2 \wedge \mathrm{d}x_3 - x_1x_3\mathrm{d}x_2 \wedge \mathrm{d}x_2 - x_2^2\mathrm{d}x_1 \wedge \mathrm{d}x_3 + x_2x_3\mathrm{d}x_1 \wedge \mathrm{d}x_2 \\ &= x_2x_3\mathrm{d}x_1 \wedge \mathrm{d}x_2 - x_2^2\mathrm{d}x_1 \wedge \mathrm{d}x_3 + x_1x_2\mathrm{d}x_2 \wedge \mathrm{d}x_3\end{aligned}$$

□

(4) $\omega_1 \wedge \omega_2 \wedge \omega_3$.

Proof. We have that

$$\begin{aligned}
\omega_1 \wedge \omega_2 \wedge \omega_3 &= (\omega_1 \wedge \omega_2) \wedge \omega_3 \\
&= (x_3^2 \mathrm{d}x_1 \wedge \mathrm{d}x_2 - x_2 x_3 \mathrm{d}x_1 \wedge \mathrm{d}x_3 + x_1 x_3 \mathrm{d}x_2 \wedge \mathrm{d}x_3) \wedge (x_1 \,\mathrm{d}x_2 - x_2 \,\mathrm{d}x_1) \\
&= x_3^2 x_1 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_2 - x_3^2 x_2 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_1 - x_2 x_3 x_1 \mathrm{d}x_1 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_2 \\
&\quad + x_2 x_3 x_2 \mathrm{d}x_1 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_1 + x_1 x_3 x_1 \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_2 - x_1 x_3 x_2 \mathrm{d}x_2 \wedge \mathrm{d}x_3 \wedge \mathrm{d}x_1 \\
&= x_1 x_2 x_3 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 - x_1 x_2 x_3 \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3 \\
&= 0
\end{aligned}$$

□

2.3.iii. Let U be an open subset of $\mathbb{R}^n$ and $f_1, \dots, f_n \in C^\infty(U)$. Show that

$$\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n = \det \left[\frac{\partial f_i}{\partial x_j} \right] \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

Proof. By Lemma 2.1.18,

$$\mathrm{d}f_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} \mathrm{d}x_j$$

for all $i = 1, \dots, n$. It follows that

$$\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n = \sum_{j=1}^{n} \frac{\partial f_1}{\partial x_j} \mathrm{d}x_j \wedge \cdots \wedge \sum_{j=1}^{n} \frac{\partial f_n}{\partial x_j} \mathrm{d}x_j$$

If we apply the distributive law for the wedge product, there will be n^n terms in the resulting sum. Every term contains the product of n partial derivatives as a scalar multiple in front of the wedge product of n one-forms. For the partial derivatives, each of the n functions f_i will be represented exactly once. However, for the one-forms (and corresponding variables of differentiation), any number from 1 through n can be represented up to n times. Thus, we need to sum terms of the form

$$\frac{\partial f_1}{\partial x_{i_1}} \cdots \frac{\partial f_n}{\partial x_{i_n}} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_n}$$

over the multi-indices of n of length n. Consequently,

$$\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n = \sum_{I} \frac{\partial f_1}{\partial x_{i_1}} \cdots \frac{\partial f_n}{\partial x_{i_n}} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_n}$$

We now consider which terms in the sum are equal to zero. By the anticommutative property of the wedge product, any repeating multi-index will lead to a term whose wedge product evaluates to zero. Thus, we can restrict our sum to the *non-repeating* multi-indices of n of length n.

Every non-repeating multi-index of n of length n is equal to the n-tuple $(\sigma(1), \dots, \sigma(n))$ for some $\sigma \in S_n$. Thus, instead of summing over the multi-indices of n of length n, we can sum over the permutations in S_n:

$$\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n = \sum_{\sigma \in S_n} \frac{\partial f_1}{\partial x_{\sigma(1)}} \cdots \frac{\partial f_n}{\partial x_{\sigma(n)}} \mathrm{d}x_{\sigma(1)} \wedge \cdots \wedge \mathrm{d}x_{\sigma(n)}$$

But by an extension of Claim 1.6.8,

$$\mathrm{d}x_{\sigma(1)} \wedge \cdots \wedge \mathrm{d}x_{\sigma(n)} = (-1)^\sigma \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

Therefore, we can factor out the one-form from the sum and equate the sum with the determinant, as desired.

$$\begin{aligned}\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n &= \left[\sum_{\sigma \in S_n} (-1)^\sigma \frac{\partial f_1}{\partial x_{\sigma(1)}} \cdots \frac{\partial f_n}{\partial x_{\sigma(n)}}\right] \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \\ &= \det \left[\frac{\partial f_i}{\partial x_j}\right] \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n\end{aligned}$$

□

2.3.iv. Let U be an open subset of $\mathbb{R}^n$. Show that every $(n-1)$-form $\omega \in \Omega^{n-1}(U)$ can be written uniquely as a sum

$$\sum_{i=1}^n f_i \, \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n$$

where $f_i \in C^\infty(U)$ and $\widehat{\mathrm{d}x_i}$ indicates that $\mathrm{d}x_i$ is to be omitted from the wedge product $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$.

Proof. Let $\omega \in \Omega^{n-1}(U)$ be arbitrary. Then ω has a decomposition

$$\omega = \sum_I f_I \, \mathrm{d}x_I$$

where we sum over the multi-indices of n of length $n-1$. However, since any wedge product with a repeat evaluates to zero (anticommutative property), we need only sum over the non-repeating multi-indices of n of length $n-1$. Moreover, all of these can be reordered so that they are strictly increasing by some $\sigma \in S_{n-1}$. The resulting sign $(-1)^\sigma$ and multiple functions f_I, if applicable, can be combined into one new function f_i and reindexed. □

2.3.v. Let $\mu = \sum_{i=1}^n x_i \, \mathrm{d}x_i$. Show that there exists an $(n-1)$-form $\omega \in \Omega^{n-1}(\mathbb{R}^n \setminus \{0\})$ with the property

$$\mu \wedge \omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

Proof. Define $1/0 = \pm\infty$ and $0 \cdot \pm\infty = 1$. Let $\omega = (1/x_1) \, \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n$. Then

$$\begin{aligned}\mu \wedge \omega &= \left(\sum_{i=1}^n x_i \, \mathrm{d}x_i\right) \wedge \left(\frac{1}{x_1} \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n\right) \\ &= \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n\end{aligned}$$

where all terms save the first cancel by the anticommutative property. □

2.3.vi. Let J be the multi-index $(j_1, \dots, j_k)$ and let $\mathrm{d}x_J = \mathrm{d}x_{j_1} \wedge \cdots \wedge \mathrm{d}x_{j_k}$. Show that $\mathrm{d}x_J = 0$ if $j_r = j_s$ for some $r \neq s$ and show that if the numbers $j_1, \dots, j_k$ are all distinct, then

$$\mathrm{d}x_J = (-1)^\sigma \, \mathrm{d}x_I$$

where $I = (i_1, \dots, i_k)$ is the strictly increasing rearrangement of $(j_1, \dots, j_k)$ and σ is the permutation

$$(j_1, \dots, j_k) \mapsto (i_1, \dots, i_k)$$

Proof. Suppose first that $j_r = j_s$ for some $r \neq s$. We wish to prove that $\mathrm{d}x_J = 0$, where "0" denotes the zero element of $\Omega^k(\mathbb{R}^n)$. To do this, we need to show that $\mathrm{d}x_J$ sends every $p \in \mathbb{R}^n$ to the zero k-tensor in $\Lambda^k(T_p^*\mathbb{R}^n) \cong \mathcal{A}^k(T_p\mathbb{R}^n)$.

Let $p \in \mathbb{R}^n$ be arbitrary. Then

$$\begin{aligned}
\mathrm{d}x_J\,(p) &= (\mathrm{d}x_{j_1})_p \wedge \cdots \wedge (\mathrm{d}x_{j_k})_p \\
&= (\mathrm{d}x_{\tau_{r,s}(j_1)})_p \wedge \cdots \wedge (\mathrm{d}x_{\tau_{r,s}(j_k)})_p \\
&= (-1)^{\tau_{r,s}}(\mathrm{d}x_{j_1})_p \wedge \cdots \wedge (\mathrm{d}x_{j_k})_p && \text{Claim 1.6.8} \\
&= -(\mathrm{d}x_{j_1})_p \wedge \cdots \wedge (\mathrm{d}x_{j_k})_p \\
&= -\,\mathrm{d}x_J\,(p) \\
2\,\mathrm{d}x_J\,(p) &= 0 \\
\mathrm{d}x_J\,(p) &= 0
\end{aligned}$$

as desired.

Now suppose that the numbers $j_1, \ldots, j_k$ are all distinct. Then like before, we need to show that $\mathrm{d}x_J$ and $(-1)^\sigma\,\mathrm{d}x_I$ send every $p \in \mathbb{R}^n$ to the same k-tensor in $\Lambda^k(T_p^*\mathbb{R}^n)$.

Let $p \in \mathbb{R}^n$ be arbitrary. Then

$$\begin{aligned}
\mathrm{d}x_J\,(p) &= (\mathrm{d}x_{j_1})_p \wedge \cdots \wedge (\mathrm{d}x_{j_k})_p \\
&= (\mathrm{d}x_{\sigma(j_1)})_p \wedge \cdots \wedge (\mathrm{d}x_{\sigma(j_k)})_p \\
&= (-1)^\sigma(\mathrm{d}x_{i_1})_p \wedge \cdots \wedge (\mathrm{d}x_{i_k})_p && \text{Claim 1.6.8} \\
&= (-1)^\sigma\,\mathrm{d}x_I\,(p)
\end{aligned}$$

as desired. $\square$