Chapter 2

Differential Forms

2.1 Notes

4/18:
- Office Hours on Wednesday, 4:00-5:00 PM.
- Plan:
  - An impressionistic overview of what (differential) forms do/are.
  - Tangent spaces.
  - Vector fields/integral curves.
  - 1-forms; a warm-up to k-forms.
- Impressionistic overview of the rest of Guillemin and Haine (2018).
  - An open subset $U \subset \mathbb{R}^n$; $n = 2$ and $n = 3$ are nice.
  - Sometimes, we'll have some functions $F : U \to V$; this is where pullbacks come into play.
  - At every point $p \in U$, we'll define a vector space (the tangent space $T_p\mathbb{R}^n$). Associated to that vector space you get our whole slew of associated spaces (the dual space $T_p^*\mathbb{R}^n$, and all of the higher exterior powers $\Lambda^k(T_p^*\mathbb{R}^n)$).
  - We let $\omega \in \Omega^k(U)$ be a k-form in the space of k-forms.
  - ω assigns (smoothly) to every point $p \in U$ an element of $\Lambda^k(T_p^*\mathbb{R}^n)$.
  - Question: What really is a k-form?
    - Answer: Something that can be integrated on k-dimensional subsets.
    - If $k = 1$, i.e., $\omega \in \Omega^1(U)$, then U can be integrated over curves.
  - If we take $k = 0$, then $\Omega^0(U) = C^\infty(U)$, i.e., the set of all smooth functions $f : U \to \mathbb{R}$.
    - Guillemin and Haine (2018) doesn't, but Klug will and we should distinguish between functions $F : U \to V$ and $f : U \to \mathbb{R}$.
  - We will soon construct a map $d : \Omega^0(U) \to \Omega^1(U)$ (the **exterior derivative**) that is rather like the gradient but not quite.
    - d is linear.
    - Maps from vector spaces are heretofore assumed to be linear unless stated otherwise.
  - The 1-forms in $\operatorname{im}(d)$ are special: $\int_\gamma \mathrm{d}f = f(\gamma(b)) - f(\gamma(a))$ only depends on the endpoints of $\gamma : [a, b] \to U$! The integral is *path-independent*.
  - A generalization of this fact is that instead of integrating along the surface M, we can integrate along the boundary curve:
    $$\int_M \mathrm{d}\omega = \int_{\partial M} \omega$$
    This is **Stokes' theorem**.

        - M is a k-dimensional subset of $U \subset \mathbb{R}^n$.
    - Note that we have all manner of functions d that we could differentiate between (because they are functions) but nobody does.

$$0 \to \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(U) \xrightarrow{d} 0$$

    - Theorem: $d^2 = d \circ d = 0$.
        - Corollary: $\operatorname{im}(d^{n-1}) \subset \ker(d^n)$.
    - We'll define $H^k_{dR}(U) = \ker(d)/\operatorname{im}(d)$.
        - These will be finite dimensional, even though all the individual vector spaces will be infinite dimensional.
        - These will tell us about the shape of U; basically, if all of these equal zero, U is simply connected. If some are nonzero, U has some holes.
    - For small values of n and k, this d will have some nice geometric interpretations (div, grad, curl, n'at).
    - We'll have additional operations on forms such as the wedge product.
- **Tangent space** (of p): The following set. *Denoted by* $\boldsymbol{T_p\mathbb{R}^n}$. *Given by*

$$T_p\mathbb{R}^n = \{(p,v) : v \in \mathbb{R}^n\}$$

    - This is naturally a vector space with addition and scalar multiplication defined as follows.

$$(p,v_1) + (p,v_2) = (p, v_1 + v_2) \qquad\qquad \lambda(p,v) = (p, \lambda v)$$

    - The point is that

$$T_p\mathbb{R}^n \neq T_q\mathbb{R}^n$$

      for $p \neq q$ even though the spaces are isomorphic.
    - Aside: $F : U \to V$ differentiable and $p \in U$ induce a map $\mathrm{d}F_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ called the "derivative at p."
        - We will see that the matrix of this map is the Jacobian.
    - Chain rule: If $U \xrightarrow{F} V \xrightarrow{G} W$, then

$$\mathrm{d}(G \circ F)_p = \mathrm{d}G_{F(p)} \circ \mathrm{d}F_p$$

- This is round 1 of our discussion on tangent spaces.
- Round 2, later on, will be submanifolds such as T_pM: The tangent space to a point p of a manifold M.
- **Vector field** (on U): A function that assigns to each $p \in U$ an element of $T_p\mathbb{R}^n$.
    - A constant vector field would be $p \mapsto (p,v)$, visualized as a field of vectors at every p all pointing the same direction. For example, we could take $v = (1,1)$. *picture*
    - Special case: $v = e_1, e_2, \dots, e_n$. Here we use the notation $e_i = \mathrm{d}/\mathrm{d}x_i$.
    - Example: $n = 2$, $U = \mathbb{R}^2 \setminus \{(0,0)\}$. We could take a vector field that spins us around in circles.
    - Notice that for all p, $\mathrm{d}/\mathrm{d}x_1\,|_p, \dots, \mathrm{d}/\mathrm{d}x_n\,|_p \in T_p\mathbb{R}^n$ are a basis.
        - Thus, any vector field v on U can be written uniquely as

$$v = f_1 \frac{\mathrm{d}}{\mathrm{d}x_1} + \cdots + f_n \frac{\mathrm{d}}{\mathrm{d}x_n}$$

          where the $f_1, \dots, f_n$ are functions $f_i : U \to \mathbb{R}$.

4/20:

- Plan:
  - Vector fields and their integral curves.
  - Lie derivatives.
  - 1-forms and k-forms.
  - $\Omega^0(U) \xrightarrow{d} \Omega^1(U)$.
- Notation.
  - $U \subset \mathbb{R}^n$.
  - v denotes a vector field on U.
    - Note that the set of all vector fields on U constitute the vector space ??.
  - $v_p \in T_p\mathbb{R}^n$.
  - $\omega_p \in \Lambda^k(T_p^*\mathbb{R}^n)$.
  - $\mathrm{d}/\mathrm{d}x_i \mid_p = (p, e_i) \in T_p\mathbb{R}^n$.
- Recall that any vector field v on U can be written uniquely as

$$v = g_1 \frac{\mathrm{d}}{\mathrm{d}x_1} + \cdots + g_n \frac{\mathrm{d}}{\mathrm{d}x_n}$$

  where the $g_i : U \to \mathbb{R}$.
- **Smooth** (vector field): A vector field v for which all g_i are smooth.
- From now on, we assume unless stated otherwise that all vector fields are smooth.
- **Lie derivative** (of f wrt. v): The function $L_v f : U \to \mathbb{R}$ defined by $p \mapsto D_{v_p}(f)(p)$, where v is a vector field on U and $f : U \to \mathbb{R}$ (always smooth).
  - Recall that $D_{v_p}(f)(p)$ denotes the directional derivative of f in the direction v_p at p.
  - As some examples, we have

$$L_{\mathrm{d}/\mathrm{d}x_i} f = \frac{\mathrm{d}f}{\mathrm{d}x_i} \qquad\qquad L_{(g_1 \frac{\mathrm{d}}{\mathrm{d}x_1} + \cdots + g_n \frac{\mathrm{d}}{\mathrm{d}x_n})} f = g_1 \frac{\mathrm{d}f}{\mathrm{d}x_1} + \cdots + g_n \frac{\mathrm{d}f}{\mathrm{d}x_n}$$

- Property.
  1. Product rule: $L_v(f_1 f_2) = (L_v f_1) f_2 + f_1 (L_v f_2)$.
- Later: Geometric meaning to the expression $L_v f = 0$.
  - Satisfied iff f is constant on the integral curves of v. As if f "flows along" the vector field.
- We define $T_p^*\mathbb{R}^n = (T_p\mathbb{R}^n)^*$.
- 1-forms:
  - A (differential) 1-form on $U \subset \mathbb{R}^n$ is a function $\omega : p \mapsto \omega_p \in T_p^*\mathbb{R}^n$.
  - A "co-vector field"
- Notation: $\mathrm{d}x_i$ is the 1-form that at p is $(p, e_i^*) \in T_p^*\mathbb{R}^n$.
- For example, if $U = \mathbb{R}^2$ and $\omega = \mathrm{d}x_1$, then we have the vector field of "unit vectors pointing to the right at each point."

- Note: Given any 1-form ω on U, we can write ω uniquely as

$$\omega = g_1 \, \mathrm{d}x_1 + \cdots + g_n \, \mathrm{d}x_n$$

  for some set of smooth $g_i : U \to \mathbb{R}$.

- Notation:
  - $\Omega^1(U)$ is the set of all smooth 1-forms.
  - Notice that $\Omega^1(U)$ is a vector space.
- Given $\omega \in \Omega^1(U)$ and a vector field v on U, we can define $\omega(v) : U \to \mathbb{R}$ by $p \mapsto \omega_p(v_p)$.
- If $U = \mathbb{R}^2$, we have that

$$\mathrm{d}x\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) = 1 \qquad\qquad \mathrm{d}x\left(\frac{\mathrm{d}}{\mathrm{d}y}\right) = 0$$

- Note that $\mathrm{d}x\,, \mathrm{d}y$ are not a basis for $\Omega^1(U)$ since the latter is infinite dimensional.
- Exterior derivative for 0/1 forms.
  - Let $d : \Omega^0(U) \to \Omega^1(U)$ take $f : U \to \mathbb{R}$ to $\frac{\partial f}{\partial x_1} \, \mathrm{d}x_1 + \cdots + \frac{\partial f}{\partial x_n} \, \mathrm{d}x_n$.
  - This represents the gradient as a 1-form.
- Check:
  1. Linear.
  2. $\mathrm{d}x_i = \mathrm{d}(x_i)$, where $x_i : \mathbb{R}^n \to \mathbb{R}$ is the i^{th} coordinate function.

4/22:

- Plan:
  - Clear up a bit of notational confusion.
  - Discuss integral curves of vectors fields.
  - k-forms.
  - Exterior derivatives $d : \Omega^k(U) \to \Omega^{k+1}(U)$ (definition and properties).
- Notation:
  - $F : \mathbb{R}^n \to \mathbb{R}^m$ smooth.
  - We are used to denoting derivatives by big D: $DF_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$ where bases of the two spaces are $e_1, \dots, e_n$ and $e_1, \dots, e_m$ has matrix equal to the Jacobian:

$$[DF_p] = \left[\tfrac{\mathrm{d}F_i}{\mathrm{d}x_j}(p)\right]$$

  - The book often uses small d: $f : U \to \mathbb{R}$ has $df_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}$, where the latter set is isomorphic to $\mathbb{R}$.
  - $df : p \to df_p \in T_p^*\mathbb{R}^n$.
  - Klug said

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \, \mathrm{d}x_i$$

  - Homework 1 defined $\mathrm{d}f = df$?
  - Sometimes three perspectives help you keep this all straight:
    1. Abstract nonsense: The definition of the derivative.

2. How do I compute it: Apply the formula.
3. What is it: E.g., magnitude of the directional derivative in the direction of steepest ascent.

- For the homework,
  - Let ω be a 1-form in $\Omega^1(U)$.
  - Let $\gamma : [a,b] \to U$ be a curve in U.
  - Then $\mathrm{d}\gamma_p = \gamma'_p : T_p\mathbb{R} \to T_{\gamma(p)}\mathbb{R}^n$ is a function that takes in points of the curve and spits out tangent vectors.
  - Integrating swallows 1-forms and spits out numbers.
  $$\int_\gamma \omega = \int_a^b \omega(\gamma'(t))\,\mathrm{d}t$$
  - Problem: If $\omega = \mathrm{d}f$, then
  $$\int_\gamma \omega = f(\gamma(b)) - f(\gamma(a))$$
  regardless of the path.
  - Question: Given a 1-form ω, is $\omega = \mathrm{d}f$ for some f?
  - Homework: Explicit U, ω, closed γ such that $\int_\gamma \omega \neq 0$ implies that $\omega \neq \mathrm{d}f$. This motivates and leads into the de Rham cohomology.
- Aside: It won't hurt (for now) to think of 1-forms as vector fields.
- Integral curves: Let $U \subset \mathbb{R}^n$, v be a (smooth) vector field on U. A curve $\gamma : (a,b) \to U$ is an **integral curve** for v if $\gamma'(t) = v_{\gamma(t)}$.
- Examples:
  - If $U = \mathbb{R}^2$ and $\gamma = \mathrm{d}/\mathrm{d}x$, then the integral curve is the line from left to right traveling at unit speed. The curve has to always have as it's tangent vector the unit vector pointing right (which is the vector at every point in the vector field).
  - Vector fields flow everything around. An integral curve is the trajectory of a particle subjected to the vector field as a force field.
- Main points:
  1. These integral curves always exist (locally) and often exist globally (cases in which they do are called **complete vector fields**).
  2. They are unique given a starting point $p \in U$.
- An incomplete vector field is one such as the "all roads lead to Rome" vector field where everything always points inward. This is because integral curves cannot be defined for all "time" (real numbers, positive and negative).
- The proofs are in the book; they require an existence/uniqueness result for ODEs and the implicit function theorem.
- Aside: $f : U \to \mathbb{R}$, v a vector field, implies that $L_v f = 0$ means that f is constant along all the integral curves of v. This also means that f is integral for v.
- **Pullback** (of 1-forms): If $F : U \to V$, $d : \Omega^0(U) \to \Omega^1(U)$, and $d : \Omega^0(V) \to \Omega^1(V)$, then we get an induced map $F^* : \Omega^0(V) \to \Omega^0(U)$. If $f : V \to \mathbb{R}$, then $f \circ F$ is involved.
  - We're basically saying that if we have $\mathrm{Hom}(A,X)$ (the set of all functions from A to X) and $\mathrm{Hom}(B,X)$, then if we have $F : A \to B$, we get an induced map $F^* : \mathrm{Hom}(B,X) \to \mathrm{Hom}(A,X)$ that is precomposed with F.

4/27:
- Announcements.
  - No class this Friday, next Monday.
  - Midterm next Friday.
    - Up through Chapter 2.
    - The exam will likely be computationally heavy.
    - Compute d, pullbacks, interior products, Lie derivatives, etc.
    - Emphasis on Chapter 2 as opposed to Chapter 1 even though it all builds on itself.
    - He'll probably cook up a few problems too.
  - There is a recorded lecture for us.
    - On Chapter 3 content.
    - We'll cover Chapter 3 in kind of an impressionistic way as it is.
  - There are also some notes on the physics stuff.
- Vector calculus operations.
  - In one dimension, you have functions, and you take derivatives.
    - The derivative operation does essentially map $\Omega^0 \to \Omega^1$ or $C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$.
  - In two dimensions, ...
    - $\mathrm{d}^2 = 0$ reflects the fact that gradient vector fields are curl-free.
  - If you want to understand the 2D-curl business...
    - $\mathrm{curl}(v) : \mathbb{R}^2 \to \mathbb{R}$ is intuitively about balls spinning around in a vector field.
    - There's also a nice formula to compute it.
    - And then there's a connection with $\mathrm{d} : \Omega^1 \to \Omega^2$.
  - In 3D, you can take top-dimensional forms (which are just functions) and bottom-dimensional forms (which are by definition functions) and you can work out an identification between them.
  - Note that $\mathrm{curl} : \mathfrak{X}(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$, where $\mathfrak{X}(\mathbb{R}^2)$ is the space of vector fields.
- The musical operator $\sharp$ identifies forms with vector fields, i.e., $\sharp : \Omega^1 \to \mathfrak{X}(\mathbb{R}^2)$.
- Properties of exterior derivatives $\mathrm{d} : \Omega^k(U) \to \Omega^{k+1}(U)$.
  1. $\mathrm{d}(\omega_1 + \omega_2) = \mathrm{d}\omega_1 + \mathrm{d}\omega_2$ and $\mathrm{d}(\lambda\omega) = \lambda\,\mathrm{d}\omega$.
  2. Product rule $\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge \mathrm{d}\omega_2$.
     - Special case $k = \ell = 0$. Then
     $$\mathrm{d}(fg) = g\,\mathrm{d}f + f\,\mathrm{d}g$$
     which is the usual product rule for gradient.
     - Claim:
     $$\mathrm{d}\left(\sum_I f_I\,\mathrm{d}x_I\right) = \sum_I \mathrm{d}f_I \wedge \mathrm{d}x_I$$
       - Let $\omega_1 \in \Omega^k$ and $\omega_2 \in \Omega^\ell$ be defined by
       $$\omega_1 = \sum_I f_I\,\mathrm{d}x_I \qquad\qquad \omega_2 = \sum_J g_J\,\mathrm{d}x_J$$
       where we're summing over all I such that $|I| = k$ and all J such that $|J| = \ell$. Then
       $$\omega_1 \wedge \omega_2 = \sum_{I,J} f_I g_J\,\mathrm{d}x_I \wedge \mathrm{d}x_J\,\mathrm{d}(\omega_1 \wedge \omega_2) \qquad = \sum_{I,J} \mathrm{d}(f_I g_J) \wedge \mathrm{d}x_I \wedge \mathrm{d}x_J$$

- Note that
$$\mathrm{d}(f_I g_J) = g_J\,\mathrm{d}f_I + f_I\,\mathrm{d}g_J$$
and
$$\mathrm{d}g_J \wedge \mathrm{d}x_I = (-1)^k\,\mathrm{d}x_I \wedge \mathrm{d}g_J$$
- These identities allow us to take the previous equation to
$$\begin{aligned}
\mathrm{d}(\omega_1 \wedge \omega_2) &= \sum_{I,J} g_J\,\mathrm{d}f_I \wedge \mathrm{d}x_I \wedge \mathrm{d}x_J + (-1)^k f_I\,\mathrm{d}x_I \wedge \mathrm{d}g_J \wedge \mathrm{d}x_J \\
&= \sum_{I,J} (\mathrm{d}f_I \wedge \mathrm{d}x_I) \wedge (g_J\,\mathrm{d}x_J) + \sum_{I,J} (f_I\,\mathrm{d}x_I) \wedge (ddg_J \wedge \mathrm{d}x_J) \\
&= \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^k \omega_1\,\mathrm{d}\omega_2
\end{aligned}$$

3. $\mathrm{d}^2 = 0$.
   - Let $\omega = \sum_I f_I\,\mathrm{d}x_I$.
   - Then
$$\begin{aligned}
\mathrm{d}^2(\omega) &= \mathrm{d}(\mathrm{d}\omega) \\
&= \mathrm{d}\left(\sum_I \mathrm{d}f_I \wedge \mathrm{d}x_I\right) \\
&= \sum_I \mathrm{d}(\mathrm{d}f_I \wedge \mathrm{d}x_I) && \text{Property 1} \\
&= \sum_I \mathrm{d}(\mathrm{d}f_I) \wedge \mathrm{d}x_I && \text{Property 2}
\end{aligned}$$
   so it suffices to just show that $\mathrm{d}^2 f = 0$ for all $f \in \Omega^0$.
   - We know that $\mathrm{d}f = \sum_{i=1}^n \partial f/\partial x_i\,\mathrm{d}x_i$. Thus,
$$\begin{aligned}
\mathrm{d}(\mathrm{d}f) &= \sum_i \mathrm{d}\left(\frac{\partial f}{\partial x_i}\right) \wedge \mathrm{d}x_i \\
&= \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i}\,\mathrm{d}x_j \wedge \mathrm{d}x_i \\
&= 0
\end{aligned}$$
   - The last equality holds because of commuting partial derivatives for smooth f, and the fact that changing order introduces a negative sign by some property.

- In fact, if we fix $\mathrm{d}^0 : \Omega^0(U) \to \Omega^1(U)$ to be the "gradient," then these properties characterize the function d on its domain and codomain. In particular, d is the unique function on its domain and codomain that satisfies these properties.
  - We define it by
$$\mathrm{d}\left(\sum_I f_I\,\mathrm{d}x_I\right) = \sum_I \mathrm{d}f_I \wedge \mathrm{d}x_I$$
  - The above properties characterize it axiomatically.
  - We can prove this uniqueness theorem.
- **Closed** (form): A form $\omega \in \Omega^k(U)$ such that $\mathrm{d}\omega = 0$.
- **Exact** (form): A form $\omega \in \Omega^k(U)$ such that $\omega = \mathrm{d}\eta$ for some $\eta \in \Omega^{k-1}(U)$.
- $\mathrm{d}^2 = 0$ implies closed and exact implies closed.
- **Poincaré lemma**: Locally closed forms are exact.