Chapter 1

Multilinear Algebra

1.1 Notes

3/28: • Motivation for the course and an overview of Guillemin and Haine (2018).

3/30: • Plan:
  - More (multi)linear algebra.
- Dual spaces.
- Let V be an n-dimensional real vector space.
- **Hom $(V, \mathbb{R})$**: The set of all homomorphisms (i.e., linear maps) from V to $\mathbb{R}$. *Also known as* $\boldsymbol{V^*}$.
- **Dual basis** (for V^*): The set of linear transformations from V to $\mathbb{R}$ defined by

$$\mathbf{e}_j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

  where $\mathbf{e}_1, \dots, \mathbf{e}_n$ is a basis of V. *Denoted by* $\mathbf{e}_1^*, \dots, \mathbf{e}_n^*$.
- Check: $\mathbf{e}_1^*, \dots, \mathbf{e}_n^*$ are a basis for V^*.
  - Are they linearly independent? Let $c_1\mathbf{e}_1^* + \cdots + c_n\mathbf{e}_n^* = 0 \in \text{Hom}(V, \mathbb{R})$. Then

$$c_i = (c_1\mathbf{e}_1^* + \cdots + c_n\mathbf{e}_n^*)(\mathbf{e}_i) = 0 \in \mathbb{R}$$

    as desired.
  - Span? Let $\varphi \in \text{Hom}(V, \mathbb{R})$. Then we can verify that

$$\varphi(\mathbf{e}_1)\mathbf{e}_1^* + \cdots + \varphi(\mathbf{e}_n)\mathbf{e}_n^* = \varphi$$

    - We prove this by verifying the previous statement on the basis of V (if two linear transformations have the same action on the basis of a vector space, they are equal).
- With a choice of basis for V, we obtain an isomorphism $\varepsilon : V \to V^*$ with the mapping $\mathbf{e}_i \mapsto \mathbf{e}_i^*$ for all i.
- The dual space is known as such because $(V^*)^* \cong V$, where $\cong$ is **canonical** (no choice of basis is needed).
- One more property of dual spaces: **functoriality**.

  - Given a linear transformation $A : V \to W$, we know that $A^* : W^* \to V^*$ where A^* is the transpose of A. In particular, if $\varphi \in W^*$, then $\varphi \circ A : V \to \mathbb{R}$.
  - Claim: A^* is linear.
- **Functoriality**: If $A : V \to W$ and $B : W \to U$, then $B^* : U^* \to W^*$ and $A^* : W^* \to V^*$. The functoriality statement is that $(B \circ A)^* = A^* \circ B^*$.
- A^* is the **pullback** (or transpose) of A.
- Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be a basis for V and $\mathbf{w}_1, \dots, \mathbf{w}_m$ be a basis for W. Then $[A]_{\mathbf{v}_1,\dots,\mathbf{v}_n}^{\mathbf{w}_1,\dots,\mathbf{w}_m} = A$ is the matrix of the linear transformation A with respect to these bases. Then if $\mathbf{v}_1^*, \dots, \mathbf{v}_n^*$ and $\mathbf{w}_1^*, \dots, \mathbf{w}_m^*$ are the corresponding dual bases, then $[A^*]_{\mathbf{v}_1^*,\dots,\mathbf{v}_n^*}^{\mathbf{w}_1^*,\dots,\mathbf{w}_m^*} = A^T$. We can and should verify this for ourselves.
- This is over the real numbers, so A^* is just the transpose because there are no complex numbers of which to take the conjugate!
- A generalization: Tensors.
- ***k*-tensor**: A **multilinear** map
$$T : \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}$$
- **Multilinear** (map T): A function T such that
$$T(\mathbf{v}_1, \dots, \mathbf{v}_i^1 + \mathbf{v}_i^2, \dots, \mathbf{v}_k) = T(\mathbf{v}_1, \dots, \mathbf{v}_i^1, \dots, \mathbf{v}_k) + T(\mathbf{v}_1, \dots, \mathbf{v}_i^2, \dots, \mathbf{v}_k)$$
$$T(\mathbf{v}_1, \dots, \lambda\mathbf{v}_i, \dots, \mathbf{v}_k) = \lambda T(\mathbf{v}_1, \dots, \mathbf{v}_i, \dots, \mathbf{v}_k)$$
for all $(\mathbf{v}_1, \dots, \mathbf{v}_k) \in V^k$.
- The determinant is an n-tensor!
- 1-tensors are just covectors.
- $\boldsymbol{L^k(V)}$: The vector space of all k-tensors on V.
- Calculating $\dim L^k(V)$. (Answer not given in this class.)
- Let $A : V \to W$. Then $A^* : L^k(W) \to L^k(V)$.
  - Check $(A \circ B)^* = B^* \circ A^*$.
- **multi-index of $\boldsymbol{n}$ of length $\boldsymbol{k}$**: A k-tuple $(i_1, \dots, i_k)$ where each $i_j \in \mathbb{N}$ satisfies $1 \leq i_j \leq n$ $(j = 1, \dots, k)$. *Denoted by* $\boldsymbol{I}$.
- Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be a basis for V.
- **Tensor product** (of $T_1 \in L^k(V)$, $T_2 \in L^l(V)$): The function from V^{k+l} to $\mathbb{R}$ defined by
$$(\mathbf{v}_1, \dots, \mathbf{v}_{k+l}) \mapsto T_1(\mathbf{v}_1, \dots, \mathbf{v}_k) T_2(\mathbf{v}_{k+1}, \dots, \mathbf{v}_{k+l})$$
Denoted by $\boldsymbol{T_1 \otimes T_2}$.
- Claims:
  1. $T_1 \otimes T_2 \in L^{k+l}(V)$.
  2. $A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$.
- $\mathbf{e}_{\boldsymbol{I}}^*$: The function $\mathbf{e}_{i_1}^* \otimes \cdots \otimes \mathbf{e}_{i_k}^*$, where $I = (i_1, \dots, i_k)$ is a multi-index of n of length k.
- Claim: Letting I range over all n^k multi-indices of n of length k, the $\mathbf{e}_I^*$ are a basis for $L^k(V)$.
- If $V = \mathbb{R}$, then $V = \mathbb{R}\mathbf{e}_1$. If $V = \mathbb{R}^2$, then $V = \mathbb{R}\mathbf{e}_1 \oplus \mathbb{R}\mathbf{e}_2$.
- We know that $L^1(V) = V^* = R\mathbf{e}_1^*$. Thus, $\mathbf{e}_1^* \otimes \mathbf{e}_2^* : V \times V \to \mathbb{R}$. Thus, for example,
$$(\mathbf{e}_1^* \otimes \mathbf{e}_2^*)((1,2),(3,4)) = \mathbf{e}_1^*(1,2) \cdot \mathbf{e}_2^*(3,4) = 1 \cdot 4 = 4$$