MATH 20510 (Analysis in $\mathbb{R}^n$ III – Accelerated) Notes

Steven Labalme

March 31, 2022

Contents

Chapter 1

Multilinear Algebra

1.1 Notes

3/28: • Motivation for the course and an overview of Guillemin and Haine (2018).

3/30: • Plan:

  - More (multi)linear algebra.

- Dual spaces.
- Let V be an n-dimensional real vector space.
- **Hom** $(\boldsymbol{V},\mathbb{R})$: The set of all homomorphisms (i.e., linear maps) from V to $\mathbb{R}$. *Also known as* $\boldsymbol{V^*}$.
- **Dual basis** (for V^*): The set of linear transformations from V to $\mathbb{R}$ defined by

$$e_j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

  where $e_1, \dots, e_n$ is a basis of V. *Denoted by* $e_1^*, \dots, e_n^*$.

- Check: $e_1^*, \dots, e_n^*$ are a basis for V^*.
  - Are they linearly independent? Let $c_1e_1^* + \cdots + c_ne_n^* = 0 \in \operatorname{Hom}(V, \mathbb{R})$. Then

$$c_i = (c_1e_1^* + \cdots + c_ne_n^*)(e_i) = 0 \in \mathbb{R}$$

    as desired.
  - Span? Let $\varphi \in \operatorname{Hom}(V, \mathbb{R})$. Then we can verify that

$$\varphi(e_1)e_1^* + \cdots + \varphi(e_n)e_n^* = \varphi$$

    - We prove this by verifying the previous statement on the basis of V (if two linear transformations have the same action on the basis of a vector space, they are equal).
- With a choice of basis for V, we obtain an isomorphism $\varepsilon : V \to V^*$ with the mapping $e_i \mapsto e_i^*$ for all i.
- The dual space is known as such because $(V^*)^* \cong V$, where $\cong$ is **canonical** (no choice of basis is needed).
- One more property of dual spaces: **functoriality**.

  - Given a linear transformation $A : V \to W$, we know that $A^* : W^* \to V^*$ where A^* is the transpose of A. In particular, if $\varphi \in W^*$, then $\varphi \circ A : V \to \mathbb{R}$.
  - Claim: A^* is linear.
- **Functoriality**: If $A : V \to W$ and $B : W \to U$, then $B^* : U^* \to W^*$ and $A^* : W^* \to V^*$. The functoriality statement is that $(B \circ A)^* = A^* \circ B^*$.
- A^* is the **pullback** (or transpose) of A.
- Let $v_1, \dots, v_n$ be a basis for V and $w_1, \dots, w_m$ be a basis for W. Then $[A]_{v_1,\dots,v_n}^{w_1,\dots,w_m} = A$ is the matrix of the linear transformation A with respect to these bases. Then if $v_1^*, \dots, v_n^*$ and $w_1^*, \dots, w_m^*$ are the corresponding dual bases, then $[A^*]_{v_1^*,\dots,v_n^*}^{w_1^*,\dots,w_m^*} = A^T$. We can and should verify this for ourselves.
- This is over the real numbers, so A^* is just the transpose because there are no complex numbers of which to take the conjugate!
- A generalization: Tensors.
- ***k*-tensor**: A **multilinear** map

$$T : \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}$$

- **Multilinear** (map T): A function T such that

$$\begin{aligned} T(v_1, \dots, v_i^1 + v_i^2, \dots, v_k) &= T(v_1, \dots, v_i^1, \dots, v_k) + T(v_1, \dots, v_i^2, \dots, v_k) \\ T(v_1, \dots, \lambda v_i, \dots, v_k) &= \lambda T(v_1, \dots, v_i, \dots, v_k) \end{aligned}$$

  for all $(v_1, \dots, v_k) \in V^k$.
- The determinant is an n-tensor!
- 1-tensors are just covectors.
- $\boldsymbol{L^k(V)}$: The vector space of all k-tensors on V.
- Calculating $\dim L^k(V)$. (Answer not given in this class.)
- Let $A : V \to W$. Then $A^* : L^k(W) \to L^k(V)$.
  - Check $(A \circ B)^* = B^* \circ A^*$.
- **Multi-index of $\boldsymbol{n}$ of length $\boldsymbol{k}$**: A k-tuple $(i_1, \dots, i_k)$ where each $i_j \in \mathbb{N}$ satisfies $1 \leq i_j \leq n$ $(j = 1, \dots, k)$. *Denoted by* $\boldsymbol{I}$.
- Let $e_1, \dots, e_n$ be a basis for V.
- **Tensor product** (of $T_1 \in L^k(V)$, $T_2 \in L^l(V)$): The function from V^{k+l} to $\mathbb{R}$ defined by

$$(v_1, \dots, v_{k+l}) \mapsto T_1(v_1, \dots, v_k) T_2(v_{k+1}, \dots, v_{k+l})$$

  Denoted by $\boldsymbol{T_1 \otimes T_2}$.
- Claims:
  1. $T_1 \otimes T_2 \in L^{k+l}(V)$.
  2. $A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$.
- $\boldsymbol{e_I^*}$: The function $e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*$, where $I = (i_1, \dots, i_k)$ is a multi-index of n of length k.
- Claim: Letting I range over all n^k multi-indices of n of length k, the e_I^* are a basis for $L^k(V)$.
- If $V = \mathbb{R}$, then $V = \mathbb{R}e_1$. If $V = \mathbb{R}^2$, then $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$.
- We know that $L^1(V) = V^* = Re_1^*$. Thus, $e_1^* \otimes e_2^* : V \times V \to \mathbb{R}$. Thus, for example,

$$(e_1^* \otimes e_2^*)((1,2),(3,4)) = e_1^*(1,2) \cdot e_2^*(3,4) = 1 \cdot 4 = 4$$

1.2 Chapter 1: Multilinear Algebar

From Guillemin and Haine (2018).

3/31:

- Guillemin and Haine (2018) defines real vector spaces, the operations on them, their basic properties, and the zero vector.
- **Linearly independent** (vectors $v_1, \dots, v_k$): A finite set of vectors $v_1, \dots, v_k \in V$ such that the map from $\mathbb{R}^k$ to V defined by $(c_1, \dots, c_k) \mapsto c_1v_1 + \cdots + c_kv_k$ is injective.
- **Spanning** (vectors $v_1, \dots, v_k$): We require that the above map is surjective.
- Guillemin and Haine (2018) defines basis, finite-dimensional vector space, dimension, subspace, linear map, and kernel.
- **Image** (of $A : V \to W$): The range space of A, a subspace of W. *Also known as* $\mathbf{im}\,(\boldsymbol{A})$.
- Guillemin and Haine (2018) defines the matrix of a linear map.
- **Inner product** (on V): A map $B : V \times V \to \mathbb{R}$ with the following three properties.
  - *Bilinearity*: For vectors $v, v_1, v_2, w \in V$ and $\lambda \in \mathbb{R}$, we have
    $$B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w)$$
    and
    $$B(\lambda v, w) = \lambda B(v, w)$$
  - *Symmetry*: For vectors $v, w \in V$, we have $B(v, w) = B(w, v)$.
  - *Positivity*: For every vector $v \in V$, we have $B(v, v) \geq 0$. Moreover, if $v \neq 0$, then $B(v, v) > 0$.
- $\boldsymbol{W}$**-coset**: A set of the form $\{v + w \mid w \in W\}$, where W is a subspace V and $v \in V$. *Denoted by* $\boldsymbol{v + W}$.
  - If $v_1 - v_2 \in W$, then $v_1 + W = v_2 + W$.
  - It follows that the distinct W-cosets decompose V into a disjoint collection of subsets of V.
- **Quotient space** (of V by W): The set of distinct W-cosets in V, along with the following definitions of vector addition and scalar multiplication.
  $$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W \qquad \lambda(v + W) = (\lambda v) + W$$
  Denoted by $\boldsymbol{V/W}$.
- **Quotient map**: The linear map $\pi : V \to V/W$ defined by
  $$\pi(v) = v + W$$
  - π is surjective.
  - Note that $\ker(\pi) = W$ since for all $w \in W$, $\pi(w) = w + W = 0 + W$, which is the zero vector in V/W.
- If V, W are finite dimensional, then
  $$\dim(V/W) = \dim(V) - \dim(W)$$
- Proposition 1.2.9: Let $A : V \to U$ be a linear map. If $W \subset \ker(A)$, then there exists a unique linear map $A^\sharp : V/W \to U$ with the property that $A = A^\sharp \circ \pi$, where $\pi : V \to V/W$ is the quotient map.
  - This proposition rephrases in terms of quotient spaces the fact that if $w \in W$, then $A(v+w) = Av$.

- **Dual space** (of V): The set of all linear functions $\ell : V \to \mathbb{R}$, along with the following definitions of vector addition and scalar multiplication.

$$(\ell_1 + \ell_2)(v) = \ell_1(v) + \ell_2(v) \qquad\qquad (\lambda\ell)(v) = \lambda \cdot \ell(v)$$

  Denoted by V^*.

- **Dual basis** (of $e_1, \dots, e_n$ a basis of V): The basis of V^* consisting of the n functions that take every $v = c_1e_1 + \cdots + c_ne_n$ to one of the c_i. *Denoted by* $\boldsymbol{e_1^*, \dots, e_n^*}$. *Given by*

$$e_i^*(v) = c_i$$

  for all $v \in V$.

- Claim 1.2.12: If V is an n-dimensional vector space with basis $e_1, \dots, e_n$, then $e_1^*, \dots, e_n^*$ is a basis of V^*.

  Proof. We will first prove that $e_1^*, \dots, e_n^*$ spans V^*. Let $\ell \in V^*$ be arbitrary. Set $\lambda_i = \ell(e_i)$ for all $i \in [n]$. Define $\ell' = \sum_{i=1}^n \lambda_i e_i^*$. Then

$$\ell'(e_j) = \sum_{i=1}^n \lambda_i e_i^*(e_j) = \lambda_j \cdot 1 = \ell(e_j)$$

  for all $j \in [n]$. Therefore, since ℓ, ℓ' take identical values on the basis of V, $\ell = \ell'$, as desired.

  We now prove that $e_1^*, \dots, e_n^*$ spans V^*. Let $\sum_{i=1}^n \lambda_i e_i^* = 0$. Then for all $j \in [n]$,

$$\lambda_j = (\sum_{i=1}^n \lambda_i e_i^*)(e_j) = 0$$

  as desired. □

- **Transpose** (of A): The map from W^* to V^* defined by $\ell \mapsto A^*\ell = \ell \circ A$ for all $\ell \in W^*$.

- Claim 1.2.15: If $e_1, \dots, e_n$ is a basis of V, $f_1, \dots, f_m$ is a basis of W, $e_1^*, \dots, e_n^*$ and $f_1^*, \dots, f_m^*$ are the corresponding dual bases, and $[a_{i,j}]$ is the $m \times n$ matrix of A with respect to $\{e_i\}, \{f_i\}$, then the linear map A^* is defined in terms of $\{f_i^*\}, \{e_i^*\}$ by the transpose matrix $(a_{j,i})$.

Bibliography

Guillemin, V., & Haine, P. J. (2018). *Differential forms* [https://math.mit.edu/classes/18.952/2018SP/files/18.952_book.pdf].