MATH 20510 (Analysis in $\mathbb{R}^n$ III – Accelerated) Notes

Steven Labalme

September 17, 2022

Weeks

List of Figures

Week 1

Tensors

1.1 Course Motivation

3/28: • Motivation for the course and an overview of Guillemin and Haine (2018).

1.2 Defining Tensors and Their Operations

3/30: • Plan:

  – More (multi)linear algebra.

• Let V be an n-dimensional real vector space.

• **Dual space** (of V): The set of all homomorphisms from V to $\mathbb{R}$. *Also known as* $\mathbf{Hom}(V,\mathbb{R})$, V^*.

  – A homomorphism of vector spaces (such as any $\varphi \in V^*$) is just a linear map or, specifically, a **linear functional**.

• **Linear functional**: A linear map from a vector field to its field of scalars (often $\mathbb{R}$ or $\mathbb{C}$).

• **Dual basis** (for V^*): The set of linear transformations from V to $\mathbb{R}$ defined by

$$e_j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

where $e_1, \dots, e_n$ is a basis of V. *Denoted by* $e_1^*, \dots, e_n^*$.

• Check: $e_1^*, \dots, e_n^*$ are a basis for V^*.

  – Are they linearly independent? Let $c_1 e_1^* + \cdots + c_n e_n^* = 0 \in \operatorname{Hom}(V,\mathbb{R})$. Then

$$c_i = (c_1 e_1^* + \cdots + c_n e_n^*)(e_i) = 0 \in \mathbb{R}$$

  as desired.

  – Span? Let $\varphi \in \operatorname{Hom}(V,\mathbb{R})$. Then we can verify that

$$\varphi(e_1)e_1^* + \cdots + \varphi(e_n)e_n^* = \varphi$$

    ■ We prove this by verifying the previous statement on the basis of V (if two linear transformations have the same action on the basis of a vector space, they are equal).

• With a choice of basis for V, we obtain an **isomorphism** $\varepsilon : V \to V^*$ defined by $e_i \mapsto e_i^*$ for all i.

- **Isomorphism** (between V, W): A bijective (equiv., invertible) linear map between two vector spaces V, W. *Denoted by* $\boldsymbol{V \cong W}$.
- The dual space is known as such because $(V^*)^* \cong V$, where the isomorphism is **canonical**.
  - The canonical isomorphism $I : V \to (V^*)^*$ is given by $v \mapsto I_v$, where $I_v : V^* \to \mathbb{R}$ is the linear functional defined by
    $$I_v(\varphi) = \varphi(v)$$
    for all $\varphi \in V^*$.
  - Some incomplete thoughts on the proof are commented out in the `.tex` file. See also this link.
- **Canonical** (map): A map between objects that arises naturally from the definition or construction of the objects.
- **Canonical** (map of vector spaces): A map between vector spaces such that no choice of bases is needed to describe it.
  - Equivalently, such a map as described on any choice of bases will be equal to the canonical map.
- **Pullback** (of A): The linear transformation from $W^* \to V^*$, where V, W are vector spaces and $A : V \to W$, defined as follows. *Also known as* **transpose**. *Denoted by* $\boldsymbol{A^*}$. *Given by*
  $$A^*(\varphi) = \varphi \circ A$$
  - This object is also known as the transpose because the matrix of A^* is the transpose of the matrix of A, provided V, W are real vector spaces.
  - Claim: A^* is linear.
- One more property of dual spaces: **functoriality**.
- **Functoriality**: If $A : V \to W$ and $B : W \to U$, then $B^* : U^* \to W^*$ and $A^* : W^* \to V^*$. The functoriality statement is that $(B \circ A)^* = A^* \circ B^*$.
- Let $v_1, \dots, v_n$ be a basis for V and $w_1, \dots, w_m$ be a basis for W. Then $[A]_{v_1,\dots,v_n}^{w_1,\dots,w_m} = A$ is the matrix of the linear transformation A with respect to these bases. Then if $v_1^*, \dots, v_n^*$ and $w_1^*, \dots, w_m^*$ are the corresponding dual bases, then $[A^*]_{v_1^*,\dots,v_n^*}^{w_1^*,\dots,w_m^*} = A^T$. We can and should verify this for ourselves.
- This is over the real numbers, so A^* is just the transpose because there are no complex numbers of which to take the conjugate!
- A generalization: Tensors.
- $\boldsymbol{k}$**-tensor**: A **multilinear** map
  $$T : \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}$$
- **Multilinear** (map T): A function T such that
  $$\begin{aligned} T(v_1, \dots, v_i^1 + v_i^2, \dots, v_k) &= T(v_1, \dots, v_i^1, \dots, v_k) + T(v_1, \dots, v_i^2, \dots, v_k) \\ T(v_1, \dots, \lambda v_i, \dots, v_k) &= \lambda T(v_1, \dots, v_i, \dots, v_k) \end{aligned}$$
  for all $(v_1, \dots, v_k) \in V^k$.
- The determinant is an n-tensor!
- 1-tensors are just covectors.
- $\boldsymbol{\mathcal{L}^k(V)}$: The vector space of all k-tensors on V.

- Calculating $\dim \mathcal{L}^k(V)$. (Answer not given in this class.)
- Let $A : V \to W$. Then $A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$.
  - Check $(A \circ B)^* = B^* \circ A^*$.
- **Multi-index of n of length k**: A k-tuple $(i_1, \dots, i_k)$ where each $i_j \in \mathbb{N}$ satisfies $1 \leq i_j \leq n$ $(j = 1, \dots, k)$. *Denoted by* $\boldsymbol{I}$.
- Let $e_1, \dots, e_n$ be a basis for V.
- **Tensor product** (of $T_1 \in \mathcal{L}^k(V)$, $T_2 \in L^l(V)$): The function from V^{k+l} to $\mathbb{R}$ defined by

$$(v_1, \dots, v_{k+l}) \mapsto T_1(v_1, \dots, v_k)T_2(v_{k+1}, \dots, v_{k+l})$$

  Denoted by $\boldsymbol{T_1 \otimes T_2}$.
- Claims:
  1. $T_1 \otimes T_2 \in L^{k+l}(V)$.
  2. $A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$.
- $\boldsymbol{e_I^*}$: The function $e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*$, where $I = (i_1, \dots, i_k)$ is a multi-index of n of length k.
- Claim: Letting I range over all n^k multi-indices of n of length k, the e_I^* are a basis for $\mathcal{L}^k(V)$.
- If $V = \mathbb{R}$, then $V = \mathrm{R}e_1$. If $V = \mathbb{R}^2$, then $V = \mathrm{R}e_1 \oplus \mathrm{R}e_2$.
- We know that $L^1(V) = V^* = Re_1^*$. Thus, $e_1^* \otimes e_2^* : V \times V \to \mathbb{R}$. Thus, for example,

$$(e_1^* \otimes e_2^*)((1, 2), (3, 4)) = e_1^*(1, 2) \cdot e_2^*(3, 4) = 1 \cdot 4 = 4$$

1.3 The Tensor Product and Permutations

4/1:
- Plan: More multilinear algebra.
  - Properties of the tensor product.
  - Sign of a permutation.
  - Alternating tensors (lead into differential forms down the road).
- Recall: V is an n-dimensional vector space over $\mathbb{R}$ with basis $e_1, \dots, e_n$. $\mathcal{L}^k(V)$ is the vector space of k-tensors on V. $\{e_I^* \mid I \text{ a multiindex of } n \text{ of length } k\}$ is a basis for $\mathcal{L}^k(V)$.
- For example, if $V = \mathbb{R}^2$ and $T \in \mathcal{L}^2(V)$, then

$$T(a_1e_1 + a_2e_2, b_1e_1 + b_2e_2) = a_1b_1T(e_1, e_1) + a_1b_2T(e_1, e_2) + a_2b_1T(e_2, e_1) + a_2b_2T(e_2, e_2)$$

  - A basis of $\mathcal{L}^2(V)$ is

$$\{e_1^* \otimes e_1^*, e_1^* \otimes e_2^*, e_2^* \otimes e_1^*, e_2^* \otimes e_2^*\}$$

  - Recall that some basic properties are

$$e_1^* \otimes e_2^*((1, 2), (3, 4)) = 1 \cdot 4 = 4 \qquad e_2^* \otimes e_1^*((1, 2), (3, 4)) = 2 \cdot 3 = 6$$

  - It follows by the initial decomposition of T that

$$T = a_1b_1e_1^* \otimes e_1^* + a_1b_2e_1^* \otimes e_2^* + a_2b_1e_2^* \otimes e_1^* + a_2b_2e_2^* \otimes e_2^*$$

- Important consequence: To know the action of T on an arbitrary pair of vectors, you need only know its action on the basis; a higher-dimensional generalization of the earlier property.

- Note that
$$e_I^*(e_J) = \delta_{IJ} = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$$
- Basic properties of the tensor product.
  1. *Right-distributive*: If $T_1 \in \mathcal{L}^k(V)$ and $T_2, T_3 \in \mathcal{L}^\ell(V)$, then
  $$T_1 \otimes (T_2 + T_3) = T_1 \otimes T_2 + T_1 \otimes T_3$$
  2. *Left-distributive*: If $T_1, T_2 \in \mathcal{L}^k(V)$ and $T_3 \in \mathcal{L}^\ell(V)$, then
  $$(T_1 + T_2) \otimes T_3 = T_1 \otimes T_3 + T_2 \otimes T_3$$
  3. *Associative*: If $T_1 \in \mathcal{L}^k(V)$, $T_2 \in \mathcal{L}^\ell(V)$, and $T_3 \in \mathcal{L}^m(V)$, then
  $$T_1 \otimes (T_2 \otimes T_3) = (T_1 \otimes T_2) \otimes T_2 = T_1 \otimes T_2 \otimes T_3$$
  4. *Scalar multiplication*: If $T_1 \in \mathcal{L}^k(V)$, $T_2 \in \mathcal{L}^\ell(V)$, and $\lambda \in \mathbb{R}$, then
  $$(\lambda T_1) \otimes T_2 = \lambda(T_1 \otimes T_2) = T_1 \otimes (\lambda T_2)$$
- Note that the tensor product is not commutative.
- Aside: Defining the sign of a permutation.
- $\boldsymbol{S_A}$: The set of all automorphisms of A (bijections from A to A), where A is a set.
- $\boldsymbol{S_n}$: The set $S_{[n]}$.
- Given $\sigma_1, \sigma_2 \in S_n$, $\sigma_1 \circ \sigma_2 \in S_n$.
  - Thus, S_n is a **group**.
- **Transposition**: A function in S_n such that
$$k \mapsto \begin{cases} j & k = i \\ i & k = j \\ k & k \neq i, j \end{cases}$$
  for some $i, j \in [n]$. *Denoted by* $\boldsymbol{\tau_{i,j}}$.
- Theorem: An element of S_n can be written as the product of transpositions (i.e., for all $\sigma \in S_n$, there exist $\tau_1, \dots, \tau_m \in S_n$ such that $\sigma = \tau_1 \circ \cdots \circ \tau_m$).
- **Sign** (of $\sigma \in S_n$): The number (mod 2) of transpositions whose product equals σ. *Denoted by* $\boldsymbol{(-1)^\sigma}$, $\mathbf{sign}(\boldsymbol{\sigma})$.
- Theorem: The sign of σ is well-defined. Additionally,
$$(-1)^{\sigma_1 \sigma_2} = (-1)^{\sigma_1} \cdot (-1)^{\sigma_2}$$
- Example: Consider the identity permutation. $(-1)^\sigma = +1$. We can think of this as the product of zero transpositions or, for instance, as the product of the two transpositions $\tau_{1,2} \circ \tau_{1,2}$. Another example would be $\tau_{2,3} \circ \tau_{1,2} \circ \tau_{1,2} \circ \tau_{2,3}$.
- Theorem: Let X_i be a rational or polynomial function for each $i \in \mathbb{N}$. Then
$$(-1)^\sigma = \prod_{i<j} \frac{X_{\sigma(i)} - X_{\sigma(j)}}{X_i - X_j}$$

- Example: For the permutation $\sigma = (1,2,3)$, we have

$$\begin{aligned}
(-1)^\sigma &= \frac{X_{\sigma(1)} - X_{\sigma(2)}}{X_1 - X_2} \cdot \frac{X_{\sigma(1)} - X_{\sigma(3)}}{X_1 - X_3} \cdot \frac{X_{\sigma(2)} - X_{\sigma(3)}}{X_2 - X_3} \\
&= \frac{X_2 - X_3}{X_1 - X_2} \cdot \frac{X_2 - X_1}{X_1 - X_3} \cdot \frac{X_3 - X_1}{X_2 - X_3} \\
&= \frac{-(X_1 - X_2)}{X_1 - X_2} \cdot \frac{-(X_1 - X_3)}{X_1 - X_3} \cdot \frac{X_2 - X_3}{X_2 - X_3} \\
&= -1 \cdot -1 \cdot 1 \\
&= +1
\end{aligned}$$

  which checks out with the fact that $\sigma = \tau_{1,2} \circ \tau_{2,3}$.

- Claims to verify with the above formula:
  1. $\operatorname{sign}(\sigma) \in \{\pm 1\}$.
  2. $\operatorname{sign}(\tau_{i,j}) = -1$.
  3. $\operatorname{sign}(\sigma_1\sigma_2) = \operatorname{sign}(\sigma_1)\operatorname{sign}(\sigma_2)$.

1.4 Chapter 1: Multilinear Algebra

From Guillemin and Haine (2018).

3/31:

- Guillemin and Haine (2018) defines real vector spaces, the operations on them, their basic properties, and the zero vector.
- **Linearly independent** (vectors $v_1, \dots, v_k$): A finite set of vectors $v_1, \dots, v_k \in V$ such that the map from $\mathbb{R}^k$ to V defined by $(c_1, \dots, c_k) \mapsto c_1 v_1 + \cdots + c_k v_k$ is injective.
- **Spanning** (vectors $v_1, \dots, v_k$): We require that the above map is surjective.
- Guillemin and Haine (2018) defines basis, finite-dimensional vector space, dimension, subspace, linear map, and kernel.
- **Image** (of $A : V \to W$): The range space of A, a subspace of W. *Also known as* $\mathbf{im}(\boldsymbol{A})$.
- Guillemin and Haine (2018) defines the matrix of a linear map.
- **Inner product** (on V): A map $B : V \times V \to \mathbb{R}$ with the following three properties.
  - *Bilinearity*: For vectors $v, v_1, v_2, w \in V$ and $\lambda \in \mathbb{R}$, we have

$$B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w)$$

    and

$$B(\lambda v, w) = \lambda B(v, w)$$

  - *Symmetry*: For vectors $v, w \in V$, we have $B(v, w) = B(w, v)$.
  - *Positivity*: For every vector $v \in V$, we have $B(v, v) \geq 0$. Moreover, if $v \neq 0$, then $B(v, v) > 0$.
- $\boldsymbol{W}$**-coset**: A set of the form $\{v + w \mid w \in W\}$, where W is a subspace V and $v \in V$. *Denoted by* $\boldsymbol{v + W}$.
  - If $v_1 - v_2 \in W$, then $v_1 + W = v_2 + W$.
  - It follows that the distinct W-cosets decompose V into a disjoint collection of subsets of V.

- **Quotient space** (of V by W): The set of distinct W-cosets in V, along with the following definitions of vector addition and scalar multiplication.

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W \qquad\qquad \lambda(v + W) = (\lambda v) + W$$

  Denoted by $\boldsymbol{V/W}$.

- **Quotient map**: The linear map $\pi : V \to V/W$ defined by

$$\pi(v) = v + W$$

  - π is surjective.
  - Note that $\ker(\pi) = W$ since for all $w \in W$, $\pi(w) = w + W = 0 + W$, which is the zero vector in V/W.

- If V, W are finite dimensional, then

$$\dim(V/W) = \dim(V) - \dim(W)$$

- Proposition 1.2.9: Let $A : V \to U$ be a linear map. If $W \subset \ker(A)$, then there exists a unique linear map $A^\sharp : V/W \to U$ with the property that $A = A^\sharp \circ \pi$, where $\pi : V \to V/W$ is the quotient map.
  - This proposition rephrases in terms of quotient spaces the fact that if $w \in W$, then $A(v+w) = Av$.
  - Another way of thinking about it is that π cuts out all of the information that A will lose anyway, and $A^\sharp$ retains all information necessary to reconstruct A.

- **Dual space** (of V): The set of all linear functions $\ell : V \to \mathbb{R}$, along with the following definitions of vector addition and scalar multiplication.

$$(\ell_1 + \ell_2)(v) = \ell_1(v) + \ell_2(v) \qquad\qquad (\lambda\ell)(v) = \lambda \cdot \ell(v)$$

  Denoted by $\boldsymbol{V^*}$.

- **Dual basis** (of $e_1, \dots, e_n$ a basis of V): The basis of V^* consisting of the n functions that take every $v = c_1e_1 + \cdots + c_ne_n$ to one of the c_i. *Denoted by* $\boldsymbol{e_1^*, \dots, e_n^*}$. *Given by*

$$e_i^*(v) = c_i$$

  for all $v \in V$.

- Claim 1.2.12: If V is an n-dimensional vector space with basis $e_1, \dots, e_n$, then $e_1^*, \dots, e_n^*$ is a basis of V^*.

  Proof. We will first prove that $e_1^*, \dots, e_n^*$ spans V^*. Let $\ell \in V^*$ be arbitrary. Set $\lambda_i = \ell(e_i)$ for all $i \in [n]$. Define $\ell' = \sum_{i=1}^n \lambda_i e_i^*$. Then

$$\ell'(e_j) = \sum_{i=1}^n \lambda_i e_i^*(e_j) = \lambda_j \cdot 1 = \ell(e_j)$$

  for all $j \in [n]$. Therefore, since ℓ, ℓ' take identical values on the basis of V, $\ell = \ell'$, as desired.

  We now prove that $e_1^*, \dots, e_n^*$ is linearly independent. Let $\sum_{i=1}^n \lambda_i e_i^* = 0$. Then for all $j \in [n]$,

$$\lambda_j = \left(\sum_{i=1}^n \lambda_i e_i^*\right)(e_j) = 0$$

  as desired. □

- **Transpose** (of A): The map from W^* to V^* defined by $\ell \mapsto \ell \circ A$ for all $\ell \in W^*$. *Denoted by* $\boldsymbol{A^*}$.
- Claim 1.2.15: If $e_1, \dots, e_n$ is a basis of V, $f_1, \dots, f_m$ is a basis of W, $e_1^*, \dots, e_n^*$ and $f_1^*, \dots, f_m^*$ are the corresponding dual bases, and $[a_{i,j}]$ is the $m \times n$ matrix of A with respect to $\{e_j\}, \{f_i\}$, then the linear map A^* is defined in terms of $\{f_i^*\}, \{e_j^*\}$ by the transpose matrix $(a_{j,i})$.

  Proof. Let $[c_{j,i}]$ be the $n \times m$ matrix of A^* with respect to $\{f_i^*\}, \{e_j^*\}$. We seek to prove that $a_{i,j} = c_{j,i}$ ($1 \leq i \leq m$, $1 \leq j \leq n$).

  By the definition of $[a_{i,j}]$ and $[c_{j,i}]$, we have that

  $$A^* f_i^* = \sum_{k=1}^{n} c_{k,i} e_k^* \qquad\qquad Ae_j = \sum_{k=1}^{m} a_{k,j} f_k$$

  It follows that

  $$[A^* f_i^*](e_j) = \left[\sum_{k=1}^{n} c_{k,i} e_k^* \right](e_j) = c_{j,i}$$

  and

  $$[A^* f_i^*](e_j) = f_i^*(Ae_j) = f_i^* \left(\sum_{k=1}^{m} a_{k,j} f_k \right) = a_{i,j}$$

  so transitivity implies the desired result. □

4/4:

- $\boldsymbol{V^k}$: The set of all k-tuples $(v_1, \dots, v_k)$ where $v_1, \dots, v_k \in V$ a vector space.
  - Note that
    $$V^k = \underbrace{V \times \cdots \times V}_{k \text{ times}}$$
    where "$\times$" denotes the Cartesian product.
- **Linear** (function in its i^{th} variable): A function $T : V^k \to \mathbb{R}$ such that the map from V to $\mathbb{R}$ defined by $v \mapsto T(v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_k)$ is linear, where all v_j save v_i are fixed.
- $\boldsymbol{k}$**-linear** (function T): A function $T : V^k \to \mathbb{R}$ that is linear in its i^{th} variable for $i = 1, \dots, k$. *Also known as* $\boldsymbol{k}$**-tensor**.
- $\boldsymbol{\mathcal{L}^k(V)}$: The set of all k-tensors in V.
  - Since the sum $T_1 + T_2$ of two k-linear functions $T_1, T_2 : V^k \to \mathbb{R}$ is just another k-linear function, and λT_1 is k-linear for all $\lambda \in \mathbb{R}$, we have that $\mathcal{L}^k(V)$ is a vector space.
- Convention[1]: 0-tensors are just the real numbers. Mathematically, we define
  $$\mathcal{L}^0(V) = \mathbb{R}$$
- Note that $\mathcal{L}^1(V) = V^*$.
- Defines multi-indices of n of length k.
- Lemma 1.3.5: If $n, k \in \mathbb{N}$, then there are exactly n^k multi-indices of n of length k.
- $\boldsymbol{T_I}$: The real number $T(e_{i_1}, \dots, e_{i_k})$, where $T \in \mathcal{L}^k(V)$, $e_1, \dots, e_n$ is a basis of V, and I is a multi-index of n of length k.
- Proposition 1.3.7: The real numbers T_I determine T, i.e., if T, T' are k-tensors and $T_I = T'_I$ for all I, then $T = T'$.

[1] See the definition of the tensor product and ensuing note for a justification of this convention (just a few lines down).

Proof. We induct on n. For the base case $n = 1$, $T \in (\mathbb{R}^k)^*$ and we have already proven this result. Now suppose inductively that the assertion is true for $n-1$. For each e_i, let T_i be the $(k-1)$-tensor defined by

$$(v_1, \dots, v_{n-1}) \mapsto T(v_1, \dots, v_{n-1}, e_i)$$

Then for an arbitrary $v = c_1e_1 + \cdots + c_ne_n$,

$$T(v_1, \dots, v_{n-1}, v) = \sum_{i=1}^{n} c_iT_i(v_1, \dots, v_{n-1})$$

so the T_i's determine T. Applying the inductive hypothesis completes the proof. ☐

- **Tensor product**: The function $\otimes : \mathcal{L}^k(V) \times \mathcal{L}^\ell(V) \to \mathcal{L}^{k+\ell}(V)$ defined by

  $$(T_1 \otimes T_2)(v_1, \dots, v_{k+\ell}) = T_1(v_1, \dots, v_k)T_2(v_{k+1}, \dots, v_{k+\ell})$$

  for all $T_1 \in \mathcal{L}^k(V)$ and $T_2 \in \mathcal{L}^\ell(V)$.
- Note that by the definition of 0-tensors as real numbers, if $a \in \mathbb{R}$ and $T \in \mathcal{L}^k(V)$, then

  $$a \otimes T = T \otimes a = aT$$

- Proposition 1.3.9: Associativity, distributivity of scalar multiplication, and left and right distributive laws for the tensor product.
- **Decomposable** (k-tensor): A k-tensor T for which there exist $\ell_1, \dots, \ell_k \in V^*$ such that

  $$T = \ell_1 \otimes \cdots \otimes \ell_k$$

- Defines e_I^*.
- Theorem 1.3.13: V a vector space with basis $e_1, \dots, e_n$ and $0 \leq k \leq n$ implies the k-tensors e_I^* form a basis of $\mathcal{L}^k(V)$.

  Proof. Spanning: Let $T \in \mathcal{L}^k(V)$ be arbitrary. Define

  $$T' = \sum_I T_Ie_I^*$$

  Since

  $$T_J' = T'(e_{j_1}, \dots, e_{j_k}) = \sum_I T_Ie_I^*(e_{j_1}, \dots, e_{j_k}) = T_Je_J^*(e_{j_1}, \dots, e_{j_k}) = T_J$$

  for all J, Proposition 1.3.7 asserts that $T = T'$. Therefore, since every $T_I \in \mathbb{R}$, $T = T' \in \operatorname{span}(e_I^*)$.

  Linear independence: Suppose

  $$T = \sum_I c_Ie_I^* = 0$$

  for some set of constants $c_I \in \mathbb{R}$. Then

  $$0 = T(e_{j_1}, \dots, e_{j_k}) = \sum_I c_Ie_I^*(e_{j_1}, \dots, e_{j_k}) = c_J$$

  for all J, as desired. ☐
- Corollary 1.3.15: If $\dim V = n$, then $\dim(\mathcal{L}^k(V)) = n^k$.

  Proof. Follows immediately from Lemma 1.3.5. ☐

- **Pullback** (of T by the map A): The k-tensor $A^*T : V^k \to \mathbb{R}$ defined by
$$(A^*T)(v_1, \dots, v_k) = T(Av_1, \dots, Av_k)$$
where V, W are finite-dimensional vector spaces, $A : V \to W$ is linear, and $T \in \mathcal{L}^k(W)$.
- Proposition 1.3.18: The map $A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$ defined by $T \mapsto A^*T$ is linear.
- Identities:
  - If $T_1 \in \mathcal{L}^k(W)$ and $T_2 \in \mathcal{L}^m(W)$, then
$$A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$$
  - If U is a vector space, $B : U \to V$ is linear, and $T \in \mathcal{L}^k(W)$, then $(AB)^*T = B^*(A^*T)$. Hence,
$$(AB)^* = B^*A^*$$

4/13:
- $\boldsymbol{\Sigma_k}$: The set containing the natural numbers 1 through k. *Given by*
$$\Sigma_k = \{1, 2, \dots, k\}$$
- **Permutation of order $\boldsymbol{k}$**: A bijection on Σ_k. *Denoted by* $\boldsymbol{\sigma}$.
- **Product** (of σ_1, σ_2): The composition $\sigma_1 \circ \sigma_2$, i.e., the map
$$i \mapsto \sigma_1(\sigma_2(i))$$
Denoted by $\boldsymbol{\sigma_1\sigma_2}$.
- **Inverse** (of σ): The permutation of order k which is the inverse bijection of σ. *Denoted by* $\boldsymbol{\sigma^{-1}}$.
- **Permutation group** (of Σ_k): The set of all permutations of order k. *Also known as* **symmetric group on $\boldsymbol{k}$ letters**. *Denoted by* $\boldsymbol{S_k}$.
- Lemma 1.4.2: The group S_k has $k!$ elements.
- **Transposition**: A permutation of order k defined by
$$\ell \mapsto \begin{cases} j & \ell = i \\ i & \ell = j \\ \ell & \ell \neq i, j \end{cases}$$
for all $\ell \in \Sigma_k$, where $i, j \in \Sigma_k$. *Denoted by* $\boldsymbol{\tau_{i,j}}$.
- **Elementary transposition**: A transposition of the form $\pi_{i,i+1}$.
- Theorem 1.4.4: Every $\sigma \in S_k$ can be written as a product of (a finite number of) transpositions.

Proof. We induct on k.

For the base case $k = 2$, the identity permutation of S_2 is the "product" of zero transpositions, and the only other permutation is a transposition (the "product" of one transposition, namely itself).

Now suppose inductively that we have proven the claim for $k - 1$. Let $\sigma \in S_k$ be arbitrary. Suppose $\sigma(k) = i$. Then $\tau_{i,k}\sigma(k) = k$. Since $(\tau_{i,k}\sigma)|_{\Sigma_{k-1}} \in S_{k-1}$, we have by the inductive hypothesis that $(\tau_{i,k}\sigma)|_{\Sigma_{k-1}} = \tau_1 \cdots \tau_m$ for some set of permutations $\tau_1, \dots, \tau_m \in S_{k-1}$. For each τ_j ($1 \leq j \leq m$), define $\tau_j' \in S_k$
$$\tau_j'(\ell) = \begin{cases} \tau_j(\ell) & \ell < k \\ \ell & \ell = k \end{cases}$$

It follows that

$$\begin{aligned} \tau_{i,k}\sigma &= \tau_1' \cdots \tau_m' \\ \sigma &= \tau_{i,k}\tau_1' \cdots \tau_m' \end{aligned}$$

as desired. ☐

- Theorem 1.4.5: Every transposition can be written as a product of elementary transpositions.

  Proof. Let $\tau_{i,j} \in S_k$, and let $i < j$ WLOG. Then we have that

  $$\tau_{i,j} = \prod_{\ell=i}^{i-1} \tau_{\ell,\ell+1}$$

  as desired. ☐

- Corollary 1.4.6: Every permutation can be written as a product of elementary transpositions.
- **Sign** (of σ): The number ± 1 assigned to σ by the expression

  $$\prod_{i<j} \frac{x_{\sigma(i)} - x_{\sigma(j)}}{x_i - x_j}$$

  where $x_1, \dots, x_k$ are coordinate functions on $\mathbb{R}^k$. *Denoted by* $\boldsymbol{(-1)^{\sigma}}$.
- Claim 1.4.9: The sign defines a group homomorphism $S_k \to \{\pm 1\}$. That is, for $\sigma_1, \sigma_2 \in S_k$, we have

  $$(-1)^{\sigma_1\sigma_2} = (-1)^{\sigma_1}(-1)^{\sigma_2}$$

  Proof. For all $i < j$, define p, q such that p is the lesser of $\sigma_2(i), \sigma_2(j)$ and q is the greater of $\sigma_2(i), \sigma_2(j)$. Formally,

  $$p = \begin{cases} \sigma_2(i) & \sigma_2(i) < \sigma_2(j) \\ \sigma_2(j) & \sigma_2(j) < \sigma_2(i) \end{cases} \qquad q = \begin{cases} \sigma_2(j) & \sigma_2(i) < \sigma_2(j) \\ \sigma_2(i) & \sigma_2(j) < \sigma_2(i) \end{cases}$$

  It follows that if $\sigma_2(i) < \sigma_2(j)$, then

  $$\frac{x_{\sigma_1\sigma_2(i)} - x_{\sigma_1\sigma_2(j)}}{x_{\sigma_2(i)} - x_{\sigma_2(j)}} = \frac{x_{\sigma_1(p)} - x_{\sigma_1(q)}}{x_p - x_q}$$

  and if $\sigma_2(j) < \sigma_2(i)$, then

  $$\frac{x_{\sigma_1\sigma_2(i)} - x_{\sigma_1\sigma_2(j)}}{x_{\sigma_2(i)} - x_{\sigma_2(j)}} = \frac{x_{\sigma_1(q)} - x_{\sigma_1(p)}}{x_q - x_p} = \frac{x_{\sigma_1(p)} - x_{\sigma_1(q)}}{x_p - x_q}$$

  Therefore,

  $$\begin{aligned} (-1)^{\sigma_1\sigma_2} &= \prod_{i<j} \frac{x_{\sigma_1\sigma_2(i)} - x_{\sigma_1\sigma_2(j)}}{x_i - x_j} \\ &= \prod_{i<j} \frac{x_{\sigma_1\sigma_2(i)} - x_{\sigma_1\sigma_2(j)}}{x_{\sigma_2(i)} - x_{\sigma_2(j)}} \cdot \frac{x_{\sigma_2(i)} - x_{\sigma_2(j)}}{x_i - x_j} \\ &= \prod_{i<j} \frac{x_{\sigma_1(p)} - x_{\sigma_1(q)}}{x_p - x_q} \cdot \prod_{i<j} \frac{x_{\sigma_2(i)} - x_{\sigma_2(j)}}{x_i - x_j} \\ &= (-1)^{\sigma_1}(-1)^{\sigma_2} \end{aligned}$$

  as desired. ☐

- Proposition 1.4.11: If σ is the product of an odd number of transpositions, then $(-1)^\sigma = -1$, and if σ is the product of an even number of transpositions, then $(-1)^\sigma = +1$.

  Proof. Follows from the fact that $(-1)^\sigma = -1$ (see Exercise 1.4.ii). $\square$

- $\boldsymbol{T^\sigma}$: The k-tensor defined by

$$T^\sigma(v_1, \dots, v_k) = T(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)})$$

  where $T \in \mathcal{L}^k(V)$, V is an n-dimensional vector space, and $\sigma \in S_k$.

- Proposition 1.4.14:

  1. If $T = \ell_1 \otimes \cdots \otimes \ell_k$ ($\ell_i \in V^*$), then $T^\sigma = \ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}$.

     Proof. If $v_1, \dots, v_k \in V$, then

$$\begin{aligned}
T^\sigma(v_1, \dots, v_k) &= T(v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)}) \\
&= [\ell_1 \otimes \cdots \otimes \ell_k](v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(k)}) \\
&= \ell_1(v_{\sigma^{-1}(1)}) \cdots \ell_k(v_{\sigma^{-1}(k)}) \\
&= \ell_{\sigma(1)}(v_1) \cdots \ell_{\sigma(k)}(v_2) \\
&= [\ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}](v_1, \dots, v_k)
\end{aligned}$$

     as desired. Note that we can justify the fourth equality by nothing that if $\sigma^{-1}(i) = q$, then the i^{th} term in the product is $\ell_{\sigma(q)}(v_q)$, so since σ is a bijection, the product can be arranged to the form on the right-hand side of equality four. $\square$

  2. The assignment $T \mapsto T^\sigma$ is a linear map from $\mathcal{L}^k(V) \to \mathcal{L}^k(V)$.

     Proof. See Exercise 1.4.iii. $\square$

  3. If $\sigma_1, \sigma_2 \in S_k$, we have $T^{\sigma_1\sigma_2} = (T^{\sigma_1})^{\sigma_2}$.

     Proof. Let $T = \ell_1 \otimes \cdots \otimes \ell_k$[2]. Then

$$T^{\sigma_1} = \ell_{\sigma_1(1)} \otimes \cdots \otimes \ell_{\sigma_1(k)} = \ell'_1 \otimes \cdots \otimes \ell'_k$$

     and thus

$$(T^{\sigma_1})^{\sigma_2} = \ell'_{\sigma_2(1)} \otimes \cdots \otimes \ell'_{\sigma_2(k)}$$

     Let $\sigma_2(i) = j$. Then since $\ell'_p = \ell_{\sigma_1(p)}$ by definition, we have that $\ell'_{\sigma_2(j)} = \ell_{\sigma_1(\sigma_2(j))}$. Therefore,

$$\begin{aligned}
(T^{\sigma_1})^{\sigma_2} &= \ell'_{\sigma_2(1)} \otimes \cdots \otimes \ell'_{\sigma_2(k)} \\
&= \ell_{\sigma_1(\sigma_2(1))} \otimes \cdots \otimes \ell_{\sigma_1(\sigma_2(k))} \\
&= \ell_{\sigma_1\sigma_2(1)} \otimes \cdots \otimes \ell_{\sigma_1\sigma_2(k)} \\
&= T^{\sigma_1\sigma_2}
\end{aligned}$$

     as desired. $\square$

- **Alternating** (k-tensor): A k-tensor $T \in \mathcal{L}^k(V)$ such that $T^\sigma = (-1)^\sigma T$ for all $\sigma \in S_k$.

- $\boldsymbol{\mathcal{A}^k(V)}$: The set of all alternating k-tensors in $\mathcal{L}^k(V)$.

  - Proposition 1.4.14(2) implies that $(T_1 + T_2)^\sigma = T_1^\sigma + T_2^\sigma$ and $(\lambda T)^\sigma = \lambda T^\sigma$; it follows that $\mathcal{A}^k(V)$ is a vector space.

[2] What gives us the right to assume T is decomposable?

- **Alternation operation**: The function from $\mathcal{L}^k(V) \to \mathcal{L}^k(V)$ defined by

$$T \mapsto \sum_{\tau \in S_k} (-1)^\tau T^\tau$$

  Denoted by **Alt**.

- Proposition 1.4.17: For $T \in \mathcal{L}^k(V)$ and $\sigma \in S_k$, we have that
  1. $\operatorname{Alt}(T)^\sigma = (-1)^\sigma \operatorname{Alt} T$.

     Proof. We have that

$$\begin{aligned}
\operatorname{Alt}(T)^\sigma &= \left(\sum_{\tau \in S_k} (-1)^\tau T^\tau \right)^\sigma \\
&= \sum_{\tau \in S_k} (-1)^\tau (T^\tau)^\sigma && \text{Proposition 1.4.14(2)} \\
&= \sum_{\tau \in S_k} (-1)^\tau T^{\tau\sigma} && \text{Proposition 1.4.14(3)} \\
&= (-1)^\sigma \sum_{\tau \in S_k} (-1)^{\tau\sigma} T^{\tau\sigma} \\
&= (-1)^\sigma \sum_{\tau\sigma \in S_k} (-1)^{\tau\sigma} T^{\tau\sigma} \\
&= (-1)^\sigma \operatorname{Alt} T
\end{aligned}$$

     as desired. □

  2. If $T \in \mathcal{A}^k(V)$, then $\operatorname{Alt} T = k!T$.

     Proof. Since $T \in \mathcal{A}^k(V)$, we know that $T^\sigma = (-1)^\sigma T$. Therefore,

$$\operatorname{Alt} T = \sum_{\tau \in S_k} (-1)^\tau T^\tau = \sum_{\tau \in S_k} (-1)^\tau (-1)^\tau T = \sum_{\tau \in S_k} T = k!T$$

     where the last equality holds because the cardinality of S_k is $k!$. □

  3. $\operatorname{Alt}(T^\sigma) = \operatorname{Alt}(T)^\sigma$.

     Proof. We have that

$$\operatorname{Alt}(T^\sigma) = \sum_{\tau \in S_k} (-1)^\tau T^{\tau\sigma} = (-1)^\sigma \sum_{\tau \in S_k} (-1)^{\tau\sigma} T^{\tau\sigma} = (-1)^\sigma \operatorname{Alt}(T) = \operatorname{Alt}(T)^\sigma$$

     as desired. □

  4. The alternation operation is linear.

     Proof. Follows by Proposition 1.4.14. □

- **Repeating** (multi-index I): A multi-index I of length k such that $i_r = i_s$ for some $r \neq s$.
- **Strictly increasing** (multi-index I): A multi-index I of length k such that $i_1 < i_2 < \cdots < i_r$.
- $\boldsymbol{I^\sigma}$: The multi-index of length k defined by

$$I^\sigma = (i_{\sigma(1)}, \dots, i_{\sigma(k)})$$

- If I is non-repeating, there is a unique $\sigma \in S_k$ such that I^σ is strictly increasing.

- $\boldsymbol{\psi_I}$: The following k-tensor. *Given by*
$$\psi_I = \operatorname{Alt}(e_I^*)$$
- Proposition 1.4.20:
  1. $\psi_{I^\sigma} = (-1)^\sigma \psi_I$.

     Proof. We have that
     $$\psi_{I^\sigma} = \operatorname{Alt}(e_{I^\sigma}^*) = \operatorname{Alt}[(e_I^*)^\sigma] = \operatorname{Alt}(e_I^*)^\sigma = (-1)^\sigma \operatorname{Alt}(e_I^*) = (-1)^\sigma \psi_I$$
     as desired. □
  2. If I is repeating, then $\psi_I = 0$.

     Proof. Suppose $I = (i_1, \dots, i_k)$ is such that $i_r = i_s$ for some distinct $r, s \in \Sigma_k$. Then $e_I^* = e_{I^{\tau_{i_r,i_s}}}^*$, so
     $$\psi_I = \psi_{I^{\tau_{i_r,i_s}}} = (-1)^{\tau_{i_r,i_s}} \psi_I = -\psi_I$$
     Therefore, we must have $\psi_I = 0$, as desired. □
  3. If I and J are strictly increasing, then
     $$\psi_I(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$$
     Proof. We have by definition that
     $$\psi_I(e_{j_1}, \dots, e_{j_k}) = \sum_\tau (-1)^\tau e_{I^\tau}^*(e_{j_1}, \dots, e_{j_k})$$
     This combined with the facts that
     $$e_{I^\tau}^*(e_{j_1}, \dots, e_{j_k}) = \begin{cases} 1 & I^\tau = J \\ 0 & I^\tau \neq J \end{cases}$$
     I^τ is strictly increasing iff $I^\tau = I$, and the above equation is nonzero iff $I^\tau = I = J$ implies the desired result. □
- Conclusion 1.4.22: If $T \in \mathcal{A}^k(V)$, then we can write T as a sum
$$T = \sum_I c_I \psi_I$$
with I's strictly increasing.

  Proof. Let $T \in \mathcal{A}^k(V)$ be arbitrary. By Theorem 1.3.13,
$$T = \sum_J a_J e_J^*$$
for some set of $a_J \in \mathbb{R}$. It follows since $\operatorname{Alt}(T) = k!T$ that
$$T = \frac{1}{k!} \sum a_J \operatorname{Alt}(e_J^*) = \sum b_J \psi_J$$
We can disregard all repeating terms in the sum since they are zero by Proposition 1.4.20(2); for every non-repeating term J, we can write $J = I^\sigma$, where I is strictly increasing and hence $\psi_J = (-1)^\sigma \psi_I$. □
- Claim 1.4.24: The c_I's of Conclusion 1.4.22 are unique.

Proof. For J strictly increasing, we have

$$T_J = T(e_{j_1}, \dots, e_{j_k}) = \sum_I c_I \psi_I(e_{j_1}, \dots, e_{j_k}) = c_J$$

□

- Proposition 1.4.26: The alternating tensors ψ_I with I strictly increasing are a basis for $\mathcal{A}^k(V)$.

  Proof. Spanning: See Conclusion 1.4.22.

  Linear independence: See Claim 1.4.24. □

- We have that

$$\dim \mathcal{A}^k(V) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$$

  - Hint in proving this claim: "Show that every strictly increasing multi-index of length k determines a k-element subset of $\{1, \dots, n\}$ and vice versa." (Guillemin & Haine, 2018, p. 16).
  - Note also that if $k > n$, every multi-index has a repeat somewhere, meaning that $\dim \mathcal{A}^k(V) = \binom{n}{k} = 0$.

Week 2

Tensor Classifications

2.1 Alternating Tensors

4/4:

- Plan:
  - More multilinear algebra.
  - Alternating k-tensors — 2 views:
    1. As a subspace of $\mathcal{L}^k(V)$.
    2. As a quotient of $\mathcal{L}^k(V)$.
  - Next time: Operators as alternating tensors.
    - Wedge products.
    - Interior products.
    - Pullbacks.
- Recall: $\dim V = n$, $e_1, \dots, e_n$ a basis, $\mathcal{L}^k(V)$ the space of k-tensors, $\sigma \in S_k$ implies $(-1)^\sigma \in \{\pm 1\}$, key property: $(-1)^{\sigma_1\sigma_2} = (-1)^{\sigma_1}(-1)^{\sigma_2}$.
- $\boldsymbol{T^\sigma}$: The k-tensor over V defined by

$$T^\sigma(v_1, \dots, v_k) = T(v_{\bar{\sigma}(1)}, \dots, v_{\bar{\sigma}(k)})$$

  where $T \in \mathcal{L}^k(V)$, $\sigma \in S_k$, and $\bar{\sigma}$ denotes the inverse of σ.
- Example: $n = 2$, $k = 2$. Let $T = e_1^* \otimes e_2^* \in \mathcal{L}^2(V)$. Let $\sigma = \tau_{1,2}$. Then $T^\sigma = e_2^* \otimes e_1^*$.
- Another property is $(e_I^*)^\sigma = e_{\sigma(I)}^*$.
- Properties:
  1. $T^{\sigma_1\sigma_2} = (T^{\sigma_1})^{\sigma_2}$.
  2. $(T_1 + T_2)^\sigma = T_1^\sigma + T_2^\sigma$.
  3. $(cT)^\sigma = cT^\sigma$.
- Thus, you can view $\sigma : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ as a linear map!
- **Alternating $\boldsymbol{k}$-tensor**: A tensor $T \in \mathcal{L}^k(V)$ such that $T^\sigma = (-1)^\sigma T$ for all $\sigma \in S_k$.
  - Equivalently, $T^\tau = -T$ for all $\tau \in S_k$.
- An example of an alternating 2-tensor when $\dim V = 2$ is $T = e_1^* \otimes e_2^* - e_2^* \otimes e_1^*$.
  - Naturally, $T^{\tau_{1,2}} = -T$, and $\tau_{1,2}$ is the unique transposition in S_2.

- $e_1^* \otimes e_2^*$ is *not* an alternating 2-tensor since $(e_1^* \otimes e_2^*)^\tau = e_2^* \otimes e_1^* \neq (-1)^\tau (e_1^* \otimes e_2^*)$.
- We can look at $n = 2$, $k = 1$ for ourselves.
- Note: If T_1, T_2 are both alternating k-tensors, then $T_1 + T_2$ is also alternating, as is cT_1 for all $c \in \mathbb{R}$.
- $\boldsymbol{\mathcal{A}^k(V)}$: The vector space of alternating k-tensors.
- $\mathbf{Alt}(\boldsymbol{T})$: The function $\text{Alt} : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ defined by

$$\text{Alt}(T) = \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma$$

- Properties:
  1. $\text{im}(\text{Alt}) = \mathcal{A}^k(V)$.
  2. $\mathcal{L}^k(V)/\ker(\text{Alt}) = \Lambda^k(V^*)$[1] is isomorphic to $\mathcal{A}^k(V)$.
  3. $\text{Alt}(T)^\sigma = (-1)^\sigma \text{Alt}(T)$.
     - Proof:

$$\begin{aligned}
\text{Alt}(T)^{\sigma'} &= \left(\sum_{\sigma \in S_k} (-1)^\sigma T^\sigma \right)^{\sigma'} \\
&= \sum_{\sigma \in S_k} (-1)^\sigma T^{\sigma\sigma'} \\
&= (-1)^{\sigma'} \sum_{\sigma \in S_k} (-1)^{\sigma'} (-1)^\sigma T^{\sigma\sigma'} \\
&= (-1)^{\sigma'} \sum_{\sigma \in S_k} (-1)^{\sigma\sigma'} T^{\sigma\sigma'} \\
&= (-1)^{\sigma'} \text{Alt}(T)
\end{aligned}$$

     - The last equality holds because summing over all σ is the same as summing over all $\sigma' \circ \sigma$.
     - This implies $\text{im}(\text{Alt}) \leq \mathcal{A}^k(V)$.
  4. If $T \in \mathcal{A}^k(T)$, $\text{Alt}(T) = k!T$.
     - We have

$$\begin{aligned}
\text{Alt}(T) &= \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma \\
&= \sum_{\sigma \in S_k} (-1)^\sigma (-1)^\sigma T \\
&= \sum_{\sigma \in S_k} T \\
&= k!T
\end{aligned}$$

       where $T^\sigma = (-1)^\sigma T$ since $T \in \mathcal{A}^k(V)$ by definition.
     - This implies that $\text{im}(\text{Alt}) = \mathcal{A}^k(V)$: $\text{Alt}(\frac{1}{k!}T) = T \in \mathcal{A}^k(V)$.
  5. $\text{Alt}(T^\sigma) = \text{Alt}(T)^\sigma$.

[1] Note that we use V^* here instead of V because $\Lambda^k(V^*)$ does not indicate some set of functions over the vector space V, but rather the k-dimensional exterior powers of the linear functionals $\ell \in V^*$ that are dual to the vectors $v \in V$. In other words, whereas $\mathcal{A}^k(V)$ denotes the set of alternating k-tensors *acting on* V, $\Lambda^k(V^*)$ denotes the vector space containing all linear combinations of all products of length k of covectors $\ell \in V^*$, where the multiplication operation is the exterior product. Also, $\Lambda^k(V)$ is already otherwise defined as the set of all k-vectors, linear combinations of k-blades, which in turn are exterior products of length k of vectors $v \in V$.

6. $\text{Alt} : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ is linear.

- Warning: Some people take $\text{Alt}(T) = \frac{1}{k!}\sum_{\sigma\in S_k}(-1)^\sigma T^\sigma$[2].
- Example: $n = k = 2$. We have

$$\text{Alt}(e_1^* \otimes e_2^*) = e_1^* \otimes e_2^* - e_2^* \otimes e_1^*$$

- **Non-repeating** (multi-index I): A multi-index I such that $i_{j_1} \neq i_{j_2}$ for all $j_1 \neq j_2$.
- **Increasing** (multi-index I): A multi-index I such that $i_1 < \cdots < i_k$.
- Claim: $\{\text{Alt}(e_I^*)\}$ where I is non-repeating and increasing is a basis for $\mathcal{A}^k(V)$. There are $\binom{n}{k}$ of these; thus, $\dim \mathcal{A}^k(V) = \binom{n}{k}$.
  - Note that we spend so much time on alternating tensors, because we can prove that all of the tensors that we'll eventually be interested in (e.g., those describing the slope of a function of interest at a given point p) *are* alternating.
  - See, for example, Exercise 2.4.i(3). Therein, a two-form is decomposed all the way to a linear combination of alternating two-forms (that become alternating tensors at any point p).
  - It is because of this curious fact that a more robust exploration of the properties of alternating tensors was called for. And this is why we're exploring them now.

2.2 Redundant Tensors and Alternatization

4/6:

- Klug will be in Texas on Monday and thus is cancelling class on Monday. Homework is now due next Friday. We'll have weekly homeworks going forward after that.
- Plan:
  - $\text{Alt} : \mathcal{L}^k(V) \twoheadrightarrow \mathcal{A}^k(V)$[3].
  - Goal: Identify $\ker(\text{Alt}) = \mathcal{I}^k(V)$, where $\mathcal{I}^k(V)$ is the space of **redundant** k-tensors[4].
  - Then: Operations on alternating tensors, e.g.,
    - Wedge product.
    - Interior product.
    - Orientations.
- Claim: $\{\text{Alt}(e_I^*) \mid I \text{ non-repeating, increasing multi-index}\}$ is a basis for $\mathcal{A}^k(V)$.
  - Left as an exercise to us.
- **Redundant** (k-tensor): A k-tensor of the form

$$\ell_1 \otimes \cdots \otimes \ell_i \otimes \ell_i \otimes \ell_{i+2} \otimes \cdots \otimes \ell_k$$

  where $\ell_1, \dots, \ell_k \in V^*$.
- $\boldsymbol{\mathcal{I}^k(V)}$: The span of all redundant k-tensors.
  - Note that not every k-tensor in $\mathcal{I}^k(V)$ is redundant.
- **Decomposable** (k-tensor): A k-tensor of the form $\ell_1 \otimes \cdots \otimes \ell_k$ for some $\ell_i \in \mathcal{L}^1(V)$.
  - It often suffices to prove things for decomposable tensors.

[2]Klug prefers this convention, but the text takes the other one.
[3]The two-headed right arrow denotes a surjective map.
[4]The $\mathcal{I}$ in $\mathcal{I}^k(V)$ stands for "ideal."

- Properties.
  1. If $T \in \mathcal{I}^k(V)$, then $\operatorname{Alt}(T) = 0$, i.e., $\mathcal{I}^k(V) \leq \ker(\operatorname{Alt})$.
     - "Proof by example": If $T = \ell_1 \otimes \ell_1 \otimes \ell_2 \otimes \ell_3$, then $T^{\tau_{1,2}} = T$. It follows from the properties of Alt that

$$\begin{aligned}
\operatorname{Alt}(T) &= \operatorname{Alt}(T^{\tau_{1,2}}) = (-1)^{\tau_{1,2}} \operatorname{Alt}(T) = -\operatorname{Alt}(T) \\
2\operatorname{Alt}(T) &= 0 \\
\operatorname{Alt}(T) &= 0
\end{aligned}$$

  2. If $T \in \mathcal{I}^r(V)$ and $T' \in \mathcal{L}^s(V)$, then

$$T \otimes T' \in \mathcal{I}^{r+s}(V)$$

     Similarly, if $T \in \mathcal{L}^r(V)$ and $T \in \mathcal{I}^s(V)$, then

$$T \otimes T' \in \mathcal{I}^{r+s}(V)$$

     - Proof: It suffices to assume that T is redundant. Obviously adding more tensors to the direct product will not change the redundancy of the initial tensor. Example: $\ell_1 \otimes \ell_1 \otimes \ell_2$ is just as redundant as $\ell_1 \otimes \ell_1 \otimes \ell_2 \otimes T$.
  3. If $T \in \mathcal{L}^k(V)$ and $\sigma \in S_k$, then

$$T^\sigma = (-1)^\sigma T + S$$

     for some $S \in \mathcal{I}^k(V)$.
     - Proof by example: It suffices to check this for decomposable tensors (a tensor is just a sum of decomposable tensors). Take $k = 2$. Let $T = \ell_1 \otimes \ell_2$. Let $\sigma = \tau_{1,2}$. Then

$$T^\sigma - (-1)^\sigma T = \ell_2 \otimes \ell_1 + \ell_1 \otimes \ell_2 = (\ell_1 + \ell_2) \otimes (\ell_1 + \ell_2) - \ell_1 \otimes \ell_1 - \ell_2 \otimes \ell_2$$

     - Actual proof: It suffices to assume T is decomposable. We induct on the number of transpositions needed to write σ as a product of **adjacent** transpositions.
     - Base case: $\sigma = \tau_{i,i+1}$. Then

$$\begin{aligned}
T^{\tau_{i,i+1}} + T = \ell_1 \otimes \cdots \otimes (\ell_i + \ell_{i+1}) \otimes (\ell_i + \ell_{i+1}) \otimes \cdots \otimes \ell_k \\
- \ell_1 \otimes \cdots \otimes \ell_i \otimes \ell_i \otimes \cdots \otimes \ell_k \\
- \ell_1 \otimes \cdots \otimes \ell_{i+1} \otimes \ell_{i+1} \otimes \cdots \otimes \ell_k
\end{aligned}$$

     - Inductive step: If $\sigma = \beta\tau$, then

$$\begin{aligned}
T^\sigma &= T^{\beta\tau} \\
&= (-1)^\tau T^\beta + \text{stuff in } \mathcal{I}^k(V) \\
&= (-1)^\tau [(-1)^\beta T + \text{stuff in } \mathcal{I}^k(V)] + \text{stuff in } \mathcal{I}^k(V)
\end{aligned}$$

  4. If $T \in \mathcal{L}^k(V)$, then

$$\operatorname{Alt}(T) = k!T + W$$

     for some $W \in \mathcal{I}^k(V)$.
     - We have that

$$\begin{aligned}
\operatorname{Alt}(T) &= \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma \\
&= \sum_{\sigma \in S_k} (-1)^\sigma [(-1)^\sigma T + S_\sigma] \\
&= \sum_{\sigma \in S_k} T + \sum_{\sigma \in S_k} (-1)^\sigma S_\sigma \\
&= k!T + W
\end{aligned}$$

5. $\mathcal{I}^k(V) = \ker(\text{Alt})$.
   - We have that $\mathcal{I}^k(V) \leq \ker(\text{Alt})$ by property 1.
   - Now suppose $T \in \ker(\text{Alt})$. Then $\text{Alt}(T) = 0$. Then by property 4,

$$\begin{aligned}\text{Alt}(T) &= k!T + W\\ 0 &= k!T + W\\ T &= -\frac{1}{k!}W \in \mathcal{I}^k(V)\end{aligned}$$

- Warning: If $T \in \mathcal{A}^r(V)$ and $T' \in \mathcal{A}^s(V)$, then we do not necessarily have $T \otimes T' \in \mathcal{A}^{r+s}(V)$.
  - Example: $e_1^*, e_2^* \in \mathcal{A}^1(V)$ have $e_1^* \otimes e_2^* \notin \mathcal{A}^2(V)$.
- **Adjacent** (transposition): A transposition of the form $\tau_{i,i+1}$.

2.3 The Wedge Product

4/8:

- Recall that $\mathcal{A}^k(V) \hookrightarrow \mathcal{L}^k(V)$[5]
- Functoriality: $(A \circ B)^* = B^* \circ A^*$.
  - A^* takes $\mathcal{L}^k(W) \to \mathcal{L}^k(V)$ and $\mathcal{A}^k(W) \to \mathcal{A}^k(V)$.
- $\dim(\Lambda^k(V^*)) = \binom{n}{k}$.
  - Special case $k = n$: $\dim \Lambda^n(V) = 1$.
  - $A : V \to V$ induces a map from $\Lambda^n(V^*) \to \Lambda^n(V^*)$ (the relevant pullback) defined by the determinant.
- Aside: $\Lambda^k(V^*)$ is "exterior powers."
- Plan: Wedge products + basis for $\Lambda^k(V^*)$.
- **Wedge product**: A function $\wedge : \Lambda^k(V^*) \times \Lambda^\ell(V^*) \to \Lambda^{k+\ell}(V)$.
  - We denote elements of $\Lambda^k(V^*)$ by ω_1, ω_2, etc.
- If $\pi : \mathcal{L}^k(V) \to \Lambda^k(V^*)$ sends $T \mapsto \omega$, $\omega_1 = \pi(T_1)$, and $\omega_2 = \pi(T_2)$, then $\omega_1 \wedge \omega_2 = \pi(T_1 \otimes T_2)$.
  - Note that $\ker(\pi) = \mathcal{I}^k(V)$.
- Properties.
  1. This is well defined, i.e., this does not depend on the choice of T_1, T_2.
     - Consider $T_1 + W_1, T_2 + W_2$ with $W_1, W_2 \in \mathcal{I}^k(V)$.
     - We check that $\pi[(T_1 + W_1) \otimes (T_2 + W_2)] = \pi(T_1 \otimes T_2)$.
     - Since $W_1 \otimes T_2, T_1 \otimes W_2, W_1 \otimes W_2 \in \mathcal{I}^{k+\ell}(V)$, we have that

$$\begin{aligned}\pi[(T_1 + W_1) \otimes (T_2 + W_2)] &= \pi(T_1 \otimes T_2 + W_1 \otimes T_2 + T_1 \otimes W_2 + W_1 \otimes W_2)\\ &= \pi(T_1 \otimes T_2) + \pi(W_1 \otimes T_2) + \pi(T_1 \otimes W_2) + \pi(W_1 \otimes W_2)\\ &= \pi(T_1 \otimes T_2)\end{aligned}$$

  2. Associative: We have that

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge \omega_2 \wedge \omega_3$$

[5]The hooked right arrow denotes an injective map.

- Follows from the definition of $\wedge$ in terms of π and properties of the tensor product.

3. Distributive: We have that

$$(\omega_1+\omega_2)\wedge\omega_3=\omega_1\wedge\omega_3+\omega_2\wedge\omega_3 \qquad \omega_1\wedge(\omega_2+\omega_3)=\omega_1\wedge\omega_2+\omega_1\wedge\omega_3$$

- Follows from the definition of $\wedge$ in terms of π and properties of the tensor product.

4. Linear: We have that

$$(c\omega_1)\wedge\omega_2=c(\omega_1\wedge\omega_2)=\omega_1\wedge(c\omega_2)$$

- Follows from the definition of $\wedge$ in terms of π and properties of the tensor product.

5. Anticommutative: We have that

$$\omega_1\wedge\omega_2=(-1)^{k\ell}\omega_2\wedge\omega_1$$

- It suffices to assume that $w_1=\ell_1\wedge\cdots\wedge\ell_k, w_2=\ell_1'\wedge\cdots\wedge\ell_\ell'$.
  - We have

$$(\ell_1\wedge\cdots\wedge\ell_k)\wedge(\ell_1'\wedge\cdots\wedge\ell_\ell')=(-1)^k(\ell_1'\wedge\cdots\wedge\ell_\ell')\wedge(\ell_1\wedge\cdots\wedge\ell_k)$$

- Let $\ell_1,\dots,\ell_k\in\Lambda^1(V^*)=V^*=\mathcal{L}^1(V)$.
- Recall that $\mathcal{I}^1(V)=\{0\}$.
- Claim: $\ell_{\sigma(1)}\wedge\cdots\wedge\ell_{\sigma(k)}=(-1)^\sigma\ell_1\wedge\cdots\wedge\ell_k$ for all $\sigma\in S_k$.
  - Recall that $T^\sigma=(-1)^\sigma T+W$ for some $W\in\mathcal{I}^k(V)$.
  - Let $T=\ell_1\otimes\cdots\otimes\ell_k$.
  - Then

$$\begin{aligned}(\ell_1\otimes\cdots\otimes\ell_k)^\sigma&=\ell_{\sigma(1)}\otimes\cdots\otimes\ell_{\sigma(k)}\\&=(-1)^\sigma\ell_1\otimes\cdots\otimes\ell_k+W\end{aligned}$$

  - Then hit both sides by π, noting that $\pi(W)=0$.

- Example:
  1. $n=2$, $k=\ell=1$. Consider $e_1^*,e_2^*\in\mathcal{L}^1(V)=V^*=\mathcal{A}^1(V)=\Lambda^1(V^*)$. Then

$$e_1^*\wedge e_2^*=(-1)e_2^*\wedge e_1^* \qquad e_1^*\wedge e_1^*=0=e_2^*\wedge e_2^*$$

  2. $n=4$. We have $e_1^*\wedge(3e_1^*+2e_2^*+3e_2^*)=3(e_1^*\wedge e_1^*)+2(e_1^*\wedge e_2^*)+3(e_1^*\wedge e_3^*)$. We also have $(e_1^*\wedge e_2^*)\wedge(e_1^*\wedge e_2^*)=0$.

- Wedging with zero.
  - We wish to further illustrate the definitions behind the on-the-surface simple concept

$$\omega\wedge 0=0$$

  - Let $\omega\in\Lambda^k(V^*)$ be arbitrary. Let 0 denote the zero function of $\Lambda^\ell(V^*)$.
  - Suppose $\omega=\pi(T_1)$ and $0=\pi(T_2)$.
  - Then

$$\begin{aligned}(\omega\wedge 0)(v_1,\dots,v_{k+\ell})&=\pi[(T_1\otimes T_2)(v_1,\dots,v_{k+\ell})]\\&=\pi[T_1(v_1,\dots,v_k)\cdot T_2(v_{k+1},\dots,v_{k+\ell})]\end{aligned}$$

  - Now there are two cases[6]. We could have $T_2(v_{k+1},\dots,v_{k+\ell})=0$, so the argument of π is 0, so everything is zero. But even if we don't have this, $0=\pi(T_2)$ implies that $T_2=0+T_3$ for some $T_3\in\mathcal{I}^\ell(V)$. Thus, under the projection function π, T_2 behaves like the zero element, regardless.
  - Thus, $\omega\wedge 0$ functions as the zero element of $\Lambda^{k+\ell}(V^*)$.
  - Another, perhaps more accurate, way of looking at this is to see that we need to prove that $T_1\otimes T_2\in\mathcal{I}^{k+\ell}(V)$. To do so, it will suffice to show that $T_2\in\mathcal{I}^\ell(V)$. But since $0=\pi(T_2)$, $T_2\in\ker\pi=\mathcal{I}^\ell(V)$, as desired.

[6]Technically, we only need the second case, but for pedagogical purposes, both are presented.

2.4 Chapter 1: Multilinear Algebra

From Guillemin and Haine (2018).

4/14:

- Having discussed $\text{im}(\text{Alt}) = \mathcal{A}^k(V)$ in some detail now, we move onto $\ker(\text{Alt})$.
- **Redundant** (decomposable k-tensor): A decomposable k-tensor $\ell_1 \otimes \cdots \otimes \ell_k$ such that for some $i \in [k-1]$, $\ell_i = \ell_{i+1}$.
- $\boldsymbol{\mathcal{I}^k(V)}$: The linear span of the set of redundant k-tensors.
- Convention: There are no redundant 1-tensors. Hence, we define

$$\mathcal{I}^1(V) = 0$$

- Proposition 1.5.2: $T \in \mathcal{I}^k(V)$ implies $\text{Alt}(T) = 0$.

  Proof. Let $T = \ell_1 \otimes \cdots \otimes \ell_k$ with $\ell_i = \ell_{i+1}$. Then if $\sigma = \tau_{i,i+1}$, we have that $T^\sigma = T$ and $(-1)^\sigma = -1$. Therefore,

$$\begin{aligned} \text{Alt}(T) &= \text{Alt}(T^\sigma) && \\ &= \text{Alt}(T)^\sigma && \text{Proposition 1.4.17(3)} \\ &= (-1)^\sigma \text{Alt}(T) && \text{Proposition 1.4.17(1)} \\ &= -\text{Alt}(T) \end{aligned}$$

  so we must have that $\text{Alt}(T) = 0$, as desired. □

- Proposition 1.5.3: $T \in \mathcal{I}^r(V)$ and $T' \in \mathcal{L}^s(V)$ imply $T \otimes T', T' \otimes T \in \mathcal{I}^{r+s}(V)$.

  Proof. We first justify why we need only prove this claim for T' decomposable. As an element of $\mathcal{L}^s(V)$, we know that $T' = \sum a_I e_I^*$ for some set of $a_I \in \mathbb{R}$. Since each e_I^* is decomposable, this means that T' is a linear combination of decomposable tensors. This combined with the fact that the tensor product is linear means that

$$T \otimes T' = T \otimes \sum a_I e_I^* = \sum a_I (T \otimes e_I^*)$$

  and similarly for $T' \otimes T$. Thus, if we can prove that each $T \otimes e_I^* \in \mathcal{I}^{r+s}(V)$, it will follow since $\mathcal{I}^k(V)$ is a vector space that $\sum a_I(T \otimes e_I^*) = T \otimes T' \in \mathcal{I}^{r+s}(V)$. In other words, we need only prove that $T \otimes T' \in \mathcal{I}^{r+s}(V)$ for T' decomposable, as desired.

  Let $T = \ell_1 \otimes \cdots \otimes \ell_r$ with $\ell_i = \ell_{i+1}$, and let $T' = \ell_1' \otimes \cdots \otimes \ell_s'$. It follows that

$$T \otimes T' = (\ell_1 \otimes \cdots \otimes \ell_i \otimes \ell_{i+1} \otimes \cdots \otimes \ell_r) \otimes (\ell_1' \otimes \cdots \otimes \ell_s')$$

  is redundant and hence in $\mathcal{I}^{r+s}(V)$, as desired. The argument is symmetric for $T' \otimes T$. □

- Proposition 1.5.4: $T \in \mathcal{L}^k(V)$ and $\sigma \in S_k$ imply

$$T^\sigma = (-1)^\sigma T + S$$

  where $S \in \mathcal{I}^k(V)$.

  Proof. As with Proposition 1.5.3, the linearity of $\sigma : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ allows us to assume that T is decomposable.

  By Theorem 1.4.5, σ can be written as a product of m elementary transpositions. To prove the claim, we induct on m.

For the base case $m = 1$, let $\sigma = \tau_{i,i+1}$. If $T_1 = \ell_1 \otimes \cdots \otimes \ell_{i-1}$ and $T_2 = \ell_{i+2} \otimes \cdots \otimes \ell_k$, then

$$\begin{aligned} T^\sigma - (-1)^\sigma T &= T_1 \otimes (\ell_{i+1} \otimes \ell_i \pm \ell_i \otimes \ell_{i+1}) \otimes T_2 \\ &= T_1 \otimes [(\ell_i + \ell_{i+1}) \otimes (\ell_i + \ell_{i+1}) \mp \ell_i \otimes \ell_i \mp \ell_{i+1} \otimes \ell_{i+1}] \otimes T_2 \end{aligned}$$

i.e., $T^\sigma - (-1)^\sigma T$ is the sum of three redundant k-tensors, and thus a redundant k-tensor in and of itself, as desired. Note that even though only the middle portion is explicitly redundant, Proposition 1.5.3 allows us to call the whole tensor product redundant.

Now suppose inductively that we have proven the claim for $m - 1$. Let $\sigma = \tau\beta$ where β is the product of $m - 1$ elementary transpositions and τ is an elementary transposition. Then

$$\begin{aligned} T^\sigma &= (T^\beta)^\tau && \text{Proposition 1.4.14(3)} \\ &= (-1)^\tau T^\beta + \cdots && \text{Base case} \\ &= (-1)^\tau (-1)^\beta T + \cdots && \text{Inductive hypothesis} \\ &= (-1)^\sigma T + \cdots && \text{Claim 1.4.9} \end{aligned}$$

where the dots are elements of $\mathcal{I}^k(V)$. □

- Corollary 1.5.6: $T \in \mathcal{L}^k(V)$ implies

  $$\operatorname{Alt}(T) = k!T + W$$

  where $W \in \mathcal{I}^k(V)$.

  Proof. By definition,

  $$\operatorname{Alt}(T) = \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma$$

  By Proposition 1.5.4,

  $$T^\sigma = (-1)^\sigma T + W_\sigma$$

  for all $\sigma \in S_k$ with each $W_\sigma \in \mathcal{I}^k(V)$. It follows by combining the above two results that

  $$\operatorname{Alt}(T) = \sum_{\sigma \in S_k} (-1)^\sigma [(-1)^\sigma T + W_\sigma] = \sum_{\sigma \in S_k} T + \sum_{\sigma \in S_k} (-1)^\sigma W_\sigma = k!T + W$$

  where $W = \sum_{\sigma \in S_k} (-1)^\sigma W_\sigma$ is an element of $\mathcal{I}^k(V)$ as a linear combination of elements of $\mathcal{I}^k(V)$. □

- Corollary 1.5.8: Let V be a vector space and $k \geq 1$. Then

  $$\mathcal{I}^k(V) = \ker(\operatorname{Alt} : \mathcal{L}^k(V) \to \mathcal{A}^k(V))$$

  Proof. Suppose first that $T \in \mathcal{I}^k(V)$. Then by Proposition 1.5.2, $\operatorname{Alt}(T) = 0$, so $T \in \ker(\operatorname{Alt})$, as desired.

  Now suppose that $T \in \ker(\operatorname{Alt})$. Then $\operatorname{Alt}(T) = 0$, so by Corollary 1.5.6,

  $$\begin{aligned} 0 &= k!T + W \\ T &= -\frac{1}{k!} W \end{aligned}$$

  Therefore, as a scalar multiple of an element of $\mathcal{I}^k(V)$, $T \in \mathcal{I}^k(V)$. □

- Theorem 1.5.9: Every $T \in \mathcal{L}^k(V)$ has a unique decomposition $T = T_1 + T_2$ where $T_1 \in \mathcal{A}^k(V)$ and $T_2 \in \mathcal{I}^k(V)$.

Proof. By Corollary 1.5.6, we have that

$$\begin{aligned}\operatorname{Alt}(T) &= k!T + W \\ T &= \underbrace{\left(\frac{1}{k!}\operatorname{Alt}(T)\right)}_{T_1} + \underbrace{\left(-\frac{1}{k!}W\right)}_{T_2}\end{aligned}$$

As to uniqueness, suppose $0 = T_1 + T_2$ where $T_1 \in \mathcal{A}^k(V)$ and $T_2 \in \mathcal{I}^k(V)$. Then

$$\begin{aligned}0 &= \operatorname{Alt}(0) = \operatorname{Alt}(T_1 + T_2) = \operatorname{Alt}(T_1) + \operatorname{Alt}(T_2) = k!T_1 + 0 = k!T_1 \\ T_1 &= 0\end{aligned}$$

so $T_2 = 0$, too. □

- $\boldsymbol{\Lambda^k(V^*)}$: The quotient of the vector space $\mathcal{L}^k(V)$ by the subspace $\mathcal{I}^k(V)$. *Given by*

$$\Lambda^k(V^*) = \mathcal{L}^k(V)/\mathcal{I}^k(V)$$

- The quotient map $\pi : \mathcal{L}^k(V) \to \Lambda^k(V^*)$ defined by $T \mapsto T + \mathcal{I}^k(V)$ is onto and has $\ker(\pi) = \mathcal{I}^k(V)$.
- Theorem 1.5.13: $\pi : \mathcal{L}^k(V) \to \Lambda^k(V^*)$ maps $\mathcal{A}^k(V)$ bijectively onto $\Lambda^k(V^*)$.

  Proof. Theorem 1.5.9 implies that every $T + \mathcal{I}^k(V)$ contains a unique $T_1 \in \mathcal{A}^k(V)$. Thus, for every element of $\Lambda^k(V^*)$, there is a unique element of $\mathcal{A}^k(V)$ which gets mapped onto it by π. □

- Note that since $\mathcal{A}^k(V)$ and $\Lambda^k(V^*)$ are in bijective correspondence, many texts do not distinguish between them. There are some advantages to making the distinction, though.
  - We can either look at $\mathcal{A}^k(V)$ as the set of all alternating tensors, or as the set of all k-tensors quotient the redundant tensors.
  - The fact that alternating and redundant tensors are orthogonal can probably be related to the fact that a redundant wedge product equals zero and only non-repeating wedges are nonzero.
- The tensor product and pullback operations give rise to similar operations on the spaces $\Lambda^k(V^*)$.
- **Wedge product**: The function $\wedge : \Lambda^{k_1}(V^*) \times \Lambda^{k_2}(V^*) \to \Lambda^{k_1+k_2}(V^*)$ defined by

$$\omega_1 \wedge \omega_2 = \pi(T_1 \otimes T_2)$$

  where for $i = 1, 2$, $\omega_i \in \Lambda^{k_i}(V^*)$, and $\omega_i = \pi(T_i)$ for some $T_i \in \mathcal{L}^{k_i}(V)$.
  - Note that it is Theorem 1.5.13 that allows us to find T_i such that $\omega_i = \pi(T_i)$.
- Claim 1.6.3: The wedge product is well-defined, i.e., it does not depend on our choices of T_i.

  Proof. We prove WLOG that $\wedge$ is well defined with respect to T_1. Suppose $\omega_1 = \pi(T_1) = \pi(T_1')$. Then by the definition of the quotient map, $T_1' = T_1 + W_1$ for some $W_1 \in \mathcal{I}^{k_1}(V)$. But this means that

$$T_1' \otimes T_2 = (T_1 + W_1) \otimes T_2 = T_1 \otimes T_2 + W_1 \otimes T_2$$

  where $W_1 \otimes T_2 \in \mathcal{I}^{k_1+k_2}(V)$ by Proposition 1.5.3. It follows that

$$\pi(T_1' \otimes T_2) = \pi(T_1 \otimes T_2)$$

  □

- The wedge product also generalizes to higher orders, obeying associativity, scalar multiplication, and distributivity.

- **Decomposable element** (of $\Lambda^k(V^*)$): An element of $\Lambda^k(V^*)$ of the form $\ell_1\wedge\cdots\wedge\ell_k$ where $\ell_1,\dots,\ell_k\in V^*$.
- Claim 1.6.8: The following wedge product identity holds for decomposable elements of $\Lambda^k(V^*)$.

$$\ell_{\sigma(1)}\wedge\cdots\wedge\ell_{\sigma(k)}=(-1)^\sigma\ell_1\wedge\cdots\wedge\ell_k$$

Proof. Let $T=\ell_1\otimes\cdots\otimes\ell_k$. It follows by Proposition 1.4.14(1) that $T^\sigma=\ell_{\sigma(1)}\otimes\cdots\otimes\ell_{\sigma(k)}$. Therefore, we have that

$$\begin{aligned}\ell_{\sigma(1)}\wedge\cdots\wedge\ell_{\sigma(k)}&=\pi(\ell_{\sigma(1)}\otimes\cdots\otimes\ell_{\sigma(k)})\\&=\pi(T^\sigma)\\&=\pi[(-1)^\sigma T+W]\\&=(-1)^\sigma\pi(T)\\&=(-1)^\sigma\pi(\ell_1\otimes\cdots\otimes\ell_k)\\&=(-1)^\sigma\ell_1\wedge\cdots\wedge\ell_k\end{aligned}$$

as desired. $\square$

- An important consequence of Claim 1.6.8 is that

$$\ell_1\wedge\ell_2=-\ell_2\wedge\ell_1$$

  - Why the wedge product is anticommutative but not the tensor product: Because every "tensor" that can be wedged is alternating (as an element of $\Lambda^k(V^*)$). Indeed, the tensor product is anticommutative for alternating tensors.
- Theorem 1.6.10: If $\omega_1\in\Lambda^r(V^*)$ and $\omega_2\in\Lambda^s(V^*)$, then

$$\omega_1\wedge\omega_2=(-1)^{rs}\omega_2\wedge\omega_1$$

  - This can be deduced from Claim 1.6.8.
  - Hint: It suffices to prove this for decomposable elements, i.e., for $\omega_1=\ell_1\wedge\cdots\wedge\ell_r$ and $\omega_2=\ell_1'\wedge\cdots\wedge\ell_s'$.
- Theorem 1.6.13: The elements

$$e_{i_1}^*\wedge\cdots\wedge e_{i_k}^*=\pi(e_I^*)=\pi(e_{i_1}^*\otimes\cdots\otimes e_{i_k}^*)$$

with I strictly increasing are basis vectors of $\Lambda^k(V^*)$.

Proof. Follows from the facts that the ψ_I for I strictly increasing constitute a basis of $\mathcal{A}^k(V)$ by Proposition 1.4.26 and π is an isomorphism $\mathcal{A}^k(V)\to\Lambda^k(V^*)$. $\square$

Week 3

Multilinear Spaces, Operations, and Conventions

3.1 Exterior Powers Basis and the Determinant

4/13: • Plan:

  - Finish multilinear algebra.
  - Basis for $\Lambda^k(V^*)$.
  - Talk a bit about pullbacks and the determinant.
  - **Orientations** of vector spaces.
  - The **interior product**.

- Basis for $\Lambda^k(V^*)$.
  - Recall that $\{\text{Alt}(e_I^*) \mid I \text{ is a nonrepeating, increasing partition of } n \text{ into } k \text{ parts}\}$ is a basis for $\mathcal{A}^k(V)$.
- Alt is an isomorphism from $\Lambda^k(V^*)$ to $\mathcal{A}^k(V)$.
- If we have an injective map from $\mathcal{A}^k(V)$ to $\mathcal{L}^k(V)$ and π a projection map from $\mathcal{L}^k(V)$ to the quotient space $\mathcal{A}^k(V^*)$ gives rise to $\pi|_{\mathcal{A}^k(V)}$.
- Claim:
  1. $\pi|_{\mathcal{A}^k(V)}$ is an isomorphism.
  2. $\pi(\text{Alt}(e_I^*)) = k!\pi(e_I^*)$.

  (2) implies that $\{\pi(e_I^*) = e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*,\ I \text{ non-repeating and increasing}\}$ is a basis for $\Lambda^k(V^*)$.
- Examples:
  1. $n = 2 = \dim V$, $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$.
     - $\Lambda^0(V^*) = \mathbb{R}$ since $\binom{n}{0} = 1$.
     - $\Lambda^1(V^*) = \mathbb{R}e_1^* \oplus \mathbb{R}e_2^*$ since $\binom{n}{1} = 2$.
     - $\Lambda^2(V^*) = \mathbb{R}e_1^* \wedge e_2^*$ since $\binom{n}{2} = 1$.
       - Note that $e_1^* \wedge e_2^* = -e_2^* \wedge e_1^*$.
     - $\Lambda^3(V^*) = 0$ since $\binom{2}{3} = 0$.
       - Note that all $e_1^* \wedge e_1^* \wedge e_2^* = 0$.
  2. $n = 3$, $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3$.

  - $\binom{n}{0} = 1$: $\Lambda^0(V^*) = \mathbb{R}$.
  - $\binom{n}{1} = 3$: $\Lambda^1(V^*) = \mathbb{R}e_1^* \oplus \mathbb{R}e_2^* \oplus \mathbb{R}e_3^*$.
  - $\binom{n}{2} = 3$: $\Lambda^2(V^*) = \mathbb{R}e_1^* \wedge e_2^* \oplus \mathbb{R}e_2^* \wedge e_3^* \oplus \mathbb{R}e_1^* \wedge e_3^*$.
  - $\binom{n}{3} = 1$: $\Lambda^3(V^*) = \mathbb{R}e_1^* \wedge e_2^* \wedge e_3^*$.
  - $\binom{n}{m} = 0$ $(m > n)$: $\Lambda^m(V^*) = \Lambda^4(V^*) = 0$.
- If $A : V \to W$, $\omega_1 \in \Lambda^k(W^*)$, $\omega_2 \in \Lambda^\ell(W^*)$, then

$$A^*(\omega_1 \wedge \omega_2) = A^*\omega_1 \wedge A^*\omega_2$$

- **Determinant**: Let $\dim V = n$. Let $A : V \to V$ be a linear transformation. This induces a pullback $A^* : \Lambda^n(V^*) \to \Lambda^n(V^*)$. The top exterior power $k = n$ implies $\binom{k}{n} = 1$. We define $\det(A)$ to be the unique real number such that $A^*(v) = \det(A)v$.
- This determinant is the one we know.
  - A^* sends $e_1^* \wedge \cdots \wedge e_n^*$ to $A^*e_1^* \wedge \cdots \wedge A^*e_n^*$ which equals $A^*(e_1^* \wedge \cdots \wedge e_n^*)$ or $\det(A)$
- Sanity check.
  1. $\det(\mathrm{id}) = 1$.
     - $\mathrm{id}(e_1^* \wedge \cdots \wedge e_n^*) = \mathrm{id}\, e_1^* \wedge \cdots \wedge \mathrm{id}\, e_n^* = 1 \cdot e_1^* \wedge \cdots \wedge e_n^*$.
  2. If A is not an isomorphism, then $\det(A) = 0$.
     - If A is not an isomorphism, then there exists $v_1 \in \ker A$ with $v_1 \neq 0$. Let $v_1^*, \dots, v_n^*$ be a basis of V^*. So the pullback of this wedge is the wedge of the pullbacks, but $A^*v_1^* = 0$, so

$$A^*(v_1^* \wedge \cdots \wedge v_n^*) = (A^*v_1^*) \wedge \cdots \wedge (A^*v_n^*) = 0 \wedge \cdots \wedge (A^*v_n^*) = 0 = 0 \cdot v_1^* \wedge \cdots \wedge v_n^*$$

  3. $\det(AB) = \det(A)\det(B)$.
     - Let $A : V \to V$ and $B : V \to V$.
     - We have $(AB)^* = B^*A^*$; in particular, $n = k$, $V = W = U = V$.
- Recall: If we pick a basis for V, $e_1, \dots, e_n$.
  - Implies $[a_{ij}] = [A]_{e_1,\dots,e_n}^{e_1,\dots,e_n}$.
- Does $\det(A) = \det([a_{ij}]) = \sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$?
  - If $A : V \to V$, we know that $A^* : \Lambda^n(V^*) \to \Lambda^n(V^*)$ takes $e_1^* \wedge \cdots \wedge e_n^* \mapsto A^*(e_1^* \wedge \cdots \wedge e_n^*)$. We WTS

$$A^*(e_1^* \wedge \cdots \wedge e_n^*) = \left[\sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}\right] e_1^* \wedge \cdots \wedge e_n^*$$

  - We have that

$$\begin{aligned}
A^*(e_1^* \wedge \cdots \wedge e_n^*) &= A^*e_1^* \wedge \cdots \wedge A^*e_n^* \\
&= \left(\sum_{i_1=1}^n a_{i_1,1} e_{i_1}^*\right) \wedge \cdots \wedge \left(\sum_{i_n=1}^n a_{i_n,n} e_{i_n}^*\right) \\
&= \sum_{i_1,\dots,i_n} a_{i_1,1} \cdots a_{i_n,n} e_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \\
&= \left[\sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}\right] e_1^* \wedge \cdots \wedge e_n^*
\end{aligned}$$

    where the sign arises from the need to reorder $e_{i_1}^* \wedge \cdots \wedge e_{i_n}^*$ and the antisymmetry of the wedge product.

3.2 The Interior Product and Orientations

4/15:
- Plan:
  - Orientations.
  - Interior product.
- **Interior product**: We know that $\Lambda^k(V^*) \cong \mathcal{A}^k(V)$. Fix $v \in V$. Define $\iota_v : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$.
  - Wrong way: We take $\iota_v : \mathcal{L}^k(V) \to \mathcal{L}^{k-1}(V)$.

$$T \mapsto \sum_{r=1}^{k} (-1)^{r-1} T(v_1, \dots, v_{r-1}, v, v_r, \dots, v_{k-1})$$

  - Right way: First define $\varphi_v : \mathcal{A}^k(V) \to \mathcal{A}^{k-1}(V)$ by

$$T \mapsto T_v(v_1, \dots, v_{k-1}) = T(v, v_1, \dots, v_{k-1})$$

    - Check:
      1. $T_{v_1+v_2} = T_{v_1} + T_{v_2}$[1].
      2. $T_{\lambda v} = \lambda T_v$.
      3. $\varphi_v^{k-1} \circ \varphi_v^k = 0$ implies $\varphi_v \circ \varphi_w = -\varphi_w \circ \varphi_v$.
- Properties:
  0. $\iota_v T \in \mathcal{L}^{k-1}(V)$.
  1. ι_v is a linear map.
     - This is all happening in the set $\mathrm{Hom}(\mathcal{L}^k(V), \mathcal{L}^{k-1}(V))$.
  2. $\iota_{v_1+v_2} = \iota_{v_1} + \iota_{v_2}$; $\iota_{\lambda v} = \lambda \iota_v$.
  3. "Product rule": If $T_1 \in \mathcal{L}^p(V)$ and $T_1 \in \mathcal{L}^q(V)$, then $\iota_v(T_1 \otimes T_2) = \iota_v T_1 \otimes T_2 + (-1)^p T_1 \otimes \iota_v T_2$.
  4. We have

$$\iota_v(\ell_1 \otimes \cdots \otimes \ell_k) = \sum_{r=1}^{k} (-1)^{r-1} \ell_r(v) \ell_1 \otimes \cdots \otimes \hat{\ell}_r \otimes \cdots \otimes \ell_k$$

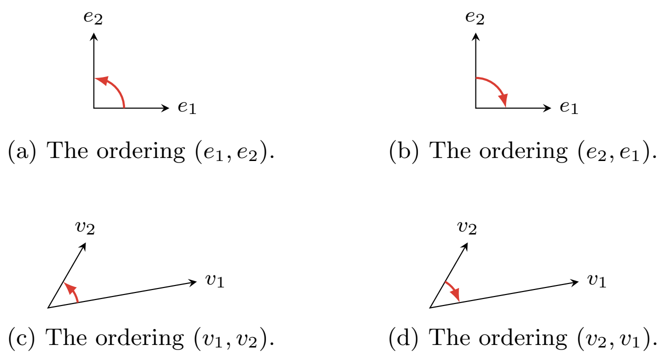
  5. $\iota_v \circ \iota_v = 0 \in \mathrm{Hom}(\mathcal{L}^k(V), \mathcal{L}^{k-2}(V))$.
     - Note that this is related to $d^2 = 0$ from the first day of class (alongside $\int_m \mathrm{d}w = \int_{\partial m} w$).
     - Proof: We induct on k. It suffices to prove the result for T decomposable.
     - Trivial base case for $k = 1$.
     - We have that

$$\begin{aligned}
(\iota_v \circ \iota_v)(\ell_1 \otimes \cdots \otimes \ell_{k-1} \otimes \ell) &= \iota_v(\iota_v T \otimes \ell + (-1)^{k-1} \ell(v) T) \\
&= \iota_v(\iota_v T \otimes \ell) + (-1)^{k-1} \ell(v) \iota_v T \\
&= (-1)^{k-2} \ell(v) \iota_v T + (-1)^{k-1} \ell(v) \iota_v T \\
&= (-1)^{k-2} \ell(v) \iota_v T - (-1)^{k-2} \ell(v) \iota_v T \\
&= 0
\end{aligned}$$

  6. If $T \in \mathcal{I}^k(V)$, then $\iota_v T \in \mathcal{I}^{k-1}(V)$.
     - Thus, ι_v induces a map $\iota_v : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$.
     - Proof: It suffices to check this for decomposables.
  7. $\iota_{v_1} \circ \iota_{v_2} = -\iota_{v_2} \circ \iota_{v_1}$.

[1] Should this be T or φ?

- Orientations.
- Motivating example: Investigating a property of bases that we will later formally define as *orientation*.

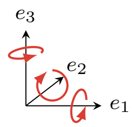


(a) The ordering (e_1, e_2). (b) The ordering (e_2, e_1).

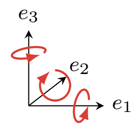
(c) The ordering (v_1, v_2). (d) The ordering (v_2, v_1).

Figure 3.1: Motivating orientations in $\mathbb{R}^2$.

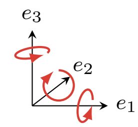
  - Consider the vector space $\mathbb{R}^2$.
  - In both Figures 3.1a and 3.1b, $\{e_1, e_2\}$ is the standard basis of $\mathbb{R}^2$. However, suppose we *order* the elements of this basis, i.e., by using a 2-tuple or ordered pair. There are two possible ways we can order the elements: (e_1, e_2) and (e_2, e_1).
  - Now consider an arbitrary basis $\{v_1, v_2\}$ of $\mathbb{R}^2$. Once again, we have two possible orderings: (v_1, v_2) and (v_2, v_1), as depicted in Figures 3.1c and 3.1d, respectively.
  - What is similar between (e_1, e_2) and (v_1, v_2), and between (e_2, e_1) and (v_2, v_1)?
    - First off, resist the temptation to regard how the vectors are labeled (i.e., with numerical subscripts) as having any bearing on the properties of the vectors themselves. "Both orientations in the first set are labeled $1, 2$, and both orientations in the second set are labeled $2, 1$" is not what we're looking for.
    - What we *are* looking for is the observation that in both Figures 3.1a and 3.1c, the second vector in the ordering is positioned counterclockwise relative to the first vector, and in both Figures 3.1b and 3.1d, the second vector in the ordering is positioned clockwise relative to the first vector. The red arrows in Figures 3.1a-3.1d indicate this relative positioning.
  - Now that we've seen the pattern in a few specific examples, it should not be too hard to see that for *any* ordered basis of $\mathbb{R}^2$ the second vector *must* be positioned either counterclockwise or clockwise relative to the first vector.
    - In fact, taking this farther, let's ask what would happen if a "basis" did not fit this rule. For example, consider the ordered "basis" (e_1, e_1). Clearly, e_1 is not positioned counterclockwise or clockwise to e_1, but rather lies directly on top of it. So, the reader might think, we have an exception! But this is not the case because, as the quotation marks around "basis" indicate, $\{e_1, e_1\}$ is *not* a basis of $\mathbb{R}^2$ (for example, $e_2 \notin \operatorname{span}\{e_1, e_1\}$).
    - Similarly, if our ordered "basis" is like $(e_1, -e_1)$, then we at least have two different basis vectors. However, once again, they are not linearly independent, nor do they span $\mathbb{R}^2$. Therefore, they are not a basis.
  - It follows that the set of ordered bases of $\mathbb{R}^2$ can be partitioned into two equivalence classes: One containing all ordered bases in which the second vector is positioned counterclockwise relative to the first, and the other containing all ordered bases in which the second vector is positioned clockwise relative to the first. All of this arises naturally; the choice of an *orientation* on $\mathbb{R}^2$ is a man-made designation of one of these equivalence classes as the "positive" one and the other as the "negative" one.
- Finding the pattern elsewhere: Three dimensions.



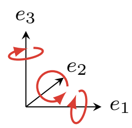
(a) The ordering (e_1, e_2, e_3).



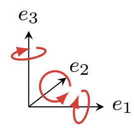
(b) The ordering (e_3, e_1, e_2).



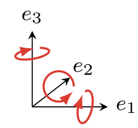
(c) The ordering (e_2, e_3, e_1).



(d) The ordering (e_1, e_3, e_2).



(e) The ordering (e_2, e_1, e_3).



(f) The ordering (e_3, e_2, e_1).

Figure 3.2: Motivating orientations in $\mathbb{R}^3$.

- Here, we will only consider orderings of standard basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$.
- Consider Figure 3.2a.
    - Similarly to in $\mathbb{R}^2$, the positioning of vector 2 relative to vector 1 defines an "orientation," which we take to be about vector 3.
        - The direction of this "orientation" is indicated by the red circular arrow wrapping around e_3 most of the way along it.
    - Differently, however, we can define additional "orientations" about vector 1 and vector 2 using, respectively, the position of vector 3 relative to vector 2, and the position of vector 1 relative to vector 3.
    - Importantly, though, all three of these "orientations" are inextricably linked (note how any time one of them inverts, they all invert), so we can treat these three "orientations" (or any one of them in particular) as one overarching *orientation* on $\mathbb{R}^3$.
- Now unlike in $\mathbb{R}^2$, we now have more than 2 ways of ordering $\{e_1, e_2, e_3\}$; in fact, we have $3! = 6$ such ways. However, we still only have two distinct orientations: Notice that all of the orientations in Figures 3.2a-3.2c are the same, and all of the orientations in Figures 3.2d-3.2f are the same.
    - Thus, we may still partition the set of ordered bases of $\mathbb{R}^3$ into two equivalence classes and arbitrarily label one to be "positive" and the other to be "negative."

- Running with it: Higher dimensional generalizations.
    - Evidently, the paradigm of orientation became much more complicated from $\mathbb{R}^2$ to $\mathbb{R}^3$. As such, it is reasonable to expect that it will become even more complicated in moving to $\mathbb{R}^4$, let alone $\mathbb{R}^n$, especially since these spaces are not so easily visualizable.
        - So how do we proceed? Well, when we've sought to express geometric notions in higher dimensions in the past, linear algebra has been the go-to model. So let's use it once again.
    - We start with two key observations:
        - For any two oriented bases $(v_1, \dots, v_n), (w_1, \dots, w_n)$ in the *same* equivalence class (i.e., both having positive orientation or both having negative orientation), there exists a unique linear transformation T with *positive* determinant such that $w_i = Tv_i$ for $i = 1, \dots, n$.
        - For any two oriented bases $(v_1, \dots, v_n), (w_1, \dots, w_n)$ in *opposite* equivalence classes (i.e., one having positive orientation and the other having negative orientation), there exists a unique linear transformation T with *negative* determinant such that $w_i = Tv_i$ for $i = 1, \dots, n$.
    - Thus, to assign an orientation on $\mathbb{R}^n$, we may take the standard basis in numerical order, i.e., $(e_1, \dots, e_n)$, and define the equivalence class of which it is an element to be the positive one. Then to check the orientation of a given basis $(v_1, \dots, v_n)$, we need only find T and calculate its determinant.

- So, what? Why do we need orientations?
  - Classical single-variable integration determines the "area under the curve" between a, b by summing the height of infinitely many, infinitely small rectangles. In this picture, we can think of $[a, b]$ as our manifold, moving in the positive direction along the x-axis as our orientation, and f as a function on the manifold.
  - For a more general manifold X, we will have a *form* (as opposed to a function) which assigns to each $p \in X$ an exterior product/alternating tensor (as opposed to a scalar). But we will still need to know in which direction we should move along the manifold, and that is the job of the orientation.
  - Additionally, we need a rigorous algebraic theory so that we can relate the orientations of domains to the orientations of their boundaries.
- Formally defining orientations: Relating orientation to the language of differential forms.
  - It is easy to define an orientation on $\mathbb{R}^1$ or any other 1-dimensional vector space. Indeed, we can take e_1 and all of its positive scalar multiples to be the positive equivalence class, and all negative multiples of e_1 to be the negative equivalence class.
    - Note that since 1×1 matrices have determinant equal to their one entry and, for all intents and purposes, function as scalars, this definition agrees with our previous ones.
    - Also, an alternate but equivalent definition of orientation on a 1-dimensional vector space is to divide it into two connected components by removing the zero element from the set and choose one of those connected components to be the "positive orientation." This is the definition we will use going forward.
  - Let V be an arbitrary n-dimensional vector space. Since $\dim \Lambda^n(V^*) = \binom{n}{n} = 1$, we can assign it an orientation based on the above rule.
    - Now let $\omega \in \Lambda^n(V^*)$ lie in the positive component. Then we say that an ordered basis $(v_1, \dots, v_n)$ of V is positive if $\omega(v_1, \dots, v_n) > 0$, and vice versa if $\omega(v_1, \dots, v_n) < 0$.
  - In particular, let $\{e_1, \dots, e_n\}$ be the standard basis of V. Then we call $e_1^* \wedge \cdots \wedge e_n^*$ the **orientation form**.
    - It has the ability to identify the standard basis as positive and some others as positive or negative, but also equals zero for a number of bases.
    - A more powerful tool is $\omega = \text{Alt}(e_1^* \wedge \cdots \wedge e_n^*) \in \Lambda^n(V^*)$. We can define an orientation on $\Lambda^n(V^*)$ based on which half of it ω lies in. Moreover, since ω will take in any ordered basis of V and return a nonzero scalar, it can provide an orientation on V.
  - *Come back after completing integration on manifolds section.*
- An interesting connection between the differential forms definition and the determinant definition, and its formalization as motivation for the new definition of the determinant given in class yesterday.
  - ω and $\det(T)$ may seem like entirely different objects at first glance, but in fact, they are almost identical.
  - Let's drop back down to $\mathbb{R}^2$ to illustrate the point. Here,

$$\omega = \text{Alt}(e_1^* \wedge e_2^*) = e_1^* \wedge e_2^* - e_2^* \wedge e_1^*$$

    - It follows that if

$$\left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\}$$

      is an arbitrary basis of $\mathbb{R}^2$, then the transformation matrix T from positively-oriented ordered basis (e_1, e_2) to (v, w) is

$$T = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$$

which has determinant

$$\det(T) = v_1w_2 - w_1v_2$$

Therefore, whether $v_1w_2 - w_1v_2$ is positive or negative determines whether (v, w) is a positively or negatively oriented basis of $\mathbb{R}^2$.

- Incidentally,

$$\begin{aligned}\omega(v, w) &= [e_1^* \wedge e_2^* - e_2^* \wedge e_1^*](v, w)\\ &= v_1 \cdot w_2 - v_2 \cdot w_1\\ &= v_1w_2 - w_1v_2\end{aligned}$$

as well.

- Therefore, evaluating the sign of $\det(T)$ is identical to evaluating the sign of $\omega(v, w)$.

- But why the relationship?
  - In fact, it's in the definition of the determinant from last class: The determinant of an endomorphism $T : V \to V$ can be interpreted as the induced action of the pullback on the top exterior power.
  - In particular,

$$\begin{aligned}\omega(v_1, \dots, v_n) &= \omega(T(e_1), \dots, T(e_n))\\ &= [T^*\omega](e_1, \dots, e_n)\\ &= \det(T) \cdot \omega(e_1, \dots, e_n)\\ &= \det(T) \cdot 1\\ &= \det(T)\end{aligned}$$

as desired.

- Summary of orientations.
  - A vector space V should have two orientations.
  - Two bases $e_1, \dots, e_n$ and $f_1, \dots, f_n$ are **orientation equivalent** if $T : V \to V$ an isomorphism has positive determinant. Otherwise, they are **orientation inequivalent**.
  - An orientation on V is a choice of equivalence classes of bases under the equivalence relation on bases.
  - $T : V \to W$ given orientations, T preserves or reverses orientations.
- Fancy orientations.
  - An orientation on a 1D vector space L is a division into two halves.
  - Def: An orientation of V is an orientation of $\Lambda^n(V^*)$.
- We can prove that they're both the same.
  - If W and V are both oriented, then V/W gets a canonical orientation.

3.3 Chapter 1: Multilinear Algebra

From Guillemin and Haine (2018).

4/14: • $\boldsymbol{\iota_v T}$: The $(k-1)$-tensor defined as follows. *Given by*

$$(v_1, \dots, v_{k-1}) \mapsto \sum_{r=1}^{k} (-1)^{r-1} T(v_1, \dots, v_{r-1}, v, v_r, \dots, v_{k-1})$$

where $T \in \mathcal{L}^k(V)$, $k \in \mathbb{N}_0$, V is a vector space, and $v \in V$.

- If $v = v_1 + v_2$, then
$$\iota_v T = \iota_{v_1} T + \iota_{v_2} T$$
- If $T = T_1 + T_2$, then
$$\iota_v T = \iota_v T_1 + \iota_v T_2$$
- Lemma 1.7.4: If $T = \ell_1 \otimes \cdots \otimes \ell_k$, then
$$\iota_v T = \sum_{r=1}^{k} (-1)^{r-1} \ell_r(v) \ell_1 \otimes \cdots \otimes \hat{\ell}_r \otimes \cdots \otimes \ell_k$$
where the hat over ℓ_r means that ℓ_r is deleted from the tensor product.
- Lemma 1.7.6: $T_1 \in \mathcal{L}^p(V)$ and $T_2 \in \mathcal{L}^q(V)$ imply
$$\iota_v(T_1 \otimes T_2) = \iota_v T_1 \otimes T_2 + (-1)^p T_1 \otimes \iota_v T_2$$
- Lemma 1.7.8: $T \in \mathcal{L}^k(V)$ implies that for all $v \in V$, we have
$$\iota_v(\iota_v T) = 0$$

  Proof. It suffices by linearity to prove this for decomposable tensors. We induct on k. For the base case $k = 1$, the claim is trivially true. Now suppose inductively that we have proven the claim for $k - 1$. Consider $\ell_1 \otimes \cdots \otimes \ell_k$. Taking $T = \ell_1 \otimes \cdots \otimes \ell_{k-1}$ and $\ell = \ell_k$, we obtain
$$\iota_v(\iota_v(T \otimes \ell)) = \iota_v(\iota_v T) \otimes \ell + (-1)^{k-2} \ell(v) \iota_v T + (-1)^{k-1} \ell(v) \iota_v T$$
  The first term is zero by the inductive hypothesis, and the second two cancel each other out, as desired. □

- Claim 1.7.10: For all $v_1, v_2 \in V$, we have that
$$\iota_{v_1} \iota_{v_2} = -\iota_{v_2} \iota_{v_1}$$

  Proof. Let $v = v_1 + v_2$. Then $\iota_v = \iota_{v_1} + \iota_{v_2}$. Therefore,
$$\begin{aligned} 0 &= \iota_v \iota_v && \text{Lemma 1.7.8} \\ &= (\iota_{v_1} + \iota_{v_2})(\iota_{v_1} + \iota_{v_2}) \\ &= \iota_{v_1}\iota_{v_1} + \iota_{v_1}\iota_{v_2} + \iota_{v_2}\iota_{v_1} + \iota_{v_2}\iota_{v_2} \\ &= \iota_{v_1}\iota_{v_2} + \iota_{v_2}\iota_{v_1} && \text{Lemma 1.7.8} \end{aligned}$$
  yielding the desired result. □

- Lemma 1.7.11: If $T \in \mathcal{L}^k(V)$ is redundant, then so is $\iota_v T$.

  Proof. Let $T = T_1 \otimes \ell \otimes \ell \otimes T_2$ where $\ell \in V^*$, $T_1 \in \mathcal{L}^p(V)$, and $T_2 \in \mathcal{L}^q(V)$. By Lemma 1.7.6, we have that
$$\iota_v T = \iota_v T_1 \otimes \ell \otimes \ell \otimes T_2 + (-1)^p T_1 \otimes \iota_v(\ell \otimes \ell) \otimes T_2 + (-1)^{p+2} T_1 \otimes \ell \otimes \ell \otimes \iota_v T_2$$
  Thus, since the first and third terms above are redundant and $\iota_v(\ell \otimes \ell) = \ell(v)\ell - \ell(v)\ell = 0$ by Lemma 1.7.4, we have the desired result. □

- $\boldsymbol{\iota_v \omega}$: The $\mathcal{I}^k(V)$-coset $\pi(\iota_v T)$, where $\omega = \pi(T)$.
- Proves that $\iota_v \omega$ does not depend on the choice of T.

- **Interior product operation**: The linear map $\iota_v : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$.
- The interior product has the following important identities.

$$\begin{aligned}
\iota_{(v_1+v_2)}\omega &= \iota_{v_1}\omega + \iota_{v_2}\omega \\
\iota_v(\omega_1 \wedge \omega_2) &= \iota_v\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge \omega_2 \\
\iota_v(\iota_v\omega) &= 0 \\
\iota_{v_1}\iota_{v_2}\omega &= -\iota_{v_2}\iota_{v_1}\omega
\end{aligned}$$

4/18:

- As we developed the pullback $A^*T \in \mathcal{L}^k(V)$, we now look to develop a pullback on $\Lambda^k(V^*)$.
- Lemma 1.8.1: If $T \in \mathcal{I}^k(W)$, then $A^*T \in \mathcal{I}^k(V)$.

Proof. It suffices to prove this for redundant k-tensors. Let $T = \ell_1 \otimes \cdots \otimes \ell_k$ be such that $\ell_i = \ell_{i+1}$. Then we have that

$$\begin{aligned}
A^*T &= A^*(\ell_1 \otimes \cdots \otimes \ell_k) \\
&= A^*\ell_1 \otimes \cdots \otimes A^*\ell_k && \text{Exercise 1.3.iii}
\end{aligned}$$

where $A^*\ell_i = A^*\ell_{i+1}$ so that $A^*T \in \mathcal{I}^k(V)$, as desired. ☐

- $\boldsymbol{A^*\omega}$: The $\mathcal{I}^k(W)$-coset $\pi(A^*T)$, where $\omega = \pi(T)$.
- Claim 1.8.3: $A^*\omega$ is well-defined.

Proof. Suppose $\omega = \pi(T) = \pi(T')$. Then $T = T' + S$ where $S \in \mathcal{I}^k(W)$. It follows that $A^*T = A^*T' + A^*S$, but since $A^*S \in \mathcal{I}^k(V)$ (Lemma 1.8.1), we have that

$$\pi(A^*T) = \pi(A^*T')$$

as desired. ☐

- Proposition 1.8.4. The map $A^* : \Lambda^k(W^*) \to \Lambda^k(V^*)$ sending $\omega \mapsto A^*\omega$ is linear. Moreover,
  1. If $\omega_i \in \Lambda^{k_i}(W^*)$ $(i = 1, 2)$, then
  $$A^*(\omega_1 \wedge \omega_2) = A^*(\omega_1) \wedge A^*(\omega_2)$$
  2. If U is a vector space and $B : U \to V$ is a linear map, then for $\omega \in \Lambda^k(W^*)$,
  $$B^*A^*\omega = (AB)^*\omega$$

  (Hint: This proposition follows immediately from Exercises 1.3.iii-1.3.iv.)
- **Determinant** (of A): The number a such that $A^*\omega = a\omega$, where $A^* : \Lambda^n(V^*) \to \Lambda^n(V^*)$. *Denoted by* $\boldsymbol{\det(A)}$.
- Proposition 1.8.7: If A and B are linear mappings of V into V, then

$$\det(AB) = \det(A)\det(B)$$

Proof. Proposition 1.8.4(2) implies that

$$\begin{aligned}
\det(AB)\omega &= (AB)^*\omega \\
&= B^*(A^*\omega) \\
&= \det(B)A^*\omega \\
&= \det(B)\det(A)\omega
\end{aligned}$$

as desired. ☐

- $\mathbf{id}_{\boldsymbol{V}}$: The identity map on V.
- Proposition 1.8.8: $\det(\mathrm{id}_V) = 1$.
  - Hint: id_V^* is the identity map on $\Lambda^n(V^*)$.
- Proposition 1.8.9: If $A : V \to V$ is not surjective, then $\det(A) = 0$.

  Proof. Let $W = \mathrm{im}(A)$. If A is not onto, $\dim W < n$, implying that $\Lambda^n(W^*) = 0$. Now let $A = \imath_W B$ where $\imath_W$ is the inclusion map of W into V and B is the mapping A regarded as a mapping from V to W. It follows by Proposition 1.8.4(1) that if $\omega \in \Lambda^n(V^*)$, then

  $$A^*\omega = B^* \imath_W^* \omega$$

  where $\imath_W^* \omega = 0$ as an element of $\Lambda^n(W^*)$. $\square$

- Deriving the typical formula for the determinant.
  - Let V, W be n-dimensional vector spaces with respective bases $e_1, \dots, e_n$ and $f_1, \dots, f_n$.
  - Denote the corresponding dual bases by $e_1^*, \dots, e_n^*$ and $f_1^*, \dots, f_n^*$.
  - Let $A : V \to W$. Recall that if the matrix of A is $[a_{i,j}]$, then the matrix of $A^* : W^* \to V^*$ is $(a_{j,i})$, i.e., if

    $$Ae_j = \sum_{i=1}^{n} a_{i,j} f_i$$

    then

    $$A^* f_j^* = \sum_{i=1}^{n} a_{j,i} e_i^*$$

  - It follows that

    $$\begin{aligned} A^*(f_1^* \wedge \cdots \wedge f_n^*) &= A^* f_1^* \wedge \cdots \wedge A^* f_n^* \\ &= \sum_{1 \le k_1, \dots, k_n \le n} (a_{1,k_1} e_{k_1}^*) \wedge \cdots \wedge (a_{n,k_n} e_{k_n}^*) \\ &= \sum_{1 \le k_1, \dots, k_n \le n} a_{1,k_1} \cdots a_{n,k_n} e_{k_1}^* \wedge \cdots \wedge e_{k_n}^* \end{aligned}$$

  - At this point, we are summing over all possible lists of length n containing the numbers between 1 and n at each index.
    - However, any list in which a number repeats will lead to a wedge product of a linear functional with itself, making that term equal to zero.
    - Thus, it is only necessary to sum over those terms that are non-repeating.
    - But the terms that are non repeating are exactly the permutations $\sigma \in S_n$.
  - Thus,

    $$\begin{aligned} A^*(f_1^* \wedge \cdots \wedge f_n^*) &= \sum_{\sigma \in S_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} (e_1^* \wedge \cdots \wedge e_n^*)^\sigma \\ &= \sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} e_1^* \wedge \cdots \wedge e_n^* \\ &= \det([a_{i,j}]) e_1^* \wedge \cdots \wedge e_n^* \end{aligned}$$

– If $V = W$ and $e_i = f_i$ ($i = 1, \dots, n$), then we may define $\omega = e_1^* \wedge \cdots \wedge e_n^* = f_1^* \wedge \cdots \wedge f_n^* \in \Lambda^k(V^*)$ to obtain

$$A^*\omega = \det([a_{i,j}])\omega$$

which proves that

$$\det(A) = \det([a_{i,j}])$$

as desired.

- **Orientation** (of ℓ): A choice of one of the disconnected components of $\ell \setminus \{0\}$, where $\ell \subset \mathbb{R}^2$ is a straight line through the origin.
- **Orientation** (of L): A choice of one of the connected components of $L \setminus \{0\}$, where L is a one-dimensional vector space.
- **Positive component** (of $L \setminus \{0\}$): The component chosen in the orientation of L. *Denoted by* $\boldsymbol{L_+}$.
- **Negative component** (of $L \setminus \{0\}$): The component chosen in the orientation of L. *Denoted by* $\boldsymbol{L_-}$.
- **Positively oriented** ($v \in L$): A vector $v \in L$ such that $v \in L_+$.
- **Orientation** (of V) An orientation of the one-dimensional vector space $\Lambda^n(V^*)$, where V is an n-dimensional vector space.
- "One important way of assigning an orientation to V is to choose a basis $e_1, \dots, e_n$ of V. Then if $e_1^*, \dots, e_n^*$ is the dual basis, we can orient $\Lambda^n(V^*)$ by requiring that $e_1^* \wedge \cdots \wedge e_n^*$ be in the positive component of $\Lambda^n(V^*)$" (Guillemin & Haine, 2018, p. 29).
- **Positively oriented** (ordered basis $e_1, \dots, e_n$ of V): An ordered basis $e_1, \dots, e_n \in V$ such that $e_1^* \wedge \cdots \wedge e_n^* \in \Lambda^n(V^*)_+$.
- Proposition 1.9.7: If $e_1, \dots, e_n$ is positively oriented, then $f_1, \dots, f_n$ is positively oriented iff $\det[a_{i,j}] > 0$ where $e_j = \sum_{i=1}^n a_{i,j} f_i$.

  Proof. We have that
  $$f_1^* \wedge \cdots \wedge f_n^* = \det[a_{i,j}] e_1^* \wedge \cdots \wedge e_n^*$$
  ◻

- Corollary 1.9.8: If $e_1, \dots, e_n$ is a positively oriented basis of V, then the basis
  $$e_1, \dots, e_{i-1}, -e_i, e_{i+1}, \dots, e_n$$
  is negatively oriented.
- Theorem 1.9.9: Given orientations on V and V/W (where $\dim V = n > 1$, $W \leq V$, and $\dim W = k < n$), one gets from these orientations a natural orientation on W.

  Proof. The orientations on V and V/W come prepackaged with a basis. We first apply an orientation to W based on these bases, and then show that any choice of basis for $V, V/W$ induces a basis with the same orientation on W. Let's begin.

  Let $r = n - k$, and let $\pi : V \to V/W$. By Exercises 1.2.i and 1.2.ii, we may choose a basis $e_1, \dots, e_n$ of V such that $e_{r+1}, \dots, e_n$ is a basis of W. It follows that $\pi(e_1), \dots, \pi(e_r)$ is a basis of V/W. WLOG[2], take $\pi(e_1), \dots, \pi(e_r)$ and $e_1, \dots, e_n$ to be positively oriented on V/W and V, respectively. Assign to W the orientation associated with $e_{r+1}, \dots, e_n$.

[2]If the first basis is negatively oriented, we may substitute $-e_1$ for e_1. If the second basis is negatively oriented, we may substitute $-e_n$ for e_n.

Now suppose $f_1, \dots, f_n$ is another basis of V such that $f_{r+1}, \dots, f_n$ is a basis of W. Let $A = [a_{i,j}]$ express $e_1, \dots, e_n$ as linear combinations of $f_1, \dots, f_n$, i.e., let

$$e_j = \sum_{i=1}^{n} a_{i,j} f_i$$

for all $j = 1, \dots, n$. Now as will be explained below, A must have the form

$$A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

where B is the $r \times r$ matrix expressing $\pi(e_1), \dots, \pi(e_r)$ as linear combinations of $\pi(f_1), \dots, \pi(f_r)$, and D is the $k \times k$ matrix expressing the basis vectors $e_{r+1}, \dots, e_n$ as linear combinations of $f_{r+1}, \dots, f_n$. We have just explained B and D. We don't particularly care about C or have a good way of defining its structure. We can, however, take the block labeled zero to be the $k \times r$ zero matrix by Proposition 1.2.9; in particular, since these components of these vectors will be fed into π and fall within W, they can moved around wherever without altering the identities of the W-cosets to which they pertain. Having justified this structure for A, we see that we can take

$$\det(A) = \det(B)\det(D)$$

It follows by Proposition 1.9.7 as well as the positivity of $\det(A)$ and $\det(B)$ that $\det(D)$ is positive, and hence the orientation of $e_{r+1}, \dots, e_n$ and $f_{r+1}, \dots, f_n$ are one and the same. $\square$

- **Orientation preserving** (map A): A bijective linear map $A : V_1 \to V_2$, where V_1, V_2 are oriented n-dimensional vector spaces, such that for all $\omega \in \Lambda^n(V_2^*)_+$, we have that $A^*\omega \in \Lambda^n(V_1^*)_+$.
- If $V_1 = V_2$, A is orientation preserving iff $\det(A) > 0$.
- Proposition 1.9.14: Let V_1, V_2, V_3 be oriented n-dimensional vector spaces, and let $A_1 : V_1 \to V_2$ and $A_2 : V_2 \to V_3$ be bijective linear maps. Then if A_1, A_2 are orientation preserving, so is $A_2 \circ A_1$.

Week 4

Differential Forms

4.1 Overview of Differential Forms

4/18:

- Office Hours on Wednesday, 4:00-5:00 PM.
- Plan:
  - An impressionistic overview of what (differential) forms do/are.
  - Tangent spaces.
  - Vector fields/integral curves.
  - 1-forms; a warm-up to k-forms.
- Impressionistic overview of the rest of Guillemin and Haine (2018).
  - An open subset $U \subset \mathbb{R}^n$; $n = 2$ and $n = 3$ are nice.
  - Sometimes, we'll have some functions $F : U \to V$; this is where pullbacks come into play.
  - At every point $p \in U$, we'll define a vector space (the tangent space $T_p\mathbb{R}^n$). Associated to that vector space you get our whole slew of associated spaces (the dual space $T_p^*\mathbb{R}^n$, and all of the higher exterior powers $\Lambda^k(T_p^*\mathbb{R}^n)$).
  - We let $\omega \in \Omega^k(U)$ be a k-form in the space of k-forms.
  - ω assigns (smoothly) to every point $p \in U$ an element of $\Lambda^k(T_p^*\mathbb{R}^n)$.
  - Question: What really is a k-form?
    - Answer: Something that can be integrated on k-dimensional subsets.
    - If $k = 1$, i.e., $\omega \in \Omega^1(U)$, then U can be integrated over curves.
  - If we take $k = 0$, then $\Omega^0(U) = C^\infty(U)$, i.e., the set of all smooth functions $f : U \to \mathbb{R}$.
    - Guillemin and Haine (2018) doesn't, but Klug will and we should distinguish between functions $F : U \to V$ and $f : U \to \mathbb{R}$.
  - We will soon construct a map $\mathrm{d} : \Omega^0(U) \to \Omega^1(U)$ (the **exterior derivative**) that is rather like the gradient but not quite.
    - d is linear.
    - Maps from vector spaces are heretofore assumed to be linear unless stated otherwise.
  - The 1-forms in im(d) are special: $\int_\gamma \mathrm{d}f = f(\gamma(b)) - f(\gamma(a))$ only depends on the endpoints of $\gamma : [a, b] \to U$! The integral is *path-independent*.
  - A generalization of this fact is that instead of integrating along the surface M, we can integrate along the boundary curve:
    $$\int_M \mathrm{d}\omega = \int_{\partial M} \omega$$
    This is **Stokes' theorem**.

- M is a k-dimensional subset of $U \subset \mathbb{R}^n$.

- Note that we have all manner of functions d that we could differentiate between (because they are functions) but nobody does.

$$0 \to \Omega^0(U) \xrightarrow{\mathrm{d}} \Omega^1(U) \xrightarrow{\mathrm{d}} \Omega^2(U) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega^n(U) \xrightarrow{\mathrm{d}} 0$$

- Theorem: $\mathrm{d}^2 = \mathrm{d} \circ \mathrm{d} = 0$.
  - Corollary: $\mathrm{im}(\mathrm{d}^{n-1}) \subset \ker(\mathrm{d}^n)$.
- We'll define $H^k_{dR}(U) = \ker(\mathrm{d}) / \mathrm{im}(\mathrm{d})$.
  - These will be finite dimensional, even though all the individual vector spaces will be infinite dimensional.
  - These will tell us about the shape of U; basically, if all of these equal zero, U is simply connected. If some are nonzero, U has some holes.
- For small values of n and k, this d will have some nice geometric interpretations (div, grad, curl, n'at).
- We'll have additional operations on forms such as the wedge product.

- **Tangent space** (of p): The following set. *Denoted by* $\boldsymbol{T_p\mathbb{R}^n}$. *Given by*

$$T_p\mathbb{R}^n = \{(p, v) : v \in \mathbb{R}^n\}$$

  - This is naturally a vector space with addition and scalar multiplication defined as follows.

$$(p, v_1) + (p, v_2) = (p, v_1 + v_2) \qquad\qquad \lambda(p, v) = (p, \lambda v)$$

  - The point is that

$$T_p\mathbb{R}^n \neq T_q\mathbb{R}^n$$

    for $p \neq q$ even though the spaces are isomorphic.
    - When in $\mathbb{R}^n$ alone, it may seem silly to define what is essentially just $\mathbb{R}^n$ again. After all, in $\mathbb{R}^n$, $(p, v) \in T_p\mathbb{R}^n$ and $(q, v) \in T_q(\mathbb{R}^n)$ both point in the same direction and are basically identical.
    - However, when we get to manifolds (see Figure 8.4), isomorphic tangent spaces may not have vectors that point in the same direction in the space *containing* the manifold!
  - Aside: $F : U \to V$ differentiable and $p \in U$ induce a map $\mathrm{d}F_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ called the "derivative at p."
    - We will see that the matrix of this map is the Jacobian.
  - Chain rule: If $U \xrightarrow{F} V \xrightarrow{G} W$, then

$$\mathrm{d}(G \circ F)_p = \mathrm{d}G_{F(p)} \circ \mathrm{d}F_p$$

- This is round 1 of our discussion on tangent spaces.

- Round 2, later on, will be submanifolds such as T_pM: The tangent space to a point p of a manifold M.

- **Vector field** (on U): A function that assigns to each $p \in U$ an element of $T_p\mathbb{R}^n$.
  - A constant vector field would be $p \mapsto (p, v)$, visualized as a field of vectors at every p all pointing the same direction. For example, we could take $v = (1, 1)$.
  - Special case: $v = e_1, e_2, \dots, e_n$. Here we use the notation $\partial/\partial x_i$ to denote the vector field with $v = e_i$.
  - Example: $n = 2$, $U = \mathbb{R}^2 \setminus \{(0, 0)\}$. We could take a vector field that spins us around in circles.

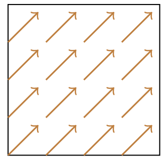


Figure 4.1: The constant vector field $v = (1, 1)$.

- Notice that for all p, $\partial/\partial x_1 |_p, \dots, \partial/\partial x_n |_p \in T_p\mathbb{R}^n$ are a basis of $T_p\mathbb{R}^n$.
  - Thus, any vector field $\boldsymbol{v}$ on U can be written uniquely as
    $$\boldsymbol{v} = f_1 \frac{\partial}{\partial x_1} + \cdots + f_n \frac{\partial}{\partial x_n}$$
    where the $f_1, \dots, f_n$ are functions $f_i : U \to \mathbb{R}$.

4.2 The Lie Derivative and 1-Forms

4/20:

- Plan:
  - Vector fields and their integral curves.
  - Lie derivatives.
  - 1-forms and k-forms.
  - $\Omega^0(U) \xrightarrow{\mathrm{d}} \Omega^1(U)$.
- Notation.
  - $U \subset \mathbb{R}^n$.
  - $\boldsymbol{v}$ denotes a vector field on U.
    - Note that the set of all vector fields on U constitute the vector space $\mathfrak{X}(U)$.
  - $\boldsymbol{v}_p \in T_p\mathbb{R}^n$.
  - $\omega_p \in \Lambda^k(T_p^*\mathbb{R}^n)$.
  - $\partial/\partial x_i |_p = (p, e_i) \in T_p\mathbb{R}^n$.
- Recall that any vector field $\boldsymbol{v}$ on U can be written uniquely as
  $$\boldsymbol{v} = g_1 \frac{\partial}{\partial x_1} + \cdots + g_n \frac{\partial}{\partial x_n}$$
  where the $g_i : U \to \mathbb{R}$.
- **Smooth** (vector field): A vector field $\boldsymbol{v}$ for which all g_i are smooth.
- From now on, we assume unless stated otherwise that all vector fields are smooth.
- **Lie derivative** (of f wrt. $\boldsymbol{v}$): The function $L_{\boldsymbol{v}}f : U \to \mathbb{R}$ defined by $p \mapsto D_{\boldsymbol{v}_p}(f)(p)$, where $\boldsymbol{v}$ is a vector field on U and $f : U \to \mathbb{R}$ (always smooth).
  - Recall that $D_{\boldsymbol{v}_p}(f)(p)$ denotes the directional derivative of f in the direction $\boldsymbol{v}_p$[1] at p.
  - As some examples, we have
    $$L_{\partial/\partial x_i} f = \frac{\partial f}{\partial x_i} \qquad L_{(g_1 \frac{\partial}{\partial x_1} + \cdots + g_n \frac{\partial}{\partial x_n})} f = g_1 \frac{\partial f}{\partial x_1} + \cdots + g_n \frac{\partial f}{\partial x_n}$$

[1] Note that by "in the direction $\boldsymbol{v}_p$," we mean in the direction v where $\boldsymbol{v}_p = (p, v)$.

- Property.
  1. Product rule: $L_{\boldsymbol{v}}(f_1f_2) = (L_{\boldsymbol{v}}f_1)f_2 + f_1(L_{\boldsymbol{v}}f_2)$.
- Later: Geometric meaning to the expression $L_{\boldsymbol{v}}f = 0$.
  - Satisfied iff f is constant on the integral curves of $\boldsymbol{v}$. As if f "flows along" the vector field.
- We define $T_p^*\mathbb{R}^n = (T_p\mathbb{R}^n)^*$.
- 1-forms:
  - A (differential) 1-form on $U \subset \mathbb{R}^n$ is a function $\omega : p \mapsto \omega_p \in T_p^*\mathbb{R}^n$.
  - A "co-vector field."
- Notation: $\mathrm{d}x_i$ is the 1-form that at p is the functional defined by

$$(p, v) \mapsto e_i^*(v)$$

  - For example, if $U = \mathbb{R}^2$ and $\omega = \mathrm{d}x_1$, then ω_p returns (as a scalar) the x_1-component of any vector v fed to it as a $(p, v) \in T_pU$ pair.
- Note: Given any 1-form ω on U, we can write ω uniquely as

$$\omega = g_1 \,\mathrm{d}x_1 + \cdots + g_n \,\mathrm{d}x_n$$

  for some set of smooth $g_i : U \to \mathbb{R}$.
- Notation:
  - $\Omega^1(U)$ is the set of all smooth 1-forms.
  - Notice that $\Omega^1(U)$ is a vector space.
- Given $\omega \in \Omega^1(U)$ and a vector field $\boldsymbol{v}$ on U, we can define $\omega(\boldsymbol{v}) : U \to \mathbb{R}$ by $p \mapsto \omega_p(\boldsymbol{v}_p)$.
- If $U = \mathbb{R}^2$, we have that

$$\mathrm{d}x\left(\frac{\partial}{\partial x}\right) = 1 \qquad\qquad \mathrm{d}x\left(\frac{\partial}{\partial y}\right) = 0$$

  - Note that in the above equation, 1 represents the identity function on U and 0 represents the zero function on $\mathbb{R}^2$.
- $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ are not a basis for $\Omega^1(U)$ since the latter is infinite dimensional.
  - In fact, at each point $p \in U$, we add n dimensions to $\Omega^1(U)$, one for each basis vector of the basis $\partial/\partial x_1 \mid_p, \ldots, \partial/\partial x_n \mid_p$ of $T_p\mathbb{R}^n$.
  - Do not confuse our ability to decompose a one-form to $\sum_{i=1}^n g_i \,\mathrm{d}x_i$ with the $\mathrm{d}x_i$ being a basis for $\Omega^1(U)$. The difference is that the g_i are functions, not constants; if the $\mathrm{d}x_i$ were a basis of $\Omega^1(U)$, then any $\omega \in \Omega^1(U)$ would be able to be decomposed into $\omega = \sum_{i=1}^n c_i \,\mathrm{d}x_i$ for $c_i \in \mathbb{R}$.
- **Exterior derivative** (for 0/1 forms): The function from $\Omega^0(U) \to \Omega^1(U)$ (recall that $\Omega^0(U) \cong C^\infty(U)$) defined as follows. *Denoted by* $\mathbf{d}$. *Given by*

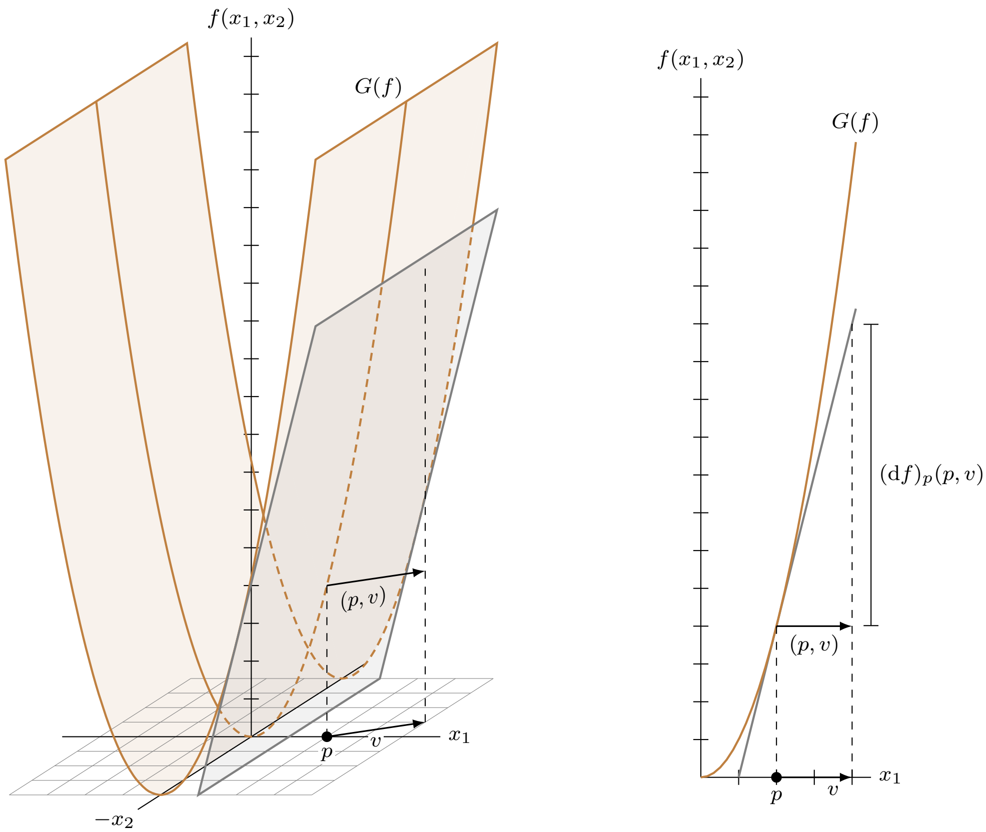
$$f \mapsto \frac{\partial f}{\partial x_1} \,\mathrm{d}x_1 + \cdots + \frac{\partial f}{\partial x_n} \,\mathrm{d}x_n$$

  - This represents the gradient as a 1-form.
- As the notation would suggest, the exterior derivative is nothing but a formalization of the familiar, intuitive concept of the differential.

– From computational calculus, we may intuitively rearrange the Leibniz derivative notation to give a quantity called the differential. For example, if $\mathrm{d}f/\mathrm{d}x = f'$, then the differential is $\mathrm{d}f = f'\,\mathrm{d}x$.

– Note, however, that the exterior derivative generalizes much more nicely than the differential, permitting many later results.

- Example: The exterior derivative d^0 for a function $f : \mathbb{R}^2 \to \mathbb{R}$.



(a) Describing a parabolic cylinder.

(b) A 2D cross section.

Figure 4.2: The exterior derivative d^0 for a function $f : \mathbb{R}^2 \to \mathbb{R}$.

– Consider the coordinate function $x_1 : \mathbb{R}^2 \to \mathbb{R}$. The graph $G(f)$ of the related function $f = x_1^2$ is a parabolic cylinder in $\mathbb{R}^3$. This graph is depicted by the brown surface in Figure 4.2a.

– Let's think about how we would interpret the differential $\mathrm{d}f$ from computational calculus. We would think of it as the infinitesimal change in f as a function of the infinitesimal change in $\mathrm{d}x$ and $\mathrm{d}y$. In this case, we would have $\mathrm{d}f = 2x\,\mathrm{d}x$, which would intuitively tell us something about how much $\mathrm{d}f$ changes (e.g., linearly, just a little bit, but more extremely at higher values of x). We might even picture a few arrows and a right triangle tangent to the graph, but these arrows are infinitely small and we can't describe them any more than via a vague picture. Indeed, this whole idea of "infinitesimals" is not well defined, and the differential is limited in its applications beyond being a small aid to our intuition and a clever bit of notation.

– Now let's build a picture of $\mathrm{d}f$ in the context of differential forms. As a one-form, $\mathrm{d}f$ should take a point $p \in \mathbb{R}^2$ and a vector $v \in \mathbb{R}^2$ and return the instantaneous rate of change of f at p in the direction v (scaled by $|v|$)[2].

[2]Technically, $\mathrm{d}f$ should take a point $p \in \mathbb{R}^2$ to a cotangent vector $(\mathrm{d}f)_p \in T_p^*\mathbb{R}^2$, which in turn takes an object $(p, v) \in T_p\mathbb{R}^2$, isolates the vector component $v \in \mathbb{R}^2$, and return the instantaneous rate of change of f at p in the direction v (scaled by $|v|$).

  - More tangibly, consider the case where $p = (2,0)$ and $v = (2,1)$. Then geometrically, $(\mathrm{d}f)_p(p,v)$ takes us to $(2,0)$ in the x_1x_2-plane, projects us up onto the surface $G(f)$ (i.e., to the point $(2,0,4) \in \mathbb{R}^3$), extends from that point on the surface a vector with x_1-component 2 and x_2-component 1 (this vector lives in $T_p\mathbb{R}^2$), and measures the distance from the tip of this vector to the tangent plane to $G(f)$ at $(2,0,4)$; this distance is 8 units long. Therefore, $(\mathrm{d}f)_p(p,v) = 8$ for p, v as defined.
- Now we know what $\mathrm{d}f$ does. But say we want to express $\mathrm{d}f$ in terms of the basis of $\Omega^1(\mathbb{R}^2)$, i.e., in terms of $\mathrm{d}x_1, \mathrm{d}x_2$, as we would want to to further work with it algebraically.
  - Applying the three properties defining the exterior derivative (see Section 2.4 of Guillemin and Haine (2018)), we can determine that

$$\begin{aligned}\mathrm{d}\big(x_1^2\big) &= \mathrm{d}(x_1 \cdot x_1)\\ &= x_1\,\mathrm{d}x_1 + (-1)^0 x_1\,\mathrm{d}x_1\\ &= 2x_1\,\mathrm{d}x_1\end{aligned}$$

    - ➢ As a sanity check, note that

$$(2x_1\,\mathrm{d}x_1)_p(p,v) = 2 \cdot x_1(p) \cdot (\mathrm{d}x_1)_p(p,v) = 2 \cdot 2 \cdot 2 = 8$$

      as expected.
  - Moreover, this should make intuitive sense. $2x_1$ is the "derivative" of x_1^2 — this means that it tells us the instantaneous rate of change of x_1^2, specifically that of it in the x_1 direction. The only thing left is scaling it appropriately to our direction vector, but $(\mathrm{d}x_1)_p$ takes care of that by isolating the x_1-component of v.
    - ➢ Notice the extreme conceptual similarity (but slight increase in rigor) between this concept and the naïve understanding of the differential.
  - This notion generalizes to functions that have nonzero rates of change in more than one direction at p via the properties of vector addition and the definition of the exterior derivative as the "gradient" expression. Think, for instance, about the paraboloid $f = x_1^2 + x_2^2$.
- Note that since $(\mathrm{d}f)_p$ is a linear transformation, the gray plane in Figure 4.2a is a decent intuitive visualization of the graph of $(\mathrm{d}f)_p$.
  - Delving further into the relationship between $(\mathrm{d}f)_p$ and the total derivative $Df(p)$ of f at p, we know that $Df(p)$ is given by the Jacobian

$$\begin{aligned}Df(p) &= \begin{bmatrix} \left.\frac{\partial f}{\partial x_1}\right|_p & \left.\frac{\partial f}{\partial x_2}\right|_p \end{bmatrix}\\ &= \begin{bmatrix} 4 & 0 \end{bmatrix}\end{aligned}$$

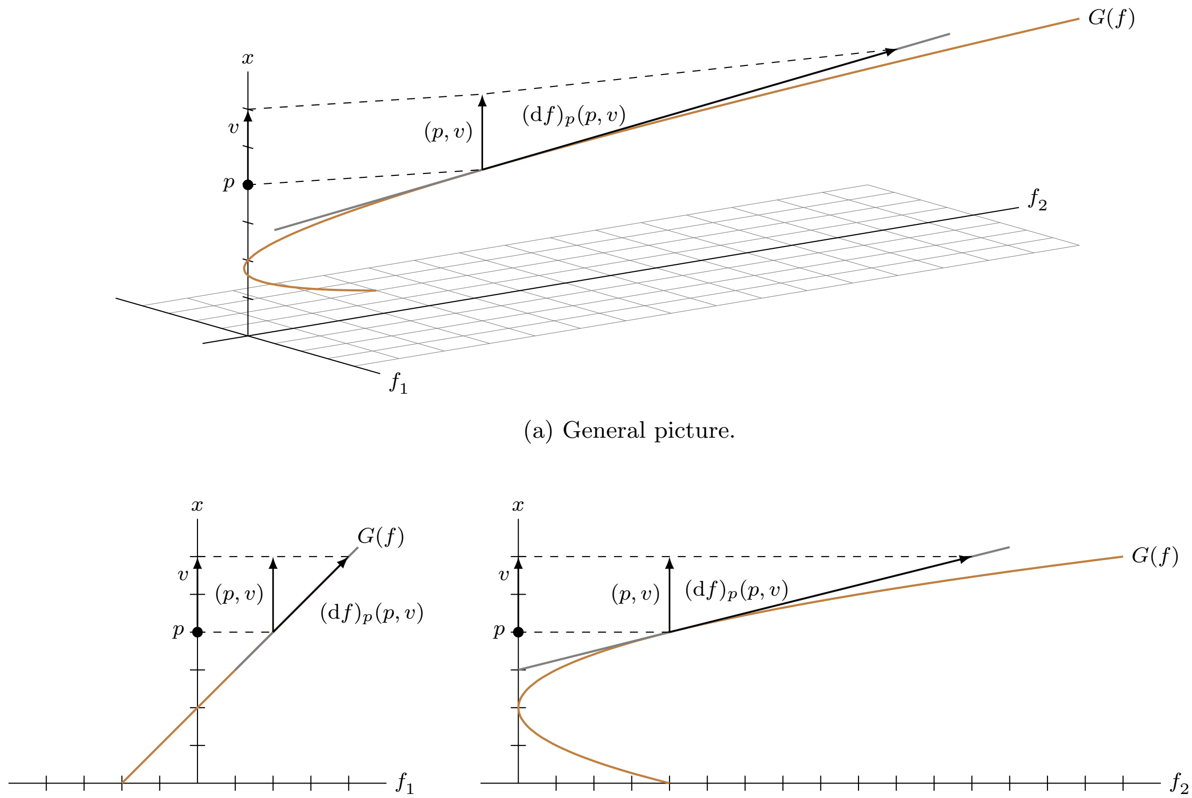
  - This matrix is very closely related to the grey plane in Figure 4.2a. In fact, if we view $Df(p)$ as a function from $\mathbb{R}^2 \to \mathbb{R}$, we realize that the grey plane is just the graph $G(Df(p))$ of $Df(p)$ translated 2 units along the x_1-axis and 4 units along the $f(x_1,x_2)$-axis so as to be tangent to $(p, f(p)) \in \mathbb{R}^3$.
  - Furthermore, this matrix is the function which relates v to $(\mathrm{d}f)_p(p,v)$. Indeed,

$$(\mathrm{d}f)_p(p,v) = \begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

    where v_1 and v_2 are the x_1- and x_2-components of v, respectively. This is exemplified by our specific example since

$$(\mathrm{d}f)_p(p,v) = \begin{bmatrix} 4 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4 \cdot 2 + 0 \cdot 1 = 8$$

    in agreement with the above.



(a) General picture.

(b) View in the $+f_2$ direction.

(c) View in the $-f_1$ direction.

Figure 4.3: The exterior derivative d^0 for a function $f : \mathbb{R} \to \mathbb{R}^2$.

- In fact, from this perspective, we see that the one-form $\mathrm{d}f$ is entirely analogous to the "unpointed" total derivative Df, which should help justify the similarity in notation.

- Example: The exterior derivative d^0 for a function $f : \mathbb{R} \to \mathbb{R}^2$.
  - Consider the parametric function $f : \mathbb{R} \to \mathbb{R}^2$ described by the relation

$$f(t) = \big(t - 2, (t - 2)^2\big)$$

  - One way to visualize the graph $G(f)$ is as a parabola "being drawn" from left to right across $\mathbb{R}^2$. In this mental movie, the third dimension of the graph is time. However, we can equally well use a spatial third dimension, as in Figure 4.3. The 3D and 2D-plus-time pictures are related as follows: Every point of the parabola drawn up until time t in the 2D plus time picture is just every point of $G(f)$ beneath the horizontal plane $z = t$.
  - Now let's move back into differential forms and describe $\mathrm{d}f$. At every point p, the graph of $(\mathrm{d}f)_p$ can be thought of geometrically as a tangent line to $G(f)$. The set of all these tangent lines (and the relation between them and the points p) is contained in $\mathrm{d}f$.
  - As a specific example, consider $p = 4$ and $v = 2$ as in Figure 4.3. Geometrically, we can see that this will lead to

$$(\mathrm{d}f)_p(p, v) = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

- In terms of linear transformations, we have that

$$
\begin{aligned}
Df(p) &= \begin{bmatrix} \left.\frac{\partial f_1}{\partial t}\right|_p \\ \left.\frac{\partial f_2}{\partial t}\right|_p \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 4 \end{bmatrix}
\end{aligned}
$$

so that

$$
(\mathrm{d}f)_p(p,v) = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}
$$

- Thus, we see that one-forms describe the slope of continuous functions that are multivariate in both domain and/or codomain[3].
- Check:
  1. d is linear.
  2. $\mathrm{d}x_i = \mathrm{d}(x_i)$, where $x_i : \mathbb{R}^n \to \mathbb{R}$ is the i^{th} coordinate function.

4.3 Integral Curves

4/22:
- Plan:
  - Clear up a bit of notational confusion.
  - Discuss integral curves of vector fields.
  - k-forms.
  - Exterior derivatives $\mathrm{d} : \Omega^k(U) \to \Omega^{k+1}(U)$ (definition and properties).
- Notation:
  - $F : \mathbb{R}^n \to \mathbb{R}^m$ smooth.
  - We are used to denoting derivatives by big D: $DF_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^m$ where bases of the two spaces are $e_1, \dots, e_n$ and $e_1, \dots, e_m$ has matrix equal to the Jacobian:

$$
[DF_p] = \left[\frac{\partial F_i}{\partial x_j}(p) \right]
$$

  - The book often uses small d: $f : U \to \mathbb{R}$ has $\mathrm{d}f_p : T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}$, where the latter set is isomorphic to $\mathbb{R}$.
  - $\mathrm{d}f$ sends $p \mapsto \mathrm{d}f_p \in T_p^*\mathbb{R}^n$.
  - Klug said

$$
\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \, \mathrm{d}x_i
$$

  - Homework 1 defined $\mathrm{d}f = \mathrm{d}f$?
  - Sometimes three perspectives help you keep this all straight:
    1. Abstract nonsense: The definition of the derivative.
    2. How do I compute it: Apply the formula.
    3. What is it: E.g., magnitude of the directional derivative in the direction of steepest ascent.
- For the homework,

[3]Then what are k-forms for?

  - Let ω be a 1-form in $\Omega^1(U)$.
  - Let $\gamma : [a,b] \to U$ be a curve in U.
  - Then $\mathrm{d}\gamma_p = \gamma_p' : T_p\mathbb{R} \to T_{\gamma(p)}\mathbb{R}^n$ is a function that takes in points of the curve and spits out tangent vectors.
  - Integrating swallows 1-forms and spits out numbers.
$$\int_\gamma \omega = \int_a^b \omega(\gamma'(t))\,\mathrm{d}t$$
  - Problem: If $\omega = \mathrm{d}f$, then
$$\int_\gamma \omega = f(\gamma(b)) - f(\gamma(a))$$
    regardless of the path.
  - Question: Given a 1-form ω, is $\omega = \mathrm{d}f$ for some f?
  - Homework: Explicit U, ω, closed γ such that $\int_\gamma \omega \neq 0$ implies that $\omega \neq \mathrm{d}f$. This motivates and leads into the de Rham cohomology.

- Aside: It won't hurt (for now) to think of 1-forms as vector fields.
- **Integral curve** (for $\boldsymbol{v}$): A curve $\gamma : (a,b) \to U$ such that
$$\gamma'(t) = \boldsymbol{v}_{\gamma(t)}$$
  where $U \subset \mathbb{R}^n$ and $\boldsymbol{v}$ is a (smooth) vector field on U.
- Examples:
  - If $U = \mathbb{R}^2$ and $\boldsymbol{v} = \partial/\partial x$, then the integral curve is the line from left to right traveling at unit speed. The curve has to always have as its tangent vector the unit vector pointing right (which is the vector at every point in the vector field).
  - Vector fields flow everything around. An integral curve is the trajectory of a particle subjected to the vector field as a *velocity* field (the vector field is not a force field or acceleration field).
- Main points:
  1. These integral curves always exist (locally) and often exist globally (cases in which they do are called **complete vector fields**).
  2. They are unique given a starting point $p \in U$.
- An incomplete vector field is one such as the "all roads lead to Rome" vector field where everything always points inward. This is because integral curves cannot be defined for all "time" (real numbers, positive and negative).
- The proofs are in the book; they require an existence/uniqueness result for ODEs and the implicit function theorem.
- Aside: $f : U \to \mathbb{R}$, $\boldsymbol{v}$ a vector field, implies that $L_{\boldsymbol{v}} f = 0$ means that f is constant along all the integral curves of $\boldsymbol{v}$. This also means that f is **integral** for v.
- **Pullback** (of 1-forms): If $F : U \to V$, $\mathrm{d} : \Omega^0(U) \to \Omega^1(U)$, and $\mathrm{d} : \Omega^0(V) \to \Omega^1(V)$, then we get an induced map $F^* : \Omega^0(V) \to \Omega^0(U)$. If $f : V \to \mathbb{R}$, then $f \circ F$ is involved.
  - We're basically saying that if we have $\mathrm{Hom}(A,X)$ (the set of all functions from A to X) and $\mathrm{Hom}(B,X)$, then if we have $F : A \to B$, we get an induced map $F^* : \mathrm{Hom}(B,X) \to \mathrm{Hom}(A,X)$ that is precomposed with F.

4.4 Chapter 2: Differential Forms

From Guillemin and Haine (2018).

5/5:
- Goals for this chapter.
  - Generalize to n dimensions the basic operations of 3D vector calculus (**divergence**, **gradient**, and **curl**).
    - div and grad are pretty straightforward, but curl is more subtle.
  - Substitute **differential forms** for **vector fields** to discover a natural generalization of the operations, in particular, where all three operations are special cases of **exterior differentiation**.
- Introducing vector fields and their dual objects (**one-forms**).
- **Tangent space** (to $\mathbb{R}^n$ at p): The set of pairs (p,v) for all $v\in\mathbb{R}^n$. *Denoted by* $\boldsymbol{T_p\mathbb{R}^n}$. *Given by*

$$T_p\mathbb{R}^n=\{(p,v)\mid v\in\mathbb{R}^n\}$$

- Operations on the tangent space.
  - Directly, we identify $T_p\mathbb{R}^n\cong\mathbb{R}^n$ by $(p,v)\mapsto v$ to make $T_p\mathbb{R}^n$ a vector space.
  - Explicitly, we define

$$(p,v_1)+(p,v_2)=(p,v_1+v_2) \qquad\qquad \lambda(p,v)=(p,\lambda v)$$

  for all $v,v_1,v_2\in\mathbb{R}^n$ and $\lambda\in\mathbb{R}$.
- **Derivative** (of f at p): The linear map from $\mathbb{R}^n\to\mathbb{R}^m$ defined by the following $m\times n$ matrix, where $U\subset\mathbb{R}^n$ is open and $f:U\to\mathbb{R}^m$ is a C^1-mapping. *Denoted by* $\boldsymbol{Df(p)}$. *Given by*

$$Df(p)=\left[\frac{\partial f_i}{\partial x_j}(p)\right]$$

- $\mathbf{d}\boldsymbol{f_p}$: The linear map from $T_p\mathbb{R}^n\to T_q\mathbb{R}^m$ defined as follows, where $U\subset\mathbb{R}^n$ open, $f:U\to\mathbb{R}^m$ is a C^1-mapping, and $q=f(p)$. *Given by*

$$\mathrm{d}f_p(p,v)=(q,Df(p)v)$$

  - Guillemin and Haine (2018) also refer to this as the "base-pointed" version of the derivative of f at p.
- The chain rule for the base-pointed version, where $U\subset\mathbb{R}^n$ open, $f:U\to\mathbb{R}^n$ is a C^1-mapping, $\operatorname{im}(f)\subset V$ open, and $g:V\to\mathbb{R}^k$ is a C^1-mapping.

$$\mathrm{d}g_q\circ\mathrm{d}f_p=\mathrm{d}(g\circ f)_p$$

- Example: The chain rule for single-variable f,g.

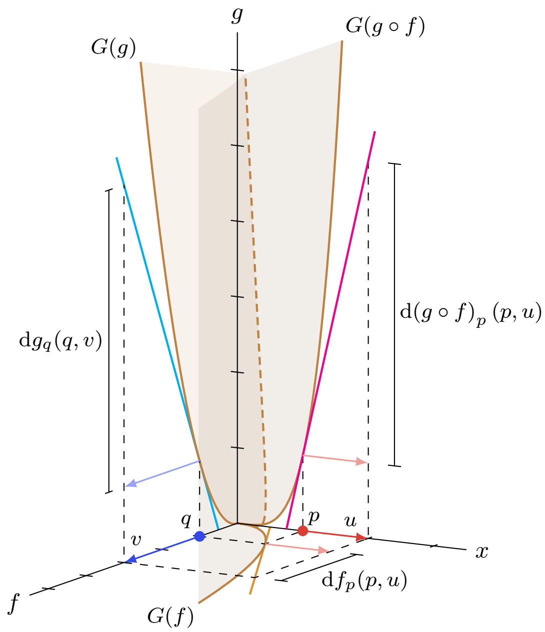


Figure 4.4: The chain rule for single-variable f, g.

- Let $U, V, W = \mathbb{R}$. Consider the functions $f : U \to V$ and $g : V \to W$, both described by the relation
$$x \mapsto x^2$$
- Since the 3 spaces U, V, W are all one-dimensional, a complete geometric representation of the actions of f, g and their composition can be realized in $\mathbb{R}^3$. This is what is depicted in Figure 4.4. Let's now state what the elements of it are.
  - The red dot on the x-axis labeled p depicts a point in U.
  - The red arrow on the x-axis labeled u depicts a vector in U.
    - ➢ Together, p and u depict the tangent vector $(p, u) \in T_pU$.
  - The blue dot on the f-axis labeled q depicts the point $f(p) \in V$.
  - The blue arrow on the f-axis labeled v depicts the vector $Df(p)(u) \in V$.
    - ➢ Together, q and v depict the tangent vector $(q, v) \in T_qV$.
  - The solid brown line in the fx-plane labeled $G(f)$ depicts the graph of f. As we would expect, it is a parabola in x, and a subset of the space $U \times V = \mathbb{R}^2 \subset \mathbb{R}^3 = U \times V \times W$.
  - The solid brown line in the gf-plane labeled $G(g)$ depicts the graph of g. Note that g does not depend on x, but rather takes f as its independent variable. Thus, $G(g)$ is a parabola *in* f, and a subset of the space $V \times W = \mathbb{R}^2 \subset \mathbb{R}^3 = U \times V \times W$. Indeed, it is the depiction of g as a function of f that facilitates composition.
  - This brings us to the solid brown line in the gx-plane labeled $G(g \circ f)$. This line depicts the graph of $g \circ f$. As we can determine from middle-school algebra, $g \circ f$ is a quartic function, and a subset of the space $U \times W = \mathbb{R}^2 \subset \mathbb{R}^3 = U \times V \times W$.
  - The orange line in the fx-plane is tangent to $G(f)$ at (p, q). It will be used to illustrate the relationship between (p, u) and $\mathrm{d}f_p(p, u)$.
  - The cyan line in the gf-plane is tangent to $G(g)$ at $(q, g(q))$. It will be used to illustrate the relationship between (q, v) and $\mathrm{d}g_q(q, v)$.
  - The magenta line in the gx-plane is tangent to $G(g \circ f)$ at $(p, g(q))$. It will be used to illustrate the relationship between (p, u) and $\mathrm{d}(g \circ f)_p\,(p, u)$.
  - The only elements left at this point are the translucent surfaces and their dashed line of intersection. Although it may not be strictly necessary to include these in the diagram,

I believe they more fully illustrate the relationship between all of the parts of this setup. Indeed, this curve in $\mathbb{R}^3$ contains all of the information conveyed by f (via its projection into the fx-plane), by g (via its projection into the gf-plane), and by $g \circ f$ (via its projection into the gx-plane). On the contrary, any one of f, g, or $g \circ f$ is missing some of the information contained in the other two. For example, given the equation $g \circ f = x^4$, there are infinitely many possible functions f, g that satisfy this equation, and one would need to specify either f or g in order to obtain the other uniquely. However, this curve from $U \to U \times V \times W$ says it all.

- The gist of this diagram is that if we want to find the slope of $g \circ f$ at p, we can go about this two ways.
  - Directly, we may plug (p, u) into the covector $\mathrm{d}(g \circ f)_p$.
    - Graphically, this is equivalent to moving (p, u) upwards parallel to the g-axis until p touches $G(g \circ f)$, and then measuring the distance from the tip of the translated vector (shown in light red) to the magenta tangent line.
  - Alternatively, we may rely solely on information about the slopes of f and g independently. Indeed, we may plug (p, u) into $\mathrm{d}f_p$, yielding (q, v), and then plug this result into $\mathrm{d}g_q$.
    - Graphically, this is equivalent to moving (p, u) outwards parallel to the f-axis until p touches $G(f)$ and then measuring both the distance from the base of the translated vector to p, yielding q, and the tip of the translated vector (also shown in light red) to the orange tangent line, yielding v. Having obtained q and v, we could project them onto the f-axis, obtaining a workable input for the next step. This next step is much the same as the first: We move (q, v) upwards parallel to the g-axis until q touches $G(g)$ and then measure the distance from the tip of the translated vector (shown in light blue) to the cyan tangent line.
  - In higher dimensions, the "measuring" described above would have to be done for every relevant component.
- In the specific example drawn, where $p = 1$, $q = f(p) = 1$, $u = 1$, and $v = Df(p)(u) = 2$, we can confirm by inspection that both $\mathrm{d}(g \circ f)_p\,(p, u)$ and $\mathrm{d}g_q(q, v)$ are 4 units long.
- From a computational point of view, we have that

$$\begin{aligned} Df &= \left[\tfrac{\partial f}{\partial x}\right] & Dg &= \left[\tfrac{\partial g}{\partial f}\right] & D(g \circ f) &= \left[\tfrac{\partial g}{\partial x}\right] \\ &= \left[2x\right] & &= \left[2f\right] & &= \left[4x^3\right] \end{aligned}$$

so that

$$\begin{aligned} Df(p) &= \left[\left.\tfrac{\partial f}{\partial x}\right|_p\right] & Dg(q) &= \left[\left.\tfrac{\partial g}{\partial f}\right|_p\right] & D(g \circ f)(p) &= \left[\left.\tfrac{\partial g}{\partial x}\right|_p\right] \\ &= \left[2\right] & &= \left[2\right] & &= \left[4\right] \end{aligned}$$

and hence

$$D(g \circ f)(p) = \left[4\right] = \left[2\right]\left[2\right] = Dg(p) \circ Df(p)$$

- **Vector field** (on $\mathbb{R}^3$): A function which attaches to each point $p \in \mathbb{R}^3$ a base-pointed arrow $(p, v) \in T_p\mathbb{R}^3$.
  - These vector fields are the typical subject of vector calculus.
- **Vector field** (on U): A function which assigns to each point $p \in U$ a vector in $T_p\mathbb{R}^n$, where $U \subset \mathbb{R}^n$ is open. *Denoted by* $\boldsymbol{v}$.
  - We denote the value of $\boldsymbol{v}$ at p by either $\boldsymbol{v}(p)$ or $\boldsymbol{v}_p$.
- **Constant** (vector field): A vector field of the form $p \mapsto (p, v)$, where $v \in \mathbb{R}^n$ is fixed.
- $\boldsymbol{\partial/\partial x_i}$: The constant vector field having $v = e_i$.

- $\boldsymbol{fv}$: The vector field defined on U as follows, where $f : U \to \mathbb{R}$. *Given by*

$$p \mapsto f(p)\boldsymbol{v}(p)$$

  - Note that we are invoking our definition of scalar multiplication on $T_p\mathbb{R}^n$ here.

- **Sum** (of $\boldsymbol{v}_1, \boldsymbol{v}_2$): The vector field on U defined as follows. *Denoted by* $\boldsymbol{v_1} + \boldsymbol{v_2}$. *Given by*

$$p \mapsto \boldsymbol{v}_1(p) + \boldsymbol{v}_2(p)$$

  - Note that we are invoking our definition of addition on $T_p\mathbb{R}^n$ here.

- The list of vectors $(\partial/\partial x_1)_p, \dots, (\partial/\partial x_n)_p$ constitutes a basis of $T_p\mathbb{R}^n$.
  - Recall that $(\partial/\partial x_i)_p = (p, e_i)$.
  - Thus, if $\boldsymbol{v}$ is a vector field on U, it has a unique decomposition

$$\boldsymbol{v} = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i}$$

    where each $g_i : U \to \mathbb{R}$.

- $\boldsymbol{C^\infty}$ (vector field): A vector field such that $g_i \in C^\infty(U)$ for all g_i's in its unique decomposition.

- **Lie derivative** (of f with respect to $\boldsymbol{v}$): The function from $U \to \mathbb{R}$ defined as follows, where $U \subset \mathbb{R}^n$, $f : U \to \mathbb{R}$ is a C^1-mapping, and $\boldsymbol{v}(p) = (p, v)$. *Denoted by* $\boldsymbol{L_v f}$. *Given by*

$$L_{\boldsymbol{v}} f(p) = Df(p)v$$

  - A more explicit formula for the Lie derivative is

$$L_{\boldsymbol{v}} f = \sum_{i=1}^{n} g_i \frac{\partial f}{\partial x_i}$$

  - The vector field decides the direction in which we take the derivative at each point. Instead of having to take a derivative everywhere in one direction at a time, we can now take a derivative in a different direction at every point!

- Lemma 2.1.11: Let U be an open subset of $\mathbb{R}^n$, $\boldsymbol{v}$ a vector field on U, and $f_1, f_2 \in C^1(U)$. Then

$$L_{\boldsymbol{v}}(f_1 \cdot f_2) = L_{\boldsymbol{v}}(f_1) \cdot f_2 + f_1 \cdot L_{\boldsymbol{v}}(f_2)$$

  Proof. See Exercise 2.1.ii. $\square$

- **Cotangent space** (to $\mathbb{R}^n$ at p): The dual vector space to $T_p\mathbb{R}^n$. *Denoted by* $\boldsymbol{T_p^*\mathbb{R}^n}$. *Given by*

$$T_p^*\mathbb{R}^n = (T_p\mathbb{R}^n)^*$$

- **Cotangent vector** (to $\mathbb{R}^n$ at p): An element of $T_p^*\mathbb{R}^n$.

- **Differential one-form** (on U): A function which assigns to each point $p \in U$ a cotangent vector. *Also known as* **one-form** (on U). *Denoted by* $\boldsymbol{\omega}$. *Given by*

$$p \mapsto \omega_p$$

- Note that by identifying $T_q\mathbb{R} \cong \mathbb{R}$, we have that $\mathrm{d}f_p \in T_p^*\mathbb{R}^n$, assuming that $f : U \to \mathbb{R}$.

  - Geometric example: Consider $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f \in C^1$. By the latter condition, we know that the graph of f is a "smooth" surface in $\mathbb{R}^3$, i.e., one without any abrupt changes in derivative (consider the graph of the piecewise function defined by $-x^2$ for $x < 0$ and x^2 for $x \geq 0$, for example). What $\mathrm{d}f_p$ does is take a point (p_1, p_2, q), where $q = f(p)$, on the surface and a vector v with tail at (p, q), and give us a number representing the magnitude of the instantaneous change of f at p in the direction v. Thus, $\mathrm{d}f_p$ contains, in a sense, all of the information concerning the rate of change of f at p.

- $\boldsymbol{\mathrm{d}f}$: The one-form on U defined as follows. *Given by*

$$p \mapsto \mathrm{d}f_p$$

  - Continuing with the geometric example: What $\mathrm{d}f$ does is take every point p across the surface and return all of the information concerning the rate of change of f at p (packaged neatly by $\mathrm{d}f_p$).

- **Pointwise product** (of ϕ with ω): The one-form on U defined as follows, where $\phi : U \to \mathbb{R}$ and ω is a one-form. *Denoted by* $\boldsymbol{\phi\omega}$. *Given by*

$$(\phi\omega)_p = \phi(p)\omega_p$$

- **Pointwise sum** (of ω_1, ω_2): The one-form on U defined as follows. *Denoted by* $\boldsymbol{\omega_1 + \omega_2}$. *Given by*

$$(\omega_1 + \omega_2)_p = (\omega_1)_p + (\omega_2)_p$$

- $\boldsymbol{x_i}$: The function from $U \to \mathbb{R}$ defined as follows. *Given by*

$$x_i(u_1, \dots, u_n) = u_i$$

  - x_i is constantly increasing in the x_i-direction, and constant in every other direction.

- $\boldsymbol{(\mathrm{d}x_i)_p}$: The linear map from $T_p\mathbb{R}^n \to \mathbb{R}$ (i.e., the cotangent vector in $T_p^*\mathbb{R}^n$) defined as follows. *Given by*

$$(\mathrm{d}x_i)_p(p, a_1x_1 + \cdots + a_nx_n) = a_i$$

  - Naturally, the instantaneous change in x_i at any point p in the direction $\boldsymbol{v}(p)$ will just be the magnitude of $\boldsymbol{v}(p)$ in the x_i-direction.
  - Note that as per the discussion associated with Figure 4.2, we can also think of $(\mathrm{d}x_i)_p$ as returning the product of the derivative of x_i at p (which will always be 1, regardless of where p is or which integer i is) and the magnitude of v in the x_i-direction. This notion can be summed up by the statement

$$\mathrm{d}x_i = \mathrm{d}(x_i) = 1\,\mathrm{d}x_i$$

  - It follows immediately that

$$(\mathrm{d}x_i)_p \left(\frac{\partial}{\partial x_j} \right)_p = \delta_{ij}$$

- Consequently, the list of cotangent vectors $(\mathrm{d}x_1)_p, \dots, (\mathrm{d}x_n)_p$ constitutes a basis of $T_p^*\mathbb{R}^n$ that is **dual** to the basis $(\partial/\partial x_1)_p, \dots, (\partial/\partial x_n)_p$ of $T_p\mathbb{R}^n$.

- $\boldsymbol{\mathrm{d}x_i}$: The one-form on U defined as follows. *Given by*

$$p \mapsto (\mathrm{d}x_i)_p$$

  - Thus, if $\omega_p \in T_p^*\mathbb{R}^n$, it has a unique decomposition

$$\omega_p = \sum_{i=1}^{n} f_i(p)(\mathrm{d}x_i)_p$$

    where every $f_i : U \to \mathbb{R}$.

- Similarly, $\omega \in \Omega^1(U)$ has a unique decomposition

$$\omega = \sum_{i=1}^{n} f_i \mathrm{d}x_i$$

- **Smooth** (one-form): A one-form for which the associated functions $f_1, \dots, f_n \in C^\infty$. *Also known as* $\boldsymbol{C^\infty}$ (one-form).
- Lemma 2.1.18: Let U be an open subset of $\mathbb{R}^n$. If $f : U \to \mathbb{R}$ is a C^∞ function, then

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \, \mathrm{d}x_i$$

- **Interior product** (of $\boldsymbol{v}$ with ω): The function on U defined as follows, where $\boldsymbol{v}$ is a vector field over U and ω is a one-form on U. *Denoted by* $\boldsymbol{\iota_v \omega}$. *Given by*

$$p \mapsto \iota_{\boldsymbol{v}(p)} \omega_p$$

  - Note that $\iota_{\boldsymbol{v}(p)} \omega_p$ denotes the interior product of the vector $\boldsymbol{v}(p)$ and the one-tensor ω_p.
  - What we are doing with this definition:
    - We first learned to take the interior product of a vector and a tensor. In particular, for every vector $v \in V$, we defined a function ι_v which took the k-tensor in question to a specifically defined $(k-1)$-tensor.
    - What we are now doing is taking a vector field (a collection of vectors indexed by the points $p \in U$) and a one-form (a collection of 1-tensors indexed by the points $p \in U$) and defining the inner product of a vector field and a one-form as the function which, at each point $p \in U$, evaluates to the inner product of the vector $\boldsymbol{v}(p)$ and the 1-tensor ω_p.
    - This is very much analogous to the step up from cotangent vectors to one-forms (which describe a set of cotangent vectors indexed by the points of a vector space).
- Examples.
  - If

$$\boldsymbol{v} = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i} \qquad\qquad \omega = \sum_{i=1}^{n} f_i \mathrm{d}x_i$$

  then

$$\iota_{\boldsymbol{v}} \omega = \sum_{i=1}^{n} f_i g_i$$

    - Proof: By definition, we know that

$$\boldsymbol{v}(p) = \sum_{i=1}^{n} g_i(p) \left. \frac{\partial}{\partial x_i} \right|_p \qquad\qquad \omega_p = \sum_{i=1}^{n} f_i(p) (\mathrm{d}x_i)_p$$

    It follows by the definition of the interior product of a vector and a tensor that

$$\begin{aligned}
\iota_{\boldsymbol{v}(p)} \omega_p &= \sum_{i=1}^{1} \omega_p(\boldsymbol{v}(p)) \\
&= \omega_p(\boldsymbol{v}(p)) \\
&= \left[\sum_{i=1}^{n} f_i(p) (\mathrm{d}x_i)_p \right] \left(\sum_{j=1}^{n} g_j(p) \left. \frac{\partial}{\partial x_j} \right|_p \right)
\end{aligned}$$

We now invoke linearity.

$$= \sum_{i,j=1}^{n} f_i(p)g_j(p) \cdot (\mathrm{d}x_i)_p \left(\frac{\partial}{\partial x_j}\right)_p$$
$$= \sum_{i,j=1}^{n} f_i(p)g_j(p) \cdot \delta_{ij}$$

All terms where $i \neq j$ are equal to zero, so only the n terms where $i = j$ remain.

$$= \sum_{i=1}^{n} f_i(p)g_i(p)$$

Thus, using the definition of $\iota_{\boldsymbol{v}}\omega$, we have by transitivity that

$$\iota_{\boldsymbol{v}}\omega = \sum_{i=1}^{n} f_i g_i$$

- Notice how the interior product is finally starting to look like a form of multiplication: In particular, we can view the inner product through a naïve lens as "taking the componentwise product of $\boldsymbol{v}$ and ω and using the fact that $(\mathrm{d}x_i)_p(\partial/\partial x_j)_p = \delta_{ij}$ to obtain this result."
- If $\boldsymbol{v}, \omega \in C^\infty$, so is $\iota_{\boldsymbol{v}}\omega$, where C^∞ refers to three different sets of smooth objects (vector fields, one-forms, and functions, respectively[4]).

- As with f, if $\phi \in C^\infty(U)$, then
$$\mathrm{d}\phi = \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} \mathrm{d}x_i$$
- It follows if $\boldsymbol{v}$ is defined as in the first example that
$$\iota_{\boldsymbol{v}}\mathrm{d}\phi = \sum_{i=1}^{n} g_i \frac{\partial \phi}{\partial x_i} = L_{\boldsymbol{v}}\phi$$

- **Integral curve** (of $\boldsymbol{v}$): A C^1 curve $\gamma : (a, b) \to U$ such that for all $t \in (a, b)$,
$$\boldsymbol{v}(\gamma(t)) = (\gamma(t), \gamma'(t))$$
where $U \subset \mathbb{R}^n$ is open and $\boldsymbol{v}$ is a vector field on U.
  - An equivalent condition if $\boldsymbol{v} = \sum_{i=1}^{n} g_i\, \partial/\partial x_i$ and $g : U \to \mathbb{R}^n$ is defined by $(g_1, \dots, g_n)$ is that γ satisfies the system of differential equations
  $$\frac{\mathrm{d}\gamma}{\mathrm{d}t} = g(\gamma(t))$$
    - Verbally, we must have "for all $1 \leq i \leq n$ that the change in γ with respect to t in the x_i-direction is equal to the x_i-component of $\boldsymbol{v}$ at every point in $\gamma((a, b)) \subset U$."

- Theorem 2.2.4 (existence of integral curves): Let $U \subset \mathbb{R}^n$ open, $\boldsymbol{v}$ a vector field on U. If $p_0 \in U$ and $a \in \mathbb{R}$, then there exist $I = (a - T, a + T)$ for some $T \in \mathbb{R}$, $U_0 = N_r(p_0) \subset U$, and, for all $p \in U_0$, an integral curve $\gamma_p : I \to U$ such that $\gamma_p(a) = p$.

- Theorem 2.2.5 (uniqueness of integral curves): Let $U \subset \mathbb{R}^n$ open, $\boldsymbol{v}$ a vector field on U, and $\gamma_1 : I_1 \to U$ and $\gamma_2 : I_2 \to U$ integral curves for $\boldsymbol{v}$. If $a \in I_1 \cap I_2$ and $\gamma_1(a) = \gamma_2(a)$, then
$$\gamma_1|_{I_1 \cap I_2} = \gamma_2|_{I_1 \cap I_2}$$

[4]Technically, these objects are all types of functions, though, so it is fair to call them all smooth.

and the curve $\gamma : I_1 \cup I_2 \to U$ defined by

$$\gamma(t) = \begin{cases} \gamma_1(t) & t \in I_1 \\ \gamma_2(t) & t \in I_2 \end{cases}$$

is an integral curve for $\boldsymbol{v}$.

- Theorem 2.2.6 (smooth dependence on initial data): Let $V \subset U \subset \mathbb{R}^n$ open, $\boldsymbol{v}$ a C^∞-vector field on V, $I \subset \mathbb{R}$ an open interval, and $a \in I$. Let $h : V \times I \to U$ have the following properties.
  1. $h(p, a) = p$.
  2. For all $p \in V$, the curve $\gamma_p : I \to U$ defined by $\gamma_p(t) = h(p, t)$ is an integral curve of $\boldsymbol{v}$.

  Then $h \in C^\infty$.
- **Autonomous** (system of ODEs): A system of ODEs that does not explicitly depend on the independent variable.
- $\mathrm{d}\gamma/\mathrm{d}t = g(\gamma(t))$ is autonomous since g does not depend on t.
- Theorem 2.2.7: Let $I = (a, b)$. For all $c \in \mathbb{R}$, define $I_c = (a-c, b-c)$. If $\gamma : I \to U$ is an integral curve, then the reparameterized curve $\gamma_c : I_c \to U$ defined by

$$\gamma_c(t) = \gamma(t + c)$$

  is an integral curve.
  - Note that this is truly just a reparameterization; we still have, for instance,

$$\gamma_c(a-c) = \gamma(a-c+c) = \gamma(a) \qquad \gamma_c(b-c) = \gamma(b-c+c) = \gamma(b)$$

- **Integral** (of the system $\mathrm{d}\gamma/\mathrm{d}t = g(\gamma(t))$): A C^1-function $\phi : U \to \mathbb{R}$ such that for every integral curve $\gamma(t)$, the function $t \mapsto \phi(\gamma(t))$ is constant.
  - To visualize this definition, consider the case where $\phi : \mathbb{R}^2 \to \mathbb{R}$.
    - Here, the graph $G(\phi)$ of ϕ is a C^1 (continuously differentiable, so continuous and with no abrupt changes in slope) surface in $\mathbb{R}^3$.
    - In particular, what this definition is saying is that if ϕ is an integral of $\boldsymbol{v}$, then projecting an integral curve in $\mathbb{R}^2 \cong \mathbb{R}^2 \times \{0\}$ up onto the surface $G(\phi)$ gives a contour line, i.e., a path along which all points are the same height above the xy-plane.
  - An alternate condition is that if $p = \gamma(t)$ and $v = \boldsymbol{v}(p) = \gamma'(t) = D\gamma(t)$, then for all t,

$$\begin{aligned} 0 &= D(\phi \circ \gamma)(t) = D\phi(\gamma(t)) \cdot D\gamma(t) = D\phi(p)v = L_{\boldsymbol{v}}\phi(p) \\ 0 &= L_{\boldsymbol{v}}\phi(\gamma(t)) \end{aligned}$$

    - To visualize this alternate condition, consider again the case where $\phi : \mathbb{R}^2 \to \mathbb{R}$.
    - Imagine you are standing on the surface $G(\phi)$ and want to walk along it. However, as you walk, you want to stay at the same height above the xy-plane. In other words, you want to walk in the direction such that your change in elevation will be zero. Naturally, at every point along $G(\phi)$ (that is not a local maximum or minimum), there will be such a direction you can walk in. The vectors indicating these directions compose $\boldsymbol{v}$. And naturally, the directional derivative/change in height of ϕ in these directions will be zero. But this directional derivative is just the Lie derivative by definition.
  - Note that taking the gradient (from vector calc) of the integral will not actually return the original vector field; rather, all the vectors in $\nabla\phi$ will be perpendicular to those in $\boldsymbol{v}$.
    - Would curl return the original vector field?

- Theorem 2.2.9: Let $U \subset \mathbb{R}^n$ open, $\phi \in C^1(U)$. Then ϕ is an integral of the system $\mathrm{d}\gamma/\mathrm{d}t = g(\gamma(t))$ iff $L_{\boldsymbol{v}}\phi = 0$.
- **Complete** (vector field): A vector field $\boldsymbol{v}$ on U such that for every $p \in U$, there exists an integral curve $\gamma : \mathbb{R} \to U$ with $\gamma(0) = p$.
  - Alternatively, for every p, there exists an integral curve that starts at p and exists for all time.
- **Maximal** (integral curve): An integral curve $\gamma : [0, b) \to U$ with $\gamma(0) = p$ such that it cannot be extended to an interval $[0, b')$ with $b' > b$.
- For a maximal curve, either...
  1. $b = +\infty$;
  2. $|\gamma(t)| \to +\infty$ as $t \to b$;
  3. The limit points of $\{\gamma(t) \mid 0 \leq t < b\}$ contain elements of the boundary of U.
- Eliminating 2 and 3, as can be done with the following lemma, provides a means of proving that γ exists for all positive time.
- Lemma 2.2.11: The scenarios 2 and 3 above cannot happen if there exists a proper C^1-function $\phi : U \to \mathbb{R}$ with $L_{\boldsymbol{v}}\phi = 0$.

  Proof. Suppose there exists $\phi \in C^1$ such that $L_{\boldsymbol{v}}\phi = 0$. Then ϕ is constant on $\gamma(t)$ (say with value $c \in \mathbb{R}$) by definition. But then since $\{c\} \subset \mathbb{R}$ is compact and $\phi \in C^1$, $\phi^{-1}(c) \subset U$ is compact and, importantly, contains $\operatorname{im}(\gamma)$. The compactness of this set implies that γ can neither "run off to infinity" as in scenario 2 or "run off the boundary" as in scenario 3. $\square$

- Theorem 2.2.12: If there exists a proper C^1-function $\phi : U \to \mathbb{R}$ with the property $L_{\boldsymbol{v}}\phi = 0$, then the vector field $\boldsymbol{v}$ is complete.

  Proof. Apply a similar argument to the interval $(-b, 0]$ and join the two results. $\square$

- Example: Let $U = \mathbb{R}^2$ and let $\boldsymbol{v}$ be the vector field

$$\boldsymbol{v} = x^3\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

  Then $\phi(x, y) = 2y^2 + x^4$ is a proper function with the above property.
  - Note that indeed, as per Theorem 2.2.12, we have that

$$\begin{aligned} L_{\boldsymbol{v}}\phi &= x^3\frac{\partial\phi}{\partial y} - y\frac{\partial\phi}{\partial x} \\ &= x^3 \cdot 4y - y \cdot 4x^3 \\ &= 0 \end{aligned}$$

- We now build up to an alternate completeness condition (Theorem 2.2.15).
- **Support** (of $\boldsymbol{v}$): The following set. *Denoted by* $\mathbf{supp}(\boldsymbol{v})$. *Given by*

$$\operatorname{supp}(\boldsymbol{v}) = \overline{\{q \in U \mid \boldsymbol{v}(q) \neq 0\}}$$

- **Compactly supported** (vector field $\boldsymbol{v}$): A vector field $\boldsymbol{v}$ for which $\operatorname{supp}(\boldsymbol{v})$ is compact.
- Theorem 2.2.15: If $\boldsymbol{v}$ is compactly supported, then $\boldsymbol{v}$ is complete.

Proof. Let $p \in U$ be such that $\boldsymbol{v}(p) = 0$. Define $\gamma_0 : (-\infty, \infty) \to U$ by $\gamma_0(t) = p$ for all $t \in (-\infty, \infty)$. Since

$$\frac{\mathrm{d}\gamma_0}{\mathrm{d}t} = 0 = \boldsymbol{v}(p) = \boldsymbol{v}(\gamma(t))$$

we know that γ_0 is an integral curve of $\boldsymbol{v}$.

Now consider an arbitrary integral curve $\gamma : (-a, b) \to U$ having the property $\gamma(t_0) = p$ for some $t_0 \in (-a, b)$. It follows by Theorem 2.2.5 that γ and γ_0 coincide on the interval $(-a, a)$.

By hypothesis, $\mathrm{supp}(\boldsymbol{v})$ is compact. Basic set theory tells us that for γ arbitrary, either $\gamma(t) \in \mathrm{supp}(\boldsymbol{v})$ for all t or there exists t_0 such that $\gamma(t_0) \in U \setminus \mathrm{supp}(\boldsymbol{v})$. But then by the definition of $\mathrm{supp}(\boldsymbol{v})$, $\boldsymbol{v}(\gamma(t_0)) = 0$. Thus, letting $p = \gamma(t_0)$, we have an associated γ_0 that γ "runs along" while outside the support. It follows that in either case, γ cannot go off to ∞ or go off the boundary of U as $t \to b$. ☐

- Intuition for Theorem 2.2.15.
  - We seek to prove that $\boldsymbol{v}$ is complete. One way we could prove that $\boldsymbol{v}$ is *not* complete is to find an integral curve that "runs off" of U at some point in positive or negative time.
  - With the introduction of the support, we can break down integral curves into two types: Those that remain in $\mathrm{supp}(\boldsymbol{v})$ for all time, necessarily always moving with some nonzero speed; and those that leave $\mathrm{supp}(\boldsymbol{v})$ eventually and "get stuck," i.e., sooner or later find themselves at a fixed point from which they cannot move for the rest of time.
  - It follows since $\mathrm{supp}(\boldsymbol{v})$ is a *compact* subset of an open set U that between $\mathrm{supp}(\boldsymbol{v})$ $\mathbb{R}^n \setminus U$, there is a buffer zone[5] of points $q \in U$ with $\boldsymbol{v}(q) = 0$. Thus, integral curves of the first kind meander forever, never leaving U, and curves of the second kind get stuck before they can. Either way, the curve can be defined for all time t.
- **Bump function**: A function $f : \mathbb{R}^n \to \mathbb{R}$ which is both smooth and compactly supported.
- Example:
  - $\Psi : \mathbb{R} \to \mathbb{R}$ defined by

$$\Psi(x) = \begin{cases} \exp\left(-\frac{1}{1-x^2}\right) & x \in (-1, 1) \\ 0 & \text{otherwise} \end{cases}$$

- $\boldsymbol{C_0^\infty(\mathbb{R}^n)}$: The vector space of all bump functions with domain $\mathbb{R}^n$.
- An application of Theorem 2.2.15.
  - Suppose $\boldsymbol{v}$ is a vector field on U and we want to inspect the integral curves of $\boldsymbol{v}$ on some $A \subset U$ compact. Let $\rho \in C_0^\infty(U)$ be such that $\rho(p) = 1$ for all $p \in N_r(A)$, where $N_r(A)$ is some neighborhood of the set A. Then the vector field $\boldsymbol{w} = \rho\boldsymbol{v}$ is compactly supported and hence complete. However, it is also identical to $\boldsymbol{v}$ on A, so its integral curves on A coincide with those of $\boldsymbol{v}$ on A.
- $\boldsymbol{f_t}$: The map from $U \to U$ defined as follows, where $\boldsymbol{v}$ is complete. *Given by*

$$f_t(p) = \gamma_p(t)$$

  where $\gamma_p : \mathbb{R} \to U$ satisfies $\gamma_p(0) = p$.
  - Note that it is the fact that $\boldsymbol{v}$ is complete that justifies the existence of an integral curve γ_p for all $p \in U$.
  - What f_t does: f_t takes every point in the set/"manifold" U and moves them, along their integral curves as defined by the vector field $\boldsymbol{v}$, to a new point at time t. There are definite parallels to a homotopy herein.

[5]Is there? Consider all roads lead to Rome over the open unit circle in $\mathbb{R}^2$. Curves can just run right off here, right, even though the support is compact?

- Properties of f_t.
  1. $\boldsymbol{v} \in C^\infty$ implies $f_t \in C^\infty$.

     Proof. By Theorem 2.2.6. ☐
  2. $f_0 = \mathrm{id}_U$.

     Proof. We have
     $$f_0(p) = \gamma_p(0) = p = \mathrm{id}_U(p)$$
     as desired. ☐
  3. $f_t \circ f_a = f_{t+a}$.

     Proof. Let $q = f_a(p)$. Since $\boldsymbol{v}$ is complete and $q \in U$, there exists γ_q such that $\gamma_q(0) = q$. It follows that $\gamma_p(a) = f_a(p) = q = \gamma_q(0)$. Thus, by Theorem 2.2.7, $\gamma_q(t)$ and $\gamma_p(t+a)$ are both integral curves of $\boldsymbol{v}$ with the same initial point. Therefore,
     $$(f_t \circ f_a)(p) = f_t(q) = \gamma_q(t) = \gamma_p(t+a) = f_{t+a}(p)$$
     for all t, as desired. ☐
  4. $f_t \circ f_{-t} = \mathrm{id}_U$.

     Proof. See properties 2 and 3. ☐
  5. $f_{-t} = f_t^{-1}$.

     Proof. See property 4. ☐
- Thus, f_t is a C^∞ **diffeomorphism**.
  - "Hence, if $\boldsymbol{v}$ is complete, it generates a **one-parameter group** f_t $(-\infty < t < \infty)$ of C^∞-diffeomorphisms of U" (Guillemin & Haine, 2018, p. 40).
- **Diffeomorphism**: An isomorphism of smooth manifolds. In particular, it is an invertible function that maps one differentiable manifold to another such that both the function and its inverse are differentiable.
- **One-parameter group**: A continuous group homomorphism $\varphi : \mathbb{R} \to G$ from the real line $\mathbb{R}$ (as an additive group) to some other topological group G.
- If $\boldsymbol{v}$ is not complete, there is an analogous result, but it is trickier to formulate.
- ***f*-related** (vector fields $\boldsymbol{v}, \boldsymbol{w}$): Two vector fields $\boldsymbol{v}, \boldsymbol{w}$ such that
  $$\mathrm{d}f_p(\boldsymbol{v}(p)) = \boldsymbol{w}(f(p))$$
  for all $p \in U$, where $\boldsymbol{v}$ is a C^∞-vector field on $U \subset \mathbb{R}^n$ open, $\boldsymbol{w}$ is a C^∞-vector field on $W \subset \mathbb{R}^m$ open, and $f : U \to W$ is a C^∞ map.
- Example: f-related $\boldsymbol{v}, \boldsymbol{w}$ for $f : \mathbb{R} \to \mathbb{R}$.
  - Let $U, W = \mathbb{R}_{\geq 0}$. Consider the function $f : U \to W$ described by the relation
    $$x \mapsto x^2$$
  - Figure 4.5 makes clear that in the same way that p and $f(p)$ are "related" by f, the vectors $\boldsymbol{v}(p)$ and $\boldsymbol{w}(f(p))$ are "related" by $\mathrm{d}f_p$. Indeed, the idea of "f-relatedness" simply implies that every vector in $\boldsymbol{v}(U)$ is so paired with a vector in $\boldsymbol{w}(W)$.
  - Let's now think about what we gain by introducing f-relatedness.

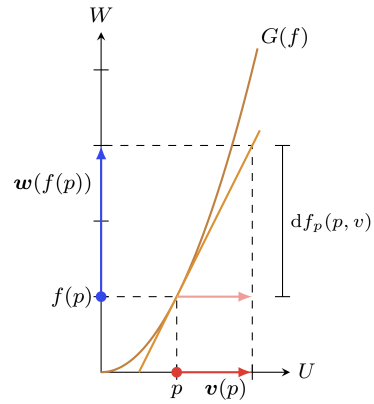


Figure 4.5: f-related $\boldsymbol{v}, \boldsymbol{w}$ for $f : \mathbb{R} \to \mathbb{R}$.

- $\mathrm{d}f_p$, as a linear transformation, takes in any vector and spits out another. $\mathrm{d}f$ attaches such a covector to each point of the domain (into which we can *later* feed whatever vector we want).
- f-relatedness allows us to *preselect* the vectors we want to feed into each $\mathrm{d}f_p$, run them through, and get the results, as indexed by $f(p)$.

– Let's now consider a specific example $\boldsymbol{v}$ and $\boldsymbol{w}$ and check that they are f-related.

- Let

$$\begin{aligned} \boldsymbol{v} : U &\to TU \\ p &\mapsto (p, e_1) \end{aligned} \qquad\qquad \begin{aligned} \boldsymbol{w} : W &\to TW \\ f(p) &\mapsto (f(p), 2p \cdot e_1) \end{aligned}$$

where TU, TW are the **tangent bundles** of U, W, respectively. Note that $\boldsymbol{v} = \partial/\partial x$.

- Let $p \in U$ be arbitrary. We know that

$$\begin{aligned} Df(p) &= \begin{bmatrix} \left.\frac{\partial f}{\partial x}\right|_p \end{bmatrix} \\ &= \begin{bmatrix} 2p \end{bmatrix} \end{aligned} \qquad\qquad \boldsymbol{v}(p) = \begin{bmatrix} 1 \end{bmatrix} \qquad\qquad \boldsymbol{w}(f(p)) = \begin{bmatrix} 2p \end{bmatrix}$$

from which it follows that

$$\mathrm{d}f_p(\boldsymbol{v}(p)) = \begin{bmatrix} 2p \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} 2p \end{bmatrix} = \boldsymbol{w}(f(p))$$

as desired.

- **Tangent bundle** (of a set U): The disjoint union of the tangent spaces of U. *Denoted by* $\boldsymbol{TU}$. *Given by*

$$TU = \bigcup_{p \in U} T_p U$$

- An alternate formulation of f-relatedness.

– In terms of coordinates,

$$w_i(q) = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} v_j(p)$$

where

$$\boldsymbol{v} = \sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i} \qquad\qquad \boldsymbol{w} = \sum_{j=1}^{m} w_j \frac{\partial}{\partial y_i}$$

for $v_i \in C^k(U)$ and $w_j \in C^k(W)$, and $f(p) = q$.

- Derivation: If $f(p) = q$, $\boldsymbol{v}(p) = (p, v)$, $v = (v_1, \dots, v_n)$, $\boldsymbol{w}(q) = (q, w)$, and $w = (w_1, \dots, w_m)$, then

$$\begin{aligned}
\boldsymbol{w}(q) &= \mathrm{d}f_p(\boldsymbol{v}(p)) \\
&= Df(p)v \\
&= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \\
\begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} &= \begin{bmatrix} \sum_{j=1}^n \frac{\partial f_1}{\partial x_j} v_j \\ \vdots \\ \sum_{j=1}^n \frac{\partial f_m}{\partial x_j} v_j \end{bmatrix}
\end{aligned}$$

so comparing i^{th} indices gives the above formula.

- If $m = n$ and f is a C^∞ diffeomorphism, then

$$\boldsymbol{w} = \sum_{i=1}^m w_i \frac{\partial}{\partial y_i}$$

where

$$w_i = \sum_{j=1}^n \left(\frac{\partial f_i}{\partial x_j} v_j \right) \circ f^{-1}$$

- Theorem 2.2.18: If $f : U \to W$ is a C^∞ diffeomorphism and $\boldsymbol{v}$ is a C^∞ vector field on U, then there exists a unique C^∞ vector field $\boldsymbol{w}$ on W having the property that $\boldsymbol{v}$ and $\boldsymbol{w}$ are f-related.

Proof. See the above. □

- **Pushforward** (of $\boldsymbol{v}$ by f): The vector field $\boldsymbol{w}$ shown to exist by Theorem 2.2.18. *Denoted by* $\boldsymbol{f_* v}$.
- Theorem 2.2.20: Let $U_1, U_2 \subset \mathbb{R}^n$ open, $\boldsymbol{v}_1, \boldsymbol{v}_2$ vector fields on U_1, U_2, and $f : U_1 \to U_2$ a C^∞ map. If $\boldsymbol{v}_1, \boldsymbol{v}_2$ are f-related, every integral curve $\gamma : I \to U_1$ of $\boldsymbol{v}_1$ gets mapped by f onto an integral curve $f \circ \gamma : I \to U_2$ of $\boldsymbol{v}_2$.

Proof. We want to show that

$$\boldsymbol{v}_2((f \circ \gamma)(t)) = \left((f \circ \gamma)(t), \tfrac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)\big|_t\right)$$

We are given that

$$\boldsymbol{v}_1(\gamma(t)) = \left(\gamma(t), \tfrac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_t\right) \qquad \mathrm{d}f_p(\boldsymbol{v}_1(p)) = \boldsymbol{v}_2(f(p))$$

Let $p = \gamma(t)$ and $q = f(p)$. Then

$$\begin{aligned}
\boldsymbol{v}_2((f \circ \gamma)(t)) &= \boldsymbol{v}_2(f(p)) \\
&= \mathrm{d}f_p(\boldsymbol{v}_1(p)) \\
&= \mathrm{d}f_p(\boldsymbol{v}_1(\gamma(t))) \\
&= \mathrm{d}f_p\left(\gamma(t), \tfrac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_t\right) \\
&= \mathrm{d}f_p\left(p, \tfrac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_t\right) \\
&= \left(q, Df(p)\left(\tfrac{\mathrm{d}\gamma}{\mathrm{d}t}\Big|_t\right)\right) \\
&= \left((f \circ \gamma)(t), \tfrac{\mathrm{d}}{\mathrm{d}t}(f \circ \gamma)\big|_t\right)
\end{aligned}$$

as desired. □

- Corollary 2.2.21: In the setting of Theorem 2.2.20, suppose $\boldsymbol{v}_1, \boldsymbol{v}_2$ are complete. Let $(f_{i,t})_{t\in\mathbb{R}} : U_i \to U_i$ be the one-parameter group of diffeomorphisms generated by $\boldsymbol{v}_i$. Then

$$f \circ f_{1,t} = f_{2,t} \circ f$$

Proof. We have that

$$(f \circ f_{1,t})(p) = (f \circ \gamma_p)(t)$$

By Theorem 2.2.20, the above is an integral curve of $\boldsymbol{v}_2$. Let $f(p) = q$. Then

$$\begin{aligned}(f_{2,t} \circ f)(p) &= f_{2,t}(q)\\ &= \gamma_q(t)\end{aligned}$$

...

Guillemin and Haine (2018) proves that if $\phi \in C^\infty(U_2)$ and $f^*\phi = \phi \circ f$, then

$$f^* L_{\boldsymbol{v}_2}\phi = L_{\boldsymbol{v}_1} f^*\phi$$

by virtue of the observations that if $f(p) = q$, then at the point p, the right-hand side above is $(\mathrm{d}\phi)_q \circ \mathrm{d}f_p(\boldsymbol{v}_1(p))$ by the chain rule and by definition the left hand side is $\mathrm{d}\phi_q(\boldsymbol{v}_2(q))$. Moreover, by definition, $\boldsymbol{v}_2(q) = \mathrm{d}f_p(\boldsymbol{v}_1(p))$ so the two sides are the same. ☐

- Theorem 2.2.22: For $i = 1,2,3$, let $U_i \subset \mathbb{R}^{n_i}$ open and $\boldsymbol{v}_i$ a vector field on U_i. For $i = 1,2$, let $f_i : U_i \to U_{i+1}$ be a C^∞ map. If $\boldsymbol{v}_1, \boldsymbol{v}_2$ are f_1-related and $\boldsymbol{v}_2, \boldsymbol{v}_3$ are f_2-related, then $\boldsymbol{v}_1, \boldsymbol{v}_3$ are $(f_2 \circ f_1)$-related. In particular, if f_1, f_2 are diffeomorphisms, we have

$$(f_2)_*(f_1)_*\boldsymbol{v}_1 = (f_2 \circ f_1)_*\boldsymbol{v}_1$$

- **Pullback** (of μ along f): The one-form on U defined as follows, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open, $f : U \to V$ is a C^∞ map, and μ is a one-form on V. *Denoted by* $\boldsymbol{f^*\mu}$. *Given by*

$$p \mapsto \mu_{f(p)} \circ \mathrm{d}f_p$$

  - This definition does stick with the theme of pullbacks being precompositions; it is just a bit more complicated because a one-form takes two inputs instead of one: a point and a tangent vector.
  - To feed a point $p \in U$ and a vector $v \in U$ into a one-form on V, we use f to send $p \mapsto f(p)$ and $\mathrm{d}f_p$ to send $v \mapsto \mathrm{d}f_p(p,v)$. Hence the above definition.

- If $\phi : V \to \mathbb{R}$ is a C^∞ map and $\mu = \mathrm{d}\phi$, then

$$\mu_q \circ \mathrm{d}f_p = \mathrm{d}\phi_q \circ \mathrm{d}f_p = \mathrm{d}(\phi \circ f)_p$$

  - In other words,

$$f^*\mu = \mathrm{d}\phi \circ f$$

- Theorem 2.2.24: If μ is a C^∞ one-form on V, its pullback $f^*\mu$ is C^∞.

Proof. See Exercise 2.2.ii. ☐

Week 5

Differentiation

5.1 k-Forms

4/25: • Definitions and examples of k-forms.

5.2 Vector Calculus Operations

4/27: • Announcements.

  - No class this Friday, next Monday.
  - Midterm next Friday.
    - Up through Chapter 2.
    - The exam will likely be computationally heavy.
    - Compute d, pullbacks, interior products, Lie derivatives, etc.
    - Emphasis on Chapter 2 as opposed to Chapter 1 even though it all builds on itself.
    - He'll probably cook up a few problems too.
  - There is a recorded lecture for us.
    - On Chapter 3 content.
    - We'll cover Chapter 3 in kind of an impressionistic way as it is.
  - There are also some notes on the physics stuff.

- Vector calculus operations.
  - In one dimension, you have functions, and you take derivatives.
    - The derivative operation does essentially map $\Omega^0 \to \Omega^1$ or $C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$.
  - In two dimensions, ...
    - $\mathrm{d}^2 = 0$ reflects the fact that gradient vector fields are curl-free.
  - If you want to understand the 2D-curl business...
    - $\mathrm{curl}(v) : \mathbb{R}^2 \to \mathbb{R}$ is intuitively about balls spinning around in a vector field.
    - There's also a nice formula to compute it.
    - And then there's a connection with $\mathrm{d} : \Omega^1 \to \Omega^2$.
  - In 3D, you can take top-dimensional forms (which are just functions) and bottom-dimensional forms (which are by definition functions) and you can work out an identification between them.
  - Note that $\mathrm{curl} : \mathfrak{X}(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$, where $\mathfrak{X}(\mathbb{R}^2)$ is the space of vector fields.

- The musical operator $\sharp$ identifies forms with vector fields, i.e., $\sharp : \Omega^1 \to \mathfrak{X}(\mathbb{R}^2)$.

- Properties of exterior derivatives $\mathrm{d} : \Omega^k(U) \to \Omega^{k+1}(U)$.
  1. $\mathrm{d}(\omega_1 + \omega_2) = \mathrm{d}\omega_1 + \mathrm{d}\omega_2$ and $\mathrm{d}(\lambda\omega) = \lambda\,\mathrm{d}\omega$.
  2. Product rule $\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge \mathrm{d}\omega_2$.
     - Special case $k = \ell = 0$. Then
     $$\mathrm{d}(fg) = g\,\mathrm{d}f + f\,\mathrm{d}g$$
     which is the usual product rule for gradient.
     - Claim:
     $$\mathrm{d}\left(\sum_I f_I\,\mathrm{d}x_I\right) = \sum_I \mathrm{d}f_I \wedge \mathrm{d}x_I$$
       - Let $\omega_1 \in \Omega^k$ and $\omega_2 \in \Omega^\ell$ be defined by
       $$\omega_1 = \sum_I f_I\,\mathrm{d}x_I \qquad\qquad \omega_2 = \sum_J g_J\,\mathrm{d}x_J$$
       where we're summing over all I such that $|I| = k$ and all J such that $|J| = \ell$. Then
       $$\begin{aligned}\omega_1 \wedge \omega_2 &= \sum_{I,J} f_I g_J\,\mathrm{d}x_I \wedge \mathrm{d}x_J\\ \mathrm{d}(\omega_1 \wedge \omega_2) &= \sum_{I,J} \mathrm{d}(f_I g_J) \wedge \mathrm{d}x_I \wedge \mathrm{d}x_J\end{aligned}$$
       - Note that
       $$\mathrm{d}(f_I g_J) = g_J\,\mathrm{d}f_I + f_I\,\mathrm{d}g_J$$
       and
       $$\mathrm{d}g_J \wedge \mathrm{d}x_I = (-1)^k\,\mathrm{d}x_I \wedge \mathrm{d}g_J$$
       - These identities allow us to take the previous equation to
       $$\begin{aligned}\mathrm{d}(\omega_1 \wedge \omega_2) &= \sum_{I,J} g_J\,\mathrm{d}f_I \wedge \mathrm{d}x_I \wedge \mathrm{d}x_J + (-1)^k f_I\,\mathrm{d}x_I \wedge \mathrm{d}g_J \wedge \mathrm{d}x_J\\ &= \sum_{I,J} (\mathrm{d}f_I \wedge \mathrm{d}x_I) \wedge (g_J\,\mathrm{d}x_J) + \sum_{I,J} (f_I\,\mathrm{d}x_I) \wedge (\mathrm{d}g_J \wedge \mathrm{d}x_J)\\ &= \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^k \omega_1\,\mathrm{d}\omega_2\end{aligned}$$
  3. $\mathrm{d}^2 = 0$.
     - Let $\omega = \sum_I f_I\,\mathrm{d}x_I$.
     - Then
     $$\begin{aligned}\mathrm{d}^2(\omega) &= \mathrm{d}(\mathrm{d}\omega)\\ &= \mathrm{d}\left(\sum_I \mathrm{d}f_I \wedge \mathrm{d}x_I\right)\\ &= \sum_I \mathrm{d}(\mathrm{d}f_I \wedge \mathrm{d}x_I) && \text{Property 1}\\ &= \sum_I \mathrm{d}(\mathrm{d}f_I) \wedge \mathrm{d}x_I && \text{Property 2}\end{aligned}$$
     so it suffices to just show that $\mathrm{d}^2 f = 0$ for all $f \in \Omega^0$.

- We know that $\mathrm{d}f = \sum_{i=1}^n \partial f/\partial x_i \, \mathrm{d}x_i$. Thus,

$$\begin{aligned}
\mathrm{d}(\mathrm{d}f) &= \sum_i \mathrm{d}\left(\frac{\partial f}{\partial x_i}\right) \wedge \mathrm{d}x_i \\
&= \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i} \, \mathrm{d}x_j \wedge \mathrm{d}x_i \\
&= 0
\end{aligned}$$

- The last equality holds because of commuting partial derivatives for smooth f, and the fact that changing order introduces a negative sign by some property.

- In fact, if we fix $\mathrm{d}^0 : \Omega^0(U) \to \Omega^1(U)$ to be the "gradient," then these properties characterize the function d on its domain and codomain. In particular, d is the unique function on its domain and codomain that satisfies these properties.
  - We define it by
$$\mathrm{d}\left(\sum_I f_I \, \mathrm{d}x_I\right) = \sum_I \mathrm{d}f_I \wedge \mathrm{d}x_I$$
  - The above properties characterize it axiomatically.
  - We can prove this uniqueness theorem.
- **Closed** (form): A form $\omega \in \Omega^k(U)$ such that $\mathrm{d}\omega = 0$.
- **Exact** (form): A form $\omega \in \Omega^k(U)$ such that $\omega = \mathrm{d}\eta$ for some $\eta \in \Omega^{k-1}(U)$.
- $\mathrm{d}^2 = 0$ implies closed and exact implies closed.
- **Poincaré lemma**: Locally closed forms are exact.

5.3 Chapter 2: Differential Forms

From Guillemin and Haine (2018).

5/5:
- As we formed the k^{th} exterior powers $\Lambda^k(V^*)$, we can form the k^{th} exterior powers $\Lambda^k(T_p^*\mathbb{R}^n)$.
  - In particular, we can take the vector space $\mathcal{L}^k(T_p\mathbb{R}^n)$ (of all k-tensors on the tangent space of p) and the span $\mathcal{I}^k(T_p\mathbb{R}^n)$ (of all redundant k-tensors on the tangent space of p) and form their quotient space $\Lambda^k(T_p^*\mathbb{R}^n)$.
  - This quotient space will be isomorphic to the set $\mathcal{A}^k(T_p\mathbb{R}^n)$ of all alternating tensors on the tangent space of p. As such, elements of $\Lambda^k(T_p^*\mathbb{R}^n)$ can be thought of as k-linear alternating tensors.
- Since $\Lambda^1(T_p^*\mathbb{R}^n) = T_p^*\mathbb{R}^n$, we can think of a one-form as a function which takes as its value at p an element of the space $\Lambda^1(T_p^*\mathbb{R}^n)$.
- ***k*-form** (on U): A function which assigns to each point $p \in U$ an element $\omega_p \in \Lambda^k(T_p^*\mathbb{R}^n)$, where $U \subset \mathbb{R}^n$ is open.
- We can use the wedge product to construct k-forms.
  - Let $\omega_1, \dots, \omega_k$ be one-forms. Then $\omega_1 \wedge \cdots \wedge \omega_k$ is the k-form whose value at p is the wedge product
$$(\omega_1 \wedge \cdots \wedge \omega_k)_p = (\omega_1)_p \wedge \cdots \wedge (\omega_k)_p$$

  - Let $f_1,\dots,f_k$ be real-valued functions in $C^\infty(U)$. Suppose $\omega_i = \mathrm{d}f_i$. Then we may obtain the k-form whose value at p is

$$(\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k)_p = (\mathrm{d}f_1)_p \wedge \cdots \wedge (\mathrm{d}f_k)_p$$

- Since $(\mathrm{d}x_1)_p,\dots,(\mathrm{d}x_n)_p$ are a basis of $T_p^*\mathbb{R}^n$, the wedge products

$$(\mathrm{d}x_I)_p = (\mathrm{d}x_{i_1})_p \wedge \cdots \wedge (\mathrm{d}x_{i_k})_p$$

  where $I = (i_1,\dots,i_k)$ is a strictly increasing multi-index of n of length k form a basis of $\Lambda^k(T_p^*\mathbb{R}^n)$.

- Thus, every $\omega_p \in \Lambda^k(T_p^*\mathbb{R}^n)$ has a unique decomposition

$$\omega_p = \sum_I c_I(\mathrm{d}x_I)_p$$

  where every $c_I \in \mathbb{R}$.

- Similarly, every k-form ω on U has a unique decomposition

$$\omega = \sum_I f_I \,\mathrm{d}x_I$$

  where every $f_I : U \to \mathbb{R}$.

- **Class $\boldsymbol{C^r}$** (k-form): A k-form ω for which every f_I in its decomposition is in $C^r(U)$.
- From here on out, we assume unless otherwise stated that all k-forms we consider are of class C^∞.
- $\boldsymbol{\Omega^k(U)}$: The set of k-forms of class C^∞ on U.
- $\boldsymbol{f\omega}$: The k-form defined as follows, where $f \in C^\infty(U)$ and $\omega \in \Omega^k(U)$. *Given by*

$$p \mapsto f(p)\omega_p$$

- **Sum** (of ω_1,ω_2): The k-form defined as follows, where $\omega_1,\omega_2 \in \Omega^k(U)$. *Denoted by* $\boldsymbol{\omega_1 + \omega_2}$. *Given by*

$$p \mapsto (\omega_1)_p + (\omega_2)_p$$

- **Wedge product** (of ω_1,ω_2): The (k_1+k_2)-form defined as follows, where $\omega_1 \in \Omega^{k_1}(U)$ and $\omega_2 \in \Omega^{k_2}(U)$. *Denoted by* $\boldsymbol{\omega_1 \wedge \omega_2}$. *Given by*

$$p \mapsto (\omega_1)_p \wedge (\omega_2)_p$$

- **Zero-form**: A function which assigns to each $p \in U$ an element of $\Lambda^0(T_p^*\mathbb{R}^n) = \mathbb{R}$. *Also known as* **real-valued function**.
- It follows from the definition of zero-forms that

$$\Omega^0(U) = C^\infty(U)$$

- **Exterior differentiation operation**: The operator from $\Omega^0(U) \to \Omega^1(U)$ which associates to a function $f \in C^\infty(U)$ the 1-form $\mathrm{d}f$. *Denoted by* $\mathbf{d}$.
- We now seek to define a generalized version of the exterior differentiation operation; in particular, we would like to define an analogous function $\mathrm{d} : \Omega^k(U) \to \Omega^{k+1}(U)$.
- Desired properties of exterior differentiation.

1. If $\omega_1, \omega_2 \in \Omega^k(U)$, then
$$\mathrm{d}(\omega_1 + \omega_2) = \mathrm{d}\omega_1 + \mathrm{d}\omega_2$$
2. If $\omega_1 \in \Omega^k(U)$ and $\omega_2 \in \Omega^\ell(U)$, then
$$\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge \mathrm{d}\omega_2$$
3. If $\omega \in \Omega^k(U)$, then
$$\mathrm{d}(\mathrm{d}\omega) = 0$$

- Consequences of these properties.
- Lemma 2.4.5: Let $U \subset \mathbb{R}^n$ open. If $f_1, \dots, f_k \in C^\infty(U)$, then
$$\mathrm{d}(\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k) = 0$$

Proof. We induct on k. For the base case $k = 1$, we have that $\mathrm{d}(\mathrm{d}f_1) = 0$ by Property 3. Now suppose inductively that we have proven the claim for $k-1$ functions; we now seek to prove it for k functions. Let $\mu = \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_{k-1}$. Then by the induction hypothesis, $\mathrm{d}\mu = 0$. Therefore,
$$\begin{aligned}
\mathrm{d}(\mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_k) &= \mathrm{d}(\mu \wedge \mathrm{d}f_k) \\
&= \mathrm{d}\mu \wedge \mathrm{d}f_k + (-1)^{k-1}\mu \wedge \mathrm{d}(\mathrm{d}f_k) && \text{Property 2} \\
&= 0
\end{aligned}$$
as desired. □

- A special case of Lemma 2.4.5 is that
$$\mathrm{d}(\mathrm{d}x_I) = 0$$
- Now since every k-form $\omega \in \Omega^k(U)$ has a unique decomposition in terms of the $\mathrm{d}x_I$, Property 2 and the above equation reveal that
$$\mathrm{d}\omega = \sum_I \mathrm{d}f_I \wedge \mathrm{d}x_I$$
- Therefore, if there exists an operator d satisfying Properties 1-3, then d necessarily has the above form. All that's left is to show that the operator defined above has these properties.
- Proposition 2.4.10: Let $U \subset \mathbb{R}^n$ be open. There is a unique operator $\mathrm{d} : \Omega^*(U) \to \Omega^{*+1}(U)$ satisfying Properties 1-3.

Proof. ... □

- **Closed** (k-form): A k-form $\omega \in \Omega^k(U)$ for which $\mathrm{d}\omega = 0$.
- **Exact** (k-form): A k-form $\omega \in \Omega^k(U)$ such that $\omega = \mathrm{d}\mu$ for some $\mu \in \Omega^{k-1}(U)$.
- Property 3 implies that every exact k-form is closed.
  - The converse is not true even for 1-forms (see Exercise 2.1.iii).
  - "It is a very interesting (and hard) question to determine if an open set U has the following property: For $k > 0$, every closed k-form is exact" (Guillemin & Haine, 2018, p. 49).
    - Note that we do not consider zero-forms since there are no (-1)-forms for which to define exactness.
- If $f \in C^\infty(U)$ and $\mathrm{d}f = 0$, then f is constant on connected components of U (see Exercise 2.2.iii).
- Lemma 2.4.16 (Poincaré lemma): If ω is a closed form on U of degree $k > 0$, then for every point $p \in U$, there exists a neighborhood of p on which ω is exact.

Proof. See Exercises 2.4.v and 2.4.vi. □

Week 6

Operations on Forms

6.1 Compact Support and Consequences

5/2:
- Plan:
  - Brouwer's fixed point theorem.
    - The classic fixed point theorem.
    - Several proofs.
  - Compactly supported forms.
  - The Poincaré lemma.
    - Allows us to define the degree of a function $F : U \to V$, where $U, V \subset \mathbb{R}^n$ open.
    - The degree will turn out to be an integer.
    - We will need F to be proper.
    - We'll eventually use the degree to give a proof of the Brouwer's fixed point theorem.
- Theorem (Brouwer's fixed point theorem): Let $B^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ be the closed unit ball in $\mathbb{R}^n$, and let $F : B^n \to B^n$ be continuous. Then there exists $x_0 \in B^n$ such that $F(x_0) = x_0$ (i.e., F has a fixed point).
  - This is a generalized form of what we proved last quarter that a map from $[0,1] \twoheadrightarrow [0,1]$ has a fixed point (IVT and an auxiliary function).
  - Think back to Sharkovsky's theorem last quarter.
  - Another interpretation of Brouwer in $\mathbb{R}^2$: Take a piece of paper, crumple it up, project it down onto where it was, and some point lies exactly above where it was.
- **Support** (of ω): The following set, where $\omega \in \Omega^k(\mathbb{R}^n)$. *Denoted by* $\mathbf{supp}(\boldsymbol{\omega})$. *Given by*

$$\operatorname{supp}(\omega) = \{p \in \mathbb{R}^n \mid \omega_p \neq 0\}$$

- Example:
  - The support of a bump function on $\mathbb{R}^1$ is the region of the line on which it is not zero.
- **Compactly supported** (form): A form ω for which $\operatorname{supp}(\omega)$ is compact.
- **Compactly supported** (form ω on U): A compactly supported form such that $\operatorname{supp}(\omega) \subset U$.
  - The idea is that we can have some crazy form, but it "dies down" when we get close to the boundary of U.
- $\boldsymbol{\Omega_c^k(U)}$: The vector space of all compactly supported k-forms on U.

  - Thus, the scalar multiple of a compactly supported form on U is still compactly supported, as is the sum of two compactly supported forms on U.

- To get a handle on the degree, we're gonna focus on the top-dimensional space $\Omega_c^n(U)$ of compactly supported forms.
- **Proper** (function): A function $F : U \to V$, where $U, V \in \mathbb{R}^n$ open, for which $F^{-1}(K)$ is compact for any F compact in V.
  - We know that the image of a compact set is compact under a *continuous* function, but we haven't said anything about the inverse image up to this point.
- Example: Sine and cosine are continuous but not proper.
  - Consider $\sin^{-1}(\{0\}) = \{\dots, -\pi, 0, \pi, \dots\}$, which is not bounded and hence not compact (by Heine-Borel).
- The pullback, when restricted to compactly supported forms, maps compactly supported forms to compactly supported forms. Symbolically,

$$F^*[\Omega_c^n(V)] \subset \Omega_c^n(U)$$

- Similarly, $\mathrm{d} : \Omega_c^{n-1}(X) \to \Omega_c^n(X)$.
- $\boldsymbol{n}^{\text{th}}$ **compactly supported de Rahm cohomology group**: The top-dimensional space of forms modulo the image of the $(n-1)$-dimensional space of forms under the exterior derivative. *Denoted by* $\boldsymbol{H_c^n(X)}$. *Given by*

$$H_c^n(X) = \frac{\Omega_c^n(X)}{\mathrm{d}\big(\Omega_c^{n-1}(X)\big)}$$

  - The top is analogous to the kernel of the appropriate d because there's no $n+1$ form so everything just gets mapped to the kernel.
- Since the pullback commutes with the exterior derivative, F will induce a map from $H_c^n(V) \to H_c^n(U)$.
  - Today, we will show that $H_c^n(X) \cong \mathbb{R}$, where the isomorphism is integration.
  - On this function, we're gonna map 1 and that will give us $\deg(F)$.
  - This is something topological: If we move/jiggle F a bit, the degree won't change. The degree is **invariant** under jiggling it around; this is the basis of topology.
  - In fact: For all $\omega \in \Omega_c^n(V)$, we have that

$$\int_U F^*\omega = \deg(F) \int_V \omega$$

  - Another thing that should be familiar from vector calculus is that this is a generalization of a classic change of coordinates integration formula. Specifically, if $F : U \to V$ is a **diffeomorphism** (smooth, bijective, smooth inverse) and $\varphi : V \to \mathbb{R}$, then

$$\int_V \varphi(y)\,\mathrm{d}y = \int_U (\varphi \circ F)(x)|\det DF(x)|\,\mathrm{d}x$$

    - Assume U, V are some bounded open subsets in $\mathbb{R}^n$, though we can get around the boundedness with a more advanced derivation.
    - This formula is just the previous formula in coordinates plus the fact that the degree of a diffeomorphism is ± 1 depending on whether or not it preserves orientation.
    - We'll use this formula over and over again to simplify the domain over which we need to integrate; it's kind of a good old u-substitution type thing.

- **Integral** (of $\omega \in \Omega_c^n(U)$): If $\omega = f\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ is a top-dimensional form, then the integral of ω over U is given as follows. *Denoted by* $\int_U \boldsymbol{\omega}$. *Given by*

$$\int_U \omega = \int_{\mathbb{R}^n} f\,\mathrm{d}x_1 \cdots \mathrm{d}x_n$$

  - Defines integration pictorially as slicing up the plane, taking a point in each region, and multiplying it's value by the area of the region, and then taking finer and finer partitions.

- Theorem (Poincaré lemma — final form): Let $\omega_1, \omega_2 \in \Omega_c^n(U)$. Then $\omega_1 \sim \omega_2$ if $\omega_1 - \omega_2 = \mathrm{d}\mu$ for some $\mu \in \Omega_c^n(U)$ (i.e., $[\omega_1] = [\omega_2]$ in $H_c^n(U)$, where we are representing equivalence classes). Let $\omega_0 \in \Omega_c^n(U)$ with $\int \omega_0 = 1$ (ω_0 is a bump function). Then $\omega \sim c\omega_0$ where c a scalar is given by $c = \int \omega$.
  - We're gonna start small by proving the Poincaré lemma for rectangles.
  - Then we'll have the lemma for general, open, connected subsets of $\mathbb{R}^n$.
  - Then we'll prove the final form above.

- To prove the Poincaré lemma, we need two steps.
  1. Poincaré lemma for rectangles: $\int \omega = 0$ iff $\omega = \mathrm{d}\mu$.
     - The backwards implication is easy.
     - The forwards implication is tricky and requires induction on dimension.
  2. Generalize from rectangles to general regions U.

- Theorem (Poincaré lemma — for rectangles): Let $Q = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$. Take $\omega \in \Omega_c^n(Q)$. Then TFAE.
  1. $\int_Q \omega = 0$.
  2. $\omega = \mathrm{d}\mu$ with $\mu \in \Omega_c^{n-1}(U)$.

  Proof ($2 \Rightarrow 1$). Let $\mu = \sum_{i=1}^n f_i\,\mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n$[1]. Then

$$\begin{aligned}
\mathrm{d}\mu &= \sum_{i=1}^n \mathrm{d}f_i \wedge \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n \\
&= \sum_{i=1}^n \left(\sum_{j=1}^n \frac{\partial f_i}{\partial x_j}\,\mathrm{d}x_j \right) \wedge \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n \\
&= \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}\,\mathrm{d}x_i \wedge \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n \\
&= \sum_{i=1}^n (-1)^{i+1} \frac{\partial f_i}{\partial x_i}\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n
\end{aligned}$$

  Now to show that $\int \mathrm{d}\mu = 0$, it suffices to check that $\int \frac{\partial f_i}{\partial x_i}\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n = 0$ for all i by the distributive property of integration over sums. The conclusion follows from the FTC and the fact that each f_i is supported in Q (i.e., each f_i is zero on the boundary of the rectangle, so the integral will look something like $f_i(b) - f_i(a) = 0 - 0 = 0$). ☐

  Proof ($1 \Rightarrow 2$). If $1 \Rightarrow 2$ on some $U \subset \mathbb{R}^n$, then $1 \Rightarrow 2$ in $U \times [a_n, b_n] \subset \mathbb{R}^{n+1}$. This inductive step gets us what we need. We'll prove it next time. ☐

- Motivation/warm up for $1 \Rightarrow 2$.

[1] Note that the carrot to delete something is universal to all fields of math, not just differential geometry.

- Let $n = 1$. Then the theorem says $f : \mathbb{R} \to \mathbb{R}$ with $\operatorname{supp}(f) \subset [a, b]$ implies TFAE.
  1. $\int_a^b f = 0$.
  2. $f = \mathrm{d}g/\mathrm{d}x$ for some $g \in \Omega_c^0([a, b])$.
- $2 \Rightarrow 1$: We just did this.
- $1 \Rightarrow 2$: We let $g(x) = \int_a^x f(t)\,\mathrm{d}t$. We can check that $\mathrm{d}g/\mathrm{d}x = f$, and that $\operatorname{supp}(g) \subset [a, b]$ (since $\int_a^a f(t)\,\mathrm{d}t = 0$ and $\int_a^b f(t)\,\mathrm{d}t = 0$; values smaller and larger are zero by definition).
- ($1 \Rightarrow 2$): We know that f starts at zero and ends at zero. We know that the integral (g) of f starts at zero and ends at zero. But then it must be that this integral is a compactly supported function whose derivative is f. Indeed, regardless of how f moves, we know that it must come back to zero, and any positive areas under the curve must be cancelled by negative areas under the curve.
- ($2 \Rightarrow 1$): We know that f starts at zero and ends at zero. We know that f is the derivative of a function g that starts at zero and ends at zero. But then the integral of f will just be the ending point of g minus the starting point of g, which are both equally zero, making the integral zero. Indeed, regardless of how g moves, any positive slopes must be cancelled by negative slopes. But these slopes *really are* one and the same as the areas inspected by the integral, as per the FTC!
- An example of two functions that illustrate the point here are $f(x) = \sin(x)$ and $g(x) = 1 - \cos(x)$ on $[0, 2\pi]$.

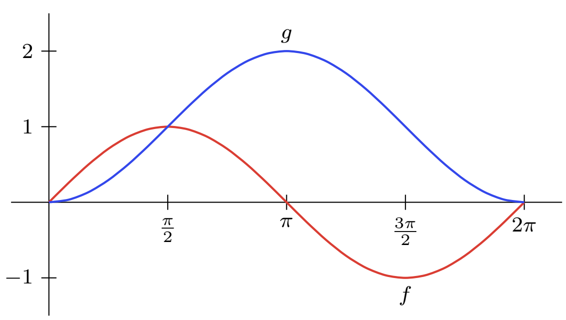


Figure 6.1: Poincaré lemma in one dimension.

6.2 The Pullback

5/4:
- Homework 3 now due Monday (the stuff will be on the exam though).
- Office hours today from 5:00-6:00.
- Exam Friday.
- Next week will be Chapter 3.
  - Integration of top-dimensional forms, i.e., if we're in 2D space, we'll integrate 2-forms; in 3D space, we'll integrate 3-forms; etc.
- Pullbacks of k-forms.
  - Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$.
  - Let $F : U \to V$ be smooth.
  - This induces $F^* : \Omega^k(V) \to \Omega^k(U)$.
  - We have $\mathrm{d}F_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$, which also induces $\mathrm{d}F_p^* : \Lambda^k(T_{F(p)}^*\mathbb{R}^m) \to \Lambda^k(T_p^*\mathbb{R}^n)$.
  - Note that F^* maps $\omega \mapsto F^*\omega$ where $F^*\omega_p = \mathrm{d}F_p^*\omega_{F(p)}$.

- In formulas, if
$$\omega = \sum_I \varphi_I \, \mathrm{d}x_I$$
then
$$F^*\omega = \sum_I F^*\varphi_I \, \mathrm{d}F_I$$
  - $\varphi_I \in V^*$.
  - Recall that $F^*\varphi_I = \varphi_I \circ F : U \to \mathbb{R}$.
  - If $I = (i_1, \dots, i_k)$, then $\mathrm{d}F_I = \mathrm{d}F_{i_1} \wedge \cdots \wedge \mathrm{d}F_{i_k}$.
  - $F_{i_j} : U \to \mathbb{R}$ sends $p \mapsto x_{i_j} \circ F(p)$, where x_{i_j} (as the ${i_j}^{\text{th}}$ component function) isolates the ${i_j}^{\text{th}}$ component of $F(x)$.
  - There is a derivation that gets you from the above abstract definition of the pullback to the below concrete form.
  - We can prove that $F^*\omega$ has the above form using properties 1-4 below.
- Note that $\mathrm{d}F_p$ is the kind of thing we worked on last quarter?
- Properties of the pullback (let $U \xrightarrow{F} V \xrightarrow{G} W$).
  1. F^* is linear.
  2. $F^*(\omega_1 \wedge \omega_2) = F^*\omega_1 \wedge F^*\omega_2$.
  3. $(F \circ G)^* = G^* \circ F^*$.
  4. $\mathrm{d} \circ F^* = F^* \circ \mathrm{d}$.

$$\begin{tikzcd}
\Omega^0(U) \arrow[d, "\mathrm{d}"'] & \Omega^0(V) \arrow[l, "F^*"'] \arrow[d, "\mathrm{d}"] \arrow[dl, bend left] \\
\Omega^1(U) & \Omega^1(V) \arrow[l, "F^*"]
\end{tikzcd}$$

Figure 6.2: Commutative diagram.

- Properties 1-3 follow from our Chapter 1 pointwise properties.
  - They also yield the explicit formula for $F^*\omega$ given above.
- Proving property 4.
  - Lemma 1: Figure 6.2 is true, i.e., property 4 holds for zero-forms.
  - Lemma 2: $\mathrm{d}F_I = F^* \, \mathrm{d}x_I$, where $I = (i_1, \dots, i_k)$.

    Proof. We have that
$$\begin{aligned}
\mathrm{d}F_I &= \mathrm{d}F_{i_1} \wedge \cdots \wedge \mathrm{d}F_{i_k} \\
&= \mathrm{d}(x_{i_1} \circ F) \wedge \cdots \wedge \mathrm{d}(x_{i_1} \circ F) \\
&= \mathrm{d}(F^* x_{i_1}) \wedge \cdots \wedge \mathrm{d}(F^* x_{i_k}) \\
&= F^* \, \mathrm{d}(x_{i_1}) \wedge \cdots \wedge F^* \, \mathrm{d}(x_{i_k}) && \text{Lemma 1} \\
&= F^* \, \mathrm{d}x_{i_1} \wedge \cdots \wedge F^* \, \mathrm{d}x_{i_k} \\
&= F^*(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}) && \text{Property 2} \\
&= F^* \, \mathrm{d}x_I
\end{aligned}$$
    as desired. □

- Let $\omega = \sum_I \varphi_I \, \mathrm{d}x_I$. Then

$$\begin{aligned}
\mathrm{d}(F^*\omega) &= \mathrm{d}\left(\sum_I F^*\varphi_I \, \mathrm{d}F_I\right) \\
&= \sum_I \mathrm{d}(F^*\varphi_I \wedge \mathrm{d}F_I) \\
&= \sum_I \mathrm{d}(F^*\varphi_I) \wedge \mathrm{d}F_I \\
&= \sum_I F^* \, \mathrm{d}\varphi_I \wedge F^* \, \mathrm{d}x_I && \text{Lemma 2} \\
&= \sum_I F^*(\mathrm{d}\varphi_I \wedge \mathrm{d}x_I) \\
&= F^* \left(\sum_I \mathrm{d}\varphi_I \wedge \mathrm{d}x_I\right) \\
&= F^* \left(\sum_I \mathrm{d}(\varphi_I \, \mathrm{d}x_I)\right) \\
&= F^*\mathrm{d} \left(\sum_I \varphi_I \, \mathrm{d}x_I\right) \\
&= F^* \, \mathrm{d}\omega
\end{aligned}$$

- $\mathrm{d}^2 = 0$ generalizes curl and all of those identities.
- Two other operations.
- **Interior product**: Given $\boldsymbol{v}$ a vector field on U, we have $\iota_{\boldsymbol{v}} : \Omega^k(U) \to \Omega^{k-1}(U)$ that sends $\omega \mapsto \iota_{\boldsymbol{v}}\omega$.
- Its properties follow from the properties of the pointwise stuff.
    1. $\iota_{\boldsymbol{v}}(\omega_1 + \omega_2) = \iota_{\boldsymbol{v}}\omega_1 + \iota_{\boldsymbol{v}}\omega_2$.
    2. $\iota_{\boldsymbol{v}}(\omega \wedge \mu) = \iota_{\boldsymbol{v}}\omega \wedge \mu + (-1)^k \omega \wedge \iota_{\boldsymbol{v}}\mu$.
    3. $\iota_{\boldsymbol{v}} \circ \iota_{\boldsymbol{w}} = -\iota_{\boldsymbol{w}} \circ \iota_{\boldsymbol{v}}$.
- **Lie derivative**: If $\boldsymbol{v}$ is a vector field on U, then $L_{\boldsymbol{v}} : \Omega^k(U) \to \Omega^k(U)$ sends $\omega \mapsto \mathrm{d}\iota_{\boldsymbol{v}}\omega + \iota_{\boldsymbol{v}} \, \mathrm{d}\omega$.
    - Note that we use ι to drop the index and d to raise it back up, and then vice versa.
- Check: Agrees with previous definition for Ω^0.
- Cartan's magic formula is what we're taking to be the definition of the Lie derivative.
- Properties.
    1. $L_{\boldsymbol{v}} \circ \mathrm{d} = \mathrm{d} \circ L_{\boldsymbol{v}}$.
    2. $L_{\boldsymbol{v}}(\omega \wedge \eta) = L_{\boldsymbol{v}}\omega \wedge \eta + \omega \wedge L_{\boldsymbol{v}}\eta$.
        - Proof:

$$\begin{aligned}
\mathrm{d}(\iota_{\boldsymbol{v}}\mathrm{d} + \mathrm{d}\iota_{\boldsymbol{v}}) &= \mathrm{d}\iota_{\boldsymbol{v}}\mathrm{d} \\
&= \iota_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}\mathrm{d} + \mathrm{d}\iota_{\boldsymbol{v}})
\end{aligned}$$

- We should find an explicit formula for the Lie derivative.
    - Your vector field will be of the form

$$\boldsymbol{v} = \sum f_i \, \partial/\partial x_i$$

    - Your form will be of the form

$$\omega = \sum \varphi_I \, \mathrm{d}x_I$$

6.3 Connections with Vector Calculus

From Klug (2022).

5/26:
- 2-dimensional analogues of class content.
  - Let $U \subset \mathbb{R}^2$ and let $\mathfrak{X}(U)$ be the vector space of vector fields on U.
  - 1-forms on U are of the form
$$f\,\mathrm{d}x + g\,\mathrm{d}y$$
  - We have an isomorphism of vector spaces $\sharp : \Omega^1(U) \to \mathfrak{X}(U)$ defined by
$$f\,\mathrm{d}x + g\,\mathrm{d}y \mapsto f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y}$$
    - The inverse of $\sharp$ is denoted $\flat$.
    - As such, these functions are referred to as the **musical operators**.
  - The exterior derivative of a function on $\mathbb{R}^2$ is
$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y$$
    - This is the **gradient**.
  - The exterior derivative of a one-form on $\mathbb{R}^2$ is
$$\mathrm{d}(f\,\mathrm{d}x + g\,\mathrm{d}y) = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y$$
    - This is related to **Green's theorem**.
    - The expression is called the **2-dimensional curl** (of a vector field), where here we are freely identifying 1-forms and vector fields via $\sharp$.
    - If we (1) make this precise and (2) prove that the intuitive definition of curl agrees with the above formula, we should gain some geometric intuition for d in this particular (co)dimension.
  - The fact that gradient vector fields are curl free, i.e., $\mathrm{curl} \circ \mathrm{grad} = 0$, reflects the fact that $\mathrm{d}^2 = 0$.
- **2-dimensional curl** (of $\boldsymbol{v} \in \mathfrak{X}(U)$): The function from $U \to \mathbb{R}$ describing the way that a ball centered at $p \in U$ would rotate (or "curl") when left in $\boldsymbol{v}$. *Denoted by* **curl($\boldsymbol{v}$)**.
- 3-dimensional analogues of class content.
  - Gradient of the zero-form $f : U \to \mathbb{R}$ where $U \subset \mathbb{R}^3$.
$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z$$
    - We have that $\sharp \circ \mathrm{d}^0$ gives the gradient, exactly as in two dimensions.
  - Curl of the one-form $f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}z$.
$$\mathrm{d}(f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}z) = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y + \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z}\right)\mathrm{d}x \wedge \mathrm{d}z$$
    - $\mathrm{curl}(\boldsymbol{v})$ is again a vector field, just with the direction at a point being the axis of rotation of a small ball placed at that point.
    - Once again, we can identify $\Omega^1(U), \Omega^2(U)$ with $\mathfrak{X}(U)$ to learn that d is curl and gradient fields are curl free as a result of $\mathrm{d}^2 = 0$.
  - Divergence of the two-form $f\,\mathrm{d}x \wedge \mathrm{d}y + g\,\mathrm{d}y \wedge \mathrm{d}z + h\,\mathrm{d}x \wedge \mathrm{d}z$.
$$\mathrm{d}(f\,\mathrm{d}x \wedge \mathrm{d}y + g\,\mathrm{d}y \wedge \mathrm{d}z + h\,\mathrm{d}x \wedge \mathrm{d}z) = \left(\frac{\partial f}{\partial z} + \frac{\partial g}{\partial x} - \frac{\partial h}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

- Modulo a sign, this is the **divergence** of a vector field in three dimensions.
- We can identify $\Omega^2(U)$ and $\Omega^3(U)$ with $\mathfrak{X}(U)$ and $\Omega^0(U)$, respectively, to learn that d is div and the fact that $\text{div} \circ \text{curl} = 0$ follows from $\mathrm{d}^2 = 0$.

- **Divergence** (of $\boldsymbol{v} \in \mathfrak{X}(U)$): The function from $U \to \mathbb{R}$ which geometrically represents the compression/stretching of objects placed in the vector field. *Denoted by* $\mathbf{div}(\boldsymbol{v})$.
- Take away: The exterior derivative packages the three operations of vector calculus, and $\mathrm{d}^2 = 0$ generalizes several simple formulas from vector calculus.

6.4 Chapter 2: Differential Forms

From Guillemin and Haine (2018).

5/5:

- **Interior product** (of $\boldsymbol{v}$ with ω): The $(k-1)$-form on U defined as follows, where $U \subset \mathbb{R}^n$ open, $\boldsymbol{v}$ a vector field on U, and $\omega \in \Omega^k(U)$. *Denoted by* $\boldsymbol{\iota_v \omega}$. *Given by*

$$p \mapsto \iota_{\boldsymbol{v}(p)}\omega_p$$

- By definition, $\iota_{\boldsymbol{v}(p)}\omega_p \in \Lambda^{k-1}(T_p^*\mathbb{R}^n)$.

5/26:

- Properties 2.5.3: The following are properties of the interior product defined above, where $U \subset \mathbb{R}^n$ open, $\boldsymbol{v}, \boldsymbol{w}$ are vector fields on U, $\omega_1, \omega_2, \omega \in \Omega^k(U)$, and $\mu \in \Omega^\ell(U)$.

  1. *Linearity in the form*: We have

  $$\iota_{\boldsymbol{v}}(\omega_1 + \omega_2) = \iota_{\boldsymbol{v}}\omega_1 + \iota_{\boldsymbol{v}}\omega_2$$

  2. *Linearity in the vector field*: We have

  $$\iota_{\boldsymbol{v}+\boldsymbol{w}}\omega = \iota_{\boldsymbol{v}}\omega + \iota_{\boldsymbol{w}}\omega$$

  3. *Derivation property*: We have

  $$\iota_{\boldsymbol{v}}(\omega \wedge \mu) = \iota_{\boldsymbol{v}}\omega \wedge \mu + (-1)^k \omega \wedge \iota_{\boldsymbol{v}}\mu$$

  4. The identity

  $$\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{w}}\omega) = -\iota_{\boldsymbol{w}}(\iota_{\boldsymbol{v}}\omega)$$

  5. The identity, as a special case of (4),

  $$\iota_{\boldsymbol{v}}(\iota_{\boldsymbol{v}}\omega) = 0$$

  6. If $\omega = \mu_1 \wedge \cdots \wedge \mu_k$ (i.e., if ω is **decomposable**), then

  $$\iota_{\boldsymbol{v}}\omega = \sum_{r=1}^{k} (-1)^{r-1} \iota_{\boldsymbol{v}}(\mu_r)\mu_1 \wedge \cdots \wedge \widehat{\mu_r} \wedge \cdots \wedge \mu_k$$

- The following are two assertions to prove, both of which are special cases of Property 2.5.3(6).
- Example 2.5.4: If $\boldsymbol{v} = \partial/\partial x_r$ and $\omega = \mathrm{d}x_I$, then

$$\iota_{\boldsymbol{v}}\omega = \sum_{i=1}^{k} (-1)^{i-1} \delta_{i,i_r} \, \mathrm{d}x_{I_r}$$

where

$$\delta_{i,i_r} = \begin{cases} 1 & i = i_r \\ 0 & i \neq i_r \end{cases} \qquad\qquad I_r = (i_1, \dots, \widehat{i_r}, \dots, i_k)$$

- Example 2.5.6: If $\boldsymbol{v} = \sum_{i=1}^n f_i\ \partial/\partial x_i$ and $\omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$, then

$$\iota_{\boldsymbol{v}}\omega = \sum_{r=1}^n (-1)^{r-1} f_r\,\mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_r} \wedge \cdots \wedge \mathrm{d}x_n$$

- **Lie derivative** (of ω with respect to $\boldsymbol{v}$): The k-form defined as follows, where $U \subset \mathbb{R}^n$ is open, $\boldsymbol{v}$ is a vector field on U, and $\omega \in \Omega^k(U)$.

$$L_{\boldsymbol{v}}\omega = \iota_{\boldsymbol{v}}(\mathrm{d}\omega) + \mathrm{d}(\iota_{\boldsymbol{v}}\omega)$$

- Properties 2.5.10: The following are properties of the Lie derivative defined above, where $U \subset \mathbb{R}^n$ open, $\boldsymbol{v}$ is a vector field on U, $\omega \in \Omega^k(U)$, and $\mu \in \Omega^\ell(U)$.
  1. *Commutativity with exterior differentiation*: We have

  $$\mathrm{d}(L_{\boldsymbol{v}}\omega) = L_{\boldsymbol{v}}(\mathrm{d}\omega)$$

  2. *Interaction with wedge products*: We have

  $$L_{\boldsymbol{v}}(\omega \wedge \mu) = L_{\boldsymbol{v}}\omega \wedge \mu + \omega \wedge L_{\boldsymbol{v}}\mu$$

- An explicit formula for $L_{\boldsymbol{v}}\omega$.
  - Let $\omega \in \Omega^k(U)$ be defined by $\omega = \sum_I f_I\,\mathrm{d}x_I$ for $f_I \in C^\infty(U)$, and let $\boldsymbol{v} = \sum_{i=1}^n g_i\ \partial/\partial x_i$ for $g_i \in C^\infty(U)$.
  - Then by the above properties,

$$\begin{aligned}
L_{\boldsymbol{v}}\omega &= L_{\boldsymbol{v}}\left(\sum_I f_I\,\mathrm{d}x_I\right) \\
&= \sum_I L_{\boldsymbol{v}}(f_I\,\mathrm{d}x_I) \\
&= \sum_I [(L_{\boldsymbol{v}}f_I)\,\mathrm{d}x_I + f_I(L_{\boldsymbol{v}}\,\mathrm{d}x_I)] \\
&= \sum_I \left[\left(\sum_{i=1}^n g_i\frac{\partial f_I}{\partial x_i}\right)\mathrm{d}x_I + f_I\left(\sum_{r=1}^k \mathrm{d}x_{i_1}\wedge\cdots\wedge L_{\boldsymbol{v}}\,\mathrm{d}x_{i_r}\wedge\cdots\wedge\mathrm{d}x_{i_k}\right)\right] \\
&= \sum_I \left[\left(\sum_{i=1}^n g_i\frac{\partial f_I}{\partial x_i}\right)\mathrm{d}x_I + f_I\left(\sum_{r=1}^k \mathrm{d}x_{i_1}\wedge\cdots\wedge \mathrm{d}L_{\boldsymbol{v}}x_{i_r}\wedge\cdots\wedge\mathrm{d}x_{i_k}\right)\right] \\
&= \sum_I \left[\left(\sum_{i=1}^n g_i\frac{\partial f_I}{\partial x_i}\right)\mathrm{d}x_I + f_I\left(\sum_{r=1}^k \mathrm{d}x_{i_1}\wedge\cdots\wedge \mathrm{d}g_{i_r}\wedge\cdots\wedge\mathrm{d}x_{i_k}\right)\right] \\
&= \sum_I \left[\left(\sum_{i=1}^n g_i\frac{\partial f_I}{\partial x_i}\right)\mathrm{d}x_I + f_I\left(\sum_{r=1}^k \mathrm{d}x_{i_1}\wedge\cdots\wedge \left(\sum_{i=1}^n\frac{\partial g_{i_r}}{\partial x_i}\,\mathrm{d}x_i\right)\wedge\cdots\wedge\mathrm{d}x_{i_k}\right)\right] \\
&= \sum_I \left[\left(\sum_{i=1}^n g_i\frac{\partial f_I}{\partial x_i}\right)\mathrm{d}x_I + f_I\left(\sum_{r=1}^k\sum_{i=1}^n \frac{\partial g_{i_r}}{\partial x_i}\,\mathrm{d}x_{i_1}\wedge\cdots\wedge\mathrm{d}x_{i_{r-1}}\wedge\mathrm{d}x_i\wedge\mathrm{d}x_{i_{r+1}}\wedge\cdots\wedge\mathrm{d}x_{i_k}\right)\right] \\
&= \sum_I \left[\left(\sum_{i=1}^n g_i\frac{\partial f_I}{\partial x_i}\right)\mathrm{d}x_I + f_I\left(\sum_{r=1}^k\sum_{\substack{i=1\\ i\notin I}}^n \frac{\partial g_{i_r}}{\partial x_i}\,\mathrm{d}x_{i_1}\wedge\cdots\wedge\mathrm{d}x_{i_{r-1}}\wedge\mathrm{d}x_i\wedge\mathrm{d}x_{i_{r+1}}\wedge\cdots\wedge\mathrm{d}x_{i_k}\right)\right]
\end{aligned}$$

- Lemma 2.5.13 (the divergence formula): Let $U \subset \mathbb{R}^n$ open, $g_1, \dots, g_n \in C^\infty(U)$, and $\boldsymbol{v} = \sum_{i=1}^n g_i\ \partial/\partial x_i$. Then

$$L_{\boldsymbol{v}}(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n) = \sum_{i=1}^n \left(\frac{\partial g_i}{\partial x_i}\right)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

- **Pullback** (of ω along f): The k-form on U defined as follows, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open, $f : U \to V$ is a C^∞ map, ω is a k-form on V, $p \in U$, and $q = f(p)$. *Denoted by* $\boldsymbol{f^*\omega}$. *Given by*

$$p \mapsto \mathrm{d}f_p^*\omega_q$$

  - Note that it is because $\mathrm{d}f_p$ is linear that we get an induced pullback $\mathrm{d}f_p^* = (\mathrm{d}f_p)^* : \Lambda^k(T_q^*\mathbb{R}^m) \to \Lambda^k(T_p^*\mathbb{R}^n)$.

- Properties 2.6.4: The following are properties of the pullback defined above, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open and $f : U \to V$ is a C^∞ map.

  1. Let $\phi \in C^\infty(V)$ be a zero-form. Since $\Lambda^0(T_p^*) = \Lambda^0(T_q^*) = \mathbb{R}$, we have that $\mathrm{d}f_p^* = \mathrm{id}_\mathbb{R}$ when $k = 0$. Hence for zero forms,
  $$(f^*\phi)(p) = (\phi \circ f)(p)$$
  for all $p \in U$.
  2. Let $\phi \in \Omega^0(U)$, and let $\mu \in \Omega^1(V)$ be the 1-form $\mu = \mathrm{d}\phi$. By the chain rule,
  $$\mathrm{d}f_p^*\mu_q = (\mathrm{d}f_p)^*\mathrm{d}\phi_q = (\mathrm{d}\phi)_q \circ \mathrm{d}f_p = \mathrm{d}(\phi \circ f)_p$$
  Hence, by property (1),
  $$f^*\mathrm{d}\phi = \mathrm{d}f^*\phi$$
  3. Let $\omega_1, \omega_2 \in \Omega^k(V)$. Then
  $$\mathrm{d}f_p^*(\omega_1 + \omega_2)_q = \mathrm{d}f_p^*(\omega_1)_q + \mathrm{d}f_p^*(\omega_2)_q$$
  so
  $$f^*(\omega_1 + \omega_2) = f^*\omega_1 + f^*\omega_2$$
  4. Since $\mathrm{d}f_p^*$ commutes with the wedge product by Proposition 1.8.4(1), if $\omega_1 \in \Omega^k(V)$ and $\omega_2 \in \Omega^\ell(V)$, then
  $$\mathrm{d}f_p^*[(\omega_1)_q \wedge (\omega_2)_q] = \mathrm{d}f_p^*(\omega_1)_q \wedge \mathrm{d}f_p^*(\omega_2)_q$$
  so
  $$f^*(\omega_1 \wedge \omega_2) = f^*\omega_1 \wedge f^*\omega_2$$
  5. Let $W \subset \mathbb{R}^k$ be open, $g : V \to W$ be a C^∞ map, $p \in U$, $q = f(p)$, and $w = g(q)$. Then $(\mathrm{d}g_q \circ \mathrm{d}f_p)^* : \Lambda^k(T_w^*) \to \Lambda^k(T_p^*)$. But since $(\mathrm{d}g_q) \circ (\mathrm{d}f)_p = \mathrm{d}(g \circ f)_p$ by the chain rule, we have that $\mathrm{d}(g \circ f)_p^* : \Lambda^k(T_w^*) \to \Lambda^k(T_p^*)$. Thus, if $\omega \in \Omega^k(W)$, then
  $$f^*(g^*\omega) = (g \circ f)^*\omega$$

- An explicit formula for $f^*\omega$.
  - Let $\omega \in \Omega^k(V)$ be given by $\omega = \sum_I \phi_I \, \mathrm{d}x_I$, where the $\phi_I \in C^\infty(V)$. Then,
  $$\begin{aligned}
  f^*\omega &= \sum_I f^*\phi_I f^*(\mathrm{d}x_I) && (1)\\
  &= \sum_I (\phi_I \circ f) f^*(\mathrm{d}x_{i_1}) \wedge \cdots \wedge f^*(\mathrm{d}x_{i_k}) && (4)\\
  &= \sum_I (\phi_I \circ f) \, \mathrm{d}f^*x_{i_1} \wedge \cdots \wedge \mathrm{d}f^*x_{i_k} && (2)\\
  &= \sum_I (\phi_I \circ f) \, \mathrm{d}(x_{i_1} \circ f) \wedge \cdots \wedge \mathrm{d}(x_{i_k} \circ f) && (2)\\
  &= \sum_I (\phi_I \circ f)\mathrm{d}f_{i_1} \wedge \cdots \wedge \mathrm{d}f_{i_k}\\
  &= \sum_I f^*\phi_I \mathrm{d}f_I
  \end{aligned}$$
  where the f_{i_j} are the i_j^{th} coordinate functions of the map f.

- Notice that we have showed in the above derivation that

$$f^*(\mathrm{d}x_I) = \mathrm{d}f_I$$

- We now prove that the pullback commutes with exterior differentiation, i.e.,

$$\mathrm{d}(f^*\omega) = f^*\mathrm{d}\omega$$

  - We have that

$$\begin{aligned}
\mathrm{d}(f^*\omega) &= \mathrm{d}\left(\sum_I f^*\phi_I \mathrm{d}f_I\right) \\
&= \sum_I \mathrm{d}(f^*\phi_I \wedge \mathrm{d}f_I) \\
&= \sum_I \left[\mathrm{d}(f^*\phi_I) \wedge \mathrm{d}f_I + (-1)^k f^*\phi_I \wedge \mathrm{d}(\mathrm{d}f_I)\right] \\
&= \sum_I \left[f^*(\mathrm{d}\phi_I) \wedge f^*(\mathrm{d}x_I) + (-1)^k f^*\phi_I \wedge 0\right] \\
&= \sum_I f^*(\mathrm{d}\phi_I) \wedge f^*(\mathrm{d}x_I) \\
&= f^* \sum_I \mathrm{d}\phi_I \wedge \mathrm{d}x_I \\
&= f^*(\mathrm{d}\omega)
\end{aligned}$$

- A special case of $f^*(\mathrm{d}x_I) = \mathrm{d}f_I$:

$$f^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n) = \det\left[\frac{\partial f_i}{\partial x_j}\right] \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

  - Let $U, V \subset \mathbb{R}^n$ open. Then for all $p \in U$,

$$\begin{aligned}
f^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n)_p &= (\mathrm{d}f_1)_p \wedge \cdots \wedge (\mathrm{d}f_n)_p \\
&= \left[\sum_{j=1}^n \left.\frac{\partial f_1}{\partial x_j}\right|_p (\mathrm{d}x_j)_p\right] \wedge \cdots \wedge \left[\sum_{j=1}^n \left.\frac{\partial f_n}{\partial x_j}\right|_p (\mathrm{d}x_j)_p\right] \\
&= \det\left[\left.\frac{\partial f_i}{\partial x_j}\right|_p\right] (\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n)_p
\end{aligned}$$

  - See the argument used in Section 1.8 to derive the typical formula for the determinant for details and context on the above.

- **Homotopy** (between f_0 and f_1): A C^∞ map from $U \times A \to V$ (where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are open, $\{0, 1\} \subset A \subset \mathbb{R}$ is an open interval, and $f_0, f_1 : U \to V$ are C^∞ maps) such that

$$\begin{aligned}
(x, 0) &\mapsto f_0(x) \\
(x, 1) &\mapsto f_1(x)
\end{aligned}$$

  Denoted by $\boldsymbol{F}$.

- **Homotopic** (maps): Two maps f_0, f_1 to which there corresponds a homotopy F. *Denoted by* $\boldsymbol{f_0 \simeq f_1}$.
  - "Intuitively, f_0 and f_1 are homotopic if there exists a family of C^∞ maps $f_t : U \to V$ where $f_t(x) = F(x, t)$ which 'smoothly deform f_0 into f_1'" (Guillemin & Haine, 2018, p. 56).

- Theorem 2.6.15: If $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ open and $f_0, f_1 : U \to V$ homotopic C^∞ maps, then for every closed form $\omega \in \Omega^k(V)$, the form $f_1^*\omega - f_0^*\omega$ is exact.
  - This theorem is closely related to the Poincaré lemma (Lemma 2.4.16) and actually implies a slightly stronger version of it.
- **Contractible** (open subset $U \subset \mathbb{R}^n$): An open subset $U \subset \mathbb{R}^n$ for which there exists a point $p_0 \in U$ such that $\mathrm{id}_U : U \to U$ is homotopic to the constant map $f_0 : U \to U$ defined by $f_0(p) = p_0$ at p_0.
  - A contractible set is so named because it can be shrunk to a single point continuously.
- Theorem 2.6.15 implies that the Poincaré lemma holds for contractible open subsets of $\mathbb{R}^n$. In particular, if U is contractible, then every closed k-form on U of degree $k > 0$ is exact.

  Proof. Let U be contractible, and let $\omega \in \Omega^k(U)$ be closed. Since U is contractible, id_U and f a constant function are homotopic. Thus, by Theorem 2.6.15, $\mathrm{id}_U^*\,\omega - f^*\omega = \omega$ is exact. ☐

- The three basic operations of 3D vector calculus are gradient, curl, and divergence. These operations are closely related to $\mathrm{d} : \Omega^k(\mathbb{R}^3) \to \Omega^{k+1}(\mathbb{R}^3)$ for $k = 0, 1, 2$, respectively.
  - Gradient and divergence generalize to higher dimensions, with gradient always equal to d^0 and divergence always equal to d^{n-1}.
  - Why we should use differential forms, even in three dimensions: **General covariance**.
    - Translations and rotations of $\mathbb{R}^3$ preserve div and curl, but $\mathrm{d}^0, \mathrm{d}^1, \mathrm{d}^2$ admit all diffeomorphisms of $\mathbb{R}^3$ as symmetries.
- **General covariance**: The desire to formulate the laws of physics in such a way that they admit as large a set of symmetries as possible.
- There are two (natural) ways to convert vector fields into forms.
- Conversion using the *inner* product.
  - Let $B(v, w) = \sum_n v_i w_i$ be the inner product on $\mathbb{R}^n$.
  - By Exercise 1.2.xi, the inner product induces a bijective linear map $L : \mathbb{R}^n \to (\mathbb{R}^n)^*$ such that $L(v) = \ell_v$ iff $\ell_v(w) = B(v, w)$.
  - By identifying $T_p\mathbb{R}^n \cong \mathbb{R}^n$, we may transfer B, L to $T_p\mathbb{R}^n$, providing an inner product B_p on $T_p\mathbb{R}^n$ and a bijective linear map $L_p : T_p\mathbb{R}^n \to T_p^*\mathbb{R}^n$.
    - Note that the only difference between L and L_p (resp. B and B_p) is that L_p eats (p, v) and focuses on v while L eats v directly.
  - The identification $p \mapsto L_p\boldsymbol{v}(p)$ constitutes the 1-form $\boldsymbol{v}^\sharp$.
    - Intuition: $\boldsymbol{v}$ is a vector field. Thus, $v = \boldsymbol{v}(p)$ is the vector in $\boldsymbol{v}$ at point p. What L_p does is take this vector (as part of (p, v)) and return the linear functional $(\ell_v)_p \in T_p^*\mathbb{R}^n$ which sends $(p, w) \mapsto (p, \ell_v(w))$. So essentially, we are identifying with every point p the linear functional that maps every vector w (as part of the ordered pair $(p, w) \in T_p\mathbb{R}^n$) to its inner product with v, $B(v, w)$ (again, as part of the ordered pair $(p, B(v, w)) \in T_p\mathbb{R}^n$).
- $\boldsymbol{v^\sharp(p)}$: The cotangent vector
$$\boldsymbol{v}^\sharp(p) = L_p\boldsymbol{v}(p)$$
- Consequences.
  - We have that
$$\boldsymbol{v} = \frac{\partial}{\partial x_i} \quad \Longleftrightarrow \quad \boldsymbol{v}^\sharp = \mathrm{d}x_i$$

- More generally,
$$\boldsymbol{v} = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} \quad \Longleftrightarrow \quad \boldsymbol{v}^\sharp = \sum_{i=1}^{n} f_i \, \mathrm{d}x_i$$

- **Gradient** (of a function f): The following vector field, as determined by $f \in C^\infty(U)$ where $U \subset \mathbb{R}^n$. *Denoted by* $\mathbf{grad}(\boldsymbol{f})$. *Given by*
$$\operatorname{grad}(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}$$
  - This gets converted by $\sharp$ into the 1-form $\sum_{i=1}^{n} \partial f/\partial x_i \, \mathrm{d}x_i = \mathrm{d}f$.
  - Thus, the gradient operation is essentially just the exterior derivative operation d^0.
- Conversion using the *interior* product.
  - Let $\boldsymbol{v} = \sum_{i=1}^{n} f_i \, \partial/\partial x_i$ be a C^∞ vector field on $U \subset \mathbb{R}^n$ open. Let $\Omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$.
  - Then
$$\iota_{\boldsymbol{v}}\Omega = \sum_{r=1}^{n} (-1)^{r-1} f_r \, \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_r} \wedge \cdots \wedge \mathrm{d}x_n$$
  - Since every $(n-1)$-form can be written uniquely as such a sum, the above equation defines a bijective correspondence between vector fields and $(n-1)$-forms.
- The d operation as an operation on vector fields.
  - We may define $\mathrm{d}(\boldsymbol{v})$ by
$$\boldsymbol{v} \mapsto \mathrm{d}\iota_{\boldsymbol{v}}\Omega$$
  - The expression on the right above can related to the **divergence** as follows.
$$\begin{aligned}
\mathrm{d}\iota_{\boldsymbol{v}}\Omega &= \iota_{\boldsymbol{v}}(\mathrm{d}(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n)) + \mathrm{d}(\iota_{\boldsymbol{v}}\Omega) \\
&= L_{\boldsymbol{v}}\Omega \\
&= \operatorname{div}(\boldsymbol{v})\Omega
\end{aligned}$$
    - The first equality follows by $\mathrm{d}^2 = 0$.
    - The second equality follows by the definition of the Lie derivative of ω with respect to $\boldsymbol{v}$.
    - The third equality follows by Lemma 2.5.13.
- **Divergence** (of a vector field $\boldsymbol{v}$): The following function from $U \to \mathbb{R}$, where $\boldsymbol{v} = \sum_{i=1}^{n} f_i \, \partial/\partial x_i$ is a vector field over U. *Denoted by* $\mathbf{div}(\boldsymbol{v})$. *Given by*
$$\operatorname{div}(\boldsymbol{v}) = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$$
  - The above correspondence between $(n-1)$-forms and vector fields converts d into the divergence operation on vector fields.
- **Curl** (of a vector field $\boldsymbol{v}$): The unique vector field $\boldsymbol{w}$ such that $\mathrm{d}\big(\boldsymbol{v}^\sharp\big) = \iota_{\boldsymbol{w}} \, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$, where $U \subset \mathbb{R}^3$ open and $\boldsymbol{v}$ is a vector field on U. *Denoted by* $\mathbf{curl}(\boldsymbol{v})$.
- We should confirm that this definition coincides with that from vector calculus. In particular, we should check that if $\boldsymbol{v} = \sum_{i=1}^{3} f_i \, \partial/\partial x_i$, then
$$\operatorname{curl}(\boldsymbol{v}) = \sum_{i=1}^{3} g_i \frac{\partial}{\partial x_i}$$

where

$$\begin{aligned} g_1 &= \frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_2} \\ g_2 &= \frac{\partial f_3}{\partial x_1} - \frac{\partial f_1}{\partial x_3} \\ g_3 &= \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1} \end{aligned}$$

- Take aways:
  - The gradient, curl, and divergence operations have differential-form analogues (i.e., $\mathrm{d}^0, \mathrm{d}^1, \mathrm{d}^2$).
  - To define the gradient, we needed the inner product. To define the divergence, we had to equip U with Ω. To define the curl, we needed both.
  - It's these additional structures that explains why diffeomorphisms preserve $\mathrm{d}^0, \mathrm{d}^1, \mathrm{d}^2$, but not $\mathrm{grad}, \mathrm{curl}, \mathrm{div}$.
- Guillemin and Haine (2018) expresses Maxwell's equations in terms of differential forms.
- Guillemin and Haine (2018) introduces symplectic geometry and Hamiltonian mechanics.

Week 7

Integration on Forms

7.1 Chapter 3: Integration on Forms

From Guillemin and Haine (2018).

5/26:

- **Change of variables formula**: If $U, V \subset \mathbb{R}^n$ open and $f : U \to V$ a C^1 diffeomorphism, then for every $\phi : V \to \mathbb{R}$ continuous, the left integral below exists iff the right integral below exists and if they are equal.

$$\int_V \phi(y)\,\mathrm{d}y \qquad\qquad \int_U (\phi \circ f)(x) |\det Df(x)|\,\mathrm{d}x$$

  - Guillemin and Haine (2018) refers us elsewhere for some types of proofs. They, instead, will focus on Lax's differential-forms-heavy proof that, nevertheless, can be modified to avoid references to differential forms and be based solely on the language of elementary multivariable calculus[1].
  - Lax's proof is also desirable since it leads into a change of variables theorem for maps other than diffeomorphisms, and involves a topological invariant (the **degree** of a map), thereby providing a first brush with topology.
- Henceforth, let f be a C^∞ diffeomorphism.
- **Support** (of ν): The following set, where $\nu \in \Omega^k(\mathbb{R}^n)$. *Denoted by* $\mathbf{supp(\boldsymbol{\nu})}$. *Given by*

$$\operatorname{supp}(\nu) = \overline{\{x \in \mathbb{R}^n : \nu_x \neq 0\}}$$

- **Compactly supported** (k-form ν): A k-form ν for which $\operatorname{supp}(\nu)$ is compact.
- $\boldsymbol{\Omega_c^k(\mathbb{R}^n)}$: The set of all C^∞ k-forms which are compactly supported.
- $\boldsymbol{\Omega_c^k(U)}$: The set of all C^∞ k-forms which are compactly supported and $\operatorname{supp}(\omega) \in U$ for all $\omega \in \Omega_c^k(U)$, where $U \subset \mathbb{R}^n$ open.
- **Integral** (of ω over $\mathbb{R}^n$): The usual integral of f over $\mathbb{R}^n$, where $\omega = f\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ is compactly supported and $f \in C_0^\infty(\mathbb{R}^n)$[2]. *Denoted by* $\boldsymbol{\int_{\mathbb{R}^n} \omega}$. *Given by*

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} f\,\mathrm{d}x$$

- **Property *P*** (possessing set U): The property of a set U that for every form $\omega \in \Omega_c^m(U)$ such that $\int_U \omega = 0$, $\omega \in \mathrm{d}\big(\Omega_c^{m-1}(U)\big)$.

[1] Guillemin and Haine (2018) recommends we read the original article; could be worthwhile if I can find it!

[2] Recall that $C_0^\infty(\mathbb{R}^n)$ is the space of all bump functions on $\mathbb{R}^n$.

- Theorem 3.2.3: Let $U \subset \mathbb{R}^{n-1}$ open and $A \subset \mathbb{R}$ an open interval. Then if U has property P, $U \times A$ does as well.

  Proof. ... □

- Theorem 3.2.2 (Poincaré lemma for rectangle): Let ω be a compactly supported n-form with $\text{supp}(\omega) \subset \text{int}(Q)$, where $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Then the following are equivalent.

  1. $\int \omega = 0$.
  2. There exists a compactly supported $(n-1)$-form μ with $\text{supp}(\mu) \subset \text{int}(Q)$ satisfying $\mathrm{d}\mu = \omega$.

  Proof $(2 \Rightarrow 1)$. As given in class on 5/2.
  One additional note: We can compute $\int_{\mathbb{R}^n} \partial f_i / \partial x_i \, \mathrm{d}x$ by Fubini's theorem. □

  Proof $(1 \Rightarrow 2)$. We induct on $\dim Q$. For the base case $n = 1$, the interval A has property P by Exercise 3.2.i. Now suppose inductively that we have proven that $A_1 \times \cdots \times A_{n-1}$ has property P, where $A_i = (a_i, b_i)$. Then by Theorem 3.2.3, $A_1 \times \cdots \times A_n = A_1 \times \cdots \times A_{n-1} \times A_n$ has property P. □

- We now seek to generalize Theorem 3.2.2 to arbitrary connected open subsets of $\mathbb{R}^n$.

- Theorem 3.3.1 (Poincaré lemma for compactly supported forms): Let $U \subset \mathbb{R}^n$ connected and open, and let let $\omega \in \Omega_c^n(U)$ satisfy $\text{supp}(\omega) \subset U$. Then the following are equivalent.

  1. $\int_{\mathbb{R}^n} \omega = 0$.
  2. There exists a compactly supported $(n-1)$-form μ with $\text{supp}(\mu) \subset U$ and $\omega = \mathrm{d}\mu$.

  Proof $(2 \Rightarrow 1)$. The support of μ is contained in a large rectangle, so the integral of $\mathrm{d}\mu$ is zero by Theorem 3.2.2. □

  Proof $(1 \Rightarrow 2)$. ... □

- **Proper** (continuous map): A continuous map $f : U \to V$, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^k$ are open, such that for every $K \subset V$ compact, the preimage $f^{-1}(K)$ is compact.

  – Proper mappings have many nice properties (see the Exercises 3.4).
  – One example is that if $f \in C^\infty$ and $\omega \in \Omega_c^k(V)$ satisfies $\text{supp}(\omega) \subset V$, then $f^*\omega \in \Omega_c^k(U)$.

- **Degree** (of f): The topological invariant of $f : U \to V$, a C^∞ map with $U, V \subset \mathbb{R}^n$ open and connected, defined as follows for all $\omega \in \Omega_c^n(V)$. *Denoted by* $\boldsymbol{\deg(f)}$. *Given by*

$$\int_U f^*\omega = \deg(f) \int_V \omega$$

- A coordinate-based formula for the degree.

  – Let $\omega = \phi(y) \, \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n$ and $x \in U$.
  – Then

$$f^*\omega = (\phi \circ f)(x) \det(Df(x)) \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

  – It follows that

$$\int_V \phi(y) \, \mathrm{d}y = \deg(f) \int_U (\phi \circ f)(x) \det(Df(x)) \, \mathrm{d}x$$

- We now seek to prove that the degree, as defined, exists for suitable functions.

Proof. ... ☐

- Proposition 3.4.4: Let $U,V,W\subset\mathbb{R}^n$ open and connected, and $f:U\to V$ and $g:V\to W$ proper C^∞ maps. Then
$$\deg(g\circ f)=\deg(g)\deg(f)$$

Proof. Let $\omega\in\Omega_c^n(W)$. Then since $(g\circ f)^*\omega=f^*(g^*\omega)$,
$$\begin{aligned}\deg(g\circ f)\int_W\omega&=\int_U(g\circ f)^*\omega\\&=\int_U f^*(g^*\omega)\\&=\deg(f)\int_V g^*\omega\\&=\deg(f)\deg(g)\int_W\omega\\ \deg(g\circ f)&=\deg(g)\deg(f)\end{aligned}$$
as desired. ☐

- Theorem 3.4.6: Let A be a non-singular $n\times n$ matrix and $f_A:\mathbb{R}^n\to\mathbb{R}^n$ the associated linear mapping. Then
$$\deg(f_A)=\begin{cases}+1 & \det(A)>0\\-1 & \det(A)<0\end{cases}$$
  - Note that the non-singularity condition allows us to ignore the case $\det(A)=0$ (since singular matrices have zero determinant).
- Theorem 3.4.7: Let $B\subset V$ compact and $A=f^{-1}(B)$. Then for all U_0 open with $A\subset U_0\subset U$, there exists V_0 open with $B\subset V_0\subset V$ and $f^{-1}(V_0)\subset U_0$.
- **Orientation-preserving** (diffeomorphism): A diffeomorphism $f:U\to V$, where $U,V\subset\mathbb{R}^n$ are open and connected, such that $\det[Df(x)]>0$ for all $x\in U$.
- **Orientation-reversing** (diffeomorphism): A diffeomorphism $f:U\to V$, where $U,V\subset\mathbb{R}^n$ are open and connected, such that $\det[Df(x)]<0$ for all $x\in U$.
  - We know that $\det[Df(x)]$ is nonzero (if it were zero at some x, one of f and its inverse would not be differentiable there, contradicting the definition of a diffeomorphism).
  - This combined with the fact that the determinant is a continuous function of x proves that its sign is the same for all $x\in U$.
  - Thus, orientation-preserving and orientation-reversing are well-defined.
- Theorem 3.5.1: The degree of f is $+1$ if f is orientation-preserving and -1 if f is orientation-reversing.

Proof. ... ☐

- Theorem 3.5.2: Let $\phi:V\to\mathbb{R}$ be a compactly supported continuous function. Then
$$\int_U(\phi\circ f)(x)|\det(Df(x))|\,\mathrm{d}x=\int_V\phi(y)\,\mathrm{d}y$$

Proof. ... ☐

- Guillemin and Haine (2018) goes through the nitty gritty analytic details of the proof.

Week 8

Manifolds and Relevant Tools

8.1 Homotopy Invariance and Applications; Manifold Definitions

5/16:

- Weekly plan:
  - Finish up Chapter 3.
    - Homotopy invariance.
    - Application: Brouwer fixed-point theorem.
  - Manifolds (in $\mathbb{R}^n$).
    - Definition of a manifold X^n.
    - Tangent spaces T_pX^n for $p \in X$.
    - (Total) derivatives of functions $f : X^n \to Y^n$. In particular, if $f : X \to Y$ sends $p \mapsto q$, then $DF_p : T_pX \to T_qY$.
  - Forms $\Omega^k(X^n)$.
  - Integrals $\int_X \omega$ and improving Stokes' theorem.
- Homotopy invariance of degree.
- **Homotopic** (functions F_0, F_1): Two maps $F_0, F_1 : X \to Y$ for which there exists some continuous map $H : X \times I \to Y$ such that

$$H(x,0) = F_0(x) \qquad\qquad H(x,1) = F_1(x)$$

  for all $x \in X$ where $I = [0,1]$. *Denoted by* $\boldsymbol{F_0 \cong F_1}$.
- **Homotopy**: The map H in the above definition.
- Example: A homotopy between two functions $F_0, F_1 : \mathbb{R} \to \mathbb{R}$.
  - Let $X, Y = \mathbb{R}$. Consider the functions $F_0, F_1 : X \to Y$ described by the relations

$$F_0(x) = x^2 \qquad\qquad F_1(x) = 2x$$

  - Let $H : X \times I \to Y$ be described by the relation

$$H(x,t) = (1-t) \cdot x^2 + t \cdot 2x$$

    - Note that x^2 and $2x$ are the relations describing F_0 and F_1, respectively, and the t terms simply provide a linear interpolation. In particular,

$$\begin{aligned} H(x,0) &= (1-0) \cdot x^2 + 0 \cdot 2x \\ &= x^2 \\ &= F_0(x) \end{aligned} \qquad \begin{aligned} H(x,0.5) &= x^2 \cdot (1-0.5) + 2x \cdot 0.5 \\ &= 0.5 \cdot x^2 + 0.5 \cdot 2x \\ &= 0.5F_0(x) + 0.5F_1(x) \end{aligned} \qquad \begin{aligned} H(x,1) &= (1-1) \cdot x^2 + 1 \cdot 2x \\ &= 2x \\ &= F_1(x) \end{aligned}$$

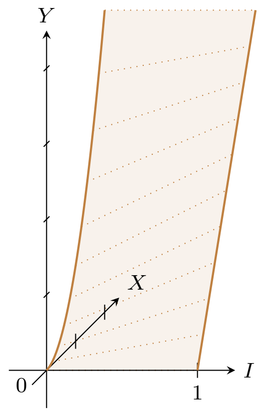


Figure 8.1: Homotopic maps.

Indeed, we can see from the above that $H(x,0) = F_0(x)$, as desired; $H(x,1) = F_1(x)$, as desired; and $H(x,0.5)$, for example, indicates the linear combination of a point that is "half $F_0(x)$ and half $F_1(x)$."

- In Figure 8.1, the parabolic brown line depicts a portion of the graph $G(F_0)$ of F_0. Similarly, the linear brown line depicts a portion of the graph $G(F_1)$ of F_1 but translated one unit along the I-axis. Lastly, the brown surface depicts a portion of the graph $G(H)$ of H.
- As we would expect for a homotopy, H is clearly continuous and interpolates between F_0 and F_1 as t moves from 0 to 1. The lines of dots indicate how several specific values of $F_0(x)$ and $F_1(x)$ are matched in bijective correspondence.

- **Proper** (homotopy): A homotopy such that for all $t \in I$, $H(\cdot,t) : X \to Y$ defined by $x \to H(x,t)$ is proper, where $X, Y \subset \mathbb{R}^n$.
- **Properly homotopic** (functions F_0, F_1): Two homotopic functions whose homotopy is proper.
- Claim: If $F_0, F_1 : U \to V$ where $U, V \subset \mathbb{R}^n$ such that F_0, F_1 are properly homotopic, then

$$\deg(F_0) = \deg(F_1)$$

- (Bad) example.
  - Consider $F_0 : \mathbb{R}^2 \to \mathbb{R}^2$ and $F_1 : \mathbb{R}^2 \to \mathbb{R}^2$ where F_0 is the constant 0 function and $F_1(z) = z^2$.
  - Then $H : \mathbb{R}^2 \times I \to \mathbb{R}^2$ may be defined by $H(z,t) = tz^2$. Clearly, this function is continuous.
  - But $\deg(F_0) = 0$ and $\deg(F_1) = 2$. This is because F_0, F_1 are not *properly* homotopic.
- Proof.
  - Let H be a proper homotopy from $F_0 \to F_1$. Let $H_t : U \to V$ send $x \to H(x,t)$ for all $t \in [0,1]$.
  - Let $\omega \in \Omega_c^n(V)$ with $\int \omega = 1$. Then

$$\deg(H_t) = \int (H_t)^*\omega = \int \varphi(H(x,t)) \det DH_t(x,t) \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

  with $\omega = \varphi \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ where the rightmost integrand is a continuously varying set of functions.
  - Since $\deg(H_t) \in \mathbb{Z}$, $\deg(H_0) = \deg(F_0)$, and $\deg(H_1) = \deg(F_1)$, then $\deg(H_t)$ is constant and the result follows.

- Theorem (Brouwer fixed-point theorem): Let $B^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ be the closed unit ball in $\mathbb{R}^n$. Let $F : B^n \to B^n$ be continuous. Then F has a fixed point.

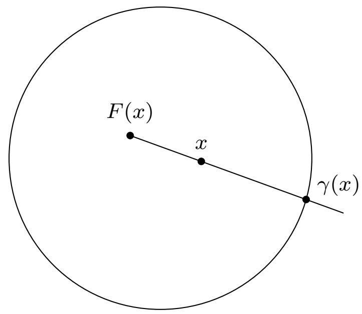


Figure 8.2: Defining γ for the Brouwer fixed point theorem.

  - Assume we have no fixed point and (trick!) consider the map $\gamma : B^n \to S^{n-1}$ which sends $x \mapsto$ the unique point on S^{n-1} that intersects the ray from $F(x)$ to x where $S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$.
  - Check:
    1. γ is continuous.
    2. For all $x \in S^{n-1}$, $\gamma(x) = x$.
  - Now we extend this to a map $\Gamma : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$x \mapsto \begin{cases} \gamma(x) & x \in B^n \\ x & |x| > 1 \end{cases}$$

  - Now for the contradiction. Notice that $\deg(\Gamma) = 1$. But Γ is not surjective (for example, $0 \notin \Gamma(\mathbb{R}^n)$). Recall that we proved earlier that degree nonzero functions are surjective.

- Manifolds (the rest of today and next time).
  - Definition of manifolds.
  - Definition of tangent spaces.
  - We want to be able to take a map $F : X \to Y$ and write $DF_p : T_pX \to T_{F(p)}Y$.
- Manifolds will have a dimension n (hence, we denote them X^n). We will now have them sit inside of some bigger thing, though, i.e., $X^n \subset \mathbb{R}^N$. For example, we'll have $S^2 \subset \mathbb{R}^3$ (the two-sphere lives most naturally in 3-space).
  - We'll also have functions $X^n \to Y^m$ where $X^n \subset \mathbb{R}^N$ and $Y^m \subset \mathbb{R}^M$.
  - We still have $\omega \in \Omega^k(X)$.
- **Smooth** (function $F : X \to Y$): A function $F : X \to Y$ where $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$ such that for all $p \in X$, there is some neighborhood $U_p \subset \mathbb{R}^N$ of p and a map $g_p : U_p \to \mathbb{R}^M$ that is smooth and agrees with F on $X \cap U_p$.

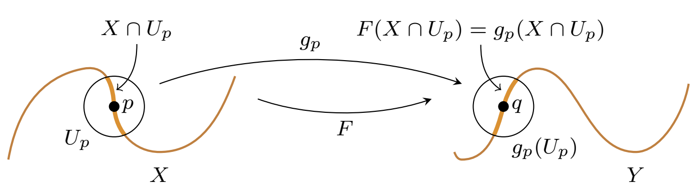


Figure 8.3: Smooth function of manifolds.

- **Diffeomorphism**: A function F that is bijective with F, F^{-1} smooth.
- **n-manifold**: A subset $X^n \subset \mathbb{R}^N$ (where $n \leq N$ are natural numbers) such that for all $p \in X$, there is a neighborhood V of p in $\mathbb{R}^N$, an open set $U \subset \mathbb{R}^n$, and a diffeomorphism $\varphi : U \to X \cap V$. *Also known as* **n-dimensional manifold**.
  - By convention, we indicate the dimension of our manifold with a superscript the first time we write it but not on subsequent writings. So X^n and X are the same thing here; we just write X^n on the first occurrence.
- **Chart**: The map φ in the above definition. *Also known as* **coordinate**, **parameterization**.
- Examples.
  1. $S^1 = \{x \in \mathbb{R}^2 \mid |x| = 1\}$.
     - For every point p on the unit circle, there is a neighborhood V such that $V \cap S^1$ maps bijectively onto $U \subset \mathbb{R}^1$ via some function φ.

8.2 Manifold Examples and Tangent Spaces

5/18:
- Plan.
  - Examples of manifolds.
  - Total derivative of smooth maps between manifolds.
    - So we'll have $F : X \to Y$, $p \in X$, $DF_p = \mathrm{d}F_p : T_pX \to T_{F(p)}Y$
    - If this is surjective, we get local properties of the map F.
    - Injective?
    - Bijective?
  - We'll then take $\mathrm{d}F_p$ and make $F^* : \Omega^k(X) \leftarrow \Omega^k(Y)$.
  - More on $\Lambda^k(T_p^*X)$ and $\omega \in \Omega^k(X)$.
- Examples of manifolds.
  1. $S^2 \subset \mathbb{R}^3$ is the two-sphere.
  2. $U \subset \mathbb{R}^n$ open.
     - The identity map $\mathrm{id} : U \to U$ is a parameterization.
  3. Given $f : \mathbb{R} \to \mathbb{R}$ smooth, its graph $\Gamma_f = \{(x, f(x)) \in \mathbb{R}^2\}$.
     - We can generalize this to graphs of functions $f : \mathbb{R}^k \to \mathbb{R}$ smooth, $\Gamma_f \subset \mathbb{R}^k$. This is a k-manifold.
  4. The torus or any other higher-genus surface in $\mathbb{R}^3$.
  5. $X_1 \subset \mathbb{R}^{N_1}$ and $X_2 \subset \mathbb{R}^{N_1}$ manifolds imply that $X_1 \times X_2 \subset \mathbb{R}^{N_1+N_2}$ is a manifold. Products of parameterizations.
     - Consider $S^1 \times S^1 \subset \mathbb{R}^4$.
     - The 2-torus T^2 is also $S^1 \times S^1$.
     - All such sets are diffeomorphic.
  6. More product manifolds.
     - $S^2 \times S^1$ (concentric spheres with the innermost glued to the outermost through the fourth dimension).
     - $S^1 \times S^1 \times S^1 = T^2 \times S^1$ where T^2 is the 2-torus.
     - Klug discusses unknotting the trefoil knot in $S^2 \times S^1$!

- Note that according to our definition of n-manifolds as *subsets* of N-space, a subset $X^n \subset \mathbb{R}^n$ of Euclidean space is *not* a manifold, even if it may be isomorphic to a manifold.
  - For example, the 2-torus $T^2 \subset \mathbb{R}^3$ is isomorphic to the unit square $[0,1] \times [0,1] \subset \mathbb{R}^2$, but we would not call the latter a manifold. To see the isomorphism, think about cutting a torus once meridionally to create a cylinder and then again longitudinally to create a plane; this plane can then be stretched or squeezed as necessary to fit atop $[0,1]^2$.
- **Cross product** (of X_1, X_2): The Cartesian product of X_1, X_2 as sets.
- We can glue together two genus 2 surfaces with an isomorphism $\varphi : S \to S$ (there are many).
  - In other words, all genus 2 surfaces are isomorphic. Thus, we can divide 2-manifolds into isomorphism classes based on their genus, which also serves as a kind of "manifold invariant."
- Manifolds as solutions to equations.
- **Submersion** (at p): A smooth function $F : U \to \mathbb{R}^k$ such that DF_p is surjective, where $U \subset \mathbb{R}^N$ is open, $N \geq k$, and $p \in \mathbb{R}^N$.
- **Regular value**: A point $q \in \mathbb{R}^k$ such that for all $p \in F^{-1}(q)$, F is a submersion at p (F being defined as above).
- Theorem: If $F : U \to \mathbb{R}^k$ smooth where $U \subset \mathbb{R}^N$ is open and $q \in \mathbb{R}^k$ is a regular value, then $F^{-1}(q) \subset U$ is an $(N-k)$-manifold.
- Example:
  1. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be defined by $(x,y,z) \mapsto x^2 + y^2 + z^2 - 1$.
     - F is smooth and $\mathbb{R}^3 \subset \mathbb{R}^3$ is open.
     - 0 is a regular value.
       - Proof: Suppose (contradiction) that 0 is not a regular value. Then there exists $p \in F^{-1}(0)$ such that F is not a submersion at p. If F is not a submersion at p, then $DF_p : \mathbb{R}^3 \to \mathbb{R}$ is not surjective. Since DF_p is linear, this must mean that DF_p is the zero map. Thus, since
$$DF_{(x,y,z)} = \begin{bmatrix} 2x & 2y & 2z \end{bmatrix}$$
         we must have that $p = (0,0,0)$. But $F(0,0,0) = -1 \neq 0$, a contradiction.
     - Thus, by the theorem, $S^2 = F^{-1}(q) \subset \mathbb{R}^3$ is a $(3-1)$-manifold or 2-manifold.
     - This theorem therefore provides a nice way of proving that a manifold is a manifold without having to find a chart for each point, as we would need to using the definition of an n-manifold alone to determine whether or not an object is a manifold.
  2. Consider $O(n)$, the set of orthogonal square $n \times n$ matrices. We have that $O(n) \subset \mathbb{R}^{n^2}$, where the latter set is the set of all $n \times n$ matrices.
     - We can find a suitable function F and check regular values so that $O(n) = F^{-1}(0)$ where $0 \in \mathbb{R}^n$.
     - Something about the dimension?
     - Is it the determinant?
- Tangent spaces and derivatives.
- Goal:
  - Define T_pX given $X^n \subset \mathbb{R}^N$.
  - Define the induced derivative $\mathrm{d}F_p : T_pX \to T_{F(p)}Y$ where $F : X^n \to Y^m$ for $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$.

- **Tangent space** (to X at p): If $\varphi : U \to X \cap V$ is a parameterization of X at p and sends $p_0 \mapsto p$, then we have a map $\mathrm{d}\varphi_{p_0} : T_{p_0}\mathbb{R}^n \to T_p\mathbb{R}^N$ where we define $T_pX = \operatorname{im}(\mathrm{d}\varphi_{p_0})$.
  - Essentially, the tangent space to an n-manifold should be n-dimensional, but it should be tilted, rotated, placed, etc. such that it is *tangent* to the n-manifold at the point in question.
- Example: Tangent space to S^2.

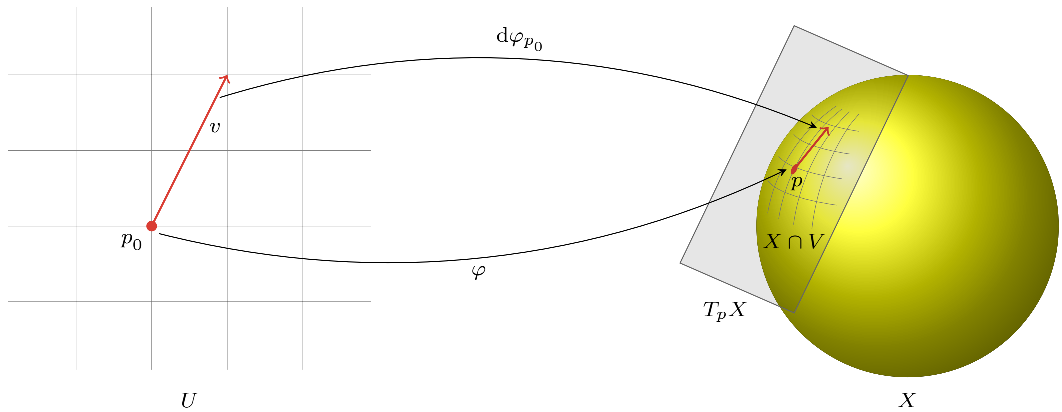


Figure 8.4: Tangent space to a manifold.

  - Consider the two-sphere $S^2 \subset \mathbb{R}^3$. We will call this manifold X.
  - As a 2-manifold, X is locally diffeomorphic to $\mathbb{R}^2$. In this specific instance, we focus in on the point $p \in X$. X is surrounded by some neighborhood $V \subset \mathbb{R}^3$ (not shown) such that $X \cap V$ may be depicted by the area on the surface of X covered in grid lines. The diffeomorphism φ maps $U \subset \mathbb{R}^2$ to $X \cap V \subset \mathbb{R}^3$ and, in particular, maps $p_0 \mapsto p$.
  - As before, we may easily define $T_{p_0}\mathbb{R}^2$ and $T_p\mathbb{R}^3$. However, neither of these spaces particularly well describe describe the set T_pX of tangent vectors to p. We may notice that T_pX is of the same dimension as $T_{p_0}\mathbb{R}^2$, and that T_pX is a subset of $T_p\mathbb{R}^3$. In fact, this is the key: We can use $\mathrm{d}\varphi_{p_0}$ to map $T_p\mathbb{R}^2$ into $T_p\mathbb{R}^3$, and the set of all vectors in the range is equal to T_pX; in particular, $T_pX = \operatorname{im}(\mathrm{d}\varphi_{p_0})$.
  - Lastly, we give a specific example of several of the objects in this picture.
    - Let $p = (2, 7\pi/6, \pi/3)$ in spherical coordinates (note that this implies that S^2 has radius 2), and $p_0 = (0,0)$ in Cartesian coordinates. Also let $U = (-2,3)^2$.
    - Map one unit in $\mathbb{R}^2$ to $\pi/18$ radians ($10°$) of longitude or lattitude across X. Then $\varphi : U \to X \cap V$ is given by
$$\varphi(x,y) = \left(2, \tfrac{7\pi}{6} + \tfrac{\pi}{18}x, \tfrac{\pi}{3} - \tfrac{\pi}{18}y\right)$$
    - If we convert from spherical to Cartesian coordinates, then
$$\varphi(x,y) = \begin{bmatrix} 2\sin\left(\frac{\pi}{3} - \frac{\pi}{18}y\right)\cos\left(\frac{7\pi}{6} + \frac{\pi}{18}x\right) \\ 2\sin\left(\frac{\pi}{3} - \frac{\pi}{18}y\right)\sin\left(\frac{7\pi}{6} + \frac{\pi}{18}x\right) \\ 2\cos\left(\frac{\pi}{3} - \frac{\pi}{18}y\right) \end{bmatrix}$$
    - It follows that
$$D\varphi(x,y) = \begin{bmatrix} -2\sin\left(\frac{\pi}{3} - \frac{\pi}{18}y\right)\sin\left(\frac{7\pi}{6} + \frac{\pi}{18}x\right)\cdot\frac{\pi}{18} & 2\cos\left(\frac{\pi}{3} - \frac{\pi}{18}y\right)\cdot -\frac{\pi}{18}\cos\left(\frac{7\pi}{6} + \frac{\pi}{18}x\right) \\ 2\sin\left(\frac{\pi}{3} - \frac{\pi}{18}y\right)\cos\left(\frac{7\pi}{6} + \frac{\pi}{18}x\right)\cdot\frac{\pi}{18} & 2\cos\left(\frac{\pi}{3} - \frac{\pi}{18}y\right)\cdot -\frac{\pi}{18}\sin\left(\frac{7\pi}{6} + \frac{\pi}{18}x\right) \\ 0 & -2\sin\left(\frac{\pi}{3} - \frac{\pi}{18}y\right)\cdot -\frac{\pi}{18} \end{bmatrix}$$
    - In particular, we have
$$\varphi(0,0) = \begin{bmatrix} -1.5 \\ -\sqrt{3}/2 \\ 1 \end{bmatrix} \qquad D\varphi(0,0) \approx \begin{bmatrix} 0.151 & 0.151 \\ -0.262 & 0.087 \\ 0 & 0.302 \end{bmatrix}$$

- Thus,

$$p = \begin{bmatrix} -1.5 \\ -\sqrt{3}/2 \\ 1 \end{bmatrix} \qquad T_pX \approx \operatorname{span}\left\{ \begin{bmatrix} 0.151 \\ -0.262 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.151 \\ 0.087 \\ 0.302 \end{bmatrix} \right\}$$

- One last note with respect to the drawing of Figure 8.4.
  - All elements of the right side of the diagram are a legitimate orthogonal projection of the elements described above.
  - To begin, we are viewing the sphere such that if the equator were to be drawn, it would have x-radius equal to 2 cm and y-radius equal to 0.5 cm. In other words, we are viewing the sphere from an angle of $\arcsin(0.5/2) = \arcsin(1/4) \approx 14.48^\circ = \theta$ above the equatorial plane.
  - We define the 3-space axes as follows: The x-axis points 1 unit toward the right of the page, the z-axis points 1 unit toward the top of the sphere, and the y-axis is the cross product $z \times x$ of these (pointing into the page). We define the axes in the plane of the page as follows: The x axis points 1 cm toward the right of the page and the y-axis points 1 cm toward the top of the page.
  - With these axis definitions, trigonometric arguments show that the projection operator P should map

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ \sin\theta \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ \cos\theta \end{bmatrix}$$

    It follows that

$$\mathcal{M}(P) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.25 & 0.968 \end{bmatrix}$$

  - Thus, to display p, the tangent vector, and the tangent plane, we need only feed p, $D\varphi(p_0)(v)$, and the basis of T_pX, respectively, into P.
  - The longitude and latitude lines are a bit trickier, since these are *functions* that need to be passed through P.
  - We'll start with longitude. Again using trigonometric arguments, we can determine that if α is the angle between the x-axis and the vertical plane containing a longitude line, then the path of the longitude line across the surface of the sphere as a function of $h = z/2$ is given by

$$h \mapsto \begin{bmatrix} -\cos\alpha\sqrt{1-h^2} \\ -\sin\alpha\sqrt{1-h^2} \\ h \end{bmatrix}$$

  - Latitude can be done similarly, but it's easier to take the equator ellipse (with half-axes 2 and 0.5) and move, scale, and shrink it upwards.
  - See handwritten pages for more info.

- An alternate definition of T_pX, assuming $X = f^{-1}(0)$ where $f : \mathbb{R}^N \to \mathbb{R}^k$ is a C^∞ map: The kernel of the surjective map $\mathrm{d}f_p : T_p\mathbb{R}^N \to T_0\mathbb{R}^k$.
  - In terms of Figure 8.4, we can use $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $x \mapsto x_1^2 + x_2^2 + x_3^2 - 1$. Then since f only has nonzero change in directions normal to $X = S^2$, every $(p, v) \in T_p\mathbb{R}^3$ with v having no component normal to X at p will be mapped to zero by $\mathrm{d}f_p$. And these are exactly the tangent vectors.
- Guaranteeing that T_pX does not depend on φ.
  - If $\mathrm{d}\varphi_{p_0}$ is injective, then $\dim T_pX = n$.
  - No, it does not depend on the choice of parameterization; yes, $\mathrm{d}\varphi_{p_0}$ is injective.

- We check both of these assertions by stating and proving that all manifolds are locally given as solutions to equations.
- We then use this to get a clearly well-defined definition of T_pX — this check agrees with our definition for all φ.

- Given $F : X \to Y$ smooth and $p \in X$, we define $\mathrm{d}F_p : T_pX \to T_{F(p)}Y$ as follows: There is some neighborhood U of $p \in \mathbb{R}^N$ and a smooth function $\tilde{F} : U \to \mathbb{R}^M$ that agrees with F on $X \cap U$. This implies $\mathrm{d}\tilde{F}_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^M$. We have $T_pX \subset T_p\mathbb{R}^N$ and $T_{F(p)}Y \subset T_{F(p)}\mathbb{R}^M$. We now define $\mathrm{d}F_p$ as the restriction of $\mathrm{d}\tilde{F}_p$ to T_pX.
- Check.
  1. Image of $\mathrm{d}F_p$ is indeed inside $T_{F(p)}Y$.
  2. Does not depend on $\tilde{F}$ or U.

8.3 Objects on Manifolds

5/20: • Today:
  - Bring all of our favorite gadgets to manifold land.
  - Tangent spaces (done).
  - (Total) derivatives $\mathrm{d}F_p : T_pX \to T_{F(p)}Y$.
  - Vector fields, integral curves, and flows.
  - Differential forms $\omega \in \Omega^k(X)$.
  - Differential forms package d, $\wedge$, L_v, maps $\cdots \xrightarrow{d} \Omega^k(X) \xrightarrow{d} \Omega^{k+1}(X) \xrightarrow{d} \cdots$, pullbacks.
- Next time:
  - $\int \omega$ (integration of forms on manifolds).
  - Relationship between $\int$, d. This implies Stokes' Theorem.
- Let $X^n \subset \mathbb{R}^N$ be a manifold[1].
- **Vector field** (on X): A function that to every $p \in X$ assigns some tangent vector $v_p \in T_pX$. *Denoted by* $\boldsymbol{v}$.
- Examples.
  - Consider a circular vector field on the unit circle $S^1 \subset \mathbb{R}^2$.
    - The integral curves follow the vector field around S^1.
    - If we take the vectors to be longer, then we'll go around faster.
  - Consider a meridional vector field on a torus $T \subset \mathbb{R}^3$.
    - The integral curves go around the donut.
  - Consider a vector field on the unit sphere $S^2 \subset \mathbb{R}^3$ that creeps out from the south pole, reaching unit length by the equator and then getting shorter and shorter towards the north pole.
    - The integral curves are the half lines of longitudes.
- Theorem (Hairy ball theorem): There is no smooth nonvanishing vector field on S^2.
  - "No matter how you comb the hair on a ball, you get some cowlicks somewhere."

[1]Thinking of our manifold as a subset of Euclidean space is a crutch. We should think of ourselves standing on the isolated manifold, of existing in its space. We can think of having our little Euclidean charts to navigate in the viscinity of every point, but we are in/on the manifold.

- **Integral curve** (of $\boldsymbol{v}$ on X): A map $\gamma : (a, b) \to X$ for which $\gamma'(t_0) = \boldsymbol{v}_{\gamma(t_0)}$.
- **k-form** (on X): A function that to each $p \in X$ assigns some $\omega_p \in \Lambda^k(T_p^* X)$.
- Remember that $T_pX \subset T_p\mathbb{R}^N$, but $\Lambda^k(T_p^*X) \not\subset \Lambda^k(T_p^*\mathbb{R}^N)$.
- **Support** (of $\omega \in \Omega^k(X)$): Defined similarly to before.
- **$\Omega_c^k(X)$**: The set of all compactly supported k-forms.
- If X is compact, then $\Omega^k(X) = \Omega_c^k(X)$.
- An example noncompact manifold.

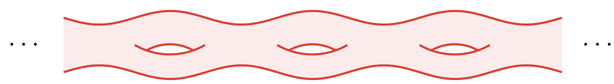


Figure 8.5: Noncompact manifold.

- **Pullback** (of ω to X): The k-form on X which sends each $p \in X$ to $\mathrm{d}F_p^* \omega_q$, where $X^n \subset \mathbb{R}^N$, $Y^m \subset \mathbb{R}^M$, $F : X \to Y$, $F(p) = q$, and $\omega \in \Omega^k(Y)$. *Denoted by* $\boldsymbol{F^*\omega}$.
  - Since F is linear, we get a bunch (one for each $p \in X$) of maps $\mathrm{d}F_p : T_pX \to T_{F(p)}Y$. But as a linear map, we can define its pullback $\mathrm{d}F_p^* = (\mathrm{d}F_p)^* : \Lambda^k(T_p^*X) \leftarrow \Lambda^k(T_{F(p)}^*Y)$. This can then transform the vectors fed into ω.
  - This definition is entirely analogous to the definition of the pullback of forms on vector spaces.
- **Smooth** (k-form on X): A k-form ω such that for all $p \in X$, there exists some open neighborhood U of p in X and a parameterization $\varphi : U_0 \to U$, where $U_0 \subset \mathbb{R}^n$, such that $\varphi^*\omega|_U$ is a smooth form on U_0.
- From now on, $\Omega^k(X)$ is the set of *smooth* k-forms on X.
- Examples of maps between manifolds.
  1. Consider S^1 as a subset of the complex plane $\mathbb{C}$ instead of $\mathbb{R}^2$. Then the map $F : S^1 \to S^1$ which sends each $z \in S^1$ to $z^2 \in S^1$ rotates all points of S^1 about the origin to the point that is twice as far from the $+x$-axis.
  2. Map $\mathbb{C} \cup \{\infty\}$ to S^2 such that 0 is the south pole, the equator is the unit circle, and the north pole is ∞. Then $F : S^2 \to S^2$ defined by $z \mapsto z^2$ is again a curious type of rotation map.
  3. Consider the 2-torus $T^2 \subset \mathbb{R}^3$. A map from T^2 to the manifold $(a, b) \subset \mathbb{R}$ could be the height map of the torus.
     - Preimages of points in (a, b) are circular submanifolds.
     - Dots are critical values.
- Determining when a vector field $\boldsymbol{v}$ on X is smooth.
  - Way 1: If for all $p \in X$, there exists V open in $\mathbb{R}^N$ and a smooth vector field $\tilde{\boldsymbol{v}}$ on V that agrees with $\boldsymbol{v}$ on $V \cap X$.
  - Way 2: As with forms, pullbacks and check on the charts (φ is a diffeomorphism).
- **Exterior derivative** (for k-forms on manifolds): The function from $\Omega^k(X) \to \Omega^{k+1}(X)$ defined as follows, where X is an n-manifold, $p \in X$, $p \in U \subset X$, $U_0 \subset \mathbb{R}^n$, and $\varphi : U_0 \to U$ is a diffeomorphism. *Denoted by* $\mathbf{d}$. *Given by*
$$(\mathrm{d}\omega)_p = [(\varphi^{-1})^* \, \mathrm{d}(\varphi^*\omega)]_p$$
  - Check: Well-defined, i.e., does not depend on the choice of φ.
  - All the familiar properties carry over.
$$\mathrm{d} \circ F^* = F^* \circ \mathrm{d} \qquad \mathrm{d}(\omega \wedge \eta) = \cdots \qquad \mathrm{d}^2 = 0 \qquad \mathrm{d} \text{ is linear}$$

8.4 Klug Meeting

- What are alternating tensors? Sure, I can define them. I also found the alternate definition of them as "two elements in the argument are the same implies $T(v_1, \dots, v_k) = 0$." But I still have no concept of what they "look like intuitively," what to make of their basis (the alternatizations of the strictly increasing dual basis vectors), or why their dimension should transform as $\binom{n}{k}$.
  - They're just an algebraic thing you need to make integration make sense.
  - We're gonna want to integrate things that are oriented, and when we change the orientation, we're gonna flip the sign. So alternating tensors capture how things change when you flip the signs.
  - We'll probably see this next week.
  - ω_p is an alternating tensor if ω is a form.
  - Covectors are 1-tensors, which makes them alternating automatically. But we don't have to worry about this with covectors because there's only one entry point.
  - Two forms use alternating 2-tensors.
  - Top dimensional forms.
  - Two forms are functions decorated by $\mathrm{d}x \wedge \mathrm{d}y$; you integrate them via the 2D integral.

$$\int_U \omega = \int_U f \,\mathrm{d}x \wedge \mathrm{d}y = \iint f \,\mathrm{d}x \,\mathrm{d}y$$

  - Don't try to figure out every little piece; just sit back and watch the theory unfold and then it will make more sense on subsequent viewings.
  - Alt and π are two isomorphisms between $\Lambda^k(V^*)$ and $\mathcal{A}^k(V)$. The alternating guys are more natural to think about; the quotient is more weird. The advantage of $\Lambda^k(V^*)$ is it makes defining $\wedge$ simpler.
  - Rep theory and algebra will introduce this stuff again in a different context and it will make more sense then.
  - We do have to deal with the nitty gritty on the homework still however. Making us suffer in a hopefully productive way. Choose things that come naturally to you, though. You'll come back later, you'll be better at learning (the ocean will rise), and it will often make so much more sense then.
- How does the idea of "it suffices to check this for decomposable tensors" typically work? It seems to often appear in cases where linearity is a factor and we can decompose an arbitrary tensor into a linear combination of the basis, which is of course composed of decomposable tensors.
- What is functoriality?
  - A fancy word people use to obfuscate things.
  - If $X \xrightarrow{F} Y \xrightarrow{Z}$, then $(G \circ F)^* = F^* \circ G^*$.
  - Just something that happens really often in math.
  - Category theory is just a language for talking about certain phenomena that arise so often that you'd want to have a language, but it's just grammar. You would never actually use it.
- What is $\Lambda^k(V^*)$, and why is it the k^{th} exterior power of V^*, and what does that even mean? The elements of it are $\mathcal{I}^k(V)$-cosets of tensors; what does one of these look like? The elements of it aren't even functions, right? They're just sets of functions?
- What is the wedge product intuitively?
- How does the tensor product we learned relate to the tensor product of two vectors and the tensor product of two vector spaces? And what are these latter quantities?

- What properties intuitively characterize decomposable tensors?
- What properties intuitively characterize redundant tensors?
- What is the interior product?
- What is the pullback?
- How did we define the determinant in terms of exterior powers?
- What are 1-forms?
- How did all that stuff we did with tensors relate to forms? Is $\mathrm{d}f$ a 2-tensor $\mathrm{d}f : U \times \mathbb{R}^n \to T^*\mathbb{R}^n$?
- On the integral: Doesn't the definition imply that the integral of $\partial/\partial x$ where $U = \mathbb{R}^2$ is the constant plane instead of the sloped plane? If we need the integral curves along it to be constant?
- I've been thinking of one-forms as mathematical objects which assign to every point p of a vector space a bundle of vectors. What are k-forms?
- What is exterior differentiation?
- PSet 2, 2.1.iii.
- Thoughts on the degree?
- How much multivariable calculus knowledge have you assumed for us? Do you believe there is value in knowing the more computational aspects of multi before looking into this?
  - Klug has never taken a course on this stuff.
  - You wouldn't need any duals if you just stuck to Euclidean space.
  - We're unifying vector calculus and multivariable calculus while generalizing to n-dimensions.
  - Instead of looking for motivation now, you kinda need to finish the whole textbook first and then reread it. At the end, you'll have theorems that make it worth it, and then you can reverse engineer.
  - John Lee trilogy of books on this math with an eye toward stuff that people care about. Point set topology. Introduction to smooth manifolds is book 2.
  - Nobody cares about point-set topology, but it's helpful for writing proofs and practicing logic.
  - We won't get to de Rahm cohomology in this course, but we should see it.
  - Klug read Lee in kind of an anxious haze believing it was gonna be important but it largely hasn't been. Any book is a linearization of an organic bloby process.
  - All the Lee books get used as the language of general relativity. If your Einstein trying to express your thoughts, you're happy to know the people who have been developing this differential forms language.
  - You want to get in the full mindset of "I could have discovered this." But it's very hard to reach that level. You can often use the stuff short of being there. Using it enough will get you to back into expert knowledge. Use it, and then backfill your knowledge.
- What do you want us to be getting out of this survey of the material?
- What do you want us to be getting out of the homework?
- How do you recommend we use the textbook? Where can we go for additional reference?
- How are we supposed to learn/motivate this stuff? Will we get to the motivation part in this course? Because I'd really learn this stuff better than just memorizing a bunch of definitions for the final, but I have basically no idea what the definitions mean.

- What resources do we have for help on homework problems we can't get?
- What will the final look like?
  - Probably just like the midterm, but he'll figure it out later.

8.5 Chapter 3: Integration on Forms

From Guillemin and Haine (2018).

5/28:

- **Critical point** (of f): A point $x \in U$ such that the derivative $Df(x) : \mathbb{R}^n \to \mathbb{R}^n$ fails to be bijective, i.e., $\det(Df(x)) = 0$.
- $\boldsymbol{C_f}$: The set of critical points of f.
  - Since $\det(Df) : U \to \mathbb{R}$ is continuous ($f \in C^\infty$ by hypothesis must be *continuously* differentiable) and $\{0\}$ is closed, $C_f = \det(Df)^{-1}(\{0\})$ is closed.
  - Consequently, $f(C_f)$ is a closed subset of V.
- **Critical value** (of f): The image of a critical point under f, i.e., an element of $f(C_f)$.
- **Regular value** (of f): An element of the range of f that is not a critical value, i.e., an element of $f(U) \setminus f(C_f)$.
  - Since $V \setminus f(U) \subset f(U) \setminus f(C_f)$, if $q \in V$ is not in the image of f, it is a regular value of f by default. More precisely, since elements of $V \setminus f(U)$ do not contain values of U, let alone any critical points of f, in their preimage, $V \setminus f(U)$ cannot contain any critical values[2].
- Theorem 3.6.2 (Sard): If $U, V \subset \mathbb{R}^n$ open and $f : U \to V$ a proper C^∞ map, then the set of regular values of f is an open dense subset of V.

6/1:

- Theorem 3.6.3: If q is a regular value of f a proper function, the set $f^{-1}(q)$ is finite. Additionally, if we let $f^{-1}(q) = \{p_1, \dots, p_n\}$, then there exist connected open neighborhoods $U_i \subset U$ of all p_i and an open neighborhood $W \subset V$ of q such that...
  1. For $i \neq j$, the sets U_i, U_j are disjoint;
  2. $f^{-1}(W) = U_1 \cup \cdots \cup U_n$;
  3. f maps every U_i diffeomorphically onto W.

  Proof. Let $p \in f^{-1}(q)$. Then p is not a critical point of f, so the derivative $Df(p)$ of f at p is bijective. It follows by the inverse function theorem that there exists a neighborhood U_p of p that f maps diffemorphically onto a neighborhood V_q of q.

  Since we can pick such an open subset for all $p \in f^{-1}(q)$, we know that the set $\{U_p \mid p \in f^{-1}(q)\}$ is an open cover of $f^{-1}(q)$. Additionally, since f is proper and $\{q\}$ is compact, $f^{-1}(q)$ is compact. Thus, as an open cover of a compact set, $\{U_p \mid p \in f^{-1}(q)\}$ has a finite subcover (which we may call $\{U_{p_1}, \dots, U_{p_N}\}$).

  Now suppose for the sake of contradiction that p_i, p_j are both elements of U_{p_i}. Then since $f(p_i) = f(p_j) = q$, f does not map U_{p_i} bijectively onto V_{q_i}. Thus, f does not map U_{p_i} diffeomorphically onto U_{q_i}, a contradiction. Therefore, every U_{p_i} contains at most one element of $f^{-1}(q)$. In particular, since U_{p_i} contains p_i by definition, it must be that every p_i is the one point in U_i. (For example, we could not have $p_1 \in U_2$ and $p_2 \in U_1$ since $p_1 \in U_1$ and $p_2 \in U_2$ by definition.) It follows that there is a bijective correspondence between the $\{U_{p_i}\}$ and the $\{p_i\}$, so it must be that $f^{-1}(q) = \{p_1, \dots, p_N\}$ is a finite set.

[2] I get the gist of this statement, but it makes no sense. It is in Guillemin and Haine (2018), regardless, though.

We now make the $\{U_{p_i}\}$ disjoint, if they are not already. Suppose, for instance, $U_{p_i} \cap U_{p_j} \neq \emptyset$. Then since there are only finitely many p_i (i.e., p_i, p_j are not infinitely close together), we may simply shrink the neighborhoods as needed. One way to do this is to redefine $U_{p_i} = U_{p_i} \cap N_r(p_i)$ and likewise for p_j, where $r = d(p_i, p_j)/2$.

Finally, by Theorem 3.4.7, there exists a connected open neighborhood $W \subset V$ of q for which $f^{-1}(W) \subset U_{p_1} \cup \cdots \cup U_{p_N}$. We lastly define every $U_i = f^{-1}(W) \cap U_{p_i}$, and it will follow from the above that these U_i have all the desired properties. □

- Theorem 3.6.4: Let q be a regular value of f, and let $f^{-1}(q) = \{p_1, \dots, p_N\}$, as above. Define $\sigma : f^{-1}(q) \to \{\pm 1\}$ by

$$\sigma_{p_i} = \begin{cases} +1 & f : U_i \to W \text{ is orientation preserving} \\ -1 & f : U_i \to W \text{ is orientation reversing} \end{cases}$$

Then

$$\deg(f) = \sum_{i=1}^{N} \sigma_{p_i}$$

Proof. Let $\omega \in \Omega_c^n(W)$ such that $\int_W \omega = 1$. Then

$$\deg(f) = \int_U f^*\omega = \sum_{i=1}^{N} \int_{U_i} f^*\omega$$

where by Theorem 3.5.1,

$$\int_{U_i} f^*\omega = \int_W \omega = \begin{cases} +1 & f \text{ is orientation preserving} \\ -1 & f \text{ is orientation reversing} \end{cases}$$

Thus, we have the desired result. □

- Theorem 3.6.6: If $f : U \to V$ is not surjective, then $\deg(f) = 0$.

Proof. We first present a hand-wavey proof based on Theorem 3.6.4. Choose $q \in V \setminus f(U)$. Then $f^{-1}(q) = \emptyset$. It follows that

$$\deg(f) = \sum_{i=1}^{0} \sigma_{p_i} = 0$$

For a more rigorous proof, consider the following. By Exercise 3.4.iii, $V \setminus f(U)$ is open. This combined with the fact that it is nonempty reveals that there exists a compactly supported n-form ω with support in $V \setminus f(U)$ and $\int_{V \setminus f(U)} \omega = 1$. Since $\omega(f(U)) = \{0\}$ as a compactly supported form on a set of points outside $f(U)$, $f^*\omega = 0$, so

$$0 = \int_U f^*\omega = \deg(f) \int_V \omega = \deg(f)$$

□

- Theorem 3.6.8: If $\deg(f) \neq 0$, then $f : U \twoheadrightarrow V$.

Proof. This is the contrapositive of Theorem 3.6.6. □

- Note that we will use Theorem 3.6.8 far more often than Theorem 3.6.6.

- **Proper homotopy**: A homotopy F between f_0, f_1 for which $F^\sharp : U \times A \to V \times A$ defined by
$$(x,t) \mapsto (F(x,t),t)$$
is proper.
  - If f_0, f_1 are properly homotopic, then f_t defined by $f_t(x) = F(x,t)$ is proper for all $t \in (0,1)$.
- Theorem 3.6.10: If f_0, f_1 are properly homotopic, then $\deg(f_0) = \deg(f_1)$.
- Theorem 3.6.13 (The Brouwer fixed point theorem): Let $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ be the closed unit ball in $\mathbb{R}^n$. If $f : B^n \to B^n$ is continuous, then f has a fixed point, i.e., there exists $x_0 \in B^n$ for which
$$f(x_0) = x_0$$
- Guillemin and Haine (2018) also proves the fundamental theorem of algebra.
- Guillemin and Haine (2018) proves Sard's theorem.

8.6 Chapter 4: Manifolds and Forms on Manifolds

From Guillemin and Haine (2018).

6/2:
- In this section, we let $X \subset \mathbb{R}^N$, $Y \subset \mathbb{R}^n$, and $f : X \to Y$ continuous unless stated otherwise.
- ***C*$^\infty$ map**: A continuous map $f : X \to Y$, where $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^n$, such that for every $p \in X$, there exists a neighborhood $U_p \subset \mathbb{R}^N$ of p and a C^∞ map $g_p : U_p \to \mathbb{R}^n$ which coincides with f on $U_p \cap X$.
- Theorem 4.1.2: If $f : X \to Y$ is a C^∞ map, then there exists a neighborhood $U \subset \mathbb{R}^N$ of X and a C^∞ map $g : U \to \mathbb{R}^n$ such that g coincides with f on X.
- Intuitively, if Y is an open subset, the set X described by the above definitions is a **manifold**.
- ***n*-manifold**: A subset $X \subset \mathbb{R}^N$ such that for every $p \in X$, there exists a neighborhood $V \subset \mathbb{R}^N$ of p, an open subset $U \subset \mathbb{R}^n$, and a diffeomorphism $\phi : U \to X \cap V$, where $N, n \in \mathbb{N}_0$ satisfy $n \leq N$.
  - An alternate interpretation is that X is an n-manifold if, locally near every point p, X "looks like" an open subset of $\mathbb{R}^n$.
- Examples.
  1. Graphs of functions $f : U \to \mathbb{R}$.
  2. Graphs of mappings $f : U \to \mathbb{R}^k$.
  3. Vector subspaces (of $\mathbb{R}^n$ or any abstract vector space V).
  4. Affine subspaces of $\mathbb{R}^n$ (e.g., cosets; subsets of the form $p + V$ where $V \leq \mathbb{R}^n$).
  5. Product manifolds.
  6. The unit n-sphere.
  7. The 2-torus.
- Guillemin and Haine (2018) also gives diffeomorphisms for the above examples.
  - Two important diffeomorphism that arise.
  - One arises in conjunction with vector subspaces. In particular, we define $\phi : \mathbb{R}^n \to V$ by
$$(x_1, \dots, x_n) \mapsto \sum_{i=1}^{n} x_i e_i$$
where $\{e_i\}$ is a basis of V.

  - One arises in conjunction with affine subspaces. In particular, we define $\tau_p : \mathbb{R}^N \to \mathbb{R}^N$, where $p \in \mathbb{R}^N$, by
$$x \mapsto p + x$$

- We now build up to regarding manifolds as the solutions to systems of equations.
- **Submersion** (at p): A C^∞ map $f : U \to \mathbb{R}^k$, where $U \subset \mathbb{R}^N$, for which $Df(p) : \mathbb{R}^N \to \mathbb{R}^k$ is surjective.
  - Note that for this linear map to be surjective, we must have $k \leq N$.
- **Regular value** (of f): A point $a \in \mathbb{R}^k$ such that for all $p \in f^{-1}(a)$, f is a submersion at p.
- **Canonical submersion**: The function defined as follows, which is a submersion at every point of its domain. *Denoted by* $\boldsymbol{\pi}$. *Given by*
$$\pi(x_1, \dots, x_n) = (x_1, \dots, x_k)$$
- Theorem B.17 (canonical submersion theorem): Let $U \subset \mathbb{R}^n$ open and $\phi : (U, p) \to (\mathbb{R}^k, 0)$ a C^∞ map plus a submersion at p. Then there exists a neighborhood $V \subset U$ of p, a neighborhood $U_0 \subset \mathbb{R}^n$ of the origin, and a diffeomorphism $g : (U_0, 0) \to (V, p)$ such that $\phi \circ g : (U_0, 0) \to (\mathbb{R}^n, 0)$ is the restriction to U_0 of the canonical submersion.
- Theorem 4.1.7: Let $n = N - k$. If a is a regular value of $f : U \to \mathbb{R}^k$, then $X = f^{-1}(a)$ is an n-manifold.

  Proof. Instead of considering f, let's consider $\tau_{-a} \circ f$ so that $a = 0$ WLOG. Indeed, when we say "f" from now on, we mean "$\tau_{-a} \circ f$."

  To prove that $X = f^{-1}(0)$ is an n-manifold, it will suffice to show that for every $p \in X$, there exists a neighborhood $V \subset \mathbb{R}^N$ of p, an open subset $U \subset \mathbb{R}^n$, and a diffeomorphism $\phi : U \to X \cap V$. Let $p \in X$ be arbitrary. Then since 0 is a regular value of f, f is a submersion at p. Thus, by the canonical submersion theorem, there exists a neighborhood $O \subset \mathbb{R}^N$ of 0, a neighborhood $U_0 \subset U$ of p, and a diffeomorphism $g : O \to U_0$ such that $f \circ g = \pi$ where π is the canonical submersion. Since $\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^n$, it follows that $\pi^{-1}(0) = \{0\} \times \mathbb{R}^n \cong \mathbb{R}^n$ (where both zeros in the previous equation refer to the $0 \in \mathbb{R}^k$). Consequently, by the definition of the diffeomorphism of vector subspaces, g maps $O \cap \pi^{-1}(0)$ diffeomorphically onto $U_0 \cap f^{-1}(0)$. However, $O \cap \pi^{-1}(0) \subset \mathbb{R}^n$ is a neighborhood of 0 and $U_0 \cap f^{-1}(0) \subset X$ is a neighborhood of p and these two neighborhoods are diffeomorphic. □

- Examples of manifolds as the solution to equations.
  1. $f : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $(x_1, \dots, x_{n+1}) \mapsto x_1^2 + \cdots + x_{n+1}^2 - 1$ and the n-sphere as $f^{-1}(0)$.
  2. Graphs.
  3. The space of orthogonal matrices.
- Note that it is not random that these manifolds arise as zero sets of submersions. In fact, "we will show that locally *every* manifold arises this way" (Guillemin & Haine, 2018, p. 115).
- Guillemin and Haine (2018) goes about proving this fact.
- Theorem 4.1.15: Let X be an n-dimensional submanifold of $\mathbb{R}^N$ and let $\ell = N - n$. Then for every $p \in X$, there exists a neighborhood $V_p \subset \mathbb{R}^N$ of p and a submersion $f : (V_p, p) \to (\mathbb{R}^\ell, 0)$ such that $X \cap V_p$ is defined by ...
- We interpret Theorem 4.1.15 as saying that $f^{-1}(a)$ is the set of solutions to the system $f_i(x) = a_i$ $(i = 1, \dots, k)$, and the regular value condition as guaranteeing that the system is an independent system of defining equations (e.g., no redundant information).
- **Parameterization** (of X at p): The function $\phi : U \to X \cap V$, where $U \subset \mathbb{R}^n$ is open, $V \subset \mathbb{R}^N$ is a neighborhood of p, and X is an n-manifold.

- **Tangent space** (to X at p): The image of the linear map $(\mathrm{d}\phi)_q : T_q\mathbb{R}^n \to T_p\mathbb{R}^N$, where ϕ is a parameterization of X at p and $\phi(q) = p$.
  - The examples given make it clear that if $X^n \subset \mathbb{R}^N$, T_pX is the subset of $T_p\mathbb{R}^N$ containing all vectors with tail at point p that are tangent to X.
- Guillemin and Haine (2018) goes through the nitty gritty details of defining the tangent space properly.
- **Vector field** (on X): A function which assigns to each $p \in X$ an element of T_pX. *Denoted by* $\boldsymbol{v}$.
- **$\boldsymbol{k}$-form** (on X): A function which assigns to each $p \in X$ an element of $\Lambda^k(T_p^*X)$. *Denoted by* $\boldsymbol{\omega}$.
- **$\boldsymbol{f}$-related** (vector fields $\boldsymbol{v}, \boldsymbol{w}$ on X): Two vector fields $\boldsymbol{v}, \boldsymbol{w}$ on X such that for all $p \in X$ and $q = f(p)$,

$$(\mathrm{d}f)_p\boldsymbol{v}(p) = \boldsymbol{w}(q)$$

  - See Figure 4.5 and the associated discussion.
- **Pushforward** (of $\boldsymbol{v}$ by f): The unique vector field $\boldsymbol{w}$ such that $\boldsymbol{v}, \boldsymbol{w}$ are f-related.
- **Pullback** (of $\boldsymbol{w}$ by f): The unique vector field $\boldsymbol{v}$ such that $\boldsymbol{v}, \boldsymbol{w}$ are f-related.
- Proposition 4.3.4: Defining the chain rule/functoriality for the pushforward and pullback.
- **Parameterizable open set**: An open subset U of X for which there exists a corresponding open set $U_0 \subset \mathbb{R}^n$ and diffeomorphism $\phi_0 : U_0 \to U$.
  - "Note that X being a manifold means that every point is contained in a parameterizable open set" (Guillemin & Haine, 2018, p. 126).
- **Smooth** (k-form on U): A k-form ω on $U \subset X$ for which there exists a parameterizable open set with parameterization ϕ_0 such that $\phi_0^*\omega$ is C^∞.
- Guillemin and Haine (2018) proves that this definition is independent of our choice of ϕ_0.
- **Smooth** (k-form on X): A k-form ω on X such that for every $p \in X$, ω is smooth on a neighborhood of p.
- Proposition 4.3.10: Let X, Y manifolds and $f : X \to Y$ a C^∞ map. If ω is a smooth k-form on Y, then the pullback $f^*\omega$ is a smooth k-form on X.
- Proposition 4.3.11: An analogous result for vector fields.
- **Unit vector** (in $T_{t_0}\mathbb{R}$): The vector $(t_0, 1) \in T_{t_0}\mathbb{R}$. *Denoted by* $\vec{\boldsymbol{u}}$.
  - This vector arises when we have an integral curve $\gamma : I \to X$, where $t_0 \in I \subset \mathbb{R}$, I being an open interval. Specifically, we will use it to define the tangent vector to γ at t_0, as follows.
- **Tangent vector** (to γ at p): The vector $\mathrm{d}\gamma_{t_0}(\vec{u}) \in T_pX$, where $p = \gamma(t_0)$.
- **Integral curve** (of $\boldsymbol{v}$): A curve $\gamma : I \to X$ such that for all $t_0 \in I$,

$$\boldsymbol{v}(\gamma(t_0)) = \mathrm{d}\gamma_{t_0}(\vec{u})$$

  where $\boldsymbol{v}$ is a vector field on X.
- Proposition 4.3.13: Integral curves get mapped from X to Y by f for f-related vector fields.
- Local existence, local uniqueness, and smooth dependence on initial data follow.
- More integral curves stuff.

- **Exterior derivative** (of ω on X): The k-form defined as follows, where ω is a smooth k-form on X, $U \subset X$ is a parameterizable open set, and $\phi_0 : U_0 \to U$ is a parameterization. *Denoted by* $\mathbf{d}\boldsymbol{\omega}$. *Given by*

$$\mathrm{d}\omega = (\phi_0^{-1})^*(\mathrm{d}(\phi_0^*(\omega)))$$

  - Essentially, what we're doing here is pulling back our k-form on X into $\mathbb{R}^n$, taking the exterior derivative there, and then pulling it back onto X.

- Theorem 4.3.22: If X, Y are manifolds and $f : X \to Y$ is smooth, then for $\omega \in \Omega^k(Y)$, we have

$$f^*(\mathrm{d}\omega) = \mathrm{d}(f^*\omega)$$

- Guillemin and Haine (2018) covers the interior product and Lie derivative in manifold-land.

Week 9

Integration of Manifolds

9.1 Orientations on Manifolds

5/23:

- Weekly plan.
  - We've got places to be — it's good to worry about what everything is, but it's also good to just think of stuff as how it historically developed.
  - Goal: Stokes' Theorem ($\int_x \mathrm{d}\omega = \int_{\partial x} \omega$).
    - We need to talk about the boundary ∂x.
    - We need to talk about the integral (integrating forms on manifolds).
    - Hidden orientation convention (we'll see in examples).
- Special cases.
  1. The fundamental theorem of calculus.
     - Take the manifold to be $X = [a, b] \in \mathbb{R}$.
     - Here, $\partial x = \{a, b\}$.
     - Take $f(x) \in \Omega^0(X)$.
     - Take $\mathrm{d}f = f'(x)\,\mathrm{d}x$ where $\mathrm{d}x \in \Omega^1(X)$.
     - So by Stokes' theorem, integrating over the whole interval $\int_a^b f'(x)\,\mathrm{d}x$ is equal to integrating over the boundary, but integration over the boundary is $f(b) - f(a)$.
       - The minus sign $f(b) - f(a)$ is where the orientation convention comes in.
  2. Green's theorem.
     - You have a region in the plane. $X = U \subset \mathbb{R}^2$ open.
     - You have a one-form $\omega = P\,\mathrm{d}x + Q\,\mathrm{d}y$.
     - The corresponding two-form is

$$
\begin{aligned}
\mathrm{d}\omega &= \left(\frac{\partial P}{\partial x}\,\mathrm{d}x + \frac{\partial P}{\partial y}\,\mathrm{d}y\right) \wedge \mathrm{d}x + \left(\frac{\partial Q}{\partial x}\,\mathrm{d}x + \frac{\partial Q}{\partial y}\,\mathrm{d}y\right) \wedge \mathrm{d}y \\
&= \frac{\partial P}{\partial x}\,\mathrm{d}x \wedge \mathrm{d}x + \frac{\partial P}{\partial y}\,\mathrm{d}y \wedge \mathrm{d}x + \frac{\partial Q}{\partial x}\,\mathrm{d}x \wedge \mathrm{d}y + \frac{\partial Q}{\partial y}\,\mathrm{d}y \wedge \mathrm{d}y \\
&= \frac{\partial P}{\partial x} \cdot 0 - \frac{\partial P}{\partial y}\,\mathrm{d}x \wedge \mathrm{d}y + \frac{\partial Q}{\partial x}\,\mathrm{d}x \wedge \mathrm{d}y + \frac{\partial Q}{\partial y} \cdot 0 \\
&= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y
\end{aligned}
$$

  - Well, if you want the double integral on the left below, Green's theorem tells us that it's equal to the line integral around the boundary on the right below.

$$\int_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y = \int_{\partial U} (P\,\mathrm{d}x + Q\,\mathrm{d}y)$$

  - We orient counterclockwise around the boundary, or we get the wrong thing!
  - If U has holes, you orient clockwise about the holes for reasons we'll talk about shortly.

3. A bit more abstract.
   - Say we have a slightly more abstract three-manifold X^3. We are given a two-form $\omega \in \Omega^2(X^3)$.
   - Let $\Sigma^2 \subset X^3$ be some two-dimensional submanifold. If we want $\int_\Sigma \omega$, we just need (as we will prove shortly) $\int_\Sigma i^*\omega$. Pulling back our two-form in this manner does just give us a top-dimensional form.
4. Many other special cases were known before the general Stokes' theorem.

- Claim: If ω is exact (i.e., $\omega = \mathrm{d}\mu$), then $\int_\Sigma \omega = 0$ for all Σ that have no boundary (i.e., $\partial\Sigma = \emptyset$).
  - Proof: Stokes plus $\partial\Sigma = \emptyset$ (the integral of a manifold over an emptyset naturally equals zero).
- **Closed** (submanifold): A submanifold with $\partial\Sigma = \emptyset$.
- This leads into the degree theory stuff we were doing on manifolds.
  - Applies to X^n, Y^n compact, closed manifolds.
  - If we have $F : X \to Y$, we have $\deg(F) = \int F^*\omega$ where $\int_Y \omega = 1$ and $F^* : \Omega^n(Y) \to \Omega^n(X)$ sends $\omega \mapsto F^*\omega$.
  - Degree is homotopy invariant.
  - Compute it with "counting preimages with sign."
  - Fun applications.
- Stokes was thinking about surfaces in three dimensions when he developed his Stokes' theorem. That's the setting he was working in.
- Three things to straighten out first.
  1. Orientations on manifolds — take the pointwise concept of an orientation of a vector space and extend it to T_pX.
     - Aside: We were happy to identify $T_p\mathbb{R}^n \cong \mathbb{R}^n \cong T_p^*\mathbb{R}^n$. However, it does not make sense to identify T_pX with $\mathbb{R}^n$ ($T_pX \ncong \mathbb{R}^n$), even for S^2 or something.
     - A choice of charts gives $T_pX \cong \mathbb{R}^n$, but we have to choose these charts; there is no natural identification.
  2. Boundaries and induced orientations.
     - If we have an orientation at every point here, we cananocially induce an orientation by taking your favorite first vector, moving it to the boundary, and then every other vector gives your orientation.
     - If we don't take this convention, we'd have to put a minus sign in Stokes' theorem and it would mess up everything else. Math has to go counterclockwise.
  3. Integration on manifolds.
     - Let $X^n \subset \mathbb{R}^N$ be an **oriented manifold**.
     - Let $\omega \in \Omega^n(X)$ be a top-dimensional form.
     - $\int_X \omega$ means:
       - ■ Step 1: Pick a set $\{U_i\}$ of orientation preserving charts that cover X. For S^2 for example, we can take a chart of the top of the sphere, and a chart of the bottom of the sphere.

- Step 2: Pick a partition of unity $\{\rho_i\}$ with ρ_i supported in U_i.
- Now define
$$\int_X \mathrm{d}\omega = \int \sum_i \rho_i \omega = \sum_i \int_{U_i'} \varphi_i^* \rho_i \omega$$
where the expression on the right is a good, old-fashioned integral in Euclidean space.

– This definition does not depend on the choices in steps 1 and 2.
  - We should go through this to see how our nice machinery of differential forms neatly gets rid of and absorbs all of the inherent ambiguity herein.

- Let's now say all that with way more words.
- **Orientation** (of $X^n \subset \mathbb{R}^N$): A function that assigns to each point $p \in X$ an orientation of T_pX (or an element of $\Lambda^n(T_p^*X)$).
  - This is a preliminary definition.
  - Flaws: We could choose different orientations at every point.
  - We need "smoothness," i.e., we need close-by points to have the same orientation.
- Note: If $\omega \in \Omega^n(X)$ and ω is nonvanishing on some $U \subset X$, then ω induces an orientation on U by assigning to $p \in U$, a form $\omega_p \in \Lambda^n(T_p^*X)$.
- **Smooth** (orientation of $X^n \subset \mathbb{R}^N$): An orientation on X such that for all $p \in X$, there is a neighborhood U and a nonvanishing form $\omega \in \Omega^n(X)$ such that the orientation on U induced by ω agrees with the given orientation.
  - There are like ten different definitions that all agree. This is the one using forms, which is most suited to our study.
- From now on, we will assume all orientations are smooth.
- Examples.
  1. Let $X = U \subset \mathbb{R}^n$. Take the orientation given by the ordered basis $\partial/\partial x_1, \dots, \partial/\partial x_n$ at each point.
     - This is why we haven't needed to talk about orientations in our discussion of Euclidean space.
     - Equivalently, this is the orientation induced by $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$.
  2. For $S^1 \subset \mathbb{R}^2$, the orientation is dual to the vector space $\boldsymbol{v} : S^1 \to TS^1$ defined by
$$\boldsymbol{v}(x, y) = ((x, y), (-y, x))$$
     - Graphically, $\boldsymbol{v}$ is the vector space on S^1 whose tangent vectors spin around it counterclockwise (tho $\boldsymbol{w}$ related to $\boldsymbol{v}$ with vectors spinning clockwise would also be acceptable to take the dual of).
     - This is for every point a choice of ordered basis of the tangent space because at every $p \in S^1$, there is an ω_p which takes any basis vector (nonzero scalar) in T_pS^1 and returns either a positive or negative scalar.
- Equivalently, we use the orientation induced by $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$.
- **Orientable** (manifold): A manifold X that can be (smoothly) oriented.

9.2 Domains and Steps to Integration

5/25:
- Plan:
  - Orientations.
  - Domains – manifolds with boundary and how we include an orientation on the boundary.
  - Integration.
- Examples of orientable manifolds.
  - A line segment.
  - $\mathbb{R}^n$ with $\partial/\partial x_1, \dots, \partial/\partial x_n$ and $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$.
  - A torus.
  - A genus 2 surface.
  - The product manifold of two orientable manifolds.
- Bad examples:
  - Möbius strip.
  - Klein bottle.
- Aside:
  - If X^n is compact, the de Rahm cohomology is

$$H^n_{\mathrm{dR}}(X) = \frac{\Omega^n(X)}{\mathrm{d}(\Omega^{n-1}(X))} = \begin{cases} \mathbb{R} & X \text{ orientable} \\ 0 & X \text{ not orientable} \end{cases}$$

- **Orientation preserving** (diffeomorphism): A diffeomorphism $F : X \to Y$, where X^n, Y^n are oriented n-manifolds, for which $\mathrm{d}F_p : T_pX \to T_{F(p)}Y$ implies $(\mathrm{d}F_p)^* : \Lambda^n(T_p^*X) \leftarrow \Lambda^n(T_{F(p)}^*Y)$ is orientation preserving for all $p \in X$.
- Examples:
  - The map from $[0,1]$ to $[0,2]$ that stretches it by a factor of 2, where we assume that both 1-manifolds are oriented in the positive direction.
  - The map that rotates $\mathbb{R}^2$ by $90°$.
  - An example of a diffemorphism that is *not* orientation preserving is $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $z \mapsto \bar{z}$.
- Boundaries.
- **Domain**: An open subset $D \subset X^n$ such that...
  1. ∂D is an $(n-1)$-manifold;
  2. $\partial D = \partial(\overline{D})$[1].
- Examples.
  - If $X = \mathbb{R}^2$, we may take $D = \{x \in \mathbb{R}^2 \mid |x| < 1\}$ (i.e., X is the open unit disk).
    - Condition 2 is here to rule out things like $\mathbb{R}^2 \setminus S^1 = D$.
  - The upper half plane $\mathbb{H}^n \subset \mathbb{R}^n$[2] where $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$.
    - $\partial\mathbb{H}^n = \mathbb{R}^1$.

[1] $\overline{D}$ is the closure of D.
[2] For us, $\mathbb{H}$ for half; later on, $\mathbb{H}$ for hyperbolic.

  - The point: $D \subset X$ and $\omega \in \Omega^n(X)$ makes it so that $\int_D \mathrm{d}\omega_0 = \int_{\partial D} \omega_0$.

- If X is oriented, it induces an orientation on D (via the restriction of the orientation form to D), which induces an orientation ∂D, which we can integrate over.

- Claim (Existence of boundary charts): If $D \subset X^n$ is a domain in a manifold, $p \in \partial D$, and X is oriented, then there exists $U \subset X$ open that is a neighborhood of p and a chart $\varphi : U_0 \to U$ that sends $0 \mapsto p$, where $U_0 \subset \mathbb{R}^n$, such that...

  1. φ is orientation preserving.
  2. $U_0 \cap \mathbb{H}^n \xrightarrow{\varphi} U \cap D$.

- Example: Boundary chart for S^2.

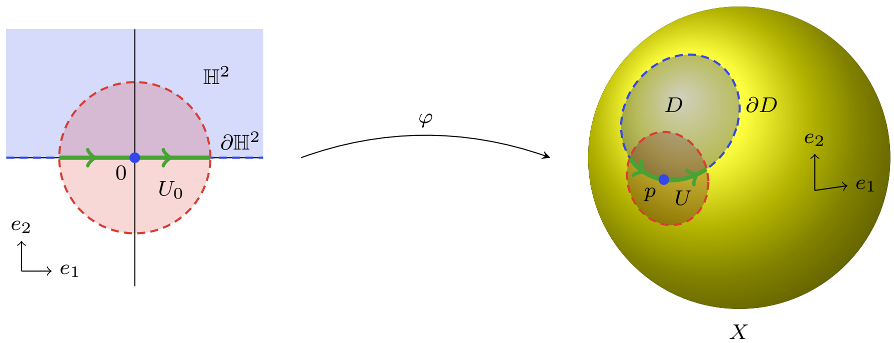


Figure 9.1: Existence of boundary charts.

  - Consider the two-sphere $S^2 \subset \mathbb{R}^3$. We will call this manifold X.
  - The shaded blue circle on the surface of X is a domain $D \subset X$. Its boundary ∂D is represented by a dashed blue line, where dashing is chosen to remind the viewer that D is open. The point p is an element of ∂D and is contained in the open neighborhood $U \subset X$. X is oriented, as indicated. φ maps the open disk $U_0 \subset \mathbb{R}^2$ to U and such that $\varphi(U_0 \cap \mathbb{H}^2) = U \cap D$. Moreover, φ is clearly orientation preserving and satisfies $\varphi(0) = p$.
  - Note that Figure 9.1 was drawn in a largely analogous manner to Figure 8.4. See handwritten pages for more info.

- Proof.

  - Proving 1: You know φ exists by the definition of a manifold; if it's not orientation preserving, compose it with a map of $\mathbb{R}^n$ that reverses all of the needed orientations.

- Using this chart φ, we get an induced orientation on ∂D from the orientation of $\partial\mathbb{H}^n = \mathbb{R}^{n-1}$.

  - See the green arrows in the Figure 9.1.
  - Check:
    1. This gives a (global) orientation of ∂D.
    2. This does not depend on the choice of chart φ.

- Note that this implies that the Klein bottle can't bound an orientable manifold because this would induce an orientation on the Klein bottle.

- Integration of forms on manifolds.

  - Take X^n to be our manifold and $\omega \in \Omega^n(X)$ to be some top-dimensional form.
  - For now, we'll let X be compact.

- We want to define $\int_X \omega$, which should be a real number.
- We got the recipe last time; now we just have to make it precise. Slogan: Break apart, compute in charts, put back together.
- Here are the steps.
  1. Pick *oriented* charts (i.e., $\{\varphi_i : U_i' \to U_i\}$ orientation preserving) so that $\{U_i\}$ covers X.
     - Example: Taking the top- and bottom-halves of S^2, as discussed last time, but we may do this however.
  2. Choice of a "partition of unity supported on the U_i."
     - Take a set $\{\rho_i : X \to \mathbb{R}\}$ with a few properties.
     (a) For all $p \in X$, $\rho_i(p) = 0$ for all but finitely many indices i.
     (b) $\sum_i \rho_i(p) = 1$; this is where the name *partition of unity* comes from.
     (c) $\operatorname{supp}(\rho_i) \subset U_i$.

9.3 Stokes' Theorem and Course Retrospective

5/27:

- Office hours: 4:00pm-5:00pm today.
- Let $W \subset X^n \subset \mathbb{R}^N$, where W is a domain and X^n is an oriented manifold.
  - Then we seek to define $\int_W \omega$.
  - There are two possibilities.
    1. $\overline{W}$ is compact implies
    $$\int_W \omega = \sum_i \int_{U_i \cap \phi_i^{-1}(W)} \varphi_i^* \rho_i \omega$$
    2. $\omega \in \Omega_c^n(X)$ and W arbitrary.
- For 1 (wrt the homework):
  - Let $\sigma_{\text{vol}} \in \Omega^n(X)$. Pointwise, $T_p\mathbb{R}^N \times T_p\mathbb{R}^N \to \mathbb{R}$, restrict to $T_pX \times T_pX \to \mathbb{R}$.
  - Let $e_1, \dots, e_n$ be an orthonormal basis for T_pX.
  - σ_{vol} is the **volume form**.
  - We take $(\sigma_{\text{vol}})|_p = e_1^* \wedge \cdots \wedge e_n^*$.
  - Check:
    - Well-defined.
    - What it is in charts.
    - $X = \Gamma_f = F^{-1}(\{0\})$.
- **Volume** (of W): The following quantity, where W is compact. *Denoted by* $\mathbf{Vol}(\boldsymbol{W})$. *Given by*
$$\operatorname{Vol}(W) = \int_W \sigma_{\text{vol}}$$
  - This the actual volume!
  - The volume is the limit (computed the right way) of inserted polygons.
  - $\ell(\gamma) = \int |\gamma|$ is the limit length of the inserted polygons.
- Properties.
  1. $\int_W (\omega_1 + \omega_2) = \int_W \omega_1 + \int_W \omega_2$.
  2. $\int_W c\omega = c \int_W \omega$.

3. $F : X \to Y$, where $X \subset W_X$ and $Y \subset W_Y$. an orientation preserving diffeomorphism implies $\int_X F^*\omega = \int_Y \omega$ and $\int_{W_X} F^*\omega = \int_{W_Y} \omega$.
4. Don't forget the orientation! $\int_{-X} \omega = -\int_X \omega$.

- Theorem (Stokes): Let $W \subset X \subset \mathbb{R}^N$, where W is a domain such that $\overline{W}$ is compact and X is oriented. Let $\mu \in \Omega^{n-1}(X)$. Then
$$\int_W \mathrm{d}\mu = \int_{\partial W} \mu$$
where ∂W is oriented.
- Example:
  - Let $W = \{(x, y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$ be the unit disk, with its boundary oriented counterclockwise. Let $X = \mathbb{R}^2$. Let $\omega = \mathrm{d}x \wedge \mathrm{d}y$. *picture*
    - We want to find $\omega_0 \in \Omega^1(X)$ such that $\mathrm{d}\omega_0 = \omega$. We could take $\omega_0 = x \wedge \mathrm{d}y$, but we'll take $\omega_0 = (-y/2)\,\mathrm{d}x + (x/2)\,\mathrm{d}y$ because it's symmetric.
    - We take as our one chart the circumference of the circle minus a point, and it will suffice to integrate over this one chart. This chart maps $(0, 2\pi) \to \partial W$ via $t \mapsto (\cos t, \sin t)$.
    - Thus, by Stokes' theorem,
$$\begin{aligned}\int_W \mathrm{d}x \wedge \mathrm{d}y &= \int_{\partial W} \left(\left(-\frac{y}{2}\right)\mathrm{d}x + \left(\frac{x}{2}\right)\mathrm{d}y\right)\\ &= \int_0^{2\pi} -\frac{\sin t}{2}\,\mathrm{d}(\cos t) + \frac{\cos t}{2}\,\mathrm{d}(\sin t)\\ &= \int_0^{2\pi} \frac{1}{2}(\sin^2 t + \cos^2 t)\,\mathrm{d}t\\ &= \pi\end{aligned}$$
    - Sanity check: $\mathrm{d}x \wedge \mathrm{d}y$ is the volume form, so integrating it should give the "volume" (area) of the unit disk, and it does!
- The proof of Stokes' theorem relies on the FTC and machinery.
  - To truly understand it, fight with Green's theorem, then jump up a dimension and prove Stokes' theorem, then jump up many dimensions and prove the generalized Stokes theorem.
- Overview of the whole year.
  - Fall:
    - $\mathbb{R}$ exists.
    - Metric spaces, open sets, etc.
    - Sequences and series, continuous functions.
    - Theme: Just a bunch of chit-chat/language: Look, I can write a proof!
  - Winter:
    - Derivatives ($\mathbb{R}$ and $\mathbb{R}^n$), familiar properties.
    - Integrals.
    - FTC.
    - Theme: Interchanging limits.
  - Spring:
    - Manifolds: Locally Euclidean. The point is that they show up a lot. Most mathematicians in this building study manifolds in one form or another.
    - Manifolds are locally solutions to equations $F : \mathbb{R}^N \to \mathbb{R}^{N-n}$.

- Manifolds are a place where you can do calculus (reducing it to linear algebra).
- Tangent spaces T_pX, linear approximations $\mathrm{d}f_p$.
- Integration/differentiation with $\Omega^k(X)$.
- Forms package with Λ^n, $\wedge$, d, $\int$, and Stokes' theorem.
- Manifolds $\Leftarrow$ local theory (esp. the degree) $\Leftarrow$ pointwise linear algebra.

- Where to go from here.
  - Don't read Guillemin and Haine (2018) again if you don't know what differential forms are.
    - Move forward and the foundations will fill themselves in.
  - More analysis.
    - Complex analysis. All your favorite stuff, but over $f : \mathbb{C} \to \mathbb{C}$. Do derivatives and integrals over closed curves. This is a beautiful theory. Since homomorphisms $f : \mathbb{C} \to \mathbb{C}$ are really rare and constricted, we can say a lot about them.
    - Functional analysis. A blend of linear algebra in infinite dimensions $\mathbb{R}^\infty$.
    - Fourier analysis. Approximating functions $f : \mathbb{R} \to \mathbb{R}$ with sine and cosine.
    - Harmonic analysis. The more general philosophical counterpart to Fourier analysis.
  - Use analysis.
    - Partial differential equations. Usually the motivation for needing to learn functional and Fourier analysis.
    - Dynamical systems. Iterating functions as with 3 $\Rightarrow$ chaos. Many Chicago mathematicians study dynamical systems.
    - Probability theory.
    - Differential geometry. The properties of surfaces and how they curve and what how they curve tells us. Pictures here! But not too much; it's still pretty abstract.
    - Analytic number theory.
    - Discrete math.
    - Analysis is a love-hate burden. Klug doesn't consider it interesting in its own right, but you always seem to need more of it to do the math you want to do.
  - Chapter 5 of Guillemin and Haine (2018).
    - The chain $\cdots \to \Omega^{k-1}(X) \xrightarrow{\mathrm{d}} \Omega^k(X) \xrightarrow{\mathrm{d}} \Omega^{k+1}(X) \to \cdots$.
    - $H^k_{\mathrm{dR}}(X)$.
    - Homological algebra.
    - It will seem random and crazy if you read it, but it's actually this whole "cohomology theory" that is pretty ubiquitous. If you invest in it, you will get something out of it pretty much regardless of the definition you go in.

- The final will be like the midterm: Computations. Write down the tangent space with this chart. Can you find a function that this is the zero of? Can you compute the degree? Can you push around a couple definitions?
  - Computing the degree can be a bit painful (see the homework), so we might just "eyeball it."

9.4 Office Hours (Klug)

- What is a k-form, and can you give some examples of them?
- What is the pullback and what does it do?
- What is $\Lambda^k(V^*)$, and why is it the k^{th} exterior power of V^*, and what does that even mean? The elements of it are $\mathcal{I}^k(V)$-cosets of tensors; what does one of these look like? The elements of it aren't even functions, right? They're just sets of functions?
- More Klug meeting questions as time allows.

9.5 Final Review Sheet

6/2:

- Computing the sign.

$$(-1)^\sigma = \prod_{i<j} \frac{X_{\sigma(i)} - X_{\sigma(j)}}{X_i - X_j}$$

- **Pullback** (of A): The linear map $A^* : W^* \to V^*$ defined as follows, where $A : V \to W$ be a linear transformation between V, W vector spaces. *Given by*

$$\ell \mapsto \ell \circ A$$

  - Essentially, we take every linear functional on W and relate it to a linear functional $\ell \circ A$ on V by having A translate vectors in V to vectors in W, which ℓ can eat.

- Every k-tensor $T : V^k \to \mathbb{R}$ has a decomposition

$$T = \sum_I T_I e_I^*$$

  - Proofs for decomposable tensors follow from the linearity of this decomposition.

- Recall decomposable and redundant k-tensors.
- **Pullback** (of T by A): Once again, A supplies values to T. So if $A : V \to W$ and $T : W^k \to \mathbb{R}$, then

$$A^*T(v_1, \dots, v_k) = T(Av_1, \dots, Av_k)$$

- Recall T^σ, defined in terms of T by σ^{-1} indices.
- **Alternating** (k-tensor): A k-tensor T such that for all $\sigma \in S^k$,

$$T^\sigma = (-1)^\sigma T$$

- **Alternation operation**: The function $\mathrm{Alt} : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ defined by

$$T \mapsto \sum_{\tau \in S^k} (-1)^\tau T^\tau$$

- Properties.
  1. $\mathrm{Alt}(T)^\sigma = (-1)^\sigma \mathrm{Alt}\, T$.
  2. $T \in \mathcal{A}^k(V)$ implies $\mathrm{Alt}\, T = k!T$.
  3. $\mathrm{Alt}(T^\sigma) = \mathrm{Alt}(T)^\sigma$.
  4. Alt is linear.

- $\mathcal{A}^k(V) \cong \Lambda^k(V^*) = \mathcal{L}^k(V)/\ker(\mathrm{Alt})$.
  - This means that $\Lambda^k(T_p^*\mathbb{R}^n) \cong \mathcal{A}^k(T_p\mathbb{R}^n)$.
  - In terms of k-forms, this must mean that we're taking k things and sending them like a one-form. The useful picture of $f : \mathbb{R}^2 \to \mathbb{R}^2$ is as $f_1 : \mathbb{R}^2 \to \mathbb{R}^1$ and separately $f_2 : \mathbb{R}^2 \to \mathbb{R}^1$. It's just more.

- Recall that $\mathcal{I}^k(V)$ is the span of all redundant k-tensors.
  - In particular, $\mathcal{I}^k(V) = \ker(\mathrm{Alt})$.

- Let $\pi : \mathcal{L}^k(V) \to \Lambda^k(V^*)$ send $T \mapsto \omega$. Then if $\omega_1 = \pi(T_1)$ and $\omega_2 = \pi(T_2)$, we have $\omega_1 \wedge \omega_2 = \pi(T_1 \otimes T_2)$.

  - Indeed, with the wedge product, we are kind of just appending forms/tensors together. The 2-tensor at a point describes the derivative of a function into 2-dimensional space.

- If $\omega_1 \in \Lambda^k(V^*)$ and $\omega_2 \in \Lambda^\ell(V^*)$, then

$$\omega_1 \wedge \omega_2 = (-1)^{k\ell} \omega_2 \wedge \omega_1$$

- The cursed product rule (for the interior product and the exterior derivative):

$$\iota_v(T_1 \otimes T_2) = \iota_v T_1 \otimes T_2 + (-1)^p T_1 \otimes \iota_v T_2$$

- **Interior product** (of a vector v and k-tensor T): The $(k-1)$-tensor

$$(\iota_v T)(v_1, \dots, v_{k-1}) = \sum_{r=1}^{k} (-1)^{r-1} T(v_1, \dots, v_{r-1}, v, v_r, \dots, v_{k-1})$$

- **Pullback** (of ω by A): As always, A supplies values to T, where $\omega = \pi(T)$. However, this time it's indirectly through the projection operation:

$$A^*\omega = \pi(A^*T)$$

- **Determinant** (of A): The number A such that $A^*\omega = a\omega$.
  - Appeal to A's actions on coordinates/bases to derive the typical formula.
- Recall the definitions of the tangent space, a vector field.
- $\partial/\partial x_i$ is the n-dimensional vector field where vectors point in the x_i-direction at every point.
- Every vector has a unique decomposition in terms of the standard basis $(x_1, \dots, x_n)$, given by a set of numbers. If we assign each point of a vector field to these numbers via functions $\{g_i\}$, we realize that every n-dimensional vector field on U has a unique decomposition

$$\boldsymbol{v} = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i}$$

  where $g_i : U \to \mathbb{R}$.

- The Lie derivative takes the directional derivative of a function f on U according to the vector field $\boldsymbol{v}$ on U. In particular, if $\boldsymbol{v}(p) = (p, v)$, then

$$L_{\boldsymbol{v}} f(p) = Df(p)v$$

  In coordinates,

$$L_{\boldsymbol{v}} f = \sum_{i=1}^{n} g_i \frac{\partial f}{\partial x_i}$$

  - If $L_{\boldsymbol{v}} f = 0$, f is an integral of $\boldsymbol{v}$.

- Differential 1-forms: Essentially, these are covector fields. We can interconvert with the musical operators.
- We need differential 1-forms to generalize to higher dimensions for U than $\mathbb{R}^3$, and differential k-forms to describe functions *into* higher-dimensional spaces.
- **Integral curve** (of $\boldsymbol{v}$): A curve $\gamma : [a, b] \to U$ such that $\gamma'(t) = \boldsymbol{v}(\gamma(t))$.

- The following related definitions for the derivative of f.

$$
\begin{aligned}
Df(p) &= \left[\left.\frac{\partial f_i}{\partial x_j}\right|_p\right] \\
\mathrm{d}f_p(p,v) &= (q, Df(p)v) \\
\mathrm{d}f(p) &= \mathrm{d}f_p
\end{aligned}
$$

- The following constructs related to forming a basis of one-forms.

$$
\begin{aligned}
x_i(v_1,\dots,v_n) &= v_i \\
(\mathrm{d}x_i)_p(p, a_1x_1+\cdots+a_nx_n) &= a_i \\
\mathrm{d}x_i(p) &= (\mathrm{d}x_i)_p
\end{aligned}
$$

- Relating the last two thoughts: All one-forms have a unique decomposition

$$\omega = \sum_{i=1}^n f_i \mathrm{d}x_i$$

- **Interior product** (of a vector field $\boldsymbol{v}$ and a one-form ω): The following expression, where the vector field and one-form are defined in coordinates as $\boldsymbol{v} = \sum_{i=1}^n g_i\, \partial/\partial x_i$ and $\omega = \sum_{i=1}^n f_i\, \mathrm{d}x_i$. *Given by*

$$\iota_{\boldsymbol{v}}\omega = \sum_{i=1}^n f_i g_i$$

- Note that

$$\iota_{\boldsymbol{v}}\mathrm{d}f = L_{\boldsymbol{v}}f$$

  as follows directly from the definitions.

- Recall that $C_0^\infty(\mathbb{R}^n)$ is the vector space of all bump functions on $\mathbb{R}^n$.

- Exterior derivative properties.
  1. Linearity.
  2. Cursed product rule (where $p = k$ is the dimension of ω_1).
  3. Special case ($k = \ell = 0$, so $\omega_1 = f$ and $\omega_2 = g$ are C^∞ functions).

  $$\mathrm{d}(fg) = g\mathrm{d}f + f\mathrm{d}g$$

  4. Formula.

  $$\mathrm{d}\left(\sum_I f_I \mathrm{d}x_i\right) = \sum_I \mathrm{d}f_I \wedge \mathrm{d}x_I$$

  5. $\mathrm{d}^2 = 0$.

- Recall closed and exact k-forms. Closed ones have $\mathrm{d}\omega = 0$; exact ones have $\omega = \mathrm{d}\mu$. Exact implies closed by $\mathrm{d}^2 = 0$.

- A k-form at a point p is an alternating k-tensor. It takes k vectors in and spits out the result of applying the best linear approximation to each of them for a k-dimensional function (a function into $\mathbb{R}^k$). It's really just best to rephrase everything in terms of alternating tensors and view the wedge product as the tensor product.

- This makes it so that we analogously have

$$\omega_p = \sum_I c_I (\mathrm{d}x_I)_p$$

and

$$\omega = \sum_I f_I \mathrm{d}x_I$$

- Recall that $\Omega_c^k(U)$ is the vector space of all compactly supported k-forms on U.
  - Recall further that the support is the set of all points at which the form is nonzero, and compact just means that the support is compact. This can have cool consequences, as with (maximal) integral curves.
- **Proper** (function $f: U \to V$): Continuous, $K \subset V$ compact implies $f^{-1}(K) \subset U$ compact.
  - Sine is not proper.
- The pullback maps compactly supported forms to compactly supported forms.
- **Integral** (of a top-dimensional form): If $\omega = f\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ is a top-dimensional form, then the integral of ω over U is given as follows. *Given by*

$$\int_U \omega = \int_{\mathbb{R}^n} f\,\mathrm{d}x_1 \cdots \mathrm{d}x_n$$

  - Evaluate with repeated integrals.
- Poincaré lemma for rectangles: Let $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Take $\omega \in \Omega_c^n(Q)$. Then TFAE.
  1. $\int_Q \omega = 0$.
  2. $\omega = \mathrm{d}\mu$ with $\mu \in \Omega_c^{n-1}(Q)$
- Intuition ($n = 1$ case).
  - The following are equivalent.
    1. $\int_a^b f = 0$.
    2. $f = g'$ for some compactly supported smooth g on $[a, b]$.
  - Let g be the bump function on $(-1, 1)$. Then starting at -1, f goes up and down to zero and down and up to 1. Naturally, $\int_a^b f = 0$, and similarly, g is compactly supported on $[-1, 1]$.
  - ($2 \Rightarrow 1$): If g is compactly supported, then $g(b) = g(a)$. Thus, $\int_a^b f = g(b) - g(a) = 0$.
  - ($1 \Rightarrow 2$): If $\int_a^b f = 0$, define $g(x) = \int_a^x f(t)\,\mathrm{d}t$. This is compactly supported (i.e., has $g(a) = 0$ and $g(b) = 0$) since

$$g(a) = \int_a^a f = 0 \qquad\qquad g(b) = \int_a^b f = 0$$

    where the left equality follows by the properties of integrals and the right follows by hypothesis 1.
- **Pullback** (of a one-form μ on V onto U by f): It looks like here, f (in the one-form form $\mathrm{d}f$) is supplying values to μ.

$$f^*\mu(p) = \mu_q \circ \mathrm{d}f_p$$

- In formulas, if
$$\omega = \sum_I \phi_I \, \mathrm{d}x_I$$
then
$$f^*\omega = \sum_I f^*\phi_I \, \mathrm{d}f_I$$
where $f^*\phi_I = \phi_I \circ f$.
- Note that $\mathrm{d} \circ f^* = f^* \circ \mathrm{d}$.

- **Lie derivative** (of the k-form ω with respect to $\boldsymbol{v}$):The k-form defined as follows, where $U \subset \mathbb{R}^n$ is open, $\boldsymbol{v} \in \mathfrak{X}(U)$, and $\omega \in \Omega^k(U)$. *Given by*
$$L_{\boldsymbol{v}}\omega = \iota_{\boldsymbol{v}}(\mathrm{d}\omega) + \mathrm{d}(\iota_{\boldsymbol{v}}\omega)$$
  - Note that we use ι to drop the index and d to raise it back up, and then vice versa.

- Properties of this Lie derviative.
  1. $L_{\boldsymbol{v}} \circ \mathrm{d} = \mathrm{d} \circ L_{\boldsymbol{v}}$.
  2. $L_{\boldsymbol{v}}(\omega \wedge \mu) = L_{\boldsymbol{v}}\omega \wedge \nu + \omega \wedge L_{\boldsymbol{v}}\mu$.

- Explicit formula for this Lie derivative: If $\omega = \sum_I f_I \, \mathrm{d}x_I$ and $\boldsymbol{v} = \sum_{i=1}^n g_i \, \partial/\partial x_i$, then
$$L_{\boldsymbol{v}}\omega = \sum_I \left[\left(\sum_{i=1}^n g_i \frac{\partial f_I}{\partial x_i} \right) \mathrm{d}x_I + f_I \left(\sum_{r=1}^k \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}g_{i_r} \wedge \cdots \wedge \mathrm{d}x_{i_k} \right) \right]$$

- Vector calc connections.
  - The musical operators.
$$\sharp(f \, \mathrm{d}x + g \, \mathrm{d}y) = f \frac{\partial}{\partial x} + g \frac{\partial}{\partial y}$$
  - The exterior derivative of a (2D) function.
$$\mathrm{d}f = \frac{\partial f}{\partial x} \, \mathrm{d}x + \frac{\partial f}{\partial y} \, \mathrm{d}y$$
  - The exterior derivative of a (2D) one-form.
$$\mathrm{d}(f \, \mathrm{d}x + g \, \mathrm{d}y) = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \mathrm{d}x \wedge \mathrm{d}y$$
  - The exterior derivative of a (3D) function.
$$\mathrm{d}f = \frac{\partial f}{\partial x} \, \mathrm{d}x + \frac{\partial f}{\partial y} \, \mathrm{d}y + \frac{\partial f}{\partial z} \, \mathrm{d}z$$
  - The exterior derivative of a (2D) one-form.
$$\mathrm{d}(f \, \mathrm{d}x + g \, \mathrm{d}y + h \, \mathrm{d}z) = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \mathrm{d}x \wedge \mathrm{d}y + \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z} \right) \mathrm{d}x \wedge \mathrm{d}z$$
  - The exterior derivative of a (3D) two-form.
$$\mathrm{d}(f \, \mathrm{d}x \wedge \mathrm{d}y + g \, \mathrm{d}y \wedge \mathrm{d}z + h \, \mathrm{d}x \wedge \mathrm{d}z) = \left(\frac{\partial f}{\partial z} + \frac{\partial g}{\partial x} - \frac{\partial h}{\partial y} \right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

- **Interior product** (of a vector field $\boldsymbol{v}$ and a k-form ω): We take every $p \in U$ to the interior product of the vector $\boldsymbol{v}(p)$ and the k-tensor ω_p.
- Thus, if $\boldsymbol{v} = \partial/\partial x_r$ and $\omega = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$, we have that

$$\begin{aligned}
[\iota_{\boldsymbol{v}}\omega(p)](v_1, \dots, v_{k-1}) &= [\iota_{\boldsymbol{v}(p)}\omega_p](v_1, \dots, v_{k-1}) \\
&= \sum_{i=1}^{k} (-1)^{i-1} \omega_p(v_1, \dots, v_{i-1}, \boldsymbol{v}(p), v_i, \dots, v_{k-1}) \\
&= \sum_{i=1}^{k} (-1)^{i-1} [\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}]_p(v_1, \dots, v_{i-1}, \boldsymbol{v}(p), v_i, \dots, v_{k-1}) \\
&= \sum_{i=1}^{k} (-1)^{i-1} [(\mathrm{d}x_{i_1})_p \wedge \cdots \wedge (\mathrm{d}x_{i_k})_p](v_1, \dots, v_{i-1}, \boldsymbol{v}(p), v_i, \dots, v_{k-1}) \\
&= \cdots
\end{aligned}$$

- **Pullback** (of the k-form ω along f): As per usual, we feed ω_q some values spit out by $\mathrm{d}f_p$. *Given by*

$$f^*\omega(p) = \mathrm{d}f_p^*\omega_q$$

- Properties of this pullback.
  1. $(f^*\phi)(p) = (\phi \circ f)(p)$.
  2. $f^*\mathrm{d}\phi = \mathrm{d}f^*\phi$.
  3. Linearity.
  4. Distributivity over the wedge product.
  5. Functoriality.
  6. $f^*(\mathrm{d}x_I) = \mathrm{d}f_I$.
  7. $\mathrm{d}(f^*\omega) = f^*\mathrm{d}\omega$.
  8. $f^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n) = \det[\partial f_i/\partial x_j]\, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$.
- Functoriality is another property just like distributivity. It's another way a function can behave.
- Explicit formula for this pullback: If $\omega = \sum_I \phi_I \,\mathrm{d}x_I$, then

$$f^*\omega = \sum_I f^*\phi_I \mathrm{d}f_I$$

- Recall homotopies.
- Recall contractible sets (rigorously, sets that are homotopic to a constant map).
- Defining $\sharp$ via the inner product and $L : \mathbb{R}^n \to (\mathbb{R}^n)^*$ defined by $L(v) = \ell_v$.
- Change of variables formula: If $f : U \to V$ a diffeomorphism and $\phi : V \to \mathbb{R}$ continuous, then

$$\int_V \phi(y)\,\mathrm{d}y = \int_U (\phi \circ f)(x) |\det Df(x)|\,\mathrm{d}x$$

- Degree theory.
  - If $f : U \to V$, then

$$\int_U f^*\omega = \deg(f) \int_V \omega$$

- A coordinate-based formula for the degree.

$$\int_V \phi(y)\,\mathrm{d}y = \deg(f)\int_U (\phi\circ f)(x)\det(Df(x))\,\mathrm{d}x$$

- $\deg(g\circ f) = \deg(g)\deg(f)$.
  - Proven from the original definition and functoriality.
- orientation preserving diffeomorphisms ($\det[Df(x)] > 0$) have $\deg(f) = +1$.
- orientation reversing diffeomorphisms ($\det[Df(x)] < 0$) have $\deg(f) = -1$.
- Properly homotopic functions have the same degree.
- If f is not surjective, then $\deg(f) = 0$.
- Computing the degree.
  - Take a regular value of f.
  - Find the points in its preimage.
  - Find disjoint neighborhoods around these points.
  - Find functions $f : U_i \to W$.
  - Figure out which are orientation preserving and which aren't.
  - Subtract the number of orientation reversing ones from the number of orientation preserving ones.

- Brouwer fixed-point theorem: If $f : B^n \to B^n$ continuous, then it has a fixed point.
- **Smooth** (function $f : X \to Y$): A function between manifolds $X^n, Y^m \subset \mathbb{R}^N, \mathbb{R}^M$, respectively, such that for all $p \in X$, there is some neighborhood $U_p \subset \mathbb{R}^N$ of p and a smooth map $g_p : U_p \to \mathbb{R}^M$ that is smooth and agrees with f on $X \cap U_p$.
- ***n*-manifold**: A subset $X^n \subset \mathbb{R}^N$ such that for all $p \in X$, there is a neighborhood $V \subset \mathbb{R}^N$ of p, an open set $U \subset \mathbb{R}^n$, and a diffeomorphism $\phi : U \to X \cap V$.
- **Parameterization**: Defined as above. *Also known as* **chart**, **coordinate**.
- Manifold examples:
  1. n-spheres.
  2. Subsets of $\mathbb{R}^n$.
  3. Graphs Γ_f.
  4. Tori.
  5. Product manifolds.
- **Submersion** (at $p \in U$): A smooth map $f : U \to \mathbb{R}^k$, where $U \subset \mathbb{R}^N$ open, such that $Df(p) : \mathbb{R}^N \to \mathbb{R}^k$ is surjective.
- Recall critical points, critical values, and regular values.
  - Remember the difference between critical points and super-critical points (flat surface in the path along a hill vs. the top of the mountain).
- $\boldsymbol{C_f}$: The set of all critical points of f.
- **Tangent space** (of p to X): Intuitively, this is exactly what you would think. It lives in $T_p\mathbb{R}^N$ and comprises all base-pointed vectors tangent to p. Rigorously, we have to relate our parameterization $\phi : U \to \mathbb{R}^N$ to $\mathrm{d}\phi_p : T_p\mathbb{R}^n \to T_p\mathbb{R}^N$. Though I guess what this is really doing is taking a curved tangent vector along the manifold and making it a straight tangent vector in the tangent line/plane/manifold.

- Recall vector fields, integral curves, k-forms, etc. on manifolds. Smoothness is defined for these objects as with functions between manifolds, i.e., by returning to $\mathbb{R}^N$ and then going back to the manifold.
- **Pullback** (of the k-form ω on X along f): If $f : X^n \to Y^m$ where $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$, then we may define it as before.
- **Exterior derivative** (at p on X): The tensor defined by

$$(\mathrm{d}\omega)_p = [(\phi^{-1})^* \,\mathrm{d}(\phi^*\omega)]_p$$

  - The properties carry over.
- Sard's theorem: The set of regular values of f is an open dense subset of V.
- The preimage of a regular value is a finite set.
- $f^{-1}(a)$, where a is a regular value, is the set of solutions to the (independent) system of equations $f_i(x) = a_i$ $(i = 1, \dots, k)$.
- The fundamental theorem of calculus as a special case of Stokes' theorem.
  - If we integrate over $X = [a, b]$, well $\partial X = \{a, b\}$, so

$$\int_a^b \mathrm{d}f = \int_X \mathrm{d}f = \int_{\partial X} f = f(b) - f(a)$$

- Green's theorem.
  - Take a one form

$$\omega = P\,\mathrm{d}x + Q\,\mathrm{d}y$$

  - Applying the exterior derivative generates a corresponding two-form:

$$\mathrm{d}\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y$$

  - Here, we have that

$$\int_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y = \int_{\partial U} P\,\mathrm{d}x + Q\,\mathrm{d}y$$

- Domain: An open subset $D \subset X^n$ such that
  1. D is an $(n-1)$=manifold.
  2. $\partial D = \partial(\overline{D})$.
- Existence of domain boundary charts and $\mathbb{H}^n$.
- The n-dimensional volume form is just $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$.
  - The volume of a manifold is equal to the integral over W of the volume form.
- Linearity property of integrals.
- Actually converting between two-forms and one-forms as in Green's theorem (boundary of a circle example).

9.6 Chapter 4: Manifolds and Forms on Manifolds

From Guillemin and Haine (2018).

- For the remainder of Chapter 4, we will look to prove manifold versions of Stokes' theorem and the divergence theorem, and develop a manifold version of degree theory.
- We confine ourselves to **orientable** manifolds for simplicity.
  - Section 4.4 is concerned with explaining orientability.
- **Orientation** (of X): A rule assigning to each $p \in X$ an orientation of T_pX.
  - Essentially, for every $p \in X$, we label one of the two components of the set $\Lambda^n(T_p^*X) \setminus \{0\}$ by $\Lambda^n(T_p^*X)_+$.
- **Plus part** (of $\Lambda^n(T_p^*X)$): The component $\Lambda^n(T_p^*X)_+$.
- **Minus part** (of $\Lambda^n(T_p^*X)$): The other component of $\Lambda^n(T_p^*X) \setminus \{0\}$. *Denoted by* $\boldsymbol{\Lambda^n(T_p^*X)_-}$.
- **Smooth** (orientation of X): An orientation of a manifold X such that for every $p \in X$, there exists a neighborhood U of p and a non-vanishing n-form $\omega \in \Omega^n(U)$ such that for all $q \in U$, $\omega_q \in \Lambda^n(T_q^*X)_+$. *Also known as* $\boldsymbol{C^\infty}$.
- **Reversed orientation** (of X): The orientation of X defined by assigning to each $p \in X$ the opposite orientation to the one already assigned.
- If X is connected and has a smooth orientation, the only smooth orientations of X are that one and its reversed orientation.
  - Rationale: Given any smooth orientation ω, the set of points where it agrees with the given orientation is open — by definition, every oriented $p \in X$ is surrounded by an open neighborhood U on which the orientation form is smooth, and the union of all these U is open. But then the set where ω agrees with the given orientation and the set where ω agrees with the reversed orientation are both open sets whose union is the connected set X. Therefore, one must be empty.
- **Volume form**: A non-vanishing form $\omega \in \Omega^n(X)$ such that one gets from ω a smooth orientation of X by requiring $\omega_p \in \Lambda^n(T_p^*X)_+$ for all $p \in X$.
  - If ω_1, ω_2 are volume forms on X, then $\omega_2 = f_{2,1}\omega_1$, where $f_{2,1}$ is an everywhere positive C^∞ function.
- **Standard orientation** (of U): The orientation defined by $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$, where $U \subset \mathbb{R}^n$ is open.
- The standard orientation of U induces a standard orientation of T_pX as follows.
  - Recall that if $X = f^{-1}(0)$, then $T_pX = \ker(\mathrm{d}f_p)$, where $\mathrm{d}f_p$ is surjective.
  - Thus, $\mathrm{d}f_p$ induces a bijective linear map from $T_p\mathbb{R}^N \setminus T_pX \to T_0\mathbb{R}^k$.
  - Since $T_p\mathbb{R}^N$ and $T_p\mathbb{R}^k$ have standard orientations by the above definition, requiring that the above map be orientation preserving gives $T_p\mathbb{R}^N \setminus T_pX$ an orientation.
  - It follows by Theorem 1.9.9 that T_pX has an orientation.
  - Note: It should be intuitively clear that the smoothness of $\mathrm{d}f_p$ implies that the orientation is smooth, but we will prove this directly, too, in the exercises.
- Theorem 4.4.9: Let X be an oriented submanifold of $\mathbb{R}^N$, B be the inner product on $\mathbb{R}^N$, $B_p : T_pX \times T_pX \to \mathbb{R}$ be the related inner product on T_pX, $e_1, \dots, e_n$ be an orthonormal basis of T_pX, $\sigma_p = e_1^* \wedge \cdots \wedge e_n^*$ be the volume element in $\Lambda^n(T_p^*X)$ associated with B_p, and σ_X be the non-vanishing n-form defined by $p \mapsto \sigma_p$. Then the form σ_X is C^∞ and hence is a volume form.

- **Riemannian volume form**: The volume form described above. *Denoted by* $\boldsymbol{\sigma_X}$.
- **Orientation preserving** (map): A diffeomorphism $f : X \to Y$, where X, Y are oriented n-manifolds, such that for all $p \in X$ and $q = f(p)$, the linear map $\mathrm{d}f_p : T_pX \to T_qY$ is orientation preserving.
- If $\omega = \sigma_Y$, then f is orientation preserving iff $f^*\omega = \sigma_X$.
- Theorem 4.4.11: If Z is an oriented n-manifold and $g : Y \to Z$ a diffeomorphism, then if both f and g are orientation preserving, so is $g \circ f$.
- If X is connected, then $\mathrm{d}f_p$ must be orientation preserving at all points $p \in X$, or orientation reversing at all points $p \in X$.
- **Oriented parameterization** (of U): A parameterization $\phi : U_0 \to U$ that is orientation preserving with respect to the standard orientation of U_0 and the given orientation on U.
- Suppose ϕ isn't oriented. Then we can still convert it to a related parametrization which is oriented, as follows.
  1. Let V_0 be the union of all connected components of U_0 on which ϕ isn't orientation preserving.
  2. Replace V_0 by
  $$V_0^\sharp = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid (x_1, \dots, x_{n-1}, -x_n) \in V_0\}$$
     - This aligns the orientations between the domain of the parameterization and U.
  3. Replace $\phi : U_0 \to U$ by the map $\psi : U_0 \setminus V_0 \cup V_0^\sharp \to U$ defined by
  $$\psi(x_1, \dots, x_n) = \phi(x_1, \dots, x_{n-1}, -x_n)$$
     - This ensures that the parameterization still maps to U (and not to other parts of the manifold or its containing space).
- Suppose $\phi_i : U_i \to U$ $(i = 0, 1)$ are two oriented parameterizations of U. Let $\psi : U_0 \to U_1$ be the diffeomorphism defined by
$$\psi = \phi_1^{-1} \circ \phi_0$$
  - Then by Theorem 4.4.11, ψ is orientation preserving as well.
  - It follows that $\mathrm{d}\psi_p$ is orientation preserving for all $p \in U_0$, and thus
  $$\det[D\psi(p)] > 0$$
  for all $p \in U_0$.
- **Smooth domain**: An open subset D of X such that
  1. The boundary ∂D is an $(n-1)$-dimensional submanifold of X;
  2. The boundary of D coincides with the boundary of the closure of D.
- Examples.
  1. The open ball in $\mathbb{R}^n$, formally defined by $B^n = \{x \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 < 1\}$, whose boundary is the $(n-1)$-sphere.
  2. The n-dimensional annulus (ball with the center removed)
  $$1 < x_1^2 + \cdots + x_n^2 < 2$$
  whose boundary consists of the union of the two spheres
  $$\begin{cases} x_1^2 + \cdots + x_n^2 = 1 \\ x_1^2 + \cdots + x_n^2 = 2 \end{cases}$$

3. Bad example: Consider $\mathbb{R}^n \setminus S^{n-1}$, i.e., n-dimensional space with the $(n-1)$-sphere missing. The boundary of this space is S^{n-1}, but since the closure of this space is just $\mathbb{R}^n$, the boundary of the closure is empty. Hence D is not a smooth domain.
4. The simplest smooth domain is $\mathbb{H}^n$, since we can identify $\partial\mathbb{H}^n$ with $\mathbb{R}^{n-1}$ via the map from $\mathbb{R}^{n-1} \to \mathbb{H}^n$ defined by
$$(x_2, \dots, x_n) \mapsto (0, x_2, \dots, x_n)$$
   - Think of how we identify the straight line $\mathbb{R}^1$ with the boundary of the $\mathbb{H}^2$, which is just the x-axis, a line.

- In fact, we can show that every bounded domain looks (locally) like Example 4, above:
- Theorem 4.4.17: Let D be a smooth domain and $p \in \partial D$. Then there exists a neighborhood $U \subset X$ of p, an open set $U_0 \subset \mathbb{R}^n$, and a diffeomorphism $\psi : U_0 \to U$ such that $\psi(U_0 \cap \mathbb{H}^n) = U \cap D$.
  - See Figure 9.1 and the associated discussion.
  - Guillemin and Haine (2018) proves Theorem 4.4.17.
- ***D*-adapted parameterizable open set**: The open set U characterized by Theorem 4.4.17.
- We now build up to the result that if X is oriented and $D \subset X$ is a smooth domain, then the boundary $Z = \partial D$ of D acquires from X a natural orientation.
  - We first prove Lemma 4.4.24.
  - We then let $V_0 = U_0 \cap \mathbb{R}^{n-1} = \partial(U_0 \cap \mathbb{H}^n)$. It will follow since $\psi|_{V_0}$ maps V_0 onto $U \cap Z$ diffeomorphically, we can orient $U \cap Z$ by requiring that this map be orientation preserving. To prove that this orientation on $U \cap Z$ is *intrinsic*, we need only show that the orientation induced on $U \cap Z$ does not depend on the choice of ψ. We can do this with Theorem 4.4.25.
  - To prove Theorem 4.4.25, we make use of Proposition 4.4.26.
  - Finally, we orient the boundary of D by requiring that for every D-adapted parameterizable open set U, the orientation of Z conincides with the orientation of $U \cap Z$ that we described above.
- Lemma 4.4.24: The diffeomorphism $\psi : U_0 \to U$ in Theorem 4.4.17 can be chosen to be orientation preserving.

  Proof. Uses the $V_0^\sharp$ trick from earlier. □

- Theorem 4.4.25: If $\psi_i : U_i \to U$ $(i = 0, 1)$ are oriented parameterizations of U with the property with the property
$$\psi_i(U_i \cap \mathbb{H}^n) = U \cap D$$
then the restrictions of each ψ_i to $V_i = U_i \cap \mathbb{R}^{n-1} = \partial(U_i \cap \mathbb{H}^n)$ induce compatible orientations on $U \cap X$.
- Proposition 4.4.26: Let $U_0, U_1 \subset \mathbb{R}^n$ open and $f : U_0 \to U_1$ an orientation preserving diffeomorphism which maps $U_0 \cap \mathbb{H}^n$ onto $U_1 \cap \mathbb{H}^n$. If
$$V_i = U_i \cap \mathbb{R}^{n-1} = \partial(U_i \cap \mathbb{H}^n)$$
for $i = 0, 1$, then the restriction $g = f|_{V_0}$ is an orientation preserving diffeomorphism which sends
$$g(V_0) = V_1$$
- We now conclude with a global version of Proposition 4.4.26.
  - What the following proposition posits in layman's terms is that if we have any two smooth domains on any two manifolds, an orientation preserving diffeomorphism between the domains is also an orientation preserving diffeomorphism of their boundaries.
- Proposition 4.4.30: For $i = 1, 2$, let X_i be an oriented manifold, $D_i \subset X_i$ a smooth domain, and $Z_i = \partial D_i$ its boundary. Then if f is an orientation preserving diffeomorphism of (X_1, D_1) onto (X_2, D_2), the restriction $g = f|_{Z_1}$ is an orientation preserving diffeomorphism of Z_1 onto Z_2.

References

Guillemin, V., & Haine, P. J. (2018). *Differential forms* [https://math.mit.edu/classes/18.952/2018SP/files/18.952_book.pdf].

Klug, M. (2022, April). *Connections with vector calculus*.