Chapter 1

Multilinear Algebra

1.1 Notes

3/28:
- Motivation for the course and an overview of Guillemin and Haine (2018).

3/30:
- Plan:
  - More (multi)linear algebra.
- Dual spaces.
- Let V be an n-dimensional real vector space.
- $\textbf{Hom}\,(\boldsymbol{V},\mathbb{R})$: The set of all homomorphisms (i.e., linear maps) from V to $\mathbb{R}$. *Also known as* $\boldsymbol{V^*}$.
- **Dual basis** (for V^*): The set of linear transformations from V to $\mathbb{R}$ defined by

$$e_j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

  where $e_1,\dots,e_n$ is a basis of V. *Denoted by* $e_1^*,\dots,e_n^*$.
- Check: $e_1^*,\dots,e_n^*$ are a basis for V^*.
  - Are they linearly independent? Let $c_1e_1^* + \cdots + c_ne_n^* = 0 \in \text{Hom}(V,\mathbb{R})$. Then

$$c_i = (c_1e_1^* + \cdots + c_ne_n^*)(e_i) = 0 \in \mathbb{R}$$

    as desired.
  - Span? Let $\varphi \in \text{Hom}(V,\mathbb{R})$. Then we can verify that

$$\varphi(e_1)e_1^* + \cdots + \varphi(e_n)e_n^* = \varphi$$

    - We prove this by verifying the previous statement on the basis of V (if two linear transformations have the same action on the basis of a vector space, they are equal).
- With a choice of basis for V, we obtain an isomorphism $\varepsilon : V \to V^*$ with the mapping $e_i \mapsto e_i^*$ for all i.
- The dual space is known as such because $(V^*)^* \cong V$, where $\cong$ is **canonical** (no choice of basis is needed).
- One more property of dual spaces: **functoriality**.

  - Given a linear transformation $A : V \to W$, we know that $A^* : W^* \to V^*$ where A^* is the transpose of A. In particular, if $\varphi \in W^*$, then $\varphi \circ A : V \to \mathbb{R}$.
  - Claim: A^* is linear.

- **Functoriality**: If $A : V \to W$ and $B : W \to U$, then $B^* : U^* \to W^*$ and $A^* : W^* \to V^*$. The functoriality statement is that $(B \circ A)^* = A^* \circ B^*$.
- A^* is the **pullback** (or transpose) of A.
- Let $v_1, \ldots, v_n$ be a basis for V and $w_1, \ldots, w_m$ be a basis for W. Then $[A]_{v_1,\ldots,v_n}^{w_1,\ldots,w_m} = A$ is the matrix of the linear transformation A with respect to these bases. Then if $v_1^*, \ldots, v_n^*$ and $w_1^*, \ldots, w_m^*$ are the corresponding dual bases, then $[A^*]_{v_1^*,\ldots,v_n^*}^{w_1^*,\ldots,w_m^*} = A^T$. We can and should verify this for ourselves.
- This is over the real numbers, so A^* is just the transpose because there are no complex numbers of which to take the conjugate!
- A generalization: Tensors.
- ***k*-tensor**: A **multilinear** map
$$T : \underbrace{V \times \cdots \times V}_{k \text{ times}} \to \mathbb{R}$$
- **Multilinear** (map T): A function T such that
$$\begin{aligned} T(v_1, \ldots, v_i^1 + v_i^2, \ldots, v_k) &= T(v_1, \ldots, v_i^1, \ldots, v_k) + T(v_1, \ldots, v_i^2, \ldots, v_k) \\ T(v_1, \ldots, \lambda v_i, \ldots, v_k) &= \lambda T(v_1, \ldots, v_i, \ldots, v_k) \end{aligned}$$
for all $(v_1, \ldots, v_k) \in V^k$.
- The determinant is an n-tensor!
- 1-tensors are just covectors.
- $\boldsymbol{L^k(V)}$: The vector space of all k-tensors on V.
- Calculating $\dim L^k(V)$. (Answer not given in this class.)
- Let $A : V \to W$. Then $A^* : L^k(W) \to L^k(V)$.
  - Check $(A \circ B)^* = B^* \circ A^*$.
- **Multi-index of $\boldsymbol{n}$ of length $\boldsymbol{k}$**: A k-tuple $(i_1, \ldots, i_k)$ where each $i_j \in \mathbb{N}$ satisfies $1 \leq i_j \leq n$ $(j = 1, \ldots, k)$. *Denoted by* $\boldsymbol{I}$.
- Let $e_1, \ldots, e_n$ be a basis for V.
- **Tensor product** (of $T_1 \in L^k(V)$, $T_2 \in L^l(V)$): The function from V^{k+l} to $\mathbb{R}$ defined by
$$(v_1, \ldots, v_{k+l}) \mapsto T_1(v_1, \ldots, v_k) T_2(v_{k+1}, \ldots, v_{k+l})$$
Denoted by $\boldsymbol{T_1 \otimes T_2}$.
- Claims:
  1. $T_1 \otimes T_2 \in L^{k+l}(V)$.
  2. $A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$.
- $\boldsymbol{e_I^*}$: The function $e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*$, where $I = (i_1, \ldots, i_k)$ is a multi-index of n of length k.
- Claim: Letting I range over all n^k multi-indices of n of length k, the e_I^* are a basis for $L^k(V)$.

- If $V=\mathbb{R}$, then $V=\mathbb{R}e_1$. If $V=\mathbb{R}^2$, then $V=\mathbb{R}e_1\oplus\mathbb{R}e_2$.
- We know that $L^1(V)=V^*=Re_1^*$. Thus, $e_1^*\otimes e_2^*:V\times V\to\mathbb{R}$. Thus, for example,

$$(e_1^*\otimes e_2^*)((1,2),(3,4))=e_1^*(1,2)\cdot e_2^*(3,4)=1\cdot 4=4$$

4/1:

- Plan: More multilinear algebra.
  - Properties of the tensor product.
  - Sign of a permutation.
  - Alternating tensors (lead into differential forms down the road).
- Recall: V is an n-dimensional vector space over $\mathbb{R}$ with basis $e_1,\dots,e_n$. $\mathcal{L}^k(V)$ is the vector space of k-tensors on V. $\{e_I^*\mid I\text{ a multiindex of }n\text{ of length }k\}$ is a basis for $\mathcal{L}^k(V)$.
- For example, if $V=\mathbb{R}^2$ and $T\in\mathcal{L}^2(V)$, then

$$T(a_1e_1+a_2e_2,b_1e_1+b_2e_2)=a_1b_1T(e_1,e_1)+a_1b_2T(e_1,e_2)+a_2b_1T(e_2,e_1)+a_2b_2T(e_2,e_2)$$

  - A basis of $\mathcal{L}^2(V)$ is

$$\{e_1^*\otimes e_1^*,e_1^*\otimes e_2^*,e_2^*\otimes e_1^*,e_2^*\otimes e_2^*\}$$

  - Recall that some basic properties are

$$e_1^*\otimes e_2^*((1,2),(3,4))=1\cdot 4=4 \qquad e_2^*\otimes e_1^*((1,2),(3,4))=2\cdot 3=6$$

  - It follows by the initial decomposition of T that

$$T=a_1b_1e_1^*\otimes e_1^*+a_1b_2e_1^*\otimes e_2^*+a_2b_1e_2^*\otimes e_1^*+a_2b_2e_2^*\otimes e_2^*$$

- Important consequence: To know the action of T on an arbitrary pair of vectors, you need only know its action on the basis; a higher-dimensional generalization of the earlier property.
- Note that

$$e_I^*(e_J)=\delta_{IJ}=\begin{cases}1 & I=J\\ 0 & I\neq J\end{cases}$$

- Basic properties of the tensor product.
  1. *Right-distributive*: If $T_1\in\mathcal{L}^k(V)$ and $T_2,T_3\in\mathcal{L}^\ell(V)$, then

$$T_1\otimes(T_2+T_3)=T_1\otimes T_2+T_1\otimes T_3$$

  2. *Left-distributive*: If $T_1,T_2\in\mathcal{L}^k(V)$ and $T_3\in\mathcal{L}^\ell(V)$, then

$$(T_1+T_2)\otimes T_3=T_1\otimes T_3+T_2\otimes T_3$$

  3. *Associative*: If $T_1\in\mathcal{L}^k(V)$, $T_2\in\mathcal{L}^\ell(V)$, and $T_3\in\mathcal{L}^m(V)$, then

$$T_1\otimes(T_2\otimes T_3)=(T_1\otimes T_2)\otimes T_2=T_1\otimes T_2\otimes T_3$$

  4. *Scalar multiplication*: If $T_1\in\mathcal{L}^k(V)$, $T_2\in\mathcal{L}^\ell(V)$, and $\lambda\in\mathbb{R}$, then

$$(\lambda T_1)\otimes T_2=\lambda(T_1\otimes T_2)=T_1\otimes(\lambda T_2)$$

- Note that the tensor product is not commutative.
- Aside: Defining the sign of a permutation.

- $\boldsymbol{S_A}$: The set of all automorphisms of A (bijections from A to A), where A is a set.
- $\boldsymbol{S_n}$: The set $S_{[n]}$.
- Given $\sigma_1, \sigma_2 \in S_n$, $\sigma_1 \circ \sigma_2 \in S_n$.
  - Thus, S_n is a **group**.
- **Transposition**: A function $\tau \in S_n$ such that

$$\tau(k) = \begin{cases} j & k = i \\ i & k = j \\ k & k \neq i, j \end{cases}$$

  for some $i, j \in [n]$. *Denoted by* $\boldsymbol{\tau_{i,j}}$.
- Theorem: An element of S_n can be written as the product of transpositions (i.e., for all $\sigma \in S_n$, there exist $\tau_1, \dots, \tau_m \in S_n$ such that $\sigma = \tau_1 \circ \cdots \circ \tau_m$).
- **Sign** (of $\sigma \in S_n$): The number (mod 2) of transpositions whose product equals σ. *Denoted by* $\boldsymbol{(-1)^\sigma}$, $\mathbf{sign}\,\boldsymbol{(\sigma)}$.
- Theorem: The sign of σ is well-defined. Additionally,

$$(-1)^{\sigma_1 \sigma_2} = (-1)^{\sigma_1} \cdot (-1)^{\sigma_2}$$

- Example: Consider the identity permutation. $(-1)^\sigma = +1$. We can think of this as the product of zero transpositions or, for instance, as the product of the two transpositions $\tau_{1,2} \circ \tau_{1,2}$. Another example would be $\tau_{2,3} \circ \tau_{1,2} \circ \tau_{1,2} \circ \tau_{2,3}$.
- Theorem: Let X_i be a rational or polynomial function for each $i \in \mathbb{N}$. Then

$$(-1)^\sigma = \prod_{i<j} \frac{X_{\sigma(i)} - X_{\sigma(j)}}{X_i - X_j}$$

- Example: For the permutation $\sigma = (1, 2, 3)$, we have

$$\begin{aligned}
(-1)^\sigma &= \frac{X_{\sigma(1)} - X_{\sigma(2)}}{X_1 - X_2} \cdot \frac{X_{\sigma(1)} - X_{\sigma(3)}}{X_1 - X_3} \cdot \frac{X_{\sigma(2)} - X_{\sigma(3)}}{X_2 - X_3} \\
&= \frac{X_2 - X_3}{X_1 - X_2} \cdot \frac{X_2 - X_1}{X_1 - X_3} \cdot \frac{X_3 - X_1}{X_2 - X_3} \\
&= \frac{-(X_1 - X_2)}{X_1 - X_2} \cdot \frac{-(X_1 - X_3)}{X_1 - X_3} \cdot \frac{X_2 - X_3}{X_2 - X_3} \\
&= -1 \cdot -1 \cdot 1 \\
&= +1
\end{aligned}$$

  which squares with the fact that $\sigma = \tau_{1,2} \circ \tau_{2,3}$.
- Claims to verify with the above formula:
  1. $\text{sign}(\sigma) \in \{\pm 1\}$.
  2. $\text{sign}(\tau_{i,j}) = -1$.
  3. $\text{sign}(\sigma_1 \sigma_2) = \text{sign}(\sigma_1)\,\text{sign}(\sigma_2)$.

4/4:
- Plan:
  - More multilinear algebra.

- Alternating k-tensors — 2 views:
  1. As a subspace of $\mathcal{L}^k(V)$.
  2. As a quotient of $\mathcal{L}^k(V)$.
- Next time: Operators as alternating tensors.
  - Wedge products.
  - Interior products.
  - Pullbacks.

- Recall: $\dim V = n$, $e_1, \dots, e_n$ a basis, $\mathcal{L}^k(V)$ the space of k-tensors, $\sigma \in S_k$ implies $(-1)^\sigma \in \{\pm 1\}$, key property: $(-1)^{\sigma_1\sigma_2} = (-1)^{\sigma_1}(-1)^{\sigma_2}$.
- $\boldsymbol{T^\sigma}$: The k-tensor over V defined by

$$T^\sigma(v_1, \dots, v_k) = T(v_{\bar{\sigma}(1)}, \dots, v_{\bar{\sigma}(k)})$$

  where $T \in \mathcal{L}^k(V)$, $\sigma \in S_k$, and $\bar{\sigma}$ denotes the inverse of σ.
- Example: $n = 2$, $k = 2$. Let $T = e_1^* \otimes e_2^* \in \mathcal{L}^2(V)$. Let $\sigma = \tau_{1,2}$. Then $T^\sigma = e_2^* \otimes e_1^*$.
- Another property is $e_I^\sigma = e_{\sigma(I)}^*$.
- Properties:
  1. $T^{\sigma_1\sigma_2} = (T^{\sigma_1})^{\sigma_2}$.
  2. $(T_1 + T_2)^\sigma = T_1^\sigma + T_2^\sigma$.
  3. $(cT)^\sigma = cT^\sigma$.
- Thus, you can view $\sigma : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ as a linear map!
- **Alternating $\boldsymbol{k}$-tensor**: A tensor $T \in \mathcal{L}^k(V)$ such that $T^\sigma = (-1)^\sigma T$ for all $\sigma \in S_k$.
  - Equivalently, $T^\tau = -T$ for all $\tau \in S_k$.
- An example of an alternating 2-tensor when $\dim V = 2$ is $T = e_1^* \otimes e_2^* - e_2^* \otimes e_1^*$.
  - Naturally, $T^\tau_{1,2} = -T$, and $\tau_{1,2}$ is the unique transposition in S_2.
- $e_1^* \otimes e_2^*$ is *not* an alternating 2-tensor since $(e_1^* \otimes e_2^*)^\tau = e_2^* \otimes e_1^* \neq (-1)^\tau (e_1^* \otimes e_2^*)$.
- We can look at $n = 2$, $k = 1$ for ourselves.
- Note: If $T - 1, T_2$ are both alternating k-tensors, then $T_1 + T_2$ is also alternating, as is cT_1 for all $c \in \mathbb{R}$.
- $\boldsymbol{\mathcal{A}^k(V)}$: The vector space of alternating k-tensors.
- **Alt $\boldsymbol{(T)}$**: The function $\text{Alt} : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ defined by

$$\text{Alt}(T) = \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma$$

- Properties:
  1. $\text{im}(\text{Alt}) = \mathcal{A}^k(V)$.
  2. $\mathcal{L}^k(V) / \ker(\text{Alt}) = \Lambda^k(V^*)$ is isomorphic to $\mathcal{A}^k(V)$.
  3. $\text{Alt}(T)^\sigma = (-1)^\sigma \, \text{Alt}(T)$.

- Proof:

$$\begin{aligned}\operatorname{Alt}(T)^{\sigma'} &= \left(\sum_{\sigma\in S_k}(-1)^\sigma T^\sigma\right)^{\sigma'}\\ &= \sum_{\sigma\in S_k}(-1)^\sigma T^{\sigma\sigma'}\\ &= (-1)^{\sigma'}\sum_{\sigma\in S_k}(-1)^{\sigma'}(-1)^\sigma T^{\sigma\sigma'}\\ &= (-1)^{\sigma'}\sum_{\sigma\in S_k}(-1)^{\sigma\sigma'}T^{\sigma\sigma'}\\ &= (-1)^{\sigma'}\operatorname{Alt}(T)\end{aligned}$$

- The last equality holds because summing over all σ is the same as summing over all $\sigma'\circ\sigma$.
- This implies $\operatorname{im}(\operatorname{Alt})\leq\mathcal{A}^k(V)$.

4. If $T\in\mathcal{A}^k(T)$, $\operatorname{Alt}(T)=k!T$.
   - We have

$$\begin{aligned}\operatorname{Alt}(T) &= \sum_{\sigma\in S_k}(-1)^\sigma T^\sigma\\ &= \sum_{\sigma\in S_k}(-1)^\sigma(-1)^\sigma T\\ &= \sum_{\sigma\in S_k}T\\ &= k!T\end{aligned}$$

   where $T^\sigma=(-1)^\sigma T$ since $T\in\mathcal{A}^k(V)$ by definition.
   - This implies that $\operatorname{im}(\operatorname{Alt})=\mathcal{A}^k(V)$: $\operatorname{Alt}(\frac{1}{k!}T)=T\in\mathcal{A}^k(V)$.
5. $\operatorname{Alt}(T^\sigma)=\operatorname{Alt}(T)^\sigma$.
6. $\operatorname{Alt}:\mathcal{L}^k(V)\to\mathcal{L}^k(V)$ is linear.

- Warning: Some people take $\operatorname{Alt}(T)=\frac{1}{k!}\sum_{\sigma\in S_k}(-1)^\sigma T^\sigma$[1].
- Example: $n=k=2$. We have

$$\operatorname{Alt}(e_1^*\otimes e_2^*)=e_1^*\otimes e_2^*-e_2^*\otimes e_1^*$$

- **Non-repeating** (multi-index I): A multi-index I such that $i_{j_1}\neq i_{j_2}$ for all $j_1\neq j_2$.
- **Increasing** (multi-index I): A multi-index I such that $i_1<\cdots<i_k$.
- Claim: $\{\operatorname{Alt}(e_I^*)\}$ where I is non-repeating and increasing is a basis for $\mathcal{A}^k(V)$. There are $\binom{n}{k}$ of these; thus, $\dim\mathcal{A}^k(V)=\binom{n}{k}$.

4/6:

- Klug will be in Texas on Monday and thus is cancelling class on Monday. Homework is now due next Friday. We'll have weekly homeworks going forward after that.
- Plan:
  - $\operatorname{Alt}:\mathcal{L}^k(V)\twoheadrightarrow\mathcal{A}^k(V)$[2].
  - Goal: Identify $\ker(\operatorname{Alt})=\mathcal{I}^k(V)$, where $\mathcal{I}^k(V)$ is the space of **redundant** k-tensors[3].

[1] Klug prefers this convention, but the text takes the other one.
[2] The two-headed right arrow denotes a surjective map.
[3] The $\mathcal{I}$ in $\mathcal{I}^k(V)$ stands for "ideal."

- Then: Operations on alternating tensors, e.g.,
  - Wedge product.
  - Interior product.
  - Orientations.

- Claim: $\{\text{Alt}(e_I^*) \mid I \text{ non-repeating, increasing multi-index}\}$ is a basis for $\mathcal{A}^k(V)$.
  - Left as an exercise to us.

- **Redundant** (k-tensor): A k-tensor of the form

$$\ell_1 \otimes \cdots \otimes \ell_i \otimes \ell_i \otimes \ell_{i+2} \otimes \cdots \otimes \ell_k$$

  where $\ell_1, \dots, \ell_k \in V^*$.

- $\boldsymbol{\mathcal{I}^k(V)}$: The span of all redundant k-tensors.
  - Note that not every k-tensor in $\mathcal{I}^k(V)$ is a redundant.

- **Decomposable** (k-tensor): A k-tensor of the form $\ell_1 \otimes \cdots \otimes \ell_k$ for $\ell_i \in \mathcal{L}^1(V)$.
  - It often suffices to prove things for decomposable tensors.

- Properties.
  1. If $T \in \mathcal{I}^k(V)$, then $\text{Alt}(T) = 0$, i.e., $\mathcal{I}^k(V) \leq \ker(\text{Alt})$.
     - "Proof by example": If $T = \ell_1 \otimes \ell_1 \otimes \ell_2 \otimes \ell_3$, then $T^{\tau_{1,2}} = T$. It follows from the properties of Alt that

$$\begin{aligned}\text{Alt}(T) &= \text{Alt}(T^{\tau_{1,2}}) = (-1)^{\tau_{1,2}} \text{Alt}(T) = -\text{Alt}(T) \\ 2\,\text{Alt}(T) &= 0 \\ \text{Alt}(T) &= 0\end{aligned}$$

  2. If $T \in \mathcal{I}^r(V)$ and $T' \in \mathcal{L}^s(V)$, then

$$T \otimes T' \in \mathcal{I}^{r+s}(V)$$

     Similarly, if $T \in \mathcal{L}^r(V)$ and $T \in \mathcal{I}^s(V)$, then

$$T \otimes T' \in \mathcal{I}^{r+s}(V)$$

     - Proof: It suffices to assume that T is redundant. Obviously adding more tensors to the direct product will not change the redundancy of the initial tensor. Example: $\ell_1 \otimes \ell_1 \otimes \ell_2$ is just as redundant as $\ell_1 \otimes \ell_1 \otimes \ell_2 \otimes T$.
  3. If $T \in \mathcal{L}^k(V)$ and $\sigma \in S_k$, then

$$T^\sigma = (-1)^\sigma T + S$$

     for some $S \in \mathcal{I}^k(V)$.
     - Proof by example: It suffices to check this for decomposable tensors (a tensor is just a sum of decomposable tensors). Take $k = 2$. Let $T = \ell_1 \otimes \ell_2$. Let $\sigma = \tau_{1,2}$. Then

$$T^\sigma - (-1)^\sigma T = \ell_2 \otimes \ell_1 + \ell_1 \otimes \ell_2 = (\ell_1 + \ell_2) \otimes (\ell_1 + \ell_2) - \ell_1 \otimes \ell_1 - \ell_2 \otimes \ell_2$$

     - Actual proof: It suffices to assume T is decomposable. We induct on the number of transpositions needed to write σ as a product of **adjacent** transpositions.

- Base case: $\sigma = \tau_{i,i+1}$. Then

$$\begin{aligned} T^{\tau_{i,i+1}} + T = \ell_1 \otimes \cdots \otimes (\ell_i + \ell_{i+1}) \otimes (\ell_i + \ell_{i+1}) \otimes \cdots \otimes \ell_k \\ - \ell_1 \otimes \cdots \otimes \ell_i \otimes \ell_i \otimes \cdots \otimes \ell_k \\ - \ell_1 \otimes \cdots \otimes \ell_{i+1} \otimes \ell_{i+1} \otimes \cdots \otimes \ell_k \end{aligned}$$

- Inductive step: If $\sigma = \beta\tau$, then

$$\begin{aligned} T^\sigma &= T^{\beta\tau} \\ &= (-1)^\tau T^\beta + \text{stuff in } \mathcal{I}^k(V) \\ &= (-1)^\tau[(-1)^\beta T + \text{stuff in } \mathcal{I}^k(V)] + \text{stuff in } \mathcal{I}^k(V) \end{aligned}$$

4. If $T \in \mathcal{L}^k(V)$, then
$$\mathrm{Alt}(T) = k!T + W$$
for some $W \in \mathcal{I}^k(V)$.
   - We have that

$$\begin{aligned} \mathrm{Alt}(T) &= \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma \\ &= \sum_{\sigma \in S_k} (-1)^\sigma [(-1)^\sigma T + S_\sigma] \\ &= \sum_{\sigma \in S_k} T + \sum_{\sigma \in S_k} (-1)^\sigma S_\sigma \\ &= k!T + W \end{aligned}$$

5. $\mathcal{I}^k(V) = \ker(\mathrm{Alt})$.
   - We have that $\mathcal{I}^k(V) \leq \ker(\mathrm{Alt})$ by property 1.
   - Now suppose $T \in \ker(\mathrm{Alt})$. Then $\mathrm{Alt}(T) = 0$. Then by property 4,

$$\begin{aligned} \mathrm{Alt}(T) &= k!T + W \\ 0 &= k!T + W \\ T &= -\frac{1}{k!}W \in \mathcal{I}^k(V) \end{aligned}$$

- Warning: If $T \in \mathcal{A}^r(V)$ and $T' \in \mathcal{A}^s(V)$, then we do not necessarily have $T \otimes T' \in \mathcal{A}^{r+s}(V)$.
  - Example: $e_1^*, e_2^* \in \mathcal{A}^1(V)$ have $e_1^* \otimes e_2^* \notin \mathcal{A}^2(V)$.
- **Adjacent** (transposition): A transposition of the form $\tau_{i,i+1}$.

4/8:

- Recall that $\mathcal{A}^k(V) \hookrightarrow \mathcal{L}^k(V)$[4]
- Functoriality: $(A \circ B)^* = B^* \circ A^*$.
  - A^* takes $\mathcal{L}^k(W) \to \mathcal{L}^k(V)$ and $\mathcal{A}^k(W) \to \mathcal{A}^k(V)$.
- $\dim(\Lambda^k(V)) = \binom{n}{k}$.
  - Special case $k = n$: $\dim \Lambda^n(V) = 1$.
  - If $A : V \to V$ induces a map $\Lambda^n(V^*) \to \Lambda^n(V^*)$ defined by the determinant.
- Aside: $\Lambda^k(V)$ is "exterior powers."

[4]The hooked right arrow denotes an injective map.

- Plan: Wedge products + basis for $\Lambda^k(V)$.
- **Wedge product**: A function $\wedge : \Lambda^k(V^*) \times \Lambda^\ell(V^*) \to \Lambda^{k+\ell}(V)$.
  - We denote elements of $\Lambda^k(V^*)$ by ω_1, ω_2, etc.
- If $\pi : \mathcal{L}^k(V) \to \Lambda^k(V^*)$ sends $T \mapsto \omega$, $\omega_1 = \pi(T_1)$, and $\omega_2 = \pi(T_2)$, then $\omega_1 \wedge \omega_2 = \pi(T_1 \otimes T_2)$.
  - Note that $\ker(\pi) = \mathcal{I}^k(V)$.
- Properties.
  1. This is well defined, i.e., this does not depend on the choice of T_1, T_2.
     - Consider $T_1 + W_1, T_2 + W_2$ with $W_1, W_2 \in \mathcal{I}^k(V)$.
     - We check that $\pi[(T_1 + W_1) \otimes (T_2 + W_2)] = \pi(T_1 \otimes T_2)$.
     - Since $W_1 \otimes T_2, T_1 \otimes W_2, W_1 \otimes W_2 \in \mathcal{I}^{k+\ell}(V)$, we have that

$$\begin{aligned}\pi[(T_1 + W_1) \otimes (T_2 + W_2)] &= \pi(T_1 \otimes T_2 + W_1 \otimes T_2 + T_1 \otimes W_2 + W_1 \otimes W_2)\\ &= \pi(T_1 \otimes T_2) + \pi(W_1 \otimes T_2) + \pi(T_1 \otimes W_2) + \pi(W_1 \otimes W_2)\\ &= \pi(T_1 \otimes T_2)\end{aligned}$$

  2. Associative: We have that

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge \omega_2 \wedge \omega_3$$

     - Follows from the definition of $\wedge$ in terms of π and properties of the tensor product.
  3. Distributive: We have that

$$(\omega_1 + \omega_2) \wedge \omega_3 = \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_3 \qquad \omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$$

     - Follows from the definition of $\wedge$ in terms of π and properties of the tensor product.
  4. Linear: We have that

$$(c\omega_1) \wedge \omega_2 = c(\omega_1 \wedge \omega_2) = \omega_1 \wedge (c\omega_2)$$

     - Follows from the definition of $\wedge$ in terms of π and properties of the tensor product.
  5. Anticommutative: We have that

$$\omega_1 \wedge \omega_2 = (-1)^{k\ell} \omega_2 \wedge \omega_1$$

     - It suffices to assume that $w_1 = \ell_1 \wedge \cdots \wedge \ell_k, w_2 = \ell'_1 \wedge \cdots \wedge \ell'_\ell$.
       - We have

$$(\ell_1 \wedge \cdots \wedge \ell_k) \wedge (\ell'_1 \wedge \cdots \wedge \ell'_\ell) = (-1)^k (\ell'_1 \wedge \cdots \wedge \ell'_\ell) \wedge (\ell_1 \wedge \cdots \wedge \ell_k)$$

     - Let $\ell_1, \dots, \ell_k \in \Lambda^1(V^*) = V^* = \mathcal{L}^1(V)$.
     - Recall that $\mathcal{I}^1(V) = \{0\}$.
     - Claim: $\ell_{\sigma(1)} \wedge \cdots \wedge \ell_{\sigma(k)} = (-1)^\sigma \ell_1 \wedge \cdots \wedge \ell_k$ for all $\sigma \in S_k$.
       - Recall that $T^\sigma = (-1)^\sigma T + W$ for some $W \in \mathcal{I}^k(V)$.
       - Let $T = \ell_1 \otimes \cdots \otimes \ell_k$.
       - Then

$$\begin{aligned}(\ell_1 \otimes \cdots \otimes \ell_k)^\sigma &= \ell_{\sigma(1)} \otimes \cdots \otimes \ell_{\sigma(k)}\\ &= (-1)^\sigma \ell_1 \otimes \cdots \otimes \ell_k + W\end{aligned}$$

       - Then hit both sides by π, noting that $\pi(W) = 0$.
- Example:
  1. $n = 2$, $k = \ell = 1$. Consider $e_1^*, e_2^* \in \mathcal{L}^1(V) = V^* = \mathcal{A}^1(V) = \Lambda^1(V^*)$. Then

$$e_1^* \wedge e_2^* = (-1) e_2^* \wedge e_1^* \qquad e_1^* \wedge e_1^* = 0 = e_2^* \wedge e_2^*$$

  2. $n = 4$. We have $e_1^* \wedge (3e_1^* + 2e_2^* + 3e_2^*) = 3(e_1^* \wedge e_1^*) + 2(e_1^* \wedge e_2^*) + 3(e_1^* \wedge e_3^*)$. We also have $(e_1^* \wedge e_2^*) \wedge (e_1^* \wedge e_2^*) = 0$.

1.2 Chapter 1: Multilinear Algebar

From Guillemin and Haine (2018).

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- Guillemin and Haine (2018) defines real vector spaces, the operations on them, their basic properties, and the zero vector.
- **Linearly independent** (vectors $v_1, \dots, v_k$): A finite set of vectors $v_1, \dots, v_k \in V$ such that the map from $\mathbb{R}^k$ to V defined by $(c_1, \dots, c_k) \mapsto c_1 v_1 + \cdots + c_k v_k$ is injective.
- **Spanning** (vectors $v_1, \dots, v_k$): We require that the above map is surjective.
- Guillemin and Haine (2018) defines basis, finite-dimensional vector space, dimension, subspace, linear map, and kernel.
- **Image** (of $A : V \to W$): The range space of A, a subspace of W. *Also known as* $\mathbf{im}\,(\boldsymbol{A})$.
- Guillemin and Haine (2018) defines the matrix of a linear map.
- **Inner product** (on V): A map $B : V \times V \to \mathbb{R}$ with the following three properties.
  - *Bilinearity*: For vectors $v, v_1, v_2, w \in V$ and $\lambda \in \mathbb{R}$, we have
    $$B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w)$$
    and
    $$B(\lambda v, w) = \lambda B(v, w)$$
  - *Symmetry*: For vectors $v, w \in V$, we have $B(v, w) = B(w, v)$.
  - *Positivity*: For every vector $v \in V$, we have $B(v, v) \geq 0$. Moreover, if $v \neq 0$, then $B(v, v) > 0$.
- $\boldsymbol{W}$**-coset**: A set of the form $\{v + w \mid w \in W\}$, where W is a subspace V and $v \in V$. *Denoted by* $\boldsymbol{v + W}$.
  - If $v_1 - v_2 \in W$, then $v_1 + W = v_2 + W$.
  - It follows that the distinct W-cosets decompose V into a disjoint collection of subsets of V.
- **Quotient space** (of V by W): The set of distinct W-cosets in V, along with the following definitions of vector addition and scalar multiplication.
  $$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W \qquad \lambda(v + W) = (\lambda v) + W$$
  Denoted by $\boldsymbol{V/W}$.
- **Quotient map**: The linear map $\pi : V \to V/W$ defined by
  $$\pi(v) = v + W$$
  - π is surjective.
  - Note that $\ker(\pi) = W$ since for all $w \in W$, $\pi(w) = w + W = 0 + W$, which is the zero vector in V/W.
- If V, W are finite dimensional, then
  $$\dim(V/W) = \dim(V) - \dim(W)$$
- Proposition 1.2.9: Let $A : V \to U$ be a linear map. If $W \subset \ker(A)$, then there exists a unique linear map $A^\sharp : V/W \to U$ with the property that $A = A^\sharp \circ \pi$, where $\pi : V \to V/W$ is the quotient map.
  - This proposition rephrases in terms of quotient spaces the fact that if $w \in W$, then $A(v+w) = Av$.

- **Dual space** (of V): The set of all linear functions $\ell : V \to \mathbb{R}$, along with the following definitions of vector addition and scalar multiplication.
$$(\ell_1 + \ell_2)(v) = \ell_1(v) + \ell_2(v) \qquad\qquad (\lambda\ell)(v) = \lambda \cdot \ell(v)$$
Denoted by V^*.
- **Dual basis** (of $e_1, \dots, e_n$ a basis of V): The basis of V^* consisting of the n functions that take every $v = c_1e_1 + \cdots + c_ne_n$ to one of the c_i. *Denoted by* $\boldsymbol{e_1^*, \dots, e_n^*}$. *Given by*
$$e_i^*(v) = c_i$$
for all $v \in V$.
- Claim 1.2.12: If V is an n-dimensional vector space with basis $e_1, \dots, e_n$, then $e_1^*, \dots, e_n^*$ is a basis of V^*.

Proof. We will first prove that $e_1^*, \dots, e_n^*$ spans V^*. Let $\ell \in V^*$ be arbitrary. Set $\lambda_i = \ell(e_i)$ for all $i \in [n]$. Define $\ell' = \sum_{i=1}^n \lambda_i e_i^*$. Then
$$\ell'(e_j) = \sum_{i=1}^n \lambda_i e_i^*(e_j) = \lambda_j \cdot 1 = \ell(e_j)$$
for all $j \in [n]$. Therefore, since ℓ, ℓ' take identical values on the basis of V, $\ell = \ell'$, as desired.

We now prove that $e_1^*, \dots, e_n^*$ spans V^*. Let $\sum_{i=1}^n \lambda_i e_i^* = 0$. Then for all $j \in [n]$,
$$\lambda_j = \left(\sum_{i=1}^n \lambda_i e_i^* \right)(e_j) = 0$$
as desired. $\square$

- **Transpose** (of A): The map from W^* to V^* defined by $\ell \mapsto A^*\ell = \ell \circ A$ for all $\ell \in W^*$.
- Claim 1.2.15: If $e_1, \dots, e_n$ is a basis of V, $f_1, \dots, f_m$ is a basis of W, $e_1^*, \dots, e_n^*$ and $f_1^*, \dots, f_m^*$ are the corresponding dual bases, and $[a_{i,j}]$ is the $m \times n$ matrix of A with respect to $\{e_i\}, \{f_i\}$, then the linear map A^* is defined in terms of $\{f_i^*\}, \{e_i^*\}$ by the transpose matrix $(a_{j,i})$.

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- $\boldsymbol{V^k}$: The set of all k-tuples $(v_1, \dots, v_k)$ where $v_1, \dots, v_k \in V$ a vector space.
  - Note that
$$V^k = \underbrace{V \oplus \cdots \oplus V}_{k \text{ times}}$$
where "$\oplus$" denotes the direct sum.
- **Linear** (function in its i^{th} variable): A function $T : V^k \to \mathbb{R}$ such that the map from V to $\mathbb{R}$ defined by $v \mapsto T(v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_k)$ is linear, where all v_j save v_i are fixed.
- $\boldsymbol{k}$**-linear** (function T): A function $T : V^k \to \mathbb{R}$ that is linear in its i^{th} variable for $i = 1, \dots, k$. *Also known as* $\boldsymbol{k}$**-tensor**.
- $\boldsymbol{\mathcal{L}^k(V)}$: The set of all k-tensors in V.
  - Since the sum $T_1 + T_2$ of two k-linear functions $T_1, T_2 : V^k \to \mathbb{R}$ is just another k-linear function, and λT_1 is k-linear for all $\lambda \in \mathbb{R}$, we have that $\mathcal{L}^k(V)$ is a vector space.
- Convention: 0-tensors are just the real numbers. Mathematically, we define
$$\mathcal{L}^0(V) = \mathbb{R}$$

- Note that $\mathcal{L}^1(V) = V^*$.
- Defines multi-indices of n of length k.
- Lemma 1.3.5: If $n, k \in \mathbb{N}$, then there are exactly n^k multi-indices of n of length k.
- $\boldsymbol{T_I}$: The real number $T(e_{i_1}, \dots, e_{i_k})$, where $T \in \mathcal{L}^k(V)$, $e_1, \dots, e_n$ is a basis of V, and I is a multi-index of n of length k.
- Proposition 1.3.7: The real numbers T_I determine T, i.e., if T, T' are k-tensors and $T_I = T'_I$ for all I, then $T = T'$.

  Proof. We induct on n. For the base case $n = 1$, $T \in (\mathbb{R}^k)^*$ and we have already proven this result. Now suppose inductively that the assertion is true for $n - 1$. For each e_i, let T_i be the $(k-1)$-tensor defined by
  $$(v_1, \dots, v_{n-1}) \mapsto T(v_1, \dots, v_{n-1}, e_i)$$
  Then for an arbitrary $v = c_1e_1 + \cdots + c_ne_n$,
  $$T(v_1, \dots, v_{n-1}, v) = \sum_{i=1}^{n} c_i T_i(v_1, \dots, v_{n-1})$$
  so the T_i's determine T. Applying the inductive hypothesis completes the proof. ☐

- **Tensor product**: The tensor $T_1 \otimes T_2$ defined by
  $$(T_1 \otimes T_2)(v_1, \dots, v_{k+\ell}) = T_1(v_1, \dots, v_k)T_2(v_{k+1}, \dots, v_{k+\ell})$$
  where $T_1 \in \mathcal{L}^k(V)$ and $T_2 \in \mathcal{L}^\ell(V)$.
- Note that by the definition of 0-tensors as real numbers, if $a \in \mathbb{R}$ and $T \in \mathcal{L}^k(V)$, then
  $$a \otimes T = T \otimes a = aT$$
- Proposition 1.3.9: Associativity, distributivity of scalar multiplication, and left and right distributive laws for the tensor product.
- **Decomposable** (k-tensor): A k-tensor T for which there exist $\ell_1, \dots, \ell_k \in V^*$ such that
  $$T = \ell_1 \otimes \cdots \otimes \ell_k$$
- Defines e_I^*.
- Theorem 1.3.13: V a vector space with basis $e_1, \dots, e_n$ and $0 \leq k \leq n$ implies the k-tensors e_I^* form a basis of $\mathcal{L}^k(V)$.

  Proof. Spanning: Let $T \in \mathcal{L}^k(V)$ be arbitrary. Define
  $$T' = \sum_I T_I e_I^*$$
  Since
  $$T'_J = T'(e_{j_1}, \dots, e_{j_k}) = \sum_I T_I e_I^*(e_{j_1}, \dots, e_{j_k}) = T_J e_J^*(e_{j_1}, \dots, e_{j_k}) = T_J$$
  for all J, Proposition 1.3.7 asserts that $T = T'$. Therefore, since every $T_I \in \mathbb{R}$, $T = T' \in \operatorname{span}(e_I^*)$.

  Linear independence: Suppose
  $$T = \sum_I c_I e_I^* = 0$$

for some set of constants $c_I \in \mathbb{R}$. Then

$$0 = T(e_{j_1}, \dots, e_{j_k}) = \sum_I c_I e_I^*(e_{j_1}, \dots, e_{j_k}) = c_J$$

for all J, as desired. $\square$

- Corollary 1.3.15: If $\dim V = n$, then $\dim(\mathcal{L}^k(V)) = n^k$.

  Proof. Follows immediately from Lemma 1.3.5. $\square$

- **Pullback** (of T by the map A): The k-tensor $A^*T : V^k \to \mathbb{R}$ defined by

  $$(A^*T)(v_1, \dots, v_k) = T(Av_1, \dots, Av_k)$$

  where V, W are finite-dimensional vector spaces, $A : V \to W$ is linear, and $T \in \mathcal{L}^k(W)$.
- Proposition 1.3.18: The map $A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$ defined by $T \mapsto A^*T$ is linear.
- Identities:
  - If $T_1 \in \mathcal{L}^k(W)$ and $T_2 \in \mathcal{L}^m(W)$, then

    $$A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$$

  - If U is a vector space, $B : U \to V$ is linear, and $T \in \mathcal{L}^k(W)$, then

    $$(AB)^*T = B^*(A^*T)$$