Week 8

Manifolds and Relevant Tools

8.1 Homotopy Invariance and Applications; Manifold Definitions

5/16:

- Weekly plan:
  - Finish up Chapter 3.
    - Homotopy invariance.
    - Application: Brouwer fixed-point theorem.
  - Manifolds (in $\mathbb{R}^n$).
    - Definition of a manifold X^n.
    - Tangent spaces T_pX^n for $p \in X$.
    - (Total) derivatives of functions $f : X^n \to Y^n$. In particular, if $f : X \to Y$ sends $p \mapsto q$, then $DF_p : T_pX \to T_qY$.
  - Forms $\Omega^k(X^n)$.
  - Integrals $\int_X \omega$ and improving Stokes' theorem.
- Homotopy invariance of degree.
- **Homotopic** (functions F_0, F_1): Two maps $F_0, F_1 : X \to Y$ for which there exists some continuous map $H : X \times I \to Y$ such that

$$H(x,0) = F_0(x) \qquad\qquad H(x,1) = F_1(x)$$

  for all $x \in X$ where $I = [0,1]$. *Denoted by* $\boldsymbol{F_0 \cong F_1}$.
- **Homotopy**: The map H in the above definition.
- Example: A homotopy between two functions $F_0, F_1 : \mathbb{R} \to \mathbb{R}$.
  - Let $X, Y = \mathbb{R}$. Consider the functions $F_0, F_1 : X \to Y$ described by the relations

$$F_0(x) = x^2 \qquad\qquad F_1(x) = 2x$$

  - Let $H : X \times I \to Y$ be described by the relation

$$H(x,t) = (1-t) \cdot x^2 + t \cdot 2x$$

    - Note that x^2 and $2x$ are the relations describing F_0 and F_1, respectively, and the t terms simply provide a linear interpolation. In particular,

$$\begin{aligned} H(x,0) &= (1-0) \cdot x^2 + 0 \cdot 2x \\ &= x^2 \\ &= F_0(x) \end{aligned} \qquad \begin{aligned} H(x,0.5) &= x^2 \cdot (1-0.5) + 2x \cdot 0.5 \\ &= 0.5 \cdot x^2 + 0.5 \cdot 2x \\ &= 0.5F_0(x) + 0.5F_1(x) \end{aligned} \qquad \begin{aligned} H(x,1) &= (1-1) \cdot x^2 + 1 \cdot 2x \\ &= 2x \\ &= F_1(x) \end{aligned}$$

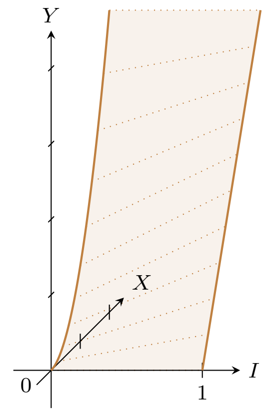

Figure 8.1: Homotopic maps.

Indeed, we can see from the above that $H(x,0) = F_0(x)$, as desired; $H(x,1) = F_1(x)$, as desired; and $H(x,0.5)$, for example, indicates the linear combination of a point that is "half $F_0(x)$ and half $F_1(x)$."

- In Figure 8.1, the parabolic brown line depicts a portion of the graph $G(F_0)$ of F_0. Similarly, the linear brown line depicts a portion of the graph $G(F_1)$ of F_1 but translated one unit along the I-axis. Lastly, the brown surface depicts a portion of the graph $G(H)$ of H.
- As we would expect for a homotopy, H is clearly continuous and interpolates between F_0 and F_1 as t moves from 0 to 1. The lines of dots indicate how several specific values of $F_0(x)$ and $F_1(x)$ are matched in bijective correspondence.

- **Proper** (homotopy): A homotopy such that for all $t \in I$, $H(\cdot,t) : X \to Y$ defined by $x \to H(x,t)$ is proper, where $X, Y \subset \mathbb{R}^n$.
- **Properly homotopic** (functions F_0, F_1): Two homotopic functions whose homotopy is proper.
- Claim: If $F_0, F_1 : U \to V$ where $U, V \subset \mathbb{R}^n$ such that F_0, F_1 are properly homotopic, then

$$\deg(F_0) = \deg(F_1)$$

- (Bad) example.
  - Consider $F_0 : \mathbb{R}^2 \to \mathbb{R}^2$ and $F_1 : \mathbb{R}^2 \to \mathbb{R}^2$ where F_0 is the constant 0 function and $F_1(z) = z^2$.
  - Then $H : \mathbb{R}^2 \times I \to \mathbb{R}^2$ may be defined by $H(z,t) = tz^2$. Clearly, this function is continuous.
  - But $\deg(F_0) = 0$ and $\deg(F_1) = 2$. This is because F_0, F_1 are not *properly* homotopic.
- Proof.
  - Let H be a proper homotopy from $F_0 \to F_1$. Let $H_t : U \to V$ send $x \to H(x,t)$ for all $t \in [0,1]$.
  - Let $\omega \in \Omega_c^n(V)$ with $\int \omega = 1$. Then

$$\deg(H_t) = \int (H_t)^* \omega = \int \varphi(H(x,t)) \det DH_t(x,t) \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$

  with $\omega = \varphi \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ where the rightmost integrand is a continuously varying set of functions.
  - Since $\deg(H_t) \in \mathbb{Z}$, $\deg(H_0) = \deg(F_0)$, and $\deg(H_1) = \deg(F_1)$, then $\deg(H_t)$ is constant and the result follows.

- Theorem (Brouwer fixed-point theorem): Let $B^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$ be the closed unit ball in $\mathbb{R}^n$. Let $F : B^n \to B^n$ be continuous. Then F has a fixed point.

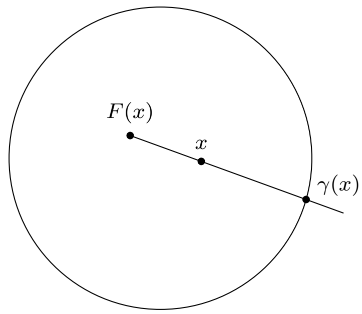

Figure 8.2: Defining γ for the Brouwer fixed point theorem.

  - Assume we have no fixed point and (trick!) consider the map $\gamma : B^n \to S^{n-1}$ which sends $x \mapsto$ the unique point on S^{n-1} that intersects the ray from $F(x)$ to x where $S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\}$.
  - Check:
    1. γ is continuous.
    2. For all $x \in S^{n-1}$, $\gamma(x) = x$.
  - Now we extend this to a map $\Gamma : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$x \mapsto \begin{cases} \gamma(x) & x \in B^n \\ x & |x| > 1 \end{cases}$$

  - Now for the contradiction. Notice that $\deg(\Gamma) = 1$. But Γ is not surjective (for example, $0 \notin \Gamma(\mathbb{R}^n)$). Recall that we proved earlier that degree nonzero functions are surjective.

- Manifolds (the rest of today and next time).
  - Definition of manifolds.
  - Definition of tangent spaces.
  - We want to be able to take a map $F : X \to Y$ and write $DF_p : T_pX \to T_{F(p)}Y$.

- Manifolds will have a dimension n (hence, we denote them X^n). We will now have them sit inside of some bigger thing, though, i.e., $X^n \subset \mathbb{R}^N$. For example, we'll have $S^2 \subset \mathbb{R}^3$ (the two-sphere lives most naturally in 3-space).
  - We'll also have functions $X^n \to Y^m$ where $X^n \subset \mathbb{R}^N$ and $Y^m \subset \mathbb{R}^M$.
  - We still have $\omega \in \Omega^k(X)$.

- **Smooth** (function $F : X \to Y$): A function $F : X \to Y$ where $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$ such that for all $p \in X$, there is some neighborhood $U_p \subset \mathbb{R}^N$ of p and a map $g_p : U_p \to \mathbb{R}^M$ that is smooth and agrees with F on $X \cap U_p$.

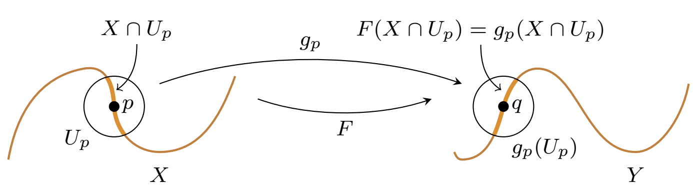

Figure 8.3: Smooth function of manifolds.

- **Diffeomorphism**: A function F that is bijective with F, F^{-1} smooth.
- **n-manifold**: A subset $X^n \subset \mathbb{R}^N$ (where $n \leq N$ are natural numbers) such that for all $p \in X$, there is a neighborhood V of p in $\mathbb{R}^N$, an open set $U \subset \mathbb{R}^n$, and a diffeomorphism $\varphi : U \to X \cap V$. *Also known as* **n-dimensional manifold**.
  - By convention, we indicate the dimension of our manifold with a superscript the first time we write it but not on subsequent writings. So X^n and X are the same thing here; we just write X^n on the first occurrence.
- **Chart**: The map φ in the above definition. *Also known as* **coordinate**, **parameterization**.
- Examples.
  1. $S^1 = \{x \in \mathbb{R}^2 \mid |x| = 1\}$.
     - For every point p on the unit circle, there is a neighborhood V such that $V \cap S^1$ maps bijectively onto $U \subset \mathbb{R}^1$ via some function φ.

8.2 Manifold Examples and Tangent Spaces

5/18:
- Plan.
  - Examples of manifolds.
  - Total derivative of smooth maps between manifolds.
    - So we'll have $F : X \to Y$, $p \in X$, $DF_p = \mathrm{d}F_p : T_pX \to T_{F(p)}Y$
    - If this is surjective, we get local properties of the map F.
    - Injective?
    - Bijective?
  - We'll then take $\mathrm{d}F_p$ and make $F^* : \Omega^k(X) \leftarrow \Omega^k(Y)$.
  - More on $\Lambda^k(T_p^*X)$ and $\omega \in \Omega^k(X)$.
- Examples of manifolds.
  1. $S^2 \subset \mathbb{R}^3$ is the two-sphere.
  2. $U \subset \mathbb{R}^n$ open.
     - The identity map $\mathrm{id} : U \to U$ is a parameterization.
  3. Given $f : \mathbb{R} \to \mathbb{R}$ smooth, its graph $\Gamma_f = \{(x, f(x)) \in \mathbb{R}^2\}$.
     - We can generalize this to graphs of functions $f : \mathbb{R}^k \to \mathbb{R}$ smooth, $\Gamma_f \subset \mathbb{R}^k$. This is a k-manifold.
  4. The torus or any other higher-genus surface in $\mathbb{R}^3$.
  5. $X_1 \subset \mathbb{R}^{N_1}$ and $X_2 \subset \mathbb{R}^{N_1}$ manifolds imply that $X_1 \times X_2 \subset \mathbb{R}^{N_1+N_2}$ is a manifold. Products of parameterizations.
     - Consider $S^1 \times S^1 \subset \mathbb{R}^4$.
     - The 2-torus T^2 is also $S^1 \times S^1$.
     - All such sets are diffeomorphic.
  6. More product manifolds.
     - $S^2 \times S^1$ (concentric spheres with the innermost glued to the outermost through the fourth dimension).
     - $S^1 \times S^1 \times S^1 = T^2 \times S^1$ where T^2 is the 2-torus.
     - Klug discusses unknotting the trefoil knot in $S^2 \times S^1$!

- Note that according to our definition of n-manifolds as *subsets* of N-space, a subset $X^n \subset \mathbb{R}^n$ of Euclidean space is *not* a manifold, even if it may be isomorphic to a manifold.
  - For example, the 2-torus $T^2 \subset \mathbb{R}^3$ is isomorphic to the unit square $[0,1] \times [0,1] \subset \mathbb{R}^2$, but we would not call the latter a manifold. To see the isomorphism, think about cutting a torus once meridionally to create a cylinder and then again longitudinally to create a plane; this plane can then be stretched or squeezed as necessary to fit atop $[0,1]^2$.
- **Cross product** (of X_1, X_2): The Cartesian product of X_1, X_2 as sets.
- We can glue together two genus 2 surfaces with an isomorphism $\varphi : S \to S$ (there are many).
  - In other words, all genus 2 surfaces are isomorphic. Thus, we can divide 2-manifolds into isomorphism classes based on their genus, which also serves as a kind of "manifold invariant."
- Manifolds as solutions to equations.
- **Submersion** (at p): A smooth function $F : U \to \mathbb{R}^k$ such that DF_p is surjective, where $U \subset \mathbb{R}^N$ is open, $N \geq k$, and $p \in \mathbb{R}^N$.
- **Regular value**: A point $q \in \mathbb{R}^k$ such that for all $p \in F^{-1}(q)$, F is a submersion at p (F being defined as above).
- Theorem: If $F : U \to \mathbb{R}^k$ smooth where $U \subset \mathbb{R}^N$ is open and $q \in \mathbb{R}^k$ is a regular value, then $F^{-1}(q) \subset U$ is an $(N-k)$-manifold.
- Example:
  1. Let $F : \mathbb{R}^3 \to \mathbb{R}$ be defined by $(x, y, z) \mapsto x^2 + y^2 + z^2 - 1$.
     - F is smooth and $\mathbb{R}^3 \subset \mathbb{R}^3$ is open.
     - 0 is a regular value.
       - Proof: Suppose (contradiction) that 0 is not a regular value. Then there exists $p \in F^{-1}(0)$ such that F is not a submersion at p. If F is not a submersion at p, then $DF_p : \mathbb{R}^3 \to \mathbb{R}$ is not surjective. Since DF_p is linear, this must mean that DF_p is the zero map. Thus, since
       $$DF_{(x,y,z)} = \begin{bmatrix} 2x & 2y & 2z \end{bmatrix}$$
       we must have that $p = (0,0,0)$. But $F(0,0,0) = -1 \neq 0$, a contradiction.
     - Thus, by the theorem, $S^2 = F^{-1}(q) \subset \mathbb{R}^3$ is a $(3-1)$-manifold or 2-manifold.
     - This theorem therefore provides a nice way of proving that a manifold is a manifold without having to find a chart for each point, as we would need to using the definition of an n-manifold alone to determine whether or not an object is a manifold.
  2. Consider $O(n)$, the set of orthogonal square $n \times n$ matrices. We have that $O(n) \subset \mathbb{R}^{n^2}$, where the latter set is the set of all $n \times n$ matrices.
     - We can find a suitable function F and check regular values so that $O(n) = F^{-1}(0)$ where $0 \in \mathbb{R}^n$.
     - Something about the dimension?
     - Is it the determinant?
- Tangent spaces and derivatives.
- Goal:
  - Define T_pX given $X^n \subset \mathbb{R}^N$.
  - Define the induced derivative $\mathrm{d}F_p : T_pX \to T_{F(p)}Y$ where $F : X^n \to Y^m$ for $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$.

- **Tangent space** (to X at p): If $\varphi : U \to X \cap V$ is a parameterization of X at p and sends $p_0 \mapsto p$, then we have a map $\mathrm{d}\varphi_{p_0} : T_{p_0}\mathbb{R}^n \to T_p\mathbb{R}^N$ where we define $T_pX = \operatorname{im}(\mathrm{d}\varphi_{p_0})$.
  - Essentially, the tangent space to an n-manifold should be n-dimensional, but it should be tilted, rotated, placed, etc. such that it is *tangent* to the n-manifold at the point in question.
- Example: Tangent space to S^2.

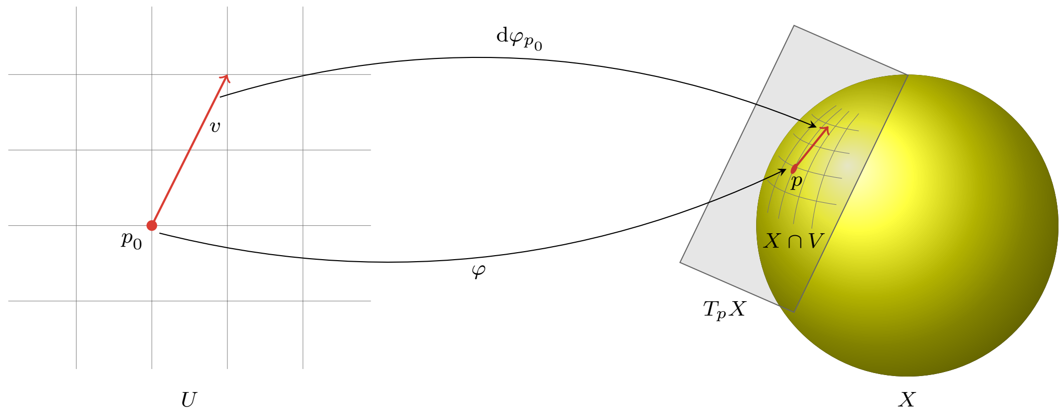

Figure 8.4: Tangent space to a manifold.

  - Consider the two-sphere $S^2 \subset \mathbb{R}^3$. We will call this manifold X.
  - As a 2-manifold, X is locally diffeomorphic to $\mathbb{R}^2$. In this specific instance, we focus in on the point $p \in X$. X is surrounded by some neighborhood $V \subset \mathbb{R}^3$ (not shown) such that $X \cap V$ may be depicted by the area on the surface of X covered in grid lines. The diffeomorphism φ maps $U \subset \mathbb{R}^2$ to $X \cap V \subset \mathbb{R}^3$ and, in particular, maps $p_0 \mapsto p$.
  - As before, we may easily define $T_{p_0}\mathbb{R}^2$ and $T_p\mathbb{R}^3$. However, neither of these spaces particularly well describe describe the set T_pX of tangent vectors to p. We may notice that T_pX is of the same dimension as $T_{p_0}\mathbb{R}^2$, and that T_pX is a subset of $T_p\mathbb{R}^3$. In fact, this is the key: We can use $\mathrm{d}\varphi_{p_0}$ to map $T_p\mathbb{R}^2$ into $T_p\mathbb{R}^3$, and the set of all vectors in the range is equal to T_pX; in particular, $T_pX = \operatorname{im}(\mathrm{d}\varphi_{p_0})$.
  - Lastly, we give a specific example of several of the objects in this picture.
    - Let $p = (2, 7\pi/6, \pi/3)$ in spherical coordinates (note that this implies that S^2 has radius 2), and $p_0 = (0,0)$ in Cartesian coordinates. Also let $U = (-2,3)^2$.
    - Map one unit in $\mathbb{R}^2$ to $\pi/18$ radians (10°) of longitude or lattitude across X. Then $\varphi : U \to X \cap V$ is given by
$$\varphi(x,y) = \left(2, \tfrac{7\pi}{6} + \tfrac{\pi}{18}x, \tfrac{\pi}{3} - \tfrac{\pi}{18}y\right)$$
    - If we convert from spherical to Cartesian coordinates, then
$$\varphi(x,y) = \begin{bmatrix} 2\sin\left(\frac{\pi}{3} - \frac{\pi}{18}y\right)\cos\left(\frac{7\pi}{6} + \frac{\pi}{18}x\right) \\ 2\sin\left(\frac{\pi}{3} - \frac{\pi}{18}y\right)\sin\left(\frac{7\pi}{6} + \frac{\pi}{18}x\right) \\ 2\cos\left(\frac{\pi}{3} - \frac{\pi}{18}y\right) \end{bmatrix}$$
    - It follows that
$$D\varphi(x,y) = \begin{bmatrix} -2\sin\left(\frac{\pi}{3} - \frac{\pi}{18}y\right)\sin\left(\frac{7\pi}{6} + \frac{\pi}{18}x\right)\cdot\frac{\pi}{18} & 2\cos\left(\frac{\pi}{3} - \frac{\pi}{18}y\right)\cdot -\frac{\pi}{18}\cos\left(\frac{7\pi}{6} + \frac{\pi}{18}x\right) \\ 2\sin\left(\frac{\pi}{3} - \frac{\pi}{18}y\right)\cos\left(\frac{7\pi}{6} + \frac{\pi}{18}x\right)\cdot\frac{\pi}{18} & 2\cos\left(\frac{\pi}{3} - \frac{\pi}{18}y\right)\cdot -\frac{\pi}{18}\sin\left(\frac{7\pi}{6} + \frac{\pi}{18}x\right) \\ 0 & -2\sin\left(\frac{\pi}{3} - \frac{\pi}{18}y\right)\cdot -\frac{\pi}{18} \end{bmatrix}$$
    - In particular, we have
$$\varphi(0,0) = \begin{bmatrix} -1.5 \\ -\sqrt{3}/2 \\ 1 \end{bmatrix} \qquad D\varphi(0,0) \approx \begin{bmatrix} 0.151 & 0.151 \\ -0.262 & 0.087 \\ 0 & 0.302 \end{bmatrix}$$

- Thus,

$$p = \begin{bmatrix} -1.5 \\ -\sqrt{3}/2 \\ 1 \end{bmatrix} \qquad\qquad T_pX \approx \operatorname{span}\left\{ \begin{bmatrix} 0.151 \\ -0.262 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.151 \\ 0.087 \\ 0.302 \end{bmatrix} \right\}$$

- One last note with respect to the drawing of Figure 8.4.
  - All elements of the right side of the diagram are a legitimate orthogonal projection of the elements described above.
  - To begin, we are viewing the sphere such that if the equator were to be drawn, it would have x-radius equal to 2 cm and y-radius equal to 0.5 cm. In other words, we are viewing the sphere from an angle of $\arcsin(0.5/2) = \arcsin(1/4) \approx 14.48^\circ = \theta$ above the equatorial plane.
  - We define the 3-space axes as follows: The x-axis points 1 unit toward the right of the page, the z-axis points 1 unit toward the top of the sphere, and the y-axis is the cross product $z \times x$ of these (pointing into the page). We define the axes in the plane of the page as follows: The x axis points 1 cm toward the right of the page and the y-axis points 1 cm toward the top of the page.
  - With these axis definitions, trigonometric arguments show that the projection operator P should map

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ \sin\theta \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ \cos\theta \end{bmatrix}$$

    It follows that

$$\mathcal{M}(P) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.25 & 0.968 \end{bmatrix}$$

  - Thus, to display p, the tangent vector, and the tangent plane, we need only feed p, $D\varphi(p_0)(v)$, and the basis of T_pX, respectively, into P.
  - The longitude and latitude lines are a bit trickier, since these are *functions* that need to be passed through P.
  - We'll start with longitude. Again using trigonometric arguments, we can determine that if α is the angle between the x-axis and the vertical plane containing a longitude line, then the path of the longitude line across the surface of the sphere as a function of $h = z/2$ is given by

$$h \mapsto \begin{bmatrix} -\cos\alpha\sqrt{1-h^2} \\ -\sin\alpha\sqrt{1-h^2} \\ h \end{bmatrix}$$

  - Latitude can be done similarly, but it's easier to take the equator ellipse (with half-axes 2 and 0.5) and move, scale, and shrink it upwards.
  - See handwritten pages for more info.

- An alternate definition of T_pX, assuming $X = f^{-1}(0)$ where $f : \mathbb{R}^N \to \mathbb{R}^k$ is a C^∞ map: The kernel of the surjective map $\mathrm{d}f_p : T_p\mathbb{R}^N \to T_0\mathbb{R}^k$.
  - In terms of Figure 8.4, we can use $f : \mathbb{R}^3 \to \mathbb{R}$ defined by $x \mapsto x_1^2 + x_2^2 + x_3^2 - 1$. Then since f only has nonzero change in directions normal to $X = S^2$, every $(p, v) \in T_p\mathbb{R}^3$ with v having no component normal to X at p will be mapped to zero by $\mathrm{d}f_p$. And these are exactly the tangent vectors.

- Guaranteeing that T_pX does not depend on φ.
  - If $\mathrm{d}\varphi_{p_0}$ is injective, then $\dim T_pX = n$.
  - No, it does not depend on the choice of parameterization; yes, $\mathrm{d}\varphi_{p_0}$ is injective.

  - We check both of these assertions by stating and proving that all manifolds are locally given as solutions to equations.
  - We then use this to get a clearly well-defined definition of T_pX — this check agrees with our definition for all φ.
- Given $F : X \to Y$ smooth and $p \in X$, we define $\mathrm{d}F_p : T_pX \to T_{F(p)}Y$ as follows: There is some neighborhood U of $p \in \mathbb{R}^N$ and a smooth function $\tilde{F} : U \to \mathbb{R}^M$ that agrees with F on $X \cap U$. This implies $\mathrm{d}\tilde{F}_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^M$. We have $T_pX \subset T_p\mathbb{R}^N$ and $T_{F(p)}Y \subset T_{F(p)}\mathbb{R}^M$. We now define $\mathrm{d}F_p$ as the restriction of $\mathrm{d}\tilde{F}_p$ to T_pX.
- Check.
  1. Image of $\mathrm{d}F_p$ is indeed inside $T_{F(p)}Y$.
  2. Does not depend on $\tilde{F}$ or U.

8.3 Objects on Manifolds

5/20:
- Today:
  - Bring all of our favorite gadgets to manifold land.
  - Tangent spaces (done).
  - (Total) derivatives $\mathrm{d}F_p : T_pX \to T_{F(p)}Y$.
  - Vector fields, integral curves, and flows.
  - Differential forms $\omega \in \Omega^k(X)$.
  - Differential forms package d, $\wedge$, L_v, maps $\cdots \xrightarrow{d} \Omega^k(X) \xrightarrow{d} \Omega^{k+1}(X) \xrightarrow{d} \cdots$, pullbacks.
- Next time:
  - $\int \omega$ (integration of forms on manifolds).
  - Relationship between $\int$, d. This implies Stokes' Theorem.
- Let $X^n \subset \mathbb{R}^N$ be a manifold[1].
- **Vector field** (on X): A function that to every $p \in X$ assigns some tangent vector $v_p \in T_pX$. *Denoted by* $\boldsymbol{v}$.
- Examples.
  - Consider a circular vector field on the unit circle $S^1 \subset \mathbb{R}^2$.
    - The integral curves follow the vector field around S^1.
    - If we take the vectors to be longer, then we'll go around faster.
  - Consider a meridional vector field on a torus $T \subset \mathbb{R}^3$.
    - The integral curves go around the donut.
  - Consider a vector field on the unit sphere $S^2 \subset \mathbb{R}^3$ that creeps out from the south pole, reaching unit length by the equator and then getting shorter and shorter towards the north pole.
    - The integral curves are the half lines of longitudes.
- Theorem (Hairy ball theorem): There is no smooth nonvanishing vector field on S^2.
  - "No matter how you comb the hair on a ball, you get some cowlicks somewhere."

[1] Thinking of our manifold as a subset of Euclidean space is a crutch. We should think of ourselves standing on the isolated manifold, of existing in its space. We can think of having our little Euclidean charts to navigate in the viscinity of every point, but we are in/on the manifold.

- **Integral curve** (of $\boldsymbol{v}$ on X): A map $\gamma : (a,b) \to X$ for which $\gamma'(t_0) = \boldsymbol{v}_{\gamma(t_0)}$.
- ***k*-form** (on X): A function that to each $p \in X$ assigns some $\omega_p \in \Lambda^k(T_p^*X)$.
- Remember that $T_pX \subset T_p\mathbb{R}^N$, but $\Lambda^k(T_p^*X) \not\subset \Lambda^k(T_p^*\mathbb{R}^N)$.
- **Support** (of $\omega \in \Omega^k(X)$): Defined similarly to before.
- $\boldsymbol{\Omega_c^k(X)}$: The set of all compactly supported k-forms.
- If X is compact, then $\Omega^k(X) = \Omega_c^k(X)$.
- An example noncompact manifold.

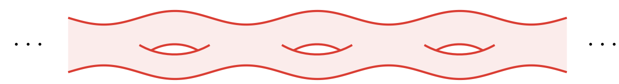

Figure 8.5: Noncompact manifold.

- **Pullback** (of ω to X): The k-form on X which sends each $p \in X$ to $\mathrm{d}F_p^*\omega_q$, where $X^n \subset \mathbb{R}^N$, $Y^m \subset \mathbb{R}^M$, $F : X \to Y$, $F(p) = q$, and $\omega \in \Omega^k(Y)$. *Denoted by* $\boldsymbol{F^*\omega}$.
  - Since F is linear, we get a bunch (one for each $p \in X$) of maps $\mathrm{d}F_p : T_pX \to T_{F(p)}Y$. But as a linear map, we can define its pullback $\mathrm{d}F_p^* = (\mathrm{d}F_p)^* : \Lambda^k(T_p^*X) \leftarrow \Lambda^k(T_{F(p)}^*Y)$. This can then transform the vectors fed into ω.
  - This definition is entirely analogous to the definition of the pullback of forms on vector spaces.
- **Smooth** (k-form on X): A k-form ω such that for all $p \in X$, there exists some open neighborhood U of p in X and a parameterization $\varphi : U_0 \to U$, where $U_0 \subset \mathbb{R}^n$, such that $\varphi^*\omega|_U$ is a smooth form on U_0.
- From now on, $\Omega^k(X)$ is the set of *smooth* k-forms on X.
- Examples of maps between manifolds.
  1. Consider S^1 as a subset of the complex plane $\mathbb{C}$ instead of $\mathbb{R}^2$. Then the map $F : S^1 \to S^1$ which sends each $z \in S^1$ to $z^2 \in S^1$ rotates all points of S^1 about the origin to the point that is twice as far from the $+x$-axis.
  2. Map $\mathbb{C} \cup \{\infty\}$ to S^2 such that 0 is the south pole, the equator is the unit circle, and the north pole is ∞. Then $F : S^2 \to S^2$ defined by $z \mapsto z^2$ is again a curious type of rotation map.
  3. Consider the 2-torus $T^2 \subset \mathbb{R}^3$. A map from T^2 to the manifold $(a,b) \subset \mathbb{R}$ could be the height map of the torus.
     - Preimages of points in (a,b) are circular submanifolds.
     - Dots are critical values.
- Determining when a vector field $\boldsymbol{v}$ on X is smooth.
  - Way 1: If for all $p \in X$, there exists V open in $\mathbb{R}^N$ and a smooth vector field $\tilde{\boldsymbol{v}}$ on V that agrees with $\boldsymbol{v}$ on $V \cap X$.
  - Way 2: As with forms, pullbacks and check on the charts (φ is a diffeomorphism).
- **Exterior derivative** (for k-forms on manifolds): The function from $\Omega^k(X) \to \Omega^{k+1}(X)$ defined as follows, where X is an n-manifold, $p \in X$, $p \in U \subset X$, $U_0 \subset \mathbb{R}^n$, and $\varphi : U_0 \to U$ is a diffeomorphism. *Denoted by* $\mathbf{d}$. *Given by*
$$(\mathrm{d}\omega)_p = [(\varphi^{-1})^*\,\mathrm{d}(\varphi^*\omega)]_p$$
  - Check: Well-defined, i.e., does not depend on the choice of φ.
  - All the familiar properties carry over.
$$\mathrm{d} \circ F^* = F^* \circ \mathrm{d} \qquad \mathrm{d}(\omega \wedge \eta) = \cdots \qquad \mathrm{d}^2 = 0 \qquad \mathrm{d} \text{ is linear}$$

8.4 Klug Meeting

- What are alternating tensors? Sure, I can define them. I also found the alternate definition of them as "two elements in the argument are the same implies $T(v_1, \dots, v_k) = 0$." But I still have no concept of what they "look like intuitively," what to make of their basis (the alternatizations of the strictly increasing dual basis vectors), or why their dimension should transform as $\binom{n}{k}$.
  - They're just an algebraic thing you need to make integration make sense.
  - We're gonna want to integrate things that are oriented, and when we change the orientation, we're gonna flip the sign. So alternating tensors capture how things change when you flip the signs.
  - We'll probably see this next week.
  - ω_p is an alternating tensor if ω is a form.
  - Covectors are 1-tensors, which makes them alternating automatically. But we don't have to worry about this with covectors because there's only one entry point.
  - Two forms use alternating 2-tensors.
  - Top dimensional forms.
  - Two forms are functions decorated by $\mathrm{d}x \wedge \mathrm{d}y$; you integrate them via the 2D integral.

$$\int_U \omega = \int_U f \,\mathrm{d}x \wedge \mathrm{d}y = \iint f \,\mathrm{d}x \,\mathrm{d}y$$

  - Don't try to figure out every little piece; just sit back and watch the theory unfold and then it will make more sense on subsequent viewings.
  - Alt and π are two isomorphisms between $\Lambda^k(V^*)$ and $\mathcal{A}^k(V)$. The alternating guys are more natural to think about; the quotient is more weird. The advantage of $\Lambda^k(V^*)$ is it makes defining $\wedge$ simpler.
  - Rep theory and algebra will introduce this stuff again in a different context and it will make more sense then.
  - We do have to deal with the nitty gritty on the homework still however. Making us suffer in a hopefully productive way. Choose things that come naturally to you, though. You'll come back later, you'll be better at learning (the ocean will rise), and it will often make so much more sense then.
- How does the idea of "it suffices to check this for decomposable tensors" typically work? It seems to often appear in cases where linearity is a factor and we can decompose an arbitrary tensor into a linear combination of the basis, which is of course composed of decomposable tensors.
- What is functoriality?
  - A fancy word people use to obfuscate things.
  - If $X \xrightarrow{F} Y \xrightarrow{Z}$, then $(G \circ F)^* = F^* \circ G^*$.
  - Just something that happens really often in math.
  - Category theory is just a language for talking about certain phenomena that arise so often that you'd want to have a language, but it's just grammar. You would never actually use it.
- What is $\Lambda^k(V^*)$, and why is it the k^{th} exterior power of V^*, and what does that even mean? The elements of it are $\mathcal{I}^k(V)$-cosets of tensors; what does one of these look like? The elements of it aren't even functions, right? They're just sets of functions?
- What is the wedge product intuitively?
- How does the tensor product we learned relate to the tensor product of two vectors and the tensor product of two vector spaces? And what are these latter quantities?

- What properties intuitively characterize decomposable tensors?
- What properties intuitively characterize redundant tensors?
- What is the interior product?
- What is the pullback?
- How did we define the determinant in terms of exterior powers?
- What are 1-forms?
- How did all that stuff we did with tensors relate to forms? Is $\mathrm{d}f$ a 2-tensor $\mathrm{d}f : U \times \mathbb{R}^n \to T^*\mathbb{R}^n$?
- On the integral: Doesn't the definition imply that the integral of $\partial/\partial x$ where $U = \mathbb{R}^2$ is the constant plane instead of the sloped plane? If we need the integral curves along it to be constant?
- I've been thinking of one-forms as mathematical objects which assign to every point p of a vector space a bundle of vectors. What are k-forms?
- What is exterior differentiation?
- PSet 2, 2.1.iii.
- Thoughts on the degree?
- How much multivariable calculus knowledge have you assumed for us? Do you believe there is value in knowing the more computational aspects of multi before looking into this?
  - Klug has never taken a course on this stuff.
  - You wouldn't need any duals if you just stuck to Euclidean space.
  - We're unifying vector calculus and multivariable calculus while generalizing to n-dimensions.
  - Instead of looking for motivation now, you kinda need to finish the whole textbook first and then reread it. At the end, you'll have theorems that make it worth it, and then you can reverse engineer.
  - John Lee trilogy of books on this math with an eye toward stuff that people care about. Point set topology. Introduction to smooth manifolds is book 2.
  - Nobody cares about point-set topology, but it's helpful for writing proofs and practicing logic.
  - We won't get to de Rahm cohomology in this course, but we should see it.
  - Klug read Lee in kind of an anxious haze believing it was gonna be important but it largely hasn't been. Any book is a linearization of an organic bloby process.
  - All the Lee books get used as the language of general relativity. If your Einstein trying to express your thoughts, you're happy to know the people who have been developing this differential forms language.
  - You want to get in the full mindset of "I could have discovered this." But it's very hard to reach that level. You can often use the stuff short of being there. Using it enough will get you to back into expert knowledge. Use it, and then backfill your knowledge.
- What do you want us to be getting out of this survey of the material?
- What do you want us to be getting out of the homework?
- How do you recommend we use the textbook? Where can we go for additional reference?
- How are we supposed to learn/motivate this stuff? Will we get to the motivation part in this course? Because I'd really learn this stuff better than just memorizing a bunch of definitions for the final, but I have basically no idea what the definitions mean.

- What resources do we have for help on homework problems we can't get?
- What will the final look like?
  - Probably just like the midterm, but he'll figure it out later.

8.5 Chapter 3: Integration on Forms

From Guillemin and Haine (2018).

5/28:
- **Critical point** (of f): A point $x \in U$ such that the derivative $Df(x) : \mathbb{R}^n \to \mathbb{R}^n$ fails to be bijective, i.e., $\det(Df(x)) = 0$.
- $\boldsymbol{C_f}$: The set of critical points of f.
  - Since $\det(Df) : U \to \mathbb{R}$ is continuous ($f \in C^\infty$ by hypothesis must be *continuously* differentiable) and $\{0\}$ is closed, $C_f = \det(Df)^{-1}(\{0\})$ is closed.
  - Consequently, $f(C_f)$ is a closed subset of V.
- **Critical value** (of f): The image of a critical point under f, i.e., an element of $f(C_f)$.
- **Regular value** (of f): An element of the range of f that is not a critical value, i.e., an element of $f(U) \setminus f(C_f)$.
  - Since $V \setminus f(U) \subset f(U) \setminus f(C_f)$, if $q \in V$ is not in the image of f, it is a regular value of f by default. More precisely, since elements of $V \setminus f(U)$ do not contain values of U, let alone any critical points of f, in their preimage, $V \setminus f(U)$ cannot contain any critical values[2].
- Theorem 3.6.2 (Sard): If $U, V \subset \mathbb{R}^n$ open and $f : U \to V$ a proper C^∞ map, then the set of regular values of f is an open dense subset of V.

6/1:
- Theorem 3.6.3: If q is a regular value of f a proper function, the set $f^{-1}(q)$ is finite. Additionally, if we let $f^{-1}(q) = \{p_1, \dots, p_n\}$, then there exist connected open neighborhoods $U_i \subset U$ of all p_i and an open neighborhood $W \subset V$ of q such that...
  1. For $i \neq j$, the sets U_i, U_j are disjoint;
  2. $f^{-1}(W) = U_1 \cup \cdots \cup U_n$;
  3. f maps every U_i diffeomorphically onto W.

  Proof. Let $p \in f^{-1}(q)$. Then p is not a critical point of f, so the derivative $Df(p)$ of f at p is bijective. It follows by the inverse function theorem that there exists a neighborhood U_p of p that f maps diffemorphically onto a neighborhood V_q of q.

  Since we can pick such an open subset for all $p \in f^{-1}(q)$, we know that the set $\{U_p \mid p \in f^{-1}(q)\}$ is an open cover of $f^{-1}(q)$. Additionally, since f is proper and $\{q\}$ is compact, $f^{-1}(q)$ is compact. Thus, as an open cover of a compact set, $\{U_p \mid p \in f^{-1}(q)\}$ has a finite subcover (which we may call $\{U_{p_1}, \dots, U_{p_N}\}$).

  Now suppose for the sake of contradiction that p_i, p_j are both elements of U_{p_i}. Then since $f(p_i) = f(p_j) = q$, f does not map U_{p_i} bijectively onto V_{q_i}. Thus, f does not map U_{p_i} diffeomorphically onto U_{q_i}, a contradiction. Therefore, every U_{p_i} contains at most one element of $f^{-1}(q)$. In particular, since U_{p_i} contains p_i by definition, it must be that every p_i is the one point in U_i. (For example, we could not have $p_1 \in U_2$ and $p_2 \in U_1$ since $p_1 \in U_1$ and $p_2 \in U_2$ by definition.) It follows that there is a bijective correspondence between the $\{U_{p_i}\}$ and the $\{p_i\}$, so it must be that $f^{-1}(q) = \{p_1, \dots, p_N\}$ is a finite set.

[2]I get the gist of this statement, but it makes no sense. It is in Guillemin and Haine (2018), regardless, though.

We now make the $\{U_{p_i}\}$ disjoint, if they are not already. Suppose, for instance, $U_{p_i} \cap U_{p_j} \neq \emptyset$. Then since there are only finitely many p_i (i.e., p_i, p_j are not infinitely close together), we may simply shrink the neighborhoods as needed. One way to do this is to redefine $U_{p_i} = U_{p_i} \cap N_r(p_i)$ and likewise for p_j, where $r = d(p_i, p_j)/2$.

Finally, by Theorem 3.4.7, there exists a connected open neighborhood $W \subset V$ of q for which $f^{-1}(W) \subset U_{p_1} \cup \cdots \cup U_{p_N}$. We lastly define every $U_i = f^{-1}(W) \cap U_{p_i}$, and it will follow from the above that these U_i have all the desired properties. □

- Theorem 3.6.4: Let q be a regular value of f, and let $f^{-1}(q) = \{p_1, \dots, p_N\}$, as above. Define $\sigma : f^{-1}(q) \to \{\pm 1\}$ by

$$\sigma_{p_i} = \begin{cases} +1 & f : U_i \to W \text{ is orientation preserving} \\ -1 & f : U_i \to W \text{ is orientation reversing} \end{cases}$$

Then

$$\deg(f) = \sum_{i=1}^{N} \sigma_{p_i}$$

Proof. Let $\omega \in \Omega_c^n(W)$ such that $\int_W \omega = 1$. Then

$$\deg(f) = \int_U f^*\omega = \sum_{i=1}^{N} \int_{U_i} f^*\omega$$

where by Theorem 3.5.1,

$$\int_{U_i} f^*\omega = \int_W \omega = \begin{cases} +1 & f \text{ is orientation preserving} \\ -1 & f \text{ is orientation reversing} \end{cases}$$

Thus, we have the desired result. □

- Theorem 3.6.6: If $f : U \to V$ is not surjective, then $\deg(f) = 0$.

Proof. We first present a hand-wavey proof based on Theorem 3.6.4. Choose $q \in V \setminus f(U)$. Then $f^{-1}(q) = \emptyset$. It follows that

$$\deg(f) = \sum_{i=1}^{0} \sigma_{p_i} = 0$$

For a more rigorous proof, consider the following. By Exercise 3.4.iii, $V \setminus f(U)$ is open. This combined with the fact that it is nonempty reveals that there exists a compactly supported n-form ω with support in $V \setminus f(U)$ and $\int_{V \setminus f(U)} \omega = 1$. Since $\omega(f(U)) = \{0\}$ as a compactly supported form on a set of points outside $f(U)$, $f^*\omega = 0$, so

$$0 = \int_U f^*\omega = \deg(f) \int_V \omega = \deg(f)$$

□

- Theorem 3.6.8: If $\deg(f) \neq 0$, then $f : U \twoheadrightarrow V$.

Proof. This is the contrapositive of Theorem 3.6.6. □

- Note that we will use Theorem 3.6.8 far more often than Theorem 3.6.6.

- **Proper homotopy**: A homotopy F between f_0, f_1 for which $F^\sharp : U \times A \to V \times A$ defined by
$$(x, t) \mapsto (F(x, t), t)$$
is proper.
  - If f_0, f_1 are properly homotopic, then f_t defined by $f_t(x) = F(x, t)$ is proper for all $t \in (0, 1)$.
- Theorem 3.6.10: If f_0, f_1 are properly homotopic, then $\deg(f_0) = \deg(f_1)$.
- Theorem 3.6.13 (The Brouwer fixed point theorem): Let $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ be the closed unit ball in $\mathbb{R}^n$. If $f : B^n \to B^n$ is continuous, then f has a fixed point, i.e., there exists $x_0 \in B^n$ for which
$$f(x_0) = x_0$$
- Guillemin and Haine (2018) also proves the fundamental theorem of algebra.
- Guillemin and Haine (2018) proves Sard's theorem.

8.6 Chapter 4: Manifolds and Forms on Manifolds

From Guillemin and Haine (2018).

6/2:

- In this section, we let $X \subset \mathbb{R}^N$, $Y \subset \mathbb{R}^n$, and $f : X \to Y$ continuous unless stated otherwise.
- ***C^∞* map**: A continuous map $f : X \to Y$, where $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^n$, such that for every $p \in X$, there exists a neighborhood $U_p \subset \mathbb{R}^N$ of p and a C^∞ map $g_p : U_p \to \mathbb{R}^n$ which coincides with f on $U_p \cap X$.
- Theorem 4.1.2: If $f : X \to Y$ is a C^∞ map, then there exists a neighborhood $U \subset \mathbb{R}^N$ of X and a C^∞ map $g : U \to \mathbb{R}^n$ such that g coincides with f on X.
- Intuitively, if Y is an open subset, the set X described by the above definitions is a **manifold**.
- ***n*-manifold**: A subset $X \subset \mathbb{R}^N$ such that for every $p \in X$, there exists a neighborhood $V \subset \mathbb{R}^N$ of p, an open subset $U \subset \mathbb{R}^n$, and a diffeomorphism $\phi : U \to X \cap V$, where $N, n \in \mathbb{N}_0$ satisfy $n \leq N$.
  - An alternate interpretation is that X is an n-manifold if, locally near every point p, X "looks like" an open subset of $\mathbb{R}^n$.
- Examples.
  1. Graphs of functions $f : U \to \mathbb{R}$.
  2. Graphs of mappings $f : U \to \mathbb{R}^k$.
  3. Vector subspaces (of $\mathbb{R}^n$ or any abstract vector space V).
  4. Affine subspaces of $\mathbb{R}^n$ (e.g., cosets; subsets of the form $p + V$ where $V \leq \mathbb{R}^n$).
  5. Product manifolds.
  6. The unit n-sphere.
  7. The 2-torus.
- Guillemin and Haine (2018) also gives diffeomorphisms for the above examples.
  - Two important diffeomorphism that arise.
  - One arises in conjunction with vector subspaces. In particular, we define $\phi : \mathbb{R}^n \to V$ by
$$(x_1, \dots, x_n) \mapsto \sum_{i=1}^n x_i e_i$$
where $\{e_i\}$ is a basis of V.

- One arises in conjunction with affine subspaces. In particular, we define $\tau_p : \mathbb{R}^N \to \mathbb{R}^N$, where $p \in \mathbb{R}^N$, by
$$x \mapsto p + x$$

- We now build up to regarding manifolds as the solutions to systems of equations.
- **Submersion** (at p): A C^∞ map $f : U \to \mathbb{R}^k$, where $U \subset \mathbb{R}^N$, for which $Df(p) : \mathbb{R}^N \to \mathbb{R}^k$ is surjective.
  - Note that for this linear map to be surjective, we must have $k \leq N$.
- **Regular value** (of f): A point $a \in \mathbb{R}^k$ such that for all $p \in f^{-1}(a)$, f is a submersion at p.
- **Canonical submersion**: The function defined as follows, which is a submersion at every point of its domain. *Denoted by* $\boldsymbol{\pi}$. *Given by*
$$\pi(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$$
- Theorem B.17 (canonical submersion theorem): Let $U \subset \mathbb{R}^n$ open and $\phi : (U, p) \to (\mathbb{R}^k, 0)$ a C^∞ map plus a submersion at p. Then there exists a neighborhood $V \subset U$ of p, a neighborhood $U_0 \subset \mathbb{R}^n$ of the origin, and a diffeomorphism $g : (U_0, 0) \to (V, p)$ such that $\phi \circ g : (U_0, 0) \to (\mathbb{R}^n, 0)$ is the restriction to U_0 of the canonical submersion.
- Theorem 4.1.7: Let $n = N - k$. If a is a regular value of $f : U \to \mathbb{R}^k$, then $X = f^{-1}(a)$ is an n-manifold.

  Proof. Instead of considering f, let's consider $\tau_{-a} \circ f$ so that $a = 0$ WLOG. Indeed, when we say "f" from now on, we mean "$\tau_{-a} \circ f$."

  To prove that $X = f^{-1}(0)$ is an n-manifold, it will suffice to show that for every $p \in X$, there exists a neighborhood $V \subset \mathbb{R}^N$ of p, an open subset $U \subset \mathbb{R}^n$, and a diffeomorphism $\phi : U \to X \cap V$. Let $p \in X$ be arbitrary. Then since 0 is a regular value of f, f is a submersion at p. Thus, by the canonical submersion theorem, there exists a neighborhood $O \subset \mathbb{R}^N$ of 0, a neighborhood $U_0 \subset U$ of p, and a diffeomorphism $g : O \to U_0$ such that $f \circ g = \pi$ where π is the canonical submersion. Since $\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^n$, it follows that $\pi^{-1}(0) = \{0\} \times \mathbb{R}^n \cong \mathbb{R}^n$ (where both zeros in the previous equation refer to the $0 \in \mathbb{R}^k$). Consequently, by the definition of the diffeomorphism of vector subspaces, g maps $O \cap \pi^{-1}(0)$ diffeomorphically onto $U_0 \cap f^{-1}(0)$. However, $O \cap \pi^{-1}(0) \subset \mathbb{R}^n$ is a neighborhood of 0 and $U_0 \cap f^{-1}(0) \subset X$ is a neighborhood of p and these two neighborhoods are diffeomorphic. $\square$

- Examples of manifolds as the solution to equations.
  1. $f : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $(x_1, \ldots, x_{n+1}) \mapsto x_1^2 + \cdots + x_{n+1}^2 - 1$ and the n-sphere as $f^{-1}(0)$.
  2. Graphs.
  3. The space of orthogonal matrices.
- Note that it is not random that these manifolds arise as zero sets of submersions. In fact, "we will show that locally *every* manifold arises this way" (Guillemin & Haine, 2018, p. 115).
- Guillemin and Haine (2018) goes about proving this fact.
- Theorem 4.1.15: Let X be an n-dimensional submanifold of $\mathbb{R}^N$ and let $\ell = N - n$. Then for every $p \in X$, there exists a neighborhood $V_p \subset \mathbb{R}^N$ of p and a submersion $f : (V_p, p) \to (\mathbb{R}^\ell, 0)$ such that $X \cap V_p$ is defined by ...
- We interpret Theorem 4.1.15 as saying that $f^{-1}(a)$ is the set of solutions to the system $f_i(x) = a_i$ $(i = 1, \ldots, k)$, and the regular value condition as guaranteeing that the system is an independent system of defining equations (e.g., no redundant information).
- **Parameterization** (of X at p): The function $\phi : U \to X \cap V$, where $U \subset \mathbb{R}^n$ is open, $V \subset \mathbb{R}^N$ is a neighborhood of p, and X is an n-manifold.

- **Tangent space** (to X at p): The image of the linear map $(\mathrm{d}\phi)_q : T_q\mathbb{R}^n \to T_p\mathbb{R}^N$, where ϕ is a parameterization of X at p and $\phi(q) = p$.
  - The examples given make it clear that if $X^n \subset \mathbb{R}^N$, T_pX is the subset of $T_p\mathbb{R}^N$ containing all vectors with tail at point p that are tangent to X.
- Guillemin and Haine (2018) goes through the nitty gritty details of defining the tangent space properly.
- **Vector field** (on X): A function which assigns to each $p \in X$ an element of T_pX. *Denoted by* $\boldsymbol{v}$.
- **k-form** (on X): A function which assigns to each $p \in X$ an element of $\Lambda^k(T_p^*X)$. *Denoted by* $\boldsymbol{\omega}$.
- **f-related** (vector fields $\boldsymbol{v}, \boldsymbol{w}$ on X): Two vector fields $\boldsymbol{v}, \boldsymbol{w}$ on X such that for all $p \in X$ and $q = f(p)$,

$$(\mathrm{d}f)_p\boldsymbol{v}(p) = \boldsymbol{w}(q)$$

  - See Figure 4.5 and the associated discussion.
- **Pushforward** (of $\boldsymbol{v}$ by f): The unique vector field $\boldsymbol{w}$ such that $\boldsymbol{v}, \boldsymbol{w}$ are f-related.
- **Pullback** (of $\boldsymbol{w}$ by f): The unique vector field $\boldsymbol{v}$ such that $\boldsymbol{v}, \boldsymbol{w}$ are f-related.
- Proposition 4.3.4: Defining the chain rule/functoriality for the pushforward and pullback.
- **Parameterizable open set**: An open subset U of X for which there exists a corresponding open set $U_0 \subset \mathbb{R}^n$ and diffeomorphism $\phi_0 : U_0 \to U$.
  - "Note that X being a manifold means that every point is contained in a parameterizable open set" (Guillemin & Haine, 2018, p. 126).
- **Smooth** (k-form on U): A k-form ω on $U \subset X$ for which there exists a parameterizable open set with parameterization ϕ_0 such that $\phi_0^*\omega$ is C^∞.
- Guillemin and Haine (2018) proves that this definition is independent of our choice of ϕ_0.
- **Smooth** (k-form on X): A k-form ω on X such that for every $p \in X$, ω is smooth on a neighborhood of p.
- Proposition 4.3.10: Let X, Y manifolds and $f : X \to Y$ a C^∞ map. If ω is a smooth k-form on Y, then the pullback $f^*\omega$ is a smooth k-form on X.
- Proposition 4.3.11: An analogous result for vector fields.
- **Unit vector** (in $T_{t_0}\mathbb{R}$): The vector $(t_0, 1) \in T_{t_0}\mathbb{R}$. *Denoted by* $\vec{\boldsymbol{u}}$.
  - This vector arises when we have an integral curve $\gamma : I \to X$, where $t_0 \in I \subset \mathbb{R}$, I being an open interval. Specifically, we will use it to define the tangent vector to γ at t_0, as follows.
- **Tangent vector** (to γ at p): The vector $\mathrm{d}\gamma_{t_0}(\vec{u}) \in T_pX$, where $p = \gamma(t_0)$.
- **Integral curve** (of $\boldsymbol{v}$): A curve $\gamma : I \to X$ such that for all $t_0 \in I$,

$$\boldsymbol{v}(\gamma(t_0)) = \mathrm{d}\gamma_{t_0}(\vec{u})$$

  where $\boldsymbol{v}$ is a vector field on X.
- Proposition 4.3.13: Integral curves get mapped from X to Y by f for f-related vector fields.
- Local existence, local uniqueness, and smooth dependence on initial data follow.
- More integral curves stuff.

- **Exterior derivative** (of ω on X): The k-form defined as follows, where ω is a smooth k-form on X, $U \subset X$ is a parameterizable open set, and $\phi_0 : U_0 \to U$ is a parameterization. *Denoted by* $\mathbf{d}\boldsymbol{\omega}$. *Given by*
$$\mathrm{d}\omega = (\phi_0^{-1})^*(\mathrm{d}(\phi_0^*(\omega)))$$
  - Essentially, what we're doing here is pulling back our k-form on X into $\mathbb{R}^n$, taking the exterior derivative there, and then pulling it back onto X.
- Theorem 4.3.22: If X, Y are manifolds and $f : X \to Y$ is smooth, then for $\omega \in \Omega^k(Y)$, we have
$$f^*(\mathrm{d}\omega) = \mathrm{d}(f^*\omega)$$
- Guillemin and Haine (2018) covers the interior product and Lie derivative in manifold-land.