Chapter 2

Differential Forms

2.1 Notes

4/18:
- Office Hours on Wednesday, 4:00-5:00 PM.
- Plan:
  - An impressionistic overview of what (differential) forms do/are.
  - Tangent spaces.
  - Vector fields/integral curves.
  - 1-forms; a warm-up to k-forms.
- Impressionistic overview of the rest of Guillemin and Haine (2018).
  - An open subset $U \subset \mathbb{R}^n$; $n = 2$ and $n = 3$ are nice.
  - Sometimes, we'll have some functions $F : U \to V$; this is where pullbacks come into play.
  - At every point $p \in U$, we'll define a vector space (the tangent space $T_p\mathbb{R}^n$). Associated to that vector space you get our whole slew of associated spaces (the dual space $T_p^*\mathbb{R}^n$, and all of the higher exterior powers $\Lambda^k(T_p^*\mathbb{R}^n)$).
  - We let $\omega \in \Omega^k(U)$ be a k-form in the space of k-forms.
  - ω assigns (smoothly) to every point $p \in U$ an element of $\Lambda^k(T_p^*\mathbb{R}^n)$.
  - Question: What really is a k-form?
    - Answer: Something that can be integrated on k-dimensional subsets.
    - If $k = 1$, i.e., $\omega \in \Omega^1(U)$, then U can be integrated over curves.
  - If we take $k = 0$, then $\Omega^0(U) = C^\infty(U)$, i.e., the set of all smooth functions $f : U \to \mathbb{R}$.
    - Guillemin and Haine (2018) doesn't, but Klug will and we should distinguish between functions $F : U \to V$ and $f : U \to \mathbb{R}$.
  - We will soon construct a map $d : \Omega^0(U) \to \Omega^1(U)$ (the **exterior derivative**) that is rather like the gradient but not quite.
    - d is linear.
    - Maps from vector spaces are heretofore assumed to be linear unless stated otherwise.
  - The 1-forms in $\text{im}(d)$ are special: $\int_\gamma \mathrm{d}f = f(\gamma(b)) - f(\gamma(a))$ only depends on the endpoints of $\gamma : [a, b] \to U$! The integral is *path-independent*.
  - A generalization of this fact is that instead of integrating along the surface M, we can integrate along the boundary curve:

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega$$

    This is **Stokes' theorem**.

  - M is a k-dimensional subset of $U \subset \mathbb{R}^n$.
- Note that we have all manner of functions d that we could differentiate between (because they are functions) but nobody does.

$$0 \to \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(U) \xrightarrow{d} 0$$

- Theorem: $d^2 = d \circ d = 0$.
  - Corollary: $\operatorname{im}(d^{n-1}) \subset \ker(d^n)$.
- We'll define $H^k_{dR}(U) = \ker(d)/\operatorname{im}(d)$.
  - These will be finite dimensional, even though all the individual vector spaces will be infinite dimensional.
  - These will tell us about the shape of U; basically, if all of these equal zero, U is simply connected. If some are nonzero, U has some holes.
- For small values of n and k, this d will have some nice geometric interpretations (div, grad, curl, n'at).
- We'll have additional operations on forms such as the wedge product.

- **Tangent space** (of p): The following set. *Denoted by* $\boldsymbol{T_p\mathbb{R}^n}$. *Given by*

$$T_p\mathbb{R}^n = \{(p, v) : v \in \mathbb{R}^n\}$$

  - This is naturally a vector space with addition and scalar multiplication defined as follows.

$$(p, v_1) + (p, v_2) = (p, v_1 + v_2) \qquad\qquad \lambda(p, v) = (p, \lambda v)$$

  - The point is that

$$T_p\mathbb{R}^n \neq T_q\mathbb{R}^n$$

    for $p \neq q$ even though the spaces are isomorphic.
  - Aside: $F : U \to V$ differentiable and $p \in U$ induce a map $\mathrm{d}F_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ called the "derivative at p."
    - We will see that the matrix of this map is the Jacobian.
  - Chain rule: If $U \xrightarrow{F} V \xrightarrow{G} W$, then

$$\mathrm{d}(G \circ F)_p = \mathrm{d}G_{F(p)} \circ \mathrm{d}F_p$$

- This is round 1 of our discussion on tangent spaces.
- Round 2, later on, will be submanifolds such as T_pM: The tangent space to a point p of a manifold M.
- **Vector field** (on U): A function that assigns to each $p \in U$ an element of $T_p\mathbb{R}^n$.
  - A constant vector field would be $p \mapsto (p, v)$, visualized as a field of vectors at every p all pointing the same direction. For example, we could take $v = (1, 1)$. *picture*
  - Special case: $v = e_1, e_2, \dots, e_n$. Here we use the notation $e_i = \mathrm{d}/\mathrm{d}x_i$.
  - Example: $n = 2$, $U = \mathbb{R}^2 \setminus \{(0, 0)\}$. We could take a vector field that spins us around in circles.
  - Notice that for all p, $\mathrm{d}/\mathrm{d}x_1 \mid_p, \dots, \mathrm{d}/\mathrm{d}x_n \mid_p \in T_p\mathbb{R}^n$ are a basis.
    - Thus, any vector field v on U can be written uniquely as

$$v = f_1 \frac{\mathrm{d}}{\mathrm{d}x_1} + \cdots + f_n \frac{\mathrm{d}}{\mathrm{d}x_n}$$

      where the $f_1, \dots, f_n$ are functions $f_i : U \to \mathbb{R}$.

4/20:

- Plan:
  - Vector fields and their integral curves.
  - Lie derivatives.
  - 1-forms and k-forms.
  - $\Omega^0(U) \xrightarrow{d} \Omega^1(U)$.
- Notation.
  - $U \subset \mathbb{R}^n$.
  - v denotes a vector field on U.
    - Note that the set of all vector fields on U constitute the vector space ??.
  - $v_p \in T_p\mathbb{R}^n$.
  - $\omega_p \in \Lambda^k(T_p^*\mathbb{R}^n)$.
  - $\mathrm{d}/\mathrm{d}x_i \mid_p = (p, e_i) \in T_p\mathbb{R}^n$.
- Recall that any vector field v on U can be written uniquely as

$$v = g_1 \frac{\mathrm{d}}{\mathrm{d}x_1} + \cdots + g_n \frac{\mathrm{d}}{\mathrm{d}x_n}$$

  where the $g_i : U \to \mathbb{R}$.
- **Smooth** (vector field): A vector field v for which all g_i are smooth.
- From now on, we assume unless stated otherwise that all vector fields are smooth.
- **Lie derivative** (of f wrt. v): The function $L_v f : U \to \mathbb{R}$ defined by $p \mapsto D_{v_p}(f)(p)$, where v is a vector field on U and $f : U \to \mathbb{R}$ (always smooth).
  - Recall that $D_{v_p}(f)(p)$ denotes the directional derivative of f in the direction v_p at p.
  - As some examples, we have

$$L_{\mathrm{d}/\mathrm{d}x_i} f = \frac{\mathrm{d}f}{\mathrm{d}x_i} \qquad\qquad L_{(g_1 \frac{\mathrm{d}}{\mathrm{d}x_1} + \cdots + g_n \frac{\mathrm{d}}{\mathrm{d}x_n})} f = g_1 \frac{\mathrm{d}f}{\mathrm{d}x_1} + \cdots + g_n \frac{\mathrm{d}f}{\mathrm{d}x_n}$$

- Property.
  1. Product rule: $L_v(f_1 f_2) = (L_v f_1) f_2 + f_1 (L_v f_2)$.
- Later: Geometric meaning to the expression $L_v f = 0$.
  - Satisfied iff f is constant on the integral curves of v. As if f "flows along" the vector field.
- We define $T_p^*\mathbb{R}^n = (T_p\mathbb{R}^n)^*$.
- 1-forms:
  - A (differential) 1-form on $U \subset \mathbb{R}^n$ is a function $\omega : p \mapsto \omega_p \in T_p^*\mathbb{R}^n$.
  - A "co-vector field"
- Notation: $\mathrm{d}x_i$ is the 1-form that at p is $(p, e_i^*) \in T_p^*\mathbb{R}^n$.
- For example, if $U = \mathbb{R}^2$ and $\omega = \mathrm{d}x_1$, then we have the vector field of "unit vectors pointing to the right at each point."

- Note: Given any 1-form ω on U, we can write ω uniquely as

$$\omega = g_1 \, \mathrm{d}x_1 + \cdots + g_n \, \mathrm{d}x_n$$

  for some set of smooth $g_i : U \to \mathbb{R}$.
- Notation:
  - $\Omega^1(U)$ is the set of all smooth 1-forms.
  - Notice that $\Omega^1(U)$ is a vector space.
- Given $\omega \in \Omega^1(U)$ and a vector field v on U, we can define $\omega(v) : U \to \mathbb{R}$ by $p \mapsto \omega_p(v_p)$.
- If $U = \mathbb{R}^2$, we have that

$$\mathrm{d}x\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) = 1 \qquad\qquad \mathrm{d}x\left(\frac{\mathrm{d}}{\mathrm{d}y}\right) = 0$$

- Note that $\mathrm{d}x \,, \mathrm{d}y$ are not a basis for $\Omega^1(U)$ since the latter is infinite dimensional.
- Exterior derivative for 0/1 forms.
  - Let $d : \Omega^0(U) \to \Omega^1(U)$ take $f : U \to \mathbb{R}$ to $\frac{\partial f}{\partial x_1} \, \mathrm{d}x_1 + \cdots + \frac{\partial f}{\partial x_n} \, \mathrm{d}x_n$.
  - This represents the gradient as a 1-form.
- Check:
  1. Linear.
  2. $\mathrm{d}x_i = \mathrm{d}(x_i)$, where $x_i : \mathbb{R}^n \to \mathbb{R}$ is the i^{th} coordinate function.