

MATH 20510 (Analysis in \mathbb{R}^n III – Accelerated) Notes

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Chapter 1

Multilinear Algebra

1.1 Notes

3/28: • Motivation for the course and an overview of Guillemin and Haine (2018).

3/30: • Plan:

– More (multi)linear algebra.

• Dual spaces.

• Let V be an n -dimensional real vector space.

• **Hom** (V, \mathbb{R}): The set of all homomorphisms (i.e., linear maps) from V to \mathbb{R} . *Also known as V^* .*

• **Dual basis** (for V^*): The set of linear transformations from V to \mathbb{R} defined by

$$e_j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

where e_1, \dots, e_n is a basis of V . *Denoted by e_1^*, \dots, e_n^* .*

• Check: e_1^*, \dots, e_n^* are a basis for V^* .

– Are they linearly independent? Let $c_1 e_1^* + \dots + c_n e_n^* = 0 \in \text{Hom}(V, \mathbb{R})$. Then

$$c_i = (c_1 e_1^* + \dots + c_n e_n^*)(e_i) = 0 \in \mathbb{R}$$

as desired.

– Span? Let $\varphi \in \text{Hom}(V, \mathbb{R})$. Then we can verify that

$$\varphi(e_1) e_1^* + \dots + \varphi(e_n) e_n^* = \varphi$$

■ We prove this by verifying the previous statement on the basis of V (if two linear transformations have the same action on the basis of a vector space, they are equal).

• With a choice of basis for V , we obtain an isomorphism $\varepsilon : V \rightarrow V^*$ with the mapping $e_i \mapsto e_i^*$ for all i .

• The dual space is known as such because $(V^*)^* \cong V$, where \cong is **canonical** (no choice of basis is needed).

• One more property of dual spaces: **functoriality**.

- Given a linear transformation $A : V \rightarrow W$, we know that $A^* : W^* \rightarrow V^*$ where A^* is the transpose of A . In particular, if $\varphi \in W^*$, then $\varphi \circ A : V \rightarrow \mathbb{R}$.
- Claim: A^* is linear.
- **Functoriality:** If $A : V \rightarrow W$ and $B : W \rightarrow U$, then $B^* : U^* \rightarrow W^*$ and $A^* : W^* \rightarrow V^*$. The functoriality statement is that $(B \circ A)^* = A^* \circ B^*$.
- A^* is the **pullback** (or transpose) of A .
- Let v_1, \dots, v_n be a basis for V and w_1, \dots, w_m be a basis for W . Then $[A]_{v_1, \dots, v_n}^{w_1, \dots, w_m} = A$ is the matrix of the linear transformation A with respect to these bases. Then if v_1^*, \dots, v_n^* and w_1^*, \dots, w_m^* are the corresponding dual bases, then $[A^*]_{v_1^*, \dots, v_n^*}^{w_1^*, \dots, w_m^*} = A^T$. We can and should verify this for ourselves.
- This is over the real numbers, so A^* is just the transpose because there are no complex numbers of which to take the conjugate!
- A generalization: Tensors.
- **k -tensor:** A **multilinear** map

$$T : \underbrace{V \times \dots \times V}_{k \text{ times}} \rightarrow \mathbb{R}$$

- **Multilinear** (map T): A function T such that

$$\begin{aligned} T(v_1, \dots, v_i^1 + v_i^2, \dots, v_k) &= T(v_1, \dots, v_i^1, \dots, v_k) + T(v_1, \dots, v_i^2, \dots, v_k) \\ T(v_1, \dots, \lambda v_i, \dots, v_k) &= \lambda T(v_1, \dots, v_i, \dots, v_k) \end{aligned}$$

for all $(v_1, \dots, v_k) \in V^k$.

- The determinant is an n -tensor!
- 1-tensors are just covectors.
- $L^k(V)$: The vector space of all k -tensors on V .
- Calculating $\dim L^k(V)$. (Answer not given in this class.)
- Let $A : V \rightarrow W$. Then $A^* : L^k(W) \rightarrow L^k(V)$.
 - Check $(A \circ B)^* = B^* \circ A^*$.
- **Multi-index of n of length k :** A k -tuple (i_1, \dots, i_k) where each $i_j \in \mathbb{N}$ satisfies $1 \leq i_j \leq n$ ($j = 1, \dots, k$). Denoted by \mathbf{I} .
- Let e_1, \dots, e_n be a basis for V .
- **Tensor product** (of $T_1 \in L^k(V)$, $T_2 \in L^l(V)$): The function from V^{k+l} to \mathbb{R} defined by

$$(v_1, \dots, v_{k+l}) \mapsto T_1(v_1, \dots, v_k) T_2(v_{k+1}, \dots, v_{k+l})$$

Denoted by $T_1 \otimes T_2$.

- Claims:
 1. $T_1 \otimes T_2 \in L^{k+l}(V)$.
 2. $A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$.
- $e_{\mathbf{I}}^*$: The function $e_{i_1}^* \otimes \dots \otimes e_{i_k}^*$, where $\mathbf{I} = (i_1, \dots, i_k)$ is a multi-index of n of length k .
- Claim: Letting \mathbf{I} range over all n^k multi-indices of n of length k , the $e_{\mathbf{I}}^*$ are a basis for $L^k(V)$.

- If $V = \mathbb{R}$, then $V = \mathbb{R}e_1$. If $V = \mathbb{R}^2$, then $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$.
- We know that $L^1(V) = V^* = \mathbb{R}e_1^*$. Thus, $e_1^* \otimes e_2^* : V \times V \rightarrow \mathbb{R}$. Thus, for example,

$$(e_1^* \otimes e_2^*)((1, 2), (3, 4)) = e_1^*(1, 2) \cdot e_2^*(3, 4) = 1 \cdot 4 = 4$$

4/1:

- Plan: More multilinear algebra.
 - Properties of the tensor product.
 - Sign of a permutation.
 - Alternating tensors (lead into differential forms down the road).
- Recall: V is an n -dimensional vector space over \mathbb{R} with basis e_1, \dots, e_n . $\mathcal{L}^k(V)$ is the vector space of k -tensors on V . $\{e_I^* \mid I \text{ a multiindex of } n \text{ of length } k\}$ is a basis for $\mathcal{L}^k(V)$.

- For example, if $V = \mathbb{R}^2$ and $T \in \mathcal{L}^2(V)$, then

$$T(a_1e_1 + a_2e_2, b_1e_1 + b_2e_2) = a_1b_1T(e_1, e_1) + a_1b_2T(e_1, e_2) + a_2b_1T(e_2, e_1) + a_2b_2T(e_2, e_2)$$

- A basis of $\mathcal{L}^2(V)$ is

$$\{e_1^* \otimes e_1^*, e_1^* \otimes e_2^*, e_2^* \otimes e_1^*, e_2^* \otimes e_2^*\}$$

- Recall that some basic properties are

$$e_1^* \otimes e_2^*((1, 2), (3, 4)) = 1 \cdot 4 = 4 \qquad e_2^* \otimes e_1^*((1, 2), (3, 4)) = 2 \cdot 3 = 6$$

- It follows by the initial decomposition of T that

$$T = a_1b_1e_1^* \otimes e_1^* + a_1b_2e_1^* \otimes e_2^* + a_2b_1e_2^* \otimes e_1^* + a_2b_2e_2^* \otimes e_2^*$$

- Important consequence: To know the action of T on an arbitrary pair of vectors, you need only know its action on the basis; a higher-dimensional generalization of the earlier property.
- Note that

$$e_I^*(e_J) = \delta_{IJ} = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$$

- Basic properties of the tensor product.

1. *Right-distributive*: If $T_1 \in \mathcal{L}^k(V)$ and $T_2, T_3 \in \mathcal{L}^\ell(V)$, then

$$T_1 \otimes (T_2 + T_3) = T_1 \otimes T_2 + T_1 \otimes T_3$$

2. *Left-distributive*: If $T_1, T_2 \in \mathcal{L}^k(V)$ and $T_3 \in \mathcal{L}^\ell(V)$, then

$$(T_1 + T_2) \otimes T_3 = T_1 \otimes T_3 + T_2 \otimes T_3$$

3. *Associative*: If $T_1 \in \mathcal{L}^k(V)$, $T_2 \in \mathcal{L}^\ell(V)$, and $T_3 \in \mathcal{L}^m(V)$, then

$$T_1 \otimes (T_2 \otimes T_3) = (T_1 \otimes T_2) \otimes T_3 = T_1 \otimes T_2 \otimes T_3$$

4. *Scalar multiplication*: If $T_1 \in \mathcal{L}^k(V)$, $T_2 \in \mathcal{L}^\ell(V)$, and $\lambda \in \mathbb{R}$, then

$$(\lambda T_1) \otimes T_2 = \lambda(T_1 \otimes T_2) = T_1 \otimes (\lambda T_2)$$

- Note that the tensor product is not commutative.
- Aside: Defining the sign of a permutation.

- S_A : The set of all automorphisms of A (bijections from A to A), where A is a set.
- S_n : The set $S_{[n]}$.
- Given $\sigma_1, \sigma_2 \in S_n$, $\sigma_1 \circ \sigma_2 \in S_n$.
 - Thus, S_n is a **group**.
- **Transposition**: A function $\tau \in S_n$ such that

$$\tau(k) = \begin{cases} j & k = i \\ i & k = j \\ k & k \neq i, j \end{cases}$$

for some $i, j \in [n]$. Denoted by $\tau_{i,j}$.

- Theorem: An element of S_n can be written as the product of transpositions (i.e., for all $\sigma \in S_n$, there exist $\tau_1, \dots, \tau_m \in S_n$ such that $\sigma = \tau_1 \circ \dots \circ \tau_m$).
- **Sign** (of $\sigma \in S_n$): The number (mod 2) of transpositions whose product equals σ . Denoted by $(-1)^\sigma$, $\text{sign}(\sigma)$.
- Theorem: The sign of σ is well-defined. Additionally,

$$(-1)^{\sigma_1 \sigma_2} = (-1)^{\sigma_1} \cdot (-1)^{\sigma_2}$$

- Example: Consider the identity permutation. $(-1)^\sigma = +1$. We can think of this as the product of zero transpositions or, for instance, as the product of the two transpositions $\tau_{1,2} \circ \tau_{1,2}$. Another example would be $\tau_{2,3} \circ \tau_{1,2} \circ \tau_{1,2} \circ \tau_{2,3}$.
- Theorem: Let X_i be a rational or polynomial function for each $i \in \mathbb{N}$. Then

$$(-1)^\sigma = \prod_{i < j} \frac{X_{\sigma(i)} - X_{\sigma(j)}}{X_i - X_j}$$

- Example: For the permutation $\sigma = (1, 2, 3)$, we have

$$\begin{aligned} (-1)^\sigma &= \frac{X_{\sigma(1)} - X_{\sigma(2)}}{X_1 - X_2} \cdot \frac{X_{\sigma(1)} - X_{\sigma(3)}}{X_1 - X_3} \cdot \frac{X_{\sigma(2)} - X_{\sigma(3)}}{X_2 - X_3} \\ &= \frac{X_2 - X_3}{X_1 - X_2} \cdot \frac{X_2 - X_1}{X_1 - X_3} \cdot \frac{X_3 - X_1}{X_2 - X_3} \\ &= \frac{-(X_1 - X_2)}{X_1 - X_2} \cdot \frac{-(X_1 - X_3)}{X_1 - X_3} \cdot \frac{X_2 - X_3}{X_2 - X_3} \\ &= -1 \cdot -1 \cdot 1 \\ &= +1 \end{aligned}$$

which squares with the fact that $\sigma = \tau_{1,2} \circ \tau_{2,3}$.

- Claims to verify with the above formula:
 1. $\text{sign}(\sigma) \in \{\pm 1\}$.
 2. $\text{sign}(\tau_{i,j}) = -1$.
 3. $\text{sign}(\sigma_1 \sigma_2) = \text{sign}(\sigma_1) \text{sign}(\sigma_2)$.

4/4:

- Plan:
 - More multilinear algebra.

- Alternating k -tensors — 2 views:
 1. As a subspace of $\mathcal{L}^k(V)$.
 2. As a quotient of $\mathcal{L}^k(V)$.
- Next time: Operators as alternating tensors.
 - Wedge products.
 - Interior products.
 - Pullbacks.
- Recall: $\dim V = n$, e_1, \dots, e_n a basis, $\mathcal{L}^k(V)$ the space of k -tensors, $\sigma \in S_k$ implies $(-1)^\sigma \in \{\pm 1\}$, key property: $(-1)^{\sigma_1 \sigma_2} = (-1)^{\sigma_1} (-1)^{\sigma_2}$.
- T^σ : The k -tensor over V defined by

$$T^\sigma(v_1, \dots, v_k) = T(v_{\bar{\sigma}(1)}, \dots, v_{\bar{\sigma}(k)})$$

where $T \in \mathcal{L}^k(V)$, $\sigma \in S_k$, and $\bar{\sigma}$ denotes the inverse of σ .

- Example: $n = 2$, $k = 2$. Let $T = e_1^* \otimes e_2^* \in \mathcal{L}^2(V)$. Let $\sigma = \tau_{1,2}$. Then $T^\sigma = e_2^* \otimes e_1^*$.
- Another property is $e_I^\sigma = e_{\sigma(I)}^*$.
- Properties:
 1. $T^{\sigma_1 \sigma_2} = (T^{\sigma_1})^{\sigma_2}$.
 2. $(T_1 + T_2)^\sigma = T_1^\sigma + T_2^\sigma$.
 3. $(cT)^\sigma = cT^\sigma$.
- Thus, you can view $\sigma : \mathcal{L}^k(V) \rightarrow \mathcal{L}^k(V)$ as a linear map!
- **Alternating k -tensor**: A tensor $T \in \mathcal{L}^k(V)$ such that $T^\sigma = (-1)^\sigma T$ for all $\sigma \in S_k$.
 - Equivalently, $T^\tau = -T$ for all $\tau \in S_k$.
- An example of an alternating 2-tensor when $\dim V = 2$ is $T = e_1^* \otimes e_2^* - e_2^* \otimes e_1^*$.
 - Naturally, $T_{1,2}^\tau = -T$, and $\tau_{1,2}$ is the unique transposition in S_2 .
- $e_1^* \otimes e_2^*$ is *not* an alternating 2-tensor since $(e_1^* \otimes e_2^*)^\tau = e_2^* \otimes e_1^* \neq (-1)^\tau (e_1^* \otimes e_2^*)$.
- We can look at $n = 2$, $k = 1$ for ourselves.
- Note: If T_1, T_2 are both alternating k -tensors, then $T_1 + T_2$ is also alternating, as is cT_1 for all $c \in \mathbb{R}$.
- $\mathcal{A}^k(V)$: The vector space of alternating k -tensors.
- **Alt** (T): The function $\text{Alt} : \mathcal{L}^k(V) \rightarrow \mathcal{L}^k(V)$ defined by

$$\text{Alt}(T) = \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma$$

- Properties:
 1. $\text{im}(\text{Alt}) = \mathcal{A}^k(V)$.
 2. $\mathcal{L}^k(V) / \ker(\text{Alt}) = \Lambda^k(V^*)$ is isomorphic to $\mathcal{A}^k(V)$.
 3. $\text{Alt}(T)^\sigma = (-1)^\sigma \text{Alt}(T)$.

– Proof:

$$\begin{aligned}
 \text{Alt}(T)^{\sigma'} &= \left(\sum_{\sigma \in S_k} (-1)^\sigma T^\sigma \right)^{\sigma'} \\
 &= \sum_{\sigma \in S_k} (-1)^\sigma T^{\sigma\sigma'} \\
 &= (-1)^{\sigma'} \sum_{\sigma \in S_k} (-1)^{\sigma'} (-1)^\sigma T^{\sigma\sigma'} \\
 &= (-1)^{\sigma'} \sum_{\sigma \in S_k} (-1)^{\sigma\sigma'} T^{\sigma\sigma'} \\
 &= (-1)^{\sigma'} \text{Alt}(T)
 \end{aligned}$$

- The last equality holds because summing over all σ is the same as summing over all $\sigma' \circ \sigma$.
- This implies $\text{im}(\text{Alt}) \leq \mathcal{A}^k(V)$.

4. If $T \in \mathcal{A}^k(V)$, $\text{Alt}(T) = k!T$.

– We have

$$\begin{aligned}
 \text{Alt}(T) &= \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma \\
 &= \sum_{\sigma \in S_k} (-1)^\sigma (-1)^\sigma T \\
 &= \sum_{\sigma \in S_k} T \\
 &= k!T
 \end{aligned}$$

where $T^\sigma = (-1)^\sigma T$ since $T \in \mathcal{A}^k(V)$ by definition.

- This implies that $\text{im}(\text{Alt}) = \mathcal{A}^k(V)$: $\text{Alt}(\frac{1}{k!}T) = T \in \mathcal{A}^k(V)$.

5. $\text{Alt}(T^\sigma) = \text{Alt}(T)^\sigma$.

6. $\text{Alt} : \mathcal{L}^k(V) \rightarrow \mathcal{L}^k(V)$ is linear.

- Warning: Some people take $\text{Alt}(T) = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma T^{\sigma[1]}$.
- Example: $n = k = 2$. We have

$$\text{Alt}(e_1^* \otimes e_2^*) = e_1^* \otimes e_2^* - e_2^* \otimes e_1^*$$

- **Non-repeating** (multi-index I): A multi-index I such that $i_{j_1} \neq i_{j_2}$ for all $j_1 \neq j_2$.
- **Increasing** (multi-index I): A multi-index I such that $i_1 < \dots < i_k$.
- Claim: $\{\text{Alt}(e_I^*)\}$ where I is non-repeating and increasing is a basis for $\mathcal{A}^k(V)$. There are $\binom{n}{k}$ of these; thus, $\dim \mathcal{A}^k(V) = \binom{n}{k}$.

1.2 Chapter 1: Multilinear Algebra

From Guillemin and Haine (2018).

- 3/31: • Guillemin and Haine (2018) defines real vector spaces, the operations on them, their basic properties, and the zero vector.

¹Klug prefers this convention, but the text takes the other one.

- **Linearly independent** (vectors v_1, \dots, v_k): A finite set of vectors $v_1, \dots, v_k \in V$ such that the map from \mathbb{R}^k to V defined by $(c_1, \dots, c_k) \mapsto c_1 v_1 + \dots + c_k v_k$ is injective.
- **Spanning** (vectors v_1, \dots, v_k): We require that the above map is surjective.
- Guillemin and Haine (2018) defines basis, finite-dimensional vector space, dimension, subspace, linear map, and kernel.
- **Image** (of $A : V \rightarrow W$): The range space of A , a subspace of W . *Also known as $\mathbf{im}(A)$.*
- Guillemin and Haine (2018) defines the matrix of a linear map.
- **Inner product** (on V): A map $B : V \times V \rightarrow \mathbb{R}$ with the following three properties.
 - *Bilinearity*: For vectors $v, v_1, v_2, w \in V$ and $\lambda \in \mathbb{R}$, we have

$$B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w)$$

and

$$B(\lambda v, w) = \lambda B(v, w)$$

- *Symmetry*: For vectors $v, w \in V$, we have $B(v, w) = B(w, v)$.
- *Positivity*: For every vector $v \in V$, we have $B(v, v) \geq 0$. Moreover, if $v \neq 0$, then $B(v, v) > 0$.
- **W-coset**: A set of the form $\{v + w \mid w \in W\}$, where W is a subspace V and $v \in V$. *Denoted by $v + W$.*
 - If $v_1 - v_2 \in W$, then $v_1 + W = v_2 + W$.
 - It follows that the distinct W -cosets decompose V into a disjoint collection of subsets of V .
- **Quotient space** (of V by W): The set of distinct W -cosets in V , along with the following definitions of vector addition and scalar multiplication.

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$$

$$\lambda(v + W) = (\lambda v) + W$$

Denoted by V/W .

- **Quotient map**: The linear map $\pi : V \rightarrow V/W$ defined by

$$\pi(v) = v + W$$

- π is surjective.
- Note that $\ker(\pi) = W$ since for all $w \in W$, $\pi(w) = w + W = 0 + W$, which is the zero vector in V/W .
- If V, W are finite dimensional, then

$$\dim(V/W) = \dim(V) - \dim(W)$$

- Proposition 1.2.9: Let $A : V \rightarrow U$ be a linear map. If $W \subset \ker(A)$, then there exists a unique linear map $A^\sharp : V/W \rightarrow U$ with the property that $A = A^\sharp \circ \pi$, where $\pi : V \rightarrow V/W$ is the quotient map.
 - This proposition rephrases in terms of quotient spaces the fact that if $w \in W$, then $A(v+w) = Av$.
- **Dual space** (of V): The set of all linear functions $\ell : V \rightarrow \mathbb{R}$, along with the following definitions of vector addition and scalar multiplication.

$$(\ell_1 + \ell_2)(v) = \ell_1(v) + \ell_2(v)$$

$$(\lambda \ell)(v) = \lambda \cdot \ell(v)$$

Denoted by V^ .*

- **Dual basis** (of e_1, \dots, e_n a basis of V): The basis of V^* consisting of the n functions that take every $v = c_1 e_1 + \dots + c_n e_n$ to one of the c_i . Denoted by e_1^*, \dots, e_n^* . Given by

$$e_i^*(v) = c_i$$

for all $v \in V$.

- Claim 1.2.12: If V is an n -dimensional vector space with basis e_1, \dots, e_n , then e_1^*, \dots, e_n^* is a basis of V^* .

Proof. We will first prove that e_1^*, \dots, e_n^* spans V^* . Let $\ell \in V^*$ be arbitrary. Set $\lambda_i = \ell(e_i)$ for all $i \in [n]$. Define $\ell' = \sum_{i=1}^n \lambda_i e_i^*$. Then

$$\ell'(e_j) = \sum_{i=1}^n \lambda_i e_i^*(e_j) = \lambda_j \cdot 1 = \ell(e_j)$$

for all $j \in [n]$. Therefore, since ℓ, ℓ' take identical values on the basis of V , $\ell = \ell'$, as desired.

We now prove that e_1^*, \dots, e_n^* spans V^* . Let $\sum_{i=1}^n \lambda_i e_i^* = 0$. Then for all $j \in [n]$,

$$\lambda_j = \left(\sum_{i=1}^n \lambda_i e_i^* \right) (e_j) = 0$$

as desired. □

- **Transpose** (of A): The map from W^* to V^* defined by $\ell \mapsto A^* \ell = \ell \circ A$ for all $\ell \in W^*$.
- Claim 1.2.15: If e_1, \dots, e_n is a basis of V , f_1, \dots, f_m is a basis of W , e_1^*, \dots, e_n^* and f_1^*, \dots, f_m^* are the corresponding dual bases, and $[a_{i,j}]$ is the $m \times n$ matrix of A with respect to $\{e_i\}, \{f_i\}$, then the linear map A^* is defined in terms of $\{f_i^*\}, \{e_i^*\}$ by the transpose matrix $(a_{j,i})$.

References

Guillemin, V., & Haine, P. J. (2018). *Differential forms* [https://math.mit.edu/classes/18.952/2018SP/files/18.952_book.pdf].