Week 9

Integration of Manifolds

9.1 Orientations on Manifolds

5/23:

- Weekly plan.
  - We've got places to be — it's good to worry about what everything is, but it's also good to just think of stuff as how it historically developed.
  - Goal: Stokes' Theorem ($\int_x \mathrm{d}\omega = \int_{\partial x} \omega$).
    - We need to talk about the boundary ∂x.
    - We need to talk about the integral (integrating forms on manifolds).
    - Hidden orientation convention (we'll see in examples).
- Special cases.
  1. The fundamental theorem of calculus.
     - Take the manifold to be $X = [a, b] \in \mathbb{R}$.
     - Here, $\partial x = \{a, b\}$.
     - Take $f(x) \in \Omega^0(X)$.
     - Take $\mathrm{d}f = f'(x)\,\mathrm{d}x$ where $\mathrm{d}x \in \Omega^1(X)$.
     - So by Stokes' theorem, integrating over the whole interval $\int_a^b f'(x)\,\mathrm{d}x$ is equal to integrating over the boundary, but integration over the boundary is $f(b) - f(a)$.
       - The minus sign $f(b) - f(a)$ is where the orientation convention comes in.
  2. Green's theorem.
     - You have a region in the plane. $X = U \subset \mathbb{R}^2$ open.
     - You have a one-form $\omega = P\,\mathrm{d}x + Q\,\mathrm{d}y$.
     - The corresponding two-form is

$$
\begin{aligned}
\mathrm{d}\omega &= \left(\frac{\partial P}{\partial x}\,\mathrm{d}x + \frac{\partial P}{\partial y}\,\mathrm{d}y\right) \wedge \mathrm{d}x + \left(\frac{\partial Q}{\partial x}\,\mathrm{d}x + \frac{\partial Q}{\partial y}\,\mathrm{d}y\right) \wedge \mathrm{d}y \\
&= \frac{\partial P}{\partial x}\,\mathrm{d}x \wedge \mathrm{d}x + \frac{\partial P}{\partial y}\,\mathrm{d}y \wedge \mathrm{d}x + \frac{\partial Q}{\partial x}\,\mathrm{d}x \wedge \mathrm{d}y + \frac{\partial Q}{\partial y}\,\mathrm{d}y \wedge \mathrm{d}y \\
&= \frac{\partial P}{\partial x} \cdot 0 - \frac{\partial P}{\partial y}\,\mathrm{d}x \wedge \mathrm{d}y + \frac{\partial Q}{\partial x}\,\mathrm{d}x \wedge \mathrm{d}y + \frac{\partial Q}{\partial y} \cdot 0 \\
&= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y
\end{aligned}
$$

- Well, if you want the double integral on the left below, Green's theorem tells us that it's equal to the line integral around the boundary on the right below.

$$\int_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y = \int_{\partial U} (P\,\mathrm{d}x + Q\,\mathrm{d}y)$$

- We orient counterclockwise around the boundary, or we get the wrong thing!
- If U has holes, you orient clockwise about the holes for reasons we'll talk about shortly.

3. A bit more abstract.
   - Say we have a slightly more abstract three-manifold X^3. We are given a two-form $\omega \in \Omega^2(X^3)$.
   - Let $\Sigma^2 \subset X^3$ be some two-dimensional submanifold. If we want $\int_\Sigma \omega$, we just need (as we will prove shortly) $\int_\Sigma i^*\omega$. Pulling back our two-form in this manner does just give us a top-dimensional form.
4. Many other special cases were known before the general Stokes' theorem.

- Claim: If ω is exact (i.e., $\omega = \mathrm{d}\mu$), then $\int_\Sigma \omega = 0$ for all Σ that have no boundary (i.e., $\partial\Sigma = \emptyset$).
  - Proof: Stokes plus $\partial\Sigma = \emptyset$ (the integral of a manifold over an emptyset naturally equals zero).
- **Closed** (submanifold): A submanifold with $\partial\Sigma = \emptyset$.
- This leads into the degree theory stuff we were doing on manifolds.
  - Applies to X^n, Y^n compact, closed manifolds.
  - If we have $F : X \to Y$, we have $\deg(F) = \int F^*\omega$ where $\int_Y \omega = 1$ and $F^* : \Omega^n(Y) \to \Omega^n(X)$ sends $\omega \mapsto F^*\omega$.
  - Degree is homotopy invariant.
  - Compute it with "counting preimages with sign."
  - Fun applications.
- Stokes was thinking about surfaces in three dimensions when he developed his Stokes' theorem. That's the setting he was working in.
- Three things to straighten out first.
  1. Orientations on manifolds — take the pointwise concept of an orientation of a vector space and extend it to T_pX.
     - Aside: We were happy to identify $T_p\mathbb{R}^n \cong \mathbb{R}^n \cong T_p^*\mathbb{R}^n$. However, it does not make sense to identify T_pX with $\mathbb{R}^n$ ($T_pX \ncong \mathbb{R}^n$), even for S^2 or something.
     - A choice of charts gives $T_pX \cong \mathbb{R}^n$, but we have to choose these charts; there is no natural identification.
  2. Boundaries and induced orientations.
     - If we have an orientation at every point here, we cananocially induce an orientation by taking your favorite first vector, moving it to the boundary, and then every other vector gives your orientation.
     - If we don't take this convention, we'd have to put a minus sign in Stokes' theorem and it would mess up everything else. Math has to go counterclockwise.
  3. Integration on manifolds.
     - Let $X^n \subset \mathbb{R}^N$ be an **oriented manifold**.
     - Let $\omega \in \Omega^n(X)$ be a top-dimensional form.
     - $\int_X \omega$ means:
       - Step 1: Pick a set $\{U_i\}$ of orientation preserving charts that cover X. For S^2 for example, we can take a chart of the top of the sphere, and a chart of the bottom of the sphere.

        - Step 2: Pick a partition of unity $\{\rho_i\}$ with ρ_i supported in U_i.
        - Now define
$$\int_X \mathrm{d}\omega = \int \sum_i \rho_i \omega = \sum_i \int_{U_i'} \varphi_i^* \rho_i \omega$$
          where the expression on the right is a good, old-fashioned integral in Euclidean space.
    - This definition does not depend on the choices in steps 1 and 2.
        - We should go through this to see how our nice machinery of differential forms neatly gets rid of and absorbs all of the inherent ambiguity herein.

- Let's now say all that with way more words.
- **Orientation** (of $X^n \subset \mathbb{R}^N$): A function that assigns to each point $p \in X$ an orientation of T_pX (or an element of $\Lambda^n(T_p^*X)$).
    - This is a preliminary definition.
    - Flaws: We could choose different orientations at every point.
    - We need "smoothness," i.e., we need close-by points to have the same orientation.
- Note: If $\omega \in \Omega^n(X)$ and ω is nonvanishing on some $U \subset X$, then ω induces an orientation on U by assigning to $p \in U$, a form $\omega_p \in \Lambda^n(T_p^*X)$.
- **Smooth** (orientation of $X^n \subset \mathbb{R}^N$): An orientation on X such that for all $p \in X$, there is a neighborhood U and a nonvanishing form $\omega \in \Omega^n(X)$ such that the orientation on U induced by ω agrees with the given orientation.
    - There are like ten different definitions that all agree. This is the one using forms, which is most suited to our study.
- From now on, we will assume all orientations are smooth.
- Examples.
    1. Let $X = U \subset \mathbb{R}^n$. Take the orientation given by the ordered basis $\partial/\partial x_1, \dots, \partial/\partial x_n$ at each point.
        - This is why we haven't needed to talk about orientations in our discussion of Euclidean space.
        - Equivalently, this is the orientation induced by $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$.
    2. For $S^1 \subset \mathbb{R}^2$, the orientation is dual to the vector space $\boldsymbol{v} : S^1 \to TS^1$ defined by
$$\boldsymbol{v}(x, y) = ((x, y), (-y, x))$$
        - Graphically, $\boldsymbol{v}$ is the vector space on S^1 whose tangent vectors spin around it counterclockwise (tho $\boldsymbol{w}$ related to $\boldsymbol{v}$ with vectors spinning clockwise would also be acceptable to take the dual of).
        - This is for every point a choice of ordered basis of the tangent space because at every $p \in S^1$, there is an ω_p which takes any basis vector (nonzero scalar) in T_pS^1 and returns either a positive or negative scalar.
- Equivalently, we use the orientation induced by $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$.
- **Orientable** (manifold): A manifold X that can be (smoothly) oriented.

9.2 Domains and Steps to Integration

5/25: • Plan:
  - Orientations.
  - Domains – manifolds with boundary and how we include an orientation on the boundary.
  - Integration.
- Examples of orientable manifolds.
  - A line segment.
  - $\mathbb{R}^n$ with $\partial/\partial x_1, \dots, \partial/\partial x_n$ and $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$.
  - A torus.
  - A genus 2 surface.
  - The product manifold of two orientable manifolds.
- Bad examples:
  - Möbius strip.
  - Klein bottle.
- Aside:
  - If X^n is compact, the de Rahm cohomology is

$$H^n_{\mathrm{dR}}(X) = \frac{\Omega^n(X)}{\mathrm{d}(\Omega^{n-1}(X))} = \begin{cases} \mathbb{R} & X \text{ orientable} \\ 0 & X \text{ not orientable} \end{cases}$$

- **Orientation preserving** (diffeomorphism): A diffeomorphism $F : X \to Y$, where X^n, Y^n are oriented n-manifolds, for which $\mathrm{d}F_p : T_pX \to T_{F(p)}Y$ implies $(\mathrm{d}F_p)^* : \Lambda^n(T_p^*X) \leftarrow \Lambda^n(T^*_{F(p)}Y)$ is orientation preserving for all $p \in X$.
- Examples:
  - The map from $[0,1]$ to $[0,2]$ that stretches it by a factor of 2, where we assume that both 1-manifolds are oriented in the positive direction.
  - The map that rotates $\mathbb{R}^2$ by $90°$.
  - An example of a diffemorphism that is *not* orientation preserving is $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $z \mapsto \bar{z}$.
- Boundaries.
- **Domain**: An open subset $D \subset X^n$ such that...
  1. ∂D is an $(n-1)$-manifold;
  2. $\partial D = \partial(\overline{D})$[1].
- Examples.
  - If $X = \mathbb{R}^2$, we may take $D = \{x \in \mathbb{R}^2 \mid |x| < 1\}$ (i.e., X is the open unit disk).
    - Condition 2 is here to rule out things like $\mathbb{R}^2 \setminus S^1 = D$.
  - The upper half plane $\mathbb{H}^n \subset \mathbb{R}^n$[2] where $\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$.
    - $\partial\mathbb{H}^n = \mathbb{R}^1$.

[1] $\overline{D}$ is the closure of D.
[2] For us, $\mathbb{H}$ for half; later on, $\mathbb{H}$ for hyperbolic.

  - The point: $D \subset X$ and $\omega \in \Omega^n(X)$ makes it so that $\int_D d\omega_0 = \int_{\partial D} \omega_0$.

- If X is oriented, it induces an orientation on D (via the restriction of the orientation form to D), which induces an orientation ∂D, which we can integrate over.
- Claim (Existence of boundary charts): If $D \subset X^n$ is a domain in a manifold, $p \in \partial D$, and X is oriented, then there exists $U \subset X$ open that is a neighborhood of p and a chart $\varphi : U_0 \to U$ that sends $0 \mapsto p$, where $U_0 \subset \mathbb{R}^n$, such that...
  1. φ is orientation preserving.
  2. $U_0 \cap \mathbb{H}^n \xrightarrow{\varphi} U \cap D$.
- Example: Boundary chart for S^2.

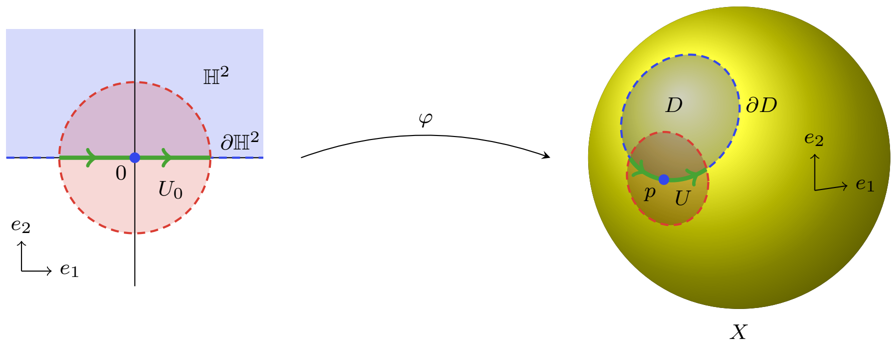

Figure 9.1: Existence of boundary charts.

  - Consider the two-sphere $S^2 \subset \mathbb{R}^3$. We will call this manifold X.
  - The shaded blue circle on the surface of X is a domain $D \subset X$. Its boundary ∂D is represented by a dashed blue line, where dashing is chosen to remind the viewer that D is open. The point p is an element of ∂D and is contained in the open neighborhood $U \subset X$. X is oriented, as indicated. φ maps the open disk $U_0 \subset \mathbb{R}^2$ to U and such that $\varphi(U_0 \cap \mathbb{H}^2) = U \cap D$. Moreover, φ is clearly orientation preserving and satisfies $\varphi(0) = p$.
  - Note that Figure 9.1 was drawn in a largely analogous manner to Figure 8.4. See handwritten pages for more info.
- Proof.
  - Proving 1: You know φ exists by the definition of a manifold; if it's not orientation preserving, compose it with a map of $\mathbb{R}^n$ that reverses all of the needed orientations.
- Using this chart φ, we get an induced orientation on ∂D from the orientation of $\partial\mathbb{H}^n = \mathbb{R}^{n-1}$.
  - See the green arrows in the Figure 9.1.
  - Check:
    1. This gives a (global) orientation of ∂D.
    2. This does not depend on the choice of chart φ.
- Note that this implies that the Klein bottle can't bound an orientable manifold because this would induce an orientation on the Klein bottle.
- Integration of forms on manifolds.
  - Take X^n to be our manifold and $\omega \in \Omega^n(X)$ to be some top-dimensional form.
  - For now, we'll let X be compact.

- We want to define $\int_X \omega$, which should be a real number.
- We got the recipe last time; now we just have to make it precise. Slogan: Break apart, compute in charts, put back together.
- Here are the steps.
  1. Pick *oriented* charts (i.e., $\{\varphi_i : U_i' \to U_i\}$ orientation preserving) so that $\{U_i\}$ covers X.
     - Example: Taking the top- and bottom-halves of S^2, as discussed last time, but we may do this however.
  2. Choice of a "partition of unity supported on the U_i."
     - Take a set $\{\rho_i : X \to \mathbb{R}\}$ with a few properties.
     (a) For all $p \in X$, $\rho_i(p) = 0$ for all but finitely many indices i.
     (b) $\sum_i \rho_i(p) = 1$; this is where the name *partition of unity* comes from.
     (c) $\text{supp}(\rho_i) \subset U_i$.

9.3 Stokes' Theorem and Course Retrospective

5/27: • Office hours: 4:00pm-5:00pm today.

- Let $W \subset X^n \subset \mathbb{R}^N$, where W is a domain and X^n is an oriented manifold.
  - Then we seek to define $\int_W \omega$.
  - There are two possibilities.
    1. $\overline{W}$ is compact implies
    $$\int_W \omega = \sum_i \int_{U_i \cap \phi_i^{-1}(W)} \varphi_i^* \rho_i \omega$$
    2. $\omega \in \Omega_c^n(X)$ and W arbitrary.
- For 1 (wrt the homework):
  - Let $\sigma_{\text{vol}} \in \Omega^n(X)$. Pointwise, $T_p\mathbb{R}^N \times T_p\mathbb{R}^N \to \mathbb{R}$, restrict to $T_pX \times T_pX \to \mathbb{R}$.
  - Let $e_1, \dots, e_n$ be an orthonormal basis for T_pX.
  - σ_{vol} is the **volume form**.
  - We take $(\sigma_{\text{vol}})|_p = e_1^* \wedge \cdots \wedge e_n^*$.
  - Check:
    - Well-defined.
    - What it is in charts.
    - $X = \Gamma_f = F^{-1}(\{0\})$.
- **Volume** (of W): The following quantity, where W is compact. *Denoted by* $\textbf{Vol}(\boldsymbol{W})$. *Given by*
$$\text{Vol}(W) = \int_W \sigma_{\text{vol}}$$
  - This the actual volume!
  - The volume is the limit (computed the right way) of inserted polygons.
  - $\ell(\gamma) = \int |\gamma|$ is the limit length of the inserted polygons.
- Properties.
  1. $\int_W (\omega_1 + \omega_2) = \int_W \omega_1 + \int_W \omega_2$.
  2. $\int_W c\omega = c \int_W \omega$.

3. $F : X \to Y$, where $X \subset W_X$ and $Y \subset W_Y$. an orientation preserving diffeomorphism implies $\int_X F^*\omega = \int_Y \omega$ and $\int_{W_X} F^*\omega = \int_{W_Y} \omega$.
4. Don't forget the orientation! $\int_{-X} \omega = -\int_X \omega$.

- Theorem (Stokes): Let $W \subset X \subset \mathbb{R}^N$, where W is a domain such that $\overline{W}$ is compact and X is oriented. Let $\mu \in \Omega^{n-1}(X)$. Then
$$\int_W \mathrm{d}\mu = \int_{\partial W} \mu$$
where ∂W is oriented.
- Example:
  - Let $W = \{(x, y) \in \mathbb{R}^2 : x^2+y^2 \leq 1\}$ be the unit disk, with its boundary oriented counterclockwise. Let $X = \mathbb{R}^2$. Let $\omega = \mathrm{d}x \wedge \mathrm{d}y$. *picture*
    - We want to find $\omega_0 \in \Omega^1(X)$ such that $\mathrm{d}\omega_0 = \omega$. We could take $\omega_0 = x \wedge \mathrm{d}y$, but we'll take $\omega_0 = (-y/2)\,\mathrm{d}x + (x/2)\,\mathrm{d}y$ because it's symmetric.
    - We take as our one chart the circumference of the circle minus a point, and it will suffice to integrate over this one chart. This chart maps $(0, 2\pi) \to \partial W$ via $t \mapsto (\cos t, \sin t)$.
    - Thus, by Stokes' theorem,
$$\begin{aligned}\int_W \mathrm{d}x \wedge \mathrm{d}y &= \int_{\partial W} \left(\left(-\frac{y}{2}\right)\mathrm{d}x + \left(\frac{x}{2}\right)\mathrm{d}y\right)\\ &= \int_0^{2\pi} -\frac{\sin t}{2}\,\mathrm{d}(\cos t) + \frac{\cos t}{2}\,\mathrm{d}(\sin t)\\ &= \int_0^{2\pi} \frac{1}{2}(\sin^2 t + \cos^2 t)\,\mathrm{d}t\\ &= \pi\end{aligned}$$
    - Sanity check: $\mathrm{d}x \wedge \mathrm{d}y$ is the volume form, so integrating it should give the "volume" (area) of the unit disk, and it does!
- The proof of Stokes' theorem relies on the FTC and machinery.
  - To truly understand it, fight with Green's theorem, then jump up a dimension and prove Stokes' theorem, then jump up many dimensions and prove the generalized Stokes theorem.
- Overview of the whole year.
  - Fall:
    - $\mathbb{R}$ exists.
    - Metric spaces, open sets, etc.
    - Sequences and series, continuous functions.
    - Theme: Just a bunch of chit-chat/language: Look, I can write a proof!
  - Winter:
    - Derivatives ($\mathbb{R}$ and $\mathbb{R}^n$), familiar properties.
    - Integrals.
    - FTC.
    - Theme: Interchanging limits.
  - Spring:
    - Manifolds: Locally Euclidean. The point is that they show up a lot. Most mathematicians in this building study manifolds in one form or another.
    - Manifolds are locally solutions to equations $F : \mathbb{R}^N \to \mathbb{R}^{N-n}$.

  - Manifolds are a place where you can do calculus (reducing it to linear algebra).
  - Tangent spaces T_pX, linear approximations $\mathrm{d}f_p$.
  - Integration/differentiation with $\Omega^k(X)$.
  - Forms package with Λ^n, $\wedge$, d, $\int$, and Stokes' theorem.
  - Manifolds $\Leftarrow$ local theory (esp. the degree) $\Leftarrow$ pointwise linear algebra.

- Where to go from here.
  - Don't read Guillemin and Haine (2018) again if you don't know what differential forms are.
    - Move forward and the foundations will fill themselves in.
  - More analysis.
    - Complex analysis. All your favorite stuff, but over $f : \mathbb{C} \to \mathbb{C}$. Do derivatives and integrals over closed curves. This is a beautiful theory. Since homomorphisms $f : \mathbb{C} \to \mathbb{C}$ are really rare and constricted, we can say a lot about them.
    - Functional analysis. A blend of linear algebra in infinite dimensions $\mathbb{R}^\infty$.
    - Fourier analysis. Approximating functions $f : \mathbb{R} \to \mathbb{R}$ with sine and cosine.
    - Harmonic analysis. The more general philosophical counterpart to Fourier analysis.
  - Use analysis.
    - Partial differential equations. Usually the motivation for needing to learn functional and Fourier analysis.
    - Dynamical systems. Iterating functions as with 3 $\Rightarrow$ chaos. Many Chicago mathematicians study dynamical systems.
    - Probability theory.
    - Differential geometry. The properties of surfaces and how they curve and what how they curve tells us. Pictures here! But not too much; it's still pretty abstract.
    - Analytic number theory.
    - Discrete math.
    - Analysis is a love-hate burden. Klug doesn't consider it interesting in its own right, but you always seem to need more of it to do the math you want to do.
  - Chapter 5 of Guillemin and Haine (2018).
    - The chain $\cdots \to \Omega^{k-1}(X) \xrightarrow{\mathrm{d}} \Omega^k(X) \xrightarrow{\mathrm{d}} \Omega^{k+1}(X) \to \cdots$.
    - $H^k_{\mathrm{dR}}(X)$.
    - Homological algebra.
    - It will seem random and crazy if you read it, but it's actually this whole "cohomology theory" that is pretty ubiquitous. If you invest in it, you will get something out of it pretty much regardless of the definition you go in.

- The final will be like the midterm: Computations. Write down the tangent space with this chart. Can you find a function that this is the zero of? Can you compute the degree? Can you push around a couple definitions?
  - Computing the degree can be a bit painful (see the homework), so we might just "eyeball it."

9.4 Office Hours (Klug)

- What is a k-form, and can you give some examples of them?
- What is the pullback and what does it do?
- What is $\Lambda^k(V^*)$, and why is it the k^{th} exterior power of V^*, and what does that even mean? The elements of it are $\mathcal{I}^k(V)$-cosets of tensors; what does one of these look like? The elements of it aren't even functions, right? They're just sets of functions?
- More Klug meeting questions as time allows.

9.5 Final Review Sheet

6/2:

- Computing the sign.

$$(-1)^\sigma = \prod_{i<j} \frac{X_{\sigma(i)} - X_{\sigma(j)}}{X_i - X_j}$$

- **Pullback** (of A): The linear map $A^* : W^* \to V^*$ defined as follows, where $A : V \to W$ be a linear transformation between V, W vector spaces. *Given by*

$$\ell \mapsto \ell \circ A$$

  - Essentially, we take every linear functional on W and relate it to a linear functional $\ell \circ A$ on V by having A translate vectors in V to vectors in W, which ℓ can eat.

- Every k-tensor $T : V^k \to \mathbb{R}$ has a decomposition

$$T = \sum_I T_I e_I^*$$

  - Proofs for decomposable tensors follow from the linearity of this decomposition.

- Recall decomposable and redundant k-tensors.
- **Pullback** (of T by A): Once again, A supplies values to T. So if $A : V \to W$ and $T : W^k \to \mathbb{R}$, then

$$A^*T(v_1, \dots, v_k) = T(Av_1, \dots, Av_k)$$

- Recall T^σ, defined in terms of T by σ^{-1} indices.
- **Alternating** (k-tensor): A k-tensor T such that for all $\sigma \in S^k$,

$$T^\sigma = (-1)^\sigma T$$

- **Alternation operation**: The function $\text{Alt} : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ defined by

$$T \mapsto \sum_{\tau \in S^k} (-1)^\tau T^\tau$$

- Properties.
  1. $\text{Alt}(T)^\sigma = (-1)^\sigma \operatorname{Alt} T$.
  2. $T \in \mathcal{A}^k(V)$ implies $\operatorname{Alt} T = k!T$.
  3. $\text{Alt}(T^\sigma) = \text{Alt}(T)^\sigma$.
  4. Alt is linear.

- $\mathcal{A}^k(V) \cong \Lambda^k(V^*) = \mathcal{L}^k(V) / \ker(\text{Alt})$.
  - This means that $\Lambda^k(T_p^*\mathbb{R}^n) \cong \mathcal{A}^k(T_p\mathbb{R}^n)$.
  - In terms of k-forms, this must mean that we're taking k things and sending them like a one-form. The useful picture of $f : \mathbb{R}^2 \to \mathbb{R}^2$ is as $f_1 : \mathbb{R}^2 \to \mathbb{R}^1$ and separately $f_2 : \mathbb{R}^2 \to \mathbb{R}^1$. It's just more.

- Recall that $\mathcal{I}^k(V)$ is the span of all redundant k-tensors.
  - In particular, $\mathcal{I}^k(V) = \ker(\text{Alt})$.

- Let $\pi : \mathcal{L}^k(V) \to \Lambda^k(V^*)$ send $T \mapsto \omega$. Then if $\omega_1 = \pi(T_1)$ and $\omega_2 = \pi(T_2)$, we have $\omega_1 \wedge \omega_2 = \pi(T_1 \otimes T_2)$.

  - Indeed, with the wedge product, we are kind of just appending forms/tensors together. The 2-tensor at a point describes the derivative of a function into 2-dimensional space.

- If $\omega_1 \in \Lambda^k(V^*)$ and $\omega_2 \in \Lambda^\ell(V^*)$, then

$$\omega_1 \wedge \omega_2 = (-1)^{k\ell}\omega_2 \wedge \omega_1$$

- The cursed product rule (for the interior product and the exterior derivative):

$$\iota_v(T_1 \otimes T_2) = \iota_v T_1 \otimes T_2 + (-1)^p T_1 \otimes \iota_v T_2$$

- **Interior product** (of a vector v and k-tensor T): The $(k-1)$-tensor

$$(\iota_v T)(v_1, \ldots, v_{k-1}) = \sum_{r=1}^{k} (-1)^{r-1} T(v_1, \ldots, v_{r-1}, v, v_r, \ldots, v_{k-1})$$

- **Pullback** (of ω by A): As always, A supplies values to T, where $\omega = \pi(T)$. However, this time it's indirectly through the projection operation:

$$A^*\omega = \pi(A^*T)$$

- **Determinant** (of A): The number A such that $A^*\omega = a\omega$.
  - Appeal to A's actions on coordinates/bases to derive the typical formula.
- Recall the definitions of the tangent space, a vector field.
- $\partial/\partial x_i$ is the n-dimensional vector field where vectors point in the x_i-direction at every point.
- Every vector has a unique decomposition in terms of the standard basis $(x_1, \ldots, x_n)$, given by a set of numbers. If we assign each point of a vector field to these numbers via functions $\{g_i\}$, we realize that every n-dimensional vector field on U has a unique decomposition

$$\boldsymbol{v} = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i}$$

where $g_i : U \to \mathbb{R}$.

- The Lie derivative takes the directional derivative of a function f on U according to the vector field $\boldsymbol{v}$ on U. In particular, if $\boldsymbol{v}(p) = (p, v)$, then

$$L_{\boldsymbol{v}} f(p) = Df(p)v$$

In coordinates,

$$L_{\boldsymbol{v}} f = \sum_{i=1}^{n} g_i \frac{\partial f}{\partial x_i}$$

  - If $L_{\boldsymbol{v}} f = 0$, f is an integral of $\boldsymbol{v}$.
- Differential 1-forms: Essentially, these are covector fields. We can interconvert with the musical operators.
- We need differential 1-forms to generalize to higher dimensions for U than $\mathbb{R}^3$, and differential k-forms to describe functions *into* higher-dimensional spaces.
- **Integral curve** (of $\boldsymbol{v}$): A curve $\gamma : [a, b] \to U$ such that $\gamma'(t) = \boldsymbol{v}(\gamma(t))$.

- The following related definitions for the derivative of f.

$$\begin{aligned} Df(p) &= \left[\left.\frac{\partial f_i}{\partial x_j}\right|_p \right] \\ \mathrm{d}f_p(p,v) &= (q, Df(p)v) \\ \mathrm{d}f(p) &= \mathrm{d}f_p \end{aligned}$$

- The following constructs related to forming a basis of one-forms.

$$\begin{aligned} x_i(v_1,\dots,v_n) &= v_i \\ (\mathrm{d}x_i)_p(p, a_1x_1+\cdots+a_nx_n) &= a_i \\ \mathrm{d}x_i(p) &= (\mathrm{d}x_i)_p \end{aligned}$$

- Relating the last two thoughts: All one-forms have a unique decomposition

$$\omega = \sum_{i=1}^n f_i \mathrm{d}x_i$$

- **Interior product** (of a vector field $\boldsymbol{v}$ and a one-form ω): The following expression, where the vector field and one-form are defined in coordinates as $\boldsymbol{v} = \sum_{i=1}^n g_i\, \partial/\partial x_i$ and $\omega = \sum_{i=1}^n f_i\, \mathrm{d}x_i$. *Given by*

$$\iota_{\boldsymbol{v}}\omega = \sum_{i=1}^n f_i g_i$$

- Note that

$$\iota_{\boldsymbol{v}}\mathrm{d}f = L_{\boldsymbol{v}}f$$

  as follows directly from the definitions.

- Recall that $C_0^\infty(\mathbb{R}^n)$ is the vector space of all bump functions on $\mathbb{R}^n$.
- Exterior derivative properties.
  1. Linearity.
  2. Cursed product rule (where $p = k$ is the dimension of ω_1).
  3. Special case ($k = \ell = 0$, so $\omega_1 = f$ and $\omega_2 = g$ are C^∞ functions).

$$\mathrm{d}(fg) = g\mathrm{d}f + f\mathrm{d}g$$

  4. Formula.

$$\mathrm{d}\left(\sum_I f_I \mathrm{d}x_i\right) = \sum_I \mathrm{d}f_I \wedge \mathrm{d}x_I$$

  5. $\mathrm{d}^2 = 0$.

- Recall closed and exact k-forms. Closed ones have $\mathrm{d}\omega = 0$; exact ones have $\omega = \mathrm{d}\mu$. Exact implies closed by $\mathrm{d}^2 = 0$.
- A k-form at a point p is an alternating k-tensor. It takes k vectors in and spits out the result of applying the best linear approximation to each of them for a k-dimensional function (a function into $\mathbb{R}^k$). It's really just best to rephrase everything in terms of alternating tensors and view the wedge product as the tensor product.

- This makes it so that we analogously have
$$\omega_p = \sum_I c_I (\mathrm{d}x_I)_p$$
and
$$\omega = \sum_I f_I \mathrm{d}x_I$$
- Recall that $\Omega_c^k(U)$ is the vector space of all compactly supported k-forms on U.
  - Recall further that the support is the set of all points at which the form is nonzero, and compact just means that the support is compact. This can have cool consequences, as with (maximal) integral curves.
- **Proper** (function $f : U \to V$): Continuous, $K \subset V$ compact implies $f^{-1}(K) \subset U$ compact.
  - Sine is not proper.
- The pullback maps compactly supported forms to compactly supported forms.
- **Integral** (of a top-dimensional form): If $\omega = f \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ is a top-dimensional form, then the integral of ω over U is given as follows. *Given by*
$$\int_U \omega = \int_{\mathbb{R}^n} f \, \mathrm{d}x_1 \cdots \mathrm{d}x_n$$
  - Evaluate with repeated integrals.
- Poincaré lemma for rectangles: Let $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Take $\omega \in \Omega_c^n(Q)$. Then TFAE.
  1. $\int_Q \omega = 0$.
  2. $\omega = \mathrm{d}\mu$ with $\mu \in \Omega_c^{n-1}(Q)$
- Intuition ($n = 1$ case).
  - The following are equivalent.
    1. $\int_a^b f = 0$.
    2. $f = g'$ for some compactly supported smooth g on $[a, b]$.
  - Let g be the bump function on $(-1, 1)$. Then starting at -1, f goes up and down to zero and down and up to 1. Naturally, $\int_a^b f = 0$, and similarly, g is compactly supported on $[-1, 1]$.
  - ($2 \Rightarrow 1$): If g is compactly supported, then $g(b) = g(a)$. Thus, $\int_a^b f = g(b) - g(a) = 0$.
  - ($1 \Rightarrow 2$): If $\int_a^b f = 0$, define $g(x) = \int_a^x f(t) \, \mathrm{d}t$. This is compactly supported (i.e., has $g(a) = 0$ and $g(b) = 0$) since
$$g(a) = \int_a^a f = 0 \qquad\qquad g(b) = \int_a^b f = 0$$
    where the left equality follows by the properties of integrals and the right follows by hypothesis 1.
- **Pullback** (of a one-form μ on V onto U by f): It looks like here, f (in the one-form form $\mathrm{d}f$) is supplying values to μ.
$$f^*\mu(p) = \mu_q \circ \mathrm{d}f_p$$

- In formulas, if
$$\omega = \sum_I \phi_I \, \mathrm{d}x_I$$
then
$$f^*\omega = \sum_I f^*\phi_I \, \mathrm{d}f_I$$
where $f^*\phi_I = \phi_I \circ f$.
- Note that $\mathrm{d} \circ f^* = f^* \circ \mathrm{d}$.

- **Lie derivative** (of the k-form ω with respect to $\boldsymbol{v}$):The k-form defined as follows, where $U \subset \mathbb{R}^n$ is open, $\boldsymbol{v} \in \mathfrak{X}(U)$, and $\omega \in \Omega^k(U)$. *Given by*
$$L_{\boldsymbol{v}}\omega = \iota_{\boldsymbol{v}}(\mathrm{d}\omega) + \mathrm{d}(\iota_{\boldsymbol{v}}\omega)$$
  - Note that we use ι to drop the index and d to raise it back up, and then vice versa.
- Properties of this Lie derviative.
  1. $L_{\boldsymbol{v}} \circ \mathrm{d} = \mathrm{d} \circ L_{\boldsymbol{v}}$.
  2. $L_{\boldsymbol{v}}(\omega \wedge \mu) = L_{\boldsymbol{v}}\omega \wedge \nu + \omega \wedge L_{\boldsymbol{v}}\mu$.
- Explicit formula for this Lie derivative: If $\omega = \sum_I f_I \, \mathrm{d}x_I$ and $\boldsymbol{v} = \sum_{i=1}^n g_i \, \partial/\partial x_i$, then
$$L_{\boldsymbol{v}}\omega = \sum_I \left[\left(\sum_{i=1}^n g_i \frac{\partial f_I}{\partial x_i} \right) \mathrm{d}x_I + f_I \left(\sum_{r=1}^k \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}g_{i_r} \wedge \cdots \wedge \mathrm{d}x_{i_k} \right) \right]$$
- Vector calc connections.
  - The musical operators.
$$\sharp(f \, \mathrm{d}x + g \, \mathrm{d}y) = f \frac{\partial}{\partial x} + g \frac{\partial}{\partial y}$$
  - The exterior derivative of a (2D) function.
$$\mathrm{d}f = \frac{\partial f}{\partial x} \, \mathrm{d}x + \frac{\partial f}{\partial y} \, \mathrm{d}y$$
  - The exterior derivative of a (2D) one-form.
$$\mathrm{d}(f \, \mathrm{d}x + g \, \mathrm{d}y) = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \mathrm{d}x \wedge \mathrm{d}y$$
  - The exterior derivative of a (3D) function.
$$\mathrm{d}f = \frac{\partial f}{\partial x} \, \mathrm{d}x + \frac{\partial f}{\partial y} \, \mathrm{d}y + \frac{\partial f}{\partial z} \, \mathrm{d}z$$
  - The exterior derivative of a (2D) one-form.
$$\mathrm{d}(f \, \mathrm{d}x + g \, \mathrm{d}y + h \, \mathrm{d}z) = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \mathrm{d}x \wedge \mathrm{d}y + \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial h}{\partial x} - \frac{\partial f}{\partial z} \right) \mathrm{d}x \wedge \mathrm{d}z$$
  - The exterior derivative of a (3D) two-form.
$$\mathrm{d}(f \, \mathrm{d}x \wedge \mathrm{d}y + g \, \mathrm{d}y \wedge \mathrm{d}z + h \, \mathrm{d}x \wedge \mathrm{d}z) = \left(\frac{\partial f}{\partial z} + \frac{\partial g}{\partial x} - \frac{\partial h}{\partial y} \right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

- **Interior product** (of a vector field $\boldsymbol{v}$ and a k-form ω): We take every $p \in U$ to the interior product of the vector $\boldsymbol{v}(p)$ and the k-tensor ω_p.
- Thus, if $\boldsymbol{v} = \partial/\partial x_r$ and $\omega = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$, we have that

$$
\begin{aligned}
[\iota_{\boldsymbol{v}}\omega(p)](v_1,\dots,v_{k-1}) &= [\iota_{\boldsymbol{v}(p)}\omega_p](v_1,\dots,v_{k-1}) \\
&= \sum_{i=1}^{k}(-1)^{i-1}\omega_p(v_1,\dots,v_{i-1},\boldsymbol{v}(p),v_i,\dots,v_{k-1}) \\
&= \sum_{i=1}^{k}(-1)^{i-1}[\mathrm{d}x_{i_1}\wedge\cdots\wedge\mathrm{d}x_{i_k}]_p(v_1,\dots,v_{i-1},\boldsymbol{v}(p),v_i,\dots,v_{k-1}) \\
&= \sum_{i=1}^{k}(-1)^{i-1}[(\mathrm{d}x_{i_1})_p\wedge\cdots\wedge(\mathrm{d}x_{i_k})_p](v_1,\dots,v_{i-1},\boldsymbol{v}(p),v_i,\dots,v_{k-1}) \\
&= \cdots
\end{aligned}
$$

- **Pullback** (of the k-form ω along f): As per usual, we feed ω_q some values spit out by $\mathrm{d}f_p$. *Given by*

$$
f^*\omega(p) = \mathrm{d}f_p^*\omega_q
$$

- Properties of this pullback.
  1. $(f^*\phi)(p) = (\phi\circ f)(p)$.
  2. $f^*\mathrm{d}\phi = \mathrm{d}f^*\phi$.
  3. Linearity.
  4. Distributivity over the wedge product.
  5. Functoriality.
  6. $f^*(\mathrm{d}x_I) = \mathrm{d}f_I$.
  7. $\mathrm{d}(f^*\omega) = f^*\mathrm{d}\omega$.
  8. $f^*(\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n) = \det[\partial f_i/\partial x_j]\,\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n$.
- Functoriality is another property just like distributivity. It's another way a function can behave.
- Explicit formula for this pullback: If $\omega = \sum_I \phi_I\,\mathrm{d}x_I$, then

$$
f^*\omega = \sum_I f^*\phi_I \mathrm{d}f_I
$$

- Recall homotopies.
- Recall contractible sets (rigorously, sets that are homotopic to a constant map).
- Defining $\sharp$ via the inner product and $L : \mathbb{R}^n \to (\mathbb{R}^n)^*$ defined by $L(v) = \ell_v$.
- Change of variables formula: If $f : U \to V$ a diffeomorphism and $\phi : V \to \mathbb{R}$ continuous, then

$$
\int_V \phi(y)\,\mathrm{d}y = \int_U (\phi\circ f)(x)|\det Df(x)|\,\mathrm{d}x
$$

- Degree theory.
  - If $f : U \to V$, then

$$
\int_U f^*\omega = \deg(f)\int_V \omega
$$

- A coordinate-based formula for the degree.

$$\int_V \phi(y)\,\mathrm{d}y = \deg(f)\int_U (\phi\circ f)(x)\det(Df(x))\,\mathrm{d}x$$

- $\deg(g\circ f) = \deg(g)\deg(f)$.
  - Proven from the original definition and functoriality.
- orientation preserving diffeomorphisms ($\det[Df(x)] > 0$) have $\deg(f) = +1$.
- orientation reversing diffeomorphisms ($\det[Df(x)] < 0$) have $\deg(f) = -1$.
- Properly homotopic functions have the same degree.
- If f is not surjective, then $\deg(f) = 0$.
- Computing the degree.
  - Take a regular value of f.
  - Find the points in its preimage.
  - Find disjoint neighborhoods around these points.
  - Find functions $f : U_i \to W$.
  - Figure out which are orientation preserving and which aren't.
  - Subtract the number of orientation reversing ones from the number of orientation preserving ones.

- Brouwer fixed-point theorem: If $f : B^n \to B^n$ continuous, then it has a fixed point.
- **Smooth** (function $f : X \to Y$): A function between manifolds $X^n, Y^m \subset \mathbb{R}^N, \mathbb{R}^M$, respectively, such that for all $p \in X$, there is some neighborhood $U_p \subset \mathbb{R}^N$ of p and a smooth map $g_p : U_p \to \mathbb{R}^M$ that is smooth and agrees with f on $X \cap U_p$.
- ***n*-manifold**: A subset $X^n \subset \mathbb{R}^N$ such that for all $p \in X$, there is a neighborhood $V \subset \mathbb{R}^N$ of p, an open set $U \subset \mathbb{R}^n$, and a diffeomorphism $\phi : U \to X \cap V$.
- **Parameterization**: Defined as above. *Also known as* **chart**, **coordinate**.
- Manifold examples:
  1. n-spheres.
  2. Subsets of $\mathbb{R}^n$.
  3. Graphs Γ_f.
  4. Tori.
  5. Product manifolds.
- **Submersion** (at $p \in U$): A smooth map $f : U \to \mathbb{R}^k$, where $U \subset \mathbb{R}^N$ open, such that $Df(p) : \mathbb{R}^N \to \mathbb{R}^k$ is surjective.
- Recall critical points, critical values, and regular values.
  - Remember the difference between critical points and super-critical points (flat surface in the path along a hill vs. the top of the mountain).
- $\boldsymbol{C_f}$: The set of all critical points of f.
- **Tangent space** (of p to X): Intuitively, this is exactly what you would think. It lives in $T_p\mathbb{R}^N$ and comprises all base-pointed vectors tangent to p. Rigorously, we have to relate our parameterization $\phi : U \to \mathbb{R}^N$ to $\mathrm{d}\phi_p : T_p\mathbb{R}^n \to T_p\mathbb{R}^N$. Though I guess what this is really doing is taking a curved tangent vector along the manifold and making it a straight tangent vector in the tangent line/plane/manifold.

- Recall vector fields, integral curves, k-forms, etc. on manifolds. Smoothness is defined for these objects as with functions between manifolds, i.e., by returning to $\mathbb{R}^N$ and then going back to the manifold.
- **Pullback** (of the k-form ω on X along f): If $f : X^n \to Y^m$ where $X \subset \mathbb{R}^N$ and $Y \subset \mathbb{R}^M$, then we may define it as before.
- **Exterior derivative** (at p on X): The tensor defined by

$$(\mathrm{d}\omega)_p = [(\phi^{-1})^* \, \mathrm{d}(\phi^*\omega)]_p$$

  - The properties carry over.
- Sard's theorem: The set of regular values of f is an open dense subset of V.
- The preimage of a regular value is a finite set.
- $f^{-1}(a)$, where a is a regular value, is the set of solutions to the (independent) system of equations $f_i(x) = a_i$ $(i = 1, \ldots, k)$.
- The fundamental theorem of calculus as a special case of Stokes' theorem.
  - If we integrate over $X = [a, b]$, well $\partial X = \{a, b\}$, so

$$\int_a^b \mathrm{d}f = \int_X \mathrm{d}f = \int_{\partial X} f = f(b) - f(a)$$

- Green's theorem.
  - Take a one form

$$\omega = P \, \mathrm{d}x + Q \, \mathrm{d}y$$

  - Applying the exterior derivative generates a corresponding two-form:

$$\mathrm{d}\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x \wedge \mathrm{d}y$$

  - Here, we have that

$$\int_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x \wedge \mathrm{d}y = \int_{\partial U} P \, \mathrm{d}x + Q \, \mathrm{d}y$$

- Domain: An open subset $D \subset X^n$ such that
  1. D is an $(n-1)$=manifold.
  2. $\partial D = \partial(\overline{D})$.
- Existence of domain boundary charts and $\mathbb{H}^n$.
- The n-dimensional volume form is just $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$.
  - The volume of a manifold is equal to the integral over W of the volume form.
- Linearity property of integrals.
- Actually converting between two-forms and one-forms as in Green's theorem (boundary of a circle example).

9.6 Chapter 4: Manifolds and Forms on Manifolds

From Guillemin and Haine (2018).

- For the remainder of Chapter 4, we will look to prove manifold versions of Stokes' theorem and the divergence theorem, and develop a manifold version of degree theory.
- We confine ourselves to **orientable** manifolds for simplicity.
  - Section 4.4 is concerned with explaining orientability.
- **Orientation** (of X): A rule assigning to each $p \in X$ an orientation of T_pX.
  - Essentially, for every $p \in X$, we label one of the two components of the set $\Lambda^n(T_p^*X) \setminus \{0\}$ by $\Lambda^n(T_p^*X)_+$.
- **Plus part** (of $\Lambda^n(T_p^*X)$): The component $\Lambda^n(T_p^*X)_+$.
- **Minus part** (of $\Lambda^n(T_p^*X)$): The other component of $\Lambda^n(T_p^*X) \setminus \{0\}$. *Denoted by* $\boldsymbol{\Lambda^n(T_p^*X)_-}$.
- **Smooth** (orientation of X): An orientation of a manifold X such that for every $p \in X$, there exists a neighborhood U of p and a non-vanishing n-form $\omega \in \Omega^n(U)$ such that for all $q \in U$, $\omega_q \in \Lambda^n(T_q^*X)_+$. *Also known as* $\boldsymbol{C^\infty}$.
- **Reversed orientation** (of X): The orientation of X defined by assigning to each $p \in X$ the opposite orientation to the one already assigned.
- If X is connected and has a smooth orientation, the only smooth orientations of X are that one and its reversed orientation.
  - Rationale: Given any smooth orientation ω, the set of points where it agrees with the given orientation is open — by definition, every oriented $p \in X$ is surrounded by an open neighborhood U on which the orientation form is smooth, and the union of all these U is open. But then the set where ω agrees with the given orientation and the set where ω agrees with the reversed orientation are both open sets whose union is the connected set X. Therefore, one must be empty.
- **Volume form**: A non-vanishing form $\omega \in \Omega^n(X)$ such that one gets from ω a smooth orientation of X by requiring $\omega_p \in \Lambda^n(T_p^*X)_+$ for all $p \in X$.
  - If ω_1, ω_2 are volume forms on X, then $\omega_2 = f_{2,1}\omega_1$, where $f_{2,1}$ is an everywhere positive C^∞ function.
- **Standard orientation** (of U): The orientation defined by $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$, where $U \subset \mathbb{R}^n$ is open.
- The standard orientation of U induces a standard orientation of T_pX as follows.
  - Recall that if $X = f^{-1}(0)$, then $T_pX = \ker(\mathrm{d}f_p)$, where $\mathrm{d}f_p$ is surjective.
  - Thus, $\mathrm{d}f_p$ induces a bijective linear map from $T_p\mathbb{R}^N \setminus T_pX \to T_0\mathbb{R}^k$.
  - Since $T_p\mathbb{R}^N$ and $T_p\mathbb{R}^k$ have standard orientations by the above definition, requiring that the above map be orientation preserving gives $T_p\mathbb{R}^N \setminus T_pX$ an orientation.
  - It follows by Theorem 1.9.9 that T_pX has an orientation.
  - Note: It should be intuitively clear that the smoothness of $\mathrm{d}f_p$ implies that the orientation is smooth, but we will prove this directly, too, in the exercises.
- Theorem 4.4.9: Let X be an oriented submanifold of $\mathbb{R}^N$, B be the inner product on $\mathbb{R}^N$, $B_p : T_pX \times T_pX \to \mathbb{R}$ be the related inner product on T_pX, $e_1, \dots, e_n$ be an orthonormal basis of T_pX, $\sigma_p = e_1^* \wedge \cdots \wedge e_n^*$ be the volume element in $\Lambda^n(T_p^*X)$ associated with B_p, and σ_X be the non-vanishing n-form defined by $p \mapsto \sigma_p$. Then the form σ_X is C^∞ and hence is a volume form.

- **Riemannian volume form**: The volume form described above. *Denoted by* $\boldsymbol{\sigma_X}$.
- **Orientation preserving** (map): A diffeomorphism $f : X \to Y$, where X, Y are oriented n-manifolds, such that for all $p \in X$ and $q = f(p)$, the linear map $\mathrm{d}f_p : T_pX \to T_qY$ is orientation preserving.
- If $\omega = \sigma_Y$, then f is orientation preserving iff $f^*\omega = \sigma_X$.
- Theorem 4.4.11: If Z is an oriented n-manifold and $g : Y \to Z$ a diffeomorphism, then if both f and g are orientation preserving, so is $g \circ f$.
- If X is connected, then $\mathrm{d}f_p$ must be orientation preserving at all points $p \in X$, or orientation reversing at all points $p \in X$.
- **Oriented parameterization** (of U): A parameterization $\phi : U_0 \to U$ that is orientation preserving with respect to the standard orientation of U_0 and the given orientation on U.
- Suppose ϕ isn't oriented. Then we can still convert it to a related parametrization which is oriented, as follows.
  1. Let V_0 be the union of all connected components of U_0 on which ϕ isn't orientation preserving.
  2. Replace V_0 by
  $$V_0^\sharp = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid (x_1, \dots, x_{n-1}, -x_n) \in V_0\}$$
     - This aligns the orientations between the domain of the parameterization and U.
  3. Replace $\phi : U_0 \to U$ by the map $\psi : U_0 \setminus V_0 \cup V_0^\sharp \to U$ defined by
  $$\psi(x_1, \dots, x_n) = \phi(x_1, \dots, x_{n-1}, -x_n)$$
     - This ensures that the parameterization still maps to U (and not to other parts of the manifold or its containing space).
- Suppose $\phi_i : U_i \to U$ $(i = 0, 1)$ are two oriented parameterizations of U. Let $\psi : U_0 \to U_1$ be the diffeomorphism defined by
$$\psi = \phi_1^{-1} \circ \phi_0$$
  - Then by Theorem 4.4.11, ψ is orientation preserving as well.
  - It follows that $\mathrm{d}\psi_p$ is orientation preserving for all $p \in U_0$, and thus
  $$\det[D\psi(p)] > 0$$
  for all $p \in U_0$.
- **Smooth domain**: An open subset D of X such that
  1. The boundary ∂D is an $(n-1)$-dimensional submanifold of X;
  2. The boundary of D coincides with the boundary of the closure of D.
- Examples.
  1. The open ball in $\mathbb{R}^n$, formally defined by $B^n = \{x \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 < 1\}$, whose boundary is the $(n-1)$-sphere.
  2. The n-dimensional annulus (ball with the center removed)
  $$1 < x_1^2 + \cdots + x_n^2 < 2$$
  whose boundary consists of the union of the two spheres
  $$\begin{cases} x_1^2 + \cdots + x_n^2 = 1 \\ x_1^2 + \cdots + x_n^2 = 2 \end{cases}$$

3. Bad example: Consider $\mathbb{R}^n \setminus S^{n-1}$, i.e., n-dimensional space with the $(n-1)$-sphere missing. The boundary of this space is S^{n-1}, but since the closure of this space is just $\mathbb{R}^n$, the boundary of the closure is empty. Hence D is not a smooth domain.
4. The simplest smooth domain is $\mathbb{H}^n$, since we can identify $\partial\mathbb{H}^n$ with $\mathbb{R}^{n-1}$ via the map from $\mathbb{R}^{n-1} \to \mathbb{H}^n$ defined by
$$(x_2, \dots, x_n) \mapsto (0, x_2, \dots, x_n)$$
   - Think of how we identify the straight line $\mathbb{R}^1$ with the boundary of the $\mathbb{H}^2$, which is just the x-axis, a line.

- In fact, we can show that every bounded domain looks (locally) like Example 4, above:
- Theorem 4.4.17: Let D be a smooth domain and $p \in \partial D$. Then there exists a neighborhood $U \subset X$ of p, an open set $U_0 \subset \mathbb{R}^n$, and a diffeomorphism $\psi : U_0 \to U$ such that $\psi(U_0 \cap \mathbb{H}^n) = U \cap D$.
  - See Figure 9.1 and the associated discussion.
  - Guillemin and Haine (2018) proves Theorem 4.4.17.
- ***D*-adapted parameterizable open set**: The open set U characterized by Theorem 4.4.17.
- We now build up to the result that if X is oriented and $D \subset X$ is a smooth domain, then the boundary $Z = \partial D$ of D acquires from X a natural orientation.
  - We first prove Lemma 4.4.24.
  - We then let $V_0 = U_0 \cap \mathbb{R}^{n-1} = \partial(U_0 \cap \mathbb{H}^n)$. It will follow since $\psi|_{V_0}$ maps V_0 onto $U \cap Z$ diffeomorphically, we can orient $U \cap Z$ by requiring that this map be orientation preserving. To prove that this orientation on $U \cap Z$ is *intrinsic*, we need only show that the orientation induced on $U \cap Z$ does not depend on the choice of ψ. We can do this with Theorem 4.4.25.
  - To prove Theorem 4.4.25, we make use of Proposition 4.4.26.
  - Finally, we orient the boundary of D by requiring that for every D-adapted parameterizable open set U, the orientation of Z conincides with the orientation of $U \cap Z$ that we described above.
- Lemma 4.4.24: The diffeomorphism $\psi : U_0 \to U$ in Theorem 4.4.17 can be chosen to be orientation preserving.

  Proof. Uses the $V_0^\sharp$ trick from earlier. □
- Theorem 4.4.25: If $\psi_i : U_i \to U$ $(i = 0, 1)$ are oriented parameterizations of U with the property with the property
$$\psi_i(U_i \cap \mathbb{H}^n) = U \cap D$$
then the restrictions of each ψ_i to $V_i = U_i \cap \mathbb{R}^{n-1} = \partial(U_i \cap \mathbb{H}^n)$ induce compatible orientations on $U \cap X$.
- Proposition 4.4.26: Let $U_0, U_1 \subset \mathbb{R}^n$ open and $f : U_0 \to U_1$ an orientation preserving diffeomorphism which maps $U_0 \cap \mathbb{H}^n$ onto $U_1 \cap \mathbb{H}^n$. If
$$V_i = U_i \cap \mathbb{R}^{n-1} = \partial(U_i \cap \mathbb{H}^n)$$
for $i = 0, 1$, then the restriction $g = f|_{V_0}$ is an orientation preserving diffeomorphism which sends
$$g(V_0) = V_1$$
- We now conclude with a global version of Proposition 4.4.26.
  - What the following proposition posits in layman's terms is that if we have any two smooth domains on any two manifolds, an orientation preserving diffeomorphism between the domains is also an orientation preserving diffeomorphism of their boundaries.
- Proposition 4.4.30: For $i = 1, 2$, let X_i be an oriented manifold, $D_i \subset X_i$ a smooth domain, and $Z_i = \partial D_i$ its boundary. Then if f is an orientation preserving diffeomorphism of (X_1, D_1) onto (X_2, D_2), the restriction $g = f|_{Z_1}$ is an orientation preserving diffeomorphism of Z_1 onto Z_2.