

# MATH 20510 (Analysis in $\mathbb{R}^n$ III – Accelerated) Notes

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# Chapter 1

## Multilinear Algebra

### 1.1 Notes

3/28: • Motivation for the course and an overview of Guillemin and Haine (2018).

3/30: • Plan:

– More (multi)linear algebra.

• Dual spaces.

• Let  $V$  be an  $n$ -dimensional real vector space.

• **Hom** ( $V, \mathbb{R}$ ): The set of all homomorphisms (i.e., linear maps) from  $V$  to  $\mathbb{R}$ . *Also known as  $V^*$ .*

• **Dual basis** (for  $V^*$ ): The set of linear transformations from  $V$  to  $\mathbb{R}$  defined by

$$e_j \mapsto \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

where  $e_1, \dots, e_n$  is a basis of  $V$ . *Denoted by  $e_1^*, \dots, e_n^*$ .*

• Check:  $e_1^*, \dots, e_n^*$  are a basis for  $V^*$ .

– Are they linearly independent? Let  $c_1 e_1^* + \dots + c_n e_n^* = 0 \in \text{Hom}(V, \mathbb{R})$ . Then

$$c_i = (c_1 e_1^* + \dots + c_n e_n^*)(e_i) = 0 \in \mathbb{R}$$

as desired.

– Span? Let  $\varphi \in \text{Hom}(V, \mathbb{R})$ . Then we can verify that

$$\varphi(e_1) e_1^* + \dots + \varphi(e_n) e_n^* = \varphi$$

■ We prove this by verifying the previous statement on the basis of  $V$  (if two linear transformations have the same action on the basis of a vector space, they are equal).

• With a choice of basis for  $V$ , we obtain an isomorphism  $\varepsilon : V \rightarrow V^*$  with the mapping  $e_i \mapsto e_i^*$  for all  $i$ .

• The dual space is known as such because  $(V^*)^* \cong V$ , where  $\cong$  is **canonical** (no choice of basis is needed).

• One more property of dual spaces: **functoriality**.

- Given a linear transformation  $A : V \rightarrow W$ , we know that  $A^* : W^* \rightarrow V^*$  where  $A^*$  is the transpose of  $A$ . In particular, if  $\varphi \in W^*$ , then  $\varphi \circ A : V \rightarrow \mathbb{R}$ .
- Claim:  $A^*$  is linear.
- **Functoriality:** If  $A : V \rightarrow W$  and  $B : W \rightarrow U$ , then  $B^* : U^* \rightarrow W^*$  and  $A^* : W^* \rightarrow V^*$ . The functoriality statement is that  $(B \circ A)^* = A^* \circ B^*$ .
- $A^*$  is the **pullback** (or transpose) of  $A$ .
- Let  $v_1, \dots, v_n$  be a basis for  $V$  and  $w_1, \dots, w_m$  be a basis for  $W$ . Then  $[A]_{v_1, \dots, v_n}^{w_1, \dots, w_m} = A$  is the matrix of the linear transformation  $A$  with respect to these bases. Then if  $v_1^*, \dots, v_n^*$  and  $w_1^*, \dots, w_m^*$  are the corresponding dual bases, then  $[A^*]_{v_1^*, \dots, v_n^*}^{w_1^*, \dots, w_m^*} = A^T$ . We can and should verify this for ourselves.
- This is over the real numbers, so  $A^*$  is just the transpose because there are no complex numbers of which to take the conjugate!
- A generalization: Tensors.
- **$k$ -tensor:** A **multilinear** map

$$T : \underbrace{V \times \cdots \times V}_{k \text{ times}} \rightarrow \mathbb{R}$$

- **Multilinear** (map  $T$ ): A function  $T$  such that

$$\begin{aligned} T(v_1, \dots, v_i^1 + v_i^2, \dots, v_k) &= T(v_1, \dots, v_i^1, \dots, v_k) + T(v_1, \dots, v_i^2, \dots, v_k) \\ T(v_1, \dots, \lambda v_i, \dots, v_k) &= \lambda T(v_1, \dots, v_i, \dots, v_k) \end{aligned}$$

for all  $(v_1, \dots, v_k) \in V^k$ .

- The determinant is an  $n$ -tensor!
- 1-tensors are just covectors.
- $L^k(V)$ : The vector space of all  $k$ -tensors on  $V$ .
- Calculating  $\dim L^k(V)$ . (Answer not given in this class.)
- Let  $A : V \rightarrow W$ . Then  $A^* : L^k(W) \rightarrow L^k(V)$ .
  - Check  $(A \circ B)^* = B^* \circ A^*$ .
- **Multi-index of  $n$  of length  $k$ :** A  $k$ -tuple  $(i_1, \dots, i_k)$  where each  $i_j \in \mathbb{N}$  satisfies  $1 \leq i_j \leq n$  ( $j = 1, \dots, k$ ). Denoted by  $\mathbf{I}$ .
- Let  $e_1, \dots, e_n$  be a basis for  $V$ .
- **Tensor product** (of  $T_1 \in L^k(V)$ ,  $T_2 \in L^l(V)$ ): The function from  $V^{k+l}$  to  $\mathbb{R}$  defined by

$$(v_1, \dots, v_{k+l}) \mapsto T_1(v_1, \dots, v_k) T_2(v_{k+1}, \dots, v_{k+l})$$

Denoted by  $T_1 \otimes T_2$ .

- Claims:
  1.  $T_1 \otimes T_2 \in L^{k+l}(V)$ .
  2.  $A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$ .
- $e_{\mathbf{I}}^*$ : The function  $e_{i_1}^* \otimes \cdots \otimes e_{i_k}^*$ , where  $\mathbf{I} = (i_1, \dots, i_k)$  is a multi-index of  $n$  of length  $k$ .
- Claim: Letting  $\mathbf{I}$  range over all  $n^k$  multi-indices of  $n$  of length  $k$ , the  $e_{\mathbf{I}}^*$  are a basis for  $L^k(V)$ .

- If  $V = \mathbb{R}$ , then  $V = \mathbb{R}e_1$ . If  $V = \mathbb{R}^2$ , then  $V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$ .
- We know that  $L^1(V) = V^* = \mathbb{R}e_1^*$ . Thus,  $e_1^* \otimes e_2^* : V \times V \rightarrow \mathbb{R}$ . Thus, for example,

$$(e_1^* \otimes e_2^*)((1, 2), (3, 4)) = e_1^*(1, 2) \cdot e_2^*(3, 4) = 1 \cdot 4 = 4$$

4/1:

- Plan: More multilinear algebra.
  - Properties of the tensor product.
  - Sign of a permutation.
  - Alternating tensors (lead into differential forms down the road).
- Recall:  $V$  is an  $n$ -dimensional vector space over  $\mathbb{R}$  with basis  $e_1, \dots, e_n$ .  $\mathcal{L}^k(V)$  is the vector space of  $k$ -tensors on  $V$ .  $\{e_I^* \mid I \text{ a multiindex of } n \text{ of length } k\}$  is a basis for  $\mathcal{L}^k(V)$ .

- For example, if  $V = \mathbb{R}^2$  and  $T \in \mathcal{L}^2(V)$ , then

$$T(a_1e_1 + a_2e_2, b_1e_1 + b_2e_2) = a_1b_1T(e_1, e_1) + a_1b_2T(e_1, e_2) + a_2b_1T(e_2, e_1) + a_2b_2T(e_2, e_2)$$

- A basis of  $\mathcal{L}^2(V)$  is

$$\{e_1^* \otimes e_1^*, e_1^* \otimes e_2^*, e_2^* \otimes e_1^*, e_2^* \otimes e_2^*\}$$

- Recall that some basic properties are

$$e_1^* \otimes e_2^*((1, 2), (3, 4)) = 1 \cdot 4 = 4 \qquad e_2^* \otimes e_1^*((1, 2), (3, 4)) = 2 \cdot 3 = 6$$

- It follows by the initial decomposition of  $T$  that

$$T = a_1b_1e_1^* \otimes e_1^* + a_1b_2e_1^* \otimes e_2^* + a_2b_1e_2^* \otimes e_1^* + a_2b_2e_2^* \otimes e_2^*$$

- Important consequence: To know the action of  $T$  on an arbitrary pair of vectors, you need only know its action on the basis; a higher-dimensional generalization of the earlier property.
- Note that

$$e_I^*(e_J) = \delta_{IJ} = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$$

- Basic properties of the tensor product.

1. *Right-distributive*: If  $T_1 \in \mathcal{L}^k(V)$  and  $T_2, T_3 \in \mathcal{L}^\ell(V)$ , then

$$T_1 \otimes (T_2 + T_3) = T_1 \otimes T_2 + T_1 \otimes T_3$$

2. *Left-distributive*: If  $T_1, T_2 \in \mathcal{L}^k(V)$  and  $T_3 \in \mathcal{L}^\ell(V)$ , then

$$(T_1 + T_2) \otimes T_3 = T_1 \otimes T_3 + T_2 \otimes T_3$$

3. *Associative*: If  $T_1 \in \mathcal{L}^k(V)$ ,  $T_2 \in \mathcal{L}^\ell(V)$ , and  $T_3 \in \mathcal{L}^m(V)$ , then

$$T_1 \otimes (T_2 \otimes T_3) = (T_1 \otimes T_2) \otimes T_3 = T_1 \otimes T_2 \otimes T_3$$

4. *Scalar multiplication*: If  $T_1 \in \mathcal{L}^k(V)$ ,  $T_2 \in \mathcal{L}^\ell(V)$ , and  $\lambda \in \mathbb{R}$ , then

$$(\lambda T_1) \otimes T_2 = \lambda(T_1 \otimes T_2) = T_1 \otimes (\lambda T_2)$$

- Note that the tensor product is not commutative.
- Aside: Defining the sign of a permutation.

- $S_A$ : The set of all automorphisms of  $A$  (bijections from  $A$  to  $A$ ), where  $A$  is a set.
- $S_n$ : The set  $S_{[n]}$ .
- Given  $\sigma_1, \sigma_2 \in S_n$ ,  $\sigma_1 \circ \sigma_2 \in S_n$ .
  - Thus,  $S_n$  is a **group**.
- **Transposition**: A function  $\tau \in S_n$  such that

$$\tau(k) = \begin{cases} j & k = i \\ i & k = j \\ k & k \neq i, j \end{cases}$$

for some  $i, j \in [n]$ . Denoted by  $\tau_{i,j}$ .

- Theorem: An element of  $S_n$  can be written as the product of transpositions (i.e., for all  $\sigma \in S_n$ , there exist  $\tau_1, \dots, \tau_m \in S_n$  such that  $\sigma = \tau_1 \circ \dots \circ \tau_m$ ).
- **Sign** (of  $\sigma \in S_n$ ): The number (mod 2) of transpositions whose product equals  $\sigma$ . Denoted by  $(-1)^\sigma$ ,  $\text{sign}(\sigma)$ .
- Theorem: The sign of  $\sigma$  is well-defined. Additionally,

$$(-1)^{\sigma_1 \sigma_2} = (-1)^{\sigma_1} \cdot (-1)^{\sigma_2}$$

- Example: Consider the identity permutation.  $(-1)^\sigma = +1$ . We can think of this as the product of zero transpositions or, for instance, as the product of the two transpositions  $\tau_{1,2} \circ \tau_{1,2}$ . Another example would be  $\tau_{2,3} \circ \tau_{1,2} \circ \tau_{1,2} \circ \tau_{2,3}$ .
- Theorem: Let  $X_i$  be a rational or polynomial function for each  $i \in \mathbb{N}$ . Then

$$(-1)^\sigma = \prod_{i < j} \frac{X_{\sigma(i)} - X_{\sigma(j)}}{X_i - X_j}$$

- Example: For the permutation  $\sigma = (1, 2, 3)$ , we have

$$\begin{aligned} (-1)^\sigma &= \frac{X_{\sigma(1)} - X_{\sigma(2)}}{X_1 - X_2} \cdot \frac{X_{\sigma(1)} - X_{\sigma(3)}}{X_1 - X_3} \cdot \frac{X_{\sigma(2)} - X_{\sigma(3)}}{X_2 - X_3} \\ &= \frac{X_2 - X_3}{X_1 - X_2} \cdot \frac{X_2 - X_1}{X_1 - X_3} \cdot \frac{X_3 - X_1}{X_2 - X_3} \\ &= \frac{-(X_1 - X_2)}{X_1 - X_2} \cdot \frac{-(X_1 - X_3)}{X_1 - X_3} \cdot \frac{X_2 - X_3}{X_2 - X_3} \\ &= -1 \cdot -1 \cdot 1 \\ &= +1 \end{aligned}$$

which squares with the fact that  $\sigma = \tau_{1,2} \circ \tau_{2,3}$ .

- Claims to verify with the above formula:
  1.  $\text{sign}(\sigma) \in \{\pm 1\}$ .
  2.  $\text{sign}(\tau_{i,j}) = -1$ .
  3.  $\text{sign}(\sigma_1 \sigma_2) = \text{sign}(\sigma_1) \text{sign}(\sigma_2)$ .

4/4:

- Plan:
  - More multilinear algebra.

- Alternating  $k$ -tensors — 2 views:
  1. As a subspace of  $\mathcal{L}^k(V)$ .
  2. As a quotient of  $\mathcal{L}^k(V)$ .
- Next time: Operators as alternating tensors.
  - Wedge products.
  - Interior products.
  - Pullbacks.
- Recall:  $\dim V = n$ ,  $e_1, \dots, e_n$  a basis,  $\mathcal{L}^k(V)$  the space of  $k$ -tensors,  $\sigma \in S_k$  implies  $(-1)^\sigma \in \{\pm 1\}$ , key property:  $(-1)^{\sigma_1 \sigma_2} = (-1)^{\sigma_1} (-1)^{\sigma_2}$ .
- $T^\sigma$ : The  $k$ -tensor over  $V$  defined by

$$T^\sigma(v_1, \dots, v_k) = T(v_{\bar{\sigma}(1)}, \dots, v_{\bar{\sigma}(k)})$$

where  $T \in \mathcal{L}^k(V)$ ,  $\sigma \in S_k$ , and  $\bar{\sigma}$  denotes the inverse of  $\sigma$ .

- Example:  $n = 2$ ,  $k = 2$ . Let  $T = e_1^* \otimes e_2^* \in \mathcal{L}^2(V)$ . Let  $\sigma = \tau_{1,2}$ . Then  $T^\sigma = e_2^* \otimes e_1^*$ .
- Another property is  $e_I^\sigma = e_{\sigma(I)}^*$ .
- Properties:
  1.  $T^{\sigma_1 \sigma_2} = (T^{\sigma_1})^{\sigma_2}$ .
  2.  $(T_1 + T_2)^\sigma = T_1^\sigma + T_2^\sigma$ .
  3.  $(cT)^\sigma = cT^\sigma$ .
- Thus, you can view  $\sigma : \mathcal{L}^k(V) \rightarrow \mathcal{L}^k(V)$  as a linear map!
- **Alternating  $k$ -tensor**: A tensor  $T \in \mathcal{L}^k(V)$  such that  $T^\sigma = (-1)^\sigma T$  for all  $\sigma \in S_k$ .
  - Equivalently,  $T^\tau = -T$  for all  $\tau \in S_k$ .
- An example of an alternating 2-tensor when  $\dim V = 2$  is  $T = e_1^* \otimes e_2^* - e_2^* \otimes e_1^*$ .
  - Naturally,  $T_{1,2}^\tau = -T$ , and  $\tau_{1,2}$  is the unique transposition in  $S_2$ .
- $e_1^* \otimes e_2^*$  is *not* an alternating 2-tensor since  $(e_1^* \otimes e_2^*)^\tau = e_2^* \otimes e_1^* \neq (-1)^\tau (e_1^* \otimes e_2^*)$ .
- We can look at  $n = 2$ ,  $k = 1$  for ourselves.
- Note: If  $T_1, T_2$  are both alternating  $k$ -tensors, then  $T_1 + T_2$  is also alternating, as is  $cT_1$  for all  $c \in \mathbb{R}$ .
- $\mathcal{A}^k(V)$ : The vector space of alternating  $k$ -tensors.
- **Alt** ( $T$ ): The function  $\text{Alt} : \mathcal{L}^k(V) \rightarrow \mathcal{L}^k(V)$  defined by

$$\text{Alt}(T) = \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma$$

- Properties:
  1.  $\text{im}(\text{Alt}) = \mathcal{A}^k(V)$ .
  2.  $\mathcal{L}^k(V) / \ker(\text{Alt}) = \Lambda^k(V^*)$  is isomorphic to  $\mathcal{A}^k(V)$ .
  3.  $\text{Alt}(T)^\sigma = (-1)^\sigma \text{Alt}(T)$ .

– Proof:

$$\begin{aligned}
 \text{Alt}(T)^{\sigma'} &= \left( \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma \right)^{\sigma'} \\
 &= \sum_{\sigma \in S_k} (-1)^\sigma T^{\sigma\sigma'} \\
 &= (-1)^{\sigma'} \sum_{\sigma \in S_k} (-1)^{\sigma'} (-1)^\sigma T^{\sigma\sigma'} \\
 &= (-1)^{\sigma'} \sum_{\sigma \in S_k} (-1)^{\sigma\sigma'} T^{\sigma\sigma'} \\
 &= (-1)^{\sigma'} \text{Alt}(T)
 \end{aligned}$$

- The last equality holds because summing over all  $\sigma$  is the same as summing over all  $\sigma' \circ \sigma$ .
- This implies  $\text{im}(\text{Alt}) \leq \mathcal{A}^k(V)$ .

4. If  $T \in \mathcal{A}^k(V)$ ,  $\text{Alt}(T) = k!T$ .

– We have

$$\begin{aligned}
 \text{Alt}(T) &= \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma \\
 &= \sum_{\sigma \in S_k} (-1)^\sigma (-1)^\sigma T \\
 &= \sum_{\sigma \in S_k} T \\
 &= k!T
 \end{aligned}$$

where  $T^\sigma = (-1)^\sigma T$  since  $T \in \mathcal{A}^k(V)$  by definition.

- This implies that  $\text{im}(\text{Alt}) = \mathcal{A}^k(V)$ :  $\text{Alt}(\frac{1}{k!}T) = T \in \mathcal{A}^k(V)$ .

5.  $\text{Alt}(T^\sigma) = \text{Alt}(T)^\sigma$ .

6.  $\text{Alt} : \mathcal{L}^k(V) \rightarrow \mathcal{L}^k(V)$  is linear.

- Warning: Some people take  $\text{Alt}(T) = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma T^\sigma$ <sup>[1]</sup>.
- Example:  $n = k = 2$ . We have

$$\text{Alt}(e_1^* \otimes e_2^*) = e_1^* \otimes e_2^* - e_2^* \otimes e_1^*$$

- **Non-repeating** (multi-index  $I$ ): A multi-index  $I$  such that  $i_{j_1} \neq i_{j_2}$  for all  $j_1 \neq j_2$ .
- **Increasing** (multi-index  $I$ ): A multi-index  $I$  such that  $i_1 < \dots < i_k$ .
- Claim:  $\{\text{Alt}(e_I^*)\}$  where  $I$  is non-repeating and increasing is a basis for  $\mathcal{A}^k(V)$ . There are  $\binom{n}{k}$  of these; thus,  $\dim \mathcal{A}^k(V) = \binom{n}{k}$ .

## 1.2 Chapter 1: Multilinear Algebra

From Guillemin and Haine (2018).

- 3/31: • Guillemin and Haine (2018) defines real vector spaces, the operations on them, their basic properties, and the zero vector.

<sup>1</sup>Klug prefers this convention, but the text takes the other one.



- **Linearly independent** (vectors  $v_1, \dots, v_k$ ): A finite set of vectors  $v_1, \dots, v_k \in V$  such that the map from  $\mathbb{R}^k$  to  $V$  defined by  $(c_1, \dots, c_k) \mapsto c_1 v_1 + \dots + c_k v_k$  is injective.
- **Spanning** (vectors  $v_1, \dots, v_k$ ): We require that the above map is surjective.
- Guillemin and Haine (2018) defines basis, finite-dimensional vector space, dimension, subspace, linear map, and kernel.
- **Image** (of  $A : V \rightarrow W$ ): The range space of  $A$ , a subspace of  $W$ . *Also known as  $\mathbf{im}(A)$ .*
- Guillemin and Haine (2018) defines the matrix of a linear map.
- **Inner product** (on  $V$ ): A map  $B : V \times V \rightarrow \mathbb{R}$  with the following three properties.
  - *Bilinearity*: For vectors  $v, v_1, v_2, w \in V$  and  $\lambda \in \mathbb{R}$ , we have

$$B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w)$$

and

$$B(\lambda v, w) = \lambda B(v, w)$$

- *Symmetry*: For vectors  $v, w \in V$ , we have  $B(v, w) = B(w, v)$ .
- *Positivity*: For every vector  $v \in V$ , we have  $B(v, v) \geq 0$ . Moreover, if  $v \neq 0$ , then  $B(v, v) > 0$ .
- **W-coset**: A set of the form  $\{v + w \mid w \in W\}$ , where  $W$  is a subspace  $V$  and  $v \in V$ . *Denoted by  $v + W$ .*
  - If  $v_1 - v_2 \in W$ , then  $v_1 + W = v_2 + W$ .
  - It follows that the distinct  $W$ -cosets decompose  $V$  into a disjoint collection of subsets of  $V$ .
- **Quotient space** (of  $V$  by  $W$ ): The set of distinct  $W$ -cosets in  $V$ , along with the following definitions of vector addition and scalar multiplication.

$$(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$$

$$\lambda(v + W) = (\lambda v) + W$$

*Denoted by  $V/W$ .*

- **Quotient map**: The linear map  $\pi : V \rightarrow V/W$  defined by

$$\pi(v) = v + W$$

- $\pi$  is surjective.
- Note that  $\ker(\pi) = W$  since for all  $w \in W$ ,  $\pi(w) = w + W = 0 + W$ , which is the zero vector in  $V/W$ .
- If  $V, W$  are finite dimensional, then

$$\dim(V/W) = \dim(V) - \dim(W)$$

- Proposition 1.2.9: Let  $A : V \rightarrow U$  be a linear map. If  $W \subset \ker(A)$ , then there exists a unique linear map  $A^\sharp : V/W \rightarrow U$  with the property that  $A = A^\sharp \circ \pi$ , where  $\pi : V \rightarrow V/W$  is the quotient map.
  - This proposition rephrases in terms of quotient spaces the fact that if  $w \in W$ , then  $A(v+w) = Av$ .
- **Dual space** (of  $V$ ): The set of all linear functions  $\ell : V \rightarrow \mathbb{R}$ , along with the following definitions of vector addition and scalar multiplication.

$$(\ell_1 + \ell_2)(v) = \ell_1(v) + \ell_2(v)$$

$$(\lambda \ell)(v) = \lambda \cdot \ell(v)$$

*Denoted by  $V^*$ .*

- **Dual basis** (of  $e_1, \dots, e_n$  a basis of  $V$ ): The basis of  $V^*$  consisting of the  $n$  functions that take every  $v = c_1 e_1 + \dots + c_n e_n$  to one of the  $c_i$ . Denoted by  $e_1^*, \dots, e_n^*$ . Given by

$$e_i^*(v) = c_i$$

for all  $v \in V$ .

- Claim 1.2.12: If  $V$  is an  $n$ -dimensional vector space with basis  $e_1, \dots, e_n$ , then  $e_1^*, \dots, e_n^*$  is a basis of  $V^*$ .

*Proof.* We will first prove that  $e_1^*, \dots, e_n^*$  spans  $V^*$ . Let  $\ell \in V^*$  be arbitrary. Set  $\lambda_i = \ell(e_i)$  for all  $i \in [n]$ . Define  $\ell' = \sum_{i=1}^n \lambda_i e_i^*$ . Then

$$\ell'(e_j) = \sum_{i=1}^n \lambda_i e_i^*(e_j) = \lambda_j \cdot 1 = \ell(e_j)$$

for all  $j \in [n]$ . Therefore, since  $\ell, \ell'$  take identical values on the basis of  $V$ ,  $\ell = \ell'$ , as desired.

We now prove that  $e_1^*, \dots, e_n^*$  spans  $V^*$ . Let  $\sum_{i=1}^n \lambda_i e_i^* = 0$ . Then for all  $j \in [n]$ ,

$$\lambda_j = \left( \sum_{i=1}^n \lambda_i e_i^* \right) (e_j) = 0$$

as desired. □

- **Transpose** (of  $A$ ): The map from  $W^*$  to  $V^*$  defined by  $\ell \mapsto A^* \ell = \ell \circ A$  for all  $\ell \in W^*$ .
  - Claim 1.2.15: If  $e_1, \dots, e_n$  is a basis of  $V$ ,  $f_1, \dots, f_m$  is a basis of  $W$ ,  $e_1^*, \dots, e_n^*$  and  $f_1^*, \dots, f_m^*$  are the corresponding dual bases, and  $[a_{i,j}]$  is the  $m \times n$  matrix of  $A$  with respect to  $\{e_i\}, \{f_i\}$ , then the linear map  $A^*$  is defined in terms of  $\{f_i^*\}, \{e_i^*\}$  by the transpose matrix  $(a_{j,i})$ .
- 4/4: •  **$V^k$** : The set of all  $k$ -tuples  $(v_1, \dots, v_k)$  where  $v_1, \dots, v_k \in V$  a vector space.

– Note that

$$V^k = \underbrace{V \oplus \dots \oplus V}_{k \text{ times}}$$

where “ $\oplus$ ” denotes the direct sum.

- **Linear** (function in its  $i^{\text{th}}$  variable): A function  $T : V^k \rightarrow \mathbb{R}$  such that the map from  $V$  to  $\mathbb{R}$  defined by  $v \mapsto T(v_1, \dots, v_{i-1}, v, v_{i+1}, \dots, v_k)$  is linear, where all  $v_j$  save  $v_i$  are fixed.
- **$k$ -linear** (function  $T$ ): A function  $T : V^k \rightarrow \mathbb{R}$  that is linear in its  $i^{\text{th}}$  variable for  $i = 1, \dots, k$ . Also known as  **$k$ -tensor**.
- **$\mathcal{L}^k(V)$** : The set of all  $k$ -tensors in  $V$ .
  - Since the sum  $T_1 + T_2$  of two  $k$ -linear functions  $T_1, T_2 : V^k \rightarrow \mathbb{R}$  is just another  $k$ -linear function, and  $\lambda T_1$  is  $k$ -linear for all  $\lambda \in \mathbb{R}$ , we have that  $\mathcal{L}^k(V)$  is a vector space.
- Convention: 0-tensors are just the real numbers. Mathematically, we define

$$\mathcal{L}^0(V) = \mathbb{R}$$

- Note that  $\mathcal{L}^1(V) = V^*$ .
- Defines multi-indices of  $n$  of length  $k$ .
- Lemma 1.3.5: If  $n, k \in \mathbb{N}$ , then there are exactly  $n^k$  multi-indices of  $n$  of length  $k$ .

- **$T_I$** : The real number  $T(e_{i_1}, \dots, e_{i_k})$ , where  $T \in \mathcal{L}^k(V)$ ,  $e_1, \dots, e_n$  is a basis of  $V$ , and  $I$  is a multi-index of  $n$  of length  $k$ .
- Proposition 1.3.7: The real numbers  $T_I$  determine  $T$ , i.e., if  $T, T'$  are  $k$ -tensors and  $T_I = T'_I$  for all  $I$ , then  $T = T'$ .

*Proof.* We induct on  $n$ . For the base case  $n = 1$ ,  $T \in (\mathbb{R}^k)^*$  and we have already proven this result. Now suppose inductively that the assertion is true for  $n - 1$ . For each  $e_i$ , let  $T_i$  be the  $(k - 1)$ -tensor defined by

$$(v_1, \dots, v_{n-1}) \mapsto T(v_1, \dots, v_{n-1}, e_i)$$

Then for an arbitrary  $v = c_1 e_1 + \dots + c_n e_n$ ,

$$T(v_1, \dots, v_{n-1}, v) = \sum_{i=1}^n c_i T_i(v_1, \dots, v_{n-1})$$

so the  $T_i$ 's determine  $T$ . Applying the inductive hypothesis completes the proof.  $\square$

- **Tensor product**: The tensor  $T_1 \otimes T_2$  defined by

$$(T_1 \otimes T_2)(v_1, \dots, v_{k+\ell}) = T_1(v_1, \dots, v_k) T_2(v_{k+1}, \dots, v_{k+\ell})$$

where  $T_1 \in \mathcal{L}^k(V)$  and  $T_2 \in \mathcal{L}^\ell(V)$ .

- Note that by the definition of 0-tensors as real numbers, if  $a \in \mathbb{R}$  and  $T \in \mathcal{L}^k(V)$ , then

$$a \otimes T = T \otimes a = aT$$

- Proposition 1.3.9: Associativity, distributivity of scalar multiplication, and left and right distributive laws for the tensor product.
- **Decomposable** ( $k$ -tensor): A  $k$ -tensor  $T$  for which there exist  $\ell_1, \dots, \ell_k \in V^*$  such that

$$T = \ell_1 \otimes \dots \otimes \ell_k$$

- Defines  $e_I^*$ .
- Theorem 1.3.13:  $V$  a vector space with basis  $e_1, \dots, e_n$  and  $0 \leq k \leq n$  implies the  $k$ -tensors  $e_I^*$  form a basis of  $\mathcal{L}^k(V)$ .

*Proof.* Spanning: Let  $T \in \mathcal{L}^k(V)$  be arbitrary. Define

$$T' = \sum_I T_I e_I^*$$

Since

$$T'_J = T'(e_{j_1}, \dots, e_{j_k}) = \sum_I T_I e_I^*(e_{j_1}, \dots, e_{j_k}) = T_J e_J^*(e_{j_1}, \dots, e_{j_k}) = T_J$$

for all  $J$ , Proposition 1.3.7 asserts that  $T = T'$ . Therefore, since every  $T_I \in \mathbb{R}$ ,  $T = T' \in \text{span}(e_I^*)$ .

Linear independence: Suppose

$$T = \sum_I c_I e_I^* = 0$$

for some set of constants  $c_I \in \mathbb{R}$ . Then

$$0 = T(e_{j_1}, \dots, e_{j_k}) = \sum_I c_I e_I^*(e_{j_1}, \dots, e_{j_k}) = c_J$$

for all  $J$ , as desired.  $\square$

- Corollary 1.3.15: If  $\dim V = n$ , then  $\dim(\mathcal{L}^k(V)) = n^k$ .

*Proof.* Follows immediately from Lemma 1.3.5. □

- **Pullback** (of  $T$  by the map  $A$ ): The  $k$ -tensor  $A^*T : V^k \rightarrow \mathbb{R}$  defined by

$$(A^*T)(v_1, \dots, v_k) = T(Av_1, \dots, Av_k)$$

where  $V, W$  are finite-dimensional vector spaces,  $A : V \rightarrow W$  is linear, and  $T \in \mathcal{L}^k(W)$ .

- Proposition 1.3.18: The map  $A^* : \mathcal{L}^k(W) \rightarrow \mathcal{L}^k(V)$  defined by  $T \mapsto A^*T$  is linear.
- Identities:

- If  $T_1 \in \mathcal{L}^k(W)$  and  $T_2 \in \mathcal{L}^m(W)$ , then

$$A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$$

- If  $U$  is a vector space,  $B : U \rightarrow V$  is linear, and  $T \in \mathcal{L}^k(W)$ , then

$$(AB)^*T = B^*(A^*T)$$

# References

Guillemin, V., & Haine, P. J. (2018). *Differential forms* [[https://math.mit.edu/classes/18.952/2018SP/files/18.952\\_book.pdf](https://math.mit.edu/classes/18.952/2018SP/files/18.952_book.pdf)].