Week 6

Operations on Forms

6.1 The Pullback

5/4:

- Klug got his flight to his wedding paid for by giving a talk at a nearby institution!
- Homework 3 now due Monday (the stuff will be on the exam though).
- Office hours today from 5:00-6:00.
- Exam Friday.
- Next week will be Chapter 3.
  - Integration of top-dimensional forms, i.e., if we're in 2D space, we'll integrate 2D forms; in 3D space, we'll integrate 3D forms, etc.
- Pullbacks of k-forms.
  - Let $F : U \to V$ be smooth where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$.
  - This induces $F^* : \Omega^k(V) \to \Omega^k(U)$.
  - We have $\mathrm{d}F_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$, which also induces $\mathrm{d}F_p^* : \Lambda^k(T^*_{F(p)}\mathbb{R}^m) \to \Lambda^k(T^*_p\mathbb{R}^n)$.
  - Note that F^* maps $\omega \mapsto F^*\omega$ where $F^*\omega_p = \mathrm{d}F_p^*\omega_{F(p)}$.
- In formulas...

$$\omega = \sum_I \varphi_I \, \mathrm{d}x_I \qquad\qquad F^*\omega = \sum_I F^*\varphi_I \, \mathrm{d}F_I$$

  - φ_I is just a function.
  - Recall that $F^*\varphi_I = \varphi_I \circ F : U \to \mathbb{R}$.
  - If $I = (i_1, \dots, i_k)$, then $\mathrm{d}F_I = \mathrm{d}F_{i_1} \wedge \cdots \wedge \mathrm{d}F_{i_k}$.
  - Recall that $F_{i_j} : U \to \mathbb{R}$ sends $x \mapsto x_{i_j}$ (the component of F).
  - There is a derivation that gets you from the above abstract definition of the pullback to the below concrete form.
- Note that $\mathrm{d}F_p$ is the kind of thing we worked on last quarter?
- Properties of the pullback (let $U \xrightarrow{F} V \xrightarrow{G} W$).
  1. F^* is linear.
  2. $F^*(\omega_1 \wedge \omega_2) = F^*\omega_1 \wedge F^*\omega_2$.

3. $(F \circ G)^* = G^* \circ F^*$.
4. $\mathrm{d} \circ F^* = F^* \circ \mathrm{d}$. *picture; Commutative diagram*

- Properties 1-3 follow from our Chapter 1 pointwise properties.
  - They also yield the explicit formula for $F^*\omega$ given above.
- Property 4:
  - First: Recall that the following diagram holds. *picture*
  - Check: $\mathrm{d}F_I = F^* \mathrm{d}x_I$ where $\mathrm{d}F_{i_1} \wedge \cdots \wedge \mathrm{d}F_{i_k}$ where $I = (i_1, \dots, i_k)$.
  - Now we prove the property by taking

$$\begin{aligned}
\mathrm{d}F_I &= F^*(\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}) \\
&= F^* \mathrm{d}x_{i_1} \wedge \cdots \wedge F^* \mathrm{d}x_{i_k} && \text{Property 2} \\
&= \mathrm{d}(F^* x_{i_1}) \wedge \cdots \wedge \mathrm{d}(F^* x_{i_k}) \\
&= \mathrm{d}F_{i_1} \wedge \cdots \wedge \mathrm{d}F_{i_k}
\end{aligned}$$

  - Now we have that if $\omega = \sum_I \varphi_I \, \mathrm{d}x_I$, then

$$\begin{aligned}
\mathrm{d}(F^*\omega) &= \mathrm{d}\left(\sum_I F^*\varphi_I \, \mathrm{d}F_I\right) \\
&= \sum_I \mathrm{d}(F^*\varphi_I \wedge \mathrm{d}F_I) \\
&= \sum_I \mathrm{d}(F^*\varphi_I) \wedge \mathrm{d}F_I \\
&= \sum_I F^* \mathrm{d}\varphi_I \wedge F^* \mathrm{d}x_I \\
&= \sum_I F^*(\mathrm{d}\varphi_I \wedge \mathrm{d}x_I) \\
&= F^*\left(\sum_I \mathrm{d}\varphi_I \wedge \mathrm{d}x_I\right) \\
&= F^* \mathrm{d}\omega
\end{aligned}$$

    where the second equality holds by the linearity of d and we insert the wedge because multiplication is the same as wedging a zero-form, the third equality holds by the product rule $\mathrm{d}^2 = 0$, the fourth equality holds because d and F^* commute for 0-forms, and the fifth equality holds by Property 2.
- $\mathrm{d}^2 = 0$ generalizes curl and all of those identities.
- Two other operations.
- **Interior product**: Given v a vector field on U, we have $\iota_v : \Omega^k(U) \to \Omega^{k-1}(U)$ that sends $\omega \mapsto \iota_v\omega$.
  - Its properties follow from the properties of the pointwise stuff.
    1. $\iota_v(\omega_1 + \omega_2) = \iota_v\omega_1 + \iota_v\omega_2$.
    2. $\iota_v(\omega_1 \wedge \omega_2) = \cdots$.
    3. $\iota_v \circ \iota_w = -\iota_w \circ \iota_v$.
- **Lie derivative**: If v is a vector field on U, then $L_v : \Omega^k(U) \to \Omega^k(U)$ sends $\omega \mapsto \mathrm{d}\iota_v\omega + \iota_v \, \mathrm{d}\omega$.
  - Note that we use ι to drop the index and d to raise it back up, and then vice versa.
- Check: Agrees with previous definition for Ω^0.

- Cartan's magic formula is what we're taking to be the definition of the Lie derivative.
- Properties.
  1. $L_v \circ \mathrm{d} = \mathrm{d} \circ L_v$.
  2. $L_v(\omega \wedge \eta) = L_v\omega \wedge \eta + \omega \wedge L_v\eta$.
     - Proof:

$$\begin{aligned}
\mathrm{d}(\iota_v \mathrm{d} + \mathrm{d}\iota_v) &= \mathrm{d}\iota_v \mathrm{d} \\
&= \iota_v(\iota_v \mathrm{d} + \mathrm{d}\iota_v)
\end{aligned}$$

- We should find an explicit formula for the Lie derivative.
  - Your vector field will be of the form

$$v = \sum f_i \, \partial/\partial x_i$$

  - Your form will be of the form

$$\omega = \sum \varphi_I \, \mathrm{d}x_I$$

6.2 Chapter 2: Differential Forms

From Guillemin and Haine (2018).

5/5: - **Interior product** (of $\boldsymbol{v}$ with ω): The $(k-1)$-form on U defined as follows, where $U \subset \mathbb{R}^n$ open, $\boldsymbol{v}$ a vector field on U, and $\omega \in \Omega^k(U)$. *Denoted by* $\boldsymbol{\iota_v \omega}$. *Given by*

$$p \mapsto \iota_{\boldsymbol{v}(p)} \omega_p$$

- By definition, $\iota_{\boldsymbol{v}(p)}\omega_p \in \Lambda^{k-1}(T_p^*\mathbb{R}^n)$.