MATH 20510 (Analysis in $\mathbb{R}^n$ III – Accelerated) Problem Sets

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Contents

1 Multilinear Algebra

From Guillemin and Haine (2018).

Chapter 1

1.2.iv. Let U, V, and W be vector spaces and let $A : V \to W$ and $B : U \to V$ be linear mappings. Show that $(AB)^* = B^* A^*$.

1.2.v. Let $V = \mathbb{R}^2$ and let W be the x_1-axis, i.e., the one-dimensional subspace

$$\{(x_1, 0) \mid x_1 \in \mathbb{R}\}$$

of $\mathbb{R}^2$.

(1) Show that the W-cosets are the lines $x_2 = a$ parallel to the x_1-axis.

(2) Show that the sum of the cosets $x_2 = a$ and $x_2 = b$ is the coset $x_2 = a + b$.

(3) Show that the scalar multiple of the coset $x_2 = c$ by the number λ is the coset $x_2 = \lambda c$.

1.2.vi. (1) Let $(V^*)^*$ be the dual of the vector space V^*. For every $v \in V$, let $\mathrm{ev}_v : V^* \to \mathbb{R}$ be the **evaluation function** $\mathrm{ev}_v(\ell) = \ell(v)$. Show that the ev_v is a linear function on V^*, i.e., an element of $(V^*)^*$, and show that the map $\mathrm{ev} = \mathrm{ev}_{(-)} : V \to (V^*)^*$ defined by $v \mapsto \mathrm{ev}_v$ is a linear map of V into $(V^*)^*$.

(2) If V is finite dimensional, show that the map ev is bijective. Conclude that there is a natural identification of V with $(V^*)^*$, i.e., that V and $(V^*)^*$ are two descriptions of the same object. (Hint: $\dim(V^*)^* = \dim V^* = \dim V$, so since $\dim(V) = \dim(\ker(A)) + \dim(\mathrm{im}(A))$, it suffices to show that ev is injective.)

1.2.xi. Let V be a vector space.

(1) Let $B : V \times V \to \mathbb{R}$ be an inner product on V. For all $v \in V$, let $\ell_v : V \to \mathbb{R}$ be the function $\ell_v(w) = B(v, w)$. Show that ℓ_v is linear, and show that the map $L : V \to V^*$ defined by $v \mapsto \ell_v$ is a linear mapping.

(2) If V is finite dimensional, prove that L is bijective. Conclude that if V has an inner product, one gets from it a natural identification of V with V^*. (Hint: Since $\dim V = \dim V^*$ and $\dim(V) = \dim(\ker(A)) + \dim(\mathrm{im}(A))$, it suffices to show that $\ker(L) = 0$. Now note that if $v \neq \mathbf{0}$, then $\ell_v(v) = B(v, v)$ is a positive number.)

1.3.i. Verify that there are exactly n^k multi-indices of length k.

1.3.ii. Prove that the map $A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(W)$ defined by $T \mapsto A^*T$ is linear.

1.3.iii. Verify that

$$A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$$

1.3.iv. Verify that

$$(AB)^*T = B^*(A^*T)$$

1.3.vii. Let T be a k-tensor and v be a vector. Define $T_v : V^{k-1} \to \mathbb{R}$ by

$$T_v(v_1, \ldots, v_{k-1}) = T(v, v_1, \ldots, v_{k-1})$$

Show that T_v is a $(k-1)$-tensor.

1.3.viii. Show that if T_1 is an r-tensor and T_2 is an s-tensor, then if $r > 0$,

$$(T_1 \otimes T_2)_v = (T_1)_v \otimes T_2$$

1.3.ix. Let $A : V \to W$ be a linear map, let $v \in V$, and let $w = Av$. Show that for all $T \in \mathcal{L}^k(W)$,

$$A^*(T_w) = (A^*T)_v$$

1.4.i. Show that there are exactly $k!$ permutations of order k. (Hint: Induction on k: Let $\sigma \in S_k$, and let $\sigma(k) = i$ ($1 \leq i \leq k$). Show that $\tau_{i,k}\sigma$ leaves k fixed and hence is, in effect, a permutation of Σ_{k-1}.)

1.4.ii. Prove that if $\tau \in S_k$ is a transposition, $(-1)^\tau = -1$. Deduce from this that if σ is the product of an odd number of transpositions, then $(-1)^\sigma = -1$, and if σ is the product of an even number of transpositions, then $(-1)^\sigma = +1$.

1.4.iii. Prove that the assignment $T \mapsto T^\sigma$ is a linear map $\mathcal{L}^k(V) \to \mathcal{L}^k(V)$.

1.4.vi. Show that every one of the six elements of S_3 is either a transposition or can be written as a product of two transpositions.

1.4.ix. Let $A : V \to W$ be a linear mapping. Show that if $T \in \mathcal{A}^k(W)$, then $A^*T \in \mathcal{A}^k(V)$.

1.5.i. A k-tensor $T \in \mathcal{L}^k(V)$ is **symmetric** if $T^\sigma = T$ for all $\sigma \in S_k$. Show that the set $\mathcal{S}^k(V)$ of symmetric k-tensors is a vector subspace of $\mathcal{L}^k(V)$.

1.6.i. Verify the following three equations, where $\lambda \in \mathbb{R}$.

(1) $\lambda(\omega_1 \wedge \omega_2) = (\lambda\omega_1) \wedge \omega_2 = \omega_1 \wedge (\lambda\omega_2)$.

(2) $(\omega_1 + \omega_2) \wedge \omega_3 = \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_3$.

(3) $\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$.

1.6.ii. Verify the following multiplicative law for the wedge product.

$$\omega_1 \wedge \omega_2 = (-1)^{rs}\omega_2 \wedge \omega_1$$

1.6.iv. If $\omega, \mu \in \Lambda^r(V^*)$, prove that

$$(\omega + \mu)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} \omega^\ell \wedge \mu^{k-\ell}$$

(Hint: As in freshman calculus, prove this binomial theorem by induction using the identity $\binom{k}{\ell} = \binom{k-1}{\ell-1} + \binom{k-1}{\ell}$.)

1.7.i. Prove that if T is the decomposable k-tensor $\ell_1 \otimes \cdots \otimes \ell_k$, then

$$\imath_v T = \sum_{r=1}^{k} (-1)^{r-1} \ell_r(v) \ell_1 \otimes \cdots \otimes \hat{\ell}_r \otimes \cdots \otimes \ell_k$$

where the hat over ℓ_r means that ℓ_r is deleted from the tensor product.

1.7.ii. Prove that if $T_1 \in \mathcal{L}^p(V)$ and $T_2 \in \mathcal{L}^q(V)$, then

$$\imath_v(T_1 \otimes T_2) = \imath_v T_1 \otimes T_2 + (-1)^p T_1 \otimes \imath_v T_2$$

1.7.iii. Show that if $T \in \mathcal{A}^k(V)$, then $\imath_v T = kT_v$, where T_v is defined as in Exercise 1.3.vii. In particular, conclude that $\imath_v T \in \mathcal{A}^{k-1}(V)$. (See Exercise 1.4.viii, which asserts that $T \in \mathcal{A}^k(V)$ implies $T_v \in \mathcal{A}^{k-1}(V)$.)

1.8.i. Verify the following assertions.

(1) The map $A^* : \Lambda^k(W^*) \to \Lambda^k(V^*)$ sending $\omega \mapsto A^*\omega$ is linear.

(2) If $\omega_i \in \Lambda^{k_i}(W^*)$ $(i = 1, 2)$, then

$$A^*(\omega_1 \wedge \omega_2) = A^*(\omega_1) \wedge A^*(\omega_2)$$

(3) If U is a vector space and $B : U \to V$ is a linear map, then for $\omega \in \Lambda^k(W^*)$,

$$B^* A^* \omega = (AB)^* \omega$$

1.8.ii. Deduce from the fact "$A : V \to V$ not surjective implies $\det(A) = 0$" a well-known fact about determinants of $n \times n$ matrices: If two columns are equal, the determinant is zero.

1.8.iv. Deduce from Exercise 1.8.i another well-known fact about determinants of $n \times n$ matrices: If $(b_{i,j})$ is the inverse of $[a_{i,j}]$, its determinant is the inverse of the determinant of $[a_{i,j}]$.

1.8.v. Extract from the formula $\det([a_{i,j}]) = \sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$ the following well-known formula for determinants of 2×2 matrices.

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

1.9.i. Prove that if $e_1, \ldots, e_n$ is a positively oriented basis of V, then the basis $e_1, \ldots, e_{i-1}, -e_i, e_{i+1}, \ldots, e_n$ is negatively oriented.

1.9.ii. Show that the argument in the proof of Theorem 1.9.9 can be modified to prove that if V and W are oriented, then these orientations induce a natural orientation on V/W.

References

Guillemin, V., & Haine, P. J. (2018). *Differential forms* [https://math.mit.edu/classes/18.952/2018SP/files/18.952_book.pdf].