Week 7

Integration on Forms

7.1 Chapter 3: Integration on Forms

From Guillemin and Haine (2018).

5/26:

- **Change of variables formula**: If $U,V\subset\mathbb{R}^n$ open and $f:U\to V$ a C^1 diffeomorphism, then for every $\phi:V\to\mathbb{R}$ continuous, the left integral below exists iff the right integral below exists and if they are equal.

$$\int_V \phi(y)\,\mathrm{d}y \qquad\qquad \int_U (\phi\circ f)(x)|\det Df(x)|\,\mathrm{d}x$$

  - Guillemin and Haine (2018) refers us elsewhere for some types of proofs. They, instead, will focus on Lax's differential-forms-heavy proof that, nevertheless, can be modified to avoid references to differential forms and be based solely on the language of elementary multivariable calculus[1].
  - Lax's proof is also desirable since it leads into a change of variables theorem for maps other than diffeomorphisms, and involves a topological invariant (the **degree** of a map), thereby providing a first brush with topology.

- Henceforth, let f be a C^∞ diffeomorphism.
- **Support** (of ν): The following set, where $\nu\in\Omega^k(\mathbb{R}^n)$. *Denoted by* $\boldsymbol{\mathrm{supp}(\nu)}$. *Given by*

$$\mathrm{supp}(\nu)=\overline{\{x\in\mathbb{R}^n:\nu_x\neq 0\}}$$

- **Compactly supported** (k-form ν): A k-form ν for which $\mathrm{supp}(\nu)$ is compact.
- $\boldsymbol{\Omega_c^k(\mathbb{R}^n)}$: The set of all C^∞ k-forms which are compactly supported.
- $\boldsymbol{\Omega_c^k(U)}$: The set of all C^∞ k-forms which are compactly supported and $\mathrm{supp}(\omega)\in U$ for all $\omega\in\Omega_c^k(U)$, where $U\subset\mathbb{R}^n$ open.
- **Integral** (of ω over $\mathbb{R}^n$): The usual integral of f over $\mathbb{R}^n$, where $\omega=f\,\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n$ is compactly supported and $f\in C_0^\infty(\mathbb{R}^n)$[2]. *Denoted by* $\boldsymbol{\int_{\mathbb{R}^n}\omega}$. *Given by*

$$\int_{\mathbb{R}^n}\omega=\int_{\mathbb{R}^n}f\,\mathrm{d}x$$

- **Property $\boldsymbol{P}$** (possessing set U): The property of a set U that for every form $\omega\in\Omega_c^m(U)$ such that $\int_U\omega=0$, $\omega\in\mathrm{d}\big(\Omega_c^{m-1}(U)\big)$.

[1]Guillemin and Haine (2018) recommends we read the original article; could be worthwhile if I can find it!
[2]Recall that $C_0^\infty(\mathbb{R}^n)$ is the space of all bump functions on $\mathbb{R}^n$.

- Theorem 3.2.3: Let $U \subset \mathbb{R}^{n-1}$ open and $A \subset \mathbb{R}$ an open interval. Then if U has property P, $U \times A$ does as well.

  Proof. ... $\square$

- Theorem 3.2.2 (Poincaré lemma for rectangle): Let ω be a compactly supported n-form with $\operatorname{supp}(\omega) \subset \operatorname{int}(Q)$, where $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Then the following are equivalent.

  1. $\int \omega = 0$.
  2. There exists a compactly supported $(n-1)$-form μ with $\operatorname{supp}(\mu) \subset \operatorname{int}(Q)$ satisfying $\mathrm{d}\mu = \omega$.

  Proof ($2 \Rightarrow 1$). As given in class on 5/2.
  One additional note: We can compute $\int_{\mathbb{R}^n} \partial f_i / \partial x_i \, \mathrm{d}x$ by Fubini's theorem. $\square$

  Proof ($1 \Rightarrow 2$). We induct on $\dim Q$. For the base case $n = 1$, the interval A has property P by Exercise 3.2.i. Now suppose inductively that we have proven that $A_1 \times \cdots \times A_{n-1}$ has property P, where $A_i = (a_i, b_i)$. Then by Theorem 3.2.3, $A_1 \times \cdots \times A_n = A_1 \times \cdots \times A_{n-1} \times A_n$ has property P. $\square$

- We now seek to generalize Theorem 3.2.2 to arbitrary connected open subsets of $\mathbb{R}^n$.
- Theorem 3.3.1 (Poincaré lemma for compactly supported forms): Let $U \subset \mathbb{R}^n$ connected and open, and let let $\omega \in \Omega_c^n(U)$ satisfy $\operatorname{supp}(\omega) \subset U$. Then the following are equivalent.

  1. $\int_{\mathbb{R}^n} \omega = 0$.
  2. There exists a compactly supported $(n-1)$-form μ with $\operatorname{supp}(\mu) \subset U$ and $\omega = \mathrm{d}\mu$.

  Proof ($2 \Rightarrow 1$). The support of μ is contained in a large rectangle, so the integral of $\mathrm{d}\mu$ is zero by Theorem 3.2.2. $\square$

  Proof ($1 \Rightarrow 2$). ... $\square$

- **Proper** (continuous map): A continuous map $f : U \to V$, where $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^k$ are open, such that for every $K \subset V$ compact, the preimage $f^{-1}(K)$ is compact.
  - Proper mappings have many nice properties (see the Exercises 3.4).
  - One example is that if $f \in C^\infty$ and $\omega \in \Omega_c^k(V)$ satisfies $\operatorname{supp}(\omega) \subset V$, then $f^*\omega \in \Omega_c^k(U)$.
- **Degree** (of f): The topological invariant of $f : U \to V$, a C^∞ map with $U, V \subset \mathbb{R}^n$ open and connected, defined as follows for all $\omega \in \Omega_c^n(V)$. *Denoted by* $\boldsymbol{\deg(f)}$. *Given by*

$$\int_U f^*\omega = \deg(f) \int_V \omega$$

- A coordinate-based formula for the degree.
  - Let $\omega = \phi(y) \, \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_n$ and $x \in U$.
  - Then
$$f^*\omega = (\phi \circ f)(x) \det(Df(x)) \, \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$$
  - It follows that
$$\int_V \phi(y) \, \mathrm{d}y = \deg(f) \int_U (\phi \circ f)(x) \det(Df(x)) \, \mathrm{d}x$$
- We now seek to prove that the degree, as defined, exists for suitable functions.

Proof. ... ☐

- Proposition 3.4.4: Let $U,V,W\subset\mathbb{R}^n$ open and connected, and $f:U\to V$ and $g:V\to W$ proper C^∞ maps. Then
$$\deg(g\circ f)=\deg(g)\deg(f)$$

Proof. Let $\omega\in\Omega_c^n(W)$. Then since $(g\circ f)^*\omega=f^*(g^*\omega)$,
$$\begin{aligned}\deg(g\circ f)\int_W\omega &= \int_U (g\circ f)^*\omega\\ &= \int_U f^*(g^*\omega)\\ &= \deg(f)\int_V g^*\omega\\ &= \deg(f)\deg(g)\int_W\omega\\ \deg(g\circ f) &= \deg(g)\deg(f)\end{aligned}$$
as desired. ☐

- Theorem 3.4.6: Let A be a non-singular $n\times n$ matrix and $f_A:\mathbb{R}^n\to\mathbb{R}^n$ the associated linear mapping. Then
$$\deg(f_A)=\begin{cases}+1 & \det(A)>0\\ -1 & \det(A)<0\end{cases}$$
  - Note that the non-singularity condition allows us to ignore the case $\det(A)=0$ (since singular matrices have zero determinant).
- Theorem 3.4.7: Let $B\subset V$ compact and $A=f^{-1}(B)$. Then for all U_0 open with $A\subset U_0\subset U$, there exists V_0 open with $B\subset V_0\subset V$ and $f^{-1}(V_0)\subset U_0$.
- **Orientation-preserving** (diffeomorphism): A diffeomorphism $f:U\to V$, where $U,V\subset\mathbb{R}^n$ are open and connected, such that $\det[Df(x)]>0$ for all $x\in U$.
- **Orientation-reversing** (diffeomorphism): A diffeomorphism $f:U\to V$, where $U,V\subset\mathbb{R}^n$ are open and connected, such that $\det[Df(x)]<0$ for all $x\in U$.
  - We know that $\det[Df(x)]$ is nonzero (if it were zero at some x, one of f and its inverse would not be differentiable there, contradicting the definition of a diffeomorphism).
  - This combined with the fact that the determinant is a continuous function of x proves that its sign is the same for all $x\in U$.
  - Thus, orientation-preserving and orientation-reversing are well-defined.
- Theorem 3.5.1: The degree of f is $+1$ if f is orientation-preserving and -1 if f is orientation-reversing.

Proof. ... ☐

- Theorem 3.5.2: Let $\phi:V\to\mathbb{R}$ be a compactly supported continuous function. Then
$$\int_U(\phi\circ f)(x)|\det(Df(x))|\,\mathrm{d}x=\int_V\phi(y)\,\mathrm{d}y$$

Proof. ... ☐

- Guillemin and Haine (2018) goes through the nitty gritty analytic details of the proof.