1 Multilinear Algebra

From Guillemin and Haine (2018).

Chapter 1

1.2.iv. Let U, V, and W be vector spaces and let $A : V \to W$ and $B : U \to V$ be linear mappings. Show that $(AB)^* = B^*A^*$.

Proof. Clearly, both $(AB)^*$ and B^*A^* send W^* to U^*. Thus, we need only verify that both maps have the same action on every element of W^*.

Let $\ell \in W^*$ be arbitrary. Then

$$(AB)^*\ell = \ell \circ AB = (\ell \circ A) \circ B = A^*\ell \circ B$$

where $A^*\ell \in V^*$. It follows in a similar fashion that

$$A^*\ell \circ B = B^*(A^*\ell) = (B^*A^*)\ell$$

where we have the last equality above by the associativity of the composition operation. Transitivity between the first and second equations above finishes the proof. □

1.2.v. Let $V = \mathbb{R}^2$ and let W be the x_1-axis, i.e., the one-dimensional subspace

$$\{(x_1, 0) \mid x_1 \in \mathbb{R}\}$$

of $\mathbb{R}^2$.

(1) Show that the W-cosets are the lines $x_2 = a$ parallel to the x_1-axis.

Proof. Let $v + W \in V/W$ be arbitrary. Let $v = (v_1, v_2)$. Then

$$\begin{aligned}
v + W &= \{v + w \mid w \in \{(x_1, 0) \mid x_1 \in \mathbb{R}\}\} \\
&= \{v + (x_1, 0) \mid x_1 \in \mathbb{R}\} \\
&= \{(v_1 + x_1, v_2) \mid x_1 \in \mathbb{R}\} \\
&= \{(x_1, v_2) \mid x_1 \in \mathbb{R}\}
\end{aligned}$$

Since every line $x_2 = a$ is a set of the form $\{(x_1, a) \mid x_1 \in \mathbb{R}\}$, we have that $v + W$ is equal to the line $x_2 = v_2$, as desired. □

(2) Show that the sum of the cosets $x_2 = a$ and $x_2 = b$ is the coset $x_2 = a + b$.

Proof. By part (1), every line $x_2 = a$ is a set of the form $(0, a) + W$. Therefore, by the definition of addition on V/W,

$$\begin{aligned}
[(0, a) + W] + [(0, b) + W] &= [(0, a) + (0, b)] + W \\
&= (0, a + b) + W
\end{aligned}$$

as desired. □

(3) Show that the scalar multiple of the coset $x_2 = c$ by the number λ is the coset $x_2 = \lambda c$.

Proof. Proceeding in a similar manner to part (2), we have that

$$\begin{aligned}
\lambda[(0, c) + W] &= [\lambda(0, c)] + W \\
&= (0, \lambda c) + W
\end{aligned}$$

as desired. □

1.2.vi. (1) Let $(V^*)^*$ be the dual of the vector space V^*. For every $v \in V$, let $\mathrm{ev}_v : V^* \to \mathbb{R}$ be the **evaluation function** $\mathrm{ev}_v(\ell) = \ell(v)$. Show that the ev_v is a linear function on V^*, i.e., an element of $(V^*)^*$, and show that the map $\mathrm{ev} = \mathrm{ev}_{(-)} : V \to (V^*)^*$ defined by $v \mapsto \mathrm{ev}_v$ is a linear map of V into $(V^*)^*$.

Proof. Let $v \in V$, $\ell_1, \ell_2, \ell \in V^*$, and $\lambda \in \mathbb{R}$ be arbitrary. Then

$$\begin{aligned} \mathrm{ev}_v(\ell_1 + \ell_2) &= (\ell_1 + \ell_2)(v) \\ &= \ell_1(v) + \ell_2(v) \\ &= \mathrm{ev}_v(\ell_1) + \mathrm{ev}_v(\ell_2) \end{aligned} \qquad \begin{aligned} \mathrm{ev}_v(\lambda\ell) &= (\lambda\ell)(v) \\ &= \lambda\ell(v) \\ &= \lambda\,\mathrm{ev}_v(\ell) \end{aligned}$$

so ev_v is linear, as desired.
Let $v_1, v_2, v \in V$, $\ell \in V^*$, and $\lambda \in \mathbb{R}$ be arbitrary. Then

$$\begin{aligned} \mathrm{ev}(v_1 + v_2)(\ell) &= \mathrm{ev}_{v_1+v_2}(\ell) \\ &= \ell(v_1 + v_2) \\ &= \ell(v_1) + \ell(v_2) \\ &= \mathrm{ev}_{v_1}(\ell) + \mathrm{ev}_{v_2}(\ell) \\ &= \mathrm{ev}(v_1)(\ell) + \mathrm{ev}(v_2)(\ell) \\ &= [\mathrm{ev}(v_1) + \mathrm{ev}(v_2)](\ell) \end{aligned} \qquad \begin{aligned} \mathrm{ev}(\lambda v)(\ell) &= \mathrm{ev}_{\lambda v}(\ell) \\ &= \ell(\lambda v) \\ &= \lambda\ell(v) \\ &= \lambda\,\mathrm{ev}_v(\ell) \\ &= \lambda\,\mathrm{ev}(v)(\ell) \\ &= [\lambda\,\mathrm{ev}(v)](\ell) \end{aligned}$$

Thus, $\mathrm{ev}(v_1 + v_2)$ and $\mathrm{ev}(v_1) + \mathrm{ev}(v_2)$, and $\mathrm{ev}(\lambda v)$ and $\lambda\,\mathrm{ev}(v)$ have the same action pairwise on every $\ell \in V^*$. Consequently, the two pairs of functions in V^* are both equal pairwise. Therefore, ev itself is linear. □

(2) If V is finite dimensional, show that the map ev is bijective. Conclude that there is a natural identification of V with $(V^*)^*$, i.e., that V and $(V^*)^*$ are two descriptions of the same object. (Hint: $\dim(V^*)^* = \dim V^* = \dim V$, so since $\dim(V) = \dim(\ker(A)) + \dim(\mathrm{im}(A))$, it suffices to show that ev is injective.)

Proof. Taking the hint, we seek to show that ev is injective. Suppose $v_1 \neq v_2$. WLOG let $v_2 \neq 0$. Let $\ell : V \to \mathbb{R}$ be defined by

$$\ell(v) = \begin{cases} \|v\| & v = \lambda v_2 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned} \ell(v_1) &\neq \ell(v_2) \\ \mathrm{ev}_{v_1}(\ell) &\neq \mathrm{ev}_{v_2}(\ell) \\ \mathrm{ev}(v_1)(\ell) &\neq \mathrm{ev}(v_2)(\ell) \end{aligned}$$

as desired. □

1.2.xi. Let V be a vector space.

(1) Let $B : V \times V \to \mathbb{R}$ be an inner product on V. For all $v \in V$, let $\ell_v : V \to \mathbb{R}$ be the function $\ell_v(w) = B(v, w)$. Show that ℓ_v is linear, and show that the map $L : V \to V^*$ defined by $v \mapsto \ell_v$ is a linear mapping.

Proof. Since

$$\begin{aligned} \ell_v(w_1 + w_2) &= B(v, w_1 + w_2) \\ &= B(w_1 + w_2, v) \\ &= B(w_1, v) + B(w_2, v) \\ &= B(v, w_1) + B(v, w_2) \\ &= \ell_v(w_1) + \ell_v(w_2) \end{aligned} \qquad \begin{aligned} \ell_v(\lambda w) &= B(v, \lambda w) \\ &= B(\lambda w, v) \\ &= \lambda B(w, v) \\ &= \lambda B(v, w) \\ &= \lambda\ell_v(w) \end{aligned}$$

we have that ℓ_v is linear, as desired. Note that each step follows either from the definition of ℓ_v or one of the three inner product properties (bilinearity, symmetry, and positivity).

Since

$$\begin{aligned}[L(v_1+v_2)](w) &= \ell_{v_1+v_2}(w) \\ &= B(v_1+v_2,w) \\ &= B(v_1,w)+B(v_2,w) \\ &= \ell_{v_1}(w)+\ell_{v_2}(w) \\ &= L(v_1)(w)+L(v_2)(w) \\ &= [L(v_1)+L(v_2)](w)\end{aligned} \qquad \begin{aligned}[L(\lambda v)](w) &= \ell_{\lambda v}(w) \\ &= B(\lambda v,w) \\ &= \lambda B(v,w) \\ &= \lambda \ell_v(w) \\ &= \lambda L(v)(w) \\ &= [\lambda L(v)](w)\end{aligned}$$

we know that the functions $L(v_1+v_2)$ and $L(v_1)+L(v_2)$ have the same action on every $w \in V$. Thus they are equal. A symmetric statement holds for $L(\lambda v)$ and $\lambda L(v)$. □

(2) If V is finite dimensional, prove that L is bijective. Conclude that if V has an inner product, one gets from it a natural identification of V with V^*. (Hint: Since $\dim V = \dim V^*$ and $\dim(V) = \dim(\ker(A)) + \dim(\operatorname{im}(A))$, it suffices to show that $\ker(L) = 0$. Now note that if $v \neq \mathbf{0}$, then $\ell_v(v) = B(v,v)$ is a positive number.)

Proof. Taking the hint, suppose $L(v) = 0 \in V^*$ for some $v \in V$. Thus, for all $w \in V$ (and, in particular, for v), we have that

$$0 = L(v)(v) = \ell_v(v) = B(v,v)$$

But then by the positivity of the inner product, $v = 0$, as desired. □

1.3.i. Verify that there are exactly n^k multi-indices of length k.

Proof. Let $(i_1,\dots,i_k)$ be a multi-index of n of length k. We independently pick each i_j to be any one of the n numbers between 1 and n, inclusive. Thus, for each of the n values of i_1, there are n possible values of i_2. For each of the n^2 values of (i_1,i_2), there are n possible values of i_3. Continuing on in this fashion inductively confirms that there are always exactly n^k multi-indices of length k. □

1.3.ii. Prove that the map $A^* : \mathcal{L}^k(W) \to \mathcal{L}^k(V)$ defined by $T \mapsto A^*T$ is linear.

Proof. We have that

$$\begin{aligned}[A^*(T_1+T_2)](v_1,\dots,v_k) &= (T_1+T_2)(Av_1,\dots,Av_k) \\ &= T_1(Av_1,\dots,Av_k)+T_2(Av_1,\dots,Av_k) \\ &= A^*T_1(v_1,\dots,v_k)+A^*T_2(v_1,\dots,v_k) \\ &= [A^*T_1+A^*T_2](v_1,\dots,v_k)\end{aligned}$$

and

$$\begin{aligned}[A^*(\lambda T)](v_1,\dots,v_k) &= (\lambda T)(Av_1,\dots,Av_k) \\ &= \lambda T(Av_1,\dots,Av_k) \\ &= \lambda(A^*T)(v_1,\dots,v_k) \\ &= [\lambda(A^*T)](v_1,\dots,v_k)\end{aligned}$$

as desired. □

1.3.iii. Verify that

$$A^*(T_1 \otimes T_2) = A^*(T_1) \otimes A^*(T_2)$$

Proof. Let $T_1 \in \mathcal{L}^k(W)$ and $T_2 \in \mathcal{L}^\ell(W)$. Then

$$\begin{aligned}[A^*(T_1 \otimes T_2)](v_1, \dots, v_{k+\ell}) &= (T_1 \otimes T_2)(Av_1, \dots, Av_{k+\ell}) \\ &= T_1(Av_1, \dots, Av_k)T_2(Av_{k+1}, \dots, Av_{k+\ell}) \\ &= (A^*T_1)(v_1, \dots, v_k)(A^*T_2)(v_{k+1}, \dots, v_{k+\ell}) \\ &= [A^*(T_1) \otimes A^*(T_2)](v_1, \dots, v_{k+\ell})\end{aligned}$$

as desired. □

1.3.iv. Verify that

$$(AB)^*T = B^*(A^*T)$$

Proof. Let U, V, W be vector spaces, $A : V \to W$, $B : U \to V$, and $T \in \mathcal{L}^k(W)$. Then

$$\begin{aligned}[(AB)^*T](v_1, \dots, v_k) &= T(ABv_1, \dots, ABv_k) \\ &= A^*T(Bv_1, \dots, Bv_k) \\ &= [B^*(A^*T)](v_1, \dots, v_k)\end{aligned}$$

as desired. □

1.3.vii. Let T be a k-tensor and v be a vector. Define $T_v : V^{k-1} \to \mathbb{R}$ by

$$T_v(v_1, \dots, v_{k-1}) = T(v, v_1, \dots, v_{k-1})$$

Show that T_v is a $(k-1)$-tensor.

Proof. For the sake of space and ease of notation, I will show only that T_v is linear in its 1^{st} variable. However, a symmetric argument would work in the generalized i^{th} case. This being established, it will follow that T_v is $(k-1)$-linear and thus a $(k-1)$-tensor, as desired. Let's begin.

We have that

$$\begin{aligned}T_v(v_1 + v_1', \dots, v_{k-1}) &= T(v, v_1 + v_1', \dots, v_{k-1}) \\ &= T(v, v_1, \dots, v_{k-1}) + T(v, v_1', \dots, v_{k-1}) \\ &= T_v(v_1, \dots, v_{k-1}) + T_v(v_1', \dots, v_{k-1})\end{aligned}$$

and

$$\begin{aligned}T_v(\lambda v_1, \dots, v_{k-1}) &= T(v, \lambda v_1, \dots, v_{k-1}) \\ &= \lambda T(v, v_1, \dots, v_{k-1}) \\ &= \lambda T_v(v_1, \dots, v_{k-1})\end{aligned}$$

as desired. □

1.3.viii. Show that if T_1 is an r-tensor and T_2 is an s-tensor, then if $r > 0$,

$$(T_1 \otimes T_2)_v = (T_1)_v \otimes T_2$$

Proof. We have that

$$\begin{aligned}[(T_1 \otimes T_2)_v](v_1, \dots, v_{r+s-1}) &= (T_1 \otimes T_2)(v, v_1, \dots, v_{r+s-1}) \\ &= T_1(v, v_1, \dots, v_{r-1})T_2(v_r, \dots, v_{r+s-1}) \\ &= (T_1)_v(v_1, \dots, v_{r-1})T_2(v_r, \dots, v_{r+s-1}) \\ &= [(T_1)_v \otimes T_2](v_1, \dots, v_{r+s-1})\end{aligned}$$

as desired. □

1.3.ix. Let $A : V \to W$ be a linear map, let $v \in V$, and let $w = Av$. Show that for all $T \in \mathcal{L}^k(W)$,

$$A^*(T_w) = (A^*T)_v$$

Proof. We have that

$$\begin{aligned}
[A^*(T_w)](v_1, \dots, v_{k-1}) &= T_w(Av_1, \dots, Av_{k-1}) \\
&= T(w, Av_1, \dots, Av_{k-1}) \\
&= T(Av, Av_1, \dots, Av_{k-1}) \\
&= (A^*T)(v, v_1, \dots, v_k) \\
&= [(A^*T)_v](v_1, \dots, v_k)
\end{aligned}$$

as desired. □

1.4.i. Show that there are exactly $k!$ permutations of order k. (Hint: Induction on k: Let $\sigma \in S_k$, and let $\sigma(k) = i$ $(1 \leq i \leq k)$. Show that $\tau_{i,k}\sigma$ leaves k fixed and hence is, in effect, a permutation of Σ_{k-1}.)

Proof. We induct on k. For the base case $k = 1$, there is clearly only $1! = 1$ possible bijection from a singleton set to itself. Now suppose inductively that we have proven the claim for $k - 1$. Let $\sigma \in S_k$ be arbitrary. Suppose $\sigma(k) = i$. It follows that $(\tau_{i,k}\sigma)(k) = \tau_{i,k}(i) = k$. Thus, since $\tau_{i,k}\sigma$ is a bijection on Σ_k, $(\tau_{i,k}\sigma)|_{\Sigma_{k-1}} \in S_{k-1}$. Consequently, by the inductive hypothesis, there are $(k-1)!$ possible permutations $(\tau_{i,k}\sigma)|_{\Sigma_{k-1}}$. Furthermore, to each of these permutations, there correspond k distinct permutations in S_k (i.e., those obtained by iterating i from 1 through k). Thus, there are $k \cdot (k - 1)! = k!$ permutations of order k, as desired. □

1.4.ii. Prove that if $\tau \in S_k$ is a transposition, $(-1)^\tau = -1$. Deduce from this that if σ is the product of an odd number of transpositions, then $(-1)^\sigma = -1$, and if σ is the product of an even number of transpositions, then $(-1)^\sigma = +1$.

Proof. We induct on k.

For the base case $k = 2$, the only possible transposition is $\tau_{1,2}$. For this transposition, we have

$$(-1)^{\tau_{1,2}} = \prod_{i<j} \frac{x_{\tau_{1,2}(i)} - x_{\tau_{1,2}(j)}}{x_i - x_j} = \frac{x_{\tau_{1,2}(1)} - x_{\tau_{1,2}(2)}}{x_1 - x_2} = \frac{x_2 - x_1}{x_1 - x_2} = -1$$

as desired.

Now suppose inductively that we have proven the claim for $k-1$. Let $\tau_{p,q} \in S_k$ with $p < q$ WLOG. We divide into two cases ($q \neq k$ and $q = k$).

If $q \neq k$, then as in Exercise 1.4.i, we can identify $\tau_{p,q}$ with an element $\tau'_{p,q} \in S_{k-1}$. By the inductive hypothesis,

$$-1 = (-1)^{\tau'_{p,q}} = \prod_{\substack{i<j \\ j \neq k}} \frac{x_{\tau_{p,q}(i)} - x_{\tau_{p,q}(j)}}{x_i - x_j}$$

It follows that

$$(-1)^{\tau_{p,q}} = \prod_{i<j} \frac{x_{\tau_{p,q}(i)} - x_{\tau_{p,q}(j)}}{x_i - x_j} = \prod_{\substack{i<j \\ j \neq k}} \frac{x_{\tau_{p,q}(i)} - x_{\tau_{p,q}(j)}}{x_i - x_j} \cdot \prod_{i=1}^{k-1} \frac{x_{\tau_{p,q}(i)} - x_{\tau_{p,q}(k)}}{x_i - x_k} = -1 \cdot 1 = -1$$

where we evaluate

$$\begin{aligned}\prod_{i=1}^{k-1}\frac{x_{\tau_{p,q}(i)}-x_{\tau_{p,q}(k)}}{x_i-x_k} &= \prod_{i=1}^{k-1}\frac{x_{\tau_{p,q}(i)}-x_k}{x_i-x_k}\\ &= \prod_{\substack{i=1\\ i\neq p,q}}^{k-1}\frac{x_{\tau_{p,q}(i)}-x_k}{x_i-x_k}\cdot\frac{x_{\tau_{p,q}(p)}-x_k}{x_p-x_k}\cdot\frac{x_{\tau_{p,q}(q)}-x_k}{x_q-x_k}\\ &= \prod_{\substack{i=1\\ i\neq p,q}}^{k-1}\frac{x_i-x_k}{x_i-x_k}\cdot\frac{x_q-x_k}{x_p-x_k}\cdot\frac{x_p-x_k}{x_q-x_k}\\ &= 1\end{aligned}$$

If $q=k$, then we divide into two subcases ($p=k-1$ and $p\neq k-1$). If $p=k-1$, then $\tau_{p,q}=\tau_{k-1,k}$. Therefore,

$$\begin{aligned}&(-1)^{\tau_{p,q}}\\ &=\prod_{i<j}\frac{x_{\tau_{k-1,k}(i)}-x_{\tau_{k-1,k}(j)}}{x_i-x_j}\\ &=\prod_{\substack{i<j\\ j<k-1}}\frac{x_{\tau_{k-1,k}(i)}-x_{\tau_{k-1,k}(j)}}{x_i-x_j}\cdot\prod_{i=1}^{k-2}\frac{x_{\tau_{k-1,k}(i)}-x_{\tau_{k-1,k}(k-1)}}{x_i-x_{k-1}}\cdot\prod_{i=1}^{k-2}\frac{x_{\tau_{k-1,k}(i)}-x_{\tau_{k-1,k}(k)}}{x_i-x_k}\cdot\frac{x_{\tau_{k-1,k}(k-1)}-x_{\tau_{k-1,k}(k)}}{x_{k-1}-x_k}\\ &=\prod_{\substack{i<j\\ j<k-1}}\frac{x_i-x_j}{x_i-x_j}\cdot\prod_{i=1}^{k-2}\frac{x_i-x_k}{x_i-x_{k-1}}\cdot\prod_{i=1}^{k-2}\frac{x_i-x_{k-1}}{x_i-x_k}\cdot\frac{x_k-x_{k-1}}{x_{k-1}-x_k}\\ &=\prod_{\substack{i<j\\ j<k-1}}\frac{x_i-x_j}{x_i-x_j}\cdot\prod_{i=1}^{k-2}\left(\frac{x_i-x_{k-1}}{x_i-x_{k-1}}\frac{x_i-x_k}{x_i-x_k}\right)\cdot\frac{x_k-x_{k-1}}{x_{k-1}-x_k}\\ &=1\cdot 1\cdot -1\\ &=-1\end{aligned}$$

If $p\neq k-1$, then $\tau_{p,q}=\tau_{p,k}=\tau_{k-1,k}\tau_{p,k-1}\tau_{k-1,k}$. By our argument for the case $q\neq k$, we know that $(-1)^{\tau_{p,k-q}}=-1$, and by our argument for the case $q=k$ and $p=k-1$, we know that $(-1)^{\tau_{k-1,k}}=-1$. Therefore, by Claim 1.4.9,

$$(-1)^{\tau_{p,q}}=(-1)^{\tau_{k-1,k}\tau_{p,k-1}\tau_{k-1,k}}=(-1)^{\tau_{k-1,k}}(-1)^{\tau_{p,k-1}}(-1)^{\tau_{k-1,k}}=-1\cdot -1\cdot -1=-1$$

as desired.

It follows by Claim 1.4.9 that if $\sigma\in S_k$ can be decomposed into $\sigma=\tau_1\cdots\tau_n$ where $n|2=1$, then

$$(-1)^\sigma=(-1)^{\tau_1\cdots\tau_n}=(-1)^{\tau_1}\cdots(-1)^{\tau_n}=\underbrace{(-1)\cdots(-1)}_{n\text{ times}}=-1$$

as desired.

The proof is symmetric for even permutations. ☐

1.4.iii. Prove that the assignment $T\mapsto T^\sigma$ is a linear map $\mathcal{L}^k(V)\to\mathcal{L}^k(V)$.

Proof. We have that

$$\begin{aligned}(T_1+T_2)^\sigma(v_1,\dots,v_k) &= (T_1+T_2)(v_{\sigma^{-1}(1)},\dots,v_{\sigma^{-1}(k)})\\ &= T_1(v_{\sigma^{-1}(1)},\dots,v_{\sigma^{-1}(k)})+T_2(v_{\sigma^{-1}(1)},\dots,v_{\sigma^{-1}(k)})\\ &= T_1^\sigma(v_1,\dots,v_k)+T_2^\sigma(v_1,\dots,v_k)\end{aligned}$$

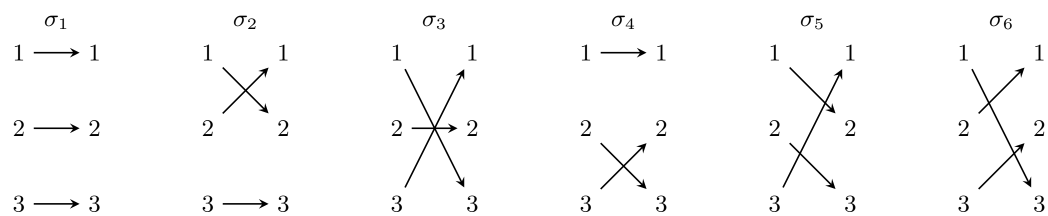
and

$$\begin{aligned}(\lambda T)^\sigma(v_1,\dots,v_k) &= (\lambda T)(v_{\sigma^{-1}(1)},\dots,v_{\sigma^{-1}(k)})\\ &= \lambda T(v_{\sigma^{-1}(1)},\dots,v_{\sigma^{-1}(k)})\\ &= \lambda T^\sigma(v_1,\dots,v_k)\end{aligned}$$

as desired. □

1.4.vi. Show that every one of the six elements of S_3 is either a transposition or can be written as a product of two transpositions.

Proof. The six elements $\sigma_1,\dots,\sigma_6 \in S_3$ are the permutations



It follows that we may write

$$\sigma_1 = \tau_{1,2}\tau_{1,2} \qquad \sigma_2 = \tau_{1,2} \qquad \sigma_3 = \tau_{1,3} \qquad \sigma_4 = \tau_{2,3} \qquad \sigma_5 = \tau_{1,2}\tau_{2,3} \qquad \sigma_6 = \tau_{1,2}\tau_{1,3}$$

□

1.4.ix. Let $A : V \to W$ be a linear mapping. Show that if $T \in \mathcal{A}^k(W)$, then $A^*T \in \mathcal{A}^k(V)$.

Proof. Since $T \in \mathcal{A}^k(W)$, we know that $T^\sigma = (-1)^\sigma T$ for all $\sigma \in S_k$. It follows that

$$\begin{aligned}(A^*T)^\sigma(v_1,\dots,v_k) &= (A^*T)(v_{\sigma^{-1}(1)},\dots,v_{\sigma^{-1}(k)})\\ &= T(Av_{\sigma^{-1}(1)},\dots,Av_{\sigma^{-1}(k)})\\ &= T^\sigma(Av_1,\dots,Av_k)\\ &= (-1)^\sigma T(Av_1,\dots,Av_k)\\ &= (-1)^\sigma A^*T(v_1,\dots,v_k)\end{aligned}$$

as desired. □

1.5.i. A k-tensor $T \in \mathcal{L}^k(V)$ is **symmetric** if $T^\sigma = T$ for all $\sigma \in S_k$. Show that the set $\mathcal{S}^k(V)$ of symmetric k-tensors is a vector subspace of $\mathcal{L}^k(V)$.

Proof. To prove that $\mathcal{S}^k(V) \leq \mathcal{L}^k(V)$, it will suffice to show that it contains the additive identity of $\mathcal{L}^k(V)$ (i.e., the zero tensor), and that it is closed under addition and scalar multiplication. Since we clearly have

$$0^\sigma(v_1,\dots,v_k) = 0(v_{\sigma^{-1}(1)},\dots,v_{\sigma^{-1}(k)}) = 0(v_1,\dots,v_k)$$

we know that $\mathcal{S}^k(V)$ contains the additive identity. Now suppose $T_1, T_2 \in \mathcal{S}^k(V)$. Then since

$$(T_1+T_2)^\sigma = T_1^\sigma + T_2^\sigma = T_1 + T_2$$

where the first equality holds because of the linearity of $\sigma : \mathcal{L}^k(V) \to \mathcal{L}^k(V)$ and the second equality holds since $T_1, T_2 \in \mathcal{S}^k(V)$, $\mathcal{S}^k(V)$ is closed under addition. Similarly, the fact that

$$(\lambda T)^\sigma = \lambda T^\sigma = \lambda T$$

confirms that $\mathcal{S}^k(V)$ is closed under scalar multiplication. □

1.6.i. Verify the following three equations, where $\lambda \in \mathbb{R}$.

(1) $\lambda(\omega_1 \wedge \omega_2) = (\lambda\omega_1) \wedge \omega_2 = \omega_1 \wedge (\lambda\omega_2)$.

Proof. We have that

$$\begin{aligned}\lambda(\omega_1 \wedge \omega_2) &= \lambda\pi(T_1 \otimes T_2)\\ &= \pi[(\lambda T_1) \otimes T_2]\\ &= (\lambda\omega_1) \wedge \omega_2\end{aligned}$$

It follows by a symmetric argument that $\lambda(\omega_1 \wedge \omega_2) = \omega_1 \wedge (\lambda\omega_2)$. □

(2) $(\omega_1 + \omega_2) \wedge \omega_3 = \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_3$.

Proof. We have that

$$\begin{aligned}(\omega_1 + \omega_2) \wedge \omega_3 &= \pi[(T_1 + T_2) \otimes T_3]\\ &= \pi[T_1 \otimes T_3 + T_2 \otimes T_3]\\ &= \omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_3\end{aligned}$$

as desired. □

(3) $\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$.

Proof. We have that

$$\begin{aligned}\omega_1 \wedge (\omega_2 + \omega_3) &= \pi[T_1 \otimes (T_2 + T_3)]\\ &= \pi[T_1 \otimes T_2 + T_1 \otimes T_3]\\ &= \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3\end{aligned}$$

as desired. □

1.6.ii. Verify the following multiplicative law for the wedge product.

$$\omega_1 \wedge \omega_2 = (-1)^{rs}\omega_2 \wedge \omega_1$$

Proof. As per Guillemin and Haine (2018), it suffices to prove this claim for decomposable elements. As such, let $\omega_1 = \ell_1 \wedge \cdots \wedge \ell_r$ and let $\omega_2 = \ell'_1 \wedge \cdots \wedge \ell'_s$. Let $\sigma \in S_{r+s}$ be the permutation

$$\sigma(x) = \begin{cases} x + s & x \leq r \\ x - r & x > r \end{cases}$$

We can write σ as a product of elementary transpositions in a systematic manner as follows.

$$\sigma = \prod_{j=s-1}^{0} \prod_{i=1}^{r} \tau_{i+j,\, i+j+1}$$

Clearly, there are rs of these transpositions, so $(-1)^\sigma = (-1)^{rs}$. Therefore, we have that

$$\begin{aligned}\omega_1 \wedge \omega_2 &= (\ell_1 \wedge \cdots \wedge \ell_r) \wedge (\ell'_1 \wedge \cdots \wedge \ell'_s)\\ &= (-1)^\sigma(\ell'_1 \wedge \cdots \wedge \ell'_s) \wedge (\ell_1 \wedge \cdots \wedge \ell_r)\\ &= (-1)^{rs}\omega_2 \wedge \omega_1\end{aligned}$$

□

1.6.iv. If $\omega, \mu \in \Lambda^r(V^*)$, prove that

$$(\omega+\mu)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} \omega^\ell \wedge \mu^{k-\ell}$$

(Hint: As in freshman calculus, prove this binomial theorem by induction using the identity $\binom{k}{\ell} = \binom{k-1}{\ell-1} + \binom{k-1}{\ell}$.)

Proof. We induct on k.

For the base case $k=1$, we have that

$$\begin{aligned}
\sum_{\ell=0}^{1} \binom{1}{\ell} \omega^\ell \wedge \mu^{1-\ell} &= \binom{1}{0} \omega^0 \wedge \mu^{1-0} + \binom{1}{1} \omega^1 \wedge \mu^{1-1} \\
&= \mu + \omega \\
&= (\omega+\mu)^1
\end{aligned}$$

as desired.

Now suppose inductively that we have proven the claim for $k-1$. Then

$$\begin{aligned}
(\omega+\mu)^k &= (\omega+\mu)^1(\omega+\mu)^{k-1} \\
&= (\omega+\mu) \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} \omega^\ell \wedge \mu^{(k-1)-\ell} \\
&= \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} \omega^{\ell+1} \wedge \mu^{(k-1)-\ell} + \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} \omega^\ell \wedge \mu^{k-\ell} \\
&= \sum_{\ell=1}^{k} \binom{k-1}{\ell-1} \omega^\ell \wedge \mu^{k-\ell} + \sum_{\ell=0}^{k-1} \binom{k-1}{\ell} \omega^\ell \wedge \mu^{k-\ell} \\
&= \binom{k-1}{k-1} \omega^{k-1} \wedge \mu^1 + \sum_{\ell=1}^{k-1} \left[\binom{k-1}{\ell-1} + \binom{k-1}{\ell} \right] \omega^\ell \wedge \mu^{k-\ell} + \binom{k-1}{0} \omega^0 \wedge \mu^k \\
&= \binom{k}{k} \omega^{k-1} \wedge \mu^1 + \sum_{\ell=1}^{k-1} \binom{k}{\ell} \omega^\ell \wedge \mu^{k-\ell} + \binom{k}{0} \omega^0 \wedge \mu^k \\
&= \sum_{\ell=0}^{k} \binom{k}{\ell} \omega^\ell \wedge \mu^{k-\ell}
\end{aligned}$$

as desired. $\square$

1.7.i. Prove that if T is the decomposable k-tensor $\ell_1 \otimes \cdots \otimes \ell_k$, then

$$\iota_v T = \sum_{r=1}^{k} (-1)^{r-1} \ell_r(v) \ell_1 \otimes \cdots \otimes \hat{\ell}_r \otimes \cdots \otimes \ell_k$$

where the hat over ℓ_r means that ℓ_r is deleted from the tensor product.

Proof. We have that

$$
\begin{aligned}
(\iota_v T)(v_1,\dots,v_{k-1}) &= \sum_{r=1}^{k}(-1)^{r-1}T(v_1,\dots,v_{r-1},v,v_r,\dots,v_{k-1})\\
&= \sum_{r=1}^{k}(-1)^{r-1}[\ell_1\otimes\cdots\otimes\ell_{r-1}\otimes\ell_r\otimes\ell_{r+1}\otimes\cdots\otimes\ell_k](v_1,\dots,v_{r-1},v,v_r,\dots,v_{k-1})\\
&= \sum_{r=1}^{k}(-1)^{r-1}\ell_1(v_1)\cdots\ell_{r-1}(v_{r-1})\ell_r(v)\ell_{r+1}(v_r)\cdots\ell_k(v_{k-1})\\
&= \sum_{r=1}^{k}(-1)^{r-1}\ell_r(v)\ell_1(v_1)\cdots\ell_{r-1}(v_{r-1})\ell_{r+1}(v_r)\cdots\ell_k(v_{k-1})\\
&= \sum_{r=1}^{k}(-1)^{r-1}\ell_r(v)[\ell_1\otimes\cdots\otimes\hat{\ell}_r\otimes\cdots\otimes\ell_k](v_1,\dots,v_{k-1})
\end{aligned}
$$

as desired. ☐

1.7.ii. Prove that if $T_1 \in \mathcal{L}^p(V)$ and $T_2 \in \mathcal{L}^q(V)$, then

$$\iota_v(T_1\otimes T_2) = \iota_v T_1 \otimes T_2 + (-1)^p T_1 \otimes \iota_v T_2$$

Proof. We have that

$$
\begin{aligned}
[\iota_v(T_1\otimes T_2)](v_1,\dots,v_{p+q-1}) &= \sum_{r=1}^{p+q}(-1)^{r-1}(T_1\otimes T_2)(v_1,\dots,v_{r-1},v,v_r,\dots,v_{p+q-1})\\
&= \sum_{r=1}^{p}(-1)^{r-1}(T_1\otimes T_2)(v_1,\dots,v_{r-1},v,v_r,\dots,v_{p+q-1})\\
&\quad+\sum_{r=p+1}^{p+q}(-1)^{r-1}(T_1\otimes T_2)(v_1,\dots,v_{r-1},v,v_r,\dots,v_{p+q-1})\\
&= \sum_{r=1}^{p}(-1)^{r-1}T_1(v_1,\dots,v_{r-1},v,v_r,\dots,v_{p-1})T_2(v_p,\dots,v_{p+q-1})\\
&\quad+\sum_{r=p+1}^{p+q}(-1)^{r-1}T_1(v_1,\dots,v_p)T_2(v_{p+1},\dots,v_{r-1},v,v_r,\dots,v_{p+q-1})\\
&= \left[\sum_{r=1}^{p}(-1)^{r-1}T_1(v_1,\dots,v_{r-1},v,v_r,\dots,v_{p-1})\right]\cdot T_2(v_p,\dots,v_{p+q-1})\\
&\quad+T_1(v_1,\dots,v_p)\cdot\sum_{r=p+1}^{p+q}(-1)^{r-1}T_2(v_{p+1},\dots,v_{r-1},v,v_r,\dots,v_{p+q-1})\\
&= \left[\sum_{r=1}^{p}(-1)^{r-1}T_1(v_1,\dots,v_{r-1},v,v_r,\dots,v_{p-1})\right]\cdot T_2(v_p,\dots,v_{p+q-1})\\
&\quad+T_1(v_1,\dots,v_p)\cdot(-1)^p\sum_{r=1}^{q}(-1)^{r-1}T_2(v_{p+1},\dots,v_{p+r-1},v,v_{p+r},\dots,v_{p+q-1})\\
&= (\iota_v T_1)(v_1,\dots,v_{p-1})\cdot T_2(v_p,\dots,v_{p+q-1})\\
&\quad+(-1)^pT_1(v_1,\dots,v_p)\cdot(\iota_vT_2)(v_{p+1},\dots,v_{p+q-1})\\
&= (\iota_vT_1\otimes T_2)(v_1,\dots,v_{p+q-1})+(-1)^p(T_1\otimes\iota_vT_2)(v_1,\dots,v_{p+q-1})\\
&= [\iota_vT_1\otimes T_2+(-1)^pT_1\otimes\iota_vT_2](v_1,\dots,v_{p+q-1})
\end{aligned}
$$

as desired. ☐

1.7.iii. Show that if $T \in \mathcal{A}^k(V)$, then $\iota_v T = kT_v$, where T_v is defined as in Exercise 1.3.vii. In particular, conclude that $\iota_v T \in \mathcal{A}^{k-1}(V)$. (See Exercise 1.4.viii, which asserts that $T \in \mathcal{A}^k(V)$ implies $T_v \in \mathcal{A}^{k-1}(V)$.)

Proof. Suppose $T \in \mathcal{A}^k(V)$. Let $\sigma \in S_k$ be the permutation that moves the r^{th} index to the first place and shifts all $r-1$ indices to its left up one. For example, if $r = 4$ and $\sigma \in S_6$, $\sigma(1,2,3,4,5,6) = (4,1,2,3,5,6)$. More relevant to our situation would be the ability of σ to do the following.

$$\sigma(v_1, v_2, v_3, v, v_4, v_5) = \sigma(v, v_1, v_2, v_3, v_4, v_5)$$

Going back to the general case, since we have

$$\sigma = \prod_{i=1}^{r-1} \tau_{i,i+1}$$

we can determine that

$$(-1)^\sigma = (-1)^{r-1}$$

Therefore, by the above and since $T^\sigma = (-1)^\sigma T$ as an alternating k-tensor,

$$\begin{aligned}
(\iota_v T)(v_1, \dots, v_{k-1}) &= \sum_{r=1}^{k} (-1)^{r-1} T(v_1, \dots, v_{r-1}, v, v_r, \dots, v_{k-1}) \\
&= \sum_{r=1}^{k} (-1)^{\sigma} T(v_1, \dots, v_{r-1}, v, v_r, \dots, v_{k-1}) \\
&= \sum_{r=1}^{k} T^{\sigma}(v_1, \dots, v_{r-1}, v, v_r, \dots, v_{k-1}) \\
&= \sum_{r=1}^{k} T(v, v_1, \dots, v_{k-1}) \\
&= \sum_{r=1}^{k} T_v(v_1, \dots, v_{k-1}) \\
&= kT_v(v_1, \dots, v_{k-1})
\end{aligned}$$

as desired.

As stated in the question, we may invoke Exercise 1.4.vii to determine that $\iota_v T = kT_v \in \mathcal{A}^{k-1}(V)$. ☐

1.8.i. Verify the following assertions.

(1) The map $A^* : \Lambda^k(W^*) \to \Lambda^k(V^*)$ sending $\omega \mapsto A^*\omega$ is linear.

Proof. We have that

$$\begin{aligned}
A^*(\omega_1 + \omega_2) &= \pi(A^*(T_1 + T_2)) \\
&= \pi(A^*T_1 + A^*T_2) \\
&= \pi(A^*T_1) + \pi(A^*T_2) \\
&= A^*\omega_1 + A^*\omega_2
\end{aligned}
\qquad
\begin{aligned}
A^*(\lambda\omega) &= \pi(A^*(\lambda T)) \\
&= \pi(\lambda A^*T) \\
&= \lambda\pi(A^*T) \\
&= \lambda A^*\omega
\end{aligned}$$

as desired. ☐

(2) If $\omega_i \in \Lambda^{k_i}(W^*)$ $(i = 1, 2)$, then

$$A^*(\omega_1 \wedge \omega_2) = A^*(\omega_1) \wedge A^*(\omega_2)$$

Proof. We have that

$$\begin{aligned} A^*(\omega_1 \wedge \omega_2) &= A^*(\pi(T_1 \otimes T_2)) \\ &= \pi(A^*(T_1 \otimes T_2)) \\ &= \pi(A^*T_1 \otimes A^*T_2) \\ &= \pi(A^*T_1) \wedge \pi(A^*T_2) \\ &= A^*(\omega_1) \wedge A^*(\omega_2) \end{aligned}$$

as desired. □

(3) If U is a vector space and $B : U \to V$ is a linear map, then for $\omega \in \Lambda^k(W^*)$,

$$B^*A^*\omega = (AB)^*\omega$$

Proof. We have that

$$\begin{aligned} B^*A^*\omega &= B^*(\pi(A^*T)) \\ &= \pi(B^*A^*T) \\ &= \pi((AB)^*T) \\ &= (AB^*)\omega \end{aligned}$$

as desired. □

1.8.ii. Deduce from the fact "$A : V \to V$ not surjective implies $\det(A) = 0$" a well-known fact about determinants of $n \times n$ matrices: If two columns are equal, the determinant is zero.

Proof. If an $n \times n$ matrix has two identical columns, then the dimension of its range space is at most $n - 1$. Thus, A is not surjective, and hence has $\det(A) = 0$. □

1.8.iv. Deduce from Exercise 1.8.i another well-known fact about determinants of $n \times n$ matrices: If $(b_{i,j})$ is the inverse of $[a_{i,j}]$, its determinant is the inverse of the determinant of $[a_{i,j}]$.

Proof. Let $(b_{i,j}) = [a_{i,j}]^{-1}$. Then

$$(b_{i,j})[a_{i,j}] = \mathrm{id}_V$$

It follows from Propositions 1.8.7 and 1.8.8 (which in turn follow from Exercise 1.8.i) that

$$\begin{aligned} \det(b_{i,j}) \det[a_{i,j}] &= \det(\mathrm{id}_V) = 1 \\ \det(b_{i,j}) &= \frac{1}{\det[a_{i,j}]} \end{aligned}$$

as desired. □

1.8.v. Extract from the formula $\det([a_{i,j}]) = \sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$ the following well-known formula for determinants of 2×2 matrices.

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Proof. The two elements of S_2 are the identity permutation (which we will refer to as σ_1) and $\tau_{1,2}$ (which we will refer to as σ_2). It follows that for the $n = 2$ case,

$$\begin{aligned}\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &= \sum_{\sigma\in S_2}(-1)^{\sigma}a_{1,\sigma(1)}a_{2,\sigma(2)} \\ &= (-1)^{\sigma_1}a_{1,\sigma_1(1)}a_{2,\sigma_1(2)} + (-1)^{\sigma_2}a_{1,\sigma_2(1)}a_{2,\sigma_2(2)} \\ &= (1)a_{1,1}a_{2,2} + (-1)a_{1,2}a_{2,1} \\ &= a_{1,1}a_{2,2} - a_{1,2}a_{2,1}\end{aligned}$$

as desired. □

1.9.i. Prove that if $e_1, \dots, e_n$ is a positively oriented basis of V, then the basis $e_1, \dots, e_{i-1}, -e_i, e_{i+1}, \dots, e_n$ is negatively oriented.

Proof. Since $e_1, \dots, e_n$ is a positively oriented basis of V, we know that $e_1^* \wedge \cdots \wedge e_n^* \in \Lambda^n(V^*)_+$. This combined with the fact that

$$e_1 \wedge \cdots \wedge e_{i-1}, -e_i, e_{i+1} \wedge \cdots \wedge e_n = -e_1^* \wedge \cdots \wedge e_n^* \notin \Lambda^n(V^*)$$

implies that the given basis is negatively oriented, as desired. □

1.9.ii. Show that the argument in the proof of Theorem 1.9.9 can be modified to prove that if V and W are oriented, then these orientations induce a natural orientation on V/W.

Proof. Let $W \leq V$, $\dim V = n > 1$, $\dim W = k < n$, and $r = n - k$. WLOG choose $e_1, \dots, e_n$ a positively oriented basis of V such that $e_{r+1}, \dots, e_n$ is a positively oriented basis of W. It follows that $\pi(e_1), \dots, \pi(e_r)$ for a basis of V/W. Assign to V/W the orientation associated with this basis. Now suppose $\pi(f_1), \dots, \pi(f_r)$ is another basis of V/W. □