

Week 3

Multilinear Spaces, Operations, and Conventions

3.1 Exterior Powers Basis and the Determinant

4/13:

- Plan:
 - Finish multilinear algebra.
 - Basis for $\Lambda^k(V^*)$.
 - Talk a bit about pullbacks and the determinant.
 - **Orientations** of vector spaces.
 - The **interior product**.
- Basis for $\Lambda^k(V^*)$.
 - Recall that $\{\text{Alt}(e_I^*) \mid I \text{ is a nonrepeating, increasing partition of } n \text{ into } k \text{ parts}\}$ is a basis for $\mathcal{A}^k(V)$.
- Alt is an isomorphism from $\Lambda^k(V^*)$ to $\mathcal{A}^k(V)$.
- If we have an injective map from $\mathcal{A}^k(V)$ to $\mathcal{L}^k(V)$ and π a projection map from $\mathcal{L}^k(V)$ to the quotient space $\mathcal{A}^k(V^*)$ gives rise to $\pi|_{\mathcal{A}^k(V)}$.
- Claim:
 1. $\pi|_{\mathcal{A}^k(V)}$ is an isomorphism.
 2. $\pi(\text{Alt}(e_I^*)) = k! \pi(e_I^*)$.
- (2) implies that $\{\pi(e_I^*) = e_{i_1}^* \wedge \cdots \wedge e_{i_k}^*, I \text{ non-repeating and increasing}\}$ is a basis for $\Lambda^k(V^*)$.
- Examples:
 1. $n = 2 = \dim V, V = \mathbb{R}e_1 \oplus \mathbb{R}e_2$.
 - $\Lambda^0(V^*) = \mathbb{R}$ since $\binom{n}{0} = 1$.
 - $\Lambda^1(V^*) = \mathbb{R}e_1^* \oplus \mathbb{R}e_2^*$ since $\binom{n}{1} = 2$.
 - $\Lambda^2(V^*) = \mathbb{R}e_1^* \wedge e_2^*$ since $\binom{n}{2} = 1$.
 - Note that $e_1^* \wedge e_2^* = -e_2^* \wedge e_1^*$.
 - $\Lambda^3(V^*) = 0$ since $\binom{2}{3} = 0$.
 - Note that all $e_1^* \wedge e_1^* \wedge e_2^* = 0$.
 2. $n = 3, V = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3$.

- $\binom{n}{0} = 1$: $\Lambda^0(V^*) = \mathbb{R}$.
- $\binom{n}{1} = 3$: $\Lambda^1(V^*) = \mathbb{R}e_1^* \oplus \mathbb{R}e_2^* \oplus \mathbb{R}e_3^*$.
- $\binom{n}{2} = 3$: $\Lambda^2(V^*) = \mathbb{R}e_1^* \wedge e_2^* \oplus \mathbb{R}e_2^* \wedge e_3^* \oplus \mathbb{R}e_1^* \wedge e_3^*$.
- $\binom{n}{3} = 1$: $\Lambda^3(V^*) = \mathbb{R}e_1^* \wedge e_2^* \wedge e_3^*$.
- $\binom{n}{m} = 0$ ($m > n$): $\Lambda^m(V^*) = \Lambda^4(V^*) = 0$.

- If $A : V \rightarrow W$, $\omega_1 \in \Lambda^k(W^*)$, $\omega_2 \in \Lambda^\ell(W^*)$, then

$$A^*(\omega_1 \wedge \omega_2) = A^*\omega_1 \wedge A^*\omega_2$$

- **Determinant:** Let $\dim V = n$. Let $A : V \rightarrow V$ be a linear transformation. This induces a pullback $A^* : \Lambda^n(V^*) \rightarrow \Lambda^n(V^*)$. The top exterior power $k = n$ implies $\binom{k}{n} = 1$. We define $\det(A)$ to be the unique real number such that $A^*(v) = \det(A)v$.
- This determinant is the one we know.

- A^* sends $e_1^* \wedge \cdots \wedge e_n^*$ to $A^*e_1^* \wedge \cdots \wedge A^*e_n^*$ which equals $A^*(e_1^* \wedge \cdots \wedge e_n^*)$ or $\det(A)$

- Sanity check.

1. $\det(\text{id}) = 1$.

- $\text{id}(e_1^* \wedge \cdots \wedge e_n^*) = \text{id}e_1^* \wedge \cdots \wedge \text{id}e_n^* = 1 \cdot e_1^* \wedge \cdots \wedge e_n^*$.

2. If A is not an isomorphism, then $\det(A) = 0$.

- If A is not an isomorphism, then there exists $v_1 \in \ker A$ with $v_1 \neq 0$. Let v_1^*, \dots, v_n^* be a basis of V^* . So the pullback of this wedge is the wedge of the pullbacks, but $A^*v_1^* = 0$, so

$$A^*(v_1^* \wedge \cdots \wedge v_n^*) = (A^*v_1^*) \wedge \cdots \wedge (A^*v_n^*) = 0 \wedge \cdots \wedge (A^*v_n^*) = 0 = 0 \cdot v_1^* \wedge \cdots \wedge v_n^*$$

3. $\det(AB) = \det(A)\det(B)$.

- Let $A : V \rightarrow V$ and $B : V \rightarrow V$.

- We have $(AB)^* = B^*A^*$; in particular, $n = k$, $V = W = U = V$.

- Recall: If we pick a basis for V , e_1, \dots, e_n .

- Implies $[a_{ij}] = [A]_{e_1, \dots, e_n}^{e_1, \dots, e_n}$.

- Does $\det(A) = \det([a_{ij}]) = \sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$?

- If $A : V \rightarrow V$, we know that $A^* : \Lambda^n(V^*) \rightarrow \Lambda^n(V^*)$ takes $e_1^* \wedge \cdots \wedge e_n^* \mapsto A^*(e_1^* \wedge \cdots \wedge e_n^*)$. We WTS

$$A^*(e_1^* \wedge \cdots \wedge e_n^*) = \left[\sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} \right] e_1^* \wedge \cdots \wedge e_n^*$$

- We have that

$$\begin{aligned} A^*(e_1^* \wedge \cdots \wedge e_n^*) &= A^*e_1^* \wedge \cdots \wedge A^*e_n^* \\ &= \left(\sum_{i_1=1}^n a_{i_1,1} e_{i_1}^* \right) \wedge \cdots \wedge \left(\sum_{i_n=1}^n a_{i_n,n} e_{i_n}^* \right) \\ &= \sum_{i_1, \dots, i_n} a_{i_1,1} \cdots a_{i_n,n} e_{i_1}^* \wedge \cdots \wedge e_{i_n}^* \\ &= \left[\sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \cdots a_{n,\sigma(n)} \right] e_1^* \wedge \cdots \wedge e_n^* \end{aligned}$$

where the sign arises from the need to reorder $e_{i_1}^* \wedge \cdots \wedge e_{i_n}^*$ and the antisymmetry of the wedge product.

3.2 The Interior Product and Orientations

4/15:

- Plan:
 - Orientations.
 - Interior product.
- **Interior product:** We know that $\Lambda^k(V^*) \cong \mathcal{A}^k(V)$. Fix $v \in V$. Define $\iota_v : \Lambda^k(V^*) \rightarrow \Lambda^{k-1}(V^*)$.
 - Wrong way: We take $\iota_v : \mathcal{L}^k(V) \rightarrow \mathcal{L}^{k-1}(V)$.

$$T \mapsto \sum_{r=1}^k (-1)^{r-1} T(v_1, \dots, v_{r-1}, v, v_r, \dots, v_{k-1})$$

- Right way: First define $\varphi_v : \mathcal{A}^k(V) \rightarrow \mathcal{A}^{k-1}(V)$ by

$$T \mapsto T_v(v_1, \dots, v_{k-1}) = T(v, v_1, \dots, v_{k-1})$$

■ Check:

1. $T_{v_1+v_2} = T_{v_1} + T_{v_2}$ ^[1].
2. $T_{\lambda v} = \lambda T_v$.
3. $\varphi_v^{k-1} \circ \varphi_v^k = 0$ implies $\varphi_v \circ \varphi_w = -\varphi_w \circ \varphi_v$.

• Properties:

0. $\iota_v T \in \mathcal{L}^{k-1}(V)$.
1. ι_v is a linear map.
 - This is all happening in the set $\text{Hom}(\mathcal{L}^k(V), \mathcal{L}^{k-1}(V))$.
2. $\iota_{v_1+v_2} = \iota_{v_1} + \iota_{v_2}$; $\iota_{\lambda v} = \lambda \iota_v$.
3. “Product rule”: If $T_1 \in \mathcal{L}^p(V)$ and $T_2 \in \mathcal{L}^q(V)$, then $\iota_v(T_1 \otimes T_2) = \iota_v T_1 \otimes T_2 + (-1)^p T_1 \otimes \iota_v T_2$.
4. We have

$$\iota_v(\ell_1 \otimes \dots \otimes \ell_k) = \sum_{r=1}^k (-1)^{r-1} \ell_r(v) \ell_1 \otimes \dots \otimes \hat{\ell}_r \otimes \dots \otimes \ell_k$$

5. $\iota_v \circ \iota_v = 0 \in \text{Hom}(\mathcal{L}^k(V), \mathcal{L}^{k-2}(V))$.
 - Note that this is related to $d^2 = 0$ from the first day of class (alongside $\int_m dw = \int_{\partial m} w$).
 - Proof: We induct on k . It suffices to prove the result for T decomposable.
 - Trivial base case for $k = 1$.
 - We have that

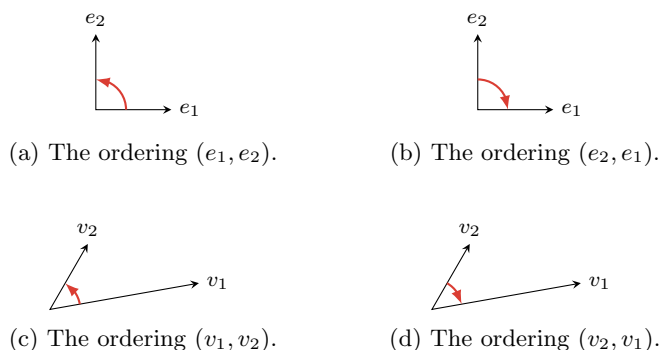
$$\begin{aligned} (\iota_v \circ \iota_v)(\ell_1 \otimes \dots \otimes \ell_{k-1} \otimes \ell) &= \iota_v(\iota_v T \otimes \ell + (-1)^{k-1} \ell(v) T) \\ &= \iota_v(\iota_v T \otimes \ell) + (-1)^{k-1} \ell(v) \iota_v T \\ &= (-1)^{k-2} \ell(v) \iota_v T + (-1)^{k-1} \ell(v) \iota_v T \\ &= (-1)^{k-2} \ell(v) \iota_v T - (-1)^{k-2} \ell(v) \iota_v T \\ &= 0 \end{aligned}$$

6. If $T \in \mathcal{I}^k(V)$, then $\iota_v T \in \mathcal{I}^{k-1}(V)$.
 - Thus, ι_v induces a map $\iota_v : \Lambda^k(V^*) \rightarrow \Lambda^{k-1}(V^*)$.
 - Proof: It suffices to check this for decomposables.

7. $\iota_{v_1} \circ \iota_{v_2} = -\iota_{v_2} \circ \iota_{v_1}$.

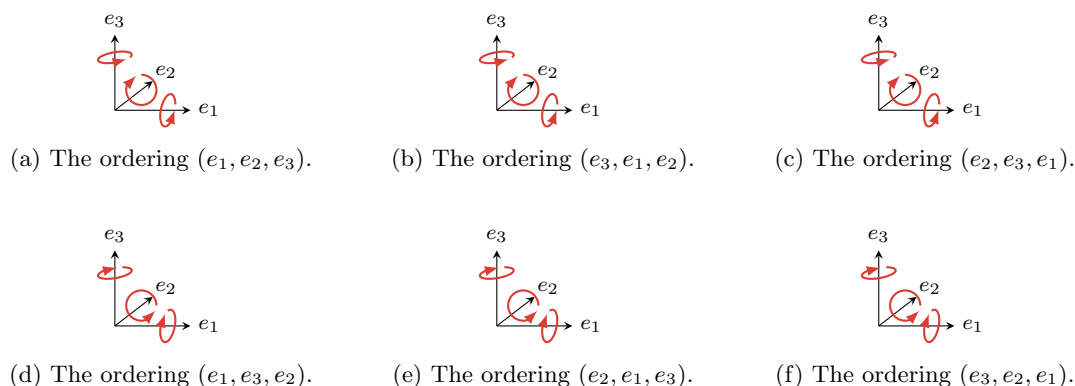
¹Should this be T or φ ?

- Orientations.
- Motivating example: Investigating a property of bases that we will later formally define as *orientation*.

Figure 3.1: Motivating orientations in \mathbb{R}^2 .

- Consider the vector space \mathbb{R}^2 .
- In both Figures 3.1a and 3.1b, $\{e_1, e_2\}$ is the standard basis of \mathbb{R}^2 . However, suppose we *order* the elements of this basis, i.e., by using a 2-tuple or ordered pair. There are two possible ways we can order the elements: (e_1, e_2) and (e_2, e_1) .
- Now consider an arbitrary basis $\{v_1, v_2\}$ of \mathbb{R}^2 . Once again, we have two possible orderings: (v_1, v_2) and (v_2, v_1) , as depicted in Figures 3.1c and 3.1d, respectively.
- What is similar between (e_1, e_2) and (v_1, v_2) , and between (e_2, e_1) and (v_2, v_1) ?
 - First off, resist the temptation to regard how the vectors are labeled (i.e., with numerical subscripts) as having any bearing on the properties of the vectors themselves. “Both orientations in the first set are labeled 1, 2, and both orientations in the second set are labeled 2, 1” is not what we’re looking for.
 - What we *are* looking for is the observation that in both Figures 3.1a and 3.1c, the second vector in the ordering is positioned counterclockwise relative to the first vector, and in both Figures 3.1b and 3.1d, the second vector in the ordering is positioned clockwise relative to the first vector. The red arrows in Figures 3.1a-3.1d indicate this relative positioning.
- Now that we’ve seen the pattern in a few specific examples, it should not be too hard to see that for *any* ordered basis of \mathbb{R}^2 the second vector *must* be positioned either counterclockwise or clockwise relative to the first vector.
 - In fact, taking this farther, let’s ask what would happen if a “basis” did not fit this rule. For example, consider the ordered “basis” (e_1, e_1) . Clearly, e_1 is not positioned counterclockwise or clockwise to e_1 , but rather lies directly on top of it. So, the reader might think, we have an exception! But this is not the case because, as the quotation marks around “basis” indicate, $\{e_1, e_1\}$ is *not* a basis of \mathbb{R}^2 (for example, $e_2 \notin \text{span}\{e_1, e_1\}$).
 - Similarly, if our ordered “basis” is like $(e_1, -e_1)$, then we at least have two different basis vectors. However, once again, they are not linearly independent, nor do they span \mathbb{R}^2 . Therefore, they are not a basis.
- It follows that the set of ordered bases of \mathbb{R}^2 can be partitioned into two equivalence classes: One containing all ordered bases in which the second vector is positioned counterclockwise relative to the first, and the other containing all ordered bases in which the second vector is positioned clockwise relative to the first. All of this arises naturally; the choice of an *orientation* on \mathbb{R}^2 is a man-made designation of one of these equivalence classes as the “positive” one and the other as the “negative” one.

- Finding the pattern elsewhere: Three dimensions.

Figure 3.2: Motivating orientations in \mathbb{R}^3 .

- Here, we will only consider orderings of standard basis $\{e_1, e_2, e_3\}$ of \mathbb{R}^3 .
- Consider Figure 3.2a.
 - Similarly to in \mathbb{R}^2 , the positioning of vector 2 relative to vector 1 defines an “orientation,” which we take to be about vector 3.
 - The direction of this “orientation” is indicated by the red circular arrow wrapping around e_3 most of the way along it.
 - Differently, however, we can define additional “orientations” about vector 1 and vector 2 using, respectively, the position of vector 3 relative to vector 2, and the position of vector 1 relative to vector 3.
 - Importantly, though, all three of these “orientations” are inextricably linked (note how any time one of them inverts, they all invert), so we can treat these three “orientations” (or any one of them in particular) as one overarching *orientation* on \mathbb{R}^3 .
- Now unlike in \mathbb{R}^2 , we now have more than 2 ways of ordering $\{e_1, e_2, e_3\}$; in fact, we have $3! = 6$ such ways. However, we still only have two distinct orientations: Notice that all of the orientations in Figures 3.2a-3.2c are the same, and all of the orientations in Figures 3.2d-3.2f are the same.
 - Thus, we may still partition the set of ordered bases of \mathbb{R}^3 into two equivalence classes and arbitrarily label one to be “positive” and the other to be “negative.”
- Running with it: Higher dimensional generalizations.
 - Evidently, the paradigm of orientation became much more complicated from \mathbb{R}^2 to \mathbb{R}^3 . As such, it is reasonable to expect that it will become even more complicated in moving to \mathbb{R}^4 , let alone \mathbb{R}^n , especially since these spaces are not so easily visualizable.
 - So how do we proceed? Well, when we’ve sought to express geometric notions in higher dimensions in the past, linear algebra has been the go-to model. So let’s use it once again.
 - We start with two key observations:
 - For any two oriented bases $(v_1, \dots, v_n), (w_1, \dots, w_n)$ in the *same* equivalence class (i.e., both having positive orientation or both having negative orientation), there exists a unique linear transformation T with *positive* determinant such that $w_i = Tv_i$ for $i = 1, \dots, n$.
 - For any two oriented bases $(v_1, \dots, v_n), (w_1, \dots, w_n)$ in *opposite* equivalence classes (i.e., one having positive orientation and the other having negative orientation), there exists a unique linear transformation T with *negative* determinant such that $w_i = Tv_i$ for $i = 1, \dots, n$.
 - Thus, to assign an orientation on \mathbb{R}^n , we may take the standard basis in numerical order, i.e., (e_1, \dots, e_n) , and define the equivalence class of which it is an element to be the positive one. Then to check the orientation of a given basis (v_1, \dots, v_n) , we need only find T and calculate its determinant.

- So, what? Why do we need orientations?
 - Classical single-variable integration determines the “area under the curve” between a, b by summing the height of infinitely many, infinitely small rectangles. In this picture, we can think of $[a, b]$ as our manifold, moving in the positive direction along the x -axis as our orientation, and f as a function on the manifold.
 - For a more general manifold X , we will have a *form* (as opposed to a function) which assigns to each $p \in X$ an exterior product/alternating tensor (as opposed to a scalar). But we will still need to know in which direction we should move along the manifold, and that is the job of the orientation.
 - Additionally, we need a rigorous algebraic theory so that we can relate the orientations of domains to the orientations of their boundaries.
- Formally defining orientations: Relating orientation to the language of differential forms.
 - It is easy to define an orientation on \mathbb{R}^1 or any other 1-dimensional vector space. Indeed, we can take e_1 and all of its positive scalar multiples to be the positive equivalence class, and all negative multiples of e_1 to be the negative equivalence class.
 - Note that since 1×1 matrices have determinant equal to their one entry and, for all intents and purposes, function as scalars, this definition agrees with our previous ones.
 - Also, an alternate but equivalent definition of orientation on a 1-dimensional vector space is to divide it into two connected components by removing the zero element from the set and choose one of those connected components to be the “positive orientation.” This is the definition we will use going forward.
 - Let V be an arbitrary n -dimensional vector space. Since $\dim \Lambda^n(V^*) = \binom{n}{n} = 1$, we can assign it an orientation based on the above rule.
 - Now let $\omega \in \Lambda^n(V^*)$ lie in the positive component. Then we say that an ordered basis (v_1, \dots, v_n) of V is positive if $\omega(v_1, \dots, v_n) > 0$, and vice versa if $\omega(v_1, \dots, v_n) < 0$.
 - In particular, let $\{e_1, \dots, e_n\}$ be the standard basis of V . Then we call $e_1^* \wedge \dots \wedge e_n^*$ the **orientation form**.
 - It has the ability to identify the standard basis as positive and some others as positive or negative, but also equals zero for a number of bases.
 - A more powerful tool is $\omega = \text{Alt}(e_1^* \wedge \dots \wedge e_n^*) \in \Lambda^n(V^*)$. We can define an orientation on $\Lambda^n(V^*)$ based on which half of it ω lies in. Moreover, since ω will take in any ordered basis of V and return a nonzero scalar, it can provide an orientation on V .
 - *Come back after completing integration on manifolds section.*
- An interesting connection between the differential forms definition and the determinant definition, and its formalization as motivation for the new definition of the determinant given in class yesterday.
 - ω and $\det(T)$ may seem like entirely different objects at first glance, but in fact, they are almost identical.
 - Let’s drop back down to \mathbb{R}^2 to illustrate the point. Here,

$$\omega = \text{Alt}(e_1^* \wedge e_2^*) = e_1^* \wedge e_2^* - e_2^* \wedge e_1^*$$

- It follows that if

$$\left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \right\}$$

is an arbitrary basis of \mathbb{R}^2 , then the transformation matrix T from positively-oriented ordered basis (e_1, e_2) to (v, w) is

$$T = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$$

which has determinant

$$\det(T) = v_1 w_2 - w_1 v_2$$

Therefore, whether $v_1 w_2 - w_1 v_2$ is positive or negative determines whether (v, w) is a positively or negatively oriented basis of \mathbb{R}^2 .

■ Incidentally,

$$\begin{aligned}\omega(v, w) &= [e_1^* \wedge e_2^* - e_2^* \wedge e_1^*](v, w) \\ &= v_1 \cdot w_2 - v_2 \cdot w_1 \\ &= v_1 w_2 - w_1 v_2\end{aligned}$$

as well.

■ Therefore, evaluating the sign of $\det(T)$ is identical to evaluating the sign of $\omega(v, w)$.

– But why the relationship?

■ In fact, it's in the definition of the determinant from last class: The determinant of an endomorphism $T : V \rightarrow V$ can be interpreted as the induced action of the pullback on the top exterior power.

■ In particular,

$$\begin{aligned}\omega(v_1, \dots, v_n) &= \omega(T(e_1), \dots, T(e_n)) \\ &= [T^* \omega](e_1, \dots, e_n) \\ &= \det(T) \cdot \omega(e_1, \dots, e_n) \\ &= \det(T) \cdot 1 \\ &= \det(T)\end{aligned}$$

as desired.

• Summary of orientations.

- A vector space V should have two orientations.
- Two bases e_1, \dots, e_n and f_1, \dots, f_n are **orientation equivalent** if $T : V \rightarrow V$ an isomorphism has positive determinant. Otherwise, they are **orientation inequivalent**.
- An orientation on V is a choice of equivalence classes of bases under the equivalence relation on bases.
- $T : V \rightarrow W$ given orientations, T preserves or reverses orientations.

• Fancy orientations.

- An orientation on a 1D vector space L is a division into two halves.
- Def: An orientation of V is an orientation of $\Lambda^n(V^*)$.

• We can prove that they're both the same.

- If W and V are both oriented, then V/W gets a canonical orientation.

3.3 Chapter 1: Multilinear Algebra

From Guillemin and Haine (2018).

4/14: • $\iota_v T$: The $(k-1)$ -tensor defined as follows. Given by

$$(v_1, \dots, v_{k-1}) \mapsto \sum_{r=1}^k (-1)^{r-1} T(v_1, \dots, v_{r-1}, v, v_r, \dots, v_{k-1})$$

where $T \in \mathcal{L}^k(V)$, $k \in \mathbb{N}_0$, V is a vector space, and $v \in V$.

- If $v = v_1 + v_2$, then

$$\iota_v T = \iota_{v_1} T + \iota_{v_2} T$$

- If $T = T_1 + T_2$, then

$$\iota_v T = \iota_v T_1 + \iota_v T_2$$

- Lemma 1.7.4: If $T = \ell_1 \otimes \cdots \otimes \ell_k$, then

$$\iota_v T = \sum_{r=1}^k (-1)^{r-1} \ell_r(v) \ell_1 \otimes \cdots \otimes \hat{\ell}_r \otimes \cdots \otimes \ell_k$$

where the hat over ℓ_r means that ℓ_r is deleted from the tensor product.

- Lemma 1.7.6: $T_1 \in \mathcal{L}^p(V)$ and $T_2 \in \mathcal{L}^q(V)$ imply

$$\iota_v(T_1 \otimes T_2) = \iota_v T_1 \otimes T_2 + (-1)^p T_1 \otimes \iota_v T_2$$

- Lemma 1.7.8: $T \in \mathcal{L}^k(V)$ implies that for all $v \in V$, we have

$$\iota_v(\iota_v T) = 0$$

Proof. It suffices by linearity to prove this for decomposable tensors. We induct on k . For the base case $k = 1$, the claim is trivially true. Now suppose inductively that we have proven the claim for $k - 1$. Consider $\ell_1 \otimes \cdots \otimes \ell_k$. Taking $T = \ell_1 \otimes \cdots \otimes \ell_{k-1}$ and $\ell = \ell_k$, we obtain

$$\iota_v(\iota_v(T \otimes \ell)) = \iota_v(\iota_v T) \otimes \ell + (-1)^{k-2} \ell(v) \iota_v T + (-1)^{k-1} \ell(v) \iota_v T$$

The first term is zero by the inductive hypothesis, and the second two cancel each other out, as desired. \square

- Claim 1.7.10: For all $v_1, v_2 \in V$, we have that

$$\iota_{v_1} \iota_{v_2} = -\iota_{v_2} \iota_{v_1}$$

Proof. Let $v = v_1 + v_2$. Then $\iota_v = \iota_{v_1} + \iota_{v_2}$. Therefore,

$$\begin{aligned} 0 &= \iota_v \iota_v \\ &= (\iota_{v_1} + \iota_{v_2})(\iota_{v_1} + \iota_{v_2}) \\ &= \iota_{v_1} \iota_{v_1} + \iota_{v_1} \iota_{v_2} + \iota_{v_2} \iota_{v_1} + \iota_{v_2} \iota_{v_2} \\ &= \iota_{v_1} \iota_{v_2} + \iota_{v_2} \iota_{v_1} \end{aligned} \quad \begin{array}{l} \text{Lemma 1.7.8} \\ \text{Lemma 1.7.8} \end{array}$$

yielding the desired result. \square

- Lemma 1.7.11: If $T \in \mathcal{L}^k(V)$ is redundant, then so is $\iota_v T$.

Proof. Let $T = T_1 \otimes \ell \otimes \ell \otimes T_2$ where $\ell \in V^*$, $T_1 \in \mathcal{L}^p(V)$, and $T_2 \in \mathcal{L}^q(V)$. By Lemma 1.7.6, we have that

$$\iota_v T = \iota_v T_1 \otimes \ell \otimes \ell \otimes T_2 + (-1)^p T_1 \otimes \iota_v(\ell \otimes \ell) \otimes T_2 + (-1)^{p+2} T_1 \otimes \ell \otimes \ell \otimes \iota_v T_2$$

Thus, since the first and third terms above are redundant and $\iota_v(\ell \otimes \ell) = \ell(v)\ell - \ell(v)\ell = 0$ by Lemma 1.7.4, we have the desired result. \square

- $\iota_v \omega$: The $\mathcal{L}^k(V)$ -coset $\pi(\iota_v T)$, where $\omega = \pi(T)$.
- Proves that $\iota_v \omega$ does not depend on the choice of T .

- **Interior product operation:** The linear map $\iota_v : \Lambda^k(V^*) \rightarrow \Lambda^{k-1}(V^*)$.
- The interior product has the following important identities.

$$\begin{aligned}\iota_{(v_1+v_2)}\omega &= \iota_{v_1}\omega + \iota_{v_2}\omega \\ \iota_v(\omega_1 \wedge \omega_2) &= \iota_v\omega_1 \wedge \omega_2 + (-1)^p\omega_1 \wedge \iota_v\omega_2 \\ \iota_v(\iota_v\omega) &= 0 \\ \iota_{v_1}\iota_{v_2}\omega &= -\iota_{v_2}\iota_{v_1}\omega\end{aligned}$$

- 4/18:
- As we developed the pullback $A^*T \in \mathcal{L}^k(V)$, we now look to develop a pullback on $\Lambda^k(V^*)$.
 - Lemma 1.8.1: If $T \in \mathcal{I}^k(W)$, then $A^*T \in \mathcal{I}^k(V)$.

Proof. It suffices to prove this for redundant k -tensors. Let $T = \ell_1 \otimes \cdots \otimes \ell_k$ be such that $\ell_i = \ell_{i+1}$. Then we have that

$$\begin{aligned}A^*T &= A^*(\ell_1 \otimes \cdots \otimes \ell_k) \\ &= A^*\ell_1 \otimes \cdots \otimes A^*\ell_k\end{aligned}$$

Exercise 1.3.iii

where $A^*\ell_i = A^*\ell_{i+1}$ so that $A^*T \in \mathcal{I}^k(V)$, as desired. □

- $A^*\omega$: The $\mathcal{I}^k(W)$ -coset $\pi(A^*T)$, where $\omega = \pi(T)$.
- Claim 1.8.3: $A^*\omega$ is well-defined.

Proof. Suppose $\omega = \pi(T) = \pi(T')$. Then $T = T' + S$ where $S \in \mathcal{I}^k(W)$. It follows that $A^*T = A^*T' + A^*S$, but since $A^*S \in \mathcal{I}^k(V)$ (Lemma 1.8.1), we have that

$$\pi(A^*T) = \pi(A^*T')$$

as desired. □

- Proposition 1.8.4. The map $A^* : \Lambda^k(W^*) \rightarrow \Lambda^k(V^*)$ sending $\omega \mapsto A^*\omega$ is linear. Moreover,
 1. If $\omega_i \in \Lambda^{k_i}(W^*)$ ($i = 1, 2$), then

$$A^*(\omega_1 \wedge \omega_2) = A^*(\omega_1) \wedge A^*(\omega_2)$$

2. If U is a vector space and $B : U \rightarrow V$ is a linear map, then for $\omega \in \Lambda^k(W^*)$,

$$B^*A^*\omega = (AB)^*\omega$$

(Hint: This proposition follows immediately from Exercises 1.3.iii-1.3.iv.)

- **Determinant** (of A): The number a such that $A^*\omega = a\omega$, where $A^* : \Lambda^n(V^*) \rightarrow \Lambda^n(V^*)$. Denoted by $\det(A)$.
- Proposition 1.8.7: If A and B are linear mappings of V into V , then

$$\det(AB) = \det(A)\det(B)$$

Proof. Proposition 1.8.4(2) implies that

$$\begin{aligned}\det(AB)\omega &= (AB)^*\omega \\ &= B^*(A^*\omega) \\ &= \det(B)A^*\omega \\ &= \det(B)\det(A)\omega\end{aligned}$$

as desired. □

- id_V : The identity map on V .
- Proposition 1.8.8: $\det(\text{id}_V) = 1$.
 - Hint: id_V^* is the identity map on $\Lambda^n(V^*)$.
- Proposition 1.8.9: If $A : V \rightarrow V$ is not surjective, then $\det(A) = 0$.

Proof. Let $W = \text{im}(A)$. If A is not onto, $\dim W < n$, implying that $\Lambda^n(W^*) = 0$. Now let $A = \iota_W B$ where ι_W is the inclusion map of W into V and B is the mapping A regarded as a mapping from V to W . It follows by Proposition 1.8.4(1) that if $\omega \in \Lambda^n(V^*)$, then

$$A^* \omega = B^* \iota_W^* \omega$$

where $\iota_W^* \omega = 0$ as an element of $\Lambda^n(W^*)$. □

- Deriving the typical formula for the determinant.
 - Let V, W be n -dimensional vector spaces with respective bases e_1, \dots, e_n and f_1, \dots, f_n .
 - Denote the corresponding dual bases by e_1^*, \dots, e_n^* and f_1^*, \dots, f_n^* .
 - Let $A : V \rightarrow W$. Recall that if the matrix of A is $[a_{i,j}]$, then the matrix of $A^* : W^* \rightarrow V^*$ is $(a_{j,i})$, i.e., if

$$Ae_j = \sum_{i=1}^n a_{i,j} f_i$$

then

$$A^* f_j^* = \sum_{i=1}^n a_{j,i} e_i^*$$

- It follows that

$$\begin{aligned} A^*(f_1^* \wedge \dots \wedge f_n^*) &= A^* f_1^* \wedge \dots \wedge A^* f_n^* \\ &= \sum_{1 \leq k_1, \dots, k_n \leq n} (a_{1,k_1} e_{k_1}^*) \wedge \dots \wedge (a_{n,k_n} e_{k_n}^*) \\ &= \sum_{1 \leq k_1, \dots, k_n \leq n} a_{1,k_1} \dots a_{n,k_n} e_{k_1}^* \wedge \dots \wedge e_{k_n}^* \end{aligned}$$

- At this point, we are summing over all possible lists of length n containing the numbers between 1 and n at each index.
 - However, any list in which a number repeats will lead to a wedge product of a linear functional with itself, making that term equal to zero.
 - Thus, it is only necessary to sum over those terms that are non-repeating.
 - But the terms that are non repeating are exactly the permutations $\sigma \in S_n$.
- Thus,

$$\begin{aligned} A^*(f_1^* \wedge \dots \wedge f_n^*) &= \sum_{\sigma \in S_n} a_{1,\sigma(1)} \dots a_{n,\sigma(n)} (e_1^* \wedge \dots \wedge e_n^*)^\sigma \\ &= \sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} \dots a_{n,\sigma(n)} e_1^* \wedge \dots \wedge e_n^* \\ &= \det([a_{i,j}]) e_1^* \wedge \dots \wedge e_n^* \end{aligned}$$

- If $V = W$ and $e_i = f_i$ ($i = 1, \dots, n$), then we may define $\omega = e_1^* \wedge \dots \wedge e_n^* = f_1^* \wedge \dots \wedge f_n^* \in \Lambda^n(V^*)$ to obtain

$$A^*\omega = \det([a_{i,j}])\omega$$

which proves that

$$\det(A) = \det([a_{i,j}])$$

as desired.

- **Orientation** (of ℓ): A choice of one of the disconnected components of $\ell \setminus \{0\}$, where $\ell \subset \mathbb{R}^2$ is a straight line through the origin.
- **Orientation** (of L): A choice of one of the connected components of $L \setminus \{0\}$, where L is a one-dimensional vector space.
- **Positive component** (of $L \setminus \{0\}$): The component chosen in the orientation of L . Denoted by L_+ .
- **Negative component** (of $L \setminus \{0\}$): The component chosen in the orientation of L . Denoted by L_- .
- **Positively oriented** ($v \in L$): A vector $v \in L$ such that $v \in L_+$.
- **Orientation** (of V): An orientation of the one-dimensional vector space $\Lambda^n(V^*)$, where V is an n -dimensional vector space.
- “One important way of assigning an orientation to V is to choose a basis e_1, \dots, e_n of V . Then if e_1^*, \dots, e_n^* is the dual basis, we can orient $\Lambda^n(V^*)$ by requiring that $e_1^* \wedge \dots \wedge e_n^*$ be in the positive component of $\Lambda^n(V^*)$ ” (Guillemin & Haine, 2018, p. 29).
- **Positively oriented** (ordered basis e_1, \dots, e_n of V): An ordered basis $e_1, \dots, e_n \in V$ such that $e_1^* \wedge \dots \wedge e_n^* \in \Lambda^n(V^*)_+$.
- Proposition 1.9.7: If e_1, \dots, e_n is positively oriented, then f_1, \dots, f_n is positively oriented iff $\det[a_{i,j}] > 0$ where $e_j = \sum_{i=1}^n a_{i,j} f_i$.

Proof. We have that

$$f_1^* \wedge \dots \wedge f_n^* = \det[a_{i,j}] e_1^* \wedge \dots \wedge e_n^*$$

□

- Corollary 1.9.8: If e_1, \dots, e_n is a positively oriented basis of V , then the basis

$$e_1, \dots, e_{i-1}, -e_i, e_{i+1}, \dots, e_n$$

is negatively oriented.

- Theorem 1.9.9: Given orientations on V and V/W (where $\dim V = n > 1$, $W \leq V$, and $\dim W = k < n$), one gets from these orientations a natural orientation on W .

Proof. The orientations on V and V/W come prepackaged with a basis. We first apply an orientation to W based on these bases, and then show that any choice of basis for $V, V/W$ induces a basis with the same orientation on W . Let's begin.

Let $r = n - k$, and let $\pi : V \rightarrow V/W$. By Exercises 1.2.i and 1.2.ii, we may choose a basis e_1, \dots, e_n of V such that e_{r+1}, \dots, e_n is a basis of W . It follows that $\pi(e_1), \dots, \pi(e_r)$ is a basis of V/W . WLOG^[2], take $\pi(e_1), \dots, \pi(e_r)$ and e_1, \dots, e_n to be positively oriented on V/W and V , respectively. Assign to W the orientation associated with e_{r+1}, \dots, e_n .

²If the first basis is negatively oriented, we may substitute $-e_1$ for e_1 . If the second basis is negatively oriented, we may substitute $-e_n$ for e_n .

Now suppose f_1, \dots, f_n is another basis of V such that f_{r+1}, \dots, f_n is a basis of W . Let $A = [a_{i,j}]$ express e_1, \dots, e_n as linear combinations of f_1, \dots, f_n , i.e., let

$$e_j = \sum_{i=1}^n a_{i,j} f_i$$

for all $j = 1, \dots, n$. Now as will be explained below, A must have the form

$$A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

where B is the $r \times r$ matrix expressing $\pi(e_1), \dots, \pi(e_r)$ as linear combinations of $\pi(f_1), \dots, \pi(f_r)$, and D is the $k \times k$ matrix expressing the basis vectors e_{r+1}, \dots, e_n as linear combinations of f_{r+1}, \dots, f_n . We have just explained B and D . We don't particularly care about C or have a good way of defining its structure. We can, however, take the block labeled zero to be the $k \times r$ zero matrix by Proposition 1.2.9; in particular, since these components of these vectors will be fed into π and fall within W , they can moved around wherever without altering the identities of the W -cosets to which they pertain. Having justified this structure for A , we see that we can take

$$\det(A) = \det(B) \det(D)$$

It follows by Proposition 1.9.7 as well as the positivity of $\det(A)$ and $\det(B)$ that $\det(D)$ is positive, and hence the orientation of e_{r+1}, \dots, e_n and f_{r+1}, \dots, f_n are one and the same. \square

- **Orientation preserving** (map A): A bijective linear map $A : V_1 \rightarrow V_2$, where V_1, V_2 are oriented n -dimensional vector spaces, such that for all $\omega \in \Lambda^n(V_2^*)_+$, we have that $A^*\omega \in \Lambda^n(V_1^*)_+$.
- If $V_1 = V_2$, A is orientation preserving iff $\det(A) > 0$.
- Proposition 1.9.14: Let V_1, V_2, V_3 be oriented n -dimensional vector spaces, and let $A_1 : V_1 \rightarrow V_2$ and $A_2 : V_2 \rightarrow V_3$ be bijective linear maps. Then if A_1, A_2 are orientation preserving, so is $A_2 \circ A_1$.