4 Integration of Forms

From Guillemin and Haine (2018).

Chapter 3

5/17: **3.2.i.** Let $f : \mathbb{R} \to \mathbb{R}$ be a compactly supported function of class C^r with support on the interval (a, b). Show that the following are equivalent.

(1) $\int_a^b f(x)\,\mathrm{d}x = 0$.

(2) There exists a function $g : \mathbb{R} \to \mathbb{R}$ of class C^{r+1} with support on (a, b) with $\mathrm{d}g/\mathrm{d}x = f$.

(Hint: Show that the function $g(x) = \int_a^x f(s)\,\mathrm{d}s$ is compactly supported.)

Proof. Suppose first that $\int_a^b f(x)\,\mathrm{d}x = 0$. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$x \mapsto \int_a^x f(s)\,\mathrm{d}s$$

By the FTC, $\mathrm{d}g/\mathrm{d}x = f$ and, hence, $g \in C^{r+1}(\mathbb{R})$. Moreover, since f is supported on (a, b), we know that $f(x) = 0$ for all $x \leq a$ and $x \geq b$. It follows that

$$g(x) = \int_a^x f(x)\,\mathrm{d}x = \int_a^x 0\,\mathrm{d}x = 0$$

for all $x \leq a$ and that

$$g(x) = \int_a^x f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^x f(x)\,\mathrm{d}x = 0 + \int_b^x 0\,\mathrm{d}x = 0$$

for all $x \geq b$. Thus, g is supported on (a, b). Moreover, since $\operatorname{supp}(g) \subset \mathbb{R}$ is closed by definition and bounded (as a subset of (a, b)), the Heine-Borel theorem proves that g is compactly supported.

Now suppose that there exists a function $g : \mathbb{R} \to \mathbb{R}$ of class C^{r+1} with support on (a, b) and with $\mathrm{d}g/\mathrm{d}x = f$. Then by the FTC,

$$\int_a^b f(x)\,\mathrm{d}x = g(b) - g(a) = 0 - 0 = 0$$

as desired. □

3.6.iii. Show that the Brouwer fixed point theorem isn't true if one replaces the closed unit ball by the open unit ball. (Hint: Let U be the open unit ball (i.e., the interior of B^n). Show that the map $h : U \to \mathbb{R}^n$ defined by

$$h(x) = \frac{x}{1 - \|x\|^2}$$

is a diffeomorphism of U onto $\mathbb{R}^n$, and show that there are lots of mappings of $\mathbb{R}^n$ onto $\mathbb{R}^n$ which do not have fixed points.)

Proof. It appears that taking the hint will not suffice to prove the claim. After all, proving that there exist continuous mappings $h : U \to \mathbb{R}^n$ with no fixed point will not negate the modified Brouwer fixed point theorem; we would need to find a continuous mapping $f : U \to U$ with no fixed points. Fortunately, this is not hard to do — let $x = (1, 0, \dots, 0) \in \mathbb{R}^n$ and choose $f : U \to U$ defined by the rule "take every $p \in U$ to the midpoint of the line $\overline{px}$." This is clearly a continuous mapping of $U \to U$ with no fixed points. □

3.6.iv. Show that the fixed point in the Brouwer theorem doesn't have to be an interior point of B^n, i.e., show that it can lie on the boundary.

Proof. Take the mapping f from the proof of Exercise 3.6.iii. There, the fixed point is x. $\square$

3.6.v. If we identify $\mathbb{C}$ with $\mathbb{R}^2$ via the mapping $(x,y) \mapsto x+iy$, we can think of a $\mathbb{C}$-linear mapping of $\mathbb{C}$ into itself, i.e., a mapping of the form $z \mapsto cz$ for a fixed $c \in \mathbb{C}$ as an $\mathbb{R}$-linear mapping of $\mathbb{R}^2$ into itself. Show that the determinant of this mapping is $|c|^2$.

Proof. Let $c = a+ib$. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the real form of the described complex mapping, i.e.,

$$f(x,y) = (\operatorname{Re}(c \cdot (x+iy)), \operatorname{Im}(c \cdot (x+iy)))$$

Then since

$$(a+ib)(x+iy) = ax + aiy + ibx - by = (ax-by) + i(bx+ay)$$

we have that

$$f(x,y) = (ax-by, bx+ay)$$

It follows that the matrix of f is

$$\mathcal{M}(f) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

The determinant of $\mathcal{M}(f)$ is hence

$$\det[\mathcal{M}(f)] = (a)(a) - (-b)(b) = a^2 + b^2 = \left(\sqrt{a^2+b^2}\right)^2 = |c|^2$$

as desired. $\square$

3.6.vi. (1) Let $f : \mathbb{C} \to \mathbb{C}$ be the mapping $f(z) = z^n$. Show that $Df(z)$ is the linear map

$$Df(z) = nz^{n-1}$$

given by multiplication by nz^{n-1}. (Hint: Argue from first principles. Show that for $h \in \mathbb{C} = \mathbb{R}^2$,

$$\frac{(z+h)^n - z^n - nz^{n-1}h}{|h|}$$

tends to zero as $|h| \to 0$.)

Proof. We have that

$$\begin{aligned}
0 &\stackrel{?}{=} \lim_{|h|\to 0} \frac{(z+h)^n - z^n - nz^{n-1}h}{|h|} \\
0 &\stackrel{?}{=} \lim_{|h|\to 0} \frac{\sum_{k=0}^n \binom{n}{k} z^{n-k}h^k - z^n - nz^{n-1}h}{|h|} \\
0 &\stackrel{?}{=} \lim_{|h|\to 0} \frac{z^n + nz^{n-1}h + \sum_{k=2}^n \binom{n}{k} z^{n-k}h^k - z^n - nz^{n-1}h}{|h|} \\
0 &\stackrel{?}{=} \lim_{|h|\to 0} \frac{\sum_{k=2}^n \binom{n}{k} z^{n-k}h^k}{|h|} \\
0 &\stackrel{?}{=} \lim_{|h|\to 0} \sum_{k=2}^n \binom{n}{k} z^{n-k}h^{k-1} \\
0 &\stackrel{?}{=} \sum_{k=2}^n \binom{n}{k} z^{n-k}0^{k-1} \\
0 &\stackrel{\checkmark}{=} 0
\end{aligned}$$

as desired. $\square$

(2) Conclude from Exercise 3.6.v that

$$\det(Df(z)) = n^2|z|^{2n-2}$$

Proof. By calling "nz^{n-1}" a linear map, we mean the linear map $x \mapsto nz^{n-1} \cdot x$ for $x \in \mathbb{C}$ and $\cdot : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ the multiplication operation on $\mathbb{C}$. Thus, in the context of Exercise 3.6.v, $c = nz^{n-1}$. It follows that

$$\det(Df(z)) = |nz^{n-1}|^2 = n^2|z^{n-1}|^2 = n^2|z^{2n-2}| = n^2|z|^{2n-2}$$

as desired. ☐

(3) Show that at every point $z \in \mathbb{C} \setminus \{0\}$, f is orientation preserving.

Proof. Let $z \in \mathbb{C} \setminus \{0\}$ be arbitrary. To prove that f is orientation preserving at z, it will suffice to show that $\det[Df(z)] > 0$. But since $n > 0$ and $|z| > 0$ for $z \neq 0$, we have by part (2) that

$$\det[Df(z)] = n^2|z|^{2n-2} > 0$$

as desired. ☐

(4) Show that every point $w \in \mathbb{C} \setminus \{0\}$ is a regular value of f and that

$$f^{-1}(w) = \{z_1, \dots, z_n\}$$

with $\sigma_{z_i} = +1$.

Proof. By part (3), $\det[Df(z)] > 0$ for all $z \in \mathbb{C} \setminus \{0\}$. Thus, no $z \in \mathbb{C} \setminus \{0\}$ is a critical point of f. Additionally,

$$\det[Df(0)] = n^2|0|^{2n-2} = 0$$

so 0 is the lone critical value of f and element of C_f. Moreover, since $f(0) = 0$, $f(C_f) = \{0\}$, so the set of regular values of f is

$$f(\mathbb{C}) \setminus f(C_f) = \mathbb{C} \setminus \{0\}$$

as desired.

Additionally, by DeMoivre's Theorem, there are exactly n roots $z_1, \dots, z_n$ of the function z^n for all z. Lastly, by part (3), f is orientation preserving at all z, including $z_1, \dots, z_n$; therefore, $\sigma_{z_i} = +1$ for all $i = 1, \dots, n$. ☐

(5) Conclude that the degree of f is n.

Proof. By part (4) and Theorem 3.6.4,

$$\deg(f) = \sum_{i=1}^{n} \sigma_{z_i} = \sum_{i=1}^{n} +1 = n$$

as desired. ☐

3.7.i. What are the set of critical points and the image of the set of critical points for the following maps from $\mathbb{R} \to \mathbb{R}$?

(1) The map $f_1(x) = (x^2 - 1)^2$.

Answer.

$$\boxed{\begin{array}{l}\text{Critical points: } -1, 0, 1 \\ \text{Critical values: } 0, 1\end{array}}$$

☐

(2) The map $f_2(x) = \sin(x) + x$.

Answer.

$$\boxed{\begin{array}{l}\text{Critical points: } \pi + 2\pi z,\ z \in \mathbb{Z} \\ \text{Critical values: } \pi + 2\pi z,\ z \in \mathbb{Z}\end{array}}$$

□

(3) The map

$$f_3(x) = \begin{cases} 0 & x \leq 0 \\ \mathrm{e}^{-1/x} & x > 0 \end{cases}$$

Answer.

$$\boxed{\begin{array}{l}\text{Critical point: } 0 \\ \text{Critical value: } 0\end{array}}$$

□

3.7.ii. (Sard's theorem for affine maps) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an **affine map**, i.e., a map of the form $f(x) = A(x) + x_0$ where $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map and $x_0 \in \mathbb{R}^n$. Prove Sard's theorem for f.

Proof. We have that

$$Df(x) = A$$

for all $x \in \mathbb{R}^n$. We divide into two cases ($\det A = 0$ and $\det A \neq 0$). If $\det A = 0$, $f(\mathbb{R}^n) \setminus f(C_f) = \emptyset$ which is open and dense in $\mathbb{R}^n$. If $\det A \neq 0$, $f(\mathbb{R}^n) \setminus f(C_f) = \mathbb{R}^n$ which is open and dense in $\mathbb{R}^n$. □