Chapter 2

Basic Topology

2.1 Notes

11/1: • Equivalence relationships are denoted $A \sim B$.
  - These are...
    - Reflexive ($A \sim A$).
    - Symmetric ($A \sim B \iff B \sim A$).
    - Transitive ($A \sim B$ & $B \sim C \implies A \sim C$).
  - Equivalence relations give rise to equivalence classes.

- **Countable** (set A): A set A such that $A \sim \mathbb{N}$, in the sense that there exists a one-to-one and onto map from $\mathbb{N} \to A$.
  - Alternatively, A can be written in the form $A = \{f(n) : n \in \mathbb{N}\}$.
- **Finite countable** vs. **infinite countable** (see Rudin (1976)).
- $\mathbb{N}$ denotes the natural numbers.
- $\mathbb{N}_0$ denotes the natural numbers including 0.
- $\mathbb{Z}$ denotes the integers.
- We know that $\mathbb{N} \sim \mathbb{Z}$: Let $f : \mathbb{N} \to \mathbb{Z}$ be defined by

$$f(n) = \begin{cases} \frac{n}{2} & n \text{ even} \\ \frac{n-1}{2} & n \text{ odd} \end{cases}$$

- More facts.
  1. Every subset of a countable set is countable.
  2. Unions of countable sets are countable.
     - If the sets E_n for some finite list of numbers are countable, then $\bigcup_n E_n$ is countable.
     - Soug goes over the diagonalization method of counting.
  3. n-fold Cartesian products of countable sets are countable (we induct on n).
     - If A is countable and B is countable, then $A \times B$ is countable.
     - If A is finite and to each $\alpha \in A$ we assign a countable set E_α, $\otimes_{\alpha \in A} E_\alpha$ is countable.
- **Metric space**: A space X along with a metric $d : X \times X \to [0, \infty)$ such that

- $d(x,y) > 0$ iff $x \neq y$, and $d(x,x) = 0$ iff $x = 0$.
- $d(x,y) = d(y,x)$.
- $d(x,y) \leq d(x,z) + d(z,y)$.

- Example ($\mathbb{R}^n$):
  - We may define d by
$$d(x,y) = \sqrt{\sum (x_i - y_i)^2}$$
  - We can also define the p-metrics (recall normed spaces) with p where 2 is.
- Example ($X_p = \{f : Y \to \mathbb{R} : 1 \leq p < \infty, \int_Y |f|^p \, \mathrm{d}y < \infty\}$):
  - This is ℓ_p.
  - Define
$$\|f - g\|_p = \left[\int_Y |f - g|^p \, \mathrm{d}y \right]^{1/p}$$
- Convergence: $x_n \to x \iff d(x_n, x) \to 0$.
- **Neighborhood**: The set of all points a distance less than r away from p. *Denoted by* $\boldsymbol{N_r(p)}$. *Given by*
$$N_r(p) = \{q \in X : d(p,q) < r\}$$
- **Limit point** (of E): A point p such that every neighborhood of p intersects E at a point other than p. *Also known as* **accumulation point**.
  - Symbolically,
$$N_r(p) \cap (E \setminus \{p\}) \neq \emptyset$$
    for all $r > 0$.
- **Isolated point** (of E): A point p such that $p \in E$ and p is not a limit point of E.
- **Closed** (set E): A set E that contains all of its limit points.
- **Interior** (point p): A point p such that there exists $N_r(p) \subset E$.
- **Open** (set E): A set E, all points of which are interior points.
- **Perfect** (set E): A set E that is closed and every point of E is a limit point of E.
- **Bounded** (set E): There exists a number M and a $y \in X$ such that $E \subset \{p : d(p,y) \leq M\}$.
- **Dense** (set E in X): A set E such that every point of X is a limit point of E or a point of E, itself.

11/3:

- Every neighborhood is an open set.
- If p is a limit point of E, every neighborhood of p contains infinitely many points of E.
  - Thus, a finite set cannot have a limit point.
  - Prove by contradiction: Suppose there is a neighborhood that contains only finitely many points of E. Then the neighborhood with radius smaller than the distance to the closest point does not contain any points of E, a contradiction.
- E is open iff E^c[1] is closed.
  - Assume E^c closed. If $p \in E$, then p is not a limit point of E^c. It follows that there exists a neighborhood of p that is entirely contained within E, so p is interior, as desired.

[1] The complement of E.

  - Suppose E is open. Let p be any limit point of E^c. Then $p \in E^c$.

- F is closed iff F^c is open.
- If $(G_\alpha)_{\alpha \in A}$ is a family of open sets in X, then the union is open.
  - Let $p \in \bigcup_{\alpha \in A} G_\alpha$. Then $p \in G_\alpha$ for some $\alpha \in A$. It follows that p is an interior point of G_α, so thus an interior point of the union of G_α with everything else.
- Finite intersections of open sets are open.
  - In the infinite case $\bigcap_{n \in \mathbb{N}}(-1/n, 1/n) = \{0\}$, an intersection of infinitely many open sets is closed.
  - However, in the finite case, just consider the neighborhood with the smallest radius and take this one.
- The intersection of closed sets is closed.
- The union of finitely many closed sets is closed.
  - These follow from the previous two by De Morgan's rule.
- Let $\bar{E} = E \cup E'$ where E' is the set of limit points of E.
- Let X be a metric space and $E \subset X$. Then
  1. $\bar{E}$ is closed.
     - WTS: $\bar{E}^c$ is open. Let $p \in \bar{E}^c$. Then p is neither in E nor is it a limit point of E. Thus, there exists a neighborhood of $\bar{E}^c$ containing entirely points of $\bar{E}^c$. Therefore, $\bar{E}^c$ is open, so $\bar{E}$ is closed.
  2. $E = \bar{E}$ iff E is closed.
     - Think $p \in \bigcap G_\alpha$?
  3. $\bar{E} \subset F$ for any closed $F \supset E$.
     - If $E \subset F$, then any limit point of E will be a limit point of F. Thus, $E' \subset F'$. Then $\bar{E} = E \cup E' \subset F \cup F' = \bar{F} = F$ where the last equality holds because F is closed.
- Types of sets.

	Closed	Open	Perfect	Bounded
$\{z \in \mathbb{Q} : \lvert z\rvert < 1\}$	N	Y	N	Y
$\{z \in \mathbb{Q} : \lvert z\rvert \leq 1\}$	Y	N	Y	Y
Nonempty finite set	Y	N	N	Y
$\mathbb{Z}$	Y	N	N	N
$\{1/n : n \in \mathbb{N}\}$	N	N	N	Y
$\mathbb{R}^2$	Y	Y	Y	N
(a, b)	N	?	N	Y

Table 2.1: Types of sets.

- **Relatively open** (set E to Y): A set $E \subset Y \subset X$ such that if $p \in E$, then there exists a Y-neighborhood of E contained in E.
- Let $N_r^X(p) = \{y \in X : d(y, p) < r\}$ be a neighborhood of p in X, and let $N_r^Y(p) = \{y \in Y : d(y, p) < r\}$ be a neighborhood of p in Y. Then $N_r^Y(p) = N_r^X(p) \cap Y$.

- E is open relative to Y iff $E = G \cap Y$ where G is open relative to X.
- Introduces the supremum.
- If $E \subset \mathbb{R}$, $E \neq \emptyset$, and E is bounded above, $\sup E < \infty$.
- Let $y = \sup E$. Then $y \in \bar{E}$.
- There exists a sequence $a_n \in A$ such that $a_n \to x = \sup A$.
- A is compact iff any open cover of the set has a finite subcover.
- Study and *know* all of these proofs.

11/5:

- Compactness: Defines compactness in terms of open covers.
- Finite sets are compact.
- Compactness is "absolute" (i.e., it is not a relative property like openness).
  - If $K \subset Y \subset X$, then K is compact relative to X iff K is compact relative to Y.
  - V is open relative to Y iff $V = G \cap Y$ where G is open relative to X.
- Compact implies closed.
  - We will show K compact implies K^c open.
  - WTS: For all $p \in K^c$, there exists $N_r(p) \subset K^c$ suh that $N_r(p) \cap K = \emptyset$.
- A closed subset of a compact set is compact.
  - Let K be compact and let $F \subset K$ be closed.
  - Take any open cover of F. Extend it to an open cover of K. Take the finite subcover of K. Naturally, this finite subcover is also a finite cover of $F \subset K$.
- F closed, K compact implies $F \cap K$ compact.
- If $(K_\alpha)_{\alpha \in A}$ is compact in X with finite intersection property (every intersection of any finite number of these sets is nonempty), then $\bigcap_{\alpha \in A} K_\alpha \neq \emptyset$.
  - Argue by contradiction.
  - Let $G_\alpha = K_\alpha^c$.
  - Assume the intersection is empty. Assume WLOG that no point of K_1 is in any of the other K_α's.
  - Then $\{G_\alpha\}_{\alpha \in A}$ be an open cover of K_1.
  - K_1 compact implies there is a finite subcover $G_{\alpha_1}, \dots, G_{\alpha_n}$. Then $K \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$. This implies that $K_1 \cap K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} = \emptyset$, a contradiction.
- Let E be an infinite subset of a compact K. Then E has a limit point in K.
  - Argue by contradiction.
  - Suppose for all $p \in K$, there exists $N_r(p)$ such that $N_r(p) \cap E = \{p\}$.
  - Consider the set $\{N_r(p) : p \in K\}$. This is an open cover of K. Thus, there exists a finite subcover of it. But since $E \subset K \subset N_{r_1}(p_1) \cup \cdots \cup N_{r_n}(p_n) = \{p_1\} \cup \cdots \cup \{p_n\}$, E is finite, a contradiction.
- **2-cell** (in $\mathbb{R}^2$): A set that is the Cartesian product of two closed intervals.
  - Generalizes to ***k*-cells**.
- Let $I_n = [a_n, b_n] \subset \mathbb{R}$ such that $I_{n+1} \subset I_n$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

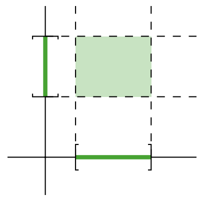

Figure 2.1: 2-cell.

- Let I_k be a k-cell in $\mathbb{R}^k$ such that $I_k \supset I_{k+1}$. Then $\bigcap_k I_k \neq \emptyset$.
- We know that $a_m \leq a_{m+n} \leq b_{m+n} \leq b_m$, so $\sup a_n \in \bigcap I_n$.

- Every k-cell is compact.

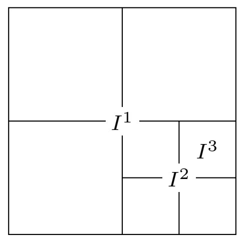

Figure 2.2: k-cells are compact.

- Argue by contradiction.
- Consider an open cover of the k-cell I^1. If it has a finite subcover, we're done. So suppose we have an open cover that doesn't have a finite subcover. Split the k-cell into 2^k chunks. At least one of the chunks I^2 must not have a finite subcover.
- Split that one into 2^k chunks. At least one of the chunks I^3 must not have a finite subcover.
- Continue.
- Thus, we have a decreasing family of k-cells, so by the previous result, their $\bigcap I^n \neq \emptyset$.
- Let $x \in \bigcap I^n$. Then the...

- Heine-Borel theorem: Let $E \subset \mathbb{R}^k$. Then TFAE[2]

1. E is closed and bounded.
2. E is compact.
3. Every infinite subset of E has a limit point in E.

- ($1 \Rightarrow 2$) E closed and bounded implies E is a closed subset of some I_k, so it's compact.
- ($2 \Rightarrow 3$) Already done.
- ($3 \Rightarrow 1$)
  - Suppose E not bounded. Then there is an infinite sequence of points in E that never converges. Contradiction.
  - Suppose E is not closed. Then there exists a sequence of points in E which "converges" to an $x_0 \notin E$.

11/8:

- Hewitt and Stromberg (1965) has harder analysis problems than Rudin (1976).

- Theorem: If P is a nonempty perfect subset of $\mathbb{R}^k$, then P is uncountable.

[2]The following are equivalent.

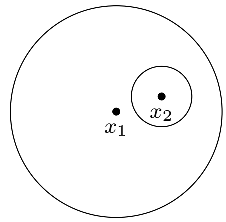

Figure 2.3: Nonempty perfect sets are uncountable.

  - P perfect implies P infinite.
  - Suppose P is countable. Let $P = \{x_1, x_2, \dots\}$.
  - Start with x_1. Take an open neighborhood V_1 of x_1. Since x_1 is a limit point of P, there will be another point $x_2 \in P$ in V_1. Choose V_2 to be a neighborhood of x_2 such that $\bar{V}_2 \subset V_1$.
  - Keep going — there is a point $x_3 \in P$ in V_2, choose an appropriate neighborhood, etc.
  - Thus, we have a sequence of closed compact sets such that $\bar{V}_n \supset \bar{V}_{n+1}$ ($n \in \mathbb{N}$). It follows that $\bigcap \bar{V}_n \neq \emptyset$.
  - We also know that $V_n \cap P \neq \emptyset$ for each n.
  - Let $K_n = \bar{V}_n \cap P$. Each K_n is compact and $K_n \supset K_{n+1}$ for each n. Therefore, by compactness, $\bigcap K_n \neq \emptyset$. But the construction implies that $\bigcap K_n = \emptyset$ because we exhausted the whole sequence of possible points $x_i \in P$.

- Corollary: Any interval is uncountable.
- The Cantor set:

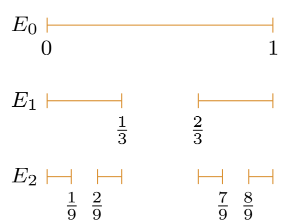

Figure 2.4: Constructing the Cantor set.

  - Let $E_0 = [0, 1]$.
  - Take out the middle third, so that $E_1 = [0, 1/3] \cup [2/3, 1]$.
  - Take out the middle thirds of the remaining intervals and keep going.
  - Thus, we are building a decreasing family of compact sets, so the overall intersection $E = \bigcap E_n$ of every set is nonempty.
  - E^n is the union of 2^n closed intervals of length $n/3$. Thus, the overall length of E^n is $(2/3)^n$.
  - Thus, we have a compact nonempty set with Lebesgue measure zero.
  - E does not contain any segment of the form

$$\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$$

    for $k, m \in \mathbb{N}$.

    - Therefore, no segment of the form (α, β) is contained in E (any segment of said form contains a segment of the above form).

- Moreover, E (the Cantor set) is perfect.
  - Let $x \in E$. WTS: For all segments S containing x, $S \cap (E \setminus \{x\}) \neq \emptyset$.
  - Let S be an arbitrary such segment...

- Consider the **Devil's staircase**.
  - $0 = \int_0^1 F'(x)\,\mathrm{d}x = F(1) - F(0) = 1$. This function does not obey the fundamental theorem of calculus. A function satisfies the fundamental theorem of calculus if and only if it is absolutely continuous.

- Connected sets (motivation):
  - In a convex set, you can connect any two points with a straight line.
  - In a nonconvex connected set, there exist points that you must connect with a curve.
  - In a disconnected set, there exist points that cannot be connected via a line whose points lie wholly in the set.

- **Connected** (set E): A set E that is not the union of two **separated** sets.

- **Separated** (sets A, B): Two sets $A, B \subset X$ that are nonempty and such that $\bar{A} \cap B = \emptyset$, and $A \cap \bar{B} = \emptyset$.

- Theorem: $E \subset \mathbb{R}$ is connected iff $x, y \in E$ and $x < z < y$ implies $z \in E$.
  - If there is a $z \notin E$ between x, y, then $\{x \in E : x < z\}$ and $\{x \in E : z < y\}$ are separated sets, so E is not connected.

2.2 Chapter 2: Basic Topology

From Rudin (1976).

11/6:

- **Countable** (set A): A set A that is in bijective correspondence with the set of all positive integers. *Also known as* **enumerable**, **denumerable**.

- **At most countable** (set A): A set A that is finite or countable.

- An alternative definition of an **infinite** set would be a set that is equivalent to one of its proper subsets.

- Theorem: Let A be the set of all sequences whose elements are the digits 0 and 1. This set A is uncountable.

  Proof. Let $E = \{s_1, s_2, \dots\}$ be an arbitrary countable subset of A, where each s_j is a sequence whose elements are the digits 0 and 1. Let s be the sequence, the n^{th} term of which is the opposite of the n^{th} term of s_n (i.e., if the n^{th} term of s_n is 0, we set the n^{th} term of s equal to 1). This guarantees that s is distinct from each of the s_j, i.e., that $s \notin E$. It follows that $E \subsetneq A$, i.e., that every countable subset of A is a proper subset of A. Therefore, A must be uncountable (for otherwise A would be a proper subset of A, a contradiction). □

  - The idea of this proof is called **Cantor's diagonalization process**.
  - Since every real number can be represented as a binary sequence of numbers, i.e., $A \sim \mathbb{R}$, the reals are uncountable.

- **Metric space**: A set X such that with any two points $p, q \in X$, there is associated a real number $d(p, q)$ such that
  1. $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$.
  2. $d(p, q) = d(q, p)$.

3. $d(p,q) \leq d(p,r) + d(r,q)$ for any $r \in X$.

- **Distance** (from $p \in X$ to $q \in X$, X a metric space): The real number $d(p,q)$.
- **Distance function**: A function $d : X \times X \to \mathbb{R}$ that sends $(p,q) \mapsto d(p,q)$. *Also known as* **metric**.
- Every subset of a metric space is a metric space in its own right under the same distance function.
- **Segment** (from a to b): The set of all real numbers x such that $a < x < b$. *Denoted by* $\boldsymbol{(a,b)}$.
- **Interval** (from a to b): The set of all real numbers x such that $a \leq x \leq b$. *Denoted by* $\boldsymbol{[a,b]}$.
- ***k*-cell**: The set of all points $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k$ whose coordinates satisfy the inequalities $a_i \leq x_i \leq b_i$ where $a_i < b_i$ for each $1 \leq i \leq k$.
  - Note that a 1-cell is an interval and a 2-cell is a rectangle.
- **Convex** (set E): A subset E of $\mathbb{R}^k$ such that

$$\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in E$$

  for all $\mathbf{x}, \mathbf{y} \in E$ and $0 < \lambda < 1$.
  - Balls and k-cells are both convex.
- The segment (a,b) is open as a subset of $\mathbb{R}^1$, but not open as a subset of $\mathbb{R}^2$.
- Since compactness is not relative, while it makes no sense to talk about *open* or *closed* metric spaces, it does make sense to talk about *compact* metric spaces.
- **Weierstrass theorem**: Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.
- Theorem: Let P be a nonempty perfect set in $\mathbb{R}^k$. Then P is uncountable.

  Proof. Since P is nonempty and perfect, there exists a limit point of P. It follows that P is infinite.

  Now suppose for the sake of contradiction that P is countable, and denote the elements of P by $\mathbf{x}_1, \mathbf{x}_2, \dots$. We now construct a sequence $\{V_n\}$ of neighborhoods, as follows. Let $V_1 = N_r(\mathbf{x}_1)$. Clearly, $V_1 \subset P$ since $\mathbf{x}_1 \in P$. It follows that since V_1 is a neighborhood that V_1 contains infinitely many points of P. Now suppose inductively that V_n has been constructed. Thus, by analogous conditions to those on V_1, we may let V_{n+1} be a neighborhood such that (i) $\bar{V}_{n+1} \subset V_n$, (ii) $\mathbf{x}_n \notin \bar{V}_{n+1}$, and (iii) $V_{n+1} \cap P$ is nonempty. By (iii), we can continue on to construct V_{n+2}, and so on and so forth.

  Let $K_n = \bar{V}_n \cap P$. Since $\bar{V}_n$ is closed and bounded, $\bar{V}_n$ is compact. Additionally, since $\mathbf{x}_n \notin K_{n+1}$ for each n, no point of P lies in $\bigcap_1^\infty K_n$. Thus, since each $K_n \subset P$, $\bigcap_1^\infty K_n$ is empty. But this contradicts our previous result that since each K_n is nonempty, compact, and such that $K_n \supset K_{n+1}$, $\bigcap_1^\infty K_n$ is nonempty. $\square$

- Corollary: Every interval $[a,b]$ is uncountable. In particular, $\mathbb{R}$ is uncountable.
- **Cantor set**: The set resulting from the following construction. Let $E_0 = [0,1]$. Remove the segment $(1/3, 2/3)$, so that $E_1 = [0,1/3] \cup [1/3, 2/3]$. Now remove the middle third of these two intervals to create E_2. Continue on indefinitely.
  - This is a perfect set in $\mathbb{R}^1$ which contains no segment.
- **Separated** (sets A, B): Two subsets A, B of a metric space X such that $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty.
- **Connected** (set E): A set E that is not the union of two nonempty separated sets.
- Separated sets are disjoint, but disjoint sets are not necessarily separated (consider $[0,1]$ and $(1,2)$).
- A subset E of $\mathbb{R}^1$ is connected if and only if it has the following property: If $x, y \in E$ and $x < z < y$, then $z \in E$.