1 Matrix Basics and Linear Systems

From Treil (2017).

Chapter 1

10/4: **1.2.** Which of the following sets (with natural addition and multiplication by a scalar) are vector spaces? Justify your answer.

a) The set of all continuous functions on the interval $[0, 1]$.

b) The set of all non-negative functions on the interval $[0, 1]$.

c) The set of all polynomials of degree *exactly* n.

d) The set of all symmetric $n \times n$ matrices, i.e., the set of matrices $A = \{a_{j,k}\}_{j,k=1}^n$ such that $A^T = A$.

1.3. True or false:

a) Every vector space contains a zero vector.

b) A vector space can have more than one zero vector.

c) An $m \times n$ matrix has m rows and n columns.

d) If f and g are polynomials of degree n, then $f + g$ is also a polynomial of degree n.

e) If f and g are polynomials of degree at most n, then $f + g$ is also a polynomial of degree at most n.

2.2. True or false:

a) Any set containing a zero vector is linearly dependent.

b) A basis must contain $\mathbf{0}$.

c) Subsets of linearly dependent sets are linearly dependent.

d) Subsets of linearly independent sets are linearly independent.

e) If $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0}$, then all scalars α_k are zero.

2.5. Let a system of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ be linearly independent but not generating. Show that it is possible to find a vector $\mathbf{v}_{r+1}$ such that the system $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}$ is linearly independent. (Hint: Take for $\mathbf{v}_{r+1}$ any vector that cannot be represented as a linear combination $\sum_{k=1}^r \alpha_k\mathbf{v}_k$ and show that the system $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}$ is linearly independent.)

2.6. Is it possible that vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent, but the vectors $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3, \mathbf{w}_3 = \mathbf{v}_3 + \mathbf{v}_1$ are linearly *independent*?

3.3. For each linear transformation below, find its matrix.

a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y)^T = (x + 2y, 2x - 5y, 7y)^T$.

b) $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3, x_4)^T = (x_1 + x_2 + x_3 + x_4, x_2 - x_4, x_1 + 3x_2 + 6x_4)^T$.

c) $T : \mathbb{P}_n \to \mathbb{P}_n$ defined by $Tf(t) = f'(t)$ (find the matrix with respect to the standard basis $1, t, t^2, \ldots, t^n$).

d) $T : \mathbb{P}_n \to \mathbb{P}_n$ defined by $Tf(t) = 2f(t) + 3f'(t) - 4f''(t)$ (again with respect to the standard basis $1, t, t^2, \ldots, t^n$).

3.6. The set $\mathbb{C}$ of complex numbers can be canonically identified with the space $\mathbb{R}^2$ by treating each $z = x + iy \in \mathbb{C}$ as a column $(x, y)^T \in \mathbb{R}^2$.

a) Treating $\mathbb{C}$ as a complex vector space, show that the multiplication by $\alpha = a + ib \in \mathbb{C}$ is a linear transformation in $\mathbb{C}$. What is its matrix?

b) Treating $\mathbb{C}$ as the real vector space $\mathbb{R}^2$, show that the multiplication by $\alpha = a+ib$ defines a linear transformation there. What is its matrix?

c) Define $T(x+iy) = 2x-y+i(x-3y)$. Show that this transformation is not a linear transformation in the complex vector space $\mathbb{C}$, but if we treat $\mathbb{C}$ as the real vector space $\mathbb{R}^2$, then it is a linear transformation there (i.e., that T is a *real linear* but not a *complex linear* transformation). Find the matrix of the real linear transformation T.

5.3. Multiply two rotation matrices T_α and T_β (it is a rare case when the multiplication is commutative, i.e., $T_\alpha T_\beta = T_\beta T_\alpha$, so the order is not essential). Deduce formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$ from here.

5.5. Find linear transformations $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ such that $AB = \mathbf{0}$ but $BA \neq \mathbf{0}$.

5.8. Find the matrix of the reflection through the line $y = -2x/3$. Perform all the multiplications.

6.3. Find all left inverses of the column $(1,2,3)^T$.

6.6. Suppose the product AB is invertible. Show that A is right invertible and B is left invertible. (Hint: You can just write formulas for right and left inverses.)

6.8. Let A be an $n \times n$ matrix. Prove that if $A^2 = \mathbf{0}$, then A is not invertible.

6.10. Write matrices of the linear transformations T_1 and T_2 in $\mathbb{F}^5$, defined as follows: T_1 interchanges the coordinates x_2 and x_4 of the vector $\mathbf{x}$, and T_2 just adds to the coordinate x_2 the quantity a times the coordinate x_4, and does not change other coordinates, i.e.,

$$T_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_4 \\ x_3 \\ x_2 \\ x_5 \end{pmatrix} \qquad\qquad T_2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 + ax_4 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

where a is some fixed number. Show that T_1 and T_2 are invertible transformations, and write the matrices of the inverses. (Hint: It may be simpler, if you first describe the inverse transformation, and then find its matrix, rather than trying to guess [or compute] the inverses of the matrices T_1, T_2.)

6.13. Let A be an invertible symmetric ($A^T = A$) matrix. Is the inverse of A symmetric? Justify.

7.3. Let X be a subspace of a vector space V, and let $\mathbf{v} \in V$, $\mathbf{v} \notin X$. Prove that if $\mathbf{x} \in X$, then $\mathbf{x} + \mathbf{v} \notin X$.

7.4. Let X and Y be subspaces of a vector space V. Using the previous exercise, show that $X \cup Y$ is a subspace if and only if $X \subset Y$ or $Y \subset X$.

7.5. What is the smallest subspace of the space of 4×4 matrices which contains all upper triangular matrices ($a_{j,k} = 0$ for all $j > k$), and all symmetric matrices ($A = A^T$)? What is the largest subspace contained in both of those subspaces?

Chapter 2

3.4. Do the polynomials $x^3 + 2x$, $x^2 + x + 1$, $x^3 + 5$ generate (span) $\mathbb{P}_3$? Justify your answer.

3.5. Can 5 vectors in $\mathbb{F}^4$ be linearly independent? Justify your answer.

3.7. Prove or disprove: If the columns of a square ($n \times n$) matrix A are linearly independent, so are the rows of $A^3 = AAA$.

5.1. True or false:

a) Every vector space that is generated by a finite set has a basis.

b) Every vector space has a (finite) basis.

c) A vector space cannot have more than one basis.

d) If a vector space has a finite basis, then the number of vectors in every basis is the same.

e) The dimension of $\mathbb{P}_n$ is n.

f) The dimension on $M_{m\times n}$ is $m+n$.

g) If vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ generate (span) the vector space V, then every vector in V can be written as a linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in only one way.

h) Every subspace of a finite-dimensional space is finite-dimensional.

i) If V is a vector space having dimension n, then V has exactly one subspace of dimension 0 and exactly one subspace of dimension n.

5.2. Prove that if V is a vector space having dimension n, then a system of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in V is linearly independent if and only if it spans V.

5.6. Consider in the space $\mathbb{R}^5$ vectors $\mathbf{v}_1 = (2, -1, 1, 5, -3)^T$, $\mathbf{v}_2 = (3, -2, 0, 0, 0)^T$, $\mathbf{v}_3 = (1, 1, 50, -921, 0)^T$. (Hint: If you do part (b) first, you can do everything without any computations.)

a) Prove that these vectors are linearly independent.

b) Complete the system of vectors to a basis.

6.1. True or false:

a) Any system of linear equations has at least one solution.

b) Any system of linear equations has at most one solution.

c) Any homogeneous system of linear equations has at least one solution.

d) Any system of n linear equations in n unknowns has at least one solution.

e) Any system of n linear equations in n unknowns has at most one solution.

f) If the homogeneous system corresponding to a given system of linear equations has a solution, then the given system has a solution.

g) If the coefficient matrix of a homogeneous system of n linear equations in n unknowns is invertible, then the system has no non-zero solutions.

h) The solution set of any system of m equations in n unknowns is a subspace of $\mathbb{R}^n$.

i) The solution set of any homogeneous system of m equations in n unknowns is a subspace of $\mathbb{R}^n$.

7.1. True or false:

a) The rank of a matrix is equal to the number of its non-zero columns.

b) The $m\times n$ zero matrix is the only $m\times n$ matrix having rank 0.

c) Elementary row operations preserve rank.

d) Elementary column operations do not necessarily preserve rank.

e) The rank of a matrix is equal to the maximum number of linearly independent columns in the matrix.

f) The rank of a matrix is equal to the maximum number of linearly independent rows in the matrix.

g) The rank of an $n\times n$ matrix is at most n.

h) An $n\times n$ matrix having rank n is invertible.

7.4. Prove that if $A : X \to Y$ and V is a subspace of X, then $\dim AV \leq \operatorname{rank} A$. ($AV$ here means the subspace V transformed by the transformation A, i.e., any vector in AV can be represented as $A\mathbf{v}$, $\mathbf{v} \in V$.) Deduce from here that $\operatorname{rank} AB \leq \operatorname{rank} A$. (Remark: Here, one can use the fact that if $V \subset W$, then $\dim V \leq \dim W$. Do you understand why it is true?)

7.6. Prove that if the product AB of two $n \times n$ matrices is invertible, then both A and B are invertible. Even if you know about determinants, do not use them (we did not cover them yet). (Hint: Use the previous 2 problems.)

7.9. If A has the same four fundamental subspaces as B, does $A = B$?

7.14. Is it possible for a real matrix A that $\operatorname{Ran} A = \operatorname{Ker} A^T$? Is it possible for a complex A?

8.3. Find the change of coordinates matrix that changes the coordinates in the basis $1, 1+t$ in $\mathbb{P}_1$ to the coordinates in the basis $1-t, 2t$.

8.6. Are the matrices $\left(\begin{smallmatrix} 1 & 3 \\ 2 & 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 2 \\ 4 & 2 \end{smallmatrix}\right)$ similar? Justify.