Chapter 9

Advanced Spectral Theory

9.1 Notes

- Let $p(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial. Let A be an $n \times n$ matrix. We let $p(A) = \sum_{i=0}^{n} a_i A^i$.

- Theorem: If A is an $n \times n$ and $p(\lambda) = \det(A - \lambda I)$, then $p(A) = 0$.

  - We know that $p(\lambda) = a(z - \lambda_1) \cdots (z - \lambda_n)$ where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues.
  - Thus $p(A) = a(A - \lambda_1 I) \cdots (A - \lambda_n I)$.
  - If you are in $\mathbb{R}^n$ and have this property, you can factorize your matrix.
  - Thus, $p(A)\mathbf{x} = \mathbf{0}$ since $\mathbf{x}$ can be decomposed into a linear combination of eigenvectors of A, which will be taken to 0 one by one by the terms of $p(A)$.

- $\sigma(B) = \{$eigenvalues of $B\}$ is known as the **spectrum** of B.

- If p is an arbitrary polynomial and A is $n \times n$, then μ is an eigenvalue of $p(A)$ if and only if $\mu = p(\lambda)$ where λ is an eigenvalue of A. In essence, $\sigma(p(A)) = p(\sigma(A))$.

- Chapter 9 will not be on the exam. We don't have to know the generalization to infinite dimensional spaces.

- If A is an $n \times n$ square matrix and $p(\lambda) = \det(A - \lambda I)$, then $p(A) = 0$.

  - Proof: WLOG, let A be an upper triangular matrix with diagonal entries equal to the eigenvalues.
  - Think of $p(z) = (-1)^n (z - \lambda_1) \cdots (z - \lambda_n)$.
  - Thus, $p(A) = (-1)^n (A - \lambda_1 I) \cdots (A - \lambda_n I)$.
  - WTS: $p(A)\mathbf{x} = 0$ for all $\mathbf{x} \in V$.
  - Let $E_k = \text{span}(e_1, \ldots, e_k)$ be the span of the first k eigenvectors of A, where $e_1, \ldots, e_n$ is a standard basis in $\mathbb{C}^n$.
  - A triangular implies $AE_k \subset E_k$. Thus, $(A - \lambda I)E_k \subset E_k$, so E_k is invariant under $A - \lambda I$ for all λ.
  - If we apply $A - \lambda_k I$ to a vector in E_k, we are left with a vector in E_{k-1}.
  - Thus, if we apply $\prod_{k=1}^{n}(A - \lambda_k I) = p(A)$ to any vector in $E_n = V$, we will kill it piece by piece down to zero.

- Let A be a square $n \times n$ matrix. Then p an arbitrary polynomial implies $\sigma(p(A)) = p(\sigma(A))$. (Any eigenvalue μ of $p(A)$ is $\mu = p(\lambda)$, where λ is an eigenvalue of A.)

  - Shows that polynomials of operators commute.

- Proof: Let λ be an eigenvalue of A. We want to show that $p(\lambda)$ is an eigenvalue of $p(A)$. This is obvious since $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x}$, so $A^k\mathbf{x} = \lambda^k\mathbf{x}$, so in particular, $p(A)\mathbf{x} = p(\lambda)\mathbf{x}$.
- On the other hand, if μ is an eigenvalue of $p(A)$, we want to show that there exists $\lambda \in \sigma(A)$ such that $\mu = p(\lambda)$.
- Consider $q(z) = p(z) - \mu$. Then $q(A) = p(A) - \mu I$. Since μ is an eigenvalue of $p(A)$, $q(A)$ is not invertible.
- Thus, $q(z) = (-1)^n (z - z_1) \cdots (z - z_n)$ and $q(A) = (-1)^k (A - z_1 I) \cdots (A - z_k I)$.
- But $q(A)$ is not invertible, so one of the $A - z_k I$ is not invertible. Take z_k such that $A - z_k I$ is not invertible. Then $z_k \in \sigma(A)$. It follows that $q(z_k) = p(z_k) - \mu = \sigma$.

- If A is $n \times n$, $\lambda_1, \ldots, \lambda_n$ are its eigenvalues, p is a polynomial, then $p(A)$ is invertible if and only if $p(\lambda_k) \neq 0$ for each $k = 1, \ldots, n$.

  - This is an immediate corollary to the previous result.

- We now build up to the **generalized eigenspace**, which is related to some "geometric" properties of the algebraic multiplicity of an eigenvalue.

- If $A : V \to V$ is a linear operator and $E \subset V$ is a subspace, E is A-invariant if $AE \subset E$.

- Facts:

  - If E is A-invariant, E is A^k-invariant.
  - Thus, E is $p(A)$-invariant.

- Consider the restriction map $A|_E$.

- A has a block-diagonalized matrix where each block corresponds to the generalized eigenvectors of a generalized eigenvalue of A.

  - Let $E_1, \ldots, E_r$ be a **basis of invariant subspaces**.
  - Let $A_k = A|_{E_k}$. Then the A_k's act independently of each other.

- **Generalized eigenvector** (of A): A vector $\mathbf{v}$ corresponding to an eigenvalue λ if there exists $k \geq 1$ such that $(A - \lambda I)^k \mathbf{v} = \mathbf{0}$.

- **Generalized eigenspace**: The set E_λ of all of the generalized eigenvectors of λ. *Given by*

$$E_k = \bigcup_{k \geq 1} \ker(A - \lambda I)^k$$

  - E_λ is a linear subspace of V.

- **Degree** (of λ): The smallest number k such that increasing k any more does not add further vectors to the generalized eigenspace. *Denoted by $\boldsymbol{d(\lambda)}$. Also known as* **depth**.

  - Symbolically, $d(\lambda)$ is the smallest number such that

$$E_\lambda = \bigcup_{k=1}^{d(\lambda)} \ker(A - \lambda I)^k$$

- Start working through the first 25 problems of Rudin (1976) (his metric spaces problems).

10/27:
- Jordan form.

- Reviews build up to generalized eigenvectors.

- Theorem: If $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$ and $E_1, \ldots, E_n$ are the corresponding generalized eigenspaces, then $E_1, \ldots, E_n$ is a basis of subspaces of U, i.e., $V = \oplus_k E_k$.

- Corollary: $A : V \to V$ can be represented as $A = D + N$ where D is diagonalizable and N is nilpotent and $ND = DN$.

    - Proof: Consider the basis of generalized eigenspaces known to exist from the theorem. Then $A = \mathrm{diag}\{A_1, \ldots, A_r\}$.
    - Let
$$N_k = A_k - \lambda_k I_{E_k}$$
    This is nilpotent.
    - Then let
$$D = \mathrm{diag}\{\lambda_1 I_{E_1}, \ldots, \lambda_n I_{E_n}\}$$
    - These two matrices satisfy the necessary properties.

- Let $\dot{\mathbf{x}} = A\mathbf{x}$.

    - Let $\mathbf{x}(t) = \mathrm{e}^{tA}$, where
$$\mathrm{e}^{tA} = \sum \frac{(tA)^k}{k!}$$
    - $\left\|\mathrm{e}^{tA}\right\| \leq \sum \frac{\|A^k\|}{k!} = \sum \frac{\|A\|^k}{k!}$.
    - Let p be a polynomial of degree k. Then
$$p(a + x) = \sum_{k=0}^{d} \frac{p^{(k)}(a)}{k!} x^k$$
    - If $A = D + N$, then...

- Nilpotent operators:

    - Let $A = \mathrm{diag}\{A_1, \ldots, A_r\}$.
    - We know that $A_k = \lambda_k I_{E_k} + N_k$ for each k.
    - Every nilpotent N can be written in the form
$$\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$$

10/28:

- The exam is long but not that hard. The only question is will you do good or very good.

- Revise the previous two homeworks, especially the last two.

- No justification for any of the true/false questions. Just circle T or F.

    - There are four problems. One is T/F (with multiple subparts); the other three are problem problems (with subparts, too).
    - Some of the questions will take you 2 seconds. Some you've already seen in the PSets.
    - The exam is supposed to be boring.

- Calculators?

    - No calculators needed. Calculators are for chem/physics exams.

      – Not a lot of computation.

- 50 minutes.

- Chloe will be proctoring.

- Remember the determinant of "special" matrices.

  - $|\det U| = 1$ if U is unitary.
  - $\det A = \pm 1$ if A is orthogonal.
  - Make a list of matrix types that are automatically diagonalizable.
  - Determinant is the product of the eigenvalues.
  - Determinant of A is equal to the conjugate of the determinant of A^*.

- Most of the exercises use the inner product.

  - Whenever you had something to prove about eigenvalues or eigenbasis, you went through diagonalization or SVD or the inner product or polar decomposition.
  - Proving eigenvalues of self-adjoint matrices are real w/ the inner product.

- Eigenvalues/eigenvectors of a projection.

  - It's implied that it's asking you the multiplicities!!!

- Know useful facts but have an idea how to prove them as well.

- Recommends against shorthanding in the exams.

- Not grading on clarity (since the exam is long).

- Max and min are for when you're sure something will be attained. Otherwise use sup and inf.

9.2   Chapter 9: Advanced Spectral Theory

10/28:
- Theorem 9.1.1 (Cayley-Hamilton): If p is the characteristic polynomial of A, $p(A) = 0$.

- Theorem 9.2.1 (Spectral Mapping Theorem): For a square matrix A and an arbitrary polynomial p, $\sigma(p(A)) = p(\sigma(A))$. In other words, μ is an eigenvalue of $p(A)$ if and only if $\mu = p(\lambda)$ for some eigenvalue λ of A.

- Corollary 9.2.2: Let A be a square matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and let p be a polynomial. Then $p(A)$ is invertible iff $p(\lambda_k) \neq 0$ for all $k = 1, \ldots, n$.

- Algebraic multiplicity is the dimension of the corresponding generalized eigenspace.