Chapter 8

Dual Spaces and Tensors

8.1 Notes

10/22:

- **Functional**: A linear bounded map $L : H \to F$, where H is finite dimensional (equivalent to $\mathbb{R}^n$).
- **Dual space**: The set of bounded linear functionals on H. *Denoted by* $\boldsymbol{H'}$, $\boldsymbol{H^*}$.
- If $l \leq p < \infty$, then
$$l^p = \left\{ (a_n)_{n \in \mathbb{N}} : \sum_{n=1}^{\infty} |a_n|^p < \infty \right\}$$
- Back to finite dimensions, $H' \approx \mathbb{R}^n$.
- Let $\mathbf{a}_1, \dots, \mathbf{a}_n$ be a basis of H. Then $L\mathbf{x} = (L\mathbf{a}_1, \dots, L\mathbf{a}_n) \approx \mathbb{R}^n$.
- Let $L((a_n)_{n \in \mathbb{N}}) = \sum_{n=1}^{\infty} a_n b_n$. Then $L((a_n)_{n \in \mathbb{N}})$ will be bounded if and only if $(b_n)_{n \in \mathbb{N}} \in l^q$ where $1 < p < q$ where $\frac{1}{q} + \frac{1}{p} = 1$.
- **Young's inequality**: The statement
$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$
- We have $|\sum a_n b_n| \leq \|a_n\|_p \|b_n\|_p$.
- Conclusion:
$$\sum \frac{|a_n||b_n|}{\|a_n\|_p \|b_n\|_q} = 1$$
- We can define H'', too. This contains linear functionals on H'.
- We know that $L(x) = \langle x, L \rangle = x(L)$. $x \in H''$.
- Riesz representation theorem: Let H have an inner product. $L \in H'$ if and only if there exists a unique $y \in H$ such that $L(x) = (x, y)$.
  - Gives us a way to identify all bounded linear functionals on H.
  - In finite dimensions, $L(x)$, where $x = \sum_1^n \alpha_i a_i$ gives us $L(x) = \sum_1^n \alpha_i L(a_i)$.

8.2 Chapter 8: Dual Spaces and Tensors

10/28:
- Linear functionals are denoted by L.
  - L is given by a $1 \times n$ matrix denoted by $[L]$.
- The collection of all $[L]$ (the dual space) is isomorphic to $\mathbb{R}^n$ via $[L] \mapsto [L]^T$.
  - However, the objects are different: Let $[I]_{\mathcal{B}\mathcal{A}}$ be the change of coordinates matrix in $\mathbb{R}^n$. We thus have that
    $$[\mathbf{v}]_{\mathcal{B}} = [I]_{\mathcal{B}\mathcal{A}}[\mathbf{v}]_{\mathcal{A}}$$
    but we also have that
    $$[L]_{\mathcal{B}} = [L]_{\mathcal{A}}[I]_{\mathcal{A}\mathcal{B}}$$
    so that
    $$[L]_{\mathcal{B}}^T = ([L]_{\mathcal{A}}[I]_{\mathcal{A}\mathcal{B}})^T = [I]_{\mathcal{A}\mathcal{B}}^T[L]_{\mathcal{A}}^T$$
  - Essentially, "if S is the change of coordinate matrix in $X\ldots$ then the change of coordinate matrix in the dual space X' is $(S^{-1})^T$" (Treil, 2017, p. 219).
- Lemma 8.1.3: Let $\mathbf{v} \in V$. If $L(\mathbf{v}) = 0$ for all $L \in V'$, then $\mathbf{v} = \mathbf{0}$. As a corollary, if $L(\mathbf{v}_1) = L(\mathbf{v}_2)$ for all $L \in V'$, then $\mathbf{v}_1 = \mathbf{v}_2$.
- The second dual V'' is canonically (i.e., in a natural way) isomorphic to V.
- **Dual basis** (to $\mathbf{b}_1, \ldots, \mathbf{b}_n \in V$): The system of vectors $\mathbf{b}_1', \ldots, \mathbf{b}_n' \in V'$ uniquely defined by the following equation. *Also known as* **biorthogonal basis**.
  $$\mathbf{b}_k'(\mathbf{b}_j) = \delta_{kj}$$
  - The k^{th} coordinate of a vector $\mathbf{v}$ in a basis $\mathbf{b}_1, \ldots, \mathbf{b}_n$ is $\mathbf{b}_k'(\mathbf{v})$.
    - This is a baby version of the **abstract non-orthogonal Fourier decomposition** of $\mathbf{v}$.
- Theorem 8.2.1 (Riesz representation theorem): Let H be an inner product space. Given a linear functional L on H, there exists a unique vector $\mathbf{y} \in H$ such that
  $$L(\mathbf{v}) = (\mathbf{v}, \mathbf{y})$$
  for all $\mathbf{v} \in H$.
- If V is a real inner product space, we can define an isomorphism from V to V' by $\mathbf{y} \mapsto L_{\mathbf{y}} = (\mathbf{v}, \mathbf{y})$.
  - If V is complex, this function is not linear since if α is complex,
    $$L_{\alpha\mathbf{y}}(\mathbf{v}) = (\mathbf{v}, \alpha\mathbf{y}) = \bar{\alpha}(\mathbf{v}, \mathbf{y}) = \bar{\alpha}L_{\mathbf{y}}(\mathbf{v})$$
  - It follows by such a mapping that $\mathbf{b}_k' = \mathbf{b}_k$ for each k.
- **Conjugate linear** (transformation): A transformation T such that
  $$T(\alpha\mathbf{x} + \beta\mathbf{y}) = \bar{\alpha}T\mathbf{x} + \bar{\beta}T\mathbf{y}$$
- It is customary to write outputs of linear functionals $L(\mathbf{v})$ in the form $\langle \mathbf{v}, L \rangle$.
  - This expression is linear in both arguments, unlike the inner product.
- Defines the dual transformation as the unique transformation such that
  $$\langle A\mathbf{x}, \mathbf{y}' \rangle = \langle \mathbf{x}, A'\mathbf{y} \rangle$$
  for all $\mathbf{x} \in X$, $\mathbf{y}' \in Y'$.

  - It's matrix in the standard bases equals A^T.

- Annihilators are denoted by $E^\perp$ here.
- Proposition 8.3.6: The annihilator of the annihilator of E equals E.
- Let $A : X \to Y$ be an operator acting from one vector space to another. Then
  1. $\ker A' = (\operatorname{range} A)^\perp$.
  2. $\ker A = (\operatorname{range} A')^\perp$.
  3. $\operatorname{range} A = (\ker A')^\perp$.
  4. $\operatorname{range} A' = (\ker A)^\perp$.