MATH 20700 (Honors Analysis in $\mathbb{R}^n$ I) Notes

Steven Labalme

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Part I

Linear Algebra

Chapter 1

Basic Notions

9/27:

- **Vector space**: Basically, a set for which you have an addition and multiplication.
- $\mathbb{F}^d$ is used for $\mathbb{R}^d$ or $\mathbb{C}^d$ in Treil (2017).
- $\mathbb{P}_n$ is the vector space of polynomials up to degree n.
- $C([0,1])$ is the set of continuous functions defined on $[0,1]$, an infinite-dimensional vector space.
- **Generating set**: A subset of a vector space, all linear combinations of which generate the vector space. *Also known as* **spanning set**.
  - Any element of VS is a linear comb. of elements of the generating set.
- **Linearly independent** (list): A list of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$ such that $\sum_{i=1}^k \alpha_i \mathbf{v}_i = 0$ implies $\alpha_i = 0$ for all i.
- **Base**: A generating set consisting of linearly independent vectors.
- Any element of a VS can be written as a *unique* linear combination of the vectors in a base.
  - If $\mathbf{x} = \sum_{i=1}^k \alpha_i \mathbf{v}_i = \sum_{i=1}^k \beta_i \mathbf{v}_i$, then $\alpha_i = \beta_i$ for all i.
- **Linear transformation**: A function $T : X \to Y$, where X, Y are VSs, such that

  $$T(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha T\mathbf{x} + \beta T\mathbf{y}$$

  for all $\mathbf{x} \in X$, $\mathbf{y} \in Y$.
- Examples of linear transformations:
  - Consider $\mathbb{P}_n$. Let $Tp_n = p_n'$. This T is linear.
  - Rotation in $\mathbb{R}^d$.
    - Think graphically about two vectors $\mathbf{x}, \mathbf{y}$.
    - Rotating and summing them is the same as summing and rotating. Same for scaling.
    - Thus, rotation is actually linear!
  - Reflection as well.
- Consider $T : \mathbb{R} \to \mathbb{R}$.
  - Any linear map on the line is a line.
  - We must have $Tx = \alpha x$: $Tx = T(1x) = xT(1) = x\alpha$.
- Consider $T : \mathbb{R}^n \to \mathbb{R}^m$ linear.

- Any linear map between $\mathbb{R}^n$ and $\mathbb{R}^m$ is linear.
- Thus, $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$, where A is an $m \times n$ matrix.

- To find A, do the same calculation as for $Tx = \alpha x$ but more carefully:
  - Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a basis.
  - So $\mathbf{x} = \sum_{i=1}^n \alpha_i \mathbf{e}_i$.
  - Thus, $T\mathbf{x} = \sum_{i=1}^n \alpha_i T(\mathbf{e}_i)$.
  - Each $T(\mathbf{e}_i)$ is part of the matrix that we multiply by the column vector representing $\mathbf{x}$.
- Multiplication of matrices is equivalent to composition of linear maps.
- Consider $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \to \mathbb{F}^r$.
  - $T_2 \circ T_1$ is equivalent to BA, if A represents T_1 and B represents T_2. In other words, $(T_2 \circ T_1)(\mathbf{x}) = BA\mathbf{x}$ for all $\mathbf{x}$.
- Recall that if $A = (\alpha_{ij})$ and $B = (\beta_{ij})$, then $(BA)_{ij} = (\sum \beta_{ik}\alpha_{kj})$.
- Properties of multiplication:

$$(AB)C = A(BC)$$
$$A(B + C) = AB + AC$$
$$(A + B)C = AC + BC$$

  - However, it is not true in general that $AB = BA$.
- **Trace** (of an $n \times n$ matrix A): The sum of the diagonal entries of A. *Denoted by* $\mathbf{tr}\,(\boldsymbol{A})$. *Given by*

$$\operatorname{tr}(A) = \sum \alpha_{ii}$$

- It is true that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.
  - Indeed, on the diagonals, multiplication is commutative; it's the other terms that mess you up in general.
- Invertibility of matrices.
  - In general, matrices are not invertible: Not every system of equations is solveable; $Ax = b$ does not always have a solution $x = A^{-1}b$.
- C is the inverse from the left: $CA = I$. B is the inverse from the right: $AB = I$. A matrix can have a left and a right inverse and still not be invertible. A matrix is invertible iff $C = B$.
- Any time we write "inverse," we do so under the assumption that it exists.
- $(AB)^{-1} = B^{-1}A^{-1}$ — easy proof by multiplication.
- If $A = (a_{ij})$, $A^T = (a_{ji})$.
  - $(A^{-1})^T = (A^T)^{-1}$.
  - $(AB)^T = B^T A^T$.
- Let X, Y VS.
  - $X \cong Y$[1] if there exists a linear $T : X \to Y$ that is one-to-one and onto.
  - Check: $A(\text{basis of } X) = \text{basis of } Y$. Prove by definition and expression of elements as linear combinations.
- **Subspace**: A subset of a vector space which happens to be a vector space, itself.

[1] "X is isomorphic to Y."

Chapter 2

Systems of Linear Equations

9/29:

- Row elimination:
  - Let
$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 3 & 1 & 2 & 7 \\ 2 & 1 & 2 & 1 \end{pmatrix}$$
  - Then the **eschelon form** matrix
$$A_e = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & -4 \end{pmatrix}$$
  - Lastly, the **reduced eschelon form** matrix
$$A_{re} = \begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$
- **Eschelon form**:
  - All zero rows are below nonzero rows.
  - For any nonzero row, its leading element is strictly to the left of the nonzero entry of the next row.
- **Reduced eschelon form**:
  - All pivots are 1.
  - Used to solve systems of the form $Ax = b$.
- **Inconsistent** (system of equations): A system with no solution.
  - If the last row is of the form $(0, \dots, 0, b)$ where $b \neq 0$, then there is no solution.
- Unique solution if A_e has a pivot in every column.
- There exists a solution for every b if there is a pivot in every row?
- Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a matrix. Then $\ker A = \{x \in \mathbb{R}^n : Ax = 0\}$ (subspace of $\mathbb{R}^n$) and $\operatorname{range} A = \{Ax : x \in \mathbb{R}^n\}$ (subspace of $\mathbb{R}^m$).
- Also consider $\ker(A^T)$ and $\operatorname{range}(A^T)$, the basis of the kernel and range, and dimension.
- Finite-dimensional vector spaces:

- A basis is a generating set (so every element of V can be written uniquely as a linear combination of the basis) the length of which is equal to the dimension of V.
- All bases of finite-dimensional vector spaces have the same number of elements.
  - Let v_1, v_2, v_3 and w_1, w_2 be two generating sets of V.
  - Then

$$\begin{aligned} v_1 &= \lambda_{11} w_1 + \lambda_{12} w_2 \\ v_2 &= \lambda_{21} w_1 + \lambda_{22} w_2 \\ v_3 &= \lambda_{31} w_1 + \lambda_{32} w_2 \end{aligned}$$

  - Suppose the only solution to $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$ is $\alpha_1 = \alpha_2 = \alpha_3 = 0$.
  - But this is not true, as we can find another one in terms of the λs.
- If you have a list of linearly independent vectors, you can complete it into a basis.
  - If there exists a vector that can't be written as a linear combination of the list, add it to the list.
- If you find any particular solution to a system $Ax = b$, and you add to it any element of $\ker A$, you will obtain another solution.
  - $Ax_1 = b$ and $Ax_h = 0$ implies that $A(x_1 + x_h) = b$.
  - $Ax_1 = b$ and $Ax_2 = b$ imply that $A(x_1 - x_2) = 0$, i.e., that $x_1 - x_2 \in \ker A$.
- If $A : \mathbb{R}^n \to \mathbb{R}^m$ and $\dim \operatorname{range} A = m$, then $Ax = b$ is solveable for all $b \in \mathbb{R}^m$.
- Let $\operatorname{rank} A = \dim \operatorname{range} A$.
- Rank theorem:
  - $\operatorname{rank} A = \operatorname{rank} A^T$.
  - Let $A : \mathbb{R}^n \to \mathbb{R}^m$. We know that $\dim \ker A + \dim \operatorname{range} A = n$.
  - $\dim \ker A^T + \operatorname{rank} A^T = m$.
  - This theorem survives linear algebra and enters functional analysis under the name **Fredholm's alternative**.

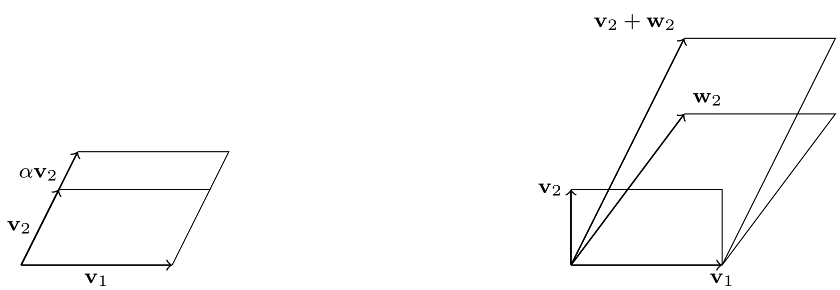
- **Fredholm's alternative**: $Ax = b$ has a solution for all $b \in \mathbb{R}^n$ iff $\dim \ker A^T = 0$.
  - $\dim \ker A^T = 0$ implies $\operatorname{rank} A^T = m$ implies $\operatorname{rank} A = m$ implies $\dim \operatorname{range} A = m$, as desired.
- **Pivot column** (of A): A column of A where A_e has pivots.
- The **pivot columns** of A give a basis for $\operatorname{range} A$.
- The pivot rows of A_e give a basis for $\operatorname{range} A^T$.
- A basis for the kernel is enough to solve $Ax = 0$.
- If you take these three things as givens, you can prove the rank theorem.

Chapter 3

Determinants

9/29:
- The determinant, geometrically, is the volume of the object (in $\mathbb{R}^3$) you get when you take linear combinations of the vectors.
- In 2D:
  - Let v_1, v_2 be two vectors. Put tail to tail and forming a parallelogram, the determinant of the matrix (v_1, v_2) is the area of said parallelogram.
  - Linearity 1: $D(av_1, v_2, \dots, v_n) = aD(v_1, \dots, v_n)$ is the same as saying that if you stretch one vector by a, you scale up the area by that much, too.
  - Linearity 2: $D(v_1, \dots, v_{k+} + v_{k-}, \dots, v_n) = D(-) + D(+)$.
  - Antisymmetry: $D(v_1, \dots, v_k, \dots, v_j, \dots, v_n) = -D(v_1, \dots, v_j, \dots, v_k, \dots, v_n)$. Interchanging columns flips the sign of the determinant.
  - Basis: $D(e_1, \dots, e_n) = 1$.
- Determinant: Denoted by $D(v_1, \dots, v_n)$, where $(v_1, \dots, v_n)$ is an $n \times n$ matrix.

10/1:
- Consider an $n \times n$ matrix A consisting of n columns containing vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^n$.
  - $D(A)$ is the volume of the solid $V = \sum_{i=1}^{n} \alpha_i v_i$.
  - $D(\mathbf{e}_1, \dots, \mathbf{e}_n) = 1$.



(a) $D(\mathbf{v}_1, \alpha\mathbf{v}_2) = \alpha D(\mathbf{v}_1, \mathbf{v}_2)$. (b) $D(\mathbf{v}_1, \mathbf{v}_2 + \mathbf{w}_2) = D(\mathbf{v}_1, \mathbf{v}_2) + D(\mathbf{v}_1, \mathbf{w}_2)$.

Figure 3.1: Visualizing properties of determinants.

- Basic properties of the determinant.
  - If A has a zero column, then $\det A = 0$: Scalar property.
  - If A has two equal columns, then $\det A = 0$: Multiply one by minus and add.

  - If A has a column which is a multiple of another, then $\det A = 0$: Pull out the multiple and then you have the previous one.
  - If columns are linearly dependent, then $\det A = 0$: Decompose it into sums, split, add back up with previous properties.
  - The determinant is preserved under column reduction.
  - $\det A^T = \det A$: Put everything in rref.
  - If A is not invertible, then $\det A = 0$ (not invertible implies linearly dependent columns, implies $\det A = 0$).
  - $\det(AB) = \det A \det B$.

- Determinant of...
  - A diagonal matrix: The product of the diagonal entries (pull out the terms, and then note that the remaining identity matrix has determinant 1).
  - An upper triangular matrix: The product of the diagonal entries (column reduction to make it into a diagonal matrix, and then the property above).

Chapter 4

Introduction to Spectral Theory

10/1:
- **Difference equation**: Like a differential equation, but instead of writing a differentials, you write differences.
- Suppose we want to solve $x_{n+1} = Ax_n$ with x_0 given.
  - You will find that $x_n = A^n x_0$.
  - This gets hard to compute, so we want to find a way to simplify the computation.
- Thus, we want to diagonalize the matrix, and this concept is inherently linked to eigenvalues and eigenvectors.
  - If you can decompose the x_0 into a linear combination of eigenvectors, then you can simplify the computation a lot:
$$x_n = \sum \alpha_i A^n v_i = \sum \alpha_i \lambda_i^n v_i$$
  - An $n \times n$ matrix will have n eigenvalues. You want n linearly independent eigenvectors, creating an eigenbasis.
- To find eigenvalues and eigenvectors, we need to solve $Ax = \lambda x$, i.e., $(A - \lambda I)x = 0$. Thus, $\ker(A - \lambda I) \neq \{0\}$, so $\det(A - \lambda I) = 0$.
- The eigenvalues of A are independent of the choice of basis of the domain of A or the range.

10/4:
- We need to know everything in Treil (2017).
  - We don't need to know the applications sections, but you should be interested.
- **Spectral theory**: Decomposing a linear operator.
- Let $A : V \to V$ be a linear operator. $\lambda \in \mathbb{C}$ is an eigenvalue if there exists $x \in V$ nonzero such that $Ax = \lambda x$.
  - Let A be an $n \times n$ matrix over $\mathbb{C}$ or $\mathbb{R}$.
  - The eigenvalues are the roots of the polynomial $\det(A - \lambda I) = 0$ in λ.
- Things we want to do:
  - Given A, find the eigenvalues and eigenvectors (solve $(A - \lambda I)x = 0$).
  - In order to simplify A, make it a diagonal matrix:
$$A = S \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} S^{-1}$$

- Eigenvalues are independent of the choice of basis.
  - From the book, we have that
$$[A]_{\mathcal{A}\mathcal{A}} = [S]_{\mathcal{A}\mathcal{B}}[B]_{\mathcal{B}\mathcal{B}}[S]_{\mathcal{A}\mathcal{B}}^{-1}$$
  - It follows that
$$A - \lambda I = [S]_{\mathcal{A}\mathcal{B}}(B - \lambda I)[S]_{\mathcal{A}\mathcal{B}}^{-1}$$
    so
$$\det(A - \lambda I) = \det\big([S]_{\mathcal{A}\mathcal{B}}(B - \lambda I)[S]_{\mathcal{A}\mathcal{B}}^{-1}\big) = \det\big([S]_{\mathcal{A}\mathcal{B}}[S]_{\mathcal{A}\mathcal{B}}^{-1}(B - \lambda I)\big) = \det(B - \lambda I)$$
- If $p(z) = (z - \lambda)^k q(z)$, then k is the **algebraic multiplicity** of λ. The **geometric multiplicity** of λ is $\dim\ker(A - \lambda I)$.
  - These terms are not always the same, but they are related.
- Diagonalization:
  - Given A that corresponds to $T : V \to V$, can we find a basis of V in which the operator is a diagonal matrix?
  - $A = SDS^{-1}$ iff there exists a basis of V consisting of the eigenvectors of A.
  - Proves $A^N = SD^N S^{-1}$ via $A^2 = SDS^{-1}SDS^{-1} = SDIDS^{-1} = SD^2S^{-1}$.
- Let A be an $n \times n$ matrix over $\mathbb{F}$. If $\lambda_1, \dots, \lambda_r$ are distinct eigenvalues, then their eigenvectors are linearly independent.
  - Prove with induction contradiction argument. Assume true for $\mathbf{v}_{r-1}$. Then
$$0 = (A - \lambda_r I)[\mathbf{v}_1 + \cdots + \mathbf{v}_r] = (\lambda_1 - \lambda_r)\mathbf{v}_1 + \cdots + (\lambda_{r-1} - \lambda_r)\mathbf{v}_{r-1}$$
  - Implies $\lambda_r = \lambda_i$ for all $i \in [r-1]$, a contradiction.
  - If A has n distinct eigenvalues, then A is diagonalizable.
- If $A : V \to V$ has n complex eigenvalues, then A is diagonalizable iff the algebraic multiplicity equals the geometric multiplicity for each eigenvalue.
- Goes through a sample diagonalization with $\left(\begin{smallmatrix} 1 & 2 \\ 8 & 1 \end{smallmatrix}\right)$.
  - We have
$$A - \lambda I = \begin{pmatrix} 1 - \lambda & 2 \\ 8 & 1 - \lambda \end{pmatrix}$$
    so
$$0 = \det(A - \lambda I) = (1 - \lambda)^2 - 16$$
  - It follows that $\lambda = 5, -3$.
  - This yields
$$\begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}^{-1}$$
    by inspection.
- As another example, consider $\left(\begin{smallmatrix} 1 & 2 \\ -2 & 1 \end{smallmatrix}\right)$.
  - Here, we have $\lambda = 1 \pm 2i$.

Chapter 5

Inner Product Spaces

10/6:

- We define

$$\ell^2(\mathbb{R}) = \left\{ \{a_n\}_{n\geq 1} \subset \mathbb{R} : \sum_1^\infty |a_n|^2 < \infty \right\}$$

- **Inner product**: A map $V \times V \to \mathbb{F}$ that takes $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \cdot \mathbf{y}$. *Denoted by* $\cdot, (\cdot, \cdot), \langle \cdot, \cdot \rangle$.
- Properties of the inner product:
  - $(\mathbf{x}, \mathbf{y}) = \overline{(\mathbf{y}, \mathbf{x})}$ (symmetry).
  - $(\alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z}) = \alpha(\mathbf{x}, \mathbf{z}) + \beta(\mathbf{y}, \mathbf{z})$ (linearity).
  - $(\mathbf{x}, \mathbf{x}) \geq 0$.
  - $(\mathbf{x}, \mathbf{x}) = 0$ iff $\mathbf{x} = 0$.
- If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n x_i y_i$$

- If $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, then

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n x_i \bar{y}_i$$

- If $f, g \in \mathbb{P}_n(t)$, then

$$(f, g) = \int_{-1}^1 f\bar{g} \, \mathrm{d}t$$

  - The conjugate of a polynomial is the polynomial with the conjugate of the coefficients of the original polynomial. Symbolically, if $f = \sum_{i=0}^n \alpha_i t^i$ is a polynomial, then $\bar{f} = \sum_{i=0}^n \bar{\alpha}_i t^i$.
- It is a fact that

$$\left| \sum^\infty a_n \bar{b}_n \right| \leq \|(a_n)_{n\geq 1}\| \|(b_n)_{n\geq 1}\|$$

- Suppose we want to define the inner product between two matrices.
  - A common one is

$$(A, B) = \operatorname{tr}(B^* A)$$

    where $B^* = \bar{B}^T = \overline{B^T}$ is the conjugate transpose.

- We define the norm as a function $V \to [0, \infty)$ given by

$$\|\mathbf{x}\| = \sqrt{(\mathbf{x}, \mathbf{x})}$$

- Properties of the norm.
  - $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$.
  - $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.
  - $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
- In $\mathbb{R}^n$,

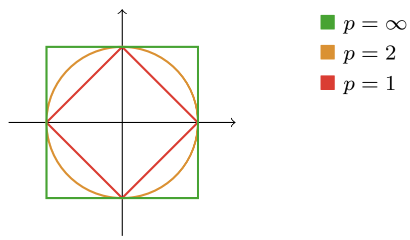


Figure 5.1: The unit ball of norms corresponding to $p = 1, 2, \infty$.

  - The standard norm is
$$\|\mathbf{x}\| = \sqrt{\sum |x_i|^2}$$
  - We can also define
$$\|\mathbf{x}\|_p = \sqrt[p]{\sum |x_i|^p}$$
  - We can even define
$$\|\mathbf{x}\|_\infty = \max |x_i|$$
  - And we can prove that all of these are valid norms.
  - Only the norm corresponding to ℓ^2 is given by an inner product, but all the other quantities are still norms as defined by the properties (see Treil (2017)).
  - Figure 5.1 shows the unit ball of each norm, i.e., the set of all points which have norm 1.
- The parallelogram rule:
$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)$$
- Orthogonality: Given $\mathbf{v}, \mathbf{w}$, if $\mathbf{v} \perp \mathbf{w}$, then $(\mathbf{v}, \mathbf{w}) = 0$.
- In particular, if $\mathbf{v} \perp \mathbf{w}$, then
$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$$
- Let E be a subspace of V. If $\mathbf{v} \perp E$, then $\mathbf{v} \perp \mathbf{e}$ for all $\mathbf{e} \in E$, i.e., $\mathbf{v} \perp$ a set of vectors spanning E.
- Any set of orthogonal vectors is linearly independent. Thus, if V is n dimensional, then $\mathbf{v}_1, \dots, \mathbf{v}_n$ orthogonal is a basis.
- Let E be a subspace of V. Take $\mathbf{v} \in V$. We want to define the projection $P_E\mathbf{v}$ of $\mathbf{v}$ onto E.
  - We have that $P_E\mathbf{v} \in E$ and $v - P_E\mathbf{v} \perp E$.
  - Additionally, we have that
$$\|\mathbf{v} - P_E\mathbf{v}\| \leq \|\mathbf{v} - \mathbf{e}\|$$
    for all $\mathbf{e} \in E$.

- Lastly, we have that $P_E\mathbf{v}$ is unique.

- If we receive a basis of a vector space, how do we create out of that a basis that is orthogonal? The process of doing this is called **Gram-Schmidt orthogonalization**.
  - We keep $\mathbf{v}_1$, subtract $P_{\mathbf{v}_1}\mathbf{v}_2$ from $\mathbf{v}_2$, subtract $P_{\{\mathbf{v}_1,\mathbf{v}_2\}}\mathbf{v}_3$ from $\mathbf{v}_3$, and on and on.
- If we are given a set of orthogonal vectors, we can normalize them by dividing each by its norm. This creates an orthonormal list. The standard basis is orthonormal.
- Let
$$E^\perp = \{v \in V : v \perp E\}$$
  - It follows that $V = E \oplus E^\perp$.
- How close can we come to solving $A\mathbf{x} = \mathbf{b}$ if we cannot solve it exactly (i.e., if the columns are not linearly independent)?
  - Let A be an $m \times n$ matrix, and let $\mathbf{b} \in \mathbb{R}^m$.
  - Then the best solution is given by minimizing $\|A\mathbf{x} - \mathbf{b}\|$. We minimize this with projections. A special case of this is least squares regression! More details in Treil (2017).

10/8:
- Soug is gonna send us a hefty amount of reading for the weekend.
- Least square approximation:
  - If we want to minimize $\|A\mathbf{x} - \mathbf{b}\|$, the best we can do is project $\mathbf{b}$ onto the range of A.
  - Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be an orthogonal basis of range A.
  - Then
$$\text{Proj}|_{\text{range } A}\mathbf{b} = \sum^k \frac{(\mathbf{b}, \mathbf{v}_k)}{\|v_k\|^2}\mathbf{v}_k$$
  - Matrix equation form:
$$\text{Projection}_{\text{range } A} = A(A^*A)^{-1}A^*$$
    if A^*A is invertible, where $A^* = \bar{A}^T$.
    - Soug never uses this though.
  - The minimum is found when $\mathbf{b} - A\mathbf{x} \perp \text{range } A$. Implies that $\mathbf{b} - A\mathbf{x} \perp \mathbf{a}_k$ for all k. Implies $(\mathbf{b} - A\mathbf{x}, \mathbf{a}_k) = \bar{\mathbf{a}}_k^T(\mathbf{b} - A\mathbf{x}) = 0$.
  - Note that we're letting $\bar{\mathbf{a}}_k^T$ be the row vector
$$\bar{\mathbf{a}}_k^T = \begin{pmatrix} \bar{a}_{1,k} & \cdots & \bar{a}_{n,k} \end{pmatrix}$$
  - We also have $\bar{A}^T(\mathbf{b} - A\mathbf{x}) = 0$, from which it follows that $A^*A\mathbf{x} = A^*\mathbf{b}$, so $\mathbf{x} = (A^*A)^{-1}A^*\mathbf{b}$. Thus, $\text{Proj}|_{\text{range } A} = Ax$, so $\text{Proj}|_{\text{range } A} = A(A^*A)^{-1}A^*\mathbf{b}$.
- Adjoint of a linear map $T : V \to W$ is the A^* discussed above.
  - First, we'll do this for matrices. And then we'll do it for any finite-dimensional vector space.
  - Let A be an $m \times n$ matrix. We claim then that
$$(A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, A^*\mathbf{y})$$
    for all $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{y} \in \mathbb{C}^m$. Proof:
$$\begin{aligned}(A\mathbf{x}, \mathbf{y}) &= \bar{\mathbf{y}}^T A\mathbf{x} \\ &= \mathbf{y}^* A\mathbf{x} \\ &= (A^*\mathbf{y})^*\mathbf{x} \\ &= (\mathbf{x}, A^*\mathbf{y})\end{aligned}$$

- Properties of the adjoint:

$$(AB)^T = B^T A^T$$
$$(AB)^* = B^* A^*$$
$$(A^*)^* = A$$

- A^* is the unique matrix B such that $(A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, B\mathbf{y})$.
- Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be a basis of V, and let $\mathbf{w}_1, \dots, \mathbf{w}_m$ be a basis of W.
- Definition of A^*: If $(A\mathbf{x}, \mathbf{y}) = (y, A^*\mathbf{x})$ for all $\mathbf{x} \in V$ and $\mathbf{y} \in W$.
- But it's not enough to define something; we have to check that it exists.
- If $[A]_{\mathcal{AB}}$, then $[A^*]_{\mathcal{AB}}$.
- More properties (give criteria for solving systems of equations):

$$\ker A^* = (\operatorname{range} A)^\perp$$
$$\ker A = (\operatorname{range} A^*)^\perp$$
$$\operatorname{range} A = (\ker A^*)^\perp$$
$$\operatorname{range} A^* = (\ker A)^\perp$$

  - Soug proves these.

- Isometries and unitary operators.
  - $U : X \to Y$ is an isometry if $\|\mathbf{x}\| = \|U\mathbf{x}\|$ for all $\mathbf{x} \in X$. It is an isometry because it preserves the distance between points.
  - It immediately follows that $\|\mathbf{x}_1 - \mathbf{x}_2\| = \|U\mathbf{x}_1 - U\mathbf{x}_2\| = \|U(\mathbf{x}_1 - \mathbf{x}_2)\|$.
  - This definition is equivalent to an inner product one: $(\mathbf{x}, \mathbf{y}) = (U\mathbf{x}, U\mathbf{y})$. This follows from the definition of the norm.
  - We have

$$(\mathbf{a}, \mathbf{b}) = \frac{1}{4} \sum_{\alpha = \pm 1, \pm i} \alpha \|\mathbf{a} + \alpha\mathbf{b}\|^2$$

    - $(a+b)^2 - (a-b)^2 = 4ab$ for any $a, b \in \mathbb{R}$, so $ab = \frac{1}{4}[(a+b)^2 - (a-b)^2]$. Thus, in a real inner product space,

$$(\mathbf{a}, \mathbf{b}) = \frac{1}{4}\left(\|\mathbf{a} + \mathbf{b}\|^2 - \|\mathbf{a} - \mathbf{b}\|^2\right)$$

    - It follows that isometries preserve inner products.
  - U is an isometry if and only if $U^*U = I$. Proof:

$$(\mathbf{x}, \mathbf{x}) = (U\mathbf{x}, U\mathbf{y}) = (U^*U\mathbf{x}, \mathbf{x})$$
$$(\mathbf{x}, \mathbf{y}) = (U\mathbf{x}, U\mathbf{y}) = (U^*U\mathbf{x}, \mathbf{y})$$
$$(\mathbf{x}, \mathbf{y}) = (U\mathbf{x}, U\mathbf{y}) = (\mathbf{x}, \mathbf{y})$$

    for all $\mathbf{y}$.
  - An isometry is unitary if it is invertible.
    - Thus, $U : X \to Y$ an isometry is unitary iff $\dim X = \dim Y$.
  - Note that it follows that $U^* = U^{-1}$ for U an isometry.
  - U unitary implies $|\det U| = 1$, so λ an eigenvalue of U implies that $|\lambda| = 1$.
  - A is diagonalizable iff it has an orthogonal basis of eigenvectors.

Chapter 6

Structure of Operators on Inner Product Spaces

10/11:

- Spectral decomposition of self-adjoint linear maps.
  - Can we write a map in term of the eigenvalues only?
  - Let $A : X \to X$ be linear and self-adjoint. Where $\dim X < \infty$.
  - Let A have eigenvalues $\lambda_1, \dots, \lambda_n$ and eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$. The there is an orthonormal basis of X consisting of eigenvectors of A. An operator is self-adjoint if $A = A^*$.
  - If A is self-adjoint, then A can be written as diagonal with the eigenvalues on the diagonal with respect to some orthonormal basis of eigenvectors.
  - Let $\mathbb{F} = \mathbb{C}$.
- If there exists an orthonormal basis $u_1, \dots, u_n$ of X such that A is triangular, then $A = UTU^*$ where U is unitary and T is upper triangular.
  - Proved with induction on $\dim X$.
  - $\dim X = 1$ is clear.
  - Assume for $\dim X = n - 1$, WTS for $\dim X = n$.
  - The subspace has a basis $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ such that A has a diagonal form.
  - Let $u \in X$ be linearly independent of $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$.
  - Let λ be the remaining eigenvalue and u the corresponding eigenvector. Let $E = \operatorname{span}(u)$. Then make the matrix λ in the upper left corner, and block diagonal with "A_{n-1}" in the bottom right corner, zeroes everywhere else.
- **Self-adjoint** (matrix A): A linear map $A : X \to X$ where $\dim X < \infty$ such that $A = A^*$.
  - Similarly, $(Ax, y) = (x, Ay)$.
  - A self-adjoint implies all eigenvalues are real, eigenvectors corresponding to different eigenvalues are orthogonal.
    - Soug proves this.
- **Strictly positive** (operator A): A self-adjoint operator $A : X \to X$ such that $(Ax, x) > 0$ for all $x \neq 0$. *Also known as* **positive definite**.
  - Implies that all eigenvalues are strictly positive.
- **Nonnegative** (operator A): A self-adjoint operator $A : X \to X$ such that $(Ax, x) \geq 0$ for all $x \neq 0$. *Also known as* **definite**.

- All eigenvalues are nonnegative.

- Suppose $A \geq 0$ is self-adjoint. Then there exists a unique self-adjoint $B \geq 0$ such that $B^2 = A$.
  - A self-adjoint is diagonal (wrt. some basis).
  - A positive means that all eigenvalues (diagonal entries) are positive.
  - Thus, take
$$B = \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sqrt{\lambda_n} \end{pmatrix}$$
  - Suppose $B^2 = A$, $C^2 = A$. Then we have an orthonormal basis corresponding to B and an orthonormal basis corresponding to C. It follows that $B^2 = C^2 = A$. Write B^2x and C^2x in terms of their bases; will necessitate that the bases are the same.

10/13:

- If we get yes/no questions, we don't have to justify.
- Cauchy-Schwarz inequality:
$$|(\mathbf{x}, \mathbf{y})| \leq \|\mathbf{x}\|\|\mathbf{y}\|$$
  - Real spaces, V vs. $(\cdot, \cdot)$ inner product.
  - Proof:
$$\begin{aligned} 0 &\leq \|\mathbf{x} + t\mathbf{y}\|^2 \\ &= t^2\|\mathbf{y}^2\| + 2t(\mathbf{x}, \mathbf{y}) + \|\mathbf{x}\|^2 \end{aligned}$$
    Thus, the discriminant must be less than zero (because the whole polynomial is positive, so the discriminant [the opposite of the x^0 term of the factored form of the polynomial] must be less than zero so the polynomial doesn't get dragged down to negative values):
$$(\mathbf{x}, \mathbf{y})^2 - \|\mathbf{x}\|^2\|\mathbf{y}\|^2 \leq 0$$
    Taking square roots of both sides proves the desired inequality.
- Recall that if $A^* = A$, then all eigenvalues are real and all eigenvectors of distinct eigenvalues are orthogonal to each other.
- **Normal** (matrix): A matrix N such that $N^*N = NN^*$.
  - Examples: Diagonal, self-adjoint, and unitary operators are all normal.
- Any normal operator in a complex vector spae has an orthonormal set of eigenvectors, e.g., $N = UDU^*$.
  - Proof: N is upper triangular wrt. some basis (because all matrices are). WTS any normal upper triangular matrix is diagonal. Done by induction on the dimension of N from $n = 2$.
  - Assume the claim for every $(n-1) \times (n-1)$ normal upper triangular matrix.
  - Let
$$N = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & & & \\ 0 & & & \\ \vdots & & N_1 & \\ 0 & & & \end{pmatrix}$$
    (we know every normal matrix can be written in this upper triangular form)
  - Then just compute NN^* and N^*N. Knowing they have to be equal, we have that $a_{12} = \cdots = a_{1n} = 0$.

  - We can also prove from the above (block diagonal multiplication) that N_1 is normal. Thus, it's diagonal, too. Therefore, the whole thing is diagonal.
- N is normal if and only if $\|N\mathbf{x}\| = \|N^*\mathbf{x}\|$.
  - Proof: $(N\mathbf{x}, N\mathbf{y}) = (N^*N\mathbf{x}, \mathbf{y}) = (NN^*\mathbf{x}, \mathbf{y}) = (N^*\mathbf{x}, N^*\mathbf{y})$. This is equivalent to the desired condition.
- If A is nonnegative and $(A\mathbf{e}_k, \mathbf{e}_k) = a_{kk}$, then

$$\sum_{i,j=1}^{n} a_{ij}\mathbf{x}_i\mathbf{x}_j$$

- **Positive definite** (matrix): An $n \times n$ self-adjoint matrix such that $(A\mathbf{x}, \mathbf{x}) > 0$ for all $\mathbf{x} \in X$.
- Let $A : X \to Y$, $\dim X = \dim Y$. Then AA^* is positive semidefinite. And there exists a unique square root $R = \sqrt{A^*A}$.
  - Proof: $(A^*A\mathbf{x}, \mathbf{x}) = (A\mathbf{x}, A\mathbf{x}) = \|A\mathbf{x}\|^2 \geq 0$.
- **Modulus** (of A): The matrix $|A| = \sqrt{A^*A}$.
- Check $\||A|\mathbf{x}\| = \|A\mathbf{x}\|$.

$$\||A|\mathbf{x}\|^2 = (|A|\mathbf{x}, |A|\mathbf{x}) = (|A|^*|A|\mathbf{x}, \mathbf{x}) = (A^*A\mathbf{x}, \mathbf{x}) = (A\mathbf{x}, A\mathbf{x}) = \|A\mathbf{x}\|^2$$

- Let $A : X \to X$ be a linear operator. Then $A = U|A|$ where U is unitary.
- Look at singular matrices.

10/15:

- Recall that if $A : X \to Y$, we have that A^*A is semidefinite, positive, and self adjoint.
  - Thus, there exists a unique matrix $R = \sqrt{A^*A} \geq 0$, which we define to be $|A| = \sqrt{A^*A}$.
- Polar form of a matrix:

$$A = U|A|$$

  - This may not be unique!
  - Proof: Suppose $A\mathbf{x} = U(|A|\mathbf{x})$. $A\mathbf{x} \in \operatorname{range} A$, and $|A|\mathbf{x} \in \operatorname{range}(|A|)$. $\mathbf{x} \in \operatorname{range}(|A|)$ implies that there exists $\mathbf{v} \in X$ such that $x = |A|\mathbf{v}$.
  - Define $U\mathbf{x} = A\mathbf{x}$. U is a well-defined linear map.
  - $\|U_0\mathbf{x}\| = \|A\mathbf{x}\| = \||A|\mathbf{v}\| = \|\mathbf{x}\|$.
  - U is an isometry.
  - $\operatorname{range}|A| \to X$.
  - Use $\ker A = \ker|A| = (\operatorname{range} A)^\perp$ to extend U_0 to U: $U = U_0 + U_1$.
- **Singular values** (of a matrix): The eigenvalues of $|A|$.
  - So if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A^*A, the singular values of A are $\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}$.
- Let $A : X \to Y$ be a linear map.
  - Let $\sigma_1, \dots, \sigma_n$ be the signular values of A. Then $\sigma_1, \dots, \sigma_n > 0$.
  - Additionally, if $\mathbf{v}_1, \dots, \mathbf{v}_n$ is an orthonormal basis of eigenvectors of A^*A, then the list of n vectors $\mathbf{w}_1, \dots, \mathbf{w}_n$ defined by $\mathbf{w}_k = 1/\sigma_k A\mathbf{v}_k$ for each $k = 1, \dots, n$ is orthonormal.

    - Proof:

$$(\mathbf{w}_k, \mathbf{w}_j) = \frac{1}{\sigma_k\sigma_k}(A\mathbf{v}_k, A\mathbf{v}_j) = \frac{1}{\sigma_k\sigma_j} = \frac{1}{\sigma_k\sigma_j}(A^*A\mathbf{v}_k, \mathbf{v}_j) = \frac{\sigma_k^2}{\sigma_k\sigma_j}(\mathbf{v}_k, \mathbf{v}_j) = 0$$

    and

$$\|\mathbf{w}_k\| = \frac{1}{\sigma_k}\|A\mathbf{v}_k\| = \frac{1}{\sigma_k}\||A|\mathbf{v}_k\| = 1$$

  - Schmidt decomposition of A:

$$A\mathbf{x} = \sum^r \sigma_k(\mathbf{x}, \mathbf{v}_k)\mathbf{w}_k$$

    - This is because $\mathbf{x} = \sum(\mathbf{x}, \mathbf{v}_k)\mathbf{v}_k$, so by the above,

$$A\mathbf{x} = \sum^n(\mathbf{x}, \mathbf{v}_k)A\mathbf{v}_k = \sum^r \sigma_k(\mathbf{x}, \mathbf{v}_k)\mathbf{w}_k$$

- **Operator norm**: $\|A\| = \max\{\|A\mathbf{x}\| : \|\mathbf{x}\| \leq 1\}$.
- Properties of the operator norm:
  - $\|A\mathbf{x}\| \leq \|A\|\|\mathbf{x}\|$.
  - $\|\alpha A\| = |\alpha|\|A\|$.
  - $\|A + B\| \leq \|A\| + \|B\|$.
  - $\|A\| \geq 0$.
  - $\|A\| = 0$ iff $A = 0$.
- **Frobenious norm**: The norm $\|A\|_2^2 = \operatorname{tr}(A^*A)$.
- The operator norm is always less than or equal to the Frobenius norm.
- If $A : \mathbb{F}^n \to \mathbb{F}^n$, then $A = W\Sigma V^*$ where σ is a diagonal matrix of nonzero singular values.
- The operator norm of A is the largest of the singular values.
- An orthogonal matrix can be decomposed to a block-diagonal matrix of rotations.

10/18:

- Soug tests what he teaches and doesn't give super tricky questions.
- Structure of orthogonal matrices.
- **Orthogonal** (matrix): A unitary matrix U with all elements real and $|\det U| = 1$.
- Theorem: Let U be an orthogonal operator on $\mathbb{R}^n$ such that $\det U = 1$. Then there exists an orthonormal basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ such that with respect to this basis,

$$U = \begin{pmatrix} R_{\phi_1} & & & \mathbf{0} \\ & \ddots & & \\ & & \mathbb{R}_{\phi_k} & \\ \mathbf{0} & & & I_{n-2k} \end{pmatrix}$$

  where each R_{ϕ_i} is a 2×2 rotation matrix.
  - If you are in $\mathbb{R}^7$ for example, you would be able to express U as a composition of at most 3 rotation maps and the identity map.
  - Each rotation map acts on two orthonormal vectors.
  - Proof: $P(\lambda)$ is the n-degree characteristic polynomial $\det(U - \lambda I) = 0$. The eigenvalues are the roots of it.

- $p(\lambda) = 0$ if and only if $p(\bar{\lambda}) = 0$.
  - $\lambda \in \mathbb{C}$ is an eigenvalue with eigenvector $\mathbf{u} \neq 0$ iff $U\mathbf{u} = \lambda\mathbf{u}$ and $U\bar{\mathbf{u}} = \bar{\lambda}\bar{\mathbf{u}}$.
- Recall that U unitary implies $|\lambda| = 1$.
  - Proof[1]: $\|U\mathbf{x}\| = \|\mathbf{x}\|$ and $U\mathbf{x} = \lambda\mathbf{x}$. Thus,

$$\|U\mathbf{x}\| = \|\lambda\mathbf{x}\| = |\lambda|\|\mathbf{x}\| = \|\mathbf{x}\|$$

    and since $\mathbf{x} \neq 0$, we can divide by $\|\mathbf{x}\|$, so $|\lambda| = 1$.
- Let $\mathbf{u} = \operatorname{Re}\mathbf{u} + \operatorname{Im}\mathbf{u}$.
- It follows that we may define

$$\mathbf{x} = \operatorname{Re}\mathbf{u} = \frac{\mathbf{u} + \bar{\mathbf{u}}}{2} \qquad \qquad \mathbf{y} = \operatorname{Im}\mathbf{u} = \frac{\mathbf{u} - \bar{\mathbf{u}}}{2}$$

- Thus, $\mathbf{u} = \mathbf{x} + i\mathbf{y}$ and $\bar{\mathbf{u}} = \mathbf{x} - i\mathbf{y}$.
- Since $U\mathbf{x} = \frac{U\mathbf{u}+U\bar{\mathbf{u}}}{2} = \frac{\lambda\mathbf{u}+\bar{\lambda}\bar{\mathbf{u}}}{2}$, $U\mathbf{y} = \operatorname{Im}(\lambda\mathbf{u}) = \operatorname{Re}(\lambda\mathbf{u})$.
- Since $|\lambda| = 1$, $\lambda = \mathrm{e}^{i\alpha}$ and $\bar{\lambda} = \mathrm{e}^{-i\alpha}$.
- It follows that $U\mathbf{x} = (\cos\alpha)\mathbf{x} - (\sin\alpha)\mathbf{y}$ and $U\mathbf{y} = (\cos\alpha)\mathbf{y}) + (\sin\alpha)\mathbf{x}$.
- Thus, since $U\mathbf{x} = \operatorname{Re}\lambda\mathbf{u}$, we have that

$$\begin{aligned}\lambda\mathbf{u} &= (\cos\alpha + i\sin\alpha)(\mathbf{x} + i\mathbf{y}) \\ &= (\cos\alpha)\mathbf{x} - (\sin\alpha)\mathbf{y} + i[(\cos\alpha)\mathbf{y} + (\sin\alpha)\mathbf{x}]\end{aligned}$$

- If E_λ is a 2 dimensional space spanned by $\mathbf{x}$ and $\mathbf{y}$ and invariant by U. Thus, any block of the desired matrix leaves its desired sub-block invariant.
- We also know that the eigenvectors of a unitary matrix corresponding to different eigenvalues are orthogonal.
- Thus, $\|\mathbf{x}\| = \|\mathbf{y}\| = \sqrt{2}/2\|\mathbf{u}\|$, $\mathbf{x} \perp \mathbf{y}$.
- Let $\mathbf{x}, \mathbf{y}$ complete the theorem to form a basis of $\mathbb{R}^n$.
- It will follow that

$$U = \begin{pmatrix} R_\alpha & \mathbf{0} \\ \mathbf{0} & U_1 \end{pmatrix}$$

  where U_1 is orthogonal, and we may repeat the process.

[1]This would be a good exam question.

Chapter 7

Bilinear and Quadratic Forms

10/18:

- **Bilinear form**: A function $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$L(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2, \mathbf{y}) = \alpha L(\mathbf{x}_1, \mathbf{y}) + \beta L(\mathbf{x}_2, \mathbf{y}) \qquad L(\mathbf{x}, \alpha\mathbf{y}_1 + \beta\mathbf{y}_2) = \alpha L(\mathbf{x}, \mathbf{y}_1) + \beta L(\mathbf{x}, \mathbf{y}_2)$$

  - $L(\mathbf{x}, \mathbf{y}) = (A\mathbf{x}, \mathbf{y})$.

- **Quadratic form**: A bilinear form $L(\mathbf{x}, \mathbf{x})$.
  - $(\mathbf{x}, \mathbf{x})$ is a polynomial of degree 2 in $\mathbf{x}_1, \ldots, \mathbf{x}_n$:

$$L(\lambda\mathbf{x}, \lambda\mathbf{x}) = (\lambda\mathbf{x}, \lambda\mathbf{x}) = \lambda^2(\mathbf{x}, \mathbf{x})$$

- We have that

$$(A\mathbf{x}, \mathbf{x}) = (A\lambda\mathbf{x}, \lambda\mathbf{x}) = \lambda^2(A\mathbf{x}, \mathbf{x}) = \sum_{j,i=1}^{n} \alpha_{j,i}\mathbf{x}_i\mathbf{x}_j$$

- The general form of a quadratic form:
  - Can any quadratic form on $\mathbb{R}^n$ be written as $(A\mathbf{x}, \mathbf{x})$?

References

Treil, S. (2017). *Linear algebra done wrong* [http://www.math.brown.edu/streil/papers/LADW/LADW_2017-09-04.pdf].