2 Eigenvalues and Eigenvectors

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Chapter 4

10/11: **1.1.** True or false:

a) Every linear operator in an n-dimensional vector space has n distinct eigenvalues.
b) If a matrix has one eigenvector, it has infinitely many eigenvectors.
c) There exists a square real matrix with no real eigenvalues.
d) There exists a square matrix with no (complex) eigenvectors.
e) Similar matrices always have the same eigenvalues.
f) Similar matrices always have the same eigenvectors.
g) A non-zero sum of two eigenvectors of a matrix A is always an eigenvector.
h) A non-zero sum of two eigenvectors of a matrix A corresponding to the same eigenvalue λ is always an eigenvectors.

1.3. Compute eigenvalues and eigenvectors of the rotation matrix

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Note that the eigenvalues (and eigenvectors) do not need to be real.

1.5. Prove that eigenvalues (counting multiplicities) of a triangular matrix coincide with its diagonal entries.

1.6. An operator A is called **nilpotent** if $A^k = \mathbf{0}$ for some k. Prove that if A is nilpotent, then $\sigma(A) = \{0\}$ (i.e., that 0 is the only eigenvalue of A).

1.7. Show that the characteristic polynomial of a block triangular matrix

$$\begin{pmatrix} A & * \\ \mathbf{0} & B \end{pmatrix}$$

where A and B are square matrices coincides with $\det(A - \lambda I)\det(B - \lambda I)$. (Hint: Use Exercise 3.11 from Chapter 3.)

1.8. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a basis in a vector space V. Assume also that the first k vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ of the basis are eigenvectors of an operator A, corresponding to an eigenvalue λ (i.e., that $A\mathbf{v}_j = \lambda\mathbf{v}_j$, $j = 1, \ldots, k$). Show that in this basis, the matrix of the operator A has block triangular form

$$\begin{pmatrix} \lambda I_k & * \\ \mathbf{0} & B \end{pmatrix}$$

where I_k is the $k \times k$ identity matrix and B is some $(n-k) \times (n-k)$ matrix.

1.10. Prove that the determinant of a matrix A is the product of its eigenvalues (counting multiplicities). (Hint: First show that $\det(A - \lambda I) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda)$, where $\lambda_1, \ldots, \lambda_n$ are eigenvalues (counting multiplicities). Then compare the free terms (terms without λ) or plug in $\lambda = 0$ to get the conclusion.)

1.11. Prove that the trace of a matrix equals the sum of its eigenvalues in three steps. First, compute the coefficient of λ^{n-1} in the right side of the equality

$$\det(A - \lambda I) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda)$$

Then show that $\det(A - \lambda I)$ can be represented as

$$\det(A - \lambda I) = (a_{1,1} - \lambda)(a_{2,2} - \lambda) \cdots (a_{n,n} - \lambda) + q(\lambda)$$

where $q(\lambda)$ is a polynomial of degree at most $n - 2$. And finally, compare the coefficients of λ^{n-1} to get the conclusion.

2.1. Let A be an $n \times n$ matrix. True or false (justify your conclusions):

a) A^T has the same eigenvalues as A.

b) A^T has the same eigenvectors as A.

c) If A is diagonalizable, then so is A^T.

2.2. Let A be a square matrix with real entries, and let λ be its complex eigenvalue. Suppose $\mathbf{v} = (v_1, \dots, v_n)^T$ is a corresponding eigenvector, i.e., $A\mathbf{v} = \lambda\mathbf{v}$. Prove that the $\bar{\lambda}$ is an eigenvalue of A and $A\bar{\mathbf{v}} = \bar{\lambda}\bar{\mathbf{v}}$, where $\bar{\mathbf{v}} = (\bar{v}_1, \dots, \bar{v}_n)^T$ is the complex conjugate of the vector $\mathbf{v}$.

2.3. Let

$$A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$$

Find A^{2004} by diagonalizing A.

2.4. Construct a matrix A with eigenvalues 1 and 3 and corresponding eigenvectors $(1, 2)^T$ and $(1, 1)^T$. Is such a matrix unique?

2.6. Consider the matrix

$$A = \begin{pmatrix} 2 & 6 & -6 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{pmatrix}$$

a) Find its eigenvalues. Is it possible to find the eigenvalues without computing?

b) Is this matrix diagonalizable? Find out without computing anything.

c) If the matrix is diagonalizable, diagonalize it.

2.8. Find all square roots of the matrix

$$A = \begin{pmatrix} 5 & 2 \\ -3 & 0 \end{pmatrix}$$

i.e., find all matrices B such that $B^2 = A$. (Hint: Finding a square root of a diagonal matrix is easy. You can leave your answer as a product.)

2.10. Let A be a 5×5 matrix with 3 eigenvalues (not counting multiplicities). Suppose we know that one eigenspace is three-dimensional. Can you say if A is diagonalizable?

2.11. Give an example of a 3×3 matrix which cannot be diagonalized. After you construct the matrix, can you make it "generic," so no special structure of the matrix can be seen?

2.13. Eigenvalues of a transposition:

a) Consider the transformation T in the space $M_{2\times 2}$ of 2×2 matrices defined by $T(A) = A^T$. Find all its eigenvalues and eigenvectors. Is it possible to diagonalize this transformation? (Hint: While it is possible to write a matrix of this linear transformation in some basis, compute the characteristic polynomial, and so on, it is easier to find eigenvalues and eigenvectors directly from the definition.)

b) Can you do the same problem but in the space of $n \times n$ matrices?

2.14. Prove that two subspaces V_1 and V_2 are linearly independent if and only if $V_1 \cap V_2 = \{\mathbf{0}\}$.