Chapter 7

Sequences and Series of Functions

7.1 Notes

11/15:

- Soug will not test on differentiation/integration assuming that we know them already.
- **Pointwise convergent** (sequence $(f_n)_{n\in\mathbb{N}}$ of functions): A sequence of functions $f_n : E \to \mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in E$.
- Can we interchange "limit" in the above definition with continuity, convergence of series, integration, differentiation, etc.?
- Examples with negative answer:
  1. Interchanging limits: Let $S_{mn} = \frac{m}{m+n}$. $S_{mn} \to 1$ as $m \to \infty$, and $S_{mn} \to 0$ as $n \to \infty$.
  2. $f_n(x) = x^2/(1+x)^n$. $f(x) = \sum_{n=1}^{\infty} f_n(x)$. If $x = 0$, then $f_n(x) = 0$ for all n and $f(x) = 0$. If $x \neq 0$, we have
  $$f(x) = \sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n} = x^2 \sum_{n=1}^{\infty} X^n = \frac{x^2}{1-X} = \frac{x^2}{1-(1/(1+x^2))} = 1 + x^2$$
  3. Consider $f_m(x) = \lim_{n\to\infty}(\cos(m\pi x))^2n$. $\lim_{m\to\infty} f_m(x)$ goes to 0 if $x \notin \mathbb{Q}$ and goes to 1 if $x \in \mathbb{Q}$. $f_m \to \chi_{\mathbb{Q}}$, where $\chi_{\mathbb{Q}}$ is the characteristic function of the rationals which is not Riemann integrable (partitions, upper and lower integrals, etc.).
  4. $f_n(x) = \sin nx/\sqrt{n} \to f(x) = 0$ for all x. However, $f_n'(x) = \sqrt{n}\cos nx \nrightarrow 0$
  5. If $0 \leq x \leq 1$, define $f_n(x) = n^2 x(1-x^2)^n$. We know that $f_n(0) = 0$. $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in (0,1]$. We can show that $\int_0^1 x(1-x^2)^n \,\mathrm{d}x = 1/(2n+2)$. Thus, $\int_0^1 f_n(x)\,\mathrm{d}x = n^2/(2n+2)$. Limit of the functions is zero, but their integrals diverge to infinity.
- **Uniformly convergent** (sequence $(f_n)_{n\in\mathbb{N}}$ of functions on E): A sequence of functions $f_n : E \to \mathbb{R}$ such that for all $\epsilon > 0$, there exists N such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$ for all $x \in E$. *Denoted by* $\boldsymbol{f_n \rightrightarrows f}$.
- Theorem: $f_n \rightrightarrows f$ iff $(f_n)_{n\in\mathbb{N}}$ is uniformly Cauchy (i.e., for all $\epsilon > 0$, there exists N such that if $n, m \geq N$ then $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in E$).
  - Let $M_n = \sup_{x\in E} |f_n(x) - f(x)|$. If $f_n \to f$ pointwise, then $f_n \rightrightarrows f$ if $M_n \to 0$.
- Theorem: If $(f_n)_{n\in\mathbb{N}}$ and $|f_n(x)| \leq M_n$, then $\sum f_n \rightrightarrows f$ if $\sum M_n < \infty$.
- Theorem: If E is a compact metric space, $f_n \rightrightarrows f$ in E, x is a limit point of E, and $\lim_{t\to x} f_n(t) = A_n$ exists, then $(A_n)_{n\in\mathbb{N}}$ converges and $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$.

- Corollary: $\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$.

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  - Fix $\epsilon > 0$. Then $f_n \rightrightarrows f$ implies that there exists some N such that $n,m \geq N$ implies $|f_n(t) - f_m(t)| < \epsilon$ for all $t \in E$.
  - x is a limit point of E and $t \to x$ implies $|A_n - A_m| < \epsilon$. Thus, $(A_n)_{n\in\mathbb{N}}$ is cauchy, so there exists A such that $A_n \to A$.
  - WTS: $|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|$, so we WTS the three terms on the right are small.
  - There exists n such that $|f(t) - f_n(t)| < \epsilon/3$ for all t since $f_n \rightrightarrows f$ by hypothesis.
  - Since t is in a small neighborhood of x, there exists n such that $|A_n - A| < \epsilon/3$.
  - We also have $|f_n(t) - A_n| < \epsilon/3$ by hypothesis.
  - This is a very important proof to understand, because proofs like this pop up often.
- Corollary: f_n continuous and $f_n \rightrightarrows f$ implies f is continuous.
- Theorem: Let K be compact. Assume

  (a) $(f_n)_{n\in\mathbb{N}} \subset C(K) = \{f : K \to \mathbb{R} \mid f \text{ continuous}\}$.

  (b) $f_n \to f$ pointwise in K and $f \in C(K)$.

  (c) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$.

  Then $f_n \rightrightarrows f$.
  - WLOG $f = 0$, $g_n = f_n - f \to 0$, $g_n \geq g_{n+1} \geq 0$.
  - For all $\epsilon > 0$, there exists N such that $n \geq N$ and $0 \leq g_n(x) \leq \epsilon$ for all $x \in K$.
  - $K_n = \{x \in K : g_n(x) \geq \epsilon\}$.
  - g_n continuous implies K_n closed. This combined with K compact implies K_n is compact.
  - g_n decreasing implies $K_n \supset K_{n+1}$. Thus, K_n is a nested family of compact sets, so $\bigcap K_n \neq \emptyset$.
  - This implies that each K_n is nonempty, contradicting the fact that each $g_n \to 0$ for all x.
  - Thus, there exists an N such that K_n is empty for all $n \geq N$. Thus $g_n(x) \leq \epsilon$ for all $x \in K$, $n \geq N$.
  - Note that the compactness of K is important. If $f : (0,1) \to \mathbb{R}$ is defined by $f(x) = 1/(nx+1)$, then $f_n \to 0$, but $f_n \not\rightrightarrows f$.
- Let $C(X) = \{f : X \to \mathbb{R} \mid f \text{ continuous, bounded}\}$ for X a metric space.
- If we define $\|f\| = \sum_{x\in X} |f(x)|$, for $f, g \in C(X)$, we may define $d(f,g) = \|f - g\|$. This definition satisfies the properties of a distance function, and $\|\cdot\|$ is a norm.
  - Thus, $C(X)$ is a complete metric space, a normed space, or, specifically, a **Banach space**.
- Theorem: $(f_n)_{n\in\mathbb{N}} \subset C(X)$ such that $\|f_n - f_m\|_{n,m\to\infty} \to 0$. Then there exists $f \in C(X)$ such that $\|f_n - f\|_{n\to\infty} \to 0$.
  - We get such a strong statement using properties of the image, not properties of the domain.
  - For all $\epsilon > 0$, there exists N such that $n, m \geq N$.
  - $|f_n(x) - f_m(x)| \leq \|f_n - f_m\| < \epsilon$ for all x.
  - Then there exists f such that $f_m(x) \to f(x)$. It follows that $|f_n(x) - f_m(x)| < \epsilon$
- Uniform convergence and integration.
- Stieltjes integral.

- Define $\alpha : \mathbb{R} \to \mathbb{R}$ nondecreasing.
- If we sum over the minimums/maximums of a partition times $\alpha(x_{i+1}) - \alpha(x_i)$ instead of $x_{i+1} - x_i$, we obtain said integral as the upper/lower limits just like the Riemann integral.
- We write $\int_a^b f(x)\,\mathrm{d}\alpha(x)$ where $\mathrm{d}\alpha(x) = \alpha(x)\,\mathrm{d}x$.

- Theorem: If α is nondecreasing on $[a,b]$, $f_n \in R(\alpha)$ such that $f_n \rightrightarrows f$ on $[a,b]$
  - We have
$$\begin{aligned}\left|\int f_n(x)\,\mathrm{d}\alpha(x) - \int f(x)\,\mathrm{d}\alpha(x)\right| &= \left|\int (f_n - f)(x)\,\mathrm{d}\alpha(x)\right|\\ &\leq \|f_n - f\|(\alpha(b) - \alpha(a))\\ &\leq \int |f_n - f|\,\mathrm{d}\alpha(x)\\ &\leq \int \|f_n - f\|\,\mathrm{d}\alpha(x) \qquad \leq \|f_n - f\| \int_a^b \mathrm{d}\alpha(x) = \|f_n - f\|(\alpha(b) - \alpha(a))\end{aligned}$$

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- Suppose $f_n \to f$ and $f_n' \to g$. When does $f' = g$?
- Theorem: If $f_n : [a,b] \to \mathbb{R}$ is differentiable, $f_n(x_0)$ converges for some $x_0 \in [a,b]$, and f_n' converges uniformly on $[a,b]$, then there exists f differentiable on $[a,b]$ such that $f_n \rightrightarrows f$ and $f_n' \rightrightarrows f'$.
  - Assume the f_n' are continuous. Then $f_n(x) - f_n(x_0) = \int_{x_0}^x f_n'(y)\,\mathrm{d}y$.
  - Since $f_n' \rightrightarrows g$, $\int_{x_0}^x f_n'(y)\,\mathrm{d}y \to \int_{x_0}^x g(y)\,\mathrm{d}y$.
  - It follows since $f_n(x_0) \to f(x_0)$ that $f_n \rightrightarrows f$.
  - By the previous theorem, if
$$f_n'(x) = \lim_{h\to 0} \frac{f_n(x+h) - f_n(x)}{h}$$
    then
$$\lim_{n\to\infty} f_n'(x) = \lim_{h\to 0}\lim_{n\to\infty} \frac{f_n(x+h) - f_n(x)}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
  - Fix $\epsilon > 0$. Then there exists N such that $n, m \geq N$ such that $|f_n(x_0) - f_m(x_0)| < \epsilon/2$ and $|f_n'(t) - f_m'(t)| < \epsilon/2$ for all $t \in [a,b]$.
  - We know that $f_n(t) - f_n(x_0) = \int_{x_0}^t f_n'(y)\,\mathrm{d}y$ and $f_m(t) - f_m(x_0) = \int_{x_0}^t f_m'(y)\,\mathrm{d}y$.
  - Thus,
$$|f_n(t) - f_n(x_0)| \leq |f_n(t) - f_n(x_0)| + |f_m(t) - f_m(x_0)| - |f_m(t) - f_m(x_0)|$$
  - Let $f_n(t) - f_n(x_0) = c_n(t - x_0)$ and $f_m(t) - f_m(x_0) = c_m(t - x_0)$.
  - ...

- Let $f : [a,b] \to \mathbb{R}$ be continuous. What conditions on f imply that f' exists?
- Suppose f is Lipschitz continuous (equivalent to saying there exists $L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$); then f' exists **almost everywhere**.
  - If f differentiable, this is equivalent to saying f bounded.
- **Almost everywhere**: Something happens almost everywhere if the set of places where it doesn't happen has measure zero.
- Suppose f is **Hölder continuous**, then f' does not exist?
- **Hölder continuous** (function f): There exists $L > 0$ such that $|f(y) - f(x)| < L|x - y|^\alpha$ where $\alpha \in (0,1)$

- Suppose f exists such that f is Hölder continuous in a neighborhood of every point in the domain. This function is not anywhere differentiable. Such a function does indeed exist (and it's Brownian motion). The construction of such a function is the essence of Stochastic analysis.
  - Probabilistically: Has mean zero, distributed as a normal function like the Gaussian, and the increments are independent of each other.
  - Analytically: It's a function that is Hölder continuous at half plus ϵ for every ϵ and it is nowhere differentiable.
- Theorem: There exists $f : \mathbb{R} \to \mathbb{R}$ continuous but nowhere differentiable.
  - This theorem is due to Weierstrass and as such, such functions are typically called Weierstrass functions.
- A general class of functions that are nowhere differentiable (not in Rudin (1976); we don't have to prove this).
  - Example 1:
    $$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$
    where $0 < a < 1$, b positive odd integer greater than 1, and $ab > 1 + \frac{3}{2}\pi$.
    - This function at every point oscillates more and more and more.
- Rudin (1976)'s simple example.
  - $\phi : [-1, 1] \to \mathbb{R}$ defined by $\phi(x) = |x|$ is not differentiable at zero.
  - Takes ϕ extends it periodically with period 2, creating a sawtooth function.
  - Repeat the behavior so that the nondifferentiability becomes more and more frequent to get
    $$f(x) = \sum_{1}^{\infty} \left(\frac{3}{4}\right)^n \phi(4^n x)$$
  - This is continuous.
  - Fix any $x \in \mathbb{R}$, $m \in \mathbb{N}$. Then $\delta_m = \pm\frac{1}{2} \cdot 4^{-m}$.
  - Then consider $4^m x$, $4^m(x + \delta_m)$.
  - Rudin asserts
    $$\left|\frac{f(x+\delta_m) - f(x)}{\delta_m}\right| \to \infty$$
    as $m \to \infty$ for all x.

11/29:

- Finding a uniformly convergent subsequence of a sequence of functions.
  - Pointwise, uniformly, bounded if there exist M_x such that $|f_n(x)| \leq M_x$ for all n, x. Uniformly bounded if there exists M such that $|f_n(x)| \leq M$ for all n, x.
- Theorem: If $(f_n)_{n\in\mathbb{N}}$ is pointwise bounded and $E \subset X$ is countable, then there exists a subsequence f_{n_k} which converges for every $x \in E$.
  - Let $E = \{x_i : i \in \mathbb{N}\}$. Consider $f_n(x_1)$. $f_{1,k}(x_1)$ converges.
  - $S_1 : f_{1,1}(x_1), f_{1,2}(x_1), f_{1,3}(x_1), f_{1,4}(x_1), \dots$.
  - $S_2 : f_{2,1}(x_2), f_{2,2}(x_2), f_{2,3}(x_2), f_{2,4}(x_2), \dots$.
  - Now consider $f_{2,k}(x_3)$.
  - $S_3 : f_{3,1}, f_{3,2}, f_{3,3}, f_{3,4}, \dots$.

- Continue on and on to $S_4, S_5, \ldots$. We know that each of these sequences converges pointwise by hypothesis.
- Now consider the diagonal sequence $f_{1,1}, f_{2,2}, f_{3,3}, f_{4,4}$.
  - This subsequence of the original sequence we may call g_k.
  - We posit that g_k converges for every $x \in E$.

- Theorem: There exists f_n which is uniformly bounded but does not converge uniformly.
  - Let $f_n(x) = \sin(2\pi x)$ for $0 \leq x \leq 2\pi$.
  - Let $f_n(x) = x^2/(x^2 + (1-nx)^2)$ on $0 \leq x \leq 1$. This sequence is uniformly bounded, converges pointwise, but $f_n(1/n) = 1$ so f_n cannot converge uniformly to zero.
- What does it mean that $f_n : [0,1] \to \mathbb{R}$ does not converge uniformly?
  - It means that there exists a subsequence of the functions evaluated at certain points that is always greater than or equal to some fixed distance away from the limit.
- Equicontinuity: $\mathcal{F}\{f : X \to \mathbb{R}\}$ for (X,d) a metric space is equicontinuous iff for all $\epsilon > 0$, there exists a $\delta > 0$ such that $d(x,y) < \delta$ implies $|f(x) - f(y)| < \epsilon$ for all $x, y \in X$, $f \in \mathcal{F}$.
- Modulus of continuity: $f : X \to \mathbb{R}$ is continuous at x. A modulus of continuity is a function $\omega_X : [0,1] \to [0,1]$ such that $|f(y) - f(x)| \leq \omega_X|y - x|$.
- The final result we'll prove: **Arzelà-Ascoli theorem**: If we have a family of functions on a compact set and we have a dense subset of that set, then if we have a sequence of functions that are equicontinuous, then they converge uniformly.