# Chapter 3

# Numerical Sequences and Series

## 3.1 Notes

11/8: - Any bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

11/10: - Read and understand the section about Cauchy sequences converging and the sup/inf.

## 3.2 Chapter 3: Numerical Sequences and Series

*From Rudin (1976).*

11/7: - Convergence of sequences is relative.
  - For example, the sequence $1/n$ for $n = 1, 2, \dots$ converges in $\mathbb{R}$, but not in $(0, \infty)$.
- **Range** (of $\{p_n\}$): The set of all points $p_n$.
  - This definition squares nicely with the formal definition of a sequence as a function $p$ defined on $\mathbb{N}$.
- Theorem 3.6a: If $\{p_n\}$ is a sequence in a compact metric space $X$, then some subsequence of $\{p_n\}$ converges to a point of $X$.
- Theorem 3.7: The subsequential limits of a sequence $\{p_n\}$ in a metric space $X$ form a closed subset of $X$.
- **Diameter** (of $E$): The supremum of the set

$$S = \{d(p, q) : p, q \in E\}$$

  where $E$ is a nonempty subset of a metric space $X$. *Denoted by* **diam $\boldsymbol{E}$**.
- Theorem 3.10:
  (a) If $\bar{E}$ is the closure of a set $E$ in a metric space $X$, then

$$\operatorname{diam} \bar{E} = \operatorname{diam} E$$

  (b) If $K_n$ is a sequence of compact sets in $X$ such that $K_n \supset K_{n+1}$ $(n = 1, 2, 3, \dots)$ and if $\lim_{n \to \infty} \operatorname{diam} K_n = 0$, then $\bigcap_1^\infty K_n$ consists of exactly one point.
- **Complete** (metric space): A metric space in which every Cauchy sequence converges.
- All compact metric spaces and all Euclidean spaces are complete.

  - The metric space $(\mathbb{Q}, |x-y|)$ is not complete.

- **Monotonically increasing** (sequence $\{s_n\}$): A sequence $\{s_n\}$ of real numbers such that $s_n \leq s_{n+1}$ for each $n \in \mathbb{N}$.
- **Monotonically decreasing** (sequence $\{s_n\}$): A sequence $\{s_n\}$ of real numbers such that $s_n \geq s_{n+1}$ for each $n \in \mathbb{N}$.
- **Monotonic sequences**: The class of all sequences that are either monotonically increasing or monotonically decreasing.
- **Upper limit** (of $\{s_n\}$): The supremum of the set $E$ of all subsequential limits of $\{s_n\}$. *Denoted by* $\boldsymbol{s^*}$, $\boldsymbol{\limsup_{n\to\infty} s_n}$.
- **Lower limit** (of $\{s_n\}$): The infimum of the set $E$ of all subsequential limits of $\{s_n\}$. *Denoted by* $\boldsymbol{s_*}$, $\boldsymbol{\liminf_{n\to\infty} s_n}$.
- Theorem 3.17: Let $\{s_n\}$ be a sequence of real numbers. Then $s^*$ has (and is the only number to have both of) the following two properties.
  (a) $s^* \in E$.
  (b) If $x > s^*$, then there is an integer $N$ such that $n \geq N$ implies $s_n < x$.

  An analogous result holds for $s_*$.

11/8:

- Series are defined in terms of sequences. Moreover, sequences can be defined in terms of series: Let $a_1 = s_1$, $a_n = s_n - s_{n-1}$ ($n \in \mathbb{N}+1$). Thus, every theorem about sequences can be stated in terms of series and vice versa, but it is nevertheless useful to consider both concepts (Rudin, 1976, p. 59).
- Theorem 3.27: Suppose $\{a_n\}$ is a monotonically decreasing sequence of nonnegative terms. Then the series $\sum_{n=1}^\infty a_n$ converges if and only if the series
$$\sum_{k=0}^\infty 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$$
  converges.
- Theorem 3.29: If $p > 1$,
$$\sum_{n=2}^\infty \frac{1}{n(\log n)^p}$$
  converges; if $p \leq 1$, the series diverges.
  - Note that $\log n = \ln n$.
  - Note that we sum from $n = 2$ since $\log 1 = 0$.
- **e**: The number
$$\mathrm{e} = \sum_{n=0}^\infty \frac{1}{n!}$$
- Theorem 3.31: $\lim_{n\to\infty}(1+1/n)^n = \mathrm{e}$.
- Theorem 3.32: e is irrational.
- Theorem 3.39: Given the power series $\sum c_n z^n$, put
$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|} \qquad\qquad R = \frac{1}{\alpha}$$
  (If $\alpha = 0$, let $R = +\infty$; if $\alpha = +\infty$, let $R = 0$.) Then $\sum c_n z^n$ converges if $|z| < R$ and diverges if $|z| > R$.

- **Radius of convergence** (of a power series): The number $R$ defined by Theorem 3.39.
- Theorem 3.41 (partial summation formula): Given two sequence $\{a_n\}, \{b_n\}$, put

$$A_n = \begin{cases} \sum_{k=0}^{n} a_k & n \geq 0 \\ 0 & n = -1 \end{cases}$$

  Then if $0 \leq p \leq q$, we have

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

- **Product** (of $\sum a_n, \sum b_n$): The series $\sum c_n$ defined by

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

  for each $n = 0, 1, 2, \dots$.
  - We motivate this definition by noting that if $\sum c_n$ is the product of $\sum a_n, \sum b_n$, then

$$\sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} c_n z^n$$

  - Setting $z = 1$ then yields the given definition.
- The product of two convergent series may diverge. However...
- Theorem 3.50 (by Mertens): Suppose (a) $\sum_{n=0}^{\infty} a_n$ converges absolutely, (b) $\sum_{n=0}^{\infty} a_n = A$, (c) $\sum_{n=0}^{\infty} b_n = B$, and (d) $\sum_{n=0}^{\infty} c_n$ is the product of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$. Then

$$\sum_{n=0}^{\infty} c_n = AB$$

- Theorem 3.51 (by Abel): If $\sum a_n, \sum b_n, \sum c_n$ converge to $A, B, C$, respectively, and $\sum c_n$ is the product of $\sum a_n, \sum b_n$, then $C = AB$.
- **Rearrangement** (of $\sum a_n$): A series $\sum a_n'$ defined by $a_n' = a_{k_n}$ for each $n \in \mathbb{N}$, where $\{k_n\}$ is a sequence in which every positive integer appears once and only once (that is, $\{k_n\}$ is a 1-1 function from $\mathbb{N}$ onto $\mathbb{N}$).
- Theorem 3.54: Let $\sum a_n$ be a series of real number which converges, but not absolutely. Suppose $-\infty \leq \alpha \leq \beta \leq \infty$. Then there exists a rearrangement $\sum a_n'$ with partial sums $s_n'$ such that

$$\liminf_{n\to\infty} s_n' = \alpha \qquad \limsup_{n\to\infty} s_n' = \beta$$

- Theorem 3.55: If $\sum a_n$ is a series of complex numbers which converges absolutely, then every rearrangement of $\sum a_n$ converges, and they all converge to the same sum.