Chapter 3

Numerical Sequences and Series

3.1 Notes

11/8:
- Any bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

3.2 Chapter 3: Numerical Sequences and Series

From Rudin (1976).

11/7:
- Convergence of sequences is relative.
  - For example, the sequence $1/n$ for $n = 1, 2, \dots$ converges in $\mathbb{R}$, but not in $(0, \infty)$.
- **Range** (of $\{p_n\}$): The set of all points p_n.
  - This definition squares nicely with the formal definition of a sequence as a function p defined on $\mathbb{N}$.
- Theorem 3.6a: If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a point of X.
- Theorem 3.7: The subsequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X.
- **Diameter** (of E): The supremum of the set

  $$S = \{d(p, q) : p, q \in E\}$$

  where E is a nonempty subset of a metric space X. *Denoted by* $\mathbf{diam}\,\boldsymbol{E}$.
- Theorem 3.10:
  (a) If $\bar{E}$ is the closure of a set E in a metric space X, then

  $$\operatorname{diam} \bar{E} = \operatorname{diam} E$$

  (b) If K_n is a sequence of compact sets in X such that $K_n \supset K_{n+1}$ $(n = 1, 2, 3, \dots)$ and if $\lim_{n\to\infty} \operatorname{diam} K_n = 0$, then $\bigcap_1^\infty K_n$ consists of exactly one point.
- **Complete** (metric space): A metric space in which every Cauchy sequence converges.
- All compact metric spaces and all Euclidean spaces are complete.
  - The metric space $(\mathbb{Q}, |x - y|)$ is not complete.

- **Monotonically increasing** (sequence $\{s_n\}$): A sequence $\{s_n\}$ of real numbers such that $s_n \leq s_{n+1}$ for each $n \in \mathbb{N}$.
- **Monotonically decreasing** (sequence $\{s_n\}$): A sequence $\{s_n\}$ of real numbers such that $s_n \geq s_{n+1}$ for each $n \in \mathbb{N}$.
- **Monotonic sequences**: The class of all sequences that are either monotonically increasing or monotonically decreasing.
- **Upper limit** (of $\{s_n\}$): The supremum of the set E of all subsequential limits of $\{s_n\}$. *Denoted by* $\boldsymbol{s^*}$, $\boldsymbol{\limsup_{n\to\infty} s_n}$.
- **Lower limit** (of $\{s_n\}$): The infimum of the set E of all subsequential limits of $\{s_n\}$. *Denoted by* $\boldsymbol{s_*}$, $\boldsymbol{\liminf_{n\to\infty} s_n}$.
- Theorem 3.17: Let $\{s_n\}$ be a sequence of real numbers. Then s^* has (and is the only number to have both of) the following two properties.

  (a) $s^* \in E$.

  (b) If $x > s^*$, then there is an integer N such that $n \geq N$ implies $s_n < x$.

  An analogous result holds for s_*.