

# MATH 20700 (Honors Analysis in $\mathbb{R}^n$ I) Notes

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Part I

Linear Algebra

# Chapter 1

## Basic Notions

### 1.1 Notes

- 9/27:
- **Vector space:** Basically, a set for which you have an addition and multiplication.
  - $\mathbb{F}^d$  is used for  $\mathbb{R}^d$  or  $\mathbb{C}^d$  in Treil (2017).
  - $\mathbb{P}_n$  is the vector space of polynomials up to degree  $n$ .
  - $C([0, 1])$  is the set of continuous functions defined on  $[0, 1]$ , an infinite-dimensional vector space.
  - **Generating set:** A subset of a vector space, all linear combinations of which generate the vector space. *Also known as spanning set.*
    - Any element of VS is a linear comb. of elements of the generating set.
  - **Linearly independent** (list): A list of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$  such that  $\sum_{i=1}^k \alpha_i \mathbf{v}_i = 0$  implies  $\alpha_i = 0$  for all  $i$ .
  - **Base:** A generating set consisting of linearly independent vectors.
  - Any element of a VS can be written as a *unique* linear combination of the vectors in a base.
    - If  $\mathbf{x} = \sum_{i=1}^k \alpha_i \mathbf{v}_i = \sum_{i=1}^k \beta_i \mathbf{v}_i$ , then  $\alpha_i = \beta_i$  for all  $i$ .
  - **Linear transformation:** A function  $T : X \rightarrow Y$ , where  $X, Y$  are VSs, such that

$$T(\alpha \mathbf{x} + \beta \mathbf{y}) = \alpha T\mathbf{x} + \beta T\mathbf{y}$$

for all  $\mathbf{x} \in X, \mathbf{y} \in Y$ .

- Examples of linear transformations:
  - Consider  $\mathbb{P}_n$ . Let  $Tp_n = p'_n$ . This  $T$  is linear.
  - Rotation in  $\mathbb{R}^d$ .
    - Think graphically about two vectors  $\mathbf{x}, \mathbf{y}$ .
    - Rotating and summing them is the same as summing and rotating. Same for scaling.
    - Thus, rotation is actually linear!
  - Reflection as well.
- Consider  $T : \mathbb{R} \rightarrow \mathbb{R}$ .
  - Any linear map on the line is a line.
  - We must have  $Tx = \alpha x$ :  $Tx = T(1x) = xT(1) = x\alpha$ .

- Consider  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  linear.
  - Any linear map between  $\mathbb{R}^n$  and  $\mathbb{R}^m$  is linear.
  - Thus,  $T(\mathbf{x}) = A\mathbf{x}$  for all  $\mathbf{x} \in \mathbb{R}^n$ , where  $A$  is an  $m \times n$  matrix.
- To find  $A$ , do the same calculation as for  $Tx = \alpha x$  but more carefully:
  - Let  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  be a basis.
  - So  $\mathbf{x} = \sum_{i=1}^n \alpha_i \mathbf{e}_i$ .
  - Thus,  $T\mathbf{x} = \sum_{i=1}^n \alpha_i T(\mathbf{e}_i)$ .
  - Each  $T(\mathbf{e}_i)$  is part of the matrix that we multiply by the column vector representing  $\mathbf{x}$ .
- Multiplication of matrices is equivalent to composition of linear maps.
- Consider  $T_1 : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $T_2 : \mathbb{R}^m \rightarrow \mathbb{R}^r$ .
  - $T_2 \circ T_1$  is equivalent to  $BA$ , if  $A$  represents  $T_1$  and  $B$  represents  $T_2$ . In other words,  $(T_2 \circ T_1)(\mathbf{x}) = BA\mathbf{x}$  for all  $\mathbf{x}$ .
- Recall that if  $A = (\alpha_{ij})$  and  $B = (\beta_{ij})$ , then  $(BA)_{ij} = (\sum \beta_{ik} \alpha_{kj})$ .
- Properties of multiplication:

$$(AB)C = A(BC)$$

$$A(B + C) = AB + AC$$

$$(A + B)C = AC + BC$$

- However, it is not true in general that  $AB = BA$ .

- **Trace** (of an  $n \times n$  matrix  $A$ ): The sum of the diagonal entries of  $A$ . Denoted by  $\text{tr}(A)$ . Given by

$$\text{tr}(A) = \sum \alpha_{ii}$$

- It is true that  $\text{tr}(AB) = \text{tr}(BA)$ .
  - Indeed, on the diagonals, multiplication is commutative; it's the other terms that mess you up in general.
- Invertibility of matrices.
  - In general, matrices are not invertible: Not every system of equations is solveable;  $Ax = b$  does not always have a solution  $x = A^{-1}b$ .
- $C$  is the inverse from the left:  $CA = I$ .  $B$  is the inverse from the right:  $AB = I$ . A matrix can have a left and a right inverse and still not be invertible. A matrix is invertible iff  $C = B$ .
- Any time we write “inverse,” we do so under the assumption that it exists.
- $(AB)^{-1} = B^{-1}A^{-1}$  — easy proof by multiplication.
- If  $A = (a_{ij})$ ,  $A^T = (a_{ji})$ .
  - $(A^{-1})^T = (A^T)^{-1}$ .
  - $(AB)^T = B^T A^T$ .
- Let  $X, Y$  VS.
  - $X \cong Y^{[1]}$  if there exists a linear  $T : X \rightarrow Y$  that is one-to-one and onto.
  - Check:  $A(\text{basis of } X) = \text{basis of } Y$ . Prove by definition and expression of elements as linear combinations.
- **Subspace**: A subset of a vector space which happens to be a vector space, itself.

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<sup>1</sup>“ $X$  is isomorphic to  $Y$ .”



## 1.2 Chapter 1: Basic Notions

From Treil (2017).

- 10/24:
- **Coordinates** (of  $\mathbf{v} \in V$  wrt. a basis  $\mathbf{v}_1, \dots, \mathbf{v}_n$  of  $V$ ): The unique scalars  $\alpha_1, \dots, \alpha_n \in \mathbb{F}$  such that  $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n$ .
  - **Spanning system**: A list of vectors that spans  $V$ . *Also known as **generating system**, **complete system**.*
  - **Trivial** (linear combination): A linear combination  $\alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n$  of vectors such that  $\alpha_k = 0$  for each  $k = 1, \dots, n$ .
  - **Transformation**: A function  $T : X \rightarrow Y$ . *Also known as **transform**, **mapping**, **map**, **operator**, and **function**.*
  - The matrix of a linear transformation  $T$  is often denoted by  $[T]$ .
  - To compute the reflection of vectors over an arbitrary line through the origin in  $\mathbb{R}^2$ , represent the overall transformation as a composition of rotating the line to be the  $x$ -axis, reflecting over the  $x$ -axis, and rotating back.
  - Theorem 1.5.1: If  $A$  is an  $m \times n$  matrix and  $B$  is an  $n \times m$  matrix, then

$$\text{tr}(AB) = \text{tr}(BA)$$

- Theorem 1.6.1: If a linear transformation is invertible, then its left and right inverses are unique and coincide.
- The column  $(1, 1)^T$  is left-invertible, with one possible left inverse being  $(1/2, 1/2)$ .
  - Note that it is not right invertible since its left inverses are not unique (see Theorem 1.6.1).
- An invertible matrix must be square.
- **Isomorphic** (vector spaces): Two vectors spaces  $V, W$  such that there exists an isomorphism  $A : V \rightarrow W$ . *Denoted by  $V \cong W$ .*
- Theorem 1.6.8:  $A : X \rightarrow Y$  is invertible if and only if for any right side  $\mathbf{b} \in Y$ , the equation

$$A\mathbf{x} = \mathbf{b}$$

has a unique solution  $\mathbf{x} \in X$ .

- Corollary 1.6.9: An  $m \times n$  matrix is invertible if and only if its columns form a basis in  $\mathbb{F}^m$ .
- **Linear span** (of  $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$ ): The set of all linear combinations of  $\mathbf{v}_1, \dots, \mathbf{v}_n$ . *Denoted by  $\mathcal{L}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ ,  $\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_n)$ .*

## Chapter 2

# Systems of Linear Equations

### 2.1 Notes

9/29: • Row elimination:

– Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 3 & 1 & 2 & 7 \\ 2 & 1 & 2 & 1 \end{pmatrix}$$

– Then the **echelon form** matrix

$$A_e = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & -4 \end{pmatrix}$$

– Lastly, the **reduced echelon form** matrix

$$A_{re} = \begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

- **echelon form:**

- All zero rows are below nonzero rows.
- For any nonzero row, its leading element is strictly to the left of the nonzero entry of the next row.

- **Reduced echelon form:**

- All pivots are 1.
- Used to solve systems of the form  $Ax = b$ .

- **Inconsistent** (system of equations): A system with no solution.

- If the last row is of the form  $(0, \dots, 0, b)$  where  $b \neq 0$ , then there is no solution.

- Unique solution if  $A_e$  has a pivot in every column.

- There exists a solution for every  $b$  if there is a pivot in every row?

- Let  $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a matrix. Then  $\ker A = \{x \in \mathbb{R}^n : Ax = 0\}$  (subspace of  $\mathbb{R}^n$ ) and  $\text{range } A = \{Ax : x \in \mathbb{R}^n\}$  (subspace of  $\mathbb{R}^m$ ).

- Also consider  $\ker(A^T)$  and  $\text{range}(A^T)$ , the basis of the kernel and range, and dimension.

- Finite-dimensional vector spaces:

- A basis is a generating set (so every element of  $V$  can be written uniquely as a linear combination of the basis) the length of which is equal to the dimension of  $V$ .
- All bases of finite-dimensional vector spaces have the same number of elements.

- Let  $v_1, v_2, v_3$  and  $w_1, w_2$  be two generating sets of  $V$ .

- Then

$$v_1 = \lambda_{11}w_1 + \lambda_{12}w_2$$

$$v_2 = \lambda_{21}w_1 + \lambda_{22}w_2$$

$$v_3 = \lambda_{31}w_1 + \lambda_{32}w_2$$

- Suppose the only solution to  $\alpha_1v_1 + \alpha_2v_2 + \alpha_3v_3 = 0$  is  $\alpha_1 = \alpha_2 = \alpha_3 = 0$ .

- But this is not true, as we can find another one in terms of the  $\lambda$ s.

- If you have a list of linearly independent vectors, you can complete it into a basis.
  - If there exists a vector that can't be written as a linear combination of the list, add it to the list.

- If you find any particular solution to a system  $Ax = b$ , and you add to it any element of  $\ker A$ , you will obtain another solution.

- $Ax_1 = b$  and  $Ax_h = 0$  implies that  $A(x_1 + x_h) = b$ .

- $Ax_1 = b$  and  $Ax_2 = b$  imply that  $A(x_1 - x_2) = 0$ , i.e., that  $x_1 - x_2 \in \ker A$ .

- If  $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $\dim \text{range } A = m$ , then  $Ax = b$  is solvable for all  $b \in \mathbb{R}^m$ .

- Let  $\text{rank } A = \dim \text{range } A$ .

- Rank theorem:

- $\text{rank } A = \text{rank } A^T$ .

- Let  $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ . We know that  $\dim \ker A + \dim \text{range } A = n$ .

- $\dim \ker A^T + \text{rank } A^T = m$ .

- This theorem survives linear algebra and enters functional analysis under the name **Fredholm's alternative**.

- **Fredholm's alternative:**  $Ax = b$  has a solution for all  $b \in \mathbb{R}^n$  iff  $\dim \ker A^T = 0$ .

- $\dim \ker A^T = 0$  implies  $\text{rank } A^T = m$  implies  $\text{rank } A = m$  implies  $\dim \text{range } A = m$ , as desired.

- **Pivot column** (of  $A$ ): A column of  $A$  where  $A_e$  has pivots.

- The **pivot columns** of  $A$  give a basis for  $\text{range } A$ .

- The pivot rows of  $A_e$  give a basis for  $\text{range } A^T$ .

- A basis for the kernel is enough to solve  $Ax = 0$ .

- If you take these three things as givens, you can prove the rank theorem.

## 2.2 Chapter 2: Systems of Linear Equations

From Treil (2017).

10/24:

- A system is inconsistent iff the echelon form of the augmented matrix has a row of the form  $(0 \ \cdots \ 0 \ b)$ .
- A solution to  $Ax = b$  is unique iff there are no free variables, i.e., iff there is a pivot in every column.
- $Ax = b$  is consistent iff the echelon form of the coefficient matrix has a pivot in every row.

- $A\mathbf{x} = \mathbf{b}$  has a unique solution for any  $\mathbf{b}$  iff the echelon form of the coefficient matrix  $A$  has a pivot in every row and column.
- Proposition 2.3.1: Let  $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathbb{F}^n$ , and let  $A = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_m]$  be an  $n \times m$  matrix with columns  $\mathbf{v}_1, \dots, \mathbf{v}_m$ . Then
  1. The system  $\mathbf{v}_1, \dots, \mathbf{v}_m$  is linearly independent iff the echelon form of  $A$  has a pivot in every column.
  2. The system  $\mathbf{v}_1, \dots, \mathbf{v}_m$  is complete iff the echelon form of  $A$  has a pivot in every row.
  3. The system  $\mathbf{v}_1, \dots, \mathbf{v}_m$  is a basis of  $\mathbb{F}^n$  iff the echelon form of  $A$  has a pivot in every column and in every row.
- Proposition 2.3.6: A matrix  $A$  is invertible if and only if its echelon form has a pivot in every column and every row.
- Corollary 2.3.7: An invertible matrix must be square ( $n \times n$ ).
- Proposition 2.3.8: If a square ( $n \times n$ ) matrix is left invertible or if it is right invertible, then it is invertible. In other words, to check the invertibility of a square matrix  $A$ , it is sufficient to check only one of the conditions  $AA^{-1} = I$ ,  $A^{-1}A = I$ .
- Any invertible matrix is row-equivalent to (can be row-reduced to) the identity matrix.
- **Homogeneous** (system of linear equations): A system of the form  $A\mathbf{x} = \mathbf{0}$ .
- Theorem 2.6.1: Let a vector  $\mathbf{x}_1$  satisfy the equation  $A\mathbf{x} = \mathbf{b}$ . and let  $H$  be the set of all solutions of the associated homogeneous system  $A\mathbf{x} = \mathbf{0}$ . Then the set

$$\{\mathbf{x}_1 + \mathbf{x}_h : \mathbf{x}_h \in H\}$$

is the set of all solutions to the equation  $A\mathbf{x} = \mathbf{b}$ .

- The pivot columns are a basis of range  $A$ . The pivot rows are a basis of range  $A^T$ . The solutions to the equation  $A\mathbf{x} = \mathbf{0}$  are a basis of  $\ker A$ .
- Theorem 2.7.3: Let  $A$  be an  $m \times n$  matrix. Then the equation  $A\mathbf{x} = \mathbf{b}$  has a solution for every  $\mathbf{b} \in \mathbb{R}^m$  iff the dual equation  $A^T\mathbf{x} = \mathbf{0}$  has a unique (only the trivial) solution.
  - Note that this is a corollary to the rank theorem.
- Change of coordinates formula:
  - Let  $T : V \rightarrow W$  be a linear transformation, and let  $\mathcal{V} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$  and  $\mathcal{W} = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$  be bases of  $V$  and  $W$ , respectively.
  - The  $m \times n$  matrix of  $T$  with respect to these bases is  $[T]_{\mathcal{W}\mathcal{V}}$ , and relates the coordinates of  $[T\mathbf{v}]_{\mathcal{W}}$  and  $[\mathbf{v}]_{\mathcal{V}}$  via

$$[T\mathbf{v}]_{\mathcal{W}} = [T]_{\mathcal{W}\mathcal{V}}[\mathbf{v}]_{\mathcal{V}}$$

- Change of coordinates matrix: If  $\mathcal{A}, \mathcal{B}$  are two bases of  $V$ , then we can convert the coordinates of a vector in  $\mathcal{B}$  to its in  $\mathcal{A}$  with the identity matrix (with respect to the appropriate bases). In particular,

$$[\mathbf{v}]_{\mathcal{B}} = [I]_{\mathcal{B}\mathcal{A}}[\mathbf{v}]_{\mathcal{A}}$$

- Note that the  $k^{\text{th}}$  column of  $[I]_{\mathcal{B}\mathcal{A}}$  is the coordinate representation in  $\mathcal{B}$  of  $\mathbf{a}_k$ , i.e.,  $[\mathbf{a}_k]_{\mathcal{B}}$ .
- The change of coordinates matrix from a basis  $\mathcal{B}$  to the standard basis  $\mathcal{S}$  is easy to compute; by the above, it's just

$$[I]_{\mathcal{S}\mathcal{B}} = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_n]$$

- It follows that  $[I]_{\mathcal{B}\mathcal{S}} = ([I]_{\mathcal{S}\mathcal{B}})^{-1}$ .

- This allows us to compute  $[I]_{\mathcal{B}\mathcal{A}}$  as  $[I]_{\mathcal{B}\mathcal{S}}[I]_{\mathcal{S}\mathcal{A}}$
- If  $T : V \rightarrow W$ ,  $\mathcal{A}, \tilde{\mathcal{A}}$  are bases of  $V$ , and  $\mathcal{B}, \tilde{\mathcal{B}}$  are bases of  $W$ , and we have  $[T]_{\mathcal{B}\mathcal{A}}$ , then

$$[T]_{\tilde{\mathcal{B}}\tilde{\mathcal{A}}} = [I]_{\tilde{\mathcal{B}}\mathcal{B}}[T]_{\mathcal{B}\mathcal{A}}[I]_{\mathcal{A}\tilde{\mathcal{A}}}$$

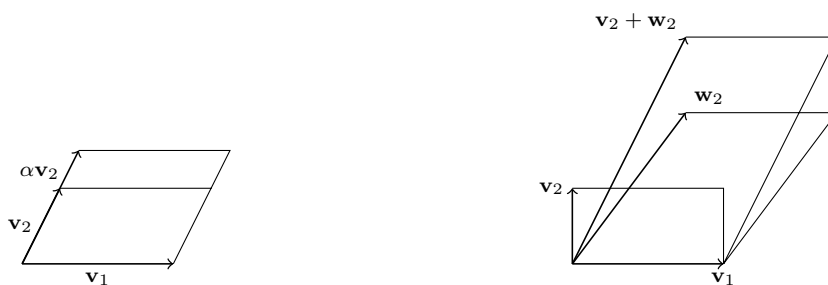
- Change of basis ends up at similarity; two operators are similar if we can change the basis of one into another.

# Chapter 3

## Determinants

### 3.1 Notes

- 9/29:
- The determinant, geometrically, is the volume of the object (in  $\mathbb{R}^3$ ) you get when you take linear combinations of the vectors.
  - In 2D:
    - Let  $v_1, v_2$  be two vectors. Put tail to tail and forming a parallelogram, the determinant of the matrix  $(v_1, v_2)$  is the area of said parallelogram.
    - Linearity 1:  $D(av_1, v_2, \dots, v_n) = aD(v_1, \dots, v_n)$  is the same as saying that if you stretch one vector by  $a$ , you scale up the area by that much, too.
    - Linearity 2:  $D(v_1, \dots, v_{k+} + v_{k-}, \dots, v_n) = D(-) + D(+)$ .
    - Antisymmetry:  $D(v_1, \dots, v_k, \dots, v_j, \dots, v_n) = -D(v_1, \dots, v_j, \dots, v_k, \dots, v_n)$ . Interchanging columns flips the sign of the determinant.
    - Basis:  $D(e_1, \dots, e_n) = 1$ .
  - Determinant: Denoted by  $D(v_1, \dots, v_n)$ , where  $(v_1, \dots, v_n)$  is an  $n \times n$  matrix.
- 10/1:
- Consider an  $n \times n$  matrix  $A$  consisting of  $n$  columns containing vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^n$ .
    - $D(A)$  is the volume of the solid  $V = \sum_{i=1}^n \alpha_i v_i$ .
    - $D(\mathbf{e}_1, \dots, \mathbf{e}_n) = 1$ .



(a)  $D(\mathbf{v}_1, \alpha\mathbf{v}_2) = \alpha D(\mathbf{v}_1, \mathbf{v}_2)$ .

(b)  $D(\mathbf{v}_1, \mathbf{v}_2 + \mathbf{w}_2) = D(\mathbf{v}_1, \mathbf{v}_2) + D(\mathbf{v}_1, \mathbf{w}_2)$ .

Figure 3.1: Visualizing properties of determinants.

- Basic properties of the determinant.
  - If  $A$  has a zero column, then  $\det A = 0$ : Scalar property.

- If  $A$  has two equal columns, then  $\det A = 0$ : Multiply one by minus and add.
- If  $A$  has a column which is a multiple of another, then  $\det A = 0$ : Pull out the multiple and then you have the previous one.
- If columns are linearly dependent, then  $\det A = 0$ : Decompose it into sums, split, add back up with previous properties.
- The determinant is preserved under column reduction.
- $\det A^T = \det A$ : Put everything in rref.
- If  $A$  is not invertible, then  $\det A = 0$  (not invertible implies linearly dependent columns, implies  $\det A = 0$ ).
- $\det(AB) = \det A \det B$ .
- Determinant of...
  - A diagonal matrix: The product of the diagonal entries (pull out the terms, and then note that the remaining identity matrix has determinant 1).
  - An upper triangular matrix: The product of the diagonal entries (column reduction to make it into a diagonal matrix, and then the property above).

## 3.2 Chapter 3: Determinants

From Treil (2017).

- 10/24: • Let  $A_{j,k}$  denote the  $(n-1) \times (n-1)$  matrix obtained from  $A$  by crossing out row  $j$  and column  $k$  and pushing it together.

- **Cofactors** (of  $A$ ): The numbers  $C_{j,k}$ , one per entry, defined by

$$C_{j,k} = (-1)^{j+k} \det A_{j,k}$$

- **Cofactor matrix** (of  $A$ ): The matrix

$$C = \{C_{j,k}\}_{j,k=1}^n$$

- Theorem 3.5.2: Let  $A$  be an invertible matrix and let  $C$  be its cofactor matrix. Then

$$A^{-1} = \frac{1}{\det A} C^T$$

- **Cramer's rule**: If  $A$  is invertible and  $A\mathbf{x} = \mathbf{b}$ , then

$$x_k = \frac{\det B_k}{\det A}$$

where  $B_k$  is obtained from  $A$  by replacing column  $k$  of  $A$  by the vector  $\mathbf{b}$ .

- **Minor** (of order  $k$  of  $A$ ): The determinant of a  $k \times k$  submatrix of  $A$ .
- Theorem 3.6.1: The rank of a nonzero matrix  $A$  is equal to the largest integer  $k$  such that there exists a nonzero minor of order  $k$ .

# Chapter 4

## Introduction to Spectral Theory

### 4.1 Notes

- 10/1:
- **Difference equation:** Like a differential equation, but instead of writing a differentials, you write differences.
  - Suppose we want to solve  $x_{n+1} = Ax_n$  with  $x_0$  given.
    - You will find that  $x_n = A^n x_0$ .
    - This gets hard to compute, so we want to find a way to simplify the computation.
  - Thus, we want to diagonalize the matrix, and this concept is inherently linked to eigenvalues and eigenvectors.
    - If you can decompose the  $x_0$  into a linear combination of eigenvectors, then you can simplify the computation a lot:
$$x_n = \sum \alpha_i A^n v_i = \sum \alpha_i \lambda_i^n v_i$$
    - An  $n \times n$  matrix will have  $n$  eigenvalues. You want  $n$  linearly independent eigenvectors, creating an eigenbasis.
  - To find eigenvalues and eigenvectors, we need to solve  $Ax = \lambda x$ , i.e.,  $(A - \lambda I)x = 0$ . Thus,  $\ker(A - \lambda I) \neq \{0\}$ , so  $\det(A - \lambda I) = 0$ .
  - The eigenvalues of  $A$  are independent of the choice of basis of the domain of  $A$  or the range.
- 10/4:
- We need to know everything in Treil (2017).
    - We don't need to know the applications sections, but you should be interested.
  - **Spectral theory:** Decomposing a linear operator.
  - Let  $A : V \rightarrow V$  be a linear operator.  $\lambda \in \mathbb{C}$  is an eigenvalue if there exists  $x \in V$  nonzero such that  $Ax = \lambda x$ .
    - Let  $A$  be an  $n \times n$  matrix over  $\mathbb{C}$  or  $\mathbb{R}$ .
    - The eigenvalues are the roots of the polynomial  $\det(A - \lambda I) = 0$  in  $\lambda$ .
  - Things we want to do:
    - Given  $A$ , find the eigenvalues and eigenvectors (solve  $(A - \lambda I)x = 0$ ).



- In order to simplify  $A$ , make it a diagonal matrix:

$$A = S \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} S^{-1}$$

- Eigenvalues are independent of the choice of basis.

- From the book, we have that

$$[A]_{\mathcal{A}\mathcal{A}} = [S]_{\mathcal{A}\mathcal{B}}[B]_{\mathcal{B}\mathcal{B}}[S]_{\mathcal{A}\mathcal{B}}^{-1}$$

- It follows that

$$A - \lambda I = [S]_{\mathcal{A}\mathcal{B}}(B - \lambda I)[S]_{\mathcal{A}\mathcal{B}}^{-1}$$

so

$$\det(A - \lambda I) = \det([S]_{\mathcal{A}\mathcal{B}}(B - \lambda I)[S]_{\mathcal{A}\mathcal{B}}^{-1}) = \det([S]_{\mathcal{A}\mathcal{B}}[S]_{\mathcal{A}\mathcal{B}}^{-1}(B - \lambda I)) = \det(B - \lambda I)$$

- If  $p(z) = (z - \lambda)^k q(z)$ , then  $k$  is the **algebraic multiplicity** of  $\lambda$ . The **geometric multiplicity** of  $\lambda$  is  $\dim \ker(A - \lambda I)$ .

- These terms are not always the same, but they are related.

- Diagonalization:

- Given  $A$  that corresponds to  $T : V \rightarrow V$ , can we find a basis of  $V$  in which the operator is a diagonal matrix?
- $A = SDS^{-1}$  iff there exists a basis of  $V$  consisting of the eigenvectors of  $A$ .
- Proves  $A^N = SD^N S^{-1}$  via  $A^2 = SDS^{-1}SDS^{-1} = SDIDS^{-1} = SD^2 S^{-1}$ .

- Let  $A$  be an  $n \times n$  matrix over  $\mathbb{F}$ . If  $\lambda_1, \dots, \lambda_r$  are distinct eigenvalues, then their eigenvectors are linearly independent.

- Prove with induction contradiction argument. Assume true for  $\mathbf{v}_{r-1}$ . Then

$$0 = (A - \lambda_r I)[\mathbf{v}_1 + \dots + \mathbf{v}_r] = (\lambda_1 - \lambda_r)\mathbf{v}_1 + \dots + (\lambda_{r-1} - \lambda_r)\mathbf{v}_{r-1}$$

- Implies  $\lambda_r = \lambda_i$  for all  $i \in [r-1]$ , a contradiction.
- If  $A$  has  $n$  distinct eigenvalues, then  $A$  is diagonalizable.

- If  $A : V \rightarrow V$  has  $n$  complex eigenvalues, then  $A$  is diagonalizable iff the algebraic multiplicity equals the geometric multiplicity for each eigenvalue.

- Goes through a sample diagonalization with  $\begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix}$ .

- We have

$$A - \lambda I = \begin{pmatrix} 1 - \lambda & 2 \\ 8 & 1 - \lambda \end{pmatrix}$$

so

$$0 = \det(A - \lambda I) = (1 - \lambda)^2 - 16$$

- It follows that  $\lambda = 5, -3$ .
- This yields

$$\begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}^{-1}$$

by inspection.

- As another example, consider  $\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ .

- Here, we have  $\lambda = 1 \pm 2i$ .

## 4.2 Chapter 4: Introduction to Spectral Theory

From Treil (2017).

10/24:

- **Spectrum** (of  $A$ ): The set of all eigenvalues of  $A$ . Denoted by  $\sigma(A)$ .
- Proposition 4.1.1: The geometric multiplicity of an eigenvalue cannot exceed its algebraic multiplicity.
- Theorem 4.2.1: A matrix  $A$  (with values in  $\mathbb{F}$ ) admits a representation  $A = SDS^{-1}$  where  $D$  is a diagonal matrix and  $S$  is invertible if and only if there exists a basis of  $\mathbb{F}^n$  of eigenvectors of  $A$ . Moreover, in this case diagonal entries of  $D$  are the eigenvalues of  $A$  and columns of  $S$  are the corresponding eigenvectors.
- Any operator on a complex vector space has  $n$  eigenvalues (counting multiplicities).
  - Think  $n$  necessary roots of the characteristic polynomial, or the necessary upper triangular representation.
- Theorem 4.2.8: Let an operator  $A : V \rightarrow V$  have exactly  $n = \dim V$  eigenvalues (counting multiplicities). Then  $A$  is diagonalizable if and only if for each eigenvalue  $\lambda$ , the dimension of the eigenspace  $\ker(A - \lambda I)$  (i.e., the geometric multiplicity of  $\lambda$ ) coincides with the algebraic multiplicity of  $\lambda$ .
- Theorem 4.2.9: A real  $n \times n$  matrix  $A$  admits a real factorization (i.e., a real representation  $A = SDS^{-1}$  where  $S$  and  $D$  are real matrices,  $D$  is diagonal, and  $S$  is invertible) if and only if it admits a complex factorization and all eigenvalues of  $A$  are real.
- Example of a nondiagonalizable matrix:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

- $p(\lambda) = (1 - \lambda)^2$ , so  $\lambda = 1$  with algebraic multiplicity 2.
- However,  $\dim \ker(A - I) = 1$  since  $A - I$  has only one pivot, hence  $2 - 1 = 1$  free variable.
- Thus, apply Theorem 4.2.8.

# Chapter 5

## Inner Product Spaces

### 5.1 Notes

- 10/6: • We define

$$\ell^2(\mathbb{R}) = \left\{ \{a_n\}_{n \geq 1} \subset \mathbb{R} : \sum_1^\infty |a_n|^2 < \infty \right\}$$

- **Inner product:** A map  $V \times V \rightarrow \mathbb{F}$  that takes  $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \cdot \mathbf{y}$ . Denoted by  $\cdot, (\cdot, \cdot), \langle \cdot, \cdot \rangle$ .

- Properties of the inner product:

- $(\mathbf{x}, \mathbf{y}) = \overline{(\mathbf{y}, \mathbf{x})}$  (symmetry).
- $(\alpha \mathbf{x} + \beta \mathbf{y}, \mathbf{z}) = \alpha(\mathbf{x}, \mathbf{z}) + \beta(\mathbf{y}, \mathbf{z})$  (linearity).
- $(\mathbf{x}, \mathbf{x}) \geq 0$ .
- $(\mathbf{x}, \mathbf{x}) = 0$  iff  $\mathbf{x} = 0$ .

- If  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ , then

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n x_i y_i$$

- If  $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ , then

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n x_i \bar{y}_i$$

- If  $f, g \in \mathbb{P}_n(t)$ , then

$$(f, g) = \int_{-1}^1 f \bar{g} dt$$

- The conjugate of a polynomial is the polynomial with the conjugate of the coefficients of the original polynomial. Symbolically, if  $f = \sum_{i=0}^n \alpha_i t^i$  is a polynomial, then  $\bar{f} = \sum_{i=0}^n \bar{\alpha}_i t^i$ .

- It is a fact that

$$\left| \sum_{n=1}^\infty a_n \bar{b}_n \right| \leq \|(a_n)_{n \geq 1}\| \|(b_n)_{n \geq 1}\|$$

- Suppose we want to define the inner product between two matrices.

- A common one is

$$(A, B) = \text{tr}(B^* A)$$

where  $B^* = \bar{B}^T = \overline{B^T}$  is the conjugate transpose.

- We define the norm as a function  $V \rightarrow [0, \infty)$  given by

$$\|\mathbf{x}\| = \sqrt{(\mathbf{x}, \mathbf{x})}$$

- Properties of the norm.

- $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$ .
- $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ .
- $\|\mathbf{x}\| = 0$  iff  $\mathbf{x} = 0$ .

- In  $\mathbb{R}^n$ ,

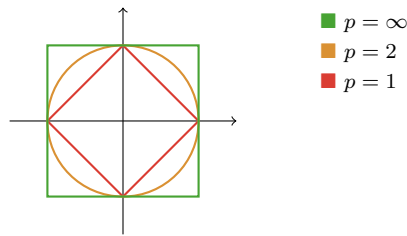


Figure 5.1: The unit ball of norms corresponding to  $p = 1, 2, \infty$ .

- The standard norm is

$$\|\mathbf{x}\| = \sqrt{\sum |x_i|^2}$$

- We can also define

$$\|\mathbf{x}\|_p = \sqrt[p]{\sum |x_i|^p}$$

- We can even define

$$\|\mathbf{x}\|_\infty = \max |x_i|$$

- And we can prove that all of these are valid norms.
- Only the norm corresponding to  $\ell^2$  is given by an inner product, but all the other quantities are still norms as defined by the properties (see Treil (2017)).
- Figure 5.1 shows the unit ball of each norm, i.e., the set of all points which have norm 1.

- The parallelogram rule:

$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)$$

- Orthogonality: Given  $\mathbf{v}, \mathbf{w}$ , if  $\mathbf{v} \perp \mathbf{w}$ , then  $(\mathbf{v}, \mathbf{w}) = 0$ .

- In particular, if  $\mathbf{v} \perp \mathbf{w}$ , then

$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$$

- Let  $E$  be a subspace of  $V$ . If  $\mathbf{v} \perp E$ , then  $\mathbf{v} \perp \mathbf{e}$  for all  $\mathbf{e} \in E$ , i.e.,  $\mathbf{v} \perp$  a set of vectors spanning  $E$ .
- Any set of orthogonal vectors is linearly independent. Thus, if  $V$  is  $n$  dimensional, then  $\mathbf{v}_1, \dots, \mathbf{v}_n$  orthogonal is a basis.
- Let  $E$  be a subspace of  $V$ . Take  $\mathbf{v} \in V$ . We want to define the projection  $P_E \mathbf{v}$  of  $\mathbf{v}$  onto  $E$ .
  - We have that  $P_E \mathbf{v} \in E$  and  $\mathbf{v} - P_E \mathbf{v} \perp E$ .
  - Additionally, we have that

$$\|\mathbf{v} - P_E \mathbf{v}\| \leq \|\mathbf{v} - \mathbf{e}\|$$

for all  $\mathbf{e} \in E$ .

- Lastly, we have that  $P_E \mathbf{v}$  is unique.
- If we receive a basis of a vector space, how do we create out of that a basis that is orthogonal? The process of doing this is called **Gram-Schmidt orthogonalization**.
  - We keep  $\mathbf{v}_1$ , subtract  $P_{\mathbf{v}_1} \mathbf{v}_2$  from  $\mathbf{v}_2$ , subtract  $P_{\{\mathbf{v}_1, \mathbf{v}_2\}} \mathbf{v}_3$  from  $\mathbf{v}_3$ , and on and on.
- If we are given a set of orthogonal vectors, we can normalize them by dividing each by its norm. This creates an orthonormal list. The standard basis is orthonormal.
- Let

$$E^\perp = \{v \in V : v \perp E\}$$

- It follows that  $V = E \oplus E^\perp$ .
- How close can we come to solving  $A\mathbf{x} = \mathbf{b}$  if we cannot solve it exactly (i.e., if the columns are not linearly independent)?
  - Let  $A$  be an  $m \times n$  matrix, and let  $\mathbf{b} \in \mathbb{R}^m$ .
  - Then the best solution is given by minimizing  $\|A\mathbf{x} - \mathbf{b}\|$ . We minimize this with projections. A special case of this is least squares regression! More details in Treil (2017).

10/8:

- Soug is gonna send us a hefty amount of reading for the weekend.
- Least square approximation:
  - If we want to minimize  $\|A\mathbf{x} - \mathbf{b}\|$ , the best we can do is project  $\mathbf{b}$  onto the range of  $A$ .
  - Let  $\mathbf{v}_1, \dots, \mathbf{v}_k$  be an orthogonal basis of range  $A$ .
  - Then

$$\text{Proj}_{\text{range } A} \mathbf{b} = \sum_{k=1}^k \frac{(\mathbf{b}, \mathbf{v}_k)}{\|\mathbf{v}_k\|^2} \mathbf{v}_k$$

- Matrix equation form:

$$\text{Projection}_{\text{range } A} = A(A^*A)^{-1}A^*$$

if  $A^*A$  is invertible, where  $A^* = \bar{A}^T$ .

■ Soug never uses this though.

- The minimum is found when  $\mathbf{b} - A\mathbf{x} \perp \text{range } A$ . Implies that  $\mathbf{b} - A\mathbf{x} \perp \mathbf{a}_k$  for all  $k$ . Implies  $(\mathbf{b} - A\mathbf{x}, \mathbf{a}_k) = \bar{\mathbf{a}}_k^T (\mathbf{b} - A\mathbf{x}) = 0$ .
- Note that we're letting  $\bar{\mathbf{a}}_k^T$  be the row vector

$$\bar{\mathbf{a}}_k^T = (\bar{a}_{1,k} \quad \cdots \quad \bar{a}_{n,k})$$

- We also have  $\bar{A}^T (\mathbf{b} - A\mathbf{x}) = 0$ , from which it follows that  $A^*A\mathbf{x} = A^*\mathbf{b}$ , so  $\mathbf{x} = (A^*A)^{-1}A^*\mathbf{b}$ . Thus,  $\text{Proj}_{\text{range } A} = Ax$ , so  $\text{Proj}_{\text{range } A} = A(A^*A)^{-1}A^*$ .
- Adjoint of a linear map  $T : V \rightarrow W$  is the  $A^*$  discussed above.
  - First, we'll do this for matrices. And then we'll do it for any finite-dimensional vector space.
  - Let  $A$  be an  $m \times n$  matrix. We claim then that

$$(A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, A^*\mathbf{y})$$

for all  $\mathbf{x} \in \mathbb{C}^n, \mathbf{y} \in \mathbb{C}^m$ . Proof:

$$\begin{aligned} (A\mathbf{x}, \mathbf{y}) &= \bar{\mathbf{y}}^T A\mathbf{x} \\ &= \mathbf{y}^* A\mathbf{x} \\ &= (A^*\mathbf{y})^* \mathbf{x} \\ &= (\mathbf{x}, A^*\mathbf{y}) \end{aligned}$$

- Properties of the adjoint:

$$(AB)^T = B^T A^T$$

$$(AB)^* = B^* A^*$$

$$(A^*)^* = A$$

- $A^*$  is the unique matrix  $B$  such that  $(A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, B\mathbf{y})$ .
- Let  $\mathbf{v}_1, \dots, \mathbf{v}_n$  be a basis of  $V$ , and let  $\mathbf{w}_1, \dots, \mathbf{w}_m$  be a basis of  $W$ .
- Definition of  $A^*$ : If  $(A\mathbf{x}, \mathbf{y}) = (y, A^*\mathbf{x})$  for all  $\mathbf{x} \in V$  and  $\mathbf{y} \in W$ .
- But it's not enough to define something; we have to check that it exists.
- If  $[A]_{AB}$ , then  $[A^*]_{AB}$ .
- More properties (give criteria for solving systems of equations):

$$\ker A^* = (\text{range } A)^\perp$$

$$\ker A = (\text{range } A^*)^\perp$$

$$\text{range } A = (\ker A^*)^\perp$$

$$\text{range } A^* = (\ker A)^\perp$$

■ Soug proves these.

• Isometries and unitary operators.

- $U : X \rightarrow Y$  is an isometry if  $\|\mathbf{x}\| = \|U\mathbf{x}\|$  for all  $\mathbf{x} \in X$ . It is an isometry because it preserves the distance between points.
- It immediately follows that  $\|\mathbf{x}_1 - \mathbf{x}_2\| = \|U\mathbf{x}_1 - U\mathbf{x}_2\| = \|U(\mathbf{x}_1 - \mathbf{x}_2)\|$ .
- This definition is equivalent to an inner product one:  $(\mathbf{x}, \mathbf{y}) = (U\mathbf{x}, U\mathbf{y})$ . This follows from the definition of the norm.
- We have

$$(\mathbf{a}, \mathbf{b}) = \frac{1}{4} \sum_{\alpha=\pm 1, \pm i} \alpha \|\mathbf{a} + \alpha \mathbf{b}\|^2$$

■  $(a+b)^2 - (a-b)^2 = 4ab$  for any  $a, b \in \mathbb{R}$ , so  $ab = \frac{1}{4}[(a+b)^2 - (a-b)^2]$ . Thus, in a real inner product space,

$$(\mathbf{a}, \mathbf{b}) = \frac{1}{4} \left( \|\mathbf{a} + \mathbf{b}\|^2 - \|\mathbf{a} - \mathbf{b}\|^2 \right)$$

■ It follows that isometries preserve inner products.

- $U$  is an isometry if and only if  $U^*U = I$ . Proof:

$$(\mathbf{x}, \mathbf{x}) = (U\mathbf{x}, U\mathbf{y}) = (U^*U\mathbf{x}, \mathbf{x})$$

$$(\mathbf{x}, \mathbf{y}) = (U\mathbf{x}, U\mathbf{y}) = (U^*U\mathbf{x}, \mathbf{y})$$

$$(\mathbf{x}, \mathbf{y}) = (U\mathbf{x}, U\mathbf{y}) = (\mathbf{x}, \mathbf{y})$$

for all  $\mathbf{y}$ .

- An isometry is unitary if it is invertible.

■ Thus,  $U : X \rightarrow Y$  an isometry is unitary iff  $\dim X = \dim Y$ .

- Note that it follows that  $U^* = U^{-1}$  for  $U$  an isometry.
- $U$  unitary implies  $|\det U| = 1$ , so  $\lambda$  an eigenvalue of  $U$  implies that  $|\lambda| = 1$ .
- $A$  is diagonalizable iff it has an orthogonal basis of eigenvectors.

## 5.2 Chapter 5: Inner Product Spaces

From Treil (2017).

- 10/24: • **Standard inner product** (on  $\mathbb{C}^n$ ): The inner product  $(\mathbf{z}, \mathbf{w})$  defined by

$$(\mathbf{z}, \mathbf{w}) = \mathbf{w}^* \mathbf{z}$$

- Corollary 5.1.5: Let  $\mathbf{x}, \mathbf{y}$  be vectors in an inner product space  $V$ . The equality  $\mathbf{x} = \mathbf{y}$  holds if and only if

$$(\mathbf{x}, \mathbf{z}) = (\mathbf{y}, \mathbf{z})$$

for all  $\mathbf{z} \in V$ .

- Corollary 5.1.6: Suppose two operator  $A, B : X \rightarrow Y$  satisfy

$$(A\mathbf{x}, \mathbf{y}) = (B\mathbf{x}, \mathbf{y})$$

for all  $\mathbf{x} \in X$  and  $\mathbf{y} \in Y$ . Then  $A = B$ .

- **Normed space:** A vector space  $V$  equipped with a norm that satisfies properties of homogeneity, the triangle inequality, non-negativity, and non-degeneracy.
- Any inner product space is naturally a normed space.
- If  $1 \leq p < \infty$ , we can define a corresponding norm on  $\mathbb{R}^n$  or  $\mathbb{C}^n$  by

$$\|\mathbf{x}\|_p = \left( \sum_{k=1}^n |x_k|^p \right)^{1/p}$$

- We can also define the norm for  $p = \infty$  by

$$\|\mathbf{x}\|_\infty = \max\{|x_k| : k = 1, \dots, n\}$$

- Note that the norm of this form for  $p = 2$  is the usual norm.
- These norms are heavily associated with Figure 5.1.

- **Minkowski inequality:** One of the triangle inequalities for norms with  $p \neq 2$ .
- Theorem 5.1.11: A norm in a normed space is obtained from some inner product if and only if it satisfies the Parallelogram Identity

$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2)$$

for all  $\mathbf{u}, \mathbf{v} \in V$ .

- It follows that norms with  $p \neq 2$  do not have associated inner products, since such norms fail to satisfy the parallelogram identity.

- Lemma 5.2.5 (Generalized Pythagorean Identity): Let  $\mathbf{v}_1, \dots, \mathbf{v}_n$  be an orthogonal system. Then

$$\left\| \sum_{k=1}^n \alpha_k \mathbf{v}_k \right\|^2 = \sum_{k=1}^n |\alpha_k|^2 \|\mathbf{v}_k\|^2$$

- Proposition 5.3.3: Let  $\mathbf{v}_1, \dots, \mathbf{v}_r$  be an orthogonal basis in  $E$ . Then the orthogonal projection  $P_E \mathbf{v}$  of a vector  $\mathbf{v}$  is given by the formula

$$P_E \mathbf{v} = \sum_{k=1}^r \frac{(\mathbf{v}, \mathbf{v}_k)}{\|\mathbf{v}_k\|^2} \mathbf{v}_k$$

– It follows that

$$\begin{aligned} P_E \mathbf{v} &= \sum_{k=1}^r \frac{\mathbf{v}_k^* \mathbf{v}}{\|\mathbf{v}_k\|^2} \mathbf{v}_k \\ &= \sum_{k=1}^r \frac{1}{\|\mathbf{v}_k\|^2} \mathbf{v}_k \mathbf{v}_k^* \mathbf{v} \\ &= \left( \sum_{k=1}^r \frac{1}{\|\mathbf{v}_k\|^2} \mathbf{v}_k \mathbf{v}_k^* \right) \mathbf{v} \end{aligned}$$

– Thus, we have that

$$P_E = \sum_{k=1}^r \frac{1}{\|\mathbf{v}_k\|^2} \mathbf{v}_k \mathbf{v}_k^*$$

- **Gram-Schmidt orthogonalization:** Let  $\mathbf{x}_1, \dots, \mathbf{x}_n$  be a linearly independent system of vectors to orthogonalize. Then  $\mathbf{v}_1 = \mathbf{x}_1$ ,  $\mathbf{v}_2 = \mathbf{x}_2 - P_{\text{span}\{\mathbf{v}_1\}} \mathbf{x}_2$ ,  $\mathbf{v}_3 = \mathbf{x}_3 - P_{\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}} \mathbf{x}_3$ , and on and on.
- To find the least squares solution to  $A\mathbf{x} = \mathbf{b}$ , solve  $A\mathbf{x} = P_{\text{range } A} \mathbf{b}$ .
  - We can do this by finding an orthogonal basis of range  $A$  and then applying the projection formula.
  - Alternatively, we can use the following formula to speed things up if  $A^*A$  is invertible:

$$P_{\text{range } A} \mathbf{b} = A(A^*A)^{-1}A^*\mathbf{b}$$

- Theorem 5.4.1: For an  $m \times n$  matrix  $A$ ,

$$\ker A = \ker(A^*A)$$

– Thus,  $A^*A$  is invertible iff  $A$  is invertible iff  $A$  is full rank. This gives us a condition on when we can use the projection formula.

- Theorem 5.6.1: An operator  $U : X \rightarrow Y$  is an isometry if and only if it preserves the inner product, i.e., if and only if

$$(\mathbf{x}, \mathbf{y}) = (U\mathbf{x}, U\mathbf{y})$$

for all  $\mathbf{x}, \mathbf{y} \in X$ .

- Lemma 5.6.2: An operator  $U : X \rightarrow Y$  is an isometry if and only if  $U^*U = I$ .
- **Unitary** (operator): An invertible isometry.
- Proposition 5.6.3: An isometry  $U : X \rightarrow Y$  is a unitary operator iff  $\dim X = \dim Y$ .
- **Orthogonal** (matrix): A unitary matrix with real entries.
- Unitary operator properties:
  1.  $U^{-1} = U^*$ .
  2.  $U$  unitary implies  $U^* = U^{-1}$  unitary.
  3. If  $\mathbf{v}_1, \dots, \mathbf{v}_n$  is orthonormal,  $U\mathbf{v}_1, \dots, U\mathbf{v}_n$  is orthonormal.
  4.  $U, V$  unitary implies  $UV$  unitary.
- A matrix  $U$  is an isometry iff its columns form an orthonormal system.
- Proposition 5.6.4: Let  $U$  be a unitary matrix. Then
  1.  $|\det U| = 1$ . In particular, if  $U$  is orthogonal, then  $\det U = \pm 1$ .
  2.  $|\lambda| = 1$  for every eigenvalue  $\lambda$  of  $U$ .
- Proposition 5.6.5: A matrix  $A$  is unitarily equivalent to a diagonal one iff it has an orthogonal (or-thonormal) basis of eigenvectors.



## Chapter 6

# Structure of Operators on Inner Product Spaces

### 6.1 Notes

10/11:

- Spectral decomposition of self-adjoint linear maps.
  - Can we write a map in term of the eigenvalues only?
  - Let  $A : X \rightarrow X$  be linear and self-adjoint. Where  $\dim X < \infty$ .
  - Let  $A$  have eigenvalues  $\lambda_1, \dots, \lambda_n$  and eigenvectors  $\mathbf{v}_1, \dots, \mathbf{v}_n$ . Then there is an orthonormal basis of  $X$  consisting of eigenvectors of  $A$ . An operator is self-adjoint if  $A = A^*$ .
  - If  $A$  is self-adjoint, then  $A$  can be written as diagonal with the eigenvalues on the diagonal with respect to some orthonormal basis of eigenvectors.
  - Let  $\mathbb{F} = \mathbb{C}$ .
- If there exists an orthonormal basis  $u_1, \dots, u_n$  of  $X$  such that  $A$  is triangular, then  $A = UTU^*$  where  $U$  is unitary and  $T$  is upper triangular.
  - Proved with induction on  $\dim X$ .
  - $\dim X = 1$  is clear.
  - Assume for  $\dim X = n - 1$ , WTS for  $\dim X = n$ .
  - The subspace has a basis  $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$  such that  $A$  has a diagonal form.
  - Let  $u \in X$  be linearly independent of  $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ .
  - Let  $\lambda$  be the remaining eigenvalue and  $u$  the corresponding eigenvector. Let  $E = \text{span}(u)$ . Then make the matrix  $\lambda$  in the upper left corner, and block diagonal with “ $A_{n-1}$ ” in the bottom right corner, zeroes everywhere else.
- **Self-adjoint** (matrix  $A$ ): A linear map  $A : X \rightarrow X$  where  $\dim X < \infty$  such that  $A = A^*$ .
  - Similarly,  $(Ax, y) = (x, Ay)$ .
  - A self-adjoint implies all eigenvalues are real, eigenvectors corresponding to different eigenvalues are orthogonal.
    - Soug proves this.
- **Strictly positive** (operator  $A$ ): A self-adjoint operator  $A : X \rightarrow X$  such that  $(Ax, x) > 0$  for all  $x \neq 0$ . Also known as **positive definite**.
  - Implies that all eigenvalues are strictly positive.

- **Nonnegative** (operator  $A$ ): A self-adjoint operator  $A : X \rightarrow X$  such that  $(Ax, x) \geq 0$  for all  $x \neq 0$ . Also known as **definite**.

- All eigenvalues are nonnegative.

- Suppose  $A \geq 0$  is self-adjoint. Then there exists a unique self-adjoint  $B \geq 0$  such that  $B^2 = A$ .

- $A$  self-adjoint is diagonal (wrt. some basis).
- $A$  positive means that all eigenvalues (diagonal entries) are positive.
- Thus, take

$$B = \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sqrt{\lambda_n} \end{pmatrix}$$

- Suppose  $B^2 = A$ ,  $C^2 = A$ . Then we have an orthonormal basis corresponding to  $B$  and an orthonormal basis corresponding to  $C$ . It follows that  $B^2 = C^2 = A$ . Write  $B^2x$  and  $C^2x$  in terms of their bases; will necessitate that the bases are the same.

10/13:

- If we get yes/no questions, we don't have to justify.
- Cauchy-Schwarz inequality:

$$|(\mathbf{x}, \mathbf{y})| \leq \|\mathbf{x}\| \|\mathbf{y}\|$$

- Real spaces,  $V$  vs.  $(\cdot, \cdot)$  inner product.
- Proof:

$$\begin{aligned} 0 &\leq \|\mathbf{x} + t\mathbf{y}\|^2 \\ &= t^2 \|\mathbf{y}\|^2 + 2t(\mathbf{x}, \mathbf{y}) + \|\mathbf{x}\|^2 \end{aligned}$$

Thus, the discriminant must be less than zero (because the whole polynomial is positive, so the discriminant [the opposite of the  $x^0$  term of the factored form of the polynomial] must be less than zero so the polynomial doesn't get dragged down to negative values):

$$(\mathbf{x}, \mathbf{y})^2 - \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 \leq 0$$

Taking square roots of both sides proves the desired inequality.

- Recall that if  $A^* = A$ , then all eigenvalues are real and all eigenvectors of distinct eigenvalues are orthogonal to each other.
- **Normal** (matrix): A matrix  $N$  such that  $N^*N = NN^*$ .
  - Examples: Diagonal, self-adjoint, and unitary operators are all normal.
- Any normal operator in a complex vector space has an orthonormal set of eigenvectors, e.g.,  $N = UDU^*$ .
  - Proof:  $N$  is upper triangular wrt. some basis (because all matrices are). WTS any normal upper triangular matrix is diagonal. Done by induction on the dimension of  $N$  from  $n = 2$ .
  - Assume the claim for every  $(n - 1) \times (n - 1)$  normal upper triangular matrix.
  - Let

$$N = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & & & \\ 0 & & & \\ \vdots & & N_1 & \\ 0 & & & \end{pmatrix}$$

(we know every normal matrix can be written in this upper triangular form)

- Then just compute  $NN^*$  and  $N^*N$ . Knowing they have to be equal, we have that  $a_{12} = \dots = a_{1n} = 0$ .
- We can also prove from the above (block diagonal multiplication) that  $N_1$  is normal. Thus, it's diagonal, too. Therefore, the whole thing is diagonal.
- $N$  is normal if and only if  $\|N\mathbf{x}\| = \|N^*\mathbf{x}\|$ .
  - Proof:  $(N\mathbf{x}, N\mathbf{y}) = (N^*N\mathbf{x}, \mathbf{y}) = (NN^*\mathbf{x}, \mathbf{y}) = (N^*\mathbf{x}, N^*\mathbf{y})$ . This is equivalent to the desired condition.
- If  $A$  is nonnegative and  $(A\mathbf{e}_k, \mathbf{e}_k) = a_{kk}$ , then

$$\sum_{i,j=1}^n a_{ij} \mathbf{x}_i \mathbf{x}_j$$

- **Positive definite** (matrix): An  $n \times n$  self-adjoint matrix such that  $(A\mathbf{x}, \mathbf{x}) > 0$  for all  $\mathbf{x} \in X$ .
- Let  $A : X \rightarrow Y$ ,  $\dim X = \dim Y$ . Then  $AA^*$  is positive semidefinite. And there exists a unique square root  $R = \sqrt{A^*A}$ .
  - Proof:  $(A^*A\mathbf{x}, \mathbf{x}) = (A\mathbf{x}, A\mathbf{x}) = \|A\mathbf{x}\|^2 \geq 0$ .
- **Modulus** (of  $A$ ): The matrix  $|A| = \sqrt{A^*A}$ .
- Check  $\| |A|\mathbf{x} \| = \|A\mathbf{x}\|$ .

$$\| |A|\mathbf{x} \|^2 = (|A|\mathbf{x}, |A|\mathbf{x}) = (|A|^*|A|\mathbf{x}, \mathbf{x}) = (A^*A\mathbf{x}, \mathbf{x}) = (A\mathbf{x}, A\mathbf{x}) = \|A\mathbf{x}\|^2$$

- Let  $A : X \rightarrow X$  be a linear operator. Then  $A = U|A|$  where  $U$  is unitary.
- Look at singular matrices.

10/15:

- Recall that if  $A : X \rightarrow Y$ , we have that  $A^*A$  is semidefinite, positive, and self adjoint.
  - Thus, there exists a unique matrix  $R = \sqrt{A^*A} \geq 0$ , which we define to be  $|A| = \sqrt{A^*A}$ .
- Polar form of a matrix:

$$A = U|A|$$

- This may not be unique!
- Proof: Suppose  $A\mathbf{x} = U(|A|\mathbf{x})$ .  $A\mathbf{x} \in \text{range } A$ , and  $|A|\mathbf{x} \in \text{range}(|A|)$ .  $\mathbf{x} \in \text{range}(|A|)$  implies that there exists  $\mathbf{v} \in X$  such that  $\mathbf{x} = |A|\mathbf{v}$ .
- Define  $U\mathbf{x} = A\mathbf{x}$ .  $U$  is a well-defined linear map.
- $\|U\mathbf{x}\| = \|A\mathbf{x}\| = \||A|\mathbf{v}\| = \|\mathbf{x}\|$ .
- $U$  is an isometry.
- $\text{range } |A| \rightarrow X$ .
- Use  $\ker A = \ker |A| = (\text{range } A)^\perp$  to extend  $U_0$  to  $U$ :  $U = U_0 + U_1$ .
- **Singular values** (of a matrix): The eigenvalues of  $|A|$ .
  - So if  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $A^*A$ , the singular values of  $A$  are  $\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}$ .
- Let  $A : X \rightarrow Y$  be a linear map.
  - Let  $\sigma_1, \dots, \sigma_n$  be the singular values of  $A$ . Then  $\sigma_1, \dots, \sigma_n > 0$ .
  - Additionally, if  $\mathbf{v}_1, \dots, \mathbf{v}_n$  is an orthonormal basis of eigenvectors of  $A^*A$ , then the list of  $n$  vectors  $\mathbf{w}_1, \dots, \mathbf{w}_n$  defined by  $\mathbf{w}_k = 1/\sigma_k A\mathbf{v}_k$  for each  $k = 1, \dots, n$  is orthonormal.

■ Proof:

$$(\mathbf{w}_k, \mathbf{w}_j) = \frac{1}{\sigma_k \sigma_j} (A\mathbf{v}_k, A\mathbf{v}_j) = \frac{1}{\sigma_k \sigma_j} = \frac{1}{\sigma_k \sigma_j} (A^* A \mathbf{v}_k, \mathbf{v}_j) = \frac{\sigma_k^2}{\sigma_k \sigma_j} (\mathbf{v}_k, \mathbf{v}_j) = 0$$

and

$$\|\mathbf{w}_k\| = \frac{1}{\sigma_k} \|A\mathbf{v}_k\| = \frac{1}{\sigma_k} \||A|\mathbf{v}_k\| = 1$$

– Schmidt decomposition of  $A$ :

$$A\mathbf{x} = \sum_{k=1}^r \sigma_k (\mathbf{x}, \mathbf{v}_k) \mathbf{w}_k$$

■ This is because  $\mathbf{x} = \sum (\mathbf{x}, \mathbf{v}_k) \mathbf{v}_k$ , so by the above,

$$A\mathbf{x} = \sum_{k=1}^n (\mathbf{x}, \mathbf{v}_k) A\mathbf{v}_k = \sum_{k=1}^r \sigma_k (\mathbf{x}, \mathbf{v}_k) \mathbf{w}_k$$

• **Operator norm:**  $\|A\| = \max\{\|A\mathbf{x}\| : \|\mathbf{x}\| \leq 1\}$ .

• Properties of the operator norm:

- $\|A\mathbf{x}\| \leq \|A\| \|\mathbf{x}\|$ .
- $\|\alpha A\| = |\alpha| \|A\|$ .
- $\|A + B\| \leq \|A\| + \|B\|$ .
- $\|A\| \geq 0$ .
- $\|A\| = 0$  iff  $A = 0$ .

• **Frobenius norm:** The norm  $\|A\|_2^2 = \text{tr}(A^* A)$ .

• The operator norm is always less than or equal to the Frobenius norm.

• If  $A : \mathbb{F}^n \rightarrow \mathbb{F}^n$ , then  $A = W\Sigma V^*$  where  $\sigma$  is a diagonal matrix of nonzero singular values.

• The operator norm of  $A$  is the largest of the singular values.

• An orthogonal matrix can be decomposed to a block-diagonal matrix of rotations.

10/18:

• Soug tests what he teaches and doesn't give super tricky questions.

• Structure of orthogonal matrices.

• **Orthogonal (matrix):** A unitary matrix  $U$  with all elements real and  $|\det U| = 1$ .

• Theorem: Let  $U$  be an orthogonal operator on  $\mathbb{R}^n$  such that  $\det U = 1$ . Then there exists an orthonormal basis  $\mathbf{v}_1, \dots, \mathbf{v}_n$  such that with respect to this basis,

$$U = \begin{pmatrix} R_{\phi_1} & & & \mathbf{0} \\ & \ddots & & \\ & & R_{\phi_k} & \\ \mathbf{0} & & & I_{n-2k} \end{pmatrix}$$

where each  $R_{\phi_i}$  is a  $2 \times 2$  rotation matrix.

- If you are in  $\mathbb{R}^7$  for example, you would be able to express  $U$  as a composition of at most 3 rotation maps and the identity map.
- Each rotation map acts on two orthonormal vectors.
- Proof:  $P(\lambda)$  is the  $n$ -degree characteristic polynomial  $\det(U - \lambda I) = 0$ . The eigenvalues are the roots of it.

- $p(\lambda) = 0$  if and only if  $p(\bar{\lambda}) = 0$ .
  - $\lambda \in \mathbb{C}$  is an eigenvalue with eigenvector  $\mathbf{u} \neq 0$  iff  $U\mathbf{u} = \lambda\mathbf{u}$  and  $U\bar{\mathbf{u}} = \bar{\lambda}\bar{\mathbf{u}}$ .
- Recall that  $U$  unitary implies  $|\lambda| = 1$ .
  - Proof<sup>[1]</sup>:  $\|U\mathbf{x}\| = \|\mathbf{x}\|$  and  $U\mathbf{x} = \lambda\mathbf{x}$ . Thus,

$$\|U\mathbf{x}\| = \|\lambda\mathbf{x}\| = |\lambda|\|\mathbf{x}\| = \|\mathbf{x}\|$$

and since  $\mathbf{x} \neq 0$ , we can divide by  $\|\mathbf{x}\|$ , so  $|\lambda| = 1$ .

- Let  $\mathbf{u} = \text{Re } \mathbf{u} + i \text{Im } \mathbf{u}$ .
- It follows that we may define

$$\mathbf{x} = \text{Re } \mathbf{u} = \frac{\mathbf{u} + \bar{\mathbf{u}}}{2} \qquad \mathbf{y} = \text{Im } \mathbf{u} = \frac{\mathbf{u} - \bar{\mathbf{u}}}{2}$$

- Thus,  $\mathbf{u} = \mathbf{x} + i\mathbf{y}$  and  $\bar{\mathbf{u}} = \mathbf{x} - i\mathbf{y}$ .
- Since  $U\mathbf{x} = \frac{U\mathbf{u} + U\bar{\mathbf{u}}}{2} = \frac{\lambda\mathbf{u} + \bar{\lambda}\bar{\mathbf{u}}}{2}$ ,  $U\mathbf{y} = \text{Im}(\lambda\mathbf{u}) = \text{Re}(\lambda\mathbf{u})$ .
- Since  $|\lambda| = 1$ ,  $\lambda = e^{i\alpha}$  and  $\bar{\lambda} = e^{-i\alpha}$ .
- It follows that  $U\mathbf{x} = (\cos \alpha)\mathbf{x} - (\sin \alpha)\mathbf{y}$  and  $U\mathbf{y} = (\cos \alpha)\mathbf{y} + (\sin \alpha)\mathbf{x}$ .
- Thus, since  $U\mathbf{x} = \text{Re } \lambda\mathbf{u}$ , we have that

$$\begin{aligned} \lambda\mathbf{u} &= (\cos \alpha + i \sin \alpha)(\mathbf{x} + i\mathbf{y}) \\ &= (\cos \alpha)\mathbf{x} - (\sin \alpha)\mathbf{y} + i[(\cos \alpha)\mathbf{y} + (\sin \alpha)\mathbf{x}] \end{aligned}$$

- If  $E_\lambda$  is a 2 dimensional space spanned by  $\mathbf{x}$  and  $\mathbf{y}$  and invariant by  $U$ . Thus, any block of the desired matrix leaves its desired sub-block invariant.
- We also know that the eigenvectors of a unitary matrix corresponding to different eigenvalues are orthogonal.
- Thus,  $\|\mathbf{x}\| = \|\mathbf{y}\| = \sqrt{2}/2\|\mathbf{u}\|$ ,  $\mathbf{x} \perp \mathbf{y}$ .
- Let  $\mathbf{x}, \mathbf{y}$  complete the theorem to form a basis of  $\mathbb{R}^n$ .
- It will follow that

$$U = \begin{pmatrix} R_\alpha & \mathbf{0} \\ \mathbf{0} & U_1 \end{pmatrix}$$

where  $U_1$  is orthogonal, and we may repeat the process.

## 6.2 Chapter 6: Structure of Operators on Inner Product Spaces

From Treil (2017).

10/24:

- Theorem 6.1.1: Let  $A : X \rightarrow X$  be an operator acting in a complex inner product space. Then there exists an orthonormal basis  $\mathbf{u}_1, \dots, \mathbf{u}_n$  of  $X$  such that the matrix of  $A$  in this basis is upper triangular. In other words, any  $n \times n$  matrix  $A$  can be represented as  $A = UTU^*$ , where  $U$  is unitary and  $T$  is upper-triangular.
- Theorem 6.1.2: Let  $A : X \rightarrow X$  be an operator acting on a real inner product space. Suppose that all eigenvalues of  $A$  are real. Then there exists an orthonormal basis  $\mathbf{u}_1, \dots, \mathbf{u}_n$  in  $X$  such that the matrix of  $A$  in this basis is upper triangular. In other words, any real  $n \times n$  matrix  $A$  with all real eigenvalues can be represented as  $T = UTU^* = UTU^T$ , where  $U$  is orthogonal and  $T$  is a real upper-triangular matrix.

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<sup>1</sup>This would be a good exam question.

- Theorem 6.2.1: Let  $A = A^*$  be a self-adjoint operator in an inner product space  $X$  (the space can be complex or real). Then all eigenvalues of  $A$  are real and there exists an orthonormal basis of eigenvectors of  $A$  in  $X$ .

Equivalently (see Theorem 6.2.2),  $A$  can be represented as  $A = UDU^*$  where  $U$  is a unitary matrix and  $D$  is a diagonal matrix with real entries. Moreover, if  $A$  is real,  $U$  can be chosen to be real, i.e., orthogonal.

- Proposition 6.2.3: Let  $A = A^*$  be a self-adjoint operator and let  $\lambda, \mathbf{u}, \mu, \mathbf{v}$  be such that  $A\mathbf{u} = \lambda\mathbf{u}$  and  $A\mathbf{v} = \mu\mathbf{v}$ . Then if  $\lambda \neq \mu$ ,  $\mathbf{u} \perp \mathbf{v}$ .
- Since complex multiplication is commutative,

$$D^*D = DD^*$$

for every diagonal matrix  $D$ .

– It follows that  $A^*A = AA^*$  if the matrix of  $A$  in some orthonormal basis is diagonal.

- Theorem 6.2.4: Any normal operator  $N$  in a complex vector space has an orthonormal basis of eigenvectors.

Equivalently, any matrix  $N$  satisfying  $N^*N = NN^*$  can be represented as  $N = UDU^*$  where  $U$  is unitary and  $D$  is diagonal.

- Proposition 6.2.5: An operator  $N : X \rightarrow X$  is normal iff

$$\|N\mathbf{x}\| = \|N^*\mathbf{x}\|$$

for all  $\mathbf{x} \in X$ .

- **Hermitian square** (of  $A$ ): The matrix  $A^*A$ .
- **Modulus** (of  $A$ ): The unique positive semidefinite square root  $\sqrt{A^*A}$ .
- Proposition 6.3.3: For a linear operator  $A : X \rightarrow Y$ ,

$$\| |A| \mathbf{x} \| = \| A \mathbf{x} \|$$

- Corollary 6.3.4:  $\ker A = \ker |A|$ .
- Theorem 6.3.5: Let  $A : X \rightarrow X$  be an operator (square matrix). Then  $A$  can be represented as

$$A = U|A|$$

where  $U$  is a unitary operator.

- **Singular value** (of  $A$ ): An eigenvalue of  $|A|$ .
  - A positive square root of an operator of  $A^*A$ .
- Proposition 6.3.6: Let  $\sigma_1, \dots, \sigma_n$  be the singular values of  $A$ , ordered such that  $\sigma_1, \dots, \sigma_r$  are the nonzero singular values, and let  $\mathbf{v}_1, \dots, \mathbf{v}_n$  be an orthonormal basis of eigenvectors of  $A^*A$ . Then the system

$$\mathbf{w}_k = \frac{1}{\sigma_k} A \mathbf{v}_k$$

for  $k = 1, \dots, r$  is orthonormal.

- **Schmidt decomposition** (of  $A$ ): The decompositions

$$A = \sum_{k=1}^r \sigma_k \mathbf{w}_k \mathbf{v}_k^*$$

and

$$A\mathbf{x} = \sum_{k=1}^r \sigma_k (\mathbf{x}, \mathbf{v}_k) \mathbf{w}_k$$

- Note that these can be verified by plugging  $\mathbf{x} = \mathbf{v}_j$  for each  $j = 1, \dots, n$  into the latter equation.

10/25:

- Lemma 6.3.7:  $A$  can be represented as the Schmidt decomposition

$$A = \sum_{k=1}^r \sigma_k \mathbf{w}_k \mathbf{v}_k^*$$

where  $\sigma_k > 0$  for any orthonormal systems  $\mathbf{v}_1, \dots, \mathbf{v}_r$  and  $\mathbf{w}_1, \dots, \mathbf{w}_r$ .

- Corollary 6.3.8: Let  $A = \sum_{k=1}^r \sigma_k \mathbf{w}_k \mathbf{v}_k^*$  be a Schmidt decomposition of  $A$ . Then

$$A^* = \sum_{k=1}^r \sigma_k \mathbf{v}_k \mathbf{w}_k^*$$

is a Schmidt decomposition of  $A^*$ .

- **Reduced singular value decomposition** (of  $A$ ): The decomposition

$$A = \tilde{W} \tilde{\Sigma} \tilde{V}^*$$

where  $A : \mathbb{F}^n \rightarrow \mathbb{F}^m$  has the Schmidt decomposition  $A = \sum_{k=1}^r \sigma_k \mathbf{w}_k \mathbf{v}_k^*$ ,  $\tilde{\Sigma} = \text{diag}\{\sigma_1, \dots, \sigma_r\}$ , and  $\tilde{V}, \tilde{W}$  are matrices with columns  $\mathbf{v}_1, \dots, \mathbf{v}_r$  and  $\mathbf{w}_1, \dots, \mathbf{w}_r$ , respectively. *Also known as **compact singular value decomposition**.*

- Note that  $\tilde{V}$  is an  $n \times r$  matrix,  $\tilde{\Sigma}$  is an  $r \times r$  matrix, and  $\tilde{W}$  is an  $m \times r$  matrix.
- Since  $\mathbf{v}_1, \dots, \mathbf{v}_r$  and  $\mathbf{w}_1, \dots, \mathbf{w}_r$  are orthonormal,  $\tilde{V}, \tilde{W}$  are isometries.
- Note that  $r = \text{rank } A$  (see Problem 6.3.1).
  - It follows that if  $A$  is invertible, then  $m = n = r$ , so  $\tilde{V}, \tilde{W}$  are unitary and  $\tilde{\Sigma}$  is an invertible diagonal matrix.
- However,  $A$  need not be invertible for us to get a representation similar to  $A = \tilde{W} \tilde{\Sigma} \tilde{V}^*$ .
  - Complete  $\mathbf{v}_1, \dots, \mathbf{v}_r$  and  $\mathbf{w}_1, \dots, \mathbf{w}_r$  to bases of  $\mathbb{F}^n$  and  $\mathbb{F}^m$ , respectively.
  - Then we get the following.
- **Singular value decomposition** (of  $A$ ): The decomposition

$$A = W \Sigma V^*$$

where  $V \in M_{n \times n}^{\mathbb{F}}$  and  $W \in M_{m \times m}^{\mathbb{F}}$  are unitary matrices with columns  $\mathbf{v}_1, \dots, \mathbf{v}_n$  and  $\mathbf{w}_1, \dots, \mathbf{w}_m$ , respectively, and  $\Sigma \in M_{m \times n}^{\mathbb{R}^+}$  is a “diagonal” matrix such that

$$\Sigma_{j,k} = \begin{cases} \sigma_k & j = k \leq r \\ 0 & \text{otherwise} \end{cases}$$

- Notice that if  $A = W\Sigma V^*$ , then

$$A^*A = (W\Sigma V^*)^*(W\Sigma V^*) = V\Sigma^*W^*W\Sigma V^* = V\Sigma^2V^*$$

proving that the singular values of  $A$ , squared, are the eigenvalues of  $A^*A$ .

- If  $A$  is invertible, the reduced SVD is the matrix form of the Schmidt decomposition is the SVD.
- If  $A = W\Sigma V^*$  is  $n \times n$ , then

$$A = \underbrace{(WV^*)}_U \underbrace{(V\Sigma V^*)}_{|A|}$$

is a polar decomposition of  $A$ .

- Consider the unit ball  $B = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq 1\}$ .
  - We want to describe  $A(B)$ , i.e., the image of the unit ball under  $A$ .
  - Let  $\mathbf{x} = (x_1, \dots, x_n)^T$  and let  $\mathbf{y} = (y_1, \dots, y_n)^T$ . If  $A = \text{diag}\{\sigma_1, \dots, \sigma_n\}$ , we have  $\mathbf{y} \in A(B)$  iff  $\mathbf{y} = A\mathbf{x}$  where  $\mathbf{x} \in B$  iff

$$\sum_{k=1}^n \frac{y_k^2}{\sigma_k^2} = \sum_{k=1}^n x_k^2 = \|\mathbf{x}\|^2 \leq 1$$

- Thus,  $A(B)$  is an ellipsoid with half-axes  $\sigma_1, \dots, \sigma_n$ .
- In the more general case, if  $A = W\Sigma V^*$ , then since  $V^*$  is unitary,  $V^*(B) = B$ .  $\Sigma V^*(B) = \Sigma(B)$  is thus by the above an ellipsoid in range  $\Sigma$  with half-axes  $\sigma_1, \dots, \sigma_r$ . Thus, since isometries don't change geometry,  $W(\Sigma(B))$  is also an ellipsoid with the same half-axes, but in range  $A$ .
- Conclusion: The image  $A(B)$  of the closed unit ball  $B$  is an ellipsoid in range  $A$  with half-axes  $\sigma_1, \dots, \sigma_r$ , where  $r$  is the number of nonzero singular values, i.e., the rank of  $A$ .
- Finding the maximum of  $\|A\mathbf{x}\|$  for  $\mathbf{x} \in B$ .
  - For a diagonal matrix  $\Sigma$  with nonnegative entries, the maximum is clearly the maximal diagonal entry: In this case if  $s_1$  is the maximal diagonal entry, then since

$$\Sigma\mathbf{x} = \sum_{k=1}^r s_k x_k \mathbf{e}_k$$

we have that

$$\|A\mathbf{x}\|^2 = \sum_{k=1}^r s_k^2 |x_k|^2 \leq s_1^2 \sum_{k=1}^r |x_k|^2 = s_1^2 \cdot \|\mathbf{x}\|^2$$

- We get the following by a similar logic to before.
- Conclusion: The maximum of  $\|A\mathbf{x}\|$  on the unit ball  $B$  is the maximal singular value of  $A$ .
- **Operator norm** (of  $A$ ): The following quantity. Denoted by  $\|A\|$ . Given by

$$\|A\| = \max\{\|A\mathbf{x}\| : \mathbf{x} \in X, \|\mathbf{x}\| \leq 1\}$$

- $\|A\|$  clearly satisfies the four properties of a norm.
- Additionally,

$$\|A\mathbf{x}\| \leq \|A\| \cdot \|\mathbf{x}\|$$

- Alternate definition: The operator norm  $\|A\|$  is the smallest number  $C \geq 0$  such that  $\|A\mathbf{x}\| \leq C\|\mathbf{x}\|$ .



- **Frobenius norm:** The following norm. *Also known as Hilbert-Schmidt norm.* Denoted by  $\|A\|_2$ . Given by

$$\|A\|_2^2 = \text{tr}(A^*A)$$

- If we let  $s_1, \dots, s_n$  be the singular values of  $A$  and let  $s_1$  be the largest value, then we have

$$\|A\|^2 = s_1^2 \leq \sum_{k=1}^n s_k^2 = \text{tr}(A^*A) = \|A\|_2^2$$

- Conclusion: The operator norm of a matrix cannot be more than its Frobenius norm.
- Suppose we want to solve  $A\mathbf{x} = \mathbf{b}$  where  $A$  is invertible, but there is some (experimental) error  $\Delta\mathbf{b}$  in  $\mathbf{b}$ . Then we are really solving for an approximate solution  $\mathbf{x} + \Delta\mathbf{x}$  to the equation

$$A(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{b} + \Delta\mathbf{b}$$

- It follows since  $A$  is invertible that  $\mathbf{x} = A^{-1}\mathbf{b}$  and  $\Delta\mathbf{x} = A^{-1}\Delta\mathbf{b}$ .
- To estimate the relative error  $\|\Delta\mathbf{x}\|/\|\mathbf{x}\|$  in the solution in comparison with the relative error  $\|\Delta\mathbf{b}\|/\|\mathbf{b}\|$  in the data, use

$$\frac{\|\Delta\mathbf{x}\|}{\|\mathbf{x}\|} = \frac{\|A^{-1}\Delta\mathbf{b}\|}{\|\mathbf{b}\|} \frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} \leq \frac{\|A^{-1}\| \cdot \|\Delta\mathbf{b}\|}{\|\mathbf{b}\|} \frac{\|A\| \cdot \|\mathbf{x}\|}{\|\mathbf{x}\|} = \|A^{-1}\| \cdot \|A\| \cdot \frac{\|\Delta\mathbf{b}\|}{\|\mathbf{b}\|}$$

- **Condition number** (of  $A$ ): The following quantity. *Given by*

$$\|A\| \cdot \|A^{-1}\|$$

- If  $s_1$  is the largest singular value of  $A$  and  $s_n$  is the smallest, then

$$\|A\| \cdot \|A^{-1}\| = s_1 \cdot \frac{1}{s_n} = \frac{s_1}{s_n}$$

- **Well-conditioned** (matrix): A matrix the condition number of which is not “too big.”
- **Ill-conditioned** (matrix): A matrix that is not well-conditioned.
- Theorem 6.5.1: Let  $U$  be an orthogonal operator on  $\mathbb{R}^n$  and let  $\det U = 1$ . Then there exists an orthonormal basis  $\mathbf{v}_1, \dots, \mathbf{v}_n$  such that the matrix of  $U$  in this basis has the block diagonal form

$$\begin{pmatrix} R_{\varphi_1} & & & 0 \\ & \ddots & & \\ & & R_{\varphi_k} & \\ 0 & & & I_{n-2k} \end{pmatrix}$$

where each  $R_{\varphi_j}$  is a two-dimensional rotation

$$R_{\varphi_j} = \begin{pmatrix} \cos \varphi_j & -\sin \varphi_j \\ \sin \varphi_j & \cos \varphi_j \end{pmatrix}$$

and  $I_{n-2k}$  represents the  $(n-2k) \times (n-2k)$  identity matrix.

- Alternate interpretation: Any rotation in  $\mathbb{R}^n$  can be represented as a composition of at most  $n/2$  commuting planar rotations.

- Theorem 6.5.2: Let  $U$  be an orthogonal operator on  $\mathbb{R}^n$  and let  $\det U = -1$ . Then there exists an orthonormal basis  $\mathbf{v}_1, \dots, \mathbf{v}_n$  such that the matrix of  $U$  in this basis has block diagonal form

$$\begin{pmatrix} R_{\varphi_1} & & & & 0 \\ & \ddots & & & \\ & & R_{\varphi_k} & & \\ & & & I_r & \\ 0 & & & & -1 \end{pmatrix}$$

where  $r = n - 2k - 1$  and each  $R_{\varphi_j}$  is a two-dimensional rotation

$$R_{\varphi_j} = \begin{pmatrix} \cos \varphi_j & -\sin \varphi_j \\ \sin \varphi_j & \cos \varphi_j \end{pmatrix}$$

- Corollary: An orthogonal  $2 \times 2$  matrix  $U$  with determinant  $-1$  is always a reflection.
- Theorem 6.5.3: Any rotation  $U$  (i.e., any orthogonal transformation  $U$  with  $\det U = 1$ ) can be represented as a product of at most  $n(n-1)/2$  elementary rotations.
- Consider the following orthonormal bases of  $\mathbb{R}^2$ .



Figure 6.1: Orientation in  $\mathbb{R}^2$ .

- Notice that a rotation will get you from the standard basis (a) to basis (b), but not from the standard basis (a) to basis (c).
- This is the motivation for defining orientation.
- More formally, we know that there is a unique linear transformation  $U$  such that  $U\mathbf{e}_k = \mathbf{v}_k$  for each  $k = 1, 2$ . In particular, the matrix of  $U$  with respect to the standard basis is orthogonal with columns  $\mathbf{v}_1, \mathbf{v}_2$ .
- By Theorems 6.5.1 and 6.5.2, if  $\det U = 1$ , then  $U$  is a rotation, and if  $\det U = -1$ , then  $U$  is not a rotation.
- **Similarly oriented** (bases  $\mathcal{A}, \mathcal{B}$ ): Two bases  $\mathcal{A}, \mathcal{B}$  of a real vector space such that the change of coordinates matrix  $[I]_{\mathcal{B}\mathcal{A}}$  has a positive determinant.
- **Differently oriented** (bases  $\mathcal{A}, \mathcal{B}$ ): Two bases  $\mathcal{A}, \mathcal{B}$  of a real vector space that are not similarly oriented (i.e.,  $[I]_{\mathcal{B}\mathcal{A}}$  has a negative determinant).
- We usually let the standard basis of  $\mathbb{R}^n$  have a **positive orientation**.
  - In an abstract vector space, we need only fix a basis and declare its orientation to be positive.
- **Continuously transformable** (bases  $\mathcal{A}, \mathcal{B}$ ): Two bases  $\mathcal{A}, \mathcal{B}$  such that  $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$  can be continuously transformed to a basis  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ . In particular, there exists a **continuous family of bases**  $\mathcal{V}(t) = \{\mathbf{v}_1(t), \dots, \mathbf{v}_n(t)\}$ ,  $t \in [a, b]$ , such that

$$\mathbf{v}_k(a) = \mathbf{a}_k \qquad \mathbf{v}_k(b) = \mathbf{b}_k$$

for each  $k = 1, \dots, n$ .

- **Continuous family of bases:** A family of bases  $\mathcal{V}(t) = \{\mathbf{v}_1(t), \dots, \mathbf{v}_n(t)\}$ ,  $t \in [a, b]$ , such that the vector-functions  $\mathbf{v}_k(t)$  are continuous (their coordinates in some bases are continuous functions) and the system  $\mathbf{v}_1(t), \dots, \mathbf{v}_n(t)$  is a basis for all  $t \in [a, b]$ .
- Theorem 6.6.1: Two bases  $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$  and  $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  have the same orientation if and only if one of the bases can be continuously transformed to the other.

## Chapter 7

# Bilinear and Quadratic Forms

### 7.1 Notes

10/18: • **Bilinear form:** A function  $L : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  such that

$$L(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2, \mathbf{y}) = \alpha L(\mathbf{x}_1, \mathbf{y}) + \beta L(\mathbf{x}_2, \mathbf{y}) \quad L(\mathbf{x}, \alpha \mathbf{y}_1 + \beta \mathbf{y}_2) = \alpha L(\mathbf{x}, \mathbf{y}_1) + \beta L(\mathbf{x}, \mathbf{y}_2)$$

$$- L(\mathbf{x}, \mathbf{y}) = (A\mathbf{x}, \mathbf{y}).$$

• **Quadratic form:** A bilinear form  $L(\mathbf{x}, \mathbf{x})$ .

$$- (\mathbf{x}, \mathbf{x}) \text{ is a polynomial of degree 2 in } \mathbf{x}_1, \dots, \mathbf{x}_n:$$

$$L(\lambda \mathbf{x}, \lambda \mathbf{x}) = (\lambda \mathbf{x}, \lambda \mathbf{x}) = \lambda^2 (\mathbf{x}, \mathbf{x})$$

• We have that

$$(A\mathbf{x}, \mathbf{x}) = (A\lambda \mathbf{x}, \lambda \mathbf{x}) = \lambda^2 (A\mathbf{x}, \mathbf{x}) = \sum_{j,i=1}^n \alpha_{j,i} \mathbf{x}_i \mathbf{x}_j$$

• The general form of a quadratic form:

$$- \text{Can any quadratic form on } \mathbb{R}^n \text{ be written as } (A\mathbf{x}, \mathbf{x})?$$

10/20: • Bilinear forms are linear in each argument when keeping the other fixed.

• Quadratic forms  $Q(\mathbf{x}) = L(\mathbf{x}, \mathbf{x})$  are quadratic polynomials in the coordinates of  $x$ .

$$- \text{In particular, } Q(\lambda \mathbf{x}) = |\lambda|^2 Q(\mathbf{x}).$$

• If  $Q$  quadratic is real, then  $Q(\mathbf{x}) = (A\mathbf{x}, \mathbf{x})$  where  $A$  is some square matrix.

$$- \text{If } \mathbf{e}_1, \dots, \mathbf{e}_n \text{ is an orthonormal basis of } \mathbb{R}^n, \text{ then there exists a unique } A = A^* \text{ such that } (A)_{ij} = L(\mathbf{e}_i, \mathbf{e}_j).$$

$$- \text{Keeping } \mathbf{x} = \sum_{i=1}^n \mathbf{x}_i \mathbf{e}_i \text{ fixed, we have}$$

$$\begin{aligned} Q(\mathbf{x}) &= L(\mathbf{x}, \mathbf{x}) \\ &= L\left(\sum_{i=1}^n \mathbf{x}_i \mathbf{e}_i, \sum_{j=1}^n \mathbf{x}_j \mathbf{e}_j\right) \\ &= \sum_{i=1}^n \mathbf{x}_i L\left(\mathbf{e}_i, \sum_{j=1}^n \mathbf{x}_j \mathbf{e}_j\right) \\ &= \sum_{i,j=1}^n \mathbf{x}_i \mathbf{x}_j \underbrace{L(\mathbf{e}_i, \mathbf{e}_j)}_{A_{ij}} \end{aligned}$$

- We have that

$$\begin{aligned}
 (A\mathbf{x}, \mathbf{x}) &= (UDU^{-1}\mathbf{x}, \mathbf{x}) \\
 &= (DU^{-1}\mathbf{x}, U^{-1}\mathbf{x}) \\
 &= \sum_{i=1}^n \lambda_i \underbrace{(U^{-1}\mathbf{x})_i}_{\mathbf{y}_i} \underbrace{(U^{-1}\mathbf{x})_i}_{\mathbf{y}_i}
 \end{aligned}$$

- Can we characterize the set  $\{\mathbf{x} : (A\mathbf{x}, \mathbf{x}) = 1\}$ ?
  - Note that this set is equivalent to  $\{\mathbf{y} : (D\mathbf{y}, \mathbf{y}) = 1\}$  by the above. This set is a rotation of the previous one. Ellipse?
- Positive quadratic form:
  - $Q$  is positive definite if  $Q(\mathbf{x}) > 0$  for all  $\mathbf{x} \neq \mathbf{0}$  and  $Q$  is positive semidefinite if  $Q(\mathbf{x}) \geq 0$  for all  $\mathbf{x} \neq \mathbf{0}$ .
  - Take a self-adjoint matrix  $A = A^*$ . It is positive definite if  $Q(\mathbf{x}) = (A\mathbf{x}, \mathbf{x})$  is positive definite.
- Theorem: If  $A = A^*$ , then
  1.  $A$  is positive definite if and only if all eigenvalues of  $A$  are positive.
  2.  $A$  is positive semidefinite if and only if all eigenvalues of  $A$  are nonnegative.
  3.  $A$  is negative semidefinite if and only if all eigenvalues of  $A$  are nonpositive.
  4.  $A$  is negative definite if and only if all eigenvalues of  $A$  are negative.
  5.  $A$  is indefinite if and only if the eigenvalues of  $A$  have positive and negative values.
- Theorem:  $A = A^*$  is positive definite iff  $\det A_k > 0$  for all  $k = 1, \dots, n$  where  $A_k$  is the upper left  $k \times k$  submatrix.
- Minimax representation of eigenvalues of a self-adjoint  $A$ .
  - Let  $E$  be a subspace of  $X$  where  $\dim X < \infty$ . We define  $\text{codim}(E) = \dim E^\perp$ .
  - Thus,  $\dim E + \text{codim } E = \dim X$ .
  - Theorem: Let  $A = A^*$ ,  $\lambda_1 \geq \dots \geq \lambda_n$  eigenvalues of  $A$ . Then

$$\lambda_k = \max_{\substack{E \text{ subspace} \\ \dim E = k}} \min_{\substack{\mathbf{x} \in E \\ \|\mathbf{x}\|=1}} (A\mathbf{x}, \mathbf{x}) = \min_{\substack{F \text{ subspace} \\ \text{codim } F = k-1}} \max_{\substack{\mathbf{x} \in F \\ \|\mathbf{x}\|=1}} (A\mathbf{x}, \mathbf{x})$$

- Proof:  $A$  diagonal equals  $(\lambda_1, \dots, \lambda_n)$ .
- An orthonormal basis of  $X$  such that  $\dim E = k$ ,  $\text{codim } F = k-1$ ,  $\dim F = n-k+1$ .
- There exists an  $\mathbf{x}_0 \neq \mathbf{0}$  such that  $\mathbf{x}_0 \in E \cap F$ .
- Note that if  $B = B^*$ , then the max and min of  $(B\mathbf{x}, \mathbf{x})$  over the unit sphere is the maximal and minimal eigenvalue of  $B$ .
- Thus,

$$\min_{\substack{\mathbf{x} \in E \\ \|\mathbf{x}\|=1}} (A\mathbf{x}, \mathbf{x}) \leq (A\mathbf{x}_0, \mathbf{x}_0) \leq \max_{\substack{\mathbf{x} \in F \\ \|\mathbf{x}\|=1}} (A\mathbf{x}, \mathbf{x})$$

- This is true for any  $E, F$  subspaces.  $\dim E = k$ ,  $\text{codim } F = k-1$ ,  $E_0 = \text{span}(\mathbf{e}_1, \dots, \mathbf{e}_k)$  and  $F_0 = \text{span}(\mathbf{e}_k, \dots, \mathbf{e}_n)$ .
- Thus,

$$\min_{\substack{E_0 \\ \|\mathbf{x}\|=1}} (A\mathbf{x}, \mathbf{x}) = \lambda_k = \max_{\substack{F_0 \\ \|\mathbf{x}\|=1}} (A\mathbf{x}, \mathbf{x})$$

■ Additionally,

$$\lambda_{k_1} \leq \max_{\dim E=k} \min_{\mathbf{x}} (A\mathbf{x}, \mathbf{x}) \leq \min_{\text{codim } F=k-1} \max_{\mathbf{x}} (A\mathbf{x}, \mathbf{x}) \leq \lambda_k$$

- Corollary: Let  $A = A^* = (a_{jk})_{1 \leq j, k \leq n}$  with eigenvalues  $\lambda_1, \dots, \lambda_n$  listed in decreasing order. Let  $\tilde{A} = (a_{j,k})_{1 \leq j, k \leq n-1}$  with eigenvalues  $\mu_1, \dots, \mu_{n-1}$  listed in decreasing order. Then  $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \geq \mu_{n-1} \geq \lambda_n$ .
  - Consider  $(A\mathbf{x}, \mathbf{x})$  on  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ , but then restrict yourself to  $\mathbf{x} \in \mathbb{R}^{n-1}$  on  $\{\mathbf{e}_1, \dots, \mathbf{e}_{n-1}\}$ .

## 7.2 Chapter 7: Bilinear and Quadratic Forms

From Treil (2017).

- 10/25: • **Bilinear form** (on  $\mathbb{R}^n$ ): A function  $L(\mathbf{x}, \mathbf{y})$  of two arguments  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  that is linear in each argument.
- Linearity in each argument:

$$L(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2, \mathbf{y}) = \alpha L(\mathbf{x}_1, \mathbf{y}) + \beta L(\mathbf{x}_2, \mathbf{y}) \quad L(\mathbf{x}, \alpha\mathbf{y}_1 + \beta\mathbf{y}_2) = \alpha L(\mathbf{x}, \mathbf{y}_1) + \beta L(\mathbf{x}, \mathbf{y}_2)$$

- If  $\mathbf{x} = (x_1, \dots, x_n)^T$  and  $\mathbf{y} = (y_1, \dots, y_n)^T$ , then

$$\begin{aligned} L(\mathbf{x}, \mathbf{y}) &= \sum_{j,k=1}^n a_{j,k} x_k y_j \\ &= (A\mathbf{x}, \mathbf{y}) \\ &= \mathbf{y}^T A\mathbf{x} \end{aligned}$$

where

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix}$$

- $A$  is uniquely determined by  $L$ .
- **Quadratic form** (on  $\mathbb{R}^n$ ): The diagonal of a bilinear form  $L$ , i.e., a bilinear form  $Q[\mathbf{x}] = L(\mathbf{x}, \mathbf{x}) = (A\mathbf{x}, \mathbf{x})$ .
  - Alternatively: A homogeneous polynomial of degree 2, i.e., a polynomial in  $x_1, \dots, x_n$  with only  $ax_k^2$  and  $cx_jx_k$  terms.
- There are infinitely many ways to write a quadratic form as  $(A\mathbf{x}, \mathbf{x})$ .
  - However, there is a unique representation  $(A\mathbf{x}, \mathbf{x})$  where  $A$  is a (real) symmetric matrix.
- **Quadratic form** (on  $\mathbb{C}^n$ ): A function of the form  $Q[\mathbf{x}] = (A\mathbf{x}, \mathbf{x})$  where  $A$  is self-adjoint.
- Lemma 7.1.1: Let  $(A\mathbf{x}, \mathbf{x})$  be real for all  $\mathbf{x} \in \mathbb{C}^n$ . Then  $A = A^*$ .
- To classify quadratic forms, consider the set of points  $\mathbf{x} \in \mathbb{R}^n$  defined by  $Q[\mathbf{x}] = 1$  for some quadratic form  $Q$ .
  - If the matrix of  $Q$  is diagonal, i.e.,  $Q[\mathbf{x}] = a_1x_1^2 + \dots + a_nx_n^2$ , then the set of points can easily be visualized.
- The standard method of diagonalizing a quadratic form is change of variables.
- Orthogonal diagonalization.

- Let  $Q[\mathbf{x}] = (A\mathbf{x}, \mathbf{x})$  in  $\mathbb{F}^n$ .
- Suppose  $\mathbf{y} = S^{-1}\mathbf{x}$  where  $S$  is an invertible  $n \times n$  matrix. Then

$$Q[\mathbf{x}] = Q[S\mathbf{y}] = (AS\mathbf{y}, S\mathbf{y}) = (S^*AS\mathbf{y}, \mathbf{y})$$

so in the new variables  $\mathbf{y}$ , the quadratic form has matrix  $S^*AS$ .

- Thus, we can let  $A = UDU^*$ , choose  $D = U^*AU$  as our new (diagonal) matrix, and let this matrix act on the variables  $\mathbf{y} = U^*\mathbf{x}$ .

- Non-orthogonal diagonalization:

- Completing the square:
  - Eliminate all  $x_i x_j$  terms by completing the square. Then substitute in a  $y_k$  for each squared term.
- Row/column operations:
  - Augment  $(A|I)$ . Row reduce  $A$  to  $D$ . Then  $I \rightarrow S^*$ .

10/28:

- **Sylvester's Law of Inertia:** For a Hermitian matrix  $A$  (i.e., for a quadratic form  $Q[\mathbf{x}] = (A\mathbf{x}, \mathbf{x})$ ) and any of its diagonalizations  $D = S^*AS$ , the number of positive, negative, and zero diagonal entries of  $D$  depends only on  $A$ , but not on a particular choice of diagonalization.
- **Positive** (subspace  $E \subset \mathbb{F}^n$  corresponding to  $A$ ): A subspace  $E$  such that  $(A\mathbf{x}, \mathbf{x}) > 0$  for all nonzero  $\mathbf{x} \in E$ . Also known as **A-positive**.
- **Negative** (subspace  $E \subset \mathbb{F}^n$  corresponding to  $A$ ): A subspace  $E$  such that  $(A\mathbf{x}, \mathbf{x}) < 0$  for all nonzero  $\mathbf{x} \in E$ . Also known as **A-negative**.
- **Neutral** (subspace  $E \subset \mathbb{F}^n$  corresponding to  $A$ ): A subspace  $E$  such that  $(A\mathbf{x}, \mathbf{x}) = 0$  for all nonzero  $\mathbf{x} \in E$ . Also known as **A-neutral**.
- Theorem 7.3.1: Let  $A$  be an  $n \times n$  Hermitian matrix, and let  $D = S^*AS$  be its diagonalization by an invertible matrix  $S$ . Then the number of positive (resp. negative) diagonal entries of  $D$  coincides with the maximal dimension of an  $A$ -positive (resp.  $A$ -negative) subspace.
- Lemma 7.3.2: Let  $D = \text{diag}\{\lambda_1, \dots, \lambda_n\}$ . Then the number of positive (resp. negative) diagonal entries of  $D$  coincides with the maximal dimension of a  $D$ -positive (resp.  $D$ -negative) subspace.
- **Positive definite** (quadratic form  $Q$ ): A quadratic form  $Q$  such that  $Q[\mathbf{x}] > 0$  for all  $\mathbf{x} \neq \mathbf{0}$ .
- **Positive semidefinite** (quadratic form  $Q$ ): A quadratic form  $Q$  such that  $Q[\mathbf{x}] \geq 0$  for all  $\mathbf{x}$ .
- **Negative definite** (quadratic form  $Q$ ): A quadratic form  $Q$  such that  $Q[\mathbf{x}] < 0$  for all  $\mathbf{x} \neq \mathbf{0}$ .
- **Negative semidefinite** (quadratic form  $Q$ ): A quadratic form  $Q$  such that  $Q[\mathbf{x}] \leq 0$  for all  $\mathbf{x}$ .
- **Indefinite** (quadratic form  $Q$ ): A quadratic form  $Q$  for which there exist  $\mathbf{x}_1, \mathbf{x}_2$  such that  $Q[\mathbf{x}_1] > 0$  and  $Q[\mathbf{x}_2] < 0$ .
- **Positive definite** (Hermitian matrix  $A$ ): A matrix  $A$  for which the corresponding quadratic form  $Q[\mathbf{x}] = (A\mathbf{x}, \mathbf{x})$  is positive definite.
  - Positive semidefinite, negative definite, negative semidefinite, and indefinite Hermitian matrices are defined similarly.
- Theorem 7.4.1: Let  $A = A^*$ . Then
  1.  $A$  is positive definite iff all eigenvalues of  $A$  are positive.
  2.  $A$  is positive semidefinite iff all eigenvalues of  $A$  are non-negative.

3.  $A$  is negative definite iff all eigenvalues of  $A$  are negative.
  4.  $A$  is negative semidefinite iff all eigenvalues of  $A$  are non-positive.
  5.  $A$  is indefinite iff it has both positive and negative eigenvalues.
- **Upper left submatrix** (of  $A$ ): A  $k \times k$  matrix  $A_k$  composed of all entries of  $A$  from row (column) 1 through  $k$  in the same arrangement.
  - Theorem 7.4.2 (Sylvester's Criterion of Positivity): A matrix  $A = A^*$  is positive definite if and only if  $\det A_k > 0$  for all  $k = 1, \dots, n$ .
    - To check if a matrix  $A$  is negative definite, check that the matrix  $-A$  is positive definite.
  - Theorem 7.4.3 (Minimax characterization of eigenvalues): Let  $A = A^*$  be an  $n \times n$  matrix and let  $\lambda_1 \geq \dots \geq \lambda_n$  be its eigenvalues taken in decreasing order. Then

$$\lambda_k = \max_{E: \dim E = k} \min_{\mathbf{x} \in E: \|\mathbf{x}\|=1} (A\mathbf{x}, \mathbf{x}) = \min_{F: \text{codim } F = k-1} \max_{\mathbf{x} \in F: \|\mathbf{x}\|=1} (A\mathbf{x}, \mathbf{x})$$

- Corollary 7.4.4 (Intertwining of eigenvalues): Let  $A = A^* = \{a_{j,k}\}_{j,k=1}^n$  be a self-adjoint matrix and let  $\tilde{A} = \{a_{j,k}\}_{j,k=1}^{n-1}$  be its submatrix of size  $(n-1) \times (n-1)$ . Let  $\lambda_1, \dots, \lambda_n$  and  $\mu_1, \dots, \mu_{n-1}$  be the eigenvalues of  $A$  and  $\tilde{A}$  respectively, taken in decreasing order. Then

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$$



# Chapter 8

## Dual Spaces and Tensors

### 8.1 Notes

10/22: • **Functional:** A linear bounded map  $L : H \rightarrow F$ , where  $H$  is finite dimensional (equivalent to  $\mathbb{R}^n$ ).

• **Dual space:** The set of bounded linear functionals on  $H$ . Denoted by  $H'$ ,  $H^*$ .

• If  $l \leq p < \infty$ , then

$$l^p = \left\{ (a_n)_{n \in \mathbb{N}} : \sum_{n=1}^{\infty} |a_n|^p < \infty \right\}$$

• Back to finite dimensions,  $H' \approx \mathbb{R}^n$ .

• Let  $\mathbf{a}_1, \dots, \mathbf{a}_n$  be a basis of  $H$ . Then  $L\mathbf{x} = (L\mathbf{a}_1, \dots, L\mathbf{a}_n) \approx \mathbb{R}^n$ .

• Let  $L((a_n)_{n \in \mathbb{N}}) = \sum_{n=1}^{\infty} a_n b_n$ . Then  $L((a_n)_{n \in \mathbb{N}})$  will be bounded if and only if  $(b_n)_{n \in \mathbb{N}} \in l^q$  where  $1 < p < q$  where  $\frac{1}{q} + \frac{1}{p} = 1$ .

• **Young's inequality:** The statement

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

• We have  $|\sum a_n b_n| \leq \|a_n\|_p \|b_n\|_q$ .

• Conclusion:

$$\sum \frac{|a_n| |b_n|}{\|a_n\|_p \|b_n\|_q} = 1$$

• We can define  $H''$ , too. This contains linear functionals on  $H'$ .

• We know that  $L(x) = \langle x, L \rangle = x(L)$ .  $x \in H''$ .

• Riesz representation theorem: Let  $H$  have an inner product.  $L \in H'$  if and only if there exists a unique  $y \in H$  such that  $L(x) = (x, y)$ .

– Gives us a way to identify all bounded linear functionals on  $H$ .

– In finite dimensions,  $L(x)$ , where  $x = \sum_1^n \alpha_i a_i$  gives us  $L(x) = \sum_1^n \alpha_i L(a_i)$ .

## 8.2 Chapter 8: Dual Spaces and Tensors

10/28:

- Linear functionals are denoted by  $L$ .
  - $L$  is given by a  $1 \times n$  matrix denoted by  $[L]$ .
- The collection of all  $[L]$  (the dual space) is isomorphic to  $\mathbb{R}^n$  via  $[L] \mapsto [L]^T$ .
  - However, the objects are different: Let  $[I]_{\mathcal{B}\mathcal{A}}$  be the change of coordinates matrix in  $\mathbb{R}^n$ . We thus have that

$$[\mathbf{v}]_{\mathcal{B}} = [I]_{\mathcal{B}\mathcal{A}}[\mathbf{v}]_{\mathcal{A}}$$

but we also have that

$$[L]_{\mathcal{B}} = [L]_{\mathcal{A}}[I]_{\mathcal{A}\mathcal{B}}$$

so that

$$[L]_{\mathcal{B}}^T = ([L]_{\mathcal{A}}[I]_{\mathcal{A}\mathcal{B}})^T = [I]_{\mathcal{A}\mathcal{B}}^T [L]_{\mathcal{A}}^T$$

- Essentially, “if  $S$  is the change of coordinate matrix in  $X \dots$  then the change of coordinate matrix in the dual space  $X'$  is  $(S^{-1})^T$ ” (Treil, 2017, p. 219).
- Lemma 8.1.3: Let  $\mathbf{v} \in V$ . If  $L(\mathbf{v}) = 0$  for all  $L \in V'$ , then  $\mathbf{v} = \mathbf{0}$ . As a corollary, if  $L(\mathbf{v}_1) = L(\mathbf{v}_2)$  for all  $L \in V'$ , then  $\mathbf{v}_1 = \mathbf{v}_2$ .
- The second dual  $V''$  is canonically (i.e., in a natural way) isomorphic to  $V$ .
- **Dual basis** (to  $\mathbf{b}_1, \dots, \mathbf{b}_n \in V$ ): The system of vectors  $\mathbf{b}'_1, \dots, \mathbf{b}'_n \in V'$  uniquely defined by the following equation. *Also known as biorthogonal basis.*

$$\mathbf{b}'_k(\mathbf{b}_j) = \delta_{kj}$$

- The  $k^{\text{th}}$  coordinate of a vector  $\mathbf{v}$  in a basis  $\mathbf{b}_1, \dots, \mathbf{b}_n$  is  $\mathbf{b}'_k(\mathbf{v})$ .
  - This is a baby version of the **abstract non-orthogonal Fourier decomposition** of  $\mathbf{v}$ .
- Theorem 8.2.1 (Riesz representation theorem): Let  $H$  be an inner product space. Given a linear functional  $L$  on  $H$ , there exists a unique vector  $\mathbf{y} \in H$  such that

$$L(\mathbf{v}) = (\mathbf{v}, \mathbf{y})$$

for all  $\mathbf{v} \in H$ .

- If  $V$  is a real inner product space, we can define an isomorphism from  $V$  to  $V'$  by  $\mathbf{y} \mapsto L_{\mathbf{y}} = (\mathbf{v}, \mathbf{y})$ .
  - If  $V$  is complex, this function is not linear since if  $\alpha$  is complex,

$$L_{\alpha\mathbf{y}}(\mathbf{v}) = (\mathbf{v}, \alpha\mathbf{y}) = \bar{\alpha}(\mathbf{v}, \mathbf{y}) = \bar{\alpha}L_{\mathbf{y}}(\mathbf{v})$$

- It follows by such a mapping that  $\mathbf{b}'_k = \mathbf{b}_k$  for each  $k$ .
- **Conjugate linear** (transformation): A transformation  $T$  such that

$$T(\alpha\mathbf{x} + \beta\mathbf{y}) = \bar{\alpha}T\mathbf{x} + \bar{\beta}T\mathbf{y}$$

- It is customary to write outputs of linear functionals  $L(\mathbf{v})$  in the form  $\langle \mathbf{v}, L \rangle$ .
  - This expression is linear in both arguments, unlike the inner product.
- Defines the dual transformation as the unique transformation such that

$$\langle A\mathbf{x}, \mathbf{y}' \rangle = \langle \mathbf{x}, A'\mathbf{y} \rangle$$

for all  $\mathbf{x} \in X, \mathbf{y}' \in Y'$ .

- It's matrix in the standard bases equals  $A^T$ .
- Annihilators are denoted by  $E^\perp$  here.
- Proposition 8.3.6: The annihilator of the annihilator of  $E$  equals  $E$ .
- Let  $A : X \rightarrow Y$  be an operator acting from one vector space to another. Then
  1.  $\ker A' = (\text{range } A)^\perp$ .
  2.  $\ker A = (\text{range } A')^\perp$ .
  3.  $\text{range } A = (\ker A')^\perp$ .
  4.  $\text{range } A' = (\ker A)^\perp$ .

## Chapter 9

# Advanced Spectral Theory

### 9.1 Notes

- 10/22:
- Let  $p(z) = \sum_{i=0}^n a_i z^i$  be a polynomial. Let  $A$  be an  $n \times n$  matrix. We let  $p(A) = \sum_{i=0}^n a_i A^i$ .
  - Theorem: If  $A$  is an  $n \times n$  and  $p(\lambda) = \det(A - \lambda I)$ , then  $p(A) = 0$ .
    - We know that  $p(\lambda) = a(z - \lambda_1) \cdots (z - \lambda_n)$  where  $\lambda_1, \dots, \lambda_n$  are the eigenvalues.
    - Thus  $p(A) = a(A - \lambda_1 I) \cdots (A - \lambda_n I)$ .
    - If you are in  $\mathbb{R}^n$  and have this property, you can factorize your matrix.
    - Thus,  $p(A)\mathbf{x} = \mathbf{0}$  since  $\mathbf{x}$  can be decomposed into a linear combination of eigenvectors of  $A$ , which will be taken to 0 one by one by the terms of  $p(A)$ .
  - $\sigma(B) = \{\text{eigenvalues of } B\}$  is known as the **spectrum** of  $B$ .
  - If  $p$  is an arbitrary polynomial and  $A$  is  $n \times n$ , then  $\mu$  is an eigenvalue of  $p(A)$  if and only if  $\mu = p(\lambda)$  where  $\lambda$  is an eigenvalue of  $A$ . In essence,  $\sigma(p(A)) = p(\sigma(A))$ .
  - Chapter 9 will not be on the exam. We don't have to know the generalization to infinite dimensional spaces.
- 10/25:
- If  $A$  is an  $n \times n$  square matrix and  $p(\lambda) = \det(A - \lambda I)$ , then  $p(A) = 0$ .
    - Proof: WLOG, let  $A$  be an upper triangular matrix with diagonal entries equal to the eigenvalues.
    - Think of  $p(z) = (-1)^n (z - \lambda_1) \cdots (z - \lambda_n)$ .
    - Thus,  $p(A) = (-1)^n (A - \lambda_1 I) \cdots (A - \lambda_n I)$ .
    - WTS:  $p(A)\mathbf{x} = 0$  for all  $\mathbf{x} \in V$ .
    - Let  $E_k = \text{span}(e_1, \dots, e_k)$  be the span of the first  $k$  eigenvectors of  $A$ , where  $e_1, \dots, e_n$  is a standard basis in  $\mathbb{C}^n$ .
    - $A$  triangular implies  $AE_k \subset E_k$ . Thus,  $(A - \lambda I)E_k \subset E_k$ , so  $E_k$  is invariant under  $A - \lambda I$  for all  $\lambda$ .
    - If we apply  $A - \lambda_k I$  to a vector in  $E_k$ , we are left with a vector in  $E_{k-1}$ .
    - Thus, if we apply  $\prod_{k=1}^n (A - \lambda_k I) = p(A)$  to any vector in  $E_n = V$ , we will kill it piece by piece down to zero.
  - Let  $A$  be a square  $n \times n$  matrix. Then  $p$  an arbitrary polynomial implies  $\sigma(p(A)) = p(\sigma(A))$ . (Any eigenvalue  $\mu$  of  $p(A)$  is  $\mu = p(\lambda)$ , where  $\lambda$  is an eigenvalue of  $A$ .)
    - Shows that polynomials of operators commute.

- Proof: Let  $\lambda$  be an eigenvalue of  $A$ . We want to show that  $p(\lambda)$  is an eigenvalue of  $p(A)$ . This is obvious since  $A\mathbf{x} = \lambda\mathbf{x}$  for some  $\mathbf{x}$ , so  $A^k\mathbf{x} = \lambda^k\mathbf{x}$ , so in particular,  $p(A)\mathbf{x} = p(\lambda)\mathbf{x}$ .
- On the other hand, if  $\mu$  is an eigenvalue of  $p(A)$ , we want to show that there exists  $\lambda \in \sigma(A)$  such that  $\mu = p(\lambda)$ .
- Consider  $q(z) = p(z) - \mu$ . Then  $q(A) = p(A) - \mu I$ . Since  $\mu$  is an eigenvalue of  $p(A)$ ,  $q(A)$  is not invertible.
- Thus,  $q(z) = (-1)^n(z - z_1) \cdots (z - z_n)$  and  $q(A) = (-1)^n(A - z_1 I) \cdots (A - z_n I)$ .
- But  $q(A)$  is not invertible, so one of the  $A - z_k I$  is not invertible. Take  $z_k$  such that  $A - z_k I$  is not invertible. Then  $z_k \in \sigma(A)$ . It follows that  $q(z_k) = p(z_k) - \mu = 0$ .
- If  $A$  is  $n \times n$ ,  $\lambda_1, \dots, \lambda_n$  are its eigenvalues,  $p$  is a polynomial, then  $p(A)$  is invertible if and only if  $p(\lambda_k) \neq 0$  for each  $k = 1, \dots, n$ .
  - This is an immediate corollary to the previous result.
- We now build up to the **generalized eigenspace**, which is related to some “geometric” properties of the algebraic multiplicity of an eigenvalue.
- If  $A : V \rightarrow V$  is a linear operator and  $E \subset V$  is a subspace,  $E$  is  $A$ -invariant if  $AE \subset E$ .
- Facts:
  - If  $E$  is  $A$ -invariant,  $E$  is  $A^k$ -invariant.
  - Thus,  $E$  is  $p(A)$ -invariant.
- Consider the restriction map  $A|_E$ .
- $A$  has a block-diagonalized matrix where each block corresponds to the generalized eigenvectors of a generalized eigenvalue of  $A$ .
  - Let  $E_1, \dots, E_r$  be a **basis of invariant subspaces**.
  - Let  $A_k = A|_{E_k}$ . Then the  $A_k$ ’s act independently of each other.
- **Generalized eigenvector** (of  $A$ ): A vector  $\mathbf{v}$  corresponding to an eigenvalue  $\lambda$  if there exists  $k \geq 1$  such that  $(A - \lambda I)^k \mathbf{v} = \mathbf{0}$ .
- **Generalized eigenspace**: The set  $E_\lambda$  of all of the generalized eigenvectors of  $\lambda$ . *Given by*

$$E_\lambda = \bigcup_{k \geq 1} \ker(A - \lambda I)^k$$

- $E_\lambda$  is a linear subspace of  $V$ .
- **Degree** (of  $\lambda$ ): The smallest number  $k$  such that increasing  $k$  any more does not add further vectors to the generalized eigenspace. *Denoted by  $d(\lambda)$ . Also known as **depth**.*
  - Symbolically,  $d(\lambda)$  is the smallest number such that

$$E_\lambda = \bigcup_{k=1}^{d(\lambda)} \ker(A - \lambda I)^k$$

- Start working through the first 25 problems of Rudin (1976) (his metric spaces problems).
- 10/27: • Jordan form.
- Reviews build up to generalized eigenvectors.

- Theorem: If  $\sigma(A) = \{\lambda_1, \dots, \lambda_n\}$  and  $E_1, \dots, E_n$  are the corresponding generalized eigenspaces, then  $E_1, \dots, E_n$  is a basis of subspaces of  $U$ , i.e.,  $V = \oplus_k E_k$ .
- Corollary:  $A : V \rightarrow V$  can be represented as  $A = D + N$  where  $D$  is diagonalizable and  $N$  is nilpotent and  $ND = DN$ .

– Proof: Consider the basis of generalized eigenspaces known to exist from the theorem. Then  $A = \text{diag}\{A_1, \dots, A_r\}$ .

– Let

$$N_k = A_k - \lambda_k I_{E_k}$$

This is nilpotent.

– Then let

$$D = \text{diag}\{\lambda_1 I_{E_1}, \dots, \lambda_n I_{E_n}\}$$

– These two matrices satisfy the necessary properties.

- Let  $\dot{\mathbf{x}} = A\mathbf{x}$ .

– Let  $\mathbf{x}(t) = e^{tA}$ , where

$$e^{tA} = \sum \frac{(tA)^k}{k!}$$

–  $\|e^{tA}\| \leq \sum \frac{\|A^k\|}{k!} = \sum \frac{\|A\|^k}{k!}$ .

– Let  $p$  be a polynomial of degree  $k$ . Then

$$p(a+x) = \sum_{k=0}^d \frac{p^{(k)}(a)}{k!} x^k$$

– If  $A = D + N$ , then...

- Nilpotent operators:

– Let  $A = \text{diag}\{A_1, \dots, A_r\}$ .

– We know that  $A_k = \lambda_k I_{E_k} + N_k$  for each  $k$ .

– Every nilpotent  $N$  can be written in the form

$$\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$$

10/28:

- The exam is long but not that hard. The only question is will you do good or very good.
- Revise the previous two homeworks, especially the last two.
- No justification for any of the true/false questions. Just circle T or F.
  - There are four problems. One is T/F (with multiple subparts); the other three are problem problems (with subparts, too).
  - Some of the questions will take you 2 seconds. Some you've already seen in the PSets.
  - The exam is supposed to be boring.
- Calculators?
  - No calculators needed. Calculators are for chem/physics exams.

- Not a lot of computation.
- 50 minutes.
- Chloe will be proctoring.
- Remember the determinant of “special” matrices.
  - $|\det U| = 1$  if  $U$  is unitary.
  - $\det A = \pm 1$  if  $A$  is orthogonal.
  - Make a list of matrix types that are automatically diagonalizable.
  - Determinant is the product of the eigenvalues.
  - Determinant of  $A$  is equal to the conjugate of the determinant of  $A^*$ .
- Most of the exercises use the inner product.
  - Whenever you had something to prove about eigenvalues or eigenbasis, you went through diagonalization or SVD or the inner product or polar decomposition.
  - Proving eigenvalues of self-adjoint matrices are real w/ the inner product.
- Eigenvalues/eigenvectors of a projection.
  - It’s implied that it’s asking you the multiplicities!!!
- Know useful facts but have an idea how to prove them as well.
- Recommends against shorthanding in the exams.
- Not grading on clarity (since the exam is long).
- Max and min are for when you’re sure something will be attained. Otherwise use sup and inf.

## 9.2 Chapter 9: Advanced Spectral Theory

- 10/28:
- Theorem 9.1.1 (Cayley-Hamilton): If  $p$  is the characteristic polynomial of  $A$ ,  $p(A) = 0$ .
  - Theorem 9.2.1 (Spectral Mapping Theorem): For a square matrix  $A$  and an arbitrary polynomial  $p$ ,  $\sigma(p(A)) = p(\sigma(A))$ . In other words,  $\mu$  is an eigenvalue of  $p(A)$  if and only if  $\mu = p(\lambda)$  for some eigenvalue  $\lambda$  of  $A$ .
  - Corollary 9.2.2: Let  $A$  be a square matrix with eigenvalues  $\lambda_1, \dots, \lambda_n$  and let  $p$  be a polynomial. Then  $p(A)$  is invertible iff  $p(\lambda_k) \neq 0$  for all  $k = 1, \dots, n$ .
  - Algebraic multiplicity is the dimension of the corresponding generalized eigenspace.

## Part II

# Point Set Topology of Metric Spaces



# Chapter 1

## The Real and Complex Number Systems

### 1.1 Notes

- 11/1:
- Spent a lot of time trying to cheer us up regarding the midterm.
  - There may be some true/false on linear algebra on the final.
  - Facts:
    1.  $\sqrt{2}$  is irrational.
    2. Archimedes principle: If  $x > 0$  and  $y \in \mathbb{R}$ , then there exists  $n$  such that  $nx > y$ .
    3. If  $x > y$ , then there exists  $q \in \mathbb{Q}$  such that  $x > q > y$ .

### 1.2 Chapter 1: The Real and Complex Number Systems

*From Rudin (1976).*

- 11/6:
- Rudin (1976) presents several interesting proofs throughout this section that may be of interest later by means of their divergence from the ones with which I am familiar.
  - **Least-upper-bound property:** The property pertaining to a set  $S$  that if  $E \subset S$ ,  $E \neq \emptyset$ , and  $E$  is bounded above, then  $\sup E \in S$ .
    - For example,  $\mathbb{Q}$  does not have the least-upper-bound property.
    - The **greatest-lower-bound property** is analogously defined.
  - Theorem: Suppose  $S$  is an ordered set with the least-upper-bound property,  $B \subset S$  is nonempty, and  $B$  is bounded below. Let  $L$  be the set of all lower bounds of  $B$ . Then  $\alpha = \sup L$  exists in  $S$ , and  $\alpha = \inf B$ . In particular,  $\inf B$  exists in  $S$ .
    - Essentially, this theorem states that any set that satisfies the least-upper-bound property satisfies the greatest lower bound property.
  - **Existence theorem:** There exists an ordered field  $\mathbb{R}$  which has the least-upper-bound property. Moreover,  $\mathbb{R}$  contains  $\mathbb{Q}$  as a subfield.
    - The second statement implies that the operations of addition and multiplication on  $\mathbb{R}$ , when applied to  $\mathbb{Q}$ , coincide with the operations of addition and multiplication on  $\mathbb{Q}$ .

- **Archimedean property** (of  $\mathbb{R}$ ): If  $x \in \mathbb{R}$ ,  $y \in \mathbb{R}$ , and  $x > 0$ , then there is a positive integer  $n$  such that  $nx > y$ .
- Rudin (1976) proves several theorems about the real numbers from the least-upper-bound property as opposed to the traditional construction of the real numbers.
- Introduces the decimal system.
- **Finite real number system**: That which has been defined thus far.
- **Extended real number system**: The set  $\mathbb{R} \cup \{+\infty, -\infty\}$  where  $+\infty, -\infty$  obey the expected properties (supremum [resp. infimum] of every set,  $x + \infty = \infty$ , etc.).
- Defines the complex field axiomatically with complex numbers in the form  $(a, b)$  for  $a, b \in \mathbb{R}$ .
  - Notes that the real numbers form a subfield of the complex field.
  - Defines  $i = (0, 1)$ , proves  $i^2 = -1$ , proves  $a + bi = (a, b)$ .
- **Schwarz inequality**: If  $a_1, \dots, a_n$  and  $b_1, \dots, b_n$  are complex numbers, then

$$\left| \sum_{j=1}^n a_j \bar{b}_j \right|^2 \leq \sum_{j=1}^n |a_j|^2 \sum_{j=1}^n |b_j|^2$$

- **Euclidean  $k$ -space**: The vector space  $\mathbb{R}^k$  over the real field.

# Chapter 2

## Basic Topology

### 2.1 Notes

- 11/1:
- Equivalence relationships are denoted  $A \sim B$ .
    - These are...
      - Reflexive ( $A \sim A$ ).
      - Symmetric ( $A \sim B \iff B \sim A$ ).
      - Transitive ( $A \sim B \ \& \ B \sim C \implies A \sim C$ ).
    - Equivalence relations give rise to equivalence classes.
  - **Countable** (set  $A$ ): A set  $A$  such that  $A \sim \mathbb{N}$ , in the sense that there exists a one-to-one and onto map from  $\mathbb{N} \rightarrow A$ .
    - Alternatively,  $A$  can be written in the form  $A = \{f(n) : n \in \mathbb{N}\}$ .
  - **Finite countable** vs. **infinite countable** (see Rudin (1976)).
  - $\mathbb{N}$  denotes the natural numbers.
  - $\mathbb{N}_0$  denotes the natural numbers including 0.
  - $\mathbb{Z}$  denotes the integers.
  - We know that  $\mathbb{N} \sim \mathbb{Z}$ : Let  $f : \mathbb{N} \rightarrow \mathbb{Z}$  be defined by

$$f(n) = \begin{cases} \frac{n}{2} & n \text{ even} \\ \frac{n-1}{2} & n \text{ odd} \end{cases}$$

- More facts.
  1. Every subset of a countable set is countable.
  2. Unions of countable sets are countable.
    - If the sets  $E_n$  for some finite list of numbers are countable, then  $\bigcup_n E_n$  is countable.
    - Soug goes over the diagonalization method of counting.
  3.  $n$ -fold Cartesian products of countable sets are countable (we induct on  $n$ ).
    - If  $A$  is countable and  $B$  is countable, then  $A \times B$  is countable.
    - If  $A$  is finite and to each  $\alpha \in A$  we assign a countable set  $E_\alpha$ ,  $\otimes_{\alpha \in A} E_\alpha$  is countable.
- **Metric space**: A space  $X$  along with a metric  $d : X \times X \rightarrow [0, \infty)$  such that

- $d(x, y) > 0$  iff  $x \neq y$ , and  $d(x, x) = 0$  iff  $x = 0$ .
- $d(x, y) = d(y, x)$ .
- $d(x, y) \leq d(x, z) + d(z, y)$ .

- Example ( $\mathbb{R}^n$ ):

- We may define  $d$  by

$$d(x, y) = \sqrt{\sum (x_i - y_i)^2}$$

- We can also define the  $p$ -metrics (recall normed spaces) with  $p$  where 2 is.

- Example ( $X_p = \{f : Y \rightarrow \mathbb{R} : 1 \leq p < \infty, \int_Y |f|^p dy < \infty\}$ ):

- This is  $\ell_p$ .
- Define

$$\|f - g\|_p = \left[ \int_Y |f - g|^p dy \right]^{1/p}$$

- Convergence:  $x_n \rightarrow x \iff d(x_n, x) \rightarrow 0$ .

- **Neighborhood**: The set of all points a distance less than  $r$  away from  $p$ . Denoted by  $N_r(p)$ . Given by

$$N_r(p) = \{q \in X : d(p, q) < r\}$$

- **Limit point** (of  $E$ ): A point  $p$  such that every neighborhood of  $p$  intersects  $E$  at a point other than  $p$ . Also known as **accumulation point**.

- Symbolically,

$$N_r(p) \cap (E \setminus \{p\}) \neq \emptyset$$

for all  $r > 0$ .

- **Isolated point** (of  $E$ ): A point  $p$  such that  $p \in E$  and  $p$  is not a limit point of  $E$ .

- **Closed** (set  $E$ ): A set  $E$  that contains all of its limit points.

- **Interior** (point  $p$ ): A point  $p$  such that there exists  $N_r(p) \subset E$ .

- **Open** (set  $E$ ): A set  $E$ , all points of which are interior points.

- **Perfect** (set  $E$ ): A set  $E$  that is closed and every point of  $E$  is a limit point of  $E$ .

- **Bounded** (set  $E$ ): There exists a number  $M$  and a  $y \in X$  such that  $E \subset \{p : d(p, y) \leq M\}$ .

- **Dense** (set  $E$  in  $X$ ): A set  $E$  such that every point of  $X$  is a limit point of  $E$  or a point of  $E$ , itself.

11/3:

- Every neighborhood is an open set.

- If  $p$  is a limit point of  $E$ , every neighborhood of  $p$  contains infinitely many points of  $E$ .

- Thus, a finite set cannot have a limit point.

- Prove by contradiction: Suppose there is a neighborhood that contains only finitely many points of  $E$ . Then the neighborhood with radius smaller than the distance to the closest point does not contain any points of  $E$ , a contradiction.

- $E$  is open iff  $E^{c[1]}$  is closed.

- Assume  $E^c$  closed. If  $p \in E$ , then  $p$  is not a limit point of  $E^c$ . It follows that there exists a neighborhood of  $p$  that is entirely contained within  $E$ , so  $p$  is interior, as desired.

---

<sup>1</sup>The complement of  $E$ .

- Suppose  $E$  is open. Let  $p$  be any limit point of  $E^c$ . Then  $p \in E^c$ .
- $F$  is closed iff  $F^c$  is open.
- If  $(G_\alpha)_{\alpha \in A}$  is a family of open sets in  $X$ , then the union is open.
  - Let  $p \in \bigcup_{\alpha \in A} G_\alpha$ . Then  $p \in G_\alpha$  for some  $\alpha \in A$ . It follows that  $p$  is an interior point of  $G_\alpha$ , so thus an interior point of the union of  $G_\alpha$  with everything else.
- Finite intersections of open sets are open.
  - In the infinite case  $\bigcap_{n \in \mathbb{N}} (-1/n, 1/n) = \{0\}$ , an intersection of infinitely many open sets is closed.
  - However, in the finite case, just consider the neighborhood with the smallest radius and take this one.
- The intersection of closed sets is closed.
- The union of finitely many closed sets is closed.
  - These follow from the previous two by De Morgan's rule.
- Let  $\bar{E} = E \cup E'$  where  $E'$  is the set of limit points of  $E$ .
- Let  $X$  be a metric space and  $E \subset X$ . Then
  1.  $\bar{E}$  is closed.
    - WTS:  $\bar{E}^c$  is open. Let  $p \in \bar{E}^c$ . Then  $p$  is neither in  $E$  nor is it a limit point of  $E$ . Thus, there exists a neighborhood of  $\bar{E}^c$  containing entirely points of  $\bar{E}^c$ . Therefore,  $\bar{E}^c$  is open, so  $\bar{E}$  is closed.
  2.  $E = \bar{E}$  iff  $E$  is closed.
    - Think  $p \in \bigcap G_\alpha$ ?
  3.  $\bar{E} \subset F$  for any closed  $F \supset E$ .
    - If  $E \subset F$ , then any limit point of  $E$  will be a limit point of  $F$ . Thus,  $E' \subset F'$ . Then  $\bar{E} = E \cup E' \subset F \cup F' = \bar{F} = F$  where the last equality holds because  $F$  is closed.
- Types of sets.

	Closed	Open	Perfect	Bounded
$\{z \in \mathbb{Q} :  z  < 1\}$	N	Y	N	Y
$\{z \in \mathbb{Q} :  z  \leq 1\}$	Y	N	Y	Y
Nonempty finite set	Y	N	N	Y
$\mathbb{Z}$	Y	N	N	N
$\{1/n : n \in \mathbb{N}\}$	N	N	N	Y
$\mathbb{R}^2$	Y	Y	Y	N
$(a, b)$	N	?	N	Y

Table 2.1: Types of sets.

- **Relatively open** (set  $E$  to  $Y$ ): A set  $E \subset Y \subset X$  such that if  $p \in E$ , then there exists a  $Y$ -neighborhood of  $E$  contained in  $E$ .
- Let  $N_r^X(p) = \{y \in X : d(y, p) < r\}$  be a neighborhood of  $p$  in  $X$ , and let  $N_r^Y(p) = \{y \in Y : d(y, p) < r\}$  be a neighborhood of  $p$  in  $Y$ . Then  $N_r^Y(p) = N_r^X(p) \cap Y$ .

- $E$  is open relative to  $Y$  iff  $E = G \cap Y$  where  $G$  is open relative to  $X$ .
- Introduces the supremum.
- If  $E \subset \mathbb{R}$ ,  $E \neq \emptyset$ , and  $E$  is bounded above,  $\sup E < \infty$ .
- Let  $y = \sup E$ . Then  $y \in \bar{E}$ .
- There exists a sequence  $a_n \in A$  such that  $a_n \rightarrow x = \sup A$ .
- $A$  is compact iff any open cover of the set has a finite subcover.
- Study and *know* all of these proofs.

11/5:

- Compactness: Defines compactness in terms of open covers.
- Finite sets are compact.
- Compactness is “absolute” (i.e., it is not a relative property like openness).
  - If  $K \subset Y \subset X$ , then  $K$  is compact relative to  $X$  iff  $K$  is compact relative to  $Y$ .
  - $V$  is open relative to  $Y$  iff  $V = G \cap Y$  where  $G$  is open relative to  $X$ .
- Compact implies closed.
  - We will show  $K$  compact implies  $K^c$  open.
  - WTS: For all  $p \in K^c$ , there exists  $N_r(p) \subset K^c$  such that  $N_r(p) \cap K = \emptyset$ .
- A closed subset of a compact set is compact.
  - Let  $K$  be compact and let  $F \subset K$  be closed.
  - Take any open cover of  $F$ . Extend it to an open cover of  $K$ . Take the finite subcover of  $K$ . Naturally, this finite subcover is also a finite cover of  $F \subset K$ .
- $F$  closed,  $K$  compact implies  $F \cap K$  compact.
- If  $(K_\alpha)_{\alpha \in A}$  is compact in  $X$  with finite intersection property (every intersection of any finite number of these sets is nonempty), then  $\bigcap_{\alpha \in A} K_\alpha \neq \emptyset$ .
  - Argue by contradiction.
  - Let  $G_\alpha = K_\alpha^c$ .
  - Assume the intersection is empty. Assume WLOG that no point of  $K_1$  is in any of the other  $K_\alpha$ ’s.
  - Then  $\{G_\alpha\}_{\alpha \in A}$  be an open cover of  $K_1$ .
  - $K_1$  compact implies there is a finite subcover  $G_{\alpha_1}, \dots, G_{\alpha_n}$ . Then  $K \subset G_{\alpha_1} \cup \dots \cup G_{\alpha_n}$ . This implies that  $K_1 \cap K_{\alpha_1} \cap \dots \cap K_{\alpha_n} = \emptyset$ , a contradiction.
- Let  $E$  be an infinite subset of a compact  $K$ . Then  $E$  has a limit point in  $K$ .
  - Argue by contradiction.
  - Suppose for all  $p \in K$ , there exists  $N_r(p)$  such that  $N_r(p) \cap E = \{p\}$ .
  - Consider the set  $\{N_r(p) : p \in K\}$ . This is an open cover of  $K$ . Thus, there exists a finite subcover of it. But since  $E \subset K \subset N_{r_1}(p_1) \cup \dots \cup N_{r_n}(p_n) = \{p_1\} \cup \dots \cup \{p_n\}$ ,  $E$  is finite, a contradiction.
- **2-cell** (in  $\mathbb{R}^2$ ): A set that is the Cartesian product of two closed intervals.
  - Generalizes to **k-cells**.
- Let  $I_n = [a_n, b_n] \subset \mathbb{R}$  such that  $I_{n+1} \subset I_n$ . Then  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ .

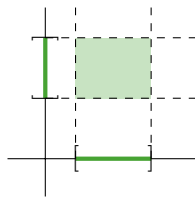
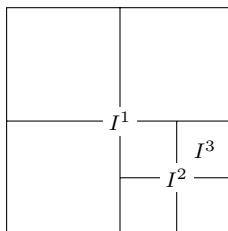


Figure 2.1: 2-cell.

- Let  $I_k$  be a  $k$ -cell in  $\mathbb{R}^k$  such that  $I_k \supset I_{k+1}$ . Then  $\bigcap_k I_k \neq \emptyset$ .
- We know that  $a_m \leq a_{m+n} \leq b_{m+n} \leq b_m$ , so  $\sup a_n \in \bigcap I_n$ .
- Every  $k$ -cell is compact.

Figure 2.2:  $k$ -cells are compact.

- Argue by contradiction.
- Consider an open cover of the  $k$ -cell  $I^1$ . If it has a finite subcover, we're done. So suppose we have an open cover that doesn't have a finite subcover. Split the  $k$ -cell into  $2^k$  chunks. At least one of the chunks  $I^2$  must not have a finite subcover.
- Split that one into  $2^k$  chunks. At least one of the chunks  $I^3$  must not have a finite subcover.
- Continue.
- Thus, we have a decreasing family of  $k$ -cells, so by the previous result, their  $\bigcap I^n \neq \emptyset$ .
- Let  $x \in \bigcap I^n$ . Then the...
- Heine-Borel theorem: Let  $E \subset \mathbb{R}^k$ . Then TFAE<sup>[2]</sup>
  1.  $E$  is closed and bounded.
  2.  $E$  is compact.
  3. Every infinite subset of  $E$  has a limit point in  $E$ .
  - $(1 \Rightarrow 2)$   $E$  closed and bounded implies  $E$  is a closed subset of some  $I_k$ , so it's compact.
  - $(2 \Rightarrow 3)$  Already done.
  - $(3 \Rightarrow 1)$ 
    - Suppose  $E$  not bounded. Then there is an infinite sequence of points in  $E$  that never converges. Contradiction.
    - Suppose  $E$  is not closed. Then there exists a sequence of points in  $E$  which “converges” to an  $x_0 \notin E$ .

11/8: • Hewitt and Stromberg (1965) has harder analysis problems than Rudin (1976).

- Theorem: If  $P$  is a nonempty perfect subset of  $\mathbb{R}^k$ , then  $P$  is uncountable.

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<sup>2</sup>The following are equivalent.

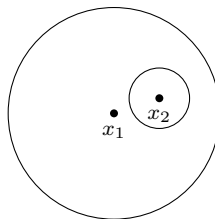


Figure 2.3: Nonempty perfect sets are uncountable.

- $P$  perfect implies  $P$  infinite.
  - Suppose  $P$  is countable. Let  $P = \{x_1, x_2, \dots\}$ .
  - Start with  $x_1$ . Take an open neighborhood  $V_1$  of  $x_1$ . Since  $x_1$  is a limit point of  $P$ , there will be another point  $x_2 \in P$  in  $V_1$ . Choose  $V_2$  to be a neighborhood of  $x_2$  such that  $\bar{V}_2 \subset V_1$ .
  - Keep going — there is a point  $x_3 \in P$  in  $V_2$ , choose an appropriate neighborhood, etc.
  - Thus, we have a sequence of closed compact sets such that  $\bar{V}_n \supset \bar{V}_{n+1}$  ( $n \in \mathbb{N}$ ). It follows that  $\bigcap \bar{V}_n \neq \emptyset$ .
  - We also know that  $V_n \cap P \neq \emptyset$  for each  $n$ .
  - Let  $K_n = \bar{V}_n \cap P$ . Each  $K_n$  is compact and  $K_n \supset K_{n+1}$  for each  $n$ . Therefore, by compactness,  $\bigcap K_n \neq \emptyset$ . But the construction implies that  $\bigcap K_n = \emptyset$  because we exhausted the whole sequence of possible points  $x_i \in P$ .
- Corollary: Any interval is uncountable.
  - The Cantor set:

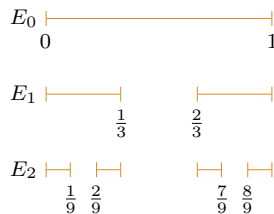


Figure 2.4: Constructing the Cantor set.

- Let  $E_0 = [0, 1]$ .
- Take out the middle third, so that  $E_1 = [0, 1/3] \cup [2/3, 1]$ .
- Take out the middle thirds of the remaining intervals and keep going.
- Thus, we are building a decreasing family of compact sets, so the overall intersection  $E = \bigcap E_n$  of every set is nonempty.
- $E^n$  is the union of  $2^n$  closed intervals of length  $n/3$ . Thus, the overall length of  $E^n$  is  $(2/3)^n$ .
- Thus, we have a compact nonempty set with Lebesgue measure zero.
- $E$  does not contain any segment of the form

$$\left( \frac{3k+1}{3^m}, \frac{3k+2}{3^m} \right)$$

for  $k, m \in \mathbb{N}$ .

- Therefore, no segment of the form  $(\alpha, \beta)$  is contained in  $E$  (any segment of said form contains a segment of the above form).



- Moreover,  $E$  (the Cantor set) is perfect.
  - Let  $x \in E$ . WTS: For all segments  $S$  containing  $x$ ,  $S \cap (E \setminus \{x\}) \neq \emptyset$ .
  - Let  $S$  be an arbitrary such segment...
- Consider the **Devil's staircase**.
  - $0 = \int_0^1 F'(x) dx = F(1) - F(0) = 1$ . This function does not obey the fundamental theorem of calculus. A function satisfies the fundamental theorem of calculus if and only if it is absolutely continuous.
- Connected sets (motivation):
  - In a convex set, you can connect any two points with a straight line.
  - In a nonconvex connected set, there exist points that you must connect with a curve.
  - In a disconnected set, there exist points that cannot be connected via a line whose points lie wholly in the set.
- **Connected** (set  $E$ ): A set  $E$  that is not the union of two **separated** sets.
- **Separated** (sets  $A, B$ ): Two sets  $A, B \subset X$  that are nonempty and such that  $\bar{A} \cap B = \emptyset$ , and  $A \cap \bar{B} = \emptyset$ .
- Theorem:  $E \subset \mathbb{R}$  is connected iff  $x, y \in E$  and  $x < z < y$  implies  $z \in E$ .
  - If there is a  $z \notin E$  between  $x, y$ , then  $\{x \in E : x < z\}$  and  $\{x \in E : z < y\}$  are separated sets, so  $E$  is not connected.

## 2.2 Chapter 2: Basic Topology

From Rudin (1976).

- 11/6:
- **Countable** (set  $A$ ): A set  $A$  that is in bijective correspondence with the set of all positive integers. Also known as **enumerable**, **denumerable**.
  - **At most countable** (set  $A$ ): A set  $A$  that is finite or countable.
  - An alternative definition of an **infinite** set would be a set that is equivalent to one of its proper subsets.
  - Theorem: Let  $A$  be the set of all sequences whose elements are the digits 0 and 1. This set  $A$  is uncountable.

*Proof.* Let  $E = \{s_1, s_2, \dots\}$  be an arbitrary countable subset of  $A$ , where each  $s_j$  is a sequence whose elements are the digits 0 and 1. Let  $s$  be the sequence, the  $n^{\text{th}}$  term of which is the opposite of the  $n^{\text{th}}$  term of  $s_n$  (i.e., if the  $n^{\text{th}}$  term of  $s_n$  is 0, we set the  $n^{\text{th}}$  term of  $s$  equal to 1). This guarantees that  $s$  is distinct from each of the  $s_j$ , i.e., that  $s \notin E$ . It follows that  $E \subsetneq A$ , i.e., that every countable subset of  $A$  is a proper subset of  $A$ . Therefore,  $A$  must be uncountable (for otherwise  $A$  would be a proper subset of  $A$ , a contradiction).  $\square$

- The idea of this proof is called **Cantor's diagonalization process**.
- Since every real number can be represented as a binary sequence of numbers, i.e.,  $A \sim \mathbb{R}$ , the reals are uncountable.
- **Metric space**: A set  $X$  such that with any two points  $p, q \in X$ , there is associated a real number  $d(p, q)$  such that
  1.  $d(p, q) > 0$  if  $p \neq q$ ;  $d(p, p) = 0$ .
  2.  $d(p, q) = d(q, p)$ .

3.  $d(p, q) \leq d(p, r) + d(r, q)$  for any  $r \in X$ .

- **Distance** (from  $p \in X$  to  $q \in X$ ,  $X$  a metric space): The real number  $d(p, q)$ .
- **Distance function**: A function  $d : X \times X \rightarrow \mathbb{R}$  that sends  $(p, q) \mapsto d(p, q)$ . Also known as **metric**.
- Every subset of a metric space is a metric space in its own right under the same distance function.
- **Segment** (from  $a$  to  $b$ ): The set of all real numbers  $x$  such that  $a < x < b$ . Denoted by  $(a, b)$ .
- **Interval** (from  $a$  to  $b$ ): The set of all real numbers  $x$  such that  $a \leq x \leq b$ . Denoted by  $[a, b]$ .
- **$k$ -cell**: The set of all points  $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k$  whose coordinates satisfy the inequalities  $a_i \leq x_i \leq b_i$  where  $a_i < b_i$  for each  $1 \leq i \leq k$ .

– Note that a 1-cell is an interval and a 2-cell is a rectangle.

- **Convex** (set  $E$ ): A subset  $E$  of  $\mathbb{R}^k$  such that

$$\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} \in E$$

for all  $\mathbf{x}, \mathbf{y} \in E$  and  $0 < \lambda < 1$ .

– Balls and  $k$ -cells are both convex.

- The segment  $(a, b)$  is open as a subset of  $\mathbb{R}^1$ , but not open as a subset of  $\mathbb{R}^2$ .
- Since compactness is not relative, while it makes no sense to talk about *open* or *closed* metric spaces, it does make sense to talk about *compact* metric spaces.
- **Weierstrass theorem**: Every bounded infinite subset of  $\mathbb{R}^k$  has a limit point in  $\mathbb{R}^k$ .
- Theorem: Let  $P$  be a nonempty perfect set in  $\mathbb{R}^k$ . Then  $P$  is uncountable.

*Proof.* Since  $P$  is nonempty and perfect, there exists a limit point of  $P$ . It follows that  $P$  is infinite.

Now suppose for the sake of contradiction that  $P$  is countable, and denote the elements of  $P$  by  $\mathbf{x}_1, \mathbf{x}_2, \dots$ . We now construct a sequence  $\{V_n\}$  of neighborhoods, as follows. Let  $V_1 = N_r(\mathbf{x}_1)$ . Clearly,  $V_1 \subset P$  since  $\mathbf{x}_1 \in P$ . It follows that since  $V_1$  is a neighborhood that  $V_1$  contains infinitely many points of  $P$ . Now suppose inductively that  $V_n$  has been constructed. Thus, by analogous conditions to those on  $V_1$ , we may let  $V_{n+1}$  be a neighborhood such that (i)  $\bar{V}_{n+1} \subset V_n$ , (ii)  $\mathbf{x}_n \notin \bar{V}_{n+1}$ , and (iii)  $V_{n+1} \cap P$  is nonempty. By (iii), we can continue on to construct  $V_{n+2}$ , and so on and so forth.

Let  $K_n = \bar{V}_n \cap P$ . Since  $\bar{V}_n$  is closed and bounded,  $\bar{V}_n$  is compact. Additionally, since  $\mathbf{x}_n \notin K_{n+1}$  for each  $n$ , no point of  $P$  lies in  $\bigcap_1^\infty K_n$ . Thus, since each  $K_n \subset P$ ,  $\bigcap_1^\infty K_n$  is empty. But this contradicts our previous result that since each  $K_n$  is nonempty, compact, and such that  $K_n \supset K_{n+1}$ ,  $\bigcap_1^\infty K_n$  is nonempty.  $\square$

- Corollary: Every interval  $[a, b]$  is uncountable. In particular,  $\mathbb{R}$  is uncountable.
- **Cantor set**: The set resulting from the following construction. Let  $E_0 = [0, 1]$ . Remove the segment  $(1/3, 2/3)$ , so that  $E_1 = [0, 1/3] \cup [2/3, 1]$ . Now remove the middle third of these two intervals to create  $E_2$ . Continue on indefinitely.
  - This is a perfect set in  $\mathbb{R}^1$  which contains no segment.
- **Separated** (sets  $A, B$ ): Two subsets  $A, B$  of a metric space  $X$  such that  $A \cap \bar{B}$  and  $\bar{A} \cap B$  are empty.
- **Connected** (set  $E$ ): A set  $E$  that is not the union of two nonempty separated sets.
- Separated sets are disjoint, but disjoint sets are not necessarily separated (consider  $[0, 1]$  and  $(1, 2)$ ).
- A subset  $E$  of  $\mathbb{R}^1$  is connected if and only if it has the following property: If  $x, y \in E$  and  $x < z < y$ , then  $z \in E$ .

## Chapter 3

# Numerical Sequences and Series

### 3.1 Notes

- 11/8:      • Any bounded sequence in  $\mathbb{R}^k$  has a convergent subsequence.
- 11/10:     • Read and understand the section about Cauchy sequences converging and the sup/inf.

### 3.2 Chapter 3: Numerical Sequences and Series

*From Rudin (1976).*

- 11/7:      • Convergence of sequences is relative.
- For example, the sequence  $1/n$  for  $n = 1, 2, \dots$  converges in  $\mathbb{R}$ , but not in  $(0, \infty)$ .
- **Range** (of  $\{p_n\}$ ): The set of all points  $p_n$ .
- This definition squares nicely with the formal definition of a sequence as a function  $p$  defined on  $\mathbb{N}$ .
- Theorem 3.6a: If  $\{p_n\}$  is a sequence in a compact metric space  $X$ , then some subsequence of  $\{p_n\}$  converges to a point of  $X$ .
- Theorem 3.7: The subsequential limits of a sequence  $\{p_n\}$  in a metric space  $X$  form a closed subset of  $X$ .
- **Diameter** (of  $E$ ): The supremum of the set

$$S = \{d(p, q) : p, q \in E\}$$

where  $E$  is a nonempty subset of a metric space  $X$ . Denoted by **diam**  $E$ .

- Theorem 3.10:
- (a) If  $\bar{E}$  is the closure of a set  $E$  in a metric space  $X$ , then
$$\text{diam } \bar{E} = \text{diam } E$$
  - (b) If  $K_n$  is a sequence of compact sets in  $X$  such that  $K_n \supset K_{n+1}$  ( $n = 1, 2, 3, \dots$ ) and if  $\lim_{n \rightarrow \infty} \text{diam } K_n = 0$ , then  $\bigcap_1^\infty K_n$  consists of exactly one point.
- **Complete** (metric space): A metric space in which every Cauchy sequence converges.
- All compact metric spaces and all Euclidean spaces are complete.

- The metric space  $(\mathbb{Q}, |x - y|)$  is not complete.
- **Monotonically increasing** (sequence  $\{s_n\}$ ): A sequence  $\{s_n\}$  of real numbers such that  $s_n \leq s_{n+1}$  for each  $n \in \mathbb{N}$ .
- **Monotonically decreasing** (sequence  $\{s_n\}$ ): A sequence  $\{s_n\}$  of real numbers such that  $s_n \geq s_{n+1}$  for each  $n \in \mathbb{N}$ .
- **Monotonic sequences**: The class of all sequences that are either monotonically increasing or monotonically decreasing.
- **Upper limit** (of  $\{s_n\}$ ): The supremum of the set  $E$  of all subsequential limits of  $\{s_n\}$ . *Denoted by  $s^*$ ,  $\limsup_{n \rightarrow \infty} s_n$ .*
- **Lower limit** (of  $\{s_n\}$ ): The infimum of the set  $E$  of all subsequential limits of  $\{s_n\}$ . *Denoted by  $s_*$ ,  $\liminf_{n \rightarrow \infty} s_n$ .*
- Theorem 3.17: Let  $\{s_n\}$  be a sequence of real numbers. Then  $s^*$  has (and is the only number to have both of) the following two properties.
  - (a)  $s^* \in E$ .
  - (b) If  $x > s^*$ , then there is an integer  $N$  such that  $n \geq N$  implies  $s_n < x$ .

An analogous result holds for  $s_*$ .

11/8:

- Series are defined in terms of sequences. Moreover, sequences can be defined in terms of series: Let  $a_1 = s_1$ ,  $a_n = s_n - s_{n-1}$  ( $n \in \mathbb{N} + 1$ ). Thus, every theorem about sequences can be stated in terms of series and vice versa, but it is nevertheless useful to consider both concepts (Rudin, 1976, p. 59).
- Theorem 3.27: Suppose  $\{a_n\}$  is a monotonically decreasing sequence of nonnegative terms. Then the series  $\sum_{n=1}^{\infty} a_n$  converges if and only if the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$$

converges.

- Theorem 3.29: If  $p > 1$ ,

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$$

converges; if  $p \leq 1$ , the series diverges.

- Note that  $\log n = \ln n$ .
- Note that we sum from  $n = 2$  since  $\log 1 = 0$ .
- **e**: The number

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

- Theorem 3.31:  $\lim_{n \rightarrow \infty} (1 + 1/n)^n = e$ .
- Theorem 3.32:  $e$  is irrational.
- Theorem 3.39: Given the power series  $\sum c_n z^n$ , put

$$\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|} \qquad R = \frac{1}{\alpha}$$

(If  $\alpha = 0$ , let  $R = +\infty$ ; if  $\alpha = +\infty$ , let  $R = 0$ .) Then  $\sum c_n z^n$  converges if  $|z| < R$  and diverges if  $|z| > R$ .

- **Radius of convergence** (of a power series): The number  $R$  defined by Theorem 3.39.
- Theorem 3.41 (partial summation formula): Given two sequence  $\{a_n\}, \{b_n\}$ , put

$$A_n = \begin{cases} \sum_{k=0}^n a_k & n \geq 0 \\ 0 & n = -1 \end{cases}$$

Then if  $0 \leq p \leq q$ , we have

$$\sum_{n=p}^q a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

- **Product** (of  $\sum a_n, \sum b_n$ ): The series  $\sum c_n$  defined by

$$c_n = \sum_{k=0}^n a_k b_{n-k}$$

for each  $n = 0, 1, 2, \dots$

- We motivate this definition by noting that if  $\sum c_n$  is the product of  $\sum a_n, \sum b_n$ , then

$$\sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} c_n z^n$$

- Setting  $z = 1$  then yields the given definition.

- The product of two convergent series may diverge. However...
- Theorem 3.50 (by Mertens): Suppose (a)  $\sum_{n=0}^{\infty} a_n$  converges absolutely, (b)  $\sum_{n=0}^{\infty} a_n = A$ , (c)  $\sum_{n=0}^{\infty} b_n = B$ , and (d)  $\sum_{n=0}^{\infty} c_n$  is the product of  $\sum_{n=0}^{\infty} a_n$  and  $\sum_{n=0}^{\infty} b_n$ . Then

$$\sum_{n=0}^{\infty} c_n = AB$$

- Theorem 3.51 (by Abel): If  $\sum a_n, \sum b_n, \sum c_n$  converge to  $A, B, C$ , respectively, and  $\sum c_n$  is the product of  $\sum a_n, \sum b_n$ , then  $C = AB$ .
- **Rearrangement** (of  $\sum a_n$ ): A series  $\sum a'_n$  defined by  $a'_n = a_{k_n}$  for each  $n \in \mathbb{N}$ , where  $\{k_n\}$  is a sequence in which every positive integer appears once and only once (that is,  $\{k_n\}$  is a 1-1 function from  $\mathbb{N}$  onto  $\mathbb{N}$ ).
- Theorem 3.54: Let  $\sum a_n$  be a series of real number which converges, but not absolutely. Suppose  $-\infty \leq \alpha \leq \beta \leq \infty$ . Then there exists a rearrangement  $\sum a'_n$  with partial sums  $s'_n$  such that

$$\liminf_{n \rightarrow \infty} s'_n = \alpha \qquad \limsup_{n \rightarrow \infty} s'_n = \beta$$

- Theorem 3.55: If  $\sum a_n$  is a series of complex numbers which converges absolutely, then every rearrangement of  $\sum a_n$  converges, and they all converge to the same sum.

# Chapter 4

## Continuity

### 4.1 Notes

- 11/8:
- Consider a function  $f : X \rightarrow Y$  whose domain and codomain are, respectively, the metric spaces  $(X, d_X)$  and  $(Y, d_Y)$ .
  - **Limit** (of  $f$  at  $p$ ): A point  $q \in Y$  such that for all  $\epsilon > 0$ , there exists  $\delta$  such that  $d_X(x, p) < \delta$  implies  $d_Y(q, f(x)) < \epsilon$ , where  $p$  is a limit point of  $X$  (otherwise,  $x \nrightarrow p$ ).
  - **Continuous** (function  $f$  at  $p$ ): A function  $f$  such that  $\lim_{x \rightarrow p} f(x) = f(p)$ .
  - $f$  is continuous on  $X$  if it is continuous at every  $p \in X$ .
  - **Uniformly continuous** (function  $f$ ): A function  $f$  such that for every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that  $d_X(x, y) < \delta$  implies  $d_Y(f(x), f(y)) < \epsilon$  for all  $x, y \in X$ .
- 11/10:
- $f, g$  continuous implies  $f + g, fg$ , and  $f/g$  continuous, the latter where  $g(x) \neq 0$ .
  - If  $f, g$  continuous, then  $h = g \circ f$  is continuous.
  - Theorem:  $f : X \rightarrow Y$  is continuous iff  $f^{-1}(V)$  is open in  $X$  for every  $V \subset Y$  open.

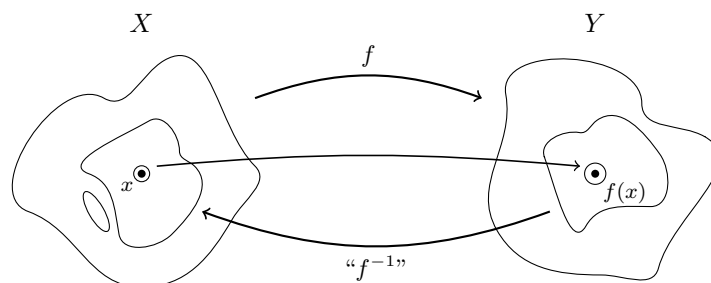


Figure 4.1: Set theoretic definition of continuity.

- This works in a general topological space, too, not just a metric space.
- Note that  $f^{-1}(V)$  is not a function defined on  $V$ ; it's a specifically defined set  $\{x \in X : f(x) \in V\}$ .
- $f$  being continuous means that open circular neighborhood of a point  $x$  in the domain maps to an area of the range encompassed by a circular neighborhood of  $f(x)$ .
- The other condition means that every open set surrounding  $f(x)$  maps to an open set of the domain surrounding  $x$ . Indeed, going off of this definition, if an open set containing  $f(x)$  maps to an open set containing  $x$ , then we can choose a neighborhood subset of the open set surrounding  $x$  and know that it will map into a neighborhood subset of the open set surrounding  $f(x)$ .

- Corollary:  $f : X \rightarrow Y$  continuous iff  $f^{-1}(C)$  closed for every  $C \subset Y$  closed.
    - We use the property that  $f^{-1}(X \subset C) = X \subset f^{-1}(C)$ .
  - Let  $f_1 : X_1 \rightarrow Y_1$  and  $f_2 : X_2 \rightarrow Y_2$ . Suppose  $f : X_1 \times X_2 \rightarrow Y_1 \times Y_2$  is defined by  $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$ . Then  $f$  is continuous iff  $f_1, f_2$  are continuous, under appropriately defined metrics.
  - Continuity and compactness.
  - Theorem:  $f : X \rightarrow Y$  continuous and  $X$  compact imply  $f(X)$  compact.
    - Let  $\{V_\alpha\}$  be an open cover of  $f(X)$ .
    - Then  $\{f^{-1}(V_\alpha)\}$  is an open cover of  $X$ .
    - Choose a finite subcover of  $\{f^{-1}(V_\alpha)\}$ . Then the corresponding  $V_\alpha$ 's form a finite subcover of  $f(X)$ .
  - If  $f : X \rightarrow \mathbb{R}^k$  is continuous and  $X$  is compact,  $f(X)$  is compact and closed/bounded.
  - If  $f : X \rightarrow \mathbb{R}$  is continuous and  $X$  is compact, then  $M = \sup_{x \in X} |f(x)| = |f(\bar{x})|$  and  $m = \inf_{x \in X} |f(x)| = |f(\underline{x})|$  where  $\bar{x}, \underline{x} \in X$ .
    - There is a subsequence  $\{x_m\}$  such that  $|f(x_m)| \rightarrow M$ . Since  $f(X)$  is compact, the limit of this sequence is in  $f(X)$ .
  - If  $f : X \rightarrow Y$  is continuous, bijective, and  $X, Y$  are compact,  $f^{-1} : Y \rightarrow X$  is continuous.
  - Uniform continuity.
  - Examples.
    - Linear functions are uniformly continuous.
    - $f : (0, 1) \rightarrow \mathbb{R}$  defined by  $f(x) = x^2$  is uniformly continuous.
    - $f : (a, \infty) \rightarrow \mathbb{R}$  defined by  $f(x) = x^2$  is *not* uniformly continuous.
    - $f : (a, \infty) \rightarrow \mathbb{R}$  defined by  $f(x) = 1/x$  is uniformly continuous if  $a > 0$ .
    - $f : (0, \infty) \rightarrow \mathbb{R}$  defined by  $f(x) = 1/x$  is *not* uniformly continuous.
  - **Lipschitz continuous** (function  $f$  on  $E \subset X$ ): A function such that  $|f(x) - f(y)| \leq L|x - y|$  for each  $x, y \in E$ .
  - Theorem:  $f : X \rightarrow Y$  continuous and  $X$  compact implies  $f$  is uniformly continuous.
    - Fix  $\epsilon > 0$ . There exists  $\delta = \delta(p) > 0$ .
    - Def. of continuity:  $q \in N_{\delta(p)}(p)$  implies  $f(q) \in N_\epsilon(f(p))$ .
    - $\{N_{\delta(p)/2}(p) : p \in X\}$  is an open cover of  $X$ . Choose a finite subcover. Let  $\delta = \min(\delta(p_1)/2, \dots, \delta(p_n)/2)$ .
    - ...
- 11/12:
- $f : X \rightarrow Y$  continuous and  $E \subset X$  connected implies  $f(E)$  connected.
    - Suppose  $f(E) = A \cup B$ ,  $A, B$  nonempty, separated subsets of  $Y$ .
    - Let  $G = E \cap f^{-1}(A)$ ,  $H = E \cap f^{-1}(B)$ . It follows that  $E = G \cup H$ , where  $G, H$  nonempty.
    - $A \subset \bar{A}$  implies  $G \subset f^{-1}(\bar{A})$  implies (inverse image def. of continuity)  $\bar{G} \subset f^{-1}(A)$  implies  $f(\bar{G}) \subset \bar{A}$ .
    - $f(H) = B$  and  $\bar{A} \cap B = \emptyset$  yields  $\bar{G} \cap H = \emptyset$ . Symmetrically,  $G \cap \bar{H} = \emptyset$ . This contradicts our assumption that  $E$  is connected.
  - Introduces monotone functions.
  - Theorem: If  $f$  is monotonic on  $(a, b)$ , then the set of points of  $(a, b)$  at which  $f$  is discontinuous is at most countable.
    - ...

## 4.2 Chapter 4: Continuity

From Rudin (1976).

- 11/8:
- **Limit** (of  $f$  at  $p$ ): The point  $q \in Y$ , if it exists, such that for every  $\epsilon > 0$ , there exists a  $\delta > 0$  such that  $d_Y(f(x), q) < \epsilon$  for all points  $x \in E$  for which  $0 < d_X(x, p) < \delta$ , where  $(X, d_X), (Y, d_Y)$  are metric spaces,  $E \subset X$ ,  $f : E \rightarrow Y$ , and  $p \in E'$ . Denoted by  $\lim_{x \rightarrow p} f(x)$ .
    - Note that we do not require that  $p \in E$ ; only that some elements of the domain  $E$  approach  $p$ .
    - We also write  $f(x) \rightarrow q$  as  $x \rightarrow p$ .
  - Theorem 4.2: Let  $X, Y, E, f$ , and  $p$  be as specified above. Then  $\lim_{x \rightarrow p} f(x) = q$  iff  $\lim_{n \rightarrow \infty} f(p_n) = q$  for every sequence  $\{p_n\}$  in  $E$  such that  $p_n \neq p$  for any  $n$  and  $\lim_{n \rightarrow \infty} p_n = p$ .
  - Rudin (1976) proves the sum, product, and quotient rules of limits from the analogous properties of series.
  - Continuity is defined.
    - Note that  $f$  *does* have to be defined at  $p$  to be continuous at  $p$  (in comparison to the fact that it can have a limit at a point  $p'$  at which it is not defined).
      - Thus, for proofs concerning continuity (as opposed to limits), we will consider functions  $f$  the domains of which are metric spaces, not *subsets* of metric spaces.
    - It follows from the definition that if  $p \in E$  is isolated, then every possible  $f$  defined on  $E$  is continuous at  $p$ .
  - Theorem 4.7: Compositions of continuous functions are continuous.
  - Theorem 4.8: Preimage definition of continuity.
  - Theorem 4.9: If  $f, g$  are complex continuous functions on  $X$ ,  $f + g$ ,  $fg$ , and  $f/g$  are continuous on  $X$ .
  - Theorem 4.10:  $\mathbf{f}$  continuous implies  $f_1, \dots, f_k$  continuous. Also,  $\mathbf{f}, \mathbf{g} : X \rightarrow \mathbb{R}^k$  continuous implies  $\mathbf{f} + \mathbf{g}$  and  $\mathbf{f} \cdot \mathbf{g}$  continuous.
- 11/9:
- Theorem 4.14:  $f$  continuous and  $X$  compact implies  $f(X)$  compact.
  - Theorem 4.15:  $\mathbf{f} : X \rightarrow \mathbb{R}^k$  continuous and  $X$  compact implies  $f(X)$  closed and bounded.
  - Theorem 4.16:  $f$  continuous and  $X$  compact implies  $f$  attains its minimum and maximum.
  - Theorem 4.17:  $f : X \rightarrow Y$  continuous, 1-1 for  $X, Y$  compact implies  $f^{-1} : Y \rightarrow X$  continuous.
  - Theorem 4.19:  $f$  continuous and  $X$  compact implies  $f$  uniformly continuous.
  - Theorem 4.20: Compactness is a necessary condition in Theorems 4.14, 4.15, 4.16, and 4.19.
  - Theorem 4.22:  $f : X \rightarrow Y$  continuous and  $E \subset X$  connected implies  $f(E)$  connected.
  - Theorem 4.23: Intermediate value theorem.
  - **Right-hand limit** (of  $f$  at  $x$ ): Denoted by  $f(x+)$ .
  - **Left-hand limit** (of  $f$  at  $x$ ): Denoted by  $f(x-)$ .
  - **Discontinuity of the first kind** (of  $f$  at  $x$ ): A discontinuity of  $f$  at  $x$  such that  $f(x+)$  and  $f(x-)$  exist. Also known as **simple discontinuity**.
  - **Discontinuity of the second kind** (of  $f$  at  $x$ ): A discontinuity of  $f$  at  $x$  that is not of the first kind (i.e., a discontinuity such that at least one of  $f(x+)$  and  $f(x-)$  does not exist).



- Theorem 4.29: If  $f$  is monotonic on  $(a, b)$ , then  $f(x+)$ ,  $f(x-)$  exist at every  $x \in (a, b)$ .
- Corollary: Monotonic functions have no discontinuities of the second kind.
- Theorem 4.30: If  $f$  is monotonic on  $(a, b)$ , then the set of points of  $(a, b)$  at which  $f$  is discontinuous is at most countable.

# Chapter 7

## Sequences and Series of Functions

### 7.1 Notes

11/15:

- Soug will not test on differentiation/integration assuming that we know them already.
- **Pointwise convergent** (sequence  $(f_n)_{n \in \mathbb{N}}$  of functions): A sequence of functions  $f_n : E \rightarrow \mathbb{R}$  such that  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  for all  $x \in E$ .
- Can we interchange “limit” in the above definition with continuity, convergence of series, integration, differentiation, etc.?
- Examples with negative answer:

1. Interchanging limits: Let  $S_{mn} = \frac{m}{m+n}$ .  $S_{mn} \rightarrow 1$  as  $m \rightarrow \infty$ , and  $S_{mn} \rightarrow 0$  as  $n \rightarrow \infty$ .
2.  $f_n(x) = x^2/(1+x)^n$ .  $f(x) = \sum_{n=1}^{\infty} f_n(x)$ . If  $x = 0$ , then  $f_n(x) = 0$  for all  $n$  and  $f(x) = 0$ . If  $x \neq 0$ , we have

$$f(x) = \sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n} = x^2 \sum_{n=1}^{\infty} X^n = \frac{x^2}{1-X} = \frac{x^2}{1-(1/(1+x^2))} = 1+x^2$$

3. Consider  $f_m(x) = \lim_{n \rightarrow \infty} (\cos(m\pi x))^{2n}$ .  $\lim_{m \rightarrow \infty} f_m(x)$  goes to 0 if  $x \notin \mathbb{Q}$  and goes to 1 if  $x \in \mathbb{Q}$ .  $f_m \rightarrow \chi_{\mathbb{Q}}$ , where  $\chi_{\mathbb{Q}}$  is the characteristic function of the rationals which is not Riemann integrable (partitions, upper and lower integrals, etc.).
  4.  $f_n(x) = \sin nx / \sqrt{n} \rightarrow f(x) = 0$  for all  $x$ . However,  $f'_n(x) = \sqrt{n} \cos nx \not\rightarrow 0$
  5. If  $0 \leq x \leq 1$ , define  $f_n(x) = n^2 x(1-x^2)^n$ . We know that  $f_n(0) = 0$ .  $\lim_{n \rightarrow \infty} f_n(x) = 0$  for all  $x \in (0, 1]$ . We can show that  $\int_0^1 x(1-x^2)^n dx = 1/(2n+2)$ . Thus,  $\int_0^1 f_n(x) dx = n^2/(2n+2)$ . Limit of the functions is zero, but their integrals diverge to infinity.
- **Uniformly convergent** (sequence  $(f_n)_{n \in \mathbb{N}}$  of functions on  $E$ ): A sequence of functions  $f_n : E \rightarrow \mathbb{R}$  such that for all  $\epsilon > 0$ , there exists  $N$  such that if  $n \geq N$ , then  $|f_n(x) - f(x)| < \epsilon$  for all  $x \in E$ . Denoted by  $f_n \rightrightarrows f$ .
  - Theorem:  $f_n \rightrightarrows f$  iff  $(f_n)_{n \in \mathbb{N}}$  is uniformly Cauchy (i.e., for all  $\epsilon > 0$ , there exists  $N$  such that if  $n, m \geq N$  then  $|f_n(x) - f_m(x)| < \epsilon$  for all  $x \in E$ ).
    - Let  $M_n = \sup_{x \in E} |f_n(x) - f(x)|$ . If  $f_n \rightarrow f$  pointwise, then  $f_n \rightrightarrows f$  if  $M_n \rightarrow 0$ .
  - Theorem: If  $(f_n)_{n \in \mathbb{N}}$  and  $|f_n(x)| \leq M_n$ , then  $\sum f_n \rightrightarrows f$  if  $\sum M_n < \infty$ .
  - Theorem: If  $E$  is a compact metric space,  $f_n \rightrightarrows f$  in  $E$ ,  $x$  is a limit point of  $E$ , and  $\lim_{t \rightarrow x} f_n(t) = A_n$  exists, then  $(A_n)_{n \in \mathbb{N}}$  converges and  $\lim_{t \rightarrow x} f(t) = \lim_{n \rightarrow \infty} A_n$ .

- Corollary:  $\lim_{t \rightarrow x} \lim_{n \rightarrow \infty} f_n(t) = \lim_{n \rightarrow \infty} \lim_{t \rightarrow x} f_n(t)$ .

11/16:

- Fix  $\epsilon > 0$ . Then  $f_n \rightrightarrows f$  implies that there exists some  $N$  such that  $n, m \geq N$  implies  $|f_n(t) - f_m(t)| < \epsilon$  for all  $t \in E$ .
- $x$  is a limit point of  $E$  and  $t \rightarrow x$  implies  $|A_n - A_m| < \epsilon$ . Thus,  $(A_n)_{n \in \mathbb{N}}$  is cauchy, so there exists  $A$  such that  $A_n \rightarrow A$ .
- WTS:  $|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|$ , so we WTS the three terms on the right are small.
- There exists  $n$  such that  $|f(t) - f_n(t)| < \epsilon/3$  for all  $t$  since  $f_n \rightrightarrows f$  by hypothesis.
- Since  $t$  is in a small neighborhood of  $x$ , there exists  $n$  such that  $|A_n - A| < \epsilon/3$ .
- We also have  $|f_n(t) - A_n| < \epsilon/3$  by hypothesis.
- This is a very important proof to understand, because proofs like this pop up often.
- Corollary:  $f_n$  continuous and  $f_n \rightrightarrows f$  implies  $f$  is continuous.
- Theorem: Let  $K$  be compact. Assume
  - (a)  $(f_n)_{n \in \mathbb{N}} \subset C(K) = \{f : K \rightarrow \mathbb{R} \mid f \text{ continuous}\}$ .
  - (b)  $f_n \rightarrow f$  pointwise in  $K$  and  $f \in C(K)$ .
  - (c)  $f_n(x) \geq f_{n+1}(x)$  for all  $x \in K$ .

Then  $f_n \rightrightarrows f$ .

- WLOG  $f = 0$ ,  $g_n = f_n - f \rightarrow 0$ ,  $g_n \geq g_{n+1} \geq 0$ .
- For all  $\epsilon > 0$ , there exists  $N$  such that  $n \geq N$  and  $0 \leq g_n(x) \leq \epsilon$  for all  $x \in K$ .
- $K_n = \{x \in K : g_n(x) \geq \epsilon\}$ .
- $g_n$  continuous implies  $K_n$  closed. This combined with  $K$  compact implies  $K_n$  is compact.
- $g_n$  decreasing implies  $K_n \supset K_{n+1}$ . Thus,  $K_n$  is a nested family of compact sets, so  $\bigcap K_n \neq \emptyset$ .
- This implies that each  $K_n$  is nonempty, contradicting the fact that each  $g_n \rightarrow 0$  for all  $x$ .
- Thus, there exists an  $N$  such that  $K_n$  is empty for all  $n \geq N$ . Thus  $g_n(x) \leq \epsilon$  for all  $x \in K$ ,  $n \geq N$ .
- Note that the compactness of  $K$  is important. If  $f : (0, 1) \rightarrow \mathbb{R}$  is defined by  $f(x) = 1/(nx + 1)$ , then  $f_n \rightarrow 0$ , but  $f_n \not\rightrightarrows f$ .
- Let  $C(X) = \{f : X \rightarrow \mathbb{R} \mid f \text{ continuous, bounded}\}$  for  $X$  a metric space.
- If we define  $\|f\| = \sum_{x \in X} |f(x)|$ , for  $f, g \in C(X)$ , we may define  $d(f, g) = \|f - g\|$ . This definition satisfies the properties of a distance function, and  $\|\cdot\|$  is a norm.
  - Thus,  $C(X)$  is a complete metric space, a normed space, or, specifically, a **Banach space**.
- Theorem:  $(f_n)_{n \in \mathbb{N}} \subset C(X)$  such that  $\|f_n - f_m\|_{n, m \rightarrow \infty} \rightarrow 0$ . Then there exists  $f \in C(X)$  such that  $\|f_n - f\|_{n \rightarrow \infty} \rightarrow 0$ .
  - We get such a strong statement using properties of the image, not properties of the domain.
  - For all  $\epsilon > 0$ , there exists  $N$  such that  $n, m \geq N$ .
  - $|f_n(x) - f_m(x)| \leq \|f_n - f_m\| < \epsilon$  for all  $x$ .
  - Then there exists  $f$  such that  $f_m(x) \rightarrow f(x)$ . It follows that  $|f_n(x) - f_m(x)| < \epsilon$
- Uniform convergence and integration.
- Stieltjes integral.

- Define  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  nondecreasing.
- If we sum over the minimums/maximums of a partition times  $\alpha(x_{i+1}) - \alpha(x_i)$  instead of  $x_{i+1} - x_i$ , we obtain said integral as the upper/lower limits just like the Riemann integral.
- We write  $\int_a^b f(x) d\alpha(x)$  where  $d\alpha(x) = \alpha(x) dx$ .
- Theorem: If  $\alpha$  is nondecreasing on  $[a, b]$ ,  $f_n \in R(\alpha)$  such that  $f_n \rightrightarrows f$  on  $[a, b]$ 
  - We have

$$\begin{aligned}
 \left| \int f_n(x) d\alpha(x) - \int f(x) d\alpha(x) \right| &= \left| \int (f_n - f)(x) d\alpha(x) \right| \\
 &\leq \|f_n - f\|(\alpha(b) - \alpha(a)) \\
 &\leq \int |f_n - f| d\alpha(x) \\
 &\leq \int \|f_n - f\| d\alpha(x) \leq \|f_n - f\| \int_a^b d\alpha(x) = \|f_n - f\|(\alpha(b) - \alpha(a))
 \end{aligned}$$

11/19:

- Suppose  $f_n \rightarrow f$  and  $f'_n \rightarrow g$ . When does  $f' = g$ ?
- Theorem: If  $f_n : [a, b] \rightarrow \mathbb{R}$  is differentiable,  $f_n(x_0)$  converges for some  $x_0 \in [a, b]$ , and  $f'_n$  converges uniformly on  $[a, b]$ , then there exists  $f$  differentiable on  $[a, b]$  such that  $f_n \rightrightarrows f$  and  $f'_n \rightrightarrows f'$ .

- Assume the  $f'_n$  are continuous. Then  $f_n(x) - f_n(x_0) = \int_{x_0}^x f'_n(y) dy$ .
- Since  $f'_n \rightrightarrows g$ ,  $\int_{x_0}^x f'_n(y) dy \rightarrow \int_{x_0}^x g(y) dy$ .
- It follows since  $f_n(x_0) \rightarrow f(x_0)$  that  $f_n \rightrightarrows f$ .
- By the previous theorem, if

$$f'_n(x) = \lim_{h \rightarrow 0} \frac{f_n(x+h) - f_n(x)}{h}$$

then

$$\lim_{n \rightarrow \infty} f'_n(x) = \lim_{h \rightarrow 0} \lim_{n \rightarrow \infty} \frac{f_n(x+h) - f_n(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

- Fix  $\epsilon > 0$ . Then there exists  $N$  such that  $n, m \geq N$  such that  $|f_n(x_0) - f_m(x_0)| < \epsilon/2$  and  $|f'_n(t) - f'_m(t)| < \epsilon/2$  for all  $t \in [a, b]$ .
- We know that  $f_n(t) - f_n(x_0) = \int_{x_0}^t f'_n(y) dy$  and  $f_m(t) - f_m(x_0) = \int_{x_0}^t f'_m(y) dy$ .
- Thus,
 
$$|f_n(t) - f_n(x_0)| \leq |f_n(t) - f_m(t)| + |f_m(t) - f_m(x_0)| + |f_m(t) - f_m(x_0)|$$
- Let  $f_n(t) - f_n(x_0) = c_n(t - x_0)$  and  $f_m(t) - f_m(x_0) = c_m(t - x_0)$ .
- ...
- Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous. What conditions on  $f$  imply that  $f'$  exists?
- Suppose  $f$  is Lipschitz continuous (equivalent to saying there exists  $L > 0$  such that  $|f(x) - f(y)| \leq L|x - y|$ ); then  $f'$  exists **almost everywhere**.
  - If  $f$  differentiable, this is equivalent to saying  $f$  bounded.
- **Almost everywhere**: Something happens almost everywhere if the set of places where it doesn't happen has measure zero.
- Suppose  $f$  is **Hölder continuous**, then  $f'$  does not exist?
- **Hölder continuous** (function  $f$ ): There exists  $L > 0$  such that  $|f(y) - f(x)| < L|x - y|^\alpha$  where  $\alpha \in (0, 1)$

- Suppose  $f$  exists such that  $f$  is Hölder continuous in a neighborhood of every point in the domain. This function is not anywhere differentiable. Such a function does indeed exist (and it's Brownian motion). The construction of such a function is the essence of Stochastic analysis.
  - Probabilistically: Has mean zero, distributed as a normal function like the Gaussian, and the increments are independent of each other.
  - Analytically: It's a function that is Hölder continuous at half plus  $\epsilon$  for every  $\epsilon$  and it is nowhere differentiable.
- Theorem: There exists  $f : \mathbb{R} \rightarrow \mathbb{R}$  continuous but nowhere differentiable.
  - This theorem is due to Weierstrass and as such, such functions are typically called Weierstrass functions.
- A general class of functions that are nowhere differentiable (not in Rudin (1976); we don't have to prove this).
  - Example 1:

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

where  $0 < a < 1$ ,  $b$  positive odd integer greater than 1, and  $ab > 1 + \frac{3}{2}\pi$ .

■ This function at every point oscillates more and more and more.

- Rudin (1976)'s simple example.
  - $\phi : [-1, 1] \rightarrow \mathbb{R}$  defined by  $\phi(x) = |x|$  is not differentiable at zero.
  - Takes  $\phi$  extends it periodically with period 2, creating a sawtooth function.
  - Repeat the behavior so that the nondifferentiability becomes more and more frequent to get

$$f(x) = \sum_1^{\infty} \left(\frac{3}{4}\right)^n \phi(4^n x)$$

- This is continuous.
- Fix any  $x \in \mathbb{R}$ ,  $m \in \mathbb{N}$ . Then  $\delta_m = \pm \frac{1}{2} \cdot 4^{-m}$ .
- Then consider  $4^m x$ ,  $4^m(x + \delta_m)$ .
- Rudin asserts

$$\left| \frac{f(x + \delta_m) - f(x)}{\delta_m} \right| \rightarrow \infty$$

as  $m \rightarrow \infty$  for all  $x$ .

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