6 Basic Topology

From Rudin (1976).

Chapter 2

11/8: **1.** Prove that the empty set is a subset of every set.

2. A complex number z is said to be **algebraic** if there are integers $a_0, \ldots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$$

Prove that the set of all algebraic numbers is countable. (Hint: For every positive integer N, there are only finitely many equations with $n + |a_0| + |a_1| + \cdots + |a_n| = N$.)

3. Prove that there exist real number which are not algebraic.

4. Is the set of all irrational real numbers countable?

5. Construct a bounded set of real numbers with exactly three limit points.

6. Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\bar{E}$ have the same limit points (recall that $\bar{E} = E \cup E'$). Do E and E' always have the same limit points?

7. Let $A_1, A_2, \ldots$ be subsets of a metric space.

(a) If $B_n = \bigcup_{i=1}^{n} A_i$, prove that $\bar{B}_n = \bigcup_{i=1}^{n} \bar{A}_i$ for $n = 1, 2, 3, \ldots$.

(b) If $B = \bigcup_{i=1}^{\infty} A_i$, prove that $\bar{B} \supset \bigcup_{i=1}^{\infty} \bar{A}_i$. Show, by an example, that this inclusion can be proper.

8. Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

9. Let E° denote the set of all interior points of a set E (see Definition 2.18e; E° is called the **interior** of E).

(a) Prove that E° is always open.

(b) Prove that E is open if and only if $E^\circ = E$.

(c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.

(d) Prove that the complement of E° is the closure of the complement of E.

(e) Do E and $\bar{E}$ always have the same interiors?

(f) Do E and E° always have the same closures?

10. Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p, q) = \begin{cases} 1 & p \neq q \\ 0 & p = q \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

11. For $x \in \mathbb{R}^1$ and $y \in \mathbb{R}^1$, define

$$\begin{aligned} d_1(x, y) &= (x - y)^2 \\ d_2(x, y) &= \sqrt{|x - y|} \\ d_3(x, y) &= |x^2 - y^2| \\ d_4(x, y) &= |x - 2y| \\ d_5(x, y) &= \frac{|x - y|}{1 + |x - y|} \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

12. Let $K \subset \mathbb{R}^1$ consist of 0 and the numbers $1/n$ for $n = 1, 2, 3, \ldots$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).

13. Construct a compact set of real numbers whose limit points form a countable set.

14. Give an exmaple of an open cover of the segment $(0, 1)$ which has no finite subcover.

15. Show that Theorem 2.36 and its Corollary become false (in $\mathbb{R}^1$, for example) if the word "compact" is replaced by "closed" or by "bounded."