

Chapter 1

Basic Notions

- 9/27:
- **Vector space:** Basically, a set for which you have an addition and multiplication.
 - \mathbb{F}^d is used for \mathbb{R}^d or \mathbb{C}^d in Treil (2017).
 - \mathbb{P}_n is the vector space of polynomials up to degree n .
 - $C([0, 1])$ is the set of continuous functions defined on $[0, 1]$, an infinite-dimensional vector space.
 - **Generating set:** A subset of a vector space, all linear combinations of which generate the vector space. *Also known as spanning set.*
 - Any element of VS is a linear comb. of elements of the generating set.
 - **Linearly independent (list):** A list of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$ such that $\sum_{i=1}^k \alpha_i \mathbf{v}_i = 0$ implies $\alpha_i = 0$ for all i .
 - **Base:** A generating set consisting of linearly independent vectors.
 - Any element of a VS can be written as a *unique* linear combination of the vectors in a base.
 - If $\mathbf{x} = \sum_{i=1}^k \alpha_i \mathbf{v}_i = \sum_{i=1}^k \beta_i \mathbf{v}_i$, then $\alpha_i = \beta_i$ for all i .
 - **Linear transformation:** A function $T : X \rightarrow Y$, where X, Y are VSs, such that

$$T(\alpha \mathbf{x} + \beta \mathbf{y}) = \alpha T\mathbf{x} + \beta T\mathbf{y}$$

for all $\mathbf{x} \in X, \mathbf{y} \in Y$.

- Examples of linear transformations:
 - Consider \mathbb{P}_n . Let $Tp_n = p'_n$. This T is linear.
 - Rotation in \mathbb{R}^d .
 - Think graphically about two vectors \mathbf{x}, \mathbf{y} .
 - Rotating and summing them is the same as summing and rotating. Same for scaling.
 - Thus, rotation is actually linear!
 - Reflection as well.
- Consider $T : \mathbb{R} \rightarrow \mathbb{R}$.
 - Any linear map on the line is a line.
 - We must have $Tx = \alpha x$: $Tx = T(1x) = xT(1) = x\alpha$.
- Consider $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ linear.

- Any linear map between \mathbb{R}^n and \mathbb{R}^m is linear.
- Thus, $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$, where A is an $m \times n$ matrix.
- To find A , do the same calculation as for $T\mathbf{x} = \alpha\mathbf{x}$ but more carefully:
 - Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a basis.
 - So $\mathbf{x} = \sum_{i=1}^n \alpha_i \mathbf{e}_i$.
 - Thus, $T\mathbf{x} = \sum_{i=1}^n \alpha_i T(\mathbf{e}_i)$.
 - Each $T(\mathbf{e}_i)$ is part of the matrix that we multiply by the column vector representing \mathbf{x} .
- Multiplication of matrices is equivalent to composition of linear maps.
- Consider $T_1 : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \rightarrow \mathbb{R}^r$.
 - $T_2 \circ T_1$ is equivalent to BA , if A represents T_1 and B represents T_2 . In other words, $(T_2 \circ T_1)(\mathbf{x}) = BA\mathbf{x}$ for all \mathbf{x} .
- Recall that if $A = (\alpha_{ij})$ and $B = (\beta_{ij})$, then $(BA)_{ij} = (\sum \beta_{ik} \alpha_{kj})$.
- Properties of multiplication:

$$\begin{aligned}(AB)C &= A(BC) \\ A(B+C) &= AB+AC \\ (A+B)C &= AC+BC\end{aligned}$$

- However, it is not true in general that $AB = BA$.
- **Trace** (of an $n \times n$ matrix A): The sum of the diagonal entries of A . Denoted by **trace** (A). Given by

$$\text{trace}(A) = \sum \alpha_{ii}$$

- It is true that $\text{trace}(AB) = \text{trace}(BA)$.
 - Indeed, on the diagonals, multiplication is commutative; it's the other terms that mess you up in general.
- Invertibility of matrices.
 - In general, matrices are not invertible: Not every system of equations is solveable; $Ax = b$ does not always have a solution $x = A^{-1}b$.
- C is the inverse from the left: $CA = I$. B is the inverse from the right: $AB = I$. A matrix can have a left and a right inverse and still not be invertible. A matrix is invertible iff $C = B$.
- Any time we write “inverse,” we do so under the assumption that it exists.
- $(AB)^{-1} = B^{-1}A^{-1}$ — easy proof by multiplication.
- If $A = (a_{ij})$, $A^T = (a_{ji})$.
 - $(A^{-1})^T = (A^T)^{-1}$.
 - $(AB)^T = B^T A^T$.
- Let X, Y VS.
 - $X \cong Y^{[1]}$ if there exists a linear $T : X \rightarrow Y$ that is one-to-one and onto.
 - Check: $A(\text{basis of } X) = \text{basis of } Y$. Prove by definition and expression of elements as linear combinations.
- **Subspace**: A subset of a vector space which happens to be a vector space, itself.

¹“ X is isomorphic to Y .”