MATH 20700 (Honors Analysis in $\mathbb{R}^n$ I) Notes

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Contents

Part I

Linear Algebra

Chapter 1

Basic Notions

9/27:

- **Vector space**: Basically, a set for which you have an addition and multiplication.
- $\mathbb{F}^d$ is used for $\mathbb{R}^d$ or $\mathbb{C}^d$ in Treil (2017).
- $\mathbb{P}_n$ is the vector space of polynomials up to degree n.
- $C([0,1])$ is the set of continuous functions defined on $[0,1]$, an infinite-dimensional vector space.
- **Generating set**: A subset of a vector space, all linear combinations of which generate the vector space. *Also known as* **spanning set**.
  - Any element of VS is a linear comb. of elements of the generating set.
- **Linearly independent** (list): A list of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V$ such that $\sum_{i=1}^k \alpha_i \mathbf{v}_i = 0$ implies $\alpha_i = 0$ for all i.
- **Base**: A generating set consisting of linearly independent vectors.
- Any element of a VS can be written as a *unique* linear combination of the vectors in a base.
  - If $\mathbf{x} = \sum_{i=1}^k \alpha_i \mathbf{v}_i = \sum_{i=1}^k \beta_i \mathbf{v}_i$, then $\alpha_i = \beta_i$ for all i.
- **Linear transformation**: A function $T : X \to Y$, where X, Y are VSs, such that

$$T(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha T\mathbf{x} + \beta T\mathbf{y}$$

  for all $\mathbf{x} \in X$, $\mathbf{y} \in Y$.
- Examples of linear transformations:
  - Consider $\mathbb{P}_n$. Let $Tp_n = p_n'$. This T is linear.
  - Rotation in $\mathbb{R}^d$.
    - Think graphically about two vectors $\mathbf{x}, \mathbf{y}$.
    - Rotating and summing them is the same as summing and rotating. Same for scaling.
    - Thus, rotation is actually linear!
  - Reflection as well.
- Consider $T : \mathbb{R} \to \mathbb{R}$.
  - Any linear map on the line is a line.
  - We must have $Tx = \alpha x$: $Tx = T(1x) = xT(1) = x\alpha$.
- Consider $T : \mathbb{R}^n \to \mathbb{R}^m$ linear.

- Any linear map between $\mathbb{R}^n$ and $\mathbb{R}^m$ is linear.
- Thus, $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$, where A is an $m \times n$ matrix.

- To find A, do the same calculation as for $Tx = \alpha x$ but more carefully:
  - Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a basis.
  - So $\mathbf{x} = \sum_{i=1}^n \alpha_i \mathbf{e}_i$.
  - Thus, $T\mathbf{x} = \sum_{i=1}^n \alpha_i T(\mathbf{e}_i)$.
  - Each $T(\mathbf{e}_i)$ is part of the matrix that we multiply by the column vector representing $\mathbf{x}$.
- Multiplication of matrices is equivalent to composition of linear maps.
- Consider $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \to \mathbb{F}^r$.
  - $T_2 \circ T_1$ is equivalent to BA, if A represents T_1 and B represents T_2. In other words, $(T_2 \circ T_1)(\mathbf{x}) = BA\mathbf{x}$ for all $\mathbf{x}$.
- Recall that if $A = (\alpha_{ij})$ and $B = (\beta_{ij})$, then $(BA)_{ij} = (\sum \beta_{ik}\alpha_{kj})$.
- Properties of multiplication:

$$(AB)C = A(BC)$$
$$A(B + C) = AB + AC$$
$$(A + B)C = AC + BC$$

  - However, it is not true in general that $AB = BA$.
- **Trace** (of an $n \times n$ matrix A): The sum of the diagonal entries of A. *Denoted by* **trace (A)**. *Given by*

$$\text{trace}(A) = \sum \alpha_{ii}$$

- It is true that $\text{trace}(AB) = \text{trace}(BA)$.
  - Indeed, on the diagonals, multiplication is commutative; it's the other terms that mess you up in general.
- Invertibility of matrices.
  - In general, matrices are not invertible: Not every system of equations is solveable; $Ax = b$ does not always have a solution $x = A^{-1}b$.
- C is the inverse from the left: $CA = I$. B is the inverse from the right: $AB = I$. A matrix can have a left and a right inverse and still not be invertible. A matrix is invertible iff $C = B$.
- Any time we write "inverse," we do so under the assumption that it exists.
- $(AB)^{-1} = B^{-1}A^{-1}$ — easy proof by multiplication.
- If $A = (a_{ij})$, $A^T = (a_{ji})$.
  - $(A^{-1})^T = (A^T)^{-1}$.
  - $(AB)^T = B^T A^T$.
- Let X, Y VS.
  - $X \cong Y$[1] if there exists a linear $T : X \to Y$ that is one-to-one and onto.
  - Check: $A(\text{basis of } X) = \text{basis of } Y$. Prove by definition and expression of elements as linear combinations.
- **Subspace**: A subset of a vector space which happens to be a vector space, itself.

[1] "X is isomorphic to Y."

Chapter 2

Systems of Linear Equations

9/29:

- Row elimination:
  - Let
$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 3 & 1 & 2 & 7 \\ 2 & 1 & 2 & 1 \end{pmatrix}$$
  - Then the **eschelon form** matrix
$$A_e = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & -4 \end{pmatrix}$$
  - Lastly, the **reduced eschelon form** matrix
$$A_{re} = \begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$
- **Eschelon form**:
  - All zero rows are below nonzero rows.
  - For any nonzero row, its leading element is strictly to the left of the nonzero entry of the next row.
- **Reduced eschelon form**:
  - All pivots are 1.
  - Used to solve systems of the form $Ax = b$.
- **Inconsistent** (system of equations): A system with no solution.
  - If the last row is of the form $(0, \dots, 0, b)$ where $b \neq 0$, then there is no solution.
- Unique solution if A_e has a pivot in every column.
- There exists a solution for every b if there is a pivot in every row?
- Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a matrix. Then $\ker A = \{x \in \mathbb{R}^n : Ax = 0\}$ (subspace of $\mathbb{R}^n$) and $\operatorname{range} A = \{Ax : x \in \mathbb{R}^n\}$ (subspace of $\mathbb{R}^m$).
- Also consider $\ker(A^T)$ and $\operatorname{range}(A^T)$, the basis of the kernel and range, and dimension.
- Finite-dimensional vector spaces:

  - A basis is a generating set (so every element of V can be written uniquely as a linear combination of the basis) the length of which is equal to the dimension of V.
  - All bases of finite-dimensional vector spaces have the same number of elements.
    - Let v_1, v_2, v_3 and w_1, w_2 be two generating sets of V.
    - Then

$$\begin{aligned} v_1 &= \lambda_{11} w_1 + \lambda_{12} w_2 \\ v_2 &= \lambda_{21} w_1 + \lambda_{22} w_2 \\ v_3 &= \lambda_{31} w_1 + \lambda_{32} w_2 \end{aligned}$$

    - Suppose the only solution to $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$ is $\alpha_1 = \alpha_2 = \alpha_3 = 0$.
    - But this is not true, as we can find another one in terms of the λs.
  - If you have a list of linearly independent vectors, you can complete it into a basis.
    - If there exists a vector that can't be written as a linear combination of the list, add it to the list.
  - If you find any particular solution to a system $Ax = b$, and you add to it any element of $\ker A$, you will obtain another solution.
    - $Ax_1 = b$ and $Ax_h = 0$ implies that $A(x_1 + x_h) = b$.
    - $Ax_1 = b$ and $Ax_2 = b$ imply that $A(x_1 - x_2) = 0$, i.e., that $x_1 - x_2 \in \ker A$.
  - If $A : \mathbb{R}^n \to \mathbb{R}^m$ and $\dim \operatorname{range} A = m$, then $Ax = b$ is solveable for all $b \in \mathbb{R}^m$.
  - Let $\operatorname{rank} A = \dim \operatorname{range} A$.
  - Rank theorem:
    - $\operatorname{rank} A = \operatorname{rank} A^T$.
    - Let $A : \mathbb{R}^n \to \mathbb{R}^m$. We know that $\dim \ker A + \dim \operatorname{range} A = n$.
    - $\dim \ker A^T + \operatorname{rank} A^T = m$.
    - This theorem survives linear algebra and enters functional analysis under the name **Fredholm's alternative**.

- **Fredholm's alternative**: $Ax = b$ has a solution for all $b \in \mathbb{R}^n$ iff $\dim \ker A^T = 0$.
  - $\dim \ker A^T = 0$ implies $\operatorname{rank} A^T = m$ implies $\operatorname{rank} A = m$ implies $\dim \operatorname{range} A = m$, as desired.
- **Pivot column** (of A): A column of A where A_e has pivots.
- The **pivot columns** of A give a basis for $\operatorname{range} A$.
- The pivot rows of A_e give a basis for $\operatorname{range} A^T$.
- A basis for the kernel is enough to solve $Ax = 0$.
- If you take these three things as givens, you can prove the rank theorem.

Chapter 3

Determinants

9/29:

- The determinant, geometrically, is the volume of the object (in $\mathbb{R}^3$) you get when you take linear combinations of the vectors.
- In 2D:
  - Let v_1, v_2 be two vectors. Put tail to tail and forming a parallelogram, the determinant of the matrix (v_1, v_2) is the area of said parallelogram.
  - Linearity 1: $D(av_1, v_2, \ldots, v_n) = aD(v_1, \ldots, v_n)$ is the same as saying that if you stretch one vector by a, you scale up the area by that much, too.
  - Linearity 2: $D(v_1, \ldots, v_{k+} + v_{k-}, \ldots, v_n) = D(-) + D(+)$.
  - Antisymmetry: $D(v_1, \ldots, v_k, \ldots, v_j, \ldots, v_n) = -D(v_1, \ldots, v_j, \ldots, v_k, \ldots, v_n)$. Interchanging columns flips the sign of the determinant.
  - Basis: $D(e_1, \ldots, e_n) = 1$.
- Determinant: Denoted by $D(v_1, \ldots, v_n)$, where $(v_1, \ldots, v_n)$ is an $n \times n$ matrix.

References

Treil, S. (2017). *Linear algebra done wrong* [http://www.math.brown.edu/streil/papers/LADW/LADW_2017-09-04.pdf].