

# MATH 20700 (Honors Analysis in $\mathbb{R}^n$ I) Notes

Steven Labalme

October 3, 2021

# Contents

<b>I</b>	<b>Linear Algebra</b>	<b>1</b>
<b>1</b>	<b>Basic Notions</b>	<b>2</b>
<b>2</b>	<b>Systems of Linear Equations</b>	<b>4</b>
<b>3</b>	<b>Determinants</b>	<b>6</b>
<b>4</b>	<b>Introduction to Spectral Theory</b>	<b>8</b>
	<b>References</b>	<b>9</b>

# List of Figures

3.1 Visualizing properties of determinants. . . . .	6
---	---

Part I

# Linear Algebra

# Chapter 1

## Basic Notions

- 9/27:
- **Vector space:** Basically, a set for which you have an addition and multiplication.
  - $\mathbb{F}^d$  is used for  $\mathbb{R}^d$  or  $\mathbb{C}^d$  in Treil (2017).
  - $\mathbb{P}_n$  is the vector space of polynomials up to degree  $n$ .
  - $C([0, 1])$  is the set of continuous functions defined on  $[0, 1]$ , an infinite-dimensional vector space.
  - **Generating set:** A subset of a vector space, all linear combinations of which generate the vector space. *Also known as spanning set.*
    - Any element of VS is a linear comb. of elements of the generating set.
  - **Linearly independent (list):** A list of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$  such that  $\sum_{i=1}^k \alpha_i \mathbf{v}_i = 0$  implies  $\alpha_i = 0$  for all  $i$ .
  - **Base:** A generating set consisting of linearly independent vectors.
  - Any element of a VS can be written as a *unique* linear combination of the vectors in a base.
    - If  $\mathbf{x} = \sum_{i=1}^k \alpha_i \mathbf{v}_i = \sum_{i=1}^k \beta_i \mathbf{v}_i$ , then  $\alpha_i = \beta_i$  for all  $i$ .
  - **Linear transformation:** A function  $T : X \rightarrow Y$ , where  $X, Y$  are VSs, such that

$$T(\alpha \mathbf{x} + \beta \mathbf{y}) = \alpha T\mathbf{x} + \beta T\mathbf{y}$$

for all  $\mathbf{x} \in X, \mathbf{y} \in Y$ .

- Examples of linear transformations:
  - Consider  $\mathbb{P}_n$ . Let  $Tp_n = p'_n$ . This  $T$  is linear.
  - Rotation in  $\mathbb{R}^d$ .
    - Think graphically about two vectors  $\mathbf{x}, \mathbf{y}$ .
    - Rotating and summing them is the same as summing and rotating. Same for scaling.
    - Thus, rotation is actually linear!
  - Reflection as well.
- Consider  $T : \mathbb{R} \rightarrow \mathbb{R}$ .
  - Any linear map on the line is a line.
  - We must have  $Tx = \alpha x$ :  $Tx = T(1x) = xT(1) = x\alpha$ .
- Consider  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  linear.

- Any linear map between  $\mathbb{R}^n$  and  $\mathbb{R}^m$  is linear.
- Thus,  $T(\mathbf{x}) = A\mathbf{x}$  for all  $\mathbf{x} \in \mathbb{R}^n$ , where  $A$  is an  $m \times n$  matrix.
- To find  $A$ , do the same calculation as for  $T\mathbf{x} = \alpha\mathbf{x}$  but more carefully:
  - Let  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  be a basis.
  - So  $\mathbf{x} = \sum_{i=1}^n \alpha_i \mathbf{e}_i$ .
  - Thus,  $T\mathbf{x} = \sum_{i=1}^n \alpha_i T(\mathbf{e}_i)$ .
  - Each  $T(\mathbf{e}_i)$  is part of the matrix that we multiply by the column vector representing  $\mathbf{x}$ .
- Multiplication of matrices is equivalent to composition of linear maps.
- Consider  $T_1 : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $T_2 : \mathbb{R}^m \rightarrow \mathbb{R}^r$ .
  - $T_2 \circ T_1$  is equivalent to  $BA$ , if  $A$  represents  $T_1$  and  $B$  represents  $T_2$ . In other words,  $(T_2 \circ T_1)(\mathbf{x}) = BA\mathbf{x}$  for all  $\mathbf{x}$ .
- Recall that if  $A = (\alpha_{ij})$  and  $B = (\beta_{ij})$ , then  $(BA)_{ij} = (\sum \beta_{ik} \alpha_{kj})$ .
- Properties of multiplication:

$$\begin{aligned}(AB)C &= A(BC) \\ A(B+C) &= AB+AC \\ (A+B)C &= AC+BC\end{aligned}$$

- However, it is not true in general that  $AB = BA$ .
- **Trace** (of an  $n \times n$  matrix  $A$ ): The sum of the diagonal entries of  $A$ . Denoted by **trace** ( $A$ ). Given by

$$\text{trace}(A) = \sum \alpha_{ii}$$

- It is true that  $\text{trace}(AB) = \text{trace}(BA)$ .
  - Indeed, on the diagonals, multiplication is commutative; it's the other terms that mess you up in general.
- Invertibility of matrices.
  - In general, matrices are not invertible: Not every system of equations is solveable;  $Ax = b$  does not always have a solution  $x = A^{-1}b$ .
- $C$  is the inverse from the left:  $CA = I$ .  $B$  is the inverse from the right:  $AB = I$ . A matrix can have a left and a right inverse and still not be invertible. A matrix is invertible iff  $C = B$ .
- Any time we write “inverse,” we do so under the assumption that it exists.
- $(AB)^{-1} = B^{-1}A^{-1}$  — easy proof by multiplication.
- If  $A = (a_{ij})$ ,  $A^T = (a_{ji})$ .
  - $(A^{-1})^T = (A^T)^{-1}$ .
  - $(AB)^T = B^T A^T$ .
- Let  $X, Y$  VS.
  - $X \cong Y^{[1]}$  if there exists a linear  $T : X \rightarrow Y$  that is one-to-one and onto.
  - Check:  $A(\text{basis of } X) = \text{basis of } Y$ . Prove by definition and expression of elements as linear combinations.
- **Subspace**: A subset of a vector space which happens to be a vector space, itself.

---

<sup>1</sup>“ $X$  is isomorphic to  $Y$ .”

## Chapter 2

# Systems of Linear Equations

9/29:

- Row elimination:

- Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 3 & 1 & 2 & 7 \\ 2 & 1 & 2 & 1 \end{pmatrix}$$

- Then the **eschelon form** matrix

$$A_e = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & -4 \end{pmatrix}$$

- Lastly, the **reduced eschelon form** matrix

$$A_{re} = \begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

- **Eschelon form:**

- All zero rows are below nonzero rows.
  - For any nonzero row, its leading element is strictly to the left of the nonzero entry of the next row.

- **Reduced eschelon form:**

- All pivots are 1.
  - Used to solve systems of the form  $Ax = b$ .

- **Inconsistent** (system of equations): A system with no solution.

- If the last row is of the form  $(0, \dots, 0, b)$  where  $b \neq 0$ , then there is no solution.

- Unique solution if  $A_e$  has a pivot in every column.

- There exists a solution for every  $b$  if there is a pivot in every row?

- Let  $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a matrix. Then  $\ker A = \{x \in \mathbb{R}^n : Ax = 0\}$  (subspace of  $\mathbb{R}^n$ ) and  $\text{range } A = \{Ax : x \in \mathbb{R}^n\}$  (subspace of  $\mathbb{R}^m$ ).

- Also consider  $\ker(A^T)$  and  $\text{range}(A^T)$ , the basis of the kernel and range, and dimension.

- Finite-dimensional vector spaces:

- A basis is a generating set (so every element of  $V$  can be written uniquely as a linear combination of the basis) the length of which is equal to the dimension of  $V$ .
- All bases of finite-dimensional vector spaces have the same number of elements.
  - Let  $v_1, v_2, v_3$  and  $w_1, w_2$  be two generating sets of  $V$ .
  - Then

$$v_1 = \lambda_{11}w_1 + \lambda_{12}w_2$$

$$v_2 = \lambda_{21}w_1 + \lambda_{22}w_2$$

$$v_3 = \lambda_{31}w_1 + \lambda_{32}w_2$$

- Suppose the only solution to  $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$  is  $\alpha_1 = \alpha_2 = \alpha_3 = 0$ .
  - But this is not true, as we can find another one in terms of the  $\lambda$ s.
- If you have a list of linearly independent vectors, you can complete it into a basis.
  - If there exists a vector that can't be written as a linear combination of the list, add it to the list.
- If you find any particular solution to a system  $Ax = b$ , and you add to it any element of  $\ker A$ , you will obtain another solution.
  - $Ax_1 = b$  and  $Ax_h = 0$  implies that  $A(x_1 + x_h) = b$ .
  - $Ax_1 = b$  and  $Ax_2 = b$  imply that  $A(x_1 - x_2) = 0$ , i.e., that  $x_1 - x_2 \in \ker A$ .
- If  $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $\dim \text{range } A = m$ , then  $Ax = b$  is solvable for all  $b \in \mathbb{R}^m$ .
- Let  $\text{rank } A = \dim \text{range } A$ .
- Rank theorem:
  - $\text{rank } A = \text{rank } A^T$ .
  - Let  $A : \mathbb{R}^n \rightarrow \mathbb{R}^m$ . We know that  $\dim \ker A + \dim \text{range } A = n$ .
  - $\dim \ker A^T + \text{rank } A^T = m$ .
  - This theorem survives linear algebra and enters functional analysis under the name **Fredholm's alternative**.

- **Fredholm's alternative:**  $Ax = b$  has a solution for all  $b \in \mathbb{R}^n$  iff  $\dim \ker A^T = 0$ .
  - $\dim \ker A^T = 0$  implies  $\text{rank } A^T = m$  implies  $\text{rank } A = m$  implies  $\dim \text{range } A = m$ , as desired.
- **Pivot column** (of  $A$ ): A column of  $A$  where  $A_e$  has pivots.
- The **pivot columns** of  $A$  give a basis for  $\text{range } A$ .
- The pivot rows of  $A_e$  give a basis for  $\text{range } A^T$ .
- A basis for the kernel is enough to solve  $Ax = 0$ .
- If you take these three things as givens, you can prove the rank theorem.



## Chapter 3

# Determinants

- 9/29:
- The determinant, geometrically, is the volume of the object (in  $\mathbb{R}^3$ ) you get when you take linear combinations of the vectors.
  - In 2D:
    - Let  $v_1, v_2$  be two vectors. Put tail to tail and forming a parallelogram, the determinant of the matrix  $(v_1, v_2)$  is the area of said parallelogram.
    - Linearity 1:  $D(av_1, v_2, \dots, v_n) = aD(v_1, \dots, v_n)$  is the same as saying that if you stretch one vector by  $a$ , you scale up the area by that much, too.
    - Linearity 2:  $D(v_1, \dots, v_{k+} + v_{k-}, \dots, v_n) = D(-) + D(+)$ .
    - Antisymmetry:  $D(v_1, \dots, v_k, \dots, v_j, \dots, v_n) = -D(v_1, \dots, v_j, \dots, v_k, \dots, v_n)$ . Interchanging columns flips the sign of the determinant.
    - Basis:  $D(e_1, \dots, e_n) = 1$ .
  - Determinant: Denoted by  $D(v_1, \dots, v_n)$ , where  $(v_1, \dots, v_n)$  is an  $n \times n$  matrix.
- 10/1:
- Consider an  $n \times n$  matrix  $A$  consisting of  $n$  columns containing vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^n$ .
    - $D(A)$  is the volume of the solid  $V = \sum_{i=1}^n \alpha_i v_i$ .
    - $D(\mathbf{e}_1, \dots, \mathbf{e}_n) = 1$ .

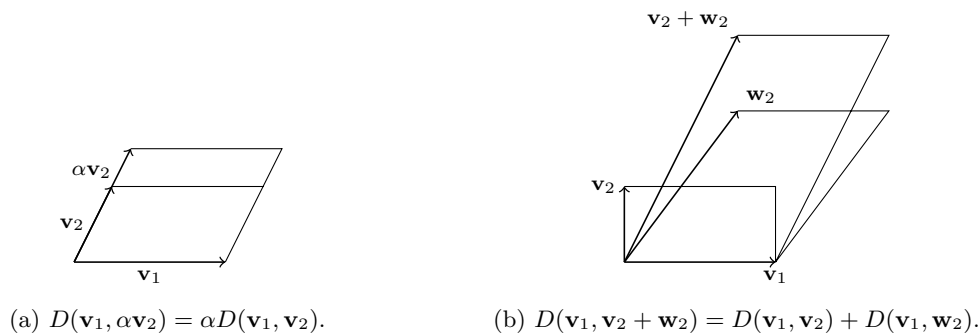


Figure 3.1: Visualizing properties of determinants.

- Basic properties of the determinant.
  - If  $A$  has a zero column, then  $\det A = 0$ : Scalar property.
  - If  $A$  has two equal columns, then  $\det A = 0$ : Multiply one by minus and add.

- If  $A$  has a column which is a multiple of another, then  $\det A = 0$ : Pull out the multiple and then you have the previous one.
  - If columns are linearly dependent, then  $\det A = 0$ : Decompose it into sums, split, add back up with previous properties.
  - The determinant is preserved under column reduction.
  - $\det A^T = \det A$ : Put everything in rref.
  - If  $A$  is not invertible, then  $\det A = 0$  (not invertible implies linearly dependent columns, implies  $\det A = 0$ ).
  - $\det(AB) = \det A \det B$ .
- Determinant of...
    - A diagonal matrix: The product of the diagonal entries (pull out the terms, and then note that the remaining identity matrix has determinant 1).
    - An upper triangular matrix: The product of the diagonal entries (column reduction to make it into a diagonal matrix, and then the property above).

## Chapter 4

# Introduction to Spectral Theory

10/1:

- **Difference equation:** Like a differential equation, but instead of writing a differentials, you write differences.
- Suppose we want to solve  $x_{n+1} = Ax_n$  with  $x_0$  given.
  - You will find that  $x_n = A^n x_0$ .
  - This gets hard to compute, so we want to find a way to simplify the computation.
- Thus, we want to diagonalize the matrix, and this concept is inherently linked to eigenvalues and eigenvectors.
  - If you can decompose the  $x_0$  into a linear combination of eigenvectors, then you can simplify the computation a lot:
$$x_n = \sum \alpha_i A^n v_i = \sum \alpha_i \lambda_i^n v_i$$
  - An  $n \times n$  matrix will have  $n$  eigenvalues. You want  $n$  linearly independent eigenvectors, creating an eigenbasis.
- To find eigenvalues and eigenvectors, we need to solve  $Ax = \lambda x$ , i.e.,  $(A - \lambda I)x = 0$ . Thus,  $\ker(A - \lambda I) \neq \{0\}$ , so  $\det(A - \lambda I) = 0$ .
- The eigenvalues of  $A$  are independent of the choice of basis of the domain of  $A$  or the range.

# References

Treil, S. (2017). *Linear algebra done wrong* [[http://www.math.brown.edu/streil/papers/LADW/LADW\\_2017-09-04.pdf](http://www.math.brown.edu/streil/papers/LADW/LADW_2017-09-04.pdf)].