Chapter 2

Basic Topology

2.1 Notes

11/1: • Equivalence relationships are denoted $A \sim B$.

  – These are...
    - Reflexive ($A \sim A$).
    - Symmetric ($A \sim B \iff B \sim A$).
    - Transitive ($A \sim B$ & $B \sim C \implies A \sim C$).
  – Equivalence relations give rise to equivalence classes.

• **Countable** (set A): A set A such that $A \sim \mathbb{N}$, in the sense that there exists a one-to-one and onto map from $\mathbb{N} \to A$.

  – Alternatively, A can be written in the form $A = \{f(n) : n \in \mathbb{N}\}$.

• **Finite countable** vs. **infinite countable** (see Rudin (1976)).

• $\mathbb{N}$ denotes the natural numbers.

• $\mathbb{N}_0$ denotes the natural numbers including 0.

• $\mathbb{Z}$ denotes the integers.

• We know that $\mathbb{N} \sim \mathbb{Z}$: Let $f : \mathbb{N} \to \mathbb{Z}$ be defined by

$$f(n) = \begin{cases} \frac{n}{2} & n \text{ even} \\ \frac{n-1}{2} & n \text{ odd} \end{cases}$$

• More facts.

  1. Every subset of a countable set is countable.
  2. Unions of countable sets are countable.
     – If the sets E_n for some finite list of numbers are countable, then $\bigcup_n E_n$ is countable.
     – Soug goes over the diagonalization method of counting.
  3. n-fold Cartesian products of countable sets are countable (we induct on n).
     – If A is countable and B is countable, then $A \times B$ is countable.
     – If A is finite and to each $\alpha \in A$ we assign a countable set E_α, $\otimes_{\alpha \in A} E_\alpha$ is countable.

• **Metric space**: A space X along with a matrix $d : X \times X \to [0, \infty)$ such that

  - $d(x, y) > 0$ iff $x \neq y$, and $d(x, x) = 0$ iff $x = 0$.
  - $d(x, y) = d(y, x)$.
  - $d(x, y) \leq d(x, z) + d(z, y)$.
- Example ($\mathbb{R}^n$):
  - We may define d by
$$d(x, y) = \sqrt{\sum (x_i - y_i)^2}$$
  - We can also define the p-metrics (recall normed spaces) with p where 2 is.
- Example ($X_p = \{f : Y \to \mathbb{R} : 1 \leq p < \infty, \int_Y |f|^p \, \mathrm{d}y < \infty\}$):
  - This is ℓ_p.
  - Define
$$\|f - g\|_p = \left[\int_Y |f - g|^p \, \mathrm{d}y\right]^{1/p}$$
- Convergence: $x_n \to x \iff d(x_n, x) \to 0$.
- **Neighborhood**: The set $N_r(p) = \{q \in X : d(p, q) < r\}$.
- **Limit point** (of E): A point p such that every neighborhood of p intersects E at a point other than p.
  - Symbolically,
$$N_r(p) \cap (E \setminus \{p\}) \neq \emptyset$$
    for all $r > 0$.
- **Isolated point** (of E): A point p such that $p \in E$ and p is not a limit point of E.
- **Closed** (set E): A set E that contains all of its limit points.
- **Interior** (point p): A point p such that there exists $N_r(p) \subset E$.
- **Open** (set E): A set E, all points of which are interior points.
- **Perfect** (set E): A set E that is closed and every point of E is a limit point of E.
- **Bounded** (set E): There exists a number M and a $y \in X$ such that $E \subset \{p : d(p, y) \leq M\}$.
- **Dense** (set E in X): A set E such that every point of X is a limit point of E or a point of E, itself.