

# Chapter 7

## Sequences and Series of Functions

### 7.1 Notes

11/15:

- Soug will not test on differentiation/integration assuming that we know them already.
- **Pointwise convergent** (sequence  $(f_n)_{n \in \mathbb{N}}$  of functions): A sequence of functions  $f_n : E \rightarrow \mathbb{R}$  such that  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  for all  $x \in E$ .
- Can we interchange “limit” in the above definition with continuity, convergence of series, integration, differentiation, etc.?
- Examples with negative answer:

1. Interchanging limits: Let  $S_{mn} = \frac{m}{m+n}$ .  $S_{mn} \rightarrow 1$  as  $m \rightarrow \infty$ , and  $S_{mn} \rightarrow 0$  as  $n \rightarrow \infty$ .
2.  $f_n(x) = x^2/(1+x)^n$ .  $f(x) = \sum_{n=1}^{\infty} f_n(x)$ . If  $x = 0$ , then  $f_n(x) = 0$  for all  $n$  and  $f(x) = 0$ . If  $x \neq 0$ , we have

$$f(x) = \sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n} = x^2 \sum_{n=1}^{\infty} X^n = \frac{x^2}{1-X} = \frac{x^2}{1-(1/(1+x^2))} = 1+x^2$$

3. Consider  $f_m(x) = \lim_{n \rightarrow \infty} (\cos(m\pi x))^2 n$ .  $\lim_{m \rightarrow \infty} f_m(x)$  goes to 0 if  $x \notin \mathbb{Q}$  and goes to 1 if  $x \in \mathbb{Q}$ .  $f_m \rightarrow \chi_{\mathbb{Q}}$ , where  $\chi_{\mathbb{Q}}$  is the characteristic function of the rationals which is not Riemann integrable (partitions, upper and lower integrals, etc.).
  4.  $f_n(x) = \sin nx / \sqrt{n} \rightarrow f(x) = 0$  for all  $x$ . However,  $f'_n(x) = \sqrt{n} \cos nx \not\rightarrow 0$
  5. If  $0 \leq x \leq 1$ , define  $f_n(x) = n^2 x(1-x^2)^n$ . We know that  $f_n(0) = 0$ .  $\lim_{n \rightarrow \infty} f_n(x) = 0$  for all  $x \in (0, 1]$ . We can show that  $\int_0^1 x(1-x^2)^n dx = 1/(2n+2)$ . Thus,  $\int_0^1 f_n(x) dx = n^2/(2n+2)$ . Limit of the functions is zero, but their integrals diverge to infinity.
- **Uniformly convergent** (sequence  $(f_n)_{n \in \mathbb{N}}$  of functions on  $E$ ): A sequence of functions  $f_n : E \rightarrow \mathbb{R}$  such that for all  $\epsilon > 0$ , there exists  $N$  such that if  $n \geq N$ , then  $|f_n(x) - f(x)| < \epsilon$  for all  $x \in E$ . Denoted by  $f_n \rightrightarrows f$ .
  - Theorem:  $f_n \rightrightarrows f$  iff  $(f_n)_{n \in \mathbb{N}}$  is uniformly Cauchy (i.e., for all  $\epsilon > 0$ , there exists  $N$  such that if  $n, m \geq N$  then  $|f_n(x) - f_m(x)| < \epsilon$  for all  $x \in E$ ).
    - Let  $M_n = \sup_{x \in E} |f_n(x) - f(x)|$ . If  $f_n \rightarrow f$  pointwise, then  $f_n \rightrightarrows f$  if  $M_n \rightarrow 0$ .
  - Theorem: If  $(f_n)_{n \in \mathbb{N}}$  and  $|f_n(x)| \leq M_n$ , then  $\sum f_n \rightrightarrows f$  if  $\sum M_n < \infty$ .
  - Theorem: If  $E$  is a compact metric space,  $f_n \rightrightarrows f$  in  $E$ ,  $x$  is a limit point of  $E$ , and  $\lim_{t \rightarrow x} f_n(t) = A_n$  exists, then  $(A_n)_{n \in \mathbb{N}}$  converges and  $\lim_{t \rightarrow x} f(t) = \lim_{n \rightarrow \infty} A_n$ .

- Corollary:  $\lim_{t \rightarrow x} \lim_{n \rightarrow \infty} f_n(t) = \lim_{n \rightarrow \infty} \lim_{t \rightarrow x} f_n(t)$ .

11/16:

- Fix  $\epsilon > 0$ . Then  $f_n \rightrightarrows f$  implies that there exists some  $N$  such that  $n, m \geq N$  implies  $|f_n(t) - f_m(t)| < \epsilon$  for all  $t \in E$ .
- $x$  is a limit point of  $E$  and  $t \rightarrow x$  implies  $|A_n - A_m| < \epsilon$ . Thus,  $(A_n)_{n \in \mathbb{N}}$  is cauchy, so there exists  $A$  such that  $A_n \rightarrow A$ .
- WTS:  $|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|$ , so we WTS the three terms on the right are small.
- There exists  $n$  such that  $|f(t) - f_n(t)| < \epsilon/3$  for all  $t$  since  $f_n \rightrightarrows f$  by hypothesis.
- Since  $t$  is in a small neighborhood of  $x$ , there exists  $n$  such that  $|A_n - A| < \epsilon/3$ .
- We also have  $|f_n(t) - A_n| < \epsilon/3$  by hypothesis.
- This is a very important proof to understand, because proofs like this pop up often.

- Corollary:  $f_n$  continuous and  $f_n \rightrightarrows f$  implies  $f$  is continuous.

- Theorem: Let  $K$  be compact. Assume

- $(f_n)_{n \in \mathbb{N}} \subset C(K) = \{f : K \rightarrow \mathbb{R} \mid f \text{ continuous}\}$ .
- $f_n \rightarrow f$  pointwise in  $K$  and  $f \in C(K)$ .
- $f_n(x) \geq f_{n+1}(x)$  for all  $x \in K$ .

Then  $f_n \rightrightarrows f$ .

- WLOG  $f = 0$ ,  $g_n = f_n - f \rightarrow 0$ ,  $g_n \geq g_{n+1} \geq 0$ .
- For all  $\epsilon > 0$ , there exists  $N$  such that  $n \geq N$  and  $0 \leq g_n(x) \leq \epsilon$  for all  $x \in K$ .
- $K_n = \{x \in K : g_n(x) \geq \epsilon\}$ .
- $g_n$  continuous implies  $K_n$  closed. This combined with  $K$  compact implies  $K_n$  is compact.
- $g_n$  decreasing implies  $K_n \supset K_{n+1}$ . Thus,  $K_n$  is a nested family of compact sets, so  $\bigcap K_n \neq \emptyset$ .
- This implies that each  $K_n$  is nonempty, contradicting the fact that each  $g_n \rightarrow 0$  for all  $x$ .
- Thus, there exists an  $N$  such that  $K_n$  is empty for all  $n \geq N$ . Thus  $g_n(x) \leq \epsilon$  for all  $x \in K$ ,  $n \geq N$ .
- Note that the compactness of  $K$  is important. If  $f : (0, 1) \rightarrow \mathbb{R}$  is defined by  $f(x) = 1/(nx + 1)$ , then  $f_n \rightarrow 0$ , but  $f_n \not\rightrightarrows f$ .

- Let  $C(X) = \{f : X \rightarrow \mathbb{R} \mid f \text{ continuous, bounded}\}$  for  $X$  a metric space.

- If we define  $\|f\| = \sum_{x \in X} |f(x)|$ , for  $f, g \in C(X)$ , we may define  $d(f, g) = \|f - g\|$ . This definition satisfies the properties of a distance function, and  $\|\cdot\|$  is a norm.

- Thus,  $C(X)$  is a complete metric space, a normed space, or, specifically, a **Banach space**.

- Theorem:  $(f_n)_{n \in \mathbb{N}} \subset C(X)$  such that  $\|f_n - f_m\|_{n, m \rightarrow \infty} \rightarrow 0$ . Then there exists  $f \in C(X)$  such that  $\|f_n - f\|_{n \rightarrow \infty} \rightarrow 0$ .

- We get such a strong statement using properties of the image, not properties of the domain.
- For all  $\epsilon > 0$ , there exists  $N$  such that  $n, m \geq N$ .
- $|f_n(x) - f_m(x)| \leq \|f_n - f_m\| < \epsilon$  for all  $x$ .
- Then there exists  $f$  such that  $f_m(x) \rightarrow f(x)$ . It follows that  $|f_n(x) - f_m(x)| < \epsilon$

- Uniform convergence and integration.

- Stieltjes integral.

- Define  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  nondecreasing.
- If we sum over the minimums/maximums of a partition times  $\alpha(x_{i+1}) - \alpha(x_i)$  instead of  $x_{i+1} - x_i$ , we obtain said integral as the upper/lower limits just like the Riemann integral.
- We write  $\int_a^b f(x) d\alpha(x)$  where  $d\alpha(x) = \alpha(x) dx$ .
- Theorem: If  $\alpha$  is nondecreasing on  $[a, b]$ ,  $f_n \in R(\alpha)$  such that  $f_n \rightrightarrows f$  on  $[a, b]$ 
  - We have

$$\begin{aligned}
 \left| \int f_n(x) d\alpha(x) - \int f(x) d\alpha(x) \right| &= \left| \int (f_n - f)(x) d\alpha(x) \right| \\
 &\leq \|f_n - f\|(\alpha(b) - \alpha(a)) \\
 &\leq \int |f_n - f| d\alpha(x) \\
 &\leq \int \|f_n - f\| d\alpha(x) \leq \|f_n - f\| \int_a^b d\alpha(x) = \|f_n - f\|(\alpha(b) - \alpha(a))
 \end{aligned}$$

11/19:

- Suppose  $f_n \rightarrow f$  and  $f'_n \rightarrow g$ . When does  $f' = g$ ?
- Theorem: If  $f_n : [a, b] \rightarrow \mathbb{R}$  is differentiable,  $f_n(x_0)$  converges for some  $x_0 \in [a, b]$ , and  $f'_n$  converges uniformly on  $[a, b]$ , then there exists  $f$  differentiable on  $[a, b]$  such that  $f_n \rightrightarrows f$  and  $f'_n \rightrightarrows f'$ .

- Assume the  $f'_n$  are continuous. Then  $f_n(x) - f_n(x_0) = \int_{x_0}^x f'_n(y) dy$ .
- Since  $f'_n \rightrightarrows g$ ,  $\int_{x_0}^x f'_n(y) dy \rightarrow \int_{x_0}^x g(y) dy$ .
- It follows since  $f_n(x_0) \rightarrow f(x_0)$  that  $f_n \rightrightarrows f$ .
- By the previous theorem, if

$$f'_n(x) = \lim_{h \rightarrow 0} \frac{f_n(x+h) - f_n(x)}{h}$$

then

$$\lim_{n \rightarrow \infty} f'_n(x) = \lim_{h \rightarrow 0} \lim_{n \rightarrow \infty} \frac{f_n(x+h) - f_n(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

- Fix  $\epsilon > 0$ . Then there exists  $N$  such that  $n, m \geq N$  such that  $|f_n(x_0) - f_m(x_0)| < \epsilon/2$  and  $|f'_n(t) - f'_m(t)| < \epsilon/2$  for all  $t \in [a, b]$ .
- We know that  $f_n(t) - f_n(x_0) = \int_{x_0}^t f'_n(y) dy$  and  $f_m(t) - f_m(x_0) = \int_{x_0}^t f'_m(y) dy$ .
- Thus,
 
$$|f_n(t) - f_n(x_0)| \leq |f_n(t) - f_m(t)| + |f_m(t) - f_m(x_0)| + |f_m(t) - f_m(x_0)|$$
- Let  $f_n(t) - f_n(x_0) = c_n(t - x_0)$  and  $f_m(t) - f_m(x_0) = c_m(t - x_0)$ .
- ...
- Let  $f : [a, b] \rightarrow \mathbb{R}$  be continuous. What conditions on  $f$  imply that  $f'$  exists?
- Suppose  $f$  is Lipschitz continuous (equivalent to saying there exists  $L > 0$  such that  $|f(x) - f(y)| \leq L|x - y|$ ); then  $f'$  exists **almost everywhere**.
  - If  $f$  differentiable, this is equivalent to saying  $f$  bounded.
- **Almost everywhere**: Something happens almost everywhere if the set of places where it doesn't happen has measure zero.
- Suppose  $f$  is **Hölder continuous**, then  $f'$  does not exist?
- **Hölder continuous** (function  $f$ ): There exists  $L > 0$  such that  $|f(y) - f(x)| < L|x - y|^\alpha$  where  $\alpha \in (0, 1)$

- Suppose  $f$  exists such that  $f$  is Hölder continuous in a neighborhood of every point in the domain. This function is not anywhere differentiable. Such a function does indeed exist (and it's Brownian motion). The construction of such a function is the essence of Stochastic analysis.
  - Probabilistically: Has mean zero, distributed as a normal function like the Gaussian, and the increments are independent of each other.
  - Analytically: It's a function that is Hölder continuous at half plus  $\epsilon$  for every  $\epsilon$  and it is nowhere differentiable.
- Theorem: There exists  $f : \mathbb{R} \rightarrow \mathbb{R}$  continuous but nowhere differentiable.
  - This theorem is due to Weierstrass and as such, such functions are typically called Weierstrass functions.
- A general class of functions that are nowhere differentiable (not in Rudin (1976); we don't have to prove this).
  - Example 1:

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

where  $0 < a < 1$ ,  $b$  positive odd integer greater than 1, and  $ab > 1 + \frac{3}{2}\pi$ .

■ This function at every point oscillates more and more and more.

- Rudin (1976)'s simple example.
  - $\phi : [-1, 1] \rightarrow \mathbb{R}$  defined by  $\phi(x) = |x|$  is not differentiable at zero.
  - Takes  $\phi$  extends it periodically with period 2, creating a sawtooth function.
  - Repeat the behavior so that the nondifferentiability becomes more and more frequent to get

$$f(x) = \sum_1^{\infty} \left(\frac{3}{4}\right)^n \phi(4^n x)$$

- This is continuous.
- Fix any  $x \in \mathbb{R}$ ,  $m \in \mathbb{N}$ . Then  $\delta_m = \pm \frac{1}{2} \cdot 4^{-m}$ .
- Then consider  $4^m x$ ,  $4^m(x + \delta_m)$ .
- Rudin asserts

$$\left| \frac{f(x + \delta_m) - f(x)}{\delta_m} \right| \rightarrow \infty$$

as  $m \rightarrow \infty$  for all  $x$ .

11/29:

- Finding a uniformly convergent subsequence of a sequence of functions.
  - Pointwise, uniformly, bounded if there exist  $M_x$  such that  $|f_n(x)| \leq M_x$  for all  $n, x$ . Uniformly bounded if there exists  $M$  such that  $|f_n(x)| \leq M$  for all  $n, x$ .
- Theorem: If  $(f_n)_{n \in \mathbb{N}}$  is pointwise bounded and  $E \subset X$  is countable, then there exists a subsequence  $f_{n_k}$  which converges for every  $x \in E$ .
  - Let  $E = \{x_i : i \in \mathbb{N}\}$ . Consider  $f_n(x_1)$ .  $f_{1,k}(x_1)$  converges.
  - $S_1 : f_{1,1}(x_1), f_{1,2}(x_1), f_{1,3}(x_1), f_{1,4}(x_1), \dots$
  - $S_2 : f_{2,1}(x_2), f_{2,2}(x_2), f_{2,3}(x_2), f_{2,4}(x_2), \dots$
  - Now consider  $f_{2,k}(x_3)$ .
  - $S_3 : f_{3,1}, f_{3,2}, f_{3,3}, f_{3,4}, \dots$

- Continue on and on to  $S_4, S_5, \dots$ . We know that each of these sequences converges pointwise by hypothesis.
  - Now consider the diagonal sequence  $f_{1,1}, f_{2,2}, f_{3,3}, f_{4,4}$ .
    - This subsequence of the original sequence we may call  $g_k$ .
    - We posit that  $g_k$  converges for every  $x \in E$ .
  - Theorem: There exists  $f_n$  which is uniformly bounded but does not converge uniformly.
    - Let  $f_n(x) = \sin(2\pi x)$  for  $0 \leq x \leq 2\pi$ .
    - Let  $f_n(x) = x^2/(x^2 + (1 - nx)^2)$  on  $0 \leq x \leq 1$ . This sequence is uniformly bounded, converges pointwise, but  $f_n(1/n) = 1$  so  $f_n$  cannot converge uniformly to zero.
  - What does it mean that  $f_n : [0, 1] \rightarrow \mathbb{R}$  does not converge uniformly?
    - It means that there exists a subsequence of the functions evaluated at certain points that is always greater than or equal to some fixed distance away from the limit.
  - Equicontinuity:  $\mathcal{F}\{f : X \rightarrow \mathbb{R}\}$  for  $(X, d)$  a metric space is equicontinuous iff for all  $\epsilon > 0$ , there exists a  $\delta > 0$  such that  $d(x, y) < \delta$  implies  $|f(x) - f(y)| < \epsilon$  for all  $x, y \in X, f \in \mathcal{F}$ .
  - Modulus of continuity:  $f : X \rightarrow \mathbb{R}$  is continuous at  $x$ . A modulus of continuity is a function  $\omega_X : [0, 1] \rightarrow [0, 1]$  such that  $|f(y) - f(x)| \leq \omega_X|y - x|$ .
  - The final result we'll prove: **Arzelà-Ascoli theorem**: If we have a family of functions on a compact set and we have a dense subset of that set, then if we have a sequence of functions that are equicontinuous, then they converge uniformly.
- 12/1:
- The final will be in this room.
  - The last PSet will not be graded, but there will be similar questions on the final.
  - No class on Friday.
  - Review from last time:
    - Equiboundedness and equicontinuity.
    - If  $E$  is a dense subset of  $X$ , then any pointwise bounded sequence has a subsequence that converges on  $E$  (diagonal argument.)
  - **Equicontinuous** (sequence  $\{f_n\}$ ): For all  $\epsilon > 0$ , there exists a  $\delta > 0$  such that  $d(x, y) < \delta$  implies  $|f_n(x) - f_n(y)| < \epsilon$  for all  $n$ .
  - Theorem: If  $K$  is a compact set and  $\{f_n\} \in C(K)$  converges uniformly on  $K$ , then the  $f_n$ 's are equicontinuous on  $K$ .
    - The  $f_n$  are uniformly Cauchy: For all  $\epsilon > 0$ , there exists  $N$  such that  $n, m \geq N$  imply  $\|f_n - f_m\| < \epsilon$  where  $\|f_n - f_m\| = \sup_{x \in K} (f_n - f_m)(x)$ .
    - If  $n \geq N$ , then  $|f_n(x) - f_n(y)| \leq |f_N(x) - f_N(y)| + 2\|f_n - f_N\|$  (since  $f_n(x) - f_n(y) = f_n(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f_n(y)$ ).
    - Thus  $|f_n(x) - f_n(y)| \leq |f_N(x) - f_N(y)| + 2\|f_n - f_N\| < 3\epsilon$  implies  $|f_i(x) - f_i(y)| < \epsilon$  if  $|x - y| < \delta$  for  $i = 1, \dots, N$ .
  - **Arzelà-Ascoli theorem**: If  $K$  is compact,  $(f_n)_{n \geq 1} \subset C(K)$  which are pointwise bounded and equicontinuous, then
    - (a)  $(f_n)_{n \geq 1}$  are uniformly bounded (equicontinuous).
    - (b) There exists  $(f_{n_k})_{k \geq 1}$  which converges uniformly on  $K$ .

- Since  $K$  is compact, you can cover it by finitely many balls of radius  $\delta$ .
- Thus  $|f_n(p_k)| \leq M = \max(M_{p_1}, \dots, M_{p_k})$  where  $K \subset \bigcup_{k \in K} B(p_k, \delta)$ .
- $K$  has a countable dense subset  $E$  (Exercise 2.25).
- $|f_n(x)| \leq M + \epsilon$  for all  $x$ .
- $\{B(x, \delta)\}_{x \in E}$  is an open cover of  $K$ .
- Thus has a finite subcover.
- ...
- If  $f_n : \mathbb{R}^n \rightarrow \mathbb{R}$  are continuous, equibounded, equicontinuous, then there exists  $f_{n_k}$  which converges locally uniformly to some  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ .
- How do you learn math?
  - In an ideal world, study by looking at theorems, thinking that you should be able to prove it, and etc.
  - Since we don't have the time to do everything ourselves, don't just get stuck in a place; move on and continue thinking if you have to.
- Let  $\dot{\phi} = f(x, t)$  and  $x(0) = c$ . Let  $\phi : \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ . Assume  $\phi$  is bounded and continuous. Then there exists a solution of the differential equation and initial condition.
  - We need to find a function  $x : [0, 1] \rightarrow \mathbb{R}$  continuous such that  $x(t) = c + \int_0^t \phi(x(s), s) ds$ .
  - Let  $t_i = i/N$ . Then  $x_n(t) = \phi(x_i, t_i)$  on  $t_i < t < t_{i+1}$ .
  - $x_n(t) = x_n(t_i) + \phi(x_i, t_i)(t - t_i)$ .
  - $\frac{x_{i+1} - x_i}{1/N} = \phi(x_i, t_i)$ .
  - $\Delta_n(t) = x'_n(t) - \phi(x_n(t), t)$  for  $\phi(x_i, t_i) - \phi(x_n(t), t)$  measures how close our solution is.
  - All of these things imply that our final formula is
 
$$x_n(t) = c + \int_0^t [\phi(x_n(s), s) + \Delta_n(s)] ds$$
  - If we know that  $x_n \rightrightarrows x$ , then  $\Delta_n \rightrightarrows 0$ .
  - We then use the A-A theorem to imply convergence.
- When we get to MATH 208, say we didn't do any multivariable calculus in MATH 207.
  - We didn't do how to integrate in  $\mathbb{R}^n$ , how to integrate by parts (Stoke's theorem), Lagrange multipliers (constraint minimization).
- Problem 4.23: Show the inequalities at the bottom first and then use those to show continuity.
  - Consider  $\lim_{t \rightarrow u} f(t)$ . Approach from two sides separately and show cancellation??? Chloe will write a solution.
  - This is a particular trick for convex functions; it's not exactly recyclable.
- The trick for Problem 4.26 is recyclable.
- Linear algebra questions on the final are easier than the midterm.
  - The last question will be the hard one this time.