Chapter 6

Structure of Operators on Inner Product Spaces

6.1 Notes

10/11:

- Spectral decomposition of self-adjoint linear maps.
  - Can we write a map in term of the eigenvalues only?
  - Let $A : X \to X$ be linear and self-adjoint. Where $\dim X < \infty$.
  - Let A have eigenvalues $\lambda_1, \dots, \lambda_n$ and eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$. The there is an orthonormal basis of X consisting of eigenvectors of A. An operator is self-adjoint if $A = A^*$.
  - If A is self-adjoint, then A can be written as diagonal with the eigenvalues on the diagonal with respect to some orthonormal basis of eigenvectors.
  - Let $\mathbb{F} = \mathbb{C}$.
- If there exists an orthonormal basis $u_1, \dots, u_n$ of X such that A is triangular, then $A = UTU^*$ where U is unitary and T is upper triangular.
  - Proved with induction on $\dim X$.
  - $\dim X = 1$ is clear.
  - Assume for $\dim X = n - 1$, WTS for $\dim X = n$.
  - The subspace has a basis $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ such that A has a diagonal form.
  - Let $u \in X$ be linearly independent of $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$.
  - Let λ be the remaining eigenvalue and u the corresponding eigenvector. Let $E = \text{span}(u)$. Then make the matrix λ in the upper left corner, and block diagonal with "A_{n-1}" in the bottom right corner, zeroes everywhere else.
- **Self-adjoint** (matrix A): A linear map $A : X \to X$ where $\dim X < \infty$ such that $A = A^*$.
  - Similarly, $(Ax, y) = (x, Ay)$.
  - A self-adjoint implies all eigenvalues are real, eigenvectors corresponding to different eigenvalues are orthogonal.
    - Soug proves this.
- **Strictly positive** (operator A): A self-adjoint operator $A : X \to X$ such that $(Ax, x) > 0$ for all $x \neq 0$. *Also known as* **positive definite**.
  - Implies that all eigenvalues are strictly positive.

- **Nonnegative** (operator A): A self-adjoint operator $A : X \to X$ such that $(Ax, x) \geq 0$ for all $x \neq 0$. *Also known as* **definite**.
  - All eigenvalues are nonnegative.
- Suppose $A \geq 0$ is self-adjoint. Then there exists a unique self-adjoint $B \geq 0$ such that $B^2 = A$.
  - A self-adjoint is diagonal (wrt. some basis).
  - A positive means that all eigenvalues (diagonal entries) are positive.
  - Thus, take
$$B = \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sqrt{\lambda_n} \end{pmatrix}$$
  - Suppose $B^2 = A$, $C^2 = A$. Then we have an orthonormal basis corresponding to B and an orthonormal basis corresponding to C. It follows that $B^2 = C^2 = A$. Write B^2x and C^2x in terms of their bases; will necessitate that the bases are the same.

10/13:

- If we get yes/no questions, we don't have to justify.
- Cauchy-Schwarz inequality:
$$|(\mathbf{x}, \mathbf{y})| \leq \|\mathbf{x}\|\|\mathbf{y}\|$$
  - Real spaces, V vs. $(\cdot, \cdot)$ inner product.
  - Proof:
$$\begin{aligned} 0 &\leq \|\mathbf{x} + t\mathbf{y}\|^2 \\ &= t^2\|\mathbf{y}^2\| + 2t(\mathbf{x}, \mathbf{y}) + \|\mathbf{x}\|^2 \end{aligned}$$
    Thus, the discriminant must be less than zero (because the whole polynomial is positive, so the discriminant [the opposite of the x^0 term of the factored form of the polynomial] must be less than zero so the polynomial doesn't get dragged down to negative values):
$$(\mathbf{x}, \mathbf{y})^2 - \|\mathbf{x}\|^2\|\mathbf{y}\|^2 \leq 0$$
    Taking square roots of both sides proves the desired inequality.
- Recall that if $A^* = A$, then all eigenvalues are real and all eigenvectors of distinct eigenvalues are orthogonal to each other.
- **Normal** (matrix): A matrix N such that $N^*N = NN^*$.
  - Examples: Diagonal, self-adjoint, and unitary operators are all normal.
- Any normal operator in a complex vector spae has an orthonormal set of eigenvectors, e.g., $N = UDU^*$.
  - Proof: N is upper triangular wrt. some basis (because all matrices are). WTS any normal upper triangular matrix is diagonal. Done by induction on the dimension of N from $n = 2$.
  - Assume the claim for every $(n-1) \times (n-1)$ normal upper triangular matrix.
  - Let
$$N = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & & & \\ 0 & & & \\ \vdots & & N_1 & \\ 0 & & & \end{pmatrix}$$
    (we know every normal matrix can be written in this upper triangular form)

  - Then just compute NN^* and N^*N. Knowing they have to be equal, we have that $a_{12} = \cdots = a_{1n} = 0$.
  - We can also prove from the above (block diagonal multiplication) that N_1 is normal. Thus, it's diagonal, too. Therefore, the whole thing is diagonal.

- N is normal if and only if $\|N\mathbf{x}\| = \|N^*\mathbf{x}\|$.
  - Proof: $(N\mathbf{x}, N\mathbf{y}) = (N^*N\mathbf{x}, \mathbf{y}) = (NN^*\mathbf{x}, \mathbf{y}) = (N^*\mathbf{x}, N^*\mathbf{y})$. This is equivalent to the desired condition.
- If A is nonnegative and $(A\mathbf{e}_k, \mathbf{e}_k) = a_{kk}$, then

$$\sum_{i,j=1}^{n} a_{ij}\mathbf{x}_i\mathbf{x}_j$$

- **Positive definite** (matrix): An $n \times n$ self-adjoint matrix such that $(A\mathbf{x}, \mathbf{x}) > 0$ for all $\mathbf{x} \in X$.
- Let $A : X \to Y$, $\dim X = \dim Y$. Then AA^* is positive semidefinite. And there exists a unique square root $R = \sqrt{A^*A}$.
  - Proof: $(A^*A\mathbf{x}, \mathbf{x}) = (A\mathbf{x}, A\mathbf{x}) = \|A\mathbf{x}\|^2 \geq 0$.
- **Modulus** (of A): The matrix $|A| = \sqrt{A^*A}$.
- Check $\||A|\mathbf{x}\| = \|A\mathbf{x}\|$.

$$\||A|\mathbf{x}\|^2 = (|A|\mathbf{x}, |A|\mathbf{x}) = (|A|^*|A|\mathbf{x}, \mathbf{x}) = (A^*A\mathbf{x}, \mathbf{x}) = (A\mathbf{x}, A\mathbf{x}) = \|A\mathbf{x}\|^2$$

- Let $A : X \to X$ be a linear operator. Then $A = U|A|$ where U is unitary.
- Look at singular matrices.

10/15:

- Recall that if $A : X \to Y$, we have that A^*A is semidefinite, positive, and self adjoint.
  - Thus, there exists a unique matrix $R = \sqrt{A^*A} \geq 0$, which we define to be $|A| = \sqrt{A^*A}$.
- Polar form of a matrix:

$$A = U|A|$$

  - This may not be unique!
  - Proof: Suppose $A\mathbf{x} = U(|A|\mathbf{x})$. $A\mathbf{x} \in \operatorname{range} A$, and $|A|\mathbf{x} \in \operatorname{range}(|A|)$. $\mathbf{x} \in \operatorname{range}(|A|)$ implies that there exists $\mathbf{v} \in X$ such that $x = |A|\mathbf{v}$.
  - Define $U\mathbf{x} = A\mathbf{x}$. U is a well-defined linear map.
  - $\|U_0\mathbf{x}\| = \|A\mathbf{x}\| = \||A|\mathbf{v}\| = \|\mathbf{x}\|$.
  - U is an isometry.
  - $\operatorname{range}|A| \to X$.
  - Use $\ker A = \ker|A| = (\operatorname{range} A)^\perp$ to extend U_0 to U: $U = U_0 + U_1$.
- **Singular values** (of a matrix): The eigenvalues of $|A|$.
  - So if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A^*A, the singular values of A are $\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}$.
- Let $A : X \to Y$ be a linear map.
  - Let $\sigma_1, \dots, \sigma_n$ be the signular values of A. Then $\sigma_1, \dots, \sigma_n > 0$.
  - Additionally, if $\mathbf{v}_1, \dots, \mathbf{v}_n$ is an orthonormal basis of eigenvectors of A^*A, then the list of n vectors $\mathbf{w}_1, \dots, \mathbf{w}_n$ defined by $\mathbf{w}_k = 1/\sigma_k A\mathbf{v}_k$ for each $k = 1, \dots, n$ is orthonormal.

- Proof:

$$(\mathbf{w}_k, \mathbf{w}_j) = \frac{1}{\sigma_k \sigma_k}(A\mathbf{v}_k, A\mathbf{v}_j) = \frac{1}{\sigma_k \sigma_j} = \frac{1}{\sigma_k \sigma_j}(A^* A\mathbf{v}_k, \mathbf{v}_j) = \frac{\sigma_k^2}{\sigma_k \sigma_j}(\mathbf{v}_k, \mathbf{v}_j) = 0$$

and

$$\|\mathbf{w}_k\| = \frac{1}{\sigma_k}\|A\mathbf{v}_k\| = \frac{1}{\sigma_k}\||A|\mathbf{v}_k\| = 1$$

- Schmidt decomposition of A:

$$A\mathbf{x} = \sum^{r} \sigma_k(\mathbf{x}, \mathbf{v}_k)\mathbf{w}_k$$

- This is because $\mathbf{x} = \sum(\mathbf{x}, \mathbf{v}_k)\mathbf{v}_k$, so by the above,

$$A\mathbf{x} = \sum^{n}(\mathbf{x}, \mathbf{v}_k)A\mathbf{v}_k = \sum^{r} \sigma_k(\mathbf{x}, \mathbf{v}_k)\mathbf{w}_k$$

- **Operator norm**: $\|A\| = \max\{\|A\mathbf{x}\| : \|\mathbf{x}\| \leq 1\}$.
- Properties of the operator norm:
  - $\|A\mathbf{x}\| \leq \|A\|\|\mathbf{x}\|$.
  - $\|\alpha A\| = |\alpha|\|A\|$.
  - $\|A + B\| \leq \|A\| + \|B\|$.
  - $\|A\| \geq 0$.
  - $\|A\| = 0$ iff $A = 0$.
- **Frobenious norm**: The norm $\|A\|_2^2 = \operatorname{tr}(A^* A)$.
- The operator norm is always less than or equal to the Frobenius norm.
- If $A : \mathbb{F}^n \to \mathbb{F}^n$, then $A = W\Sigma V^*$ where σ is a diagonal matrix of nonzero singular values.
- The operator norm of A is the largest of the singular values.
- An orthogonal matrix can be decomposed to a block-diagonal matrix of rotations.

10/18:

- Soug tests what he teaches and doesn't give super tricky questions.
- Structure of orthogonal matrices.
- **Orthogonal** (matrix): A unitary matrix U with all elements real and $|\det U| = 1$.
- Theorem: Let U be an orthogonal operator on $\mathbb{R}^n$ such that $\det U = 1$. Then there exists an orthonormal basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ such that with respect to this basis,

$$U = \begin{pmatrix} R_{\phi_1} & & & \mathbf{0} \\ & \ddots & & \\ & & \mathbb{R}_{\phi_k} & \\ \mathbf{0} & & & I_{n-2k} \end{pmatrix}$$

  where each R_{ϕ_i} is a 2×2 rotation matrix.
  - If you are in $\mathbb{R}^7$ for example, you would be able to express U as a composition of at most 3 rotation maps and the identity map.
  - Each rotation map acts on two orthonormal vectors.
  - Proof: $P(\lambda)$ is the n-degree characteristic polynomial $\det(U - \lambda I) = 0$. The eigenvalues are the roots of it.

- $p(\lambda) = 0$ if and only if $p(\bar{\lambda}) = 0$.
  - $\lambda \in \mathbb{C}$ is an eigenvalue with eigenvector $\mathbf{u} \neq 0$ iff $U\mathbf{u} = \lambda\mathbf{u}$ and $U\bar{\mathbf{u}} = \bar{\lambda}\bar{\mathbf{u}}$.
- Recall that U unitary implies $|\lambda| = 1$.
  - Proof[1]: $\|U\mathbf{x}\| = \|\mathbf{x}\|$ and $U\mathbf{x} = \lambda\mathbf{x}$. Thus,
$$\|U\mathbf{x}\| = \|\lambda\mathbf{x}\| = |\lambda|\|\mathbf{x}\| = \|\mathbf{x}\|$$
and since $\mathbf{x} \neq 0$, we can divide by $\|\mathbf{x}\|$, so $|\lambda| = 1$.
- Let $\mathbf{u} = \operatorname{Re}\mathbf{u} + \operatorname{Im}\mathbf{u}$.
- It follows that we may define
$$\mathbf{x} = \operatorname{Re}\mathbf{u} = \frac{\mathbf{u}+\bar{\mathbf{u}}}{2} \qquad \mathbf{y} = \operatorname{Im}\mathbf{u} = \frac{\mathbf{u}-\bar{\mathbf{u}}}{2}$$
- Thus, $\mathbf{u} = \mathbf{x} + i\mathbf{y}$ and $\bar{\mathbf{u}} = \mathbf{x} - i\mathbf{y}$.
- Since $U\mathbf{x} = \frac{U\mathbf{u}+U\bar{\mathbf{u}}}{2} = \frac{\lambda\mathbf{u}+\bar{\lambda}\bar{\mathbf{u}}}{2}$, $U\mathbf{y} = \operatorname{Im}(\lambda\mathbf{u}) = \operatorname{Re}(\lambda\mathbf{u})$.
- Since $|\lambda| = 1$, $\lambda = e^{i\alpha}$ and $\bar{\lambda} = e^{-i\alpha}$.
- It follows that $U\mathbf{x} = (\cos\alpha)\mathbf{x} - (\sin\alpha)\mathbf{y}$ and $U\mathbf{y} = (\cos\alpha)\mathbf{y}) + (\sin\alpha)\mathbf{x}$.
- Thus, since $U\mathbf{x} = \operatorname{Re}\lambda\mathbf{u}$, we have that
$$\begin{aligned}\lambda\mathbf{u} &= (\cos\alpha + i\sin\alpha)(\mathbf{x} + i\mathbf{y})\\ &= (\cos\alpha)\mathbf{x} - (\sin\alpha)\mathbf{y} + i[(\cos\alpha)\mathbf{y} + (\sin\alpha)\mathbf{x}]\end{aligned}$$
- If E_λ is a 2 dimensional space spanned by $\mathbf{x}$ and $\mathbf{y}$ and invariant by U. Thus, any block of the desired matrix leaves its desired sub-block invariant.
- We also know that the eigenvectors of a unitary matrix corresponding to different eigenvalues are orthogonal.
- Thus, $\|\mathbf{x}\| = \|\mathbf{y}\| = \sqrt{2}/2\|\mathbf{u}\|$, $\mathbf{x} \perp \mathbf{y}$.
- Let $\mathbf{x}, \mathbf{y}$ complete the theorem to form a basis of $\mathbb{R}^n$.
- It will follow that
$$U = \begin{pmatrix} R_\alpha & \mathbf{0} \\ \mathbf{0} & U_1 \end{pmatrix}$$
where U_1 is orthogonal, and we may repeat the process.

6.2 Chapter 6: Structure of Operators on Inner Product Spaces

From Treil (2017).

10/24:

- Theorem 6.1.1: Let $A : X \to X$ be an operator acting in a complex inner product space. Then there exists an orthonormal basis $\mathbf{u}_1, \dots, \mathbf{u}_n$ of X such that the matrix of A in this basis is upper triangular. In other words, any $n \times n$ matrix A can be represented as $A = UTU^*$, where U is unitary and T is upper-triangular.
- Theorem 6.1.2: Let $A : X \to X$ be an operator acting on a real inner product space. Suppose that all eigenvalues of A are real. Then there exists an orthonormal basis $\mathbf{u}_1, \dots, \mathbf{u}_n$ in X such that the matrix of A in this basis is upper triangular. In other words, any real $n \times n$ matrix A with all real eigenvalues can be represented as $T = UTU^* = UTU^T$, where U is orthogonal and T is a real upper-triangular matrix.

[1] This would be a good exam question.

- Theorem 6.2.1: Let $A = A^*$ be a self-adjoint operator in an inner product space X (the space can be complex or real). Then all eigenvalues of A are real and there exists an orthonormal basis of eigenvectors of A in X.

  Equivalently (see Theorem 6.2.2), A can be represented as $A = UDU^*$ where U is a unitary matrix and D is a diagonal matrix with real entries. Moreover, if A is real, U can be chosen to be real, i.e., orthogonal.

- Proposition 6.2.3: Let $A = A^*$ be a self-adjoint operator and let $\lambda, \mathbf{u}, \mu, \mathbf{v}$ be such that $A\mathbf{u} = \lambda\mathbf{u}$ and $A\mathbf{v} = \mu\mathbf{v}$. Then if $\lambda \neq \mu$, $\mathbf{u} \perp \mathbf{v}$.

- Since complex multiplication is commutative,

  $$D^*D = DD^*$$

  for every diagonal matrix D.

  - It follows that $A^*A = AA^*$ if the matrix of A in some orthonormal basis is diagonal.

- Theorem 6.2.4: Any normal operator N in a complex vector space has an orthonormal basis of eigenvectors.

  Equivalently, any matrix N satisfying $N^*N = NN^*$ can be represented as $N = UDU^*$ where U is unitary and D is diagonal.

- Proposition 6.2.5: An operator $N : X \to X$ is normal iff

  $$\|N\mathbf{x}\| = \|N^*\mathbf{x}\|$$

  for all $\mathbf{x} \in X$.

- **Hermitian square** (of A): The matrix A^*A.

- **Modulus** (of A): The unique positive semidefinite square root $\sqrt{A^*A}$.

- Proposition 6.3.3: For a linear operator $A : X \to Y$,

  $$\||A|\mathbf{x}\| = \|A\mathbf{x}\|$$

- Corollary 6.3.4: $\ker A = \ker |A|$.

- Theorem 6.3.5: Let $A : X \to X$ be an operator (square matrix). Then A can be represented as

  $$A = U|A|$$

  where U is a unitary operator.

- **Singular value** (of A): An eigenvalue of $|A|$.

  - A positive square root of an operator of A^*A.

- Proposition 6.3.6: Let $\sigma_1, \ldots, \sigma_n$ be the singular values of A, ordered such that $\sigma_1, \ldots, \sigma_r$ are the nonzero singular values, and let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be an orthonormal basis of eigenvectors of A^*A. Then the system

  $$\mathbf{w}_k = \frac{1}{\sigma_k} A\mathbf{v}_k$$

  for $k = 1, \ldots, r$ is orthonormal.

- **Schmidt decomposition** (of A): The decompositions

$$A = \sum_{k=1}^{r} \sigma_k \mathbf{w}_k \mathbf{v}_k^*$$

and

$$A\mathbf{x} = \sum_{k=1}^{r} \sigma_k (\mathbf{x}, \mathbf{v}_k) \mathbf{w}_k$$

  - Note that these can be verified by plugging $\mathbf{x} = \mathbf{v}_j$ for each $j = 1, \dots, n$ into the latter equation.