7 Basic Topology II

From Rudin (1976).

Chapter 2

11/15: **16.** Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space, with $d(p,q) = |p-q|$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$, but that E is not compact. Is E open in $\mathbb{Q}$?

17. Let E be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0,1]$? Is E compact? Is E perfect?

18. Is there a nonempty perfect set in $\mathbb{R}^1$ which contains no rational number?

19. (a) If A and B are disjoint closed sets in some metric space X, prove that they are separated.

(b) Prove the same for disjoint open sets.

(c) Fix $p \in X$ and $\delta > 0$. Define A to be the set of all $q \in X$ for which $d(p,q) < \delta$, and define B similarly with $>$ in place of $<$. Prove that A and B are separated.

(d) Prove that every connected metric space with at least two points is uncountable. (Hint: Use (c).)

20. Are closures and interiors of connected sets always connected? (Hint: Look at subsets of $\mathbb{R}^2$.)

21. Let A and B be separated subsets of some $\mathbb{R}^k$, suppose $\mathbf{a} \in A$ and $\mathbf{b} \in B$, and define

$$\mathbf{p}(t) = (1-t)\mathbf{a} + t\mathbf{b}$$

for all $t \in \mathbb{R}^1$. Let $A_0 = \mathbf{p}^{-1}(A)$, $B_0 = \mathbf{p}^{-1}(B)$.

(a) Prove that A_0 and B_0 are separated subsets of $\mathbb{R}^1$.

(b) Prove that there exists $t_0 \in (0,1)$ such that $p(t_0) \notin A \cup B$.

(c) Prove that every convex subset of $\mathbb{R}^k$ is connected.

22. A metric space is called **separable** if it contains a countable dense subset. Show that $\mathbb{R}^k$ is separable. (Hint: Consider the set of points which only have rational coordinates.)

23. A collection $\{V_\alpha\}$ of open subsets of X is said to be a **base** for X if the following is true: For every $x \in X$ and every open set $G \subset X$ such that $x \in G$, we have $x \in V_\alpha \subset G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$. Prove that every separable metric space has a *countable* base. (Hint: Take all neighborhoods with rational radius and center in some countable dense subset of X.)

24. Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable. (Hint: Fix $\delta > 0$ and pick $x_1 \in X$. Having chosen $x_1, \dots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for each $i = 1, \dots, j$. Show that this process must stop after a finite number of steps, and that X can therefore be covered by finitely many neighborhoods of radius δ. Take $\delta = 1/n$ $(n = 1, 2, 3, \dots)$ and consider the centers of the corresponding neighborhoods.)

25. Prove that every compact metric space K has a countable base, and that K is therefore separable. (Hint: For every positive integer n, there are finitely many neighborhoods of radius $1/n$ whose union covers K.)

26. Let X be a metric space in which every infinite subset has a limit point. Prove that X is compact. (Hint: By Exercises 2.23 and 2.24, X has a countable base. It follows that every open cover of X has a *countable* subcover $\{G_n\}_{n \in \mathbb{N}}$. If no finite subcollection of $\{G_n\}$ covers X, then the complement F_n of $\bigcup_1^n G_i$ is nonempty for each n, but $\bigcap F_n$ is empty. If E is a set which contains a point from each F_n, consider a limit point of E, and obtain a contradiction.)

27. Define a point p in a metric space X to be a **condensation point** of a set $E \subset X$ if every neighborhood of p contains uncountably many points of E. Suppose $E \subset \mathbb{R}^k$ is uncountable, and let P be the set of all condensation points of E. Prove that P is perfect and that at most countably many points of E are not in P. In other words, show that $P^c \cap E$ is at most countable. (Hint: Let $\{V_n\}$ be a countable base of $\mathbb{R}^k$, let W be the union of those V_n for which $E \cap V_n$ is at most countable, and show that $P = W^c$.)

28. Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable. Corollary: Every countable closed set in $\mathbb{R}^k$ has isolated points. (Hint: Use Exercise 2.27.)

29. Prove that every open set in $\mathbb{R}^1$ is the union of an at most countable collection of disjoint segments (Hint: Use Exercise 22.)

30. Imitate the proof of Theorem 2.43 to obtain the following result: If $\mathbb{R}^k = \bigcup_1^\infty F_n$, where each F_n is a closed subset of $\mathbb{R}^k$, then at least one F_n has a nonempty interior. Equivalent statement: If G_n is a dense open subset of $\mathbb{R}^k$ for each $n \in \mathbb{N}$, then $\bigcap_1^\infty G_n$ is not empty (in fact, it is dense in $\mathbb{R}^k$). This is a special case of Baire's theorem; see Exercise 3.22 for the general case.