Chapter 7

Sequences and Series of Functions

7.1 Notes

11/15:

- Soug will not test on differentiation/integration assuming that we know them already.
- **Pointwise convergent** (sequence $(f_n)_{n\in\mathbb{N}}$ of functions): A sequence of functions $f_n : E \to \mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in E$.
- Can we interchange "limit" in the above definition with continuity, convergence of series, integration, differentiation, etc.?
- Examples with negative answer:
  1. Interchanging limits: Let $S_{mn} = \frac{m}{m+n}$. $S_{mn} \to 1$ as $m \to \infty$, and $S_{mn} \to 0$ as $n \to \infty$.
  2. $f_n(x) = x^2/(1+x)^n$. $f(x) = \sum_{n=1}^{\infty} f_n(x)$. If $x = 0$, then $f_n(x) = 0$ for all n and $f(x) = 0$. If $x \neq 0$, we have
  $$f(x) = \sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n} = x^2 \sum_{n=1}^{\infty} X^n = \frac{x^2}{1-X} = \frac{x^2}{1-(1/(1+x^2))} = 1 + x^2$$
  3. Consider $f_m(x) = \lim_{n\to\infty} (\cos(m\pi x))^2 n$. $\lim_{m\to\infty} f_m(x)$ goes to 0 if $x \notin \mathbb{Q}$ and goes to 1 if $x \in \mathbb{Q}$. $f_m \to \chi_{\mathbb{Q}}$, where $\chi_{\mathbb{Q}}$ is the characteristic function of the rationals which is not Riemann integrable (partitions, upper and lower integrals, etc.).
  4. $f_n(x) = \sin nx/\sqrt{n} \to f(x) = 0$ for all x. However, $f_n'(x) = \sqrt{n}\cos nx \nrightarrow 0$
  5. If $0 \leq x \leq 1$, define $f_n(x) = n^2 x(1-x^2)^n$. We know that $f_n(0) = 0$. $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in (0,1]$. We can show that $\int_0^1 x(1-x^2)^n \,\mathrm{d}x = 1/(2n+2)$. Thus, $\int_0^1 f_n(x)\,\mathrm{d}x = n^2/(2n+2)$. Limit of the functions is zero, but their integrals diverge to infinity.
- **Uniformly convergent** (sequence $(f_n)_{n\in\mathbb{N}}$ of functions on E): A sequence of functions $f_n : E \to \mathbb{R}$ such that for all $\epsilon > 0$, there exists N such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$ for all $x \in E$. *Denoted by* $\boldsymbol{f_n \rightrightarrows f}$.
- Theorem: $f_n \rightrightarrows f$ iff $(f_n)_{n\in\mathbb{N}}$ is uniformly Cauchy (i.e., for all $\epsilon > 0$, there exists N such that if $n, m \geq N$ then $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in E$).
  - Let $M_n = \sup_{x\in E} |f_n(x) - f(x)|$. If $f_n \to f$ pointwise, then $f_n \rightrightarrows f$ if $M_n \to 0$.
- Theorem: If $(f_n)_{n\in\mathbb{N}}$ and $|f_n(x)| \leq M_n$, then $\sum f_n \rightrightarrows f$ if $\sum M_n < \infty$.
- Theorem: If E is a compact metric space, $f_n \rightrightarrows f$ in E, x is a limit point of E, and $\lim_{t\to x} f_n(t) = A_n$ exists, then $(A_n)_{n\in\mathbb{N}}$ converges and $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$.

- Corollary: $\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$.

11/16:
  - Fix $\epsilon > 0$. Then $f_n \rightrightarrows f$ implies that there exists some N such that $n, m \geq N$ implies $|f_n(t) - f_m(t)| < \epsilon$ for all $t \in E$.
  - x is a limit point of E and $t \to x$ implies $|A_n - A_m| < \epsilon$. Thus, $(A_n)_{n\in\mathbb{N}}$ is cauchy, so there exists A such that $A_n \to A$.
  - WTS: $|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|$, so we WTS the three terms on the right are small.
  - There exists n such that $|f(t) - f_n(t)| < \epsilon/3$ for all t since $f_n \rightrightarrows f$ by hypothesis.
  - Since t is in a small neighborhood of x, there exists n such that $|A_n - A| < \epsilon/3$.
  - We also have $|f_n(t) - A_n| < \epsilon/3$ by hypothesis.
  - This is a very important proof to understand, because proofs like this pop up often.

- Corollary: f_n continuous and $f_n \rightrightarrows f$ implies f is continuous.

- Theorem: Let K be compact. Assume

  (a) $(f_n)_{n\in\mathbb{N}} \subset C(K) = \{f : K \to \mathbb{R} \mid f \text{ continuous}\}$.
  (b) $f_n \to f$ pointwise in K and $f \in C(K)$.
  (c) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$.

  Then $f_n \rightrightarrows f$.

  - WLOG $f = 0$, $g_n = f_n - f \to 0$, $g_n \geq g_{n+1} \geq 0$.
  - For all $\epsilon > 0$, there exists N such that $n \geq N$ and $0 \leq g_n(x) \leq \epsilon$ for all $x \in K$.
  - $K_n = \{x \in K : g_n(x) \geq \epsilon\}$.
  - g_n continuous implies K_n closed. This combined with K compact implies K_n is compact.
  - g_n decreasing implies $K_n \supset K_{n+1}$. Thus, K_n is a nested family of compact sets, so $\bigcap K_n \neq \emptyset$.
  - This implies that each K_n is nonempty, contradicting the fact that each $g_n \to 0$ for all x.
  - Thus, there exists an N such that K_n is empty for all $n \geq N$. Thus $g_n(x) \leq \epsilon$ for all $x \in K$, $n \geq N$.
  - Note that the compactness of K is important. If $f : (0,1) \to \mathbb{R}$ is defined by $f(x) = 1/(nx+1)$, then $f_n \to 0$, but $f_n \not\rightrightarrows f$.

- Let $C(X) = \{f : X \to \mathbb{R} \mid f \text{ continuous, bounded}\}$ for X a metric space.

- If we define $\|f\| = \sum_{x\in X} |f(x)|$, for $f, g \in C(X)$, we may define $d(f,g) = \|f - g\|$. This definition satisfies the properties of a distance function, and $\|\cdot\|$ is a norm.

  - Thus, $C(X)$ is a complete metric space, a normed space, or, specifically, a **Banach space**.

- Theorem: $(f_n)_{n\in\mathbb{N}} \subset C(X)$ such that $\|f_n - f_m\|_{n,m\to\infty} \to 0$. Then there exists $f \in C(X)$ such that $\|f_n - f\|_{n\to\infty} \to 0$.

  - We get such a strong statement using properties of the image, not properties of the domain.
  - For all $\epsilon > 0$, there exists N such that $n, m \geq N$.
  - $|f_n(x) - f_m(x)| \leq \|f_n - f_m\| < \epsilon$ for all x.
  - Then there exists f such that $f_m(x) \to f(x)$. It follows that $|f_n(x) - f_m(x)| < \epsilon$

- Uniform convergence and integration.

- Stieltjes integral.

- Define $\alpha : \mathbb{R} \to \mathbb{R}$ nondecreasing.
- If we sum over the minimums/maximums of a partition times $\alpha(x_{i+1}) - \alpha(x_i)$ instead of $x_{i+1} - x_i$, we obtain said integral as the upper/lower limits just like the Riemann integral.
- We write $\int_a^b f(x)\,\mathrm{d}\alpha(x)$ where $\mathrm{d}\alpha(x) = \alpha(x)\,\mathrm{d}x$.

- Theorem: If α is nondecreasing on $[a, b]$, $f_n \in R(\alpha)$ such that $f_n \rightrightarrows f$ on $[a, b]$
  - We have

$$\begin{aligned}
&\left|\int f_n(x)\,\mathrm{d}\alpha(x) - \int f(x)\,\mathrm{d}\alpha(x)\right| \\
&\qquad = \left|\int (f_n - f)(x)\,\mathrm{d}\alpha(x)\right| \\
&\qquad \leq \|f_n - f\|(\alpha(b) - \alpha(a)) \\
&\qquad \leq \int |f_n - f|\,\mathrm{d}\alpha(x) \\
&\qquad \leq \int \|f_n - f\|\,\mathrm{d}\alpha(x) \qquad \leq \|f_n - f\| \int_a^b \mathrm{d}\alpha(x) = \|f_n - f\|(\alpha(b) - \alpha(a))
\end{aligned}$$