9   Sequences and Series of Functions

From Rudin (1976).

Chapter 7

12/10:   **1.** Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

> *Proof.* Let $\{f_n\}$ be an arbitrary uniformly convergent sequence of bounded functions. To prove that it is uniformly bounded, it will suffice to find a number M such that $|f_n(x)| < M$ for all $x \in E$ and $n \in \mathbb{N}$. Let f be the function such that $f_n \rightrightarrows f$, and let $M_n = \sup_{x \in E} |f_n(x) - f(x)|$ for each $n \in \mathbb{N}$ (the boundedness of each f_n implies that such an M_n always exists). Thus, based on the last two definitions, we can invoke Theorem 7.9 to learn that $M_n \to 0$ as $n \to \infty$. But since $\{M_n\}$ converges, Theorem 3.2c implies that $\{M_n\}$ is bounded, say by $M/2$. Taking M to be our M yields that for an arbitrary $x \in E$ and $n \in \mathbb{N}$,
>
> $$|f_n(x)| \le M_n \le \frac{M}{2} < M$$
>
> as desired.   □

3. Construct sequences $\{f_n\}, \{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converge on E).

> *Proof.* Let
>
> $$f_n(x) = x \qquad\qquad\qquad g_n(x) = \frac{1}{n}$$
>
> for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then $\{f_n\}$ converges uniformly to $f(x) = x$ (by choosing $N = 1$ for any ϵ) and $\{g_n\}$ converges uniformly to $g(x) = 0$ (by choosing $1/N < \epsilon$ with the Archimedean principle). However, while $\{f_n g_n\}$ converges pointwise to $(fg)(x) = 0$ by Theorem 3.3c, it does not converge uniformly since for any n, choosing $x = n$ yields $(f_n g_n)(x) = 1$.   □

4. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?

> *Proof.* <u>Absolute convergence values</u>: The series converges absolutely for any
>
> $$x \in (-\infty, -1) \cup \left(\bigcup_{k=1}^{\infty} \left(-\frac{1}{k^2}, -\frac{1}{(k+1)^2} \right) \right) \cup (0, \infty)$$
>
> We prove this via casework as follows.
>
> Let $x \in (0, \infty)$. Then we have
>
> $$\sum_{n=1}^{\infty} \left| \frac{1}{1 + n^2 x} \right| = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x} \le \sum_{n=1}^{\infty} \frac{1}{n^2 x} = \frac{1}{x} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{c}{x}$$
>
> where $c \in \mathbb{R}$ is finite by Theorem 3.28. Therefore, since the sum is monotonically increasing and bounded, Theorem 3.14 implies that the sum overall converges, as desired.

Let $x \in (-\infty, -1)$. Then we have

$$n^2 x + 1 < n^2 x + n^2 = n^2(x+1)$$

Since $x < -1$,

$$n^2 x + 1 < 0 \qquad\qquad\qquad n^2(x+1) < 0$$

for all $n \in \mathbb{N}$. Thus,

$$n^2 x + 1 < n^2(x+1)$$
$$\frac{n^2 x + 1}{n^2(x+1)} > 1$$
$$\frac{1}{n^2(x+1)} < \frac{1}{n^2 x + 1}$$
$$\left| \frac{1}{n^2 x + 1} \right| < \left| \frac{1}{n^2(x+1)} \right|$$

for all $n \in \mathbb{N}$. It follows that

$$\sum_{n=1}^{\infty} \left| \frac{1}{n^2 x + 1} \right| < \sum_{n=1}^{\infty} \left| \frac{1}{n^2(x+1)} \right| = \frac{1}{x+1} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{c}{x+1}$$

where $c \in \mathbb{R}$ is finite by Theorem 3.28. Therefore, since the sum is monotonically increasing and bounded, Theorem 3.14 implies that the sum overall converges, as desired.

Let $x \in (-1/k^2, -1/(k+1)^2)$. For right now, we consider only the sum for $n \geq \sqrt{2}(k+1)$, leaving finitely many terms out of the sum. Let $\delta = 1/(k+1)^2$. It follows that

$$n \geq \sqrt{2}(k+1) \qquad\qquad\qquad x < -\frac{1}{(k+1)^2}$$
$$n \geq \sqrt{\frac{2}{1/(k+1)^2}} \qquad\qquad\qquad -x > \frac{1}{(k+1)^2}$$
$$n^2 \geq \frac{2}{\delta}$$
$$\frac{\delta}{2} \geq \frac{1}{n^2}$$

Additionally, since $n \geq \sqrt{2}(k+1) > k$ (hence $n^2 \geq (k+1)^2$) and $x < -1/(k+1)^2$, we have that

$$n^2 x < (k+1)^2 \cdot -\frac{1}{(k+1)^2}$$
$$n^2 x < -1$$
$$n^2 x + 1 < 0$$

Thus, for $n \geq \sqrt{2}(k+1)$, we have that

$$\left| \frac{1}{1 + n^2 x} \right| = \frac{1}{n^2(-x) - 1} < \frac{1}{n^2 \delta - 1} = \frac{1}{n^2} \cdot \frac{1}{\delta - 1/n^2} \leq \frac{1}{n^2} \cdot \frac{1}{\delta - \delta/2} = \frac{2}{\delta n^2}$$

Therefore, since $|f_n(x)| \leq M_n = 2/\delta n^2$ and $\sum M_n$ converges by Theorem 3.28, the comparison test implies that $\sum |f_n(x)|$ converges, as desired. Adding on the finitely many terms we left out of the summation will not change this fact.

Note that the series diverges for $x = 0$ since each term becomes 1 in this case. Additionally, the series fails to exist for $x = -1/k^2$ ($k \in \mathbb{N}$) since the kth term is undefined in this case.

Uniform convergence intervals: The series converges uniformly on any

$$[a,b] \subset (-\infty, -1) \cup \left(\bigcup_{k=1}^{\infty} \left(-\frac{1}{k^2}, -\frac{1}{(k+1)^2} \right) \right) \cup (0, \infty)$$

This is because any such interval will be a subset of either $(-\infty, -1)$, $(0, \infty)$, or a set of the form $(-1/k^2, -1/(k+1)^2)$ ($k \in \mathbb{N}$). Thus, we may take as $\sum M_n$ the supremum on $[a,b]$ of the appropriate bound derived above (either c/x, $c/(x+1)$, or $2c/\delta$, respectively; all supremums of which will exist by the definition of $[a,b]$) and apply Theorem 7.10.

Non-uniform convergence intervals: Any interval containing one or more of the points in the set $\{0\} \cup \{-1/n^2\}_{n=1}^{\infty}$, by the above.

Points of continuity: The series is continuous at all points at which it converges.

Let x be a point at which f converges. Then by the first part of the proof, x is an element of an open set G. Thus, let $N_{2r}(x) \subset G$, and consider $[x - r, x + r]$. By the above, f converges uniformly on this interval. Additionally, each f_n is continuous on this interval by definition. Thus, by Theorem 7.12, f is continuous at x, as desired.

Boundedness: f is not bounded.

If we suppose for the sake of contradiction that f is bounded by m, we nevertheless find that

$$f(\tfrac{1}{4m^2}) > \sum_{n=1}^{2m} \frac{1}{1 + \frac{n^2}{4m^2}} = \sum_{n=1}^{2m} \frac{(2m)^2}{(2m)^2 + n^2} \geq \sum_{n=1}^{2m} \frac{(2m)^2}{(2m)^2 + (2m)^2} = \sum_{n=1}^{2m} \frac{1}{2} = m$$

$\square$

5. Let

$$f_n(x) = \begin{cases} 0 & x < \frac{1}{n+1} \\ \sin^2 \frac{\pi}{x} & \frac{1}{n+1} \leq x \leq \frac{1}{n} \\ 0 & \frac{1}{n} < x \end{cases}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that absolute convergence, even for all x, does not imply uniform convergence.

Proof. To prove that $\{f_n\}$ converges pointwise to the continuous function f defined by $f(x) = 0$ for all $x \in \mathbb{R}$, it will suffice to show that for every $\epsilon > 0$ and for every $x \in \mathbb{R}$, there exists an integer N such that if $n \geq N$, then $|f_n(x)| < \epsilon$. Let $\epsilon > 0$ and $x \in \mathbb{R}$ be arbitrary. We divide into three cases $(x \in \{1/n\}_{n=1}^{\infty}, x \in [0,1] \setminus \{1/n\}_{n=1}^{\infty}$, and $x \notin [0,1])$.

If $x \in \{1/n\}_{n=1}^{\infty}$, let $x = 1/k$. Then by the definition of $f_n(x)$, we have that

$$f_i(x) = \begin{cases} 0 & i < k - 1 \\ \sin^2 \frac{\pi}{1/k} = \sin^2 k\pi = 0 & i = k - 1, k \\ 0 & i > k \end{cases}$$

Thus, choose $N = 1$. It follows that if $n \geq N$, then

$$|f_n(x)| = 0 < \epsilon$$

as desired.

If $x \in [0,1] \setminus \{1/n\}_{n=1}^{\infty}$, let $x \in (1/[(N-1)+1], 1/(N-1))$ where $N \in \mathbb{N}$. Choose this N to be our N. It follows that if $n \geq N$, then

$$\frac{1}{n} \leq \frac{1}{N} = \frac{1}{(N-1)+1} < x$$

so by definition,
$$|f_n(x)| = 0 < \epsilon$$
as desired.

If $x \notin [0,1]$, then either $x < 1/(n+1)$ for all $n \in \mathbb{N}$ or $x > 1/n$ for all $n \in \mathbb{N}$. Either way, we choosing $N = 1$ yields that if $n \geq N$, then
$$|f_n(x)| = 0 < \epsilon$$
as desired.

To prove that $\{f_n\}$ does not converge uniformly to f, Theorem 7.9 tells us that it will suffice to show that if $M_n = \sup_{x \in \mathbb{R}} |f_n(x) - f(x)|$, then $M_n \nrightarrow 0$ as $n \to \infty$. Let $n \in \mathbb{N}$ be arbitrary. Since $n < n + 1/2 < n + 1$ and hence $1/(n+1) \leq 2/(2n+1) \leq 1/n$, we have by the properties of the sine function that

$$f_n(\tfrac{2}{2n+1}) = \sin^2\left[\frac{\pi}{2/(2n+1)}\right] = \sin^2\left[\frac{2n+1}{2}\pi\right] = \sin^2\left[\left(n + \frac{1}{2}\right)\pi\right] = 1$$

and that $f_n(x) \leq 1$ everywhere else. Thus, $M_n = 1$ for all $n \in \mathbb{N}$. But then $M_n \nrightarrow 0$ as $n \to \infty$, as desired.

It follows by an argument symmetric to the above that while $\sum f_n$ converges absolutely to

$$f(x) = \begin{cases} 0 & x \leq 0 \\ \sin^2 \frac{\pi}{x} & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$$

$M_n = 1$ for all $n \in \mathbb{N}$. $\qquad\square$

6. Prove that the series
$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$
converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

Proof. Let $[a,b]$ be an arbitrary bounded interval, and let $f_n(x) = (-1)^n \frac{x^2+n}{n^2}$. To prove that the series converges uniformly on $[a,b]$, Theorem 7.8 tells us that it will suffice to show that for every $\epsilon > 0$, there exists an N such that if $n, m \geq N$ (WLOG let $n \leq m$) and $x \in [a,b]$, then

$$\left| \sum_{i=n}^{m} f_i(x) \right| < \epsilon$$

Let $\epsilon > 0$ be arbitrary. Define $m = \max(|a|, |b|)$ (note that since $a \neq b$ by definition, $m > 0$). By consecutive applications of Theorem 3.43, we know that both $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converge. Thus, by consecutive applications of Theorem 3.22, there exist integers N_1, N_2 such that $m \geq n \geq N_1$ implies the left result below and $m \geq n \geq N_2$ implies the right result below.

$$\left| \sum_{k=n}^{m} (-1)^k \frac{1}{k^2} \right| < \frac{\epsilon}{2m^2} \qquad\qquad \left| \sum_{k=n}^{m} (-1)^k \frac{1}{k} \right| < \frac{\epsilon}{2}$$

Choose $N = \max(N_1, N_2)$. Now let $n, m \geq N$ with WLOG $n \leq m$, and let $x \in [a,b]$. It follows that

$$\left| \sum_{k=n}^{m} f_k(x) \right| = \left| \sum_{k=n}^{m} (-1)^k \frac{x^2 + k}{k^2} \right|$$

$$= \left| x^2 \sum_{k=n}^{m}(-1)^k \frac{1}{k^2} + \sum_{k=n}^{m}(-1)^k \frac{1}{k} \right|$$

$$\leq |x^2| \cdot \left| \sum_{k=n}^{m}(-1)^k \frac{1}{k^2} \right| + \left| \sum_{k=n}^{m}(-1)^k \frac{1}{k} \right|$$

$$\leq m^2 \cdot \left| \sum_{k=n}^{m}(-1)^k \frac{1}{k^2} \right| + \left| \sum_{k=n}^{m}(-1)^k \frac{1}{k} \right|$$

$$< m^2 \cdot \frac{\epsilon}{2m^2} + \frac{\epsilon}{2}$$

$$= \epsilon$$

as desired.

To prove that the series does not converge absolutely for any value of x, let $x \in \mathbb{R}$ be arbitrary. Then

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{x^2+n}{n^2} \right| = x^2 \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n}$$

$$= cx^2 + \infty \qquad\qquad \text{Theorem 3.28}$$

$$= \infty$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

7. For $n = 1, 2, 3, \ldots$ and x real, put

$$f_n(x) = \frac{x}{1+nx^2}$$

Show that $\{f_n\}$ converges uniformly to a function f and that the equation

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

is correct if $x \neq 0$ but false if $x = 0$.

Proof. To prove that $\{f_n\}$ converges uniformly to f defined by $f(x) = 0$ ($x \in \mathbb{R}$), Theorem 7.9 tells us that it will suffice to show that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$ and that the sequence $\{M_n\}$ defined by $M_n = \sup_{x\in\mathbb{R}} |f_n(x)|$ tends to zero as $n \to \infty$. Since

$$f_n(x) = \frac{x}{1+nx^2} < \frac{x}{nx^2} = \frac{1}{x} \cdot \frac{1}{n} \to 0$$

as $n \to \infty$ for all $x \neq 0$ and $f_n(0) = 0$ for all n, $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$, as desired. Additionally, by the Schwarz inequality, if a_1, a_2, b_1, b_2 are real numbers, then

$$|a_1 b_1 + a_2 b_2|^2 \leq (|a_1|^2 + |a_2|^2)(|b_1|^2 + |b_2|^2)$$

It follows that

$$|2\sqrt{n}x|^2 = |\underbrace{1}_{a_1} \cdot \underbrace{\sqrt{n}x}_{b_1} + \underbrace{\sqrt{n}x}_{a_2} \cdot \underbrace{1}_{b_2}|^2 \leq (|1|^2 + |\sqrt{n}x|^2)(|\sqrt{n}x|^2 + |1|^2) = (1+nx^2)^2$$

$$|2\sqrt{n}x| \leq |1+nx^2|$$

$$\frac{1}{|1+nx^2|} \leq \frac{1}{2\sqrt{n}|x|}$$

$$\frac{|x|}{|1+nx^2|} \leq \frac{1}{2\sqrt{n}}$$

$$\left| \frac{x}{1+nx^2} \right| \leq \frac{1}{2\sqrt{n}}$$

for all $x \neq 0$, $n \in \mathbb{N}$. This combined with the facts that $f_n(0) = 0 < \frac{1}{2\sqrt{n}}$ for all $n \in \mathbb{N}$ and $f_n(1/\sqrt{n}) = 1/2\sqrt{n}$ for all $n \in \mathbb{N}$ implies that $M_n = 1/2\sqrt{n}$. Thus, $M_n \to 0$ as $n \to \infty$, as desired. $f'(x) = 0$ for all $x \in \mathbb{R}$. Additionally,

$$f_n'(x) = \frac{1 - nx^2}{(1 + nx^2)^2} \leq \frac{1 - nx^2}{(nx^2)^2} = \frac{1}{x^4} \cdot \frac{1}{n^2} - \frac{1}{x^2} \cdot \frac{1}{n} \to 0$$

as $n \to \infty$ for all $x \neq 0$, as desired. However, $f_n'(0) = 1$ for all $n \in \mathbb{N}$, as desired. $\qquad \square$

8. If
$$I(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

if $\{x_n\}$ is a sequence of distinct points of (a, b), and if $\sum |c_n|$ converges, prove that the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n)$$

converges uniformly on $[a, b]$, and that f is continuous for every $x \neq x_n$.

Proof. Let $f_n(x) = c_n I(x - x_n)$ for all $n \in \mathbb{N}$. To prove that f converges uniformly on $[a, b]$, Theorem 7.10 tells us that it will suffice to show that $|f_n(x)| \leq M_n$ for all $x \in [a, b]$ and $\sum M_n$ converges. Let $M_n = c_n$ for all $n \in \mathbb{N}$. Then for any $x \in [a, b]$,

$$|f_n(x)| = c_n I(x - x_n) \leq c_n = M_n$$

as desired. Additionally, $\sum M_n = \sum c_n$ converges, as desired. This completes the proof.

For the second part of the proof, let $x \notin \{x_n\}$. Then every f_n is continuous at x by definition. Thus, f is a uniformly convergent sequence of functions continuous at x, so by Theorem 7.12, f is continuous at x. $\qquad \square$

9. Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that
$$\lim_{n \to \infty} f_n(x_n) = f(x)$$
for every sequence of points $x_n \in E$ such that $x_n \to x$ and $x \in E$. Is the converse of this true?

Proof. Let $\{x_n\} \subset E$ be an arbitrary sequence of points that converges to some $x \in E$. To prove that $\lim_{n \to \infty} f_n(x_n) = f(x)$, it will suffice to show that for every $\epsilon > 0$, there exists an N such that if $n \geq N$, then $|f_n(x_n) - f(x)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Since $\{f_n\}$ is a uniformly convergent sequence of continuous functions, Theorem 7.12 implies that f is a continuous function. Thus, there exists a $\delta > 0$ such that if $y \in E$ and $|y - x| < \delta$, then $|f(y) - f(x)| < \epsilon/2$. Additionally, since $x_n \to x$, there exists an N_1 such that if $n \geq N_1$, $|x_n - x| < \delta$. Furthermore, since $f_n \rightrightarrows f$, there exists N_2 such that if $n \geq N_2$, then $|f_n(y) - f(y)| < \epsilon/2$ for all $y \in E$. In particular, $|f_n(x_n) - f(x_n)| < \epsilon/2$. Choose $N = \max(N_1, N_2)$. Let $n \geq N$ be arbitrary. Then

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$
$$= \epsilon$$

as desired.

No, it is not true in general that if $\{f_n\}$ is a sequence of continuous functions for which $\lim_{n \to \infty} f_n(x_n) = f(x)$ for every sequence of points $x_n \in E$ such that $x_n \to x$ and $x \in E$, then f_n converges uniformly. Consider the sequence of functions from Exercise 7.5. This is a sequence of continuous functions for which $\lim_{n \to \infty} f_n(x_n) = f(x)$ for any sequence $\{x_n\}$ of the desired type since we can always choose N large enough so that the moving "hump" and neighborhood of x containing all remaining x_n are separated forever more. Moreover, by Exercise 7.5, $\{f_n\}$ does not converge uniformly, as desired. $\qquad \square$

10. Letting (x) denote the fractional part of the real number x (see Exercise 4.16 for the definition), consider the function
$$f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2}$$
defined on $\mathbb{R}$. Find all discontinuities of f, and show that they form a countable dense set. Show that f is nevertheless Riemann-integrable on every bounded interval.

11. Suppose $\{f_n\}, \{g_n\}$ are defined on E and that

(a) $\sum f_n$ has uniformly bounded partial sums;

(b) $g_n \to 0$ uniformly on E;

(c) $g_1(x) \geq g_2(x) \geq g_3(x) \geq \cdots$ for every $x \in E$.

Prove that $\sum f_n g_n$ converges uniformly on E. (Hint: Compare with Theorem 3.42.)

12. Suppose g and f_n ($n \in \mathbb{N}$) are defined on $(0, \infty)$, are Riemann-integrable on $[t, T]$ whenever $0 < t < T < \infty$, $|f_n| \leq g$, $f_n \to f$ uniformly on every compact subset of $(0, \infty)$, and
$$\int_0^{\infty} g(x)\, \mathrm{d}x < \infty$$

Prove that
$$\lim_{n \to \infty} \int_0^{\infty} f_n(x)\, \mathrm{d}x = \int_0^{\infty} f(x)\, \mathrm{d}x$$

(See Exercises 6.7-6.8 for the relevant definitions.) This is a rather weak form of Lebesgue's dominated convergence theorem (Theorem 11.32). Even in the context of the Riemann integral, uniform convergence can be replaced by pointwise convergence if it is assumed that $f \in \mathscr{R}$. (See Cunningham (1967) and Kestelman (1970).)

13. Assume that $\{f_n\}$ is a sequence of monotonically increasing functions on $\mathbb{R}^1$ with $0 \leq f_n(x) \leq 1$ for all x and all n.

(a) Prove that there is a function f and a sequence $\{n_k\}$ such that
$$f(x) = \lim_{k \to \infty} f_{n_k}(x)$$
for every $x \in \mathbb{R}^1$. The existence of such a pointwise convergent subsequence is usually called **Helly's selection theorem**. (Hint: (i) Some subsequence $\{f_{n_i}\}$ converges at all rational points r, say, to $f(r)$. (ii) Define $f(x)$ for any $x \in \mathbb{R}^1$ to be $\sup f(r)$, the sup being taken over all $r \leq x$. (iii) Show that $f_{n_i}(x) \to f(x)$ at every x at which f is continuous. [This is where monotonicity is strongly used.] (iv) A subsequence of $\{f_{n_i}\}$ converges at every point of discontinuity of f since there are at most countably many such points.)

(b) If, moreover, f is continuous, prove that $f_{n_k} \to f$ uniformly on compact sets. (Hint: Modify your proof of (iii) appropriately.)

14. Let f be a continuous real function on $\mathbb{R}^1$ with the following properties: $0 \leq f(t) \leq 1$, $f(t+2) = f(t)$ for every t, and
$$f(t) = \begin{cases} 0 & 0 \leq t \leq \frac{1}{3} \\ 1 & \frac{2}{3} \leq t \leq 1 \end{cases}$$

Put $\Phi(t) = (x(t), y(t))$, where
$$x(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1}t) \qquad\qquad y(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n}t)$$

Prove that Φ is continuous and that Φ maps $I = [0, 1]$ onto the unit square $I^2 \subset \mathbb{R}^2$. In fact, show that Φ maps the Cantor set onto I^2. (Hint: Each $(x_0, y_0) \in I^2$ has the form

$$x_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n-1} \qquad\qquad y_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n}$$

where each a_i is 0 or 1. If

$$t_0 = \sum_{i=1}^{\infty} 3^{-i-1}(2a_i)$$

show that $f(3^k t_0) = a_k$, and hence that $x(t_0) = x_0$, $y_0(t_0) = y_0$.) This simple example of a so-called **space-filling curve** is due to Schoenberg (1938).

Proof. Save for last. $\square$

15. Suppose f is a real continuous function on $\mathbb{R}^1$, $f_n(t) = f(nt)$ for $n \in \mathbb{N}$, and $\{f_n\}$ is equicontinuous on $[0, 1]$. What conclusion can you draw about f?

16. Suppose $\{f_n\}$ is an equicontinuous sequence of functions on a compact set K, and $\{f_n\}$ converges pointwise on K. Prove that $\{f_n\}$ converges uniformly on K.

18. Let $\{f_n\}$ be a uniformly bounded sequence of functions which are Riemann-integrable on $[a, b]$ and put

$$F_n(x) = \int_a^x f_n(t)\, \mathrm{d}t$$

for $x \in [a, b]$. Prove that there exists a subsequence $\{F_{n_k}\}$ which converges uniformly on $[a, b]$.

19. Let K be a compact metric space, let S be a subset of $\mathscr{C}(K)$. Prove that S is compact (with respect to the metric defined in Section 7.14) if and only if S is uniformly closed, pointwise bounded, and equicontinuous. (If S is not equicontinuous, then S contains a sequence which has no equicontinuous subsequence, hence has no subsequence that converges uniformly on K.)

20. If f is continuous on $[0, 1]$ and if

$$\int_0^1 f(x) x^n\, \mathrm{d}x = 0$$

for $n = 0, 1, 2, \ldots$, prove that $f(x) = 0$ on $[0, 1]$. (Hint: The integral of the product of f with any polynomial is zero. Use the Weierstrass theorem to show that $\int_0^1 f^2(x)\, \mathrm{d}x = 0$.)

22. Assume $f \in \mathscr{R}(\alpha)$ on $[a, b]$, and prove that there are polynomials P_n such that

$$\lim_{n \to \infty} \int_a^b |f - P_n|^2\, \mathrm{d}\alpha = 0$$

(Compare with Exercise 6.12.)

23. Put $P_0 = 0$, and define for $n = 0, 1, 2, \ldots$

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2}$$

Prove that

$$\lim_{n \to \infty} P_n(x) = |x|$$

uniformly on $[-1, 1]$. This makes it possible to prove the Stone-Weierstrass theorem without first proving Theorem 7.26. (Hint: Use the identity

$$|x| - P_{n+1}(x) = [|x| - P_n(x)] \left[1 - \frac{|x| + P_n(x)}{2} \right]$$

to prove that $0 \leq P_n(x) \leq P_{n+1}(x) \leq |x|$ if $|x| \leq 1$ and that

$$|x| - P_n(x) \leq |x| \left(1 - \frac{|x|}{2}\right)^n < \frac{2}{n+1}$$

if $|x| < 1$.)

25. Suppose ϕ is a continuous bounded real function in the strip defined by $0 \leq x \leq 1$, $-\infty < y < \infty$. Prove that the initial-value problem

$$y' = \phi(x, y) \qquad\qquad\qquad y(0) = c$$

has a solution. Note that the hypotheses of this existence theorem are less stringent than those of the corresponding uniqueness theorem; see Exercise 5.27. (Hint: Fix n. For $i = 0, \ldots, n$, put $x_i = i/n$. Let f_n be a continuous function on $[0, 1]$ such that $f_n(0) = c$, let

$$f_n'(t) = \phi(x_i, f_n(x_i))$$

if $x_i < t < x_{i+1}$, and put

$$\Delta_n(t) = f_n'(t) - \phi(t, f_n(t))$$

except at points x_i, where $\Delta_n(t) = 0$. Then

$$f_n(x) = c + \int_0^x [\phi(t, f_n(t)) + \Delta_n(t)] \, dt$$

Choose $M < \infty$ so that $|\phi| \leq M$. Verify the following assertions.

(a) $|f_n'| \leq M$, $|\Delta_n| \leq 2M$, $\Delta_n \in \mathscr{R}$, and $|f_n| \leq |c| + M = M_1$ say, on $[0, 1]$, for all n.

(b) $\{f_n\}$ is equicontinuous on $[0, 1]$ since $|f_n'| \leq M$.

(c) Some $\{f_{n_k}\}$ converges to some f, uniformly on $[0, 1]$.

(d) Since ϕ is uniformly continuous on the rectangle $0 \leq x \leq 1$, $|y| \leq M_1$,

$$\phi(t, f_{n_k}(t)) \to \phi(t, f(t))$$

uniformly on $[0, 1]$.

(e) $\Delta_n(t) \to 0$ uniformly on $[0, 1]$ since

$$\Delta_n(t) = \phi(x_i, f_n(x_i)) - \phi(t, f_n(t))$$

in (x_i, x_{i+1}).

(f) Hence

$$f(x) = c + \int_0^x \phi(t, f(t)) \, dt$$

This f is a solution of the given problem.)

26. Prove an analogous existence theorem for the initial-value problem

$$\mathbf{y}' = \mathbf{\Phi}(\mathbf{x}, \mathbf{y}) \qquad\qquad\qquad \mathbf{y}(0) = \mathbf{c}$$

where now $\mathbf{c} \in \mathbb{R}^k$, $\mathbf{y} \in \mathbb{R}^k$, and $\mathbf{\Phi}$ is a continuous bounded mapping of the part of $\mathbb{R}^{k+1}$ defined by $0 \leq x \leq 1$, $\mathbf{y} \in \mathbb{R}^k$ into $\mathbb{R}^k$. Compare Exercise 5.28. (Hint: Use the vector-valued version of Theorem 7.25.)