3 Inner Product Spaces

From Treil (2017).

Chapter 5

10/18: **3.2.** Apply Gram-Schmidt orthogonalization to the system of vectors $(1,2,3)^T$, $(1,3,1)^T$. Write the matrix of the orthogonal projection onto the 2-dimensional subspace spanned by these vectors.

3.5. Find the orthogonal projection of a vector $(1,1,1,1)^T$ onto the subspace spanned by the vectors $\mathbf{v}_1 = (1,3,1,1)^T$ and $\mathbf{v}_2 = (2,-1,1,0)^T$ (note that $\mathbf{v}_1 \perp \mathbf{v}_2$).

3.6. Find the distance from a vector $(1,2,3,4)$ to the subspace spanned by the vectors $\mathbf{v}_1 = (1,-1,1,0)^T$ and $\mathbf{v}_2 = (1,2,1,1)^T$ (note that $\mathbf{v}_1 \perp \mathbf{v}_2$). Can you find the distance without actually computing the projection? That would simplify the calculations.

3.7. True or false: If E is a subspace of V, then $\dim E + \dim(E^\perp) = \dim V$? Justify.

3.8. Let P be the orthogonal projection onto a subspace E of an inner product space V, let $\dim V = n$, and let $\dim E = r$. Find the eigenvalues and the eigenvectors (eigenspaces). Find the algebraic and geometric multiplicities of each eigenvalue.

3.9. Using eigenvalues to compute determinants:

a) Find the matrix of the orthogonal projection onto the one-dimensional subspace in $\mathbb{R}^n$ spanned by the vector $(1,\ldots,1)^T$.
b) Let A be the $n \times n$ matrix with all entries equal to 1. Compute its eigenvalues and their multiplicities (use the previous problem).
c) Compute eigenvalues (and multiplicities) of the matrix $A - I$, i.e., of the matrix with zeroes on the main diagonal and ones everywhere else.
d) Compute $\det(A - I)$.

3.11. Let $P = P_E$ be the matrix of an orthogonal projection onto a subspace E. Show that

a) The matrix P is self-adjoint, meaning that $P^* = P$.
b) $P^2 = P$.

3.13. Suppose P is the orthogonal projection onto a subspace E, and Q is the orthogonal projection onto the orthogonal complement $E^\perp$.

a) What are $P + Q$ and PQ?
b) Show that $P - Q$ is its own inverse.

4.2. Find the matrix of the orthogonal projection P onto the column space of

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{pmatrix}$$

Use two methods: Gram-Schmidt orthogonalization and the formula for the projection. Compare the results.

4.4. Fit a plane $z = a + bx + cy$ to four points $(1,1,3)$, $(0,3,6)$, $(2,1,5)$, and $(0,0,0)$. To do that

a) Find 4 equations with 3 unknowns a, b, c such that the plane passes through all 4 points (this system does not have to have a solution).
b) Find the least squares solution of the system.

4.5. Minimal norm solution. Let an equation $A\mathbf{x} = \mathbf{b}$ have a solution, and let A have a non-trivial kernel (so the solution is not unique). Prove that

a) There exists a unique solution $\mathbf{x}_0$ of $A\mathbf{x} = \mathbf{b}$ minimizing the norm $\|\mathbf{x}\|$, i.e., that there exists a unique $\mathbf{x}_0$ such that $A\mathbf{x}_0 = \mathbf{b}$ and $\|\mathbf{x}_0\| \leq \|\mathbf{x}\|$ for any $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{b}$.

b) $\mathbf{x}_0 = P_{(\ker A)^\perp}\mathbf{x}$ for any $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{b}$.

5.2. Find matrices of orthogonal projections onto all 4 fundamental subspaces of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 2 & 4 & 3 \end{pmatrix}$$

Note that you only really need to compute 2 of the projections. If you pick an appropriate 2, the other 2 are easy to obtain from them (recall how the projections onto E and $E^\perp$ are related).

5.3. Let A be an $m \times n$ matrix. Show that $\ker A = \ker(A^*A)$. (Hint: To do this, you need to prove 2 inclusions, namely $\ker(A^*A) \subset \ker A$ and $\ker A \subset \ker(A^*A)$. One of the inclusions is trivial, and for the other one, use the fact that $\|A\mathbf{x}\|^2 = (A\mathbf{x}, A\mathbf{x}) = (A^*A\mathbf{x}, \mathbf{x})$.)

5.4. Use the equality $\ker A = \ker(A^*A)$ to prove that

a) $\operatorname{rank} A = \operatorname{rank}(A^*A)$.

b) If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, A is left invertible. (Hint: You can just write a formula for the left inverse.)

5.6. Let a matrix P be self-adjoint ($P^* = P$) and let $P^2 = P$. Show that P is the matrix of an orthogonal projection. (Hint: Consider the decomposition $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$, $\mathbf{x}_1 \in \operatorname{range} P$ and $\mathbf{x}_2 \perp \operatorname{range} P$, and show that $P\mathbf{x}_1 = \mathbf{x}_1$, $P\mathbf{x}_2 = \mathbf{0}$. For one of the equalities, you will need self-adjointness; for the other one, the property $P^2 = P$.)

6.1. Orthogonally diagonalize the following matrices

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

i.e., for each matrix A, find a unitary matrix U and a diagonal matrix D such that $A = UDU^*$.

6.2. True or false: A matrix is unitarily equivalent to a diagonal one if and only if it has an orthogonal basis of eigenvectors.

6.5. Let $U : X \to X$ be a linear transformation on a finite-dimensional inner product space. True or false:

a) If $\|U\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in X$, then U is unitary.

b) If $\|U\mathbf{e}_k\| = \|\mathbf{e}_k\|$ for each $k = 1, \ldots, n$ for some orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$, then U is unitary.

6.6. Let A and B be unitarily equivalent $n \times n$ matrices.

a) Prove that $\operatorname{tr}(A^*A) = \operatorname{tr}(B^*B)$.

b) Use (a) to prove that

$$\sum_{j,k=1}^{n} |A_{j,k}|^2 = \sum_{j,k=1}^{n} |B_{j,k}|^2$$

c) Use (b) to prove that the matrices

$$\begin{pmatrix} 1 & 2 \\ 2 & i \end{pmatrix} \qquad \begin{pmatrix} i & 4 \\ 1 & 1 \end{pmatrix}$$

are not unitarily equivalent.

6.7. Which of the following pairs of matrices are unitarily equivalent? (Hint: It is easy to eliminate matrices that are not unitarily equivalent: Remember that unitarily equivalent matrices are similar, and recall that the trace, determinant, and eigenvalues of similar matrices coincide. Also, the previous problem helps in eliminating non-unitarily equivalent matrices. Finally, a matrix is unitarily equivalent to a diagonal one if and only if it has an orthogonal basis of eigenvectors.)

a)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

b)

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$$

c)

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

d)

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{pmatrix}$$

e)

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

6.9. Let U be a 3×3 orthogonal matrix with $\det U = 1$. Prove that

a) 1 is an eigenvalue of U.

b) If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is an orthonormal basis, such that $U\mathbf{v}_1 = \mathbf{v}_1$ (remember that 1 is an eigenvalue), then in this basis, the matrix of U is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

where α is some angle. (Hint: Show that since $\mathbf{v}_1$ is an eigenvector of U, all entries below 1 must be zero, and since $\mathbf{v}_1$ is also an eigenvector of U^* [why?], all entries right of 1 must also be zero. Then show that the lower right 2×2 matrix is an orthogonal one with determinant 1, and use the previous problem.)

8.1. Prove the following formula.

$$(\mathbf{x}, \mathbf{y})_{\mathbb{R}} = \operatorname{Re}(\mathbf{x}, \mathbf{y})_{\mathbb{C}}$$

Namely, show that if

$$\mathbf{x} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

where $z_k = x_k + iy_k$, $w_k = u_k + iv_k$, $x_k, y_k, u_k, v_k \in \mathbb{R}$, then

$$\operatorname{Re}\left(\sum_{k=1}^{n} z_k \bar{w}_k\right) = \sum_{k=1}^{n} x_k u_k + \sum_{k=1}^{n} y_k v_k$$

8.4. Show that if U is an orthogonal transformation satisfying $U^2 = -I$, then $U^* = -U$.