6 Basic Topology

From Rudin (1976).

Chapter 2

11/8: **1.** Prove that the empty set is a subset of every set.

Proof. Let A be a set. Then $x \in A$ for all $x \in \emptyset$ is vacuously true. Thus, $\emptyset \subset A$. ☐

2. A complex number z is said to be **algebraic** if there are integers $a_0, \dots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$$

Prove that the set of all algebraic numbers is countable. (Hint: For every positive integer N, there are only finitely many equations with $n + |a_0| + |a_1| + \cdots + |a_n| = N$.)

Proof. Define a family of sets $\{A_N\}$ such that each A_N is the set of all complex zeroes of all polynomials $\sum_{k=0}^{n} a_k z^{n-k}$ with integer coefficients $a_0, \dots, a_n$, not all zero, satisfying the equation $n + |a_0| + \cdots + |a_n| = N$. Symbolically, let each

$$A_N = \{z \in \mathbb{C} \mid \textstyle\sum_{k=0}^{n} a_k z^{n-k} = 0,\ a_0, \dots, a_n \in \mathbb{Z},\ \exists\ a_i : a_i \neq 0,\ n + |a_0| + \cdots + |a_n| = N\}$$

Since there are only finitely many equations with $n + |a_0| + \cdots + |a_n| = N$ for each N by the hint, there are only finitely many corresponding polynomials $\sum_{k=0}^{n} a_k z^{n-k}$ for each N. By the fundamental theorem of arithmetic, every polynomial p has at most $\deg p$ distinct solutions. Thus, since each A_N is the union of finitely many finite sets, each A_N is finite.

Consider the set $A = \bigcup_{N=1}^{\infty} A_N$. Since every algebraic number is a zero of a polynomial with integer coefficients, not all zero, whose coefficients' absolute values and degree add up to *some* positive integer N, A is the set of all algebraic numbers. Moreover, as the union of an at most countable number of at most countable sets, the Corollary to Theorem 2.12 implies that A is at most countable. Additionally, since the set of solutions to $a_0 z + a_1 = 0$ for $a_0, a_1 \in \mathbb{Z}$, $a_0 \neq 0$ is both a subset of the algebraic numbers and equal to $\mathbb{Q}$ (a countable set), A is at least countable. Therefore, A is countable, as desired. ☐

3. Prove that there exist real numbers which are not algebraic.

Proof. Suppose for the sake of contradiction that every real number is algebraic. Then if A is the set of all complex algebraic numbers, $\mathbb{R} \subset A$. Thus, since $\mathbb{R}$ is infinite and A is countable (by Problem 2.2), Theorem 2.8 implies that $\mathbb{R}$ is countable, a contradiction. ☐

4. Is the set of all irrational real numbers countable?

Proof. No.

Suppose for the sake of contradiction that $\mathbb{R} \setminus \mathbb{Q}$ is countable. Then since $\mathbb{R} \setminus \mathbb{Q}$ and $\mathbb{Q}$ are at most countable, the Corollary to Theorem 2.12 implies that $(\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q} = \mathbb{R}$ is at most countable, contradicting the fact that $\mathbb{R}$ is uncountable. ☐

5. Construct a bounded set of real numbers with exactly three limit points.

Proof. Let $A = \bigcup_{i=0}^{2} \{1/n + i : n \in \mathbb{N}\}$. Then A has limit points at $0, 1, 2$ and nowhere else. ☐

6. Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\bar{E}$ have the same limit points (recall that $\bar{E} = E \cup E'$). Do E and E' always have the same limit points?

Proof. To prove that E' is closed, it will suffice to show that it contains all of its limit points. Let p be an arbitrary limit point of E'. To show that $p \in E'$, it will suffice to verify that p is a limit point of E, i.e., that every neighborhood of p intersects E at a point other than p. Let $N_r(p)$ be an arbitrary neighborhood of p. Since p is a limit point of E', $N_r(p) \cap E'$ is infinite (2.20). Thus, we can choose a point $x \in N_r(p) \cap E'$ such that $x \neq p$. It follows that $x \in E'$, so it must be that every neighborhood of x has infinite intersection with E (2.20). In particular, since $N_r(p)$ is open and $x \in N_r(p)$, x is an interior point of $N_r(p)$, so we can choose a neighborhood N of x such that $N \subset N_r(p)$. The last two statements combined imply that $N \cap E$ is infinite. In particular, since $N \cap E \subset N \subset N_r(p)$, there exist infinitely many points of E in $N_r(p)$; choosing any one of these that is not equal to p completes the proof.

To prove that E and $\bar{E}$ have the same limit points, it will suffice to show that every limit point of E is a limit point of $\bar{E}$ and that every limit point of $\bar{E}$ is a limit point of E. The latter was accomplished by the above. Thus, let p be an arbitrary limit point of E. To prove that p is a limit point of $\bar{E}$, it will suffice to show that every neighborhood of p intersects $\bar{E}$ at some point other than p. Consider an arbitrary neighborhood $N_r(p)$ of p. Since p is a limit point of E, $N_r(p) \cap (E \setminus \{p\}) \neq \emptyset$. Therefore, we have that

$$\begin{aligned}
N_r(p) \cap (\bar{E} \setminus \{p\}) &= N_r(p) \cap [(E \cup E') \setminus \{p\}] \\
&= N_r(p) \cap [(E \setminus \{p\}) \cup (E' \setminus \{p\})] \\
&= [N_r(p) \cap (E \setminus \{p\})] \cup [N_r(p) \cap (E' \setminus \{p\})] \\
&\supset N_r(p) \cap (E \setminus \{p\}) \\
&\neq \emptyset
\end{aligned}$$

as desired.

No, E and E' do not always have the same limit points. Let $E = \{1/n : n \in \mathbb{N}\}$. Then $E' = \{0\}$, but since E' is finite, $E'' = \emptyset$. $\square$

7. Let $A_1, A_2, \dots$ be subsets of a metric space.

(a) If $B_n = \bigcup_{i=1}^n A_i$, prove that $\bar{B}_n = \bigcup_{i=1}^n \bar{A}_i$ for $n = 1, 2, 3, \dots$.

Proof. Let $n \in \mathbb{N}$ be arbitrary.

Suppose first that $x \in \bar{B}_n$. We divide into two cases ($x \in B_n$ and $x \in B_n'$). If $x \in B_n$, then $x \in A_i$ for some $i = 1, \dots, n$. It follows that $x \in A_i \cup A_i' = \bar{A}_i \subset \bigcup_{i=1}^n \bar{A}_i$, as desired. On the other hand, if $x \in B_n'$, then $N_r(x) \cap (B_n \setminus \{p\}) \neq \emptyset$ for every $r > 0$. Now suppose for the sake of contradiction that $x \notin A_i'$ for any $i = 1, \dots, n$. Then there exist neighborhoods $N_{r_1}(x), \dots, N_{r_n}(x)$ of x such that no $N_{r_i}(x)$ contains a point of A_i other than p. Let $0 < r_j \leq r_i$ for each $i = 1, \dots, n$. It follows that

$$\begin{aligned}
\emptyset &= \bigcup_{i=1}^n N_{r_j}(x) \cap (A_i \setminus \{p\}) \\
&= N_{r_j}(x) \cap \left[\bigcup_{i=1}^n (A_i \setminus \{p\}) \right] \\
&= N_{r_j}(x) \cap \left[\left(\bigcup_{i=1}^n A_i \right) \setminus \{p\} \right] \\
&= N_{r_j}(x) \cap [B_n \setminus \{p\}]
\end{aligned}$$

a contradiction. Therefore, $x \in A_i'$ for some $i = 1, \dots, n$. It follows that $x \in A_i \cup A_i' = \bar{A}_i \subset \bigcup_{i=1}^n \bar{A}_i$, as desired.

Now suppose that $x \in \bigcup_{i=1}^n \bar{A}_i$. Then $x \in \bar{A}_i$ for some $i = 1, \dots, n$. We divide into two cases ($x \in A_i$ and $x \in A_i'$). If $x \in A_i$, then $x \in \bigcup_{i=1}^n A_i = B_n \subset B_n \cup B_n' = \bar{B}_n$, as desired. On the other hand, if $x \in A_i'$, then every neighborhood of x contains a point $q \neq x$ of A_i. But since $A_i \subset \bigcup_{i=1}^n A_i = B_n$, it follows that every neighborhood of x contains a point $q \neq x$ of B_n. Thus, $x \in B_n' \subset B_n \cup B_n' = \bar{B}_n$, as desired. ☐

(b) If $B = \bigcup_{i=1}^\infty A_i$, prove that $\bar{B} \supset \bigcup_{i=1}^\infty \bar{A}_i$. Show, by an example, that this inclusion can be proper.

Proof. Let $x \in \bigcup_{i=1}^\infty \bar{A}_i$ be arbitrary. Then $x \in \bar{A}_i$ for some i. We divide into two cases ($x \in A_i$ and $x \in A_i'$). If $x \in A_i$, then $x \in \bigcup_{i=1}^\infty A_i = B \subset B \cup B' = \bar{B}$, as desired. On the other hand, if $x \in A_i'$, then every neighborhood of x contains a point $q \neq x$ of A_i. But since $A_i \subset \bigcup_{i=1}^\infty A_i = B$, it follows that every neighborhood of x contains a point $q \neq x$ of B. Thus, $x \in B' \subset B \cup B' = \bar{B}$, as desired.

Define the family of sets $\{A_n\}$ by $A_n = \{1/n\}$ for each $n \in \mathbb{N}$. Then since each A_n is finite, each $\bar{A}_n = \emptyset$, so $\bigcup_{i=1}^\infty \bar{A}_i = \emptyset$. However, $B = \bigcup_{i=1}^\infty A_i$ has zero as a limit point, so

$$\bar{B} \supset \{0\} \supsetneq \emptyset = \bigcup_{i=1}^\infty \bar{A}_i$$

as desired. ☐

8. Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

Proof. Yes, every point of every open set $E \subset \mathbb{R}^2$ is a limit point of E. Let E be an arbitrary open subset of $\mathbb{R}^2$. Let $x \in E$ be arbitrary. Since $x \in E$ open, x is an interior point of E, meaning that there exists $N_r(x) \subset E$. Now to prove that x is a limit point of E, it will suffice to show that every neighborhood of x contains a point $q \neq x$ of E. Let $N_s(x)$ be an arbitrary neighborhood of x. If $x = (x_1, x_2)$ and $m = \min(r, s)$, choose $q = (x_1 + m/2, x_2 + m/2)$. Since $r, s > 0$ by definition, $q \neq x$. Additionally,

$$\begin{aligned} |q - x|^2 &= (x_1 + m/2 - x_1)^2 + (x_2 + m/2 - x_2)^2 \\ &= m^2/2 \\ &< m^2 \end{aligned}$$

Taking square roots reveals that $|q - x| < r$ and $|q - x| < s$. It follows that $q \in N_r(x) \subset E$ and $q \in N_s(x)$, as desired.

No, every point of every closed set $E \subset \mathbb{R}^2$ is not a limit point of E. Let E be a nonempty finite set. Then by the table on Rudin (1976, p. 33), E is closed but not perfect, implying that E is a closed set not every point of which is a limit point of it (in fact, the fact that not every point of every closed set is a limit point of E is the whole motivation for defining perfect sets!). ☐

9. Let E° denote the set of all interior points of a set E (see Definition 2.18e; E° is called the **interior** of E).

(a) Prove that E° is always open.

Proof. Let $x \in E^\circ$ be arbitrary. Then since x is an interior point of E, there exists a neighborhood $N(x)$ of x such that $N(x) \subset E$. By Theorem 2.19, $N(x)$ is open. It follows from Theorem 2.24 that $\bigcup_{x \in E^\circ} N(x)$ is open. We now prove that $E^\circ = \bigcup_{x \in E^\circ} N(x)$. The inclusion in one direction is obvious. In the other, let $y \in \bigcup_{x \in E^\circ} N(x)$. Then $y \in N(x)$ for some x. It follows since each $N(x)$ is open that there exists a neighborhood N of y such that $N \subset N(x)$. But since $N(x) \subset E$ by definition, we have both that $y \in E$ and that $N \subset E$. Thus, y is an interior point of E, so $y \in E^\circ$, as desired. ☐

(b) Prove that E is open if and only if $E^\circ = E$.

Proof. Suppose first that E is open. Let $x \in E^\circ$ be arbitrary. Then since x is an interior point of E, x is naturally a point of E. On the other hand, let $x \in E$. Then since E is open, x is an interior point of E, so $x \in E^\circ$, as desired.

Now suppose that $E^\circ = E$. Then since E° is open by part (a), E is open. □

(c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.

Proof. Let $x \in G$ be arbitrary. Since G is open, there exists a neighborhood N of x such that $N \subset G$. But since $G \subset E$, $N \subset E$. Thus, x is an interior point of E, so $x \in E^\circ$, as desired. □

(d) Prove that the complement of E° is the closure of the complement of E.

Proof. Let $x \in (E^\circ)^c$. Then $x \notin E^\circ$. We divide into two cases ($x \notin E$ and $x \in E$). If $x \notin E$, then $x \in E^c$. It follows that $x \in E^c \cup (E^c)' = \overline{E^c}$, as desired. On the other hand, if $x \in E$ (but $x \notin E^\circ$), then there exists no neighborhood of x that is a subset of E. In other words, every neighborhood of x contains some point of E^c. This combined with the fact that $x \notin E^c$ implies that $x \in (E^c)'$. Therefore, $x \in E^c \cup (E^c)' = \overline{E^c}$, as desired.

Let $x \in \overline{E^c}$. We divide into two cases ($x \in E^c$ and $x \in (E^c)'$). If $x \in E^c$, then $x \notin E$. It follows that $x \notin E^\circ \subset E$. Therefore, $x \in (E^\circ)^c$, as desired. On the other hand, if $x \in (E^c)'$, then every neighborhood of x contains a point of E^c. This combined with the fact that $x \in E$ ($x \notin E^c$ in this case) implies that no neighborhood N of x exists such that $N \subset E$. Therefore, x is not an interior point of E, i.e., $x \notin E^\circ$; it follows that $x \in (E^\circ)^c$, as desired. □

(e) Do E and $\bar{E}$ always have the same interiors?

Proof. No.

Consider $\mathbb{Q} \subset \mathbb{R}$. Since $\mathbb{Q}$ is disconnected at every point, $\mathbb{Q}^\circ = \emptyset$ but $(\bar{\mathbb{Q}})^\circ = \mathbb{R}^\circ = \mathbb{R}$. □

(f) Do E and E° always have the same closures?

Proof. No.

Consider $\mathbb{Q} \subset \mathbb{R}$. As before, we have that $\bar{\mathbb{Q}} = \mathbb{R}$ while $\bar{\mathbb{Q}^\circ} = \bar{\emptyset} = \emptyset$. □

10. Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p,q) = \begin{cases} 1 & p \neq q \\ 0 & p = q \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

Proof. To prove that d is a metric, it will suffice to show that $d(p,q) > 0$ if $p \neq q$, $d(p,p) = 0$, $d(p,q) = d(q,p)$, and $d(p,q) \leq d(p,r) + d(r,q)$ for any $r \in X$. Let's begin. Let $p \neq q$. Then by the definition of d, $d(p,q) = 1 > 0$, as desired. Let $p \in X$. Then by the definition of d, $d(p,p) = 0$, as desired. Let $p, q \in X$. We divide into two cases ($p = q$ and $p \neq q$). If $p = q$, then $d(p,q) = 0 = d(q,p)$. If $p \neq q$, then $d(p,q) = 1 = d(q,p)$, as desired. Let $p, q, r \in X$. We divide into two cases ($p = q$ and $p \neq q$). If $p = q$, then $d(p,q) = 0$ must be less than the sum of two numbers that are either 0 or 1. If $p \neq q$, then $d(p,q) = 1$. However, since r cannot equal the distinct p and q, at least on of $d(p,r)$ and $d(r,q)$ equals 1, so the inequality holds here, too, as desired.

Every subset is open. Let $E \subset X$, and let $x \in E$. Then by the definition of d, $N_1(x) = \{y \in X : d(y,x) < 1\} = \{x\} \subset E$. Thus, every point of E is an interior point, as desired.

Every subset is closed. Let $E \subset X$. By the previous result, E^c is open. Thus, by Theorem 2.23, E is closed.

Only finite sets are compact. We know that every finite set is compact (choose an open cover $\{G_\alpha\}$ of E finite; map every $x \in E$ to some G_α that contains it; choose the range of this map as the finite subcover). If E is infinite, however, choose the open cover $\{\{x\}\}_{x \in E}$. We know that all of these sets are open (because every set is open). Additionally, since each one only contains one element of E, we need all infinitely many of them to cover E. Thus, this infinite E is not compact. □

11. For $x \in \mathbb{R}^1$ and $y \in \mathbb{R}^1$, define

$$\begin{aligned} d_1(x,y) &= (x-y)^2 \\ d_2(x,y) &= \sqrt{|x-y|} \\ d_3(x,y) &= |x^2-y^2| \\ d_4(x,y) &= |x-2y| \\ d_5(x,y) &= \frac{|x-y|}{1+|x-y|} \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

Proof. d_1 is not a metric. Let $x=2$, $y=0$, $z=1$. Then

$$d_1(2,0) = (2-0)^2 = 4 > 2 = (2-1)^2 + (1-0)^2 = d_1(2,1) + d_1(1,0)$$

so d_1 does not obey the triangle inequality.

d_2 is a metric. If $x \neq y$, then $x-y \neq 0$, so $d_2(x,y) = \sqrt{|x-y|} > 0$, as desired. For each x, $d_2(x,x) = \sqrt{|x-x|} = \sqrt{0} = 0$, as desired. For all x,y, $d_2(x,y) = \sqrt{|x-y|} = \sqrt{|y-x|} = d_2(y,x)$, as desired. For all x,y,z,

$$\begin{aligned} d_2(x,y) &= \sqrt{|x-y|} \\ &\leq \sqrt{|x-z|+|z-y|} \\ &\leq \sqrt{|x-z|} + \sqrt{|z-y|} \\ &= d_2(x,z) + d_2(z,y) \end{aligned}$$

as desired.

d_3 is not a metric. Let $x=1$, $y=-1$. Then $x \neq y$, but

$$d_3(1,-1) = |1^2 - (-1)^2| = 0$$

d_4 is not a metric. Let $x=2$, $y=1$. Then $x \neq y$, but

$$d_4(2,1) = |2-2(1)| = 0$$

d_5 is a metric. If $x \neq y$, then $x-y \neq 0$, so $d_5(x,y) = |x-y|/(1+|x-y|) > 0$, as desired. For each x, $d_5(x,x) = |x-x|/(1+|x-x|) = 0$, as desired. For all x,y, $d_5(x,y) = |x-y|/(1+|x-y|) = |y-x|/(1+|y-x|) = d(y,x)$. For all x,y,z,

$$\begin{aligned} d(x,y) &= \frac{|x-y|}{1+|x-y|} \\ &\leq \frac{|x-z|+|z-y|}{1+|x-z|+|z-y|} \\ &= \frac{|x-z|}{1+|x-z|+|z-y|} + \frac{|z-y|}{1+|x-z|+|z-y|} \\ &\leq \frac{|x-z|}{1+|x-z|} + \frac{|z-y|}{1+|z-y|} \\ &= d(x,z) + d(z,y) \end{aligned}$$

as desired. $\square$

12. Let $K \subset \mathbb{R}^1$ consist of 0 and the numbers $1/n$ for $n = 1,2,3,\dots$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).

Proof. Let $\{G_\alpha\}$ be an arbitrary open cover of K. Then $0 \in G_\alpha$ for some α. Since G_α is open, 0 is an interior point of it, so there exists a neighborhood $N_r(0)$ such that $N_r(0) \subset G_\alpha$. Since $r > 0$ by definition, if we let $x = r$ and $y = 1$, the Archimedean property implies there exists a positive integer m such that $mr > 1$. It follows that $1/m < r$, so every $1/n$ such that $n \geq m$ is an element of $N_r(0) \subset G_\alpha$. Since G_α contains 0 and infinitely many of the $1/n$, let this G_α be part of our finite subcover. For the remaining entries in our finite subcover, choose for each of the finitely many $1/n$ such that $n < m$ a G_β that contains it. □

13. Construct a compact set of real numbers whose limit points form a countable set.

Proof. Consider the family of sets $\{K_i\}$ defined by

$$K_i = \{1/i\} \cup \{1/i + 1/n : n \in \mathbb{N}\}$$

for each $i \in \mathbb{N}$ and $i = +\infty$. Let

$$K = \bigcup_{i=1}^{+\infty} K_i$$

K is bounded with lower bound $0 \in K_\infty$ and upper bound $2 = 1/1 + 1/1 \in K_1$. Additionally, K is closed with limit points $K' = K_\infty$. Thus, if we define $f : \mathbb{N} \to K'$ by

$$f(n) = \begin{cases} 0 & n = 1 \\ \frac{1}{n-1} & n > 1 \end{cases}$$

we will have a bijection between the natural number and K', proving that K' is countable, as desired. □

14. Give an example of an open cover of the segment $(0, 1)$ which has no finite subcover.

Proof. Choose $\{G_i\}_{i=3}^{\infty}$ defined by

$$G_i = \left(\frac{1}{i}, \frac{1}{i-2}\right)$$

Every segment is open in $\mathbb{R}$. Additionally, $\{G_i\}$ is a cover since if $x \in (0, 1)$, then we can modify the Archimedean property to choose the smallest integer n such that $1/n < x$. It follows that $x \leq \frac{1}{n-1} < \frac{1}{n-2}$, so $x \in (1/n, 1/(n-2))$, as desired. Lastly, $\{G_i\}$ has no finite subcover: if it did, we could use the betweeness of the reals to choose an $x < 1/i$ where $(1/i, 1/(i-2))$ is the smallest segment in the finite subcover. It would follow that $x \in (0, 1)$ but x is not an element of any set in the cover, a contradiction. □

15. Show that Theorem 2.36 and its Corollary become false (in $\mathbb{R}^1$, for example) if the word "compact" is replaced by "closed" or by "bounded."

Proof. Suppose first that "compact" is replaced by "closed." Consider the collection of sets $\{K_n\}_{n=1}^{\infty}$ defined by

$$K_n = n\mathbb{N}$$

for each n, where by $n\mathbb{N}$ we mean all the natural number multiples of n (e.g., $3\mathbb{N} = \{3, 6, 9, ...\}$). Clearly any finite collection of these sets will intersect at the least common multiple of the relevant n's. However, the intersection of all such sets will be the empty set since for any possible natural number n in the intersection, $n \notin (n + 1)\mathbb{N} = K_{n+1}$.

Now suppose that "compact" is replaced by "bounded." Consider the collection of sets $\{K_n\}_{n=1}^{\infty}$ defined by

$$K_n = (0, 1/n)$$

for each n. This family of sets satisfies the constraints of both the modified Theorem 2.36 and its Corollary. However, $\bigcap_{n=1}^{\infty} K_n = \emptyset$ since by the Archimedean principle, we can always find a $1/n$ smaller than any x in any of the sets, and thus a set in the intersection that does not contain said x. □