MATH 20700 (Honors Analysis in $\mathbb{R}^n$ I) Notes

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Part I

Linear Algebra

Chapter 1

Basic Notions

1.1 Notes

9/27:

- **Vector space**: Basically, a set for which you have an addition and multiplication.
- $\mathbb{F}^d$ is used for $\mathbb{R}^d$ or $\mathbb{C}^d$ in Treil (2017).
- $\mathbb{P}_n$ is the vector space of polynomials up to degree n.
- $C([0,1])$ is the set of continuous functions defined on $[0,1]$, an infinite-dimensional vector space.
- **Generating set**: A subset of a vector space, all linear combinations of which generate the vector space. *Also known as* **spanning set**.
  - Any element of VS is a linear comb. of elements of the generating set.
- **Linearly independent** (list): A list of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$ such that $\sum_{i=1}^{k} \alpha_i \mathbf{v}_i = 0$ implies $\alpha_i = 0$ for all i.
- **Base**: A generating set consisting of linearly independent vectors.
- Any element of a VS can be written as a *unique* linear combination of the vectors in a base.
  - If $\mathbf{x} = \sum_{i=1}^{k} \alpha_i \mathbf{v}_i = \sum_{i=1}^{k} \beta_i \mathbf{v}_i$, then $\alpha_i = \beta_i$ for all i.
- **Linear transformation**: A function $T : X \to Y$, where X, Y are VSs, such that

$$T(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha T\mathbf{x} + \beta T\mathbf{y}$$

  for all $\mathbf{x} \in X$, $\mathbf{y} \in Y$.
- Examples of linear transformations:
  - Consider $\mathbb{P}_n$. Let $Tp_n = p_n'$. This T is linear.
  - Rotation in $\mathbb{R}^d$.
    - Think graphically about two vectors $\mathbf{x}, \mathbf{y}$.
    - Rotating and summing them is the same as summing and rotating. Same for scaling.
    - Thus, rotation is actually linear!
  - Reflection as well.
- Consider $T : \mathbb{R} \to \mathbb{R}$.
  - Any linear map on the line is a line.
  - We must have $Tx = \alpha x$: $Tx = T(1x) = xT(1) = x\alpha$.

- Consider $T : \mathbb{R}^n \to \mathbb{R}^m$ linear.
  - Any linear map between $\mathbb{R}^n$ and $\mathbb{R}^m$ is linear.
  - Thus, $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$, where A is an $m \times n$ matrix.
- To find A, do the same calculation as for $Tx = \alpha x$ but more carefully:
  - Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a basis.
  - So $\mathbf{x} = \sum_{i=1}^n \alpha_i \mathbf{e}_i$.
  - Thus, $T\mathbf{x} = \sum_{i=1}^n \alpha_i T(\mathbf{e}_i)$.
  - Each $T(\mathbf{e}_i)$ is part of the matrix that we multiply by the column vector representing $\mathbf{x}$.
- Multiplication of matrices is equivalent to composition of linear maps.
- Consider $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \to \mathbb{F}^r$.
  - $T_2 \circ T_1$ is equivalent to BA, if A represents T_1 and B represents T_2. In other words, $(T_2 \circ T_1)(\mathbf{x}) = BA\mathbf{x}$ for all $\mathbf{x}$.
- Recall that if $A = (\alpha_{ij})$ and $B = (\beta_{ij})$, then $(BA)_{ij} = (\sum \beta_{ik}\alpha_{kj})$.
- Properties of multiplication:

$$\begin{aligned}(AB)C &= A(BC)\\ A(B + C) &= AB + AC\\ (A + B)C &= AC + BC\end{aligned}$$

  - However, it is not true in general that $AB = BA$.
- **Trace** (of an $n \times n$ matrix A): The sum of the diagonal entries of A. *Denoted by* $\mathbf{tr}\,(\boldsymbol{A})$. *Given by*

$$\operatorname{tr}(A) = \sum \alpha_{ii}$$

- It is true that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.
  - Indeed, on the diagonals, multiplication is commutative; it's the other terms that mess you up in general.
- Invertibility of matrices.
  - In general, matrices are not invertible: Not every system of equations is solveable; $Ax = b$ does not always have a solution $x = A^{-1}b$.
- C is the inverse from the left: $CA = I$. B is the inverse from the right: $AB = I$. A matrix can have a left and a right inverse and still not be invertible. A matrix is invertible iff $C = B$.
- Any time we write "inverse," we do so under the assumption that it exists.
- $(AB)^{-1} = B^{-1}A^{-1}$ — easy proof by multiplication.
- If $A = (a_{ij})$, $A^T = (a_{ji})$.
  - $(A^{-1})^T = (A^T)^{-1}$.
  - $(AB)^T = B^T A^T$.
- Let X, Y VS.
  - $X \cong Y$[1] if there exists a linear $T : X \to Y$ that is one-to-one and onto.
  - Check: $A(\text{basis of } X) = \text{basis of } Y$. Prove by definition and expression of elements as linear combinations.
- **Subspace**: A subset of a vector space which happens to be a vector space, itself.

[1] "X is isomorphic to Y."

1.2 Chapter 1: Basic Notions

From Treil (2017).

10/24:

- **Coordinates** (of $\mathbf{v} \in V$ wrt. a basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ of V): The unique scalars $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ such that $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$.
- **Spanning system**: A list of vectors that spans V. *Also known as* **generating system**, **complete system**.
- **Trivial** (linear combination): A linear combination $\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$ of vectors such that $\alpha_k = 0$ for each $k = 1, \dots, n$.
- **Transformation**: A function $T : X \to Y$. *Also known as* **transform**, **mapping**, **map**, **operator**, and **function**.
- The matrix of a linear transformation T is often denoted by $[\boldsymbol{T}]$.
- To compute the reflection of vectors over an arbitrary line through the origin in $\mathbb{R}^2$, represent the overall transformation as a composition of rotating the line to be the x-axis, reflecting over the x-axis, and rotating back.
- Theorem 1.5.1: If A is an $m \times n$ matrix and B is an $n \times m$ matrix, then

$$\operatorname{tr}(AB) = \operatorname{tr}(BA)$$

- Theorem 1.6.1: If a linear transformation is invertible, then its left and right inverses are unique and coincide.
- The column $(1, 1)^T$ is left-invertible, with one possible left inverse being $(1/2, 1/2)$.
  - Note that it is not right invertible since its left inverses are not unique (see Theorem 1.6.1).
- An invertible matrix must be square.
- **Isomorphic** (vector spaces): Two vectors spaces V, W such that there exists an isomorphism $A : V \to W$. *Denoted by* $\boldsymbol{V \cong W}$.
- Theorem 1.6.8: $A : X \to Y$ is invertible if and only if for any right side $\mathbf{b} \in Y$, the equation

$$A\mathbf{x} = \mathbf{b}$$

  has a unique solution $\mathbf{x} \in X$.
- Corollary 1.6.9: An $m \times n$ matrix is invertible if and only if its columns form a basis in $\mathbb{F}^m$.
- **Linear span** (of $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$): The set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_n$. *Denoted by* $\boldsymbol{\mathcal{L}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}}$, $\mathbf{span}\,(\mathbf{v}_1, \dots, \mathbf{v}_n)$.

Chapter 2

Systems of Linear Equations

2.1 Notes

9/29:

- Row elimination:
  - Let
$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 3 & 1 & 2 & 7 \\ 2 & 1 & 2 & 1 \end{pmatrix}$$
  - Then the **echelon form** matrix
$$A_e = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & -4 \end{pmatrix}$$
  - Lastly, the **reduced echelon form** matrix
$$A_{re} = \begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$
- **echelon form**:
  - All zero rows are below nonzero rows.
  - For any nonzero row, its leading element is strictly to the left of the nonzero entry of the next row.
- **Reduced echelon form**:
  - All pivots are 1.
  - Used to solve systems of the form $Ax = b$.
- **Inconsistent** (system of equations): A system with no solution.
  - If the last row is of the form $(0, \dots, 0, b)$ where $b \neq 0$, then there is no solution.
- Unique solution if A_e has a pivot in every column.
- There exists a solution for every b if there is a pivot in every row?
- Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a matrix. Then $\ker A = \{x \in \mathbb{R}^n : Ax = 0\}$ (subspace of $\mathbb{R}^n$) and $\operatorname{range} A = \{Ax : x \in \mathbb{R}^n\}$ (subspace of $\mathbb{R}^m$).
- Also consider $\ker(A^T)$ and $\operatorname{range}(A^T)$, the basis of the kernel and range, and dimension.

- Finite-dimensional vector spaces:
  - A basis is a generating set (so every element of V can be written uniquely as a linear combination of the basis) the length of which is equal to the dimension of V.
  - All bases of finite-dimensional vector spaces have the same number of elements.
    - Let v_1, v_2, v_3 and w_1, w_2 be two generating sets of V.
    - Then

$$\begin{aligned} v_1 &= \lambda_{11} w_1 + \lambda_{12} w_2 \\ v_2 &= \lambda_{21} w_1 + \lambda_{22} w_2 \\ v_3 &= \lambda_{31} w_1 + \lambda_{32} w_2 \end{aligned}$$

    - Suppose the only solution to $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$ is $\alpha_1 = \alpha_2 = \alpha_3 = 0$.
    - But this is not true, as we can find another one in terms of the λs.
  - If you have a list of linearly independent vectors, you can complete it into a basis.
    - If there exists a vector that can't be written as a linear combination of the list, add it to the list.
  - If you find any particular solution to a system $Ax = b$, and you add to it any element of $\ker A$, you will obtain another solution.
    - $Ax_1 = b$ and $Ax_h = 0$ implies that $A(x_1 + x_h) = b$.
    - $Ax_1 = b$ and $Ax_2 = b$ imply that $A(x_1 - x_2) = 0$, i.e., that $x_1 - x_2 \in \ker A$.
  - If $A : \mathbb{R}^n \to \mathbb{R}^m$ and $\dim \operatorname{range} A = m$, then $Ax = b$ is solveable for all $b \in \mathbb{R}^m$.
  - Let $\operatorname{rank} A = \dim \operatorname{range} A$.
  - Rank theorem:
    - $\operatorname{rank} A = \operatorname{rank} A^T$.
    - Let $A : \mathbb{R}^n \to \mathbb{R}^m$. We know that $\dim \ker A + \dim \operatorname{range} A = n$.
    - $\dim \ker A^T + \operatorname{rank} A^T = m$.
    - This theorem survives linear algebra and enters functional analysis under the name **Fredholm's alternative**.
- **Fredholm's alternative**: $Ax = b$ has a solution for all $b \in \mathbb{R}^n$ iff $\dim \ker A^T = 0$.
  - $\dim \ker A^T = 0$ implies $\operatorname{rank} A^T = m$ implies $\operatorname{rank} A = m$ implies $\dim \operatorname{range} A = m$, as desired.
- **Pivot column** (of A): A column of A where A_e has pivots.
- The **pivot columns** of A give a basis for $\operatorname{range} A$.
- The pivot rows of A_e give a basis for $\operatorname{range} A^T$.
- A basis for the kernel is enough to solve $Ax = 0$.
- If you take these three things as givens, you can prove the rank theorem.

2.2 Chapter 2: Systems of Linear Equations

From Treil (2017).

10/24:
- A system is inconsistent iff the echelon form of the augmented matrix has a row of the form $\begin{pmatrix} 0 & \cdots & 0 & b \end{pmatrix}$.
- A solution to $A\mathbf{x} = \mathbf{b}$ is unique iff there are no free variables, i.e., iff there is a pivot in every column.
- $A\mathbf{x} = \mathbf{b}$ is consistent iff the echelon form of the coefficient matrix has a pivot in every row.

- $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$ iff the echelon form of the coefficient matrix A has a pivot in every row and column.
- Proposition 2.3.1: Let $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{F}^n$, and let $A = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_m \end{bmatrix}$ be an $n \times m$ matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_m$. Then
  1. The system $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is linearly independent iff the echelon form of A has a pivot in every column.
  2. The system $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is complete iff the echelon form of A has a pivot in every row.
  3. The system $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is a basis of $\mathbb{F}^n$ iff the echelon form of A has a pivot in every column and in every row.
- Proposition 2.3.6: A matrix A is invertible if and only if its echelon form has a pivot in every column and every row.
- Corollary 2.3.7: An invertible matrix must be square ($n \times n$).
- Proposition 2.3.8: If a square ($n \times n$) matrix is left invertible or if it is right invertible, then it is invertible. In other words, to check the invertibility of a square matrix A, it is sufficient to check only one of the conditions $AA^{-1} = I$, $A^{-1}A = I$.
- Any invertible matrix is row-equivalent to (can be row-reduced to) to the identity matrix.
- **Homogeneous** (system of linear equations): A system of the form $A\mathbf{x} = \mathbf{0}$.
- Theorem 2.6.1: Let a vector $\mathbf{x}_1$ satisfy the equation $A\mathbf{x} = \mathbf{b}$. and let H be the set of all solutions of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$. Then the set

$$\{\mathbf{x}_1 + \mathbf{x}_h : \mathbf{x}_h \in H\}$$

  is the set of all solutions to the equation $A\mathbf{x} = \mathbf{b}$.
- The pivot columns are a basis of $\operatorname{range} A$. The pivot rows are a basis of $\operatorname{range} A^T$. The solutions to the equation $A\mathbf{x} = \mathbf{0}$ are a basis of $\ker A$.
- Theorem 2.7.3: Let A be an $m \times n$ matrix. Then the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$ iff the dual equation $A^T\mathbf{x} = \mathbf{0}$ has a unique (only the trivial) solution.
  - Note that this is a corollary to the rank theorem.
- Change of coordinates formula:
  - Let $T : V \to W$ be a linear transformation, and let $\mathcal{V} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\mathcal{W} = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ be bases of V and W, respectively.
  - The $m \times n$ matrix of T with respect to these bases is $[T]_{\mathcal{W}\mathcal{V}}$, and relates the coordinates of $[T\mathbf{v}]_{\mathcal{W}}$ and $[\mathbf{v}]_{\mathcal{V}}$ via

$$[T\mathbf{v}]_{\mathcal{W}} = [T]_{\mathcal{W}\mathcal{V}}[\mathbf{v}]_{\mathcal{V}}$$

  - Change of coordinates matrix: If $\mathcal{A}, \mathcal{B}$ are two bases of V, then we can convert the coordinates of a vector in $\mathcal{B}$ to its in $\mathcal{A}$ with the identity matrix (with respect to the appropriate bases). In particular,

$$[\mathbf{v}]_{\mathcal{B}} = [I]_{\mathcal{B}\mathcal{A}}[\mathbf{v}]_{\mathcal{A}}$$

    - Note that the k^{th} column of $[I]_{\mathcal{B}\mathcal{A}}$ is the coordinate representation in $\mathcal{B}$ of $\mathbf{a}_k$, i.e., $[\mathbf{a}_k]_{\mathcal{B}}$.
  - The change of coordinates matrix from a basis $\mathcal{B}$ to the standard basis $\mathcal{S}$ is easy to compute; by the above, it's just

$$[I]_{\mathcal{S}\mathcal{B}} = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix}$$

    - It follows that $[I]_{\mathcal{B}\mathcal{S}} = ([I]_{\mathcal{S}\mathcal{B}})^{-1}$.

- This allows us to compute $[I]_{\mathcal{BA}}$ as $[I]_{\mathcal{BS}}[I]_{\mathcal{SA}}$

– If $T : V \to W$, $\mathcal{A}, \tilde{\mathcal{A}}$ are bases of V, and $\mathcal{B}, \tilde{\mathcal{B}}$ are bases of W, and we have $[T]_{\mathcal{BA}}$, then

$$[T]_{\tilde{\mathcal{B}}\tilde{\mathcal{A}}} = [I]_{\tilde{\mathcal{B}}\mathcal{B}}[T]_{\mathcal{BA}}[I]_{\mathcal{A}\tilde{\mathcal{A}}}$$

- Change of basis ends up at similarity; two operators are similar if we can change the basis of one into another.

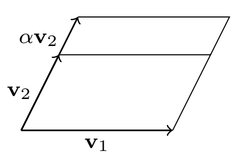
Chapter 3

Determinants

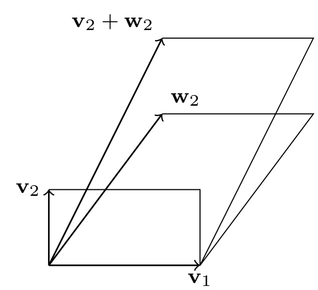
3.1 Notes

9/29:
- The determinant, geometrically, is the volume of the object (in $\mathbb{R}^3$) you get when you take linear combinations of the vectors.
- In 2D:
  - Let v_1, v_2 be two vectors. Put tail to tail and forming a parallelogram, the determinant of the matrix (v_1, v_2) is the area of said parallelogram.
  - Linearity 1: $D(av_1, v_2, \dots, v_n) = aD(v_1, \dots, v_n)$ is the same as saying that if you stretch one vector by a, you scale up the area by that much, too.
  - Linearity 2: $D(v_1, \dots, v_{k+} + v_{k-}, \dots, v_n) = D(-) + D(+)$.
  - Antisymmetry: $D(v_1, \dots, v_k, \dots, v_j, \dots, v_n) = -D(v_1, \dots, v_j, \dots, v_k, \dots, v_n)$. Interchanging columns flips the sign of the determinant.
  - Basis: $D(e_1, \dots, e_n) = 1$.
- Determinant: Denoted by $D(v_1, \dots, v_n)$, where $(v_1, \dots, v_n)$ is an $n \times n$ matrix.

10/1:
- Consider an $n \times n$ matrix A consisting of n columns containing vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^n$.
  - $D(A)$ is the volume of the solid $V = \sum_{i=1}^{n} \alpha_i v_i$.
  - $D(\mathbf{e}_1, \dots, \mathbf{e}_n) = 1$.



(a) $D(\mathbf{v}_1, \alpha\mathbf{v}_2) = \alpha D(\mathbf{v}_1, \mathbf{v}_2)$.



(b) $D(\mathbf{v}_1, \mathbf{v}_2 + \mathbf{w}_2) = D(\mathbf{v}_1, \mathbf{v}_2) + D(\mathbf{v}_1, \mathbf{w}_2)$.

Figure 3.1: Visualizing properties of determinants.

- Basic properties of the determinant.
  - If A has a zero column, then $\det A = 0$: Scalar property.

  - If A has two equal columns, then $\det A = 0$: Multiply one by minus and add.
  - If A has a column which is a multiple of another, then $\det A = 0$: Pull out the multiple and then you have the previous one.
  - If columns are linearly dependent, then $\det A = 0$: Decompose it into sums, split, add back up with previous properties.
  - The determinant is preserved under column reduction.
  - $\det A^T = \det A$: Put everything in rref.
  - If A is not invertible, then $\det A = 0$ (not invertible implies linearly dependent columns, implies $\det A = 0$).
  - $\det(AB) = \det A \det B$.
- Determinant of...
  - A diagonal matrix: The product of the diagonal entries (pull out the terms, and then note that the remaining identity matrix has determinant 1).
  - An upper triangular matrix: The product of the diagonal entries (column reduction to make it into a diagonal matrix, and then the property above).

3.2 Chapter 3: Determinants

From Treil (2017).

10/24:
- Let $A_{j,k}$ denote the $(n-1) \times (n-1)$ matrix obtained from A by crossing out row j and column k and pushing it together.
- **Cofactors** (of A): The numbers $C_{j,k}$, one per entry, defined by

$$C_{j,k} = (-1)^{j+k} \det A_{j,k}$$

- **Cofactor matrix** (of A): The matrix

$$C = \{C_{j,k}\}_{j,k=1}^{n}$$

- Theorem 3.5.2: Let A be an invertible matrix and let C be its cofactor matrix. Then

$$A^{-1} = \frac{1}{\det A} C^T$$

- **Cramer's rule**: If A is invertible and $A\mathbf{x} = \mathbf{b}$, then

$$x_k = \frac{\det B_k}{\det A}$$

  where B_k is obtained from A by replacing column k of A by the vector $\mathbf{b}$.
- **Minor** (of order k of A): The determinant of a $k \times k$ submatrix of A.
- Theorem 3.6.1: The rank of a nonzero matrix A is equal to the largest integer k such that there exists a nonzero minor of order k.

Chapter 4

Introduction to Spectral Theory

4.1 Notes

10/1:
- **Difference equation**: Like a differential equation, but instead of writing a differentials, you write differences.
- Suppose we want to solve $x_{n+1} = Ax_n$ with x_0 given.
  - You will find that $x_n = A^n x_0$.
  - This gets hard to compute, so we want to find a way to simplify the computation.
- Thus, we want to diagonalize the matrix, and this concept is inherently linked to eigenvalues and eigenvectors.
  - If you can decompose the x_0 into a linear combination of eigenvectors, then you can simplify the computation a lot:
  $$x_n = \sum \alpha_i A^n v_i = \sum \alpha_i \lambda_i^n v_i$$
  - An $n \times n$ matrix will have n eigenvalues. You want n linearly independent eigenvectors, creating an eigenbasis.
- To find eigenvalues and eigenvectors, we need to solve $Ax = \lambda x$, i.e., $(A-\lambda I)x = 0$. Thus, $\ker(A-\lambda I) \neq \{0\}$, so $\det(A - \lambda I) = 0$.
- The eigenvalues of A are independent of the choice of basis of the domain of A or the range.

10/4:
- We need to know everything in Treil (2017).
  - We don't need to know the applications sections, but you should be interested.
- **Spectral theory**: Decomposing a linear operator.
- Let $A : V \to V$ be a linear operator. $\lambda \in \mathbb{C}$ is an eigenvalue if there exists $x \in V$ nonzero such that $Ax = \lambda x$.
  - Let A be an $n \times n$ matrix over $\mathbb{C}$ or $\mathbb{R}$.
  - The eigenvalues are the roots of the polynomial $\det(A - \lambda I) = 0$ in λ.
- Things we want to do:
  - Given A, find the eigenvalues and eigenvectors (solve $(A - \lambda I)x = 0$).

  - In order to simplify A, make it a diagonal matrix:

$$A = S \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} S^{-1}$$

- Eigenvalues are independent of the choice of basis.
  - From the book, we have that
$$[A]_{\mathcal{A}\mathcal{A}} = [S]_{\mathcal{A}\mathcal{B}}[B]_{\mathcal{B}\mathcal{B}}[S]_{\mathcal{A}\mathcal{B}}^{-1}$$
  - It follows that
$$A - \lambda I = [S]_{\mathcal{A}\mathcal{B}}(B - \lambda I)[S]_{\mathcal{A}\mathcal{B}}^{-1}$$
    so
$$\det(A - \lambda I) = \det\bigl([S]_{\mathcal{A}\mathcal{B}}(B - \lambda I)[S]_{\mathcal{A}\mathcal{B}}^{-1}\bigr) = \det\bigl([S]_{\mathcal{A}\mathcal{B}}[S]_{\mathcal{A}\mathcal{B}}^{-1}(B - \lambda I)\bigr) = \det(B - \lambda I)$$
- If $p(z) = (z - \lambda)^k q(z)$, then k is the **algebraic multiplicity** of λ. The **geometric multiplicity** of λ is $\dim\ker(A - \lambda I)$.
  - These terms are not always the same, but they are related.
- Diagonalization:
  - Given A that corresponds to $T : V \to V$, can we find a basis of V in which the operator is a diagonal matrix?
  - $A = SDS^{-1}$ iff there exists a basis of V consisting of the eigenvectors of A.
  - Proves $A^N = SD^N S^{-1}$ via $A^2 = SDS^{-1}SDS^{-1} = SDIDS^{-1} = SD^2S^{-1}$.
- Let A be an $n \times n$ matrix over $\mathbb{F}$. If $\lambda_1, \dots, \lambda_r$ are distinct eigenvalues, then their eigenvectors are linearly independent.
  - Prove with induction contradiction argument. Assume true for $\mathbf{v}_{r-1}$. Then
$$0 = (A - \lambda_r I)[\mathbf{v}_1 + \cdots + \mathbf{v}_r] = (\lambda_1 - \lambda_r)\mathbf{v}_1 + \cdots + (\lambda_{r-1} - \lambda_r)\mathbf{v}_{r-1}$$
  - Implies $\lambda_r = \lambda_i$ for all $i \in [r-1]$, a contradiction.
  - If A has n distinct eigenvalues, then A is diagonalizable.
- If $A : V \to V$ has n complex eigenvalues, then A is diagonalizable iff the algebraic multiplicity equals the geometric multiplicity for each eigenvalue.
- Goes through a sample diagonalization with $\left(\begin{smallmatrix} 1 & 2 \\ 8 & 1 \end{smallmatrix}\right)$.
  - We have
$$A - \lambda I = \begin{pmatrix} 1 - \lambda & 2 \\ 8 & 1 - \lambda \end{pmatrix}$$
    so
$$0 = \det(A - \lambda I) = (1 - \lambda)^2 - 16$$
  - It follows that $\lambda = 5, -3$.
  - This yields
$$\begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}^{-1}$$
    by inspection.
- As another example, consider $\left(\begin{smallmatrix} 1 & 2 \\ -2 & 1 \end{smallmatrix}\right)$.
  - Here, we have $\lambda = 1 \pm 2i$.

4.2 Chapter 4: Introduction to Spectral Theory

From Treil (2017).

10/24:

- **Spectrum** (of A): The set of all eigenvalues of A. *Denoted by* $\boldsymbol{\sigma(A)}$.
- Proposition 4.1.1: The geometric multiplicity of an eigenvalue cannot exceed its algebraic multiplicity.
- Theorem 4.2.1: A matrix A (with values in $\mathbb{F}$) admits a representation $A = SDS^{-1}$ where D is a diagonal matrix and S is invertible if and only if there exists a basis of $\mathbb{F}^n$ of eigenvectors of A. Moreover, in this case diagonal entries of D are the eigenvalues of A and columns of S are the corresponding eigenvectors.
- Any operator on a complex vector space has n eigenvalues (counting multiplicities).
  - Think n necessary roots of the characteristic polynomial, or the necessary upper triangular representation.
- Theorem 4.2.8: Let an operator $A : V \to V$ have exactly $n = \dim V$ eigenvalues (counting multiplicities). Then A is diagonalizable if and only if for each eigenvalue λ, the dimension of the eigenspace $\ker(A - \lambda I)$ (i.e., the geometric multiplicity of λ) coincides with the algebraic multiplicity of λ.
- Theorem 4.2.9: A real $n \times n$ matrix A admits a real factorization (i.e., a real representation $A = SDS^{-1}$ where S and D are real matrices, D is diagonal, and S is invertible) if and only if it admits a complex factorization and all eigenvalues of A are real.
- Example of a nondiagonalizable matrix:
$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
  - $p(\lambda) = (1-\lambda)^2$, so $\lambda = 1$ with algebraic multiplicity 2.
  - However, $\dim\ker(A - I) = 1$ since $A - I$ has only one pivot, hence $2 - 1 = 1$ free variable.
  - Thus, apply Theorem 4.2.8.

Chapter 5

Inner Product Spaces

5.1 Notes

10/6:

- We define
$$\ell^2(\mathbb{R}) = \left\{ \{a_n\}_{n\geq 1} \subset \mathbb{R} : \sum_1^\infty |a_n|^2 < \infty \right\}$$
- **Inner product**: A map $V \times V \to \mathbb{F}$ that takes $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \cdot \mathbf{y}$. *Denoted by* $\cdot, (\cdot, \cdot), \langle \cdot, \cdot \rangle$.
- Properties of the inner product:
  - $(\mathbf{x}, \mathbf{y}) = \overline{(\mathbf{y}, \mathbf{x})}$ (symmetry).
  - $(\alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z}) = \alpha(\mathbf{x}, \mathbf{z}) + \beta(\mathbf{y}, \mathbf{z})$ (linearity).
  - $(\mathbf{x}, \mathbf{x}) \geq 0$.
  - $(\mathbf{x}, \mathbf{x}) = 0$ iff $\mathbf{x} = 0$.
- If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then
$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n x_i y_i$$
- If $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, then
$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n x_i \bar{y}_i$$
- If $f, g \in \mathbb{P}_n(t)$, then
$$(f, g) = \int_{-1}^1 f\bar{g} \, \mathrm{d}t$$
  - The conjugate of a polynomial is the polynomial with the conjugate of the coefficients of the original polynomial. Symbolically, if $f = \sum_{i=0}^n \alpha_i t^i$ is a polynomial, then $\bar{f} = \sum_{i=0}^n \bar{\alpha}_i t^i$.
- It is a fact that
$$\left| \sum^\infty a_n \bar{b}_n \right| \leq \|(a_n)_{n\geq 1}\| \|(b_n)_{n\geq 1}\|$$
- Suppose we want to define the inner product between two matrices.
  - A common one is
$$(A, B) = \operatorname{tr}(B^* A)$$
where $B^* = \bar{B}^T = \overline{B^T}$ is the conjugate transpose.

- We define the norm as a function $V \to [0, \infty)$ given by

$$\|\mathbf{x}\| = \sqrt{(\mathbf{x}, \mathbf{x})}$$

- Properties of the norm.
  - $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$.
  - $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.
  - $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
- In $\mathbb{R}^n$,

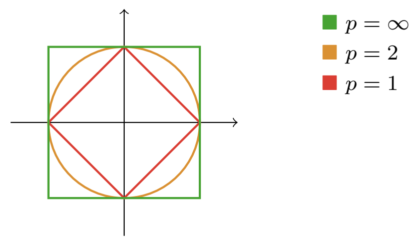


Figure 5.1: The unit ball of norms corresponding to $p = 1, 2, \infty$.

  - The standard norm is
  $$\|\mathbf{x}\| = \sqrt{\sum |x_i|^2}$$
  - We can also define
  $$\|\mathbf{x}\|_p = \sqrt[p]{\sum |x_i|^p}$$
  - We can even define
  $$\|\mathbf{x}\|_\infty = \max |x_i|$$
  - And we can prove that all of these are valid norms.
  - Only the norm corresponding to ℓ^2 is given by an inner product, but all the other quantities are still norms as defined by the properties (see Treil (2017)).
  - Figure 5.1 shows the unit ball of each norm, i.e., the set of all points which have norm 1.
- The parallelogram rule:
$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)$$
- Orthogonality: Given $\mathbf{v}, \mathbf{w}$, if $\mathbf{v} \perp \mathbf{w}$, then $(\mathbf{v}, \mathbf{w}) = 0$.
- In particular, if $\mathbf{v} \perp \mathbf{w}$, then
$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$$
- Let E be a subspace of V. If $\mathbf{v} \perp E$, then $\mathbf{v} \perp \mathbf{e}$ for all $\mathbf{e} \in E$, i.e., $\mathbf{v} \perp$ a set of vectors spanning E.
- Any set of orthogonal vectors is linearly independent. Thus, if V is n dimensional, then $\mathbf{v}_1, \dots, \mathbf{v}_n$ orthogonal is a basis.
- Let E be a subspace of V. Take $\mathbf{v} \in V$. We want to define the projection $P_E\mathbf{v}$ of $\mathbf{v}$ onto E.
  - We have that $P_E\mathbf{v} \in E$ and $v - P_E\mathbf{v} \perp E$.
  - Additionally, we have that
  $$\|\mathbf{v} - P_E\mathbf{v}\| \leq \|\mathbf{v} - \mathbf{e}\|$$
  for all $\mathbf{e} \in E$.

- Lastly, we have that $P_E\mathbf{v}$ is unique.

- If we receive a basis of a vector space, how do we create out of that a basis that is orthogonal? The process of doing this is called **Gram-Schmidt orthogonalization**.
  - We keep $\mathbf{v}_1$, subtract $P_{\mathbf{v}_1}\mathbf{v}_2$ from $\mathbf{v}_2$, subtract $P_{\{\mathbf{v}_1,\mathbf{v}_2\}}\mathbf{v}_3$ from $\mathbf{v}_3$, and on and on.
- If we are given a set of orthogonal vectors, we can normalize them by dividing each by its norm. This creates an orthonormal list. The standard basis is orthonormal.
- Let
$$E^\perp = \{v \in V : v \perp E\}$$
  - It follows that $V = E \oplus E^\perp$.
- How close can we come to solving $A\mathbf{x} = \mathbf{b}$ if we cannot solve it exactly (i.e., if the columns are not linearly independent)?
  - Let A be an $m \times n$ matrix, and let $\mathbf{b} \in \mathbb{R}^m$.
  - Then the best solution is given by minimizing $\|A\mathbf{x} - \mathbf{b}\|$. We minimize this with projections. A special case of this is least squares regression! More details in Treil (2017).

10/8:

- Soug is gonna send us a hefty amount of reading for the weekend.
- Least square approximation:
  - If we want to minimize $\|A\mathbf{x} - \mathbf{b}\|$, the best we can do is project $\mathbf{b}$ onto the range of A.
  - Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be an orthogonal basis of range A.
  - Then
$$\mathrm{Proj}|_{\text{range } A}\mathbf{b} = \sum^k \frac{(\mathbf{b}, \mathbf{v}_k)}{\|v_k\|^2}\mathbf{v}_k$$
  - Matrix equation form:
$$\mathrm{Projection}_{\text{range } A} = A(A^*A)^{-1}A^*$$
    if A^*A is invertible, where $A^* = \bar{A}^T$.
    - Soug never uses this though.
  - The minimum is found when $\mathbf{b} - A\mathbf{x} \perp \text{range } A$. Implies that $\mathbf{b} - A\mathbf{x} \perp \mathbf{a}_k$ for all k. Implies $(\mathbf{b} - A\mathbf{x}, \mathbf{a}_k) = \bar{\mathbf{a}}_k{}^T(\mathbf{b} - A\mathbf{x}) = 0$.
  - Note that we're letting $\bar{\mathbf{a}}_k^T$ be the row vector
$$\bar{\mathbf{a}}_k^T = \begin{pmatrix} \bar{a}_{1,k} & \cdots & \bar{a}_{n,k} \end{pmatrix}$$
  - We also have $\bar{A}^T(\mathbf{b} - A\mathbf{x}) = 0$, from which it follows that $A^*A\mathbf{x} = A^*\mathbf{b}$, so $\mathbf{x} = (A^*A)^{-1}A^*\mathbf{b}$. Thus, $\mathrm{Proj}|_{\text{range } A} = Ax$, so $\mathrm{Proj}|_{\text{range } A} = A(A^*A)^{-1}A^*\mathbf{b}$.
- Adjoint of a linear map $T : V \to W$ is the A^* discussed above.
  - First, we'll do this for matrices. And then we'll do it for any finite-dimensional vector space.
  - Let A be an $m \times n$ matrix. We claim then that
$$(A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, A^*\mathbf{y})$$
    for all $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{y} \in \mathbb{C}^m$. Proof:
$$\begin{aligned}(A\mathbf{x}, \mathbf{y}) &= \bar{\mathbf{y}}^T A\mathbf{x}\\ &= \mathbf{y}^* A\mathbf{x}\\ &= (A^*\mathbf{y})^*\mathbf{x}\\ &= (\mathbf{x}, A^*\mathbf{y})\end{aligned}$$

- Properties of the adjoint:

$$(AB)^T = B^T A^T$$
$$(AB)^* = B^* A^*$$
$$(A^*)^* = A$$

- A^* is the unique matrix B such that $(A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, B\mathbf{y})$.
- Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be a basis of V, and let $\mathbf{w}_1, \dots, \mathbf{w}_m$ be a basis of W.
- Definition of A^*: If $(A\mathbf{x}, \mathbf{y}) = (y, A^*\mathbf{x})$ for all $\mathbf{x} \in V$ and $\mathbf{y} \in W$.
- But it's not enough to define something; we have to check that it exists.
- If $[A]_{\mathcal{AB}}$, then $[A^*]_{\mathcal{AB}}$.
- More properties (give criteria for solving systems of equations):

$$\ker A^* = (\text{range } A)^\perp$$
$$\ker A = (\text{range } A^*)^\perp$$
$$\text{range } A = (\ker A^*)^\perp$$
$$\text{range } A^* = (\ker A)^\perp$$

  - Soug proves these.

- Isometries and unitary operators.
  - $U : X \to Y$ is an isometry if $\|\mathbf{x}\| = \|U\mathbf{x}\|$ for all $\mathbf{x} \in X$. It is an isometry because it preserves the distance between points.
  - It immediately follows that $\|\mathbf{x}_1 - \mathbf{x}_2\| = \|U\mathbf{x}_1 - U\mathbf{x}_2\| = \|U(\mathbf{x}_1 - \mathbf{x}_2)\|$.
  - This definition is equivalent to an inner product one: $(\mathbf{x}, \mathbf{y}) = (U\mathbf{x}, U\mathbf{y})$. This follows from the definition of the norm.
  - We have
$$(\mathbf{a}, \mathbf{b}) = \frac{1}{4} \sum_{\alpha = \pm 1, \pm i} \alpha \|\mathbf{a} + \alpha \mathbf{b}\|^2$$
    - $(a+b)^2 - (a-b)^2 = 4ab$ for any $a, b \in \mathbb{R}$, so $ab = \frac{1}{4}[(a+b)^2 - (a-b)^2]$. Thus, in a real inner product space,
$$(\mathbf{a}, \mathbf{b}) = \frac{1}{4}\left(\|\mathbf{a} + \mathbf{b}\|^2 - \|\mathbf{a} - \mathbf{b}\|^2\right)$$
    - It follows that isometries preserve inner products.
  - U is an isometry if and only if $U^*U = I$. Proof:

$$(\mathbf{x}, \mathbf{x}) = (U\mathbf{x}, U\mathbf{y}) = (U^*U\mathbf{x}, \mathbf{x})$$
$$(\mathbf{x}, \mathbf{y}) = (U\mathbf{x}, U\mathbf{y}) = (U^*U\mathbf{x}, \mathbf{y})$$
$$(\mathbf{x}, \mathbf{y}) = (U\mathbf{x}, U\mathbf{y}) = (\mathbf{x}, \mathbf{y})$$

    for all $\mathbf{y}$.
  - An isometry is unitary if it is invertible.
    - Thus, $U : X \to Y$ an isometry is unitary iff $\dim X = \dim Y$.
  - Note that it follows that $U^* = U^{-1}$ for U an isometry.
  - U unitary implies $|\det U| = 1$, so λ an eigenvalue of U implies that $|\lambda| = 1$.
  - A is diagonalizable iff it has an orthogonal basis of eigenvectors.

5.2 Chapter 5: Inner Product Spaces

From Treil (2017).

10/24: - **Standard inner product** (on $\mathbb{C}^n$): The inner product $(\mathbf{z},\mathbf{w})$ defined by

$$(\mathbf{z},\mathbf{w}) = \mathbf{w}^*\mathbf{z}$$

- Corollary 5.1.5: Let $\mathbf{x},\mathbf{y}$ be vectors in an inner product space V. The equality $\mathbf{x}=\mathbf{y}$ holds if and only if

$$(\mathbf{x},\mathbf{z}) = (\mathbf{y},\mathbf{z})$$

  for all $\mathbf{z}\in V$.

- Corollary 5.1.6: Suppose two operator $A,B : X\to Y$ satisfy

$$(A\mathbf{x},\mathbf{y}) = (B\mathbf{x},\mathbf{y})$$

  for all $\mathbf{x}\in x$ and $\mathbf{y}\in Y$. Then $A=B$.

- **Normed space**: A vector space V equipped with a norm that satisfies properties of homogeneity, the triangle inequality, non-negativity, and non-degeneracy.
- Any inner product space is naturally a normed space.
- If $1\leq p<\infty$, we can define a corresponding norm on $\mathbb{R}^n$ or $\mathbb{C}^n$ by

$$\|\mathbf{x}\|_p = \left(\sum_{k=1}^n |x_k|^p\right)^{1/p}$$

- We can also define the norm for $p=\infty$ by

$$\|\mathbf{x}\|_\infty = \max\{|x_k| : k=1,\dots,n\}$$

  - Note that the norm of this form for $p=2$ is the usual norm.
  - These norms are heavily associated with Figure 5.1.
- **Minkowski inequality**: One of the triangle inequalities for norms with $p\neq 2$.
- Theorem 5.1.11: A norm in a normed space is obtained from some inner product if and only if it satisfies the Parallelogram Identity

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2)$$

  for all $\mathbf{u},\mathbf{v}\in V$.
  - It follows that norms with $p\neq 2$ do not have associated inner products, since such norms fail to satisfy the parallelogram identity.
- Lemma 5.2.5 (Generalized Pythagorean Identity): Let $\mathbf{v}_1,\dots,\mathbf{v}_n$ be an orthogonal system. Then

$$\left\|\sum_{k=1}^n \alpha_k\mathbf{v}_k\right\|^2 = \sum_{k=1}^n |\alpha_k|^2\|\mathbf{v}_k\|^2$$

- Proposition 5.3.3: Let $\mathbf{v}_1,\dots,\mathbf{v}_r$ be an orthogonal basis in E. Then the orthogonal projection $P_E\mathbf{v}$ of a vector $\mathbf{v}$ is given by the formula

$$P_E\mathbf{v} = \sum_{k=1}^r \frac{(\mathbf{v},\mathbf{v}_k)}{\|\mathbf{v}_k\|^2}\mathbf{v}_k$$

- It follows that

$$\begin{aligned}
P_E\mathbf{v} &= \sum_{k=1}^{r} \frac{\mathbf{v}_k^*\mathbf{v}}{\|\mathbf{v}_k\|^2}\mathbf{v}_k \\
&= \sum_{k=1}^{r} \frac{1}{\|\mathbf{v}_k\|^2}\mathbf{v}_k\mathbf{v}_k^*\mathbf{v} \\
&= \left(\sum_{k=1}^{r} \frac{1}{\|\mathbf{v}_k\|^2}\mathbf{v}_k\mathbf{v}_k^*\right)\mathbf{v}
\end{aligned}$$

- Thus, we have that

$$P_E = \sum_{k=1}^{r} \frac{1}{\|\mathbf{v}_k\|^2}\mathbf{v}_k\mathbf{v}_k^*$$

- **Gram-Schmidt orthogonalization**: Let $\mathbf{x}_1,\dots,\mathbf{x}_n$ be a linearly independent system of vectors to orthogonalize. Then $\mathbf{v}_1 = \mathbf{x}_1$, $\mathbf{v}_2 = \mathbf{x}_2 - P_{\operatorname{span}\{\mathbf{v}_1\}}\mathbf{x}_2$, $\mathbf{v}_3 = \mathbf{x}_3 - P_{\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2\}}\mathbf{x}_3$, and on and on.
- To find the least squares solution to $A\mathbf{x} = \mathbf{b}$, solve $A\mathbf{x} = P_{\operatorname{range} A}\mathbf{b}$.
  - We can do this by finding an orthogonal basis of range A and then applying the projection formula.
  - Alternatively, we can use the following formula to speed things up if A^*A is invertible:

$$P_{\operatorname{range} A}\mathbf{b} = A(A^*A)^{-1}A^*\mathbf{b}$$

- Theorem 5.4.1: For an $m \times n$ matrix A,

$$\ker A = \ker(A^*A)$$

  - Thus, A^*A is invertible iff A is invertible iff A is full rank. This gives us a condition on when we can use the projection formula.
- Theorem 5.6.1: An operator $U : X \to Y$ is an isometry if and only if it preserves the inner product, i.e., if and only if

$$(\mathbf{x},\mathbf{y}) = (U\mathbf{x}, U\mathbf{y})$$

  for all $\mathbf{x},\mathbf{y} \in X$.
- Lemma 5.6.2: An operator $U : X \to Y$ is an isometry if and only if $U^*U = I$.
- **Unitary** (operator): An invertible isometry.
- Proposition 5.6.3: An isometry $U : X \to Y$ is a unitary operator iff $\dim X = \dim Y$.
- **Orthogonal** (matrix): A unitary matrix with real entries.
- Unitary operator properties:
  1. $U^{-1} = U^*$.
  2. U unitary implies $U^* = U^{-1}$ unitary.
  3. If $\mathbf{v}_1,\dots,\mathbf{v}_n$ is orthonormal, $U\mathbf{v}_1,\dots,U\mathbf{v}_n$ is orthonormal.
  4. U, V unitary implies UV unitary.
- A matrix U is an isometry iff its columns form an orthonormal system.
- Proposition 5.6.4: Let U be a unitary matrix. Then
  1. $|\det U| = 1$. In particular, if U is orthogonal, then $\det U = \pm 1$.
  2. $|\lambda| = 1$ for every eigenvalue λ of U.
- Proposition 5.6.5: A matrix A is unitarily equivalent to a diagonal one iff it has an orthogonal (orthonormal) basis of eigenvectors.

Chapter 6

Structure of Operators on Inner Product Spaces

6.1 Notes

10/11:

- Spectral decomposition of self-adjoint linear maps.
  - Can we write a map in term of the eigenvalues only?
  - Let $A : X \to X$ be linear and self-adjoint. Where $\dim X < \infty$.
  - Let A have eigenvalues $\lambda_1, \dots, \lambda_n$ and eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$. The there is an orthonormal basis of X consisting of eigenvectors of A. An operator is self-adjoint if $A = A^*$.
  - If A is self-adjoint, then A can be written as diagonal with the eigenvalues on the diagonal with respect to some orthonormal basis of eigenvectors.
  - Let $\mathbb{F} = \mathbb{C}$.
- If there exists an orthonormal basis $u_1, \dots, u_n$ of X such that A is triangular, then $A = UTU^*$ where U is unitary and T is upper triangular.
  - Proved with induction on $\dim X$.
  - $\dim X = 1$ is clear.
  - Assume for $\dim X = n - 1$, WTS for $\dim X = n$.
  - The subspace has a basis $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ such that A has a diagonal form.
  - Let $u \in X$ be linearly independent of $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$.
  - Let λ be the remaining eigenvalue and u the corresponding eigenvector. Let $E = \operatorname{span}(u)$. Then make the matrix λ in the upper left corner, and block diagonal with "A_{n-1}" in the bottom right corner, zeroes everywhere else.
- **Self-adjoint** (matrix A): A linear map $A : X \to X$ where $\dim X < \infty$ such that $A = A^*$.
  - Similarly, $(Ax, y) = (x, Ay)$.
  - A self-adjoint implies all eigenvalues are real, eigenvectors corresponding to different eigenvalues are orthogonal.
    - Soug proves this.
- **Strictly positive** (operator A): A self-adjoint operator $A : X \to X$ such that $(Ax, x) > 0$ for all $x \neq 0$. *Also known as* **positive definite**.
  - Implies that all eigenvalues are strictly positive.

- **Nonnegative** (operator A): A self-adjoint operator $A : X \to X$ such that $(Ax, x) \geq 0$ for all $x \neq 0$. *Also known as* **definite**.
  - All eigenvalues are nonnegative.
- Suppose $A \geq 0$ is self-adjoint. Then there exists a unique self-adjoint $B \geq 0$ such that $B^2 = A$.
  - A self-adjoint is diagonal (wrt. some basis).
  - A positive means that all eigenvalues (diagonal entries) are positive.
  - Thus, take
$$B = \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sqrt{\lambda_n} \end{pmatrix}$$
  - Suppose $B^2 = A$, $C^2 = A$. Then we have an orthonormal basis corresponding to B and an orthonormal basis corresponding to C. It follows that $B^2 = C^2 = A$. Write B^2x and C^2x in terms of their bases; will necessitate that the bases are the same.

10/13:

- If we get yes/no questions, we don't have to justify.
- Cauchy-Schwarz inequality:
$$|(\mathbf{x}, \mathbf{y})| \leq \|\mathbf{x}\|\|\mathbf{y}\|$$
  - Real spaces, V vs. $(\cdot, \cdot)$ inner product.
  - Proof:
$$\begin{aligned} 0 &\leq \|\mathbf{x} + t\mathbf{y}\|^2 \\ &= t^2\|\mathbf{y}^2\| + 2t(\mathbf{x}, \mathbf{y}) + \|\mathbf{x}\|^2 \end{aligned}$$

    Thus, the discriminant must be less than zero (because the whole polynomial is positive, so the discriminant [the opposite of the x^0 term of the factored form of the polynomial] must be less than zero so the polynomial doesn't get dragged down to negative values):
$$(\mathbf{x}, \mathbf{y})^2 - \|\mathbf{x}\|^2\|\mathbf{y}\|^2 \leq 0$$

    Taking square roots of both sides proves the desired inequality.
- Recall that if $A^* = A$, then all eigenvalues are real and all eigenvectors of distinct eigenvalues are orthogonal to each other.
- **Normal** (matrix): A matrix N such that $N^*N = NN^*$.
  - Examples: Diagonal, self-adjoint, and unitary operators are all normal.
- Any normal operator in a complex vector spae has an orthonormal set of eigenvectors, e.g., $N = UDU^*$.
  - Proof: N is upper triangular wrt. some basis (because all matrices are). WTS any normal upper triangular matrix is diagonal. Done by induction on the dimension of N from $n = 2$.
  - Assume the claim for every $(n-1) \times (n-1)$ normal upper triangular matrix.
  - Let
$$N = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & & & \\ 0 & & & \\ \vdots & & N_1 & \\ 0 & & & \end{pmatrix}$$

    (we know every normal matrix can be written in this upper triangular form)

- Then just compute NN^* and N^*N. Knowing they have to be equal, we have that $a_{12} = \cdots = a_{1n} = 0$.
- We can also prove from the above (block diagonal multiplication) that N_1 is normal. Thus, it's diagonal, too. Therefore, the whole thing is diagonal.

- N is normal if and only if $\|N\mathbf{x}\| = \|N^*\mathbf{x}\|$.
  - Proof: $(N\mathbf{x}, N\mathbf{y}) = (N^*N\mathbf{x}, \mathbf{y}) = (NN^*\mathbf{x}, \mathbf{y}) = (N^*\mathbf{x}, N^*\mathbf{y})$. This is equivalent to the desired condition.
- If A is nonnegative and $(A\mathbf{e}_k, \mathbf{e}_k) = a_{kk}$, then

$$\sum_{i,j=1}^{n} a_{ij}\mathbf{x}_i\mathbf{x}_j$$

- **Positive definite** (matrix): An $n \times n$ self-adjoint matrix such that $(A\mathbf{x}, \mathbf{x}) > 0$ for all $\mathbf{x} \in X$.
- Let $A : X \to Y$, $\dim X = \dim Y$. Then AA^* is positive semidefinite. And there exists a unique square root $R = \sqrt{A^*A}$.
  - Proof: $(A^*A\mathbf{x}, \mathbf{x}) = (A\mathbf{x}, A\mathbf{x}) = \|A\mathbf{x}\|^2 \geq 0$.
- **Modulus** (of A): The matrix $|A| = \sqrt{A^*A}$.
- Check $\||A|\mathbf{x}\| = \|A\mathbf{x}\|$.

$$\||A|\mathbf{x}\|^2 = (|A|\mathbf{x}, |A|\mathbf{x}) = (|A|^*|A|\mathbf{x}, \mathbf{x}) = (A^*A\mathbf{x}, \mathbf{x}) = (A\mathbf{x}, A\mathbf{x}) = \|A\mathbf{x}\|^2$$

- Let $A : X \to X$ be a linear operator. Then $A = U|A|$ where U is unitary.
- Look at singular matrices.

10/15:

- Recall that if $A : X \to Y$, we have that A^*A is semidefinite, positive, and self adjoint.
  - Thus, there exists a unique matrix $R = \sqrt{A^*A} \geq 0$, which we define to be $|A| = \sqrt{A^*A}$.
- Polar form of a matrix:

$$A = U|A|$$

  - This may not be unique!
  - Proof: Suppose $A\mathbf{x} = U(|A|\mathbf{x})$. $A\mathbf{x} \in \operatorname{range} A$, and $|A|\mathbf{x} \in \operatorname{range}(|A|)$. $\mathbf{x} \in \operatorname{range}(|A|)$ implies that there exists $\mathbf{v} \in X$ such that $x = |A|\mathbf{v}$.
  - Define $U\mathbf{x} = A\mathbf{x}$. U is a well-defined linear map.
  - $\|U_0\mathbf{x}\| = \|A\mathbf{x}\| = \||A|\mathbf{v}\| = \|\mathbf{x}\|$.
  - U is an isometry.
  - $\operatorname{range}|A| \to X$.
  - Use $\ker A = \ker|A| = (\operatorname{range} A)^\perp$ to extend U_0 to U: $U = U_0 + U_1$.
- **Singular values** (of a matrix): The eigenvalues of $|A|$.
  - So if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A^*A, the singular values of A are $\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}$.
- Let $A : X \to Y$ be a linear map.
  - Let $\sigma_1, \dots, \sigma_n$ be the signular values of A. Then $\sigma_1, \dots, \sigma_n > 0$.
  - Additionally, if $\mathbf{v}_1, \dots, \mathbf{v}_n$ is an orthonormal basis of eigenvectors of A^*A, then the list of n vectors $\mathbf{w}_1, \dots, \mathbf{w}_n$ defined by $\mathbf{w}_k = 1/\sigma_k A\mathbf{v}_k$ for each $k = 1, \dots, n$ is orthonormal.

  - Proof:

$$(\mathbf{w}_k,\mathbf{w}_j)=\frac{1}{\sigma_k\sigma_k}(A\mathbf{v}_k,A\mathbf{v}_j)=\frac{1}{\sigma_k\sigma_j}=\frac{1}{\sigma_k\sigma_j}(A^*A\mathbf{v}_k,\mathbf{v}_j)=\frac{\sigma_k^2}{\sigma_k\sigma_j}(\mathbf{v}_k,\mathbf{v}_j)=0$$

and

$$\|\mathbf{w}_k\|=\frac{1}{\sigma_k}\|A\mathbf{v}_k\|=\frac{1}{\sigma_k}\||A|\mathbf{v}_k\|=1$$

- Schmidt decomposition of A:

$$A\mathbf{x}=\sum^r\sigma_k(\mathbf{x},\mathbf{v}_k)\mathbf{w}_k$$

  - This is because $\mathbf{x}=\sum(\mathbf{x},\mathbf{v}_k)\mathbf{v}_k$, so by the above,

$$A\mathbf{x}=\sum^n(\mathbf{x},\mathbf{v}_k)A\mathbf{v}_k=\sum^r\sigma_k(\mathbf{x},\mathbf{v}_k)\mathbf{w}_k$$

- **Operator norm**: $\|A\|=\max\{\|A\mathbf{x}\|:\|\mathbf{x}\|\leq 1\}$.
- Properties of the operator norm:
  - $\|A\mathbf{x}\|\leq\|A\|\|\mathbf{x}\|$.
  - $\|\alpha A\|=|\alpha|\|A\|$.
  - $\|A+B\|\leq\|A\|+\|B\|$.
  - $\|A\|\geq 0$.
  - $\|A\|=0$ iff $A=0$.
- **Frobenius norm**: The norm $\|A\|_2^2=\operatorname{tr}(A^*A)$.
- The operator norm is always less than or equal to the Frobenius norm.
- If $A:\mathbb{F}^n\to\mathbb{F}^n$, then $A=W\Sigma V^*$ where σ is a diagonal matrix of nonzero singular values.
- The operator norm of A is the largest of the singular values.
- An orthogonal matrix can be decomposed to a block-diagonal matrix of rotations.

10/18:

- Soug tests what he teaches and doesn't give super tricky questions.
- Structure of orthogonal matrices.
- **Orthogonal** (matrix): A unitary matrix U with all elements real and $|\det U|=1$.
- Theorem: Let U be an orthogonal operator on $\mathbb{R}^n$ such that $\det U=1$. Then there exists an orthonormal basis $\mathbf{v}_1,\dots,\mathbf{v}_n$ such that with respect to this basis,

$$U=\begin{pmatrix}R_{\phi_1} & & & \mathbf{0}\\ & \ddots & & \\ & & \mathbb{R}_{\phi_k} & \\ \mathbf{0} & & & I_{n-2k}\end{pmatrix}$$

where each R_{ϕ_i} is a 2×2 rotation matrix.

  - If you are in $\mathbb{R}^7$ for example, you would be able to express U as a composition of at most 3 rotation maps and the identity map.
  - Each rotation map acts on two orthonormal vectors.
  - Proof: $P(\lambda)$ is the n-degree characteristic polynomial $\det(U-\lambda I)=0$. The eigenvalues are the roots of it.

- $p(\lambda) = 0$ if and only if $p(\bar{\lambda}) = 0$.
  - $\lambda \in \mathbb{C}$ is an eigenvalue with eigenvector $\mathbf{u} \neq 0$ iff $U\mathbf{u} = \lambda\mathbf{u}$ and $U\bar{\mathbf{u}} = \bar{\lambda}\bar{\mathbf{u}}$.
- Recall that U unitary implies $|\lambda| = 1$.
  - Proof[1]: $\|U\mathbf{x}\| = \|\mathbf{x}\|$ and $U\mathbf{x} = \lambda\mathbf{x}$. Thus,
$$\|U\mathbf{x}\| = \|\lambda\mathbf{x}\| = |\lambda|\|\mathbf{x}\| = \|\mathbf{x}\|$$
    and since $\mathbf{x} \neq 0$, we can divide by $\|\mathbf{x}\|$, so $|\lambda| = 1$.
- Let $\mathbf{u} = \operatorname{Re}\mathbf{u} + \operatorname{Im}\mathbf{u}$.
- It follows that we may define
$$\mathbf{x} = \operatorname{Re}\mathbf{u} = \frac{\mathbf{u} + \bar{\mathbf{u}}}{2} \qquad \mathbf{y} = \operatorname{Im}\mathbf{u} = \frac{\mathbf{u} - \bar{\mathbf{u}}}{2}$$
- Thus, $\mathbf{u} = \mathbf{x} + i\mathbf{y}$ and $\bar{\mathbf{u}} = \mathbf{x} - i\mathbf{y}$.
- Since $U\mathbf{x} = \frac{U\mathbf{u}+U\bar{\mathbf{u}}}{2} = \frac{\lambda\mathbf{u}+\bar{\lambda}\bar{\mathbf{u}}}{2}$, $U\mathbf{y} = \operatorname{Im}(\lambda\mathbf{u}) = \operatorname{Re}(\lambda\mathbf{u})$.
- Since $|\lambda| = 1$, $\lambda = \mathrm{e}^{i\alpha}$ and $\bar{\lambda} = \mathrm{e}^{-i\alpha}$.
- It follows that $U\mathbf{x} = (\cos\alpha)\mathbf{x} - (\sin\alpha)\mathbf{y}$ and $U\mathbf{y} = (\cos\alpha)\mathbf{y}) + (\sin\alpha)\mathbf{x}$.
- Thus, since $U\mathbf{x} = \operatorname{Re}\lambda\mathbf{u}$, we have that
$$\begin{aligned}\lambda\mathbf{u} &= (\cos\alpha + i\sin\alpha)(\mathbf{x} + i\mathbf{y}) \\ &= (\cos\alpha)\mathbf{x} - (\sin\alpha)\mathbf{y} + i[(\cos\alpha)\mathbf{y} + (\sin\alpha)\mathbf{x}]\end{aligned}$$
- If E_λ is a 2 dimensional space spanned by $\mathbf{x}$ and $\mathbf{y}$ and invariant by U. Thus, any block of the desired matrix leaves its desired sub-block invariant.
- We also know that the eigenvectors of a unitary matrix corresponding to different eigenvalues are orthogonal.
- Thus, $\|\mathbf{x}\| = \|\mathbf{y}\| = \sqrt{2}/2\|\mathbf{u}\|$, $\mathbf{x} \perp \mathbf{y}$.
- Let $\mathbf{x}, \mathbf{y}$ complete the theorem to form a basis of $\mathbb{R}^n$.
- It will follow that
$$U = \begin{pmatrix} R_\alpha & \mathbf{0} \\ \mathbf{0} & U_1 \end{pmatrix}$$
  where U_1 is orthogonal, and we may repeat the process.

6.2 Chapter 6: Structure of Operators on Inner Product Spaces

From Treil (2017).

10/24:

- Theorem 6.1.1: Let $A : X \to X$ be an operator acting in a complex inner product space. Then there exists an orthonormal basis $\mathbf{u}_1, \dots, \mathbf{u}_n$ of X such that the matrix of A in this basis is upper triangular. In other words, any $n \times n$ matrix A can be represented as $A = UTU^*$, where U is unitary and T is upper-triangular.
- Theorem 6.1.2: Let $A : X \to X$ be an operator acting on a real inner product space. Suppose that all eigenvalues of A are real. Then there exists an orthonormal basis $\mathbf{u}_1, \dots, \mathbf{u}_n$ in X such that the matrix of A in this basis is upper triangular. In other words, any real $n \times n$ matrix A with all real eigenvalues can be represented as $T = UTU^* = UTU^T$, where U is orthogonal and T is a real upper-triangular matrix.

[1]This would be a good exam question.

- Theorem 6.2.1: Let $A = A^*$ be a self-adjoint operator in an inner product space X (the space can be complex or real). Then all eigenvalues of A are real and there exists an orthonormal basis of eigenvectors of A in X.

  Equivalently (see Theorem 6.2.2), A can be represented as $A = UDU^*$ where U is a unitary matrix and D is a diagonal matrix with real entries. Moreover, if A is real, U can be chosen to be real, i.e., orthogonal.
- Proposition 6.2.3: Let $A = A^*$ be a self-adjoint operator and let $\lambda, \mathbf{u}, \mu, \mathbf{v}$ be such that $A\mathbf{u} = \lambda\mathbf{u}$ and $A\mathbf{v} = \mu\mathbf{v}$. Then if $\lambda \neq \mu$, $\mathbf{u} \perp \mathbf{v}$.
- Since complex multiplication is commutative,
  $$D^*D = DD^*$$
  for every diagonal matrix D.
  - It follows that $A^*A = AA^*$ if the matrix of A in some orthonormal basis is diagonal.
- Theorem 6.2.4: Any normal operator N in a complex vector space has an orthonormal basis of eigenvectors.

  Equivalently, any matrix N satisfying $N^*N = NN^*$ can be represented as $N = UDU^*$ where U is unitary and D is diagonal.
- Proposition 6.2.5: An operator $N : X \to X$ is normal iff
  $$\|N\mathbf{x}\| = \|N^*\mathbf{x}\|$$
  for all $\mathbf{x} \in X$.
- **Hermitian square** (of A): The matrix A^*A.
- **Modulus** (of A): The unique positive semidefinite square root $\sqrt{A^*A}$.
- Proposition 6.3.3: For a linear operator $A : X \to Y$,
  $$\||A|\mathbf{x}\| = \|A\mathbf{x}\|$$
- Corollary 6.3.4: $\ker A = \ker |A|$.
- Theorem 6.3.5: Let $A : X \to X$ be an operator (square matrix). Then A can be represented as
  $$A = U|A|$$
  where U is a unitary operator.
- **Singular value** (of A): An eigenvalue of $|A|$.
  - A positive square root of an operator of A^*A.
- Proposition 6.3.6: Let $\sigma_1, \dots, \sigma_n$ be the singular values of A, ordered such that $\sigma_1, \dots, \sigma_r$ are the nonzero singular values, and let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be an orthonormal basis of eigenvectors of A^*A. Then the system
  $$\mathbf{w}_k = \frac{1}{\sigma_k} A\mathbf{v}_k$$
  for $k = 1, \dots, r$ is orthonormal.

- **Schmidt decomposition** (of A): The decompositions

$$A = \sum_{k=1}^{r} \sigma_k \mathbf{w}_k \mathbf{v}_k^*$$

and

$$A\mathbf{x} = \sum_{k=1}^{r} \sigma_k (\mathbf{x}, \mathbf{v}_k) \mathbf{w}_k$$

  - Note that these can be verified by plugging $\mathbf{x} = \mathbf{v}_j$ for each $j = 1, \dots, n$ into the latter equation.

10/25: - Lemma 6.3.7: A can be represented as the Schmidt decomposition

$$A = \sum_{k=1}^{r} \sigma_k \mathbf{w}_k \mathbf{v}_k^*$$

where $\sigma_k > 0$ for any orthonormal systems $\mathbf{v}_1, \dots, \mathbf{v}_r$ and $\mathbf{w}_1, \dots, \mathbf{w}_r$.

- Corollary 6.3.8: Let $A = \sum_{k=1}^{r} \sigma_k \mathbf{w}_k \mathbf{v}_k^*$ be a Schmidt decomposition of A. Then

$$A^* = \sum_{k=1}^{r} \sigma_k \mathbf{v}_k \mathbf{w}_k^*$$

is a Schmidt decomposition of A^*.

- **Reduced singular value decomposition** (of A): The decomposition

$$A = \tilde{W} \tilde{\Sigma} \tilde{V}^*$$

where $A : \mathbb{F}^n \to \mathbb{F}^m$ has the Schmidt decomposition $A = \sum_{k=1}^{r} \sigma_k \mathbf{w}_k \mathbf{v}_k^*$, $\tilde{\Sigma} = \operatorname{diag}\{\sigma_1, \dots, \sigma_r\}$, and $\tilde{V}, \tilde{W}$ are matrices with columns $\mathbf{v}_1, \dots, \mathbf{v}_r$ and $\mathbf{w}_1, \dots, \mathbf{w}_r$, respectively. *Also known as* **compact singular value decomposition**.

  - Note that $\tilde{V}$ is an $n \times r$ matrix, $\tilde{\Sigma}$ is an $r \times r$ matrix, and $\tilde{W}$ is an $m \times r$ matrix.
  - Since $\mathbf{v}_1, \dots, \mathbf{v}_r$ and $\mathbf{w}_1, \dots, \mathbf{w}_r$ are orthonormal, $\tilde{V}, \tilde{W}$ are isometries.

- Note that $r = \operatorname{rank} A$ (see Problem 6.3.1).
  - It follows that if A is invertible, then $m = n = r$, so $\tilde{V}, \tilde{W}$ are unitary and $\tilde{\Sigma}$ is an invertible diagonal matrix.

- However, A need not be invertible for us to get a representation similar to $A = \tilde{W} \tilde{\Sigma} \tilde{V}^*$.
  - Complete $\mathbf{v}_1, \dots, \mathbf{v}_r$ and $\mathbf{w}_1, \dots, \mathbf{w}_r$ to bases of $\mathbb{F}^n$ and $\mathbb{F}^m$, respectively.
  - Then we get the following.

- **Singular value decomposition** (of A): The decomposition

$$A = W \Sigma V^*$$

where $V \in M_{n \times n}^{\mathbb{F}}$ and $W \in M_{m \times m}^{\mathbb{F}}$ are unitary matrices with columns $\mathbf{v}_1, \dots, \mathbf{v}_n$ and $\mathbf{w}_1, \dots, \mathbf{w}_n$, respectively, and $\Sigma \in M_{m \times n}^{\mathbb{R}^+}$ is a "diagonal" matrix such that

$$\Sigma_{j,k} = \begin{cases} \sigma_k & j = k \leq r \\ 0 & \text{otherwise} \end{cases}$$

- Notice that if $A = W\Sigma V^*$, then
$$A^*A = (W\Sigma V^*)^*(W\Sigma V^*) = V\Sigma^* W^* W\Sigma V^* = V\Sigma^2 V^*$$
proving that the singular values of A, squared, are the eigenvalues of A^*A.
- If A is invertible, the reduced SVD is the matrix form of the Schmidt decomposition is the SVD.
- If $A = W\Sigma V^*$ is $n \times n$, then
$$A = (\underbrace{WV^*}_{U})(\underbrace{V\Sigma V^*}_{|A|})$$
is a polar decomposition of A.
- Consider the unit ball $B = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq 1\}$.
  - We want to describe $A(B)$, i.e., the image of the unit ball under A.
  - Let $\mathbf{x} = (x_1, \dots, x_n)^T$ and let $\mathbf{y} = (y_1, \dots, y_n)^T$. If $A = \operatorname{diag}\{\sigma_1, \dots, \sigma_n\}$, we have $\mathbf{y} \in A(B)$ iff $\mathbf{y} = A\mathbf{x}$ where $\mathbf{x} \in B$ iff
$$\sum_{k=1}^n \frac{y_k^2}{\sigma_k^2} = \sum_{k=1}^n x_k^2 = \|\mathbf{x}\|^2 \leq 1$$
  - Thus, $A(B)$ is an ellipsoid with half-axes $\sigma_1, \dots, \sigma_n$.
  - In the more general case, if $A = W\Sigma V^*$, then since V^* is unitary, $V^*(B) = B$. $\Sigma V^*(B) = \Sigma(B)$ is thus by the above an ellipsoid in range Σ with half-axes $\sigma_1, \dots, \sigma_r$. Thus, since isometries don't change geometry, $W(\Sigma(B))$ is also an ellipsoid with the same half-axes, but in range A.
- Conclusion: The image $A(B)$ of the closed unit ball B is an ellipsoid in range A with half-axes $\sigma_1, \dots, \sigma_r$, where r is the number of nonzero singular values, i.e., the rank of A.
- Finding the maximum of $\|A\mathbf{x}\|$ for $\mathbf{x} \in B$.
  - For a diagonal matrix Σ with nonnegative entries, the maximum is clearly the maximal diagonal entry: In this case if s_1 is the maximal diagonal entry, then since
$$\Sigma\mathbf{x} = \sum_{k=1}^r s_k x_k \mathbf{e}_k$$
we have that
$$\|A\mathbf{x}\|^2 = \sum_{k=1}^r s_k^2 |x_k|^2 \leq s_1^2 \sum_{k=1}^r |x_k|^2 = s_1^2 \cdot \|\mathbf{x}\|^2$$
  - We get the following by a similar logic to before.
- Conclusion: The maximum of $\|A\mathbf{x}\|$ on the unit ball B is the maximal singular value of A.
- **Operator norm** (of A): The following quantity. *Denoted by* $\|\boldsymbol{A}\|$. *Given by*
$$\|A\| = \max\{\|A\mathbf{x}\| : \mathbf{x} \in X, \|\mathbf{x}\| \leq 1\}$$
  - $\|A\|$ clearly satisfies the four properties of a norm.
  - Additionally,
$$\|A\mathbf{x}\| \leq \|A\| \cdot \|\mathbf{x}\|$$
  - Alternate definition: The operator norm $\|A\|$ is the smallest number $C \geq 0$ such that $\|A\mathbf{x}\| \leq C\|\mathbf{x}\|$.

- **Frobenius norm**: The following norm. *Also known as* **Hilbert-Schmidt norm**. *Denoted by* $\|\boldsymbol{A}\|_{\mathbf{2}}$. *Given by*
$$\|A\|_2^2 = \operatorname{tr}(A^*A)$$
  - If we let $s_1, \dots, s_n$ be the singular values of A and let s_1 be the largest value, then we have
$$\|A\|^2 = s_1^2 \leq \sum_{k=1}^{n} s_k^2 = \operatorname{tr}(A^*A) = \|A\|_2^2$$
- Conclusion: The operator norm of a matrix cannot be more than its Frobenius norm.
- Suppose we want to solve $A\mathbf{x} = \mathbf{b}$ where A is invertible, but there is some (experimental) error $\Delta\mathbf{b}$ in $\mathbf{b}$. Then we are really solving for an approximate solution $\mathbf{x} + \Delta\mathbf{x}$ to the equation
$$A(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{b} + \Delta\mathbf{b}$$
  - It follows since A is invertible that $\mathbf{x} = A^{-1}\mathbf{b}$ and $\Delta\mathbf{x} = A^{-1}\Delta\mathbf{b}$.
  - To estimate the relative error $\|\Delta\mathbf{x}\|/\|\mathbf{x}\|$ in the solution in comparison with the relative error $\|\Delta\mathbf{b}\|/\|\mathbf{b}\|$ in the data, use
$$\frac{\|\Delta\mathbf{x}\|}{\|\mathbf{x}\|} = \frac{\left\|A^{-1}\Delta\mathbf{b}\right\|}{\|\mathbf{b}\|}\frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} \leq \frac{\left\|A^{-1}\right\| \cdot \|\Delta\mathbf{b}\|}{\|\mathbf{b}\|}\frac{\|A\| \cdot \|\mathbf{x}\|}{\|\mathbf{x}\|} = \left\|A^{-1}\right\| \cdot \|A\| \cdot \frac{\|\Delta\mathbf{b}\|}{\|\mathbf{b}\|}$$
- **Condition number** (of A): The following quantity. *Given by*
$$\|A\| \cdot \left\|A^{-1}\right\|$$
  - If s_1 is the largest singular value of A and s_n is the smallest, then
$$\|A\| \cdot \left\|A^{-1}\right\| = s_1 \cdot \frac{1}{s_n} = \frac{s_1}{s_n}$$
- **Well-conditioned** (matrix): A matrix the condition number of which is not "too big."
- **Ill-conditioned** (matrix): A matrix that is not well-conditioned.
- Theorem 6.5.1: Let U be an orthogonal operator on $\mathbb{R}^n$ and let $\det U = 1$. Then there exists an orthonormal basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ such that the matrix of U in this basis has the block diagonal form
$$\begin{pmatrix} R_{\varphi_1} & & & 0 \\ & \ddots & & \\ & & R_{\varphi_k} & \\ 0 & & & I_{n-2k} \end{pmatrix}$$
where each R_{φ_j} is a two-dimensional rotation
$$R_{\varphi_j} = \begin{pmatrix} \cos\varphi_j & -\sin\varphi_j \\ \sin\varphi_j & \cos\varphi_j \end{pmatrix}$$
and I_{n-2k} represents the $(n-2k) \times (n-2k)$ identity matrix.
  - Alternate interpretation: Any rotation in $\mathbb{R}^n$ can be represented as a composition of at most $n/2$ commuting planar rotations.

- Theorem 6.5.2: Let U be an orthogonal operator on $\mathbb{R}^n$ and let $\det U = -1$. Then there exists an orthonormal basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ such that the matrix of U in this basis has block diagonal form

$$\begin{pmatrix} R_{\varphi_1} & & & & 0 \\ & \ddots & & & \\ & & R_{\varphi_k} & & \\ & & & I_r & \\ 0 & & & & -1 \end{pmatrix}$$

  where $r = n - 2k - 1$ and each R_{φ_j} is a two-dimensional rotation

$$R_{\varphi_j} = \begin{pmatrix} \cos\varphi_j & -\sin\varphi_j \\ \sin\varphi_j & \cos\varphi_j \end{pmatrix}$$

- Corollary: An orthogonal 2×2 matrix U with determinant -1 is always a reflection.
- Theorem 6.5.3: Any rotation U (i.e., any orthogonal transformation U with $\det U = 1$) can be represented as a product of at most $n(n-1)/2$ elementary rotations.
- Consider the following orthonormal bases of $\mathbb{R}^2$.

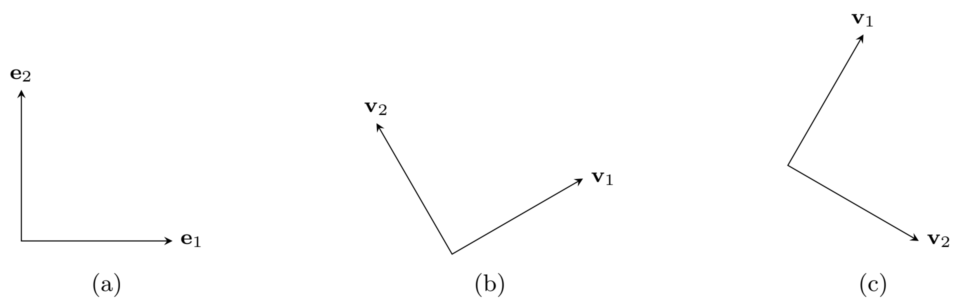


Figure 6.1: Orientation in $\mathbb{R}^2$.

  - Notice that a rotation will get you from the standard basis (a) to basis (b), but not from the standard basis (a) to basis (c).
  - This is the motivation for defining orientation.
  - More formally, we know that there is a unique linear transformation U such that $U\mathbf{e}_k = \mathbf{v}_k$ for each $k = 1, 2$. In particular, the matrix of U with respect to the standard basis is orthogonal with columns $\mathbf{v}_1, \mathbf{v}_2$.
  - By Theorems 6.5.1 and 6.5.2, if $\det U = 1$, then U is a rotation, and if $\det U = -1$, then U is not a rotation.
- **Similarly oriented** (bases $\mathcal{A}, \mathcal{B}$): Two bases $\mathcal{A}, \mathcal{B}$ of a real vector space such that the change of coordinates matrix $[I]_{\mathcal{BA}}$ has a positive determinant.
- **Differently oriented** (bases $\mathcal{A}, \mathcal{B}$): Two bases $\mathcal{A}, \mathcal{B}$ of a real vector space that are not similarly oriented (i.e., $[I]_{\mathcal{BA}}$ has a negative determinant).
- We usually let the standard basis of $\mathbb{R}^n$ have a **positive orientation**.
  - In an abstract vector space, we need only fix a basis and declare its orientation to be positive.
- **Continuously transformable** (bases $\mathcal{A}, \mathcal{B}$): Two bases $\mathcal{A}, \mathcal{B}$ such that $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ can be continuously transformed to a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$. In particular, there exists a **continuous family of bases** $\mathcal{V}(t) = \{\mathbf{v}_1(t), \dots, \mathbf{v}_n(t)\}$, $t \in [a, b]$, such that

$$\mathbf{v}_k(a) = \mathbf{a}_k \qquad \mathbf{v}_k(b) = \mathbf{b}_k$$

  for each $k = 1, \dots, n$.

- **Continuous family of bases**: A family of bases $\mathcal{V}(t) = \{\mathbf{v}_1(t), \dots, \mathbf{v}_n(t)\}$, $t \in [a, b]$, such that the vector-functions $\mathbf{v}_k(t)$ are continuous (their coordinates in some bases are continuous functions) and the system $\mathbf{v}_1(t), \dots, \mathbf{v}_n(t)$ is a basis for all $t \in [a, b]$.
- Theorem 6.6.1: Two bases $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ and $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ have the same orientation if and only if one of the bases can be continuously transformed to the other.

Chapter 7

Bilinear and Quadratic Forms

7.1 Notes

10/18: - **Bilinear form**: A function $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$L(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2, \mathbf{y}) = \alpha L(\mathbf{x}_1, \mathbf{y}) + \beta L(\mathbf{x}_2, \mathbf{y}) \qquad L(\mathbf{x}, \alpha\mathbf{y}_1 + \beta\mathbf{y}_2) = \alpha L(\mathbf{x}, \mathbf{y}_1) + \beta L(\mathbf{x}, \mathbf{y}_2)$$

  - $L(\mathbf{x}, \mathbf{y}) = (A\mathbf{x}, \mathbf{y})$.
- **Quadratic form**: A bilinear form $L(\mathbf{x}, \mathbf{x})$.
  - $(\mathbf{x}, \mathbf{x})$ is a polynomial of degree 2 in $\mathbf{x}_1, \dots, \mathbf{x}_n$:

$$L(\lambda\mathbf{x}, \lambda\mathbf{x}) = (\lambda\mathbf{x}, \lambda\mathbf{x}) = \lambda^2(\mathbf{x}, \mathbf{x})$$

- We have that

$$(A\mathbf{x}, \mathbf{x}) = (A\lambda\mathbf{x}, \lambda\mathbf{x}) = \lambda^2(A\mathbf{x}, \mathbf{x}) = \sum_{j,i=1}^{n} \alpha_{j,i}\mathbf{x}_i\mathbf{x}_j$$

- The general form of a quadratic form:
  - Can any quadratic form on $\mathbb{R}^n$ be written as $(A\mathbf{x}, \mathbf{x})$?

10/20: - Bilinear forms are linear in each argument when keeping the other fixed.
- Quadratic forms $Q(\mathbf{x}) = L(\mathbf{x}, \mathbf{x})$ are quadratic polynomials in the coordinates of x.
  - In particular, $Q(\lambda\mathbf{x}) = |\lambda|^2 Q(\mathbf{x})$.
- If Q quadratic is real, then $Q(\mathbf{x}) = (A\mathbf{x}, \mathbf{x})$ where A is some square matrix.
  - If $\mathbf{e}_1, \dots, \mathbf{e}_n$ is an orthonormal basis of $\mathbb{R}^n$, then there exists a unique $A = A^*$ such that $(A)_{ij} = L(\mathbf{e}_i, \mathbf{e}_j)$.
  - Keeping $\mathbf{x} = \sum_{i=1}^n \mathbf{x}_i, \mathbf{e}_i$ foxed, we have

$$\begin{aligned}
Q(\mathbf{x}) &= L(\mathbf{x}, \mathbf{x}) \\
&= L(\sum^n \mathbf{x}_i\mathbf{e}_i, \sum^n \mathbf{x}_j\mathbf{e}_j) \\
&= \sum_{i=1}^n \mathbf{x}_i L(\mathbf{e}_i, \sum^n \mathbf{x}_j\mathbf{e}_j) \\
&= \sum_{i,j=1}^n \mathbf{x}_i\mathbf{x}_j \underbrace{L(\mathbf{e}_i, \mathbf{e}_j)}_{A_{ij}}
\end{aligned}$$

- We have that

$$\begin{aligned}(A\mathbf{x},\mathbf{x}) &= (UDU^{-1}\mathbf{x},\mathbf{x}) \\ &= (DU^{-1}\mathbf{x},U^{-1}\mathbf{x}) \\ &= \sum_{i=1}^{n}\lambda_i(\underbrace{U^{-1}\mathbf{x}}_{\mathbf{y}_i})_i(\underbrace{U^{-1}\mathbf{x}}_{\mathbf{y}_i})_i\end{aligned}$$

- Can we characterize the set $\{\mathbf{x} : (A\mathbf{x},\mathbf{x}) = 1\}$?
  - Note that this set is equivalent to $\{\mathbf{y} : (D\mathbf{y},\mathbf{y}) = 1\}$ by teh above. This set is a rotation of the previous one. Ellipse?
- Positive quadratic form:
  - Q is positive definite if $Q(\mathbf{x}) > \mathbf{0}$ for all $\mathbf{x} \neq \mathbf{0}$ and Q is positive semidefinite if $Q(\mathbf{x}) \geq \mathbf{0}$ for all $\mathbf{x} \neq \mathbf{0}$.
  - Take a self-adjoint matrix $A = A^*$. It is positive definite if $Q(\mathbf{x}) = (A\mathbf{x},\mathbf{x})$ is positive definite.
- Theorem: If $A = A^*$, then
  1. A is positive definite if and only if all eigenvalues of A are positive.
  2. A is positive semidefinite if and only if all eigenvalues of A are nonnegative.
  3. A is negative semidefinite if and only if all eigenvalues of A are nonpositive.
  4. A is negative definite if and only if all eigenvalues of A are negative.
  5. A is indefinite if and only if the eigenvalues of A have positive and negative values.
- Theorem: $A = A^*$ is positive definite iff $\det A_k > 0$ for all $k = 1, \ldots, n$ where A_k is the upper left $k \times k$ submatrix.
- Minimax representation of eigenvalues of a self-adjoint A.
  - Let E be a subspace of X where $\dim X < \infty$. We define $\operatorname{codim}(E) = \dim E^{\perp}$.
  - Thus, $\dim E + \operatorname{codim} E = \dim X$.
  - Theorem: Let $A = A^*$, $\lambda_1 \geq \cdots \geq \lambda_n$ eigenvalues of A. Then

$$\lambda_k = \max_{\substack{\text{E subspace} \\ \dim E = k}} \min_{\substack{\mathbf{x}\in E \\ \|\mathbf{x}\|=1}} (A\mathbf{x},\mathbf{x}) = \min_{\substack{\text{F subspace} \\ \operatorname{codim} F = k-1}} \max_{\substack{\mathbf{x}\in F \\ \|\mathbf{x}\|=1}} (A\mathbf{x},\mathbf{x})$$

    - Proof: A diagonal equals $(\lambda_1, \ldots, \lambda_n)$.
    - An orthonormal basis of X such that $\dim E = k$, $\operatorname{codim} F = k-1$, $\dim F = n-k+1$.
    - There exists an $\mathbf{x}_0 \neq \mathbf{0}$ such that $\mathbf{x}_0 \in E \cap F$.
    - Note that if $B = B^*$, then the max and min of $(B\mathbf{x},\mathbf{x})$ over the unit sphere is the maximal and minimal eigenvalue of B.
    - Thus,

$$\min_{\substack{\mathbf{x}\in E \\ \|\mathbf{x}\|=1}} (A\mathbf{x},\mathbf{x}) \leq (A\mathbf{x}_0,\mathbf{x}_0) \leq \max_{\substack{\mathbf{x}\in F \\ \|\mathbf{x}\|=1}} (A\mathbf{x},\mathbf{x})$$

    - This is true for any E, F subspaces. $\dim E = k$, $\operatorname{codim} F = k-1$, $E_0 = \operatorname{span}(\mathbf{e}_1, \ldots, \mathbf{e}_k)$ and $F_0 = \operatorname{span}(\mathbf{e}_k, \ldots, \mathbf{e}_n)$.
    - Thus,

$$\min_{\substack{E_0 \\ \|\mathbf{x}\|=1}} (A\mathbf{x},\mathbf{x}) = \lambda_k = \max_{\substack{F_0 \\ \|\mathbf{x}\|=1}} (A\mathbf{x},\mathbf{x})$$

- Additionally,

$$\lambda_{k_1} \leq \max_{\dim E=k}^{E} \min_{\mathbf{x}}(A\mathbf{x}, \mathbf{x}) \leq \min_{\substack{F \\ \operatorname{codim} F=k-1}} \max_{\mathbf{x}}(A\mathbf{x}, \mathbf{x}) \leq \lambda_k$$

- Corollary: Let $A = A^* = (a_{jk})_{1 \leq j,k \leq n}$ with eigenvalues $\lambda_1, \dots, \lambda_n$ listed in decreasing order. Let $\tilde{A} = (a_{j,k})_{1 \leq j,k \leq n-1}$ with eigenvalues $\mu_1, \dots, \mu_{n-1}$ listed in decreasing order. Then $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n$.
  - Consider $(A\mathbf{x}, \mathbf{x})$ on $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, but then restrict yourself to $\mathbf{x} \in \mathbb{R}^{n-1}$ on $\{\mathbf{e}_1, \dots, \mathbf{e}_{n-1}\}$.

7.2 Chapter 7: Bilinear and Quadratic Forms

From Treil (2017).

10/25:

- **Bilinear form** (on $\mathbb{R}^n$): A function $L(\mathbf{x}, \mathbf{y})$ of two arguments $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ that is linear in each argument.
  - Linearity in each argument:

$$L(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2, \mathbf{y}) = \alpha L(\mathbf{x}_1, \mathbf{y}) + \beta L(\mathbf{x}_2, \mathbf{y}) \qquad L(\mathbf{x}, \alpha\mathbf{y}_1 + \beta\mathbf{y}_2) = \alpha L(\mathbf{x}, \mathbf{y}_1) + \beta L(\mathbf{x}, \mathbf{y}_2)$$

- If $\mathbf{x} = (x_1, \dots, x_n)^T$ and $\mathbf{y} = (y_1, \dots, y_n)^T$, then

$$\begin{aligned} L(\mathbf{x}, \mathbf{y}) &= \sum_{j,k=1}^{n} a_{j,k} x_k y_j \\ &= (A\mathbf{x}, \mathbf{y}) \\ &= \mathbf{y}^T A\mathbf{x} \end{aligned}$$

  where

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix}$$

  - A is uniquely determined by L.
- **Quadratic form** (on $\mathbb{R}^n$): The diagonal of a bilinear form L, i.e., a bilinear form $Q[\mathbf{x}] = L(\mathbf{x}, \mathbf{x}) = (A\mathbf{x}, \mathbf{x})$.
  - Alternatively: A homogeneous polynomial of degree 2, i.e., a polynomial in $x_1, \dots, x_n$ with only ax_k^2 and cx_jx_k terms.
- There are infinitely many ways to write a quadratic form as $(A\mathbf{x}, \mathbf{x})$.
  - However, there is a unique representation $(A\mathbf{x}, \mathbf{x})$ where A is a (real) symmetric matrix.
- **Quadratic form** (on $\mathbb{C}^n$): A function of the form $Q[\mathbf{x}] = (A\mathbf{x}, \mathbf{x})$ where A is self-adjoint.
- Lemma 7.1.1: Let $(A\mathbf{x}, \mathbf{x})$ be real for all $\mathbf{x} \in \mathbb{C}^n$. Then $A = A^*$.
- To classify quadratic forms, consider the set of points $\mathbf{x} \in \mathbb{R}^n$ defined by $Q[\mathbf{x}] = 1$ for some quadratic form Q.
  - If the matrix of Q is diagonal, i.e., $Q[\mathbf{x}] = a_1x_1^2 + \cdots + a_nx_n^2$, then the set of points can easily be visualized.
- The standard method of diagonalizing a quadratic form is change of variables.
- Orthogonal diagonalization.

- Let $Q[\mathbf{x}] = (A\mathbf{x}, \mathbf{x})$ in $\mathbb{F}^n$.
- Suppose $\mathbf{y} = S^{-1}\mathbf{x}$ where S is an invertible $n \times n$ matrix. Then

$$Q[\mathbf{x}] = Q[S\mathbf{y}] = (AS\mathbf{y}, S\mathbf{y}) = (S^*AS\mathbf{y}, \mathbf{y})$$

  so in the new variables $\mathbf{y}$, the quadratic form has matrix S^*AS.
- Thus, we can let $A = UDU^*$, choose $D = U^*AU$ as our new (diagonal) matrix, and let this matrix act on the variables $\mathbf{y} = U^*\mathbf{x}$.

- Non-orthogonal diagonalization:
  - Completing the square:
    - Eliminate all x_ix_j terms by completing the square. Then substitute in a y_k for each squared term.
  - Row/column operations:
    - Augment $(A|I)$. Row reduce A to D. Then $I \to S^*$.

10/28:

- **Silvester's Law of Inertia**: For a Hermitian matrix A (i.e., for a quadratic form $Q[\mathbf{x}] = (A\mathbf{x}, \mathbf{x})$) and any of its diagonalizations $D = S^*AS$, the number of positive, negative, and zero diagonal entries of D depends only on A, but not on a particular choice of diagonalization.

- **Positive** (subspace $E \subset \mathbb{F}^n$ corresponding to A): A subspace E such that $(A\mathbf{x}, \mathbf{x}) > 0$ for all nonzero $\mathbf{x} \in E$. *Also known as* ***A*-positive**.

- **Negative** (subspace $E \subset \mathbb{F}^n$ corresponding to A): A subspace E such that $(A\mathbf{x}, \mathbf{x}) < 0$ for all nonzero $\mathbf{x} \in E$. *Also known as* ***A*-negative**.

- **Neutral** (subspace $E \subset \mathbb{F}^n$ corresponding to A): A subspace E such that $(A\mathbf{x}, \mathbf{x}) = 0$ for all nonzero $\mathbf{x} \in E$. *Also known as* ***A*-neutral**.

- Theorem 7.3.1: Let A be an $n \times n$ Hermitian matrix, and let $D = S^*AS$ be its diagonalization by an invertible matrix S. Then the number of positive (resp. negative) diagonal entries of D coincides with the maximal dimension of an A-positive (resp. A-negative) subspace.

- Lemma 7.3.2: Let $D = \text{diag}\{\lambda_1, \dots, \lambda_n\}$. Then the number of positive (resp. negative) diagonal entries of D coincides with the maximal dimension of a D-positive (resp. D-negative) subspace.

- **Positive definite** (quadratic form Q): A quadratic form Q such that $Q[\mathbf{x}] > 0$ for all $\mathbf{x} \neq \mathbf{0}$.

- **Positive semidefinite** (quadratic form Q): A quadratic form Q such that $Q[\mathbf{x}] \geq 0$ for all $\mathbf{x}$.

- **Negative definite** (quadratic form Q): A quadratic form Q such that $Q[\mathbf{x}] < 0$ for all $\mathbf{x} \neq \mathbf{0}$.

- **Negative semidefinite** (quadratic form Q): A quadratic form Q such that $Q[\mathbf{x}] \leq 0$ for all $\mathbf{x}$.

- **Indefinite** (quadratic form Q): A quadratic form Q for which there exist $\mathbf{x}_1, \mathbf{x}_2$ such that $Q[\mathbf{x}_1] > 0$ and $Q[\mathbf{x}_2] < 0$.

- **Positive definite** (Hermitian matrix A): A matrix A for which the corresponding quadratic form $Q[\mathbf{x}] = (A\mathbf{x}, \mathbf{x})$ is positive definite.
  - Positive semidefinite, negative definite, negative semidefinite, and indefinite Hermitian matrices are defined similarly.

- Theorem 7.4.1: Let $A = A^*$. Then
  1. A is positive definite iff all eigenvalues of A are positive.
  2. A is positive semidefinite iff all eigenvalues of A are non-negative.

3. A is negative definite iff all eigenvalues of A are negative.
4. A is negative semidefinite iff all eigenvalues of A are non-positive.
5. A is indefinite iff it has both positive and negative eigenvalues.

- **Upper left submatrix** (of A): A $k \times k$ matrix A_k composed of all entries of A from row (column) 1 through k in the same arrangement.
- Theorem 7.4.2 (Silvester's Criterion of Positivity): A matrix $A = A^*$ is positive definite if and only if $\det A_k > 0$ for all $k = 1, \ldots, n$.
  - To check if a matrix A is negative definite, check that the matrix $-A$ is positive definite.
- Theorem 7.4.3 (Minimax characterization of eigenvalues): Let $A = A^*$ be an $n \times n$ matrix and let $\lambda_1 \geq \cdots \geq \lambda_n$ be its eigenvalues taken in decreasing order. Then

$$\lambda_k = \max_{E:\dim E = k} \min_{\mathbf{x} \in E: \|\mathbf{x}\| = 1} (A\mathbf{x}, \mathbf{x}) = \min_{F:\operatorname{codim} F = k-1} \max_{\mathbf{x} \in F: \|\mathbf{x}\| = 1} (A\mathbf{x}, \mathbf{x})$$

- Corollary 7.4.4 (Intertwining of eigenvalues): Let $A = A^* = \{a_{j,k}\}_{j,k=1}^n$ be a self-adjoint matrix and let $\tilde{A} = \{a_{j,k}\}_{j,k=1}^{n-1}$ be its submatrix of size $(n-1) \times (n-1)$. Let $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_{n-1}$ be the eigenvalues of A and $\tilde{A}$ respectively, taken in decreasing order. Then

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$$

Chapter 8

Dual Spaces and Tensors

8.1 Notes

10/22:

- **Functional**: A linear bounded map $L : H \to F$, where H is finite dimensional (equivalent to $\mathbb{R}^n$).
- **Dual space**: The set of bounded linear functionals on H. *Denoted by* $\boldsymbol{H'}$, $\boldsymbol{H^*}$.
- If $l \leq p < \infty$, then
$$l^p = \left\{ (a_n)_{n\in\mathbb{N}} : \sum_{n=1}^{\infty} |a_n|^p < \infty \right\}$$
- Back to finite dimensions, $H' \approx \mathbb{R}^n$.
- Let $\mathbf{a}_1, \dots, \mathbf{a}_n$ be a basis of H. Then $L\mathbf{x} = (L\mathbf{a}_1, \dots, L\mathbf{a}_n) \approx \mathbb{R}^n$.
- Let $L((a_n)_{n\in\mathbb{N}}) = \sum_{n=1}^{\infty} a_n b_n$. Then $L((a_n)_{n\in\mathbb{N}})$ will be bounded if and only if $(b_n)_{n\in\mathbb{N}} \in l^q$ where $1 < p < q$ where $\frac{1}{q} + \frac{1}{p} = 1$.
- **Young's inequality**: The statement
$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$
- We have $|\sum a_n b_n| \leq \|a_n\|_p \|b_n\|_p$.
- Conclusion:
$$\sum \frac{|a_n||b_n|}{\|a_n\|_p \|b_n\|_q} = 1$$
- We can define H'', too. This contains linear functionals on H'.
- We know that $L(x) = \langle x, L \rangle = x(L)$. $x \in H''$.
- Riesz representation theorem: Let H have an inner product. $L \in H'$ if and only if there exists a unique $y \in H$ such that $L(x) = (x, y)$.
  - Gives us a way to identify all bounded linear functionals on H.
  - In finite dimensions, $L(x)$, where $x = \sum_1^n \alpha_i a_i$ gives us $L(x) = \sum_1^n \alpha_i L(a_i)$.

8.2 Chapter 8: Dual Spaces and Tensors

10/28:

- Linear functionals are denoted by L.
  - L is given by a $1 \times n$ matrix denoted by $[L]$.
- The collection of all $[L]$ (the dual space) is isomorphic to $\mathbb{R}^n$ via $[L] \mapsto [L]^T$.
  - However, the objects are different: Let $[I]_{\mathcal{BA}}$ be the change of coordinates matrix in $\mathbb{R}^n$. We thus have that
$$[\mathbf{v}]_{\mathcal{B}} = [I]_{\mathcal{BA}}[\mathbf{v}]_{\mathcal{A}}$$
    but we also have that
$$[L]_{\mathcal{B}} = [L]_{\mathcal{A}}[I]_{\mathcal{AB}}$$
    so that
$$[L]_{\mathcal{B}}^T = ([L]_{\mathcal{A}}[I]_{\mathcal{AB}})^T = [I]_{\mathcal{AB}}^T[L]_{\mathcal{A}}^T$$
  - Essentially, "if S is the change of coordinate matrix in X... then the change of coordinate matrix in the dual space X' is $(S^{-1})^T$" (Treil, 2017, p. 219).
- Lemma 8.1.3: Let $\mathbf{v} \in V$. If $L(\mathbf{v}) = 0$ for all $L \in V'$, then $\mathbf{v} = \mathbf{0}$. As a corollary, if $L(\mathbf{v}_1) = L(\mathbf{v}_2)$ for all $L \in V'$, then $\mathbf{v}_1 = \mathbf{v}_2$.
- The second dual V'' is canonically (i.e., in a natural way) isomorphic to V.
- **Dual basis** (to $\mathbf{b}_1, \dots, \mathbf{b}_n \in V$): The system of vectors $\mathbf{b}_1', \dots, \mathbf{b}_n' \in V'$ uniquely defined by the following equation. *Also known as* **biorthogonal basis**.
$$\mathbf{b}_k'(\mathbf{b}_j) = \delta_{kj}$$
  - The k^{th} coordinate of a vector $\mathbf{v}$ in a basis $\mathbf{b}_1, \dots, \mathbf{b}_n$ is $\mathbf{b}_k'(\mathbf{v})$.
    - This is a baby version of the **abstract non-orthogonal Fourier decomposition** of $\mathbf{v}$.
- Theorem 8.2.1 (Riesz representation theorem): Let H be an inner product space. Given a linear functional L on H, there exists a unique vector $\mathbf{y} \in H$ such that
$$L(\mathbf{v}) = (\mathbf{v}, \mathbf{y})$$
  for all $\mathbf{v} \in H$.
- If V is a real inner product space, we can define an isomorphism from V to V' by $\mathbf{y} \mapsto L_{\mathbf{y}} = (\mathbf{v}, \mathbf{y})$.
  - If V is complex, this function is not linear since if α is complex,
$$L_{\alpha\mathbf{y}}(\mathbf{v}) = (\mathbf{v}, \alpha\mathbf{y}) = \bar{\alpha}(\mathbf{v}, \mathbf{y}) = \bar{\alpha}L_{\mathbf{y}}(\mathbf{v})$$
  - It follows by such a mapping that $\mathbf{b}_k' = \mathbf{b}_k$ for each k.
- **Conjugate linear** (transformation): A transformation T such that
$$T(\alpha\mathbf{x} + \beta\mathbf{y}) = \bar{\alpha}T\mathbf{x} + \bar{\beta}T\mathbf{y}$$
- It is customary to write outputs of linear functionals $L(\mathbf{v})$ in the form $\langle \mathbf{v}, L \rangle$.
  - This expression is linear in both arguments, unlike the inner product.
- Defines the dual transformation as the unique transformation such that
$$\langle A\mathbf{x}, \mathbf{y}' \rangle = \langle \mathbf{x}, A'\mathbf{y} \rangle$$
  for all $\mathbf{x} \in X$, $\mathbf{y}' \in Y'$.

  - It's matrix in the standard bases equals A^T.

- Annihilators are denoted by $E^\perp$ here.
- Proposition 8.3.6: The annihilator of the annihilator of E equals E.
- Let $A : X \to Y$ be an operator acting from one vector space to another. Then
  1. $\ker A' = (\operatorname{range} A)^\perp$.
  2. $\ker A = (\operatorname{range} A')^\perp$.
  3. $\operatorname{range} A = (\ker A')^\perp$.
  4. $\operatorname{range} A' = (\ker A)^\perp$.

Chapter 9

Advanced Spectral Theory

9.1 Notes

10/22:

- Let $p(z) = \sum_{i=0}^n a_i z^i$ be a polynomial. Let A be an $n \times n$ matrix. We let $p(A) = \sum_{i=0}^n a_i A^i$.
- Theorem: If A is an $n \times n$ and $p(\lambda) = \det(A - \lambda I)$, then $p(A) = 0$.
  - We know that $p(\lambda) = a(z - \lambda_1) \cdots (z - \lambda_n)$ where $\lambda_1, \dots, \lambda_n$ are the eigenvalues.
  - Thus $p(A) = a(A - \lambda_1 I) \cdots (A - \lambda_n I)$.
  - If you are in $\mathbb{R}^n$ and have this property, you can factorize your matrix.
  - Thus, $p(A)\mathbf{x} = \mathbf{0}$ since $\mathbf{x}$ can be decomposed into a linear combination of eigenvectors of A, which will be taken to 0 one by one by the terms of $p(A)$.
- $\sigma(B) = \{\text{eigenvalues of } B\}$ is known as the **spectrum** of B.
- If p is an arbitrary polynomial and A is $n \times n$, then μ is an eigenvalue of $p(A)$ if and only if $\mu = p(\lambda)$ where λ is an eigenvalue of A. In essence, $\sigma(p(A)) = p(\sigma(A))$.
- Chapter 9 will not be on the exam. We don't have to know the generalization to infinite dimensional spaces.

10/25:

- If A is an $n \times n$ square matrix and $p(\lambda) = \det(A - \lambda I)$, then $p(A) = 0$.
  - Proof: WLOG, let A be an upper triangular matrix with diagonal entries equal to the eigenvalues.
  - Think of $p(z) = (-1)^n (z - \lambda_1) \cdots (z - \lambda_n)$.
  - Thus, $p(A) = (-1)^n (A - \lambda_1 I) \cdots (A - \lambda_n I)$.
  - WTS: $p(A)\mathbf{x} = 0$ for all $\mathbf{x} \in V$.
  - Let $E_k = \operatorname{span}(e_1, \dots, e_k)$ be the span of the first k eigenvectors of A, where $e_1, \dots, e_n$ is a standard basis in $\mathbb{C}^n$.
  - A triangular implies $AE_k \subset E_k$. Thus, $(A - \lambda I)E_k \subset E_k$, so E_k is invariant under $A - \lambda I$ for all λ.
  - If we apply $A - \lambda_k I$ to a vector in E_k, we are left with a vector in E_{k-1}.
  - Thus, if we apply $\prod_{k=1}^n (A - \lambda_k I) = p(A)$ to any vector in $E_n = V$, we will kill it piece by piece down to zero.
- Let A be a square $n \times n$ matrix. Then p an arbitrary polynomial implies $\sigma(p(A)) = p(\sigma(A))$. (Any eigenvalue μ of $p(A)$ is $\mu = p(\lambda)$, where λ is an eigenvalue of A.)
  - Shows that polynomials of operators commute.

- Proof: Let λ be an eigenvalue of A. We want to show that $p(\lambda)$ is an eigenvalue of $p(A)$. This is obvious since $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x}$, so $A^k\mathbf{x} = \lambda^k\mathbf{x}$, so in particular, $p(A)\mathbf{x} = p(\lambda)\mathbf{x}$.
- On the other hand, if μ is an eigenvalue of $p(A)$, we want to show that there exists $\lambda \in \sigma(A)$ such that $\mu = p(\lambda)$.
- Consider $q(z) = p(z) - \mu$. Then $q(A) = p(A) - \mu I$. Since μ is an eigenvalue of $p(A)$, $q(A)$ is not invertible.
- Thus, $q(z) = (-1)^n(z - z_1)\cdots(z - z_n)$ and $q(A) = (-1)^k(A - z_1 I)\cdots(A - z_k I)$.
- But $q(A)$ is not invertible, so one of the $A - z_k I$ is not invertible. Take z_k such that $A - z_k I$ is not invertible. Then $z_k \in \sigma(A)$. It follows that $q(z_k) = p(z_k) - \mu = \sigma$.

- If A is $n \times n$, $\lambda_1, \dots, \lambda_n$ are its eigenvalues, p is a polynomial, then $p(A)$ is invertible if and only if $p(\lambda_k) \neq 0$ for each $k = 1, \dots, n$.
  - This is an immediate corollary to the previous result.
- We now build up to the **generalized eigenspace**, which is related to some "geometric" properties of the algebraic multiplicity of an eigenvalue.
- If $A : V \to V$ is a linear operator and $E \subset V$ is a subspace, E is A-invariant if $AE \subset E$.
- Facts:
  - If E is A-invariant, E is A^k-invariant.
  - Thus, E is $p(A)$-invariant.
- Consider the restriction map $A|_E$.
- A has a block-diagonalized matrix where each block corresponds to the generalized eigenvectors of a generalized eigenvalue of A.
  - Let $E_1, \dots, E_r$ be a **basis of invariant subspaces**.
  - Let $A_k = A|_{E_k}$. Then the A_k's act independently of each other.
- **Generalized eigenvector** (of A): A vector $\mathbf{v}$ corresponding to an eigenvalue λ if there exists $k \geq 1$ such that $(A - \lambda I)^k\mathbf{v} = \mathbf{0}$.
- **Generalized eigenspace**: The set E_λ of all of the generalized eigenvectors of λ. *Given by*
$$E_k = \bigcup_{k \geq 1} \ker(A - \lambda I)^k$$
  - E_λ is a linear subspace of V.
- **Degree** (of λ): The smallest number k such that increasing k any more does not add further vectors to the generalized eigenspace. *Denoted by* $\boldsymbol{d(\lambda)}$. *Also known as* **depth**.
  - Symbolically, $d(\lambda)$ is the smallest number such that
$$E_\lambda = \bigcup_{k=1}^{d(\lambda)} \ker(A - \lambda I)^k$$
- Start working through the first 25 problems of Rudin (1976) (his metric spaces problems).

10/27:
- Jordan form.
- Reviews build up to generalized eigenvectors.

- Theorem: If $\sigma(A) = \{\lambda_1, \dots, \lambda_n\}$ and $E_1, \dots, E_n$ are the corresponding generalized eigenspaces, then $E_1, \dots, E_n$ is a basis of subspaces of U, i.e., $V = \oplus_k E_k$.
- Corollary: $A : V \to V$ can be represented as $A = D + N$ where D is diagonalizable and N is nilpotent and $ND = DN$.
  - Proof: Consider the basis of generalized eigenspaces known to exist from the theorem. Then $A = \operatorname{diag}\{A_1, \dots, A_r\}$.
  - Let
    $$N_k = A_k - \lambda_k I_{E_k}$$
    This is nilpotent.
  - Then let
    $$D = \operatorname{diag}\{\lambda_1 I_{E_1}, \dots, \lambda_n I_{E_n}\}$$
  - These two matrices satisfy the necessary properties.
- Let $\dot{\mathbf{x}} = A\mathbf{x}$.
  - Let $\mathbf{x}(t) = \mathrm{e}^{tA}$, where
    $$\mathrm{e}^{tA} = \sum \frac{(tA)^k}{k!}$$
  - $\left\|\mathrm{e}^{tA}\right\| \leq \sum \frac{\|A^k\|}{k!} = \sum \frac{\|A\|^k}{k!}$.
  - Let p be a polynomial of degree k. Then
    $$p(a+x) = \sum_{k=0}^{d} \frac{p^{(k)}(a)}{k!} x^k$$
  - If $A = D + N$, then...
- Nilpotent operators:
  - Let $A = \operatorname{diag}\{A_1, \dots, A_r\}$.
  - We know that $A_k = \lambda_k I_{E_k} + N_k$ for each k.
  - Every nilpotent N can be written in the form
    $$\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$$

10/28:

- The exam is long but not that hard. The only question is will you do good or very good.
- Revise the previous two homeworks, especially the last two.
- No justification for any of the true/false questions. Just circle T or F.
  - There are four problems. One is T/F (with multiple subparts); the other three are problem problems (with subparts, too).
  - Some of the questions will take you 2 seconds. Some you've already seen in the PSets.
  - The exam is supposed to be boring.
- Calculators?
  - No calculators needed. Calculators are for chem/physics exams.

  - Not a lot of computation.
- 50 minutes.
- Chloe will be proctoring.
- Remember the determinant of "special" matrices.
  - $|\det U| = 1$ if U is unitary.
  - $\det A = \pm 1$ if A is orthogonal.
  - Make a list of matrix types that are automatically diagonalizable.
  - Determinant is the product of the eigenvalues.
  - Determinant of A is equal to the conjugate of the determinant of A^*.
- Most of the exercises use the inner product.
  - Whenever you had something to prove about eigenvalues or eigenbasis, you went through diagonalization or SVD or the inner product or polar decomposition.
  - Proving eigenvalues of self-adjoint matrices are real w/ the inner product.
- Eigenvalues/eigenvectors of a projection.
  - It's implied that it's asking you the multiplicities!!!
- Know useful facts but have an idea how to prove them as well.
- Recommends against shorthanding in the exams.
- Not grading on clarity (since the exam is long).
- Max and min are for when you're sure something will be attained. Otherwise use sup and inf.

9.2 Chapter 9: Advanced Spectral Theory

10/28:

- Theorem 9.1.1 (Cayley-Hamilton): If p is the characteristic polynomial of A, $p(A) = 0$.
- Theorem 9.2.1 (Spectral Mapping Theorem): For a square matrix A and an arbitrary polynomial p, $\sigma(p(A)) = p(\sigma(A))$. In other words, μ is an eigenvalue of $p(A)$ if and only if $\mu = p(\lambda)$ for some eigenvalue λ of A.
- Corollary 9.2.2: Let A be a square matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ and let p be a polynomial. Then $p(A)$ is invertible iff $p(\lambda_k) \neq 0$ for all $k = 1, \dots, n$.
- Algebraic multiplicity is the dimension of the corresponding generalized eigenspace.

Part II

Point Set Topology of Metric Spaces

Chapter 1

The Real and Complex Number Systems

1.1 Notes

11/1:
- Spent a lot of time trying to cheer us up regarding the midterm.
- There may be some true/false on linear algebra on the final.
- Facts:
  1. $\sqrt{2}$ is irrational.
  2. Archimedes principle: If $x > 0$ and $y \in \mathbb{R}$, then there exists n such that $nx > y$.
  3. If $x > y$, then there exists $q \in \mathbb{Q}$ such that $x > q > y$.

1.2 Chapter 1: The Real and Complex Number Systems

From Rudin (1976).

11/6:
- Rudin (1976) presents several interesting proofs throughout this section that may be of interest later by means of their divergence from the ones with which I am familiar.
- **Least-upper-bound property**: The property pertaining to a set S that if $E \subset S$, $E \neq \emptyset$, and E is bounded above, then $\sup E \in S$.
  - For example, $\mathbb{Q}$ does not have the least-upper-bound property.
  - The **greatest-lower-bound property** is analogously defined.
- Theorem: Suppose S is an ordered set with the least-upper-bound property, $B \subset S$ is nonempty, and B is bounded below. Let L be the set of all lower bounds of B. Then $\alpha = \sup L$ exists in S, and $\alpha = \inf B$. In particular, $\inf B$ exists in S.
  - Essentially, this theorem states that any set that satisfies the least-upper-bound property satisfies the greatest-lower-bound property.
- **Existence theorem**: There exists an ordered field $\mathbb{R}$ which has the least-upper-bound property. Moreover, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.
  - The second statement implies that the operations of addition and multiplication on $\mathbb{R}$, when applied to $\mathbb{Q}$, coincide with the operations of addition and multiplication on $\mathbb{Q}$.

- **Archimedean property** (of $\mathbb{R}$): If $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $x > 0$, then there is a positive integer n such that $nx > y$.
- Rudin (1976) proves several theorems about the real numbers from the least-upper-bound property as opposed to the traditional construction of the real numbers.
- Introduces the decimal system.
- **Finite real number system**: That which has been defined thus far.
- **Extended real number system**: The set $\mathbb{R} \cup \{+\infty, -\infty\}$ where $+\infty, -\infty$ obey the expected properties (supremum [resp. infimum] of every set, $x + \infty = \infty$, etc.).
- Defines the complex field axiomatically with complex numbers in the form (a, b) for $a, b \in \mathbb{R}$.
  - Notes that the real numbers form a subfield of the complex field.
  - Defines $i = (0, 1)$, proves $i^2 = -1$, proves $a + bi = (a, b)$.
- **Schwarz inequality**: If $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are complex numbers, then

$$\left| \sum_{j=1}^{n} a_j \bar{b}_j \right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$$

- **Euclidean *k*-space**: The vector space $\mathbb{R}^k$ over the real field.

Chapter 2

Basic Topology

2.1 Notes

11/1:

- Equivalence relationships are denoted $A \sim B$.
  - These are...
    - Reflexive ($A \sim A$).
    - Symmetric ($A \sim B \iff B \sim A$).
    - Transitive ($A \sim B$ & $B \sim C \implies A \sim C$).
  - Equivalence relations give rise to equivalence classes.
- **Countable** (set A): A set A such that $A \sim \mathbb{N}$, in the sense that there exists a one-to-one and onto map from $\mathbb{N} \to A$.
  - Alternatively, A can be written in the form $A = \{f(n) : n \in \mathbb{N}\}$.
- **Finite countable** vs. **infinite countable** (see Rudin (1976)).
- $\mathbb{N}$ denotes the natural numbers.
- $\mathbb{N}_0$ denotes the natural numbers including 0.
- $\mathbb{Z}$ denotes the integers.
- We know that $\mathbb{N} \sim \mathbb{Z}$: Let $f : \mathbb{N} \to \mathbb{Z}$ be defined by

$$f(n) = \begin{cases} \frac{n}{2} & n \text{ even} \\ \frac{n-1}{2} & n \text{ odd} \end{cases}$$

- More facts.
  1. Every infinite subset of a countable set is countable.
  2. Unions of countable sets are countable.
     - If the sets E_n for some at most countable list of numbers are countable, then $\bigcup_n E_n$ is countable.
     - Soug goes over the diagonalization method of counting.
  3. n-fold Cartesian products of countable sets are countable (we induct on n).
     - If A is countable and B is countable, then $A \times B$ is countable.
     - If A is finite and to each $\alpha \in A$ we assign a countable set E_α, $\otimes_{\alpha \in A} E_\alpha$ is countable.
- **Metric space**: A space X along with a metric $d : X \times X \to [0, \infty)$ such that

  - $d(x,y) > 0$ iff $x \neq y$, and $d(x,x) = 0$ iff $x = 0$.
  - $d(x,y) = d(y,x)$.
  - $d(x,y) \leq d(x,z) + d(z,y)$.
- Example ($\mathbb{R}^n$):
  - We may define d by
$$d(x,y) = \sqrt[2]{\sum (x_i - y_i)^2}$$
  - We can also define the p-metrics (recall normed spaces) with p where the 2's are.
- Example ($X_p = \{f : Y \to \mathbb{R} : 1 \leq p < \infty, \int_Y |f|^p \,\mathrm{d}y < \infty\}$):
  - This is ℓ_p.
  - Define
$$\|f-g\|_p = \left[\int_Y |f-g|^p \,\mathrm{d}y\right]^{1/p}$$
- Convergence: $x_n \to x \iff d(x_n, x) \to 0$.
- **Neighborhood**: The set of all points a distance less than r away from p. *Denoted by* $\boldsymbol{N_r(p)}$. *Given by*
$$N_r(p) = \{q \in X : d(p,q) < r\}$$
- **Limit point** (of E): A point p such that every neighborhood of p intersects E at a point other than p. *Also known as* **accumulation point**.
  - Symbolically,
$$N_r(p) \cap (E \setminus \{p\}) \neq \emptyset$$
    for all $r > 0$.
- **Isolated point** (of E): A point p such that $p \in E$ and p is not a limit point of E.
- **Closed** (set E): A set E that contains all of its limit points.
- **Interior** (point p): A point p such that there exists $N_r(p) \subset E$.
- **Open** (set E): A set E, all points of which are interior points.
- **Perfect** (set E): A set E that is closed and every point of E is a limit point of E.
- **Bounded** (set E): There exists a number M and a $y \in X$ such that $E \subset \{p : d(p,y) \leq M\}$.
- **Dense** (set E in X): A set E such that every point of X is a limit point of E or a point of E, itself.

11/3:

- Every neighborhood is an open set.
- If p is a limit point of E, every neighborhood of p contains infinitely many points of E.
  - Thus, a finite set cannot have a limit point.
  - Prove by contradiction: Suppose there is a neighborhood that contains only finitely many points of E. Then the neighborhood with radius smaller than the distance to the closest point does not contain any points of E, a contradiction.
- E is open iff E^c[1] is closed.
  - Assume E^c closed. If $p \in E$, then p is not a limit point of E^c. It follows that there exists a neighborhood of p that is entirely contained within E, so p is interior, as desired.

[1] The complement of E.

  - Suppose E is open. Let p be any limit point of E^c. Then $p \in E^c$.

- F is closed iff F^c is open.

- If $(G_\alpha)_{\alpha \in A}$ is a family of open sets in X, then the union is open.
  - Let $p \in \bigcup_{\alpha \in A} G_\alpha$. Then $p \in G_\alpha$ for some $\alpha \in A$. It follows that p is an interior point of G_α, so thus an interior point of the union of G_α with everything else.

- Finite intersections of open sets are open.
  - In the infinite case $\bigcap_{n \in \mathbb{N}}(-1/n, 1/n) = \{0\}$, an intersection of infinitely many open sets is closed.
  - However, in the finite case, just consider the neighborhood with the smallest radius and take this one.

- The intersection of closed sets is closed.

- The union of finitely many closed sets is closed.
  - These follow from the previous two by De Morgan's rule.

- Let $\bar{E} = E \cup E'$ where E' is the set of limit points of E.

- Let X be a metric space and $E \subset X$. Then
  1. $\bar{E}$ is closed.
     - WTS: $\bar{E}^c$ is open. Let $p \in \bar{E}^c$. Then p is neither in E nor is it a limit point of E. Thus, there exists a neighborhood of $\bar{E}^c$ containing entirely points of $\bar{E}^c$. Therefore, $\bar{E}^c$ is open, so $\bar{E}$ is closed.
  2. $E = \bar{E}$ iff E is closed.
     - $\bar{E}$ is closed (by the above), so $E = \bar{E}$ is closed.
     - E is closed implies $E' \subset E$, so $E = E \cup E' = \bar{E}$.
  3. $\bar{E} \subset F$ for any closed $F \supset E$.
     - If $E \subset F$, then any limit point of E will be a limit point of F. Thus, $E' \subset F'$. Then $\bar{E} = E \cup E' \subset F \cup F' = \bar{F} = F$ where the last equality holds because F is closed.

- Types of sets.

	Closed	Open	Perfect	Bounded
$\{z \in \mathbb{Q} : \lvert z\rvert < 1\}$	N	Y	N	Y
$\{z \in \mathbb{Q} : \lvert z\rvert \leq 1\}$	Y	N	Y	Y
Nonempty finite set	Y	N	N	Y
$\mathbb{Z}$	Y	N	N	N
$\{1/n : n \in \mathbb{N}\}$	N	N	N	Y
$\mathbb{R}^2$	Y	Y	Y	N
(a, b)	N	?	N	Y

Table 2.1: Types of sets.

- **Relatively open** (set E to Y): A set $E \subset Y \subset X$ such that if $p \in E$, then there exists a Y-neighborhood of E contained in E.

- Let $N_r^X(p) = \{y \in X : d(y,p) < r\}$ be a neighborhood of p in X, and let $N_r^Y(p) = \{y \in Y : d(y,p) < r\}$ be a neighborhood of p in Y. Then $N_r^Y(p) = N_r^X(p) \cap Y$.

- E is open relative to Y iff $E = G \cap Y$ where G is open relative to X.
- Introduces the supremum.
- If $E \subset \mathbb{R}$, $E \neq \emptyset$, and E is bounded above, $\sup E < \infty$.
- Let $y = \sup E$. Then $y \in \bar{E}$.
- There exists a sequence $a_n \in A$ such that $a_n \to x = \sup A$.
- A is compact iff any open cover of the set has a finite subcover.
- Study and *know* all of these proofs.

11/5:

- Compactness: Defines compactness in terms of open covers.
- Finite sets are compact.
- Compactness is "absolute" (i.e., it is not a relative property like openness).
  - If $K \subset Y \subset X$, then K is compact relative to X iff K is compact relative to Y.
- V is open relative to Y iff $V = G \cap Y$ where G is open relative to X.
- Compact implies closed.
  - We will show K compact implies K^c open.
  - WTS: For all $p \in K^c$, there exists $N_r(p) \subset K^c$ such that $N_r(p) \cap K = \emptyset$.
  - Let $p \in K^c$.
  - Define an open cover of K by $G = \{N_{d(p,q)/2}(q) : q \in K\}$.
  - Since K is compact, there exists a finite subcover $\{N_{r_i}(q_i)\} \subset G$ of K.
  - Let $r = \min r_i$.
  - Then $N_r(p)$ does not intersect any $N_{r_i}(q_i)$, i.e., $N_r(p)$ does not contain any point of K, as desired.
- A closed subset of a compact set is compact.
  - Let K be compact and let $F \subset K$ be closed.
  - Take any open cover of F. Extend it to an open cover of K. Take the finite subcover of K. Naturally, this finite subcover is also a finite cover of $F \subset K$.
- F closed, K compact implies $F \cap K$ compact.
  - $F \cap K$ is closed (F, K are closed).
  - $F \cap K$ closed $\subset K$ compact implies $F \cap K$ closed.
- If $(K_\alpha)_{\alpha \in A}$ is compact in X with finite intersection property (every intersection of any finite number of these sets is nonempty), then $\bigcap_{\alpha \in A} K_\alpha \neq \emptyset$.
  - Argue by contradiction.
  - Let $G_\alpha = K_\alpha^c$.
  - Assume the intersection is empty. Assume WLOG that no point of K_1 is in any of the other K_α's.
  - Then $\{G_\alpha\}_{\alpha \in A}$ be an open cover of K_1.
  - K_1 compact implies there is a finite subcover $G_{\alpha_1}, \dots, G_{\alpha_n}$.
  - Then
    $$K_1 \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n} = K_{\alpha_1}^c \cup \cdots \cup K_{\alpha_n}^c = (K_{\alpha_1} \cap \cdots \cap K_{\alpha_n})^c$$
    where the last equality holds by De Morgan's law.

- This implies that $K_1 \cap (K_{\alpha_1} \cap \cdots \cap K_{\alpha_n}) = \emptyset$, contradicting the finite intersection property.

- Let E be an infinite subset of a compact K. Then E has a limit point in K.
  - Argue by contradiction.
  - Suppose for all $p \in K$, there exists $N_r(p)$ such that $N_r(p) \cap E = \{p\}$.
  - Consider the set $\{N_r(p) : p \in K\}$. This is an open cover of K. Thus, there exists a finite subcover of it. But since $E \subset K \subset N_{r_1}(p_1) \cup \cdots \cup N_{r_n}(p_n) = \{p_1\} \cup \cdots \cup \{p_n\}$, E is finite, a contradiction.

- **2-cell** (in $\mathbb{R}^2$): A set that is the Cartesian product of two closed intervals.

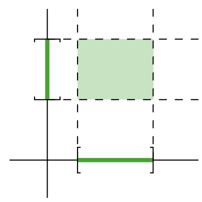


Figure 2.1: 2-cell.

  - Generalizes to ***k*-cells**.

- Let $I_n = [a_n, b_n] \subset \mathbb{R}$ such that $I_{n+1} \subset I_n$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.
  - We know that $a_n \leq a_{m+n} \leq b_{m+n} \leq b_n$ for all m, so $\sup a_n \leq b_m$ for all m (and $\sup a_n \geq a_m$ for all m by definition). Thus, $\sup a_n \in \bigcap I_n$.

- Let I_k be a k-cell in $\mathbb{R}^k$ such that $I_k \supset I_{k+1}$. Then $\bigcap_k I_k \neq \emptyset$.
  - Use the previous result once in each dimension to construct $\mathbf{x} = (x_1, \dots, x_k) \in \bigcap_k I_k \neq \emptyset$.

- Every k-cell is compact.

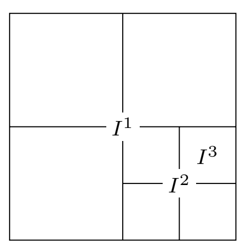


Figure 2.2: k-cells are compact.

  - Argue by contradiction.
  - Consider an open cover $\{G_\alpha\}$ of the k-cell I^1. If it has a finite subcover, we're done. So suppose it doesn't have a finite subcover. Split the k-cell into 2^k chunks. At least one of the chunks I^2 must not have a finite subcover or I^1 would have a finite subcover.
  - Split that one into 2^k chunks. At least one of the chunks I^3 must not have a finite subcover.
  - Continue.
  - Thus, we have a decreasing family of k-cells, so by the previous result, their $\bigcap I^n \neq \emptyset$.
  - Let $\mathbf{x} \in \bigcap I^n$. Naturally, $\mathbf{x} \in G_\alpha$ for some α. Since G_α is open, there exists $N_r(\mathbf{x}) \subset G_\alpha$.
  - However, since the I^n keep shrinking in size forever, we can find an $I^n \subset N_r(\mathbf{x}) \subset G_\alpha$, contradicting the supposition that I^n cannot be covered by finitely many (let alone 1) G_α's.

- Heine-Borel theorem: Let $E \subset \mathbb{R}^k$. Then TFAE[2]
  1. E is closed and bounded.
  2. E is compact.
  3. Every infinite subset of E has a limit point in E.
  - ($1 \Rightarrow 2$) E closed and bounded implies E is a closed subset of some I_k, so it's compact.
  - ($2 \Rightarrow 3$) Already done.
  - ($3 \Rightarrow 1$)
    - Suppose E not bounded. Then there is an infinite sequence of points in E that never converges. Contradiction.
    - Suppose E is not closed. Then there exists a sequence of points in E which "converges" to an $x_0 \notin E$.

11/8:
- Hewitt and Stromberg (1965) has harder analysis problems than Rudin (1976).
- Theorem: If P is a nonempty perfect subset of $\mathbb{R}^k$, then P is uncountable.

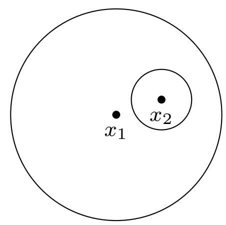


Figure 2.3: Nonempty perfect sets are uncountable.

  - P perfect implies P infinite.
  - Suppose P is countable. Let $P = \{x_1, x_2, \dots\}$.
  - Start with x_1. Take an open neighborhood V_1 of x_1. Since x_1 is a limit point of P, there will be another point $x_2 \in P$ in V_1. Choose V_2 to be a neighborhood of x_2 such that $\bar{V}_2 \subset V_1$ and $x_1 \notin \bar{V}_2$.
  - Keep going — there is a point $x_3 \in P$ in V_2, choose an appropriate neighborhood, etc.
  - Thus, we have a sequence of closed compact sets such that $\bar{V}_n \supset \bar{V}_{n+1}$ ($n \in \mathbb{N}$). It follows that $\bigcap \bar{V}_n \neq \emptyset$.
  - We also know that $V_n \cap P \neq \emptyset$ for each n.
  - Let $K_n = \bar{V}_n \cap P$. Each K_n is compact and $K_n \supset K_{n+1}$ for each n. Therefore, by compactness, $\bigcap K_n \neq \emptyset$. But the construction implies that $\bigcap K_n = \emptyset$ because we exhausted the whole sequence of possible points $x_i \in P$.
- Corollary: Any interval is uncountable.
- The Cantor set:
  - Let $E_0 = [0, 1]$.
  - Take out the middle third, so that $E_1 = [0, 1/3] \cup [2/3, 1]$.
  - Take out the middle thirds of the remaining intervals and keep going.
  - Thus, we are building a decreasing family of compact sets, so the overall intersection $E = \bigcap E_n$ of every set is nonempty.

[2]The following are equivalent.

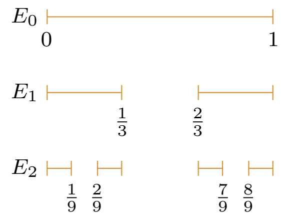


Figure 2.4: Constructing the Cantor set.

- E^n is the union of 2^n closed intervals of length $n/3$. Thus, the overall length of E^n is $(2/3)^n$.
- Thus, we have a compact nonempty set with Lebesgue measure zero.
- E does not contain any segment of the form
$$\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right)$$
for $k, m \in \mathbb{N}$.
  - Therefore, no segment of the form (α, β) is contained in E (any segment of said form contains a segment of the above form).
- Moreover, E (the Cantor set) is perfect.
  - Let $x \in E$. WTS: For all segments S containing x, $S \cap (E \setminus \{x\}) \neq \emptyset$.
  - Let S be an arbitrary such segment...

- Consider the **Devil's staircase**.
  - $0 = \int_0^1 F'(x)\,\mathrm{d}x = F(1) - F(0) = 1$. This function does not obey the fundamental theorem of calculus. A function satisfies the fundamental theorem of calculus if and only if it is absolutely continuous.

- Connected sets (motivation):
  - In a convex set, you can connect any two points with a straight line.
  - In a nonconvex connected set, there exist points that you must connect with a curve.
  - In a disconnected set, there exist points that cannot be connected via a line whose points lie wholly in the set.

- **Connected** (set E): A set E that is not the union of two **separated** sets.
- **Separated** (sets A, B): Two sets $A, B \subset X$ that are nonempty and such that $\bar{A} \cap B = \emptyset$, and $A \cap \bar{B} = \emptyset$.
- Theorem: $E \subset \mathbb{R}$ is connected iff $x, y \in E$ and $x < z < y$ implies $z \in E$.
  - If there is a $z \notin E$ between x, y, then $\{x \in E : x < z\}$ and $\{x \in E : z < y\}$ are separated sets, so E is not connected.

2.2 Chapter 2: Basic Topology

From Rudin (1976).

11/6:
- **Countable** (set A): A set A that is in bijective correspondence with the set of all positive integers. *Also known as* **enumerable**, **denumerable**.
- **At most countable** (set A): A set A that is finite or countable.

- An alternative definition of an **infinite** set would be a set that is equivalent to one of its proper subsets.
- Theorem 2.8: Infinite subsets of countable sets are countable.
- Theorem 2.12: $\{E_n\}$ a countable family of countable sets implies $\bigcup E_n$ is countable.
- Corollary: A at most countable, B_α at most countable for all $\alpha \in A$ implies $\bigcup_\alpha B_\alpha$ is at most countable.
- Theorem 2.13: Finite Cartesian products of countable sets are countable.
- Corollary: $\mathbb{Q}$ is countable.
- Theorem 2.14: Let A be the set of all sequences whose elements are the digits 0 and 1. This set A is uncountable.

  Proof. Let $E = \{s_1, s_2, \dots\}$ be an arbitrary countable subset of A, where each s_j is a sequence whose elements are the digits 0 and 1. Let s be the sequence, the n^{th} term of which is the opposite of the n^{th} term of s_n (i.e., if the n^{th} term of s_n is 0, we set the n^{th} term of s equal to 1). This guarantees that s is distinct from each of the s_j, i.e., that $s \notin E$. It follows that $E \subsetneq A$, i.e., that every countable subset of A is a proper subset of A. Therefore, A must be uncountable (for otherwise A would be a proper subset of A, a contradiction). $\square$

  - The idea of this proof is called **Cantor's diagonalization process**.
  - Since every real number can be represented as a binary sequence of numbers, i.e., $A \sim \mathbb{R}$, the reals are uncountable.
- **Metric space**: A set X such that with any two points $p, q \in X$, there is associated a real number $d(p, q)$ such that
  1. $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$.
  2. $d(p, q) = d(q, p)$.
  3. $d(p, q) \leq d(p, r) + d(r, q)$ for any $r \in X$.
- **Distance** (from $p \in X$ to $q \in X$, X a metric space): The real number $d(p, q)$.
- **Distance function**: A function $d : X \times X \to \mathbb{R}$ that sends $(p, q) \mapsto d(p, q)$. *Also known as* **metric**.
- Every subset of a metric space is a metric space in its own right under the same distance function.
- **Segment** (from a to b): The set of all real numbers x such that $a < x < b$. *Denoted by* $\boldsymbol{(a, b)}$.
- **Interval** (from a to b): The set of all real numbers x such that $a \leq x \leq b$. *Denoted by* $\boldsymbol{[a, b]}$.
- $\boldsymbol{k}$**-cell**: The set of all points $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k$ whose coordinates satisfy the inequalities $a_i \leq x_i \leq b_i$ where $a_i < b_i$ for each $1 \leq i \leq k$.
  - Note that a 1-cell is an interval and a 2-cell is a **rectangle**.
- **Convex** (set E): A subset E of $\mathbb{R}^k$ such that

  $$\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} \in E$$

  for all $\mathbf{x}, \mathbf{y} \in E$ and $0 < \lambda < 1$.
  - Balls and k-cells are both convex.
- Theorem 2.19: Every neighborhood is open.
- Theorem 2.20: p a limit point of E implies $N_r(p)$ contains infinitely many points of E.
- Corollary: Finite sets have no limit points.

- The segment (a,b) is open as a subset of $\mathbb{R}^1$, but not open as a subset of $\mathbb{R}^2$.
- Theorem 2.22: $\{E_\alpha\}$ a collection of sets implies
$$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha E_\alpha^c$$
- Theorem 2.23: E open iff E^c closed.
- Corollary: F closed iff F^c open.
- Theorem 2.24:
  (a) $\bigcup G_\alpha$ open for any collection $\{G_\alpha\}$ of open sets.
  (b) $\bigcap F_\alpha$ closed for any collection $\{F_\alpha\}$ of closed sets.
  (c) $\bigcap G_\alpha$ open for any finite collection $\{G_\alpha\}$ of open sets.
  (d) $\bigcup F_\alpha$ closed for any finite collection $\{F_\alpha\}$ of closed sets.
- Theorem 2.27: $E \subset X$ a metric space implies
  (a) $\bar{E}$ closed.
  (b) $E = \bar{E}$ iff E closed.
  (c) $\bar{E} \subset F$ for every closed $F \supset E$.
- Theorem 2.28: $E \subset \mathbb{R}$ nonempty and bounded above implies $\sup E \in \bar{E}$.
- Theorem 2.30: $Y \subset X$ implies $E \subset Y$ open wrt Y iff $E = Y \cap G$ for some open $G \subset X$.
- Theorem 2.33: $K \subset Y \subset X$ implies K compact wrt. X iff K compact wrt. Y.
- Since compactness is not relative, while it makes no sense to talk about *open* or *closed* metric spaces, it does make sense to talk about *compact* metric spaces.
- Theorem 2.34: $K \subset X$ (K compact, X a metric space) implies K closed.
- Theorem 2.35: $F \subset K$ (F closed, K compact) implies F compact.
- Corollary: F closed and K compact imply $F \cap K$ compact.
- Theorem 2.36: $\{K_\alpha\}$ a collection of compact subsets of X a metric space with the intersection of any finite subcollection of $\{K_\alpha\}$ nonempty implies $\bigcap K_\alpha$ nonempty.
- **Decreasing** (sequence of sets $\{K_n\}$): A sequence of sets $\{K_n\}$ such that $K_n \supset K_{n+1}$ for all $n \in \mathbb{N}$.
- Corollary: $\{K_n\}$ a decreasing sequence of nonempty compact sets implies $\bigcap_1^\infty K_n \neq \emptyset$.
- Theorem 2.37: $E \subset K$ (E infinite, K compact) implies E has a limit point in K.
- Theorem 2.38: $\{I_n\}$ a decreasing sequence of intervals in $\mathbb{R}^1$ implies $\bigcap_1^\infty I_n \neq \emptyset$.
- Theorem 2.39: $\{I_n\}$ a decreasing sequence of k-cells implies $\bigcap_1^\infty I_n \neq \emptyset$.
- Theorem 2.40: k-cells are compact.
- Theorem 2.41 (Heine-Borel Theorem): The following are equivalent for any $E \subset \mathbb{R}^k$.
  (a) E closed and bounded.
  (b) E compact.
  (c) Every infinite subset of E has a limit point in E.

- Theorem 2.42 (Weierstrass Theorem): Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.
- Theorem 2.43: P a nonempty perfect set in $\mathbb{R}^k$ implies P is uncountable.

  Proof. Since P is nonempty and perfect, there exists a limit point of P. It follows that P is infinite.

  Now suppose for the sake of contradiction that P is countable, and denote the elements of P by $\mathbf{x}_1, \mathbf{x}_2, \dots$. We now construct a sequence $\{V_n\}$ of neighborhoods, as follows. Let $V_1 = N_r(\mathbf{x}_1)$. Clearly, $V_1 \subset P$ since $\mathbf{x}_1 \in P$. It follows that since V_1 is a neighborhood that V_1 contains infinitely many points of P. Now suppose inductively that V_n has been constructed. Thus, by analogous conditions to those on V_1, we may let V_{n+1} be a neighborhood such that (i) $\bar{V}_{n+1} \subset V_n$, (ii) $\mathbf{x}_n \notin \bar{V}_{n+1}$, and (iii) $V_{n+1} \cap P$ is nonempty. By (iii), we can continue on to construct V_{n+2}, and so on and so forth.

  Let $K_n = \bar{V}_n \cap P$. Since $\bar{V}_n$ is closed and bounded, $\bar{V}_n$ is compact. Additionally, since $\mathbf{x}_n \notin K_{n+1}$ for each n, no point of P lies in $\bigcap_1^\infty K_n$. Thus, since each $K_n \subset P$, $\bigcap_1^\infty K_n$ is empty. But this contradicts our previous result that since each K_n is nonempty, compact, and such that $K_n \supset K_{n+1}$, $\bigcap_1^\infty K_n$ is nonempty. □

- Corollary: Every interval $[a, b]$ is uncountable. In particular, $\mathbb{R}$ is uncountable.
- **Cantor set**: The set resulting from the following construction. Let $E_0 = [0, 1]$. Remove the segment $(1/3, 2/3)$, so that $E_1 = [0, 1/3] \cup [1/3, 2/3]$. Now remove the middle third of these two intervals to create E_2. Continue on indefinitely.
  - This is a perfect set in $\mathbb{R}^1$ which contains no segment.
- **Separated** (sets A, B): Two subsets A, B of a metric space X such that $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty.
- **Connected** (set E): A set E that is not the union of two nonempty separated sets.
- Separated sets are disjoint, but disjoint sets are not necessarily separated (consider $[0, 1]$ and $(1, 2)$).
- Theorem 2.47: $E \subset \mathbb{R}^1$ connected iff $x, y \in E$ and $x < z < y$ implies $z \in E$ for all such $x, y, z \in E$.

Chapter 3

Numerical Sequences and Series

3.1 Notes

11/8: - Any bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

11/10: - Read and understand the section about Cauchy sequences converging and the sup/inf.

3.2 Chapter 3: Numerical Sequences and Series

From Rudin (1976).

11/7: - Convergence of sequences is relative.
    - For example, the sequence $1/n$ for $n = 1, 2, \dots$ converges in $\mathbb{R}$, but not in $(0, \infty)$.
- **Range** (of $\{p_n\}$): The set of all points p_n.
    - This definition squares nicely with the formal definition of a sequence as a function p defined on $\mathbb{N}$.
- Theorem 3.2: $\{p_n\} \subset X$ a metric space implies
    (a) $\{p_n\}$ converges to $p \in X$ iff every $N_r(p)$ contains all but finitely many p_n.
    (b) $p, p' \in X$, $p_n \to p$, and $p_n \to p'$ implies $p = p'$.
    (c) $\{p_n\}$ converges implies $\{p_n\}$ is bounded.
    (d) $E \subset X$ and p a limit point of E implies there exists $\{p_n\} \subset E$ such that $p = \lim_{n\to\infty} p_n$.
- Theorem 3.3: Let $\{s_n\}, \{t_n\} \subset \mathbb{C}$, $\lim_{n\to\infty} s_n = s$, and $\lim_{n\to\infty} t_n = t$. Then
    (a) $\lim_{n\to\infty}(s_n + t_n) = s + t$.
    (b) $\lim_{n\to\infty} cs_n = cs$, $\lim_{n\to\infty}(c + s_n) = c + s$ for any $c \in \mathbb{C}$.
    (c) $\lim_{n\to\infty} s_n t_n = st$.
    (d) $\lim_{n\to\infty} 1/s_n = 1/s$, provided $s_n \neq 0$ ($n \in \mathbb{N}$) and $s \neq 0$.
- Theorem 3.4:
    (a) $\{\mathbf{x}_n\} \subset \mathbb{R}^k$ and $\mathbf{x}_n = (\alpha_{1,n}, \dots, \alpha_{k,n})$ ($n \in \mathbb{N}$) implies $\mathbf{x}_n \to \mathbf{x} = (\alpha_1, \dots, \alpha_k)$ iff $\lim_{n\to\infty} \alpha_{j,n} = \alpha_j$ for each $1 \leq j \leq k$.
    (b) $\{\mathbf{x}_n\}, \{\mathbf{y}_n\} \subset \mathbb{R}^k$, $\{\beta_n\} \subset \mathbb{R}$, and $\mathbf{x}_n \to \mathbf{x}$, $\mathbf{y}_n \to \mathbf{y}$, $\beta_n \to \beta$ imply

$$\lim_{n\to\infty}(\mathbf{x}_n + \mathbf{y}_n) = \mathbf{x} + \mathbf{y} \qquad \lim_{n\to\infty} \mathbf{x}_n \cdot \mathbf{y}_n = \mathbf{x} \cdot \mathbf{y} \qquad \lim_{n\to\infty} \beta_n \mathbf{x}_n = \beta \mathbf{x}$$

- Theorem 3.6:
  (a) $\{p_n\} \subset X$ compact implies some subsequence of $\{p_n\}$ converges to a point of X.
  (b) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.
- Theorem 3.7: The subsequential limits of $\{p_n\} \subset X$ form a closed subset of X.
- **Diameter** (of E): The supremum of the set
$$S = \{d(p,q) : p, q \in E\}$$
where E is a nonempty subset of a metric space X. *Denoted by* **diam $\boldsymbol{E}$**.
- Theorem 3.10:
  (a) $E \subset X$ implies
$$\operatorname{diam} \bar{E} = \operatorname{diam} E$$
  (b) $\{K_n\} \subset X$ a decreasing sequence of compact sets and $\lim_{n\to\infty} \operatorname{diam} K_n = 0$ imply $\bigcap_1^\infty K_n$ consists of exactly one point.
- Theorem 3.11:
  (a) Every convergent sequence in X a metric space is Cauchy.
  (b) $\{p_n\} \subset X$ ($\{p_n\}$ Cauchy, X compact) implies $\{p_n\}$ converges to some point of X.
  (c) Every Cauchy sequence converges in $\mathbb{R}^k$.
- **Complete** (metric space): A metric space in which every Cauchy sequence converges.
- Rephrasing Theorem 3.11b-c: All compact metric spaces and all Euclidean spaces are complete.
  - The metric space $(\mathbb{Q}, |x-y|)$ is not complete.
- **Monotonically increasing** (sequence $\{s_n\}$): A sequence $\{s_n\}$ of real numbers such that $s_n \leq s_{n+1}$ for each $n \in \mathbb{N}$.
- **Monotonically decreasing** (sequence $\{s_n\}$): A sequence $\{s_n\}$ of real numbers such that $s_n \geq s_{n+1}$ for each $n \in \mathbb{N}$.
- **Monotonic sequences**: The class of all sequences that are either monotonically increasing or monotonically decreasing.
- Theorem 3.14: $\{s_n\}$ monotonic converges iff it is bounded.
- **Upper limit** (of $\{s_n\}$): The supremum of the set E of all subsequential limits of $\{s_n\}$. *Denoted by* $\boldsymbol{s^*}$, $\boldsymbol{\limsup_{n\to\infty} s_n}$.
- **Lower limit** (of $\{s_n\}$): The infimum of the set E of all subsequential limits of $\{s_n\}$. *Denoted by* $\boldsymbol{s_*}$, $\boldsymbol{\liminf_{n\to\infty} s_n}$.
- Theorem 3.17: $\{s_n\} \subset \mathbb{R}$ implies s^* has (and is the only number to have both of) the following two properties.
  (a) $s^* \in E$.
  (b) If $x > s^*$, then there is an integer N such that $n \geq N$ implies $s_n < x$.

  An analogous result holds for s_*.
- Theorem 3.19: $s_n \leq t_n$ for all $n \geq N$ implies
$$\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n \qquad \limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n$$

- Theorem 3.20:
  (a) $p > 0$ implies $\lim_{n\to\infty} 1/n^p = 0$.
  (b) $p > 0$ implies $\lim_{n\to\infty} \sqrt[n]{p} = 1$.
  (c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.
  (d) $p > 0$, $\alpha \in \mathbb{R}$ implies $\lim_{n\to\infty} n^\alpha/(1+p)^n = 0$.
  (e) $|x| < 1$ implies $\lim_{n\to\infty} x^n = 0$.

11/8:

- Series are defined in terms of sequences. Moreover, sequences can be defined in terms of series: Let $a_1 = s_1$, $a_n = s_n - s_{n-1}$ ($n \in \mathbb{N}+1$). Thus, every theorem about sequences can be stated in terms of series and vice versa, but it is nevertheless useful to consider both concepts (Rudin, 1976, p. 59).
- Theorem 3.22: $\sum a_n$ converges iff for every $\epsilon > 0$, there is an N such that $m \geq n \geq N$ implies

$$\left|\sum_{k=n}^{m} a_k\right| \leq \epsilon$$

  - Analogous to Theorem 3.11.
- Theorem 3.23: $\sum a_n$ converges implies $\lim_{n\to\infty} a_n = 0$.
- Theorem 3.24: $\{a_n\} \subset \mathbb{R}$ such that $a_n \geq 0$ ($n \in \mathbb{N}$) implies $\sum a_n$ converges iff its partial sums form a bounded sequence.
- Theorem 3.25 (Comparison test):
  (a) $|a_n| \leq c_n$ for all $n \geq N_0$ and $\sum c_n$ converges implies $\sum a_n$ converges.
  (b) $a_n \geq d_n \geq 0$ for all $n \geq N_0$ and $\sum d_n$ diverges implies $\sum a_n$ diverges.
- Theorem 3.26: $0 \leq x < 1$ implies

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

  $x \geq 1$ implies the series diverges.
- Theorem 3.27: $\{a_n\}$ a monotonically decreasing sequence of nonnegative terms implies the series $\sum_{n=1}^{\infty} a_n$ converges iff the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$$

  converges.
- Theorem 3.29: $p > 1$ implies

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$$

  converges; $p \leq 1$ implies the series diverges.
  - Note that $\log n = \ln n$.
  - Note that we sum from $n = 2$ since $\log 1 = 0$.
- **e**: The number

$$\mathrm{e} = \sum_{n=0}^{\infty} \frac{1}{n!}$$

- Theorem 3.31: $\lim_{n\to\infty}(1+1/n)^n = \mathrm{e}$.

- Theorem 3.32: e is irrational.
- Theorem 3.33 (Root test): Given $\sum a_n$, put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then
  (a) $\alpha < 1$ implies $\sum a_n$ converges.
  (b) $\alpha > 1$ implies $\sum a_n$ diverges.
  (c) $\alpha = 1$ implies nothing; the test is inconclusive.
- Theorem 3.34 (Ratio test): The series $\sum a_n \dots$
  (a) converges if $\limsup_{n\to\infty} |a_{n+1}/a_n| < 1$;
  (b) diverges if $|a_{n+1}/a_n| \geq 1$ for all $n \geq N_0$.
- Theorem 3.37: $\{c_n\} \subset \mathbb{R}^+$ implies

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} \sqrt[n]{c_n} \qquad \limsup_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$$

- Theorem 3.39: Given the power series $\sum c_n z^n$, put

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|} \qquad R = \frac{1}{\alpha}$$

  (If $\alpha = 0$, let $R = +\infty$; if $\alpha = +\infty$, let $R = 0$.) Then $\sum c_n z^n$ converges if $|z| < R$ and diverges if $|z| > R$.
- **Radius of convergence** (of a power series): The number R defined by Theorem 3.39.
- Theorem 3.41 (partial summation formula): Given two sequence $\{a_n\}, \{b_n\}$, put

$$A_n = \begin{cases} \sum_{k=0}^n a_k & n \geq 0 \\ 0 & n = -1 \end{cases}$$

  Then if $0 \leq p \leq q$, we have

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

- Theorem 3.42: If the partial sums A_n of $\sum a_n$ form a bounded sequence and $\{b_n\}$ is a monotonically decreasing sequence such that $b_n \to 0$, then $\sum a_n b_n$ converges.
- Theorem 3.43: If $\{c_n\}$ is an alternating series that is absolutely monotonically decreasing such that $c_n \to 0$, then $\sum c_n$ converges.
- Theorem 3.44: If the radius of convergence of $\sum c_n z^n$ is 1, $\{c_n\}$ is monotonically decreasing, and $c_n \to 0$, then $\sum c_n z^n$ converges at every point on the circle $|z| = 1$ except possibly at $z = 1$.
- Theorem 3.45: $\sum a_n$ converges absolutely implies $\sum a_n$ converges.
- Theorem 3.47: $\sum a_n = A$, $\sum b_n = B$, $c \in \mathbb{R}$ implies $\sum(a_n + b_n) = A + B$ and $\sum c a_n = cA$.
- **Product** (of $\sum a_n, \sum b_n$): The series $\sum c_n$ defined by

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

  for each $n = 0, 1, 2, \dots$.

  - We motivate this definition by noting that if $\sum c_n$ is the product of $\sum a_n, \sum b_n$, then

$$\sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} c_n z^n$$

  - Setting $z = 1$ then yields the given definition.
- The product of two convergent series may diverge. However...
- Theorem 3.50: Suppose (a) $\sum_{n=0}^{\infty} a_n$ converges absolutely, (b) $\sum_{n=0}^{\infty} a_n = A$, (c) $\sum_{n=0}^{\infty} b_n = B$, and (d) $\sum_{n=0}^{\infty} c_n$ is the product of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$. Then

$$\sum_{n=0}^{\infty} c_n = AB$$

- Theorem 3.51: If $\sum a_n, \sum b_n, \sum c_n$ converge to A, B, C, respectively, and $\sum c_n$ is the product of $\sum a_n, \sum b_n$, then $C = AB$.
- **Rearrangement** (of $\sum a_n$): A series $\sum a_n'$ defined by $a_n' = a_{k_n}$ for each $n \in \mathbb{N}$, where $\{k_n\}$ is a sequence in which every positive integer appears once and only once (that is, $\{k_n\}$ is a 1-1 function from $\mathbb{N}$ onto $\mathbb{N}$).
- Theorem 3.54: Let $\sum a_n$ be a series of real number which converges, but not absolutely. Suppose $-\infty \leq \alpha \leq \beta \leq \infty$. Then there exists a rearrangement $\sum a_n'$ with partial sums s_n' such that

$$\liminf_{n\to\infty} s_n' = \alpha \qquad \limsup_{n\to\infty} s_n' = \beta$$

- Theorem 3.55: If $\sum a_n$ is a series of complex numbers which converges absolutely, then every rearrangement of $\sum a_n$ converges, and they all converge to the same sum.

Chapter 4

Continuity

4.1 Notes

11/8: • Consider a function $f : X \to Y$ whose domain and codomain are, respectively, the metric spaces (X, d_X) and (Y, d_Y).

- **Limit** (of f at p): A point $q \in Y$ such that for all $\epsilon > 0$, there exists δ such that $d_X(x, p) < \delta$ implies $d_Y(q, f(x)) < \epsilon$, where p is a limit point of X (otherwise, $x \nrightarrow p$).
- **Continuous** (function f at p): A function f such that $\lim_{x\to p} f(x) = f(p)$.
- f is continuous on X if it is continuous at every $p \in X$.
- **Uniformly continuous** (function f): A function f such that for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \epsilon$ for all $x, y \in X$.

11/10: • f, g continuous implies $f + g$, fg, and f/g continuous, the latter where $g(x) \neq 0$.

- If f, g continuous, then $h = g \circ f$ is continuous.
- Theorem: $f : X \to Y$ is continuous iff $f^{-1}(V)$ is open in X for every $V \subset Y$ open.

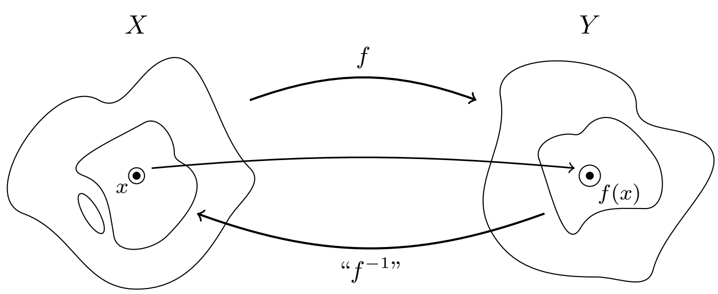


Figure 4.1: Set theoretic definition of continuity.

  - This works in a general topological space, too, not just a metric space.
  - Note that $f^{-1}(V)$ is not a function defined on V; it's a specifically defined set $\{x \in X : f(x) \in V\}$.
  - f being continuous means that open circular neighborhood of a point x in the domain maps to an area of the range encompassed by a circular neighborhood of $f(x)$.
  - The other condition means that every open set surrounding $f(x)$ maps to an open set of the domain surrounding x. Indeed, going off of this definition, if an open set containing $f(x)$ maps to an open set containing x, then we can choose a neighborhood subset of the open set surrounding x and know that it will map into a neighborhood subset of the open set surrounding $f(x)$.

- Corollary: $f : X \to Y$ continuous iff $f^{-1}(C)$ closed for every $C \subset Y$ closed.
  - We use the property that $f^{-1}(X \subset C) = X \subset f^{-1}(C)$.
- Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$. Suppose $f : X_1 \times X_2 \to Y_1 \times Y_2$ is defined by $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$. Then f is continuous iff f_1, f_2 are continuous, under appropriately defined metrics.
- Continuity and compactness.
- Theorem: $f : X \to Y$ continuous and X compact imply $f(X)$ compact.
  - Let $\{V_\alpha\}$ be an open cover of $f(X)$.
  - Then $\{f^{-1}(V_\alpha)\}$ is an open cover of X..
  - Choose a finite subcover of $\{f^{-1}(V_\alpha)\}$. Then the corresponding V_α's form a finite subcover of $f(X)$.
- If $f : X \to \mathbb{R}^k$ is continuous and X is compact, $f(X)$ is compact and closed/bounded.
- If $f : X \to \mathbb{R}$ is continuous and X is compact, then $M = \sup_{x \in X} |f(x)| = |f(\bar{x})|$ and $m = \inf x \in X|f(x)| = |f(\underline{x})|$ where $\bar{x}, \underline{x} \in X$.
  - There is a subsequence $\{x_m\}$ such that $|f(x_m)| \to M$. Since $f(X)$ is compact, the limit of this sequence is in $f(X)$.
- If $f : X \to Y$ is continuous, bijective, and X, Y are compact, $f^{-1} : Y \to X$ is continuous.
- Uniform continuity.
- Examples.
  - Linear functions are uniformly continuous.
  - $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = x^2$ is uniformly continuous.
  - $f : (a, \infty) \to \mathbb{R}$ defined by $f(x) = x^2$ is *not* uniformly continuous.
  - $f : (a, \infty) \to \mathbb{R}$ defined by $f(x) = 1/x$ is uniformly continuous if $a > 0$.
  - $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = 1/x$ is *not* uniformly continuous.
- **Lipschitz continuous** (function f on $E \subset X$): A function such that $|f(x) - f(y)| \leq L|x - y|$ for each $x, y \in E$.
- Theorem: $f : X \to Y$ continuous and X compact implies f is uniformly continuous.
  - Fix $\epsilon > 0$. There exists $\delta = \delta(p) > 0$.
  - Def. of continuity: $q \in N_{\delta(p)}(p)$ implies $f(q) \in N_\epsilon(f(p))$.
  - $\{N_{\delta(p)/2}(p) : p \in X\}$ is an open cover of X. Choose a finite subcover. Let $\delta = \min(\delta(p_1)/2, \dots, \delta(p_n)/2)$.
  - ...

11/12:

- $f : X \to Y$ continuous and $E \subset X$ connected implies $f(E)$ connected.
  - Suppose $f(E) = A \cup B$, A, B nonempty, separated subsets of Y.
  - Let $G = E \cap f^{-1}(A)$, $H = E \cap f^{-1}(B)$. It follows that $E = G \cup H$, where G, H nonempty.
  - $A \subset \bar{A}$ implies $G \subset f^{-1}(\bar{A})$ implies (inverse image def. of continuity) $\bar{G} \subset f^{-1}(A)$ implies $f(\bar{G}) \subset \bar{A}$.
  - $f(H) = B$ and $\bar{A} \cap B = \emptyset$ yields $\bar{G} \cap H = \emptyset$. Symmetrically, $G \cap \bar{H} = \emptyset$. This contradicts our assumption that E is connected.
- Introduces monotone functions.
- Theorem: If f is monotonic on (a, b), then the set of points of (a, b) at which f is discontinuous is at most countable.
  - ...

4.2 Chapter 4: Continuity

From Rudin (1976).

11/8:
- **Limit** (of f at p): The point $q \in Y$, if it exists, such that for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d_Y(f(x), q) < \epsilon$ for all points $x \in E$ for which $0 < d_X(x, p) < \delta$, where $(X, d_X), (Y, d_Y)$ are metric spaces, $E \subset X$, $f : E \to Y$, and $p \in E'$. *Denoted by* $\boldsymbol{\lim_{x \to p} f(x)}$.
  - Note that we do not require that $p \in E$; only that some elements of the domain E approach p.
  - We also write $f(x) \to q$ as $x \to p$.
- Theorem 4.2: Let X, Y, E, f, and p be as specified above. Then $\lim_{x \to p} f(x) = q$ iff $\lim_{n \to \infty} f(p_n) = q$ for every sequence $\{p_n\}$ in E such that $p_n \neq p$ for any n and $\lim_{n \to \infty} p_n = p$.
- Rudin (1976) proves the sum, product, and quotient rules of limits from the analogous properties of series.
- Continuity is defined.
  - Note that f *does* have to be defined at p to be continuous at p (in comparison to the fact that it can have a limit at a point p' at which it is not defined).
    - Thus, for proofs concerning continuity (as opposed to limits), we will consider functions f the domains of which are metric spaces, not *subsets* of metric spaces.
  - It follows from the definition that if $p \in E$ is isolated, then every possible f defined on E is continuous at p.
- Theorem 4.7: Compositions of continuous functions are continuous.
- Theorem 4.8: Preimage definition of continuity.
- Theorem 4.9: If f, g are complex continuous functions on X, $f + g$, fg, and f/g are continuous on X.
- Theorem 4.10: $\mathbf{f}$ continuous implies $f_1, \dots, f_k$ continuous. Also, $\mathbf{f}, \mathbf{g} : X \to \mathbb{R}^k$ continuous implies $\mathbf{f} + \mathbf{g}$ and $\mathbf{f} \cdot \mathbf{g}$ continuous.

11/9:
- Theorem 4.14: f continuous and X compact implies $f(X)$ compact.
- Theorem 4.15: $\mathbf{f} : X \to \mathbb{R}^k$ continuous and X compact implies $f(X)$ closed and bounded.
- Theorem 4.16: f continuous and X compact implies f attains its minimum and maximum.
- Theorem 4.17: $f : X \to Y$ continuous, 1-1 for X, Y compact implies $f^{-1} : Y \to X$ continuous.
- Theorem 4.19: f continuous and X compact implies f uniformly continuous.
- Theorem 4.20: Compactness is a necessary condition in Theorems 4.14, 4.15, 4.16, and 4.19.
- Theorem 4.22: $f : X \to Y$ continuous and $E \subset X$ connected implies $f(E)$ connected.
- Theorem 4.23: Intermediate value theorem.
- **Right-hand limit** (of f at x): *Denoted by* $\boldsymbol{f(x+)}$.
- **Left-hand limit** (of f at x): *Denoted by* $\boldsymbol{f(x-)}$.
- **Discontinuity of the first kind** (of f at x): A discontinuity of f at x such that $f(x+)$ and $f(x-)$ exist. *Also known as* **simple discontinuity**.
- **Discontinuity of the second kind** (of f at x): A discontinuity of f at x that is not of the first kind (i.e., a discontinuity such that at least one of $f(x+)$ and $f(x-)$ does not exist).

- Theorem 4.29: If f is monotonic on (a, b), then $f(x+), f(x-)$ exist at every $x \in (a, b)$.
- Corollary: Monotonic functions have no discontinuities of the second kind.
- Theorem 4.30: If f is monotonic on (a, b), then the set of points of (a, b) at which f is discontinuous is at most countable.

11/20: *Proof.* Suppose first that f is increasing. Let E be the set of points at which f is discontinuous. By Theorem 4.29, for every $x \in E$, $f(x-), f(x+)$ exist. Thus, we may pick a rational number $r(x)$ such that $f(x-) < r(x) < f(x+)$. Moreover, since $x_1 < x_2$ implies $f(x_1+) \leq f(x_2-)$, we have that $r(x_1) \neq r(x_2)$. Having established an injective function from E to the rationals $\mathbb{Q}$, we know that E is at most countable. The argument where f is decreasing is symmetric. $\square$

- Gives an example of a function with discontinuities that are not isolated.

Chapter 5

Differentiation

5.1 Chapter 5: Differentiation

From Rudin (1976).

12/5:

- Let f be a real-valued function defined on $[a,b]$.
- **Derivative** (of f at x): The limit $\lim_{t\to x}\phi(t)$, provided that said limit exists, where $\phi:(a,b)\setminus\{x\}\to\mathbb{R}$ is defined by
$$\phi(t)=\frac{f(t)-f(x)}{t-x}$$
Denoted by $\boldsymbol{f'(x)}$.
- **Derivative** (of f): The real function defined on X that evaluates to $f'(x)$ everywhere on its domain, where
$$X=\{x\in[a,b]:f'(x)\text{ exists}\}$$
Denoted by $\boldsymbol{f'}$.
- Theorem 5.2: Differentiability at x implies continuity at x.
  - The converse is not true.
- Theorem 5.3: Sum, product, and quotient rules of derivatives.
- Theorem 5.4 (Chain Rule): Suppose f is continuous on $[a,b]$, $f'(x)$ exists at some point $x\in[a,b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If $h(t)=g(f(t))$ for all $t\in[a,b]$, then h is differentiable at x and
$$h'(x)=g'(f(x))f'(x)$$
Proof. Let $y=f(x)$. Since f is differentiable at x and g is differentiable at $f(x)$, we have that
$$\begin{aligned}\frac{f(t)-f(x)}{t-x}&=f'(x)+u(t) & \qquad \frac{g(s)-g(y)}{s-y}&=g'(y)+v(s)\\ f(t)-f(x)&=(t-x)[f'(x)+u(t)] & g(s)-g(y)&=(s-y)[g'(y)+v(s)]\end{aligned}$$
where $t\in[a,b]$, $s\in I$, $u(t)\to 0$ as $t\to x$, and $v(s)\to 0$ as $s\to y$. Let $s=f(t)$. Then
$$\begin{aligned}h(t)-h(x)&=g(f(t))-g(f(x))\\&=[f(t)-f(x)]\cdot[g'(f(x))+v(s)]\\&=(t-x)\cdot[f'(x)+u(t)]\cdot[g'(f(x))+v(s)]\\ \frac{h(t)-h(x)}{t-x}&=[f'(x)+u(t)]\cdot[g'(f(x))+v(s)]\end{aligned}$$

Thus, since as $t \to x$, $s = f(t) \to f(x) = y$ by the continuity of f, we have that

$$\begin{aligned} h'(x) &= \lim_{t \to x} \frac{h(t) - h(x)}{t - x} \\ &= \lim_{t \to x} [f'(x) + u(t)] \cdot [g'(f(x)) + v(s)] \\ &= [f'(x) + 0] \cdot [g'(f(x)) + 0] \\ &= g'(f(x)) f'(x) \end{aligned}$$

as desired. □

- **Local maximum** (of $f : X \to \mathbb{R}$): A point $p \in X$ for which there exists a $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in X$ with $d(p, q) < \delta$.
- Theorem 5.8: $f(x)$ a local maximum and f' exists implies $f'(x) = 0$.
- Theorem 5.9 (Generalized or Cauchy Mean Value Theorem): f, g continuous on $[a, b]$, differentiable on (a, b) imply there exists $x \in (a, b)$ such that

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$$

- Theorem 5.10 (Mean Value Theorem): f continuous on $[a, b]$, differentiable on (a, b) implies there exists $x \in (a, b)$ such that

$$f(b) - f(a) = (b - a)f'(x)$$

Proof. Take $g(x) = x$ in Theorem 5.9. □

- Theorem 5.11: Suppose f is differentiable in (a, b).
  (a) If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotonically increasing.
  (b) If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.
  (c) If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is monotonically decreasing.
- Theorem 5.12: f differentiable on $[a, b]$ and $f'(a) < \lambda < f'(b)$ implies there exists $x \in (a, b)$ such that $f'(x) = \lambda$.
- Corollary: f differentiable on $[a, b]$ implies f' has no simple discontinuities on $[a, b]$.
  - But it may have discontinuities of the second kind.
- Theorem 5.13 (L'Hôpital's Rule): f, g differentiable on (a, b), $g'(x) \neq 0$ for all $x \in (a, b)$, $f'(x)/g'(x) \to A$, and $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ or $g(x) \to +\infty$ as $x \to a$ implies $f(x)/g(x) \to A$ as $x \to a$, where $-\infty \leq a < b \leq +\infty$.
- $\boldsymbol{n}^{\textbf{th}}$ **derivative** (of f at x): The derivative of the $(n-1)^{\text{th}}$ derivative of f at x, if it exists. *Denoted by* $\boldsymbol{f^{(n)}(x)}$.
  - $f^{(n)}(x)$ exists iff $f^{(n-1)}$ exists in some $N_r(x)$ and $f^{(n-1)'}(x)$ exists.
  - We customarily denote the first few higher order derivatives with repeated primes, e.g., $f''(x)$ is the second derivative of f.
- Theorem 5.15 (Taylor's Theorem): f defined on $[a, b]$, $n \in \mathbb{N}$, $f^{(n-1)}$ continuous on $[a, b]$, $f^{(n)}(t)$ defined on (a, b), $\alpha, \beta \in [a, b]$ such that $\alpha \neq \beta$, and

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k$$

implies there exists $x \in (\alpha, \beta)$ such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n$$

  - For $n = 1$, this is the mean value theorem.
  - "In general, the theorem shows that f can be approximated by a polynomial of degree $n-1$ and that the last equation above allows us to estimate the error, if we know bounds on $|f^{(n)}(x)|$" (Rudin, 1976, p. 111).

- **Derivative** (of $\mathbf{f}$ at x): The point $\mathbf{f}'(x) \in \mathbb{R}^k$, if it exists, such that

$$\lim_{t \to x} \left\| \frac{\mathbf{f}(t) - \mathbf{f}(x)}{t - x} - \mathbf{f}'(x) \right\| = 0$$

- Theorems 5.2-5.3 remain valid for vector-valued functions.
- If $\mathbf{f} = (f_1, \dots, f_k)$, then $\mathbf{f}'(x)$ exists iff $f_i'(x)$ $(i = 1, \dots, k)$ exists and

$$\mathbf{f}' = (f_1', \dots, f_k')$$

- Theorem 5.19: $\mathbf{f} : [a, b] \to \mathbb{R}^k$ continuous and $\mathbf{f}$ differentiable on (a, b) implies there exists $x \in (a, b)$ such that

$$\|\mathbf{f}(b) - \mathbf{f}(a)\| \leq (b - a)\|\mathbf{f}'(x)\|$$

Chapter 6

The Riemann-Stieltjes Integral

6.1 Chapter 6: The Riemann-Stieltjes Integral

From Rudin (1976).

12/5: • **Partition** (of $[a,b]$): A finite set P of points $x_0,\dots,x_n \in [a,b]$ such that

$$a = x_0 \leq \cdots \leq x_n = b$$

  - Let $\Delta x_i = x_i - x_{i-1}$.

- Let $f : [a,b] \to \mathbb{R}$ be bounded.
  - We define

$$M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\} \qquad m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$$
$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i \qquad L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$$

  for each partition P of $[a,b]$.

- **Upper Riemann integral** (of f over $[a,b]$): The following quantity. *Denoted by* $\bar{\int}_a^b f\,\mathrm{d}x$. *Given by*

$$\inf\{U(P,f) : P \text{ partitions } [a,b]\}$$

- **Lower Riemann integral** (of f over $[a,b]$): The following quantity. *Denoted by* $\underline{\int}_a^b f\,\mathrm{d}x$. *Given by*

$$\inf\{U(P,f) : P \text{ partitions } [a,b]\}$$

- The upper and lower Riemann integrals always exist since the boundedness of f on $[a,b]$ implies that the set of all lower and upper sums of f on $[a,b]$ is bounded.
- **Riemann-integrable** (f on $[a,b]$): A function f for which

$$\bar{\int_a^b} f\,\mathrm{d}x = \underline{\int_a^b} f\,\mathrm{d}x$$

- $\mathscr{R}$: The set of all Riemann-integrable functions.
- **Riemann integral** (of f on $[a,b]$): The common value of the lower and upper Riemann integrals over $[a,b]$ of a Riemann-integrable function on $[a,b]$. *Denoted by* $\int_a^b f\,\mathrm{d}x$, $\int_a^b f(x)\,\mathrm{d}x$. *Given by*

$$\bar{\int_a^b} f\,\mathrm{d}x = \underline{\int_a^b} f\,\mathrm{d}x$$

- Defining the Riemann-Stieltjes integral.
- Let $\alpha : [a,b] \to \mathbb{R}$ be monotonically increasing.
- Let $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$ be so defined for every partition P of $[a,b]$.
- Let

$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i \qquad L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

$$\overline{\int_a^b} f \, \mathrm{d}\alpha = \inf U(P,f,\alpha) \qquad \underline{\int_a^b} f \, \mathrm{d}\alpha = \sup L(P,f,\alpha)$$

- **Riemann-Stieltjes integral** (of f with respect to α over $[a,b]$): The common value, when it exists, of $\overline{\int_a^b} f \, \mathrm{d}\alpha$ and $\underline{\int_a^b} f \, \mathrm{d}\alpha$. *Also known as* **Stieltjes integral**. *Denoted by* $\boldsymbol{\int_a^b f \, \mathrm{d}\alpha}$, $\boldsymbol{\int_a^b f(x) \, \mathrm{d}\alpha(x)}$.
- $\boldsymbol{\mathscr{R}(\alpha)}$: The set of all Riemann-Stieltjes integrable functions with respect to α.
- Note that taking $\alpha(x) = x$ reveals that the Riemann integral is a special case of the Riemann-Stieltjes integral.
- **Refinement** (of P): A partition of the same interval as P that contains every point of P. *Denoted by* $\boldsymbol{P^*}$.
- **Common refinement** (of P_1, P_2): The set $P^* = P_1 \cup P_2$.
- Theorem 6.4: P^* a refinement of P implies

$$L(P,f,\alpha) \leq L(P^*,f,\alpha) \qquad U(P^*,f,\alpha) \leq U(P,f,\alpha)$$

- Theorem 6.5: $\underline{\int_a^b} f \, \mathrm{d}x \leq \overline{\int_a^b} f \, \mathrm{d}x$.
- Theorem 6.6: $f \in \mathscr{R}(\alpha)$ iff for every $\epsilon > 0$, there exists a partition P such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

- Theorem 6.7:
  (a) If $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$, then $U(P^*,f,\alpha) - L(P^*,f,\alpha) < \epsilon$ for all $P^* \supset P$.
  (b) If $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$ and $s_i, t_i \in [x_{i-1}, x_i]$, then

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i < \epsilon$$

  (c) If $f \in \mathscr{R}(\alpha)$ and the hypotheses of (b) hold, then

$$\left| \sum_{i=1}^{n} f(t_i) \Delta\alpha_i - \int_a^b f \, \mathrm{d}\alpha \right| < \epsilon$$

- Theorem 6.8: f continuous on $[a,b]$ implies $f \in \mathscr{R}(\alpha)$ on $[a,b]$.
- Theorem 6.9: f monotonic on $[a,b]$ and α continuous on $[a,b]$ imply $f \in \mathscr{R}(\alpha)$.
- Theorem 6.10: f bounded on $[a,b]$ with only finitely many discontinuities on $[a,b]$ and α continuous at every point at which f is discontinuous implies $f \in \mathscr{R}(\alpha)$.

- Theorem 6.11: $f \in \mathscr{R}(\alpha)$ on $[a,b]$, $m \leq f \leq M$, ϕ continuous on $[m,M]$, and $h(x) = \phi(f(x))$ on $[a,b]$ implies $h \in \mathscr{R}(\alpha)$ on $[a,b]$.
- Theorem 6.12:
  (a) $f_1, f_2 \in \mathscr{R}(\alpha)$ on $[a,b]$ and $c \in \mathbb{R}$ imply $f_1 + f_2 \in \mathscr{R}(\alpha)$ and $cf_1 \in \mathscr{R}(\alpha)$ with
  $$\int_a^b (f_1 + f_2)\,\mathrm{d}\alpha = \int_a^b f_1\,\mathrm{d}\alpha + \int_a^b f_2\,\mathrm{d}\alpha$$
  $$\int_a^b cf_1\,\mathrm{d}\alpha = c\int_a^b f\,\mathrm{d}\alpha$$
  (b) $f_1(x) \leq f_2(x)$ on $[a,b]$ implies
  $$\int_a^b f_1\,\mathrm{d}\alpha \leq \int_a^b f_2\,\mathrm{d}\alpha$$
  (c) $f \in \mathscr{R}(\alpha)$ on $[a,b]$ and $a < c < b$ implies $f \in \mathscr{R}(\alpha)$ on $[a,c]$ and on $[c,b]$ and
  $$\int_a^c f\,\mathrm{d}\alpha + \int_c^b f\,\mathrm{d}\alpha = \int_a^b f\,\mathrm{d}\alpha$$
  (d) $f \in \mathscr{R}(\alpha)$ on $[a,b]$ and $|f(x)| \leq M$ on $[a,b]$ implies
  $$\left|\int_a^b f\,\mathrm{d}\alpha\right| \leq M[\alpha(b) - \alpha(a)]$$
  (e) $f \in \mathscr{R}(\alpha_1)$, $f \in \mathscr{R}(\alpha_2)$, and $c \in \mathbb{R}$ imply $f \in \mathscr{R}(\alpha_1 + \alpha_2)$ and $f \in \mathscr{R}(c\alpha_1)$ with
  $$\int_a^b f\,\mathrm{d}(\alpha_1 + \alpha_2) = \int_a^b f\,\mathrm{d}\alpha_1 + \int_a^b f\,\mathrm{d}\alpha_2$$
  $$\int_a^b f\,\mathrm{d}(c\alpha_1) = c\int_a^b f\,\mathrm{d}\alpha_1$$
- Theorem 6.13: $f, g \in \mathscr{R}(\alpha)$ on $[a,b]$ implies
  (a) $fg \in \mathscr{R}(\alpha)$;
  (b) $|f| \in \mathscr{R}(\alpha)$ with
  $$\left|\int_a^b f\,\mathrm{d}\alpha\right| \leq \int_a^b |f|\,\mathrm{d}\alpha$$
- **Unit step function**: The function $I : \mathbb{R} \to \mathbb{R}$ defined by
  $$I(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$
- Theorem 6.15: $a < s < b$, f bounded on $[a,b]$ and continuous at s, and $\alpha(x) = I(x - s)$ imply
  $$\int_a^b f\,\mathrm{d}\alpha = f(s)$$
- Theorem 6.16: $c_n \geq 0$, $\sum c_n$ converges, $\{s_n\} \subset (a,b)$, $\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$, and f continuous on $[a,b]$ implies
  $$\int_a^b f\,\mathrm{d}\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$$

- Theorem 6.17: α monotonically increasing, $\alpha' \in \mathscr{R}$ on $[a,b]$, and f bounded on $[a,b]$ implies $f \in \mathscr{R}(\alpha)$ iff $f\alpha' \in \mathscr{R}$, and $f\alpha' \in \mathscr{R}$ implies

$$\int_a^b f \, \mathrm{d}\alpha = \int_a^b f(x)\alpha'(x) \, \mathrm{d}x$$

- Rudin (1976) gives an example of the physical significance of Theorems 6.15-6.17.
- Theorem 6.19 (change of variable): Suppose φ is a strictly increasing continuous function that maps an interval $[A,B]$ onto $[a,b]$. Suppose α is monotonically increasing on $[a,b]$ and $f \in \mathscr{R}(\alpha)$ on $[a,b]$. Define β and g on $[A,B]$ by

$$\beta(y) = \alpha(\varphi(y)) \qquad\qquad g(y) = f(\varphi(y))$$

  Then $g \in \mathscr{R}(\beta)$ and

$$\int_A^B g \, \mathrm{d}\beta = \int_a^b f \, \mathrm{d}\alpha$$

- Theorem 6.20: $f \in \mathscr{R}$ on $[a,b]$ and continuous at $x_0 \in [a,b]$, $a \leq x \leq b$, and $F(x) = \int_a^x f(t) \, \mathrm{d}t$ implies F continuous on $[a,b]$, F differentiable at x_0, and

$$F'(x_0) = f(x_0)$$

- Theorem 6.21 (Fundamental Theorem of Calculus): $f \in \mathscr{R}$ on $[a,b]$ and F differentiable on $[a,b]$ such that $F' = f$ imply

$$\int_a^b f(x) \, \mathrm{d}x = F(b) - F(a)$$

- Theorem 6.22 (Integration by Parts): F, G differentiable on $[a,b]$ and $(F' = f), (G' = g) \in \mathscr{R}$ imply

$$\int_a^b F(x)g(x) \, \mathrm{d}x = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x) \, \mathrm{d}x$$

- $\int_a^b \mathbf{f} \, \mathrm{d}\alpha$: The point in $\mathbb{R}^k$ whose j^{th} coordinate is $\int_a^b f_j \, \mathrm{d}\alpha$.
- Theorems 6.12a, 6.12c, 6.12e, 6.17, 6.20, and 6.21 are valid for vector-valued functions.
- Theorem 6.24: Theorem 6.21 for vector-valued functions.
- Theorem 6.25: Theorem 6.13b for vector-valued functions.
- **Curve** (in $\mathbb{R}^k$ on $[a,b]$): A continuous mapping $\gamma : [a,b] \to \mathbb{R}^k$.
  - Note that we define a curve in $\mathbb{R}^k$ to be a function instead of a subset of points in $\mathbb{R}^k$ that are the range of such a function since different curves may have the same range.
- **Arc**: A curve γ that is 1-1.
- **Closed curve**: A curve γ such that $\gamma(a) = \gamma(b)$.
- Let $\Lambda(P,\gamma) = \sum_{i=1}^n |\gamma(x_i) - \gamma(x_{i-1})|$ be so defined for every partition P of $[a,b]$.
- **Length** (of γ): The following quantity. *Denoted by* $\boldsymbol{\Lambda(\gamma)}$. *Given by*

$$\Lambda(\gamma) = \sup \Lambda(P,\gamma)$$

- **Rectifiable** (curve): A curve γ such that $\Lambda(\gamma) < \infty$.
- **Continuously differentiable** (curve): A curve γ whose derivative γ' is continuous.
- Theorem 6.27: γ' continuous on $[a,b]$ implies γ rectifiable and

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)| \, \mathrm{d}t$$

Chapter 7

Sequences and Series of Functions

7.1 Notes

11/15:

- Soug will not test on differentiation/integration assuming that we know them already.
- **Pointwise convergent** (sequence $(f_n)_{n\in\mathbb{N}}$ of functions): A sequence of functions $f_n : E \to \mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in E$.
- Can we interchange "limit" in the above definition with continuity, convergence of series, integration, differentiation, etc.?
- Examples with negative answer:
  1. Interchanging limits: Let $S_{mn} = \frac{m}{m+n}$. $S_{mn} \to 1$ as $m \to \infty$, and $S_{mn} \to 0$ as $n \to \infty$.
  2. $f_n(x) = x^2/(1+x)^n$. $f(x) = \sum_{n=1}^{\infty} f_n(x)$. If $x = 0$, then $f_n(x) = 0$ for all n and $f(x) = 0$. If $x \neq 0$, we have
  $$f(x) = \sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n} = x^2 \sum_{n=1}^{\infty} X^n = \frac{x^2}{1-X} = \frac{x^2}{1-(1/(1+x^2))} = 1 + x^2$$
  3. Consider $f_m(x) = \lim_{n\to\infty}(\cos(m\pi x))^2 n$. $\lim_{m\to\infty} f_m(x)$ goes to 0 if $x \notin \mathbb{Q}$ and goes to 1 if $x \in \mathbb{Q}$. $f_m \to \chi_{\mathbb{Q}}$, where $\chi_{\mathbb{Q}}$ is the characteristic function of the rationals which is not Riemann integrable (partitions, upper and lower integrals, etc.).
  4. $f_n(x) = \sin nx/\sqrt{n} \to f(x) = 0$ for all x. However, $f_n'(x) = \sqrt{n}\cos nx \nrightarrow 0$
  5. If $0 \leq x \leq 1$, define $f_n(x) = n^2 x(1-x^2)^n$. We know that $f_n(0) = 0$. $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in (0,1]$. We can show that $\int_0^1 x(1-x^2)^n \, \mathrm{d}x = 1/(2n+2)$. Thus, $\int_0^1 f_n(x) \, \mathrm{d}x = n^2/(2n+2)$. Limit of the functions is zero, but their integrals diverge to infinity.
- **Uniformly convergent** (sequence $(f_n)_{n\in\mathbb{N}}$ of functions on E): A sequence of functions $f_n : E \to \mathbb{R}$ such that for all $\epsilon > 0$, there exists N such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$ for all $x \in E$. *Denoted by* $\boldsymbol{f_n \rightrightarrows f}$.
- Theorem: $f_n \rightrightarrows f$ iff $(f_n)_{n\in\mathbb{N}}$ is uniformly Cauchy (i.e., for all $\epsilon > 0$, there exists N such that if $n, m \geq N$ then $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in E$).
- Theorem: Let $M_n = \sup_{x\in E} |f_n(x) - f(x)|$. If $f_n \to f$ pointwise and $M_n \to 0$, then $f_n \rightrightarrows f$.
- Theorem: If $(f_n)_{n\in\mathbb{N}}$ and $|f_n(x)| \leq M_n$, then $\sum f_n \rightrightarrows f$ if $\sum M_n < \infty$.
- Theorem: If E is a compact metric space, $f_n \rightrightarrows f$ in E, x is a limit point of E, and $\lim_{t\to x} f_n(t) = A_n$ exists, then $(A_n)_{n\in\mathbb{N}}$ converges and $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$.

- Corollary: $\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$.

11/16:

  - Fix $\epsilon > 0$. Then $f_n \rightrightarrows f$ implies that there exists some N such that $n, m \geq N$ implies $|f_n(t) - f_m(t)| < \epsilon$ for all $t \in E$.
  - x is a limit point of E and $t \to x$ implies $|A_n - A_m| < \epsilon$. Thus, $(A_n)_{n\in\mathbb{N}}$ is cauchy, so there exists A such that $A_n \to A$.
  - WTS: $|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|$, so we WTS the three terms on the right are small.
  - There exists n such that $|f(t) - f_n(t)| < \epsilon/3$ for all t since $f_n \rightrightarrows f$ by hypothesis.
  - Since t is in a small neighborhood of x, there exists n such that $|A_n - A| < \epsilon/3$.
  - We also have $|f_n(t) - A_n| < \epsilon/3$ by hypothesis.
  - This is a very important proof to understand, because proofs like this pop up often.

- Corollary: f_n continuous and $f_n \rightrightarrows f$ implies f is continuous.

- Theorem: Let K be compact. Assume

  (a) $(f_n)_{n\in\mathbb{N}} \subset C(K) = \{f : K \to \mathbb{R} \mid f \text{ continuous}\}$.
  (b) $f_n \to f$ pointwise in K and $f \in C(K)$.
  (c) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$.

  Then $f_n \rightrightarrows f$.

  - WLOG $f = 0$, $g_n = f_n - f \to 0$, $g_n \geq g_{n+1} \geq 0$.
  - For all $\epsilon > 0$, there exists N such that $n \geq N$ and $0 \leq g_n(x) \leq \epsilon$ for all $x \in K$.
  - $K_n = \{x \in K : g_n(x) \geq \epsilon\}$.
  - g_n continuous implies K_n closed. This combined with K compact implies K_n is compact.
  - g_n decreasing implies $K_n \supset K_{n+1}$. Thus, K_n is a nested family of compact sets, so $\bigcap K_n \neq \emptyset$.
  - This implies that each K_n is nonempty, contradicting the fact that each $g_n \to 0$ for all x.
  - Thus, there exists an N such that K_n is empty for all $n \geq N$. Thus $g_n(x) \leq \epsilon$ for all $x \in K$, $n \geq N$.
  - Note that the compactness of K is important. If $f : (0,1) \to \mathbb{R}$ is defined by $f(x) = 1/(nx+1)$, then $f_n \to 0$, but $f_n \not\rightrightarrows f$.

- Let $C(X) = \{f : X \to \mathbb{R} \mid f \text{ continuous, bounded}\}$ for X a metric space.

- If we define $\|f\| = \sum_{x\in X} |f(x)|$, for $f, g \in C(X)$, we may define $d(f,g) = \|f - g\|$. This definition satisfies the properties of a distance function, and $\|\cdot\|$ is a norm.

  - Thus, $C(X)$ is a complete metric space, a normed space, or, specifically, a **Banach space**.

- Theorem: $(f_n)_{n\in\mathbb{N}} \subset C(X)$ such that $\|f_n - f_m\|_{n,m\to\infty} \to 0$. Then there exists $f \in C(X)$ such that $\|f_n - f\|_{n\to\infty} \to 0$.

  - We get such a strong statement using properties of the image, not properties of the domain.
  - For all $\epsilon > 0$, there exists N such that $n, m \geq N$.
  - $|f_n(x) - f_m(x)| \leq \|f_n - f_m\| < \epsilon$ for all x.
  - Then there exists f such that $f_m(x) \to f(x)$. It follows that $|f_n(x) - f_m(x)| < \epsilon$

- Uniform convergence and integration.

- Stieltjes integral.

  - Define $\alpha : \mathbb{R} \to \mathbb{R}$ nondecreasing.
  - If we sum over the minimums/maximums of a partition times $\alpha(x_{i+1}) - \alpha(x_i)$ instead of $x_{i+1} - x_i$, we obtain said integral as the upper/lower limits just like the Riemann integral.
  - We write $\int_a^b f(x)\,\mathrm{d}\alpha(x)$ where $\mathrm{d}\alpha(x) = \alpha(x)\,\mathrm{d}x$.
- Theorem: If α is nondecreasing on $[a,b]$, $f_n \in R(\alpha)$ such that $f_n \rightrightarrows f$ on $[a,b]$
  - We have

$$\begin{aligned}\left|\int f_n(x)\,\mathrm{d}\alpha(x) - \int f(x)\,\mathrm{d}\alpha(x)\right| &= \left|\int (f_n - f)(x)\,\mathrm{d}\alpha(x)\right| \\ &\leq \|f_n - f\|(\alpha(b) - \alpha(a)) \\ &\leq \int |f_n - f|\,\mathrm{d}\alpha(x) \\ &\leq \int \|f_n - f\|\,\mathrm{d}\alpha(x) \qquad \leq \|f_n - f\| \int_a^b \mathrm{d}\alpha(x) = \|f_n - f\|(\alpha(b) - \alpha(a))\end{aligned}$$

11/19: - Suppose $f_n \to f$ and $f_n' \to g$. When does $f' = g$?
- Theorem: If $f_n : [a,b] \to \mathbb{R}$ is differentiable, $f_n(x_0)$ converges for some $x_0 \in [a,b]$, and f_n' converges uniformly on $[a,b]$, then there exists f differentiable on $[a,b]$ such that $f_n \rightrightarrows f$ and $f_n' \rightrightarrows f'$.
  - Assume the f_n' are continuous. Then $f_n(x) - f_n(x_0) = \int_{x_0}^x f_n'(y)\,\mathrm{d}y$.
  - Since $f_n' \rightrightarrows g$, $\int_{x_0}^x f_n'(y)\,\mathrm{d}y \to \int_{x_0}^x g(y)\,\mathrm{d}y$.
  - It follows since $f_n(x_0) \to f(x_0)$ that $f_n \rightrightarrows f$.
  - By the previous theorem, if

$$f_n'(x) = \lim_{h\to 0} \frac{f_n(x+h) - f_n(x)}{h}$$

    then

$$\lim_{n\to\infty} f_n'(x) = \lim_{h\to 0}\lim_{n\to\infty} \frac{f_n(x+h) - f_n(x)}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

  - Fix $\epsilon > 0$. Then there exists N such that $n, m \geq N$ such that $|f_n(x_0) - f_m(x_0)| < \epsilon/2$ and $|f_n'(t) - f_m'(t)| < \epsilon/2$ for all $t \in [a,b]$.
  - We know that $f_n(t) - f_n(x_0) = \int_{x_0}^t f_n'(y)\,\mathrm{d}y$ and $f_m(t) - f_m(x_0) = \int_{x_0}^t f_m'(y)\,\mathrm{d}y$.
  - Thus,

$$|f_n(t) - f_n(x_0)| \leq |f_n(t) - f_n(x_0)| + |f_m(t) - f_m(x_0)| - |f_m(t) - f_m(x_0)|$$

  - Let $f_n(t) - f_n(x_0) = c_n(t - x_0)$ and $f_m(t) - f_m(x_0) = c_m(t - x_0)$.
  - ...
- Let $f : [a,b] \to \mathbb{R}$ be continuous. What conditions on f imply that f' exists?
- Suppose f is Lipschitz continuous (equivalent to saying there exists $L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$); then f' exists **almost everywhere**.
  - If f differentiable, this is equivalent to saying f bounded.
- **Almost everywhere**: Something happens almost everywhere if the set of places where it doesn't happen has measure zero.
- Suppose f is **Hölder continuous**, then f' does not exist?
- **Hölder continuous** (function f): There exists $L > 0$ such that $|f(y) - f(x)| < L|x - y|^\alpha$ where $\alpha \in (0,1)$

- Suppose f exists such that f is Hölder continuous in a neighborhood of every point in the domain. This function is not anywhere differentiable. Such a function does indeed exist (and it's Brownian motion). The construction of such a function is the essence of Stochastic analysis.
  - Probabilistically: Has mean zero, distributed as a normal function like the Gaussian, and the increments are independent of each other.
  - Analytically: It's a function that is Hölder continuous at half plus ϵ for every ϵ and it is nowhere differentiable.
- Theorem: There exists $f:\mathbb{R}\to\mathbb{R}$ continuous but nowhere differentiable.
  - This theorem is due to Weierstrass and as such, such functions are typically called Weierstrass functions.
- A general class of functions that are nowhere differentiable (not in Rudin (1976); we don't have to prove this).
  - Example 1:
    $$f(x)=\sum_{n=0}^{\infty}a^n\cos(b^n\pi x)$$
    where $0<a<1$, b positive odd integer greater than 1, and $ab>1+\frac{3}{2}\pi$.
    - This function at every point oscillates more and more and more.
- Rudin (1976)'s simple example.
  - $\phi:[-1,1]\to\mathbb{R}$ defined by $\phi(x)=|x|$ is not differentiable at zero.
  - Takes ϕ extends it periodically with period 2, creating a sawtooth function.
  - Repeat the behavior so that the nondifferentiability becomes more and more frequent to get
    $$f(x)=\sum_{1}^{\infty}\left(\frac{3}{4}\right)^n\phi(4^nx)$$
  - This is continuous.
  - Fix any $x\in\mathbb{R}$, $m\in\mathbb{N}$. Then $\delta_m=\pm\frac{1}{2}\cdot 4^{-m}$.
  - Then consider 4^mx, $4^m(x+\delta_m)$.
  - Rudin asserts
    $$\left|\frac{f(x+\delta_m)-f(x)}{\delta_m}\right|\to\infty$$
    as $m\to\infty$ for all x.

11/29:

- Finding a uniformly convergent subsequence of a sequence of functions.
  - Pointwise, uniformly, bounded if there exist M_x such that $|f_n(x)|\leq M_x$ for all n,x. Uniformly bounded if there exists M such that $|f_n(x)|\leq M$ for all n,x.
- Theorem: If $(f_n)_{n\in\mathbb{N}}$ is pointwise bounded and $E\subset X$ is countable, then there exists a subsequence f_{n_k} which converges for every $x\in E$.
  - Let $E=\{x_i:i\in\mathbb{N}\}$. Consider $f_n(x_1)$. $f_{1,k}(x_1)$ converges.
  - $S_1:f_{1,1}(x_1),f_{1,2}(x_1),f_{1,3}(x_1),f_{1,4}(x_1),\dots$.
  - $S_2:f_{2,1}(x_2),f_{2,2}(x_2),f_{2,3}(x_2),f_{2,4}(x_2),\dots$.
  - Now consider $f_{2,k}(x_3)$.
  - $S_3:f_{3,1},f_{3,2},f_{3,3},f_{3,4},\dots$.

  - Continue on and on to $S_4, S_5, \dots$. We know that each of these sequences converges pointwise by hypothesis.
  - Now consider the diagonal sequence $f_{1,1}, f_{2,2}, f_{3,3}, f_{4,4}$.
    - This subsequence of the original sequence we may call g_k.
    - We posit that g_k converges for every $x \in E$.
- Theorem: There exists f_n which is uniformly bounded but does not converge uniformly.
  - Let $f_n(x) = \sin(2\pi x)$ for $0 \leq x \leq 2\pi$.
  - Let $f_n(x) = x^2/(x^2 + (1-nx)^2)$ on $0 \leq x \leq 1$. This sequence is uniformly bounded, converges pointwise, but $f_n(1/n) = 1$ so f_n cannot converge uniformly to zero.
- What does it mean that $f_n : [0,1] \to \mathbb{R}$ does not converge uniformly?
  - It means that there exists a subsequence of the functions evaluated at certain points that is always greater than or equal to some fixed distance away from the limit.
- Equicontinuity: $\mathcal{F}\{f : X \to \mathbb{R}\}$ for (X, d) a metric space is equicontinuous iff for all $\epsilon > 0$, there exists a $\delta > 0$ such that $d(x,y) < \delta$ implies $|f(x) - f(y)| < \epsilon$ for all $x, y \in X$, $f \in \mathcal{F}$.
- Modulus of continuity: $f : X \to \mathbb{R}$ is continuous at x. A modulus of continuity is a function $\omega_X : [0,1] \to [0,1]$ such that $|f(y) - f(x)| \leq \omega_X |y - x|$.
- The final result we'll prove: **Arzelà-Ascoli theorem**: If we have a family of functions on a compact set and we have a dense subset of that set, then if we have a sequence of functions that are equicontinuous, then they converge uniformly.

12/1:

- The final will be in this room.
- The last PSet will not be graded, but there will be similar questions on the final.
- No class on Friday.
- Review from last time:
  - Equiboundedness and equicontinuity.
  - If E is a dense subset of X, then any pointwise bounded sequence has a subsequence that converges on E (diagonal argument.)
- **Equicontinuous** (sequence $\{f_n\}$): For all $\epsilon > 0$, there exists a $\delta > 0$ such that $d(x,y) < \delta$ implies $|f_n(x) - f_n(y)| < \epsilon$ for all n.
- Theorem: If K is a compact set and $\{f_n\} \in C(K)$ converges uniformly on K, then the f_n's are equicontinuous on K.
  - The f_n are uniformly Cauchy: For all $\epsilon > 0$, there exists N such that $n, m \geq N$ imply $\|f_n - f_m\| < \epsilon$ where $\|f_n - f_m\| = \sup_{x \in K}(f_n - f_m)(x)$.
  - If $n \geq N$, then $|f_n(x) - f_n(y)| \leq |f_N(x) - f_N(y)| + 2\|f_n - f_N\|$ (since $f_n(x) - f_n(y) = f_n(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f_n(y)$).
  - Thus $|f_n(x) - f_n(y)| \leq |f_N(x) - f_N(y)| + 2\|f_n - f_N\| < 3\epsilon$ implies $|f_i(x) - f_i(y)| < \epsilon$ if $|x - y| < \delta$ for $i = 1, \dots, N$.
- **Arzelà-Ascoli theorem**: If K is compact, $(f_n)_{n \geq 1} \subset C(K)$ which are pointwise bounded and equicontinuous, then
  (a) $(f_n)_{n \geq 1}$ are uniformly bounded (equicontinuous).
  (b) There exists $(f_{n_k})_{k \geq 1}$ which converges uniformly on K.

- Since K is compact, you can cover it by finitely many balls of radius δ.
- Thus $|f_n(p_k)| \leq M = \max(M_{p_1}, \ldots, M_{p_k})$ where $K \subset \bigcup_{k\in K} B(p_{n_k}, \delta)$.
- K has a countable dense subset E (Exercise 2.25).
- $|f_n(x)| \leq M + \epsilon$ for all x.
- $\{B(x,\delta)\}_{x\in E}$ is an open cover of K.
- Thus has a finite subcover.
- ...

- If $f_n : \mathbb{R}^n \to \mathbb{R}$ are continuous, equibounded, equicontinuous, then there exists f_{n_k} which converges locally uniformly to some $f : \mathbb{R}^n \to \mathbb{R}$.
- How do you learn math?
  - In an ideal world, study by looking at theorems, thinking that you should be able to prove it, and etc.
  - Since we don't have the time to do everything ourselves, don't just get stuck in a place; move on and continue thinking if you have to.
- Let $\dot{\phi} = f(x,t)$ and $x(0) = c$. Let $\phi : \mathbb{R} \times [0,1] \to \mathbb{R}$. Assume ϕ is bounded and continuous. Then there exists a solution of the differential equation and initial condition.
  - We need to find a function $x : [0,1] \to \mathbb{R}$ continuous such that $x(t) = c + \int_0^t \phi(x(s),s)\,\mathrm{d}s$.
  - Let $t_i = i/N$. Then $x_n(t) = \phi(x_i, t_i)$ on $t_i < t < t_{i+1}$.
  - $x_n(t) = x_n(t_i) + \phi(x_i, t_i)(t - t_i)$.
  - $\frac{x_{i+1}-x_i}{1/N} = \phi(x_i, t_i)$.
  - $\Delta_n(t) = x_n'(t) - \phi(x_n(t), t)$ for $\phi(x_i, t_i) - \phi(x_n(t), t)$ measures how close our solution is.
  - All of these things imply that our final formula is

$$x_n(t) = c + \int_0^t [\phi(x_n(s), s) + \Delta_n(s)]\,\mathrm{d}s$$

  - If we know that $x_n \rightrightarrows x$, then $\Delta_n \rightrightarrows 0$.
  - We then use the A-A theorem to imply convergence.
- When we get to MATH 208, say we didn't do any multivariable calculus in MATH 207.
  - We didn't do how to integrate in $\mathbb{R}^n$, how to integrate by parts (Stoke's theorem), Lagrange multipliers (constraint minimization).
- Problem 4.23: Show the inequalities at the bottom first and then use those to show continuity.
  - Consider $\lim_{t\to u} f(t)$. Approach from two sides separately and show cancellation??? Chloe will write a solution.
  - This is a particular trick for convex functions; it's not exactly recyclable.
- The trick for Problem 4.26 is recyclable.
- Linear algebra questions on the final are easier than the midterm.
  - The last question will be the hard one this time.

7.2 Sequences and Series of Functions

From Rudin (1976).

12/6:

- Let f be a complex-valued function.
- **Limit** (of $\{f_n\}$): The function $f : E \to \mathbb{C}$ defined by
$$f(x) = \lim_{n\to\infty} f_n(x)$$
for all $x \in E$, where $\{f_n\}$ is a sequence of functions such that $\lim_{n\to\infty} f_n(x)$ exists for all $x \in E$. *Also known as* **limit function**.
- **Sum** (of $\{f_n\}$): The function $f : E \to \mathbb{C}$ defined by
$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$
for all $x \in E$, where $\{f_n\}$ is a sequence of functions such that $\sum_{n=1}^{\infty} f_n(x)$ exists for all $x \in E$.
- Motivation for this chapter: Which properties of functions are preserved under the limit and summation operations?
- Continuity example:
  - A function is continuous at x if $\lim_{t\to x} f(t) = f(x)$.
  - Hence, the limit of a sequence of continuous functions is continuous at x if
$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$$
  - This switching of the limits is not always possible: If $s_{m,n} = m/(m+n)$, then
$$\lim_{n\to\infty}\lim_{m\to\infty} s_{m,n} = 1 \neq 0 = \lim_{m\to\infty}\lim_{n\to\infty} s_{m,n}$$
  - For an example of a sequence of continuous functions converging to a discontinuous function, see the example 2 (11/15 class notes).
- **Pointwise convergent** (sequence $\{f_n\}$): A sequence of functions $\{f_n\}$ for which there exists a function f such that for every $\epsilon > 0$ and for every $x \in E$, there exists an integer N such that if $n \geq N$, then
$$|f_n(x) - f(x)| < \epsilon$$
- **Uniformly convergent** (sequence $\{f_n\}$): A sequence of functions $\{f_n\}$ for which there exists a function f such that for every $\epsilon > 0$, there exists an integer N such that if $n \geq N$, then
$$|f_n(x) - f(x)| < \epsilon$$
for all $x \in E$.
- Theorem 7.8: Cauchy criterion for uniform convergence.
- Theorem 7.9: If $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in E$ and $M_n = \sup_{x\in E} |f_n(x) - f(x)|$, then $f_n \to f$ uniformly on E iff $M_n \to 0$ as $n \to \infty$.
- Theorem 7.10 (Weierstrass M-test): $|f_n(x)| \leq M_n$ for all $x \in E$ and $\sum M_n$ converges implies $\sum f_n$ converges uniformly.

- Theorem 7.11: Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E, and suppose that
$$\lim_{t\to x} f_n(t) = A_n$$
for each $n \in \mathbb{N}$. Then $\{A_n\}$ converges and
$$\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$$

  Proof. See IBL Theorem 17.6. □

  - It follows that in this case,
  $$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$$
- Theorem 7.12: A uniformly convergent sequence of continuous functions converges to a continuous function.
- Theorem 7.13: K compact, $\{f_n\}$ a sequence of continuous functions on K that converges pointwise to a continuous function f on K, and $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, $n \in \mathbb{N}$ implies $f_n \to f$ uniformly on K.
- $\boldsymbol{\mathscr{C}(X)}$: The set of all complex-valued, continuous, bounded functions with domain X a metric space.
- **Supremum norm** (of $f \in \mathscr{C}(X)$): The following value. *Denoted by* $\boldsymbol{\|f\|}$. *Given by*
$$\|f\| = \sup_{x\in X} |f(x)|$$
- Properties of the supremum norm.
  - $\|f\| < \infty$ (f is bounded).
  - $\|f\| = 0$ iff $f = 0$.
  - $\|f+g\| \leq \|f\| + \|g\|$.
- The above properties plus the definition $d(f,g) = \|f-g\|$ for any $f, g \in \mathscr{C}(X)$ makes $\mathscr{C}(X)$ a metric space!
- Rephrasing Theorem 7.9: A sequence $\{f_n\}$ converges to f with respect to the metric of $\mathscr{C}(X)$ if and only if $f_n \to f$ uniformly on X.
  - Thus, closed subsets $\mathscr{A} \subset \mathscr{C}(X)$ are sometimes called **uniformly closed**.
  - The closure of a subset $\mathscr{A} \subset \mathscr{C}(X)$ can similarly be called the **uniform closure**.
- Theorem 7.15: The above metric makes $\mathscr{C}(X)$ into a complete metric space.
- Theorem 7.16: $f_n \in \mathscr{R}(\alpha)$ on $[a,b]$ and $f_n \to f$ uniformly on $[a,b]$ imply $f \in \mathscr{R}(\alpha)$ on $[a,b]$ and
$$\int_a^b f \,\mathrm{d}\alpha = \lim_{n\to\infty}\int_a^b f_n \,\mathrm{d}\alpha$$

  Proof. See IBL Theorem 17.7; Rudin (1976)'s is much slicker though. □

- Theorem 7.17: Suppose $\{f_n\}$ are differentiable on $[a,b]$, $\{f_n(x_0)\}$ converges for some $x_0 \in [a,b]$, and $\{f_n'\}$ converges uniformly on $[a,b]$. Then $\{f_n\}$ converges uniformly on $[a,b]$ to a function f such that
$$f'(x) = \lim_{n\to\infty} f_n'(x)$$
for all $x \in [a,b]$.

Proof. Long and complicated, and given in class. □

- Assuming the continuity of the $\{f_n'\}$ admits a proof like IBL Theorem 17.8.

- Theorem 7.18: There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ which is nowhere differentiable.

Proof. Long and complicated, and given in class. □

- **Pointwise bounded** (sequence $\{f_n\}$): A sequence of functions $\{f_n\}$ for which there exists a finite-valued function ϕ defined on E such that $|f_n(x)| < \phi(x)$ for all $x \in E$.
- **Uniformly bounded** (sequence $\{f_n\}$): A sequence of functions $\{f_n\}$ for which there exists a number M such that $|f_n(x)| < M$ for all $x \in E$.
- **Equicontinuous** (family of functions $\mathscr{F}$): A family of functions $\mathscr{F}$ such that for every $\epsilon > 0$, there exists a $\delta > 0$ such that
$$|f(x) - f(y)| < \epsilon$$
whenever $d(x, y) < \delta$, $x, y \in E$, and $f \in \mathscr{F}$.
  - Every member of an equicontinuous family is uniformly continuous.
- Theorem 7.23: $\{f_n\}$ pointwise bounded and defined on a countable set E implies $\{f_n\}$ has a pointwise convergent subsequence $\{f_{n_k}\}$.

Proof. Diagonalization argument from class. □

- Theorem 7.24: K compact and $\{f_n\} \subset \mathscr{C}(K)$ uniformly convergent implies $\{f_n\}$ equicontinuous on K.
- Theorem 7.25 (Arzelà-Ascoli Theorem): If K is compact and $\{f_n\} \subset \mathscr{C}(K)$ is pointwise bounded and equicontinuous on K, then
  1. $\{f_n\}$ is uniformly bounded on K;
  2. $\{f_n\}$ contains a uniformly convergent subsequence.
- Theorem 7.26 (Weierstrass Approximation Theorem): f continuous on $[a, b]$ implies there exists a sequence of polynomials P_n such that
$$\lim_{n\to\infty} P_n(x) = f(x)$$
uniformly on $[a, b]$.
- Corollary 7.27: For every interval $[-a, a]$, there is a sequence of real polynomials $\{P_n\}$ such that $P_n(0) = 0$ and such that
$$\lim_{n\to\infty} P_n(x) = |x|$$
uniformly on $[-a, a]$.
- **Algebra**: A family $\mathscr{A}$ of complex functions defined on a set E such that for all $f, g \in \mathscr{A}$ and $c \in \mathbb{C}$,
  (i) $f + g \in \mathscr{A}$;
  (ii) $fg \in \mathscr{A}$;
  (iii) $cf \in \mathscr{A}$.
- **Uniformly closed** (algebra $\mathscr{A}$): An algebra $\mathscr{A}$ such that if $\{f_n\} \subset \mathscr{A}$ and $f_n \to f$ uniformly on E, then $f \in \mathscr{A}$.
- **Uniform closure** (of an algebra $\mathscr{A}$): The set $\mathscr{B}$ of all functions which are limits of uniformly convergent sequences of members of $\mathscr{A}$.

- Rephrasing Theorem 7.26: The set of all continuous functions on $[a, b]$ is the uniform closure of the set of polynomials on $[a, b]$.
- Theorem 7.29: Let $\mathscr{B}$ be the uniform closure of an algebra $\mathscr{A}$ of bounded functions. Then $\mathscr{B}$ is a uniformly closed algebra.
- **Separating points** (family $\mathscr{A}$ on E): A family of functions $\mathscr{A}$ on a set E such that to every pair of distinct points $x_1, x_2 \in E$, there corresponds a function $f \in \mathscr{A}$ such that $f(x_1) \neq f(x_2)$.
    - Example of a family that separates points on $\mathbb{R}^1$: the algebra of all polynomials in one variable.
    - Example of a family that does not separate points on $[-1, 1]$: the set of all even polynomials in one variable (since $f(x) = f(-x)$ for every even function f).
- **Vanishing at no point** (family $\mathscr{A}$ on E): A family of functions $\mathscr{A}$ such that to each $x \in E$, there corresponds a function $g \in \mathscr{A}$ such that $g(x) \neq 0$.
- Theorem 7.31: $\mathscr{A}$ an algebra on E that separates points and vanishes at no point, $x_1, x_2 \in E$ distinct, and $c_1, c_2 \in \mathbb{C}$ imply $\mathscr{A}$ contains a function f such that

$$f(x_1) = c_1 \qquad\qquad f(x_2) = c_2$$

- Theorem 7.32 (Stone-Weierstrass Theorem): $\mathscr{A}$ an algebra of real continuous functions on K compact that separates points of K and vanishes at no point of K implies the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all real continuous functions on K.
    - Theorem 7.32 holds on complex algebras with the additional hypothesis that $\mathscr{A}$ is **self-adjoint** (see Theorem 7.33).
- **Self-adjoint** (algebra $\mathscr{A}$): An algebra $\mathscr{A}$ such that if $f \in \mathscr{A}$, then its **complex conjugate** $\bar{f} \in \mathscr{A}$.
- **Complex conjugate** (of f): The function $\bar{f}$ defined by $\bar{f}(x) = \overline{f(x)}$.
- Theorem 7.33: $\mathscr{A}$ a self-adjoint algebra of complex continuous functions on K compact that separates points of K and vanishes at no point of K implies the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all complex continuous functions on K.
    - In other words, $\mathscr{A}$ is dense in $\mathscr{C}(K)$.

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