Chapter 7

Sequences and Series of Functions

7.1 Notes

- Soug will not test on differentiation/integration assuming that we know them already.

- **Pointwise convergent** (sequence $(f_n)_{n\in\mathbb{N}}$ of functions): A sequence of functions $f_n : E \to \mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in E$.

- Can we interchange "limit" in the above definition with continuity, convergence of series, integration, differentiation, etc.?

- Examples with negative answer:

  1. Interchanging limits: Let $S_{mn} = \frac{m}{m+n}$. $S_{mn} \to 1$ as $m \to \infty$, and $S_{mn} \to 0$ as $n \to \infty$.

  2. $f_n(x) = x^2/(1+x)^n$. $f(x) = \sum_{n=1}^{\infty} f_n(x)$. If $x = 0$, then $f_n(x) = 0$ for all n and $f(x) = 0$. If $x \neq 0$, we have

  $$f(x) = \sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n} = x^2 \sum_{n=1}^{\infty} X^n = \frac{x^2}{1-X} = \frac{x^2}{1-(1/(1+x^2))} = 1 + x^2$$

  3. Consider $f_m(x) = \lim_{n\to\infty}(\cos(m\pi x))^2 n$. $\lim_{m\to\infty} f_m(x)$ goes to 0 if $x \notin \mathbb{Q}$ and goes to 1 if $x \in \mathbb{Q}$. $f_m \to \chi_{\mathbb{Q}}$, where $\chi_{\mathbb{Q}}$ is the characteristic function of the rationals which is not Riemann integrable (partitions, upper and lower integrals, etc.).

  4. $f_n(x) = \sin nx/\sqrt{n} \to f(x) = 0$ for all x. However, $f'_n(x) = \sqrt{n} \cos nx \nrightarrow 0$

  5. If $0 \leq x \leq 1$, define $f_n(x) = n^2 x(1-x^2)^n$. We know that $f_n(0) = 0$. $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in (0,1]$. We can show that $\int_0^1 x(1-x^2)^n \, \mathrm{d}x = 1/(2n+2)$. Thus, $\int_0^1 f_n(x) \, \mathrm{d}x = n^2/(2n+2)$. Limit of the functions is zero, but their integrals diverge to infinity.

- **Uniformly convergent** (sequence $(f_n)_{n\in\mathbb{N}}$ of functions on E): A sequence of functions $f_n : E \to \mathbb{R}$ such that for all $\epsilon > 0$, there exists N such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$ for all $x \in E$. *Denoted by $\boldsymbol{f_n \rightrightarrows f}$.*

- Theorem: $f_n \rightrightarrows f$ iff $(f_n)_{n\in\mathbb{N}}$ is uniformly Cauchy (i.e., for all $\epsilon > 0$, there exists N such that if $n, m \geq N$ then $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in E$).

  - Let $M_n = \sup_{x\in E} |f_n(x) - f(x)|$. If $f_n \to f$ pointwise, then $f_n \rightrightarrows f$ if $M_n \to 0$.

- Theorem: If $(f_n)_{n\in\mathbb{N}}$ and $|f_n(x)| \leq M_n$, then $\sum f_n \rightrightarrows f$ if $\sum M_n < \infty$.

- Theorem: If E is a compact metric space, $f_n \rightrightarrows f$ in E, x is a limit point of E, and $\lim_{t\to x} f_n(t) = A_n$ exists, then $(A_n)_{n\in\mathbb{N}}$ converges and $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$.

- Corollary: $\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t)$.

11/16:
  - Fix $\epsilon > 0$. Then $f_n \rightrightarrows f$ implies that there exists some N such that $n, m \geq N$ implies $|f_n(t) - f_m(t)| < \epsilon$ for all $t \in E$.
  - x is a limit point of E and $t \to x$ implies $|A_n - A_m| < \epsilon$. Thus, $(A_n)_{n \in \mathbb{N}}$ is cauchy, so there exists A such that $A_n \to A$.
  - WTS: $|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|$, so we WTS the three terms on the right are small.
  - There exists n such that $|f(t) - f_n(t)| < \epsilon/3$ for all t since $f_n \rightrightarrows f$ by hypothesis.
  - Since t is in a small neighborhood of x, there exists n such that $|A_n - A| < \epsilon/3$.
  - We also have $|f_n(t) - A_n| < \epsilon/3$ by hypothesis.
  - This is a very important proof to understand, because proofs like this pop up often.

- Corollary: f_n continuous and $f_n \rightrightarrows f$ implies f is continuous.

- Theorem: Let K be compact. Assume

  (a) $(f_n)_{n \in \mathbb{N}} \subset C(K) = \{f : K \to \mathbb{R} \mid f \text{ continuous}\}$.
  (b) $f_n \to f$ pointwise in K and $f \in C(K)$.
  (c) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$.

  Then $f_n \rightrightarrows f$.

  - WLOG $f = 0$, $g_n = f_n - f \to 0$, $g_n \geq g_{n+1} \geq 0$.
  - For all $\epsilon > 0$, there exists N such that $n \geq N$ and $0 \leq g_n(x) \leq \epsilon$ for all $x \in K$.
  - $K_n = \{x \in K : g_n(x) \geq \epsilon\}$.
  - g_n continuous implies K_n closed. This combined with K compact implies K_n is compact.
  - g_n decreasing implies $K_n \supset K_{n+1}$. Thus, K_n is a nested family of compact sets, so $\bigcap K_n \neq \emptyset$.
  - This implies that each K_n is nonempty, contradicting the fact that each $g_n \to 0$ for all x.
  - Thus, there exists an N such that K_n is empty for all $n \geq N$. Thus $g_n(x) \leq \epsilon$ for all $x \in K$, $n \geq N$.
  - Note that the compactness of K is important. If $f : (0,1) \to \mathbb{R}$ is defined by $f(x) = 1/(nx+1)$, then $f_n \to 0$, but $f_n \not\rightrightarrows f$.

- Let $C(X) = \{f : X \to \mathbb{R} \mid f \text{ continuous, bounded}\}$ for X a metric space.

- If we define $\|f\| = \sum_{x \in X} |f(x)|$, for $f, g \in C(X)$, we may define $d(f, g) = \|f - g\|$. This definition satisfies the properties of a distance function, and $\|\cdot\|$ is a norm.

  - Thus, $C(X)$ is a complete metric space, a normed space, or, specifically, a **Banach space**.

- Theorem: $(f_n)_{n \in \mathbb{N}} \subset C(X)$ such that $\|f_n - f_m\|_{n,m \to \infty} \to 0$. Then there exists $f \in C(X)$ such that $\|f_n - f\|_{n \to \infty} \to 0$.

  - We get such a strong statement using properties of the image, not properties of the domain.
  - For all $\epsilon > 0$, there exists N such that $n, m \geq N$.
  - $|f_n(x) - f_m(x)| \leq \|f_n - f_m\| < \epsilon$ for all x.
  - Then there exists f such that $f_m(x) \to f(x)$. It follows that $|f_n(x) - f_m(x)| < \epsilon$

- Uniform convergence and integration.

- Stieltjes integral.

- – Define $\alpha : \mathbb{R} \to \mathbb{R}$ nondecreasing.
- – If we sum over the minimums/maximums of a partition times $\alpha(x_{i+1}) - \alpha(x_i)$ instead of $x_{i+1} - x_i$, we obtain said integral as the upper/lower limits just like the Riemann integral.
- – We write $\int_a^b f(x) \, d\alpha(x)$ where $d\alpha(x) = \alpha(x) \, dx$.

- • Theorem: If α is nondecreasing on $[a, b]$, $f_n \in R(\alpha)$ such that $f_n \rightrightarrows f$ on $[a, b]$

  - – We have

$$\left| \int f_n(x) \, d\alpha(x) - \int f(x) \, d\alpha(x) \right| = \left| \int (f_n - f)(x) \, d\alpha(x) \right|$$

$$\leq \|f_n - f\|(\alpha(b) - \alpha(a))$$

$$\leq \int |f_n - f| \, d\alpha(x)$$

$$\leq \int \|f_n - f\| \, d\alpha(x) \qquad \leq \|f_n - f\| \int_a^b d\alpha(x) = \|f_n - f\|(\alpha(b) - \alpha(a))$$

11/19:

- • Suppose $f_n \to f$ and $f_n' \to g$. When does $f' = g$?

- • Theorem: If $f_n : [a, b] \to \mathbb{R}$ is differentiable, $f_n(x_0)$ converges for some $x_0 \in [a, b]$, and f_n' converges uniformly on $[a, b]$, then there exists f differentiable on $[a, b]$ such that $f_n \rightrightarrows f$ and $f_n' \rightrightarrows f'$.

  - – Assume the f_n' are continuous. Then $f_n(x) - f_n(x_0) = \int_{x_0}^x f_n'(y) \, dy$.
  - – Since $f_n' \rightrightarrows g$, $\int_{x_0}^x f_n'(y) \, dy \to \int_{x_0}^x g(y) \, dy$.
  - – It follows since $f_n(x_0) \to f(x_0)$ that $f_n \rightrightarrows f$.
  - – By the previous theorem, if

$$f_n'(x) = \lim_{h \to 0} \frac{f_n(x+h) - f_n(x)}{h}$$

  then

$$\lim_{n \to \infty} f_n'(x) = \lim_{h \to 0} \lim_{n \to \infty} \frac{f_n(x+h) - f_n(x)}{h} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

  - – Fix $\epsilon > 0$. Then there exists N such that $n, m \geq N$ such that $|f_n(x_0) - f_m(x_0)| < \epsilon/2$ and $|f_n'(t) - f_m'(t)| < \epsilon/2$ for all $t \in [a, b]$.
  - – We know that $f_n(t) - f_n(x_0) = \int_{x_0}^t f_n'(y) \, dy$ and $f_m(t) - f_m(x_0) = \int_{x_0}^t f_m'(y) \, dy$.
  - – Thus,

$$|f_n(t) - f_n(x_0)| \leq |f_n(t) - f_n(x_0)| + |f_m(t) - f_m(x_0)| - |f_m(t) - f_m(x_0)|$$

  - – Let $f_n(t) - f_n(x_0) = c_n(t - x_0)$ and $f_m(t) - f_m(x_0) = c_m(t - x_0)$.
  - – ...

- • Let $f : [a, b] \to \mathbb{R}$ be continuous. What conditions on f imply that f' exists?

- • Suppose f is Lipschitz continuous (equivalent to saying there exists $L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$); then f' exists **almost everywhere**.

  - – If f differentiable, this is equivalent to saying f bounded.

- • **Almost everywhere**: Something happens almost everywhere if the set of places where it doesn't happen has measure zero.

- • Suppose f is **Hölder continuous**, then f' does not exist?

- • **Hölder continuous** (function f): There exists $L > 0$ such that $|f(y) - f(x)| < L|x - y|^\alpha$ where $\alpha \in (0, 1)$

- Suppose f exists such that f is Hölder continuous in a neighborhood of every point in the domain. This function is not anywhere differentiable. Such a function does indeed exist (and it's Brownian motion). The construction of such a function is the essence of Stochastic analysis.

  – Probabilistically: Has mean zero, distributed as a normal function like the Gaussian, and the increments are independent of each other.

  – Analytically: It's a function that is Hölder continuous at half plus ϵ for every ϵ and it is nowhere differentiable.

- Theorem: There exists $f : \mathbb{R} \to \mathbb{R}$ continuous but nowhere differentiable.

  – This theorem is due to Weierstrass and as such, such functions are typically called Weierstrass functions.

- A general class of functions that are nowhere differentiable (not in Rudin (1976); we don't have to prove this).

  – Example 1:
  $$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$
  where $0 < a < 1$, b positive odd integer greater than 1, and $ab > 1 + \frac{3}{2}\pi$.

    ■ This function at every point oscillates more and more and more.

- Rudin (1976)'s simple example.

  – $\phi : [-1, 1] \to \mathbb{R}$ defined by $\phi(x) = |x|$ is not differentiable at zero.

  – Takes ϕ extends it periodically with period 2, creating a sawtooth function.

  – Repeat the behavior so that the nondifferentiability becomes more and more frequent to get
  $$f(x) = \sum_{1}^{\infty} \left(\frac{3}{4}\right)^n \phi(4^n x)$$

  – This is continuous.

  – Fix any $x \in \mathbb{R}$, $m \in \mathbb{N}$. Then $\delta_m = \pm \frac{1}{2} \cdot 4^{-m}$.

  – Then consider $4^m x$, $4^m(x + \delta_m)$.

  – Rudin asserts
  $$\left| \frac{f(x + \delta_m) - f(x)}{\delta_m} \right| \to \infty$$
  as $m \to \infty$ for all x.

11/29:
- Finding a uniformly convergent subsequence of a sequence of functions.

  – Pointwise, uniformly, bounded if there exist M_x such that $|f_n(x)| \le M_x$ for all n, x. Uniformly bounded if there exists M such that $|f_n(x)| \le M$ for all n, x.

- Theorem: If $(f_n)_{n \in \mathbb{N}}$ is pointwise bounded and $E \subset X$ is countable, then there exists a subsequence f_{n_k} which converges for every $x \in E$.

  – Let $E = \{x_i : i \in \mathbb{N}\}$. Consider $f_n(x_1)$. $f_{1,k}(x_1)$ converges.

  – $S_1 : f_{1,1}(x_1), f_{1,2}(x_1), f_{1,3}(x_1), f_{1,4}(x_1), \ldots$.

  – $S_2 : f_{2,1}(x_2), f_{2,2}(x_2), f_{2,3}(x_2), f_{2,4}(x_2), \ldots$.

  – Now consider $f_{2,k}(x_3)$.

  – $S_3 : f_{3,1}, f_{3,2}, f_{3,3}, f_{3,4}, \ldots$.

- Continue on and on to $S_4, S_5, \ldots$. We know that each of these sequences converges pointwise by hypothesis.
    - Now consider the diagonal sequence $f_{1,1}, f_{2,2}, f_{3,3}, f_{4,4}$.
        - ■ This subsequence of the original sequence we may call g_k.
        - ■ We posit that g_k converges for every $x \in E$.

- Theorem: There exists f_n which is uniformly bounded but does not converge uniformly.

    - Let $f_n(x) = \sin(2\pi x)$ for $0 \le x \le 2\pi$.
    - Let $f_n(x) = x^2/(x^2 + (1 - nx)^2)$ on $0 \le x \le 1$. This sequence is uniformly bounded, converges pointwise, but $f_n(1/n) = 1$ so f_n cannot converge uniformly to zero.

- What does it mean that $f_n : [0, 1] \to \mathbb{R}$ does not converge uniformly?

    - It means that there exists a subsequence of the functions evaluated at certain points that is always greater than or equal to some fixed distance away from the limit.

- Equicontinuity: $\mathcal{F}\{f : X \to \mathbb{R}\}$ for (X, d) a metric space is equicontinuous iff for all $\epsilon > 0$, there exists a $\delta > 0$ such that $d(x, y) < \delta$ implies $|f(x) - f(y)| < \epsilon$ for all $x, y \in X$, $f \in \mathcal{F}$.

- Modulus of continuity: $f : X \to \mathbb{R}$ is continuous at x. A modulus of continuity is a function $\omega_X : [0, 1] \to [0, 1]$ such that $|f(y) - f(x)| \le \omega_X |y - x|$.

- The final result we'll prove: **Arzelà-Ascoli theorem**: If we have a family of functions on a compact set and we have a dense subset of that set, then if we have a sequence of functions that are equicontinuous, then they converge uniformly.

12/1:
- The final will be in this room.

- The last PSet will not be graded, but there will be similar questions on the final.

- No class on Friday.

- Review from last time:

    - Equiboundedness and equicontinuity.
    - If E is a dense subset of X, then any pointwise bounded sequence has a subsequence that converges on E (diagonal argument.)

- **Equicontinuous** (sequence $\{f_n\}$): For all $\epsilon > 0$, there exists a $\delta > 0$ such that $d(x, y) < \delta$ implies $|f_n(x) - f_n(y)| < \epsilon$ for all n.

- Theorem: If K is a compact set and $\{f_n\} \in C(K)$ converges uniformly on K, then the f_n's are equicontinuous on K.

    - The f_n are uniformly Cauchy: For all $\epsilon > 0$, there exists N such that $n, m \ge N$ imply $\|f_n - f_m\| < \epsilon$ where $\|f_n - f_m\| = \sup_{x \in K}(f_n - f_m)(x)$.
    - If $n \ge N$, then $|f_n(x) - f_n(y)| \le |f_N(x) - f_N(y)| + 2\|f_n - f_N\|$ (since $f_n(x) - f_n(y) = f_n(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f_n(y)$).
    - Thus $|f_n(x) - f_n(y)| \le |f_N(x) - f_N(y)| + 2\|f_n - f_N\| < 3\epsilon$ implies $|f_i(x) - f_i(y)| < \epsilon$ if $|x - y| < \delta$ for $i = 1, \ldots, N$.

- **Arzelà-Ascoli theorem**: If K is compact, $(f_n)_{n \ge 1} \subset C(K)$ which are pointwise bounded and equicontinuous, then

    (a) $(f_n)_{n \ge 1}$ are uniformly bounded (equicontinuous).
    (b) There exists $(f_{n_k})_{k \ge 1}$ which converges uniformly on K.

- Since K is compact, you can cover it by finitely many balls of radius δ.
- Thus $|f_n(p_k)| \leq M = \max(M_{p_1}, \ldots, M_{p_k})$ where $K \subset \bigcup_{k \in K} B(p_{n_k}, \delta)$.
- K has a countable dense subset E (Exercise 2.25).
- $|f_n(x)| \leq M + \epsilon$ for all x.
- $\{B(x, \delta)\}_{x \in E}$ is an open cover of K.
- Thus has a finite subcover.
- ...

- If $f_n : \mathbb{R}^n \to \mathbb{R}$ are continuous, equibounded, equicontinuous, then there exists f_{n_k} which converges locally uniformly to some $f : \mathbb{R}^n \to \mathbb{R}$.

- How do you learn math?

  - In an ideal world, study by looking at theorems, thinking that you should be able to prove it, and etc.
  - Since we don't have the time to do everything ourselves, don't just get stuck in a place; move on and continue thinking if you have to.

- Let $\dot{\phi} = f(x, t)$ and $x(0) = c$. Let $\phi : \mathbb{R} \times [0, 1] \to \mathbb{R}$. Assume ϕ is bounded and continuous. Then there exists a solution of the differential equation and initial condition.

  - We need to find a function $x : [0, 1] \to \mathbb{R}$ continuous such that $x(t) = c + \int_0^t \phi(x(s), s)\, ds$.
  - Let $t_i = i/N$. Then $x_n(t) = \phi(x_i, t_i)$ on $t_i < t < t_{i+1}$.
  - $x_n(t) = x_n(t_i) + \phi(x_i, t_i)(t - t_i)$.
  - $\frac{x_{i+1} - x_i}{1/N} = \phi(x_i, t_i)$.
  - $\Delta_n(t) = x'_n(t) - \phi(x_n(t), t)$ for $\phi(x_i, t_i) - \phi(x_n(t), t)$ measures how close our solution is.
  - All of these things imply that our final formula is

$$x_n(t) = c + \int_0^t [\phi(x_n(s), s) + \Delta_n(s)]\, ds$$

  - If we know that $x_n \rightrightarrows x$, then $\Delta_n \rightrightarrows 0$.
  - We then use the A-A theorem to imply convergence.

- When we get to MATH 208, say we didn't do any multivariable calculus in MATH 207.

  - We didn't do how to integrate in $\mathbb{R}^n$, how to integrate by parts (Stoke's theorem), Lagrange multipliers (constraint minimization).