

Chapter 9

Advanced Spectral Theory

9.1 Notes

- 10/22:
- Let $p(z) = \sum_{i=0}^n a_i z^i$ be a polynomial. Let A be an $n \times n$ matrix. We let $p(A) = \sum_{i=0}^n a_i A^i$.
 - Theorem: If A is an $n \times n$ and $p(\lambda) = \det(A - \lambda I)$, then $p(A) = 0$.
 - We know that $p(\lambda) = a(z - \lambda_1) \cdots (z - \lambda_n)$ where $\lambda_1, \dots, \lambda_n$ are the eigenvalues.
 - Thus $p(A) = a(A - \lambda_1 I) \cdots (A - \lambda_n I)$.
 - If you are in \mathbb{R}^n and have this property, you can factorize your matrix.
 - Thus, $p(A)\mathbf{x} = \mathbf{0}$ since \mathbf{x} can be decomposed into a linear combination of eigenvectors of A , which will be taken to 0 one by one by the terms of $p(A)$.
 - $\sigma(B) = \{\text{eigenvalues of } B\}$ is known as the **spectrum** of B .
 - If p is an arbitrary polynomial and A is $n \times n$, then μ is an eigenvalue of $p(A)$ if and only if $\mu = p(\lambda)$ where λ is an eigenvalue of A . In essence, $\sigma(p(A)) = p(\sigma(A))$.
 - Chapter 9 will not be on the exam. We don't have to know the generalization to infinite dimensional spaces.
- 10/25:
- If A is an $n \times n$ square matrix and $p(\lambda) = \det(A - \lambda I)$, then $p(A) = 0$.
 - Proof: WLOG, let A be an upper triangular matrix with diagonal entries equal to the eigenvalues.
 - Think of $p(z) = (-1)^n (z - \lambda_1) \cdots (z - \lambda_n)$.
 - Thus, $p(A) = (-1)^n (A - \lambda_1 I) \cdots (A - \lambda_n I)$.
 - WTS: $p(A)\mathbf{x} = 0$ for all $\mathbf{x} \in V$.
 - Let $E_k = \text{span}(e_1, \dots, e_k)$ be the span of the first k eigenvectors of A , where e_1, \dots, e_n is a standard basis in \mathbb{C}^n .
 - A triangular implies $AE_k \subset E_k$. Thus, $(A - \lambda I)E_k \subset E_k$, so E_k is invariant under $A - \lambda I$ for all λ .
 - If we apply $A - \lambda_k I$ to a vector in E_k , we are left with a vector in E_{k-1} .
 - Thus, if we apply $\prod_{k=1}^n (A - \lambda_k I) = p(A)$ to any vector in $E_n = V$, we will kill it piece by piece down to zero.
 - Let A be a square $n \times n$ matrix. Then p an arbitrary polynomial implies $\sigma(p(A)) = p(\sigma(A))$. (Any eigenvalue μ of $p(A)$ is $\mu = p(\lambda)$, where λ is an eigenvalue of A .)
 - Shows that polynomials of operators commute.

- Proof: Let λ be an eigenvalue of A . We want to show that $p(\lambda)$ is an eigenvalue of $p(A)$. This is obvious since $A\mathbf{x} = \lambda\mathbf{x}$ for some \mathbf{x} , so $A^k\mathbf{x} = \lambda^k\mathbf{x}$, so in particular, $p(A)\mathbf{x} = p(\lambda)\mathbf{x}$.
- On the other hand, if μ is an eigenvalue of $p(A)$, we want to show that there exists $\lambda \in \sigma(A)$ such that $\mu = p(\lambda)$.
- Consider $q(z) = p(z) - \mu$. Then $q(A) = p(A) - \mu I$. Since μ is an eigenvalue of $p(A)$, $q(A)$ is not invertible.
- Thus, $q(z) = (-1)^n(z - z_1) \cdots (z - z_n)$ and $q(A) = (-1)^n(A - z_1 I) \cdots (A - z_n I)$.
- But $q(A)$ is not invertible, so one of the $A - z_k I$ is not invertible. Take z_k such that $A - z_k I$ is not invertible. Then $z_k \in \sigma(A)$. It follows that $q(z_k) = p(z_k) - \mu = 0$.
- If A is $n \times n$, $\lambda_1, \dots, \lambda_n$ are its eigenvalues, p is a polynomial, then $p(A)$ is invertible if and only if $p(\lambda_k) \neq 0$ for each $k = 1, \dots, n$.
 - This is an immediate corollary to the previous result.
- We now build up to the **generalized eigenspace**, which is related to some “geometric” properties of the algebraic multiplicity of an eigenvalue.
- If $A : V \rightarrow V$ is a linear operator and $E \subset V$ is a subspace, E is A -invariant if $AE \subset E$.
- Facts:
 - If E is A -invariant, E is A^k -invariant.
 - Thus, E is $p(A)$ -invariant.
- Consider the restriction map $A|_E$.
- A has a block-diagonalized matrix where each block corresponds to the generalized eigenvectors of a generalized eigenvalue of A .
 - Let E_1, \dots, E_r be a **basis of invariant subspaces**.
 - Let $A_k = A|_{E_k}$. Then the A_k ’s act independently of each other.
- **Generalized eigenvector** (of A): A vector \mathbf{v} corresponding to an eigenvalue λ if there exists $k \geq 1$ such that $(A - \lambda I)^k \mathbf{v} = \mathbf{0}$.
- **Generalized eigenspace**: The set E_λ of all of the generalized eigenvectors of λ . *Given by*

$$E_\lambda = \bigcup_{k \geq 1} \ker(A - \lambda I)^k$$

- E_λ is a linear subspace of V .
- **Degree** (of λ): The smallest number k such that increasing k any more does not add further vectors to the generalized eigenspace. *Denoted by $d(\lambda)$. Also known as **depth**.*
 - Symbolically, $d(\lambda)$ is the smallest number such that

$$E_\lambda = \bigcup_{k=1}^{d(\lambda)} \ker(A - \lambda I)^k$$

- Start working through the first 25 problems of Rudin (1976) (his metric spaces problems).
- 10/27: • Jordan form.
- Reviews build up to generalized eigenvectors.

- Theorem: If $\sigma(A) = \{\lambda_1, \dots, \lambda_n\}$ and E_1, \dots, E_n are the corresponding generalized eigenspaces, then E_1, \dots, E_n is a basis of subspaces of U , i.e., $V = \oplus_k E_k$.
- Corollary: $A : V \rightarrow V$ can be represented as $A = D + N$ where D is diagonalizable and N is nilpotent and $ND = DN$.

– Proof: Consider the basis of generalized eigenspaces known to exist from the theorem. Then $A = \text{diag}\{A_1, \dots, A_r\}$.

– Let

$$N_k = A_k - \lambda_k I_{E_k}$$

This is nilpotent.

– Then let

$$D = \text{diag}\{\lambda_1 I_{E_1}, \dots, \lambda_n I_{E_n}\}$$

– These two matrices satisfy the necessary properties.

- Let $\dot{\mathbf{x}} = A\mathbf{x}$.

– Let $\mathbf{x}(t) = e^{tA}$, where

$$e^{tA} = \sum \frac{(tA)^k}{k!}$$

– $\|e^{tA}\| \leq \sum \frac{\|A^k\|}{k!} = \sum \frac{\|A\|^k}{k!}$.

– Let p be a polynomial of degree k . Then

$$p(a+x) = \sum_{k=0}^d \frac{p^{(k)}(a)}{k!} x^k$$

– If $A = D + N$, then...

- Nilpotent operators:

– Let $A = \text{diag}\{A_1, \dots, A_r\}$.

– We know that $A_k = \lambda_k I_{E_k} + N_k$ for each k .

– Every nilpotent N can be written in the form

$$\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$$

9.2 Chapter 9: Advanced Spectral Theory

10/28:

- Theorem 9.1.1 (Cayley-Hamilton): If p is the characteristic polynomial of A , $p(A) = 0$.
- Theorem 9.2.1 (Spectral Mapping Theorem): For a square matrix A and an arbitrary polynomial p , $\sigma(p(A)) = p(\sigma(A))$. In other words, μ is an eigenvalue of $p(A)$ if and only if $\mu = p(\lambda)$ for some eigenvalue λ of A .
- Corollary 9.2.2: Let A be a square matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ and let p be a polynomial. Then $p(A)$ is invertible iff $p(\lambda_k) \neq 0$ for all $k = 1, \dots, n$.
- Algebraic multiplicity is the dimension of the corresponding generalized eigenspace.