8 Continuity

From Rudin (1976).

Chapter 4

11/29: **1.** Suppose f is a real function defined on $\mathbb{R}^1$ which satisfies

$$\lim_{h\to 0}[f(x+h)-f(x-h)]=0$$

for every $x\in\mathbb{R}^1$. Does this imply that f is continuous?

2. If f is a continuous mapping of a metric space X into a metric space Y, prove that

$$f(\bar{E})\subset\overline{f(E)}$$

for every set $E\subset X$ ($\bar{E}$ denotes the closure of E). Show, by an example, that $f(\bar{E})$ can be a proper subset of $\overline{f(E)}$.

3. Let f be a continuous real function on a metric space X. Let $Z(f)$ (the **zero set** of f) be the set of all $p\in X$ at which $f(p)=0$. Prove that $Z(f)$ is closed.

4. Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $g(p)=f(p)$ for all $p\in E$, prove that $g(p)=f(p)$ for all $p\in X$. (In other words, a continuous mapping is determined by its values on a dense subset of its domain.)

5. If f is a real continuous function defined on a closed set $E\subset\mathbb{R}^1$, prove that there exist continuous real functions g on $\mathbb{R}^1$ such that $g(x)=f(x)$ for all $x\in E$. (Such functions g are called **continuous extensions** of f from E to $\mathbb{R}^1$.) Show that the result becomes false if the word "closed" is omitted. Extend the result to vector-valued functions. (Hint: Let the graph of g be a straight line on each of the segments which constitute the complement of E [compare Exercise 2.29]. The result remains true if $\mathbb{R}^1$ is replaced by any metric space, but the proof is not so simple.)

6. Let (X,d_X) and (Y,d_Y) be metric spaces, and let $f:E\to Y$ where E is a compact subset of X. Consider the **graph** $G\subset X\times Y$ of f, where the metric on $X\times Y$ is $d=d_X+d_Y$, i.e., $d[(x_1,y_1),(x_2,y_2)]=d_X(x_1,x_2)+d_Y(y_1,y_2)$. Show that f is continuous if and only if G is compact. (Hint: There are several ways of doing this. There is a "topological" proof that only uses the fact that compact sets are closed in a metric space, and the fact that a function is continuous if and only if pre-images of closed sets are closed. Another way to go about it is to use sequential compactness [i.e., any sequence contained in a compact set has a convergent subsequence].)

7. If $E\subset X$ and if f is a function defined on X, the **restriction** of f to E is the function g whose domain of definition is E such that $g(p)=f(p)$ for $p\in E$. Define f and g on $\mathbb{R}^2$ by

$$f(x,y)=\begin{cases}0 & (x,y)=(0,0)\\ \frac{xy^2}{x^2+y^4} & (x,y)\neq(0,0)\end{cases}\qquad\qquad g(x,y)=\begin{cases}0 & (x,y)=(0,0)\\ \frac{xy^2}{x^2+y^6} & (x,y)\neq(0,0)\end{cases}$$

Prove that f is bounded on $\mathbb{R}^2$, that g is unbounded in every neighborhood of $(0,0)$, and that f is not continuous at $(0,0)$; nevertheless, the restrictions of both f and g to every straight line in $\mathbb{R}^2$ are continuous!

8. Let f be a real uniformly continuous function on the bounded set E in $\mathbb{R}^1$. Prove that f is bounded on E. Show that the conclusion is false if boundedness of E is omitted from the hypothesis.

9. Show that the requirement in the definition of uniform continuity can be rephrased as follows, in terms of diameters of sets: To every $\epsilon>0$, there exists a $\delta>0$ such that $\operatorname{diam} f(E)<\epsilon$ for all $E\subset X$ with $\operatorname{diam} E<\delta$.

10. Complete the details of the following alternative proof of Theorem 4.19: If f is not uniformly continuous, then for some $\epsilon > 0$, there are sequences $\{p_n\}, \{q_n\}$ in X such that $d_X(p_n, q_n) \to 0$ but $d_Y(f(p_n), f(q_n)) > \epsilon$. Use Theorem 2.37 to obtain a contradiction.

11. Suppose f is a uniformly continuous mapping of a metric space X into a metric space Y and prove that $\{f(x_n)\}$ is a Cauchy sequence in Y for every Cauchy sequence $\{x_n\}$ in X. Use this result to give an alternative proof of the theorem stated in Exercise 4.13.

12. A uniformly continuous function of a uniformly continuous function is uniformly continuous. State this more precisely and prove it.

13. Let E be a dense subset of a metric space X and let f be a uniformly continuous *real* function defined on E. Prove that f has a continuous extension from E to X (see Exercise 4.5 for terminology). Uniqueness follows from Exercise 4.4. (Hint: For each $p \in X$ and each positive integer n, let $V_n(p)$ be the set of all $q \in E$ with $d(p, q) < 1/n$. Use Exercise 4.9 to show that the intersection of the closures of the sets $f(V_1(p)), f(V_2(p)), \ldots$ consists of a single point, say $g(p)$, of $\mathbb{R}^1$. Prove that the function g so defined on X is the desired extension of f.) Could the range space $\mathbb{R}^1$ be replaced by $\mathbb{R}^k$? By any compact metric space? By any complete metric space? By any metric space?

14. Let $I = [0, 1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

15. Call a mapping of X into Y open if $f(V)$ is an open set in Y whenever V is an open set in X. Prove that every continuous open mapping of $\mathbb{R}^1$ into $\mathbb{R}^1$ is monotonic.