4 Inner Product Phenomena and Intro to Bilinear Forms

From Treil (2017).

Chapter 6

10/25: **1.1.** Use the upper-triangular representation of an operator to give an alternative proof of the fact that the determinant is the product and the trace is the sum of the eigenvalues counting multiplicities.

2.1. True or false:

a) Every unitary operator $U : X \to X$ is normal.
b) A matrix is unitary if and only if it is invertible.
c) If two matrices are unitarily equivalent, then they are also similar.
d) The sum of self-adjoint operators is self-adjoint.
e) The adjoint of a unitary operator is unitary.
f) The adjoint of a normal operator is normal.
g) If all eigenvalues of a linear operator are 1, then the operator must be unitary or orthogonal.
h) If all eigenvalues of a normal operator are 1, then the operator is the identity.
i) A linear operator may preserve norm but not the inner product.

2.2. True or false (justify your conclusion): The sum of normal operators is normal.

2.3. Show that an operator that is unitarily equivalent to a diagonal one is normal.

2.5. True or false (justify): Any self-adjoint matrix has a self-adjoint square root.

2.6. Orthogonally diagonalize the matrix

$$A = \begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix}$$

i.e., represent it as $A = UDU^*$, where D is diagonal and U is unitary. Additionally, among all square roots of A, i.e., among all matrices B such that $B^2 = A$, find one that has positive eigenvalues. You can leave B as a product.

2.7. True or false (justify your conclusions):

a) A product of two self-adjoint matrices is self-adjoint.
b) If A is self-adjoint, then A^k is self-adjoint.

2.8. Let A be an $m \times n$ matrix. Prove that

a) A^*A is self-adjoint.
b) All eigenvalues of A^*A are nonnegative.
c) $A^*A + I$ is invertible.

2.10. Orthogonally diagonalize the rotation matrix

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

where α is not a multiple of π. Note that you will get complex eigenvalues in this case.

2.13. Prove that a normal operator with unimodular eigenvalues (i.e., with all eigenvalues satisfying $|\lambda_k| = 1$) is unitary. (Hint: Consider diagonalization.)

2.14. Prove that a normal operator with real eigenvalues is self-adjoint.

2.15. Show by example that the conclusion of Theorem 2.2 fails for *complex* symmetric matrices. Namely,

a) Construct a (diagonalizable) 2×2 complex symmetric matrix not admitting an orthogonal basis of eigenvectors.

b) Construct a 2×2 complex symmetric matrix which cannot be diagonalized.

3.1. Show that the number of nonzero singular values of a matrix A coincides with its rank.

3.2. Find Schmidt decompositions $A = \sum_{k=1}^{r} s_k \mathbf{w}_k \mathbf{v}_k^*$ for the following matrices A.

$$\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}$$

3.3. Let A be an invertible matrix, and let $A = W\Sigma V^*$ be its singular value decomposition. Find a singular value decomposition for A^* and A^{-1}.

3.5. Find the singular value decomposition of the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$$

Use it to find

a) $\max_{\|\mathbf{x}\| \le 1} \|A\mathbf{x}\|$ and the vectors where the maximum is attained.

b) $\min_{\|\mathbf{x}\| = 1} \|A\mathbf{x}\|$ and the vectors where the minimum is attained.

c) The image $A(B)$ of the closed unit ball $B = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| \le 1\}$ in $\mathbb{R}^2$. Describe $A(B)$ geometrically.

3.6. Show that for a square matrix A, $|\det A| = \det |A|$.

3.7. True or false:

a) The singular values of a matrix are also eigenvalues of the matrix.

b) The singular values of a matrix A are eigenvalues of A^*A.

c) If s is a singular value of a matrix A and c is a scalar, then $|c|s$ is a singular value of cA.

d) The singular values of any linear operator are nonnegative.

e) The singular values of a self-adjoint matrix coincide with its eigenvalues.

3.8. Let A be an $m \times n$ matrix. Prove that *nonzero* eigenvalues of the matrices A^*A and AA^* (counting multiplicities) coincide. Can you say when zero eigenvalues of A^*A and zero eigenvalues of AA^* have the same multiplicity?

3.9. Let s be the largest singular value of an operator A, and let λ be the eigenvalue of A with the largest absolute value. Show that $|\lambda| \le s$.

3.11. Show that the operator norm of a matrix A coincides with its Frobenius norm if and only if the matrix has rank one. (Hint: The previous problem might help.)

3.12. For the matrix

$$A = \begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix}$$

describe the inverse image of the unit ball, i.e., the set of all $\mathbf{x} \in \mathbb{R}^2$ such that $\|A\mathbf{x}\| \le 1$. Use its singular value decomposition.

4.2. Let A be a normal operator, and let $\lambda_1, \ldots, \lambda_n$ be its eigenvalues (counting multiplicities). Show that singular values of A are $|\lambda_1|, \ldots, |\lambda_n|$.

4.3. Find the singular values, norm, and condition number of the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

You can do this problem with practically no computations if you use the previous problem and can answer the following questions:

a) What are singular values (eigenvalues) of an orthogonal projection P_E onto some subspace E?

b) What is the matrix of the orthogonal projection onto the subspace spanned by the vector $(1,1,1)^T$?

c) How are the eigenvalues of the operators T and $aT + bI$ where $a, b \in \mathbb{F}$ related?

Of course you can also just honestly do the computations.

6.1. Let R_α be the rotation through α, so its matrix in the standard basis is

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Find the matrix of R_α in the basis $\mathbf{v}_1, \mathbf{v}_2$ where $\mathbf{v}_1 = \mathbf{e}_2, \mathbf{v}_2 = \mathbf{e}_1$.

6.2. Let R_α be the rotation matrix

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Show that the 2×2 identity matrix I_2 can be continuously transformed through invertible matrices into R_α.

6.3. Let U be an $n \times n$ orthogonal matrix with $\det U > 0$. Show that the $n \times n$ identity matrix I_n can be continuously transformed through invertible matrices into U. (Hint: Use the previous problem and the representation of a rotation in $\mathbb{R}^n$ as a product of planar rotations [see Section 5].)

Chapter 7

1.1. Find the matrix of the bilinear form L on $\mathbb{R}^3$ defined by

$$L(\mathbf{x}, \mathbf{y}) = x_1y_1 + 2x_1y_2 + 14x_1y_3 - 5x_2y_1 + 2x_2y_2 - 3x_2y_3 + 8x_3y_1 + 19x_3y_2 - 2x_3y_3$$

1.2. Define the bilinear form L on $\mathbb{R}^2$ by

$$L(\mathbf{x}, \mathbf{y}) = \det[\mathbf{x}, \mathbf{y}]$$

i.e., to compute $L(\mathbf{x}, \mathbf{y})$, we form a 2×2 matrix with columns $\mathbf{x}, \mathbf{y}$ and compute its determinant. Find the matrix of L.

1.3. Find the matrix of the quadratic form Q on $\mathbb{R}^3$ defined by

$$Q[\mathbf{x}] = x_1^2 + 2x_1x_2 - 3x_1x_3 - 9x_2^2 + 6x_2x_3 + 13x_3^2$$

2.1. Diagonalize the quadratic form with the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

Use two methods: completion of squares and row operations. Which one do you like better? Can you say if the matrix A is positive definite or not?