Chapter 8

Dual Spaces and Tensors

10/22:

- **Functional**: A linear bounded map $L : H \to F$, where H is finite dimensional (equivalent to $\mathbb{R}^n$).
- **Dual space**: The set of bounded linear functionals on H. *Denoted by* $\boldsymbol{H'}$, $\boldsymbol{H^*}$.
- If $l \leq p < \infty$, then
$$l^p = \left\{ (a_n)_{n \in \mathbb{N}} : \sum_{n=1}^{\infty} |a_n|^p < \infty \right\}$$
- Back to finite dimensions, $H' \approx \mathbb{R}^n$.
- Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be a basis of H. Then $L\mathbf{x} = (L\mathbf{a}_1, \ldots, L\mathbf{a}_n) \approx \mathbb{R}^n$.
- Let $L((a_n)_{n \in \mathbb{N}}) = \sum_{n=1}^{\infty} a_n b_n$. Then $L((a_n)_{n \in \mathbb{N}})$ will be bounded if and only if $(b_n)_{n \in \mathbb{N}} \in l^q$ where $1 < p < q$ where $\frac{1}{q} + \frac{1}{p} = 1$.
- **Young's inequality**: The statement
$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$
- We have $|\sum a_n b_n| \leq \|a_n\|_p \|b_n\|_p$.
- Conclusion:
$$\sum \frac{|a_n||b_n|}{\|a_n\|_p \|b_n\|_q} = 1$$
- We can define H'', too. This contains linear functionals on H'.
- We know that $L(x) = \langle x, L \rangle = x(L)$. $x \in H''$.
- Riesz representation theorem: Let H have an inner product. $L \in H'$ if and only if there exists a unique $y \in H$ such that $L(x) = (x, y)$.
  - Gives us a way to identify all bounded linear functionals on H.
  - In finite dimensions, $L(x)$, where $x = \sum_1^n \alpha_i a_i$ gives us $L(x) = \sum_1^n \alpha_i L(a_i)$.