

Chapter 1

The Real and Complex Number Systems

1.1 Notes

- 11/1:
- Spent a lot of time trying to cheer us up regarding the midterm.
 - There may be some true/false on linear algebra on the final.
 - Facts:
 1. $\sqrt{2}$ is irrational.
 2. Archimedes principle: If $x > 0$ and $y \in \mathbb{R}$, then there exists n such that $nx > y$.
 3. If $x > y$, then there exists $q \in \mathbb{Q}$ such that $x > q > y$.

1.2 Chapter 1: The Real and Complex Number Systems

From Rudin (1976).

- 11/6:
- Rudin (1976) presents several interesting proofs throughout this section that may be of interest later by means of their divergence from the ones with which I am familiar.
 - **Least-upper-bound property:** The property pertaining to a set S that if $E \subset S$, $E \neq \emptyset$, and E is bounded above, then $\sup E \in S$.
 - For example, \mathbb{Q} does not have the least-upper-bound property.
 - The **greatest-lower-bound property** is analogously defined.
 - Theorem: Suppose S is an ordered set with the least-upper-bound property, $B \subset S$ is nonempty, and B is bounded below. Let L be the set of all lower bounds of B . Then $\alpha = \sup L$ exists in S , and $\alpha = \inf B$. In particular, $\inf B$ exists in S .
 - Essentially, this theorem states that any set that satisfies the least-upper-bound property satisfies the greatest-lower-bound property.
 - **Existence theorem:** There exists an ordered field \mathbb{R} which has the least-upper-bound property. Moreover, \mathbb{R} contains \mathbb{Q} as a subfield.
 - The second statement implies that the operations of addition and multiplication on \mathbb{R} , when applied to \mathbb{Q} , coincide with the operations of addition and multiplication on \mathbb{Q} .

- **Archimedean property** (of \mathbb{R}): If $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $x > 0$, then there is a positive integer n such that $nx > y$.
- Rudin (1976) proves several theorems about the real numbers from the least-upper-bound property as opposed to the traditional construction of the real numbers.
- Introduces the decimal system.
- **Finite real number system**: That which has been defined thus far.
- **Extended real number system**: The set $\mathbb{R} \cup \{+\infty, -\infty\}$ where $+\infty, -\infty$ obey the expected properties (supremum [resp. infimum] of every set, $x + \infty = \infty$, etc.).
- Defines the complex field axiomatically with complex numbers in the form (a, b) for $a, b \in \mathbb{R}$.
 - Notes that the real numbers form a subfield of the complex field.
 - Defines $i = (0, 1)$, proves $i^2 = -1$, proves $a + bi = (a, b)$.
- **Schwarz inequality**: If a_1, \dots, a_n and b_1, \dots, b_n are complex numbers, then

$$\left| \sum_{j=1}^n a_j \bar{b}_j \right|^2 \leq \sum_{j=1}^n |a_j|^2 \sum_{j=1}^n |b_j|^2$$

- **Euclidean k -space**: The vector space \mathbb{R}^k over the real field.