Chapter 7

Bilinear and Quadratic Forms

7.1 Notes

10/18:

- **Bilinear form**: A function $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$L(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2, \mathbf{y}) = \alpha L(\mathbf{x}_1, \mathbf{y}) + \beta L(\mathbf{x}_2, \mathbf{y}) \qquad L(\mathbf{x}, \alpha\mathbf{y}_1 + \beta\mathbf{y}_2) = \alpha L(\mathbf{x}, \mathbf{y}_1) + \beta L(\mathbf{x}, \mathbf{y}_2)$$

  - $L(\mathbf{x}, \mathbf{y}) = (A\mathbf{x}, \mathbf{y})$.
- **Quadratic form**: A bilinear form $L(\mathbf{x}, \mathbf{x})$.
  - $(\mathbf{x}, \mathbf{x})$ is a polynomial of degree 2 in $\mathbf{x}_1, \dots, \mathbf{x}_n$:

$$L(\lambda\mathbf{x}, \lambda\mathbf{x}) = (\lambda\mathbf{x}, \lambda\mathbf{x}) = \lambda^2(\mathbf{x}, \mathbf{x})$$

- We have that

$$(A\mathbf{x}, \mathbf{x}) = (A\lambda\mathbf{x}, \lambda\mathbf{x}) = \lambda^2(A\mathbf{x}, \mathbf{x}) = \sum_{j,i=1}^{n} \alpha_{j,i}\mathbf{x}_i\mathbf{x}_j$$

- The general form of a quadratic form:
  - Can any quadratic form on $\mathbb{R}^n$ be written as $(A\mathbf{x}, \mathbf{x})$?

10/20:

- Bilinear forms are linear in each argument when keeping the other fixed.
- Quadratic forms $Q(\mathbf{x}) = L(\mathbf{x}, \mathbf{x})$ are quadratic polynomials in the coordinates of x.
  - In particular, $Q(\lambda\mathbf{x}) = |\lambda|^2 Q(\mathbf{x})$.
- If Q quadratic is real, then $Q(\mathbf{x}) = (A\mathbf{x}, \mathbf{x})$ where A is some square matrix.
  - If $\mathbf{e}_1, \dots, \mathbf{e}_n$ is an orthonormal basis of $\mathbb{R}^n$, then there exists a unique $A = A^*$ such that $(A)_{ij} = L(\mathbf{e}_i, \mathbf{e}_j)$.
  - Keeping $\mathbf{x} = \sum_{i=1}^n \mathbf{x}_i, \mathbf{e}_i$ foxed, we have

$$\begin{aligned}
Q(\mathbf{x}) &= L(\mathbf{x}, \mathbf{x}) \\
&= L(\sum^n \mathbf{x}_i\mathbf{e}_i, \sum^n \mathbf{x}_j\mathbf{e}_j) \\
&= \sum_{i=1}^n \mathbf{x}_i L(\mathbf{e}_i, \sum^n \mathbf{x}_j\mathbf{e}_j) \\
&= \sum_{i,j=1}^n \mathbf{x}_i\mathbf{x}_j \underbrace{L(\mathbf{e}_i, \mathbf{e}_j)}_{A_{ij}}
\end{aligned}$$

- We have that

$$\begin{aligned}(A\mathbf{x},\mathbf{x}) &= (UDU^{-1}\mathbf{x},\mathbf{x}) \\ &= (DU^{-1}\mathbf{x},U^{-1}\mathbf{x}) \\ &= \sum_{i=1}^{n}\lambda_i(\underbrace{U^{-1}\mathbf{x}}_{\mathbf{y}_i})_i(\underbrace{U^{-1}\mathbf{x}}_{\mathbf{y}_i})_i\end{aligned}$$

- Can we characterize the set $\{\mathbf{x} : (A\mathbf{x},\mathbf{x}) = 1\}$?
  - Note that this set is equivalent to $\{\mathbf{y} : (D\mathbf{y},\mathbf{y}) = 1\}$ by teh above. This set is a rotation of the previous one. Ellipse?
- Positive quadratic form:
  - Q is positive definite if $Q(\mathbf{x}) > \mathbf{0}$ for all $\mathbf{x} \neq \mathbf{0}$ and Q is positive semidefinite if $Q(\mathbf{x}) \geq \mathbf{0}$ for all $\mathbf{x} \neq \mathbf{0}$.
  - Take a self-adjoint matrix $A = A^*$. It is positive definite if $Q(\mathbf{x}) = (A\mathbf{x},\mathbf{x})$ is positive definite.
- Theorem: If $A = A^*$, then
  1. A is positive definite if and only if all eigenvalues of A are positive.
  2. A is positive semidefinite if and only if all eigenvalues of A are nonnegative.
  3. A is negative semidefinite if and only if all eigenvalues of A are nonpositive.
  4. A is negative definite if and only if all eigenvalues of A are negative.
  5. A is indefinite if and only if the eigenvalues of A have positive and negative values.
- Theorem: $A = A^*$ is positive definite iff $\det A_k > 0$ for all $k = 1, \dots, n$ where A_k is the upper left $k \times k$ submatrix.
- Minimax representation of eigenvalues of a self-adjoint A.
  - Let E be a subspace of X where $\dim X < \infty$. We define $\operatorname{codim}(E) = \dim E^{\perp}$.
  - Thus, $\dim E + \operatorname{codim} E = \dim X$.
  - Theorem: Let $A = A^*$, $\lambda_1 \geq \cdots \geq \lambda_n$ eigenvalues of A. Then

$$\lambda_k = \max_{\substack{\text{E subspace} \\ \dim E = k}} \min_{\substack{\mathbf{x}\in E \\ \|\mathbf{x}\|=1}} (A\mathbf{x},\mathbf{x}) = \min_{\substack{\text{F subspace} \\ \operatorname{codim} F = k-1}} \max_{\substack{\mathbf{x}\in F \\ \|\mathbf{x}\|=1}} (A\mathbf{x},\mathbf{x})$$

    - Proof: A diagonal equals $(\lambda_1, \dots, \lambda_n)$.
    - An orthonormal basis of X such that $\dim E = k$, $\operatorname{codim} F = k - 1$, $\dim F = n - k + 1$.
    - There exists an $\mathbf{x}_0 \neq \mathbf{0}$ such that $\mathbf{x}_0 \in E \cap F$.
    - Note that if $B = B^*$, then the max and min of $(B\mathbf{x},\mathbf{x})$ over the unit sphere is the maximal and minimal eigenvalue of B.
    - Thus,

$$\min_{\substack{\mathbf{x}\in E \\ \|\mathbf{x}\|=1}} (A\mathbf{x},\mathbf{x}) \leq (A\mathbf{x}_0,\mathbf{x}_0) \leq \max_{\substack{\mathbf{x}\in F \\ \|\mathbf{x}\|=1}} (A\mathbf{x},\mathbf{x})$$

    - This is true for any E, F subspaces. $\dim E = k$, $\operatorname{codim} F = k - 1$, $E_0 = \operatorname{span}(\mathbf{e}_1, \dots, \mathbf{e}_k)$ and $F_0 = \operatorname{span}(\mathbf{e}_k, \dots, \mathbf{e}_n)$.
    - Thus,

$$\min_{\substack{E_0 \\ \|\mathbf{x}\|=1}} (A\mathbf{x},\mathbf{x}) = \lambda_k = \max_{\substack{F_0 \\ \|\mathbf{x}\|=1}} (A\mathbf{x},\mathbf{x})$$

- ■ Additionally,

$$\lambda_{k_1} \leq \max{}_{\dim E=k}^{E} \min_{\mathbf{x}} (A\mathbf{x},\mathbf{x}) \leq \min_{\substack{F \\ \operatorname{codim} F=k-1}} \max_{\mathbf{x}} (A\mathbf{x},\mathbf{x}) \leq \lambda_k$$

- Corollary: Let $A = A^* = (a_{jk})_{1 \leq j,k \leq n}$ with eigenvalues $\lambda_1, \dots, \lambda_n$ listed in decreasing order. Let $\tilde{A} = (a_{j,k})_{1 \leq j,k \leq n-1}$ with eigenvalues $\mu_1, \dots, \mu_{n-1}$ listed in decreasing order. Then $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n$.
  - Consider $(A\mathbf{x},\mathbf{x})$ on $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, but then restrict yourself to $\mathbf{x} \in \mathbb{R}^{n-1}$ on $\{\mathbf{e}_1, \dots, \mathbf{e}_{n-1}\}$.

7.2 Chapter 7: Bilinear and Quadratic Forms

From Treil (2017).

10/25: - **Bilinear form** (on $\mathbb{R}^n$): A function $L(\mathbf{x},\mathbf{y})$ of two arguments $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$ that is linear in each argument.
  - Linearity in each argument:

$$L(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2, \mathbf{y}) = \alpha L(\mathbf{x}_1,\mathbf{y}) + \beta L(\mathbf{x}_2,\mathbf{y}) \qquad L(\mathbf{x}, \alpha\mathbf{y}_1 + \beta\mathbf{y}_2) = \alpha L(\mathbf{x},\mathbf{y}_1) + \beta L(\mathbf{x},\mathbf{y}_2)$$

- If $\mathbf{x} = (x_1, \dots, x_n)^T$ and $\mathbf{y} = (y_1, \dots, y_n)^T$, then

$$\begin{aligned} L(\mathbf{x},\mathbf{y}) &= \sum_{j,k=1}^{n} a_{j,k} x_k y_j \\ &= (A\mathbf{x},\mathbf{y}) \\ &= \mathbf{y}^T A\mathbf{x} \end{aligned}$$

  where

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix}$$

  - A is uniquely determined by L.
- **Quadratic form** (on $\mathbb{R}^n$): The diagonal of a bilinear form L, i.e., a bilinear form $Q[\mathbf{x}] = L(\mathbf{x},\mathbf{x}) = (A\mathbf{x},\mathbf{x})$.
  - Alternatively: A homogeneous polynomial of degree 2, i.e., a polynomial in $x_1, \dots, x_n$ with only ax_k^2 and cx_jx_k terms.
- There are infinitely many ways to write a quadratic form as $(A\mathbf{x},\mathbf{x})$.
  - However, there is a unique representation $(A\mathbf{x},\mathbf{x})$ where A is a (real) symmetric matrix.
- **Quadratic form** (on $\mathbb{C}^n$): A function of the form $Q[\mathbf{x}] = (A\mathbf{x},\mathbf{x})$ where A is self-adjoint.
- Lemma 7.1.1: Let $(A\mathbf{x},\mathbf{x})$ be real for all $\mathbf{x} \in \mathbb{C}^n$. Then $A = A^*$.
- To classify quadratic forms, consider the set of points $\mathbf{x} \in \mathbb{R}^n$ defined by $Q[\mathbf{x}] = 1$ for some quadratic form Q.
  - If the matrix of Q is diagonal, i.e., $Q[\mathbf{x}] = a_1x_1^2 + \cdots + a_nx_n^2$, then the set of points can easily be visualized.
- The standard method of diagonalizing a quadratic form is change of variables.
- Orthogonal diagonalization.

- Let $Q[\mathbf{x}] = (A\mathbf{x}, \mathbf{x})$ in $\mathbb{F}^n$.
- Suppose $\mathbf{y} = S^{-1}\mathbf{x}$ where S is an invertible $n \times n$ matrix. Then

$$Q[\mathbf{x}] = Q[S\mathbf{y}] = (AS\mathbf{y}, S\mathbf{y}) = (S^*AS\mathbf{y}, \mathbf{y})$$

  so in the new variables $\mathbf{y}$, the quadratic form has matrix S^*AS.
- Thus, we can let $A = UDU^*$, choose $D = U^*AU$ as our new (diagonal) matrix, and let this matrix act on the variables $\mathbf{y} = U^*\mathbf{x}$.

- Non-orthogonal diagonalization:
  - Completing the square:
    - Eliminate all x_ix_j terms by completing the square. Then substitute in a y_k for each squared term.
  - Row/column operations:
    - Augment $(A|I)$. Row reduce A to D. Then $I \to S^*$.