Chapter 4

Continuity

4.1 Notes

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- Consider a function $f : X \to Y$ whose domain and codomain are, respectively, the metric spaces (X, d_X) and (Y, d_Y).
- **Limit** (of f at p): A point $q \in Y$ such that for all $\epsilon > 0$, there exists δ such that $d_X(x,p) < \delta$ implies $d_Y(q, f(x)) < \epsilon$, where p is a limit point of X (otherwise, $x \not\to p$).
- **Continuous** (function f at p): A function f such that $\lim_{x \to p} f(x) = f(p)$.
- f is continuous on X if it is continuous at every $p \in X$.
- **Uniformly continuous** (function f): A function f such that for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d_X(x,y) < \delta$ implies $d_Y(f(x), f(y)) < \epsilon$ for all $x, y \in X$.

4.2 Chapter 4: Continuity

From Rudin (1976).

- **Limit** (of f at p): The point $q \in Y$, if it exists, such that for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d_Y(f(x), q) < \epsilon$ for all points $x \in E$ for which $0 < d_X(x,p) < \delta$, where $(X, d_X), (Y, d_Y)$ are metric spaces, $E \subset X$, $f : E \to Y$, and $p \in E'$. *Denoted by* $\boldsymbol{\lim_{x \to p} f(x)}$.
  - Note that we do not require that $p \in E$; only that some elements of the domain E approach p.
  - We also write $f(x) \to q$ as $x \to p$.
- Theorem 4.2: Let X, Y, E, f, and p be as specified above. Then $\lim_{x \to p} f(x) = q$ iff $\lim_{n \to \infty} f(p_n) = q$ for every sequence $\{p_n\}$ in E such that $p_n \neq p$ for any n and $\lim_{n \to \infty} p_n = p$.
- Rudin (1976) proves the sum, product, and quotient rules of limits from the analogous properties of series.
- Continuity is defined.
  - Note that f *does* have to be defined at p to be continuous at p (in comparison to the fact that it can have a limit at a point p' at which it is not defined).
    - Thus, for proofs concerning continuity (as opposed to limits), we will consider functions f the domains of which are metric spaces, not *subsets* of metric spaces.
  - It follows from the definition that if $p \in E$ is isolated, then every possible f defined on E is continuous at p.
- Theorem 4.7: Compositions of continuous functions are continuous.

- Theorem 4.8: Preimage definition of continuity.
- Theorem 4.9: If f, g are complex continuous functions on X, $f + g$, fg, and f/g are continuous on X.
- Theorem 4.10: $\mathbf{f}$ continuous implies $f_1, \dots, f_k$ continuous. Also, $\mathbf{f}, \mathbf{g} : X \to \mathbb{R}^k$ continuous implies $\mathbf{f} + \mathbf{g}$ and $\mathbf{f} \cdot \mathbf{g}$ continuous.

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- Theorem 4.14: f continuous and X compact implies $f(X)$ compact.
- Theorem 4.15: $\mathbf{f} : X \to \mathbb{R}^k$ continuous and X compact implies $f(X)$ closed and bounded.
- Theorem 4.16: f continuous and X compact implies f attains its minimum and maximum.
- Theorem 4.17: $f : X \to Y$ continuous, 1-1 for X, Y compact implies $f^{-1} : Y \to X$ continuous.
- Theorem 4.19: f continuous and X compact implies f uniformly continuous.
- Theorem 4.20: Compactness is a necessary condition in Theorems 4.14, 4.15, 4.16, and 4.19.
- Theorem 4.22: $f : X \to Y$ continuous and $E \subset X$ connected implies $f(E)$ connected.
- Theorem 4.23: Intermediate value theorem.
- **Right-hand limit** (of f at x): *Denoted by* $\boldsymbol{f(x+)}$.
- **Left-hand limit** (of f at x): *Denoted by* $\boldsymbol{f(x-)}$.
- **Discontinuity of the first kind** (of f at x): A discontinuity of f at x such that $f(x+)$ and $f(x-)$ exist. *Also known as* **simple discontinuity**.
- **Discontinuity of the second kind** (of f at x): A discontinuity of f at x that is not of the first kind (i.e., a discontinuity such that at least one of $f(x+)$ and $f(x-)$ does not exist).
- Theorem 4.29: If f is monotonic on (a, b), then $f(x+), f(x-)$ exist at every $x \in (a, b)$.
- Corollary: Monotonic functions have no discontinuities of the second kind.
- Theorem 4.30: If f is monotonic on (a, b), then the set of points of (a, b) at which f is discontinuous is at most countable.