

Chapter 5

Inner Product Spaces

10/6:

- We define

$$\ell^2(\mathbb{R}) = \left\{ \{a_n\}_{n \geq 1} \subset \mathbb{R} : \sum_1^\infty |a_n|^2 < \infty \right\}$$

- **Inner product:** A map $V \times V \rightarrow \mathbb{F}$ that takes $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \cdot \mathbf{y}$. Denoted by $\cdot, (\cdot, \cdot), \langle \cdot, \cdot \rangle$.

- Properties of the inner product:

- $(\mathbf{x}, \mathbf{y}) = \overline{(\mathbf{y}, \mathbf{x})}$ (symmetry).
- $(\alpha \mathbf{x} + \beta \mathbf{y}, \mathbf{z}) = \alpha(\mathbf{x}, \mathbf{z}) + \beta(\mathbf{y}, \mathbf{z})$ (linearity).
- $(\mathbf{x}, \mathbf{x}) \geq 0$.
- $(\mathbf{x}, \mathbf{x}) = 0$ iff $\mathbf{x} = 0$.

- If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n x_i y_i$$

- If $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, then

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n x_i \bar{y}_i$$

- If $f, g \in \mathbb{P}_n(t)$, then

$$(f, g) = \int_{-1}^1 f \bar{g} dt$$

- The conjugate of a polynomial is the polynomial with the conjugate of the coefficients of the original polynomial. Symbolically, if $f = \sum_{i=0}^n \alpha_i t^i$ is a polynomial, then $\bar{f} = \sum_{i=0}^n \bar{\alpha}_i t^i$.

- It is a fact that

$$\left| \sum_{n=1}^{\infty} a_n \bar{b}_n \right| \leq \|(a_n)_{n \geq 1}\| \|(b_n)_{n \geq 1}\|$$

- Suppose we want to define the inner product between two matrices.

- A common one is

$$(A, B) = \text{tr}(B^* A)$$

where $B^* = \bar{B}^T = \overline{B^T}$ is the conjugate transpose.

- We define the norm as a function $V \rightarrow [0, \infty)$ given by

$$\|\mathbf{x}\| = \sqrt{(\mathbf{x}, \mathbf{x})}$$

- Properties of the norm.

- $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$.
- $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.
- $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.

- In \mathbb{R}^n ,

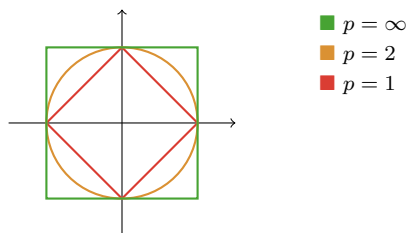


Figure 5.1: The unit ball of norms corresponding to $p = 1, 2, \infty$.

- The standard norm is

$$\|\mathbf{x}\| = \sqrt{\sum |x_i|^2}$$

- We can also define

$$\|\mathbf{x}\|_p = \sqrt[p]{\sum |x_i|^p}$$

- We can even define

$$\|\mathbf{x}\|_\infty = \max |x_i|$$

- And we can prove that all of these are valid norms.
- Only the norm corresponding to ℓ^2 is given by an inner product, but all the other quantities are still norms as defined by the properties (see Treil (2017)).
- Figure 5.1 shows the unit ball of each norm, i.e., the set of all points which have norm 1.

- The parallelogram rule:

$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)$$

- Orthogonality: Given \mathbf{v}, \mathbf{w} , if $\mathbf{v} \perp \mathbf{w}$, then $(\mathbf{v}, \mathbf{w}) = 0$.

- In particular, if $\mathbf{v} \perp \mathbf{w}$, then

$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$$

- Let E be a subspace of V . If $\mathbf{v} \perp E$, then $\mathbf{v} \perp \mathbf{e}$ for all $\mathbf{e} \in E$, i.e., $\mathbf{v} \perp$ a set of vectors spanning E .
- Any set of orthogonal vectors is linearly independent. Thus, if V is n dimensional, then $\mathbf{v}_1, \dots, \mathbf{v}_n$ orthogonal is a basis.
- Let E be a subspace of V . Take $\mathbf{v} \in V$. We want to define the projection $P_E \mathbf{v}$ of \mathbf{v} onto E .
 - We have that $P_E \mathbf{v} \in E$ and $\mathbf{v} - P_E \mathbf{v} \perp E$.
 - Additionally, we have that

$$\|\mathbf{v} - P_E \mathbf{v}\| \leq \|\mathbf{v} - \mathbf{e}\|$$

for all $\mathbf{e} \in E$.

- Lastly, we have that $P_E \mathbf{v}$ is unique.
- If we receive a basis of a vector space, how do we create out of that a basis that is orthogonal? The process of doing this is called **Gram-Schmidt orthogonalization**.
 - We keep \mathbf{v}_1 , subtract $P_{\mathbf{v}_1} \mathbf{v}_2$ from \mathbf{v}_2 , subtract $P_{\{\mathbf{v}_1, \mathbf{v}_2\}} \mathbf{v}_3$ from \mathbf{v}_3 , and on and on.
- If we are given a set of orthogonal vectors, we can normalize them by dividing each by its norm. This creates an orthonormal list. The standard basis is orthonormal.
- Let
$$E^\perp = \{v \in V : v \perp E\}$$
 - It follows that $V = E \oplus E^\perp$.
- How close can we come to solving $A\mathbf{x} = \mathbf{b}$ if we cannot solve it exactly (i.e., if the columns are not linearly independent)?
 - Let A be an $m \times n$ matrix, and let $\mathbf{b} \in \mathbb{R}^m$.
 - Then the best solution is given by minimizing $\|A\mathbf{x} - \mathbf{b}\|$. We minimize this with projections. A special case of this is least squares regression! More details in Treil (2017).