Chapter 5

Differentiation

5.1 Chapter 5: Differentiation

From Rudin (1976).

12/5:

- Let f be a real-valued function defined on $[a,b]$.
- **Derivative** (of f at x): The limit $\lim_{t\to x}\phi(t)$, provided that said limit exists, where $\phi:(a,b)\setminus\{x\}\to\mathbb{R}$ is defined by
$$\phi(t)=\frac{f(t)-f(x)}{t-x}$$
Denoted by $\boldsymbol{f'(x)}$.
- **Derivative** (of f): The real function defined on X that evaluates to $f'(x)$ everywhere on its domain, where
$$X=\{x\in[a,b]:f'(x)\text{ exists}\}$$
Denoted by $\boldsymbol{f'}$.
- Theorem 5.2: Differentiability at x implies continuity at x.
  - The converse is not true.
- Theorem 5.3: Sum, product, and quotient rules of derivatives.
- Theorem 5.4 (Chain Rule): Suppose f is continuous on $[a,b]$, $f'(x)$ exists at some point $x\in[a,b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If $h(t)=g(f(t))$ for all $t\in[a,b]$, then h is differentiable at x and
$$h'(x)=g'(f(x))f'(x)$$
Proof. Let $y=f(x)$. Since f is differentiable at x and g is differentiable at $f(x)$, we have that
$$\begin{aligned}\frac{f(t)-f(x)}{t-x}&=f'(x)+u(t) & \frac{g(s)-g(y)}{s-y}&=g'(y)+v(s)\\ f(t)-f(x)&=(t-x)[f'(x)+u(t)] & g(s)-g(y)&=(s-y)[g'(y)+v(s)]\end{aligned}$$
where $t\in[a,b]$, $s\in I$, $u(t)\to 0$ as $t\to x$, and $v(s)\to 0$ as $s\to y$. Let $s=f(t)$. Then
$$\begin{aligned}h(t)-h(x)&=g(f(t))-g(f(x))\\ &=[f(t)-f(x)]\cdot[g'(f(x))+v(s)]\\ &=(t-x)\cdot[f'(x)+u(t)]\cdot[g'(f(x))+v(s)]\\ \frac{h(t)-h(x)}{t-x}&=[f'(x)+u(t)]\cdot[g'(f(x))+v(s)]\end{aligned}$$

Thus, since as $t \to x$, $s = f(t) \to f(x) = y$ by the continuity of f, we have that

$$\begin{aligned} h'(x) &= \lim_{t \to x} \frac{h(t) - h(x)}{t - x} \\ &= \lim_{t \to x} [f'(x) + u(t)] \cdot [g'(f(x)) + v(s)] \\ &= [f'(x) + 0] \cdot [g'(f(x)) + 0] \\ &= g'(f(x)) f'(x) \end{aligned}$$

as desired. □

- **Local maximum** (of $f : X \to \mathbb{R}$): A point $p \in X$ for which there exists a $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in X$ with $d(p, q) < \delta$.
- Theorem 5.8: $f(x)$ a local maximum and f' exists implies $f'(x) = 0$.
- Theorem 5.9 (Generalized or Cauchy Mean Value Theorem): f, g continuous on $[a, b]$, differentiable on (a, b) imply there exists $x \in (a, b)$ such that

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$$

- Theorem 5.10 (Mean Value Theorem): f continuous on $[a, b]$, differentiable on (a, b) implies there exists $x \in (a, b)$ such that

$$f(b) - f(a) = (b - a)f'(x)$$

Proof. Take $g(x) = x$ in Theorem 5.9. □

- Theorem 5.11: Suppose f is differentiable in (a, b).
  (a) If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotonically increasing.
  (b) If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.
  (c) If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is monotonically decreasing.
- Theorem 5.12: f differentiable on $[a, b]$ and $f'(a) < \lambda < f'(b)$ implies there exists $x \in (a, b)$ such that $f'(x) = \lambda$.
- Corollary: f differentiable on $[a, b]$ implies f' has no simple discontinuities on $[a, b]$.
  - But it may have discontinuities of the second kind.
- Theorem 5.13 (L'Hôpital's Rule): f, g differentiable on (a, b), $g'(x) \neq 0$ for all $x \in (a, b)$, $f'(x)/g'(x) \to A$, and $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$ or $g(x) \to +\infty$ as $x \to a$ implies $f(x)/g(x) \to A$ as $x \to a$, where $-\infty \leq a < b \leq +\infty$.
- $\boldsymbol{n^{\text{th}}}$ **derivative** (of f at x): The derivative of the $(n-1)^{\text{th}}$ derivative of f at x, if it exists. *Denoted by* $\boldsymbol{f^{(n)}(x)}$.
  - $f^{(n)}(x)$ exists iff $f^{(n-1)}$ exists in some $N_r(x)$ and $f^{(n-1)'}(x)$ exists.
  - We customarily denote the first few higher order derivatives with repeated primes, e.g., $f''(x)$ is the second derivative of f.
- Theorem 5.15 (Taylor's Theorem): f defined on $[a, b]$, $n \in \mathbb{N}$, $f^{(n-1)}$ continuous on $[a, b]$, $f^{(n)}(t)$ defined on (a, b), $\alpha, \beta \in [a, b]$ such that $\alpha \neq \beta$, and

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k$$

implies there exists $x \in (\alpha, \beta)$ such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n$$

  - For $n = 1$, this is the mean value theorem.
  - "In general, the theorem shows that f can be approximated by a polynomial of degree $n-1$ and that the last equation above allows us to estimate the error, if we know bounds on $|f^{(n)}(x)|$" (Rudin, 1976, p. 111).

- **Derivative** (of $\mathbf{f}$ at x): The point $\mathbf{f}'(x) \in \mathbb{R}^k$, if it exists, such that

$$\lim_{t \to x} \left\| \frac{\mathbf{f}(t) - \mathbf{f}(x)}{t - x} - \mathbf{f}'(x) \right\| = 0$$

- Theorems 5.2-5.3 remain valid for vector-valued functions.
- If $\mathbf{f} = (f_1, \ldots, f_k)$, then $\mathbf{f}'(x)$ exists iff $f_i'(x)$ $(i = 1, \ldots, k)$ exists and

$$\mathbf{f}' = (f_1', \ldots, f_k')$$

- Theorem 5.19: $\mathbf{f} : [a, b] \to \mathbb{R}^k$ continuous and $\mathbf{f}$ differentiable on (a, b) implies there exists $x \in (a, b)$ such that

$$\|\mathbf{f}(b) - \mathbf{f}(a)\| \leq (b - a)\|\mathbf{f}'(x)\|$$