Chapter 6

The Riemann-Stieltjes Integral

6.1 Chapter 6: The Riemann-Stieltjes Integral

From Rudin (1976).

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- **Partition** (of $[a,b]$): A finite set P of points $x_0, \dots, x_n \in [a,b]$ such that

$$a = x_0 \leq \cdots \leq x_n = b$$

  - Let $\Delta x_i = x_i - x_{i-1}$.
- Let $f : [a,b] \to \mathbb{R}$ be bounded.
  - We define

$$M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\} \qquad m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$$

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i \qquad L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$$

    for each partition P of $[a,b]$.
- **Upper Riemann integral** (of f over $[a,b]$): The following quantity. *Denoted by* $\overline{\int_a^b} f\,\mathrm{d}x$. *Given by*

$$\inf\{U(P,f) : P \text{ partitions } [a,b]\}$$

- **Lower Riemann integral** (of f over $[a,b]$): The following quantity. *Denoted by* $\underline{\int_a^b} f\,\mathrm{d}x$. *Given by*

$$\inf\{U(P,f) : P \text{ partitions } [a,b]\}$$

- The upper and lower Riemann integrals always exist since the boundedness of f on $[a,b]$ implies that the set of all lower and upper sums of f on $[a,b]$ is bounded.
- **Riemann-integrable** (f on $[a,b]$): A function f for which

$$\overline{\int_a^b} f\,\mathrm{d}x = \underline{\int_a^b} f\,\mathrm{d}x$$

- $\mathscr{R}$: The set of all Riemann-integrable functions.
- **Riemann integral** (of f on $[a,b]$): The common value of the lower and upper Riemann integrals over $[a,b]$ of a Riemann-integrable function on $[a,b]$. *Denoted by* $\int_a^b f\,\mathrm{d}x$, $\int_a^b f(x)\,\mathrm{d}x$. *Given by*

$$\overline{\int_a^b} f\,\mathrm{d}x = \underline{\int_a^b} f\,\mathrm{d}x$$

- Defining the Riemann-Stieltjes integral.
- Let $\alpha : [a,b] \to \mathbb{R}$ be monotonically increasing.
- Let $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$ be so defined for every partition P of $[a,b]$.
- Let

$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i \qquad\qquad L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

$$\overline{\int_a^b} f \,\mathrm{d}\alpha = \inf U(P,f,\alpha) \qquad\qquad \underline{\int_a^b} f \,\mathrm{d}\alpha = \sup L(P,f,\alpha)$$

- **Riemann-Stieltjes integral** (of f with respect to α over $[a,b]$): The common value, when it exists, of $\overline{\int_a^b} f \,\mathrm{d}\alpha$ and $\underline{\int_a^b} f \,\mathrm{d}\alpha$. *Also known as* **Stieltjes integral**. *Denoted by* $\boldsymbol{\int_a^b f \,\mathrm{d}\alpha}$, $\boldsymbol{\int_a^b f(x) \,\mathrm{d}\alpha(x)}$.
- $\boldsymbol{\mathscr{R}(\alpha)}$: The set of all Riemann-Stieltjes integrable functions with respect to α.
- Note that taking $\alpha(x) = x$ reveals that the Riemann integral is a special case of the Riemann-Stieltjes integral.
- **Refinement** (of P): A partition of the same interval as P that contains every point of P. *Denoted by* $\boldsymbol{P^*}$.
- **Common refinement** (of P_1, P_2): The set $P^* = P_1 \cup P_2$.
- Theorem 6.4: P^* a refinement of P implies

$$L(P,f,\alpha) \leq L(P^*,f,\alpha) \qquad\qquad U(P^*,f,\alpha) \leq U(P,f,\alpha)$$

- Theorem 6.5: $\underline{\int_a^b} f \,\mathrm{d}x \leq \overline{\int_a^b} f \,\mathrm{d}x$.
- Theorem 6.6: $f \in \mathscr{R}(\alpha)$ iff for every $\epsilon > 0$, there exists a partition P such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

- Theorem 6.7:
  (a) If $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$, then $U(P^*,f,\alpha) - L(P^*,f,\alpha) < \epsilon$ for all $P^* \supset P$.
  (b) If $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$ and $s_i, t_i \in [x_{i-1}, x_i]$, then

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i < \epsilon$$

  (c) If $f \in \mathscr{R}(\alpha)$ and the hypotheses of (b) hold, then

$$\left| \sum_{i=1}^{n} f(t_i) \Delta\alpha_i - \int_a^b f \,\mathrm{d}\alpha \right| < \epsilon$$

- Theorem 6.8: f continuous on $[a,b]$ implies $f \in \mathscr{R}(\alpha)$ on $[a,b]$.
- Theorem 6.9: f monotonic on $[a,b]$ and α continuous on $[a,b]$ imply $f \in \mathscr{R}(\alpha)$.
- Theorem 6.10: f bounded on $[a,b]$ with only finitely many discontinuities on $[a,b]$ and α continuous at every point at which f is discontinuous implies $f \in \mathscr{R}(\alpha)$.

- Theorem 6.11: $f \in \mathscr{R}(\alpha)$ on $[a,b]$, $m \leq f \leq M$, ϕ continuous on $[m,M]$, and $h(x) = \phi(f(x))$ on $[a,b]$ implies $h \in \mathscr{R}(\alpha)$ on $[a,b]$.
- Theorem 6.12:
  (a) $f_1, f_2 \in \mathscr{R}(\alpha)$ on $[a,b]$ and $c \in \mathbb{R}$ imply $f_1 + f_2 \in \mathscr{R}(\alpha)$ and $cf_1 \in \mathscr{R}(\alpha)$ with
  $$\int_a^b (f_1 + f_2)\,\mathrm{d}\alpha = \int_a^b f_1\,\mathrm{d}\alpha + \int_a^b f_2\,\mathrm{d}\alpha$$
  $$\int_a^b cf_1\,\mathrm{d}\alpha = c\int_a^b f\,\mathrm{d}\alpha$$
  (b) $f_1(x) \leq f_2(x)$ on $[a,b]$ implies
  $$\int_a^b f_1\,\mathrm{d}\alpha \leq \int_a^b f_2\,\mathrm{d}\alpha$$
  (c) $f \in \mathscr{R}(\alpha)$ on $[a,b]$ and $a < c < b$ implies $f \in \mathscr{R}(\alpha)$ on $[a,c]$ and on $[c,b]$ and
  $$\int_a^c f\,\mathrm{d}\alpha + \int_c^b f\,\mathrm{d}\alpha = \int_a^b f\,\mathrm{d}\alpha$$
  (d) $f \in \mathscr{R}(\alpha)$ on $[a,b]$ and $|f(x)| \leq M$ on $[a,b]$ implies
  $$\left|\int_a^b f\,\mathrm{d}\alpha\right| \leq M[\alpha(b) - \alpha(a)]$$
  (e) $f \in \mathscr{R}(\alpha_1)$, $f \in \mathscr{R}(\alpha_2)$, and $c \in \mathbb{R}$ imply $f \in \mathscr{R}(\alpha_1 + \alpha_2)$ and $f \in \mathscr{R}(c\alpha_1)$ with
  $$\int_a^b f\,\mathrm{d}(\alpha_1 + \alpha_2) = \int_a^b f\,\mathrm{d}\alpha_1 + \int_a^b f\,\mathrm{d}\alpha_2$$
  $$\int_a^b f\,\mathrm{d}(c\alpha_1) = c\int_a^b f\,\mathrm{d}\alpha_1$$
- Theorem 6.13: $f, g \in \mathscr{R}(\alpha)$ on $[a,b]$ implies
  (a) $fg \in \mathscr{R}(\alpha)$;
  (b) $|f| \in \mathscr{R}(\alpha)$ with
  $$\left|\int_a^b f\,\mathrm{d}\alpha\right| \leq \int_a^b |f|\,\mathrm{d}\alpha$$
- **Unit step function**: The function $I : \mathbb{R} \to \mathbb{R}$ defined by
  $$I(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$
- Theorem 6.15: $a < s < b$, f bounded on $[a,b]$ and continuous at s, and $\alpha(x) = I(x - s)$ imply
  $$\int_a^b f\,\mathrm{d}\alpha = f(s)$$
- Theorem 6.16: $c_n \geq 0$, $\sum c_n$ converges, $\{s_n\} \subset (a,b)$, $\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$, and f continuous on $[a,b]$ implies
  $$\int_a^b f\,\mathrm{d}\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$$

- Theorem 6.17: α monotonically increasing, $\alpha' \in \mathscr{R}$ on $[a,b]$, and f bounded on $[a,b]$ implies $f \in \mathscr{R}(\alpha)$ iff $f\alpha' \in \mathscr{R}$, and $f\alpha' \in \mathscr{R}$ implies
$$\int_a^b f \,\mathrm{d}\alpha = \int_a^b f(x)\alpha'(x)\,\mathrm{d}x$$
- Rudin (1976) gives an example of the physical significance of Theorems 6.15-6.17.
- Theorem 6.19 (change of variable): Suppose φ is a strictly increasing continuous function that maps an interval $[A,B]$ onto $[a,b]$. Suppose α is monotonically increasing on $[a,b]$ and $f \in \mathscr{R}(\alpha)$ on $[a,b]$. Define β and g on $[A,B]$ by
$$\beta(y) = \alpha(\varphi(y)) \qquad\qquad g(y) = f(\varphi(y))$$
Then $g \in \mathscr{R}(\beta)$ and
$$\int_A^B g\,\mathrm{d}\beta = \int_a^b f\,\mathrm{d}\alpha$$
- Theorem 6.20: $f \in \mathscr{R}$ on $[a,b]$ and continuous at $x_0 \in [a,b]$, $a \leq x \leq b$, and $F(x) = \int_a^x f(t)\,\mathrm{d}t$ implies F continuous on $[a,b]$, F differentiable at x_0, and
$$F'(x_0) = f(x_0)$$
- Theorem 6.21 (Fundamental Theorem of Calculus): $f \in \mathscr{R}$ on $[a,b]$ and F differentiable on $[a,b]$ such that $F' = f$ imply
$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$
- Theorem 6.22 (Integration by Parts): F,G differentiable on $[a,b]$ and $(F' = f),(G' = g) \in \mathscr{R}$ imply
$$\int_a^b F(x)g(x)\,\mathrm{d}x = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\,\mathrm{d}x$$
- $\int_a^b \mathbf{f}\,\mathrm{d}\alpha$: The point in $\mathbb{R}^k$ whose j^{th} coordinate is $\int_a^b f_j\,\mathrm{d}\alpha$.
- Theorems 6.12a, 6.12c, 6.12e, 6.17, 6.20, and 6.21 are valid for vector-valued functions.
- Theorem 6.24: Theorem 6.21 for vector-valued functions.
- Theorem 6.25: Theorem 6.13b for vector-valued functions.
- **Curve** (in $\mathbb{R}^k$ on $[a,b]$): A continuous mapping $\gamma : [a,b] \to \mathbb{R}^k$.
  - Note that we define a curve in $\mathbb{R}^k$ to be a function instead of a subset of points in $\mathbb{R}^k$ that are the range of such a function since different curves may have the same range.
- **Arc**: A curve γ that is 1-1.
- **Closed curve**: A curve γ such that $\gamma(a) = \gamma(b)$.
- Let $\Lambda(P,\gamma) = \sum_{i=1}^n |\gamma(x_i) - \gamma(x_{i-1})|$ be so defined for every partition P of $[a,b]$.
- **Length** (of γ): The following quantity. *Denoted by* $\boldsymbol{\Lambda(\gamma)}$. *Given by*
$$\Lambda(\gamma) = \sup \Lambda(P,\gamma)$$
- **Rectifiable** (curve): A curve γ such that $\Lambda(\gamma) < \infty$.
- **Continuously differentiable** (curve): A curve γ whose derivative γ' is continuous.
- Theorem 6.27: γ' continuous on $[a,b]$ implies γ rectifiable and
$$\Lambda(\gamma) = \int_a^b |\gamma'(t)|\,\mathrm{d}t$$