# Chapter 1

# The Real and Complex Number Systems

## 1.1 Notes

11/1:

- Spent a lot of time trying to cheer us up regarding the midterm.
- There may be some true/false on linear algebra on the final.
- Facts:
  1. $\sqrt{2}$ is irrational.
  2. Archimedes principle: If $x > 0$ and $y \in \mathbb{R}$, then there exists $n$ such that $nx > y$.
  3. If $x > y$, then there exists $q \in \mathbb{Q}$ such that $x > q > y$.

## 1.2 Chapter 1: The Real and Complex Number Systems

*From Rudin (1976).*

11/6:

- Rudin (1976) presents several interesting proofs throughout this section that may be of interest later by means of their divergence from the ones with which I am familiar.
- **Least-upper-bound property**: The property pertaining to a set $S$ that if $E \subset S$, $E \neq \emptyset$, and $E$ is bounded above, then $\sup E \in S$.
  - For example, $\mathbb{Q}$ does not have the least-upper-bound property.
  - The **greatest-lower-bound property** is analogously defined.
- Theorem: Suppose $S$ is an ordered set with the least-upper-bound property, $B \subset S$ is nonempty, and $B$ is bounded below. Let $L$ be the set of all lower bounds of $B$. Then $\alpha = \sup L$ exists in $S$, and $\alpha = \inf B$. In particular, $\inf B$ exists in $S$.
  - Essentially, this theorem states that any set that satisfies the least-upper-bound property satisfies the greatest lower bound property.
- **Existence theorem**: There exists an ordered field $\mathbb{R}$ which has the least-upper-bound property. Moreover, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.
  - The second statement implies that the operations of addition and multiplication on $\mathbb{R}$, when applied to $\mathbb{Q}$, coincide with the operations of addition and multiplication on $\mathbb{Q}$.

- **Archimedean property** (of $\mathbb{R}$): If $x \in \mathbb{R}$, $y \in \mathbb{R}$, and $x > 0$, then there is a positive integer $n$ such that $nx > y$.
- Rudin (1976) proves several theorems about the real numbers from the least-upper-bound property as opposed to the traditional construction of the real numbers.
- Introduces the decimal system.
- **Finite real number system**: That which has been defined thus far.
- **Extended real number system**: The set $\mathbb{R} \cup \{+\infty, -\infty\}$ where $+\infty, -\infty$ obey the expected properties (supremum [resp. infimum] of every set, $x + \infty = \infty$, etc.).
- Defines the complex field axiomatically with complex numbers in the form $(a, b)$ for $a, b \in \mathbb{R}$.
  - Notes that the real numbers form a subfield of the complex field.
  - Defines $i = (0, 1)$, proves $i^2 = -1$, proves $a + bi = (a, b)$.
- **Schwarz inequality**: If $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are complex numbers, then

$$\left| \sum_{j=1}^{n} a_j \bar{b}_j \right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$$

- **Euclidean *k*-space**: The vector space $\mathbb{R}^k$ over the real field.