Chapter 5

Inner Product Spaces

10/6:

- We define
$$\ell^2(\mathbb{R}) = \left\{ \{a_n\}_{n\geq 1} \subset \mathbb{R} : \sum_1^\infty |a_n|^2 < \infty \right\}$$
- **Inner product**: A map $V \times V \to \mathbb{F}$ that takes $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \cdot \mathbf{y}$. *Denoted by* $\cdot, (\cdot, \cdot), \langle \cdot, \cdot \rangle$.
- Properties of the inner product:
  - $(\mathbf{x}, \mathbf{y}) = \overline{(\mathbf{y}, \mathbf{x})}$ (symmetry).
  - $(\alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z}) = \alpha(\mathbf{x}, \mathbf{z}) + \beta(\mathbf{y}, \mathbf{z})$ (linearity).
  - $(\mathbf{x}, \mathbf{x}) \geq 0$.
  - $(\mathbf{x}, \mathbf{x}) = 0$ iff $\mathbf{x} = 0$.
- If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then
$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n x_i y_i$$
- If $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, then
$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n x_i \bar{y}_i$$
- If $f, g \in \mathbb{P}_n(t)$, then
$$(f, g) = \int_{-1}^1 f\bar{g} \, \mathrm{d}t$$
  - The conjugate of a polynomial is the polynomial with the conjugate of the coefficients of the original polynomial. Symbolically, if $f = \sum_{i=0}^n \alpha_i t^i$ is a polynomial, then $\bar{f} = \sum_{i=0}^n \bar{\alpha}_i t^i$.
- It is a fact that
$$\left| \sum^\infty a_n \bar{b}_n \right| \leq \|(a_n)_{n\geq 1}\| \|(b_n)_{n\geq 1}\|$$
- Suppose we want to define the inner product between two matrices.
  - A common one is
$$(A, B) = \mathrm{tr}(B^* A)$$
where $B^* = \bar{B}^T = \overline{B^T}$ is the conjugate transpose.

- We define the norm as a function $V \to [0, \infty)$ given by

$$\|\mathbf{x}\| = \sqrt{(\mathbf{x}, \mathbf{x})}$$

- Properties of the norm.
  - $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$.
  - $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.
  - $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
- In $\mathbb{R}^n$,

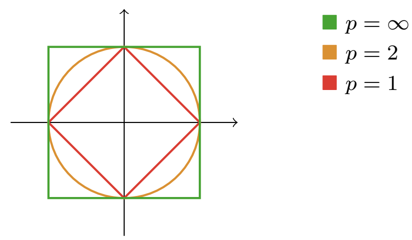

Figure 5.1: The unit ball of norms corresponding to $p = 1, 2, \infty$.

  - The standard norm is
$$\|\mathbf{x}\| = \sqrt{\sum |x_i|^2}$$
  - We can also define
$$\|\mathbf{x}\|_p = \sqrt[p]{\sum |x_i|^p}$$
  - We can even define
$$\|\mathbf{x}\|_\infty = \max |x_i|$$
  - And we can prove that all of these are valid norms.
  - Only the norm corresponding to ℓ^2 is given by an inner product, but all the other quantities are still norms as defined by the properties (see Treil (2017)).
  - Figure 5.1 shows the unit ball of each norm, i.e., the set of all points which have norm 1.
- The parallelogram rule:
$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)$$
- Orthogonality: Given $\mathbf{v}, \mathbf{w}$, if $\mathbf{v} \perp \mathbf{w}$, then $(\mathbf{v}, \mathbf{w}) = 0$.
- In particular, if $\mathbf{v} \perp \mathbf{w}$, then
$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$$
- Let E be a subspace of V. If $\mathbf{v} \perp E$, then $\mathbf{v} \perp \mathbf{e}$ for all $\mathbf{e} \in E$, i.e., $\mathbf{v} \perp$ a set of vectors spanning E.
- Any set of orthogonal vectors is linearly independent. Thus, if V is n dimensional, then $\mathbf{v}_1, \dots, \mathbf{v}_n$ orthogonal is a basis.
- Let E be a subspace of V. Take $\mathbf{v} \in V$. We want to define the projection $P_E\mathbf{v}$ of $\mathbf{v}$ onto E.
  - We have that $P_E\mathbf{v} \in E$ and $v - P_E\mathbf{v} \perp E$.
  - Additionally, we have that
$$\|\mathbf{v} - P_E\mathbf{v}\| \leq \|\mathbf{v} - \mathbf{e}\|$$
    for all $\mathbf{e} \in E$.

  - Lastly, we have that $P_E\mathbf{v}$ is unique.
- If we receive a basis of a vector space, how do we create out of that a basis that is orthogonal? The process of doing this is called **Gram-Schmidt orthogonalization**.
  - We keep $\mathbf{v}_1$, subtract $P_{\mathbf{v}_1}\mathbf{v}_2$ from $\mathbf{v}_2$, subtract $P_{\{\mathbf{v}_1,\mathbf{v}_2\}}\mathbf{v}_3$ from $\mathbf{v}_3$, and on and on.
- If we are given a set of orthogonal vectors, we can normalize them by dividing each by its norm. This creates an orthonormal list. The standard basis is orthonormal.
- Let
$$E^\perp = \{v \in V : v \perp E\}$$
  - It follows that $V = E \oplus E^\perp$.
- How close can we come to solving $A\mathbf{x} = \mathbf{b}$ if we cannot solve it exactly (i.e., if the columns are not linearly independent)?
  - Let A be an $m \times n$ matrix, and let $\mathbf{b} \in \mathbb{R}^m$.
  - Then the best solution is given by minimizing $\|A\mathbf{x} - \mathbf{b}\|$. We minimize this with projections. A special case of this is least squares regression! More details in Treil (2017).

10/8:

- Soug is gonna send us a hefty amount of reading for the weekend.
- Least square approximation:
  - If we want to minimize $\|A\mathbf{x} - \mathbf{b}\|$, the best we can do is project $\mathbf{b}$ onto the range of A.
  - Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be an orthogonal basis of range A.
  - Then
$$\text{Proj}|_{\text{range } A}\mathbf{b} = \sum^{k} \frac{(\mathbf{b}, \mathbf{v}_k)}{\|v_k\|^2}\mathbf{v}_k$$
  - Matrix equation form:
$$\text{Projection}_{\text{range } A} = A(A^*A)^{-1}A^*$$
    if A^*A is invertible, where $A^* = \bar{A}^T$.
    - Soug never uses this though.
  - The minimum is found when $\mathbf{b} - A\mathbf{x} \perp \text{range } A$. Implies that $\mathbf{b} - A\mathbf{x} \perp \mathbf{a}_k$ for all k. Implies $(\mathbf{b} - A\mathbf{x}, \mathbf{a}_k) = \bar{\mathbf{a}}_k^T(\mathbf{b} - A\mathbf{x}) = 0$.
  - Note that we're letting $\bar{\mathbf{a}}_k^T$ be the row vector
$$\bar{\mathbf{a}}_k^T = \begin{pmatrix} \bar{a}_{1,k} & \cdots & \bar{a}_{n,k} \end{pmatrix}$$
  - We also have $\bar{A}^T(\mathbf{b} - A\mathbf{x}) = 0$, from which it follows that $A^*A\mathbf{x} = A^*\mathbf{b}$, so $\mathbf{x} = (A^*A)^{-1}A^*\mathbf{b}$. Thus, $\text{Proj}|_{\text{range } A} = Ax$, so $\text{Proj}|_{\text{range } A} = A(A^*A)^{-1}A^*\mathbf{b}$.
- Adjoint of a linear map $T : V \to W$ is the A^* discussed above.
  - First, we'll do this for matrices. And then we'll do it for any finite-dimensional vector space.
  - Let A be an $m \times n$ matrix. We claim then that
$$(A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, A^*\mathbf{y})$$
    for all $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{y} \in \mathbb{C}^m$. Proof:
$$\begin{aligned}(A\mathbf{x}, \mathbf{y}) &= \bar{\mathbf{y}}^T A\mathbf{x} \\ &= \mathbf{y}^* A\mathbf{x} \\ &= (A^*\mathbf{y})^*\mathbf{x} \\ &= (\mathbf{x}, A^*\mathbf{y})\end{aligned}$$

- Properties of the adjoint:

$$(AB)^T = B^T A^T$$
$$(AB)^* = B^* A^*$$
$$(A^*)^* = A$$

- A^* is the unique matrix B such that $(A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, B\mathbf{y})$.
- Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be a basis of V, and let $\mathbf{w}_1, \dots, \mathbf{w}_m$ be a basis of W.
- Definition of A^*: If $(A\mathbf{x}, \mathbf{y}) = (y, A^*\mathbf{x})$ for all $\mathbf{x} \in V$ and $\mathbf{y} \in W$.
- But it's not enough to define something; we have to check that it exists.
- If $[A]_{\mathcal{AB}}$, then $[A^*]_{\mathcal{AB}}$.
- More properties (give criteria for solving systems of equations):

$$\ker A^* = (\operatorname{range} A)^\perp$$
$$\ker A = (\operatorname{range} A^*)^\perp$$
$$\operatorname{range} A = (\ker A^*)^\perp$$
$$\operatorname{range} A^* = (\ker A)^\perp$$

  - Soug proves these.

- Isometries and unitary operators.
  - $U : X \to Y$ is an isometry if $\|\mathbf{x}\| = \|U\mathbf{x}\|$ for all $\mathbf{x} \in X$. It is an isometry because it preserves the distance between points.
  - It immediately follows that $\|\mathbf{x}_1 - \mathbf{x}_2\| = \|U\mathbf{x}_1 - U\mathbf{x}_2\| = \|U(\mathbf{x}_1 - \mathbf{x}_2)\|$.
  - This definition is equivalent to an inner product one: $(\mathbf{x}, \mathbf{y}) = (U\mathbf{x}, U\mathbf{y})$. This follows from the definition of the norm.
  - We have

$$(\mathbf{a}, \mathbf{b}) = \frac{1}{4} \sum_{\alpha = \pm 1, \pm i} \alpha \|\mathbf{a} + \alpha \mathbf{b}\|^2$$

    - $(a+b)^2 - (a-b)^2 = 4ab$ for any $a, b \in \mathbb{R}$, so $ab = \frac{1}{4}[(a+b)^2 - (a-b)^2]$. Thus, in a real inner product space,

$$(\mathbf{a}, \mathbf{b}) = \frac{1}{4} \left(\|\mathbf{a} + \mathbf{b}\|^2 - \|\mathbf{a} - \mathbf{b}\|^2 \right)$$

    - It follows that isometries preserve inner products.
  - U is an isometry if and only if $U^*U = I$. Proof:

$$(\mathbf{x}, \mathbf{x}) = (U\mathbf{x}, U\mathbf{y}) = (U^*U\mathbf{x}, \mathbf{x})$$
$$(\mathbf{x}, \mathbf{y}) = (U\mathbf{x}, U\mathbf{y}) = (U^*U\mathbf{x}, \mathbf{y})$$
$$(\mathbf{x}, \mathbf{y}) = (U\mathbf{x}, U\mathbf{y}) = (\mathbf{x}, \mathbf{y})$$

    for all $\mathbf{y}$.
  - An isometry is unitary if it is invertible.
    - Thus, $U : X \to Y$ an isometry is unitary iff $\dim X = \dim Y$.
  - Note that it follows that $U^* = U^{-1}$ for U an isometry.
  - U unitary implies $|\det U| = 1$, so λ an eigenvalue of U implies that $|\lambda| = 1$.
  - A is diagonalizable iff it has an orthogonal basis of eigenvectors.

10/11: - Spectral decomposition of self-adjoint linear maps.

- Can we write a map in term of the eigenvalues only?
- Let $A : X \to X$ be linear and self-adjoint. Where $\dim X < \infty$.
- Let A have eigenvalues $\lambda_1, \dots, \lambda_n$ and eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$. The there is an orthonormal basis of X consisting of eigenvectors of A. An operator is self-adjoint if $A = A^*$.
- If A is self-adjoint, then A can be written as diagonal with the eigenvalues on the diagonal with respect to some orthonormal basis of eigenvectors.
- Let $\mathbb{F} = \mathbb{C}$.

- If there exists an orthonormal basis $u_1, \dots, u_n$ of X such that A is triangular, then $A = UTU^*$ where U is unitary and T is upper triangular.
  - Proved with induction on $\dim X$.
  - $\dim X = 1$ is clear.
  - Assume for $\dim X = n - 1$, WTS for $\dim X = n$.
  - The subspace has a basis $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ such that A has a diagonal form.
  - Let $u \in X$ be linearly independent of $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$.
  - Let λ be the remaining eigenvalue and u the corresponding eigenvector. Let $E = \operatorname{span}(u)$. Then make the matrix λ in the upper left corner, and block diagonal with "A_{n-1}" in the bottom right corner, zeroes everywhere else.
- **Self-adjoint** (matrix A): A linear map $A : X \to X$ where $\dim X < \infty$ such that $A = A^*$.
  - Similarly, $(Ax, y) = (x, Ay)$.
  - A self-adjoint implies all eigenvalues are real, eigenvectors corresponding to different eigenvalues are orthogonal.
    - Soug proves this.
- **Strictly positive** (operator A): A self-adjoint operator $A : X \to X$ such that $(Ax, x) > 0$ for all $x \neq 0$. *Also known as* **positive definite**.
  - Implies that all eigenvalues are strictly positive.
- **Nonnegative** (operator A): A self-adjoint operator $A : X \to X$ such that $(Ax, x) \geq 0$ for all $x \neq 0$. *Also known as* **definite**.
  - All eigenvalues are nonnegative.
- Suppose $A \geq 0$ is self-adjoint. Then there exists a unique self-adjoint $B \geq 0$ such that $B^2 = A$.
  - A self-adjoint is diagonal (wrt. some basis).
  - A positive means that all eigenvalues (diagonal entries) are positive.
  - Thus, take
$$B = \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sqrt{\lambda_n} \end{pmatrix}$$
  - Suppose $B^2 = A$, $C^2 = A$. Then we have an orthonormal basis corresponding to B and an orthonormal basis corresponding to C. It follows that $B^2 = C^2 = A$. Write B^2x and C^2x in terms of their bases; will necessitate that the bases are the same.

10/13:

- If we get yes/no questions, we don't have to justify.
- Cauchy-Schwarz inequality:
$$|(\mathbf{x}, \mathbf{y})| \leq \|\mathbf{x}\| \|\mathbf{y}\|$$

  - Real spaces, V vs. $(\cdot, \cdot)$ inner product.
  - Proof:

$$\begin{aligned} 0 &\leq \|\mathbf{x} + t\mathbf{y}\|^2 \\ &= t^2\|\mathbf{y}^2\| + 2t(\mathbf{x}, \mathbf{y}) + \|\mathbf{x}\|^2 \end{aligned}$$

    Thus, the discriminant must be less than zero (because the whole polynomial is positive, so the discriminant [the opposite of the x^0 term of the factored form of the polynomial] must be less than zero so the polynomial doesn't get dragged down to negative values):

$$(\mathbf{x}, \mathbf{y})^2 - \|\mathbf{x}\|^2\|\mathbf{y}\|^2 \leq 0$$

    Taking square roots of both sides proves the desired inequality.

- Recall that if $A^* = A$, then all eigenvalues are real and all eigenvectors of distinct eigenvalues are orthogonal to each other.
- **Normal** (matrix): A matrix N such that $N^*N = NN^*$.
  - Examples: Diagonal, self-adjoint, and unitary operators are all normal.
- Any normal operator in a complex vector spae has an orthonormal set of eigenvectors, e.g., $N = UDU^*$.
  - Proof: N is upper triangular wrt. some basis (because all matrices are). WTS any normal upper triangular matrix is diagonal. Done by induction on the dimension of N from $n = 2$.
  - Assume the claim for every $(n-1) \times (n-1)$ normal upper triangular matrix.
  - Let

$$N = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & & & \\ 0 & & & \\ \vdots & & N_1 & \\ 0 & & & \end{pmatrix}$$

    (we know every normal matrix can be written in this upper triangular form)
  - Then just compute NN^* and N^*N. Knowing they have to be equal, we have that $a_{12} = \cdots = a_{1n} = 0$.
  - We can also prove from the above (block diagonal multiplication) that N_1 is normal. Thus, it's diagonal, too. Therefore, the whole thing is diagonal.
- N is normal if and only if $\|N\mathbf{x}\| = \|N^*\mathbf{x}\|$.
  - Proof: $(N\mathbf{x}, N\mathbf{y}) = (N^*N\mathbf{x}, \mathbf{y}) = (NN^*\mathbf{x}, \mathbf{y}) = (N^*\mathbf{x}, N^*\mathbf{y})$. This is equivalent to the desired condition.
- If A is nonnegative and $(A\mathbf{e}_k, \mathbf{e}_k) = a_{kk}$, then

$$\sum_{i,j=1}^{n} a_{ij}\mathbf{x}_i\mathbf{x}_j$$

- **Positive definite** (matrix): An $n \times n$ self-adjoint matrix such that $(A\mathbf{x}, \mathbf{x}) > 0$ for all $\mathbf{x} \in X$.
- Let $A : X \to Y$, $\dim X = \dim Y$. Then AA^* is positive semidefinite. And there exists a unique square root $R = \sqrt{A^*A}$.
  - Proof: $(A^*A\mathbf{x}, \mathbf{x}) = (A\mathbf{x}, A\mathbf{x}) = \|A\mathbf{x}\|^2 \geq 0$.
- **Modulus** (of A): The matrix $|A| = \sqrt{A^*A}$.

- Check $\||A|\mathbf{x}\| = \|A\mathbf{x}\|$.

$$\||A|\mathbf{x}\|^2 = (|A|\mathbf{x}, |A|\mathbf{x}) = (|A|^*|A|\mathbf{x}, \mathbf{x}) = (A^*A\mathbf{x}, \mathbf{x}) = (A\mathbf{x}, A\mathbf{x}) = \|A\mathbf{x}\|^2$$

- Let $A : X \to X$ be a linear operator. Then $A = U|A|$ where U is unitary.
- Look at singular matrices.