

Chapter 2

Basic Topology

2.1 Notes

11/1:

- Equivalence relationships are denoted $A \sim B$.
 - These are...
 - Reflexive ($A \sim A$).
 - Symmetric ($A \sim B \iff B \sim A$).
 - Transitive ($A \sim B \ \& \ B \sim C \implies A \sim C$).
 - Equivalence relations give rise to equivalence classes.
- **Countable** (set A): A set A such that $A \sim \mathbb{N}$, in the sense that there exists a one-to-one and onto map from $\mathbb{N} \rightarrow A$.
 - Alternatively, A can be written in the form $A = \{f(n) : n \in \mathbb{N}\}$.
- **Finite countable** vs. **infinite countable** (see Rudin (1976)).
- \mathbb{N} denotes the natural numbers.
- \mathbb{N}_0 denotes the natural numbers including 0.
- \mathbb{Z} denotes the integers.
- We know that $\mathbb{N} \sim \mathbb{Z}$: Let $f : \mathbb{N} \rightarrow \mathbb{Z}$ be defined by

$$f(n) = \begin{cases} \frac{n}{2} & n \text{ even} \\ \frac{n-1}{2} & n \text{ odd} \end{cases}$$

- More facts.
 1. Every subset of a countable set is countable.
 2. Unions of countable sets are countable.
 - If the sets E_n for some finite list of numbers are countable, then $\bigcup_n E_n$ is countable.
 - Soug goes over the diagonalization method of counting.
 3. n -fold Cartesian products of countable sets are countable (we induct on n).
 - If A is countable and B is countable, then $A \times B$ is countable.
 - If A is finite and to each $\alpha \in A$ we assign a countable set E_α , $\otimes_{\alpha \in A} E_\alpha$ is countable.
- **Metric space**: A space X along with a matrix $d : X \times X \rightarrow [0, \infty)$ such that

- $d(x, y) > 0$ iff $x \neq y$, and $d(x, x) = 0$ iff $x = 0$.
- $d(x, y) = d(y, x)$.
- $d(x, y) \leq d(x, z) + d(z, y)$.

- Example (\mathbb{R}^n):

- We may define d by

$$d(x, y) = \sqrt{\sum (x_i - y_i)^2}$$

- We can also define the p -metrics (recall normed spaces) with p where 2 is.

- Example ($X_p = \{f : Y \rightarrow \mathbb{R} : 1 \leq p < \infty, \int_Y |f|^p dy < \infty\}$):

- This is ℓ_p .
- Define

$$\|f - g\|_p = \left[\int_Y |f - g|^p dy \right]^{1/p}$$

- Convergence: $x_n \rightarrow x \iff d(x_n, x) \rightarrow 0$.

- **Neighborhood**: The set of all points a distance less than r away from p . Denoted by $N_r(p)$. Given by

$$N_r(p) = \{q \in X : d(p, q) < r\}$$

- **Limit point** (of E): A point p such that every neighborhood of p intersects E at a point other than p . Also known as **accumulation point**.

- Symbolically,

$$N_r(p) \cap (E \setminus \{p\}) \neq \emptyset$$

for all $r > 0$.

- **Isolated point** (of E): A point p such that $p \in E$ and p is not a limit point of E .

- **Closed** (set E): A set E that contains all of its limit points.

- **Interior** (point p): A point p such that there exists $N_r(p) \subset E$.

- **Open** (set E): A set E , all points of which are interior points.

- **Perfect** (set E): A set E that is closed and every point of E is a limit point of E .

- **Bounded** (set E): There exists a number M and a $y \in X$ such that $E \subset \{p : d(p, y) \leq M\}$.

- **Dense** (set E in X): A set E such that every point of X is a limit point of E or a point of E , itself.

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- Every neighborhood is an open set.

- If p is a limit point of E , every neighborhood of p contains infinitely many points of E .

- Thus, a finite set cannot have a limit point.

- Prove by contradiction: Suppose there is a neighborhood that contains only finitely many points of E . Then the neighborhood with radius smaller than the distance to the closest point does not contain any points of E , a contradiction.

- E is open iff $E^{c[1]}$ is closed.

- Assume E^c closed. If $p \in E$, then p is not a limit point of E^c . It follows that there exists a neighborhood of p that is entirely contained within E , so p is interior, as desired.

¹The complement of E .

- Suppose E is open. Let p be any limit point of E^c . Then $p \in E^c$.
- F is closed iff F^c is open.
- If $(G_\alpha)_{\alpha \in A}$ is a family of open sets in X , then the union is open.
 - Let $p \in \bigcup_{\alpha \in A} G_\alpha$. Then $p \in G_\alpha$ for some $\alpha \in A$. It follows that p is an interior point of G_α , so thus an interior point of the union of G_α with everything else.
- Finite intersections of open sets are open.
 - In the infinite case $\bigcap_{n \in \mathbb{N}} (-1/n, 1/n) = \{0\}$, an intersection of infinitely many open sets is closed.
 - However, in the finite case, just consider the neighborhood with the smallest radius and take this one.
- The intersection of closed sets is closed.
- The union of finitely many closed sets is closed.
 - These follow from the previous two by De Morgan's rule.
- Let $\bar{E} = E \cup E'$ where E' is the set of limit points of E .
- Let X be a metric space and $E \subset X$. Then
 1. \bar{E} is closed.
 - WTS: \bar{E}^c is open. Let $p \in \bar{E}^c$. Then p is neither in E nor is it a limit point of E . Thus, there exists a neighborhood of \bar{E}^c containing entirely points of \bar{E}^c . Therefore, \bar{E}^c is open, so \bar{E} is closed.
 2. $E = \bar{E}$ iff E is closed.
 - Think $p \in \bigcap G_\alpha$?
 3. $\bar{E} \subset F$ for any closed $F \supset E$.
 - If $E \subset F$, then any limit point of E will be a limit point of F . Thus, $E' \subset F'$. Then $\bar{E} = E \cup E' \subset F \cup F' = \bar{F} = F$ where the last equality holds because F is closed.
- Types of sets.

	Closed	Open	Perfect	Bounded
$\{z \in \mathbb{Q} : z < 1\}$	N	Y	N	Y
$\{z \in \mathbb{Q} : z \leq 1\}$	Y	N	Y	Y
Nonempty finite set	Y	N	N	Y
\mathbb{Z}	Y	N	N	N
$\{1/n : n \in \mathbb{N}\}$	N	N	N	Y
\mathbb{R}^2	Y	Y	Y	N
(a, b)	N	?	N	Y

Table 2.1: Types of sets.

- **Relatively open** (set E to Y): A set $E \subset Y \subset X$ such that if $p \in E$, then there exists a Y -neighborhood of E contained in E .
- Let $N_r^X(p) = \{y \in X : d(y, p) < r\}$ be a neighborhood of p in X , and let $N_r^Y(p) = \{y \in Y : d(y, p) < r\}$ be a neighborhood of p in Y . Then $N_r^Y(p) = N_r^X(p) \cap Y$.

- E is open relative to Y iff $E = G \cap Y$ where G is open relative to X .
- Introduces the supremum.
- If $E \subset \mathbb{R}$, $E \neq \emptyset$, and E is bounded above, $\sup E < \infty$.
- Let $y = \sup E$. Then $y \in \bar{E}$.
- There exists a sequence $a_n \in A$ such that $a_n \rightarrow x = \sup A$.
- A is compact iff any open cover of the set has a finite subcover.
- Study and *know* all of these proofs.

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- Compactness: Defines compactness in terms of open covers.
- Finite sets are compact.
- Compactness is “absolute” (i.e., it is not a relative property like openness).
 - If $K \subset Y \subset X$, then K is compact relative to X iff K is compact relative to Y .
 - V is open relative to Y iff $V = G \cap Y$ where G is open relative to X .
- Compact implies closed.
 - We will show K compact implies K^c open.
 - WTS: For all $p \in K^c$, there exists $N_r(p) \subset K^c$ such that $N_r(p) \cap K = \emptyset$.
- A closed subset of a compact set is compact.
 - Let K be compact and let $F \subset K$ be closed.
 - Take any open cover of F . Extend it to an open cover of K . Take the finite subcover of K . Naturally, this finite subcover is also a finite cover of $F \subset K$.
- F closed, K compact implies $F \cap K$ compact.
- If $(K_\alpha)_{\alpha \in A}$ is compact in X with finite intersection property (every intersection of any finite number of these sets is nonempty), then $\bigcap_{\alpha \in A} K_\alpha \neq \emptyset$.
 - Argue by contradiction.
 - Let $G_\alpha = K_\alpha^c$.
 - Assume the intersection is empty. Assume WLOG that no point of K_1 is in any of the other K_α ’s.
 - Then $\{G_\alpha\}_{\alpha \in A}$ be an open cover of K_1 .
 - K_1 compact implies there is a finite subcover $G_{\alpha_1}, \dots, G_{\alpha_n}$. Then $K \subset G_{\alpha_1} \cup \dots \cup G_{\alpha_n}$. This implies that $K_1 \cap K_{\alpha_1} \cap \dots \cap K_{\alpha_n} = \emptyset$, a contradiction.
- Let E be an infinite subset of a compact K . Then E has a limit point in K .
 - Argue by contradiction.
 - Suppose for all $p \in K$, there exists $N_r(p)$ such that $N_r(p) \cap E = \{p\}$.
 - Consider the set $\{N_r(p) : p \in K\}$. This is an open cover of K . Thus, there exists a finite subcover of it. But since $E \subset K \subset N_{r_1}(p_1) \cup \dots \cup N_{r_n}(p_n) = \{p_1\} \cup \dots \cup \{p_n\}$, E is finite, a contradiction.
- **2-cell** (in \mathbb{R}^2): A set that is the Cartesian product of two closed intervals.
 - Generalizes to **k-cells**.
- Let $I_n = [a_n, b_n] \subset \mathbb{R}$ such that $I_{n+1} \subset I_n$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

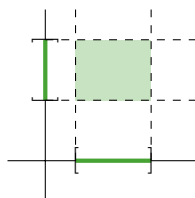
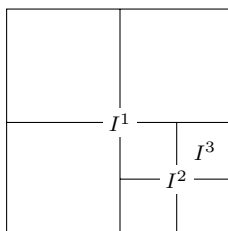


Figure 2.1: 2-cell.

- Let I_k be a k -cell in \mathbb{R}^k such that $I_k \supset I_{k+1}$. Then $\bigcap_k I_k \neq \emptyset$.
- We know that $a_m \leq a_{m+n} \leq b_{m+n} \leq b_m$, so $\sup a_n \in \bigcap I_n$.
- Every k -cell is compact.

Figure 2.2: k -cells are compact.

- Argue by contradiction.
- Consider an open cover of the k -cell I^1 . If it has a finite subcover, we're done. So suppose we have an open cover that doesn't have a finite subcover. Split the k -cell into 2^k chunks. At least one of the chunks I^2 must not have a finite subcover.
- Split that one into 2^k chunks. At least one of the chunks I^3 must not have a finite subcover.
- Continue.
- Thus, we have a decreasing family of k -cells, so by the previous result, their $\bigcap I^n \neq \emptyset$.
- Let $x \in \bigcap I^n$. Then the...
- Heine-Borel theorem: Let $E \subset \mathbb{R}^k$. Then TFAE^[2]
 1. E is closed and bounded.
 2. E is compact.
 3. Every infinite subset of E has a limit point in E .
 - $(1 \Rightarrow 2)$ E closed and bounded implies E is a closed subset of some I_k , so it's compact.
 - $(2 \Rightarrow 3)$ Already done.
 - $(3 \Rightarrow 1)$
 - Suppose E not bounded. Then there is an infinite sequence of points in E that never converges. Contradiction.
 - Suppose E is not closed. Then there exists a sequence of points in E which “converges” to an $x_0 \notin E$.

²The following are equivalent.

2.2 Chapter 2: Basic Topology

From Rudin (1976).

- 11/6:
- **Countable** (set A): A set A that is in bijective correspondence with the set of all positive integers. *Also known as enumerable, denumerable.*
 - **At most countable** (set A): A set A that is finite or countable.
 - An alternative definition of an **infinite** set would be a set that is equivalent to one of its proper subsets.
 - Theorem: Let A be the set of all sequences whose elements are the digits 0 and 1. This set A is uncountable.

Proof. Let $E = \{s_1, s_2, \dots\}$ be an arbitrary countable subset of A , where each s_j is a sequence whose elements are the digits 0 and 1. Let s be the sequence, the n^{th} term of which is the opposite of the n^{th} term of s_n (i.e., if the n^{th} term of s_n is 0, we set the n^{th} term of s equal to 1). This guarantees that s is distinct from each of the s_j , i.e., that $s \notin E$. It follows that $E \subsetneq A$, i.e., that every countable subset of A is a proper subset of A . Therefore, A must be uncountable (for otherwise A would be a proper subset of A , a contradiction). \square

- The idea of this proof is called **Cantor's diagonalization process**.
- Since every real number can be represented as a binary sequence of numbers, i.e., $A \sim \mathbb{R}$, the reals are uncountable.
- **Metric space:** A set X such that with any two points $p, q \in X$, there is associated a real number $d(p, q)$ such that
 1. $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$.
 2. $d(p, q) = d(q, p)$.
 3. $d(p, q) \leq d(p, r) + d(r, q)$ for any $r \in X$.
- **Distance** (from $p \in X$ to $q \in X$, X a metric space): The real number $d(p, q)$.
- **Distance function:** A function $d : X \times X \rightarrow \mathbb{R}$ that sends $(p, q) \mapsto d(p, q)$. *Also known as metric.*
- Every subset of a metric space is a metric space in its own right under the same distance function.
- **Segment** (from a to b): The set of all real numbers x such that $a < x < b$. *Denoted by (a, b) .*
- **Interval** (from a to b): The set of all real numbers x such that $a \leq x \leq b$. *Denoted by $[a, b]$.*
- **k -cell:** The set of all points $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k$ whose coordinates satisfy the inequalities $a_i \leq x_i \leq b_i$ where $a_i < b_i$ for each $1 \leq i \leq k$.
 - Note that a 1-cell is an interval and a 2-cell is a rectangle.
- **Convex** (set E): A subset E of \mathbb{R}^k such that

$$\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} \in E$$

for all $\mathbf{x}, \mathbf{y} \in E$ and $0 < \lambda < 1$.

- Balls and k -cells are both convex.
- The segment (a, b) is open as a subset of \mathbb{R}^1 , but not open as a subset of \mathbb{R}^2 .
- Since compactness is not relative, while it makes no sense to talk about *open* or *closed* metric spaces, it does make sense to talk about *compact* metric spaces.

- **Weierstrass theorem:** Every bounded infinite subset of \mathbb{R}^k has a limit point in \mathbb{R}^k .
- **Theorem:** Let P be a nonempty perfect set in \mathbb{R}^k . Then P is uncountable.

Proof. Since P is nonempty and perfect, there exists a limit point of P . It follows that P is infinite.

Now suppose for the sake of contradiction that P is countable, and denote the elements of P by $\mathbf{x}_1, \mathbf{x}_2, \dots$. We now construct a sequence $\{V_n\}$ of neighborhoods, as follows. Let $V_1 = N_r(\mathbf{x}_1)$. Clearly, $V_1 \subset P$ since $\mathbf{x}_1 \in P$. It follows that since V_1 is a neighborhood that V_1 contains infinitely many points of P . Now suppose inductively that V_n has been constructed. Thus, by analogous conditions to those on V_1 , we may let V_{n+1} be a neighborhood such that (i) $\bar{V}_{n+1} \subset V_n$, (ii) $\mathbf{x}_n \notin \bar{V}_{n+1}$, and (iii) $V_{n+1} \cap P$ is nonempty. By (iii), we can continue on to construct V_{n+2} , and so on and so forth.

Let $K_n = \bar{V}_n \cap P$. Since \bar{V}_n is closed and bounded, \bar{V}_n is compact. Additionally, since $\mathbf{x}_n \notin K_{n+1}$ for each n , no point of P lies in $\bigcap_1^\infty K_n$. Thus, since each $K_n \subset P$, $\bigcap_1^\infty K_n$ is empty. But this contradicts our previous result that since each K_n is nonempty, compact, and such that $K_n \supset K_{n+1}$, $\bigcap_1^\infty K_n$ is nonempty. \square

- **Corollary:** Every interval $[a, b]$ is uncountable. In particular, \mathbb{R} is uncountable.
- **Cantor set:** The set resulting from the following construction. Let $E_0 = [0, 1]$. Remove the segment $(1/3, 2/3)$, so that $E_1 = [0, 1/3] \cup [2/3, 1]$. Now remove the middle third of these two intervals to create E_2 . Continue on indefinitely.
 - This is a perfect set in \mathbb{R}^1 which contains no segment.
- **Separated** (sets A, B): Two subsets A, B of a metric space X such that $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty.
- **Connected** (set E): A set E that is not the union of two nonempty separated sets.
- Separated sets are disjoint, but disjoint sets are not necessarily separated (consider $[0, 1]$ and $(1, 2)$).
- A subset E of \mathbb{R}^1 is connected if and only if it has the following property: If $x, y \in E$ and $x < z < y$, then $z \in E$.