Chapter 1

Basic Notions

1.1 Notes

9/27:

- **Vector space**: Basically, a set for which you have an addition and multiplication.
- $\mathbb{F}^d$ is used for $\mathbb{R}^d$ or $\mathbb{C}^d$ in Treil (2017).
- $\mathbb{P}_n$ is the vector space of polynomials up to degree n.
- $C([0,1])$ is the set of continuous functions defined on $[0,1]$, an infinite-dimensional vector space.
- **Generating set**: A subset of a vector space, all linear combinations of which generate the vector space. *Also known as* **spanning set**.
  - Any element of VS is a linear comb. of elements of the generating set.
- **Linearly independent** (list): A list of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V$ such that $\sum_{i=1}^k \alpha_i \mathbf{v}_i = 0$ implies $\alpha_i = 0$ for all i.
- **Base**: A generating set consisting of linearly independent vectors.
- Any element of a VS can be written as a *unique* linear combination of the vectors in a base.
  - If $\mathbf{x} = \sum_{i=1}^k \alpha_i \mathbf{v}_i = \sum_{i=1}^k \beta_i \mathbf{v}_i$, then $\alpha_i = \beta_i$ for all i.
- **Linear transformation**: A function $T : X \to Y$, where X, Y are VSs, such that

$$T(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha T\mathbf{x} + \beta T\mathbf{y}$$

  for all $\mathbf{x} \in X$, $\mathbf{y} \in Y$.
- Examples of linear transformations:
  - Consider $\mathbb{P}_n$. Let $Tp_n = p_n'$. This T is linear.
  - Rotation in $\mathbb{R}^d$.
    - Think graphically about two vectors $\mathbf{x}, \mathbf{y}$.
    - Rotating and summing them is the same as summing and rotating. Same for scaling.
    - Thus, rotation is actually linear!
  - Reflection as well.
- Consider $T : \mathbb{R} \to \mathbb{R}$.
  - Any linear map on the line is a line.
  - We must have $Tx = \alpha x$: $Tx = T(1x) = xT(1) = x\alpha$.

- Consider $T : \mathbb{R}^n \to \mathbb{R}^m$ linear.
  - Any linear map between $\mathbb{R}^n$ and $\mathbb{R}^m$ is linear.
  - Thus, $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$, where A is an $m \times n$ matrix.
- To find A, do the same calculation as for $Tx = \alpha x$ but more carefully:
  - Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a basis.
  - So $\mathbf{x} = \sum_{i=1}^n \alpha_i \mathbf{e}_i$.
  - Thus, $T\mathbf{x} = \sum_{i=1}^n \alpha_i T(\mathbf{e}_i)$.
  - Each $T(\mathbf{e}_i)$ is part of the matrix that we multiply by the column vector representing $\mathbf{x}$.
- Multiplication of matrices is equivalent to composition of linear maps.
- Consider $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \to \mathbb{F}^r$.
  - $T_2 \circ T_1$ is equivalent to BA, if A represents T_1 and B represents T_2. In other words, $(T_2 \circ T_1)(\mathbf{x}) = BA\mathbf{x}$ for all $\mathbf{x}$.
- Recall that if $A = (\alpha_{ij})$ and $B = (\beta_{ij})$, then $(BA)_{ij} = (\sum \beta_{ik}\alpha_{kj})$.
- Properties of multiplication:

$$(AB)C = A(BC)$$
$$A(B + C) = AB + AC$$
$$(A + B)C = AC + BC$$

  - However, it is not true in general that $AB = BA$.
- **Trace** (of an $n \times n$ matrix A): The sum of the diagonal entries of A. *Denoted by* $\mathbf{tr}\,(\boldsymbol{A})$. *Given by*

$$\operatorname{tr}(A) = \sum \alpha_{ii}$$

- It is true that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.
  - Indeed, on the diagonals, multiplication is commutative; it's the other terms that mess you up in general.
- Invertibility of matrices.
  - In general, matrices are not invertible: Not every system of equations is solveable; $Ax = b$ does not always have a solution $x = A^{-1}b$.
- C is the inverse from the left: $CA = I$. B is the inverse from the right: $AB = I$. A matrix can have a left and a right inverse and still not be invertible. A matrix is invertible iff $C = B$.
- Any time we write "inverse," we do so under the assumption that it exists.
- $(AB)^{-1} = B^{-1}A^{-1}$ — easy proof by multiplication.
- If $A = (a_{ij})$, $A^T = (a_{ji})$.
  - $(A^{-1})^T = (A^T)^{-1}$.
  - $(AB)^T = B^T A^T$.
- Let X, Y VS.
  - $X \cong Y$[1] if there exists a linear $T : X \to Y$ that is one-to-one and onto.
  - Check: A(basis of X) = basis of Y. Prove by definition and expression of elements as linear combinations.
- **Subspace**: A subset of a vector space which happens to be a vector space, itself.

[1] "X is isomorphic to Y."

1.2 Chapter 1: Basic Notions

From Treil (2017).

10/24:

- **Coordinates** (of $\mathbf{v} \in V$ wrt. a basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ of V): The unique scalars $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ such that $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$.
- **Spanning system**: A list of vectors that spans V. *Also known as* **generating system**, **complete system**.
- **Trivial** (linear combination): A linear combination $\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$ of vectors such that $\alpha_k = 0$ for each $k = 1, \dots, n$.
- **Transformation**: A function $T : X \to Y$. *Also known as* **transform**, **mapping**, **map**, **operator**, and **function**.
- The matrix of a linear transformation T is often denoted by $[\boldsymbol{T}]$.
- To compute the reflection of vectors over an arbitrary line through the origin in $\mathbb{R}^2$, represent the overall transformation as a composition of rotating the line to be the x-axis, reflecting over the x-axis, and rotating back.
- Theorem 1.5.1: If A is an $m \times n$ matrix and B is an $n \times m$ matrix, then
$$\operatorname{tr}(AB) = \operatorname{tr}(BA)$$
- Theorem 1.6.1: If a linear transformation is invertible, then its left and right inverses are unique and coincide.
- The column $(1, 1)^T$ is left-invertible, with one possible left inverse being $(1/2, 1/2)$.
  - Note that it is not right invertible since its left inverses are not unique (see Theorem 1.6.1).
- An invertible matrix must be square.
- **Isomorphic** (vector spaces): Two vectors spaces V, W such that there exists an isomorphism $A : V \to W$. *Denoted by* $\boldsymbol{V \cong W}$.
- Theorem 1.6.8: $A : X \to Y$ is invertible if and only if for any right side $\mathbf{b} \in Y$, the equation
$$A\mathbf{x} = \mathbf{b}$$
has a unique solution $\mathbf{x} \in X$.
- Corollary 1.6.9: An $m \times n$ matrix is invertible if and only if its columns form a basis in $\mathbb{F}^m$.
- **Linear span** (of $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$): The set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_n$. *Denoted by* $\boldsymbol{\mathcal{L}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}}$, $\mathbf{span}\,\boldsymbol{(\mathbf{v}_1, \dots, \mathbf{v}_n)}$.