# Chapter 4

# Continuity

## 4.1 Notes

11/8:
- Consider a function $f : X \to Y$ whose domain and codomain are, respectively, the metric spaces $(X, d_X)$ and $(Y, d_Y)$.
- **Limit** (of $f$ at $p$): A point $q \in Y$ such that for all $\epsilon > 0$, there exists $\delta$ such that $d_X(x, p) < \delta$ implies $d_Y(q, f(x)) < \epsilon$, where $p$ is a limit point of $X$ (otherwise, $x \nrightarrow p$).
- **Continuous** (function $f$ at $p$): A function $f$ such that $\lim_{x\to p} f(x) = f(p)$.
- $f$ is continuous on $X$ if it is continuous at every $p \in X$.
- **Uniformly continuous** (function $f$): A function $f$ such that for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \epsilon$ for all $x, y \in X$.

11/10:
- $f, g$ continuous implies $f + g$, $fg$, and $f/g$ continuous, the latter where $g(x) \neq 0$.
- If $f, g$ continuous, then $h = g \circ f$ is continuous.
- Theorem: $f : X \to Y$ is continuous iff $f^{-1}(V)$ is open in $X$ for every $V \subset Y$ open.

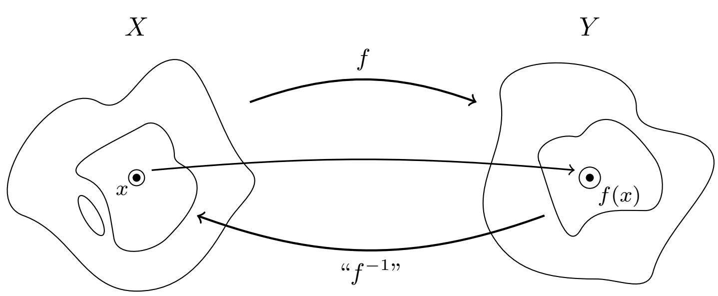


Figure 4.1: Set theoretic definition of continuity.

  - This works in a general topological space, too, not just a metric space.
  - Note that $f^{-1}(V)$ is not a function defined on $V$; it's a specifically defined set $\{x \in X : f(x) \in V\}$.
  - $f$ being continuous means that open circular neighborhood of a point $x$ in the domain maps to an area of the range encompassed by a circular neighborhood of $f(x)$.
  - The other condition means that every open set surrounding $f(x)$ maps to an open set of the domain surrounding $x$. Indeed, going off of this definition, if an open set containing $f(x)$ maps to an open set containing $x$, then we can choose a neighborhood subset of the open set surrounding $x$ and know that it will map into a neighborhood subset of the open set surrounding $f(x)$.

- Corollary: $f : X \to Y$ continuous iff $f^{-1}(C)$ closed for every $C \subset Y$ closed.
  - We use the property that $f^{-1}(X \subset C) = X \subset f^{-1}(C)$.
- Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$. Suppose $f : X_1 \times X_2 \to Y_1 \times Y_2$ is defined by $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$. Then $f$ is continuous iff $f_1, f_2$ are continuous, under appropriately defined metrics.
- Continuity and compactness.
- Theorem: $f : X \to Y$ continuous and $X$ compact imply $f(X)$ compact.
  - Let $\{V_\alpha\}$ be an open cover of $f(X)$.
  - Then $\{f^{-1}(V_\alpha)\}$ is an open cover of $X$..
  - Choose a finite subcover of $\{f^{-1}(V_\alpha)\}$. Then the corresponding $V_\alpha$'s form a finite subcover of $f(X)$.
- If $f : X \to \mathbb{R}^k$ is continuous and $X$ is compact, $f(X)$ is compact and closed/bounded.
- If $f : X \to \mathbb{R}$ is continuous and $X$ is compact, then $M = \sup_{x \in X} |f(x)| = |f(\bar{x})|$ and $m = \inf x \in X |f(x)| = |f(\underline{x})|$ where $\bar{x}, \underline{x} \in X$.
  - There is a subsequence $\{x_m\}$ such that $|f(x_m)| \to M$. Since $f(X)$ is compact, the limit of this sequence is in $f(X)$.
- If $f : X \to Y$ is continuous, bijective, and $X, Y$ are compact, $f^{-1} : Y \to X$ is continuous.
- Uniform continuity.
- Examples.
  - Linear functions are uniformly continuous.
  - $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = x^2$ is uniformly continuous.
  - $f : (a, \infty) \to \mathbb{R}$ defined by $f(x) = x^2$ is *not* uniformly continuous.
  - $f : (a, \infty) \to \mathbb{R}$ defined by $f(x) = 1/x$ is uniformly continuous if $a > 0$.
  - $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = 1/x$ is *not* uniformly continuous.
- **Lipschitz continuous** (function $f$ on $E \subset X$): A function such that $|f(x) - f(y)| \leq L|x - y|$ for each $x, y \in E$.
- Theorem: $f : X \to Y$ continuous and $X$ compact implies $f$ is uniformly continuous.
  - Fix $\epsilon > 0$. There exists $\delta = \delta(p) > 0$.
  - Def. of continuity: $q \in N_{\delta(p)}(p)$ implies $f(q) \in N_\epsilon(f(p))$.
  - $\{N_{\delta(p)/2}(p) : p \in X\}$ is an open cover of $X$. Choose a finite subcover. Let $\delta = \min(\delta(p_1)/2, \ldots, \delta(p_n)/2)$.
  - ...

11/12:

- $f : X \to Y$ continuous and $E \subset X$ connected implies $f(E)$ connected.
  - Suppose $f(E) = A \cup B$, $A, B$ nonempty, separated subsets of $Y$.
  - Let $G = E \cap f^{-1}(A)$, $H = E \cap f^{-1}(B)$. It follows that $E = G \cup H$, where $G, H$ nonempty.
  - $A \subset \bar{A}$ implies $G \subset f^{-1}(\bar{A})$ implies (inverse image def. of continuity) $\bar{G} \subset f^{-1}(A)$ implies $f(\bar{G}) \subset \bar{A}$.
  - $f(H) = B$ and $\bar{A} \cap B = \emptyset$ yields $\bar{G} \cap H = \emptyset$. Symmetrically, $G \cap \bar{H} = \emptyset$. This contradicts our assumption that $E$ is connected.
- Introduces monotone functions.
- Theorem: If $f$ is monotonic on $(a, b)$, then the set of points of $(a, b)$ at which $f$ is discontinuous is at most countable.
  - ...

## 4.2 Chapter 4: Continuity

*From Rudin (1976).*

11/8:

- **Limit** (of $f$ at $p$): The point $q \in Y$, if it exists, such that for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d_Y(f(x), q) < \epsilon$ for all points $x \in E$ for which $0 < d_X(x, p) < \delta$, where $(X, d_X), (Y, d_Y)$ are metric spaces, $E \subset X$, $f : E \to Y$, and $p \in E'$. *Denoted by* $\boldsymbol{\lim_{x \to p} f(x)}$.
  - Note that we do not require that $p \in E$; only that some elements of the domain $E$ approach $p$.
  - We also write $f(x) \to q$ as $x \to p$.
- Theorem 4.2: Let $X$, $Y$, $E$, $f$, and $p$ be as specified above. Then $\lim_{x \to p} f(x) = q$ iff $\lim_{n \to \infty} f(p_n) = q$ for every sequence $\{p_n\}$ in $E$ such that $p_n \neq p$ for any $n$ and $\lim_{n \to \infty} p_n = p$.
- Rudin (1976) proves the sum, product, and quotient rules of limits from the analogous properties of series.
- Continuity is defined.
  - Note that $f$ *does* have to be defined at $p$ to be continuous at $p$ (in comparison to the fact that it can have a limit at a point $p'$ at which it is not defined).
    - Thus, for proofs concerning continuity (as opposed to limits), we will consider functions $f$ the domains of which are metric spaces, not *subsets* of metric spaces.
  - It follows from the definition that if $p \in E$ is isolated, then every possible $f$ defined on $E$ is continuous at $p$.
- Theorem 4.7: Compositions of continuous functions are continuous.
- Theorem 4.8: Preimage definition of continuity.
- Theorem 4.9: If $f, g$ are complex continuous functions on $X$, $f + g$, $fg$, and $f/g$ are continuous on $X$.
- Theorem 4.10: $\mathbf{f}$ continuous implies $f_1, \dots, f_k$ continuous. Also, $\mathbf{f}, \mathbf{g} : X \to \mathbb{R}^k$ continuous implies $\mathbf{f} + \mathbf{g}$ and $\mathbf{f} \cdot \mathbf{g}$ continuous.

11/9:

- Theorem 4.14: $f$ continuous and $X$ compact implies $f(X)$ compact.
- Theorem 4.15: $\mathbf{f} : X \to \mathbb{R}^k$ continuous and $X$ compact implies $f(X)$ closed and bounded.
- Theorem 4.16: $f$ continuous and $X$ compact implies $f$ attains its minimum and maximum.
- Theorem 4.17: $f : X \to Y$ continuous, 1-1 for $X, Y$ compact implies $f^{-1} : Y \to X$ continuous.
- Theorem 4.19: $f$ continuous and $X$ compact implies $f$ uniformly continuous.
- Theorem 4.20: Compactness is a necessary condition in Theorems 4.14, 4.15, 4.16, and 4.19.
- Theorem 4.22: $f : X \to Y$ continuous and $E \subset X$ connected implies $f(E)$ connected.
- Theorem 4.23: Intermediate value theorem.
- **Right-hand limit** (of $f$ at $x$): *Denoted by* $\boldsymbol{f(x+)}$.
- **Left-hand limit** (of $f$ at $x$): *Denoted by* $\boldsymbol{f(x-)}$.
- **Discontinuity of the first kind** (of $f$ at $x$): A discontinuity of $f$ at $x$ such that $f(x+)$ and $f(x-)$ exist. *Also known as* **simple discontinuity**.
- **Discontinuity of the second kind** (of $f$ at $x$): A discontinuity of $f$ at $x$ that is not of the first kind (i.e., a discontinuity such that at least one of $f(x+)$ and $f(x-)$ does not exist).

- Theorem 4.29: If $f$ is monotonic on $(a,b)$, then $f(x+), f(x-)$ exist at every $x \in (a,b)$.
- Corollary: Monotonic functions have no discontinuities of the second kind.
- Theorem 4.30: If $f$ is monotonic on $(a,b)$, then the set of points of $(a,b)$ at which $f$ is discontinuous is at most countable.

11/20: *Proof.* Suppose first that $f$ is increasing. Let $E$ be the set of points at which $f$ is discontinuous. By Theorem 4.29, for every $x \in E$, $f(x-), f(x+)$ exist. Thus, we may pick a rational number $r(x)$ such that $f(x-) < r(x) < f(x+)$. Moreover, since $x_1 < x_2$ implies $f(x_1+) \leq f(x_2-)$, we have that $r(x_1) \neq r(x_2)$. Having established an injective function from $E$ to the rationals $\mathbb{Q}$, we know that $E$ is at most countable. The argument where $f$ is decreasing is symmetric. $\square$

- Gives an example of a function with discontinuities that are not isolated.