MATH 20700 (Honors Analysis in $\mathbb{R}^n$ I) Problem Sets

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October 13, 2021

Contents

1 Matrix Basics and Linear Systems

From Treil (2017).

Chapter 1

10/4: **1.2.** Which of the following sets (with natural addition and multiplication by a scalar) are vector spaces? Justify your answer.

a) The set of all continuous functions on the interval $[0, 1]$.

Answer. This IS a vector space.

Commutativity: If f, g are continuous functions on $[0, 1]$, then $f + g$ is continuous on $[0, 1]$ with $f + g = g + f$.

Associativity: If f, g, h are continuous functions on $[0, 1]$, then $(f + g) + h$ and $f + (g + h)$ are continuous functions on $[0, 1]$ with $(f + g) + h = f + (g + h)$.

Zero vector: Let $\mathbf{0} : [0, 1] \to [0, 1]$ be defined by $\mathbf{0}(x) = 0$ for all $x \in [0, 1]$. Then if f is any continuous function on $[0, 1]$, $f + \mathbf{0} = f$.

Additive inverse: Let f be a continuous function on $[0, 1]$. Define $g : [0, 1] \to [0, 1]$ by $g(x) = -f(x)$ for all $x \in [0, 1]$. Clearly g is still continuous on $[0, 1]$, and $f + g = \mathbf{0}$.

Multiplicative identity: Let f be a continuous function on $[0, 1]$. Then naturally $1f = f$.

Multiplicative associativity: Let f be a continuous function on $[0, 1]$, and let $\alpha, \beta \in \mathbb{F}$. Then $(\alpha\beta)f = \alpha(\beta f)$.

Distributive (vectors): Let f, g be continuous on $[0, 1]$, and let $\alpha \in \mathbb{F}$. Then $\alpha(f + g)$ and $\alpha f + \alpha g$ are continuous on $[0, 1]$ and equal.

Distributive (scalars): Let f be continuous on $[0, 1]$, and let $\alpha, \beta \in \mathbb{F}$. Then $(\alpha + \beta)f$ and $\alpha f + \beta f$ are continuous on $[0, 1]$ and equal. □

b) The set of all non-negative functions on the interval $[0, 1]$.

Answer. This IS NOT a vector space.

Not closed under inverses — $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = 1$ for all x would be a non-negative function on this interval, and $g : [0, 1] \to \mathbb{R}$ defined by $g(x) = -1$ for all x is naturally its inverse, but not an element of the set. □

c) The set of all polynomials of degree *exactly* n.

Answer. This IS NOT a vector space.

Not closed under summation — the inverse of x^n is $-x^n$, but their sum is 0, a polynomial of degree 0. □

d) The set of all symmetric $n \times n$ matrices, i.e., the set of matrices $A = \{a_{j,k}\}_{j,k=1}^n$ such that $A^T = A$.

Answer. This IS a vector space.

The condition for symmetric is $a_{j,k} = a_{k,j}$. Assume this is true for A and B. Then naturally

$$\begin{aligned}(a + b)_{j,k} &= a_{j,k} + b_{j,k} \\ &= a_{k,j} + b_{k,j} \\ &= (a + b)_{k,j}\end{aligned}$$

A symmetric argument verifies scalar multiplication. □

1.3. True or false:

a) Every vector space contains a zero vector.

Answer. True.
By definition. □

b) A vector space can have more than one zero vector.

Answer. False.
Suppose for the sake of contradiction that $0, 0'$ are two distinct zero vectors. Then

$$0 = 0 + 0' = 0'$$

a contradiction. □

c) An $m \times n$ matrix has m rows and n columns.

Answer. True.
By definition. □

d) If f and g are polynomials of degree n, then $f + g$ is also a polynomial of degree n.

Answer. False.
x^n and $-x^n$ are both polynomials of degree n, but their sum (0) is a polynomial of degree 0. □

e) If f and g are polynomials of degree at most n, then $f + g$ is also a polynomial of degree at most n.

Answer. True.
Suppose for the sake of contradiction that there exist f, g of degree at most n such that $f + g$ has degree $m > n$. Then $f + g$ has an ax^m term. Since f has degree n, it has no bx^m term. Thus, $(f + g) - f = g$ retains the ax^m term, and is of degree $m > n$, a contradiction. □

2.2. True or false:

a) Any set containing a zero vector is linearly dependent.

Answer. True.
Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be a list of vectors. If $\mathbf{v}_i = 0$, then

$$0\mathbf{v}_1 + \cdots + 0\mathbf{v}_{i-1} + 1\mathbf{v}_i + 0\mathbf{v}_{i+1} + \cdots + 0\mathbf{v}_n = \mathbf{0}$$

even though one of the coefficients isn't 0. Thus, the list is linearly dependent. □

b) A basis must contain $\mathbf{0}$.

Answer. False.
$\{1\}$ is a basis of $\mathbb{R}^1$. □

c) Subsets of linearly dependent sets are linearly dependent.

Proof. False.
$\{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\}$ is linearly dependent, but $\{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\}$ is linearly independent. □

d) Subsets of linearly independent sets are linearly independent.

Proof. True.
Suppose for the sake of contradiction that there exists a linearly dependent subset of a linearly independent list. Then there are nonzero coefficients that make a linear combination of the linearly dependent equal to zero. Thus, if we pair these coefficients to their respective vectors in a sum of the whole list, and use zero everywhere else, we will have a set of coefficients, not all zero, that make the supposedly linearly independent list sum to zero, a contradiction. □

e) If $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0}$, then all scalars α_k are zero.

Answer. False.
Let $\mathbf{v}_1, \mathbf{v}_2$ be defined by

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Then $1\mathbf{v}_1 + 1\mathbf{v}_2 = \mathbf{0}$. □

2.5. Let a system of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ be linearly independent but not generating. Show that it is possible to find a vector $\mathbf{v}_{r+1}$ such that the system $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r, \mathbf{v}_{r+1}$ is linearly independent. (Hint: Take for $\mathbf{v}_{r+1}$ any vector that cannot be represented as a linear combination $\sum_{k=1}^{r} \alpha_k\mathbf{v}_k$ and show that the system $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r, \mathbf{v}_{r+1}$ is linearly independent.)

Answer. Let $\mathbf{v}_{r+1}$ be any vector that cannot be represented as a linear combination $\sum_{k=1}^{r} \alpha_k\mathbf{v}_k$ (we are guaranteed that one exists, because otherwise $\mathbf{v}_1, \dots, \mathbf{v}_r$ would be generating). Now suppose for the sake of contradiction that the new list is linearly dependent. Then there exist coefficients $\alpha_1, \dots, \alpha_{r+1}$, not all zero, such that

$$\alpha_1\mathbf{v}_1 + \cdots + \alpha_{r+1}\mathbf{v}_{r+1} = \mathbf{0}$$

We divide into two cases ($\alpha_{r+1} = 0$ and $\alpha_{r+1} \neq 0$). If $\alpha_{r+1} = 0$, then it must be at least one of $\alpha_1, \dots, \alpha_r$ is nonzero by hypothesis. But then

$$\begin{aligned} \mathbf{0} &= \alpha_1\mathbf{v}_1 + \cdots + \alpha_r\mathbf{v}_r + \alpha_{r+1}\mathbf{v}_{r+1} \\ \mathbf{0} - 0\mathbf{v}_{r+1} &= \alpha_1\mathbf{v}_1 + \cdots + \alpha_r\mathbf{v}_r \\ \mathbf{0} &= \alpha_1\mathbf{v}_1 + \cdots + \alpha_r\mathbf{v}_r \end{aligned}$$

for a set of coefficients $\alpha_1, \dots, \alpha_r \in \mathbb{F}$, not all zero, contradicting the fact that $\mathbf{v}_1, \dots, \mathbf{v}_r$ are linearly independent. On the other hand, if $\alpha_{r+1} \neq 0$, then

$$\mathbf{v}_{r+1} = -\frac{\alpha_1}{\alpha_{r+1}}\mathbf{v}_1 - \cdots - \frac{\alpha_r}{\alpha_{r+1}}\mathbf{v}_r$$

so $\mathbf{v}_{r+1}$ can be expressed as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_r$, a contradiction. □

2.6. Is it possible that vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent, but the vectors $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3, \mathbf{w}_3 = \mathbf{v}_3 + \mathbf{v}_1$ are linearly *independent*?

Answer. No.

Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent. Then there exist coefficients $\alpha_1, \alpha_2, \alpha_3$, not all zero, such that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{0}$$

To prove that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ must be linearly dependent, as defined, it will suffice to show that there exist coefficients $\beta_1, \beta_2, \beta_3$, not all zero, such that

$$\beta_1\mathbf{w}_1 + \beta_2\mathbf{w}_2 + \beta_3\mathbf{w}_3 = \mathbf{0}$$

But we have that

$$\begin{aligned} \beta_1\mathbf{w}_1 + \beta_2\mathbf{w}_2 + \beta_3\mathbf{w}_3 &= \beta_1(\mathbf{v}_1 + \mathbf{v}_2) + \beta_2(\mathbf{v}_2 + \mathbf{v}_3) + \beta_3(\mathbf{v}_3 + \mathbf{v}_1) \\ &= (\beta_1 + \beta_3)\mathbf{v}_1 + (\beta_1 + \beta_2)\mathbf{v}_2 + (\beta_2 + \beta_3)\mathbf{v}_3 \end{aligned}$$

so to have $(\beta_1 + \beta_3)\mathbf{v}_1 + (\beta_1 + \beta_2)\mathbf{v}_2 + (\beta_2 + \beta_3)\mathbf{v}_3 = \mathbf{0}$, we need only require that

$$\beta_1 + \beta_3 = \alpha_1 \qquad \beta_1 + \beta_2 = \alpha_2 \qquad \beta_2 + \beta_3 = \alpha_3$$

Thus, choose

$$\beta_1 = \frac{1}{2}(\alpha_1 + \alpha_2 - \alpha_3) \qquad \beta_2 = \frac{1}{2}(-\alpha_1 + \alpha_2 + \alpha_3) \qquad \beta_3 = \frac{1}{2}(\alpha_1 - \alpha_2 + \alpha_3)$$

Lastly, note that we do not have $\beta_1 = \beta_2 = \beta_3 = 0$ because if we did, we could prove from that condition that $\alpha_1 = \alpha_2 = \alpha_3 = 0$, a contradiction. □

3.3. For each linear transformation below, find its matrix.

a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y)^T = (x + 2y, 2x - 5y, 7y)^T$.

Answer.

$$\begin{pmatrix} 1 & 2 \\ 2 & -5 \\ 0 & 7 \end{pmatrix}$$

□

b) $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3, x_4)^T = (x_1 + x_2 + x_3 + x_4, x_2 - x_4, x_1 + 3x_2 + 6x_4)^T$.

Answer.

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 3 & 0 & 6 \end{pmatrix}$$

□

c) $T : \mathbb{P}_n \to \mathbb{P}_n$ defined by $Tf(t) = f'(t)$ (find the matrix with respect to the standard basis $1, t, t^2, \ldots, t^n$).

Answer.

$$\begin{pmatrix} 0 & 1 & 0 & & 0 \\ 0 & 0 & 2 & & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & & n \\ 0 & 0 & 0 & & 0 \end{pmatrix}$$

□

d) $T : \mathbb{P}_n \to \mathbb{P}_n$ defined by $Tf(t) = 2f(t) + 3f'(t) - 4f''(t)$ (again with respect to the standard basis $1, t, t^2, \ldots, t^n$).

Answer.

$$\begin{pmatrix} 2 & 3 & -8 & & 0 \\ 0 & 2 & 6 & \ddots & 0 \\ 0 & 0 & 2 & \ddots & -4n(n-1) \\ \vdots & \vdots & & \ddots & 3n \\ 0 & 0 & 0 & & 2 \end{pmatrix}$$

□

3.6. The set $\mathbb{C}$ of complex numbers can be canonically identified with the space $\mathbb{R}^2$ by treating each $z = x + iy \in \mathbb{C}$ as a column $(x, y)^T \in \mathbb{R}^2$.

a) Treating $\mathbb{C}$ as a complex vector space, show that the multiplication by $\alpha = a + ib \in \mathbb{C}$ is a linear transformation in $\mathbb{C}$. What is its matrix?

Answer. Let $T : \mathbb{C} \to \mathbb{C}$ be defined by $Tx = \alpha x$. Then

$$\begin{aligned} T(x+y) &= \alpha(x+y) \\ &= \alpha x + \alpha y \\ &= Tx + Ty \end{aligned} \qquad \begin{aligned} T(\beta x) &= \alpha(\beta x) \\ &= \beta(\alpha x) \\ &= \beta Tx \end{aligned}$$

so T is linear. The matrix of T is $[\alpha]$. □

b) Treating $\mathbb{C}$ as the real vector space $\mathbb{R}^2$, show that the multiplication by $\alpha = a + ib$ defines a linear transformation there. What is its matrix?

Answer. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x,y)^T = (ax - by, ay + bx)^T$. Then

$$\begin{aligned} T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) &= T\left(\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}\right) \\ &= \begin{pmatrix} a(x_1 + y_1) - b(x_2 + y_2) \\ a(x_2 + y_2) + b(x_1 + y_1) \end{pmatrix} \\ &= \begin{pmatrix} ax_1 - bx_2 \\ ax_2 + bx_1 \end{pmatrix} + \begin{pmatrix} ay_1 - by_2 \\ ay_2 + by_1 \end{pmatrix} \\ &= T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + T\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \end{aligned} \qquad \begin{aligned} T\left(c\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) &= T\left(\begin{pmatrix} cx_1 \\ cx_2 \end{pmatrix}\right) \\ &= \begin{pmatrix} a(cx_1) - b(cx_2) \\ a(cx_2) + b(cx_1) \end{pmatrix} \\ &= c\begin{pmatrix} ax_1 - bx_2 \\ ax_2 + bx_1 \end{pmatrix} \\ &= cT\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

so T is linear. The matrix of T is

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

□

c) Define $T(x + iy) = 2x - y + i(x - 3y)$. Show that this transformation is not a linear transformation in the complex vector space $\mathbb{C}$, but if we treat $\mathbb{C}$ as the real vector space $\mathbb{R}^2$, then it is a linear transformation there (i.e., that T is a *real linear* but not a *complex linear* transformation). Find the matrix of the real linear transformation T.

Answer. To prove that T is not complex linear, note that

$$T(i \cdot 1) = T(i) = -1 - 3i \neq -1 + 2i = i(2 + i) = iT(1)$$

We can verify the T is real linear with the following.

$$\begin{aligned} T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) &= T\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} \\ &= \begin{pmatrix} 2(x_1 + y_1) - (x_2 + y_2) \\ (x_1 + y_1) - 3(x_2 + y_2) \end{pmatrix} \\ &= \begin{pmatrix} 2x_1 - x_2 \\ x_1 - 3x_2 \end{pmatrix} + \begin{pmatrix} 2y_1 - y_2 \\ y_1 - 3y_2 \end{pmatrix} \\ &= T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + T\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \end{aligned} \qquad \begin{aligned} T\left(c\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) &= T\begin{pmatrix} cx_1 \\ cx_2 \end{pmatrix} \\ &= \begin{pmatrix} 2(cx_1) - (cx_2) \\ (cx_1) - 3(cx_2) \end{pmatrix} \\ &= c\begin{pmatrix} 2x_1 - x_2 \\ x_1 - 3x_2 \end{pmatrix} \\ &= cT\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

The matrix of the real linear transformation is the following.

$$\begin{pmatrix} 2 & -1 \\ 1 & -3 \end{pmatrix}$$

□

5.3. Multiply two rotation matrices T_α and T_β (it is a rare case when the multiplication is commutative, i.e., $T_\alpha T_\beta = T_\beta T_\alpha$, so the order is not essential). Deduce formulas for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$ from here.

Answer.

$$\begin{aligned} T_\alpha T_\beta &= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \\ &= \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta \end{pmatrix} \end{aligned}$$

Since $T_{\alpha+\beta} = T_\alpha T_\beta$, we have that

$$\begin{aligned} \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{aligned}$$

□

5.5. Find linear transformations $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ such that $AB = \mathbf{0}$ but $BA \neq \mathbf{0}$.

Answer. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$$

Then

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad BA = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

□

5.8. Find the matrix of the reflection through the line $y = -2x/3$. Perform all the multiplications.

Answer. The reflection matrix T can be obtained by composing a rotation of $\mathbb{R}^2$ such that $y = -2x/3$ lines up with the x-axis, a reflection over the x-axis (a super simple reflection), and a rotation back. Let γ be the angle between the x-axis and the line $y = -2x/3$. Then

$$\begin{aligned} T &= R_{-\gamma} T_0 R_\gamma \\ &= \begin{pmatrix} \cos(-\gamma) & -\sin(-\gamma) \\ \sin(-\gamma) & \cos(-\gamma) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} \\ &= \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \end{pmatrix} \begin{pmatrix} \frac{3}{\sqrt{13}} & -\frac{2}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{13}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{5}{13} & -\frac{12}{13} \\ -\frac{12}{13} & -\frac{5}{13} \end{pmatrix} \end{aligned}$$

□

6.3. Find all left inverses of the column $(1, 2, 3)^T$.

Answer. The set of all left inverses of $(1,2,3)^T$ is the set of all 1×3 matrices (a,b,c) such that $(a,b,c)\cdot(1,2,3)^T=(1)$. In other words, it's the set of all (a,b,c) such that $a+2b+3c=1$. ☐

6.6. Suppose the product AB is invertible. Show that A is right invertible and B is left invertible. (Hint: You can just write formulas for right and left inverses.)

Answer. If AB is invertible, then there exists $(AB)^{-1}$. It follows that $(AB)(AB)^{-1}=A(B(AB)^{-1})=I$, so A is right invertible, and $(AB)^{-1}(AB)=((AB)^{-1}A)B=I$, so B is left invertible. ☐

6.8. Let A be an $n\times n$ matrix. Prove that if $A^2=\mathbf{0}$, then A is not invertible.

Answer. Suppose for the sake of contradiction there exists an A^{-1}. Then

$$I=AAA^{-1}A^{-1}=A^2A^{-2}=\mathbf{0}A^{-2}=\mathbf{0}$$

a contradiction. ☐

6.10. Write matrices of the linear transformations T_1 and T_2 in $\mathbb{F}^5$, defined as follows: T_1 interchanges the coordinates x_2 and x_4 of the vector $\mathbf{x}$, and T_2 just adds to the coordinate x_2 the quantity a times the coordinate x_4, and does not change other coordinates, i.e.,

$$T_1\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=\begin{pmatrix}x_1\\x_4\\x_3\\x_2\\x_5\end{pmatrix}\qquad\qquad T_2\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}=\begin{pmatrix}x_1\\x_2+ax_4\\x_3\\x_4\\x_5\end{pmatrix}$$

where a is some fixed number. Show that T_1 and T_2 are invertible transformations, and write the matrices of the inverses. (Hint: It may be simpler, if you first describe the inverse transformation, and then find its matrix, rather than trying to guess [or compute] the inverses of the matrices T_1, T_2.)

Answer.

$$T_1=\begin{pmatrix}1&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\0&0&0&0&1\end{pmatrix}\qquad\qquad T_2=\begin{pmatrix}1&0&0&0&0\\0&1&0&a&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}$$

The inverse transformation of T_1 exchanges x_2 and x_4 back, leaving everything else the same. The inverse transformation of T_2 subtracts ax_4 from the second slot, leaving everything else the same. Thus,

$$T_1^{-1}=\begin{pmatrix}1&0&0&0&0\\0&0&0&1&0\\0&0&1&0&0\\0&1&0&0&0\\0&0&0&0&1\end{pmatrix}\qquad\qquad T_2^{-1}=\begin{pmatrix}1&0&0&0&0\\0&1&0&-a&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}$$

☐

6.13. Let A be an invertible symmetric ($A^T=A$) matrix. Is the inverse of A symmetric? Justify.

Answer. We have that

$$\begin{aligned}A^{-1}&=((A^{-1})^T)^T\\&=((A^T)^{-1})^T\\&=(A^{-1})^T\end{aligned}$$

as desired. ☐

7.3. Let X be a subspace of a vector space V, and let $\mathbf{v} \in V$, $\mathbf{v} \notin X$. Prove that if $\mathbf{x} \in X$, then $\mathbf{x} + \mathbf{v} \notin X$.

Answer. Suppose for the sake of contradiction that $\mathbf{x} + \mathbf{v} \in X$. Then $\mathbf{x} + \mathbf{v}$ can be expressed as a linear combination of a basis of X. Similarly, $\mathbf{x}$ can be expressed as a linear combination of a basis of X. But this implies that $\mathbf{v} = \mathbf{x} + \mathbf{v} - \mathbf{x}$ can be written as a linear combination of the basis of X, a contradiction since $\mathbf{v} \notin X$, so it shouldn't be able to be written as a linear combination of a basis of X. $\square$

7.4. Let X and Y be subspaces of a vector space V. Using the previous exercise, show that $X \cup Y$ is a subspace if and only if $X \subset Y$ or $Y \subset X$.

Answer. Suppose that $X \cup Y$ is a subspace of V. Suppose for the sake of contradiction that $X \not\subset Y$ and $Y \not\subset X$. Then there exists $\mathbf{x} \in X$ such that $\mathbf{x} \notin Y$ and $\mathbf{y} \in Y$ such that $\mathbf{y} \notin X$. Consider $\mathbf{x} + \mathbf{y}$. Since $\mathbf{x} \in X$ and $\mathbf{y} \notin X$, we have by 7.3 that $\mathbf{x} + \mathbf{y} \notin X$. Similarly, we have that $\mathbf{x} + \mathbf{y} \notin Y$. But this implies that $\mathbf{x} + \mathbf{y} \notin X \cup Y$, contradiction the hypothesis that $X \cup Y$ is a subspace (and thus closed under addition).

Suppose that $X \subset Y$. To prove that $X \cup Y$ is a subspace, it will suffice to check that $\mathbf{v} \in X \cup Y$ implies $\alpha\mathbf{v} \in X \cup Y$, and $\mathbf{v}, \mathbf{w} \in X \cup Y$ implies $\mathbf{v} + \mathbf{w} \in X \cup Y$. Let $\mathbf{v} \in X \cup Y$. Then $\mathbf{v} \in X$ or $\mathbf{v} \in Y$. Either way, the fact that X and Y are subspaces guarantees that $\alpha\mathbf{v} \in X \cup Y$. Now let $\mathbf{v}, \mathbf{w} \in X \cup Y$. Since $X \subset Y$, this implies that $\mathbf{v}, \mathbf{w} \in Y$, so $\mathbf{v} + \mathbf{w} \in Y$, so $\mathbf{v} + \mathbf{w} \in X \cup Y$. The proof is symmetric if $Y \subset X$. $\square$

7.5. What is the smallest subspace of the space of 4×4 matrices which contains all upper triangular matrices ($a_{j,k} = 0$ for all $j > k$), and all symmetric matrices ($A = A^T$)? What is the largest subspace contained in both of those subspaces?

Answer. Out of the vector space V of 4×4 matrices, the smallest subspace which contains all upper triangular matrices and all symmetric matrices is V, itself. This is because any matrix can be decomposed into the sum of a symmetric matrix and an upper triangular matrix (fix the values in the lower triangle, and modify the upper triangle as needed with the upper triangular matrix), so every 4×4 matrix is in this subspace.

The largest subspace contained in both the subspace of upper triangular matrices and the subspace of all symmetric matrices is the subspace of all diagonal matrices. Adding another dimension by making a value *below* the diagonal nonzero makes the matrix in question not upper triangular, and adding another dimension by making a value *above* the diagonal nonzero makes the matrix not symmetric (as we would have to add a value below the diagonal to make it so and that would run into the problem described first). $\square$

Chapter 2

3.4. Do the polynomials $x^3 + 2x$, $x^2 + x + 1$, $x^3 + 5$ generate (span) $\mathbb{P}_3$? Justify your answer.

Answer. $1, x, x^2, x^3$ is the standard basis of $\mathbb{P}_3$. Thus, it spans $\mathbb{P}_3$. But since the given list has fewer vectors, Proposition 3.5 asserts that it cannot span $\mathbb{P}_3$. $\square$

3.5. Can 5 vectors in $\mathbb{F}^4$ be linearly independent? Justify your answer.

Answer. No — see Proposition 3.2. $\square$

3.7. Prove or disprove: If the columns of a square ($n \times n$) matrix A are linearly independent, so are the rows of $A^3 = AAA$.

Answer. Suppose A is $n \times n$ with linearly independent columns. Then by Proposition 3.1, A_e has a pivot in every column. But since A_e is square, this means it also has a pivot in every row. It follows by 3.6 that A is invertible. Thus A^{-1} exists. Consequently, A^{-3} is the inverse of A^3 since

$$A^3A^{-3} = AAAA^{-1}A^{-1}A^{-1} = I \qquad A^{-3}A^3 = A^{-1}A^{-1}A^{-1}AAA = I$$

so A^3 is invertible. Thus, 3.6 implies A_e^3 has a pivot in every row and column. But this implies that $(A^3)^T$ has a pivot in every row and column, meaning by 3.1 that the columns of $(A^3)^T$ are linearly independent, i.e., the rows of A^3 are linearly independent. ☐

5.1. True or false:

a) Every vector space that is generated by a finite set has a basis.

Answer. True.
See Proposition 2.8, Chapter 1. ☐

b) Every vector space has a (finite) basis.

Answer. False.
Consider the vector space of polynomials of any degree. ☐

c) A vector space cannot have more than one basis.

Answer. False.
Both 1 and 2 are bases of $\mathbb{R}^1$. ☐

d) If a vector space has a finite basis, then the number of vectors in every basis is the same.

Answer. True.
See Proposition 3.3, Chapter 2 ☐

e) The dimension of $\mathbb{P}_n$ is n.

Answer. False.
The standard basis of $\mathbb{P}_n$ is $1, t, t^2, \dots, t^n$, which has $n+1$ vectors. Thus, $\dim \mathbb{P}_n = n+1$. ☐

f) The dimension on $M_{m\times n}$ is $m+n$.

Answer. False.
The standard basis of $M_{m\times n}$ is the set of all matrices with a 1 in one slot and a 0 everywhere else. Thus, $\dim M_{m\times n} = m \times n$. ☐

g) If vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ generate (span) the vector space V, then every vector in V can be written as a linear combination of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ in only one way.

Answer. False.
The vectors $1, 2 \in \mathbb{R}^1$ span $\mathbb{R}^1$, but $3 = 1 + 2$ and $3 = -1(1) + 2(2)$. ☐

h) Every subspace of a finite-dimensional space is finite-dimensional.

Answer. True.
See Theorem 5.5. ☐

i) If V is a vector space having dimension n, then V has exactly one subspace of dimension 0 and exactly one subspace of dimension n.

Answer. True.
$\{0\}$ is THE unique VS of dimension 0 and a subspace of every vector space, so that part is true. On the other hand, any subspace of $\dim n$ has a basis consisting of n linearly independent, spanning elements of V. But any such list is also a basis of V, so the subspace is V. ☐

5.2. Prove that if V is a vector space having dimension n, then a system of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in V is linearly independent if and only if it spans V.

Answer. Suppose first that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is linearly independent. Then the $n \times n$ matrix A with these vectors as columns has a pivot in every column by 3.1. But since A is square, this means that it has a pivot in every row. Thus, by 3.1 again, the columns (i.e, the list $\mathbf{v}_1, \ldots, \mathbf{v}_n$) spans V.

The proof is the same in the reverse direction. □

5.6. Consider in the space $\mathbb{R}^5$ vectors $\mathbf{v}_1 = (2, -1, 1, 5, -3)^T$, $\mathbf{v}_2 = (3, -2, 0, 0, 0)^T$, $\mathbf{v}_3 = (1, 1, 50, -921, 0)^T$. (Hint: If you do part (b) first, you can do everything without any computations.)

a) Prove that these vectors are linearly independent.

Answer. If we add in $\mathbf{e}_1$ and $\mathbf{e}_3$ to the mix, then we can create the matrix

$$A = \begin{pmatrix} 1 & 3 & 0 & 1 & 2 \\ 0 & -2 & 0 & 1 & -1 \\ 0 & 0 & 1 & 50 & 1 \\ 0 & 0 & 0 & -921 & 5 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix}$$

A is already in eschelon form ($A = A_e$) and $A = A_e$ has a pivot in every column, so 3.1 implies that the vectors of A are linearly independent. □

b) Complete the system of vectors to a basis.

Answer. Using the same matrix as above, we can see that A has a pivot in every row and column, so 3.1 implies that its columns form a basis. Thus, the two vectors we added complete the system to a basis of $\mathbb{R}^5$. □

6.1. True or false:

a) Any system of linear equations has at least one solution.

Answer. False.
$y = x$ and $y = x + 1$ has no solution. □

b) Any system of linear equations has at most one solution.

Answer. False.
$y = x$ and $y = x$ has infinite solutions. □

c) Any homogeneous system of linear equations has at least one solution.

Answer. True.
$\mathbf{0}$ is always a solution. □

d) Any system of n linear equations in n unknowns has at least one solution.

Answer. False.
$y = x$ and $y = x + 1$ is a system of 2 linear equations in 2 unknowns but has no solution. □

e) Any system of n linear equations in n unknowns has at most one solution.

Answer. False.
$y = x$ and $y = x$ is a system of 2 linear equations in 2 unknowns but has infinite solutions. □

f) If the homogeneous system corresponding to a given system of linear equations has a solution, then the given system has a solution.

Answer. False.

$y = x$ and $y = x$ is the homogeneous system corresponding to $y = x$ and $y = x + 1$, and it has a solution, but the system itself does not. □

g) If the coefficient matrix of a homogeneous system of n linear equations in n unknowns is invertible, then the system has no non-zero solutions.

Answer. True.

Invertible implies pivots in every row/column by 3.1. This implies that A_{re} gives $\mathbf{0}$ as a particular solution, and the only solution to $A\mathbf{x} = \mathbf{b} = \mathbf{0}$. Thus, 6.1 implies that the set of all solutions is $\{\mathbf{x} + \mathbf{y} : \mathbf{x} \in \{\mathbf{0}\}, \mathbf{y} \in \{\mathbf{0}\}\} = \{\mathbf{0}\}$. □

h) The solution set of any system of m equations in n unknowns is a subspace of $\mathbb{R}^n$.

Answer. False.

The system $x + y = 1$ and $2x + y = 1$ has one solution in $\mathbb{R}^2$, namely $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. Since $\mathbf{0} \notin \{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\}$, the solution set is not a *subspace* of $\mathbb{R}^2$, a contradiction. □

i) The solution set of any homogeneous system of m equations in n unknowns is a subspace of $\mathbb{R}^n$.

Answer. True.

Let X be the solution set and let A be the coefficient matrix. The answer to Problem 6.1c shows that $\mathbf{0} \in X$. If $\mathbf{x} \in X$ and $\alpha \in \mathbb{F}$, then $A(\alpha\mathbf{x}) = \alpha A\mathbf{x} = \alpha\mathbf{0} = \mathbf{0}$, so $\alpha\mathbf{x} \in X$. If $\mathbf{x}, \mathbf{y} \in X$, then $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$, so $\mathbf{x} + \mathbf{y} \in X$. □

7.1. True or false:

a) The rank of a matrix is equal to the number of its non-zero columns.

Answer. False.

The rank of a matrix is equal to the number of its pivot columns since each pivot column of A is a vector in the basis of $\operatorname{Ran} A$. In particular, note that non-zero columns can still be linearly dependent. □

b) The $m \times n$ zero matrix is the only $m \times n$ matrix having rank 0.

Answer. True.

Suppose for the sake of contradiction that there exists a nonzero matrix with rank 0. The first column from the left with a nonzero entry will be a pivot column. Thus, this column will be part of the basis of $\operatorname{Ran} A$. But since this column exists, $\operatorname{Ran} A \geq 1$, a contradiction. □

c) Elementary row operations preserve rank.

Answer. True.

Elementary row operations, as left multiplictions by invertible matrices, do not affect linear independence. □

d) Elementary column operations do not necessarily preserve rank.

Answer. False.

Elementary column operations are the same as elementary row operations on the transpose, which we know preserve rank by the above. □

e) The rank of a matrix is equal to the maximum number of linearly independent columns in the matrix.

Answer. True.

Each pivot column is linearly independent, and the rank is equal to the number of pivot columns. □

f) The rank of a matrix is equal to the maximum number of linearly independent rows in the matrix.

Answer. True.
Each pivot row is linearly independent, and the rank is equal to the number of pivot rows/columns. ☐

g) The rank of an $n \times n$ matrix is at most n.

Answer. True.
Each linearly independent column contributes $+1$ to the rank, and since an $n \times n$ matrix can have at most n *columns*, it certainly cannot have more than n *linearly independent columns*. ☐

h) An $n \times n$ matrix having rank n is invertible.

Answer. True.
If an $n \times n$ matrix has rank n, then it has n pivot columns. But this implies by 3.6 that it is invertible. ☐

7.4. Prove that if $A : X \to Y$ and V is a subspace of X, then $\dim AV \leq \operatorname{rank} A$. ($AV$ here means the subspace V transformed by the transformation A, i.e., any vector in AV can be represented as $A\mathbf{v}$, $\mathbf{v} \in V$.) Deduce from here that $\operatorname{rank} AB \leq \operatorname{rank} A$. (Remark: Here, one can use the fact that if $V \subset W$, then $\dim V \leq \dim W$. Do you understand why it is true?)

Answer. We have that $AV \subset AX$, and that $AX = \operatorname{Ran} A$. Thus, by the hint, since $AV \subset \operatorname{Ran} A$, we have that $\dim AV \leq \dim \operatorname{Ran} A$. But this implies that $\dim AV \leq \operatorname{rank} A$, as desired.

The column space of B will be a subspace of X. Additionally, we naturally have that $\operatorname{Ran} AB = A \cdot C(B)$, where $C(B)$ is the colunn space of B ($AB\mathbf{x} \in A \cdot C(B)$ since $B\mathbf{x} \in C(B)$ and vice versa). Thus, by the previous result, $\operatorname{rank} AB = \dim \operatorname{Ran} AB = \dim A \cdot C(B) \leq \operatorname{rank} A$, as desired. ☐

7.6. Prove that if the product AB of two $n \times n$ matrices is invertible, then both A and B are invertible. Even if you know about determinants, do not use them (we did not cover them yet). (Hint: Use the previous 2 problems.)

Answer. If AB is invertible, then it has a pivot in every column and row. Thus, $\operatorname{rank} AB = n$. It follows by Problem 7.4 that $n = \operatorname{rank} AB \leq \operatorname{rank} A \leq n$, implying that $\operatorname{rank} A = n$. Similarly, Problem 7.5 implies that $\operatorname{rank} B = n$. But these two results imply that A and B both have pivots in every colunn and row, i.e., both are invertible. ☐

7.9. If A has the same four fundamental subspaces as B, does $A = B$?

Answer. No — consider the following two matrices.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Both of these matrices have

$$\operatorname{Ker} X = \{\mathbf{0}\} \qquad \operatorname{Ran} X = \mathbb{R}^2 \qquad \operatorname{Ker} X^T = \{\mathbf{0}\} \qquad \operatorname{Ran} X^T = \mathbb{R}^2$$

where $X = A$ or B. However, we also clearly have $A \neq B$. ☐

7.14. Is it possible for a real matrix A that $\operatorname{Ran} A = \operatorname{Ker} A^T$? Is it possible for a complex A?

Answer. Suppose for the sake of contradiction that for a real $m \times n$ matrix $A : V \to W$, $\operatorname{Ran} A = \operatorname{Ker} A^T$. Then $A\mathbf{v} \in \operatorname{Ran} A = \operatorname{Ker} A^T$ for all $\mathbf{v} \in V$. It follows that $A^T(A\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$. Thus, $A^T A = 0$. Consequently,

$$\begin{aligned} 0 &= \operatorname{tr}(0) \\ &= \operatorname{tr}(A^T A) \\ &= \sum_{j=1}^{n} (A^T A)_{ii} \\ &= \sum_{j=1}^{n} \sum_{i=1}^{m} A_{ij}^2 \end{aligned}$$

It follows that $A_{ij} = 0$ for all i, j, i.e., that $A = 0$. But this implies that $\operatorname{Ran} A = \{\mathbf{0}\} \neq W = \operatorname{Ker} A^T$, a contradiction.

It is possible for a complex matrix: Consider

$$A = \begin{pmatrix} 0 & 1 \\ 0 & i \end{pmatrix}$$

Clearly

$$\operatorname{Ran} A = \operatorname{span} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

and it can be shown that $\operatorname{Ker} A^T$ is the same. ☐

8.3. Find the change of coordinates matrix that changes the coordinates in the basis $1, 1+t$ in $\mathbb{P}_1$ to the coordinates in the basis $1-t, 2t$.

Answer. Let $\mathcal{A} = \{1, 1+t\}$, $\mathcal{B} = \{1-t, 2t\}$, and $\mathcal{S} = \{1, t\}$. Then following the procedure from Treil (2017), we have that

$$[I]_{\mathcal{S}\mathcal{A}} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad [I]_{\mathcal{S}\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$$

so

$$\begin{aligned} [I]_{\mathcal{B}\mathcal{A}} &= [I]_{\mathcal{B}\mathcal{S}}[I]_{\mathcal{S}\mathcal{A}} \\ &= ([I]_{\mathcal{S}\mathcal{B}})^{-1}[I]_{\mathcal{S}\mathcal{A}} \\ &= \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \end{aligned}$$

☐

8.6. Are the matrices $\left(\begin{smallmatrix} 1 & 3 \\ 2 & 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 2 \\ 4 & 2 \end{smallmatrix}\right)$ similar? Justify.

Answer. We will first prove that if A and B are similar, then $\operatorname{tr}(A) = \operatorname{tr}(B)$. Let A, B be similar. Then $A = Q^{-1}BQ$, so

$$\begin{aligned} \operatorname{tr}(A) &= \operatorname{tr}(Q^{-1}BQ) \\ &= \operatorname{tr}(Q^{-1}QB) \\ &= \operatorname{tr}(B) \end{aligned}$$

as desired.

Now observe that $\operatorname{tr}\left(\begin{smallmatrix} 1 & 3 \\ 2 & 2 \end{smallmatrix}\right) = 3$ while $\operatorname{tr}\left(\begin{smallmatrix} 0 & 2 \\ 4 & 2 \end{smallmatrix}\right) = 2$. Thus, by the contrapositive of the lemma, we have that the two matrices aren't similar. ☐

2 Eigenvalues and Eigenvectors

From Treil (2017).

Chapter 4

10/11: **1.1.** True or false:

a) Every linear operator in an n-dimensional vector space has n distinct eigenvalues.

Answer. False.
The identity linear operator I_2 in $\mathbb{R}^2$ has the sole eigenvalue $\lambda = 1$, since $I_2\mathbf{x} = 1\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^2$. □

b) If a matrix has one eigenvector, it has infinitely many eigenvectors.

Answer. True.
Let $A\mathbf{x} = \lambda\mathbf{x}$. Then $\alpha\mathbf{x}$ is also an eigenvector of A for any $\alpha \in \mathbb{F}$ since

$$A(\alpha\mathbf{x}) = \alpha A\mathbf{x} = \alpha\lambda\mathbf{x} = \lambda(\alpha\mathbf{x})$$

□

c) There exists a square real matrix with no real eigenvalues.

Answer. True.
Consider

$$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

for which we have $\lambda = 1 \pm 2i$. Since the two eigenvalues $1 + 2i$ and $1 - 2i$ are distinct, and the square matrix given is 2×2, there are no more eigenvalues. Therefore, every eigenvalue of this matrix is not real. □

d) There exists a square matrix with no (complex) eigenvectors.

Answer. False.
Let $\mathbf{x}$ be an eigenvector of A. If $\mathbf{x}$ is complex, then we are done. If $\mathbf{x}$ is real, then multiply $\mathbf{x}$ by the scalar i. It follows by the proof of part (b) that $i\mathbf{x}$ is an eigenvector if A. □

e) Similar matrices always have the same eigenvalues.

Answer. True.
The characteristic polynomials of similar matrices coincide. □

f) Similar matrices always have the same eigenvectors.

Answer. False.
The matrix

$$\begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix}$$

has eigenvectors

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

while its similar matrix

$$\begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix}$$

has eigenvectors

$$\begin{pmatrix}1\\0\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix}$$

Note that since similar matrices refer to the same linear transformation, a single linear transformation technically only has one set of eigenvectors (albeit possibly expressed in different bases). ☐

g) A non-zero sum of two eigenvectors of a matrix A is always an eigenvector.

Answer. False.
Consider

$$\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}$$

with eigenvalues $\lambda = 1, 2$ and respective eigenvectors

$$\begin{pmatrix}1\\0\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix}$$

Note that

$$\begin{pmatrix}1 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix}$$

where the "**b**" vector is not a scalar multiple of the "**x**" vector. ☐

h) A non-zero sum of two eigenvectors of a matrix A corresponding to the same eigenvalue λ is always an eigenvector.

Answer. True.
Let $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = \lambda\mathbf{y}$. Then

$$\begin{aligned} A(\alpha\mathbf{x} + \beta\mathbf{y}) &= \alpha A\mathbf{x} + \beta A\mathbf{y} \\ &= \alpha\lambda\mathbf{x} + \beta\lambda\mathbf{y} \\ &= \lambda(\alpha\mathbf{x} + \beta\mathbf{y}) \end{aligned}$$

as desired. ☐

1.3. Compute eigenvalues and eigenvectors of the rotation matrix

$$\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$$

Note that the eigenvalues (and eigenvectors) do not need to be real.

Answer. The characteristic polynomial of $A - \lambda I$ is

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= (\cos\alpha - \lambda)^2 + \sin^2\alpha \\ -\sin^2\alpha &= (\cos\alpha - \lambda)^2 \\ \pm i\sin\alpha &= \pm\cos\alpha - \lambda \\ \lambda &= \cos\alpha + i\sin\alpha = \mathrm{e}^{i\alpha} \\ &= \cos\alpha - i\sin\alpha = \mathrm{e}^{-i\alpha} \end{aligned}$$

Thus, $\lambda = \mathrm{e}^{i\alpha}, \mathrm{e}^{-i\alpha}$. It follows by solving the systems of equations

$$\begin{aligned} x_1\cos\alpha - x_2\sin\alpha &= \mathrm{e}^{i\alpha}x_1 \\ x_1\sin\alpha + x_2\cos\alpha &= \mathrm{e}^{i\alpha}x_2 \end{aligned} \qquad \begin{aligned} y_1\cos\alpha - y_2\sin\alpha &= \mathrm{e}^{-i\alpha}y_1 \\ y_1\sin\alpha + y_2\cos\alpha &= \mathrm{e}^{-i\alpha}y_2 \end{aligned}$$

that the eigenvectors are

$$x = \begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad y = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

□

1.5. Prove that eigenvalues (counting multiplicities) of a triangular matrix coincide with its diagonal entries.

Answer. Since the determinant of a triangular matrix is the product of its diagonal entries, we have that

$$\det(A - \lambda I) = (a_{1,1} - \lambda)(a_{2,2} - \lambda) \cdots (a_{n,n} - \lambda)$$

But this polynomial is zero only if and only if λ is a diagonal entry, so the eigenvalues must be the diagonal entries. □

1.6. An operator A is called **nilpotent** if $A^k = \mathbf{0}$ for some k. Prove that if A is nilpotent, then $\sigma(A) = \{0\}$ (i.e., that 0 is the only eigenvalue of A).

Answer. Suppose for the sake of contradiction that λ is a nonzero eigenvalue of A with corresponding eigenvector $\mathbf{x}$. Then since $A\mathbf{x} = \lambda\mathbf{x}$, $A^k\mathbf{x} = \lambda^k\mathbf{x} \neq \mathbf{0} = 0\mathbf{x}$, so $A^k \neq 0$, a contradiction. □

1.7. Show that the characteristic polynomial of a block triangular matrix

$$\begin{pmatrix} A & * \\ \mathbf{0} & B \end{pmatrix}$$

where A and B are square matrices coincides with $\det(A - \lambda I)\det(B - \lambda I)$. (Hint: Use Exercise 3.11 from Chapter 3.)

Answer. It follows from Chapter 3, Exercise 3.11 that

$$\begin{aligned} \det\left(\begin{pmatrix} A & * \\ \mathbf{0} & B \end{pmatrix} - \lambda I\right) &= \det\begin{pmatrix} A - \lambda I & * \\ \mathbf{0} & B - \lambda I \end{pmatrix} \\ &= \det(A - \lambda I)\det(B - \lambda I) \end{aligned}$$

as desired. □

1.8. Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be a basis in a vector space V. Assume also that the first k vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ of the basis are eigenvectors of an operator A, corresponding to an eigenvalue λ (i.e., that $A\mathbf{v}_j = \lambda\mathbf{v}_j$, $j = 1, \dots, k$). Show that in this basis, the matrix of the operator A has block triangular form

$$\begin{pmatrix} \lambda I_k & * \\ \mathbf{0} & B \end{pmatrix}$$

where I_k is the $k \times k$ identity matrix and B is some $(n-k) \times (n-k)$ matrix.

Answer. We will first show that if $\mathbf{v}_i$ is an eigenvector of A *and* a part of the basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ of V, then it's matrix with respect to $\mathbf{v}_1, \dots, \mathbf{v}_n$ has zeros in every slot except the i^{th} slot, which is 1. This is easily shown as follows.

$$\begin{aligned} A\mathbf{v}_i &= \lambda\mathbf{v}_i \\ A\mathbf{v}_i &= \lambda(0\mathbf{v}_1 + \cdots + 0\mathbf{v}_{i-1} + 1\mathbf{v}_i + 0\mathbf{v}_{i+1} + \cdots + 0\mathbf{v}_n) \end{aligned}$$

This combined with the observations that the i^{th} column of A is equal to $A\mathbf{v}_i$ and $A\mathbf{v}_i = \lambda\mathbf{v}_i$ proves that

$$A = \begin{pmatrix} A\mathbf{v}_1 & \cdots & A\mathbf{v}_n \end{pmatrix} = \begin{pmatrix} \lambda\mathbf{v}_1 & \cdots & \lambda\mathbf{v}_k & A\mathbf{v}_{k+1} & \cdots & A\mathbf{v}_n \end{pmatrix} = \begin{pmatrix} \lambda I_k & * \\ \mathbf{0} & B \end{pmatrix}$$

as desired. □

1.10. Prove that the determinant of a matrix A is the product of its eigenvalues (counting multiplicities). (Hint: First show that $\det(A - \lambda I) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$, where $\lambda_1, \dots, \lambda_n$ are eigenvalues (counting multiplicities). Then compare the free terms (terms without λ) or plug in $\lambda = 0$ to get the conclusion.)

Answer. We can row reduce A to an upper triangular matrix A_r without changing its determinant. We know that the determinant of the row-reduced matrix is equal to the product of its diagonal entries, and we know that the product of the diagonal entries of an upper-triangular matrix is equal to the product of its eigenvalues, so therefore, the determinant of A is equal to the product of the eigenvalues. □

1.11. Prove that the trace of a matrix equals the sum of its eigenvalues in three steps. First, compute the coefficient of λ^{n-1} in the right side of the equality

$$\det(A - \lambda I) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$$

Then show that $\det(A - \lambda I)$ can be represented as

$$\det(A - \lambda I) = (a_{1,1} - \lambda)(a_{2,2} - \lambda)\cdots(a_{n,n} - \lambda) + q(\lambda)$$

where $q(\lambda)$ is a polynomial of degree at most $n - 2$. And finally, compare the coefficients of λ^{n-1} to get the conclusion.

Answer. Consider $(\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$. We have that every λ^{n-1} term in the expansion of this product must take the λ from $n - 1$ of the terms and the λ_i from the remaining term. Thus, our expansion should contain the terms $\lambda_1\lambda^{n-1}, \dots, \lambda_n\lambda^{n-1}$, which, when we sum, gives $(\lambda_1 + \cdots + \lambda_n)\lambda^{n-1}$.

In the permutation sum form of the determinant, we have that $(a_{1,1} - \lambda)(a_{2,2} - \lambda)\cdots(a_{n,n} - \lambda)$ will be one of the terms in the sum. In particular, it is the *only* term to contain all the λ-containing entries in the matrix, so it solely determines the λ^n term. Additionally, the term containing the next-highest number of λ's must contain $n - 2$ λ's, not $n - 1$, since any product with $n - 1$ diagonal entries and 1 non-diagonal entry necessarily contains two terms that are in the same row or column. Thus, the term given solely determines the λ^{n-1} term as well. All of the other terms, having degree at most λ^{n-2}, can be defined equal to $q(\lambda)$.

Therefore, since the first part of the proof gives

$$(\lambda_1 + \cdots + \lambda_n)\lambda^{n-1}$$

as the λ^{n-1} term, and the second part of the proof (by a similar argument) gives

$$(a_{1,1} + a_{2,2} + \cdots + a_{n,n})\lambda^{n-1}$$

as the λ^{n-1} term, we have by comparing terms (rigorously, subtract all terms of other degrees to preserve the equality) that

$$\operatorname{tr} A = a_{1,1} + a_{2,2} + \cdots + a_{n,n} = \lambda_1 + \cdots + \lambda_n$$

as desired. □

2.1. Let A be an $n \times n$ matrix. True or false (justify your conclusions):

a) A^T has the same eigenvalues as A.

Answer. True.
Since $\det B = \det B^T$ for any matrix B and the transpose operation does not affect the diagonal, we have that

$$\begin{aligned}\det(A - \lambda I) &= \det\big((A - \lambda I)^T\big)\\ &= \det\big(A^T - (\lambda I)^T\big)\\ &= \det\big(A^T - \lambda I\big)\end{aligned}$$

as desired. □

b) A^T has the same eigenvectors as A.

Answer. False.
Let

$$A = \begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix}$$

Then we can calculate that A has eigenvectors

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

but A^T has eigenvectors

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

□

c) If A is diagonalizable, then so is A^T.

Answer. True.
Suppose $A = SDS^{-1}$. Then

$$\begin{aligned} A^T &= (SDS^{-1})^T \\ &= (S^{-1})^T D^T S^T \\ &= (S^{-1})^T D((S^{-1})^T)^{-1} \end{aligned}$$

as desired. □

2.2. Let A be a square matrix with real entries, and let λ be its complex eigenvalue. Suppose $\mathbf{v} = (v_1, \ldots, v_n)^T$ is a corresponding eigenvector, i.e., $A\mathbf{v} = \lambda\mathbf{v}$. Prove that the $\bar{\lambda}$ is an eigenvalue of A and $A\bar{\mathbf{v}} = \bar{\lambda}\bar{\mathbf{v}}$, where $\bar{\mathbf{v}} = (\bar{v}_1, \ldots, \bar{v}_n)^T$ is the complex conjugate of the vector $\mathbf{v}$.

Answer. Let $\mathbf{v} = \mathbf{a} + i\mathbf{b}$ where $a_j = \operatorname{Re} v_j$ and $b_j = \operatorname{Im} v_j$. It follows that

$$A\mathbf{a} + iA\mathbf{b} = A\mathbf{v} = \lambda\mathbf{v} = \lambda\mathbf{a} + i\lambda\mathbf{b}$$

This combined with the fact that all entries in $A, \mathbf{a}, \mathbf{b}$ are real implies by matching corresponding parts that

$$A\mathbf{a} = \lambda\mathbf{a} \qquad A\mathbf{b} = \lambda\mathbf{b}$$

Therefore,

$$A\bar{\mathbf{v}} = A(\mathbf{a} - i\mathbf{b}) = A\mathbf{a} - iA\mathbf{b} = \lambda\mathbf{a} - i\lambda\mathbf{b} = \lambda(\mathbf{a} - i\mathbf{b}) = \lambda\bar{\mathbf{v}}$$

as desired. □

2.3. Let

$$A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$$

Find A^{2004} by diagonalizing A.

Answer. We have that

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= (4-\lambda)(2-\lambda) - 3 \\ &= \lambda^2 - 6\lambda + 5 \\ &= (\lambda - 5)(\lambda - 1) \end{aligned}$$

Thus, $\lambda = 5, 1$. It follows by inspection that

$$x_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad x_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Consequently,

$$S = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} \qquad S^{-1} = \frac{1}{4}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}$$

Hence

$$A = \frac{1}{4}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}$$

Therefore,

$$\begin{aligned} A^{2004} &= \frac{1}{4}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 5^{2004} \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 5^{2004} & 5^{2004} \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 1 + 3 \cdot 5^{2004} & -3 + 3 \cdot 5^{2004} \\ -1 + 5^{2004} & 3 + 5^{2004} \end{pmatrix} \end{aligned}$$

□

2.4. Construct a matrix A with eigenvalues 1 and 3 and corresponding eigenvectors $(1,2)^T$ and $(1,1)^T$. Is such a matrix unique?

Answer. Let

$$\begin{aligned} A &= \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 6 & -3 \end{pmatrix} \\ &= \begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix} \end{aligned}$$

Suppose A' has eigenvalues $1, 3$ with corresponding eigenvectors $(1,2)^T$ and $(1,1)^T$. Then since the eigenvectors are linearly independent and form a basis of $\mathbb{R}^2$, Theorem 2.1 implies that A' is diagonal with diagonal matrix equal to the middle matrix in the first line above and change of basis matrices equal to the other two in the first line above. Therefore, $A = A'$. □

2.6. Consider the matrix

$$A = \begin{pmatrix} 2 & 6 & -6 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{pmatrix}$$

a) Find its eigenvalues. Is it possible to find the eigenvalues without computing?

Answer. It's eigenvalues are $\lambda = 2, 5, 4$, since this is an upper-triangular matrix and those are the diagonal entries. □

b) Is this matrix diagonalizable? Find out without computing anything.

Answer. Yes. Since the eigenvalues are all distinct and there are 3 for this 3×3 matrix, Corollary 2.3 implies that A is diagonalizable. □

c) If the matrix is diagonalizable, diagonalize it.

Answer. If $\lambda_1 = 2$, $\lambda_2 = 5$, and $\lambda_3 = 4$, then the corresponding eigenvectors are

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad x_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \qquad x_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

It follows that

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

□

2.8. Find all square roots of the matrix

$$A = \begin{pmatrix} 5 & 2 \\ -3 & 0 \end{pmatrix}$$

i.e., find all matrices B such that $B^2 = A$. (Hint: Finding a square root of a diagonal matrix is easy. You can leave your answer as a product.)

Answer. We have that

$$A = \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 1 & 1 \end{pmatrix}$$

Therefore, we have four possibilities for B:

$$B_1 = \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 1 & 1 \end{pmatrix}$$
$$B_2 = \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -\sqrt{3} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 1 & 1 \end{pmatrix}$$
$$B_3 = \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 1 & 1 \end{pmatrix}$$
$$B_4 = \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -\sqrt{3} & 0 \\ 0 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 1 & 1 \end{pmatrix}$$

□

2.10. Let A be a 5×5 matrix with 3 eigenvalues (not counting multiplicities). Suppose we know that one eigenspace is three-dimensional. Can you say if A is diagonalizable?

Answer. Yes, it is diagonalizable. Let $\lambda_1, \lambda_2, \lambda_3$ be the 3 eigenvalues of A, let $\mathbf{v}_1, \mathbf{v}_2$ be the eigenvectors corresponding to λ_1, λ_2, and let $\mathbf{v}_{3a}, \mathbf{v}_{3b}, \mathbf{v}_{3c}$ be a basis of the eigenvectors corresponding to λ_3. Since the eigenspace of λ_3 is three dimensional, we know that $\mathbf{v}_{3a}, \mathbf{v}_{3b}, \mathbf{v}_{3c}$ is linearly independent. Additionally, we have by consecutive applications of Theorem 2.2 that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_{3a}$, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_{3b}$, and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_{3c}$ are linearly independent lists. Hence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_{3a}, \mathbf{v}_{3b}, \mathbf{v}_{3c}$ is a linearly independent list of length 5, so it must form a basis of $\mathbb{F}^5$. Therefore, by Theorem 2.1, A is diagonalizable. □

2.11. Give an example of a 3×3 matrix which cannot be diagonalized. After you construct the matrix, can you make it "generic," so no special structure of the matrix can be seen?

Answer. Generalizing from the given example, we can show that

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

is not diagonalizable. Applying row operations can put the matrix in the more generic form

$$\begin{pmatrix} 0 & 1 & 4 \\ 1 & 2 & 3 \\ -1 & 0 & 4 \end{pmatrix}$$

□

2.13. Eigenvalues of a transposition:

a) Consider the transformation T in the space $M_{2\times 2}$ of 2×2 matrices defined by $T(A) = A^T$. Find all its eigenvalues and eigenvectors. Is it possible to diagonalize this transformation? (Hint: While it is possible to write a matrix of this linear transformation in some basis, compute the characteristic polynomial, and so on, it is easier to find eigenvalues and eigenvectors directly from the definition.)

Answer. The symmetric matrices are eigenvectors of this transformation with eigenvalue 1. A basis of them would be

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The antisymmetric matrices are eigenvectors of this transformation with eigenvalue -1. A basis of them would be

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Since these four matrices are linearly independent, there exists a basis of $M_{2\times 2}$ of eigenvectors of T. Therefore, T is diagonalizable. □

b) Can you do the same problem but in the space of $n \times n$ matrices?

Answer. Yes. A basis of the $n \times n$ symmetric matrices includes all of the matrices that are zero everywhere except for one 1 in a diagonal entry, and all of the matrices that are zero everywhere except for two 1's in off-diagonal symmetric positions. There are $\frac{n}{2}(n+1)$ of these basis "vectors." A basis of the $n \times n$ antisymmetric matrices includes all of the matrices that are zero everywhere except for a -1 in an off-diagonal position in the upper triangle and a 1 in the symmetric position in the lower triangle. There are $\frac{n}{2}(n-1)$ of these. Together, we have

$$\frac{n}{2}(n+1) + \frac{n}{2}(n-1) = n^2$$

basis "vectors," meaning that we have a complete eigenbasis of $M_{n\times n}$. □

2.14. Prove that two subspaces V_1 and V_2 are linearly independent if and only if $V_1 \cap V_2 = \{\mathbf{0}\}$.

Answer. Suppose first that V_1, V_2 are linearly independent. Let $\mathbf{v}_{11}, \dots, \mathbf{v}_{1n}$ be a basis of V_1, and let $\mathbf{v}_{21}, \dots, \mathbf{v}_{2m}$ be a basis of V_2. Then by Lemma 2.7, $\mathbf{v}_{11}, \dots, \mathbf{v}_{1n}, \mathbf{v}_{21}, \dots, \mathbf{v}_{2m}$ is linearly independent. Now suppose $\mathbf{v} \in V_1 \cap V_2$. Since $\mathbf{v} \in V_1$, $\mathbf{v} = \alpha_{11}\mathbf{v}_{11} + \cdots + \alpha_{1n}\mathbf{v}_{1n}$. Similarly, $\mathbf{v} = \alpha_{21}\mathbf{v}_{21} + \cdots + \alpha_{2m}\mathbf{v}_{2m}$. Thus,

$$\mathbf{0} = \mathbf{v} - \mathbf{v} = \alpha_{11}\mathbf{v}_{11} + \cdots + \alpha_{1n}\mathbf{v}_{1n} - \alpha_{21}\mathbf{v}_{21} - \cdots - \alpha_{2m}\mathbf{v}_{2m}$$

But since $\mathbf{v}_{11}, \dots, \mathbf{v}_{1n}, \mathbf{v}_{21}, \dots, \mathbf{v}_{2m}$ is linearly independent, it follows that all the α's are 0. Therefore, $\mathbf{v} = 0\mathbf{v}_{11} + \cdots + 0\mathbf{v}_{1n} = \mathbf{0}$, so $V_1 \cap V_2 \subset \{\mathbf{0}\}$. The inclusion in the other direction is obvious, since V_1, V_2 are subspaces.

Now suppose that $V_1 \cap V_2 = \{\mathbf{0}\}$. To prove that V_1, V_2 are linearly independent, it will suffice to show that $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0}$ where $\mathbf{v}_i \in V_i$ for all i implies $\mathbf{v}_i = \mathbf{0}$ for all i. Let $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0}$ where $\mathbf{v}_i \in V_i$ for all i. Suppose for the sake of contradiction that $\mathbf{v}_1 \neq \mathbf{0}$. Then we must have $\mathbf{v}_2 = -\mathbf{v}_1 \neq \mathbf{0}$. But by closure under scalar multiplication, this implies that $-1 \cdot -\mathbf{v}_1 = \mathbf{v}_1 \in V_2$ since $\mathbf{v}_2 \in V_2$. Therefore, $\mathbf{v}_1 \in V_1 \cap V_2$ as well, a contradiction. The proof is symmetric if we let $\mathbf{v}_2 \neq \mathbf{0}$ first. □

3 Inner Product Spaces

From Treil (2017).

Chapter 5

10/18: **3.2.** Apply Gram-Schmidt orthogonalization to the system of vectors $(1,2,3)^T$, $(1,3,1)^T$. Write the matrix of the orthogonal projection onto the 2-dimensional subspace spanned by these vectors.

3.5. Find the orthogonal projection of a vector $(1,1,1,1)^T$ onto the subspace spanned by the vectors $\mathbf{v}_1 = (1,3,1,1)^T$ and $\mathbf{v}_2 = (2,-1,1,0)^T$ (note that $\mathbf{v}_1 \perp \mathbf{v}_2$).

3.6. Find the distance from a vector $(1,2,3,4)$ to the subspace spanned by the vectors $\mathbf{v}_1 = (1,-1,1,0)^T$ and $\mathbf{v}_2 = (1,2,1,1)^T$ (note that $\mathbf{v}_1 \perp \mathbf{v}_2$). Can you find the distance without actually computing the projection? That would simplify the calculations.

3.7. True or false: If E is a subspace of V, then $\dim E + \dim(E^\perp) = \dim V$? Justify.

3.8. Let P be the orthogonal projection onto a subspace E of an inner product space V, let $\dim V = n$, and let $\dim E = r$. Find the eigenvalues and the eigenvectors (eigenspaces). Find the algebraic and geometric multiplicities of each eigenvalue.

3.9. Using eigenvalues to compute determinants:

a) Find the matrix of the orthogonal projection onto the one-dimensional subspace in $\mathbb{R}^n$ spanned by the vector $(1,\dots,1)^T$.

b) Let A be the $n \times n$ matrix with all entries equal to 1. Compute its eigenvalues and their multiplicities (use the previous problem).

c) Compute eigenvalues (and multiplicities) of the matrix $A - I$, i.e., of the matrix with zeroes on the main diagonal and ones everywhere else.

d) Compute $\det(A - I)$.

3.11. Let $P = P_E$ be the matrix of an orthogonal projection onto a subspace E. Show that

a) The matrix P is self-adjoint, meaning that $P^* = P$.

b) $P^2 = P$.

3.13. Suppose P is the orthogonal projection onto a subspace E, and Q is the orthogonal projection onto the orthogonal complement $E^\perp$.

a) What are $P + Q$ and PQ?

b) Show that $P - Q$ is its own inverse.

4.2. Find the matrix of the orthogonal projection P onto the column space of

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{pmatrix}$$

Use two methods: Gram-Schmidt orthogonalization and the formula for the projection. Compare the results.

4.4. Fit a plane $z = a + bx + cy$ to four points $(1,1,3)$, $(0,3,6)$, $(2,1,5)$, and $(0,0,0)$. To do that

a) Find 4 equations with 3 unknowns a, b, c such that the plane passes through all 4 points (this system does not have to have a solution).

b) Find the least squares solution of the system.

4.5. Minimal norm solution. Let an equation $A\mathbf{x} = \mathbf{b}$ have a solution, and let A have a non-trivial kernel (so the solution is not unique). Prove that

a) There exists a unique solution $\mathbf{x}_0$ of $A\mathbf{x} = \mathbf{b}$ minimizing the norm $\|\mathbf{x}\|$, i.e., that there exists a unique $\mathbf{x}_0$ such that $A\mathbf{x}_0 = \mathbf{b}$ and $\|\mathbf{x}_0\| \leq \|\mathbf{x}\|$ for any $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{b}$.

b) $\mathbf{x}_0 = P_{(\ker A)^\perp}\mathbf{x}$ for any $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{b}$.

5.2. Find matrices of orthogonal projections onto all 4 fundamental subspaces of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 2 & 4 & 3 \end{pmatrix}$$

Note that you only really need to compute 2 of the projections. If you pick an appropriate 2, the other 2 are easy to obtain from them (recall how the projections onto E and $E^\perp$ are related).

5.3. Let A be an $m \times n$ matrix. Show that $\ker A = \ker(A^*A)$. (Hint: To do this, you need to prove 2 inclusions, namely $\ker(A^*A) \subset \ker A$ and $\ker A \subset \ker(A^*A)$. One of the inclusions is trivial, and for the other one, use the fact that $\|A\mathbf{x}\|^2 = (A\mathbf{x}, A\mathbf{x}) = (A^*A\mathbf{x}, \mathbf{x})$.)

5.4. Use the equality $\ker A = \ker(A^*A)$ to prove that

a) $\operatorname{rank} A = \operatorname{rank}(A^*A)$.

b) If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, A is left invertible. (Hint: You can just write a formula for the left inverse.)

5.6. Let a matrix P be self-adjoint ($P^* = P$) and let $P^2 = P$. Show that P is the matrix of an orthogonal projection. (Hint: Consider the decomposition $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$, $\mathbf{x}_1 \in \operatorname{range} P$ and $\mathbf{x}_2 \perp \operatorname{range} P$, and show that $P\mathbf{x}_1 = \mathbf{x}_1$, $P\mathbf{x}_2 = \mathbf{0}$. For one of the equalities, you will need self-adjointness; for the other one, the property $P^2 = P$.)

6.1. Orthogonally diagonalize the following matrices

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

i.e., for each matrix A, find a unitary matrix U and a diagonal matrix D such that $A = UDU^*$.

6.2. True or false: A matrix is unitarily equivalent to a diagonal one if and only if it has an orthogonal basis of eigenvectors.

6.5. Let $U : X \to X$ be a linear transformation on a finite-dimensional inner product space. True or false:

a) If $\|U\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in X$, then U is unitary.

b) If $\|U\mathbf{e}_k\| = \|\mathbf{e}_k\|$ for each $k = 1, \ldots, n$ for some orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$, then U is unitary.

6.6. Let A and B be unitarily equivalent $n \times n$ matrices.

a) Prove that $\operatorname{tr}(A^*A) = \operatorname{tr}(B^*B)$.

b) Use (a) to prove that

$$\sum_{j,k=1}^{n} |A_{j,k}|^2 = \sum_{j,k=1}^{n} |B_{j,k}|^2$$

c) Use (b) to prove that the matrices

$$\begin{pmatrix} 1 & 2 \\ 2 & i \end{pmatrix} \qquad \begin{pmatrix} i & 4 \\ 1 & 1 \end{pmatrix}$$

are not unitarily equivalent.

6.7. Which of the following pairs of matrices are unitarily equivalent? (Hint: It is easy to eliminate matrices that are not unitarily equivalent: Remember that unitarily equivalent matrices are similar, and recall that the trace, determinant, and eigenvalues of similar matrices coincide. Also, the previous problem helps in eliminating non-unitarily equivalent matrices. Finally, a matrix is unitarily equivalent to a diagonal one if and only if it has an orthogonal basis of eigenvectors.)

a)

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

b)

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$$

c)

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

d)

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{pmatrix}$$

e)

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

6.9. Let U be a 3×3 orthogonal matrix with $\det U = 1$. Prove that

a) 1 is an eigenvalue of U.

b) If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is an orthonormal basis, such that $U\mathbf{v}_1 = \mathbf{v}_1$ (remember that 1 is an eigenvalue), then in this basis, the matrix of U is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}$$

where α is some angle. (Hint: Show that since $\mathbf{v}_1$ is an eigenvector of U, all entries below 1 must be zero, and since $\mathbf{v}_1$ is also an eigenvector of U^* [why?], all entries right of 1 must also be zero. Then show that the lower right 2×2 matrix is an orthogonal one with determinant 1, and use the previous problem.)

8.1. Prove the following formula.

$$(\mathbf{x}, \mathbf{y})_{\mathbb{R}} = \operatorname{Re}(\mathbf{x}, \mathbf{y})_{\mathbb{C}}$$

Namely, show that if

$$\mathbf{x} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

where $z_k = x_k + iy_k$, $w_k = u_k + iv_k$, $x_k, y_k, u_k, v_k \in \mathbb{R}$, then

$$\operatorname{Re}\left(\sum_{k=1}^{n} z_k \bar{w}_k\right) = \sum_{k=1}^{n} x_k u_k + \sum_{k=1}^{n} y_k v_k$$

8.4. Show that if U is an orthogonal transformation satisfying $U^2 = -I$, then $U^* = -U$.

References

Treil, S. (2017). *Linear algebra done wrong* [http://www.math.brown.edu/streil/papers/LADW/LADW_2017-09-04.pdf].