Chapter 7

Sequences and Series of Functions

7.1 Notes

11/15:
- Soug will not test on differentiation/integration assuming that we know them already.
- **Pointwise convergent** (sequence $(f_n)_{n\in\mathbb{N}}$ of functions): A sequence of functions $f_n : E \to \mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in E$.
- Can we interchange "limit" in the above definition with continuity, convergence of series, integration, differentiation, etc.?
- Examples with negative answer:
  1. Interchanging limits: Let $S_{mn} = \frac{m}{m+n}$. $S_{mn} \to 1$ as $m \to \infty$, and $S_{mn} \to 0$ as $n \to \infty$.
  2. $f_n(x) = x^2/(1+x)^n$. $f(x) = \sum_{n=1}^{\infty} f_n(x)$. If $x = 0$, then $f_n(x) = 0$ for all n and $f(x) = 0$. If $x \neq 0$, we have
  $$f(x) = \sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n} = x^2 \sum_{n=1}^{\infty} X^n = \frac{x^2}{1-X} = \frac{x^2}{1-(1/(1+x^2))} = 1 + x^2$$
  3. Consider $f_m(x) = \lim_{n\to\infty} (\cos(m\pi x))^2 n$. $\lim_{m\to\infty} f_m(x)$ goes to 0 if $x \notin \mathbb{Q}$ and goes to 1 if $x \in \mathbb{Q}$. $f_m \to \chi_{\mathbb{Q}}$, where $\chi_{\mathbb{Q}}$ is the characteristic function of the rationals which is not Riemann integrable (partitions, upper and lower integrals, etc.).
  4. $f_n(x) = \sin nx/\sqrt{n} \to f(x) = 0$ for all x. However, $f_n'(x) = \sqrt{n} \cos nx \nrightarrow 0$
  5. If $0 \leq x \leq 1$, define $f_n(x) = n^2 x(1-x^2)^n$. We know that $f_n(0) = 0$. $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in (0,1]$. We can show that $\int_0^1 x(1-x^2)^n \,\mathrm{d}x = 1/(2n+2)$. Thus, $\int_0^1 f_n(x) \,\mathrm{d}x = n^2/(2n+2)$. Limit of the functions is zero, but their integrals diverge to infinity.
- **Uniformly convergent** (sequence $(f_n)_{n\in\mathbb{N}}$ of functions on E): A sequence of functions $f_n : E \to \mathbb{R}$ such that for all $\epsilon > 0$, there exists N such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$ for all $x \in E$. *Denoted by* $\boldsymbol{f_n \rightrightarrows f}$.
- Theorem: $f_n \rightrightarrows f$ iff $(f_n)_{n\in\mathbb{N}}$ is uniformly Cauchy (i.e., for all $\epsilon > 0$, there exists N such that if $n, m \geq N$ then $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in E$).
- Theorem: Let $M_n = \sup_{x\in E} |f_n(x) - f(x)|$. If $f_n \to f$ pointwise and $M_n \to 0$, then $f_n \rightrightarrows f$.
- Theorem: If $(f_n)_{n\in\mathbb{N}}$ and $|f_n(x)| \leq M_n$, then $\sum f_n \rightrightarrows f$ if $\sum M_n < \infty$.
- Theorem: If E is a compact metric space, $f_n \rightrightarrows f$ in E, x is a limit point of E, and $\lim_{t\to x} f_n(t) = A_n$ exists, then $(A_n)_{n\in\mathbb{N}}$ converges and $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$.

- Corollary: $\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$.

11/16:
  - Fix $\epsilon > 0$. Then $f_n \rightrightarrows f$ implies that there exists some N such that $n,m \geq N$ implies $|f_n(t) - f_m(t)| < \epsilon$ for all $t \in E$.
  - x is a limit point of E and $t \to x$ implies $|A_n - A_m| < \epsilon$. Thus, $(A_n)_{n\in\mathbb{N}}$ is cauchy, so there exists A such that $A_n \to A$.
  - WTS: $|f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|$, so we WTS the three terms on the right are small.
  - There exists n such that $|f(t) - f_n(t)| < \epsilon/3$ for all t since $f_n \rightrightarrows f$ by hypothesis.
  - Since t is in a small neighborhood of x, there exists n such that $|A_n - A| < \epsilon/3$.
  - We also have $|f_n(t) - A_n| < \epsilon/3$ by hypothesis.
  - This is a very important proof to understand, because proofs like this pop up often.

- Corollary: f_n continuous and $f_n \rightrightarrows f$ implies f is continuous.

- Theorem: Let K be compact. Assume

  (a) $(f_n)_{n\in\mathbb{N}} \subset C(K) = \{f : K \to \mathbb{R} \mid f \text{ continuous}\}$.
  (b) $f_n \to f$ pointwise in K and $f \in C(K)$.
  (c) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$.

  Then $f_n \rightrightarrows f$.

  - WLOG $f = 0$, $g_n = f_n - f \to 0$, $g_n \geq g_{n+1} \geq 0$.
  - For all $\epsilon > 0$, there exists N such that $n \geq N$ and $0 \leq g_n(x) \leq \epsilon$ for all $x \in K$.
  - $K_n = \{x \in K : g_n(x) \geq \epsilon\}$.
  - g_n continuous implies K_n closed. This combined with K compact implies K_n is compact.
  - g_n decreasing implies $K_n \supset K_{n+1}$. Thus, K_n is a nested family of compact sets, so $\bigcap K_n \neq \emptyset$.
  - This implies that each K_n is nonempty, contradicting the fact that each $g_n \to 0$ for all x.
  - Thus, there exists an N such that K_n is empty for all $n \geq N$. Thus $g_n(x) \leq \epsilon$ for all $x \in K$, $n \geq N$.
  - Note that the compactness of K is important. If $f : (0,1) \to \mathbb{R}$ is defined by $f(x) = 1/(nx+1)$, then $f_n \to 0$, but $f_n \not\rightrightarrows f$.

- Let $C(X) = \{f : X \to \mathbb{R} \mid f \text{ continuous, bounded}\}$ for X a metric space.

- If we define $\|f\| = \sum_{x\in X} |f(x)|$, for $f, g \in C(X)$, we may define $d(f,g) = \|f - g\|$. This definition satisfies the properties of a distance function, and $\|\cdot\|$ is a norm.

  - Thus, $C(X)$ is a complete metric space, a normed space, or, specifically, a **Banach space**.

- Theorem: $(f_n)_{n\in\mathbb{N}} \subset C(X)$ such that $\|f_n - f_m\|_{n,m\to\infty} \to 0$. Then there exists $f \in C(X)$ such that $\|f_n - f\|_{n\to\infty} \to 0$.

  - We get such a strong statement using properties of the image, not properties of the domain.
  - For all $\epsilon > 0$, there exists N such that $n, m \geq N$.
  - $|f_n(x) - f_m(x)| \leq \|f_n - f_m\| < \epsilon$ for all x.
  - Then there exists f such that $f_m(x) \to f(x)$. It follows that $|f_n(x) - f_m(x)| < \epsilon$

- Uniform convergence and integration.

- Stieltjes integral.

  - Define $\alpha : \mathbb{R} \to \mathbb{R}$ nondecreasing.
  - If we sum over the minimums/maximums of a partition times $\alpha(x_{i+1}) - \alpha(x_i)$ instead of $x_{i+1} - x_i$, we obtain said integral as the upper/lower limits just like the Riemann integral.
  - We write $\int_a^b f(x)\,\mathrm{d}\alpha(x)$ where $\mathrm{d}\alpha(x) = \alpha(x)\,\mathrm{d}x$.
- Theorem: If α is nondecreasing on $[a,b]$, $f_n \in R(\alpha)$ such that $f_n \rightrightarrows f$ on $[a,b]$
  - We have
  $$\begin{aligned}\left|\int f_n(x)\,\mathrm{d}\alpha(x) - \int f(x)\,\mathrm{d}\alpha(x)\right| &= \left|\int (f_n - f)(x)\,\mathrm{d}\alpha(x)\right| \\ &\leq \|f_n - f\|(\alpha(b) - \alpha(a)) \\ &\leq \int |f_n - f|\,\mathrm{d}\alpha(x) \\ &\leq \int \|f_n - f\|\,\mathrm{d}\alpha(x) \qquad \leq \|f_n - f\| \int_a^b \mathrm{d}\alpha(x) = \|f_n - f\|(\alpha(b) - \alpha(a))\end{aligned}$$

11/19:

- Suppose $f_n \to f$ and $f_n' \to g$. When does $f' = g$?
- Theorem: If $f_n : [a,b] \to \mathbb{R}$ is differentiable, $f_n(x_0)$ converges for some $x_0 \in [a,b]$, and f_n' converges uniformly on $[a,b]$, then there exists f differentiable on $[a,b]$ such that $f_n \rightrightarrows f$ and $f_n' \rightrightarrows f'$.
  - Assume the f_n' are continuous. Then $f_n(x) - f_n(x_0) = \int_{x_0}^x f_n'(y)\,\mathrm{d}y$.
  - Since $f_n' \rightrightarrows g$, $\int_{x_0}^x f_n'(y)\,\mathrm{d}y \to \int_{x_0}^x g(y)\,\mathrm{d}y$.
  - It follows since $f_n(x_0) \to f(x_0)$ that $f_n \rightrightarrows f$.
  - By the previous theorem, if
  $$f_n'(x) = \lim_{h\to 0} \frac{f_n(x+h) - f_n(x)}{h}$$
  then
  $$\lim_{n\to\infty} f_n'(x) = \lim_{h\to 0}\lim_{n\to\infty} \frac{f_n(x+h) - f_n(x)}{h} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
  - Fix $\epsilon > 0$. Then there exists N such that $n, m \geq N$ such that $|f_n(x_0) - f_m(x_0)| < \epsilon/2$ and $|f_n'(t) - f_m'(t)| < \epsilon/2$ for all $t \in [a,b]$.
  - We know that $f_n(t) - f_n(x_0) = \int_{x_0}^t f_n'(y)\,\mathrm{d}y$ and $f_m(t) - f_m(x_0) = \int_{x_0}^t f_m'(y)\,\mathrm{d}y$.
  - Thus,
  $$|f_n(t) - f_n(x_0)| \leq |f_n(t) - f_n(x_0)| + |f_m(t) - f_m(x_0)| - |f_m(t) - f_m(x_0)|$$
  - Let $f_n(t) - f_n(x_0) = c_n(t - x_0)$ and $f_m(t) - f_m(x_0) = c_m(t - x_0)$.
  - ...
- Let $f : [a,b] \to \mathbb{R}$ be continuous. What conditions on f imply that f' exists?
- Suppose f is Lipschitz continuous (equivalent to saying there exists $L > 0$ such that $|f(x) - f(y)| \leq L|x - y|$); then f' exists **almost everywhere**.
  - If f differentiable, this is equivalent to saying f bounded.
- **Almost everywhere**: Something happens almost everywhere if the set of places where it doesn't happen has measure zero.
- Suppose f is **Hölder continuous**, then f' does not exist?
- **Hölder continuous** (function f): There exists $L > 0$ such that $|f(y) - f(x)| < L|x - y|^\alpha$ where $\alpha \in (0,1)$

- Suppose f exists such that f is Hölder continuous in a neighborhood of every point in the domain. This function is not anywhere differentiable. Such a function does indeed exist (and it's Brownian motion). The construction of such a function is the essence of Stochastic analysis.
  - Probabilistically: Has mean zero, distributed as a normal function like the Gaussian, and the increments are independent of each other.
  - Analytically: It's a function that is Hölder continuous at half plus ϵ for every ϵ and it is nowhere differentiable.
- Theorem: There exists $f : \mathbb{R} \to \mathbb{R}$ continuous but nowhere differentiable.
  - This theorem is due to Weierstrass and as such, such functions are typically called Weierstrass functions.
- A general class of functions that are nowhere differentiable (not in Rudin (1976); we don't have to prove this).
  - Example 1:
    $$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$
    where $0 < a < 1$, b positive odd integer greater than 1, and $ab > 1 + \frac{3}{2}\pi$.
    - This function at every point oscillates more and more and more.
- Rudin (1976)'s simple example.
  - $\phi : [-1, 1] \to \mathbb{R}$ defined by $\phi(x) = |x|$ is not differentiable at zero.
  - Takes ϕ extends it periodically with period 2, creating a sawtooth function.
  - Repeat the behavior so that the nondifferentiability becomes more and more frequent to get
    $$f(x) = \sum_{1}^{\infty} \left(\frac{3}{4}\right)^n \phi(4^n x)$$
  - This is continuous.
  - Fix any $x \in \mathbb{R}$, $m \in \mathbb{N}$. Then $\delta_m = \pm\frac{1}{2} \cdot 4^{-m}$.
  - Then consider $4^m x$, $4^m(x + \delta_m)$.
  - Rudin asserts
    $$\left|\frac{f(x+\delta_m) - f(x)}{\delta_m}\right| \to \infty$$
    as $m \to \infty$ for all x.

11/29:

- Finding a uniformly convergent subsequence of a sequence of functions.
  - Pointwise, uniformly, bounded if there exist M_x such that $|f_n(x)| \leq M_x$ for all n, x. Uniformly bounded if there exists M such that $|f_n(x)| \leq M$ for all n, x.
- Theorem: If $(f_n)_{n\in\mathbb{N}}$ is pointwise bounded and $E \subset X$ is countable, then there exists a subsequence f_{n_k} which converges for every $x \in E$.
  - Let $E = \{x_i : i \in \mathbb{N}\}$. Consider $f_n(x_1)$. $f_{1,k}(x_1)$ converges.
  - $S_1 : f_{1,1}(x_1), f_{1,2}(x_1), f_{1,3}(x_1), f_{1,4}(x_1), \dots$.
  - $S_2 : f_{2,1}(x_2), f_{2,2}(x_2), f_{2,3}(x_2), f_{2,4}(x_2), \dots$.
  - Now consider $f_{2,k}(x_3)$.
  - $S_3 : f_{3,1}, f_{3,2}, f_{3,3}, f_{3,4}, \dots$.

  - Continue on and on to $S_4, S_5, \dots$. We know that each of these sequences converges pointwise by hypothesis.
  - Now consider the diagonal sequence $f_{1,1}, f_{2,2}, f_{3,3}, f_{4,4}$.
    - This subsequence of the original sequence we may call g_k.
    - We posit that g_k converges for every $x \in E$.

- Theorem: There exists f_n which is uniformly bounded but does not converge uniformly.
  - Let $f_n(x) = \sin(2\pi x)$ for $0 \leq x \leq 2\pi$.
  - Let $f_n(x) = x^2/(x^2 + (1 - nx)^2)$ on $0 \leq x \leq 1$. This sequence is uniformly bounded, converges pointwise, but $f_n(1/n) = 1$ so f_n cannot converge uniformly to zero.

- What does it mean that $f_n : [0, 1] \to \mathbb{R}$ does not converge uniformly?
  - It means that there exists a subsequence of the functions evaluated at certain points that is always greater than or equal to some fixed distance away from the limit.

- Equicontinuity: $\mathcal{F}\{f : X \to \mathbb{R}\}$ for (X, d) a metric space is equicontinuous iff for all $\epsilon > 0$, there exists a $\delta > 0$ such that $d(x, y) < \delta$ implies $|f(x) - f(y)| < \epsilon$ for all $x, y \in X$, $f \in \mathcal{F}$.

- Modulus of continuity: $f : X \to \mathbb{R}$ is continuous at x. A modulus of continuity is a function $\omega_X : [0, 1] \to [0, 1]$ such that $|f(y) - f(x)| \leq \omega_X|y - x|$.

- The final result we'll prove: **Arzelà-Ascoli theorem**: If we have a family of functions on a compact set and we have a dense subset of that set, then if we have a sequence of functions that are equicontinuous, then they converge uniformly.

12/1:

- The final will be in this room.
- The last PSet will not be graded, but there will be similar questions on the final.
- No class on Friday.
- Review from last time:
  - Equiboundedness and equicontinuity.
  - If E is a dense subset of X, then any pointwise bounded sequence has a subsequence that converges on E (diagonal argument.)

- **Equicontinuous** (sequence $\{f_n\}$): For all $\epsilon > 0$, there exists a $\delta > 0$ such that $d(x, y) < \delta$ implies $|f_n(x) - f_n(y)| < \epsilon$ for all n.

- Theorem: If K is a compact set and $\{f_n\} \in C(K)$ converges uniformly on K, then the f_n's are equicontinuous on K.
  - The f_n are uniformly Cauchy: For all $\epsilon > 0$, there exists N such that $n, m \geq N$ imply $\|f_n - f_m\| < \epsilon$ where $\|f_n - f_m\| = \sup_{x \in K}(f_n - f_m)(x)$.
  - If $n \geq N$, then $|f_n(x) - f_n(y)| \leq |f_N(x) - f_N(y)| + 2\|f_n - f_N\|$ (since $f_n(x) - f_n(y) = f_n(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f_n(y)$).
  - Thus $|f_n(x) - f_n(y)| \leq |f_N(x) - f_N(y)| + 2\|f_n - f_N\| < 3\epsilon$ implies $|f_i(x) - f_i(y)| < \epsilon$ if $|x - y| < \delta$ for $i = 1, \dots, N$.

- **Arzelà-Ascoli theorem**: If K is compact, $(f_n)_{n \geq 1} \subset C(K)$ which are pointwise bounded and equicontinuous, then
  (a) $(f_n)_{n \geq 1}$ are uniformly bounded (equicontinuous).
  (b) There exists $(f_{n_k})_{k \geq 1}$ which converges uniformly on K.

  - Since K is compact, you can cover it by finitely many balls of radius δ.
  - Thus $|f_n(p_k)| \leq M = \max(M_{p_1}, \dots, M_{p_k})$ where $K \subset \bigcup_{k \in K} B(p_{n_k}, \delta)$.
  - K has a countable dense subset E (Exercise 2.25).
  - $|f_n(x)| \leq M + \epsilon$ for all x.
  - $\{B(x, \delta)\}_{x \in E}$ is an open cover of K.
  - Thus has a finite subcover.
  - ...

- If $f_n : \mathbb{R}^n \to \mathbb{R}$ are continuous, equibounded, equicontinuous, then there exists f_{n_k} which converges locally uniformly to some $f : \mathbb{R}^n \to \mathbb{R}$.
- How do you learn math?
  - In an ideal world, study by looking at theorems, thinking that you should be able to prove it, and etc.
  - Since we don't have the time to do everything ourselves, don't just get stuck in a place; move on and continue thinking if you have to.
- Let $\dot{\phi} = f(x, t)$ and $x(0) = c$. Let $\phi : \mathbb{R} \times [0, 1] \to \mathbb{R}$. Assume ϕ is bounded and continuous. Then there exists a solution of the differential equation and initial condition.
  - We need to find a function $x : [0, 1] \to \mathbb{R}$ continuous such that $x(t) = c + \int_0^t \phi(x(s), s) \, \mathrm{d}s$.
  - Let $t_i = i/N$. Then $x_n(t) = \phi(x_i, t_i)$ on $t_i < t < t_{i+1}$.
  - $x_n(t) = x_n(t_i) + \phi(x_i, t_i)(t - t_i)$.
  - $\frac{x_{i+1} - x_i}{1/N} = \phi(x_i, t_i)$.
  - $\Delta_n(t) = x_n'(t) - \phi(x_n(t), t)$ for $\phi(x_i, t_i) - \phi(x_n(t), t)$ measures how close our solution is.
  - All of these things imply that our final formula is

$$x_n(t) = c + \int_0^t [\phi(x_n(s), s) + \Delta_n(s)] \, \mathrm{d}s$$

  - If we know that $x_n \rightrightarrows x$, then $\Delta_n \rightrightarrows 0$.
  - We then use the A-A theorem to imply convergence.
- When we get to MATH 208, say we didn't do any multivariable calculus in MATH 207.
  - We didn't do how to integrate in $\mathbb{R}^n$, how to integrate by parts (Stoke's theorem), Lagrange multipliers (constraint minimization).
- Problem 4.23: Show the inequalities at the bottom first and then use those to show continuity.
  - Consider $\lim_{t \to u} f(t)$. Approach from two sides separately and show cancellation??? Chloe will write a solution.
  - This is a particular trick for convex functions; it's not exactly recyclable.
- The trick for Problem 4.26 is recyclable.
- Linear algebra questions on the final are easier than the midterm.
  - The last question will be the hard one this time.

7.2 Sequences and Series of Functions

From Rudin (1976).

12/6:

- Let f be a complex-valued function.
- **Limit** (of $\{f_n\}$): The function $f : E \to \mathbb{C}$ defined by
$$f(x) = \lim_{n\to\infty} f_n(x)$$
for all $x \in E$, where $\{f_n\}$ is a sequence of functions such that $\lim_{n\to\infty} f_n(x)$ exists for all $x \in E$. *Also known as* **limit function**.
- **Sum** (of $\{f_n\}$): The function $f : E \to \mathbb{C}$ defined by
$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$
for all $x \in E$, where $\{f_n\}$ is a sequence of functions such that $\sum_{n=1}^{\infty} f_n(x)$ exists for all $x \in E$.
- Motivation for this chapter: Which properties of functions are preserved under the limit and summation operations?
- Continuity example:
  - A function is continuous at x if $\lim_{t\to x} f(t) = f(x)$.
  - Hence, the limit of a sequence of continuous functions is continuous at x if
$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$$
  - This switching of the limits is not always possible: If $s_{m,n} = m/(m+n)$, then
$$\lim_{n\to\infty}\lim_{m\to\infty} s_{m,n} = 1 \neq 0 = \lim_{m\to\infty}\lim_{n\to\infty} s_{m,n}$$
  - For an example of a sequence of continuous functions converging to a discontinuous function, see the example 2 (11/15 class notes).
- **Pointwise convergent** (sequence $\{f_n\}$): A sequence of functions $\{f_n\}$ for which there exists a function f such that for every $\epsilon > 0$ and for every $x \in E$, there exists an integer N such that if $n \geq N$, then
$$|f_n(x) - f(x)| < \epsilon$$
- **Uniformly convergent** (sequence $\{f_n\}$): A sequence of functions $\{f_n\}$ for which there exists a function f such that for every $\epsilon > 0$, there exists an integer N such that if $n \geq N$, then
$$|f_n(x) - f(x)| < \epsilon$$
for all $x \in E$.
- Theorem 7.8: Cauchy criterion for uniform convergence.
- Theorem 7.9: If $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in E$ and $M_n = \sup_{x\in E}|f_n(x) - f(x)|$, then $f_n \to f$ uniformly on E iff $M_n \to 0$ as $n \to \infty$.
- Theorem 7.10 (Weierstrass M-test): $|f_n(x)| \leq M_n$ for all $x \in E$ and $\sum M_n$ converges implies $\sum f_n$ converges uniformly.

- Theorem 7.11: Suppose $f_n \to f$ uniformly on a set E in a metric space. Let x be a limit point of E, and suppose that
$$\lim_{t \to x} f_n(t) = A_n$$
for each $n \in \mathbb{N}$. Then $\{A_n\}$ converges and
$$\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n$$

Proof. See IBL Theorem 17.6. □

  - It follows that in this case,
$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t)$$

- Theorem 7.12: A uniformly convergent sequence of continuous functions converges to a continuous function.
- Theorem 7.13: K compact, $\{f_n\}$ a sequence of continuous functions on K that converges pointwise to a continuous function f on K, and $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, $n \in \mathbb{N}$ implies $f_n \to f$ uniformly on K.
- $\boldsymbol{\mathscr{C}(X)}$: The set of all complex-valued, continuous, bounded functions with domain X a metric space.
- **Supremum norm** (of $f \in \mathscr{C}(X)$): The following value. *Denoted by* $\boldsymbol{\|f\|}$. *Given by*
$$\|f\| = \sup_{x \in X} |f(x)|$$
- Properties of the supremum norm.
  - $\|f\| < \infty$ (f is bounded).
  - $\|f\| = 0$ iff $f = 0$.
  - $\|f + g\| \leq \|f\| + \|g\|$.
- The above properties plus the definition $d(f, g) = \|f - g\|$ for any $f, g \in \mathscr{C}(X)$ makes $\mathscr{C}(X)$ a metric space!
- Rephrasing Theorem 7.9: A sequence $\{f_n\}$ converges to f with respect to the metric of $\mathscr{C}(X)$ if and only if $f_n \to f$ uniformly on X.
  - Thus, closed subsets $\mathscr{A} \subset \mathscr{C}(X)$ are sometimes called **uniformly closed**.
  - The closure of a subset $\mathscr{A} \subset \mathscr{C}(X)$ can similarly be called the **uniform closure**.
- Theorem 7.15: The above metric makes $\mathscr{C}(X)$ into a complete metric space.
- Theorem 7.16: $f_n \in \mathscr{R}(\alpha)$ on $[a, b]$ and $f_n \to f$ uniformly on $[a, b]$ imply $f \in \mathscr{R}(\alpha)$ on $[a, b]$ and
$$\int_a^b f \, \mathrm{d}\alpha = \lim_{n \to \infty} \int_a^b f_n \, \mathrm{d}\alpha$$

Proof. See IBL Theorem 17.7; Rudin (1976)'s is much slicker though. □

- Theorem 7.17: Suppose $\{f_n\}$ are differentiable on $[a, b]$, $\{f_n(x_0)\}$ converges for some $x_0 \in [a, b]$, and $\{f_n'\}$ converges uniformly on $[a, b]$. Then $\{f_n\}$ converges uniformly on $[a, b]$ to a function f such that
$$f'(x) = \lim_{n \to \infty} f_n'(x)$$
for all $x \in [a, b]$.

Proof. Long and complicated, and given in class. □

  - Assuming the continuity of the $\{f_n'\}$ admits a proof like IBL Theorem 17.8.

- Theorem 7.18: There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ which is nowhere differentiable.

  Proof. Long and complicated, and given in class. □

- **Pointwise bounded** (sequence $\{f_n\}$): A sequence of functions $\{f_n\}$ for which there exists a finite-valued function ϕ defined on E such that $|f_n(x)| < \phi(x)$ for all $x \in E$.
- **Uniformly bounded** (sequence $\{f_n\}$): A sequence of functions $\{f_n\}$ for which there exists a number M such that $|f_n(x)| < M$ for all $x \in E$.
- **Equicontinuous** (family of functions $\mathscr{F}$): A family of functions $\mathscr{F}$ such that for every $\epsilon > 0$, there exists a $\delta > 0$ such that
$$|f(x) - f(y)| < \epsilon$$
whenever $d(x, y) < \delta$, $x, y \in E$, and $f \in \mathscr{F}$.
  - Every member of an equicontinuous family is uniformly continuous.
- Theorem 7.23: $\{f_n\}$ pointwise bounded and defined on a countable set E implies $\{f_n\}$ has a pointwise convergent subsequence $\{f_{n_k}\}$.

  Proof. Diagonalization argument from class. □

- Theorem 7.24: K compact and $\{f_n\} \subset \mathscr{C}(K)$ uniformly convergent implies $\{f_n\}$ equicontinuous on K.
- Theorem 7.25 (Arzelà-Ascoli Theorem): If K is compact and $\{f_n\} \subset \mathscr{C}(K)$ is pointwise bounded and equicontinuous on K, then
  1. $\{f_n\}$ is uniformly bounded on K;
  2. $\{f_n\}$ contains a uniformly convergent subsequence.
- Theorem 7.26 (Weierstrass Approximation Theorem): f continuous on $[a, b]$ implies there exists a sequence of polynomials P_n such that
$$\lim_{n\to\infty} P_n(x) = f(x)$$
uniformly on $[a, b]$.
- Corollary 7.27: For every interval $[-a, a]$, there is a sequence of real polynomials $\{P_n\}$ such that $P_n(0) = 0$ and such that
$$\lim_{n\to\infty} P_n(x) = |x|$$
uniformly on $[-a, a]$.
- **Algebra**: A family $\mathscr{A}$ of complex functions defined on a set E such that for all $f, g \in \mathscr{A}$ and $c \in \mathbb{C}$,
  (i) $f + g \in \mathscr{A}$;
  (ii) $fg \in \mathscr{A}$;
  (iii) $cf \in \mathscr{A}$.
- **Uniformly closed** (algebra $\mathscr{A}$): An algebra $\mathscr{A}$ such that if $\{f_n\} \subset \mathscr{A}$ and $f_n \to f$ uniformly on E, then $f \in \mathscr{A}$.
- **Uniform closure** (of an algebra $\mathscr{A}$): The set $\mathscr{B}$ of all functions which are limits of uniformly convergent sequences of members of $\mathscr{A}$.

- Rephrasing Theorem 7.26: The set of all continuous functions on $[a, b]$ is the uniform closure of the set of polynomials on $[a, b]$.
- Theorem 7.29: Let $\mathscr{B}$ be the uniform closure of an algebra $\mathscr{A}$ of bounded functions. Then $\mathscr{B}$ is a uniformly closed algebra.
- **Separating points** (family $\mathscr{A}$ on E): A family of functions $\mathscr{A}$ on a set E such that to every pair of distinct points $x_1, x_2 \in E$, there corresponds a function $f \in \mathscr{A}$ such that $f(x_1) \neq f(x_2)$.
  - Example of a family that separates points on $\mathbb{R}^1$: the algebra of all polynomials in one variable.
  - Example of a family that does not separate points on $[-1, 1]$: the set of all even polynomials in one variable (since $f(x) = f(-x)$ for every even function f).
- **Vanishing at no point** (family $\mathscr{A}$ on E): A family of functions $\mathscr{A}$ such that to each $x \in E$, there corresponds a function $g \in \mathscr{A}$ such that $g(x) \neq 0$.
- Theorem 7.31: $\mathscr{A}$ an algebra on E that separates points and vanishes at no point, $x_1, x_2 \in E$ distinct, and $c_1, c_2 \in \mathbb{C}$ imply $\mathscr{A}$ contains a function f such that

$$f(x_1) = c_1 \qquad\qquad f(x_2) = c_2$$

- Theorem 7.32 (Stone-Weierstrass Theorem): $\mathscr{A}$ an algebra of real continuous functions on K compact that separates points of K and vanishes at no point of K implies the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all real continuous functions on K.
  - Theorem 7.32 holds on complex algebras with the additional hypothesis that $\mathscr{A}$ is **self-adjoint** (see Theorem 7.33).
- **Self-adjoint** (algebra $\mathscr{A}$): An algebra $\mathscr{A}$ such that if $f \in \mathscr{A}$, then its **complex conjugate** $\bar{f} \in \mathscr{A}$.
- **Complex conjugate** (of f): The function $\bar{f}$ defined by $\bar{f}(x) = \overline{f(x)}$.
- Theorem 7.33: $\mathscr{A}$ a self-adjoint algebra of complex continuous functions on K compact that separates points of K and vanishes at no point of K implies the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all complex continuous functions on K.
  - In other words, $\mathscr{A}$ is dense in $\mathscr{C}(K)$.