

Chapter 4

Continuity

4.1 Notes

- 11/8:
- Consider a function $f : X \rightarrow Y$ whose domain and codomain are, respectively, the metric spaces (X, d_X) and (Y, d_Y) .
 - **Limit** (of f at p): A point $q \in Y$ such that for all $\epsilon > 0$, there exists δ such that $d_X(x, p) < \delta$ implies $d_Y(q, f(x)) < \epsilon$, where p is a limit point of X (otherwise, $x \nrightarrow p$).
 - **Continuous** (function f at p): A function f such that $\lim_{x \rightarrow p} f(x) = f(p)$.
 - f is continuous on X if it is continuous at every $p \in X$.
 - **Uniformly continuous** (function f): A function f such that for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \epsilon$ for all $x, y \in X$.
- 11/10:
- f, g continuous implies $f + g, fg$, and f/g continuous, the latter where $g(x) \neq 0$.
 - If f, g continuous, then $h = g \circ f$ is continuous.
 - Theorem: $f : X \rightarrow Y$ is continuous iff $f^{-1}(V)$ is open in X for every $V \subset Y$ open.

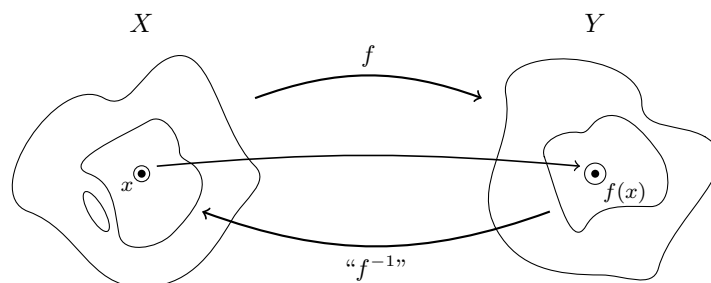


Figure 4.1: Set theoretic definition of continuity.

- This works in a general topological space, too, not just a metric space.
- Note that $f^{-1}(V)$ is not a function defined on V ; it's a specifically defined set $\{x \in X : f(x) \in V\}$.
- f being continuous means that open circular neighborhood of a point x in the domain maps to an area of the range encompassed by a circular neighborhood of $f(x)$.
- The other condition means that every open set surrounding $f(x)$ maps to an open set of the domain surrounding x . Indeed, going off of this definition, if an open set containing $f(x)$ maps to an open set containing x , then we can choose a neighborhood subset of the open set surrounding x and know that it will map into a neighborhood subset of the open set surrounding $f(x)$.

- Corollary: $f : X \rightarrow Y$ continuous iff $f^{-1}(C)$ closed for every $C \subset Y$ closed.
 - We use the property that $f^{-1}(X \subset C) = X \subset f^{-1}(C)$.
 - Let $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$. Suppose $f : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is defined by $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$. Then f is continuous iff f_1, f_2 are continuous, under appropriately defined metrics.
 - Continuity and compactness.
 - Theorem: $f : X \rightarrow Y$ continuous and X compact imply $f(X)$ compact.
 - Let $\{V_\alpha\}$ be an open cover of $f(X)$.
 - Then $\{f^{-1}(V_\alpha)\}$ is an open cover of X .
 - Choose a finite subcover of $\{f^{-1}(V_\alpha)\}$. Then the corresponding V_α 's form a finite subcover of $f(X)$.
 - If $f : X \rightarrow \mathbb{R}^k$ is continuous and X is compact, $f(X)$ is compact and closed/bounded.
 - If $f : X \rightarrow \mathbb{R}$ is continuous and X is compact, then $M = \sup_{x \in X} |f(x)| = |f(\bar{x})|$ and $m = \inf_{x \in X} |f(x)| = |f(\underline{x})|$ where $\bar{x}, \underline{x} \in X$.
 - There is a subsequence $\{x_m\}$ such that $|f(x_m)| \rightarrow M$. Since $f(X)$ is compact, the limit of this sequence is in $f(X)$.
 - If $f : X \rightarrow Y$ is continuous, bijective, and X, Y are compact, $f^{-1} : Y \rightarrow X$ is continuous.
 - Uniform continuity.
 - Examples.
 - Linear functions are uniformly continuous.
 - $f : (0, 1) \rightarrow \mathbb{R}$ defined by $f(x) = x^2$ is uniformly continuous.
 - $f : (a, \infty) \rightarrow \mathbb{R}$ defined by $f(x) = x^2$ is *not* uniformly continuous.
 - $f : (a, \infty) \rightarrow \mathbb{R}$ defined by $f(x) = 1/x$ is uniformly continuous if $a > 0$.
 - $f : (0, \infty) \rightarrow \mathbb{R}$ defined by $f(x) = 1/x$ is *not* uniformly continuous.
 - **Lipschitz continuous** (function f on $E \subset X$): A function such that $|f(x) - f(y)| \leq L|x - y|$ for each $x, y \in E$.
 - Theorem: $f : X \rightarrow Y$ continuous and X compact implies f is uniformly continuous.
 - Fix $\epsilon > 0$. There exists $\delta = \delta(p) > 0$.
 - Def. of continuity: $q \in N_{\delta(p)}(p)$ implies $f(q) \in N_\epsilon(f(p))$.
 - $\{N_{\delta(p)/2}(p) : p \in X\}$ is an open cover of X . Choose a finite subcover. Let $\delta = \min(\delta(p_1)/2, \dots, \delta(p_n)/2)$.
 - ...
- 11/12:
- $f : X \rightarrow Y$ continuous and $E \subset X$ connected implies $f(E)$ connected.
 - Suppose $f(E) = A \cup B$, A, B nonempty, separated subsets of Y .
 - Let $G = E \cap f^{-1}(A)$, $H = E \cap f^{-1}(B)$. It follows that $E = G \cup H$, where G, H nonempty.
 - $A \subset \bar{A}$ implies $G \subset f^{-1}(\bar{A})$ implies (inverse image def. of continuity) $\bar{G} \subset f^{-1}(A)$ implies $f(\bar{G}) \subset \bar{A}$.
 - $f(H) = B$ and $\bar{A} \cap B = \emptyset$ yields $\bar{G} \cap H = \emptyset$. Symmetrically, $G \cap \bar{H} = \emptyset$. This contradicts our assumption that E is connected.
 - Introduces monotone functions.
 - Theorem: If f is monotonic on (a, b) , then the set of points of (a, b) at which f is discontinuous is at most countable.
 - ...

4.2 Chapter 4: Continuity

From Rudin (1976).

- 11/8:
- **Limit** (of f at p): The point $q \in Y$, if it exists, such that for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d_Y(f(x), q) < \epsilon$ for all points $x \in E$ for which $0 < d_X(x, p) < \delta$, where $(X, d_X), (Y, d_Y)$ are metric spaces, $E \subset X$, $f : E \rightarrow Y$, and $p \in E'$. Denoted by $\lim_{x \rightarrow p} f(x)$.
 - Note that we do not require that $p \in E$; only that some elements of the domain E approach p .
 - We also write $f(x) \rightarrow q$ as $x \rightarrow p$.
 - Theorem 4.2: Let X, Y, E, f , and p be as specified above. Then $\lim_{x \rightarrow p} f(x) = q$ iff $\lim_{n \rightarrow \infty} f(p_n) = q$ for every sequence $\{p_n\}$ in E such that $p_n \neq p$ for any n and $\lim_{n \rightarrow \infty} p_n = p$.
 - Rudin (1976) proves the sum, product, and quotient rules of limits from the analogous properties of series.
 - Continuity is defined.
 - Note that f *does* have to be defined at p to be continuous at p (in comparison to the fact that it can have a limit at a point p' at which it is not defined).
 - Thus, for proofs concerning continuity (as opposed to limits), we will consider functions f the domains of which are metric spaces, not *subsets* of metric spaces.
 - It follows from the definition that if $p \in E$ is isolated, then every possible f defined on E is continuous at p .
 - Theorem 4.7: Compositions of continuous functions are continuous.
 - Theorem 4.8: Preimage definition of continuity.
 - Theorem 4.9: If f, g are complex continuous functions on X , $f + g$, fg , and f/g are continuous on X .
 - Theorem 4.10: \mathbf{f} continuous implies f_1, \dots, f_k continuous. Also, $\mathbf{f}, \mathbf{g} : X \rightarrow \mathbb{R}^k$ continuous implies $\mathbf{f} + \mathbf{g}$ and $\mathbf{f} \cdot \mathbf{g}$ continuous.
- 11/9:
- Theorem 4.14: f continuous and X compact implies $f(X)$ compact.
 - Theorem 4.15: $\mathbf{f} : X \rightarrow \mathbb{R}^k$ continuous and X compact implies $f(X)$ closed and bounded.
 - Theorem 4.16: f continuous and X compact implies f attains its minimum and maximum.
 - Theorem 4.17: $f : X \rightarrow Y$ continuous, 1-1 for X, Y compact implies $f^{-1} : Y \rightarrow X$ continuous.
 - Theorem 4.19: f continuous and X compact implies f uniformly continuous.
 - Theorem 4.20: Compactness is a necessary condition in Theorems 4.14, 4.15, 4.16, and 4.19.
 - Theorem 4.22: $f : X \rightarrow Y$ continuous and $E \subset X$ connected implies $f(E)$ connected.
 - Theorem 4.23: Intermediate value theorem.
 - **Right-hand limit** (of f at x): Denoted by $f(x+)$.
 - **Left-hand limit** (of f at x): Denoted by $f(x-)$.
 - **Discontinuity of the first kind** (of f at x): A discontinuity of f at x such that $f(x+)$ and $f(x-)$ exist. Also known as **simple discontinuity**.
 - **Discontinuity of the second kind** (of f at x): A discontinuity of f at x that is not of the first kind (i.e., a discontinuity such that at least one of $f(x+)$ and $f(x-)$ does not exist).

- Theorem 4.29: If f is monotonic on (a, b) , then $f(x+)$, $f(x-)$ exist at every $x \in (a, b)$.
- Corollary: Monotonic functions have no discontinuities of the second kind.
- Theorem 4.30: If f is monotonic on (a, b) , then the set of points of (a, b) at which f is discontinuous is at most countable.