Chapter 7

Bilinear and Quadratic Forms

10/18:

- **Bilinear form**: A function $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$L(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2, \mathbf{y}) = \alpha L(\mathbf{x}_1, \mathbf{y}) + \beta L(\mathbf{x}_2, \mathbf{y}) \qquad L(\mathbf{x}, \alpha \mathbf{y}_1 + \beta \mathbf{y}_2) = \alpha L(\mathbf{x}, \mathbf{y}_1) + \beta L(\mathbf{x}, \mathbf{y}_2)$$

  - $L(\mathbf{x}, \mathbf{y}) = (A\mathbf{x}, \mathbf{y})$.

- **Quadratic form**: A bilinear form $L(\mathbf{x}, \mathbf{x})$.
  - $(\mathbf{x}, \mathbf{x})$ is a polynomial of degree 2 in $\mathbf{x}_1, \ldots, \mathbf{x}_n$:

$$L(\lambda \mathbf{x}, \lambda \mathbf{x}) = (\lambda \mathbf{x}, \lambda \mathbf{x}) = \lambda^2 (\mathbf{x}, \mathbf{x})$$

- We have that

$$(A\mathbf{x}, \mathbf{x}) = (A\lambda \mathbf{x}, \lambda \mathbf{x}) = \lambda^2 (A\mathbf{x}, \mathbf{x}) = \sum_{j,i=1}^{n} \alpha_{j,i} \mathbf{x}_i \mathbf{x}_j$$

- The general form of a quadratic form:
  - Can any quadratic form on $\mathbb{R}^n$ be written as $(A\mathbf{x}, \mathbf{x})$?

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- Bilinear forms are linear in each argument when keeping the other fixed.
- Quadratic forms $Q(\mathbf{x}) = L(\mathbf{x}, \mathbf{x})$ are quadratic polynomials in the coordinates of x.
  - In particular, $Q(\lambda \mathbf{x}) = |\lambda|^2 Q(\mathbf{x})$.
- If Q quadratic is real, then $Q(\mathbf{x}) = (A\mathbf{x}, \mathbf{x})$ where A is some square matrix.
  - If $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is an orthonormal basis of $\mathbb{R}^n$, then there exists a unique $A = A^*$ such that $(A)_{ij} = L(\mathbf{e}_i, \mathbf{e}_j)$.
  - Keeping $\mathbf{x} = \sum_{i=1}^{n} \mathbf{x}_i, \mathbf{e}_i$ foxed, we have

$$\begin{aligned} Q(\mathbf{x}) &= L(\mathbf{x}, \mathbf{x}) \\ &= L(\sum^{n} \mathbf{x}_i \mathbf{e}_i, \sum^{n} \mathbf{x}_j \mathbf{e}_j) \\ &= \sum_{i=1}^{n} \mathbf{x}_i L(\mathbf{e}_i, \sum^{n} \mathbf{x}_j \mathbf{e}_j) \\ &= \sum_{i,j=1}^{n} \mathbf{x}_i \mathbf{x}_j \underbrace{L(\mathbf{e}_i, \mathbf{e}_j)}_{A_{ij}} \end{aligned}$$

- We have that

$$\begin{aligned}(A\mathbf{x},\mathbf{x}) &= (UDU^{-1}\mathbf{x},\mathbf{x})\\ &= (DU^{-1}\mathbf{x},U^{-1}\mathbf{x})\\ &= \sum_{i=1}^{n}\lambda_i(\underbrace{U^{-1}\mathbf{x}}_{\mathbf{y}_i})_i(\underbrace{U^{-1}\mathbf{x}}_{\mathbf{y}_i})_i\end{aligned}$$

- Can we characterize the set $\{\mathbf{x} : (A\mathbf{x},\mathbf{x}) = 1\}$?
  - Note that this set is equivalent to $\{\mathbf{y} : (D\mathbf{y},\mathbf{y}) = 1\}$ by teh above. This set is a rotation of the previous one. Ellipse?
- Positive quadratic form:
  - Q is positive definite if $Q(\mathbf{x}) > \mathbf{0}$ for all $\mathbf{x} \neq \mathbf{0}$ and Q is positive semidefinite if $Q(\mathbf{x}) \geq \mathbf{0}$ for all $\mathbf{x} \neq \mathbf{0}$.
  - Take a self-adjoint matrix $A = A^*$. It is positive definite if $Q(\mathbf{x}) = (A\mathbf{x},\mathbf{x})$ is positive definite.
- Theorem: If $A = A^*$, then
  1. A is positive definite if and only if all eigenvalues of A are positive.
  2. A is positive semidefinite if and only if all eigenvalues of A are nonnegative.
  3. A is negative semidefinite if and only if all eigenvalues of A are nonpositive.
  4. A is negative definite if and only if all eigenvalues of A are negative.
  5. A is indefinite if and only if the eigenvalues of A have positive and negative values.
- Theorem: $A = A^*$ is positive definite iff $\det A_k > 0$ for all $k = 1,\dots,n$ where A_k is the upper left $k\times k$ submatrix.
- Minimax representation of eigenvalues of a self-adjoint A.
  - Let E be a subspace of X where $\dim X < \infty$. We define $\operatorname{codim}(E) = \dim E^\perp$.
  - Thus, $\dim E + \operatorname{codim} E = \dim X$.
  - Theorem: Let $A = A^*$, $\lambda_1 \geq \cdots \geq \lambda_n$ eigenvalues of A. Then

$$\lambda_k = \max_{\substack{\text{E subspace}\\ \dim E=k}}\ \min_{\substack{\mathbf{x}\in E\\ \|\mathbf{x}\|=1}}(A\mathbf{x},\mathbf{x}) = \min_{\substack{\text{F subspace}\\ \operatorname{codim} F=k-1}}\ \max_{\substack{\mathbf{x}\in F\\ \|\mathbf{x}\|=1}}(A\mathbf{x},\mathbf{x})$$

    - Proof: A diagonal equals $(\lambda_1,\dots,\lambda_n)$.
    - An orthonormal basis of X such that $\dim E = k$, $\operatorname{codim} F = k-1$, $\dim F = n-k+1$.
    - There exists an $\mathbf{x}_0 \neq \mathbf{0}$ such that $\mathbf{x}_0 \in E \cap F$.
    - Note that if $B = B^*$, then the max and min of $(B\mathbf{x},\mathbf{x})$ over the unit sphere is the maximal and minimal eigenvalue of B.
    - Thus,

$$\min_{\substack{\mathbf{x}\in E\\ \|\mathbf{x}\|=1}}(A\mathbf{x},\mathbf{x}) \leq (A\mathbf{x}_0,\mathbf{x}_0) \leq \max_{\substack{\mathbf{x}\in F\\ \|\mathbf{x}\|=1}}(A\mathbf{x},\mathbf{x})$$

    - This is true for any E,F subspaces. $\dim E = k$, $\operatorname{codim} F = k-1$, $E_0 = \operatorname{span}(\mathbf{e}_1,\dots,\mathbf{e}_k)$ and $F_0 = \operatorname{span}(\mathbf{e}_k,\dots,\mathbf{e}_n)$.
    - Thus,

$$\min_{\substack{E_0\\ \|\mathbf{x}\|=1}}(A\mathbf{x},\mathbf{x}) = \lambda_k = \max_{\substack{F_0\\ \|\mathbf{x}\|=1}}(A\mathbf{x},\mathbf{x})$$

- Additionally,

$$\lambda_{k_1} \leq \max_{\dim E=k}^{E} \min_{\mathbf{x}}(A\mathbf{x},\mathbf{x}) \leq \min_{\substack{F \\ \operatorname{codim} F=k-1}} \max_{\mathbf{x}}(A\mathbf{x},\mathbf{x}) \leq \lambda_k$$

- Corollary: Let $A = A^* = (a_{jk})_{1\leq j,k\leq n}$ with eigenvalues $\lambda_1,\dots,\lambda_n$ listed in decreasing order. Let $\tilde{A} = (a_{j,k})_{1\leq j,k\leq n-1}$ with eigenvalues $\mu_1,\dots,\mu_{n-1}$ listed in decreasing order. Then $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n$.
  - Consider $(A\mathbf{x},\mathbf{x})$ on $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$, but then restrict yourself to $\mathbf{x}\in\mathbb{R}^{n-1}$ on $\{\mathbf{e}_1,\dots,\mathbf{e}_{n-1}\}$.