Chapter 9

Advanced Spectral Theory

10/22:

- Let $p(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial. Let A be an $n \times n$ matrix. We let $p(A) = \sum_{i=0}^{n} a_i A^i$.
- Theorem: If A is an $n \times n$ and $p(\lambda) = \det(A - \lambda I)$, then $p(A) = 0$.
  - We know that $p(\lambda) = a(z - \lambda_1) \cdots (z - \lambda_n)$ where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues.
  - Thus $p(A) = a(A - \lambda_1 I) \cdots (A - \lambda_n I)$.
  - If you are in $\mathbb{R}^n$ and have this property, you can factorize your matrix.
  - Thus, $p(A)\mathbf{x} = \mathbf{0}$ since $\mathbf{x}$ can be decomposed into a linear combination of eigenvectors of A, which will be taken to 0 one by one by the terms of $p(A)$.
- $\sigma(B) = \{\text{eigenvalues of } B\}$ is known as the **spectrum** of B.
- If p is an arbitrary polynomial and A is $n \times n$, then μ is an eigenvalue of $p(A)$ if and only if $\mu = p(\lambda)$ where λ is an eigenvalue of A. In essence, $\sigma(p(A)) = p(\sigma(A))$.
- Chapter 9 will not be on the exam. We don't have to know the generalization to infinite dimensional spaces.

10/25:

- If A is an $n \times n$ square matrix and $p(\lambda) = \det(A - \lambda I)$, then $p(A) = 0$.
  - Proof: WLOG, let A be an upper triangular matrix with diagonal entries equal to the eigenvalues.
  - Think of $p(z) = (-1)^n (z - \lambda_1) \cdots (z - \lambda_n)$.
  - Thus, $p(A) = (-1)^n (A - \lambda_1 I) \cdots (A - \lambda_n I)$.
  - WTS: $p(A)\mathbf{x} = 0$ for all $\mathbf{x} \in V$.
  - Let $E_k = \operatorname{span}(e_1, \ldots, e_k)$ be the span of the first k eigenvectors of A, where $e_1, \ldots, e_n$ is a standard basis in $\mathbb{C}^n$.
  - A triangular implies $AE_k \subset E_k$. Thus, $(A - \lambda I)E_k \subset E_k$, so E_k is invariant under $A - \lambda I$ for all λ.
  - If we apply $A - \lambda_k I$ to a vector in E_k, we are left with a vector in E_{k-1}.
  - Thus, if we apply $\prod_{k=1}^{n} (A - \lambda_k I) = p(A)$ to any vector in $E_n = V$, we will kill it piece by piece down to zero.
- Let A be a square $n \times n$ matrix. Then p an arbitrary polynomial implies $\sigma(p(A)) = p(\sigma(A))$. (Any eigenvalue μ of $p(A)$ is $\mu = p(\lambda)$, where λ is an eigenvalue of A.)
  - Shows that polynomials of operators commute.
  - Proof: Let λ be an eigenvalue of A. We want to show that $p(\lambda)$ is an eigenvalue of $p(A)$. This is obvious since $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x}$, so $A^k\mathbf{x} = \lambda^k\mathbf{x}$, so in particular, $p(A)\mathbf{x} = p(\lambda)\mathbf{x}$.

  - On the other hand, if μ is an eigenvalue of $p(A)$, we want to show that there exists $\lambda \in \sigma(A)$ such that $\mu = p(\lambda)$.
  - Consider $q(z) = p(z) - \mu$. Then $q(A) = p(A) - \mu I$. Since μ is an eigenvalue of $p(A)$, $q(A)$ is not invertible.
  - Thus, $q(z) = (-1)^n (z - z_1) \cdots (z - z_n)$ and $q(A) = (-1)^k (A - z_1 I) \cdots (A - z_k I)$.
  - But $q(A)$ is not invertible, so one of the $A - z_k I$ is not invertible. Take z_k such that $A - z_k I$ is not invertible. Then $z_k \in \sigma(A)$. It follows that $q(z_k) = p(z_k) - \mu = \sigma$.
- If A is $n \times n$, $\lambda_1, \dots, \lambda_n$ are its eigenvalues, p is a polynomial, then $p(A)$ is invertible if and only if $p(\lambda_k) \neq 0$ for each $k = 1, \dots, n$.
  - This is an immediate corollary to the previous result.
- We now build up to the **generalized eigenspace**, which is related to some "geometric" properties of the algebraic multiplicity of an eigenvalue.
- If $A : V \to V$ is a linear operator and $E \subset V$ is a subspace, E is A-invariant if $AE \subset E$.
- Facts:
  - If E is A-invariant, E is A^k-invariant.
  - Thus, E is $p(A)$-invariant.
- Consider the restriction map $A|_E$.
- A has a block-diagonalized matrix where each block corresponds to the generalized eigenvectors of a generalized eigenvalue of A.
  - Let $E_1, \dots, E_r$ be a **basis of invariant subspaces**.
  - Let $A_k = A|_{E_k}$. Then the A_k's act independently of each other.
- **Generalized eigenvector** (of A): A vector $\mathbf{v}$ corresponding to an eigenvalue λ if there exists $k \geq 1$ such that $(A - \lambda I)^k \mathbf{v} = \mathbf{0}$.
- **Generalized eigenspace**: The set E_λ of all of the generalized eigenvectors of λ. *Given by*

$$E_k = \bigcup_{k \geq 1} \ker(A - \lambda I)^k$$

  - E_λ is a linear subspace of V.
- **Degree** (of λ): The smallest number k such that increasing k any more does not add further vectors to the generalized eigenspace. *Denoted by* $\boldsymbol{d(\lambda)}$.
  - Symbolically, $d(\lambda)$ is the smallest number such that

$$E_\lambda = \bigcup_{k=1}^{d(\lambda)} \ker(A - \lambda I)^k$$

- Start working through the first 25 problems of Rudin (1976) (his metric spaces problems).

10/27:
- Jordan form.
- Reviews build up to generalized eigenvectors.
- Theorem: If $\sigma(A) = \{\lambda_1, \dots, \lambda_n\}$ and $E_1, \dots, E_n$ are the corresponding generalized eigenspaces, then $E_1, \dots, E_n$ is a basis of subspaces of U, i.e., $V = \oplus_k E_k$.

- Corollary: $A : V \to V$ can be represented as $A = D + N$ where D is diagonalizable and N is nilpotent and $ND = DN$.
  - Proof: Consider the basis of generalized eigenspaces known to exist from the theorem. Then $A = \operatorname{diag}\{A_1, \dots, A_r\}$.
  - Let
    $$N_k = A_k - \lambda_k I_{E_k}$$
    This is nilpotent.
  - Then let
    $$D = \operatorname{diag}\{\lambda_1 I_{E_1}, \dots, \lambda_n I_{E_n}\}$$
  - These two matrices satisfy the necessary properties.
- Let $\dot{\mathbf{x}} = A\mathbf{x}$.
  - Let $\mathbf{x}(t) = \mathrm{e}^{tA}$, where
    $$\mathrm{e}^{tA} = \sum \frac{(tA)^k}{k!}$$
  - $\left\|\mathrm{e}^{tA}\right\| \leq \sum \frac{\left\|A^k\right\|}{k!} = \sum \frac{\|A\|^k}{k!}$.
  - Let p be a polynomial of degree k. Then
    $$p(a + x) = \sum_{k=0}^{d} \frac{p^{(k)}(a)}{k!} x^k$$
  - If $A = D + N$, then...
- Nilpotent operators:
  - Let $A = \operatorname{diag}\{A_1, \dots, A_r\}$.
  - We know that $A_k = \lambda_k I_{E_k} + N_k$ for each k.
  - Every nilpotent N can be written in the form
    $$\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$$