Chapter 7

Sequences and Series of Functions

7.1 Notes

11/15:

- Soug will not test on differentiation/integration assuming that we know them already.
- **Pointwise convergent** (sequence $(f_n)_{n\in\mathbb{N}}$ of functions): A sequence of functions $f_n : E \to \mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in E$.
- Can we interchange "limit" in the above definition with continuity, convergence of series, integration, differentiation, etc.?
- Examples with negative answer:
  1. Interchanging limits: Let $S_{mn} = \frac{m}{m+n}$. $S_{mn} \to 1$ as $m \to \infty$, and $S_{mn} \to 0$ as $n \to \infty$.
  2. $f_n(x) = x^2/(1+x)^n$. $f(x) = \sum_{n=1}^{\infty} f_n(x)$. If $x = 0$, then $f_n(x) = 0$ for all n and $f(x) = 0$. If $x \neq 0$, we have
  $$f(x) = \sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n} = x^2 \sum_{n=1}^{\infty} X^n = \frac{x^2}{1-X} = \frac{x^2}{1-(1/(1+x^2))} = 1 + x^2$$
  3. Consider $f_m(x) = \lim_{n\to\infty}(\cos(m\pi x))^2 n$. $\lim_{m\to\infty} f_m(x)$ goes to 0 if $x \notin \mathbb{Q}$ and goes to 1 if $x \in \mathbb{Q}$. $f_m \to \chi_{\mathbb{Q}}$, where $\chi_{\mathbb{Q}}$ is the characteristic function of the rationals which is not Riemann integrable (partitions, upper and lower integrals, etc.).
  4. $f_n(x) = \sin nx/\sqrt{n} \to f(x) = 0$ for all x. However, $f_n'(x) = \sqrt{n}\cos nx \nrightarrow 0$
  5. If $0 \leq x \leq 1$, define $f_n(x) = n^2 x(1-x^2)^n$. We know that $f_n(0) = 0$. $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in (0,1]$. We can show that $\int_0^1 x(1-x^2)^n \,\mathrm{d}x = 1/(2n+2)$. Thus, $\int_0^1 f_n(x)\,\mathrm{d}x = n^2/(2n+2)$. Limit of the functions is zero, but their integrals diverge to infinity.
- **Uniformly convergent** (sequence $(f_n)_{n\in\mathbb{N}}$ of functions on E): A sequence of functions $f_n : E \to \mathbb{R}$ such that for all $\epsilon > 0$, there exists N such that if $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$ for all $x \in E$. *Denoted by* $\boldsymbol{f_n \rightrightarrows f}$.
- Theorem: $f_n \rightrightarrows f$ iff $(f_n)_{n\in\mathbb{N}}$ is uniformly Cauchy (i.e., for all $\epsilon > 0$, there exists N such that if $n, m \geq N$ then $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in E$).
  - Let $M_n = \sup_{x\in E} |f_n(x) - f(x)|$. If $f_n \to f$ pointwise, then $f_n \rightrightarrows f$ if $M_n \to 0$.
- Theorem: If $(f_n)_{n\in\mathbb{N}}$ and $|f_n(x)| \leq M_n$, then $\sum f_n \rightrightarrows f$ if $\sum M_n < \infty$.
- Theorem: If E is a compact metric space, $f_n \rightrightarrows f$ in E, x is a limit point of E, and $\lim_{t\to x} f_n(t) = A_n$ exists, then $(A_n)_{n\in\mathbb{N}}$ converges and $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$.
- Corollary: $\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty}\lim_{t\to x} f_n(t)$.