Chapter 4

Introduction to Spectral Theory

4.1 Notes

10/1:

- **Difference equation**: Like a differential equation, but instead of writing a differentials, you write differences.
- Suppose we want to solve $x_{n+1} = Ax_n$ with x_0 given.
  - You will find that $x_n = A^n x_0$.
  - This gets hard to compute, so we want to find a way to simplify the computation.
- Thus, we want to diagonalize the matrix, and this concept is inherently linked to eigenvalues and eigenvectors.
  - If you can decompose the x_0 into a linear combination of eigenvectors, then you can simplify the computation a lot:
    $$x_n = \sum \alpha_i A^n v_i = \sum \alpha_i \lambda_i^n v_i$$
  - An $n \times n$ matrix will have n eigenvalues. You want n linearly independent eigenvectors, creating an eigenbasis.
- To find eigenvalues and eigenvectors, we need to solve $Ax = \lambda x$, i.e., $(A - \lambda I)x = 0$. Thus, $\ker(A - \lambda I) \neq \{0\}$, so $\det(A - \lambda I) = 0$.
- The eigenvalues of A are independent of the choice of basis of the domain of A or the range.

10/4:

- We need to know everything in Treil (2017).
  - We don't need to know the applications sections, but you should be interested.
- **Spectral theory**: Decomposing a linear operator.
- Let $A : V \to V$ be a linear operator. $\lambda \in \mathbb{C}$ is an eigenvalue if there exists $x \in V$ nonzero such that $Ax = \lambda x$.
  - Let A be an $n \times n$ matrix over $\mathbb{C}$ or $\mathbb{R}$.
  - The eigenvalues are the roots of the polynomial $\det(A - \lambda I) = 0$ in λ.
- Things we want to do:
  - Given A, find the eigenvalues and eigenvectors (solve $(A - \lambda I)x = 0$).

  - In order to simplify A, make it a diagonal matrix:

$$A = S \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} S^{-1}$$

- Eigenvalues are independent of the choice of basis.
  - From the book, we have that
$$[A]_{\mathcal{A}\mathcal{A}} = [S]_{\mathcal{A}\mathcal{B}}[B]_{\mathcal{B}\mathcal{B}}[S]_{\mathcal{A}\mathcal{B}}^{-1}$$
  - It follows that
$$A - \lambda I = [S]_{\mathcal{A}\mathcal{B}}(B - \lambda I)[S]_{\mathcal{A}\mathcal{B}}^{-1}$$
    so
$$\det(A - \lambda I) = \det\big([S]_{\mathcal{A}\mathcal{B}}(B - \lambda I)[S]_{\mathcal{A}\mathcal{B}}^{-1}\big) = \det\big([S]_{\mathcal{A}\mathcal{B}}[S]_{\mathcal{A}\mathcal{B}}^{-1}(B - \lambda I)\big) = \det(B - \lambda I)$$
- If $p(z) = (z - \lambda)^k q(z)$, then k is the **algebraic multiplicity** of λ. The **geometric multiplicity** of λ is $\dim\ker(A - \lambda I)$.
  - These terms are not always the same, but they are related.
- Diagonalization:
  - Given A that corresponds to $T : V \to V$, can we find a basis of V in which the operator is a diagonal matrix?
  - $A = SDS^{-1}$ iff there exists a basis of V consisting of the eigenvectors of A.
  - Proves $A^N = SD^N S^{-1}$ via $A^2 = SDS^{-1}SDS^{-1} = SDIDS^{-1} = SD^2S^{-1}$.
- Let A be an $n \times n$ matrix over $\mathbb{F}$. If $\lambda_1, \dots, \lambda_r$ are distinct eigenvalues, then their eigenvectors are linearly independent.
  - Prove with induction contradiction argument. Assume true for $\mathbf{v}_{r-1}$. Then
$$0 = (A - \lambda_r I)[\mathbf{v}_1 + \cdots + \mathbf{v}_r] = (\lambda_1 - \lambda_r)\mathbf{v}_1 + \cdots + (\lambda_{r-1} - \lambda_r)\mathbf{v}_{r-1}$$
  - Implies $\lambda_r = \lambda_i$ for all $i \in [r-1]$, a contradiction.
  - If A has n distinct eigenvalues, then A is diagonalizable.
- If $A : V \to V$ has n complex eigenvalues, then A is diagonalizable iff the algebraic multiplicity equals the geometric multiplicity for each eigenvalue.
- Goes through a sample diagonalization with $\left(\begin{smallmatrix} 1 & 2 \\ 8 & 1 \end{smallmatrix}\right)$.
  - We have
$$A - \lambda I = \begin{pmatrix} 1 - \lambda & 2 \\ 8 & 1 - \lambda \end{pmatrix}$$
    so
$$0 = \det(A - \lambda I) = (1 - \lambda)^2 - 16$$
  - It follows that $\lambda = 5, -3$.
  - This yields
$$\begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}^{-1}$$
    by inspection.
- As another example, consider $\left(\begin{smallmatrix} 1 & 2 \\ -2 & 1 \end{smallmatrix}\right)$.
  - Here, we have $\lambda = 1 \pm 2i$.

4.2 Chapter 4: Introduction to Spectral Theory

From Treil (2017).

10/24:
- **Spectrum** (of A): The set of all eigenvalues of A. *Denoted by* $\boldsymbol{\sigma(A)}$.
- Proposition 4.1.1: The geometric multiplicity of an eigenvalue cannot exceed its algebraic multiplicity.
- Theorem 4.2.1: A matrix A (with values in $\mathbb{F}$) admits a representation $A = SDS^{-1}$ where D is a diagonal matrix and S is invertible if and only if there exists a basis of $\mathbb{F}^n$ of eigenvectors of A. Moreover, in this case diagonal entries of D are the eigenvalues of A and columns of S are the corresponding eigenvectors.
- Any operator on a complex vector space has n eigenvalues (counting multiplicities).
  - Think n necessary roots of the characteristic polynomial, or the necessary upper triangular representation.
- Theorem 4.2.8: Let an operator $A : V \to V$ have exactly $n = \dim V$ eigenvalues (counting multiplicities). Then A is diagonalizable if and only if for each eigenvalue λ, the dimension of the eigenspace $\ker(A - \lambda I)$ (i.e., the geometric multiplicity of λ) coincides with the algebraic multiplicity of λ.
- Theorem 4.2.9: A real $n \times n$ matrix A admits a real factorization (i.e., a real representation $A = SDS^{-1}$ where S and D are real matrices, D is diagonal, and S is invertible) if and only if it admits a complex factorization and all eigenvalues of A are real.
- Example of a nondiagonalizable matrix:
$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
  - $p(\lambda) = (1 - \lambda)^2$, so $\lambda = 1$ with algebraic multiplicity 2.
  - However, $\dim\ker(A - I) = 1$ since $A - I$ has only one pivot, hence $2 - 1 = 1$ free variable.
  - Thus, apply Theorem 4.2.8.