4 Inner Product Phenomena and Intro to Bilinear Forms

From Treil (2017).

Chapter 6

10/25: **1.1.** Use the upper-triangular representation of an operator to give an alternative proof of the fact that the determinant is the product and the trace is the sum of the eigenvalues counting multiplicities.

Answer. Let $A : V \to V$ be an operator. Then by Theorem 6.1.1, there exists a basis of V such that the matrix of A with respect to this basis is upper triangular. Since this matrix is upper triangular, the eigenvalues of A are exactly its diagonal entries. This combined with the fact that the determinant of an upper triangular matrix is the product of its diagonal entries proves that the determinant of A is the product of its eigenvalues. Similarly, the trace of A as the sum of the diagonal entries of A must be the sum of the eigenvalues of A, as desired. □

2.1. True or false:

a) Every unitary operator $U : X \to X$ is normal.

Answer. True.
Let $U : X \to X$ be unitary. Then

$$U^*U = I = UU^*$$

as desired. □

b) A matrix is unitary if and only if it is invertible.

Answer. False.
Consider the matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

A is invertible with inverse

$$A^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

but A is not unitary since A is not an isometry:

$$\|A\mathbf{x}\| = \|2\mathbf{x}\| = 2\|\mathbf{x}\| \neq \|\mathbf{x}\|$$

for any $\mathbf{x} \in \mathbb{F}^2$. □

c) If two matrices are unitarily equivalent, then they are also similar.

Answer. True.
Suppose that $A = UBU^*$. Then since $U^* = U^{-1}$, $A = UBU^{-1}$, so A, B are similar. □

d) The sum of self-adjoint operators is self-adjoint.

Answer. True.
If $A = A^*$ and $B = B^*$, then

$$(A + B)^* = A^* + B^* = A + B$$

as desired. □

e) The adjoint of a unitary operator is unitary.

Answer. True.
See property 2 of unitary operators (Treil, 2017, p. 148). □

f) The adjoint of a normal operator is normal.

Answer. True.
Let N be normal. Then $N^*N = NN^*$. This combined with the fact that $N = (N^*)^*$ implies that

$$(N^*)^*N^* = NN^* = N^*N = N^*(N^*)^*$$

as desired. $\square$

g) If all eigenvalues of a linear operator are 1, then the operator must be unitary or orthogonal.

Answer. False.
Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Clearly all eigenvalues of this matrix are 1. However,

$$A^*A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \neq I$$

so A is not unitary. $\square$

h) If all eigenvalues of a normal operator are 1, then the operator is the identity.

Answer. True.
Suppose N is a normal operator with all eigenvalues equal to 1. Then by Theorem 6.2.4, $N = UDU^*$ where $D = I$ (because of the condition on the eigenvalues). It follows that

$$N = UIU^* = UU^* = I$$

as desired. $\square$

i) A linear operator may preserve norm but not the inner product.

Answer. False.
Suppose U is a linear operator that preserves norm. Then U is an isometry. It follows by Theorem 5.6.1 that U preserves the inner product. $\square$

2.2. True or false (justify your conclusion): The sum of normal operators is normal.

Answer. False.
Let

$$N = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We know that N, M are normal since

$$\begin{aligned} NN^* &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= N^*N \end{aligned} \qquad \begin{aligned} MM^* &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= M^*M \end{aligned}$$

Then we have

$$\begin{aligned}
(N+M)(N+M)^* &= \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} \\
&\neq \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \\
&= (N+M)^*(N+M)
\end{aligned}$$

□

2.3. Show that an operator that is unitarily equivalent to a diagonal one is normal.

Answer. Let $A = UDU^*$. Then

$$\begin{aligned}
NN^* &= (UDU^*)(UDU^*)^* & N^*N &= (UDU^*)^*(UDU^*) \\
&= (UDU^*)(UD^*U^*) & &= (UD^*U^*)(UDU^*) \\
&= UDD^*U^* & &= UD^*DU^*
\end{aligned}$$

Additionally, we have that $D^*D = DD^*$ (Treil, 2017, p. 167), completing the proof. □

2.5. True or false (justify): Any self-adjoint matrix has a self-adjoint square root.

Answer. False.

Consider the trivially self-adjoint matrix

$$\begin{pmatrix} -1 \end{pmatrix}$$

The square roots of this matrix are $\begin{pmatrix} i \end{pmatrix}$ and $\begin{pmatrix} -i \end{pmatrix}$, neither of which is self-adjoint. □

2.6. Orthogonally diagonalize the matrix

$$A = \begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix}$$

i.e., represent it as $A = UDU^*$, where D is diagonal and U is unitary. Additionally, among all square roots of A, i.e., among all matrices B such that $B^2 = A$, find one that has positive eigenvalues. You can leave B as a product.

Answer. From the characteristic polynomial, we find that $\lambda_1 = 8$ and $\lambda_2 = 3$. It follows by inspection that the corresponding eigenvectors are

$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \mathbf{x}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

These vectors are already orthogonal, so we need only normalize them to get

$$U = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$$

Therefore, we have that

$$A = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}\begin{pmatrix} 8 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$$

From here, we can easily let

$$B = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 2\sqrt{2} & 0 \\ 0 & \sqrt{3} \end{pmatrix} \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$$

□

2.7. True or false (justify your conclusions):

a) A product of two self-adjoint matrices is self-adjoint.

Answer. False.
Let

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Clearly $A = A^*$ and $B = B^*$. However,

$$AB = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = (AB)^*$$

□

b) If A is self-adjoint, then A^k is self-adjoint.

Answer. True.
Suppose $A = A^*$. Then

$$(A^k)^* = (\underbrace{A \cdots A}_{k \text{ times}})^* = \underbrace{A^* \cdots A^*}_{k \text{ times}} = \underbrace{A \cdots A}_{k \text{ times}} = A^k$$

as desired. □

2.8. Let A be an $m \times n$ matrix. Prove that

a) A^*A is self-adjoint.

Answer. We have that

$$(A^*A)^* = A^*(A^*)^* = A^*A$$

as desired. □

b) All eigenvalues of A^*A are nonnegative.

Answer. Let λ be an eigenvalue of A^*A with corresponding nonzero eigenvector $\mathbf{x}$. Then

$$0 \leq (A\mathbf{x}, A\mathbf{x}) = (A^*A\mathbf{x}, \mathbf{x}) = (\lambda\mathbf{x}, \mathbf{x}) = \lambda(\mathbf{x}, \mathbf{x}) = \lambda\|\mathbf{x}\|^2$$

$$\frac{0}{\|\mathbf{x}\|^2} = 0 \leq \lambda$$

as desired. □

c) $A^*A + I$ is invertible.

Answer. To show that $A^*A + I$ is invertible, it will suffice to show that $\ker(A^*A + I) = \{\mathbf{0}\}$. One inclusion is obvious. However, for the other one, suppose $(A^*A + I)\mathbf{x} = \mathbf{0}$. Then

$$\begin{aligned} 0 &= (\mathbf{0}, \mathbf{x}) \\ &= ((A^*A + I)\mathbf{x}, \mathbf{x}) \\ &= (A^*A\mathbf{x} + \mathbf{x}, \mathbf{x}) \\ &= (A^*A\mathbf{x}, \mathbf{x}) + (\mathbf{x}, \mathbf{x}) \\ &= (A\mathbf{x}, A\mathbf{x}) + (\mathbf{x}, \mathbf{x}) \\ &= \|A\mathbf{x}\|^2 + \|\mathbf{x}\|^2 \end{aligned}$$

Therefore, $\|\mathbf{x}\| = 0$, so $\mathbf{x} = \mathbf{0}$, so $\mathbf{x} \in \{\mathbf{0}\}$, as desired. □

2.10. Orthogonally diagonalize the rotation matrix

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

where α is not a multiple of π. Note that you will get complex eigenvalues in this case.

Answer. We have from Problem 4.1.3 that $\lambda_1 = \mathrm{e}^{i\alpha}$ and $\lambda_2 = \mathrm{e}^{-i\alpha}$, and that

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad \mathbf{x}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

where $\mathbf{x}_1, \mathbf{x}_2$ are already orthogonal. Thus, normalizing gives us

$$R_\alpha = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & i/\sqrt{2} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{i\alpha} & 0 \\ 0 & \mathrm{e}^{-i\alpha} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} \\ 1/\sqrt{2} & -i/\sqrt{2} \end{pmatrix}$$

□

2.13. Prove that a normal operator with unimodular eigenvalues (i.e., with all eigenvalues satisfying $|\lambda_k| = 1$) is unitary. (Hint: Consider diagonalization.)

Answer. Let N be normal with unimodular eigenvalues. To prove that N is unitary, it will suffice to show that $NN^* = I$. First off, we have by Theorem 6.2.4 that $N = UDU^*$ where U is unitary and D is diagonal. Thus,

$$NN^* = UDU^*(UDU^*)^* = UDU^*UD^*U^* = UDD^*U = UIU^* = I$$

as desired. Note that $DD^* = I$ since each value along the diagonal of DD^* has $d_{jj}\bar{d}_{jj} = |d_{jj}|^2 = 1$. □

2.14. Prove that a normal operator with real eigenvalues is self-adjoint.

Answer. Let N be normal with all real eigenvalues. By Theorem 6.2.4, $N = UDU^*$ where D is real. Then

$$N^* = (UDU^*)^* = UD^*U^* = UDU^* = N$$

as desired. □

2.15. Show by example that the conclusion of Theorem 2.2 fails for *complex* symmetric matrices. Namely,

a) Construct a (diagonalizable) 2×2 complex symmetric matrix not admitting an orthogonal basis of eigenvectors.

Answer. Suppose A is our final matrix. We will apply the constraints sequentially to narrow down possible values of A and then pick one. Let's begin.
Since A is diagonalizable, $A = SDS^{-1}$ where D is a diagonal matrix and S is a matrix of eigenvectors of A. Since A is symmetric, $A = A^T$. It follows from these two conditions that

$$\begin{aligned} SDS^{-1} &= (SDS^{-1})^T \\ SDS^{-1} &= (S^T)^{-1}D^TS^T \\ S^TSD &= DS^TS \end{aligned}$$

Since $(S^TS)^T = S^T(S^T)^T = S^TS$ (so S^TS is symmetric), D is diagonal, and both are 2×2, we can represent them as

$$S^TS = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \qquad D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$$

for some $a, b, c, d_1, d_2 \in \mathbb{C}$. Thus, the above condition implies that

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$
$$\begin{pmatrix} ad_1 & bd_2 \\ bd_1 & cd_2 \end{pmatrix} = \begin{pmatrix} ad_1 & bd_1 \\ bd_2 & cd_2 \end{pmatrix}$$

i.e., that $bd_1 = bd_2$. It follows that either $d_1 = d_2$, or $b = 0$. Since we would like the freedom to choose distinct values, we will choose a solution for which $b = 0$. The overall conclusion is that $S^T S$ is diagonal, which implies that $\mathbf{x}_2^T \mathbf{x}_1 = 0$.

We now invoke the last given condition: that the eigenvectors $\mathbf{x}_1, \mathbf{x}_2$ are not orthogonal, i.e., $\mathbf{x}_2^* \mathbf{x}_1 \neq 0$.

To summarize, our final matrix is of the form

$$A = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix}^{-1}$$

We need to choose $\mathbf{x}_1, \mathbf{x}_2$ such that $\mathbf{x}_2^T \mathbf{x}_1 = 0$, $\mathbf{x}_2^* \mathbf{x}_1 \neq 0$, and (of course) $\mathbf{x}_1, \mathbf{x}_2$ are linearly independent. And we need to choose d_1, d_2 such that the final matrix is complex (and it'd be nice if they put A in an easily readable form). For the eigenvectors, we can find the following two satisfactory eigenvectors by inspection.

$$\mathbf{x}_1 = \begin{pmatrix} 2 \\ i \end{pmatrix} \qquad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$$

For the corresponding eigenvalues, it is easy to see that 3 and -3 nicely fit the bill, yielding

$$A = \begin{pmatrix} 2 & 1 \\ i & 2i \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ i & 2i \end{pmatrix}^{-1} = \begin{pmatrix} 5 & 4i \\ 4i & -5 \end{pmatrix}$$

as our final diagonalizable 2×2 complex symmetric matrix that does not admit an orthogonal basis of eigenvectors. □

b) Construct a 2×2 complex symmetric matrix which cannot be diagonalized.

Answer. We have

$$\begin{pmatrix} 0 & 1 \\ 1 & i \end{pmatrix}$$

as a complex symmetric matrix that cannot be diagonalized. □

3.1. Show that the number of nonzero singular values of a matrix A coincides with its rank.

Answer. By Problem 5.5.4a, $\operatorname{rank} A = \operatorname{rank} A^*A$. Additionally, since A^*A is self-adjoint by Problem 6.2.8a, we have by Theorem 6.2.1 that A^*A is similar to a diagonal matrix D. Since similar matrices have the same rank, $\operatorname{rank}(A^*A) = \operatorname{rank}(D)$. But $\operatorname{rank}(D)$ is just the number of nonzero entries on the diagonal, i.e., the number of eigenvalues of A^*A. Therefore, since the singular values of A are the square roots of the eigenvalues of A^*A, the number of nonzero singular values of A equals the number of nonzero eigenvalues of A^*A. □

3.2. Find Schmidt decompositions $A = \sum_{k=1}^{r} s_k \mathbf{w}_k \mathbf{v}_k^*$ for the following matrices A.

$$\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix}$$

Answer. Left matrix: We have

$$A^*A = \begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 6 \\ 6 & 13 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 16 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$$

so that $\sigma_1 = 4$ and $\sigma_2 = 1$, and

$$\mathbf{v}_1 = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$$

Then

$$\begin{aligned} \mathbf{w}_1 &= \frac{1}{\sigma_1} A\mathbf{v}_1 \\ &= \frac{1}{4} \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \\ &= \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \end{aligned} \qquad \begin{aligned} \mathbf{w}_2 &= \frac{1}{\sigma_2} A\mathbf{v}_2 \\ &= \frac{1}{1} \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \\ &= \begin{pmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \end{aligned}$$

Thus, we have as a Schmidt decomposition

$$\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} = 4 \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} + 1 \begin{pmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$$

Middle matrix: We have

$$A^*A = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 90 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$$

so that $\sigma_1 = 3\sqrt{10}$ and $\sigma_2 = \sqrt{10}$, and

$$\mathbf{v}_1 = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \qquad \mathbf{v}_2 = \begin{pmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$

Then

$$\mathbf{w}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \qquad \mathbf{w}_2 = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

Thus, we have as a Schmidt decomposition

$$\begin{pmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{pmatrix} = 3\sqrt{10} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} + \sqrt{10} \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}$$

Right matrix: We have

$$A^*A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so that $\sigma_1 = \sqrt{2}$ and $\sigma_2 = \sqrt{3}$, and

$$\mathbf{v}_1 = \mathbf{e}_1 \qquad \mathbf{v}_2 = \mathbf{e}_2$$

Then

$$\mathbf{w}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} \qquad \mathbf{w}_2 = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

Thus, we have as a Schmidt decomposition

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} = \sqrt{2}\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} + \sqrt{3}\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix}$$

□

3.3. Let A be an invertible matrix, and let $A = W\Sigma V^*$ be its singular value decomposition. Find a singular value decomposition for A^* and A^{-1}.

Answer. Observe that

$$A^* = (W\Sigma V^*)^* = (V^*)^*\Sigma^* W^* = V\Sigma W^*$$

where $\Sigma^* = \Sigma$ since all singular values are real numbers. Also observe that if Σ^{-1} is the matrix equal to Σ except with all diagonal entries inverted (which leaves them as real numbers), then

$$(W\Sigma V^*)(V\Sigma^{-1}W^*) = I \qquad\qquad (V\Sigma^{-1}W^*)(W\Sigma V^*) = I$$

Thus, we have that

$$A^* = V\Sigma W^* \qquad\qquad A^{-1} = V\Sigma^{-1}W^*$$

□

3.5. Find the singular value decomposition of the matrix

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$$

Answer. We have from Problem 6.3.2 that a Schmidt decomposition of A is

$$A = 4\begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}\begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} + 1\begin{pmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}\begin{pmatrix} -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$$

Thus, the singular value decomposition is

$$A = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$$

□

Use it to find

a) $\max_{\|\mathbf{x}\|\leq 1}\|A\mathbf{x}\|$ and the vectors where the maximum is attained.

Answer. We have that $\max_{\|\mathbf{x}\|\leq 1}\|A\mathbf{x}\| = \|A\| = 4$. We know that the unit vector that maximizes Σ is $\pm\mathbf{e}_1$, so we want to find $\mathbf{x}$ such that $V^*\mathbf{x} = \pm\mathbf{e}_1$. But then $\mathbf{x} = \pm V\mathbf{e}_1$, i.e., $\mathbf{x}$ equals plus or minus the first column of V. Therefore, the vectors where the maximum is attained are

$$\mathbf{x}_1 = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \qquad\qquad \mathbf{x}_2 = \begin{pmatrix} -1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix}$$

□

b) $\min_{\|\mathbf{x}\|=1}\|A\mathbf{x}\|$ and the vectors where the minimum is attained.

Answer. By a similar argument to before, $\min_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| = 1$ and

$$\mathbf{y}_1 = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \qquad \mathbf{y}_2 = \begin{pmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix}$$

□

c) The image $A(B)$ of the closed unit ball $B = \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| \leq 1\}$ in $\mathbb{R}^2$. Describe $A(B)$ geometrically.

Answer. $A(B)$ will be an ellipse in $\mathbb{R}^2$ centered at the origin with half-axes of length 4 and 1 pointing in the directions $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{y}_1, \mathbf{y}_2$, respectively. □

3.6. Show that for a square matrix A, $|\det A| = \det |A|$.

Answer. By Theorem 6.3.5, $A = U|A|$ where U is an isometry. Note that U is unitary in this case as well since U is square (see Proposition 5.6.3). Also note that $\det |A|$ is nonnegative since every eigenvalue of $|A|$ (i.e., the singular values) are nonnegative by definition. Thus,

$$\begin{aligned} |\det A| &= |\det(U|A|)| & \\ &= |\det U| \cdot |\det |A|| & \text{Theorem 3.3.5} \\ &= 1 \cdot |\det |A|| & \text{Proposition 5.6.4} \\ &= \det |A| & \end{aligned}$$

as desired. □

3.7. True or false:

a) The singular values of a matrix are also eigenvalues of the matrix.

Answer. False.
Consider the left matrix in Problem 6.3.2. Since this matrix is upper triangular, it is clear that it's eigenvalue is 2. However, we computed its singular values to be 4 and 1. □

b) The singular values of a matrix A are eigenvalues of A^*A.

Answer. False.
Consider the left matrix in Problem 6.3.2. By the diagonalization of A^*A performed in the answer to that question, the eigenvalues of A^*A are 16 and 1. However, we computed its singular values to be 4 and 1. □

c) If s is a singular value of a matrix A and c is a scalar, then $|c|s$ is a singular value of cA.

Answer. True.
Suppose s is a singular value of A. Then s^2 is an eigenvalue of A^*A, i.e., there exists a nonzero vector $\mathbf{v}$ such that $A^*A\mathbf{v} = s^2\mathbf{v}$. It follows that

$$(cA)^*(cA)\mathbf{v} = c^2 A^*A\mathbf{v} = c^2 s^2 \mathbf{v}$$

so c^2s^2 is an eigenvalue of $(cA)^*(cA)$. Therefore, $\sqrt{c^2s^2} = |c|s$ is a singular value of cA, as desired. □

d) The singular values of any linear operator are nonnegative.

Answer. True.
By definition. □

e) The singular values of a self-adjoint matrix coincide with its eigenvalues.

Answer. False.

Consider the self-adjoint 1×1 matrix

$$A = \begin{pmatrix} -1 \end{pmatrix}$$

The eigenvalue of A is -1, but the singular value is 1. $\square$

3.8. Let A be an $m \times n$ matrix. Prove that *nonzero* eigenvalues of the matrices A^*A and AA^* (counting multiplicities) coincide. Can you say when zero eigenvalues of A^*A and zero eigenvalues of AA^* have the same multiplicity?

Answer. Let A be an $m \times n$ matrix with SVD $A = W\Sigma V^*$, and let $\sigma_1, \dots, \sigma_n$ be the singular values of A arranged such that $\sigma_1, \dots, \sigma_r$ are the nonzero singular values. Then

$$\begin{aligned} A^*A &= (W\Sigma V^*)^*(W\Sigma V^*) \\ &= (V^*)^*\Sigma^* W^* W\Sigma V^* \\ &= V\Sigma^*\Sigma V^* \end{aligned} \qquad \begin{aligned} AA^* &= (W\Sigma V^*)(W\Sigma V^*)^* \\ &= W\Sigma V^* V\Sigma^* W^* \\ &= W\Sigma\Sigma^* W^* \end{aligned}$$

Let's investigate the structure of $\Sigma^*\Sigma$ and $\Sigma\Sigma^*$. By definition, Σ is of the form

$$\begin{array}{c} \\ 1 \\ \vdots \\ r \\ r+1 \\ \vdots \\ m \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & \cdots & r & r+1 & \cdots & n \end{array} \\ \left(\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_r & \\ \hline & & & \\ & 0 & & 0 \\ & & & \end{array}\right) \end{array}$$

It is thus easy to see that

$$\Sigma^*\Sigma = \begin{array}{c} \\ 1 \\ \vdots \\ r \\ r+1 \\ \vdots \\ n \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & \cdots & r & r+1 & \cdots & n \end{array} \\ \left(\begin{array}{ccc|c} \sigma_1^2 & & & \\ & \ddots & & 0 \\ & & \sigma_r^2 & \\ \hline & & & \\ & 0 & & 0 \\ & & & \end{array}\right) \end{array} \qquad \Sigma\Sigma^* = \begin{array}{c} \\ 1 \\ \vdots \\ r \\ r+1 \\ \vdots \\ m \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & \cdots & r & r+1 & \cdots & m \end{array} \\ \left(\begin{array}{ccc|c} \sigma_1^2 & & & \\ & \ddots & & 0 \\ & & \sigma_r^2 & \\ \hline & & & \\ & 0 & & 0 \\ & & & \end{array}\right) \end{array}$$

i.e., that $\Sigma^*\Sigma$ and $\Sigma\Sigma^*$ are proper diagonal matrices whose entries are the squares of the singular values. This combined with the fact that V, W are unitary means that $V(\Sigma^*\Sigma)V^*$ and $W(\Sigma\Sigma^*)W$ are orthogonal diagonalizations of A^*A and AA^*, respectively. Hence the diagonal entries of $\Sigma^*\Sigma$ are the eigenvalues of A^*A and the diagonal entries of $\Sigma\Sigma^*$ are the the eigenvalues of AA^*. Therefore, from the last equations above, it is clear that the nonzero eigenvalues of A^*A and AA^* always coincide, and the zero eigenvalues of A^*A and AA^* coincide iff $m = n$. $\square$

3.9. Let s be the largest singular value of an operator A, and let λ be the eigenvalue of A with the largest absolute value. Show that $|\lambda| \leq s$.

Answer. Let $\mathbf{v}$ be the normal eigenvector corresponding to λ. Then we have that

$$|\lambda| = |\lambda|\|\mathbf{v}\| = \|\lambda\mathbf{v}\| = \|A\mathbf{v}\| \leq \|A\| \cdot \|\mathbf{v}\| = s$$

as desired. $\square$

3.11. Show that the operator norm of a matrix A coincides with its Frobenius norm if and only if the matrix has rank one. (Hint: The previous problem might help.)

Answer. Let $\sigma_1, \dots, \sigma_n$ be the singular values of A arranged in descending order.

Suppose first that the $\|A\| = \|A\|_2$. Then

$$\sigma_1^2 = \|A\|^2 = \|A\|_2^2 = \operatorname{tr}(A^*A) = \sum_{k=1}^{n} \sigma_k^2$$

It follows that $\sigma_2, \dots, \sigma_n$ are all zero. Therefore, since A only has one nonzero singular value, Problem 6.3.1 asserts that A has rank one.

The proof is symmetric in the other direction. $\square$

3.12. For the matrix

$$A = \begin{pmatrix} 2 & -3 \\ 0 & 2 \end{pmatrix}$$

describe the inverse image of the unit ball, i.e., the set of all $\mathbf{x} \in \mathbb{R}^2$ such that $\|A\mathbf{x}\| \leq 1$. Use its singular value decomposition.

Answer. The inverse image of the unit ball under A is equal to the image of the unit ball under A^{-1}. We have that

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$$

Thus, by problem 6.3.5, the SVD of A^{-1} is

$$A^{-1} = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/4 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}$$

Thus, the inverse image will be an ellipse in $\mathbb{R}^2$ with half-axes 1 and $\frac{1}{4}$ pointing in the directions $\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -2 \\ 1 \end{smallmatrix}\right)$, respectively. $\square$

4.2. Let A be a normal operator, and let $\lambda_1, \dots, \lambda_n$ be its eigenvalues (counting multiplicities). Show that singular values of A are $|\lambda_1|, \dots, |\lambda_n|$.

Answer. Since A is normal, we have by Theorem 6.2.4 that $A = UDU^*$ where U is unitary and D is diagonal. It follows that

$$A^*A = (UDU^*)^*(UDU^*) = UD^*DU^*$$

Consider λ_j for some $j \in \{1, \dots, n\}$. We know that λ_j is a diagonal entry of D. Thus, $\bar{\lambda_j}\lambda_j = |\lambda_j|^2$ is the corresponding diagonal entry of D^*D. It follows since the singular values of A are the eigenvalues of $|A| = \sqrt{A^*A}$, i.e., the square roots of the eigenvalues of A^*A that $\sigma_j = \sqrt{|\lambda_j|^2} = |\lambda_j|$, as desired. $\square$

4.3. Find the singular values, norm, and condition number of the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

You can do this problem with practically no computations if you use the previous problem and can answer the following questions:

a) What are singular values (eigenvalues) of an orthogonal projection P_E onto some subspace E?

   Answer. 1 and 0, with respective multiplicities $\dim E$ and $\dim E^\perp$. Note that the singular values and eigenvalues coincide here because P_E is self-adjoint. $\square$

b) What is the matrix of the orthogonal projection onto the subspace spanned by the vector $(1, 1, 1)^T$?

Answer. From Problem 5.3.9a, the matrix of this projection is

$$\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

□

c) How are the eigenvalues of the operators T and $aT + bI$ where $a, b \in \mathbb{F}$ related?

Answer. Suppose λ is an eigenvalue of T. Then there exists a nonzero vector $\mathbf{v}$ such that $T\mathbf{v} = \lambda\mathbf{v}$. It follows that

$$\begin{aligned}(aT + bI)\mathbf{v} &= aT\mathbf{v} + b\mathbf{v} \\ &= a\lambda\mathbf{v} + b\mathbf{v} \\ &= (a\lambda + b)\mathbf{v}\end{aligned}$$

i.e., that $a\lambda + b$ is an eigenvalue of $aT + bI$. □

Of course you can also just honestly do the computations.

Answer. Let P_E denote the matrix provided as an answer to question (b) above. Then $A = 3P_E + 1I$. Therefore, since question (a) provides the eigenvalues to P_E as 1 and 0 (with multiplicities 2 and 1, respectively), question (c) posits that the eigenvalues of A are $3(1) + 1 = 4$ and $3(0) + 1 = 1$ (with multiplicities 2 and 1, respectively), and that these values are in fact the singular values.

It follows that $\|A\| = 4$ and the condition number is $\|A\| \cdot \|A^{-1}\| = 4/1 = 4$. □

6.1. Let R_α be the rotation through α, so its matrix in the standard basis is

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Find the matrix of R_α in the basis $\mathbf{v}_1, \mathbf{v}_2$ where $\mathbf{v}_1 = \mathbf{e}_2, \mathbf{v}_2 = \mathbf{e}_1$.

Answer. We define

$$[I]_{\mathcal{E}\mathcal{V}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

It follows that

$$[I]_{\mathcal{V}\mathcal{E}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Therefore,

$$\begin{aligned}[R_\alpha]_{\mathcal{V}\mathcal{V}} &= [I]_{\mathcal{V}\mathcal{E}}[R_\alpha]_{\mathcal{E}\mathcal{E}}[I]_{\mathcal{E}\mathcal{V}} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\end{aligned}$$

□

6.2. Let R_α be the rotation matrix

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Show that the 2×2 identity matrix I_2 can be continuously transformed through invertible matrices into R_α.

Answer. Let $V(t)$ be defined by

$$V(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

Choose $a = 0$ and $b = \alpha$. Then $V(t)$ is continuous because each component is continuous, the inverse of $V(t)$ is $V(-t)$, and clearly $V(a) = V(0) = I$ and $V(b) = V(\alpha) = R_\alpha$. □

6.3. Let U be an $n \times n$ orthogonal matrix with $\det U > 0$. Show that the $n \times n$ identity matrix I_n can be continuously transformed through invertible matrices into U. (Hint: Use the previous problem and the representation of a rotation in $\mathbb{R}^n$ as a product of planar rotations [see Section 5].)

Answer. Since U is an orthogonal matrix with $\det U = 1 > 0$, Theorem 6.5.1 asserts that there exists a basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ such that the matrix of U in this basis has the block diagonal form

$$V(t) = \begin{pmatrix} R_{\varphi_1} & & & 0 \\ & \ddots & & \\ & & R_{\varphi_k} & \\ 0 & & & I_{n-2k} \end{pmatrix}$$

Thus, let $V(t)$ be defined by

$$V(t) = \begin{pmatrix} R_{\varphi_1 t} & & & 0 \\ & \ddots & & \\ & & R_{\varphi_k t} & \\ 0 & & & I_{n-2k} \end{pmatrix}$$

Choose $a = 0$ and $b = 1$. It will follow from Problem 6.6.2 that V is a continuous transformation satisfying all the necessary properties. □

Chapter 7

1.1. Find the matrix of the bilinear form L on $\mathbb{R}^3$ defined by

$$L(\mathbf{x}, \mathbf{y}) = x_1y_1 + 2x_1y_2 + 14x_1y_3 - 5x_2y_1 + 2x_2y_2 - 3x_2y_3 + 8x_3y_1 + 19x_3y_2 - 2x_3y_3$$

Answer. We have that

$$\begin{aligned}
&= x_1y_1 + 2x_1y_2 + 14x_1y_3 - 5x_2y_1 + 2x_2y_2 - 3x_2y_3 + 8x_3y_1 + 19x_3y_2 - 2x_3y_3 \\
&= L(\mathbf{x}, \mathbf{y}) \\
&= \mathbf{y}^T A\mathbf{x} \\
&= \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&= \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix} \begin{pmatrix} a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 \\ a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 \\ a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 \end{pmatrix} \\
&= y_1a_{1,1}x_1 + y_1a_{1,2}x_2 + y_1a_{1,3}x_3 + y_2a_{2,1}x_1 + y_2a_{2,2}x_2 + y_2a_{2,3}x_3 + y_3a_{3,1}x_1 + y_3a_{3,2}x_2 + y_3a_{3,3}x_3 \\
&= a_{1,1}x_1y_1 + a_{2,1}x_1y_2 + a_{3,1}x_1y_3 + a_{1,2}x_2y_1 + a_{2,2}x_2y_2 + a_{3,2}x_2y_3 + a_{1,3}x_3y_1 + a_{2,3}x_3y_2 + a_{3,3}x_3y_3
\end{aligned}$$

It follows from comparing terms that

$$A = \begin{pmatrix} 1 & -5 & 8 \\ 2 & 2 & 19 \\ 14 & -3 & -2 \end{pmatrix}$$

□

1.2. Define the bilinear form L on $\mathbb{R}^2$ by

$$L(\mathbf{x}, \mathbf{y}) = \det[\mathbf{x}, \mathbf{y}]$$

i.e., to compute $L(\mathbf{x}, \mathbf{y})$, we form a 2×2 matrix with columns $\mathbf{x}, \mathbf{y}$ and compute its determinant. Find the matrix of L.

Answer. We have that

$$\begin{aligned}
&= a_{1,1}x_1y_1 + a_{2,1}x_1y_2 + a_{1,2}x_2y_1 + a_{2,2}x_2y_2 \\
&= \mathbf{y}^T A\mathbf{x} \\
&= L(\mathbf{x}, \mathbf{y}) \\
&= \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \\
&= x_1y_2 - y_1x_2 \\
&= 0x_1y_1 + 1x_1y_2 - 1x_2y_1 + 0x_2y_2
\end{aligned}$$

It follows from comparing terms that

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

□

1.3. Find the matrix of the quadratic form Q on $\mathbb{R}^3$ defined by

$$Q[\mathbf{x}] = x_1^2 + 2x_1x_2 - 3x_1x_3 - 9x_2^2 + 6x_2x_3 + 13x_3^2$$

Answer. We have that

$$\begin{aligned}
&= x_1^2 + 2x_1x_2 - 3x_1x_3 - 9x_2^2 + 6x_2x_3 + 13x_3^2 \\
&= Q[\mathbf{x}] \\
&= (A\mathbf{x}, \mathbf{x}) \\
&= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{1,2} & a_{2,2} & a_{2,3} \\ a_{1,3} & a_{2,3} & a_{3,3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 \\ a_{1,2}x_1 + a_{2,2}x_2 + a_{2,3}x_3 \\ a_{1,3}x_1 + a_{2,3}x_2 + a_{3,3}x_3 \end{pmatrix} \\
&= x_1a_{1,1}x_1 + x_1a_{1,2}x_2 + x_1a_{1,3}x_3 + x_2a_{1,2}x_1 + x_2a_{2,2}x_2 + x_2a_{2,3}x_3 + x_3a_{1,3}x_1 + x_3a_{2,3}x_2 + x_3a_{3,3}x_3 \\
&= a_{1,1}x_1^2 + 2a_{1,2}x_1x_2 + 2a_{1,3}x_1x_3 + a_{2,2}x_2^2 + 2a_{2,3}x_2x_3 + a_{3,3}x_3^2
\end{aligned}$$

It follows from comparing terms that

$$A = \begin{pmatrix} 1 & 1 & -3/2 \\ 1 & -9 & 3 \\ -3/2 & 3 & 13 \end{pmatrix}$$

□

2.1. Diagonalize the quadratic form with the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

Use two methods: completion of squares and row operations. Which one do you like better? Can you say if the matrix A is positive definite or not?

Answer. Completion of squares: If A has the above form, then

$$\begin{aligned}Q[\mathbf{x}] &= (A\mathbf{x}, \mathbf{x}) \\ &= a_{1,1}x_1^2 + 2a_{1,2}x_1x_2 + 2a_{1,3}x_1x_3 + a_{2,2}x_2^2 + 2a_{2,3}x_2x_3 + a_{3,3}x_3^2 \\ &= x_1^2 + 4x_1x_2 + 2x_1x_3 + 3x_2^2 + 4x_2x_3 + x_3^2 \\ &= (x_1 + 2x_2 + x_3)^2 - x_2^2 \\ &= y_1^2 - y_2^2\end{aligned}$$

where $y_1 = x_1 + 2x_2 + x_3$, $y_2 = x_2$, and $y_3 = 0$. It follows that

$$D = \begin{pmatrix}1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0\end{pmatrix} \qquad S^* = \begin{pmatrix}1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{pmatrix}$$

the latter equation coming from the fact that $\mathbf{y} = S^*\mathbf{x}$.

Row operations: We can row reduce

$$(A|I) = \left(\begin{array}{ccc|ccc}1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1\end{array}\right)$$

to

$$(D|S^*) = \left(\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 2 & 1 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0\end{array}\right)$$

getting the same result as before.

Right now, I believe I prefer completion of squares. The matrix is not positive definite since it has an eigenvalue (diagonal entry) less than zero. $\square$