Chapter 2

Systems of Linear Equations

2.1 Notes

9/29:
- Row elimination:
  - Let
$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 3 & 1 & 2 & 7 \\ 2 & 1 & 2 & 1 \end{pmatrix}$$
  - Then the **echelon form** matrix
$$A_e = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & -4 \end{pmatrix}$$
  - Lastly, the **reduced echelon form** matrix
$$A_{re} = \begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$
- **echelon form**:
  - All zero rows are below nonzero rows.
  - For any nonzero row, its leading element is strictly to the left of the nonzero entry of the next row.
- **Reduced echelon form**:
  - All pivots are 1.
  - Used to solve systems of the form $Ax = b$.
- **Inconsistent** (system of equations): A system with no solution.
  - If the last row is of the form $(0, \ldots, 0, b)$ where $b \neq 0$, then there is no solution.
- Unique solution if A_e has a pivot in every column.
- There exists a solution for every b if there is a pivot in every row?
- Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a matrix. Then $\ker A = \{x \in \mathbb{R}^n : Ax = 0\}$ (subspace of $\mathbb{R}^n$) and $\operatorname{range} A = \{Ax : x \in \mathbb{R}^n\}$ (subspace of $\mathbb{R}^m$).
- Also consider $\ker(A^T)$ and $\operatorname{range}(A^T)$, the basis of the kernel and range, and dimension.

- Finite-dimensional vector spaces:
  - A basis is a generating set (so every element of V can be written uniquely as a linear combination of the basis) the length of which is equal to the dimension of V.
  - All bases of finite-dimensional vector spaces have the same number of elements.
    - Let v_1, v_2, v_3 and w_1, w_2 be two generating sets of V.
    - Then
$$\begin{aligned} v_1 &= \lambda_{11}w_1 + \lambda_{12}w_2 \\ v_2 &= \lambda_{21}w_1 + \lambda_{22}w_2 \\ v_3 &= \lambda_{31}w_1 + \lambda_{32}w_2 \end{aligned}$$
    - Suppose the only solution to $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$ is $\alpha_1 = \alpha_2 = \alpha_3 = 0$.
    - But this is not true, as we can find another one in terms of the λs.
  - If you have a list of linearly independent vectors, you can complete it into a basis.
    - If there exists a vector that can't be written as a linear combination of the list, add it to the list.
  - If you find any particular solution to a system $Ax = b$, and you add to it any element of $\ker A$, you will obtain another solution.
    - $Ax_1 = b$ and $Ax_h = 0$ implies that $A(x_1 + x_h) = b$.
    - $Ax_1 = b$ and $Ax_2 = b$ imply that $A(x_1 - x_2) = 0$, i.e., that $x_1 - x_2 \in \ker A$.
  - If $A : \mathbb{R}^n \to \mathbb{R}^m$ and $\dim \operatorname{range} A = m$, then $Ax = b$ is solveable for all $b \in \mathbb{R}^m$.
  - Let $\operatorname{rank} A = \dim \operatorname{range} A$.
  - Rank theorem:
    - $\operatorname{rank} A = \operatorname{rank} A^T$.
    - Let $A : \mathbb{R}^n \to \mathbb{R}^m$. We know that $\dim \ker A + \dim \operatorname{range} A = n$.
    - $\dim \ker A^T + \operatorname{rank} A^T = m$.
    - This theorem survives linear algebra and enters functional analysis under the name **Fredholm's alternative**.
- **Fredholm's alternative**: $Ax = b$ has a solution for all $b \in \mathbb{R}^n$ iff $\dim \ker A^T = 0$.
  - $\dim \ker A^T = 0$ implies $\operatorname{rank} A^T = m$ implies $\operatorname{rank} A = m$ implies $\dim \operatorname{range} A = m$, as desired.
- **Pivot column** (of A): A column of A where A_e has pivots.
- The **pivot columns** of A give a basis for $\operatorname{range} A$.
- The pivot rows of A_e give a basis for $\operatorname{range} A^T$.
- A basis for the kernel is enough to solve $Ax = 0$.
- If you take these three things as givens, you can prove the rank theorem.

2.2 Chapter 2: Systems of Linear Equations

From Treil (2017).

10/24:
- A system is inconsistent iff the echelon form of the augmented matrix has a row of the form $\begin{pmatrix} 0 & \cdots & 0 & b \end{pmatrix}$.
- A solution to $A\mathbf{x} = \mathbf{b}$ is unique iff there are no free variables, i.e., iff there is a pivot in every column.
- $A\mathbf{x} = \mathbf{b}$ is consistent iff the echelon form of the coefficient matrix has a pivot in every row.

- $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$ iff the echelon form of the coefficient matrix A has a pivot in every row and column.
- Proposition 2.3.1: Let $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathbb{F}^n$, and let $A = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_m \end{bmatrix}$ be an $n \times m$ matrix with columns $\mathbf{v}_1, \dots, \mathbf{v}_m$. Then
  1. The system $\mathbf{v}_1, \dots, \mathbf{v}_m$ is linearly independent iff the echelon form of A has a pivot in every column.
  2. The system $\mathbf{v}_1, \dots, \mathbf{v}_m$ is complete iff the echelon form of A has a pivot in every row.
  3. The system $\mathbf{v}_1, \dots, \mathbf{v}_m$ is a basis of $\mathbb{F}^n$ iff the echelon form of A has a pivot in every column and in every row.
- Proposition 2.3.6: A matrix A is invertible if and only if its echelon form has a pivot in every column and every row.
- Corollary 2.3.7: An invertible matrix must be square ($n \times n$).
- Proposition 2.3.8: If a square ($n \times n$) matrix is left invertible or if it is right invertible, then it is invertible. In other words, to check the invertibility of a square matrix A, it is sufficient to check only one of the conditions $AA^{-1} = I$, $A^{-1}A = I$.
- Any invertible matrix is row-equivalent to (can be row-reduced to) to the identity matrix.
- **Homogeneous** (system of linear equations): A system of the form $A\mathbf{x} = \mathbf{0}$.
- Theorem 2.6.1: Let a vector $\mathbf{x}_1$ satisfy the equation $A\mathbf{x} = \mathbf{b}$. and let H be the set of all solutions of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$. Then the set

$$\{\mathbf{x}_1 + \mathbf{x}_h : \mathbf{x}_h \in H\}$$

  is the set of all solutions to the equation $A\mathbf{x} = \mathbf{b}$.
- The pivot columns are a basis of $\operatorname{range} A$. The pivot rows are a basis of $\operatorname{range} A^T$. The solutions to the equation $A\mathbf{x} = \mathbf{0}$ are a basis of $\ker A$.
- Theorem 2.7.3: Let A be an $m \times n$ matrix. Then the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$ iff the dual equation $A^T\mathbf{x} = \mathbf{0}$ has a unique (only the trivial) solution.
  - Note that this is a corollary to the rank theorem.
- Change of coordinates formula:
  - Let $T : V \to W$ be a linear transformation, and let $\mathcal{V} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $\mathcal{W} = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ be bases of V and W, respectively.
  - The $m \times n$ matrix of T with respect to these bases is $[T]_{\mathcal{W}\mathcal{V}}$, and relates the coordinates of $[T\mathbf{v}]_{\mathcal{W}}$ and $[\mathbf{v}]_{\mathcal{V}}$ via

$$[T\mathbf{v}]_{\mathcal{W}} = [T]_{\mathcal{W}\mathcal{V}}[\mathbf{v}]_{\mathcal{V}}$$

  - Change of coordinates matrix: If $\mathcal{A}, \mathcal{B}$ are two bases of V, then we can convert the coordinates of a vector in $\mathcal{B}$ to its in $\mathcal{A}$ with the identity matrix (with respect to the appropriate bases). In particular,

$$[\mathbf{v}]_{\mathcal{B}} = [I]_{\mathcal{B}\mathcal{A}}[\mathbf{v}]_{\mathcal{A}}$$

    - Note that the k^{th} column of $[I]_{\mathcal{B}\mathcal{A}}$ is the coordinate representation in $\mathcal{B}$ of $\mathbf{a}_k$, i.e., $[\mathbf{a}_k]_{\mathcal{B}}$.
  - The change of coordinates matrix from a basis $\mathcal{B}$ to the standard basis $\mathcal{S}$ is easy to compute; by the above, it's just

$$[I]_{\mathcal{S}\mathcal{B}} = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix}$$

    - It follows that $[I]_{\mathcal{B}\mathcal{S}} = ([I]_{\mathcal{S}\mathcal{B}})^{-1}$.

    - This allows us to compute $[I]_{\mathcal{B}\mathcal{A}}$ as $[I]_{\mathcal{B}\mathcal{S}}[I]_{\mathcal{S}\mathcal{A}}$
  - If $T : V \to W$, $\mathcal{A}, \tilde{\mathcal{A}}$ are bases of V, and $\mathcal{B}, \tilde{\mathcal{B}}$ are bases of W, and we have $[T]_{\mathcal{B}\mathcal{A}}$, then

$$[T]_{\tilde{\mathcal{B}}\tilde{\mathcal{A}}} = [I]_{\tilde{\mathcal{B}}\mathcal{B}}[T]_{\mathcal{B}\mathcal{A}}[I]_{\mathcal{A}\tilde{\mathcal{A}}}$$

- Change of basis ends up at similarity; two operators are similar if we can change the basis of one into another.