2 Eigenvalues and Eigenvectors

From Treil (2017).

Chapter 4

10/11: **1.1.** True or false:

a) Every linear operator in an n-dimensional vector space has n distinct eigenvalues.

Answer. False.
The identity linear operator I_2 in $\mathbb{R}^2$ has the sole eigenvalue $\lambda = 1$, since $I_2\mathbf{x} = 1\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^2$. □

b) If a matrix has one eigenvector, it has infinitely many eigenvectors.

Answer. True.
Let $A\mathbf{x} = \lambda\mathbf{x}$. Then $\alpha\mathbf{x}$ is also an eigenvector of A for any $\alpha \in \mathbb{F}$ since

$$A(\alpha\mathbf{x}) = \alpha A\mathbf{x} = \alpha\lambda\mathbf{x} = \lambda(\alpha\mathbf{x})$$

□

c) There exists a square real matrix with no real eigenvalues.

Answer. True.
Consider

$$\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

for which we have $\lambda = 1 \pm 2i$. Since the two eigenvalues $1 + 2i$ and $1 - 2i$ are distinct, and the square matrix given is 2×2, there are no more eigenvalues. Therefore, every eigenvalue of this matrix is not real. □

d) There exists a square matrix with no (complex) eigenvectors.

Answer. False.
Let $\mathbf{x}$ be an eigenvector of A. If $\mathbf{x}$ is complex, then we are done. If $\mathbf{x}$ is real, then multiply $\mathbf{x}$ by the scalar i. It follows by the proof of part (b) that $i\mathbf{x}$ is an eigenvector if A. □

e) Similar matrices always have the same eigenvalues.

Answer. True.
The characteristic polynomials of similar matrices coincide. □

f) Similar matrices always have the same eigenvectors.

Answer. False.
The matrix

$$\begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix}$$

has eigenvectors

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

while its similar matrix

$$\begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix}$$

has eigenvectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Note that since similar matrices refer to the same linear transformation, a single linear transformation technically only has one set of eigenvectors (albeit possibly expressed in different bases). □

g) A non-zero sum of two eigenvectors of a matrix A is always an eigenvector.

Answer. False.
Consider

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

with eigenvalues $\lambda = 1, 2$ and respective eigenvectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Note that

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

where the "**b**" vector is not a scalar multiple of the "**x**" vector. □

h) A non-zero sum of two eigenvectors of a matrix A corresponding to the same eigenvalue λ is always an eigenvector.

Answer. True.
Let $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = \lambda\mathbf{y}$. Then

$$\begin{aligned} A(\alpha\mathbf{x} + \beta\mathbf{y}) &= \alpha A\mathbf{x} + \beta A\mathbf{y} \\ &= \alpha\lambda\mathbf{x} + \beta\lambda\mathbf{y} \\ &= \lambda(\alpha\mathbf{x} + \beta\mathbf{y}) \end{aligned}$$

as desired. □

1.3. Compute eigenvalues and eigenvectors of the rotation matrix

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Note that the eigenvalues (and eigenvectors) do not need to be real.

Answer. The characteristic polynomial of $A - \lambda I$ is

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= (\cos\alpha - \lambda)^2 + \sin^2\alpha \\ -\sin^2\alpha &= (\cos\alpha - \lambda)^2 \\ \pm i\sin\alpha &= \pm\cos\alpha - \lambda \\ \lambda &= \cos\alpha + i\sin\alpha = \mathrm{e}^{i\alpha} \\ &= \cos\alpha - i\sin\alpha = \mathrm{e}^{-i\alpha} \end{aligned}$$

Thus, $\lambda = \mathrm{e}^{i\alpha}, \mathrm{e}^{-i\alpha}$. It follows by solving the systems of equations

$$\begin{aligned} x_1\cos\alpha - x_2\sin\alpha &= \mathrm{e}^{i\alpha}x_1 \\ x_1\sin\alpha + x_2\cos\alpha &= \mathrm{e}^{i\alpha}x_2 \end{aligned} \qquad \begin{aligned} y_1\cos\alpha - y_2\sin\alpha &= \mathrm{e}^{-i\alpha}y_1 \\ y_1\sin\alpha + y_2\cos\alpha &= \mathrm{e}^{-i\alpha}y_2 \end{aligned}$$

that the eigenvectors are

$$x = \begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad y = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

□

1.5. Prove that eigenvalues (counting multiplicities) of a triangular matrix coincide with its diagonal entries.

Answer. Since the determinant of a triangular matrix is the product of its diagonal entries, we have that

$$\det(A - \lambda I) = (a_{1,1} - \lambda)(a_{2,2} - \lambda) \cdots (a_{n,n} - \lambda)$$

But this polynomial is zero only if and only if λ is a diagonal entry, so the eigenvalues must be the diagonal entries. □

1.6. An operator A is called **nilpotent** if $A^k = \mathbf{0}$ for some k. Prove that if A is nilpotent, then $\sigma(A) = \{0\}$ (i.e., that 0 is the only eigenvalue of A).

Answer. Suppose for the sake of contradiction that λ is a nonzero eigenvalue of A with corresponding eigenvector $\mathbf{x}$. Then since $A\mathbf{x} = \lambda\mathbf{x}$, $A^k\mathbf{x} = \lambda^k\mathbf{x} \neq \mathbf{0} = 0\mathbf{x}$, so $A^k \neq 0$, a contradiction. □

1.7. Show that the characteristic polynomial of a block triangular matrix

$$\begin{pmatrix} A & * \\ \mathbf{0} & B \end{pmatrix}$$

where A and B are square matrices coincides with $\det(A - \lambda I)\det(B - \lambda I)$. (Hint: Use Exercise 3.11 from Chapter 3.)

Answer. It follows from Chapter 3, Exercise 3.11 that

$$\begin{aligned} \det\left(\begin{pmatrix} A & * \\ \mathbf{0} & B \end{pmatrix} - \lambda I\right) &= \det\begin{pmatrix} A - \lambda I & * \\ \mathbf{0} & B - \lambda I \end{pmatrix} \\ &= \det(A - \lambda I)\det(B - \lambda I) \end{aligned}$$

as desired. □

1.8. Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be a basis in a vector space V. Assume also that the first k vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ of the basis are eigenvectors of an operator A, corresponding to an eigenvalue λ (i.e., that $A\mathbf{v}_j = \lambda\mathbf{v}_j$, $j = 1, \dots, k$). Show that in this basis, the matrix of the operator A has block triangular form

$$\begin{pmatrix} \lambda I_k & * \\ \mathbf{0} & B \end{pmatrix}$$

where I_k is the $k \times k$ identity matrix and B is some $(n-k) \times (n-k)$ matrix.

Answer. We will first show that if $\mathbf{v}_i$ is an eigenvector of A *and* a part of the basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ of V, then it's matrix with respect to $\mathbf{v}_1, \dots, \mathbf{v}_n$ has zeros in every slot except the i^{th} slot, which is 1. This is easily shown as follows.

$$\begin{aligned} A\mathbf{v}_i &= \lambda\mathbf{v}_i \\ A\mathbf{v}_i &= \lambda(0\mathbf{v}_1 + \cdots + 0\mathbf{v}_{i-1} + 1\mathbf{v}_i + 0\mathbf{v}_{i+1} + \cdots + 0\mathbf{v}_n) \end{aligned}$$

This combined with the observations that the i^{th} column of A is equal to $A\mathbf{v}_i$ and $A\mathbf{v}_i = \lambda\mathbf{v}_i$ proves that

$$A = \begin{pmatrix} A\mathbf{v}_1 & \cdots & A\mathbf{v}_n \end{pmatrix} = \begin{pmatrix} \lambda\mathbf{v}_1 & \cdots & \lambda\mathbf{v}_k & A\mathbf{v}_{k+1} & \cdots & A\mathbf{v}_n \end{pmatrix} = \begin{pmatrix} \lambda I_k & * \\ \mathbf{0} & B \end{pmatrix}$$

as desired. □

1.10. Prove that the determinant of a matrix A is the product of its eigenvalues (counting multiplicities). (Hint: First show that $\det(A - \lambda I) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$, where $\lambda_1, \dots, \lambda_n$ are eigenvalues (counting multiplicities). Then compare the free terms (terms without λ) or plug in $\lambda = 0$ to get the conclusion.)

Answer. We know that the roots of the characteristic polynomial $\det(A - \lambda I)$ of A are exactly the eigenvalues $\lambda_1, \dots, \lambda_n$ of A. In other words, $\det(A - \lambda I)$ must go to zero exactly when $\lambda = \lambda_i$ for some i. Thus, $\det(A - \lambda I)$ must be of the form

$$c(\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$$

for some $c \in \mathbb{F}$. But since λ only occurs in $A - \lambda I$ with the coefficient -1, and the λ^n term is solely generated by the term in the permutation sum that is the product of the diagonal entries, the λ^n term must have coefficient $(-1)^n$. Additionally, the polynomial above will have λ^n have coefficient $(-1)^n$. Thus, we must have $c = 1$, and we have proven the hint. Therefore,

$$\begin{aligned}\det A &= \det(A - 0I)\\ &= (\lambda_1 - 0)\cdots(\lambda_n - 0)\\ &= \lambda_1\cdots\lambda_n\end{aligned}$$

as desired. □

1.11. Prove that the trace of a matrix equals the sum of its eigenvalues in three steps. First, compute the coefficient of λ^{n-1} in the right side of the equality

$$\det(A - \lambda I) = (\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$$

Then show that $\det(A - \lambda I)$ can be represented as

$$\det(A - \lambda I) = (a_{1,1} - \lambda)(a_{2,2} - \lambda)\cdots(a_{n,n} - \lambda) + q(\lambda)$$

where $q(\lambda)$ is a polynomial of degree at most $n - 2$. And finally, compare the coefficients of λ^{n-1} to get the conclusion.

Answer. Consider $(\lambda_1 - \lambda)\cdots(\lambda_n - \lambda)$. We have that every λ^{n-1} term in the expansion of this product must take the λ from $n-1$ of the terms and the λ_i from the remaining term. Thus, our expansion should contain the terms $\lambda_1\lambda^{n-1}, \dots, \lambda_n\lambda^{n-1}$, which, when we sum, gives $(-1)^n(\lambda_1 + \cdots + \lambda_n)\lambda^{n-1}$.

In the permutation sum form of the determinant, we have that $(a_{1,1} - \lambda)(a_{2,2} - \lambda)\cdots(a_{n,n} - \lambda)$ will be one of the terms in the sum. In particular, it is the *only* term to contain all the λ-containing entries in the matrix, so it solely determines the λ^n term. Additionally, the term containing the next-highest number of λ's must contain $n - 2$ λ's, not $n - 1$, since any product with $n - 1$ diagonal entries and 1 non-diagonal entry necessarily contains two terms that are in the same row or column. Thus, the term given solely determines the λ^{n-1} term as well. All of the other terms, having degree at most λ^{n-2}, can be defined equal to $q(\lambda)$.

Therefore, since the first part of the proof gives

$$(\lambda_1 + \cdots + \lambda_n)\lambda^{n-1}$$

as the λ^{n-1} term, and the second part of the proof (by a similar argument) gives

$$(a_{1,1} + a_{2,2} + \cdots + a_{n,n})\lambda^{n-1}$$

as the λ^{n-1} term, we have by comparing terms (rigorously, subtract all terms of other degrees to preserve the equality) that

$$\operatorname{tr} A = a_{1,1} + a_{2,2} + \cdots + a_{n,n} = \lambda_1 + \cdots + \lambda_n$$

as desired. □

2.1. Let A be an $n \times n$ matrix. True or false (justify your conclusions):

a) A^T has the same eigenvalues as A.

Answer. True.
Since $\det B = \det B^T$ for any matrix B and the transpose operation does not affect the diagonal, we have that

$$\begin{aligned}\det(A - \lambda I) &= \det\big((A - \lambda I)^T\big)\\ &= \det\big(A^T - (\lambda I)^T\big)\\ &= \det\big(A^T - \lambda I\big)\end{aligned}$$

as desired. □

b) A^T has the same eigenvectors as A.

Answer. False.
Let

$$A = \begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix}$$

Then we can calculate that A has eigenvectors

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

but A^T has eigenvectors

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

□

c) If A is diagonalizable, then so is A^T.

Answer. True.
Suppose $A = SDS^{-1}$. Then

$$\begin{aligned}A^T &= (SDS^{-1})^T\\ &= (S^{-1})^T D^T S^T\\ &= (S^{-1})^T D((S^{-1})^T)^{-1}\end{aligned}$$

as desired. □

2.2. Let A be a square matrix with real entries, and let λ be its complex eigenvalue. Suppose $\mathbf{v} = (v_1, \dots, v_n)^T$ is a corresponding eigenvector, i.e., $A\mathbf{v} = \lambda\mathbf{v}$. Prove that the $\bar{\lambda}$ is an eigenvalue of A and $A\bar{\mathbf{v}} = \bar{\lambda}\bar{\mathbf{v}}$, where $\bar{\mathbf{v}} = (\bar{v}_1, \dots, \bar{v}_n)^T$ is the complex conjugate of the vector $\mathbf{v}$.

Answer. Let $\mathbf{v} = \mathbf{a} + i\mathbf{b}$ where $a_j = \operatorname{Re} v_j$ and $b_j = \operatorname{Im} v_j$. It follows that

$$A\mathbf{a} + iA\mathbf{b} = A\mathbf{v} = \lambda\mathbf{v} = \lambda\mathbf{a} + i\lambda\mathbf{b}$$

This combined with the fact that all entries in $A, \mathbf{a}, \mathbf{b}$ are real implies by matching corresponding parts that

$$A\mathbf{a} = \lambda\mathbf{a} \qquad A\mathbf{b} = \lambda\mathbf{b}$$

Therefore,

$$A\bar{\mathbf{v}} = A(\mathbf{a} - i\mathbf{b}) = A\mathbf{a} - iA\mathbf{b} = \lambda\mathbf{a} - i\lambda\mathbf{b} = \lambda(\mathbf{a} - i\mathbf{b}) = \lambda\bar{\mathbf{v}}$$

as desired. □

2.3. Let

$$A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$$

Find A^{2004} by diagonalizing A.

Answer. We have that

$$\begin{aligned} 0 &= \det(A - \lambda I) \\ &= (4-\lambda)(2-\lambda) - 3 \\ &= \lambda^2 - 6\lambda + 5 \\ &= (\lambda - 5)(\lambda - 1) \end{aligned}$$

Thus, $\lambda = 5, 1$. It follows by inspection that

$$x_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad x_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

Consequently,

$$S = \begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} \qquad S^{-1} = \frac{1}{4}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}$$

Hence

$$A = \frac{1}{4}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}$$

Therefore,

$$\begin{aligned} A^{2004} &= \frac{1}{4}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 5^{2004} \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} -1 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 5^{2004} & 5^{2004} \end{pmatrix} \\ &= \frac{1}{4}\begin{pmatrix} 1 + 3 \cdot 5^{2004} & -3 + 3 \cdot 5^{2004} \\ -1 + 5^{2004} & 3 + 5^{2004} \end{pmatrix} \end{aligned}$$

□

2.4. Construct a matrix A with eigenvalues 1 and 3 and corresponding eigenvectors $(1,2)^T$ and $(1,1)^T$. Is such a matrix unique?

Answer. Let

$$\begin{aligned} A &= \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 6 & -3 \end{pmatrix} \\ &= \begin{pmatrix} 5 & -2 \\ 4 & -1 \end{pmatrix} \end{aligned}$$

Suppose A' has eigenvalues $1, 3$ with corresponding eigenvectors $(1,2)^T$ and $(1,1)^T$. Then since the eigenvectors are linearly independent and form a basis of $\mathbb{R}^2$, Theorem 2.1 implies that A' is diagonal with diagonal matrix equal to the middle matrix in the first line above and change of basis matrices equal to the other two in the first line above. Therefore, $A = A'$. □

2.6. Consider the matrix

$$A = \begin{pmatrix} 2 & 6 & -6 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{pmatrix}$$

a) Find its eigenvalues. Is it possible to find the eigenvalues without computing?

Answer. It's eigenvalues are $\lambda = 2, 5, 4$, since this is an upper-triangular matrix and those are the diagonal entries. □

b) Is this matrix diagonalizable? Find out without computing anything.

Answer. Yes. Since the eigenvalues are all distinct and there are 3 for this 3×3 matrix, Corollary 2.3 implies that A is diagonalizable. □

c) If the matrix is diagonalizable, diagonalize it.

Answer. If $\lambda_1 = 2$, $\lambda_2 = 5$, and $\lambda_3 = 4$, then the corresponding eigenvectors are

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad x_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \qquad x_3 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

It follows that

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

□

2.8. Find all square roots of the matrix

$$A = \begin{pmatrix} 5 & 2 \\ -3 & 0 \end{pmatrix}$$

i.e., find all matrices B such that $B^2 = A$. (Hint: Finding a square root of a diagonal matrix is easy. You can leave your answer as a product.)

Answer. We have that

$$A = \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 1 & 1 \end{pmatrix}$$

Therefore, we have four possibilities for B:

$$B_1 = \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 1 & 1 \end{pmatrix}$$
$$B_2 = \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -\sqrt{3} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 1 & 1 \end{pmatrix}$$
$$B_3 = \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 1 & 1 \end{pmatrix}$$
$$B_4 = \begin{pmatrix} -1 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -\sqrt{3} & 0 \\ 0 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} -3 & -2 \\ 1 & 1 \end{pmatrix}$$

□

2.10. Let A be a 5×5 matrix with 3 eigenvalues (not counting multiplicities). Suppose we know that one eigenspace is three-dimensional. Can you say if A is diagonalizable?

Answer. Yes, it is diagonalizable. Let $\lambda_1, \lambda_2, \lambda_3$ be the 3 eigenvalues of A, let $\mathbf{v}_1, \mathbf{v}_2$ be the eigenvectors corresponding to λ_1, λ_2, and let $\mathbf{v}_{3a}, \mathbf{v}_{3b}, \mathbf{v}_{3c}$ be a basis of the eigenvectors corresponding to λ_3. Since the eigenspace of λ_3 is three dimensional, we know that $\mathbf{v}_{3a}, \mathbf{v}_{3b}, \mathbf{v}_{3c}$ is linearly independent. Additionally, we have by consecutive applications of Theorem 2.2 that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_{3a}$, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_{3b}$, and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_{3c}$ are linearly independent lists. Hence $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_{3a}, \mathbf{v}_{3b}, \mathbf{v}_{3c}$ is a linearly independent list of length 5, so it must form a basis of $\mathbb{F}^5$. Therefore, by Theorem 2.1, A is diagonalizable. □

2.11. Give an example of a 3×3 matrix which cannot be diagonalized. After you construct the matrix, can you make it "generic," so no special structure of the matrix can be seen?

Answer. Generalizing from the given example, we can show that

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

is not diagonalizable. Applying row operations can put the matrix in the more generic form

$$\begin{pmatrix} 0 & 1 & 4 \\ 1 & 2 & 3 \\ -1 & 0 & 4 \end{pmatrix}$$

□

2.13. Eigenvalues of a transposition:

a) Consider the transformation T in the space $M_{2\times 2}$ of 2×2 matrices defined by $T(A) = A^T$. Find all its eigenvalues and eigenvectors. Is it possible to diagonalize this transformation? (Hint: While it is possible to write a matrix of this linear transformation in some basis, compute the characteristic polynomial, and so on, it is easier to find eigenvalues and eigenvectors directly from the definition.)

Answer. The symmetric matrices are eigenvectors of this transformation with eigenvalue 1. A basis of them would be

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The antisymmetric matrices are eigenvectors of this transformation with eigenvalue -1. A basis of them would be

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Since these four matrices are linearly independent, there exists a basis of $M_{2\times 2}$ of eigenvectors of T. Therefore, T is diagonalizable. □

b) Can you do the same problem but in the space of $n \times n$ matrices?

Answer. Yes. A basis of the $n \times n$ symmetric matrices includes all of the matrices that are zero everywhere except for one 1 in a diagonal entry, and all of the matrices that are zero everywhere except for two 1's in off-diagonal symmetric positions. There are $\frac{n}{2}(n+1)$ of these basis "vectors." A basis of the $n \times n$ antisymmetric matrices includes all of the matrices that are zero everywhere except for a -1 in an off-diagonal position in the upper triangle and a 1 in the symmetric position in the lower triangle. There are $\frac{n}{2}(n-1)$ of these. Together, we have

$$\frac{n}{2}(n+1) + \frac{n}{2}(n-1) = n^2$$

basis "vectors," meaning that we have a complete eigenbasis of $M_{n\times n}$. □

2.14. Prove that two subspaces V_1 and V_2 are linearly independent if and only if $V_1 \cap V_2 = \{\mathbf{0}\}$.

Answer. Suppose first that V_1, V_2 are linearly independent. Let $\mathbf{v}_{11}, \dots, \mathbf{v}_{1n}$ be a basis of V_1, and let $\mathbf{v}_{21}, \dots, \mathbf{v}_{2m}$ be a basis of V_2. Then by Lemma 2.7, $\mathbf{v}_{11}, \dots, \mathbf{v}_{1n}, \mathbf{v}_{21}, \dots, \mathbf{v}_{2m}$ is linearly independent. Now suppose $\mathbf{v} \in V_1 \cap V_2$. Since $\mathbf{v} \in V_1$, $\mathbf{v} = \alpha_{11}\mathbf{v}_{11} + \cdots + \alpha_{1n}\mathbf{v}_{1n}$. Similarly, $\mathbf{v} = \alpha_{21}\mathbf{v}_{21} + \cdots + \alpha_{2m}\mathbf{v}_{2m}$. Thus,

$$\mathbf{0} = \mathbf{v} - \mathbf{v} = \alpha_{11}\mathbf{v}_{11} + \cdots + \alpha_{1n}\mathbf{v}_{1n} - \alpha_{21}\mathbf{v}_{21} - \cdots - \alpha_{2m}\mathbf{v}_{2m}$$

But since $\mathbf{v}_{11}, \dots, \mathbf{v}_{1n}, \mathbf{v}_{21}, \dots, \mathbf{v}_{2m}$ is linearly independent, it follows that all the α's are 0. Therefore, $\mathbf{v} = 0\mathbf{v}_{11} + \cdots + 0\mathbf{v}_{1n} = \mathbf{0}$, so $V_1 \cap V_2 \subset \{\mathbf{0}\}$. The inclusion in the other direction is obvious, since V_1, V_2 are subspaces.

Now suppose that $V_1 \cap V_2 = \{\mathbf{0}\}$. To prove that V_1, V_2 are linearly independent, it will suffice to show that $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0}$ where $\mathbf{v}_i \in V_i$ for all i implies $\mathbf{v}_i = \mathbf{0}$ for all i. Let $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0}$ where $\mathbf{v}_i \in V_i$ for all i. Suppose for the sake of contradiction that $\mathbf{v}_1 \neq \mathbf{0}$. Then we must have $\mathbf{v}_2 = -\mathbf{v}_1 \neq \mathbf{0}$. But by closure under scalar multiplication, this implies that $-1 \cdot -\mathbf{v}_1 = \mathbf{v}_1 \in V_2$ since $\mathbf{v}_2 \in V_2$. Therefore, $\mathbf{v}_1 \in V_1 \cap V_2$ as well, a contradiction. The proof is symmetric if we let $\mathbf{v}_2 \neq \mathbf{0}$ first. $\square$