MATH 20700 (Honors Analysis in $\mathbb{R}^n$ I) Notes

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Part I

Linear Algebra

Chapter 1

Basic Notions

1.1 Notes

9/27:
- **Vector space**: Basically, a set for which you have an addition and multiplication.
- $\mathbb{F}^d$ is used for $\mathbb{R}^d$ or $\mathbb{C}^d$ in Treil (2017).
- $\mathbb{P}_n$ is the vector space of polynomials up to degree n.
- $C([0,1])$ is the set of continuous functions defined on $[0,1]$, an infinite-dimensional vector space.
- **Generating set**: A subset of a vector space, all linear combinations of which generate the vector space. *Also known as* **spanning set**.
  - Any element of VS is a linear comb. of elements of the generating set.
- **Linearly independent** (list): A list of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$ such that $\sum_{i=1}^k \alpha_i \mathbf{v}_i = 0$ implies $\alpha_i = 0$ for all i.
- **Base**: A generating set consisting of linearly independent vectors.
- Any element of a VS can be written as a *unique* linear combination of the vectors in a base.
  - If $\mathbf{x} = \sum_{i=1}^k \alpha_i \mathbf{v}_i = \sum_{i=1}^k \beta_i \mathbf{v}_i$, then $\alpha_i = \beta_i$ for all i.
- **Linear transformation**: A function $T : X \to Y$, where X, Y are VSs, such that

  $$T(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha T\mathbf{x} + \beta T\mathbf{y}$$

  for all $\mathbf{x} \in X$, $\mathbf{y} \in Y$.
- Examples of linear transformations:
  - Consider $\mathbb{P}_n$. Let $Tp_n = p_n'$. This T is linear.
  - Rotation in $\mathbb{R}^d$.
    - Think graphically about two vectors $\mathbf{x}, \mathbf{y}$.
    - Rotating and summing them is the same as summing and rotating. Same for scaling.
    - Thus, rotation is actually linear!
  - Reflection as well.
- Consider $T : \mathbb{R} \to \mathbb{R}$.
  - Any linear map on the line is a line.
  - We must have $Tx = \alpha x$: $Tx = T(1x) = xT(1) = x\alpha$.

- Consider $T : \mathbb{R}^n \to \mathbb{R}^m$ linear.
  - Any linear map between $\mathbb{R}^n$ and $\mathbb{R}^m$ is linear.
  - Thus, $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$, where A is an $m \times n$ matrix.
- To find A, do the same calculation as for $Tx = \alpha x$ but more carefully:
  - Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a basis.
  - So $\mathbf{x} = \sum_{i=1}^n \alpha_i \mathbf{e}_i$.
  - Thus, $T\mathbf{x} = \sum_{i=1}^n \alpha_i T(\mathbf{e}_i)$.
  - Each $T(\mathbf{e}_i)$ is part of the matrix that we multiply by the column vector representing $\mathbf{x}$.
- Multiplication of matrices is equivalent to composition of linear maps.
- Consider $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ and $T_2 : \mathbb{R}^m \to \mathbb{F}^r$.
  - $T_2 \circ T_1$ is equivalent to BA, if A represents T_1 and B represents T_2. In other words, $(T_2 \circ T_1)(\mathbf{x}) = BA\mathbf{x}$ for all $\mathbf{x}$.
- Recall that if $A = (\alpha_{ij})$ and $B = (\beta_{ij})$, then $(BA)_{ij} = (\sum \beta_{ik}\alpha_{kj})$.
- Properties of multiplication:

$$
\begin{aligned}
(AB)C &= A(BC) \\
A(B + C) &= AB + AC \\
(A + B)C &= AC + BC
\end{aligned}
$$

  - However, it is not true in general that $AB = BA$.
- **Trace** (of an $n \times n$ matrix A): The sum of the diagonal entries of A. *Denoted by* $\mathbf{tr}\,(\boldsymbol{A})$. *Given by*

$$\operatorname{tr}(A) = \sum \alpha_{ii}$$

- It is true that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.
  - Indeed, on the diagonals, multiplication is commutative; it's the other terms that mess you up in general.
- Invertibility of matrices.
  - In general, matrices are not invertible: Not every system of equations is solveable; $Ax = b$ does not always have a solution $x = A^{-1}b$.
- C is the inverse from the left: $CA = I$. B is the inverse from the right: $AB = I$. A matrix can have a left and a right inverse and still not be invertible. A matrix is invertible iff $C = B$.
- Any time we write "inverse," we do so under the assumption that it exists.
- $(AB)^{-1} = B^{-1}A^{-1}$ — easy proof by multiplication.
- If $A = (a_{ij})$, $A^T = (a_{ji})$.
  - $(A^{-1})^T = (A^T)^{-1}$.
  - $(AB)^T = B^T A^T$.
- Let X, Y VS.
  - $X \cong Y$[1] if there exists a linear $T : X \to Y$ that is one-to-one and onto.
  - Check: $A(\text{basis of } X) = \text{basis of } Y$. Prove by definition and expression of elements as linear combinations.
- **Subspace**: A subset of a vector space which happens to be a vector space, itself.

[1] "X is isomorphic to Y."

1.2 Chapter 1: Basic Notions

From Treil (2017).

10/24:

- **Coordinates** (of $\mathbf{v} \in V$ wrt. a basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ of V): The unique scalars $\alpha_1, \dots, \alpha_n \in \mathbb{F}$ such that $\mathbf{v} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$.
- **Spanning system**: A list of vectors that spans V. *Also known as* **generating system**, **complete system**.
- **Trivial** (linear combination): A linear combination $\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n$ of vectors such that $\alpha_k = 0$ for each $k = 1, \dots, n$.
- **Transformation**: A function $T : X \to Y$. *Also known as* **transform**, **mapping**, **map**, **operator**, and **function**.
- The matrix of a linear transformation T is often denoted by $[\boldsymbol{T}]$.
- To compute the reflection of vectors over an arbitrary line through the origin in $\mathbb{R}^2$, represent the overall transformation as a composition of rotating the line to be the x-axis, reflecting over the x-axis, and rotating back.
- Theorem 1.5.1: If A is an $m \times n$ matrix and B is an $n \times m$ matrix, then

$$\operatorname{tr}(AB) = \operatorname{tr}(BA)$$

- Theorem 1.6.1: If a linear transformation is invertible, then its left and right inverses are unique and coincide.
- The column $(1, 1)^T$ is left-invertible, with one possible left inverse being $(1/2, 1/2)$.
  - Note that it is not right invertible since its left inverses are not unique (see Theorem 1.6.1).
- An invertible matrix must be square.
- **Isomorphic** (vector spaces): Two vectors spaces V, W such that there exists an isomorphism $A : V \to W$. *Denoted by* $\boldsymbol{V \cong W}$.
- Theorem 1.6.8: $A : X \to Y$ is invertible if and only if for any right side $\mathbf{b} \in Y$, the equation

$$A\mathbf{x} = \mathbf{b}$$

  has a unique solution $\mathbf{x} \in X$.
- Corollary 1.6.9: An $m \times n$ matrix is invertible if and only if its columns form a basis in $\mathbb{F}^m$.
- **Linear span** (of $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$): The set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_n$. *Denoted by* $\boldsymbol{\mathcal{L}\{}\mathbf{v}_1, \dots, \mathbf{v}_n\boldsymbol{\}}$, $\mathbf{span}\,(\mathbf{v}_1, \dots, \mathbf{v}_n)$.

Chapter 2

Systems of Linear Equations

2.1 Notes

9/29:

- Row elimination:
  - Let
$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 3 & 1 & 2 & 7 \\ 2 & 1 & 2 & 1 \end{pmatrix}$$
  - Then the **echelon form** matrix
$$A_e = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & -4 \end{pmatrix}$$
  - Lastly, the **reduced echelon form** matrix
$$A_{re} = \begin{pmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$
- **echelon form**:
  - All zero rows are below nonzero rows.
  - For any nonzero row, its leading element is strictly to the left of the nonzero entry of the next row.
- **Reduced echelon form**:
  - All pivots are 1.
  - Used to solve systems of the form $Ax = b$.
- **Inconsistent** (system of equations): A system with no solution.
  - If the last row is of the form $(0, \ldots, 0, b)$ where $b \neq 0$, then there is no solution.
- Unique solution if A_e has a pivot in every column.
- There exists a solution for every b if there is a pivot in every row?
- Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a matrix. Then $\ker A = \{x \in \mathbb{R}^n : Ax = 0\}$ (subspace of $\mathbb{R}^n$) and $\operatorname{range} A = \{Ax : x \in \mathbb{R}^n\}$ (subspace of $\mathbb{R}^m$).
- Also consider $\ker(A^T)$ and $\operatorname{range}(A^T)$, the basis of the kernel and range, and dimension.

- Finite-dimensional vector spaces:
  - A basis is a generating set (so every element of V can be written uniquely as a linear combination of the basis) the length of which is equal to the dimension of V.
  - All bases of finite-dimensional vector spaces have the same number of elements.
    - Let v_1, v_2, v_3 and w_1, w_2 be two generating sets of V.
    - Then

$$\begin{aligned} v_1 &= \lambda_{11}w_1 + \lambda_{12}w_2 \\ v_2 &= \lambda_{21}w_1 + \lambda_{22}w_2 \\ v_3 &= \lambda_{31}w_1 + \lambda_{32}w_2 \end{aligned}$$

    - Suppose the only solution to $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$ is $\alpha_1 = \alpha_2 = \alpha_3 = 0$.
    - But this is not true, as we can find another one in terms of the λs.
  - If you have a list of linearly independent vectors, you can complete it into a basis.
    - If there exists a vector that can't be written as a linear combination of the list, add it to the list.
  - If you find any particular solution to a system $Ax = b$, and you add to it any element of $\ker A$, you will obtain another solution.
    - $Ax_1 = b$ and $Ax_h = 0$ implies that $A(x_1 + x_h) = b$.
    - $Ax_1 = b$ and $Ax_2 = b$ imply that $A(x_1 - x_2) = 0$, i.e., that $x_1 - x_2 \in \ker A$.
  - If $A : \mathbb{R}^n \to \mathbb{R}^m$ and $\dim \operatorname{range} A = m$, then $Ax = b$ is solveable for all $b \in \mathbb{R}^m$.
  - Let $\operatorname{rank} A = \dim \operatorname{range} A$.
  - Rank theorem:
    - $\operatorname{rank} A = \operatorname{rank} A^T$.
    - Let $A : \mathbb{R}^n \to \mathbb{R}^m$. We know that $\dim \ker A + \dim \operatorname{range} A = n$.
    - $\dim \ker A^T + \operatorname{rank} A^T = m$.
    - This theorem survives linear algebra and enters functional analysis under the name **Fredholm's alternative**.
- **Fredholm's alternative**: $Ax = b$ has a solution for all $b \in \mathbb{R}^n$ iff $\dim \ker A^T = 0$.
  - $\dim \ker A^T = 0$ implies $\operatorname{rank} A^T = m$ implies $\operatorname{rank} A = m$ implies $\dim \operatorname{range} A = m$, as desired.
- **Pivot column** (of A): A column of A where A_e has pivots.
- The **pivot columns** of A give a basis for $\operatorname{range} A$.
- The pivot rows of A_e give a basis for $\operatorname{range} A^T$.
- A basis for the kernel is enough to solve $Ax = 0$.
- If you take these three things as givens, you can prove the rank theorem.

2.2 Chapter 2: Systems of Linear Equations

From Treil (2017).

10/24:
- A system is inconsistent iff the echelon form of the augmented matrix has a row of the form $\begin{pmatrix} 0 & \cdots & 0 & b \end{pmatrix}$.
- A solution to $A\mathbf{x} = \mathbf{b}$ is unique iff there are no free variables, i.e., iff there is a pivot in every column.
- $A\mathbf{x} = \mathbf{b}$ is consistent iff the echelon form of the coefficient matrix has a pivot in every row.

- $A\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b}$ iff the echelon form of the coefficient matrix A has a pivot in every row and column.
- Proposition 2.3.1: Let $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathbb{F}^n$, and let $A = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_m \end{bmatrix}$ be an $n \times m$ matrix with columns $\mathbf{v}_1, \dots, \mathbf{v}_m$. Then
  1. The system $\mathbf{v}_1, \dots, \mathbf{v}_m$ is linearly independent iff the echelon form of A has a pivot in every column.
  2. The system $\mathbf{v}_1, \dots, \mathbf{v}_m$ is complete iff the echelon form of A has a pivot in every row.
  3. The system $\mathbf{v}_1, \dots, \mathbf{v}_m$ is a basis of $\mathbb{F}^n$ iff the echelon form of A has a pivot in every column and in every row.
- Proposition 2.3.6: A matrix A is invertible if and only if its echelon form has a pivot in every column and every row.
- Corollary 2.3.7: An invertible matrix must be square ($n \times n$).
- Proposition 2.3.8: If a square ($n \times n$) matrix is left invertible or if it is right invertible, then it is invertible. In other words, to check the invertibility of a square matrix A, it is sufficient to check only one of the conditions $AA^{-1} = I$, $A^{-1}A = I$.
- Any invertible matrix is row-equivalent to (can be row-reduced to) to the identity matrix.
- **Homogeneous** (system of linear equations): A system of the form $A\mathbf{x} = \mathbf{0}$.
- Theorem 2.6.1: Let a vector $\mathbf{x}_1$ satisfy the equation $A\mathbf{x} = \mathbf{b}$. and let H be the set of all solutions of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$. Then the set
  $$\{\mathbf{x}_1 + \mathbf{x}_h : \mathbf{x}_h \in H\}$$
  is the set of all solutions to the equation $A\mathbf{x} = \mathbf{b}$.
- The pivot columns are a basis of range A. The pivot rows are a basis of range A^T. The solutions to the equation $A\mathbf{x} = \mathbf{0}$ are a basis of $\ker A$.
- Theorem 2.7.3: Let A be an $m \times n$ matrix. Then the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$ iff the dual equation $A^T\mathbf{x} = \mathbf{0}$ has a unique (only the trivial) solution.
  - Note that this is a corollary to the rank theorem.
- Change of coordinates formula:
  - Let $T : V \to W$ be a linear transformation, and let $\mathcal{V} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ and $\mathcal{W} = \{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ be bases of V and W, respectively.
  - The $m \times n$ matrix of T with respect to these bases is $[T]_{\mathcal{W}\mathcal{V}}$, and relates the coordinates of $[T\mathbf{v}]_{\mathcal{W}}$ and $[\mathbf{v}]_{\mathcal{V}}$ via
    $$[T\mathbf{v}]_{\mathcal{W}} = [T]_{\mathcal{W}\mathcal{V}}[\mathbf{v}]_{\mathcal{V}}$$
  - Change of coordinates matrix: If $\mathcal{A}, \mathcal{B}$ are two bases of V, then we can convert the coordinates of a vector in $\mathcal{B}$ to its in $\mathcal{A}$ with the identity matrix (with respect to the appropriate bases). In particular,
    $$[\mathbf{v}]_{\mathcal{B}} = [I]_{\mathcal{B}\mathcal{A}}[\mathbf{v}]_{\mathcal{A}}$$
    - Note that the k^{th} column of $[I]_{\mathcal{B}\mathcal{A}}$ is the coordinate representation in $\mathcal{B}$ of $\mathbf{a}_k$, i.e., $[\mathbf{a}_k]_{\mathcal{B}}$.
  - The change of coordinates matrix from a basis $\mathcal{B}$ to the standard basis $\mathcal{S}$ is easy to compute; by the above, it's just
    $$[I]_{\mathcal{S}\mathcal{B}} = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix}$$
    - It follows that $[I]_{\mathcal{B}\mathcal{S}} = ([I]_{\mathcal{S}\mathcal{B}})^{-1}$.

        - This allows us to compute $[I]_{\mathcal{B}\mathcal{A}}$ as $[I]_{\mathcal{B}\mathcal{S}}[I]_{\mathcal{S}\mathcal{A}}$
    - If $T : V \to W$, $\mathcal{A}, \tilde{\mathcal{A}}$ are bases of V, and $\mathcal{B}, \tilde{\mathcal{B}}$ are bases of W, and we have $[T]_{\mathcal{B}\mathcal{A}}$, then

$$[T]_{\tilde{\mathcal{B}}\tilde{\mathcal{A}}} = [I]_{\tilde{\mathcal{B}}\mathcal{B}}[T]_{\mathcal{B}\mathcal{A}}[I]_{\mathcal{A}\tilde{\mathcal{A}}}$$

- Change of basis ends up at similarity; two operators are similar if we can change the basis of one into another.

Chapter 3

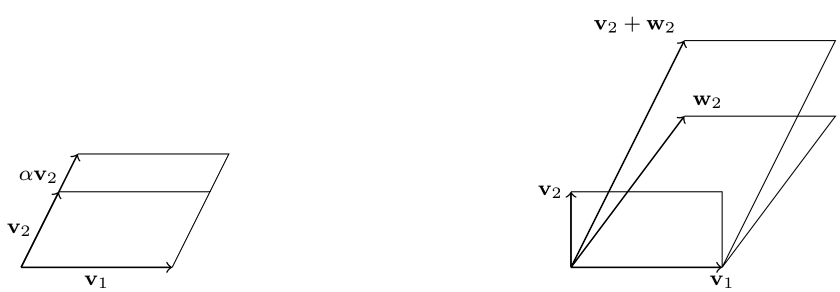
Determinants

3.1 Notes

9/29: - The determinant, geometrically, is the volume of the object (in $\mathbb{R}^3$) you get when you take linear combinations of the vectors.

- In 2D:
  - Let v_1, v_2 be two vectors. Put tail to tail and forming a parallelogram, the determinant of the matrix (v_1, v_2) is the area of said parallelogram.
  - Linearity 1: $D(av_1, v_2, \dots, v_n) = aD(v_1, \dots, v_n)$ is the same as saying that if you stretch one vector by a, you scale up the area by that much, too.
  - Linearity 2: $D(v_1, \dots, v_{k+} + v_{k-}, \dots, v_n) = D(-) + D(+)$.
  - Antisymmetry: $D(v_1, \dots, v_k, \dots, v_j, \dots, v_n) = -D(v_1, \dots, v_j, \dots, v_k, \dots, v_n)$. Interchanging columns flips the sign of the determinant.
  - Basis: $D(e_1, \dots, e_n) = 1$.
- Determinant: Denoted by $D(v_1, \dots, v_n)$, where $(v_1, \dots, v_n)$ is an $n \times n$ matrix.

10/1: - Consider an $n \times n$ matrix A consisting of n columns containing vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^n$.
  - $D(A)$ is the volume of the solid $V = \sum_{i=1}^{n} \alpha_i v_i$.
  - $D(\mathbf{e}_1, \dots, \mathbf{e}_n) = 1$.



(a) $D(\mathbf{v}_1, \alpha\mathbf{v}_2) = \alpha D(\mathbf{v}_1, \mathbf{v}_2)$. (b) $D(\mathbf{v}_1, \mathbf{v}_2 + \mathbf{w}_2) = D(\mathbf{v}_1, \mathbf{v}_2) + D(\mathbf{v}_1, \mathbf{w}_2)$.

Figure 3.1: Visualizing properties of determinants.

- Basic properties of the determinant.
  - If A has a zero column, then $\det A = 0$: Scalar property.

  - If A has two equal columns, then $\det A = 0$: Multiply one by minus and add.
  - If A has a column which is a multiple of another, then $\det A = 0$: Pull out the multiple and then you have the previous one.
  - If columns are linearly dependent, then $\det A = 0$: Decompose it into sums, split, add back up with previous properties.
  - The determinant is preserved under column reduction.
  - $\det A^T = \det A$: Put everything in rref.
  - If A is not invertible, then $\det A = 0$ (not invertible implies linearly dependent columns, implies $\det A = 0$).
  - $\det(AB) = \det A \det B$.
- Determinant of...
  - A diagonal matrix: The product of the diagonal entries (pull out the terms, and then note that the remaining identity matrix has determinant 1).
  - An upper triangular matrix: The product of the diagonal entries (column reduction to make it into a diagonal matrix, and then the property above).

3.2 Chapter 3: Determinants

From Treil (2017).

10/24:

- Let $A_{j,k}$ denote the $(n-1) \times (n-1)$ matrix obtained from A by crossing out row j and column k and pushing it together.
- **Cofactors** (of A): The numbers $C_{j,k}$, one per entry, defined by

$$C_{j,k} = (-1)^{j+k} \det A_{j,k}$$

- **Cofactor matrix** (of A): The matrix

$$C = \{C_{j,k}\}_{j,k=1}^{n}$$

- Theorem 3.5.2: Let A be an invertible matrix and let C be its cofactor matrix. Then

$$A^{-1} = \frac{1}{\det A} C^T$$

- **Cramer's rule**: If A is invertible and $A\mathbf{x} = \mathbf{b}$, then

$$x_k = \frac{\det B_k}{\det A}$$

  where B_k is obtained from A by replacing column k of A by the vector $\mathbf{b}$.
- **Minor** (of order k of A): The determinant of a $k \times k$ submatrix of A.
- Theorem 3.6.1: The rank of a nonzero matrix A is equal to the largest integer k such that there exists a nonzero minor of order k.

Chapter 4

Introduction to Spectral Theory

4.1 Notes

10/1:

- **Difference equation**: Like a differential equation, but instead of writing a differentials, you write differences.
- Suppose we want to solve $x_{n+1} = Ax_n$ with x_0 given.
  - You will find that $x_n = A^n x_0$.
  - This gets hard to compute, so we want to find a way to simplify the computation.
- Thus, we want to diagonalize the matrix, and this concept is inherently linked to eigenvalues and eigenvectors.
  - If you can decompose the x_0 into a linear combination of eigenvectors, then you can simplify the computation a lot:
  $$x_n = \sum \alpha_i A^n v_i = \sum \alpha_i \lambda_i^n v_i$$
  - An $n \times n$ matrix will have n eigenvalues. You want n linearly independent eigenvectors, creating an eigenbasis.
- To find eigenvalues and eigenvectors, we need to solve $Ax = \lambda x$, i.e., $(A - \lambda I)x = 0$. Thus, $\ker(A - \lambda I) \neq \{0\}$, so $\det(A - \lambda I) = 0$.
- The eigenvalues of A are independent of the choice of basis of the domain of A or the range.

10/4:

- We need to know everything in Treil (2017).
  - We don't need to know the applications sections, but you should be interested.
- **Spectral theory**: Decomposing a linear operator.
- Let $A : V \to V$ be a linear operator. $\lambda \in \mathbb{C}$ is an eigenvalue if there exists $x \in V$ nonzero such that $Ax = \lambda x$.
  - Let A be an $n \times n$ matrix over $\mathbb{C}$ or $\mathbb{R}$.
  - The eigenvalues are the roots of the polynomial $\det(A - \lambda I) = 0$ in λ.
- Things we want to do:
  - Given A, find the eigenvalues and eigenvectors (solve $(A - \lambda I)x = 0$).

- In order to simplify A, make it a diagonal matrix:

$$A = S \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} S^{-1}$$

- Eigenvalues are independent of the choice of basis.
  - From the book, we have that
$$[A]_{\mathcal{A}\mathcal{A}} = [S]_{\mathcal{A}\mathcal{B}}[B]_{\mathcal{B}\mathcal{B}}[S]_{\mathcal{A}\mathcal{B}}^{-1}$$
  - It follows that
$$A - \lambda I = [S]_{\mathcal{A}\mathcal{B}}(B - \lambda I)[S]_{\mathcal{A}\mathcal{B}}^{-1}$$
so
$$\det(A - \lambda I) = \det\big([S]_{\mathcal{A}\mathcal{B}}(B - \lambda I)[S]_{\mathcal{A}\mathcal{B}}^{-1}\big) = \det\big([S]_{\mathcal{A}\mathcal{B}}[S]_{\mathcal{A}\mathcal{B}}^{-1}(B - \lambda I)\big) = \det(B - \lambda I)$$
- If $p(z) = (z - \lambda)^k q(z)$, then k is the **algebraic multiplicity** of λ. The **geometric multiplicity** of λ is $\dim\ker(A - \lambda I)$.
  - These terms are not always the same, but they are related.
- Diagonalization:
  - Given A that corresponds to $T : V \to V$, can we find a basis of V in which the operator is a diagonal matrix?
  - $A = SDS^{-1}$ iff there exists a basis of V consisting of the eigenvectors of A.
  - Proves $A^N = SD^NS^{-1}$ via $A^2 = SDS^{-1}SDS^{-1} = SDIDS^{-1} = SD^2S^{-1}$.
- Let A be an $n \times n$ matrix over $\mathbb{F}$. If $\lambda_1, \dots, \lambda_r$ are distinct eigenvalues, then their eigenvectors are linearly independent.
  - Prove with induction contradiction argument. Assume true for $\mathbf{v}_{r-1}$. Then
$$0 = (A - \lambda_r I)[\mathbf{v}_1 + \cdots + \mathbf{v}_r] = (\lambda_1 - \lambda_r)\mathbf{v}_1 + \cdots + (\lambda_{r-1} - \lambda_r)\mathbf{v}_{r-1}$$
  - Implies $\lambda_r = \lambda_i$ for all $i \in [r-1]$, a contradiction.
  - If A has n distinct eigenvalues, then A is diagonalizable.
- If $A : V \to V$ has n complex eigenvalues, then A is diagonalizable iff the algebraic multiplicity equals the geometric multiplicity for each eigenvalue.
- Goes through a sample diagonalization with $\left(\begin{smallmatrix} 1 & 2 \\ 8 & 1 \end{smallmatrix}\right)$.
  - We have
$$A - \lambda I = \begin{pmatrix} 1-\lambda & 2 \\ 8 & 1-\lambda \end{pmatrix}$$
so
$$0 = \det(A - \lambda I) = (1 - \lambda)^2 - 16$$
  - It follows that $\lambda = 5, -3$.
  - This yields
$$\begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}^{-1}$$
by inspection.
- As another example, consider $\left(\begin{smallmatrix} 1 & 2 \\ -2 & 1 \end{smallmatrix}\right)$.
  - Here, we have $\lambda = 1 \pm 2i$.

4.2 Chapter 4: Introduction to Spectral Theory

From Treil (2017).

10/24:

- **Spectrum** (of A): The set of all eigenvalues of A. *Denoted by* $\boldsymbol{\sigma(A)}$.
- Proposition 4.1.1: The geometric multiplicity of an eigenvalue cannot exceed its algebraic multiplicity.
- Theorem 4.2.1: A matrix A (with values in $\mathbb{F}$) admits a representation $A = SDS^{-1}$ where D is a diagonal matrix and S is invertible if and only if there exists a basis of $\mathbb{F}^n$ of eigenvectors of A. Moreover, in this case diagonal entries of D are the eigenvalues of A and columns of S are the corresponding eigenvectors.
- Any operator on a complex vector space has n eigenvalues (counting multiplicities).
  - Think n necessary roots of the characteristic polynomial, or the necessary upper triangular representation.
- Theorem 4.2.8: Let an operator $A : V \to V$ have exactly $n = \dim V$ eigenvalues (counting multiplicities). Then A is diagonalizable if and only if for each eigenvalue λ, the dimension of the eigenspace $\ker(A - \lambda I)$ (i.e., the geometric multiplicity of λ) coincides with the algebraic multiplicity of λ.
- Theorem 4.2.9: A real $n \times n$ matrix A admits a real factorization (i.e., a real representation $A = SDS^{-1}$ where S and D are real matrices, D is diagonal, and S is invertible) if and only if it admits a complex factorization and all eigenvalues of A are real.
- Example of a nondiagonalizable matrix:
$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
  - $p(\lambda) = (1 - \lambda)^2$, so $\lambda = 1$ with algebraic multiplicity 2.
  - However, $\dim \ker(A - I) = 1$ since $A - I$ has only one pivot, hence $2 - 1 = 1$ free variable.
  - Thus, apply Theorem 4.2.8.

Chapter 5

Inner Product Spaces

5.1 Notes

10/6:

- We define

$$\ell^2(\mathbb{R}) = \left\{ \{a_n\}_{n\geq 1} \subset \mathbb{R} : \sum_{1}^{\infty} |a_n|^2 < \infty \right\}$$

- **Inner product**: A map $V \times V \to \mathbb{F}$ that takes $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \cdot \mathbf{y}$. *Denoted by* $\cdot, (\cdot, \cdot), \langle \cdot, \cdot \rangle$.
- Properties of the inner product:
  - $(\mathbf{x}, \mathbf{y}) = \overline{(\mathbf{y}, \mathbf{x})}$ (symmetry).
  - $(\alpha\mathbf{x} + \beta\mathbf{y}, \mathbf{z}) = \alpha(\mathbf{x}, \mathbf{z}) + \beta(\mathbf{y}, \mathbf{z})$ (linearity).
  - $(\mathbf{x}, \mathbf{x}) \geq 0$.
  - $(\mathbf{x}, \mathbf{x}) = 0$ iff $\mathbf{x} = 0$.
- If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} x_i y_i$$

- If $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, then

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} x_i \bar{y}_i$$

- If $f, g \in \mathbb{P}_n(t)$, then

$$(f, g) = \int_{-1}^{1} f\bar{g} \, \mathrm{d}t$$

  - The conjugate of a polynomial is the polynomial with the conjugate of the coefficients of the original polynomial. Symbolically, if $f = \sum_{i=0}^{n} \alpha_i t^i$ is a polynomial, then $\bar{f} = \sum_{i=0}^{n} \bar{\alpha}_i t^i$.
- It is a fact that

$$\left| \sum^{\infty} a_n \bar{b}_n \right| \leq \|(a_n)_{n\geq 1}\| \|(b_n)_{n\geq 1}\|$$

- Suppose we want to define the inner product between two matrices.
  - A common one is

$$(A, B) = \operatorname{tr}(B^* A)$$

    where $B^* = \bar{B}^T = \overline{B^T}$ is the conjugate transpose.

- We define the norm as a function $V \to [0, \infty)$ given by

$$\|\mathbf{x}\| = \sqrt{(\mathbf{x}, \mathbf{x})}$$

- Properties of the norm.
  - $\|\alpha \mathbf{x}\| = |\alpha| \|\mathbf{x}\|$.
  - $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.
  - $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.
- In $\mathbb{R}^n$,

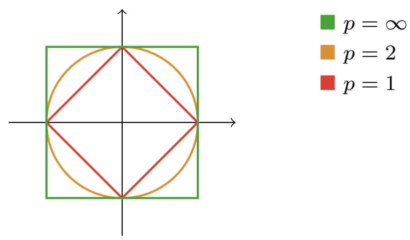


Figure 5.1: The unit ball of norms corresponding to $p = 1, 2, \infty$.

  - The standard norm is
$$\|\mathbf{x}\| = \sqrt{\sum |x_i|^2}$$
  - We can also define
$$\|\mathbf{x}\|_p = \sqrt[p]{\sum |x_i|^p}$$
  - We can even define
$$\|\mathbf{x}\|_\infty = \max |x_i|$$
  - And we can prove that all of these are valid norms.
  - Only the norm corresponding to ℓ^2 is given by an inner product, but all the other quantities are still norms as defined by the properties (see Treil (2017)).
  - Figure 5.1 shows the unit ball of each norm, i.e., the set of all points which have norm 1.
- The parallelogram rule:
$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)$$
- Orthogonality: Given $\mathbf{v}, \mathbf{w}$, if $\mathbf{v} \perp \mathbf{w}$, then $(\mathbf{v}, \mathbf{w}) = 0$.
- In particular, if $\mathbf{v} \perp \mathbf{w}$, then
$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$$
- Let E be a subspace of V. If $\mathbf{v} \perp E$, then $\mathbf{v} \perp \mathbf{e}$ for all $\mathbf{e} \in E$, i.e., $\mathbf{v} \perp$ a set of vectors spanning E.
- Any set of orthogonal vectors is linearly independent. Thus, if V is n dimensional, then $\mathbf{v}_1, \dots, \mathbf{v}_n$ orthogonal is a basis.
- Let E be a subspace of V. Take $\mathbf{v} \in V$. We want to define the projection $P_E \mathbf{v}$ of $\mathbf{v}$ onto E.
  - We have that $P_E \mathbf{v} \in E$ and $v - P_E \mathbf{v} \perp E$.
  - Additionally, we have that
$$\|\mathbf{v} - P_E \mathbf{v}\| \leq \|\mathbf{v} - \mathbf{e}\|$$
    for all $\mathbf{e} \in E$.

- Lastly, we have that $P_E\mathbf{v}$ is unique.

- If we receive a basis of a vector space, how do we create out of that a basis that is orthogonal? The process of doing this is called **Gram-Schmidt orthogonalization**.
  - We keep $\mathbf{v}_1$, subtract $P_{\mathbf{v}_1}\mathbf{v}_2$ from $\mathbf{v}_2$, subtract $P_{\{\mathbf{v}_1,\mathbf{v}_2\}}\mathbf{v}_3$ from $\mathbf{v}_3$, and on and on.
- If we are given a set of orthogonal vectors, we can normalize them by dividing each by its norm. This creates an orthonormal list. The standard basis is orthonormal.
- Let
$$E^\perp = \{v \in V : v \perp E\}$$
  - It follows that $V = E \oplus E^\perp$.
- How close can we come to solving $A\mathbf{x} = \mathbf{b}$ if we cannot solve it exactly (i.e., if the columns are not linearly independent)?
  - Let A be an $m \times n$ matrix, and let $\mathbf{b} \in \mathbb{R}^m$.
  - Then the best solution is given by minimizing $\|A\mathbf{x} - \mathbf{b}\|$. We minimize this with projections. A special case of this is least squares regression! More details in Treil (2017).

10/8:

- Soug is gonna send us a hefty amount of reading for the weekend.
- Least square approximation:
  - If we want to minimize $\|A\mathbf{x} - \mathbf{b}\|$, the best we can do is project $\mathbf{b}$ onto the range of A.
  - Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be an orthogonal basis of range A.
  - Then
$$\text{Proj}|_{\text{range } A}\mathbf{b} = \sum^k \frac{(\mathbf{b}, \mathbf{v}_k)}{\|v_k\|^2}\mathbf{v}_k$$
  - Matrix equation form:
$$\text{Projection}_{\text{range } A} = A(A^*A)^{-1}A^*$$
    if A^*A is invertible, where $A^* = \bar{A}^T$.
    - Soug never uses this though.
  - The minimum is found when $\mathbf{b} - A\mathbf{x} \perp \text{range } A$. Implies that $\mathbf{b} - A\mathbf{x} \perp \mathbf{a}_k$ for all k. Implies $(\mathbf{b} - A\mathbf{x}, \mathbf{a}_k) = \bar{\mathbf{a}}_k^T(\mathbf{b} - A\mathbf{x}) = 0$.
  - Note that we're letting $\bar{\mathbf{a}}_k^T$ be the row vector
$$\bar{\mathbf{a}}_k^T = \begin{pmatrix} \bar{a}_{1,k} & \cdots & \bar{a}_{n,k} \end{pmatrix}$$
  - We also have $\bar{A}^T(\mathbf{b} - A\mathbf{x}) = 0$, from which it follows that $A^*A\mathbf{x} = A^*\mathbf{b}$, so $\mathbf{x} = (A^*A)^{-1}A^*\mathbf{b}$. Thus, $\text{Proj}|_{\text{range } A} = Ax$, so $\text{Proj}|_{\text{range } A} = A(A^*A)^{-1}A^*\mathbf{b}$.
- Adjoint of a linear map $T : V \to W$ is the A^* discussed above.
  - First, we'll do this for matrices. And then we'll do it for any finite-dimensional vector space.
  - Let A be an $m \times n$ matrix. We claim then that
$$(A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, A^*\mathbf{y})$$
    for all $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{y} \in \mathbb{C}^m$. Proof:
$$\begin{aligned}(A\mathbf{x}, \mathbf{y}) &= \bar{\mathbf{y}}^T A\mathbf{x}\\ &= \mathbf{y}^* A\mathbf{x}\\ &= (A^*\mathbf{y})^*\mathbf{x}\\ &= (\mathbf{x}, A^*\mathbf{y})\end{aligned}$$

- Properties of the adjoint:

$$(AB)^T = B^T A^T$$
$$(AB)^* = B^* A^*$$
$$(A^*)^* = A$$

- A^* is the unique matrix B such that $(A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, B\mathbf{y})$.
- Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be a basis of V, and let $\mathbf{w}_1, \dots, \mathbf{w}_m$ be a basis of W.
- Definition of A^*: If $(A\mathbf{x}, \mathbf{y}) = (y, A^*\mathbf{x})$ for all $\mathbf{x} \in V$ and $\mathbf{y} \in W$.
- But it's not enough to define something; we have to check that it exists.
- If $[A]_{\mathcal{AB}}$, then $[A^*]_{\mathcal{AB}}$.
- More properties (give criteria for solving systems of equations):

$$\ker A^* = (\text{range}\, A)^\perp$$
$$\ker A = (\text{range}\, A^*)^\perp$$
$$\text{range}\, A = (\ker A^*)^\perp$$
$$\text{range}\, A^* = (\ker A)^\perp$$

  - Soug proves these.

- Isometries and unitary operators.
  - $U : X \to Y$ is an isometry if $\|\mathbf{x}\| = \|U\mathbf{x}\|$ for all $\mathbf{x} \in X$. It is an isometry because it preserves the distance between points.
  - It immediately follows that $\|\mathbf{x}_1 - \mathbf{x}_2\| = \|U\mathbf{x}_1 - U\mathbf{x}_2\| = \|U(\mathbf{x}_1 - \mathbf{x}_2)\|$.
  - This definition is equivalent to an inner product one: $(\mathbf{x}, \mathbf{y}) = (U\mathbf{x}, U\mathbf{y})$. This follows from the definition of the norm.
  - We have
  $$(\mathbf{a}, \mathbf{b}) = \frac{1}{4} \sum_{\alpha = \pm 1, \pm i} \alpha \|\mathbf{a} + \alpha \mathbf{b}\|^2$$
    - $(a+b)^2 - (a-b)^2 = 4ab$ for any $a, b \in \mathbb{R}$, so $ab = \frac{1}{4}[(a+b)^2 - (a-b)^2]$. Thus, in a real inner product space,
    $$(\mathbf{a}, \mathbf{b}) = \frac{1}{4}\left(\|\mathbf{a} + \mathbf{b}\|^2 - \|\mathbf{a} - \mathbf{b}\|^2\right)$$
    - It follows that isometries preserve inner products.
  - U is an isometry if and only if $U^*U = I$. Proof:
  $$(\mathbf{x}, \mathbf{x}) = (U\mathbf{x}, U\mathbf{y}) = (U^*U\mathbf{x}, \mathbf{x})$$
  $$(\mathbf{x}, \mathbf{y}) = (U\mathbf{x}, U\mathbf{y}) = (U^*U\mathbf{x}, \mathbf{y})$$
  $$(\mathbf{x}, \mathbf{y}) = (U\mathbf{x}, U\mathbf{y}) = (\mathbf{x}, \mathbf{y})$$
  for all $\mathbf{y}$.
  - An isometry is unitary if it is invertible.
    - Thus, $U : X \to Y$ an isometry is unitary iff $\dim X = \dim Y$.
  - Note that it follows that $U^* = U^{-1}$ for U an isometry.
  - U unitary implies $|\det U| = 1$, so λ an eigenvalue of U implies that $|\lambda| = 1$.
  - A is diagonalizable iff it has an orthogonal basis of eigenvectors.

5.2 Chapter 5: Inner Product Spaces

From Treil (2017).

10/24:

- **Standard inner product** (on $\mathbb{C}^n$): The inner product $(\mathbf{z},\mathbf{w})$ defined by
$$(\mathbf{z},\mathbf{w}) = \mathbf{w}^*\mathbf{z}$$
- Corollary 5.1.5: Let $\mathbf{x},\mathbf{y}$ be vectors in an inner product space V. The equality $\mathbf{x}=\mathbf{y}$ holds if and only if
$$(\mathbf{x},\mathbf{z}) = (\mathbf{y},\mathbf{z})$$
for all $\mathbf{z}\in V$.
- Corollary 5.1.6: Suppose two operator $A,B : X \to Y$ satisfy
$$(A\mathbf{x},\mathbf{y}) = (B\mathbf{x},\mathbf{y})$$
for all $\mathbf{x}\in x$ and $\mathbf{y}\in Y$. Then $A=B$.
- **Normed space**: A vector space V equipped with a norm that satisfies properties of homogeneity, the triangle inequality, non-negativity, and non-degeneracy.
- Any inner product space is naturally a normed space.
- If $1 \leq p < \infty$, we can define a corresponding norm on $\mathbb{R}^n$ or $\mathbb{C}^n$ by
$$\|\mathbf{x}\|_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{1/p}$$
- We can also define the norm for $p=\infty$ by
$$\|\mathbf{x}\|_\infty = \max\{|x_k| : k=1,\dots,n\}$$
  - Note that the norm of this form for $p=2$ is the usual norm.
  - These norms are heavily associated with Figure 5.1.
- **Minkowski inequality**: One of the triangle inequalities for norms with $p\neq 2$.
- Theorem 5.1.11: A norm in a normed space is obtained from some inner product if and only if it satisfies the Parallelogram Identity
$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2)$$
for all $\mathbf{u},\mathbf{v}\in V$.
  - It follows that norms with $p\neq 2$ do not have associated inner products, since such norms fail to satisfy the parallelogram identity.
- Lemma 5.2.5 (Generalized Pythagorean Identity): Let $\mathbf{v}_1,\dots,\mathbf{v}_n$ be an orthogonal system. Then
$$\left\|\sum_{k=1}^{n}\alpha_k\mathbf{v}_k\right\|^2 = \sum_{k=1}^{n}|\alpha_k|^2\|\mathbf{v}_k\|^2$$
- Proposition 5.3.3: Let $\mathbf{v}_1,\dots,\mathbf{v}_r$ be an orthogonal basis in E. Then the orthogonal projection $P_E\mathbf{v}$ of a vector $\mathbf{v}$ is given by the formula
$$P_E\mathbf{v} = \sum_{k=1}^{r}\frac{(\mathbf{v},\mathbf{v}_k)}{\|\mathbf{v}_k\|^2}\mathbf{v}_k$$

  - It follows that

$$\begin{aligned}
P_E\mathbf{v} &= \sum_{k=1}^{r} \frac{\mathbf{v}_k^*\mathbf{v}}{\|\mathbf{v}_k\|^2}\mathbf{v}_k \\
&= \sum_{k=1}^{r} \frac{1}{\|\mathbf{v}_k\|^2}\mathbf{v}_k\mathbf{v}_k^*\mathbf{v} \\
&= \left(\sum_{k=1}^{r} \frac{1}{\|\mathbf{v}_k\|^2}\mathbf{v}_k\mathbf{v}_k^*\right)\mathbf{v}
\end{aligned}$$

  - Thus, we have that

$$P_E = \sum_{k=1}^{r} \frac{1}{\|\mathbf{v}_k\|^2}\mathbf{v}_k\mathbf{v}_k^*$$

- **Gram-Schmidt orthogonalization**: Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ be a linearly independent system of vectors to orthogonalize. Then $\mathbf{v}_1 = \mathbf{x}_1$, $\mathbf{v}_2 = \mathbf{x}_2 - P_{\text{span}\{\mathbf{v}_1\}}\mathbf{x}_2$, $\mathbf{v}_3 = \mathbf{x}_3 - P_{\text{span}\{\mathbf{v}_1,\mathbf{v}_2\}}\mathbf{x}_3$, and on and on.
- To find the least squares solution to $A\mathbf{x} = \mathbf{b}$, solve $A\mathbf{x} = P_{\text{range}\,A}\mathbf{b}$.
  - We can do this by finding an orthogonal basis of range A and then applying the projection formula.
  - Alternatively, we can use the following formula to speed things up if A^*A is invertible:

$$P_{\text{range}\,A}\mathbf{b} = A(A^*A)^{-1}A^*\mathbf{b}$$

- Theorem 5.4.1: For an $m \times n$ matrix A,

$$\ker A = \ker(A^*A)$$

  - Thus, A^*A is invertible iff A is invertible iff A is full rank. This gives us a condition on when we can use the projection formula.
- Theorem 5.6.1: An operator $U : X \to Y$ is an isometry if and only if it preserves the inner product, i.e., if and only if

$$(\mathbf{x}, \mathbf{y}) = (U\mathbf{x}, U\mathbf{y})$$

  for all $\mathbf{x}, \mathbf{y} \in X$.
- Lemma 5.6.2: An operator $U : X \to Y$ is an isometry if and only if $U^*U = I$.
- **Unitary** (operator): An invertible isometry.
- Proposition 5.6.3: An isometry $U : X \to Y$ is a unitary operator iff $\dim X = \dim Y$.
- **Orthogonal** (matrix): A unitary matrix with real entries.
- Unitary operator properties:
  1. $U^{-1} = U^*$.
  2. U unitary implies $U^* = U^{-1}$ unitary.
  3. If $\mathbf{v}_1, \dots, \mathbf{v}_n$ is orthonormal, $U\mathbf{v}_1, \dots, U\mathbf{v}_n$ is orthonormal.
  4. U, V unitary implies UV unitary.
- A matrix U is an isometry iff its columns form an orthonormal system.
- Proposition 5.6.4: Let U be a unitary matrix. Then
  1. $|\det U| = 1$. In particular, if U is orthogonal, then $\det U = \pm 1$.
  2. $|\lambda| = 1$ for every eigenvalue λ of U.
- Proposition 5.6.5: A matrix A is unitarily equivalent to a diagonal one iff it has an orthogonal (orthonormal) basis of eigenvectors.

Chapter 6

Structure of Operators on Inner Product Spaces

6.1 Notes

10/11:
- Spectral decomposition of self-adjoint linear maps.
  - Can we write a map in term of the eigenvalues only?
  - Let $A : X \to X$ be linear and self-adjoint. Where $\dim X < \infty$.
  - Let A have eigenvalues $\lambda_1, \ldots, \lambda_n$ and eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. The there is an orthonormal basis of X consisting of eigenvectors of A. An operator is self-adjoint if $A = A^*$.
  - If A is self-adjoint, then A can be written as diagonal with the eigenvalues on the diagonal with respect to some orthonormal basis of eigenvectors.
  - Let $\mathbb{F} = \mathbb{C}$.
- If there exists an orthonormal basis $u_1, \ldots, u_n$ of X such that A is triangular, then $A = UTU^*$ where U is unitary and T is upper triangular.
  - Proved with induction on $\dim X$.
  - $\dim X = 1$ is clear.
  - Assume for $\dim X = n - 1$, WTS for $\dim X = n$.
  - The subspace has a basis $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$ such that A has a diagonal form.
  - Let $u \in X$ be linearly independent of $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$.
  - Let λ be the remaining eigenvalue and u the corresponding eigenvector. Let $E = \operatorname{span}(u)$. Then make the matrix λ in the upper left corner, and block diagonal with "A_{n-1}" in the bottom right corner, zeroes everywhere else.
- **Self-adjoint** (matrix A): A linear map $A : X \to X$ where $\dim X < \infty$ such that $A = A^*$.
  - Similarly, $(Ax, y) = (x, Ay)$.
  - A self-adjoint implies all eigenvalues are real, eigenvectors corresponding to different eigenvalues are orthogonal.
    - Soug proves this.
- **Strictly positive** (operator A): A self-adjoint operator $A : X \to X$ such that $(Ax, x) > 0$ for all $x \neq 0$. *Also known as* **positive definite**.
  - Implies that all eigenvalues are strictly positive.

- **Nonnegative** (operator A): A self-adjoint operator $A : X \to X$ such that $(Ax, x) \geq 0$ for all $x \neq 0$. *Also known as* **definite**.
  - All eigenvalues are nonnegative.
- Suppose $A \geq 0$ is self-adjoint. Then there exists a unique self-adjoint $B \geq 0$ such that $B^2 = A$.
  - A self-adjoint is diagonal (wrt. some basis).
  - A positive means that all eigenvalues (diagonal entries) are positive.
  - Thus, take
$$B = \begin{pmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \sqrt{\lambda_n} \end{pmatrix}$$
  - Suppose $B^2 = A$, $C^2 = A$. Then we have an orthonormal basis corresponding to B and an orthonormal basis corresponding to C. It follows that $B^2 = C^2 = A$. Write B^2x and C^2x in terms of their bases; will necessitate that the bases are the same.

10/13:

- If we get yes/no questions, we don't have to justify.
- Cauchy-Schwarz inequality:
$$|(\mathbf{x}, \mathbf{y})| \leq \|\mathbf{x}\|\|\mathbf{y}\|$$
  - Real spaces, V vs. $(\cdot, \cdot)$ inner product.
  - Proof:
$$\begin{aligned} 0 &\leq \|\mathbf{x} + t\mathbf{y}\|^2 \\ &= t^2\|\mathbf{y}^2\| + 2t(\mathbf{x}, \mathbf{y}) + \|\mathbf{x}\|^2 \end{aligned}$$
    Thus, the discriminant must be less than zero (because the whole polynomial is positive, so the discriminant [the opposite of the x^0 term of the factored form of the polynomial] must be less than zero so the polynomial doesn't get dragged down to negative values):
$$(\mathbf{x}, \mathbf{y})^2 - \|\mathbf{x}\|^2\|\mathbf{y}\|^2 \leq 0$$
    Taking square roots of both sides proves the desired inequality.
- Recall that if $A^* = A$, then all eigenvalues are real and all eigenvectors of distinct eigenvalues are orthogonal to each other.
- **Normal** (matrix): A matrix N such that $N^*N = NN^*$.
  - Examples: Diagonal, self-adjoint, and unitary operators are all normal.
- Any normal operator in a complex vector spae has an orthonormal set of eigenvectors, e.g., $N = UDU^*$.
  - Proof: N is upper triangular wrt. some basis (because all matrices are). WTS any normal upper triangular matrix is diagonal. Done by induction on the dimension of N from $n = 2$.
  - Assume the claim for every $(n-1) \times (n-1)$ normal upper triangular matrix.
  - Let
$$N = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & & & \\ 0 & & & \\ \vdots & & N_1 & \\ 0 & & & \end{pmatrix}$$
    (we know every normal matrix can be written in this upper triangular form)

- Then just compute NN^* and N^*N. Knowing they have to be equal, we have that $a_{12} = \cdots = a_{1n} = 0$.
- We can also prove from the above (block diagonal multiplication) that N_1 is normal. Thus, it's diagonal, too. Therefore, the whole thing is diagonal.

- N is normal if and only if $\|N\mathbf{x}\| = \|N^*\mathbf{x}\|$.
  - Proof: $(N\mathbf{x}, N\mathbf{y}) = (N^*N\mathbf{x}, \mathbf{y}) = (NN^*\mathbf{x}, \mathbf{y}) = (N^*\mathbf{x}, N^*\mathbf{y})$. This is equivalent to the desired condition.
- If A is nonnegative and $(A\mathbf{e}_k, \mathbf{e}_k) = a_{kk}$, then

$$\sum_{i,j=1}^{n} a_{ij}\mathbf{x}_i\mathbf{x}_j$$

- **Positive definite** (matrix): An $n \times n$ self-adjoint matrix such that $(A\mathbf{x}, \mathbf{x}) > 0$ for all $\mathbf{x} \in X$.
- Let $A : X \to Y$, $\dim X = \dim Y$. Then AA^* is positive semidefinite. And there exists a unique square root $R = \sqrt{A^*A}$.
  - Proof: $(A^*A\mathbf{x}, \mathbf{x}) = (A\mathbf{x}, A\mathbf{x}) = \|A\mathbf{x}\|^2 \geq 0$.
- **Modulus** (of A): The matrix $|A| = \sqrt{A^*A}$.
- Check $\||A|\mathbf{x}\| = \|A\mathbf{x}\|$.

$$\||A|\mathbf{x}\|^2 = (|A|\mathbf{x}, |A|\mathbf{x}) = (|A|^*|A|\mathbf{x}, \mathbf{x}) = (A^*A\mathbf{x}, \mathbf{x}) = (A\mathbf{x}, A\mathbf{x}) = \|A\mathbf{x}\|^2$$

- Let $A : X \to X$ be a linear operator. Then $A = U|A|$ where U is unitary.
- Look at singular matrices.

10/15:
- Recall that if $A : X \to Y$, we have that A^*A is semidefinite, positive, and self adjoint.
  - Thus, there exists a unique matrix $R = \sqrt{A^*A} \geq 0$, which we define to be $|A| = \sqrt{A^*A}$.
- Polar form of a matrix:

$$A = U|A|$$

  - This may not be unique!
  - Proof: Suppose $A\mathbf{x} = U(|A|\mathbf{x})$. $A\mathbf{x} \in \operatorname{range} A$, and $|A|\mathbf{x} \in \operatorname{range}(|A|)$. $\mathbf{x} \in \operatorname{range}(|A|)$ implies that there exists $\mathbf{v} \in X$ such that $x = |A|\mathbf{v}$.
  - Define $U\mathbf{x} = A\mathbf{x}$. U is a well-defined linear map.
  - $\|U_0\mathbf{x}\| = \|A\mathbf{x}\| = \||A|\mathbf{v}\| = \|\mathbf{x}\|$.
  - U is an isometry.
  - range $|A| \to X$.
  - Use $\ker A = \ker |A| = (\operatorname{range} A)^\perp$ to extend U_0 to U: $U = U_0 + U_1$.
- **Singular values** (of a matrix): The eigenvalues of $|A|$.
  - So if $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A^*A, the singular values of A are $\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}$.
- Let $A : X \to Y$ be a linear map.
  - Let $\sigma_1, \dots, \sigma_n$ be the signular values of A. Then $\sigma_1, \dots, \sigma_n > 0$.
  - Additionally, if $\mathbf{v}_1, \dots, \mathbf{v}_n$ is an orthonormal basis of eigenvectors of A^*A, then the list of n vectors $\mathbf{w}_1, \dots, \mathbf{w}_n$ defined by $\mathbf{w}_k = 1/\sigma_k A\mathbf{v}_k$ for each $k = 1, \dots, n$ is orthonormal.

  - ■ Proof:

$$(\mathbf{w}_k, \mathbf{w}_j) = \frac{1}{\sigma_k \sigma_k}(A\mathbf{v}_k, A\mathbf{v}_j) = \frac{1}{\sigma_k \sigma_j} = \frac{1}{\sigma_k \sigma_j}(A^* A\mathbf{v}_k, \mathbf{v}_j) = \frac{\sigma_k^2}{\sigma_k \sigma_j}(\mathbf{v}_k, \mathbf{v}_j) = 0$$

    and

$$\|\mathbf{w}_k\| = \frac{1}{\sigma_k}\|A\mathbf{v}_k\| = \frac{1}{\sigma_k}\||A|\mathbf{v}_k\| = 1$$

  - Schmidt decomposition of A:

$$A\mathbf{x} = \sum^{r} \sigma_k (\mathbf{x}, \mathbf{v}_k)\mathbf{w}_k$$

    - ■ This is because $\mathbf{x} = \sum (\mathbf{x}, \mathbf{v}_k)\mathbf{v}_k$, so by the above,

$$A\mathbf{x} = \sum^{n} (\mathbf{x}, \mathbf{v}_k) A\mathbf{v}_k = \sum^{r} \sigma_k (\mathbf{x}, \mathbf{v}_k)\mathbf{w}_k$$

- **Operator norm**: $\|A\| = \max\{\|A\mathbf{x}\| : \|\mathbf{x}\| \leq 1\}$.
- Properties of the operator norm:
  - $\|A\mathbf{x}\| \leq \|A\|\|\mathbf{x}\|$.
  - $\|\alpha A\| = |\alpha|\|A\|$.
  - $\|A + B\| \leq \|A\| + \|B\|$.
  - $\|A\| \geq 0$.
  - $\|A\| = 0$ iff $A = 0$.
- **Frobenius norm**: The norm $\|A\|_2^2 = \operatorname{tr}(A^* A)$.
- The operator norm is always less than or equal to the Frobenius norm.
- If $A : \mathbb{F}^n \to \mathbb{F}^n$, then $A = W \Sigma V^*$ where σ is a diagonal matrix of nonzero singular values.
- The operator norm of A is the largest of the singular values.
- An orthogonal matrix can be decomposed to a block-diagonal matrix of rotations.

10/18:
- Soug tests what he teaches and doesn't give super tricky questions.
- Structure of orthogonal matrices.
- **Orthogonal** (matrix): A unitary matrix U with all elements real and $|\det U| = 1$.
- Theorem: Let U be an orthogonal operator on $\mathbb{R}^n$ such that $\det U = 1$. Then there exists an orthonormal basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ such that with respect to this basis,

$$U = \begin{pmatrix} R_{\phi_1} & & & \mathbf{0} \\ & \ddots & & \\ & & \mathbb{R}_{\phi_k} & \\ \mathbf{0} & & & I_{n-2k} \end{pmatrix}$$

  where each R_{ϕ_i} is a 2×2 rotation matrix.
  - If you are in $\mathbb{R}^7$ for example, you would be able to express U as a composition of at most 3 rotation maps and the identity map.
  - Each rotation map acts on two orthonormal vectors.
  - Proof: $P(\lambda)$ is the n-degree characteristic polynomial $\det(U - \lambda I) = 0$. The eigenvalues are the roots of it.

- $p(\lambda) = 0$ if and only if $p(\bar{\lambda}) = 0$.
  - $\lambda \in \mathbb{C}$ is an eigenvalue with eigenvector $\mathbf{u} \neq 0$ iff $U\mathbf{u} = \lambda\mathbf{u}$ and $U\bar{\mathbf{u}} = \bar{\lambda}\bar{\mathbf{u}}$.
- Recall that U unitary implies $|\lambda| = 1$.
  - Proof[1]: $\|U\mathbf{x}\| = \|\mathbf{x}\|$ and $U\mathbf{x} = \lambda\mathbf{x}$. Thus,

$$\|U\mathbf{x}\| = \|\lambda\mathbf{x}\| = |\lambda|\|\mathbf{x}\| = \|\mathbf{x}\|$$

    and since $\mathbf{x} \neq 0$, we can divide by $\|\mathbf{x}\|$, so $|\lambda| = 1$.
- Let $\mathbf{u} = \operatorname{Re}\mathbf{u} + \operatorname{Im}\mathbf{u}$.
- It follows that we may define

$$\mathbf{x} = \operatorname{Re}\mathbf{u} = \frac{\mathbf{u} + \bar{\mathbf{u}}}{2} \qquad\qquad \mathbf{y} = \operatorname{Im}\mathbf{u} = \frac{\mathbf{u} - \bar{\mathbf{u}}}{2}$$

- Thus, $\mathbf{u} = \mathbf{x} + i\mathbf{y}$ and $\bar{\mathbf{u}} = \mathbf{x} - i\mathbf{y}$.
- Since $U\mathbf{x} = \frac{U\mathbf{u}+U\bar{\mathbf{u}}}{2} = \frac{\lambda\mathbf{u}+\bar{\lambda}\bar{\mathbf{u}}}{2}$, $U\mathbf{y} = \operatorname{Im}(\lambda\mathbf{u}) = \operatorname{Re}(\lambda\mathbf{u})$.
- Since $|\lambda| = 1$, $\lambda = \mathrm{e}^{i\alpha}$ and $\bar{\lambda} = \mathrm{e}^{-i\alpha}$.
- It follows that $U\mathbf{x} = (\cos\alpha)\mathbf{x} - (\sin\alpha)\mathbf{y}$ and $U\mathbf{y} = (\cos\alpha)\mathbf{y}) + (\sin\alpha)\mathbf{x}$.
- Thus, since $U\mathbf{x} = \operatorname{Re}\lambda\mathbf{u}$, we have that

$$\begin{aligned}\lambda\mathbf{u} &= (\cos\alpha + i\sin\alpha)(\mathbf{x} + i\mathbf{y})\\ &= (\cos\alpha)\mathbf{x} - (\sin\alpha)\mathbf{y} + i[(\cos\alpha)\mathbf{y} + (\sin\alpha)\mathbf{x}]\end{aligned}$$

- If E_λ is a 2 dimensional space spanned by $\mathbf{x}$ and $\mathbf{y}$ and invariant by U. Thus, any block of the desired matrix leaves its desired sub-block invariant.
- We also know that the eigenvectors of a unitary matrix corresponding to different eigenvalues are orthogonal.
- Thus, $\|\mathbf{x}\| = \|\mathbf{y}\| = \sqrt{2}/2\|\mathbf{u}\|$, $\mathbf{x} \perp \mathbf{y}$.
- Let $\mathbf{x}, \mathbf{y}$ complete the theorem to form a basis of $\mathbb{R}^n$.
- It will follow that

$$U = \begin{pmatrix} R_\alpha & \mathbf{0} \\ \mathbf{0} & U_1 \end{pmatrix}$$

  where U_1 is orthogonal, and we may repeat the process.

6.2 Chapter 6: Structure of Operators on Inner Product Spaces

From Treil (2017).

10/24:

- Theorem 6.1.1: Let $A : X \to X$ be an operator acting in a complex inner product space. Then there exists an orthonormal basis $\mathbf{u}_1, \dots, \mathbf{u}_n$ of X such that the matrix of A in this basis is upper triangular. In other words, any $n \times n$ matrix A can be represented as $A = UTU^*$, where U is unitary and T is upper-triangular.
- Theorem 6.1.2: Let $A : X \to X$ be an operator acting on a real inner product space. Suppose that all eigenvalues of A are real. Then there exists an orthonormal basis $\mathbf{u}_1, \dots, \mathbf{u}_n$ in X such that the matrix of A in this basis is upper triangular. In other words, any real $n \times n$ matrix A with all real eigenvalues can be represented as $T = UTU^* = UTU^T$, where U is orthogonal and T is a real upper-triangular matrix.

[1] This would be a good exam question.

- Theorem 6.2.1: Let $A = A^*$ be a self-adjoint operator in an inner product space X (the space can be complex or real). Then all eigenvalues of A are real and there exists an orthonormal basis of eigenvectors of A in X.

  Equivalently (see Theorem 6.2.2), A can be represented as $A = UDU^*$ where U is a unitary matrix and D is a diagonal matrix with real entries. Moreover, if A is real, U can be chosen to be real, i.e., orthogonal.

- Proposition 6.2.3: Let $A = A^*$ be a self-adjoint operator and let $\lambda, \mathbf{u}, \mu, \mathbf{v}$ be such that $A\mathbf{u} = \lambda\mathbf{u}$ and $A\mathbf{v} = \mu\mathbf{v}$. Then if $\lambda \neq \mu$, $\mathbf{u} \perp \mathbf{v}$.

- Since complex multiplication is commutative,

$$D^*D = DD^*$$

  for every diagonal matrix D.

  - It follows that $A^*A = AA^*$ if the matrix of A in some orthonormal basis is diagonal.

- Theorem 6.2.4: Any normal operator N in a complex vector space has an orthonormal basis of eigenvectors.

  Equivalently, any matrix N satisfying $N^*N = NN^*$ can be represented as $N = UDU^*$ where U is unitary and D is diagonal.

- Proposition 6.2.5: An operator $N : X \to X$ is normal iff

$$\|N\mathbf{x}\| = \|N^*\mathbf{x}\|$$

  for all $\mathbf{x} \in X$.

- **Hermitian square** (of A): The matrix A^*A.
- **Modulus** (of A): The unique positive semidefinite square root $\sqrt{A^*A}$.
- Proposition 6.3.3: For a linear operator $A : X \to Y$,

$$\||A|\mathbf{x}\| = \|A\mathbf{x}\|$$

- Corollary 6.3.4: $\ker A = \ker |A|$.
- Theorem 6.3.5: Let $A : X \to X$ be an operator (square matrix). Then A can be represented as

$$A = U|A|$$

  where U is a unitary operator.

- **Singular value** (of A): An eigenvalue of $|A|$.

  - A positive square root of an operator of A^*A.

- Proposition 6.3.6: Let $\sigma_1, \dots, \sigma_n$ be the singular values of A, ordered such that $\sigma_1, \dots, \sigma_r$ are the nonzero singular values, and let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be an orthonormal basis of eigenvectors of A^*A. Then the system

$$\mathbf{w}_k = \frac{1}{\sigma_k} A\mathbf{v}_k$$

  for $k = 1, \dots, r$ is orthonormal.

- **Schmidt decomposition** (of A): The decompositions

$$A = \sum_{k=1}^{r} \sigma_k \mathbf{w}_k \mathbf{v}_k^*$$

and

$$A\mathbf{x} = \sum_{k=1}^{r} \sigma_k (\mathbf{x}, \mathbf{v}_k) \mathbf{w}_k$$

  - Note that these can be verified by plugging $\mathbf{x} = \mathbf{v}_j$ for each $j = 1, \dots, n$ into the latter equation.

10/25:

- Lemma 6.3.7: A can be represented as the Schmidt decomposition

$$A = \sum_{k=1}^{r} \sigma_k \mathbf{w}_k \mathbf{v}_k^*$$

where $\sigma_k > 0$ for any orthonormal systems $\mathbf{v}_1, \dots, \mathbf{v}_r$ and $\mathbf{w}_1, \dots, \mathbf{w}_r$.

- Corollary 6.3.8: Let $A = \sum_{k=1}^{r} \sigma_k \mathbf{w}_k \mathbf{v}_k^*$ be a Schmidt decomposition of A. Then

$$A^* = \sum_{k=1}^{r} \sigma_k \mathbf{v}_k \mathbf{w}_k^*$$

is a Schmidt decomposition of A^*.

- **Reduced singular value decomposition** (of A): The decomposition

$$A = \tilde{W} \tilde{\Sigma} \tilde{V}^*$$

where $A : \mathbb{F}^n \to \mathbb{F}^m$ has the Schmidt decomposition $A = \sum_{k=1}^{r} \sigma_k \mathbf{w}_k \mathbf{v}_k^*$, $\tilde{\Sigma} = \operatorname{diag}\{\sigma_1, \dots, \sigma_r\}$, and $\tilde{V}, \tilde{W}$ are matrices with columns $\mathbf{v}_1, \dots, \mathbf{v}_r$ and $\mathbf{w}_1, \dots, \mathbf{w}_r$, respectively. *Also known as* **compact singular value decomposition**.

  - Note that $\tilde{V}$ is an $n \times r$ matrix, $\tilde{\Sigma}$ is an $r \times r$ matrix, and $\tilde{W}$ is an $m \times r$ matrix.
  - Since $\mathbf{v}_1, \dots, \mathbf{v}_r$ and $\mathbf{w}_1, \dots, \mathbf{w}_r$ are orthonormal, $\tilde{V}, \tilde{W}$ are isometries.

- Note that $r = \operatorname{rank} A$ (see Problem 6.3.1).

  - It follows that if A is invertible, then $m = n = r$, so $\tilde{V}, \tilde{W}$ are unitary and $\tilde{\Sigma}$ is an invertible diagonal matrix.

- However, A need not be invertible for us to get a representation similar to $A = \tilde{W} \tilde{\Sigma} \tilde{V}^*$.

  - Complete $\mathbf{v}_1, \dots, \mathbf{v}_r$ and $\mathbf{w}_1, \dots, \mathbf{w}_r$ to bases of $\mathbb{F}^n$ and $\mathbb{F}^m$, respectively.
  - Then we get the following.

- **Singular value decomposition** (of A): The decomposition

$$A = W \Sigma V^*$$

where $V \in M_{n \times n}^{\mathbb{F}}$ and $W \in M_{m \times m}^{\mathbb{F}}$ are unitary matrices with columns $\mathbf{v}_1, \dots, \mathbf{v}_n$ and $\mathbf{w}_1, \dots, \mathbf{w}_n$, respectively, and $\Sigma \in M_{m \times n}^{\mathbb{R}^+}$ is a "diagonal" matrix such that

$$\Sigma_{j,k} = \begin{cases} \sigma_k & j = k \leq r \\ 0 & \text{otherwise} \end{cases}$$

- Notice that if $A = W\Sigma V^*$, then
$$A^*A = (W\Sigma V^*)^*(W\Sigma V^*) = V\Sigma^*W^*W\Sigma V^* = V\Sigma^2 V^*$$
proving that the singular values of A, squared, are the eigenvalues of A^*A.
- If A is invertible, the reduced SVD is the matrix form of the Schmidt decomposition is the SVD.
- If $A = W\Sigma V^*$ is $n \times n$, then
$$A = (\underbrace{WV^*}_{U})(\underbrace{V\Sigma V^*}_{|A|})$$
is a polar decomposition of A.
- Consider the unit ball $B = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leq 1\}$.
  - We want to describe $A(B)$, i.e., the image of the unit ball under A.
  - Let $\mathbf{x} = (x_1, \dots, x_n)^T$ and let $\mathbf{y} = (y_1, \dots, y_n)^T$. If $A = \operatorname{diag}\{\sigma_1, \dots, \sigma_n\}$, we have $\mathbf{y} \in A(B)$ iff $\mathbf{y} = A\mathbf{x}$ where $\mathbf{x} \in B$ iff
$$\sum_{k=1}^{n} \frac{y_k^2}{\sigma_k^2} = \sum_{k=1}^{n} x_k^2 = \|\mathbf{x}\|^2 \leq 1$$
  - Thus, $A(B)$ is an ellipsoid with half-axes $\sigma_1, \dots, \sigma_n$.
  - In the more general case, if $A = W\Sigma V^*$, then since V^* is unitary, $V^*(B) = B$. $\Sigma V^*(B) = \Sigma(B)$ is thus by the above an ellipsoid in range Σ with half-axes $\sigma_1, \dots, \sigma_r$. Thus, since isometries don't change geometry, $W(\Sigma(B))$ is also an ellipsoid with the same half-axes, but in range A.
- Conclusion: The image $A(B)$ of the closed unit ball B is an ellipsoid in range A with half-axes $\sigma_1, \dots, \sigma_r$, where r is the number of nonzero singular values, i.e., the rank of A.
- Finding the maximum of $\|A\mathbf{x}\|$ for $\mathbf{x} \in B$.
  - For a diagonal matrix Σ with nonnegative entries, the maximum is clearly the maximal diagonal entry: In this case if s_1 is the maximal diagonal entry, then since
$$\Sigma\mathbf{x} = \sum_{k=1}^{r} s_k x_k \mathbf{e}_k$$
we have that
$$\|A\mathbf{x}\|^2 = \sum_{k=1}^{r} s_k^2 |x_k|^2 \leq s_1^2 \sum_{k=1}^{r} |x_k|^2 = s_1^2 \cdot \|\mathbf{x}\|^2$$
  - We get the following by a similar logic to before.
- Conclusion: The maximum of $\|A\mathbf{x}\|$ on the unit ball B is the maximal singular value of A.
- **Operator norm** (of A): The following quantity. *Denoted by* $\|\boldsymbol{A}\|$. *Given by*
$$\|A\| = \max\{\|A\mathbf{x}\| : \mathbf{x} \in X, \|\mathbf{x}\| \leq 1\}$$
  - $\|A\|$ clearly satisfies the four properties of a norm.
  - Additionally,
$$\|A\mathbf{x}\| \leq \|A\| \cdot \|\mathbf{x}\|$$
  - Alternate definition: The operator norm $\|A\|$ is the smallest number $C \geq 0$ such that $\|A\mathbf{x}\| \leq C\|\mathbf{x}\|$.

- **Frobenius norm**: The following norm. *Also known as* **Hilbert-Schmidt norm**. *Denoted by* $\|\boldsymbol{A}\|_{\boldsymbol{2}}$. *Given by*

$$\|A\|_2^2 = \operatorname{tr}(A^*A)$$

  - If we let $s_1, \dots, s_n$ be the singular values of A and let s_1 be the largest value, then we have

$$\|A\|^2 = s_1^2 \leq \sum_{k=1}^{n} s_k^2 = \operatorname{tr}(A^*A) = \|A\|_2^2$$

- Conclusion: The operator norm of a matrix cannot be more than its Frobenius norm.
- Suppose we want to solve $A\mathbf{x} = \mathbf{b}$ where A is invertible, but there is some (experimental) error $\Delta\mathbf{b}$ in $\mathbf{b}$. Then we are really solving for an approximate solution $\mathbf{x} + \Delta\mathbf{x}$ to the equation

$$A(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{b} + \Delta\mathbf{b}$$

  - It follows since A is invertible that $\mathbf{x} = A^{-1}\mathbf{b}$ and $\Delta\mathbf{x} = A^{-1}\Delta\mathbf{b}$.
  - To estimate the relative error $\|\Delta\mathbf{x}\|/\|\mathbf{x}\|$ in the solution in comparison with the relative error $\|\Delta\mathbf{b}\|/\|\mathbf{b}\|$ in the data, use

$$\frac{\|\Delta\mathbf{x}\|}{\|\mathbf{x}\|} = \frac{\left\|A^{-1}\Delta\mathbf{b}\right\|}{\|\mathbf{b}\|}\frac{\|A\mathbf{x}\|}{\|\mathbf{x}\|} \leq \frac{\left\|A^{-1}\right\| \cdot \|\Delta\mathbf{b}\|}{\|\mathbf{b}\|}\frac{\|A\| \cdot \|\mathbf{x}\|}{\|\mathbf{x}\|} = \left\|A^{-1}\right\| \cdot \|A\| \cdot \frac{\|\Delta\mathbf{b}\|}{\|\mathbf{b}\|}$$

- **Condition number** (of A): The following quantity. *Given by*

$$\|A\| \cdot \left\|A^{-1}\right\|$$

  - If s_1 is the largest singular value of A and s_n is the smallest, then

$$\|A\| \cdot \left\|A^{-1}\right\| = s_1 \cdot \frac{1}{s_n} = \frac{s_1}{s_n}$$

- **Well-conditioned** (matrix): A matrix the condition number of which is not "too big."
- **Ill-conditioned** (matrix): A matrix that is not well-conditioned.
- Theorem 6.5.1: Let U be an orthogonal operator on $\mathbb{R}^n$ and let $\det U = 1$. Then there exists an orthonormal basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ such that the matrix of U in this basis has the block diagonal form

$$\begin{pmatrix} R_{\varphi_1} & & & 0 \\ & \ddots & & \\ & & R_{\varphi_k} & \\ 0 & & & I_{n-2k} \end{pmatrix}$$

  where each R_{φ_j} is a two-dimensional rotation

$$R_{\varphi_j} = \begin{pmatrix} \cos\varphi_j & -\sin\varphi_j \\ \sin\varphi_j & \cos\varphi_j \end{pmatrix}$$

  and I_{n-2k} represents the $(n-2k) \times (n-2k)$ identity matrix.

  - Alternate interpretation: Any rotation in $\mathbb{R}^n$ can be represented as a composition of at most $n/2$ commuting planar rotations.

- Theorem 6.5.2: Let U be an orthogonal operator on $\mathbb{R}^n$ and let $\det U = -1$. Then there exists an orthonormal basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ such that the matrix of U in this basis has block diagonal form
$$\begin{pmatrix} R_{\varphi_1} & & & & 0 \\ & \ddots & & & \\ & & R_{\varphi_k} & & \\ & & & I_r & \\ 0 & & & & -1 \end{pmatrix}$$
where $r = n - 2k - 1$ and each R_{φ_j} is a two-dimensional rotation
$$R_{\varphi_j} = \begin{pmatrix} \cos\varphi_j & -\sin\varphi_j \\ \sin\varphi_j & \cos\varphi_j \end{pmatrix}$$
- Corollary: An orthogonal 2×2 matrix U with determinant -1 is always a reflection.
- Theorem 6.5.3: Any rotation U (i.e., any orthogonal transformation U with $\det U = 1$) can be represented as a product of at most $n(n-1)/2$ elementary rotations.
- Consider the following orthonormal bases of $\mathbb{R}^2$.

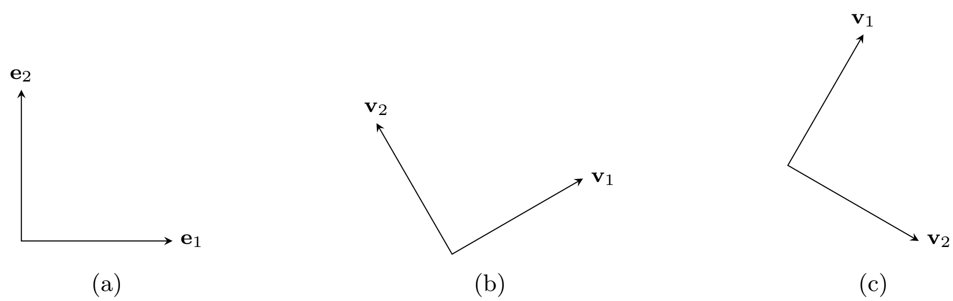


Figure 6.1: Orientation in $\mathbb{R}^2$.

  - Notice that a rotation will get you from the standard basis (a) to basis (b), but not from the standard basis (a) to basis (c).
  - This is the motivation for defining orientation.
  - More formally, we know that there is a unique linear transformation U such that $U\mathbf{e}_k = \mathbf{v}_k$ for each $k = 1, 2$. In particular, the matrix of U with respect to the standard basis is orthogonal with columns $\mathbf{v}_1, \mathbf{v}_2$.
  - By Theorems 6.5.1 and 6.5.2, if $\det U = 1$, then U is a rotation, and if $\det U = -1$, then U is not a rotation.
- **Similarly oriented** (bases $\mathcal{A}, \mathcal{B}$): Two bases $\mathcal{A}, \mathcal{B}$ of a real vector space such that the change of coordinates matrix $[I]_{\mathcal{B}\mathcal{A}}$ has a positive determinant.
- **Differently oriented** (bases $\mathcal{A}, \mathcal{B}$): Two bases $\mathcal{A}, \mathcal{B}$ of a real vector space that are not similarly oriented (i.e., $[I]_{\mathcal{B}\mathcal{A}}$ has a negative determinant).
- We usually let the standard basis of $\mathbb{R}^n$ have a **positive orientation**.
  - In an abstract vector space, we need only fix a basis and declare its orientation to be positive.
- **Continuously transformable** (bases $\mathcal{A}, \mathcal{B}$): Two bases $\mathcal{A}, \mathcal{B}$ such that $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ can be continuously transformed to a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$. In particular, there exists a **continuous family of bases** $\mathcal{V}(t) = \{\mathbf{v}_1(t), \dots, \mathbf{v}_n(t)\}$, $t \in [a, b]$, such that
$$\mathbf{v}_k(a) = \mathbf{a}_k \qquad\qquad \mathbf{v}_k(b) = \mathbf{b}_k$$
for each $k = 1, \dots, n$.

- **Continuous family of bases**: A family of bases $\mathcal{V}(t) = \{\mathbf{v}_1(t), \dots, \mathbf{v}_n(t)\}$, $t \in [a, b]$, such that the vector-functions $\mathbf{v}_k(t)$ are continuous (their coordinates in some bases are continuous functions) and the system $\mathbf{v}_1(t), \dots, \mathbf{v}_n(t)$ is a basis for all $t \in [a, b]$.
- Theorem 6.6.1: Two bases $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ and $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ have the same orientation if and only if one of the bases can be continuously transformed to the other.

Chapter 7

Bilinear and Quadratic Forms

7.1 Notes

10/18:

- **Bilinear form**: A function $L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$L(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2, \mathbf{y}) = \alpha L(\mathbf{x}_1, \mathbf{y}) + \beta L(\mathbf{x}_2, \mathbf{y}) \qquad L(\mathbf{x}, \alpha\mathbf{y}_1 + \beta\mathbf{y}_2) = \alpha L(\mathbf{x}, \mathbf{y}_1) + \beta L(\mathbf{x}, \mathbf{y}_2)$$

  - $L(\mathbf{x}, \mathbf{y}) = (A\mathbf{x}, \mathbf{y})$.

- **Quadratic form**: A bilinear form $L(\mathbf{x}, \mathbf{x})$.
  - $(\mathbf{x}, \mathbf{x})$ is a polynomial of degree 2 in $\mathbf{x}_1, \dots, \mathbf{x}_n$:

$$L(\lambda\mathbf{x}, \lambda\mathbf{x}) = (\lambda\mathbf{x}, \lambda\mathbf{x}) = \lambda^2(\mathbf{x}, \mathbf{x})$$

- We have that

$$(A\mathbf{x}, \mathbf{x}) = (A\lambda\mathbf{x}, \lambda\mathbf{x}) = \lambda^2(A\mathbf{x}, \mathbf{x}) = \sum_{j,i=1}^{n} \alpha_{j,i}\mathbf{x}_i\mathbf{x}_j$$

- The general form of a quadratic form:
  - Can any quadratic form on $\mathbb{R}^n$ be written as $(A\mathbf{x}, \mathbf{x})$?

10/20:

- Bilinear forms are linear in each argument when keeping the other fixed.
- Quadratic forms $Q(\mathbf{x}) = L(\mathbf{x}, \mathbf{x})$ are quadratic polynomials in the coordinates of x.
  - In particular, $Q(\lambda\mathbf{x}) = |\lambda|^2 Q(\mathbf{x})$.
- If Q quadratic is real, then $Q(\mathbf{x}) = (A\mathbf{x}, \mathbf{x})$ where A is some square matrix.
  - If $\mathbf{e}_1, \dots, \mathbf{e}_n$ is an orthonormal basis of $\mathbb{R}^n$, then there exists a unique $A = A^*$ such that $(A)_{ij} = L(\mathbf{e}_i, \mathbf{e}_j)$.
  - Keeping $\mathbf{x} = \sum_{i=1}^n \mathbf{x}_i, \mathbf{e}_i$ foxed, we have

$$\begin{aligned}
Q(\mathbf{x}) &= L(\mathbf{x}, \mathbf{x}) \\
&= L(\sum^n \mathbf{x}_i\mathbf{e}_i, \sum^n \mathbf{x}_j\mathbf{e}_j) \\
&= \sum_{i=1}^n \mathbf{x}_i L(\mathbf{e}_i, \sum^n \mathbf{x}_j\mathbf{e}_j) \\
&= \sum_{i,j=1}^n \mathbf{x}_i\mathbf{x}_j \underbrace{L(\mathbf{e}_i, \mathbf{e}_j)}_{A_{ij}}
\end{aligned}$$

- We have that

$$\begin{aligned}(A\mathbf{x},\mathbf{x}) &= (UDU^{-1}\mathbf{x},\mathbf{x})\\ &= (DU^{-1}\mathbf{x},U^{-1}\mathbf{x})\\ &= \sum_{i=1}^{n}\lambda_i(\underbrace{U^{-1}\mathbf{x}}_{\mathbf{y}_i})_i(\underbrace{U^{-1}\mathbf{x}}_{\mathbf{y}_i})_i\end{aligned}$$

- Can we characterize the set $\{\mathbf{x} : (A\mathbf{x},\mathbf{x}) = 1\}$?
  - Note that this set is equivalent to $\{\mathbf{y} : (D\mathbf{y},\mathbf{y}) = 1\}$ by teh above. This set is a rotation of the previous one. Ellipse?
- Positive quadratic form:
  - Q is positive definite if $Q(\mathbf{x}) > \mathbf{0}$ for all $\mathbf{x} \neq \mathbf{0}$ and Q is positive semidefinite if $Q(\mathbf{x}) \geq \mathbf{0}$ for all $\mathbf{x} \neq \mathbf{0}$.
  - Take a self-adjoint matrix $A = A^*$. It is positive definite if $Q(\mathbf{x}) = (A\mathbf{x},\mathbf{x})$ is positive definite.
- Theorem: If $A = A^*$, then
  1. A is positive definite if and only if all eigenvalues of A are positive.
  2. A is positive semidefinite if and only if all eigenvalues of A are nonnegative.
  3. A is negative semidefinite if and only if all eigenvalues of A are nonpositive.
  4. A is negative definite if and only if all eigenvalues of A are negative.
  5. A is indefinite if and only if the eigenvalues of A have positive and negative values.
- Theorem: $A = A^*$ is positive definite iff $\det A_k > 0$ for all $k = 1,\dots,n$ where A_k is the upper left $k \times k$ submatrix.
- Minimax representation of eigenvalues of a self-adjoint A.
  - Let E be a subspace of X where $\dim X < \infty$. We define $\operatorname{codim}(E) = \dim E^\perp$.
  - Thus, $\dim E + \operatorname{codim} E = \dim X$.
  - Theorem: Let $A = A^*$, $\lambda_1 \geq \cdots \geq \lambda_n$ eigenvalues of A. Then

$$\lambda_k = \max_{\substack{\text{E subspace}\\ \dim E=k}}\ \min_{\substack{\mathbf{x}\in E\\ \|\mathbf{x}\|=1}}(A\mathbf{x},\mathbf{x}) = \min_{\substack{\text{F subspace}\\ \operatorname{codim} F=k-1}}\ \max_{\substack{\mathbf{x}\in F\\ \|\mathbf{x}\|=1}}(A\mathbf{x},\mathbf{x})$$

    - Proof: A diagonal equals $(\lambda_1,\dots,\lambda_n)$.
    - An orthonormal basis of X such that $\dim E = k$, $\operatorname{codim} F = k-1$, $\dim F = n-k+1$.
    - There exists an $\mathbf{x}_0 \neq \mathbf{0}$ such that $\mathbf{x}_0 \in E \cap F$.
    - Note that if $B = B^*$, then the max and min of $(B\mathbf{x},\mathbf{x})$ over the unit sphere is the maximal and minimal eigenvalue of B.
    - Thus,

$$\min_{\substack{\mathbf{x}\in E\\ \|\mathbf{x}\|=1}}(A\mathbf{x},\mathbf{x}) \leq (A\mathbf{x}_0,\mathbf{x}_0) \leq \max_{\substack{\mathbf{x}\in F\\ \|\mathbf{x}\|=1}}(A\mathbf{x},\mathbf{x})$$

    - This is true for any E, F subspaces. $\dim E = k$, $\operatorname{codim} F = k-1$, $E_0 = \operatorname{span}(\mathbf{e}_1,\dots,\mathbf{e}_k)$ and $F_0 = \operatorname{span}(\mathbf{e}_k,\dots,\mathbf{e}_n)$.
    - Thus,

$$\min_{\substack{E_0\\ \|\mathbf{x}\|=1}}(A\mathbf{x},\mathbf{x}) = \lambda_k = \max_{\substack{F_0\\ \|\mathbf{x}\|=1}}(A\mathbf{x},\mathbf{x})$$

■ Additionally,

$$\lambda_{k_1} \leq \max{}_{\dim E=k}^{E} \min_{\mathbf{x}}(A\mathbf{x},\mathbf{x}) \leq \min_{\substack{F \\ \operatorname{codim} F=k-1}} \max_{\mathbf{x}}(A\mathbf{x},\mathbf{x}) \leq \lambda_k$$

- Corollary: Let $A = A^* = (a_{jk})_{1\leq j,k\leq n}$ with eigenvalues $\lambda_1,\dots,\lambda_n$ listed in decreasing order. Let $\tilde{A} = (a_{j,k})_{1\leq j,k\leq n-1}$ with eigenvalues $\mu_1,\dots,\mu_{n-1}$ listed in decreasing order. Then $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n$.
  - Consider $(A\mathbf{x},\mathbf{x})$ on $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$, but then restrict yourself to $\mathbf{x} \in \mathbb{R}^{n-1}$ on $\{\mathbf{e}_1,\dots,\mathbf{e}_{n-1}\}$.

7.2 Chapter 7: Bilinear and Quadratic Forms

From Treil (2017).

10/25:
- **Bilinear form** (on $\mathbb{R}^n$): A function $L(\mathbf{x},\mathbf{y})$ of two arguments $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$ that is linear in each argument.
  - Linearity in each argument:

$$L(\alpha\mathbf{x}_1 + \beta\mathbf{x}_2, \mathbf{y}) = \alpha L(\mathbf{x}_1,\mathbf{y}) + \beta L(\mathbf{x}_2,\mathbf{y}) \qquad L(\mathbf{x}, \alpha\mathbf{y}_1 + \beta\mathbf{y}_2) = \alpha L(\mathbf{x},\mathbf{y}_1) + \beta L(\mathbf{x},\mathbf{y}_2)$$

- If $\mathbf{x} = (x_1,\dots,x_n)^T$ and $\mathbf{y} = (y_1,\dots,y_n)^T$, then

$$\begin{aligned}
L(\mathbf{x},\mathbf{y}) &= \sum_{j,k=1}^{n} a_{j,k}x_ky_j \\
&= (A\mathbf{x},\mathbf{y}) \\
&= \mathbf{y}^T A\mathbf{x}
\end{aligned}$$

  where

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix}$$

  - A is uniquely determined by L.
- **Quadratic form** (on $\mathbb{R}^n$): The diagonal of a bilinear form L, i.e., a bilinear form $Q[\mathbf{x}] = L(\mathbf{x},\mathbf{x}) = (A\mathbf{x},\mathbf{x})$.
  - Alternatively: A homogeneous polynomial of degree 2, i.e., a polynomial in $x_1,\dots,x_n$ with only ax_k^2 and cx_jx_k terms.
- There are infinitely many ways to write a quadratic form as $(A\mathbf{x},\mathbf{x})$.
  - However, there is a unique representation $(A\mathbf{x},\mathbf{x})$ where A is a (real) symmetric matrix.
- **Quadratic form** (on $\mathbb{C}^n$): A function of the form $Q[\mathbf{x}] = (A\mathbf{x},\mathbf{x})$ where A is self-adjoint.
- Lemma 7.1.1: Let $(A\mathbf{x},\mathbf{x})$ be real for all $\mathbf{x} \in \mathbb{C}^n$. Then $A = A^*$.
- To classify quadratic forms, consider the set of points $\mathbf{x} \in \mathbb{R}^n$ defined by $Q[\mathbf{x}] = 1$ for some quadratic form Q.
  - If the matrix of Q is diagonal, i.e., $Q[\mathbf{x}] = a_1x_1^2 + \cdots + a_nx_n^2$, then the set of points can easily be visualized.
- The standard method of diagonalizing a quadratic form is change of variables.
- Orthogonal diagonalization.

  - Let $Q[\mathbf{x}] = (A\mathbf{x}, \mathbf{x})$ in $\mathbb{F}^n$.
  - Suppose $\mathbf{y} = S^{-1}\mathbf{x}$ where S is an invertible $n \times n$ matrix. Then

$$Q[\mathbf{x}] = Q[S\mathbf{y}] = (AS\mathbf{y}, S\mathbf{y}) = (S^*AS\mathbf{y}, \mathbf{y})$$

    so in the new variables $\mathbf{y}$, the quadratic form has matrix S^*AS.
  - Thus, we can let $A = UDU^*$, choose $D = U^*AU$ as our new (diagonal) matrix, and let this matrix act on the variables $\mathbf{y} = U^*\mathbf{x}$.

- Non-orthogonal diagonalization:
  - Completing the square:
    - Eliminate all x_ix_j terms by completing the square. Then substitute in a y_k for each squared term.
  - Row/column operations:
    - Augment $(A|I)$. Row reduce A to D. Then $I \to S^*$.

Chapter 8

Dual Spaces and Tensors

10/22:

- **Functional**: A linear bounded map $L : H \to F$, where H is finite dimensional (equivalent to $\mathbb{R}^n$).
- **Dual space**: The set of bounded linear functionals on H. *Denoted by* $\boldsymbol{H'}$, $\boldsymbol{H^*}$.
- If $l \leq p < \infty$, then
$$l^p = \left\{ (a_n)_{n \in \mathbb{N}} : \sum_{n=1}^{\infty} |a_n|^p < \infty \right\}$$
- Back to finite dimensions, $H' \approx \mathbb{R}^n$.
- Let $\mathbf{a}_1, \ldots, \mathbf{a}_n$ be a basis of H. Then $L\mathbf{x} = (L\mathbf{a}_1, \ldots, L\mathbf{a}_n) \approx \mathbb{R}^n$.
- Let $L((a_n)_{n \in \mathbb{N}}) = \sum_{n=1}^{\infty} a_n b_n$. Then $L((a_n)_{n \in \mathbb{N}})$ will be bounded if and only if $(b_n)_{n \in \mathbb{N}} \in l^q$ where $1 < p < q$ where $\frac{1}{q} + \frac{1}{p} = 1$.
- **Young's inequality**: The statement
$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$
- We have $|\sum a_n b_n| \leq \|a_n\|_p \|b_n\|_p$.
- Conclusion:
$$\sum \frac{|a_n||b_n|}{\|a_n\|_p \|b_n\|_q} = 1$$
- We can define H'', too. This contains linear functionals on H'.
- We know that $L(x) = \langle x, L \rangle = x(L)$. $x \in H''$.
- Riesz representation theorem: Let H have an inner product. $L \in H'$ if and only if there exists a unique $y \in H$ such that $L(x) = (x, y)$.
  - Gives us a way to identify all bounded linear functionals on H.
  - In finite dimensions, $L(x)$, where $x = \sum_1^n \alpha_i a_i$ gives us $L(x) = \sum_1^n \alpha_i L(a_i)$.

Chapter 9

Advanced Spectral Theory

10/22:

- Let $p(z) = \sum_{i=0}^{n} a_i z^i$ be a polynomial. Let A be an $n \times n$ matrix. We let $p(A) = \sum_{i=0}^{n} a_i A^i$.
- Theorem: If A is an $n \times n$ and $p(\lambda) = \det(A - \lambda I)$, then $p(A) = 0$.
  - We know that $p(\lambda) = a(z - \lambda_1) \cdots (z - \lambda_n)$ where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues.
  - Thus $p(A) = a(A - \lambda_1 I) \cdots (A - \lambda_n I)$.
  - If you are in $\mathbb{R}^n$ and have this property, you can factorize your matrix.
  - Thus, $p(A)\mathbf{x} = \mathbf{0}$ since $\mathbf{x}$ can be decomposed into a linear combination of eigenvectors of A, which will be taken to 0 one by one by the terms of $p(A)$.
- $\sigma(B) = \{\text{eigenvalues of } B\}$ is known as the **spectrum** of B.
- If p is an arbitrary polynomial and A is $n \times n$, then μ is an eigenvalue of $p(A)$ if and only if $\mu = p(\lambda)$ where λ is an eigenvalue of A. In essence, $\sigma(p(A)) = p(\sigma(A))$.
- Chapter 9 will not be on the exam. We don't have to know the generalization to infinite dimensional spaces.

10/25:

- If A is an $n \times n$ square matrix and $p(\lambda) = \det(A - \lambda I)$, then $p(A) = 0$.
  - Proof: WLOG, let A be an upper triangular matrix with diagonal entries equal to the eigenvalues.
  - Think of $p(z) = (-1)^n (z - \lambda_1) \cdots (z - \lambda_n)$.
  - Thus, $p(A) = (-1)^n (A - \lambda_1 I) \cdots (A - \lambda_n I)$.
  - WTS: $p(A)\mathbf{x} = 0$ for all $\mathbf{x} \in V$.
  - Let $E_k = \operatorname{span}(e_1, \ldots, e_k)$ be the span of the first k eigenvectors of A, where $e_1, \ldots, e_n$ is a standard basis in $\mathbb{C}^n$.
  - A triangular implies $AE_k \subset E_k$. Thus, $(A - \lambda I)E_k \subset E_k$, so E_k is invariant under $A - \lambda I$ for all λ.
  - If we apply $A - \lambda_k I$ to a vector in E_k, we are left with a vector in E_{k-1}.
  - Thus, if we apply $\prod_{k=1}^{n} (A - \lambda_k I) = p(A)$ to any vector in $E_n = V$, we will kill it piece by piece down to zero.
- Let A be a square $n \times n$ matrix. Then p an arbitrary polynomial implies $\sigma(p(A)) = p(\sigma(A))$. (Any eigenvalue μ of $p(A)$ is $\mu = p(\lambda)$, where λ is an eigenvalue of A.)
  - Shows that polynomials of operators commute.
  - Proof: Let λ be an eigenvalue of A. We want to show that $p(\lambda)$ is an eigenvalue of $p(A)$. This is obvious since $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x}$, so $A^k\mathbf{x} = \lambda^k\mathbf{x}$, so in particular, $p(A)\mathbf{x} = p(\lambda)\mathbf{x}$.

  - On the other hand, if μ is an eigenvalue of $p(A)$, we want to show that there exists $\lambda \in \sigma(A)$ such that $\mu = p(\lambda)$.
  - Consider $q(z) = p(z) - \mu$. Then $q(A) = p(A) - \mu I$. Since μ is an eigenvalue of $p(A)$, $q(A)$ is not invertible.
  - Thus, $q(z) = (-1)^n(z - z_1)\cdots(z - z_n)$ and $q(A) = (-1)^k(A - z_1 I)\cdots(A - z_k I)$.
  - But $q(A)$ is not invertible, so one of the $A - z_k I$ is not invertible. Take z_k such that $A - z_k I$ is not invertible. Then $z_k \in \sigma(A)$. It follows that $q(z_k) = p(z_k) - \mu = \sigma$.
- If A is $n \times n$, $\lambda_1, \dots, \lambda_n$ are its eigenvalues, p is a polynomial, then $p(A)$ is invertible if and only if $p(\lambda_k) \neq 0$ for each $k = 1, \dots, n$.
  - This is an immediate corollary to the previous result.
- We now build up to the **generalized eigenspace**, which is related to some "geometric" properties of the algebraic multiplicity of an eigenvalue.
- If $A : V \to V$ is a linear operator and $E \subset V$ is a subspace, E is A-invariant if $AE \subset E$.
- Facts:
  - If E is A-invariant, E is A^k-invariant.
  - Thus, E is $p(A)$-invariant.
- Consider the restriction map $A|_E$.
- A has a block-diagonalized matrix where each block corresponds to the generalized eigenvectors of a generalized eigenvalue of A.
  - Let $E_1, \dots, E_r$ be a **basis of invariant subspaces**.
  - Let $A_k = A|_{E_k}$. Then the A_k's act independently of each other.
- **Generalized eigenvector** (of A): A vector $\mathbf{v}$ corresponding to an eigenvalue λ if there exists $k \geq 1$ such that $(A - \lambda I)^k \mathbf{v} = \mathbf{0}$.
- **Generalized eigenspace**: The set E_λ of all of the generalized eigenvectors of λ. *Given by*

$$E_k = \bigcup_{k \geq 1} \ker(A - \lambda I)^k$$

  - E_λ is a linear subspace of V.
- **Degree** (of λ): The smallest number k such that increasing k any more does not add further vectors to the generalized eigenspace. *Denoted by* $\boldsymbol{d(\lambda)}$.
  - Symbolically, $d(\lambda)$ is the smallest number such that

$$E_\lambda = \bigcup_{k=1}^{d(\lambda)} \ker(A - \lambda I)^k$$

- Start working through the first 25 problems of Rudin (1976) (his metric spaces problems).

10/27:
- Jordan form.
- Reviews build up to generalized eigenvectors.
- Theorem: If $\sigma(A) = \{\lambda_1, \dots, \lambda_n\}$ and $E_1, \dots, E_n$ are the corresponding generalized eigenspaces, then $E_1, \dots, E_n$ is a basis of subspaces of U, i.e., $V = \oplus_k E_k$.

- Corollary: $A : V \to V$ can be represented as $A = D + N$ where D is diagonalizable and N is nilpotent and $ND = DN$.
  - Proof: Consider the basis of generalized eigenspaces known to exist from the theorem. Then $A = \text{diag}\{A_1, \dots, A_r\}$.
  - Let
    $$N_k = A_k - \lambda_k I_{E_k}$$
    This is nilpotent.
  - Then let
    $$D = \text{diag}\{\lambda_1 I_{E_1}, \dots, \lambda_n I_{E_n}\}$$
  - These two matrices satisfy the necessary properties.
- Let $\dot{\mathbf{x}} = A\mathbf{x}$.
  - Let $\mathbf{x}(t) = \mathrm{e}^{tA}$, where
    $$\mathrm{e}^{tA} = \sum \frac{(tA)^k}{k!}$$
  - $\left\|\mathrm{e}^{tA}\right\| \leq \sum \frac{\left\|A^k\right\|}{k!} = \sum \frac{\|A\|^k}{k!}$.
  - Let p be a polynomial of degree k. Then
    $$p(a + x) = \sum_{k=0}^{d} \frac{p^{(k)}(a)}{k!} x^k$$
  - If $A = D + N$, then...
- Nilpotent operators:
  - Let $A = \text{diag}\{A_1, \dots, A_r\}$.
  - We know that $A_k = \lambda_k I_{E_k} + N_k$ for each k.
  - Every nilpotent N can be written in the form
    $$\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$$

References

Rudin, W. (1976). *Principles of mathematical analysis* (A. A. Arthur & S. L. Langman, Eds.; Third). McGraw-Hill.

Treil, S. (2017). *Linear algebra done wrong* [http://www.math.brown.edu/streil/papers/LADW/LADW_2017-09-04.pdf].