9 Sequences and Series of Functions

From Rudin (1976).

Chapter 7

12/10: **1.** Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

3. Construct sequences $\{f_n\}, \{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converge on E).

4. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2 x}$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?

5. Let

$$f_n(x) = \begin{cases} 0 & x < \frac{1}{n+1} \\ \sin^2 \frac{\pi}{x} & \frac{1}{n+1} \leq x \leq \frac{1}{n} \\ 0 & \frac{1}{n} < x \end{cases}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that absolute convergence, even for all x, does not imply uniform convergence.

6. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

7. For $n = 1, 2, 3, \ldots$ and x real, put

$$f_n(x) = \frac{x}{1+nx^2}$$

Show that $\{f_n\}$ converges uniformly to a function f and that the equation

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

is correct if $x \neq 0$ but false if $x = 0$.

8. If

$$I(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

if $\{x_n\}$ is a sequence of distinct points of (a, b), and if $\sum |c_n|$ converges, prove that the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n)$$

converges uniformly on $[a, b]$, and that f is continuous for every $x \neq x_n$.

9. Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E. Prove that

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \to x$ and $x \in E$. Is the converse of this true?

10. Letting (x) denote the fractional part of the real number x (see Exercise 4.16 for the definition), consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2}$$

defined on $\mathbb{R}$. Find all discontinuities of f, and show that they form a countable dense set. Show that f is nevertheless Riemann-integrable on every bounded interval.

11. Suppose $\{f_n\}, \{g_n\}$ are defined on E and that

(a) $\sum f_n$ has uniformly bounded partial sums;

(b) $g_n \to 0$ uniformly on E;

(c) $g_1(x) \geq g_2(x) \geq g_3(x) \geq \cdots$ for every $x \in E$.

Prove that $\sum f_n g_n$ converges uniformly on E. (Hint: Compare with Theorem 3.42.)

12. Suppose g and f_n ($n \in \mathbb{N}$) are defined on $(0, \infty)$, are Riemann-integrable on $[t, T]$ whenever $0 < t < T < \infty$, $|f_n| \leq g$, $f_n \to f$ uniformly on every compact subset of $(0, \infty)$, and

$$\int_0^{\infty} g(x)\,\mathrm{d}x < \infty$$

Prove that

$$\lim_{n\to\infty} \int_0^{\infty} f_n(x)\,\mathrm{d}x = \int_0^{\infty} f(x)\,\mathrm{d}x$$

(See Exercises 6.7-6.8 for the relevant definitions.) This is a rather weak form of Lebesgue's dominated convergence theorem (Theorem 11.32). Even in the context of the Riemann integral, uniform convergence can be replaced by pointwise convergence if it is assumed that $f \in \mathscr{R}$. (See Cunningham (1967) and Kestelman (1970).)

13. Assume that $\{f_n\}$ is a sequence of monotonically increasing functions on $\mathbb{R}^1$ with $0 \leq f_n(x) \leq 1$ for all x and all n.

(a) Prove that there is a function f and a sequence $\{n_k\}$ such that

$$f(x) = \lim_{k\to\infty} f_{n_k}(x)$$

for every $x \in \mathbb{R}^1$. The existence of such a pointwise convergent subsequence is usually called **Helly's selection theorem**. (Hint: (i) Some subsequence $\{f_{n_i}\}$ converges at all rational points r, say, to $f(r)$. (ii) Define $f(x)$ for any $x \in \mathbb{R}^1$ to be $\sup f(r)$, the sup being taken over all $r \leq x$. (iii) Show that $f_{n_i}(x) \to f(x)$ at every x at which f is continuous. [This is where monotonicity is strongly used.] (iv) A subsequence of $\{f_{n_i}\}$ converges at every point of discontinuity of f since there are at most countably many such points.)

(b) If, moreover, f is continuous, prove that $f_{n_k} \to f$ uniformly on compact sets. (Hint: Modify your proof of (iii) appropriately.)

14. Let f be a continuous real function on $\mathbb{R}^1$ with the following properties: $0 \leq f(t) \leq 1$, $f(t+2) = f(t)$ for every t, and

$$f(t) = \begin{cases} 0 & 0 \leq t \leq \frac{1}{3} \\ 1 & \frac{2}{3} \leq t \leq 1 \end{cases}$$

Put $\Phi(t) = (x(t), y(t))$, where

$$x(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1}t) \qquad y(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n}t)$$

Prove that Φ is continuous and that Φ maps $I = [0,1]$ onto the unit square $I^2 \subset \mathbb{R}^2$. In fact, show that Φ maps the Cantor set onto I^2. (Hint: Each $(x_0, y_0) \in I^2$ has the form

$$x_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n-1} \qquad\qquad y_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n}$$

where each a_i is 0 or 1. If

$$t_0 = \sum_{i=1}^{\infty} 3^{-i-1}(2a_i)$$

show that $f(3^k t_0) = a_k$, and hence that $x(t_0) = x_0$, $y_0(t_0) = y_0$.) This simple example of a so-called **space-filling curve** is due to Schoenberg (1938).

15. Suppose f is a real continuous function on $\mathbb{R}^1$, $f_n(t) = f(nt)$ for $n \in \mathbb{N}$, and $\{f_n\}$ is equicontinuous on $[0,1]$. What conclusion can you draw about f?

16. Suppose $\{f_n\}$ is an equicontinuous sequence of functions on a compact set K, and $\{f_n\}$ converges pointwise on K. Prove that $\{f_n\}$ converges uniformly on K.

18. Let $\{f_n\}$ be a uniformly bounded sequence of functions which are Riemann-integrable on $[a,b]$ and put

$$F_n(x) = \int_a^x f_n(t)\,\mathrm{d}t$$

for $x \in [a,b]$. Prove that there exists a subsequence $\{F_{n_k}\}$ which converges uniformly on $[a,b]$.

19. Let K be a compact metric space, let S be a subset of $\mathscr{C}(K)$. Prove that S is compact (with respect to the metric defined in Section 7.14) if and only if S is uniformly closed, pointwise bounded, and equicontinuous. (If S is not equicontinuous, then S contains a sequence which has no equicontinuous subsequence, hence has no subsequence that converges uniformly on K.)

20. If f is continuous on $[0,1]$ and if

$$\int_0^1 f(x)x^n\,\mathrm{d}x = 0$$

for $n = 0, 1, 2, \ldots$, prove that $f(x) = 0$ on $[0,1]$. (Hint: The integral of the product of f with any polynomial is zero. Use the Weierstrass theorem to show that $\int_0^1 f^2(x)\,\mathrm{d}x = 0$.)

22. Assume $f \in \mathscr{R}(\alpha)$ on $[a,b]$, and prove that there are polynomials P_n such that

$$\lim_{n\to\infty} \int_a^b |f - P_n|^2\,\mathrm{d}\alpha = 0$$

(Compare with Exercise 6.12.)

23. Put $P_0 = 0$, and define for $n = 0, 1, 2, \ldots$

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2}$$

Prove that

$$\lim_{n\to\infty} P_n(x) = |x|$$

uniformly on $[-1,1]$. This makes it possible to prove the Stone-Weierstrass theorem without first proving Theorem 7.26. (Hint: Use the identity

$$|x| - P_{n+1}(x) = [|x| - P_n(x)]\left[1 - \frac{|x| + P_n(x)}{2}\right]$$

to prove that $0 \le P_n(x) \le P_{n+1}(x) \le |x|$ if $|x| \le 1$ and that

$$|x| - P_n(x) \le |x| \left(1 - \frac{|x|}{2}\right)^n < \frac{2}{n+1}$$

if $|x| < 1$.)

25. Suppose ϕ is a continuous bounded real function in the strip defined by $0 \le x \le 1$, $-\infty < y < \infty$. Prove that the initial-value problem

$$y' = \phi(x, y) \qquad\qquad y(0) = c$$

has a solution. Note that the hypotheses of this existence theorem are less stringent than those of the corresponding uniqueness theorem; see Exercise 5.27. (Hint: Fix n. For $i = 0, \dots, n$, put $x_i = i/n$. Let f_n be a continuous function on $[0, 1]$ such that $f_n(0) = c$, let

$$f_n'(t) = \phi(x_i, f_n(x_i))$$

if $x_i < t < x_{i+1}$, and put

$$\Delta_n(t) = f_n'(t) - \phi(t, f_n(t))$$

except at points x_i, where $\Delta_n(t) = 0$. Then

$$f_n(x) = c + \int_0^x [\phi(t, f_n(t)) + \Delta_n(t)] \, \mathrm{d}t$$

Choose $M < \infty$ so that $|\phi| \le M$. Verify the following assertions.

(a) $|f_n'| \le M$, $|\Delta_n| \le 2M$, $\Delta_n \in \mathscr{R}$, and $|f_n| \le |c| + M = M_1$ say, on $[0, 1]$, for all n.

(b) $\{f_n\}$ is equicontinuous on $[0, 1]$ since $|f_n'| \le M$.

(c) Some $\{f_{n_k}\}$ converges to some f, uniformly on $[0, 1]$.

(d) Since ϕ is uniformly continuous on the rectangle $0 \le x \le 1$, $|y| \le M_1$,

$$\phi(t, f_{n_k}(t)) \to \phi(t, f(t))$$

uniformly on $[0, 1]$.

(e) $\Delta_n(t) \to 0$ uniformly on $[0, 1]$ since

$$\Delta_n(t) = \phi(x_i, f_n(x_i)) - \phi(t, f_n(t))$$

in (x_i, x_{i+1}).

(f) Hence

$$f(x) = c + \int_0^x \phi(t, f(t)) \, \mathrm{d}t$$

This f is a solution of the given problem.)

26. Prove an analogous existence theorem for the initial-value problem

$$\mathbf{y}' = \mathbf{\Phi}(\mathbf{x}, \mathbf{y}) \qquad\qquad \mathbf{y}(0) = \mathbf{c}$$

where now $\mathbf{c} \in \mathbb{R}^k$, $\mathbf{y} \in \mathbb{R}^k$, and $\mathbf{\Phi}$ is a continuous bounded mapping of the part of $\mathbb{R}^{k+1}$ defined by $0 \le x \le 1$, $\mathbf{y} \in \mathbb{R}^k$ into $\mathbb{R}^k$. Compare Exercise 5.28. (Hint: Use the vector-valued version of Theorem 7.25.)