

# MATH 20700 (Honors Analysis in $\mathbb{R}^n$ I) Notes

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Part I

# Linear Algebra

# Chapter 1

## Basic Notions

- 9/27:
- **Vector space:** Basically, a set for which you have an addition and multiplication.
  - $\mathbb{F}^d$  is used for  $\mathbb{R}^d$  or  $\mathbb{C}^d$  in Treil (2017).
  - $\mathbb{P}_n$  is the vector space of polynomials up to degree  $n$ .
  - $C([0, 1])$  is the set of continuous functions defined on  $[0, 1]$ , an infinite-dimensional vector space.
  - **Generating set:** A subset of a vector space, all linear combinations of which generate the vector space. *Also known as spanning set.*
    - Any element of VS is a linear comb. of elements of the generating set.
  - **Linearly independent (list):** A list of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$  such that  $\sum_{i=1}^k \alpha_i \mathbf{v}_i = 0$  implies  $\alpha_i = 0$  for all  $i$ .
  - **Base:** A generating set consisting of linearly independent vectors.
  - Any element of a VS can be written as a *unique* linear combination of the vectors in a base.
    - If  $\mathbf{x} = \sum_{i=1}^k \alpha_i \mathbf{v}_i = \sum_{i=1}^k \beta_i \mathbf{v}_i$ , then  $\alpha_i = \beta_i$  for all  $i$ .
  - **Linear transformation:** A function  $T : X \rightarrow Y$ , where  $X, Y$  are VSs, such that

$$T(\alpha \mathbf{x} + \beta \mathbf{y}) = \alpha T\mathbf{x} + \beta T\mathbf{y}$$

for all  $\mathbf{x} \in X, \mathbf{y} \in Y$ .

- Examples of linear transformations:
  - Consider  $\mathbb{P}_n$ . Let  $Tp_n = p'_n$ . This  $T$  is linear.
  - Rotation in  $\mathbb{R}^d$ .
    - Think graphically about two vectors  $\mathbf{x}, \mathbf{y}$ .
    - Rotating and summing them is the same as summing and rotating. Same for scaling.
    - Thus, rotation is actually linear!
  - Reflection as well.
- Consider  $T : \mathbb{R} \rightarrow \mathbb{R}$ .
  - Any linear map on the line is a line.
  - We must have  $Tx = \alpha x$ :  $Tx = T(1x) = xT(1) = x\alpha$ .
- Consider  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  linear.

- Any linear map between  $\mathbb{R}^n$  and  $\mathbb{R}^m$  is linear.
- Thus,  $T(\mathbf{x}) = A\mathbf{x}$  for all  $\mathbf{x} \in \mathbb{R}^n$ , where  $A$  is an  $m \times n$  matrix.
- To find  $A$ , do the same calculation as for  $T\mathbf{x} = \alpha\mathbf{x}$  but more carefully:
  - Let  $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  be a basis.
  - So  $\mathbf{x} = \sum_{i=1}^n \alpha_i \mathbf{e}_i$ .
  - Thus,  $T\mathbf{x} = \sum_{i=1}^n \alpha_i T(\mathbf{e}_i)$ .
  - Each  $T(\mathbf{e}_i)$  is part of the matrix that we multiply by the column vector representing  $\mathbf{x}$ .
- Multiplication of matrices is equivalent to composition of linear maps.
- Consider  $T_1 : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $T_2 : \mathbb{R}^m \rightarrow \mathbb{R}^r$ .
  - $T_2 \circ T_1$  is equivalent to  $BA$ , if  $A$  represents  $T_1$  and  $B$  represents  $T_2$ . In other words,  $(T_2 \circ T_1)(\mathbf{x}) = BA\mathbf{x}$  for all  $\mathbf{x}$ .
- Recall that if  $A = (\alpha_{ij})$  and  $B = (\beta_{ij})$ , then  $(BA)_{ij} = (\sum \beta_{ik} \alpha_{kj})$ .
- Properties of multiplication:

$$\begin{aligned}(AB)C &= A(BC) \\ A(B+C) &= AB+AC \\ (A+B)C &= AC+BC\end{aligned}$$

- However, it is not true in general that  $AB = BA$ .
- **Trace** (of an  $n \times n$  matrix  $A$ ): The sum of the diagonal entries of  $A$ . Denoted by **trace** ( $A$ ). Given by

$$\text{trace}(A) = \sum \alpha_{ii}$$

- It is true that  $\text{trace}(AB) = \text{trace}(BA)$ .
  - Indeed, on the diagonals, multiplication is commutative; it's the other terms that mess you up in general.
- Invertibility of matrices.
  - In general, matrices are not invertible: Not every system of equations is solveable;  $Ax = b$  does not always have a solution  $x = A^{-1}b$ .
- $C$  is the inverse from the left:  $CA = I$ .  $B$  is the inverse from the right:  $AB = I$ . A matrix can have a left and a right inverse and still not be invertible. A matrix is invertible iff  $C = B$ .
- Any time we write “inverse,” we do so under the assumption that it exists.
- $(AB)^{-1} = B^{-1}A^{-1}$  — easy proof by multiplication.
- If  $A = (a_{ij})$ ,  $A^T = (a_{ji})$ .
  - $(A^{-1})^T = (A^T)^{-1}$ .
  - $(AB)^T = B^T A^T$ .
- Let  $X, Y$  VS.
  - $X \cong Y^{[1]}$  if there exists a linear  $T : X \rightarrow Y$  that is one-to-one and onto.
  - Check:  $A(\text{basis of } X) = \text{basis of } Y$ . Prove by definition and expression of elements as linear combinations.
- **Subspace**: A subset of a vector space which happens to be a vector space, itself.

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<sup>1</sup>“ $X$  is isomorphic to  $Y$ .”

# References

Treil, S. (2017). *Linear algebra done wrong* [[http://www.math.brown.edu/streil/papers/LADW/LADW\\_2017-09-04.pdf](http://www.math.brown.edu/streil/papers/LADW/LADW_2017-09-04.pdf)].