

Chapter 5

Inner Product Spaces

10/6:

- We define

$$\ell^2(\mathbb{R}) = \left\{ \{a_n\}_{n \geq 1} \subset \mathbb{R} : \sum_1^\infty |a_n|^2 < \infty \right\}$$

- **Inner product:** A map $V \times V \rightarrow \mathbb{F}$ that takes $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} \cdot \mathbf{y}$. Denoted by $\cdot, (\cdot, \cdot), \langle \cdot, \cdot \rangle$.

- Properties of the inner product:

- $(\mathbf{x}, \mathbf{y}) = \overline{(\mathbf{y}, \mathbf{x})}$ (symmetry).
- $(\alpha \mathbf{x} + \beta \mathbf{y}, \mathbf{z}) = \alpha(\mathbf{x}, \mathbf{z}) + \beta(\mathbf{y}, \mathbf{z})$ (linearity).
- $(\mathbf{x}, \mathbf{x}) \geq 0$.
- $(\mathbf{x}, \mathbf{x}) = 0$ iff $\mathbf{x} = 0$.

- If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n x_i y_i$$

- If $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, then

$$(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n x_i \bar{y}_i$$

- If $f, g \in \mathbb{P}_n(t)$, then

$$(f, g) = \int_{-1}^1 f \bar{g} dt$$

- The conjugate of a polynomial is the polynomial with the conjugate of the coefficients of the original polynomial. Symbolically, if $f = \sum_{i=0}^n \alpha_i t^i$ is a polynomial, then $\bar{f} = \sum_{i=0}^n \bar{\alpha}_i t^i$.

- It is a fact that

$$\left| \sum_{n=1}^{\infty} a_n \bar{b}_n \right| \leq \|(a_n)_{n \geq 1}\| \|(b_n)_{n \geq 1}\|$$

- Suppose we want to define the inner product between two matrices.

- A common one is

$$(A, B) = \text{tr}(B^* A)$$

where $B^* = \bar{B}^T = \overline{B^T}$ is the conjugate transpose.

- We define the norm as a function $V \rightarrow [0, \infty)$ given by

$$\|\mathbf{x}\| = \sqrt{(\mathbf{x}, \mathbf{x})}$$

- Properties of the norm.

- $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$.
- $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.
- $\|\mathbf{x}\| = 0$ iff $\mathbf{x} = 0$.

- In \mathbb{R}^n ,

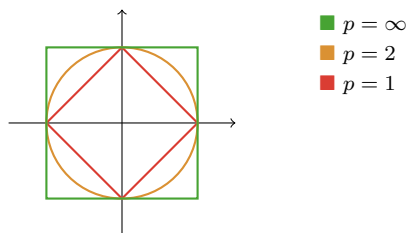


Figure 5.1: The unit ball of norms corresponding to $p = 1, 2, \infty$.

- The standard norm is

$$\|\mathbf{x}\| = \sqrt{\sum |x_i|^2}$$

- We can also define

$$\|\mathbf{x}\|_p = \sqrt[p]{\sum |x_i|^p}$$

- We can even define

$$\|\mathbf{x}\|_\infty = \max |x_i|$$

- And we can prove that all of these are valid norms.
- Only the norm corresponding to ℓ^2 is given by an inner product, but all the other quantities are still norms as defined by the properties (see Treil (2017)).
- Figure 5.1 shows the unit ball of each norm, i.e., the set of all points which have norm 1.

- The parallelogram rule:

$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)$$

- Orthogonality: Given \mathbf{v}, \mathbf{w} , if $\mathbf{v} \perp \mathbf{w}$, then $(\mathbf{v}, \mathbf{w}) = 0$.

- In particular, if $\mathbf{v} \perp \mathbf{w}$, then

$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2$$

- Let E be a subspace of V . If $\mathbf{v} \perp E$, then $\mathbf{v} \perp \mathbf{e}$ for all $\mathbf{e} \in E$, i.e., $\mathbf{v} \perp$ a set of vectors spanning E .
- Any set of orthogonal vectors is linearly independent. Thus, if V is n dimensional, then $\mathbf{v}_1, \dots, \mathbf{v}_n$ orthogonal is a basis.
- Let E be a subspace of V . Take $\mathbf{v} \in V$. We want to define the projection $P_E \mathbf{v}$ of \mathbf{v} onto E .
 - We have that $P_E \mathbf{v} \in E$ and $\mathbf{v} - P_E \mathbf{v} \perp E$.
 - Additionally, we have that

$$\|\mathbf{v} - P_E \mathbf{v}\| \leq \|\mathbf{v} - \mathbf{e}\|$$

for all $\mathbf{e} \in E$.

- Lastly, we have that $P_E \mathbf{v}$ is unique.
- If we receive a basis of a vector space, how do we create out of that a basis that is orthogonal? The process of doing this is called **Gram-Schmidt orthogonalization**.
 - We keep \mathbf{v}_1 , subtract $P_{\mathbf{v}_1} \mathbf{v}_2$ from \mathbf{v}_2 , subtract $P_{\{\mathbf{v}_1, \mathbf{v}_2\}} \mathbf{v}_3$ from \mathbf{v}_3 , and on and on.
- If we are given a set of orthogonal vectors, we can normalize them by dividing each by its norm. This creates an orthonormal list. The standard basis is orthonormal.
- Let

$$E^\perp = \{v \in V : v \perp E\}$$

- It follows that $V = E \oplus E^\perp$.
- How close can we come to solving $A\mathbf{x} = \mathbf{b}$ if we cannot solve it exactly (i.e., if the columns are not linearly independent)?
 - Let A be an $m \times n$ matrix, and let $\mathbf{b} \in \mathbb{R}^m$.
 - Then the best solution is given by minimizing $\|A\mathbf{x} - \mathbf{b}\|$. We minimize this with projections. A special case of this is least squares regression! More details in Treil (2017).

10/8:

- Soug is gonna send us a hefty amount of reading for the weekend.
- Least square approximation:
 - If we want to minimize $\|A\mathbf{x} - \mathbf{b}\|$, the best we can do is project \mathbf{b} onto the range of A .
 - Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be an orthogonal basis of range A .
 - Then

$$\text{Proj}_{\text{range } A} \mathbf{b} = \sum_{k=1}^k \frac{(\mathbf{b}, \mathbf{v}_k)}{\|\mathbf{v}_k\|^2} \mathbf{v}_k$$

- Matrix equation form:

$$\text{Projection}_{\text{range } A} = A(A^*A)^{-1}A^*$$

if A^*A is invertible, where $A^* = \bar{A}^T$.

■ Soug never uses this though.

- The minimum is found when $\mathbf{b} - A\mathbf{x} \perp \text{range } A$. Implies that $\mathbf{b} - A\mathbf{x} \perp \mathbf{a}_k$ for all k . Implies $(\mathbf{b} - A\mathbf{x}, \mathbf{a}_k) = \bar{\mathbf{a}}_k^T (\mathbf{b} - A\mathbf{x}) = 0$.
- Note that we're letting $\bar{\mathbf{a}}_k^T$ be the row vector

$$\bar{\mathbf{a}}_k^T = (\bar{a}_{1,k} \quad \cdots \quad \bar{a}_{n,k})$$

- We also have $\bar{A}^T (\mathbf{b} - A\mathbf{x}) = 0$, from which it follows that $A^*A\mathbf{x} = A^*\mathbf{b}$, so $\mathbf{x} = (A^*A)^{-1}A^*\mathbf{b}$. Thus, $\text{Proj}_{\text{range } A} = Ax$, so $\text{Proj}_{\text{range } A} = A(A^*A)^{-1}A^*\mathbf{b}$.

- Adjoint of a linear map $T : V \rightarrow W$ is the A^* discussed above.
 - First, we'll do this for matrices. And then we'll do it for any finite-dimensional vector space.
 - Let A be an $m \times n$ matrix. We claim then that

$$(A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, A^*\mathbf{y})$$

for all $\mathbf{x} \in \mathbb{C}^n, \mathbf{y} \in \mathbb{C}^m$. Proof:

$$\begin{aligned} (A\mathbf{x}, \mathbf{y}) &= \bar{\mathbf{y}}^T A\mathbf{x} \\ &= \mathbf{y}^* A\mathbf{x} \\ &= (A^*\mathbf{y})^* \mathbf{x} \\ &= (\mathbf{x}, A^*\mathbf{y}) \end{aligned}$$

- Properties of the adjoint:

$$(AB)^T = B^T A^T$$

$$(AB)^* = B^* A^*$$

$$(A^*)^* = A$$

- A^* is the unique matrix B such that $(A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, B\mathbf{y})$.
- Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be a basis of V , and let $\mathbf{w}_1, \dots, \mathbf{w}_m$ be a basis of W .
- Definition of A^* : If $(A\mathbf{x}, \mathbf{y}) = (y, A^*\mathbf{x})$ for all $\mathbf{x} \in V$ and $\mathbf{y} \in W$.
- But it's not enough to define something; we have to check that it exists.
- If $[A]_{AB}$, then $[A^*]_{AB}$.
- More properties (give criteria for solving systems of equations):

$$\ker A^* = (\text{range } A)^\perp$$

$$\ker A = (\text{range } A^*)^\perp$$

$$\text{range } A = (\ker A^*)^\perp$$

$$\text{range } A^* = (\ker A)^\perp$$

■ Soug proves these.

• Isometries and unitary operators.

- $U : X \rightarrow Y$ is an isometry if $\|\mathbf{x}\| = \|U\mathbf{x}\|$ for all $\mathbf{x} \in X$. It is an isometry because it preserves the distance between points.
- It immediately follows that $\|\mathbf{x}_1 - \mathbf{x}_2\| = \|U\mathbf{x}_1 - U\mathbf{x}_2\| = \|U(\mathbf{x}_1 - \mathbf{x}_2)\|$.
- This definition is equivalent to an inner product one: $(\mathbf{x}, \mathbf{y}) = (U\mathbf{x}, U\mathbf{y})$. This follows from the definition of the norm.
- We have

$$(\mathbf{a}, \mathbf{b}) = \frac{1}{4} \sum_{\alpha=\pm 1, \pm i} \alpha \|\mathbf{a} + \alpha \mathbf{b}\|^2$$

■ $(a+b)^2 - (a-b)^2 = 4ab$ for any $a, b \in \mathbb{R}$, so $ab = \frac{1}{4}[(a+b)^2 - (a-b)^2]$. Thus, in a real inner product space,

$$(\mathbf{a}, \mathbf{b}) = \frac{1}{4} \left(\|\mathbf{a} + \mathbf{b}\|^2 - \|\mathbf{a} - \mathbf{b}\|^2 \right)$$

■ It follows that isometries preserve inner products.

- U is an isometry if and only if $U^*U = I$. Proof:

$$(\mathbf{x}, \mathbf{x}) = (U\mathbf{x}, U\mathbf{y}) = (U^*U\mathbf{x}, \mathbf{x})$$

$$(\mathbf{x}, \mathbf{y}) = (U\mathbf{x}, U\mathbf{y}) = (U^*U\mathbf{x}, \mathbf{y})$$

$$(\mathbf{x}, \mathbf{y}) = (U\mathbf{x}, U\mathbf{y}) = (\mathbf{x}, \mathbf{y})$$

for all \mathbf{y} .

- An isometry is unitary if it is invertible.

■ Thus, $U : X \rightarrow Y$ an isometry is unitary iff $\dim X = \dim Y$.

- Note that it follows that $U^* = U^{-1}$ for U an isometry.
- U unitary implies $|\det U| = 1$, so λ an eigenvalue of U implies that $|\lambda| = 1$.
- A is diagonalizable iff it has an orthogonal basis of eigenvectors.