Chapter 3

Numerical Sequences and Series

3.1 Notes

11/8: • Any bounded sequence in $\mathbb{R}^k$ has a convergent subsequence.

11/10: • Read and understand the section about Cauchy sequences converging and the sup/inf.

3.2 Chapter 3: Numerical Sequences and Series

From Rudin (1976).

11/7: • Convergence of sequences is relative.

  - For example, the sequence $1/n$ for $n = 1, 2, \dots$ converges in $\mathbb{R}$, but not in $(0, \infty)$.

- **Range** (of $\{p_n\}$): The set of all points p_n.
  - This definition squares nicely with the formal definition of a sequence as a function p defined on $\mathbb{N}$.
- Theorem 3.2: $\{p_n\} \subset X$ a metric space implies
  (a) $\{p_n\}$ converges to $p \in X$ iff every $N_r(p)$ contains all but finitely many p_n.
  (b) $p, p' \in X$, $p_n \to p$, and $p_n \to p'$ implies $p = p'$.
  (c) $\{p_n\}$ converges implies $\{p_n\}$ is bounded.
  (d) $E \subset X$ and p a limit point of E implies there exists $\{p_n\} \subset E$ such that $p = \lim_{n\to\infty} p_n$.
- Theorem 3.3: Let $\{s_n\}, \{t_n\} \subset \mathbb{C}$, $\lim_{n\to\infty} s_n = s$, and $\lim_{n\to\infty} t_n = t$. Then
  (a) $\lim_{n\to\infty}(s_n + t_n) = s + t$.
  (b) $\lim_{n\to\infty} cs_n = cs$, $\lim_{n\to\infty}(c + s_n) = c + s$ for any $c \in \mathbb{C}$.
  (c) $\lim_{n\to\infty} s_n t_n = st$.
  (d) $\lim_{n\to\infty} 1/s_n = 1/s$, provided $s_n \neq 0$ $(n \in \mathbb{N})$ and $s \neq 0$.
- Theorem 3.4:
  (a) $\{\mathbf{x}_n\} \subset \mathbb{R}^k$ and $\mathbf{x}_n = (\alpha_{1,n}, \dots, \alpha_{k,n})$ $(n \in \mathbb{N})$ implies $\mathbf{x}_n \to \mathbf{x} = (\alpha_1, \dots, \alpha_k)$ iff $\lim_{n\to\infty} \alpha_{j,n} = \alpha_j$ for each $1 \leq j \leq k$.
  (b) $\{\mathbf{x}_n\}, \{\mathbf{y}_n\} \subset \mathbb{R}^k$, $\{\beta_n\} \subset \mathbb{R}$, and $\mathbf{x}_n \to \mathbf{x}$, $\mathbf{y}_n \to \mathbf{y}$, $\beta_n \to \beta$ imply

$$\lim_{n\to\infty}(\mathbf{x}_n + \mathbf{y}_n) = \mathbf{x} + \mathbf{y} \qquad \lim_{n\to\infty} \mathbf{x}_n \cdot \mathbf{y}_n = \mathbf{x} \cdot \mathbf{y} \qquad \lim_{n\to\infty} \beta_n \mathbf{x}_n = \beta \mathbf{x}$$

- Theorem 3.6:
  (a) $\{p_n\} \subset X$ compact implies some subsequence of $\{p_n\}$ converges to a point of X.
  (b) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.
- Theorem 3.7: The subsequential limits of $\{p_n\} \subset X$ form a closed subset of X.
- **Diameter** (of E): The supremum of the set
$$S = \{d(p,q) : p, q \in E\}$$
where E is a nonempty subset of a metric space X. *Denoted by* $\textbf{diam}\,\boldsymbol{E}$.
- Theorem 3.10:
  (a) $E \subset X$ implies
$$\operatorname{diam} \bar{E} = \operatorname{diam} E$$
  (b) $\{K_n\} \subset X$ a decreasing sequence of compact sets and $\lim_{n\to\infty} \operatorname{diam} K_n = 0$ imply $\bigcap_1^\infty K_n$ consists of exactly one point.
- Theorem 3.11:
  (a) Every convergent sequence in X a metric space is Cauchy.
  (b) $\{p_n\} \subset X$ ($\{p_n\}$ Cauchy, X compact) implies $\{p_n\}$ converges to some point of X.
  (c) Every Cauchy sequence converges in $\mathbb{R}^k$.
- **Complete** (metric space): A metric space in which every Cauchy sequence converges.
- Rephrasing Theorem 3.11b-c: All compact metric spaces and all Euclidean spaces are complete.
  - The metric space $(\mathbb{Q}, |x-y|)$ is not complete.
- **Monotonically increasing** (sequence $\{s_n\}$): A sequence $\{s_n\}$ of real numbers such that $s_n \leq s_{n+1}$ for each $n \in \mathbb{N}$.
- **Monotonically decreasing** (sequence $\{s_n\}$): A sequence $\{s_n\}$ of real numbers such that $s_n \geq s_{n+1}$ for each $n \in \mathbb{N}$.
- **Monotonic sequences**: The class of all sequences that are either monotonically increasing or monotonically decreasing.
- Theorem 3.14: $\{s_n\}$ monotonic converges iff it is bounded.
- **Upper limit** (of $\{s_n\}$): The supremum of the set E of all subsequential limits of $\{s_n\}$. *Denoted by* $\boldsymbol{s^*}$, $\boldsymbol{\limsup_{n\to\infty} s_n}$.
- **Lower limit** (of $\{s_n\}$): The infimum of the set E of all subsequential limits of $\{s_n\}$. *Denoted by* $\boldsymbol{s_*}$, $\boldsymbol{\liminf_{n\to\infty} s_n}$.
- Theorem 3.17: $\{s_n\} \subset \mathbb{R}$ implies s^* has (and is the only number to have both of) the following two properties.
  (a) $s^* \in E$.
  (b) If $x > s^*$, then there is an integer N such that $n \geq N$ implies $s_n < x$.

  An analogous result holds for s_*.
- Theorem 3.19: $s_n \leq t_n$ for all $n \geq N$ implies
$$\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n \qquad \limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n$$

- Theorem 3.20:
  (a) $p > 0$ implies $\lim_{n\to\infty} 1/n^p = 0$.
  (b) $p > 0$ implies $\lim_{n\to\infty} \sqrt[n]{p} = 1$.
  (c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.
  (d) $p > 0$, $\alpha \in \mathbb{R}$ implies $\lim_{n\to\infty} n^\alpha/(1+p)^n = 0$.
  (e) $|x| < 1$ implies $\lim_{n\to\infty} x^n = 0$.

11/8:

- Series are defined in terms of sequences. Moreover, sequences can be defined in terms of series: Let $a_1 = s_1$, $a_n = s_n - s_{n-1}$ ($n \in \mathbb{N}+1$). Thus, every theorem about sequences can be stated in terms of series and vice versa, but it is nevertheless useful to consider both concepts (Rudin, 1976, p. 59).
- Theorem 3.22: $\sum a_n$ converges iff for every $\epsilon > 0$, there is an N such that $m \geq n \geq N$ implies

$$\left|\sum_{k=n}^{m} a_k\right| \leq \epsilon$$

  - Analogous to Theorem 3.11.
- Theorem 3.23: $\sum a_n$ converges implies $\lim_{n\to\infty} a_n = 0$.
- Theorem 3.24: $\{a_n\} \subset \mathbb{R}$ such that $a_n \geq 0$ ($n \in \mathbb{N}$) implies $\sum a_n$ converges iff its partial sums form a bounded sequence.
- Theorem 3.25 (Comparison test):
  (a) $|a_n| \leq c_n$ for all $n \geq N_0$ and $\sum c_n$ converges implies $\sum a_n$ converges.
  (b) $a_n \geq d_n \geq 0$ for all $n \geq N_0$ and $\sum d_n$ diverges implies $\sum a_n$ diverges.
- Theorem 3.26: $0 \leq x < 1$ implies

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

  $x \geq 1$ implies the series diverges.
- Theorem 3.27: $\{a_n\}$ a monotonically decreasing sequence of nonnegative terms implies the series $\sum_{n=1}^{\infty} a_n$ converges iff the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$$

  converges.
- Theorem 3.29: $p > 1$ implies

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$$

  converges; $p \leq 1$ implies the series diverges.
  - Note that $\log n = \ln n$.
  - Note that we sum from $n = 2$ since $\log 1 = 0$.
- **e**: The number

$$\mathrm{e} = \sum_{n=0}^{\infty} \frac{1}{n!}$$

- Theorem 3.31: $\lim_{n\to\infty}(1 + 1/n)^n = \mathrm{e}$.

- Theorem 3.32: e is irrational.
- Theorem 3.33 (Root test): Given $\sum a_n$, put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then
  (a) $\alpha < 1$ implies $\sum a_n$ converges.
  (b) $\alpha > 1$ implies $\sum a_n$ diverges.
  (c) $\alpha = 1$ implies nothing; the test is inconclusive.
- Theorem 3.34 (Ratio test): The series $\sum a_n \ldots$
  (a) converges if $\limsup_{n\to\infty} |a_{n+1}/a_n| < 1$;
  (b) diverges if $|a_{n+1}/a_n| \geq 1$ for all $n \geq N_0$.
- Theorem 3.37: $\{c_n\} \subset \mathbb{R}^+$ implies

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} \sqrt[n]{c_n} \qquad \limsup_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$$

- Theorem 3.39: Given the power series $\sum c_n z^n$, put

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|} \qquad R = \frac{1}{\alpha}$$

  (If $\alpha = 0$, let $R = +\infty$; if $\alpha = +\infty$, let $R = 0$.) Then $\sum c_n z^n$ converges if $|z| < R$ and diverges if $|z| > R$.
- **Radius of convergence** (of a power series): The number R defined by Theorem 3.39.
- Theorem 3.41 (partial summation formula): Given two sequence $\{a_n\}, \{b_n\}$, put

$$A_n = \begin{cases} \sum_{k=0}^{n} a_k & n \geq 0 \\ 0 & n = -1 \end{cases}$$

  Then if $0 \leq p \leq q$, we have

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

- Theorem 3.42: If the partial sums A_n of $\sum a_n$ form a bounded sequence and $\{b_n\}$ is a monotonically decreasing sequence such that $b_n \to 0$, then $\sum a_n b_n$ converges.
- Theorem 3.43: If $\{c_n\}$ is an alternating series that is absolutely monotonically decreasing such that $c_n \to 0$, then $\sum c_n$ converges.
- Theorem 3.44: If the radius of convergence of $\sum c_n z^n$ is 1, $\{c_n\}$ is monotonically decreasing, and $c_n \to 0$, then $\sum c_n z^n$ converges at every point on the circle $|z| = 1$ except possibly at $z = 1$.
- Theorem 3.45: $\sum a_n$ converges absolutely implies $\sum a_n$ converges.
- Theorem 3.47: $\sum a_n = A$, $\sum b_n = B$, $c \in \mathbb{R}$ implies $\sum (a_n + b_n) = A + B$ and $\sum c a_n = cA$.
- **Product** (of $\sum a_n, \sum b_n$): The series $\sum c_n$ defined by

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

  for each $n = 0, 1, 2, \ldots$.

- We motivate this definition by noting that if $\sum c_n$ is the product of $\sum a_n, \sum b_n$, then

$$\sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} c_n z^n$$

- Setting $z = 1$ then yields the given definition.

- The product of two convergent series may diverge. However...
- Theorem 3.50: Suppose (a) $\sum_{n=0}^{\infty} a_n$ converges absolutely, (b) $\sum_{n=0}^{\infty} a_n = A$, (c) $\sum_{n=0}^{\infty} b_n = B$, and (d) $\sum_{n=0}^{\infty} c_n$ is the product of $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$. Then

$$\sum_{n=0}^{\infty} c_n = AB$$

- Theorem 3.51: If $\sum a_n, \sum b_n, \sum c_n$ converge to A, B, C, respectively, and $\sum c_n$ is the product of $\sum a_n, \sum b_n$, then $C = AB$.
- **Rearrangement** (of $\sum a_n$): A series $\sum a'_n$ defined by $a'_n = a_{k_n}$ for each $n \in \mathbb{N}$, where $\{k_n\}$ is a sequence in which every positive integer appears once and only once (that is, $\{k_n\}$ is a 1-1 function from $\mathbb{N}$ onto $\mathbb{N}$).
- Theorem 3.54: Let $\sum a_n$ be a series of real number which converges, but not absolutely. Suppose $-\infty \leq \alpha \leq \beta \leq \infty$. Then there exists a rearrangement $\sum a'_n$ with partial sums s'_n such that

$$\liminf_{n\to\infty} s'_n = \alpha \qquad\qquad \limsup_{n\to\infty} s'_n = \beta$$

- Theorem 3.55: If $\sum a_n$ is a series of complex numbers which converges absolutely, then every rearrangement of $\sum a_n$ converges, and they all converge to the same sum.