8 Algebras

3/3: **8.1.** Let $T_i \in \mathrm{End}_A(M_i)$ for $i = 1, 2$, and let M_1, M_2 be $A[X]$-modules for an arbitrary ring A. Let $S : M_1 \to M_2$ be a function.

(i) Prove that S is an $A[X]$-module homomorphism iff…
  (a) S is an A-module homomorphism;
  (b) $T_2 S = S T_1$.

(ii) Prove that S is an $A[X]$-module isomorphism iff…
  (a) S is an A-module isomorphism;
  (b) $T_2 = S T_1 S^{-1}$.

8.2. Consider (M, T) with $M = A^n$ and $T(a_1, \ldots, a_n) = (0, a_1, \ldots, a_{n-1})$. Prove that the corresponding $A[X]$-module is isomorphic to $A[X]/(X^n)$.

8.3. Let V be a finite dimensional vector space and $T : V \to V$ be a linear transformation. Consider the pair (V, T). Why is V a finitely generated torsion $F[X]$-module?

8.4. $T : V \to V$ is diagonalizable if there is a basis $e_1, \ldots, e_n$ of V consisting of eigenvectors of T, i.e., $Te_i = a_i e_i$ for some $a_i \in F$.

(i) What is the minimal polynomial of T?

(ii) What condition on $a_1, \ldots, a_n$ is necessary and sufficient for the existence of a cyclic vector for T?

8.5. Let V be an n-dimensional vector space. Let $T \in \mathrm{End}_F(V)$. Let $A = \{S \in \mathrm{End}_F(V) : ST = TS\}$. *Hint*: We may regard V as an $F[X]$-module. Identify A with $\mathrm{End}_{F[X]}(V)$. And then use the rational canonical from.

(i) Show that the dimension of A (as an F-vector space) is greater than or equal to n.

(ii) Show that the equality is attained iff T has a cyclic vector.

8.6. Let $f \in R[X]$ be a monic polynomial of degree n. Let M be a free R-module with basis $e_1, \ldots, e_n$.

(i) Show that there is a unique R-module homomorphism $T : M \to M$ such that $T(e_i) = e_{i+1}$ for all $i = 1, \ldots, n-1$ and $f(T)e_1 = 0$.

(ii) Show that $f(T)v = 0$ for all $v \in M$.

(iii) Let $b \in R$. Define $S : M \to M$ by $S(v) = bv - Tv$ for all $v \in M$. Compute $\Lambda^k(S)e_1 \cdots e_k$. for all $k = 1, \ldots, n$ inductively and deduce that $\det(S) = f(b)$.

8.7. Let V be a vector space over a field F. Let $v_1, \ldots, v_r \in V$.

(i) Prove that if $v_1, \ldots, v_r$ are linearly dependent, then $v_1 \cdots v_r \in \Lambda^r(V)$ equals zero.

(ii) Prove that if $v_1, \ldots, v_r$ are linearly independent, then $v_1 \cdots v_r \in \Lambda^r(V)$ is nonzero.

(iii) Prove that if W is a linear subspace of V and $w_1, \ldots, w_r$ is a basis of W, then the one-dimensional subspace $Fw_1 \cdots w_r$ of $\Lambda^r(V)$ depends only on W, i.e., it does not depend on the choice of the basis $w_1, \ldots, w_r$. It is conventional to refer to this one dimensional subspace as $\det(W) \subset \Lambda^r(V)$.

(iv) If W_1, W_2 are both r-dimensional subspaces of V, ad if the one-dimensional subspaces $\det(W_1)$ and $\det(W_2)$ of $\Lambda^r(V)$ are equal to each other, show that $W_1 = W_2$.

8.8. Let V be a vector space of dimension 4, and let $\omega \in \Lambda^2(V)$ be nonzero. Prove that $\omega^2 = 0$ iff $F\omega = \det(W)$ for a two-dimensional subspace $W \subset V$.

8.9. Prove that the characteristic polynomial is monic of degree n. Prove that the coefficient of λ^{n-1} in the characteristic polynomial of L is the negative of the trace of L, which is defined to be the sum of the diagonal terms of the matrix that represents L when a basis $e_1, \ldots, e_n$ is specified.

8.10. Deduce the Cayley-Hamilton theorem for fields from Problem 8.6 and the fact that every torsion $F[X]$-module is the direct sum of cyclic modules.

8.11. (i) Show that the Cayley-Hamilton theorem for fields implies the theorem for integral domains as well.

(ii) Show that the Cayley-Hamilton theorem for the polynomial ring $\mathbb{Z}[X_1, \ldots, X_{n^2}]$ implies the theorem for all $L : R^n \to R^n$ where R is a commutative ring.