3 Properties of Ideals

When solving a particular problem Y, you may appeal to the result of any problem X that has occurred before Y (earlier problem sheets included) whether or not you submitted a solution of problem X.

1/25: **3.1.** How many maximal ideals does the ring $\mathbb{Z}/a\mathbb{Z}$ possess under the following conditions?

(i) $a = 81$.

(ii) $a = 44$.

(iii) $a = 42$.

3.2. Given ring homomorphisms $f : R \to A$ and $g : R \to B$, check that $h(v) = (f(v), g(v))$ for all $v \in R$ gives a ring homomorphism $h : R \to A \times B$.

3.3. In particular, let I_1, I_2 be ideals of a commutative ring R, and let $\pi_i : R \to R/I_i$ $(i = 1, 2)$ be canonical surjections. Consider the ring homomorphism $h : R \to (R/I_1)\times(R/I_2)$ given by $h(a) = (\pi_1(a), \pi_2(a))$ for all $a \in R$.

(i) Describe $\ker(h)$ in terms of I_1, I_2.

(ii) Prove that $A \Longrightarrow B \Longrightarrow C \Longrightarrow A$.

(A) h is a surjection.

(B) $(0, 1)$ is in the image of h.

(C) $I_1 + I_2 = R$.

(iii) Assume that $I_1+I_2 = R$. Prove that $I_1I_2 = I_1\cap I_2$. Deduce that $\phi : R/(I_1I_2) \to (R/I_1)\times(R/I_2)$ is an isomorphism.

3.4. Prove that a nonzero ideal $I \subset F[[X]]$, where F is a field, is the principal ideal generated by X^n for some $n \geq 0$. (This is a continuation of Exercise 7.2.3c of Dummit and Foote (2004), addressed in HW2 Q2.2.)

3.5. Recall that $R[X, Y] := R[X][Y]$. Regard R as a subring of $R[X, Y]$. Let R be a commutative ring.

The **universal property of $\boldsymbol{R[X, Y]}$** states: Let A be commutative. Given a ring homomorphism $\alpha : R \to A$ and $x, y \in A$, prove that there is a unique ring homomorphism $\beta : R[X, Y] \to A$ that satisfies $\beta(c) = \alpha(c)$ for all $c \in R$, $\beta(X) = x$, and $\beta(Y) = y$.

Deduce this statement from the universal property of $R[X]$.

3.6. (i) For any $a \in R$, we may define the ring homomorphism $\phi : R[X] \to R$ by $\phi(f(X)) = f(a)$. Prove that $\ker\phi$ is a principal ideal, and find a generator of this ideal.

(ii) Let $g \in R[X]$. Define $\phi : R[X, Y] \to R[X]$ by $\phi(f(X, Y)) = f(X, g(X))$. Prove that $\ker\phi$ is a principal ideal, and find a generator of this ideal.

3.7. Let a, b be elements of R a commutative ring, and let a be a unit of R. Consider the ring homomorphism $\phi : R[X] \to R[X]$ given by $\phi(f) = f(aX + b)$. Prove that ϕ is an isomorphism. *Hint*: It's inverse can be written down explicitly.

3.8. Let R be an integral domain. Prove that every isomorphism $\phi : R[X] \to R[X]$ that satisfies $\phi(c) = c$ for all $c \in R$ is of the type given in Q3.7.

3.9. (i) Exercise 7.1.11 of Dummit and Foote (2004): Prove that if R is an integral domain and $x^2 = 1$ for some $x \in R$, then $x = \pm 1$.

(ii) Deduce that $\{a^2 \mid 0 \neq a \in \mathbb{F}_p\}$ has cardinality $(p-1)/2$. Here, p is an odd prime, and $\mathbb{F}_p$ is the field of cardinality p.

3.10. Prove that there are exactly four rings of cardinality p^2, where p is a prime ($p = 2$ is included). Identify which of them is a field, which is a product of two fields, and find a nonzero nilpotent in both of the remaining cases.

Hint: First show that there are only two possibilities for the characteristic of such a ring. If the characteristic is an odd prime p, show that there is some θ in the ring with the two properties: (i) $\theta^2 \in \mathbb{F}_p$ and (ii) $1, \theta$ form a basis for the given ring viewed as an $\mathbb{F}_p$ vector space. Now apply the previous problem.