1 Rings, Subrings, and Ring Homomorphisms

1/11: **1.1.** Let R be a ring with identity. Show that R is a singleton if and only if $0_R = 1_R$.

Products

1.2. Let X, Y be sets and let R be a ring. Recall that pointwise addition and multiplication turns R^X and R^Y into rings. Let $f : X \to Y$ be a function. Define $f^* : R^Y \to R^X$ by $f^*(g) = g \circ f$ for all $g : Y \to R$. Prove that f^* is a ring homomorphism.

1.3. Let $Y \subset X$. Define $\phi : R^Y \to R^X$ by the following rule: For any function $g : Y \to R \in R^Y$, let $\phi(g) : X \to R$ send

$$x \mapsto \begin{cases} g(x) & x \in Y \\ 0 & x \notin Y \end{cases}$$

State whether the assertions (i) and (ii) below are *true* or *false*. No proof required.

Warning: Make sure to use the definitions of "ring homomorphism" and "subring" from class!

(i) ϕ is a ring homomorphism.

(ii) The image of ϕ is a subring of R^X.

1.4. For any ring R, define the set $\Delta(R)$ by

$$\Delta(R) = \{(a, a) : a \in R\}$$

Note that $\Delta(R)$ is a subring of $R \times R$. Prove that if B is a subring of $\mathbb{Q} \times \mathbb{Q}$ that contains $\Delta(\mathbb{Q})$, then B is either $\Delta(\mathbb{Q})$ or $\mathbb{Q} \times \mathbb{Q}$.

Basic Properties

1.7. Let $f : R_1 \to R_2$ be a ring homomorphism, and let R_3 be a subring of R_2. Prove that $f^{-1}(R_3)$ is a subring of R_1.

1.9. Show that $A \cap B$ is a subring of R if both A, B are subrings of R.

Recall the following lemma from MATH 25700: Let $(A, +)$ be an abelian group, and let $a \in A$. Then there is a unique group homomorphism $f : \mathbb{Z} \to A$ such that $f(1) = a$. Additionally, $f(n) = na$ for all $n \in \mathbb{Z}$.

1.10. Let 1_R denote the multiplicative identity of a ring R. The above lemma then defines $na \in R$ for every $a \in R$ and $n \in \mathbb{Z}$. In particular, we define $n_R = n(1_R)$ for every integer $n \in \mathbb{Z}$. Prove that $n_R \cdot a = na$ for every $a \in R$ and $n \in \mathbb{Z}$.

1.11. With notation as above, show that $f : \mathbb{Z} \to R$ given by $f(n) = n_R$ is a ring homomorphism.

The commutativity of a ring is required for all the identities of high school algebra. The next two problems (1.12 and 1.13) are instances.

1.12. Prove that the following are equivalent.

(i) R is a commutative ring.

(ii) $(a + b)(a - b) = a^2 - b^2$ for all $a, b \in R$.

(iii) $(a + b)^2 = a^2 + 2ab + b^2$ for all $a, b \in R$.

1.14. For this problem, you only have to state whether each of the nine assertions (i), ..., (ix) is *true* or *false*. No proofs are required.

Given sets X, Y, the set of all functions $f : Y \to X$ is denoted by X^Y. Let $(A, +)$ be an abelian group. Given functions $f, g : Y \to A$, define $f + g : Y \to A$ by pointwise addition, i.e., let

$$(f + g)(y) = f(y) + g(y)$$

for all $y \in Y$.

(i) The above binary operation $+$ on A^Y gives A^Y the structure of an abelian group.

For (ii) and (iii) below, we continue with $Y = A$ where $(A, +)$ is an abelian group. In an attempt to give A^A the structure of a ring — for functions $f, g : A \to A$ — we take $\circ$ as the second binary operation. Here, $(f \circ g)(a) = f(g(a))$ for all $a \in A$.

(ii) The right distributive law, i.e., $(f + g) \circ h = f \circ h + g \circ h$ holds for all functions $f, g, h : A \to A$.

(iii) The left distributive law, i.e., $f \circ (g + h) = f \circ g + f \circ h$ holds for all functions $f, g, h : A \to A$.

(iv) The identity function $\mathrm{id}_A : A \to A$ given by $\mathrm{id}_A(a) = a$ for all $a \in A$ satisfies

$$\mathrm{id}_A \circ f = f = f \circ \mathrm{id}_A$$

for all $f : A \to A$.

If you have solved the above problems correctly, you would have seen that $(A^A, +, \circ)$ is *not* a ring. In an endeavor to produce a ring employing the same binary operations $+$ and $\circ$, we replace A^A by its subset $\mathrm{End}(A) = \{f : A \to A : f \text{ is a group homomorphism}\}$.

(v) For $f, g \in \mathrm{End}(A)$, both $f + g$ and $f \circ g$ belong to $\mathrm{End}(A)$.

(vi) The left and right distributive laws hold for $(\mathrm{End}(A), +, \circ)$.

(vii) $(\mathrm{End}(A), +, \circ)$ is a ring (with two-sided multiplicative identity).

(viii) $(\mathrm{End}(A), +, \circ)$ is a commutative ring for all abelian groups $(A, +)$.

(ix) If $A = \mathbb{Z} \times \mathbb{Z}$, then $\mathrm{End}(A)$ is isomorphic to the ring of 2×2 matrices with integer coefficients.