Week 7

???

7.1 Zorn's Lemma and Intro to Modules Over PIDs

2/13:
- Picking up from last time with Zorn's lemma.

- **Partially ordered set**: A set together with a binary relation indicating that, for certain pairs of elements in the set, one of the elements precedes the other in the ordering. *Also known as* **poset**. *Denoted by* $\boldsymbol{P}$.

    - The domain of the **partial order** may be a proper subset of $P \times P$.

- **Partial order**: The binary relation on a poset.

- **Maximal** ($f \in P$): An element $f \in P$ such that for all $q \in P$, the statement $q > f$ is false.

- Example.

    - Let X be a set with $|X| \geq 2^{[1]}$.
    - Define a poset $P = \{A \subsetneq X\}$ with corresponding partial order defined by taking subsets. In particular, if $A \subset B$, write $A \leq B$.
    - For any $x \in X$, $X - \{x\}$ is a maximal element of P.

- **Chain**: A subset of a poset P such that if c_1, c_2 are in said subset, then implies $c_1 \leq c_2$ or $c_2 \leq c_1$. *Denoted by* $\boldsymbol{C}$.

    - In other words, a chain is a subset of a poset that is a **totally ordered set**.

- **Totally ordered set**: A set together with a binary relation indicating that, for any pair of elements in the set, one of the elements precedes the other in the ordering.

- Observation: If F is a subset of a nonempty finite chain C, then there exists $c \in F$ such that $c \geq q$ for all $q \in F$.

- **Upper bound** (of C): An element $p \in P$ such that $p \geq c$ for all $c \in C$.

- **Zorn's lemma**: Let P be a poset that satisfies

    (i) $P \neq \emptyset$;
    (ii) Every chain $C \subset P$ has an upper bound.

    Then P has a maximal element.

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[1]Nori denotes cardinality by $\#X$.

- We will not prove Zorn's lemma. It rarely if ever gets proven in an undergraduate course, maybe in a logic course.

    - And by "prove" we mean "deduce Zorn's lemma from the Axiom of Choice."

- We now investigate a situation in which Zorn's lemma gets applied.

- Let M be a finitely generated A-module.

    - Let $v_1, \ldots, v_r \in M$ be elements such that such that $M = Av_1 + \cdots + Av_r$.
    - Before we prove the proposition that requires Zorn's lemma, we will need one more definition: that of a **maximal submodule**.

- **Maximal submodule** (of M): A submodule of M that is a maximal element of the poset

$$P = \{N \subsetneq M : N \text{ is an } A\text{-submodule}\}$$

- Proposition: Every nonzero finitely generated A-module M has a maximal submodule.

    Proof. To prove that M has a maximal submodule, it will suffice show that there exists a maximal element of the poset
    $$P = \{N \subsetneq M : N \text{ is an } A\text{-submodule}\}$$
    To do this, Zorn's lemma tells us that it will suffice to confirm that $P \neq \emptyset$ and that every chain $C \subset P$ has an upper bound. Let's begin.

    We first confirm that $P \neq \emptyset$. By hypothesis, M is nonzero. Thus, the zero A-submodule is a proper subset of M, so $0 \in P$ and hence P is nonempty.

    We now confirm that every chain $C \subset P$ has an upper bound. Let $C \subset P$ be an arbitrary chain. Define
    $$\mathcal{N}_C = \bigcup \{N : N \in C\}$$
    We will first verify that $\mathcal{N}_C \in P$, and then we will show that $\mathcal{N}_C$ is an upper bound of C. Let's begin. To verify that $\mathcal{N}_C \in P$, it will suffice to demonstrate that $\mathcal{N}_C$ is an A-submodule of M and that $\mathcal{N}_C \subsetneq M$.

    To demonstrate that $\mathcal{N}_C$ is an A-submodule, Proposition 10.1 tells us that it will suffice to show that $\mathcal{N}_C \neq \emptyset$ and $n_1 + an_2 \in \mathcal{N}_C$ for all $a \in A$ and $n_1, n_2 \in \mathcal{N}_C$. Since P is nonempty, $\mathcal{N}_C$ is nonempty by definition, as desired. Additionally, let $n_1, n_2 \in \mathcal{N}_C$ be arbitrary. It follows by the definition of $\mathcal{N}_C$ that there exist $N_1, N_2 \in C$ such that $n_i \in N_i$ ($i = 1, 2$). WLOG, assume $N_1 \subset N_2$. Then $n_1, n_2 \in N_2$. It follows since N_2 is an A-submodule that $n_1 + an_2 \in N_2 \subset \mathcal{N}_C$ for all $a \in A$, as desired.

    We know that $\mathcal{N}_C \subset M$. Thus, if $\mathcal{N}_C \not\subsetneq M$, then we must have $\mathcal{N}_C = M$. Suppose for the sake of contradiction that $\mathcal{N}_C = M$. Recall that $M = Av_1 + \cdots + Av_r$. Since the v_i are elements of M and $\mathcal{N}_C = M$, it follows that $v_i \in \mathcal{N}_C$ ($i = 1, \ldots, r$). Thus, as before, there must exist $N_1, \ldots, N_r \in C$, not necessarily distinct, such that $v_i \in N_i$ ($i = 1, \ldots, r$). It follows by the observation from earlier that there is an $i \in [r]$ such that for all $j \in [r]$, $N_j \subset N_i$. Consequently, $v_j \in N_j \subset N_i$ ($j = 1, \ldots, r$). But N_i is an A-submodule, so $M = Av_1 + \cdots + Av_r \subset N_i \subset M$. But this means that $N_i = M$, contradicting the assumption that $N_i \subsetneq P$ (since $N_i \in P$). Therefore, $\mathcal{N}_C \subsetneq M$, as desired.

    It follows that $\mathcal{N}_C \in P$, as desired. Lastly, we have by its definition that $N \subset \mathcal{N}_C$ for all $N \in C$, meaning that $\mathcal{N}_C$ is an upper bound of C by definition. Therefore, by Zorn's lemma, P has a maximal element, and hence M has a maximal submodule, as desired. $\qquad\square$

- Corollary: Every nonzero commutative ring R has a maximal ideal.

    Proof. Consider R as an R-module. Then $R = (1)$ is finitely generated. This combined with the fact that it is nonzero by hypothesis allows us to invoke the above proposition, learning that R has a maximal submodule N. But by the observation from Lecture 6.1, N is a left ideal, which is equivalent to a two-sided ideal in a commutative ring. Maximality transfers over as well (as we can confirm), proving that N is the desired maximal ideal of R. $\qquad\square$

- Remark: Suppose that J is a two-sided ideal of A. Let M be an A-module such that for all $a \in J$ and $m \in M$, we have $am = 0$. Then M may be regarded as an (A/J)-module in a natural manner.

  - In particular, we may take $\rho : A \to \mathrm{End}(M, +)$ to be a ring homomorphism.
  - We can factor $\rho = \bar{\rho} \circ \pi$, where $\pi : A \to A/J$ and $\bar{\rho} : A/J \to \mathrm{End}(M, +)$. It follows that $\bar{\rho}$ is a ring homomorphism. Therefore, M is an A/J-module.
  - This remark will be used!
  - Review annihilators from Section 10.1!

- Remark: Given a left ideal $I \subset A$ and an A-module M, we get a whole lot of modules because each element of M generates one. In particular, we note that $Im \subset Am \subset M$, where both Im, Am are submodules for all $m \in M$.

- **Product** (of modules): The A-submodule of M defined as follows. *Denoted by $\boldsymbol{IM}$. Given by*

$$IM = \sum_{m \in M} Im$$

- It follows that M/IM is an A-module, but also one with a special property: $a(M/IM) = 0$ for all $a \in I$.

  - If A is commutative, then M/IM is an A/I-module.

- Proposition: Let R be a nonzero commutative ring. If $R^m \cong R^n$ as R-modules, then $m = n$.

  Proof. Let $I \subset R$ be a maximal ideal. (We know that one exists by the above corollary.) If $f : R^m \to R^n$ is an isomorphism of R-modules, then f restricts to $I(R^m) \to I(R^n)$. This gives rise to the isomorphism $\bar{f} : R^m/I(R^m) \to R^n/I(R^n)$ of R-modules, in fact of R/I modules. It follows that R/I is a field, so $m = n$. $\qquad\square$

- Classifying modules up to isomorphism under commutative rings.

  - This is a hard problem, and there are still many open problems in this field today.
  - We will not go into this, though.

- We now move on to modules over PIDs.

  - Nori will go *much* slower than the book.
  - Do you have any recommended resources??
  - Do we need to read and understand Chapters 10-11 to start on Chapter 12??

- Objective: Let R be a PID. Classify all finitely generated R-modules up to isomorphism.

  - Our first result in this field was that submodules of R^n are equal to R^m for $m \leq n$.
  - Where this is applicable: $\mathbb{Z}$ and $F[X]$.
    - Go back and check out $\mathbb{Z}$-modules and $F[X]$-modules in Section 10.1!

- **Torsion module**: An R-module M such that for all $m \in M$, there exists $0 \neq a \in R$ such that $am = 0$.

- **Torsion-free module**: An R-module M such that for all nonzero $m \in M$ and for all nonzero $a \in R$, we have $am \neq 0$.

- Theorem: If M is a finitely generated torsion-free R-module, then $M \cong R^n$ for some n.

  - With a little work, we could prove this. But Nori will postpone it.

- **p-primary** (module): An R-module M such that for all $m \in M$, there exists $k \geq 0$ for which $p^k m = 0$, where p is prime in R.

- We want to classify these up to isomorphism.

  – Nori can state these today, but will not have time to prove it until another day.
  – Something that gets annihilated by p is a $\mathbb{Z}/(p)$-module. The moment you go from $k = 1$ to $k = 2$, things get interesting.

- Examples: $R/(p^{n_1}) \oplus \cdots \oplus R/(p^{n_k})$, where $n_1 \geq \cdots \geq n_k \geq 1$.

  – Note that $k = 0$ is allowed.

- Uniqueness will take some time, but existence can be given as an exercise now.

- M/pM is an $R/(p)$-vector space. $pM/p^2 M$ is an $R/(p)$-vector space as well. So is $p^k M/p^{k+1} M$.

  – Use $d_0, d_1, \ldots, d_k$ to denote the dimensions of the vector spaces.
  – $d_0, \ldots, d_k$ is a decreasing sequence of nonnegative integers.

7.2   Office Hours (Nori)

- Homework questions.

  – See pictures + unnumbered lemma.
  – Example of the kernel being bigger than (f).
  – A ring homomorphism $\mathbb{Z}[X] \to \mathbb{R}$ must be evaluation by the universal property of polynomial rings.
  – Factoring enables a constraint on a.

- Lecture 6.1: Proposition proof?

- Lecture 6.1: $(2) \subsetneq \mathbb{Z}$ example?

- Lecture 6.1: The end of the theorem proof.

- Lecture 6.2: Does the first theorem you proved not appear in the book until Chapter 12?

- Lecture 6.2: What is A in the proof?

- Resources for the proofs in Week 6?

- Lecture 7.1: Quotient stuff.

- Recommended resources for modules over PIDs? Chapter 12?

  – We should be able to read chapter 12, since chapter 11 is just vector spaces.
  – Nori's doing Chapter 12 in the classical manner (pre-1970). Dummit and Foote (2004) just does it in the first few pages as the **elementary divisor theorem**.

- HW6: So you want us to solve 1, 10, 13 for our own edification, but we don't need to write up a solution? Will we ever be responsible for the content therein?

  – We'll need to understand them to move forward.
  – Q6.4-Q6.5 are particularly important (good for number theory).

7.3   Office Hours (Ray)

- Universal properties save you from having to do pages upon pages of ring homomorphism checks (think Q3.10).

- Algebra: Chapter 0 by Paolo Aluffi for learning quotienting by polynomials.

  – Universal properties show up on page 30.

  – Read stuff before as needed.

  – Has a chapter called universal properties of polynomial rings. Universal properties of quotients, too.

- Direct sums and direct products.

  – Let M, N be R-modules. Then $M \times N$ is an R-module defined by the Cartesian product of the sets and with **diagonal** module action $r(m, n) = (rm, rn)$ (diagonal meaning we just act on two elements).

  – $M \oplus N = M \times N$.

  – For infinite sets, we get a difference. Indeed, $\prod_{i=1}^{\infty} M_i \neq \bigoplus_{i=1}^{\infty} M_i$.

7.4   Classifying Modules Over PIDs

2/15:
- We pick up from yesterday, classifying finitely generated R-modules M up to isomorphism when R is a PID.

- In particular, we begin with a further investigation of the properties of torsion modules.

- **Lift** (of $x \in M/M'$): The choice of an element $y \in M$ such that $\pi(y) = x$.

- Lemma:

  (i) $\operatorname{Tor}(M)$ is an R-submodule of M.

  Proof. To prove that $\operatorname{Tor}(M)$ is an R-submodule of M, Proposition 10.1 tells us that it will suffice to show that $\operatorname{Tor}(M) \neq \emptyset$ and that $x + ry \in \operatorname{Tor}(M)$ for all $r \in R$, $x, y \in \operatorname{Tor}(M)$. Consider $0 \in M$. By definition, $r \cdot 0 = 0$. Thus, $0 \in \operatorname{Tor}(M)$ as desired. Additionally, let $r \in R$ and $x, y \in \operatorname{Tor}(M)$ be arbitrary. Since $x, y \in \operatorname{Tor}(M)$, there exist nonzero $a, b \in R$ such that $ax = 0$ and $by = 0$. Because R is an integral domain (as a PID), a, b nonzero implies that $ab \neq 0$. Thus, since
  $$ab(x + ry) = abx + abry = b(ax) + ar(by) = b(0) + ar(0) = 0$$
  we have that $x + ry \in \operatorname{Tor}(M)$, as desired. $\square$

  (ii) The quotient module $M/\operatorname{Tor}(M)$ is torsion-free.

  Proof. To prove that $M/\operatorname{Tor}(M)$ is torsion-free, it will suffice to show that every torsion element of $M/\operatorname{Tor}(M)$ is 0. Let's begin. Let $v \in M/\operatorname{Tor}(M)$ be an arbitrary torsion element. Then there exists $a \in R$ nonzero such that $av = 0$. Now lift $v \in M/\operatorname{Tor}(M)$ to $w \in M$. The constraint $av = 0 = 0 + \operatorname{Tor}(M)$ from the quotient module implies that $0 = a\pi(w) = \pi(aw)$, hence $aw \in \operatorname{Tor}(M)$. Thus, there exists $b \in R$ nonzero such that $b(aw) = 0$. It follows that $(ba)w = 0$, where $ba \neq 0$ since $a, b \neq 0$ by the fact that R is an integral domain. Thus, $w \in \operatorname{Tor}(M)$, and hence $v = \pi(w) = 0$, as desired. $\square$

- We now give some claims that will be useful later today, but whose proofs we will delay until next lecture.

- The first one pertains to the properties of finitely generated torsion-free modules over an integral domain.

- Lemma: Let R be an integral domain, and let M be a finitely generated R-module. Then there exists a submodule $M' \subset M$ such that...

  (i) $M' \cong R^h$ for some $h \geq 0$;

  (ii) There exists a nonzero $a \in R$ such that $aM \subset M'$ (equivalently, $a(M/M') = 0$).

- The next two pertain to the properties of finitely generated modules over a PID.

- Corollary: Every finitely generated torsion-free module M over a PID R is isomorphic to R^h for some $h \in \mathbb{Z}_{\geq 0}$.

- Theorem: Let M be a finitely generated R-module, where R is a PID. Then...

  (i) $\operatorname{Tor}(M) \oplus R^h \cong M$ for some $h \geq 0$;

  (ii) $\operatorname{Tor}(M)$ is finitely generated.

- **Rank** (of a module): The number h pertaining to an R-module M, where $M/\operatorname{Tor}(M) \cong R^h$. *Denoted by* **rank($M$)**.

  - It follows by the proposition from last lecture (Lecture 7.1) that rank is well-defined.

- Corollary: Finitely generated R-modules M_1 and M_2 are isomorphic to each other iff

  (i) M_1 and M_2 have the same rank;

  (ii) $\operatorname{Tor}(M_1)$ is isomorphic to $\operatorname{Tor}(M_2)$.

  Proof. Suppose first that $\phi : M_1 \to M_2$ is an isomorphism. Then naturally they will have the same ranks and torsion submodules.

  On the other hand, if $\operatorname{rank}(M_1) = \operatorname{rank}(M_2)$, then $M_1/\operatorname{Tor}(M_1) \cong M_2/\operatorname{Tor}(M_2)$. This combined with the hypothesis that $\operatorname{Tor}(M_1) \cong \operatorname{Tor}(M_2)$ implies that

  $$\operatorname{Tor}(M_1) \oplus M_1/\operatorname{Tor}(M_1) \cong \operatorname{Tor}(M_2) \oplus M_2/\operatorname{Tor}(M_2)$$
  $$M_1 \cong M_2$$

  where the second line follows from the preceding theorem. $\qquad\square$

- The classification of finitely generated R-modules (R a PID) is completed by the following results.

- **p-primary component** (of a module): The submodule of a module M consisting of those $m \in M$ such that $p^k m = 0$ for some $k \in \mathbb{Z}_{\geq 0}$. *Denoted by* **$M_{(p)}$**.

  - Showing that $M_{(p)}$ is a submodule of M can be accomplished with the submodule criterion (Proposition 10.1), just like in the first lemma proven today.

- Notation and observations.

  1. Let $M_1, \ldots, M_k$ be submodules of M. Then $T : \prod_{i=1}^{k} M_i \to M$ defined by

  $$T(m_1, \ldots, m_k) = m_1 + \cdots + m_k$$

  is not injective in general.

    - For example, if $k = 2$, then $\ker(T) \cong M_1 \cap M_2$ in general.
    - Thus, some care is required in our selection of submodules if we want $\ker(T) = 0$.

  2. Obtaining a natural R-module homomorphism $T : \oplus_{i \in I} M_i \to M$ defined as above.

    - We have that $\oplus_{i \in I} M_i \subset \prod_{i \in I} M_i$ in general. Here's why:
    - Given a finite subset $F \subset I$, we may regard $\prod_{i \in F} M_i$ as a submodule of $\prod_{i \in I} M_i$ by taking the entries in the ith place to be zero for all $i \notin F$.

- – The direct sum is simply the union of the submodules $\prod_{i \in F} M_i$ taken over all finite $F \subset I$.
  - – We define T on the overall direct sum one submodule $\prod_{i \in F} M_i$ at a time.

- Proposition: The natural R-module homomorphism $T : \oplus_{(p)} M_{(p)} \to \text{Tor}(M)$ is an isomorphism, where the direct sum is indexed by the set of nonzero prime ideals of R.

  Proof. Let F be a set of r distinct primes $p_1, \ldots, p_r$ (i.e., the prime ideals $(p_1), \ldots, (p_r)$ are pairwise distinct sets). Let $(m_1, \ldots, m_r) \in \prod_{(p) \in F} M_{(p)}$. Then as per the notation and observations section above, T is defined such that
  $$T(m_1, \ldots, m_r) = m_1 + \cdots + m_r$$
  We first prove that T is injective. Let $(m_1, \ldots, m_r) \in \ker(T)$ be arbitrary. Then $T(m_1, \ldots, m_r) = m_1 + \cdots + m_r = 0$. By hypothesis, there exist $k_1, \ldots, k_r$ such that $p_i^{k_i} m_i = 0$ $(i = 1, \ldots, r)$. Define $a = p_2^{k_2} \cdots p_r^{k_r}$. It follows that $am_2 = \cdots = am_r = 0$. Thus,
  $$a(0) = 0$$
  $$a(m_1 + \cdots + m_r) = 0$$
  $$am_1 + \cdots + am_r = 0$$
  $$am_1 = -(am_2 + \cdots + am_r)$$
  $$= -(0 + \cdots + 0)$$
  $$= 0$$

  Additionally, $\gcd(a, p_1^{k_1}) = 1$ by definition, so $1 \in (a, p_1^{k_1})$. It follows that there exist $b, c \in R$ such that $ba + cp_1^{k_1} = 1$. This combined with the facts that $am_1 = 0$ and $p_1^{k_1} m_1 = 0$ implies that
  $$m_1 = 1 \cdot m_1 = (ba + cp_1^{k_1})m_1 = b(am_1) + c(p_1^{k_1} m_1) = b(0) + c(0) = 0$$

  A symmetric argument shows that all $m_i = 0$, i.e., $(m_1, \ldots, m_r) = (0, \ldots, 0)$. Therefore, $\ker(T) = 0$, as desired.

  We now prove that T is surjective. Let $m \in \text{Tor}(M)$ be arbitrary. Consider the submodule $N = Am \subset M$. To prove that m is the sum of elements, each from a p-primary component of M, it will suffice to prove that stronger condition that every element in N is the sum of elements, each from a p-primary component of M. Equivalently, it will suffice to show that N is the isomorphic to the sum of its p-primary components, since the p-primary components of N are contained in those of M. Define $I = \{a \in R : am = 0\}$. Notice that $I = \ker(l_a)$, where $l_a : R \to N$ is the left multiplication homomorphism. It follows by the FIT that there exists an isomorphism $\overline{l_a} : R/I \to N$. Thus, we need only show that R/I is isomorphic to the direct sum of its p-primary components. But the Chinese Remainder Theorem takes care of this for us since I is a nonzero ideal. $\square$

- In view of the last proposition, our final task will be to classify finitely generated p-primary modules.

- We begin with some definitions.

- **p-primary** (module): An R-module M such that $M = M_{(p)}$ for some prime $p \in R$.

- **Annihilator** (of a module): The set of all $a \in R$ such that $am = 0$ for all $m \in M$. *Denoted by* **Ann(M)**. *Given by*
  $$\text{Ann}(M) = \{a \in R : am = 0 \,\forall\, m \in M\}$$

- **Annihilator** (of an element): The set of all $a \in R$ such that $am = 0$ pertaining to a specific $m \in M$. *Denoted by* **Ann(m)**. *Given by*
  $$\text{Ann}(m) = \{a \in R : am = 0\}$$

- Consider $l_m : R \to M$ defined by $l_m(a) = am$.

  - – By the FIT, there exists a module isomorphism $\overline{l_m} : R/\text{Ann}(m) \to Rm$.

- ker$(l_m) = $ Ann(m).

- **Cyclic** (module): An R-module M for which there exists $m \in M$ such that $M = Rm$.

  - Cyclic modules are isomorphic to $R/$ Ann(m) for a similar reason to the above ($Rm = M$ here).

- With these definitions out of the way, we seek to show that every finitely generated R-module is the direct sum of cyclic modules.

- To prove this result, we will need the following lemma.

- Lemma: Let $M' = Re$ be a cyclic submodule of M. We assume that...

  (i) Ann$(e) = (p^n)$;

  (ii) $p^n M = 0$.

  Then every $v \in M/M'$ has a lift $w \in M$ such that Ann$(w) = $ Ann(v).

  Proof. Let $v \in M/M'$ be arbitrary. Since $p^n M = 0$, $p^n(M/M') = 0$ and hence Ann$(v) = (p^k)$ for some $k \leq n$. Now let $w \in M$ be an arbitrary lift of v. We will prove that this w satisfies all necessary constraints.

  To prove that Ann$(w) \subset $ Ann(v), let $a \in $ Ann(w) be arbitrary. Then $aw = 0$. It follows that $0 = \pi(aw) = a\pi(w) = av$. Therefore, $a \in Ann(v)$ as well.

  To prove that Ann$(v) \subset $ Ann(w) $\square$

- Proposition: For every finitely generated p-primary module M, there exist $e_1, \ldots, e_s$ such that M is the direct sum of the cyclic submodules Re_i.

  Proof. Since M is finitely generated, we know that $M = Rv_1 + \cdots + Rv_r$. We induct on r.

  For the base case $r = 1$, M is cyclic by definition.

  Now suppose that we have proven the claim for some lower cases. Again with the (p^n) issue. $\square$

7.5   Rational Canonical Form and Proofs of Earlier Lemmas

2/17:
- Theorem: Every finitely generated R-module M (where R is a PID) is isomorphic to Tor$(M) \oplus R^h$ for some $h \in \mathbb{Z}_{\geq 0}$, where $h = $ rank(M).

- Recall the following theorem.

- Theorem: Let R be a PID. Then

  (1) Every finitely generated p-primary R-module is a finite direct sum of cyclic modules (which are isomorphic to $R/p^h R$ for some $h \in \mathbb{N}$).

  (2) Every torsion module M is the direct sum of its p-primary components.

- Corollary: Every finitely generated torsion R-module is isomorphic to the finite direct sum of cyclic p-primary modules where p is an element of a finite set of primes. *picture*

- M finitely generated implies that $M_{(p)}$ is finitely generated.

- Said aloud that only finite primes p satisfy $M_{(p)} \neq 0$.

- Theorem (Rational canonical form): Let R be a PID. Then every finitely generated R-torsion module is isomorphic to
$$R/(a_1) \oplus \cdots \oplus R/(a_\ell)$$
  where $a_2 \mid a_1$, $a_3 \mid a_2$, $\ldots$, $a_\ell \mid a_{\ell-1}$.

- Observe: The principal ideal (a_1) is exactly the annihilator of M, i.e.,

$$(a_1) = \{\alpha \in R : \alpha m = 0 \; \forall \; m \in M\}$$

  - Later, (a_1) will play the role of a minimal polynomial, and the product will play the role of the characteristic polynomial.

Proof of theorem. Let $p_1, \ldots, p_\ell$ be ?? the set of primes for which $M_{(p)} \neq 0$. Let

$$M_{(p_i)} \cong R/(p_i^{m_i,1}) \times R/(p_i^{m_i,2}) \times \cdots$$

where $m_{i,1} \geq m_{i,2} \geq \cdots$ are such that there exists N for which $m_{i,N} = 0$. Then

$$M/(p_j) \cong R/(p_j^{m_j,1})^\times \times R/(p_j^{m_j,2})^\times$$

Then we apply the Chinese Remainder Theorem. Define

$$a_r = \prod_{i=1}^{\ell} p_i^{m_i,r}$$

where $a_{r+1} \mid a_r$ because $m_{i,j}$ is ?? in j. Use the CRT to imply that

$$\prod_{i=1}^{\ell} R/(p_i^{m_i,r}) \cong R/(a_r)$$

$\square$

- That concludes torsion modules over PIDs; we now do torsion modules over fields, which should be easier.

- **R-linearly independent** (elements of M): A set of elements $u_1, \ldots, u_\ell \in M$ such that the constraints

$$(a_1, \ldots, a_\ell) \in R^\ell \qquad\qquad \sum_{i=1}^{\ell} a_i u_i = 0$$

imply that $(a_1, \ldots, a_\ell) = 0$. Equivalently, $H : R^\ell \to M$ defined by

$$H(a_1, \ldots, a_\ell) = \sum_{i=1}^{\ell} a_i u_i$$

is 1-1, i.e., $R^\ell \cong H(M)$.

- Lemma: Let R be an integral domain, and let M be a finitely generated R-module. Then there exists a submodule $M' \subset M$ such that...

  (i) $M' \cong R^h$ for some $h \geq 0$;

  Proof. Let $S \subset M$ be a finite generating set. Select $T \subset S$ such that (i) T is linearly independent and (ii) $T \subsetneq W \subset S$ implies that W is *not* linearly independent. In other words, we are picking T to be a maximal linear independence set. Now suppose $|T| = h$ so that $T = \{u_1, \ldots, u_h\}$. Then by definition,

$$M' = \sum_{i=1}^{h} Ru_i \cong R^h$$

  where the latter isomorphism follows from Proposition 10.5. $\square$

(ii) There exists a nonzero $a \in R$ such that $aM \subset M'$ (equivalently, $a(M/M') = 0$).

Proof. Pick $w \in S$ such that $w \notin T$. Then since we picked T to be a *maximal* linear independence set, $T \cup \{w\}$ is linearly *dependent*. It follows that there exists a nonzero $(a_1, \ldots, a_{h+1}) \in R^{h+1}$ such that

$$a_1 u_1 + \cdots + a_h u_h + a_{h+1} w = 0$$

If $a_{h+1} = 0$, then $(a_1, \ldots, a_h) \neq 0$ makes $a_1 u_1 + \cdots + a_h u_h = 0$, contradicting the assumed linear independence of T. Thus, $a_{h+1} \neq 0$. It follows that

$$a_{h+1} w = -\sum_{i=1}^{h} a_i u_i \in M'$$

We may repeat this process for any $w \in S - T$ to obtain a nonzero a_w such that $a_w w \in M'$. Additionally, if $w \in T$, take $a_w = 1$. Now define

$$a = \prod_{w \in S} a_w$$

Since R is an integral domain by hypothesis and each a_w in the above product is nonzero, a is nonzero. Moreover, by its construction, $aw \in M'$ for all $w \in S$. Therefore,

$$aM = a \left(\sum_{s \in S} As \right) \subset M'$$

as desired. $\square$

- Note that you can make stronger statements than the above; you'll just have to use Zorn's lemma to do so.

- We now return to PID-land.

- Corollary: Every finitely generated torsion-free module M over a PID R is isomorphic to R^h for some $h \in \mathbb{Z}_{\geq 0}$.

  Proof. Apply the lemma to obtain a submodule M' of M such that $M' \cong R^h$ and a nonzero $a \in R$ such that $aM \subset M'$. Consider $H : M \to M'$ defined by $H(m) = am$. Since H is just left-multiplication, H is an R-module homomorphism. Additionally, since M is torsion free, $am = 0$ iff $m = 0$ so we have $\ker H = 0$. Thus, since H is injective, $M \cong H(M) \subset M' \cong R^h$. Furthermore, since R is a PID, the submodule $H(M)$ of R^h must be isomorphic to R^n for some $0 \leq n \leq h$ by the Theorem from Week 6. It follows by transitivity that $M \cong H(M) \cong R^n$, as desired. $\square$

- Takeaway: The torsion-free part is far easier to handle than the torsion part.

- Theorem: Let M be a finitely generated R-module, where R is a PID. Then. . .

  (i) $\text{Tor}(M) \oplus R^h \cong M$ for some $h \geq 0$;

      Proof. To prove that $\text{Tor}(M) \oplus R^h \cong M$, the second theorem from Lecture 6.3 tells us that it will suffice to show that $M/\text{Tor}(M) \cong R^h$ for some $h \geq 0$. By part (ii) of the lemma from last time (Lecture 7.2), we have that $M/\text{Tor}(M)$ is torsion-free. This combined with the fact that $M/\text{Tor}(M)$ is a finitely generated (since M is finitely generated) module over a PID allows us to invoke the above corollary, yielding the desired result.
      Note that the isomorphism $T : \text{Tor}(M) \oplus R^h \to M$ is given by

      $$T(m, (a_1, \ldots, a_h)) = m + \sum a_i e_i$$

      where $e_1, \ldots, e_h$ generate R^h. $\square$

(ii) $\text{Tor}(M)$ is finitely generated.

Proof. Since M is finitely generated, part (i) implies that $\text{Tor}(M) \oplus R^h$ is finitely generated. Now consider the projection $\pi : \text{Tor}(M) \oplus R^h \to \text{Tor}(M)$. Since it is a surjection, the (finite number of) images of the generators of $\text{Tor}(M) \oplus R^h$ generate $\text{Tor}(M)$. $\qquad\square$

- Nori reproves the claim that $M/\text{Tor}(M)$ is torsion-free (see the first lemma from last lecture).

- If $\pi : M \to M/M'$ and $S : M/M' \to R_h$ is an isomorphism, then there exists $\varphi : R^h \to M$ such that the diagram commutes, i.e., $S\pi\varphi = \text{id}_{R^h}$.

- Next week is going to be straight linear algebra.

- Nori would try to do tensors in one week (the last week), but it'd be ridiculous to do something on Friday and put it on a test on Tuesday.

- Imaginary quadratic fields, curves, Dedekind domains, etc.

- Content from this week in the book.

    - Section 12.1.

        - The material before Theorem **??** is OMITTED from the course.
        - Theorem **??** is also OMITTED from the course.
        - The rest of this section will be covered.
        - The main theorems are: The existence theorem (Theorem **??**) and the uniqueness theorem (Theorem **??**)

    - Section 12.2 deals with the PID $F[X]$ and its applications to linear algebra; this will be covered on Monday next week.

7.6   Office Hours (Callum)

- Problem 6.5?

    - Go with the explicit route, not the universal property of the ring of fractions route.
    - Explicit: Define
    $$F(v) = \frac{1}{a}f(av)$$
    - We need to prove that $1/af(av) = 1/bf(bv)$ for valid a, b. Multiply both sides by ab and use commutativity. Thus, $F(v)$ is well defined.

- Problem 6.8?

    - The hardest one. Doesn't really use any of the previous parts.
    - Define $\phi : A \oplus M \to A^2$ to be the isomorphism. Consider $(1, 0) \in A \oplus M$. In particular, let $\phi(1, 0) = (a, b)$. We know that it will generate a copy of A in A^2. Essentially, $A(a, b) = A^2$. We know that $\phi^{-1} : A^2 \to A \oplus M$ and $P : A \oplus M \to A$. Suppose $P \circ \phi^{-1} : (1, 0) \mapsto c$ and $(0, 1) \mapsto d$.
    - Consider
    $$A \hookrightarrow A \oplus M \xrightarrow{\phi} A^2 \xrightarrow{\phi^{-1}} A \oplus M \xrightarrow{P} A$$
    which is the identity on A. Then
    $$1 \mapsto (1, 0) \mapsto (a, b) = a(1, 0) + b(0, 1) \mapsto ac + bd$$
    so $ac + bd = 1$.

- Consider the matrix

$$\begin{pmatrix} a & d \\ b & c \end{pmatrix}$$

  - Determinant??
  - $(-d, c)$
  - So thus, $M = A(-d, c)$??

- $(-d, c) \in A^2$ defines a map from $A^2 \to M$ with kernel A. $(-d, c) \in \ker(P \circ \phi^{-1})$. Thus, $\phi^{-1}(-d, c) \in \{0\} \oplus M \cong M$.

- Thus, at this point, we may define a map

$$A \hookrightarrow A^2 \xrightarrow{\phi^{-1}} A \oplus M \xrightarrow{P} M$$

  by

$$1 \mapsto (-d, c)$$

  and this should be an isomorphism.

- $(-d, c)$ generates a submodule of A^2 that is isomorphic to M.

- Injectivity follows from that of all of the components.

- Surjectivity: Pull m back to $(0, m)$ and then $\phi(0, m) \in A^2$. The subset of A^2 equal to all $\phi(0, m)$ is equal to

$$\{(u, v) \in A^2 : \phi^{-1}(u, v) \in 0 \oplus M\} = \{(u, v) \in A^2 : uc + vd = 0\}$$

- We want to find $k \in A$ such that $(u, v) = k(-d, c)$. In other words, we want $u = -kd$ and $v = kc$. $ua = -kda = k(1 - bc) = k - kbc = k - bv$. Thus, $k = ua + bv$. Now we have to substitute that back in and show that it works.

- Thus, we have that

$$kc = ua + bvc = uac + b(1 - ad) = v + uac - vad = v + a(bc - ad)$$

- Saying $A \cong M$ is kind of like saying that there's a change of basis. That's why matrices keep coming up.

- Summary of what we did.

  1. We have

$$A \hookrightarrow A \oplus M \xrightarrow{\phi} A^2 \xrightarrow{\phi^{-1}} A \oplus M \xrightarrow{P} A$$

     and this is the identity.

  2. We define $(1, 0) \mapsto (a, b)$, which will generate a copy of A in A^2.
  3. We now need to find a basis vector corresponding to M (which we hope is A).
  4. $\{(1, 0), (0, 1)\}$ is the standard basis for A^2.
  5. We need to solve for x, y such that

$$\begin{pmatrix} a & x \\ b & y \end{pmatrix}$$

     is invertible.

  6. $\{\phi^{-1}(1, 0), \phi^{-1}(0, 1)\}$ is another basis of A^2.
  7. We want $ac + bd = 1$.

7.7   Chapter 11: Vector Spaces

From Dummit and Foote (2004).

Section 11.1: Definitions and Basic Theory

2/20:
- Reviewing Labalme (2021) is probably a good idea.
  - Many of Dummit and Foote (2004)'s proofs more elegant, though.

- Goal of this chapter:
  - Brief overview of results that will be used later on; more in-depth (even introductory level) linear algebra topics, such as Gauss-Jordan elimination, row echelon forms, etc., will not be covered.
  - Only finite-dimensional vector spaces are discussed in the text; some stuff on infinite dimensional vector spaces is included in the exercises
  - Characteristic polynomials and eigenvalues: Next chapter.

- Module terminology vs. vector space terminology.

Terminology for R any Ring	Terminology for R a Field
M is an R-module	M is a vector space over R
m is an element of M	m is a vector in M
α is a ring element	α is a scalar
N is a submodule of M	N is a subspace of M
M/N is a quotient module	M/N is a quotient space
M is a free module of rank n	M is a vector space of dimension n
M is a finitely generated module	M is a finite dimensional vector space
M is a nonzero cyclic module	M is a 1-dimensional vector space
$\varphi : M \to N$ is an R-module homomorphism	$\varphi : M \to N$ is a linear transformation
M and N are isomorphic as R-modules	M and N are isomorphic vector spaces
The subset A of M generates M	The subset A of M spans M
$M = RA$	Each element of M is a linear combination of elements of A, i.e., $M = \text{Span}(A)$

Table 7.1: Module vs. vector space terminology.

- In this chapter, F denotes a field and V denotes a vector space over F.

- **Linearly independent** (subset $S \subset V$): A subset S of V for which the equation $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$ with $\alpha_1, \ldots, \alpha_n \in F$ and $v_1, \ldots, v_n \in S$ implies $\alpha_1 = \cdots = \alpha_n = 0$.

- **Basis**: An ordered set of linearly independent vectors which span V. *Also known as* **ordered basis**.
  - In particular, two bases will be considered different even if one is simply a rearrangement of the other.

- Examples.
  1. $V = F[X]$.
     - Basis: $1, X, X^2, \ldots$ is linearly independent by definition since a polynomial is zero iff all of its coefficients are 0.
  2. The collection of solutions of a linear, homogeneous, constant coefficient differential equation over $\mathbb{C}$.
     - A vector space since differentiation is a linear operator.
     - Elements are linearly independent if they are linearly independent as functions.
       - Example: e^t, e^{2t} are easily seen to be solutions of the equation $y'' - 3y' + 2y = 0$.

- ∎ They are linearly independent since $ae^t + be^{2t} = 0$ implies $a + b = 0$ ($t = 0$) and $ae + be^2 = 0$ ($t = 1$), and the only solution to this system of two equations is $a = b = 0$.
    - ∎ It is a theorem of differential equations that these elements span the set of solutions of this equation.

- Vector spaces are free modules.

    Proposition 11.1. Assume the set $\mathcal{A} = \{v_1, \ldots, v_n\}$ spans the vector space V but no proper subset of $\mathcal{A}$ spans V. Then $\mathcal{A}$ is a basis of V. In particular, any finitely generated (i.e., finitely spanned) vector space over F is a free F-module.

    Proof. Given. □

- Example.

    1. Consider $F[X]/(f)$, where $f = X^n + a_{n-1}X^{n-1} + \cdots + a_0$.
        - (f) is a subspace of $F[X]$.
        - Euclidean Algorithm: Every $a \in F[X]$ can be written uniquely in the form $qf + r$ where $0 \le \deg(r) \le n - 1$. Thus, every element of the quotient is represented by a polynomial r of degree $\le n - 1$.
        - It follows that $\overline{1}, \overline{X}, \overline{X^2}, \ldots, \overline{X^{n-1}}$ spans $F[X]/(f)$.

- Spanning sets contain bases.

    Corollary 11.2. Assume the finite set $\mathcal{A}$ spans the vector space V. Then $\mathcal{A}$ contains a basis of V.

    Proof. Given. □

- A new property of bases.

    Theorem 11.3 (Replacement Theorem)**.** Assume $\mathcal{A} = \{a_1, \ldots, a_n\}$ is a basis for V containing n elements and $\{b_1, \ldots, b_m\}$ is a set of linearly independent vectors in V. Then there is an ordering $a_1, \ldots, a_n$ such that for each $k \in \{1, \ldots, m\}$, the set

    $$\{b_1, \ldots, b_k, a_{k+1}, \ldots, a_n\}$$

    is a basis of V. In other words, the elements $b_1, \ldots, b_m$ can be used to successively replace the elements of the basis $\mathcal{A}$, still retaining a basis. In particular, $n \ge m$.

    Proof. Given. □

- Linear independence, span, and cardinality.

    Corollary 11.4.

    1. Suppose V has a finite basis with n elements. Any set of linearly independent vectors has $\le n$ elements. Any spanning set has $\ge n$ elements.
    2. If V has some finite basis, then any two bases of V have the same cardinality.

    Proof. Given. □

- **Dimension**: The cardinality of any basis of V. *Denoted by* $\dim_F V$, $\dim V$.

- **Finite dimensional** (vector space): A vector space V that is finitely generated.

- **Infinite dimensional** (vector space): A vector space V that is not finitely generated.

  - We write $\dim V = \infty$ for these.

- Examples.

  1. The dimension of the solution space to $y'' - 3y' + 2y = 0$ is 2.
     - Recall from above that a basis is $\mathrm{e}^t, \mathrm{e}^{2t}$.
     - In general, it is a theorem in differential equations that the space of solutions of an nth order linear, homogeneous, constant coefficient differential equation of degree n over $\mathbb{C}$ is a vector space over $\mathbb{C}$ of dimension n.
  2. The dimension of $F[X]/(f)$ is $\deg(f)$.
     - $F[X]$ and (f) are infinite dimensional vector spaces.

- Linearly independent lists and bases.

  Corollary 11.5 (Building-Up Lemma)**.** If A is a set of linearly independent vectors in the finite dimensional space V, then there exists a basis of V containing A.

  Proof. Given. $\square$

- Characterizing finite dimensional vector spaces.

  Theorem 11.6. If V is an n-dimensional vector space over F, then $V \cong F^n$. In particular, any two finite dimensional vector spaces over F of the same dimension are isomorphic.

  Proof. Given. $\square$

- Examples.

  1. Bases of $\mathbb{F}_q{}^k$.
     - Dummit and Foote (2004) justifies that the number of distinct bases of $\mathbb{F}_q{}^k$ is

       $$(q^k - 1)(q^k - q)(q^k - q^2) \cdots (q^k - q^{k-1})$$

     - For every vector $v \in \mathbb{F}_q{}^k$, there are $q - 1$ other linearly dependent vectors (corresponding to the q $\mathbb{F}$-multiples of it).
  2. Subspaces of $\mathbb{F}_q{}^n$.
     - Dummit and Foote (2004) justifies that the number of distinct k-dimensional subspaces of $\mathbb{F}_q{}^n$ is

       $$\frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \cdots (q^k - q^{k-1})}$$

- Dimension of the quotient space.

  Theorem 11.7. Let V be a vector space over F, and let W be a subspace of V. Then V/W is a vector space with $\dim V = \dim W + \dim V/W$ (where if one side is infinite, then both are).

  Proof. Given. $\square$

- Dimension of the kernel and image of a linear transformation.

  Corollary 11.8. Let $\varphi : V \to U$ be a linear transformation of vector spaces over F. Then $\ker \varphi$ is a subspace of V, $\varphi(V)$ is a subspace of U, and $\dim V = \dim \ker \varphi + \dim \varphi(V)$.

  Proof. Given. $\square$

- Classifying isomorphic operator.

    Corollary 11.9. Let $\varphi : V \to W$ be a linear transformation of vector spaces of the same finite dimension. Then the following are equivalent.

    1. φ is an isomorphism.
    2. φ is injective, i.e., $\ker \varphi = 0$.
    3. φ is surjective, i.e., $\varphi(V) = W$.
    4. φ sends a basis of V to a basis of W.

    Proof. Given.   □

- **Null space** (of a linear transformation): The kernel of the linear transformation.

- **Nullity** (of a linear transformation): The dimension of the kernel of the linear transformation.

- **Rank** (of a linear transformation): The dimension of the image of the linear transformation.

- **Nonsingular** (linear transformation): A linear transformation φ for which $\ker \varphi = 0$.

- **General linear group**: The group of all nonsingular linear transformations from $V \to V$ under the group operation of composition. *Denoted by $\boldsymbol{GL(V)}$.*

    - Dummit and Foote (2004) justifies that if $V = \mathbb{F}_q^{\ n}$, then

    $$|GL(V)| = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})$$

Exercises

4. Prove that the space of real-valued functions on the closed interval $[a, b]$ is an infinite dimensional vector space over $\mathbb{R}$, where $a < b$.

5. Prove that the space of continuous real-valued functions on the closed interval $[a, b]$ is an infinite dimensional vector space over $\mathbb{R}$, where $a < b$.

10. Prove that any vector space V has a basis (by convention, the null set is the basis for the zero space). *Hint*: Let $\mathcal{S}$ be the set of subsets of V consisting of linearly independent vectors, partially ordered under inclusion; apply Zorn's Lemma to $\mathcal{S}$ and show that a maximal element of $\mathcal{S}$ is a basis.

11. Refine your argument in the preceding exercise to prove that any set of linearly independent vectors of V is contained in a basis of V.

12. If F is a field with a finite or countable number of elements and V is an infinite dimensional vector space over F with basis $\mathcal{B}$, prove that the cardinality of V equals the cardinality of $\mathcal{B}$. Deduce in this case that any two bases of V have the same cardinality.

13. Prove that as vector spaces over $\mathbb{Q}$, $\mathbb{R}^n \cong \mathbb{R}$ for all $n \in \mathbb{Z}^+$. Note that, in particular, this means that $\mathbb{R}^n$ and $\mathbb{R}$ are isomorphic as additive abelian groups.

14. Let $\mathcal{A}$ be a basis for the infinite dimensional vector space V. Prove that V is isomorphic to the direct sum of copies of the field F indexed by the set $\mathcal{A}$. Prove that the direct product of copies of F indexed by $\mathcal{A}$ is a vector space over F and it has strictly larger dimension than the dimension of V (see the exercises in Section 10.3 for the definitions of direct sum and direct product over infinitely many modules).

Section 11.2: The Matrix of a Linear Transformation

- Assumptions for this section.

    - V, W are vector spaces over the field F.
    - $\mathcal{B} = \{v_1, \ldots, v_n\}$ is an (ordered) basis of V, and $\mathcal{E} = \{w_1, \ldots, w_m\}$ is an (ordered) basis of W.
    - $\varphi \in \mathrm{Hom}(V, W)$.

- **Matrix** (of φ with respect to the bases $\mathcal{B}, \mathcal{E}$): The $m \times n$ matrix whose i, j entry is α_{ij}, where

$$\varphi(v_j) = \sum_{i=1}^{m} \alpha_{ij} w_i$$

  Denoted by $\boldsymbol{M_{\mathcal{B}}^{\mathcal{E}}(\varphi)}$.

- Dummit and Foote (2004) reviews how to recover φ from $M_{\mathcal{B}}^{\mathcal{E}}(\varphi)$.

    - The equivalence of matrix multiplying and linear transforming is sometimes denoted

$$[\varphi(v)]_{\mathcal{E}} = M_{\mathcal{B}}^{\mathcal{E}}(\varphi)[v]_{\mathcal{B}}$$

- **Representation** (of φ with respect to the bases $\mathcal{B}, \mathcal{E}$): The matrix $A = (a_{ij})$ associated with φ.

- Examples.

    1. Computing a matrix with respect to the standard bases of $\mathbb{R}^3, \mathbb{R}^2$.
    2. The matrix of the differentiation operator $\varphi : V \to V$ on the 2-dimensional space of solutions V to $y'' - 3y' + 2y = 0$.

        - Since

$$\varphi(v_1) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^t\right) = \mathrm{e}^t = v_1 \qquad\qquad \varphi(v_2) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{2t}\right) = 2\mathrm{e}^{2t} = 2v_2$$

        the representation of φ is

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

    3. Computing a matrix with respect to the standard bases of $\mathbb{Q}^3, \mathbb{Q}^3$.

- Isomorphism between the space of linear transformations and the space of matrices.

    Theorem 11.10. Let V be a vector space over F of dimension n and let W be a vector space over F of dimension m, with respective bases $\mathcal{B}, \mathcal{E}$. Then the map $\mathrm{Hom}_F(V, W) \to M_{m \times n}(F)$ from the space of linear transformations from V to W to the space of $m \times n$ matrices with coefficients in F defined by $\varphi \mapsto M_{\mathcal{B}}^{\mathcal{E}}(\varphi)$ is a vector space isomorphism. In particular, there is a bijective correspondence between linear transformations and their associated matrices with respect to a fixed choice of bases.

    Proof. Given.   □

- There is no *natural* isomorphism between $\mathrm{Hom}_F(V, W)$ and $M_{m \times n}(F)$.

    - This is because the choices of bases are arbitrary (there is no natural choice of them).

- Dimension of the space of linear transformations.

    Corollary 11.11. The dimension of $\mathrm{Hom}_F(V, W)$ is $(\dim V)(\dim W)$.

    Proof. Given.   □

- **Nonsingular** (matrix): An $m \times n$ matrix A such that $Ax = 0$ with $x \in F^n$ implies that $x = 0$. *Also known as* **invertible**.

- Nonsingular linear transformations vs. nonsingular matrices.

  - Independent of the choice of bases, a matrix is nonsingular iff the corresponding linear transformation is nonsingular.

- Dummit and Foote (2004) uses the definition of the matrix to deduce the formula for matrix multiplication.

- Relating matrix multiplication to linear transformation composition.

  Theorem 11.12. Let U, V, W be finite dimensional vector spaces over F with ordered bases $\mathcal{D}, \mathcal{B}, \mathcal{E}$, and assume $\psi : U \to V$ and $\varphi : V \to W$ are linear transformations. Then

  $$M_{\mathcal{D}}^{\mathcal{E}}(\varphi \circ \psi) = M_{\mathcal{B}}^{\mathcal{E}}(\varphi) M_{\mathcal{D}}^{\mathcal{B}}(\psi)$$

  In words, the product of the matrices representing the linear transformations φ, ψ is the matrix representing the composite linear transformation $\varphi \circ \psi$.

- Properties of matrix multiplication.

  Corollary 11.13. Matrix multiplication is associative and distributive (whenever the dimensions are such as to make products defined). An $n \times m$ matrix A is nonsingular if and only if it is invertible.

  Proof. Given.   □

- Ring-like properties of $M_n(F)$, as induced by those of $\operatorname{Hom}_F(V, V)$.

  Corollary 11.14.

  1. If $\mathcal{B}$ is a basis of the n-dimensional space V, the map $\varphi \mapsto M_{\mathcal{B}}^{\mathcal{B}}(\varphi)$ is a ring and a vector space isomorphism of $\operatorname{Hom}_F(V, V)$ onto the space $M_n(F)$ of $n \times n$ matrices with coefficients in F.
  2. $GL(V) \cong GL_n(F)$, where $\dim V = n$. In particular, if F is a finite field, the order of the finite group $GL_n(F)$ (which equals $|GL(V)|$) is given by the formula at the end of Section 11.1.

  Proof. Given.   □

- **Row rank** (of a matrix): The maximal number of linearly independent rows of the matrix, where the rows are considered as vectors in affine m-space.

- **Column rank** (of a matrix): The maximal number of linearly independent columns of the matrix, where the columns are considered as vectors in affine n-space.

- Relating ranks.

  - The rank of ψ equals the column rank of $M_{\mathcal{B}}^{\mathcal{E}}(\psi)$.

- **Similar** (matrices): Two $n \times n$ matrices A, B for which there exists an invertible $n \times n$ matrix P such that $P^{-1}AP = B$.

- **Similar** (linear transformations): Two linear transformations $\varphi, \psi : V \to V$ for which there exists a nonsingular linear transformation ξ such that $\xi^{-1}\varphi\xi = \psi$.

  - This is an equivalence relation whose equivalence classes are the orbits of $GL(V)$ acting by conjugation on $\operatorname{Hom}_F(V, V)$.

- **Transition** (matrix from $\mathcal{B}$ to $\mathcal{E}$): The matrix defined as follows, where I is the identity transformation. *Also known as* **change of basis** (matrix). *Denoted by $\boldsymbol{P}$. Given by*

$$P = M_\mathcal{B}^\mathcal{E}(I)$$

  - $P = M_\mathcal{B}^\mathcal{E}(I)$ satisfies $P^{-1} M_\mathcal{B}^\mathcal{B}(I) P = M_\mathcal{E}^\mathcal{E}(\varphi)$.
    - If $\mathcal{B} \neq \mathcal{E}$, then P is not the identity matrix.
  - Note that we need *ordered* bases to have a unique $P = M_\mathcal{B}^\mathcal{E}(I)$!

- **Change of basis**: The similarity action of $M_\mathcal{B}^\mathcal{E}(I)$ on $M_\mathcal{B}^\mathcal{B}(\varphi)$.

- Dummit and Foote (2004) proves that any two similar matrices represent the same linear transformation with respect to two different choices of bases.

- Example of similarity given.

- **Canonical forms**: The study of the simplest possible matrix representing a given linear transformation (and which basis to choose to realize it).

- We now move on to linear transformations on tensor products of vector spaces.

- Return to later.

- **Idempotent** (linear transformation): A linear transformation ψ satisfying $\psi^2 = \psi$.

  - Characterized in Exercise 11.2.11.

Section 11.3: Dual Vector Spaces

- **Dual space** (of a vector space): The space of linear transformations from V to F. *Denoted by $\boldsymbol{V^*}$.*

- **Linear functional**: An element of V^*.

- **Dual basis** (to a basis of V): The basis related to a basis $\{v_1, \ldots, v_n\}$ of V by

$$v_i^*(v_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

for $1 \leq j \leq n$. *Denoted by $\boldsymbol{\{v_1^*, \ldots, v_n^*\}}$.*

- The dual basis to a basis of V is a basis of V^*.

  Proposition 11.18. With notations as above, $\{v_1^*, \ldots, v_n^*\}$ is a basis of V^*. In particular, if V is finite dimensional, then V^* has the same dimension as V.

  Proof. Given. $\square$

- If V is infinite dimensional, then $\dim V < \dim V^*$.

- **Algebraic** (dual space to V): The dual space V^* taken for V of arbitrary dimension.

- If V has additional structure (e.g., a topology), we can get other types of dual spaces, such as the following.

- **Continuous** (dual of V): A dual of V in which the linear functionals must be continuous.

- Example.

  1. Let $V = C([a, b], \mathbb{R})$.
     - If $a < b$, then V is infinite dimensional.

- For each $g \in V$, the function $\varphi_g : V \to \mathbb{R}$ defined by

$$\varphi_g(f) = \int_a^b f(t)g(t) \, \mathrm{d}t$$

is a linear functional on V.

- **Double dual** (of V): The dual of V^*. *Also known as* **second dual**. *Denoted by* $\boldsymbol{V^{**}}$.

- For finite dimensional V, $\dim V = \dim V^{**}$ and hence $V \cong V^{**}$.

   - There is a **natural** (i.e., basis independent/coordinate free) isomorphism.
     - ■ More detail on this is given.
   - This is different for infinite dimensional V, as per the above.

- Existence of a natural map $V \to V^{**}$.

   Theorem 11.19. There is a natural injective linear transformation from V to V^{**}. If V is finite dimensional, then this linear transformation is an isomorphism.

   Proof. Given.   □

- $\boldsymbol{\varphi^*}$: The induced function from $W^* \to V^*$ defined by

$$f \mapsto f \circ \varphi$$

   - This is just the **pullback** or **dual map**.

- Pullback: Linearity and matrix.

   Theorem 11.20. With notations as above, φ^* is a linear transformation from W^* to V^* and $M_{\mathcal{E}^*}^{\mathcal{B}^*}(\varphi^*)$ is the transpose of the matrix $M_{\mathcal{B}}^{\mathcal{E}}(\varphi)$.

   Proof. Given.   □

- A partial statement of the rank-nullity theorem.

   Corollary 11.21. For any matrix A, the row rank of A equals the column rank of A.

   Proof. Given.   □

- **Annihilator** (of S in V): The set of all $v \in V$ for which $f(v) = 0$ for all $f \in S \subset V^*$. *Denoted by* **Ann($\boldsymbol{S}$)**. *Given by*
$$\mathrm{Ann}(S) = \{v \in V : f(v) = 0 \; \forall \; f \in S\}$$