Week 8

???

8.1 Linear Algebra Review and Rational Canonical Form

2/20:

- Nori's change of heart.
  - We've all seen linear algebra; thus, we'll speedrun it and then do exterior algebra and determinants. That's where we'll finish.
- The following is part 1 of a linear algebra course.
- Let F be a field.
- **Vector space**: An F-module.
- **Linearly independent** (subset $S \subset V$): Same definition we're familiar with.
- **Spanning** (subset $S \subset V$): A subset S of V that is a set of generators of V.
- S is a **basis** implies that S generates V and is linearly independent.
- Every linearly independent subset of V can be extended to a basis.
- Every spanning set S contains a basis.
  - Any maximal linearly independent subset of S is a basis.
- S_1, S_2 are bases for V implies that $|S_1| = |S_2|$.
  - The replacement theorem in Dummit and Foote (2004) is a good way to prove this.
- We are now done with part 1; this is part 2 of a linear algebra course.
- Let $T : V \to V$ be a linear transformation.
- Let A be a ring. What is an $A[X]$-module M?
  - It is an abelian group $(M, +)$ and a ring homomorphism $\rho : A[X] \to \operatorname{End}(M, +)$.
  - Since $A \hookrightarrow A[X]$, $\rho|_A$ turns M into an A-module.
  - Since $aX = Xa$, $\rho(a)\rho(X) = \rho(X)\rho(a)$.
  - But since we consider M to be a module, we write $a := \rho(a)$: Thus, $a\rho(X)m = \rho(X)am$ for all $m \in M$.
  - Note that $\rho(X) \in \operatorname{End}_A(M)$ (which is the set of all A-module homomorphisms).
  - Additionally, $\rho(X) : M \to M$ is an A-module homomorphism.

- Put $\rho(X) = T$. Thus, an $A[X]$-module is a pair (M, T), where M is an A-module and $T \in \mathrm{End}_A(M)$.
- Conversely, such (M, T) gives rise to an $A[X]$-module.
  - In particular, the action is
  $$\left(\sum_{n=0}^{\ell} a_n X^n\right) m = \sum_{n=0}^{\ell} a_n T^n m$$
- Take $A = F$ a field concerned with (V, T) where V is any F-vector space and $T : V \to V$ is a linear transformation.
  - This induces a module over $F[X]$.
- V finite dimensional induces $\rho : F[X] \to \mathrm{End}_F(V) \cong M_n(F)$ defined by $X \mapsto T$.
  - $\rho(X) = T$ and $\rho(c) = c$ for all $c \in F$.
- Let n^2 be the dimension of the F-vector space??
- Then $\ker(\rho) = (f)$ for some f be monic of degree $d \leq n^2$.

$$\begin{tikzcd}
F[X] \arrow[r, "\rho"] \arrow[d] & \mathrm{End}_F(V) \\
F[X]/(f) \arrow[ur, hook, "\bar{\rho}"] &
\end{tikzcd}$$

Figure 8.1: $F[X]$-module actions.

  - We have the constraint on the degree of f by the isomorphism from Lecture 3.1.
- **Minimal polynomial** (of T): The polynomial f that generates $\ker(\rho)$.
  - In particular, V is a torsion $F[X]$-module $(f \cdot g)$.
- **Cyclic vector**: A vector $v \in V$ belonging to (V, T) such that $v, Tv, T^2v, \dots$ spans V.
- Assume $v, Tv, T^2v, \dots, T^{k-1}v$ are linearly independent, but $v, Tv, \dots, T^k v$ are not.
  - Then
  $$T^k v = a_0 v + a_1 Tv + \cdots + a_{k-1} T^{k-1} v$$
  where all $a_i \in F$ and not all $a_i = 0$.
  - It follows that $T^m k \in \langle v, Tv, \dots, T^{k-1}v \rangle = W$ a vector space.
  - Let $g(X) = X^k - (a_{k-1}X^{k-1} + \cdots + a_1 X + a_0)$. Then $g(T)v = 0$. This implies that g is the minimal polynomial of T.
  - It follows that $T^h g(T)v = 0$. Thus, $g(T)T^h v = 0$ for all h.
  - Lastly, it follows that $g(T)w = 0$ for all $w \in W$.
  - Assume v is a cyclic vector. Then $W = V$. It follows that $g(T)v = 0$ for all $v \in V$.
  - The original assumption posits that no polynomial of degree less than or equal to $k - 1$ can annihilate v.
- Consider $V = F[X]/(f)$. Let $\deg(f) = d$, let $T : V \to V$, and let T be the "multiply by X" linear transformation. It follows that if $v_i = \overline{X^{i-1}}$ $(i = 1, \dots, d)$, then
$$Tv_i = v_{i+1}$$
for $i = 1, \dots, d - 1$ and
$$Tv_d = -(a_0 v_1 + a_1 v_2 + \cdots + a_{d-1} v_d)$$

- If $d = 3$, then we have
$$M(T) = \begin{pmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{pmatrix}$$
- The above matrix is called the **companion matrix** of f (monic of degree 3).

- **Rational canonical form**: The form (V, T) given by
$$F[X]/(f_1) \oplus \cdots \oplus F[X]/(f_s)$$
where $f_2 \mid f_1, \ldots, f_s \mid f_{s-1}$ and $\deg(f_s) > 0$.
  - When $V = 0$, then $s = 0$. In this case, f_1 is the minimal polynomial of T.
  - The form consisting of a block diagonal matrix of companion matrices.
- **Jordan canonical form**:
  - Has to do with p-primary components!
- There's one more canonical form, too.
- Since no one knows what canonical forms are and we very much need them for what Nori was planning to do, Nori will change his plans. No tensors in the last week, either.
- p-primary components: When $p = X - a$, $a \in F$.
- (V, T) is p**-primary** if there exists an n such that $(T - a)^n v = 0$ for all $v \in V$.
- $1_V : V \to V$ is the identity.
- $a \cdot 1_V = a_V : V \to V$.
- $(T - a_v)^n = 0 \in \mathrm{End}_F(V)$.
- We're now doing generalized eigenspaces ?? lol.
- The p-primary component is as the generalized a-eigenspace.
  - $(T - a)v = 0$, i.e., $Tv = av$ is the a-eigenspace; the eigenspaces are components of the generalized eigenspaces.
- Let $V = F[X]/(X - a)^n$. Let $v_1 = 1$, $v_2 = \overline{X - a}$, $\ldots$, $v_n = \overline{(X - a)^{n-1}}$.
  - We know that $X(X - a)^r = (X - a + a)(X - a)^r = (X - a)^{r+1} + a(X - a)^r$.
  - Nori writes Jordan blocks as
$$\begin{pmatrix} a & 0 & 0 & 0 \\ 1 & a & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 1 & a \end{pmatrix}$$
not with 1's in the superdiagonal.
    - Thus, the *last* generalized eigenvector is an eigenvector here, instead of the *first*.