# 4 Applications of Fraction Rings

Throughout this assignment, $R$ will denote a *commutative* ring.

2/1: **4.1.** Let $R$ be a ring, and let $f \in R$ be an element which is not a zero divisor. Recall that we defined $R_f = D^{-1}R$ for $D = \{1, f, f^2, \dots\}$. Prove that

$$R_f \cong R[X]/(fX - 1)$$

using the universal property of the ring of fractions.

**4.2.** Let $\mathbb{Z}[i] = \mathbb{Z}[X]/(X^2 + 1)$ denote the ring of **Gaussian integers**. Recall from class that $\mathbb{Z}[i]$ is a Euclidean domain with norm $N : \mathbb{Z}[i] \to \mathbb{Z}_{\geq 0}$ defined by $N(a + bi) = a^2 + b^2$.

(a) Let $R$ be a Euclidean domain with norm $N$ which satisfies $N(xy) = N(x)N(y)$ for all $x, y \in R$. Prove that $a \in R$ is a unit iff $N(a) = 1$. (Hint: Start by computing $N(1)$.)

(b) Using part (a), find the units in $\mathbb{Z}[i]$.

(c) Prove that $\mathrm{Frac}(\mathbb{Z}[i]) = \mathbb{Q}[i]$.

**4.3.** (a) For $a, b \in \mathbb{Z}$, prove that $a^2 - 2b^2 = 0$ iff $a = b = 0$.

(b) Prove that $\mathbb{Q}[\sqrt{2}] = \mathbb{Q}[X]/(X^2 - 2)$ is a field.

**4.4.** Let $D$ be a multiplicative subset of an integral domain $R$. Now $R$ is a subring of $D^{-1}R$. Let $J$ be an ideal of $D^{-1}R$. Put $I = R \cap J$.

(a) Is $I$ an ideal of $R$?

(b) Prove that if $I \neq R$, then $I \cap D = \emptyset$.

(c) Let $b \in J$. Is it true that $b = d^{-1}a$ for some $d \in D$ and $a \in I$?

(d) Prove that if $I$ is an ideal in $R$, then $I^e = \{s^{-1}x \in D^{-1}R \mid s \in D,\ x \in I\}$ is an ideal in $D^{-1}R$.

(e) Using part (c), prove that if $J$ is an ideal of $D^{-1}R$, then $J = (R \cap J)^e$. Therefore, we have a surjective map of sets

$$\{\text{Ideals in } R\} \to \{\text{Ideals in } D^{-1}R\}$$

given by $I \mapsto I^e$. Note that the right inverse is given by $J \mapsto R \cap J$. Is this map a bijection?

(f) If $R$ is a PID, is $D^{-1}R$ a PID?

**4.5.** (a) Let $D = \{n \in \mathbb{Z} : 2 \nmid n\}$. Recall that we defined

$$\mathbb{Z}_{(2)} = D^{-1}R = \{a/b \in \mathbb{Q} : 2 \nmid b\}$$

Write down all of the ideals in $\mathbb{Z}_{(2)}$. You can use the fact that the ideals in $\mathbb{Z}$ are $(n) = n\mathbb{Z}$ for $n \in \mathbb{Z}$, and the previous question. Which of these ideals are maximal? For each maximal ideal $M \in \mathbb{Z}_{(2)}$, what is the field $\mathbb{Z}_{(2)}/M$?

(b) Let $D = \{2^n \mid n \in \mathbb{Z}_{\geq 0}\}$ and let $R = D^{-1}\mathbb{Z}$. Write down the ideals in $R$. Which of these ideals are maximal?

**4.6.** (a) Define $M_2 : \{\text{commutative rings}\} \to \{\text{sets}\}$ by

$$M_2(R) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in R \right\}$$

Show that for any $R$, there is a natural bijection between the set $M_2(R)$ and the set $S_1$ of ring homomorphisms between $\mathbb{Z}[X, Y, Z, W]$ and $R$. Note that notationally,

$$S_1 = \mathrm{Hom}_{\mathrm{ring}}(\mathbb{Z}[X, Y, Z, W], R)$$

One sometimes says that $\mathbb{Z}[X, Y, Z, W]$ represents the function $M_2$.

(b) (**You do not need to turn in part (b)**, but you are encouraged to think about it.)

Actually, $M_2(R)$ can be naturally given a ring structure: Addition and multiplication are defined using the same procedure as $M_2(\mathbb{R})$ (or with any other field you may have seen). Hence, it makes sense to talk about the units of $M_2(R)$.

Define the set $GL_2(R)$ to be the units of $M_2(R)$, i.e.,

$$GL_2(R) = M_2(R)^\times$$

Show that for any $R$, there is a natural bijection between $GL_2(R)$ and the set $S_2$ defined by

$$S_2 = \mathrm{Hom}_{\mathrm{ring}}(\mathbb{Z}[X, Y, Z, W]_{XW-YZ}, R)$$

Note that $\mathbb{Z}[X, Y, Z, W]_{XW-YZ}$ denotes the **localization** of $\mathbb{Z}[X, Y, Z, W]$ by the multiplicative set generated by $XW - YZ$ (that is, the multiplicative set $(1, XW - YZ, (XW - YZ)^2, \dots)$). (Hint: Use the universal property.)

One sometimes says $\mathbb{Z}[X, Y, Z, W]_{XW-YZ}$ represents the function $GL_2$.

**4.7.** Let $\mathbb{Q}(X)$ denote the field of fractions of $\mathbb{Q}[X]$. By the universal property of a polynomial ring, we know that giving a ring homomorphism $\varphi : \mathbb{Q}[X] \to \mathbb{R}$ is equivalent to choosing an element $r \in \mathbb{R}$ and setting $\varphi(X) = r$. Which ring homomorphisms $\varphi : \mathbb{Q}[X] \to \mathbb{R}$ extend to ring homomorphisms $\tilde{\varphi} : \mathbb{Q}(X) \to \mathbb{R}$? These ring homomorphisms should satisfy the following commutative diagram.

$$\begin{tikzcd}
\mathbb{Q}[X] \arrow[r, "\varphi"] \arrow[d, "X \mapsto X/1"'] & \mathbb{R} \\
\mathbb{Q}(X) \arrow[ur, "\tilde{\varphi}"'] &
\end{tikzcd}$$

**4.8.** $F$ is a field. Let $R$ be the smallest subring of $F[X]$ such that (a) $F \subset R$ and (b) both $X^2$ and $X^3$ belong to $R$.

(a) Use the identity $(X^2)^3 = (X^3)^2$ to deduce that $R$ is *not* a UFD.

(b) Exhibit an ideal $I$ of $R$ that is not a principal ideal.

**4.9.** Mimic Euclid's proof of the infinitude of primes in $\mathbb{Z}$ to show that $F[X]$ has infinitely many primes for every field $F$.

**4.10.** Let $R$ be an integral domain and let $d$ be the degree of a nonzero $f \in R[X]$. Prove that $\{a \in R \mid f(a) = 0\}$ is finite. *Hint*: Case 1 — first prove this when $R$ is a field. Case 2 — reduce to case 1 by looking at the fraction field of $R$.