Week 8

???

8.1 Linear Algebra Review and Rational Canonical Form

2/20:

- Nori's change of heart.
  - We've all seen linear algebra; thus, we'll speedrun it and then do exterior algebra and determinants. That's where we'll finish.
- The following is part 1 of a linear algebra course.
- Let F be a field.
- **Vector space**: An F-module.
- **Linearly independent** (subset $S \subset V$): Same definition we're familiar with.
- **Spanning** (subset $S \subset V$): A subset S of V that is a set of generators of V.
- S is a **basis** implies that S generates V and is linearly independent.
- Every linearly independent subset of V can be extended to a basis.
- Every spanning set S contains a basis.
  - Any maximal linearly independent subset of S is a basis.
- S_1, S_2 are bases for V implies that $|S_1| = |S_2|$.
  - The replacement theorem in Dummit and Foote (2004) is a good way to prove this.
- We are now done with part 1; this is part 2 of a linear algebra course.
- Let $T : V \to V$ be a linear transformation.
- Let A be a ring. What is an $A[X]$-module M?
  - It is an abelian group $(M, +)$ and a ring homomorphism $\rho : A[X] \to \text{End}(M, +)$.
  - Since $A \hookrightarrow A[X]$, $\rho|_A$ turns M into an A-module.
  - Since $aX = Xa$, $\rho(a)\rho(X) = \rho(X)\rho(a)$.
  - But since we consider M to be a module, we write $a := \rho(a)$: Thus, $a\rho(X)m = \rho(X)am$ for all $m \in M$.
  - Note that $\rho(X) \in \text{End}_A(M)$ (which is the set of all A-module homomorphisms).
  - Additionally, $\rho(X) : M \to M$ is an A-module homomorphism.

- Put $\rho(X) = T$. Thus, an $A[X]$-module is a pair (M, T), where M is an A-module and $T \in \mathrm{End}_A(M)$.
- Conversely, such (M, T) gives rise to an $A[X]$-module.
  - In particular, the action is
  $$\left(\sum_{n=0}^{\ell} a_n X^n\right) m = \sum_{n=0}^{\ell} a_n T^n m$$
- Take $A = F$ a field concerned with (V, T) where V is any F-vector space and $T : V \to V$ is a linear transformation.
  - This induces a module over $F[X]$.
- V finite dimensional induces $\rho : F[X] \to \mathrm{End}_F(V) \cong M_n(F)$ defined by $X \mapsto T$.
  - $\rho(X) = T$ and $\rho(c) = c$ for all $c \in F$.
- Let n^2 be the dimension of the F-vector space??
- Then $\ker(\rho) = (f)$ for some f be monic of degree $d \leq n^2$.

$$\begin{tikzcd}
F[X] \arrow[r, "\rho"] \arrow[d] & \mathrm{End}_F(V) \\
F[X]/(f) \arrow[ur, hook, "\bar{\rho}"] &
\end{tikzcd}$$

Figure 8.1: $F[X]$-module actions.

  - We have the constraint on the degree of f by the isomorphism from Lecture 3.1.
- **Minimal polynomial** (of T): The polynomial f that generates $\ker(\rho)$.
  - In particular, V is a torsion $F[X]$-module $(f \cdot g)$.
- **Cyclic vector**: A vector $v \in V$ belonging to (V, T) such that $v, Tv, T^2v, \dots$ spans V.
- Assume $v, Tv, T^2v, \dots, T^{k-1}v$ are linearly independent, but $v, Tv, \dots, T^k v$ are not.
  - Then
  $$T^k v = a_0 v + a_1 Tv + \cdots + a_{k-1} T^{k-1} v$$
  where all $a_i \in F$ and not all $a_i = 0$.
  - It follows that $T^m k \in \langle v, Tv, \dots, T^{k-1}v \rangle = W$ a vector space.
  - Let $g(X) = X^k - (a_{k-1}X^{k-1} + \cdots + a_1 X + a_0)$. Then $g(T)v = 0$. This implies that g is the minimal polynomial of T.
  - It follows that $T^h g(T) v = 0$. Thus, $g(T)T^h v = 0$ for all h.
  - Lastly, it follows that $g(T)w = 0$ for all $w \in W$.
  - Assume v is a cyclic vector. Then $W = V$. It follows that $g(T)v = 0$ for all $v \in V$.
  - The original assumption posits that no polynomial of degree less than or equal to $k - 1$ can annihilate v.
- Consider $V = F[X]/(f)$. Let $\deg(f) = d$, let $T : V \to V$, and let T be the "multiply by X" linear transformation. It follows that if $v_i = \overline{X^{i-1}}$ $(i = 1, \dots, d)$, then
  $$Tv_i = v_{i+1}$$
  for $i = 1, \dots, d-1$ and
  $$Tv_d = -(a_0 v_1 + a_1 v_2 + \cdots + a_{d-1} v_d)$$

  - If $d = 3$, then we have
$$M(T) = \begin{pmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{pmatrix}$$
  - The above matrix is called the **companion matrix** of f (monic of degree 3).
- **Rational canonical form**: The form (V, T) given by
$$F[X]/(f_1) \oplus \cdots \oplus F[X]/(f_s)$$
where $f_2 \mid f_1, \ldots, f_s \mid f_{s-1}$ and $\deg(f_s) > 0$.
  - When $V = 0$, then $s = 0$. In this case, f_1 is the minimal polynomial of T.
  - The form consisting of a block diagonal matrix of companion matrices.
- **Jordan canonical form**:
  - Has to do with p-primary components!
- There's one more canonical form, too.
- Since no one knows what canonical forms are and we very much need them for what Nori was planning to do, Nori will change his plans. No tensors in the last week, either.
- p-primary components: When $p = X - a$, $a \in F$.
- (V, T) is p**-primary** if there exists an n such that $(T - a)^n v = 0$ for all $v \in V$.
- $1_V : V \to V$ is the identity.
- $a \cdot 1_V = a_V : V \to V$.
- $(T - a_v)^n = 0 \in \mathrm{End}_F(V)$.
- We're now doing generalized eigenspaces ?? lol.
- The p-primary component is as the generalized a-eigenspace.
  - $(T - a)v = 0$, i.e., $Tv = av$ is the a-eigenspace; the eigenspaces are components of the generalized eigenspaces.
- Let $V = F[X]/(X - a)^n$. Let $v_1 = 1$, $v_2 = \overline{X - a}$, $\ldots$, $v_n = \overline{(X - a)^{n-1}}$.
  - We know that $X(X - a)^r = (X - a + a)(X - a)^r = (X - a)^{r+1} + a(X - a)^r$.
  - Nori writes Jordan blocks as
$$\begin{pmatrix} a & 0 & 0 & 0 \\ 1 & a & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 1 & a \end{pmatrix}$$
not with 1's in the superdiagonal.
    - Thus, the *last* generalized eigenvector is an eigenvector here, instead of the *first*.

8.2 Office Hours (Nori)

- Midterm: We never covered the universal property of a quotient in class, did we?
  - That's the special lemma from last office hours.
- PSet 7: 7.3 and 7.4 typos.
- Wednesday lecture?
  - Seventh week summary will suffice.
- What do you need us to know about the rational canonical form? Should I still read Dummit and Foote (2004), Section 12.2 or is that no longer necessary?
  - Nori will probably push ahead with 12.2. Thus I should read it. He's not sure what he'll do beyond that, though, since he doesn't want to jam tensors into the last week.
  - I will need tensor products for representation theory, regardless, so if I want to take it, I should self-study it.
  - No chance tensor products will be covered next quarter.
  - Serre is a terrific mathematician whose wife is a super chemist, and that's why he wrote his book on representation theory (and wrote it in a less terse manner than usual).
  - No tensors means no exterior algebra, too.
  - Nori hasn't read any of Dummit and Foote (2004).
  - The transfer theory of groups arises in a later chapter, and that's important for representation theory, though.
- Nori doesn't think any teacher pays attention to what courses are supposed to cover as stated in the course catalog.
  - We will never do modules, multilinear and quadratic forms.
  - p-adic field and Galois theory.
  - Nori thinks the proof of Theorem 12.4 is very difficult to follow for a first-timer.
  - Solvable groups were supposed to be a MATH 25700 topic, but got cut because of 9-week quarters.
  - Syclotomic fields have applications to the representation theory of finite groups; there are theorems of representation theory that you need syclotomic fields to prove.
  - Emil Artin: Galois Theory is worth looking up.
  - Gauss and constructions of 17-gons also needs syclotomic fields.