7 Classifying Representations of the Symmetric Group

12/1:

1. Prove that the representation $(n-k, 1, \dots, 1)$ is isomorphic to a wedge power of the standard representation of S_n.
2. Deduce from the hook length formula that if V is an irreducible representation of S_n for $n \geq 5$ and $\dim(V) < n$, then $\dim(V) = 1$ or $\dim(V) = n-1$.
3. Find the pairs of Young tableaux corresponding to the following permutations.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 4 & 3 & 5 & 7 & 6 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 5 & 3 & 6 \end{pmatrix}$$

4. Prove that the character table of S_n is filled with rational numbers.
5. Let $X_i : S_n \to \mathbb{C}$ send each $\sigma \in S_n$ to the number of i-cycles in σ. Clearly X_i is a class function. Moreover, any polynomial $P(X_1, \dots, X_n)$ is a class function called a **character polynomial**.
   (a) Find a basis of class functions consisting of character polynomials for S_2, S_3, and S_4.
   (b) Do the same for S_5.
   (c) Find a character polynomial which equals the character of $(n-1, 1)$.
6. Consider a subgroup $H = \{e, (12)(34), (13)(24), (14)(23)\}$ in S_4. Decompose $\operatorname{Ind}_H^G \chi$ into irreducibles for all irreducible characters χ of H.