Week 8

???

8.1 Specht Modules are Irreducible and Well-Defined

11/13:
- Announcements.
  - This week's homework is the next to last one.
- Review.
  - Miracuously, we can understand all representations of S_n.
  - We start with partitions λ that are defined a certain way. We visualize them with Young diagrams.
  - The number of partitions of n is equal to the number of conjugacy classes in S_n is equal to the number of irreps in S_n.
    - It is a special feature of S_n that this is true.
  - How do we construct the irreducible representation V_λ due to λ?
    - Consider $(4,2,1)' = (3,2,1,1)$ as an example (recall the definition of an inverse partition).
    - Take Vandermonde determinants (recall the explicit definition of these, too).
    - Then we define $V_\lambda = \mathbb{C}[S_n]$, take Vandermonde determinant of variables corresponding to the first column, so that $\Delta(x_1,\dots,x_{\lambda_1'})\Delta(x_{\lambda_1'+1},\dots,x_{\lambda_2'})\cdots\Delta(x_{\lambda_{k-1}'+1},\dots,\lambda_k')$.
    - Thus, we let $\mathbb{C}[S_n]$ act on $(x_1-x_2)(x_1-x_3)(x_2-x_3)(x_4-x_5)$.
- One more example.
  - $\lambda = (2,2)$.
  - Let $\mathbb{C}[S_4]$ act on $(x_1-x_2)(x_3-x_4)$.
  - Then
$$V_\lambda = \langle (x_1-x_2)(x_3-x_4),(x_1-x_3)(x_2-x_4),(x_1-x_4)(x_2-x_3)\rangle$$
  - But we're expecting a 2D representation. Indeed, we get one because if we define the first term above to be a and the second to be b, then the third is $b-a$. Thus, there are only two linearly independent polynomials herein.
  - Now we calculate entries in the character table as follows: See how representatives of conjugacy classes like (12) and (123) acts on a,b via matrices, and then calculate traces of these matrices.
    - For example, using the definitions of a,b from above, we can see that
$$\begin{aligned}(12)\cdot a &= (12)\cdot(x_1-x_2)(x_3-x_4) = (x_2-x_2)(x_3-x_4) = -(x_1-x_2)(x_3-x_4) = -a\\ (12)\cdot b &= (12)\cdot(x_1-x_3)(x_2-x_4) = (x_2-x_3)(x_1-x_4) = (x_1-x_4)(x_2-x_3) = b-a\end{aligned}$$

- In matrix form, the above equations become

$$\begin{bmatrix} -a \\ b-a \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}}_{\rho(12)} \begin{bmatrix} a \\ b \end{bmatrix}$$

- Thus, $\chi(12) = \operatorname{tr}(\rho(12)) = 0$.
- Similarly, we can calculate that

$$\begin{bmatrix} b-a \\ -a \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}}_{\rho(123)} \begin{bmatrix} a \\ b \end{bmatrix}$$

  so $\chi(123) = -1$.

  - One of the HW problems is to do exactly this for S_4 just for practice.

- Today: A theorem that does...??
- Note that V_λ is called a **Specht module** and one of these polynomials on the above blackboard is a **Specht polynomial**.
- Theorem 1: V_λ is irreducible.

Proof. $d(\lambda)$ is the degree of a Specht polynomial and is given by

$$\sum_{i=1}^{k'} \frac{\lambda_i'(\lambda_i'-1)}{2}$$

Let $R_d \subset \mathbb{C}[x_1,\dots,x_n]$, where R_d are just polynomials of degree d. Clearly, by definition, $V_\lambda \subset R_d$. Note that as a definition, $R_d = S^d(V_{\text{perm}}^*)$. We now claim that $\operatorname{Hom}_{S_n}(V_\lambda, R_d) \cong \mathbb{C}$.

This claim implies the theorem because if we assume that $V_\lambda = \bigoplus W_i^{n_i}$ and $R_d = \bigoplus W_i^{m_i}$ where the W_i are all irreps. Then we have a nice way to compute this Hom from previous classes, explicitly, that the only homomorphisms $W_i \to W_i$. Thus, $\dim \operatorname{Hom} = \sum n_i m_i$. What this claim implies is that $\dim \operatorname{Hom} = 1$. Additionally, we are in a subrepresentation, so $n_i \leq m_i$ for all i. Thus, we must have $n_i = 1, m_i = 1$ for some i and that $n_j, m_j = 0$ for all other j. Restated, WLOG let $1 \leq n_i$. Then since $n_i \leq m_i$ and $n_i m_i = 1$, we have $m_i = 1$ and all other $n_j, m_j = 0$. Thus, $V_\lambda = W_i$ is irreducible!

Now we actually have to prove the claim. Let $f \in \operatorname{Hom}_{S_n}(V_\lambda, R_d)$ be arbitrary. Consider

$$f(\Delta(x_1,\dots,x_{\lambda_1'})\Delta(x_{\lambda_1'+1},\dots,x_{\lambda_2'})\dots)$$

where the argument is the general Specht polynomial from above. $f(x)$ is a polynomial of degree d; call $f(x)$ by $P(x_1,\dots,x_n)$. It is antisymmetric in $x_1,\dots,x_{\lambda_1'}$. It's also antisymmetric in $x_{\lambda_1'+1},\dots,x_{\lambda_2'}$. In fact, it's antisymmetric in all such sets all the way up to $x_{\lambda_{k'-1}'+1},\dots,x_{\lambda_{k'}'}$. It follows that $P(x_1,\dots,x_n)$ is divisible by $\Delta(x_1,\dots,x_{\lambda_i'})$, etc., i.e., all Vandermonde determinants. Thus, $P(x_1,\dots,x_n)$ is divisible by the product, which is the Specht polynomials. It follows that $P(x_1,\dots,x_n) = u \cdot$Specht polynomial, from which it follows that $f = uI$. This implies the claim via the isomorphism $f \mapsto u$! ☐

- Corollary: If $d' < d$, then $\operatorname{Hom}(V_\lambda, R_d') = 0$.
- Theorem 2: Let λ_1, λ_2 be partitions of n. Then $V_{\lambda_1} \cong V_{\lambda_2}$ iff $\lambda_1 = \lambda_2$.

Proof. Suppose that $V_{\lambda_1} \cong V_{\lambda_2}$.

Then $d(\lambda_1) = d(\lambda_2)$ (take the columns and compute the degree of the Specht polynomial). (If not, WLOG let $d(\lambda_1) > d(\lambda_2)$. Then $V_{\lambda_1} \cong V_{\lambda_2} \hookrightarrow R_{d(\lambda_2)}$. But then by the above corollary, this overall injective embedding is the zero map, a contradiction.)

Let $d := d(\lambda_1) = d(\lambda_2)$. At this point, we have $V_{\lambda_1} \hookrightarrow R_d$ and $V_{\lambda_2} \hookrightarrow R_d$. It follows that $V_{\lambda_1} = V_{\lambda_2}$ as a subspace of R_d. Essentially, since we have the isomorphism $V_{\lambda_1} \cong V_{\lambda_2}$, we can construct the second embedding by factoring through the first; but then this second embedding should just give the same image.

Claim: Polynomials in $V_{\lambda_1}, V_{\lambda_2}$ (which we can think of as subspaces/explicit polynomials) have no monomials in common. For this, it's enough to understand monomials in one V_{λ_1}. Which monomials appear in V_λ? Here's an example. We will do a representative example instead of a formal group. Consider $\lambda = (5, 4, 2, 2)$ and S_{13}. $\lambda' = (4, 4, 2, 2, 1)$. Our Specht polynomial is

$$\Delta(x_1, x_2, x_3, x_4)\Delta(x_5, x_6, x_7, x_8)\Delta(x_9, x_{10})\Delta(x_{11}, x_{12})$$

since $\Delta(x_{13}) = 1$. We have that

$$\Delta(x_1, x_2, x_3, x_4) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{vmatrix} = \sum_{\sigma \in S_4} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)}^0 x_{\sigma(3)}^2 x_{\sigma(4)}^3$$

Then we will have for each column, a number of variables in power each of $0, \ldots, 3$; on and on. Now we count the number of variables in power $0, \ldots, 3$ to get 5,4,2,2. Thus, every monomial will have 5 variables in power 0, 4 variables in power 1, 2 variables in power 2, and 2 variables in power 3. Thus, from every monomial, we immediately reconstruct λ. It means that we can reconstruct from any monomial this representation, so this implies that we must have $\lambda_1 = \lambda_2$. $\square$

- Corollary: V_λ's are all irreps of S_n

Proof. They are pairwise isomorphic and their number equals n. $\square$