Week 2

Constructing Representations

2.1 The Tensor Product

10/2:

- Before lecture, I chatted with a few people about tensor products and the exterior and symmetric powers.
  - Patrick: A **tensor** $v \otimes w$ is just an element of a vector space, indexed differently than in a column.
  - Raman: There is no canonical way to transform tensors into column vectors.
- Course logistics.
  - OH: T 5:30-6:30(+) and W 5:30-6:30(+). We can also meet one-on-one.
  - HW is due Thursdays at midnight.
- Today: Constructing new representations from old.
  - Rudenko will skim through tensor products really quickly.
- Reminder: Last time, we talked about how representation theory is really quite simple. If G is a finite group and $\mathbb{F} = \mathbb{C}$, there exist a finite set $V_1, \dots, V_s$ of irreps up to isomorphism, and every finite-dimensional representation $V \cong V_1^{n_1} \oplus \cdots \oplus V_s^{n_s}$.
- If V is a representation of G, then there are loads of things we can do with it.
  - We can construct the dual representation V^*.
  - We can construct the representation $V \otimes V$.
  - We can construct symmetric powers.
  - We can construct wedge powers.
  - There are more, but this is enough for now.
- Even when we take a very simple group and representation, there are some very interesting things that can fall out.
  - Example: If you take the symmetric powers of S_3 — as in PSet 1, Q4 — you get something really interesting.
- Now, we go to linear algebra.
- Let V, W be vector spaces over a field F. How do we produce a new vector space out of these?
- $\mathrm{Hom}_F(V, W)$ is the vector space of linear maps $f : V \to W$!
  - $\dim \mathrm{Hom}_{\mathbb{F}}(V, W) = (\dim V)(\dim W)$.

- Can we make $\text{Hom}_F(V,W)$ into a representation of G? Yes!

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "\rho_V(g)"'] & W \arrow[d, "\rho_W(g)"] \\ V \arrow[r, "gL"'] & W \end{tikzcd}$$

Figure 2.1: Commutative diagram, linear maps space representation.

  - Suppose that V,W are G-reps, which gives us $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$.
  - Suppose also that we have $L \in \text{Hom}_F(V,W)$.
  - Now infer from the commutative diagram that it will work to define $gL = \rho_W(g) \circ L \circ \rho_V(g)^{-1}$.
  - This is pretty standard.
- Recall that there is a different space $\text{Hom}_G(V,W)$ of morphisms of G-representations (see Figure 1.2 and the associated discussion).
  - This is a very very small subspace of $\text{Hom}_F(V,W)$.
- Special case of the above construction: **Dual representation**.
  - Consider $\text{Hom}_F(V,F)$. This the **dual vector space**.
  - Basic fact 1: Let $e_1,\dots,e_n$ be a basis of V. Then V^* has a corresponding basis $e^1,\dots,e^n$ known as its **dual basis**.
    - Computing coordinates already depends on a basis, and having bases is super nice.
    - Corollary: $\dim V = \dim V^*$.
    - This is the first time **canonical** comes into linear algebra. Canonical (nobody understands what it means) basically means that something doesn't depend on choices.
    - In particular, V,V^* are isomorphic because they have the same dimension, but for no more natural reason. They can be the same representation, or they can be different.
  - Basic fact 2: If V is finite-dimensional, then $(V^*)^* \cong V$. The formula for this isomorphism is canonical, because it does not depend on a choice of basis. In particular, choose the map $V \to (V^*)^*$ sending v to the map sending $\varphi \in V^*$ to $\varphi(v)$.
  - If V is infinite dimensional, none of this is true and you are in the realm of functional analysis.
  - Ok, so all of this was good information about the dual *space*, but what is the dual *representation*??[1] Does it matter, and do we need to know for now?
    - Defined below in the notes on the reading from Fulton and Harris (2004).
- **Dual vector space** (of V): The vector space defined as follows, given that V is a vector space over F. *Denoted by* $\boldsymbol{V^*}$. *Given by*
$$V^* = \text{Hom}_F(V,F)$$
- **Dual basis** (of V^* to $e_1,\dots,e_n$): The basis defined as follows for $i=1,\dots,n$, where $e_1,\dots,e_n$ is a basis of V. *Denoted by* $\boldsymbol{e^1,\dots,e^n}$. *Given by*
$$e^i(x_1e_1 + \cdots + x_ne_n) = x_i$$
- We now move onto the tensor product.
  - The tensor product is very hard to understand. If you learn about it and you feel you don't understand it, that's typical; nobody understands it at first.

[1]This question is answered in this week's OH, and a formal definition of the dual representation is given on Fulton and Harris (2004, p. 4) and transcribed in Section 2.8 of these notes.

  - For now, we'll discuss two ways of thinking about tensor products that won't bring us any comfort.
- Let V, W be two vector spaces over a field F.
- Abstract definition of the tensor product.
  - We have discussed maps from $V \to W$, but there is another related space.
  - Indeed, we can look at the space of **bilinear maps** from $V \times W \to F$.
  - Let V have basis $e_1, \dots, e_n$ and W have basis $f_1, \dots, f_m$.
  - Notice that every bilinear map f can be defined as a linear combination of the $f(e_i, f_j)$. In other words, the $f(e_i, f_j)$ form the basis of a function space.
    - This "bilinear maps space" has dimension nm.
  - Now, one way to understand a tensor product: Is this "bilinear maps space" actually some other space? It is! It is $(V \otimes W)^*$.
  - Bilinear maps are linear maps from where? From $V \otimes W$!
- **Bilinear** (map): A function $f : V \times W \to Z$ that satisfies the following constraints, where V, W, Z are vector spaces over F, $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, and $\lambda \in F$. *Constraints*

$$
\begin{aligned}
f(v_1 + v_2, w) &= f(v_1, w) + f(v_2, w) & \qquad f(\lambda v, w) &= \lambda f(v, w) \\
f(v, w_1 + w_2) &= f(v, w_1) + f(v, w_2) & \qquad f(v, \lambda w) &= \lambda f(v, w)
\end{aligned}
$$

- We now look at a much more elementary definition of the tensor product.
- Explicit definition of the tensor product.
  - Start off with the huge, easy-to-work-with vector space with basis consisting of pairs of elements $(v, w) \in V \times W$.
    - For example, even if V, W are one dimensional, this is like all pairs of real numbers $(1, 0)$, $(2, 0)$, (π, e), etc. *as basis vectors*; it's huge.
  - Then, we quotient it by the space of all elements satisfying the relations $\lambda(v, w) = (\lambda v, w) = (v, \lambda w)$, $(v_1 + v_2, w) = (v_1, w) + (v_2, w)$, and the like. These elements will be linear combinations of basis vectors of the following form: $\lambda(v, w) - (\lambda v, w)$, $\lambda(v, w) - (v, \lambda w)$, and $(v_1 + v_2, w) - (v_1, w) - (v_2, w)$.
    - This forces these relationships to be true. For example, in the final quotient space, we can still construct the element $\lambda(v, w) - (\lambda v, w)$. But its inclusion in the quotiented-out subspace will imply that in the quotient space, $\lambda(v, w) - (\lambda v, w) = 0$. It follows from here that $\lambda(v, w) = (\lambda v, w)$, as desired.
  - What do these relations do?
    - Essentially, they allows us to treat tensor multiplication much like real multiplication, endowing the operation with distributivity, etc.
    - For example, the rule $(v_1 + v_2, w) = (v_1, w) + (v_2, w)$ becomes, in tensor product notation, $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$.
  - Here's an example of this construction.
    - Let $V = W$ be the one-dimensional vector space over the finite field $F_2 = \mathbb{Z}/2\mathbb{Z}$.
    - Thus, the elements of V are $\{0, 1\}$ (which is, literally, all linear combinations $a0 + b1$ where $a, b \in F_2$ as well; this hearkens back to V's definition as an F_2-module).
    - Then the easy-to-work-with vector space we're talking about is the 4-dimensional **free** vector space $U = \operatorname{span}(0 \otimes 0, 0 \otimes 1, 1 \otimes 0, 1 \otimes 1)$.

- Note that in this space, for example, $(0+1)\otimes 0 \neq 0\otimes 0 + 1\otimes 0$; representing the basis as column vectors, this is equivalent the obvious observation that
$$\begin{bmatrix}0\\0\\1\\0\end{bmatrix} \neq \begin{bmatrix}1\\0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$
- But we want such relationships to hold true in our conceptual "tensor product space." Thus, we quotient it by the subspace spanning all elements of the form $(a+b)\otimes c - a\otimes c - b\otimes c$.
- By direct computation, this subspace is $\operatorname{span}(0\otimes 0, 0\otimes 1)$:
$$\begin{aligned}
&(0+0)\otimes 0 - 0\otimes 0 - 0\otimes 0 = -0\otimes 0 && (0+0)\otimes 1 - 0\otimes 1 - 0\otimes 1 = -0\otimes 1\\
&(0+1)\otimes 0 - 0\otimes 0 - 1\otimes 0 = -0\otimes 0 && (0+1)\otimes 1 - 0\otimes 1 - 1\otimes 1 = -0\otimes 1\\
&(1+1)\otimes 0 - 1\otimes 0 - 1\otimes 0 = 0\otimes 0 && (1+1)\otimes 1 - 1\otimes 1 - 1\otimes 1 = 0\otimes 1
\end{aligned}$$
  - Note that once we've considered $(a+b)\otimes c$, we don't need to consider $(b+a)\otimes c$ because of the commutativity of addition in V. That is, it is axiomatic that $a+b=b+a$ for all $a,b\in V$.
  - Additionally, in the last line above, we are using the facts that $1+1=2=0$ in F_2 and $a\otimes b + a\otimes b = 2a\otimes b = 0$ in any F_2-module to simplify the expressions.
  - Furthermore, note that since $-1 = 1 \in F_2$, $-0\otimes 0 = 1(0\otimes 0) = 0\otimes 0 \in V\otimes V$.
- Similarly, the subspace corresponding to $a\otimes(b+c) - a\otimes b - a\otimes c$ is $\operatorname{span}(0\otimes 0, 1\otimes 0)$. Thus, altogether, we quotient out the subspace $X = \operatorname{span}(0\otimes 0, 0\otimes 1, 1\otimes 0)$. This leaves us with a 1-dimensional $V\otimes V$, as expected for the tensor product of two one-dimensional vector spaces. It is interesting to note that the one vector we didn't quotient out $(1\otimes 1)$ is analogous to $e_1\otimes e_1$ since $e_1\in V$ might as well be defined $e_1 := 1$.
- Now let's see how well this quotienting worked. First off, a bit of notation: let $\pi : U \to V\otimes V$ be the projection $\pi : v \mapsto v + X$, and denote elements $\pi(v_1\otimes v_2) = v_1\otimes v_2 + X \in V\otimes V$ by $v_1 \otimes_\pi v_2$ for now to differentiate them from elements of U.
- Let $(0+1)\otimes_\pi 0 = (0+1)\otimes 0 + X$ be an element of the quotient space $V\otimes V$. Certainly, the elements $0\otimes_\pi 0$ and $1\otimes_\pi 0$ are also elements of this quotient space. Moreover, there is no reason we can't form the linear combination $(0+1)\otimes_\pi 0 - 0\otimes_\pi 0 - 1\otimes_\pi 0$. Indeed, when we do, we notice that this element lies in the quotiented-out subspace X. Thus,
$$(0+1)\otimes_\pi 0 - 0\otimes_\pi 0 - 1\otimes_\pi 0 = [(0+1)\otimes 0 - 0\otimes 0 - 1\otimes 0] + X = 0 + X = 0$$
- But
$$(0+1)\otimes_\pi 0 - 0\otimes_\pi 0 - 1\otimes_\pi 0 = 0 \quad\Longrightarrow\quad (0+1)\otimes_\pi 0 = 0\otimes_\pi 0 + 1\otimes_\pi 0$$
as desired.
- Note that this construction also gives us nice things like $0\otimes_\pi 0 = 0$, $0\otimes_\pi 1 = 0$, etc. which were not true in U!
- It should not be concluded, though, that all we need to quotient out of U for any V is $\operatorname{span}(0\otimes 0, 0\otimes v, v\otimes 0)$ for every $v\in V$; indeed, $V=\mathbb{R}$, for example, will require us to quotient out elements such as $4\otimes 7 - 2\otimes 7 - 2\otimes 7$, which can't even be expressed as a single simple tensor.

- **Free** (vector space): A vector space that has a basis consisting of linearly independent elements.
  - Example: Think of $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ as a $\mathbb{C}$-module. A free version $F(V)$ of V is infinite dimensional with every $v\in V$ a linearly independent basis vector. Elements of $F(V)$ are of the form $a_1v_1 + \cdots + a_kv_k$ for $a_1,\dots,a_k\in\mathbb{C}$ and $v_1,\dots,v_k\in V$. If $u = v+w$ where $u,v,w\in V$ are all nonzero, then $u\neq v+w$ in $F(V)$ because they are all linearly independent basis vectors.

  - Example: What we formally start with in the example above is the free F_2-module $V \times V$, *not* the Cartesian product vector space $V \times V$.
  - A terrific explanation of free vector spaces is available here.

- The two definitions above (abstract and explicit) constitute a first approximation to what the tensor product is.

- Example tensor product space.
  - Suppose $V = \mathbb{C}e_1 + \mathbb{C}e_2$. We want to look at $V \otimes V$.
  - A priori[2], it's spanned by $(ae_1 + be_2) \otimes (ce_1 + de_2) = ace_1 \otimes e_1 + ade_1 \otimes e_2 + bce_2 \otimes e_1 + cde_2 \otimes e_2$.
  - Thus, $V_1 \otimes V_2$ has 4-element basis $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$.

- Takeaway: What is true in general is that if V has basis $e_1, \dots, e_n$ and W has basis $f_1, \dots, f_m$, then $V \otimes W$ has basis $e_i \otimes f_j$ ($i = 1, \dots, n$ and $j = 1, \dots, m$).

- Having discussed the tensor product of vector spaces, let's think about the tensor product of *representations*.
  - Suppose $g : V \to V$ and $g : W \to W$.
    - We're starting to make notation sloppy.
  - How does $g : V \otimes W \to V \otimes W$? Well, we just send $v \otimes w \mapsto (gv) \otimes (gw)$.
    - Why is this map well-defined?
    - We invoke the **universal property of the tensor product operation**.
    - This guarantees us that given g — which is effectively a map from $V \times W \to V \otimes W$, as defined — there nevertheless exists a complete extension $\tilde{g} : V \otimes W \to V \otimes W$.
  - As a matrix, this map is pretty strange!
    - Example: Let $g : V \to V$ be a 2×2 matrix. What is the matrix of $g : V \otimes V \to V \otimes V$?
    - If
$$\rho_V(g) = g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} =: A$$
then we have
$$\begin{aligned} g(e_1 \otimes e_1) &= ge_1 \otimes ge_1 \\ &= (ae_1 + ce_2) \otimes (ae_1 + ce_2) \\ &= a^2 e_1 \otimes e_1 + ace_1 \otimes e_2 + ace_2 \otimes e_1 + c^2 e_2 \otimes e_2 \end{aligned}$$
    - Evaluating similarly for all basis vectors, we get a very curious block matrix:
$$\begin{array}{c} \\ e_1 \otimes e_1 \\ e_1 \otimes e_2 \\ e_2 \otimes e_1 \\ e_2 \otimes e_2 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 \otimes e_1 & e_1 \otimes e_2 & e_2 \otimes e_1 & e_2 \otimes e_2 \end{array} \\ \begin{bmatrix} a^2 & ab & ab & b^2 \\ ac & ad & bc & bd \\ ac & bc & ad & bd \\ c^2 & cd & cd & d^2 \end{bmatrix} \end{array} = \left[\begin{array}{c|c} aA & bA \\ \hline cA & dA \end{array}\right]$$
    - Notice how, for example, this takes the tensor $e_1 \otimes e_1$, represented as $(1, 0, 0, 0)$, to the tensor $a^2 e_1 \otimes e_1 + ace_1 \otimes e_2 + ace_2 \otimes e_1 + c^2 e_2 \otimes e_2$, represented as (a^2, ac, ac, c^2).
    - Does this construction imply a canonical way to convert from tensors to column vectors??
  - Classically, this is called the **Kronecker product** of two matrices.
  - People discovered all of this stuff separately before they unified it as tensor math.

$$\begin{tikzcd} V \times W \arrow[r, "\otimes"] \arrow[dr, "h"'] & V \otimes W \arrow[d, "\tilde{h}"] \\ & Z \end{tikzcd}$$

Figure 2.2: Universal property, tensor product operation.

- **Universal property of the tensor product operation**: For every bilinear map $h : V \times W \to Z$, there exists a *unique* linear map $\tilde{h} : V \otimes W \to Z$ such that $h = \tilde{h} \circ \otimes$.

  Proof. See the solid explanation here. Alternatively, here's my write up.

  Let $V = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$, $W = \mathbb{C}f_1 \oplus \cdots \oplus \mathbb{C}f_m$, Z, and $h : V \times W \to Z$ be arbitrary. Define $\tilde{h} : V \otimes W \to Z$ by
  $$\tilde{h}(e_i \otimes f_j) := h(e_i, f_j)$$
  for $i = 1, \dots, n$ and $j = 1, \dots, m$. Since a linear map is wholly defined by its action on the basis of its domain, this set of equations suffices to define $\tilde{h}$ on all of $V \otimes W$.

  <u>Existence</u>: To prove that $\tilde{h}$ satisfies the "universal property," it will suffice to show that $h = \tilde{h} \circ \otimes$. Let $(v, w) \in V \times W$ be arbitrary, and suppose $v = \sum_{i=1}^n a_i e_i \in V$, and $w = \sum_{i=1}^n b_i f_i \in W$. Then
  $$\begin{aligned} [\tilde{h} \circ \otimes](v, w) &= \tilde{h}(v \otimes w) \\ &= \sum_{i=1}^n \sum_{j=1}^m a_i b_i \tilde{h}(e_i \otimes f_i) \\ &= \sum_{i=1}^n \sum_{j=1}^m a_i b_i h(e_i, f_j) \\ &= h(v, w) \end{aligned}$$
  as desired.

  <u>Uniqueness</u>: Now suppose $\tilde{g} : V \otimes W \to Z$ also satisfies the "universal property," that is, $h = \tilde{g} \circ \otimes$. Then by definition,
  $$\tilde{h}(e_i \otimes f_j) = h(e_i, f_j) = \tilde{g}(e_i \otimes f_j)$$
  for $i = 1, \dots, n$ and $j = 1, \dots, m$. But since a linear map is wholly defined by its action on the basis of its domain, it follows that $\tilde{h} = \tilde{g}$, as desired. $\square$

- **Kronecker product** (of A, B): The matrix product defined as follows. *Denoted by* $\boldsymbol{A} \otimes \boldsymbol{B}$. *Given by*
  $$A \otimes B = \begin{array}{cc} & n \\ n & \begin{bmatrix} A \end{bmatrix} \end{array} \otimes \begin{array}{cc} & m \\ m & \begin{bmatrix} B \end{bmatrix} \end{array} = \begin{array}{cc} & nm \\ nm & \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix} \end{array}$$
  - The Kronecker product is *not* commutative, but the matrices you get are related by conjugacy and by permuting the columns.

- Vector spaces of the same dimension are all alike, but vector space representations are very interesting. By the end of this course, we'll understand what's going on.

- How we understand tensor stuff: Look at the abstract definition, look at the concrete definition, look at 5 examples, and then go in a circle. Repeat again and again until it makes sense.

[2] I.e., it follows from some logic. In particular, it follows from the logic that any element $v \in V$ is of the form $v = ae_1 + be_2$, so of course all $v \otimes v$ must be of the given form for choices of a, b, c, d.

  - Rudenko is just trying to tell us all relevant words so that they will fit together later.
- Fact: If V, W finite-dimensional, $\text{Hom}_F(V,W) \cong V \otimes W^* \cong V^* \otimes W$.
  - Tensor products are very nice spaces from which to construct maps.
  - Let's construct a reverse map, then.
  - Take $\alpha \otimes w \in V^* \otimes W$, where $\alpha : V \to F$ by definition. Send $\alpha \otimes w$ to the map $v \mapsto \alpha(v)w$. This is a *canonical* map!! We can show that they span everything.
    - For example, if we want to choose $\alpha \otimes w$ mapping to the matrix with a 1 in the upper left-hand corner and zeroes everywhere else, let $w = e_1 \in W$ and let $\alpha = e^1 \in V^*$.
    - We can do similarly for all other such matrices, mapping this basis of $\text{Hom}_F(V,W)$ to $e^i \otimes e_j$ ($i = 1, \dots, n$ and $j = 1, \dots, m$).
    - Note that this also allows us to define a (noncanonical) inverse map.
  - This inverse map from $\text{Hom}_F(V,W) \to V^* \otimes W$ is clearly a bit harder to work out.
  - Hidden in this story is why trace is invariant under matrix conjugation, e.g., why $\text{tr}\left(SAS^{-1}\right) = \text{tr}(A)$.
    - If we take $\text{Hom}_F(V,V)$, then this is isomorphic to $V^* \otimes V$.
    - There is a very natural map from these isomorphic spaces to F.
    - This map is defined by the trace (which sends $\text{Hom}_F(V,V) \to F$), and $\alpha \otimes v \mapsto \alpha(v)$ (which sends $V^* \otimes V \to F$). We can prove this.
    - Moreover, this map is canonical, as well.
    - This is why the main property of the trace is that it's invariant under conjugation, i.e., because SAS^{-1} and A both map to the same element of $V^* \otimes V$. This fact is hidden in the story very nicely.
- Tensor products are hard, it will be a pain, we will understand them very well, but it will not be nice for now.
- Symmetric products and wedge powers will be discussed briefly next time.
  - There is a nice description in Serre (1977) that we can use for the homework.
- Extra homework: Please read about tensor products in whatever textbook you like, try some examples, and repeat.

2.2 Office Hours

10/3:
- Problem 2a:
  - $\Lambda^2 V$ is *exterior powers*.
  - The exact canonical isomorphism we need is briefly discussed on Fulton and Harris (2004, p. 473).
  - I.e., we have to construct isomorphisms between the structures that don't rely on the choice of any basis. Recall the classic example of $V \cong V^{**}$, as explained in this well-written MSE post. Recall that the isomorphism from $V \to V^*$ defined by sending each element of the basis of V to the corresponding element of the dual basis of V^* is *not* canonical because *it involves choosing bases*. Definitions of canonical maps are available in MATH20510Notes, p. 2.
  - From a quick look at this, it looks like the proof may be analogous to the classic middle-school algebra identity $(v + w)^2 = v^2 + 2vw + w^2$.
  - The second exterior power $\Lambda^2 V$ of a finite-dimensional vector space V is the dual space of the vector space of alternating bilinear forms on V. Elements of $\Lambda^2 V$ are called 2-vectors.
- Problem 2b:

  - S^2V is *symmetric powers*.
  - The exact canonical isomorphism we need is briefly discussed on Fulton and Harris (2004, p. 473).

- Problem 3a:
  - This is the determinant of the multiplication table, in relation to that theorem that you showed us at the end of the first class? Yep!

- Problem 3b:
  - So a circulant matrix is a matrix like the multiplication table from (a)? Yep!
  - Is $\zeta = e^{2\pi i/n}$? Sort of. It can be any n^{th} root of unity.

- Problem 4d:
  - We'll cover higher symmetric powers in class tomorrow.
  - However, it basically just means that we're now working with elements of the form $e_1 \otimes e_2 \otimes e_3 \in S^3V$ and on and on.

- Problem 5a:
  - Is $V^\vee = V^*$? Yes. This is "vee check," and is a notation that some people prefer.

- Problem 5b:
  - Is "tr" the trace function of the linear map corresponding to L? Yes.
  - What is L?
    - An element of $V \otimes V^*$ is a linear combination of elements of the form $v \otimes \alpha$, not necessarily just one of these "decomposable" products.
    - There is an isomorphism $V \otimes V^* \cong \operatorname{Hom}(V)$.
    - Consider the matrix
    $$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
    It sends $e_1 \mapsto e_1$ and $e_2 \mapsto 0$. Thus, it is well-matched with $e_1 \otimes e^1$, which also grabs e_1 (with e^1) and sends it to e_1.
    - Consider the matrix
    $$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
    It sends $e_1 \mapsto 0$ and $e_2 \mapsto e_1$. Thus, it is well-matched with $e_1 \otimes e^2$, which also grabs e_2 (with e^2) and sends it to e_1.
    - In full,
    $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ae_1 \otimes e^1 + be_1 \otimes e^2 + ce_2 \otimes e^1 + de_2 \otimes e^2$$
    - This map *is* canonical! This is because the bases must be chosen to even begin talking about matrices.
    - If you change the matrix, the bases change, too??
    - Takeaway: We have to walk backwards from matrix to linear transformation to representation in $V \otimes V^*$ to a scalar in F.

- Problem 5c:
  - So trace of such a map is equal to the dimension of its image? Yes.

2.3 Wedge and Symmetric Powers

10/4:
- OH slightly later today at 5:45-6:45 PM.
- Recap: Last time, we built new reps from old.
  - This stuff can't be learned in 1.5 lectures; he can point us around, but we have to learn it ourselves.
- Tensor product review.
  - Given V,W, make $V\otimes_F W$.
  - This vector space is hard to describe directly, so we more often talk about its dual $(V\otimes W)^*$ because this is actually easier to describe.
  - If you want to work with $V\otimes W$ hands-on, you can do the following.
    - Start with the following easy-to-work-with vector space: The (probably infinite-dimensional) vector space where each $v\otimes w$ is a basis vector for all $v\in V$ and $w\in W$.
    - Then quotient it by relations to force them to hold in the final space.
  - If V has basis $e_1,\dots,e_n$ and W has basis $f_1,\dots,f_m$, then $e_i\otimes f_j$ is a basis of $V\otimes W$.
  - Interesting fact 1: If V,W are finite dimensional, $V^*\otimes W\cong\operatorname{Hom}(V,W)$.
  - If we want to work with the tensor product in practice in *rep theory*, the only thing we need to know is the basis of the tensor product space, which can tell us how any map $\rho(g)$ acts on both sides of a $v\otimes w\in V\otimes W$. From here, we recover the Kronecker product of matrices.
  - So many things are explained by the concept of tensor products!
  - A tensor in *physics* is something with lots of indices that changes in some way.
    - It does come from the math concept.
    - We'll get a huge basis because we have a massive product like $V\otimes\cdots\otimes V\otimes V^*\otimes\cdots\otimes V^*$.
- Last 2 useful notions: Wedge powers and symmetric powers.
  - Again, it's much easier to think about the dual space.
- Consider the space $V^{\otimes n}$.
  - $\dim V^{\otimes n}=(\dim V)^n$.
  - $(V^{\otimes n})^*$ are *linear* maps $f:V^{\otimes n}\to F$.
    - By contrast, $(V^n)^*$ is the space of all **polylinear** maps $f:V^n\to F$.
  - This distinction is subtle but important.
    - Note, for instance, that $\dim V^{\otimes n}\neq\dim V^n$ and likewise for the duals.
    - The distinction comes out fully when considering that if, for example, $V=\mathbb{R}^3$, then $V^2\cong\mathbb{R}^6$ and any map in $(V^2)^*$ is determined by its action on $(e_1,0),(e_2,0),(e_3,0),(0,e_1),(0,e_2),(0,e_3)$. By contrast, any map in $(V^{\otimes 2})^*$ is determined by its action on $(e_1,e_1),(e_1,e_2),(e_1,e_3),(e_2,e_1)$, $(e_2,e_2),(e_2,e_3),(e_3,e_1),(e_3,e_2),(e_3,e_3)$.
    - Important note: What $(V^{\otimes 2})^*$ does is consider these nine elements of V^2 as the basis of another space. This is what it truly means when we say "a bilinear map on V^2 is a linear map on $V^{\otimes 2}$."
    - Takeaway: Polylinearity changes the basis upon which a function $f:V^n\to F$ fundamentally acts.
  - A polylinear map may be **symmetric**, **antisymmetric**, or[3] neither.
  - These maps form vector spaces and the dimension is actually pretty meaningful.

[3] This is an exclusive "or."

- **Symmetric** (polylinear map): A polylinear map $f : V^n \to F$ that satisfies the following property.
  Constraint
$$f(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = f(v_1, \dots, v_n)$$
- **Antisymmetric** (polylinear map): A polylinear map $f : V^n \to F$ that satisfies the following property.
  Constraint
$$f(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = (-1)^\sigma f(v_1, \dots, v_n)$$
- Suppose you take $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$[4].
  - Consider a symmetric polylinear map $f : V \times V \times V \to \mathbb{C}$.
  - To compute it, we'll need the action of f on the basis of V^3. In particular, we'll need...
$$f(x_1e_1 + y_1e_2, x_2e_1 + y_2e_2, x_3e_1 + y_3e_2) = x_1x_2x_3f(e_1, e_1, e_1) + x_1x_2y_3f(e_1, e_1, e_2) + \cdots$$
    - Consider the term $x_1x_2y_3f(e_1, e_1, e_2)$.
    - Somewhere in there, you'll also have a $x_1y_2x_3f(e_1, e_2, e_1)$ term as well.
    - However, because f is symmetric, you know by symmetry that these "bases" are the same, so you don't count them as 2 towards the dimension but as 1.
    - Thus, dim = 4 for symmetric maps.
  - What about antisymmetric maps?
  - Suppose $g : V^3 \to \mathbb{C}$ is an antisymmetric polylinear map.
    - Consider $g(e_1, e_1, e_1)$. Suppose you apply (12). Interchanging the first two indices (for instance) obviously won't do anything, so we'll get
$$\begin{aligned} g(e_1, e_1, e_1) &= (-1)^{(12)} g(e_1, e_1, e_1) \\ g(e_1, e_1, e_1) &= -g(e_1, e_1, e_1) \\ 2g(e_1, e_1, e_1) &= 0 \\ g(e_1, e_1, e_1) &= 0 \end{aligned}$$
    - But what about $g(e_1, e_1, e_2)$? We could apply (23) and get $g(e_1, e_2, e_1)$, right? So it appears that we would just be shrinking two options into one. Technically, this is true, but what's more important is that applying (12) again yields the same thing, meaning that $g(e_1, e_1, e_2) = g(e_1, e_2, e_1) = 0$.
    - And thus, since V has dimension 2 but g takes three vectors, any argument passed to g will always be linearly dependent. Thus, $g = 0$ and, in fact, the space of antisymmetric maps on V^3 has dimension 0.
  - Takeaway: It's not a rule that $V^{\otimes m} \cong S^mV \oplus \Lambda^mV$ for *any* $m \in \mathbb{N}$.
- Mathematically, there's a more natural object to work with than symmetric and antisymmetric maps.
  - Wedge powers and symmetric powers!
  - Given V and $n \in \mathbb{N}$, we can construct S^nV and Λ^nV. $(S^nV)^*$ is symmetric polylinear maps taking n arguments from V. $(\Lambda^nV)^*$ is antisymmetric polylinear maps taking n arguments from V.
- How about a concrete way to see these? We can relate them to tensor powers.
  - Take a tensor power $V^{\otimes n}$, then look at those tensors which are symmetric and antisymmetric under permutation.
  - Example: Let V be the same as before. Then $V^{\otimes 2}$ has dim = 4.

[4] Note that this notation allows you to define a vector space *and* its basis in one go! I.e., the alternative is saying "Let V be a complex vector space with basis e_1, e_2."

    - Take as basis elements for S^2V those that don't change when you change the coordinates.
    - Take as basis elements for Λ^2V those that flip sign when you change the coordinates.
    - In this case, the basis of $V^{\otimes 2}$ is $e_1\otimes e_1, e_1\otimes e_2, e_2\otimes e_1, e_2\otimes e_2$. The basis of S^2V will be $e_1\otimes e_1, e_1\otimes e_2+e_2\otimes e_1, e_2\otimes e_2$. The basis of Λ^2V will be $e_1\otimes e_2-e_2\otimes e_1$. Notice that these bases are identical (up to scaling) with those in Serre (1977) and those produced by applying the **symmetrization** and **alternation** operators to the basis of $V^{\otimes 2}$.
  - S^2V and Λ^2V can form a *direct* sum because the dimensions match and they don't intersect.
  - Everything we're doing is representations, so $g(v_1\otimes\cdots\otimes v_n)=gv_1\otimes\cdots\otimes gv_n$.

- Relating this to something that we've seen but that is a little confusing.
  - The product notation is suggestive for symmetric vectors; you can commute $e_1\cdot e_2\in S^2V$, for instance.
  - This allows us to, for example, shrink $e_1\otimes e_1$ to $2e_1^2$[5], but $e_1\otimes e_2+e_2\otimes e_1$ only to $e_1\cdot e_2$.
  - Note that $e_1\wedge e_2=e_1\otimes e_2-e_2\otimes e_1$ by definition.
  - Fact/exercise: Let V be a vector space of dimension n. V^* is the dual space, and hence a function space. In particular, if $V=\mathbb{R}^k$, then elements of V^* are of the form

$$f(x_1,\dots,x_k)=a_1x_1+\cdots+a_kx_k$$

    where $a_i=f(e_i)$.
    - In other words, elements of V^* are polynomials of the above form in elements $(x_1,\dots,x_k)\in\mathbb{R}^k$.
  - Consequences.
    - $(\Lambda^kV)^*=\Lambda^kV^*$.
    - S^nV^* is the vector space of homogeneous polynomials of degree n.
      - Prove this by taking higher degree polynomials and just keeping pushing through.
      - For example, $e^1\cdot e^2\in S^2(\mathbb{R}^3)^*$ and

$$[e^2\cdot e^3](x_1,x_2,x_3)=e^2(x_1,x_2,x_3)\cdot e^3(x_1,x_2,x_3)=x_2\cdot x_3=x_2x_3$$

      - Evidently, x_1x_2 is a degree $n=2$ polynomial!
  - Wedge powers now.
  - By convention, $\Lambda^0V=F$ and $\Lambda^1V=V$. But then you get to Λ^2V and Λ^3V. They grow but then shrink down as the power approaches $\dim V$.
  - Fact: The dimension of wedge powers Λ^iV is $\binom{k}{i}$ for $\dim V=k$. Figuring out why this is the case is another good exercise.

    Proof. A basis vector $e_{j_1}\wedge\cdots\wedge e_{j_i}$ must be the wedge product of i distinct elements of a basis of V because if there are any repeats, the wedge product will equal zero. Thus, the number of possible wedge products (and hence dimension of Λ^iV) is equal to the number of the ways you can choose i elements from the k possible basis vectors. $\square$

- An interesting connection between wedge powers and the determinant.
  - Let $V=\mathbb{C}e_1\oplus\cdots\oplus\mathbb{C}e_n$.
  - Recall that Λ^nV^* is the space of antisymmetric polylinear functions $V\times\cdots\times V\to F$ taking n arguments from V, and it has a single basis vector $e^1\wedge\cdots\wedge e^n$.
  - Let $v_1=\sum a_{i1}e_i$, $v_2=\sum a_{i2}e_i$, etc.
  - Let $f\in\Lambda^nV^*$, so that f is an alternating polylinear map that takes n arguments.

[5] Why the 2 coefficient? Because technically, the symmetrization operator takes $e_1\otimes e_1\mapsto e_1\otimes e_1+e_1\otimes e_1=2e_1\cdot e_1=2e_1^2$.

- Since f is polylinear, we have that

$$f(v_1,\dots,v_n)=\sum_{i_1,\dots,i_n=1}^{n}a_{i_11}\cdots a_{i_nn}f(e_{i_1},\dots,e_{i_n})$$

- Because of antisymmetry, we need only look at elements where the indices are all different. Thus, the above equals

$$\sum_{\sigma\in S_n}a_{\sigma(1)1}\cdots a_{\sigma(n)n}f(e_{\sigma(1)},\dots,e_{\sigma(n)})$$

- Additionally, $f(e_{\sigma(1)},\dots,e_{\sigma(n)})=(-1)^\sigma f(e_1,\dots,e_n)$ for any $\sigma\in S_n$. Moreover, $f(e_1,\dots,e_n)\in\mathbb{C}$ by definition, so define a constant $\lambda:=f(e_1,\dots,e_n)$. Thus, the above equals

$$\lambda\sum_{\sigma\in S_n}a_{\sigma(1)1}\cdots a_{\sigma(n)n}$$

- But the term following the λ is just the determinant of the $n\times n$ matrix (a_{ij}). Thus, all said,

$$f(v_1,\dots,v_n)=\lambda\det(v_1\mid\cdots\mid v_n)$$

- Implication: Wedge powers are something like the determinant.
  - In particular, because Λ^nV^* has only a single basis vector as mentioned above, $f=\lambda e^1\wedge\cdots\wedge e^n$. It follows that $e^1\wedge\cdots\wedge e^n=\det$.
- Takeaway: Wedge powers are something interesting; there's a reason to study them.

- The basis of the wedge powers consists of wedge monomials $e_{j_1}\wedge\cdots\wedge e_{j_i}$. Moreover, no need to have the same list twice, so choose some way of indexing them, e.g., increasing indexes.
  - This is why we do *increasing* bases! There's no particular reason, it's just an arbitrary way of making sure we don't do the same thing twice! We could just as well choose decreasing or any other means of guaranteeing that we don't have duplicates.
- Now let's relate all of this exterior and symmetric product stuff back to representation theory.
  - Let $V=\mathbb{C}e_1\oplus\cdots\oplus\mathbb{C}e_n$.
  - Let $G\circlearrowright V$ via the homomorphism $G\to GL(V)\cong GL_n(\mathbb{C})$.
  - Focusing more on the *matrix* aspect this time, note that under this homomorphism, $g\mapsto A_g$ subject to the homomorphism constraints $A_e=E_n$, etc.
  - Consider the set $\{A_{g_1},\dots,A_{g_k}\}$ of all matrices in the image of the homomorphism. If we transpose all of them, will they still obey the homomorphism constraints?
    - Nope!
    - Indeed, if we do this, we'll get in trouble. More specifically, transposition is not a representation because $A_{g_1}^TA_{g_2}^T\neq A_{g_1g_2}^T=A_{g_2}^TA_{g_1}^T$.
  - It's the same story with inverses.
  - *However*, combining the two operations, we get

$$(A_{g_1g_2}^T)^{-1}=(A_{g_1}^T)^{-1}(A_{g_2}^T)^{-1}$$

    - This is exactly when we take a representation and then go to the dual[6].
  - This will be on next week's homework![7]
  - Takeaway: This is an application of Λ^jV^* to representation theory, $j\neq k,n$.

[6]Relation to MATH 20510 when we discussed dual matrices and pullbacks of matrices.
[7]I don't believe a related problem ended up appearing in PSet 2.

- Another relation: An application of $\Lambda^n V^*$ to representation theory.
  - Suppose we have a representation $G \circlearrowright V$ that we want to flatten into $G \circlearrowright \mathbb{C}$. How can we turn a relation between a group of matrices into a relation between a group of numbers?
  - Use the determinant!
  - Indeed, we already know that

$$\det(A_e) = 1 \qquad \det(A_{g_1g_2}) = (\det A_{g_1})(\det A_{g_2}) \qquad \det\big(A_{g^{-1}}\big) = \det(A_g)^{-1}$$

  - In particular, we make formal the transition $G \to GL_j(\mathbb{C}) \to \mathbb{C}$ with the **top wedge power** $\Lambda^n V^*$, i.e., by composing with $\rho : G \to GL_j(\mathbb{C})$ the map sending $(v_1 \mid \cdots \mid v_n) \mapsto v_1 \wedge \cdots \wedge v_n$.
- A last note.
  - Don't think that we're limited to top wedge powers.
  - Recall that we can define tensor products of matrices via the Kronecker product. Well, we can prove that

$$A_{g_1g_2}^{\otimes 2} = A_{g_1}^{\otimes 2} A_{g_2}^{\otimes 2}$$

    and the like as well!
  - Similarly, we can define Λ^2 of a matrix.
    - We'll get into some weird Kronecker product stuff again, but we can sort through it.
    - This will not actually be done in this course, though.
- Plan for Friday and next time.
  - Prove the theorem that every representation is a sum of irreducible representations.
  - He will use projectors.
  - Then a horror story.
  - Then associative algebra.

2.4 Office Hours

10/5:
- Problem 2a:
  - $(V \oplus W) \otimes (V \oplus W) \stackrel{?}{=} V \otimes V \oplus V \otimes W \oplus W \otimes V \oplus W \otimes W$.
  - Check linearity in all terms and then with universal property. Check antisymmetric, linear, injective, surjective; dimensions are the same, so no need to check *both* injectivity and surjectivity (surjectivity is easier to check). We can go to basis to check various properties; we can't use a basis to write the map, but we can use bases to check surjectivity and the like.
- Problem 3a:
  - Bezout and Gauss's lemma is good to learn on my own. Put polynomials in each variable. Throw some stuff about these results into my answers.
  - Relearn polynomial division.
  - $(1,1,1,1)$, $(1,1,-1,-1)$, $(1,-1,1,-1)$, and $(1,-1,-1,1)$.
  - This is a symmetric matrix.
  - The upper-left and lower-right blocks of this matrix match; so do the lower-left and upper-right.
  - When the eigenvalue is equal to zero, the determinant is equal to zero. So look for eigenvectors to calculate eigenvalues, and then just express the determinant as a product of these.
- Problem 3b:

- Corresponding eigenvalue is $\sum_{i=1}^{n} x_i z^{i-1}$.
- Can I use representation theory to do this? What group has a multiplication table like this? $\mathbb{Z}/n\mathbb{Z}$. The elements of $\mathbb{Z}/n\mathbb{Z}$ are of the form $\{1, \zeta, \dots, \zeta^{n-1}\}$.
- If that's an eigenvector, then it's a subrepresentation; it is a space that is fixed under the action of the matrix.
- Other eigenvectors: $(1,1,1)$, $(1,z^2,z)$.
- We don't need to do induction or anything fancy like that; we can just do dots. As long as your argument is complete and clear, you're good.

- Problem 4a:
  - See FH 1.3. Standard rep, not wedge. Treat τ,σ (generators of the action) on the basis vectors.
  - If both fix, it's the trivial; if one flips, you have alternating; if both flip, you have standard.
  - $(2,1)\oplus(1,1,1)$. Use problem 2.
  - The action of τ on this basis vector can be computed:
$$\tau(\alpha\wedge\beta) = 1\alpha\wedge\beta$$
  - Having obtained an eigenvalue of 1, we can rule out the standard representation.

- Problem 4b:
  - $\{\alpha\otimes\alpha\otimes\alpha, \alpha\otimes\alpha\otimes\beta+\alpha\otimes\beta\otimes\alpha+\beta\otimes\alpha\otimes\alpha, \beta\otimes\beta\otimes\beta\}$.
  - See Exercise 1.2 in Fulton and Harris (2004).

- Problem 5a:
  - Consider an alternate basis $f_1,\dots,f_n$ and dual basis $f^1,\dots,f^n$. Consider the element $f_1\otimes f^1+\cdots+f_n\otimes f^n\in V\otimes V^\vee$. We want to prove that it equals the one asked about in the question.
  - Under the isomorphism to $\mathrm{Hom}(V,V)$, we send $e_1\otimes e^1$ to $[v\mapsto e^1(v)e_1]$. More generally, we end $e_i\otimes e^i$ to $[v\mapsto e^i(v)e_i]$. Adding all these maps together yields the map $[v\mapsto e^1(v)e_1+\cdots+e^n(v)e_n]$, which is just the identity $1\in\mathrm{Hom}(V,V)$, regardless of basis.

- Problem 5b:
  - Example:
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to ae_1\otimes e^1+be_1\otimes e^2+ce_2\otimes e^1+de_2\otimes e^2$$
  - Evaluating this gives
$$e^1(ae_1)+e^2(be_1)+e^1(ce_2)+e^2(de_2) = a+d$$
    since it's only when the indices match (i.e., along the diagonal) that we get a nonzero value.

- Problem 5c:
  - P should have a block-diagonal matrix corresponding to the decomposition $V=W\oplus W^0$. P is the identity on $\mathrm{Im}(P)$. So if our basis is vectors spanning W and then vectors spanning W^0, the matrix should be the identity and then the zero matrix. That should do the trick. How rigorous does this need to be?
  - Let $e_1,\dots,e_k$ be an orthonormal basis of $\mathrm{Im}(P)$. Extend this basis to an orthonormal basis $e_1,\dots,e_n$ of V.

- Problem 5d:
  - Trivial representation: All $g\in G$ get mapped to $1\in GL(V)$.

  - Part (a) gives us the identity in $\mathrm{Hom}(V,V)$.
  - So we have $\rho : G \to GL(V)$.
  - Is any line acceptable? Span of the identity function? Rudenko: It depends on V. It has *infinitely many* trivial subrepresentations.
  - Example: $G \circlearrowright \mathbb{C}^2$. with $\rho(g) = I_2$.
  - Dual representation: Defined analogously to the $\mathrm{Hom}_F(V,W)$ representation. We also need an inverse.

- Psets will likely get easier; right now, we have to relearn a lot of old stuff and we are being challenged with harder problems. As the questions become more based on course content and thus will get easier.
- He'll do hard PSets, easy exams, and everything is curved; he agrees that this is a hard pset, and probably harder than necessary.

2.5 Complete Reducibility

10/6:

- Let G be a finite group.
- We want to study finite dimensional representations over $\mathbb{C}$.
  - Characteristic F, $|G| = 1$.
  - What is this stuff about characteristic??
- Theorem: Any f.d. representation can be decomposed into a sum of irreps via

$$V = V_1^{n_1} \oplus \cdots \oplus V_k^{n_k}$$

  Moreover, this decomposition is unique.

  - See Proposition 1.8.
- Example: We have already seen S_2, S_3 in the homework; now, let $G = \mathbb{Z}/n\mathbb{Z}$.
  - Consider $V_0, \dots, V_{n-1}$.
  - Let V_i be a 1-dimensional rep.
    - We have $\rho : G \to C^\times$ defined by $[k] \mapsto (\mathrm{e}^{2\pi i/n})^k$.
  - These are all 1-dimensional representations up to isomorphism.
- Example: Let $G = \mathbb{Z}$. What is a representation of $\mathbb{Z}$? We just need to say what happens to 1.
  - For example, if the map $G \to GL_n(\mathbb{C})$ sends $1 \mapsto A$, then $2 \mapsto A^2$, and on and on.
  - A place where you run into trouble: $n = 2$ and

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

  - The matrix has a fixed subrepresentation (i.e., eigenvector $(1,1)$).
    - More specifically, $\mathbb{C}(1,0) \hookrightarrow V$ is a 1D subrepresentation.
  - The theorem basically tells us that $V = \mathbb{C}(1,0) + \mathbb{C}w$.
  - This is an example of how things can go wrong. How??
- Proving the theorem; we need a miracle!
- Existence: We need a lemma.

- Lemma: Let G be finite, $F = \mathbb{C}$, and V a G-representation. Let $W \leq V$ be a subrepresentation or **invariant subspace**. Then there exists another invariant subspace $W' \subset V$ such that $V = W \oplus W'$.
  - See Theorem 1 (Rudenko replicates all aspects of the "limited conditions" proof).
  - This lemma implies existence.
  - Two proofs: One that only works over complex numbers. He suggests we read about it. Name??
  - He'll do the slightly less intuitive one, which involves **projectors**.
- **Projector**: A linear map $P : V \to V$ such that $P^2 = P$, that is, is **idempotent**.
  - Example: Consider $W := \mathrm{Im}(P) \leq V$. $P|_W$ does nothing; it's the identity.
  - A good mental picture: Things are falling from 3D space onto some smaller space.
  - On the kernel.
    - Importantly, $\mathrm{Ker}\, P \cap \mathrm{Im}\, P = 0$.
    - It follows that $V = \mathrm{Im}(P) \oplus \mathrm{Ker}(P)$.
    - Within the space, $v = (w, w') = w + w'$. What the projector does is $(w, w') \mapsto (w, 0)$.
  - What else can we say about projectors?
    - There is a correspondence between projectors and direct sum decompositions.
- So to prove the lemma, we need a projector $P : V \to V$ with image W and certain properties.
  - More specifically, the goal is to find a projector P...
    1. With image W;
    2. That is a morphism of G-reps.
  - On the second condition, that is, we want $P(gv) = gP(v)$. In this case, $\mathrm{Ker}(P)$ will be a shuffle??
- Strategy.
  - Take any projector $P_0 : V \to W$. And then you can get g-projectors $gP = gPg^{-1}$.
  - So define a new projector
  $$P = \frac{1}{|G|} \sum_{g \in G} \underbrace{gP_0g^{-1}}_{gP_0}$$
  One didn't work, so we hope the average will work, and it will!
  - For any $w \in W$, we can prove that the sum thing does fix W's, so it is a projector!
  - $P(hv)$ example.
  - Note: This computation will be done again later in a different context; this averaging construction is *central* to representation theory.
- Constraints we used in the proof: G is finite (or compact), $|G|$ is invertible.
  - Only when we get into **modular representation theory** is where we get into trouble; this theorem actually kills **extensions**, which are very interesting but are not in finite group rep theory.
- Hermitian inner product isn't common here, but it shows up in physics. We will talk a bit more about inner products later, though.
- Now for the other part of the original proof: Uniqueness.
- Schur's Lemma: Let G be a finite group, let the field $F = \mathbb{C}$, and let V, W be irreps over F. Let $f \in \mathrm{Hom}_G(V, W)$, which we may recall is the space of morphisms between G-reps V and W, that is, all $h : V \to W$ satisfying $h(gv) = gh(v)$. Then...

1. $f = 0$ if $V \ncong W$. If $V \cong W$, then these maps are isomorphisms.
2. In particular, if V is an irrep and $f : V \to V$ is such that $f(gv) = gf(v)$, then $f(v) = \lambda v$.

Altogether, we have that

$$\mathrm{Hom}_G(V, W) \cong \begin{cases} 0 & V \ncong W \\ \mathbb{C} & V \cong W \end{cases}$$

- The statement "$\mathrm{Hom}_G(V, W) \cong \mathbb{C}$" reflects the fact that the left space is the space of all scalar isomorphisms λI for scalars $\lambda \in \mathbb{C}$, which happens to be isomorphic to $\mathbb{C}$.
- Gist: If you want a certain kind of matrix between certain spaces, in some cases, you'll just fail.

- Proof.
  - See Lemma 1.7 or Proposition 4.
  - For $f : V \to V$, consider $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$. The latter two are subrepresentations of V, W, respectively. $\mathrm{Ker}(f) = V$ implies $f = 0$; symmetric with $\mathrm{Im}(f)$. If nonzero, then $\mathrm{Ker} = 0$ and $\mathrm{Im} = W$, implies f is an isomorphism.
- Schur's Lemma is the easiest step to learn in the whole story.
- Last 2 minutes: Finish the proof of the original theorem.
  - Don't worry if we're confused by this last line; it will be repeated later in a much more powerful way.
  - Analogous to Proposition 1.8.
  - I might have missed some stuff here??
- Plan for next week.
  - Character theory.
  - Serre (1977) is still the best source for tracking lecture content for right now.
- This would have been an interesting but wholly nonessential lecture to pay attention to, since I already did all of the readings.

2.6 S Chapter 1: Generalities on Linear Representations

From Serre (1977).

Section 1.5: Tensor Product of Two Representations

10/4:
- **Tensor product** (of V_1, V_2): The vector space W that (a) is furnished with a map $V_1 \times V_2 \to W$ sending $(x_1, x_2) \mapsto x_1 \cdot x_2$ and (b) satisfies the following two conditions.
  - (i) $x_1 \cdot x_2$ is bilinear.
  - (ii) If (e_{i_1}) is a basis of V_1 and (e_{i_2}) is a basis of V_2, the family of products $e_{i_1} \cdot e_{i_2}$ is a basis of W.

  Denoted by $\boldsymbol{V_1 \otimes V_2}$.
  - It can be shown that such a space exists and is unique up to isomorphism (see proof here).
- This definition allows us to say some things quite expediently. For example, (ii) implies that

$$\dim(V_1 \otimes V_2) = \dim(V_1) \cdot \dim(V_2)$$

- **Tensor product** (of ρ^1, ρ^2): The representation $\rho : G \to GL(V_1 \otimes V_2)$ defined as follows for all $s \in G$, $x_1 \in V_1$, and $x_2 \in V_2$, where $\rho^1 : G \to GL(V_1)$ and $\rho^2 : G \to GL(V_2)$ are representations. *Given by*

$$[\rho_s^1 \otimes \rho_s^2](x_1 \cdot x_2) = \rho_s^1(x_1) \cdot \rho_s^2(x_2)$$

- A more formal write up of the matrix translation of this definition.
  - Let (e_{i_1}) be a basis for V_1, and let (e_{i_2}) be a basis for V_2.
  - Let $r_{i_1 j_1}(s)$ be the matrix of ρ_s^1 with respect to this basis, and let $r_{i_2 j_2}(s)$ be the matrix of ρ_s^2 with respect to this basis.
  - It follows that

$$\rho_s^1(e_{j_1}) = \sum_{i_1} r_{i_1 j_1}(s) e_{i_1} \qquad\qquad \rho_s^2(e_{j_2}) = \sum_{i_2} r_{i_2 j_2}(s) e_{i_2}$$

  - Therefore,

$$[\rho_s^1 \otimes \rho_s^2](e_{j_1} \cdot e_{j_2}) = \sum_{i_1, i_2} r_{i_1 j_1}(s) r_{i_2 j_2}(s) e_{i_1} \cdot e_{i_2}$$

    and

$$\mathcal{M}(\rho_s^1 \otimes \rho_s^2) = (r_{i_1 j_1}(s) r_{i_2 j_2}(s))$$

- Aside on quantum chemistry to come back to later; I can't quite connect the dots yet.

Section 1.6: Symmetric Square and Alternating Square

- Herein, we investigate the tensor product when $V_1 = V_2 = V$.
- Let (e_i) be a basis of V.
- Define the automorphism $\theta : V \otimes V \to V \otimes V$ by

$$\theta(e_i \cdot e_j) = e_j \cdot e_i$$

  for all 2-indices (i, j).
- Properties of θ.
  - Since θ is linear, it follows that

$$\theta(x \cdot y) = y \cdot x$$

    for all $x, y \in V$.
    - Implication: θ is independent of the chosen basis (e_i)!
  - $\theta^2 = 1$, where 1 is the identity map on $V \otimes V$.
- Assertion: $V \otimes V$ decomposes into

$$V \otimes V = S^2(V) \oplus \Lambda^2(V)$$

  - Rudenko: We do not have to worry about proving this... yet, at least.
- **Symmetric square representation**: The subspace of $V \otimes V$ containing all elements z satisfying $\theta(z) = z$. *Denoted by* $\boldsymbol{S^2 V}$, $\boldsymbol{S^2(V)}$, $\mathbb{S}^2 \boldsymbol{V}$, $\textbf{Sym}^{\,2}\boldsymbol{(V)}$.
  - Basis: $(e_i \cdot e_j + e_j \cdot e_i)_{i \leq j}$.
    - Rudenko: How do we know everything is linearly independent? Well, when we add two linearly independent vectors out of a set, the sum is still linearly independent from everything else!

- Example when $\dim V = 2$: The basis of $V \otimes V$ is $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$, where all four of these vectors are linearly independent. So naturally, the basis of the corresponding symmetric square representation — which is $2e_1 \otimes e_1, e_1 \otimes e_2 + e_2 \otimes e_1, 2e_2 \otimes e_2$ — will still be a linearly independent list of vectors.

  - Dimension: If $\dim V = n$, then

$$\dim S^2(V) = \frac{n(n+1)}{2}$$

- **Alternating square representation**: The subspace of $V \otimes V$ containing all elements z satisfying $\theta(z) = -z$. *Denoted by* $\mathbf{\Lambda^2 V}$, $\mathbf{\Lambda^2(V)}$, $\textbf{Alt}^{\,2}\boldsymbol{(V)}$.
  - Basis: $(e_i \cdot e_j - e_j \cdot e_i)_{i<j}$.
  - Dimension: If $\dim V = n$, then

$$\dim \Lambda^2(V) = \frac{n(n-1)}{2}$$

2.7 FH Appendix B: On Multilinear Algebra

From Fulton and Harris (2004).

Section B.1: Tensor Products

10/5:

- **Tensor product** (of V, W over F): A vector space U equipped with a bilinear map $V \times W \to U$ sending $v \times w \to v \otimes w$ that is universal, i.e., for any bilinear map $\beta : V \times W \to Z$, there is a unique linear map from $U \to Z$ that takes $v \otimes w \mapsto \beta(v, w)$. *Denoted by* $\boldsymbol{V \otimes W}$, $\boldsymbol{V \otimes_F W}$.
  - The so-called *universal property* determines the tensor product up to canonical isomorphism.
- One construction of $V \otimes W$: From the basis $\{e_i \otimes f_j\}$.
  - This construction is **functorial**, implying that linear maps from $f : V \to V'$ and $g : W \to W'$ determine a linear map $f \otimes g : V \otimes W \to V' \otimes W'$, namely that defined by $f \otimes g : v \otimes w \to fv \otimes gw$.
- Definition of the $\boldsymbol{n}$**-fold tensor product**.
- **Multilinear** (map): A map from a Cartesian product $V_1 \times \cdots \times V_n$ of vector spaces to a vector space U such that when all but one of the factors V_i are fixed, the resulting map from $V_i \to U$ is linear.
- Properties of the tensor product.
  1. *Commutativity*:
  $$V \otimes W \cong W \otimes V$$
  by $v \otimes w \mapsto w \otimes v$.
  2. *Distributivity*:
  $$(V_1 \oplus V_2) \otimes W \cong (V_1 \otimes W) \oplus (V_2 \otimes W)$$
  by $(v_1, v_2) \otimes w \mapsto (v_1 \otimes w, v_2 \otimes w)$.
  3. *Associativity*:
  $$(U \otimes V) \otimes W \cong U \otimes (V \otimes W) \cong U \otimes V \otimes W$$
  by $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w) \mapsto u \otimes v \otimes w$.
- **Tensor power** (of V to n): The tensor product defined as follows. *Denoted by* $\boldsymbol{V^{\otimes n}}$. *Given by*

$$V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}$$

  - Convention: $V^{\otimes 0} = F$.
- Analogous construction of the tensor product for generalized algebras and modules.

Section B.2: Exterior and Symmetric Powers

- **Alternating** (multilinear map): A multilinear map β such that $\beta(v_1, \dots, v_n) = 0$ whenever $v_i = v_j$ for some $i, j \in [n]$.
  - Implication: $\beta(v_1, \dots, v_n)$ changes sign whenever two of the vectors are interchanged.
    - Follows from the definition and the **standard polarization**.
  - Implication:
  $$\beta(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = (-1)^\sigma \beta(v_1, \dots, v_n)$$
  for all $\sigma \in S_n$.
- **Standard polarization**: The equality
$$\beta(v, w) + \beta(w, v) = \beta(v + w, v + w) - \beta(v, v) - \beta(w, w) = 0 - 0 - 0 = 0$$
- **Exterior powers** (of V): The vector space U equipped with an alternating multilinear map $V \times \cdots \times V \to \Lambda^n V$ sending $v_1 \times \cdots \times v_n \mapsto v_1 \wedge \cdots \wedge v_n$ that is universal, i.e., for any alternating multilinear map $\beta : V^n \to Z$, there is a unique linear map from U to Z that takes $v_1 \wedge \cdots \wedge v_n \mapsto \beta(v_1, \dots, v_n)$. *Denoted by* $\boldsymbol{\Lambda^n V}$.
  - Convention: $\Lambda^0 V = F$.
- Quotient space construction of the exterior powers.
- Projecting from $V^{\otimes n} \to \Lambda^n V$: Define $\pi : V^{\otimes n} \to \Lambda^n V$ by
$$\pi(v_1 \otimes \cdots \otimes v_n) = v_1 \wedge \cdots \wedge v_n$$
- Basis for the exterior powers.
- There is a canonical linear map $\Lambda^a V \otimes \Lambda^b W \to \Lambda^{a+b}(V \oplus W)$, which takes $(v_1 \wedge \cdots \wedge v_a) \otimes (w_1 \wedge \cdots \wedge w_b) \mapsto v_1 \wedge \cdots \wedge v_a \wedge w_1 \wedge \cdots \wedge w_b$.
  - This determines (how??) an isomorphism
  $$\Lambda^n(V \oplus W) \cong \bigoplus_{a=0}^{n} \Lambda^a V \otimes \Lambda^{n-a} W$$
  - This isomorphism plus induction on n can justify (how??) the basis for $\Lambda^n V$ as the increasing indices.
- **Symmetric** (multilinear map): A multilinear map β such that $\beta(v_1, \dots, v_n)$ is unchanged when any two factors are interchanged, that is
$$\beta(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = \beta(v_1, \dots, v_n)$$
for all $\sigma \in S_n$.
- **Symmetric powers** (of V): The vector space U equipped with a symmetric multilinear map $V \times \cdots \times V \to S^n V$ sending $v_1 \times \cdots \times v_n \mapsto v_1 \cdot \ldots \cdot v_n$ that is universal, i.e., for any symmetric multilinear map $\beta : V^n \to Z$, there is a unique linear map from U to Z that takes $v_1 \cdot \ldots \cdot v_n \mapsto \beta(v_1, \dots, v_n)$. *Denoted by* $\boldsymbol{S^n V}$.
  - Convention: $S^0 V = F$.
- Quotient space construction of the symmetric powers.
  - Quotient out all $v_1 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$, that is, those elements of $V^{\otimes n}$ in which σ permutes two successive factors. How does this work??

- Projecting from $V^{\otimes n} \to S^n V$: Define $\pi : V^{\otimes n} \to S^n V$ by

$$\pi(v_1 \otimes \cdots \otimes v_n) = v_1 \cdot \ldots \cdot v_n$$

- Basis for the symmetric powers.
  - It follows from the basis construction that $S^n V$ can be regarded as the space of homogeneous polynomials of degree n in the variable e_i, since each element is of the form $e_{i_1} \cdot \ldots \cdot e_{i_n}$ and we can add them.

- Canonical isomorphism:

$$S^n(V \oplus W) \cong \bigoplus_{a=0}^{n} S^a V \otimes S^{n-a} W$$

- More on $\Lambda^n V, S^n V$ as subspaces of $V^{\otimes n}$.
  - We inject $\iota : \Lambda^n V \to V^{\otimes n}$ with

$$\iota(v_1 \wedge \cdots \wedge v_n) = \sum_{\sigma \in S_n} (-1)^\sigma v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

    - This relates to Rudenko's note that $v_1 \wedge v_2 = v_1 \otimes v_2 - v_2 \otimes v_2$!
    - There are some more advanced notes on the implications of ι; $[\iota \circ \pi/n!](V^{\otimes n}) = V^{\otimes n}$ is brought up.
  - We inject $\iota : S^n V \to V^{\otimes n}$ with

$$\iota(v_1 \cdot \ldots \cdot v_n) = \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

    - More, related advanced notes; includes the $1/n!$ thing again.

- **Wedge product**: The function $\Lambda^m V \otimes \Lambda^n V \to \Lambda^{m+n} V$ defined as follows. *Denoted by* $\wedge$. *Given by*

$$(v_1 \wedge \cdots \wedge v_m) \otimes (v_{m+1} \wedge \cdots \wedge v_{m+n}) \mapsto v_1 \wedge \cdots \wedge v_m \wedge v_{m+1} \wedge \cdots \wedge v_{m+n}$$

- Properties of the wedge product.
  1. *Associativity*:
  $$(v_1 \wedge v_2) \wedge v_3 = v_1 \wedge (v_2 \wedge v_3) = v_1 \wedge v_2 \wedge v_3$$
  2. *Skew-commutativity*:
  $$v_1 \wedge v_2 = -v_2 \wedge v_2$$

- Note that both of the above properties hold in higher-dimensional cases as well.

- Commutativity of the products.

$$\begin{tikzcd} \Lambda^m V \otimes \Lambda^n V \arrow[r, "\wedge"] \arrow[d, hook, "\iota\otimes\iota"'] & \Lambda^{m+n} V \arrow[d, hook, "\iota"] \\ V^{\otimes m} \otimes V^{\otimes n} \arrow[r, "f_1"'] & V^{\otimes(m+n)} \end{tikzcd}$$

(a) Wedge product.

$$\begin{tikzcd} S^m V \otimes S^n V \arrow[r, "\cdot"] \arrow[d, hook, "\iota\otimes\iota"'] & S^{m+n} V \arrow[d, hook, "\iota"] \\ V^{\otimes m} \otimes V^{\otimes n} \arrow[r, "f_2"'] & V^{\otimes(m+n)} \end{tikzcd}$$

(b) Symmetric product.

Figure 2.3: Commutative diagram, wedge and symmetric products.

- f_1 is defined by

$$(v_1 \otimes \cdots \otimes v_m) \otimes (v_{m+1} \otimes \cdots \otimes v_{m+n}) \mapsto \sum_{\sigma \in G} (-1)^\sigma v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)} \otimes v_{\sigma(m+1)} \otimes \cdots \otimes v_{\sigma(m+n)}$$

  where G is the subgroup of S_{m+n} preserving the order of the subsets $\{1, \dots, m\}$ and $\{m+1, \dots, m+n\}$.
- f_2 is defined analogously.

- The above mappings all commute with linear maps of vector spaces.
  - Example: Our definition $g(v \otimes w) = gv \otimes gw$ could be redrawn as $[g \circ \otimes](v, w) = [\otimes \circ g](v, w)$, where the latter $g : (v, w) \mapsto (gv, gw)$ by abuse of notation.
- Tensor, exterior, and symmetric algebras.

2.8 FH Chapter 1: Representations of Finite Groups

From Fulton and Harris (2004).

10/1: • Starts with a justification for beginning their investigation of rep theory with finite groups.

Section 1.1: Definitions

- Definition of a **representation**.
- ρ "gives V the structure of a G-module!" (Fulton & Harris, 2004, p. 3).
- When there is little ambiguity about ρ, we call V itself a representation of G.
  - This is what Rudenko has been doing in class!
- We also often write $g \cdot v$ for $\rho(g)(v)$, and g for $\rho(g)$.
- **Degree** (of ρ): The dimension of V.
- ***G*-linear** (map): See class notes. *Also known as* **map**, **morphism**.
- The **kernel**, **image**, and **cokernel** of φ are all G-submodules.
- **Kernel** (of a map φ): The vector subspace containing all $v \in V$ for which $\varphi(v) = 0$. *Denoted by* $\mathbf{Ker}\,\boldsymbol{\varphi}$.
- **Image** (of a map φ): The vector subspace containing all $w \in W$ for which there exists $v \in V$ such that $\varphi(v) = w$. *Denoted by* $\mathbf{Im}\,\boldsymbol{\varphi}$.
- **Cokernel** (of a map): The quotient space $W/\operatorname{Im}\varphi$. *Denoted by* $\mathbf{Coker}\,\boldsymbol{\varphi}$.
- Definitions of **subrepresentation**, **irreducible** (representation), and **direct sum** (of representations).
- **Tensor product** (of V, W): The representation with the space $V \otimes W$ where $g(v \otimes w) = gv \otimes gw$.

10/5: – The n^{th} tensor power is also a representation by this rule.
  - The n^{th} exterior and symmetric powers are subrepresentations of the n^{th} tensor power.
- **Natural pairing** (between V^*, V): The pairing defined as follows for all $v^* \in V^*$ and $v \in V$. *Denoted by* $\langle\ ,\ \rangle$. *Given by*

$$\langle v^*, v \rangle = v^*(v) = (v^*)^T v$$

- **Dual representation**: The representation from $G \to GL(V^*)$ defined as follows. *Denoted by* $\boldsymbol{\rho^*}$. *Given by*
$$\rho^*(g) = \rho(g^{-1})^T$$
  - We should — and do — have
$$\langle \rho^*(g)(v^*), \rho(g)(v) \rangle = \langle v^*, v \rangle$$
  - Indeed,
$$\begin{aligned}
\langle \rho^*(g)(v^*), \rho(g)(v) \rangle &= \rho^*(g)(v^*)^T \rho(g) v \\
&= [\rho(g^{-1})^T (v^*)]^T \rho(g) v \\
&= (v^*)^T \rho(g^{-1})^T \rho(g) v \\
&= (v^*)^T v \\
&= \langle v^*, v \rangle
\end{aligned}$$

- $\text{Hom}(V, W)$ is a representation.
  - Definition via the commutative diagram (from class): $g(L)v = [g \circ L \circ g^{-1}]v$.
  - Definition via the isomorphic space $V^* \otimes W$ and the dual representation:
$$\begin{aligned}
g(v^* \otimes w) &= gv^* \otimes gw \\
&= (g^{-1})^T v^* \otimes gw \\
&= [(g^{-1})^T v^*](gw) \\
&= [(g^{-1})^T v^*]^T gw \\
&= (v^*)^T g^{-1} gw \\
&= (v^*)^T w \\
&= v^*(w)
\end{aligned}$$

- The rules for normal vector spaces hold for representations as well, e.g.,
$$V \otimes (U \oplus W) = (V \otimes U) \oplus (V \otimes W) \qquad \Lambda^k(V \oplus W) = \bigoplus_{a+b=k} \Lambda^a V \oplus \Lambda^b W \qquad \Lambda^k(V^*) = \Lambda^k(V)^*$$

- Definition of **permutation representation** and **regular representation**.

Section 1.2: Complete Reducibility, Schur's Lemma

- **Indecomposable** (representation): See class notes. *Also known as* **irreducible**.
- Proof of Theorem 1 as in Serre (1977).
  - The method is called "integration over the group (with respect to an invariant measure on the group)" (Fulton & Harris, 2004, p. 6).
- **Complete reducibility**: The property that any representation is a direct sum of irreducible representations. *Also known as* **semisimplicity**.
  - Stated here as a corollary; proven as Theorem 2 in Serre (1977).
- The following lemma has several consequences, among which is that it determines how much a representation's direct-sum decomposition is unique.

  Lemma 1.7 (Schur's Lemma). *If V and W are irreducible representations of G and $\varphi : V \to W$ is a G-module homomorphism, then...*

1. Either φ is an isomorphism, or $\varphi = 0$;

2. If $V = W$, then $\varphi = \lambda I$ for some $\lambda \in \mathbb{C}$, I being the identity.

Proof. Suppose for the sake of contradiction that φ is neither an isomorphism nor zero. Then it has a nontrivial kernel and image, both of which are necessarily invariant under the representation[8]. Therefore, neither V nor W are irreducible representations of G, a contradiction.

Since $\mathbb{C}$ is algebraically closed, φ must have an eigenvalue λ. Equivalently, for some $\lambda \in \mathbb{C}$, $\varphi - \lambda I$ has nonzero kernel. But then by part (1), we must have $\varphi - \lambda I = 0$, implying that $\varphi = \lambda I$, as desired. ☐

- Complete reducibility.

  Proposition 1.8. *For any representation V of a finite group G, there is a decomposition*

  $$V = V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}$$

  where the V_i are distinct irreducible representations. The decomposition of V into a direct sum of the k factors is unique, as are the V_i that occur and their multiplicities a_i.

  Proof. Let W be another representation of G, possibly of different dimension. Let $\varphi : V \to W$ be a map of representations. Restrict φ to $V_i^{\oplus a_i}$, a subrepresentation of V. It follows from Schur's Lemma that this restriction either maps into the $W_j^{\oplus b_j}$ satisfying $W_j \cong V_i$, or it does not map it at all.

  Uniqueness for the decomposition of V follows by applying Schur's Lemma to the identity map on V. ☐

- Goals going forward.
  1. Describe all the irreducible representations of G.
     - We can find all *irreducible* representations of G, then describe *any* representation as a linear combination of these.
  2. Find techniques for giving the direct sum decomposition and the multiplicities of an arbitrary representation.
  3. **Plethysm**: Describe the decompositions, with multiplicities, of representations derived from a given representation V, such as $V \otimes V$, V^*, $\Lambda^k V$, $S^k V$, and $\Lambda^k(\Lambda^1 V)$.
     - Note: If V decomposes into two representations, these representations decompose accordingly, e.g., if $V = U \oplus W$, we may invoke the earlier identity to learn that $\Lambda^k V = \oplus_{i+j=k} \Lambda^i U \otimes \Lambda^j W$.
     - **Clebsch-Gordon problem**: Decompose $V \otimes W$, given two irreducible representations V and W.

Section 1.3: Examples — Abelian Groups, S_3

- Classifying the irreducible representations of abelian groups.
  - Let G be an arbitrary finite abelian group, and let V be an irreducible representation of it.
  - Observe that since $gh = hg$ for all $g, h \in G$, we have

    $$\begin{aligned}\rho_V(gh) &= \rho_V(hg) \\ \rho_V(g) \circ \rho_V(h) &= \rho_V(h) \circ \rho_V(g)\end{aligned}$$

  - Thus, each $\rho_V(g)$ is a morphism of G-representations.
  - It follows by Schur's Lemma that each $\rho_V(g) = \lambda_g I$.

[8] See the proof of Schur's Lemma in Serre (1977) for an explanation of this fact; Proposition 4.

- Consequently, every subspace of V is invariant under $\rho_V(g)$ for all $v \in V$. Therefore, V must be one dimensional, hence isomorphic to $\mathbb{C}$.

- Classifying the irreducible representations of S_3.
  - There exist two one-dimensional representations of S_3 (and of every other nontrivial symmetric group).
    - **Trivial representation** (irreducible).
    - **Alternating representation** (irreducible).
  - Using the fact that S_3 is a permutation group, we can locate the...
    - Permutation representation (reducible);
    - **Standard represenatation** (irreducible).
  - Let W be an arbitrary representation of S_3.
    - Easily done with **character theory**, but we'll only get there later.
  - Since the representation theory of finite abelian groups was just proven to be very simple, we'll start by looking at the action of the finite abelian subgroup $A_3 \cong \mathbb{Z}/3\mathbb{Z} \subset S_3$ on W.
    - Let τ be a generator of A_3. Explicitly, this means $\tau = (1,2,3)$ or $\tau = (1,3,2)$.
    - By complete reducibility (Proposition 1.8), W decomposes into a direct sum of irreducible representations:
$$W = \bigoplus V_i$$
    - Since A_3 is abelian, each V_i is one-dimensional. Thus, we may let $V_i = \operatorname{span}(v_i)$.
    - As subrepresentations of W, each V_i is stable under $\rho_W(\tau)$. In other words, each v_i is an *eigenvector* of $\rho_W(\tau)$. Thus, the eigenvectors of $\rho_W(\tau)$ span W![9]
    - What are the corresponding eigenvalues? Let ω denote the eigenvalue corresponding to v_i. Then
$$\begin{aligned}
\omega^3 v_i &= [\rho_W(\tau)]^3 v_i = \rho_W(\tau^3) v_i = \rho_W(e) v_i = 1 v_i \\
\omega^3 &= 1 \\
\omega &= \left(\mathrm{e}^{2\pi i/3}\right)^j
\end{aligned}$$
      where $j = 0,1,2$.
  - Here are two example representations of A_3.
    1. The permutational representation of A_3 is given by
$$\rho_{\mathbb{C}^3}(e) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \rho_{\mathbb{C}^3}(\tau) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad \rho_{\mathbb{C}^3}(\tau^2) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$
       - Note here that the eigenvalues and corresponding eigenvectors of $\rho_{\mathbb{C}^3}(\tau)$ are
$$\begin{aligned}
&\lambda_1 = 1 && \lambda_2 = \mathrm{e}^{2\pi i/3} && \lambda_3 = \mathrm{e}^{4\pi i/3} \\
&v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} && v_2 = \begin{bmatrix} \mathrm{e}^{2\pi i/3} \\ 1 \\ \mathrm{e}^{4\pi i/3} \end{bmatrix} && v_3 = \begin{bmatrix} 1 \\ \mathrm{e}^{2\pi i/3} \\ \mathrm{e}^{4\pi i/3} \end{bmatrix}
\end{aligned}$$
    2. Separately, we have
$$\rho_{\mathbb{C}^n}(e) = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \quad \rho_{\mathbb{C}^n}(\tau) = \begin{bmatrix} \mathrm{e}^{2\pi i/3} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathrm{e}^{2\pi i/3} \end{bmatrix} \quad \rho_{\mathbb{C}^n}(\tau^2) = \begin{bmatrix} \mathrm{e}^{4\pi i/3} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathrm{e}^{4\pi i/3} \end{bmatrix}$$

[9]Errata: This implication is very confusingly reversed in Fulton and Harris (2004).

        - Here, every vector is an eigenvector and there are infinitely many decompositions into direct sums of the one-dimensional representations.

- Now we want to see how the remainder of S_3 acts on W.
    - Let σ be an arbitrary transposition in S_3.
    - Note: $\{\sigma, \tau\}$ generates S_3.
    - Recall the relationship $\sigma\tau\sigma = \tau^2$.
    - The action of σ on the eigenvectors of τ: Let v be an arbitrary eigenvector of τ, with corresponding eigenvalue ω^j. Notice that

$$\tau(\sigma(v)) = \sigma(\tau^2(v)) = \sigma(\omega^{2j}v) = \omega^{2j}\sigma(v)$$

    - Takeaway: v an eigenvector for τ with eigenvalue ω^j implies $\sigma(v)$ an eigenvector for τ with eigenvalue ω^{2j}.
- Exercise 1.10 (A basis for the standard representation of S_3): Verify that with $\sigma = (12)$ and $\tau = (123)$, the standard representation has a basis $\alpha = (\omega, 1, \omega^2), \beta = (1, \omega, \omega^2)$, with

$$\tau\alpha = \omega\alpha \qquad \tau\beta = \omega^2\beta \qquad \sigma\alpha = \beta \qquad \sigma\beta = \alpha$$

    - $1 + \omega + \omega^2 = 0$ in the complex plane.
    - We do, indeed, get

$$\begin{aligned} \tau\alpha &= (\omega^2, \omega, 1) & \tau\beta &= (\omega^2, 1, \omega) & \sigma\alpha &= (1, \omega, \omega^2) & \sigma\beta &= (\omega, 1, \omega^2) \\ &= \omega(\omega, 1, \omega^2) & &= \omega^2(1, \omega, \omega^2) & &= \beta & &= \alpha \\ &= \omega\alpha & &= \omega^2\beta & & & & \end{aligned}$$

    - We also get — per the aforementioned rule — that

$$\tau\alpha = \omega\alpha$$

    but

$$\tau(\sigma\alpha) = \tau\beta = \omega^2\beta = \omega^2(\sigma\alpha)$$

    for instance, as predicted.
    - Note that both α, β are orthogonal to $(1, 1, 1)$, but while they are linearly independent, they are not orthogonal to each other. This is fine, because they're computationally simple, but it is noteworthy.
    - Also note that we derived α, β as eigenvalues v_2, v_3 of the permutational representation just above!
- If the eigenvalue ω of v is not equal to 1, then the eigenvalue ω^2 of $\sigma(v)$ is not equal to 1.
- Thus, if the eigenvalue of v is not 1, then v and $\sigma(v)$ are linearly independent (two linearly dependent eigenvectors necessarily have the same eigenvalue).
    - Indeed, in this case, v and $\sigma(v)$ span a 2D subspace V' that is invariant under S_3!! This is because $v = \sigma(\sigma(v))$ as well.
    - In fact, V' is isomorphic to the standard representation!
    - Note that V' does not decompose further because any eigenvectors of τ are not eigenvectors of σ and vice versa (see below computation of the eigenvectors of σ).
- What if the eigenvalue of v (under τ) is 1?
    - If $\sigma(v)$ is not linearly independent of v, then the two span a 1D subrepresentation of W, isomorphic to the trivial representation (if $\sigma(v) = v$) and isomorphic to the alternating representation (if $\sigma(v) = -v$).
        - ➢ How we know that only the trivial and alternating representations are possible: If $\sigma(v) \in \text{span}(v)$, then v is an eigenvector of σ. Thus, $\sigma(v) = \lambda v$. Since $\lambda^2 v = \sigma^2(v) = e(v) = v$, we know that $\lambda^2 = 1$, so $\lambda = \pm 1$.

- If $\sigma(v)$ is linearly independent of v, then $v+\sigma(v)$ spans a 1D subrepresentation of W isomorphic to the trivial representation and $v-\sigma(v)$ spans a 1D subrepresentation of W isomorphic to the alternating representation.
  - If the eigenvalue of v under τ is 1, then the eigenvalue of $\sigma(v)$ under τ is $1^2 = 1$. Thus, τ is the identity map on span$[v, \sigma(v)]$.
  - On the other hand, in the basis $\{v, \sigma(v)\}$, we can represent σ with the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

  - This matrix has eigenvalues and corresponding eigenvectors

$$\begin{aligned} \lambda_1 &= 1 & \lambda_2 &= -1 \\ v_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} = v + \sigma(v) & v_2 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} = v - \sigma(v) \end{aligned}$$

  - Thus, we see that the 2D subrepresentation spanned by $v, \sigma(v)$ decomposes into a trivial ($\lambda_1 = 1$, $v_1 = v+\sigma(v)$) and alternating ($\lambda_2 = -1$, $v_2 = v-\sigma(v)$) irreducible representation.
- It follows that the only three irreps of S_3 are the trivial, alternating, and standard ones.

- Using the above approach to find the decomposition of the tensor product.
  - Let V be the standard two-dimensional representation. Recall that the basis of V is $\{\alpha, \beta\}$.
  - It follows that the basis of $V \otimes V$ is $\{\alpha\otimes\alpha, \alpha\otimes\beta, \beta\otimes\alpha, \beta\otimes\beta\}$.
  - These are eigenvectors for τ, and we can find their corresponding eigenvalues via direct computation:

$$\begin{aligned} \tau(\alpha\otimes\alpha) &= \tau\alpha\otimes\tau\alpha & \tau(\alpha\otimes\beta) &= \tau\alpha\otimes\tau\beta \\ &= (\omega\alpha)\otimes(\omega\alpha) & &= (\omega\alpha)\otimes(\omega^2\beta) \\ &= \omega^2\alpha\otimes\alpha & &= 1\alpha\otimes\beta \end{aligned}$$

$$\begin{aligned} \tau(\beta\otimes\alpha) &= \tau\beta\otimes\tau\alpha & \tau(\beta\otimes\beta) &= \tau\beta\otimes\tau\beta \\ &= (\omega^2\beta)\otimes(\omega\alpha) & &= (\omega^2\beta)\otimes(\omega^2\beta) \\ &= 1\beta\otimes\alpha & &= \omega\beta\otimes\beta \end{aligned}$$

  - Similarly, we can calculate the effect of σ.

$$\begin{aligned} \sigma(\alpha\otimes\alpha) &= \sigma\alpha\otimes\sigma\alpha & \sigma(\alpha\otimes\beta) &= \sigma\alpha\otimes\sigma\beta & \sigma(\beta\otimes\alpha) &= \sigma\beta\otimes\sigma\alpha & \sigma(\beta\otimes\beta) &= \sigma\beta\otimes\sigma\beta \\ &= \beta\otimes\beta & &= \beta\otimes\alpha & &= \alpha\otimes\beta & &= \alpha\otimes\alpha \end{aligned}$$

  - Because the transformations for $\alpha\otimes\alpha$ and $\beta\otimes\beta$ are directly analogous to the untensored case of α and β, these basis vectors span a subrepresentation isomorphic to the standard representation.
  - Because $\sigma(\alpha\otimes\beta) = \beta\otimes\alpha$ is linearly independent of $\alpha\otimes\beta$ (they are literally different basis vectors), $\alpha\otimes\beta+\beta\otimes\alpha$ spans a trivial representation and $\alpha\otimes\beta-\beta\otimes\alpha$ spans an alternating representation.
  - Altogether, we get that if $V = (2,1)$, then

$$V\otimes V \cong (2,1)\oplus(3)\oplus(1,1,1)$$