Week 1

Introduction to Representation Theory

1.1 Motivating and Defining Representations

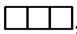
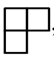
9/27:

- Rudenko would happily approve my final substitution, but it's not his call; it's Boller's.
- HW will be due every week on Wednesday or thereabouts.
  - Submit in paper in a mailbox, location TBA.
  - First HW due next Wednesday.
- Midterm eventually and an in-class final.
- Grading scheme in the syllabus.
- OH not available MW after class (Rudenko has to run to something else), but F after class, we can ask him anything.
  - Regular OH MTh, time TBA.
- There is no specific book for the course.
  - First 8 lectures come from Serre (1977); amazing book but very concise; gets confusing later on. Most lectures are made up by Rudenko.
- Course outline.
  1. Character theory: Beautiful, not too hard.
  2. Non-commutative algebra: More abstract/general approach to the same thing.
  3. Advanced topics, S_n.
- This course's focus: Representations of finite groups in finite dimensions over $\mathbb{C}$.
- This course is for math-inclined people (not quite physics) and lays the foundation for all other Rep Theory.
  - The ideas would be presented in a very different way in Physics Rep Theory.
- We can always ask questions and stop him to correct mistakes during class.
- Why we care about representations.
  - Start with a group G, finite. For example, let $G \equiv S_1$.

- People started to play with S_4 (permutations of roots of a polynomial of degree 4) in Galois theory.
  - Galois theory primer: Consider a polynomial like $x^4 + 3x + 1 = 0$; the roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ satisfy tons of equations, e.g., $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$ since 1 is the x^0 term.
- But groups also occur in much more natural places, e.g., isometries of $\mathbb{R}^3$ that preserve a tetrahedron.
- S_4 is also orientation-preserving isometries of $\mathbb{R}^3$ that preserve a cube.
- Many things lead to the same group!
- Theory of abstract groups developed far later than any of these perspectives; was developed to unify them.

- Recall group actions: Take $G, X = \{x_1, \dots, x_n\}$ both finite. We want $G \circlearrowright X$, which is a homomorphism $A : G \to S_n$.
- What can we do now?
  - We can look at orbits, which are smaller pieces.
  - We can look at the stabilizer.
  - We can identify orbits with cosets.
  - If we understand all possible subgroups, we understand all possible actions.
- This story is not boring, but it's simplistic.
- Rudenko doesn't assume we remember everything (phew!).
- Main definition (general to start, then we simplify).
- **Group representation** (of G on V): A group homomorphism $G \to GL(V)$, for G a group, V a finite-dimensional vector space over some field $\mathbb{F}$ with basis $\{e_1, \dots, e_n\}$, and $GL(V)$ the set of isomorphic linear maps $L : V \to V$. *Denoted by* $\boldsymbol{\rho}$.
  - Recall that $GL(V) \cong GL_n(\mathbb{F})$ is the set of all $n \times n$ invertible matrices.
- For every element $g \in G$, $g \mapsto \rho(g) = A_g$. Essentially, you're mapping to elements that satisfy certain equations.
  - For example, $A_e = E_n$, $A_{g_1g_2} = A_{g_1}A_{g_2}$, and $A_{g^{-1}} = {A_g}^{-1}$.
  - Thus, representations are a "concrete way to think about groups."
  - If you don't understand abstract group G, let us compare it to a group that we do understand! Like a group can *act* via S_n, we can *represent* a group in a vector space.
- In this course, G is finite, $\mathbb{F} = \mathbb{C}$, and V is finite dimensional.
  - This is the most simple case, but also a very interesting one. The theory is much, much easier, so we can get much more complicated, but this is a good place to start.
  - We could make G compact, but we're not gonna go that far.
- Examples to get an idea of what's going on.
  1. $\deg \rho = 1$ (means $\dim V = 1$). Then $\rho : G \to GL_1(V) \cong \mathbb{C}^\times$. The codomain is referred to as the **character** of the group??
     - An example group homomorphism $S_n \to \mathbb{C}^\times$ is the sign function $\sigma \to \text{sign}(\sigma) = \{\pm 1\}$.
     - Another example is the **trivial representation**, $G \to \mathbb{C}^\times$ and $g \mapsto 1$.
  2. Smallest one: Let $G = S_3$. The structure is already pretty rich, and this will be part of the homework.

- **Trivial representation** again.
- **Alternating representation**.
- **Standard representation**.
- **Regular representation**.

- **Trivial representation**: The representation $\rho : G \to GL(V)$ sending $g \mapsto 1$ for all $g \in G$. *Denoted by* , **(3)**.
  - The boxes notation is too much of a detour to explain now.
  - Note that $1 \in GL(V)$ is the identity map on V!
  - Also note that you may define *a* trivial representation on any space V, but *the* trivial *irreducible* representation is necessarily on $\mathbb{C}$. (We'll define **irreducible** and deal much more with this specific trivial representation later.)
- **Alternating representation**: The representation $\rho : G \to GL(V)$ sending $g \mapsto \text{sign}(g)$ for all $g \in G$. *Denoted by* , **(1, 1, 1)**.
- **Standard representation**: The representation $\rho : S_n \to GL(V)$ sending $\sigma \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)})$, where $V = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_1 + \cdots + x_n = 0\}$ is a $(n-1)$-dimensional vector space. *Denoted by* , **(2, 1)**.
  - A 2D representation like rotating a triangle.
  - This gives something with real numbers.
  - Example: $S_3 \circlearrowright V$ by $\sigma((x_1, x_2, x_3)) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$.
- **Regular representation**: The representation $\rho : G \to \text{Hom}(\mathbb{C}^n)$ defined by $g \mapsto \sigma_g$, where $G = \{g_1, \dots, g_n\}$, $\{e_{g_1}, \dots, e_{g_n}\}$ is a basis of $\mathbb{C}^n$, $\cdot$ is the group action of $\rho(G) \circlearrowright \mathbb{C}^n$ defined by $\rho(g) \cdot e_g = e_{gg_i}$, and $\sigma_g(e_{g_i}) = \rho(g) \cdot e_g = e_{gg_i}$.
  - This is a permutation of vectors.
  - Thus, for S_3, it will already be 6-dimensional (it's very high dimensional).
- How do we know that representation theory is tractable? Sure, we can define all these things, but how do we know that it will lead anywhere? Here's an example.
  - Let $G = \mathbb{Z}/2\mathbb{Z} = \{e, g\}$, $V = \mathbb{C}^n$, $\rho : G \to GL_n(\mathbb{C}) \cong GL(\mathbb{C}^n)$, and $A = \rho(g)$ an $n \times n$ matrix over $\mathbb{C}$. Since $g^2 = e$, we know for example that $A^2 = E_n$.
  - But how do we find the matrices A? If we look at eigenvalues of A, there are only two possibilities: ± 1. The structure of A can be very complicated with Jordan normal form and all that, but in fact, these are the **semisimple matrices**, so it's not that bad.
  - Since $A^2 = E$, we know that $(A - E)(A + E) = 0$. Consider $(A - E) : V \to V$. Naturally, it has $\text{Ker}(A - E)$ and $\text{Im}(A - E)$. In this case, Rudenko claims that $\text{Ker}(A - E) \cap \text{Im}(A - E) = \{0\}$. Here's the proof to back up that claim.

    Proof. Let $v \in \text{Ker}(A - E) \cap \text{Im}(A - E)$ be arbitrary. Since $v \in \text{Im}(A - E)$, there exists $w \in V$ such that $v = (A - E)w = Aw - w$. Since $v \in \text{Ker}(A - E)$, we have $(A - E)v = 0$, so $Av = v$. It follows that $A(Aw - w) = Aw - w$ but also $A(Aw - w) = Ew - Aw = w - Aw$. Thus,

    $$\begin{aligned} Aw - w &= w - Aw \\ 2Aw &= 2w \\ Aw &= w \end{aligned}$$

    But then $w \in \text{Ker}(A - E)$, so $v = (A - E)w = 0$. $\square$

- This combined with the fact that every vector in a vector space is in either the image or the kernel of a linear map[1] implies that $V = \operatorname{Ker}(A-E) \oplus \operatorname{Im}(A-E)$.
- Let $\operatorname{Ker}(A-E)$ have basis $e_1, \dots, e_k$ and let $\operatorname{Im}(A-E)$ have basis $e_{k+1}, \dots, n$; then all A are of the following form.

$$
\begin{array}{c c}
 & \begin{array}{cccc} 1 & k & k+1 & n \end{array} \\
\begin{array}{c} 1 \\ \\ k \\ k+1 \\ \\ n \end{array} &
\left[\begin{array}{ccc|ccc}
1 & & & & & \\
 & \ddots & & & & \\
 & & 1 & & & \\
\hline
 & & & -1 & & \\
 & & & & \ddots & \\
 & & & & & -1
\end{array}\right]
\end{array}
$$

- Next time, we will discuss sums of representations, of which this is an example of the theory.

- The same kind of thing, **simple representations**, happens with all finite groups?? This is where we're going. It's not rocket science; in fact, we'll see it next week.
- Last thing for today: A remarkable story.
  - The story of representation theory started quite different.
  - A beautiful theorem that we can prove now!
  - Frobenius determinant.
  - Think of $G = \{g_1, \dots, g_n\}$. Picture its multiplication table.
  - In every row and column, you see each element once.
  - Let's associate to the multiplication table an actual determinant in the linear algebra sense. Consider elements $x_{g_1}, \dots, x_{g_n}$. Define the $n \times n$ matrix $(x_{g_i g_j})$. Take its determinant. It will be a polynomial in n variables, i.e., an element of the ring $\mathbb{Z}[x_{g_1}, \dots, x_{g_n}]$.
  - Example: Consider
$$\begin{vmatrix} e & g \\ g & e \end{vmatrix}$$
    - The determinant is $x_e^2 - x_g^2 = (x_e - x_g)(x_e + x_g)$.
  - Example: $G = \mathbb{Z}/3\mathbb{Z}$.
    - If the elements are e, g, g^2 and we map these, respectively, to variables a, b, c, we get the matrix
$$\begin{bmatrix} e & g & g^2 \\ g & g^2 & e \\ g^2 & e & g \end{bmatrix} \mapsto \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$
    - The determinant is $3abc - a^3 - b^3 - c^3 = (a+b+c)(a^2+b^2+c^2-ab-bc-ac) = (a+b+c)(a+\zeta b+\zeta^2 c)(a+\zeta^2 b+\zeta c)$ where $\zeta \in \mu_3$ is a third root of unity, i.e., we have that $\zeta^3 = 1$.
  - Frobenius's theorem: If G is a finite group and we take this Frobenius determinant, then this determinant is equal to ${P_1}^{d_1} \cdots {P_k}^{d_k}$, where $P_1, \dots, P_k$ are irreducible polynomials in $x_g, \dots, x_{g_j}$, $\deg P_i = d_i$, and k is the number of conjugacy classes.
  - Example: Take S_3; we'll get a polynomial of degree $|S_3| = 6$ but the Frobenius determinant $FD = (x_{g_1} + \cdots + x_{g_k})(x_{g_1} \pm \cdots)(\text{some pol. of deg } 2)^2$
  - The proof is remarkable and deep and uses what would become character theory. These polynomials are related to representations and the number of simplest irreducible representations. The theory that came out came as a way to understand this miracle. We'll forget FD's for now, but then come back and prove it later.

[1] See Theorem 3.6 of Axler (2015).

1.2 Key Definitions and Category Theory Primer

9/29:
- OH: TW 4:30 or 5:00 most likely; he will confirm later.
- Today: Definitions in greater generality.
- As before, let G be a finite group and V be a finite-dimensional vector space.
- Goal of this course: Understand representations of G, that is...
  - Homomorphisms $\rho : G \to GL(V) \cong GL_n(\mathbb{C})$;
  - That send $g \mapsto A_g \in GL_n(\mathbb{C})$;
  - And satisfy $A_e = E$, $A_{g_1}A_{g_2} = A_{g_1g_2}$, and $A_{g^{-1}} = {A_g}^{-1}$.
- What are some things we might want to do?
  - Build new representations from old? Investigate and/or classify irreducible representations?
  - Before we can see if any of this works or not, we need a ton of definitions: Sum, equality, etc.
- Rudenko will start today's lecture with some general thoughts on the **category** of representations.
  - Categories are things that now permeate math.
- **Category**: A *class* (not a set) of *objects* (some things; you don't know anything about them), and then a bunch of properties.

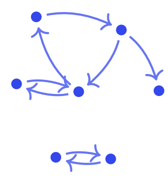


Figure 1.1: The general structure of a category.

  - Objects a, b in category C are denoted by $a, b \in \mathbf{Ob}(\boldsymbol{C})$.
  - There are also **morphisms** between the objects. These are drawn as arrows and lie in $\text{Hom}(a, b)$.
  - There is also composition $\circ : \text{Hom}(a, b) \times \text{Hom}(b, c) \to \text{Hom}(a, c)$.
    - ■ This notation rigorously defines composition, i.e., as a binary operation on functions.
  - Properties.
    1. Associativity.
    2. Existence of a unit: For any object a, there exists $\text{id}_a \in \text{Hom}(a, a)$ such that any morphism pre- or post-composed to this identity yields the same morphism.
       - ■ Example: If $f \in \text{Hom}(a, b)$, then $\text{id}_b \circ f = f = f \circ \text{id}_a$.
  - Rudenko: So a category is basically two pieces of data and a bunch of properties.
- Examples of categories.
  - Category of sets and maps between them.
  - Category of vector spaces over $\mathbb{F}$ where $\text{Ob}(C)$ is the vector spaces and $\text{Hom}(V, W)$ is filled with *linear* maps because you don't just want maps — you want maps that respect the structure.
  - Category of groups where $\text{Hom}(G_1, G_2)$ is the set of group homomorphisms.
  - Category of topological spaces and continuous maps.
  - Category of abelian groups.
  - Trivial category and the identity map; thus, categories need not be chonky.

- Comments on category theory.
  - We'll see some pretty significant category theory at the end of the course.
  - We'll see categories in every course we take; some people try to avoid them. Rudenko doesn't want to go into the material in depth, but he wants to use language from it.
  - Surprisingly, even under the stripped-down of axioms of category theory, you can say quite a lot.
  - Why any of this discussion of category theory matters: If you know the basics of category theory, you can guess the definitions of direct sum, equality, etc. for representations.
- **Category of representations**. *Denoted by* $\mathbf{Rep}_{\boldsymbol{G}}$.
- Take two G-representations V, W; how do we define a map between them?
  - Recall that V, W are vector spaces.
- **Morphism** (of G-representations): A map $f : V \to W$ such that...
  1. f is linear;
  2. f respects the structure of the representations; explicitly, for every $g \in G$, $\rho_W(g) \circ f = f \circ \rho_V(g)$[2].
- $\mathbf{Hom}_{\boldsymbol{G}}\boldsymbol{(V, W)}$: The set of all morphisms of G-representations from V to W.
- On constraint 2, above: This condition is summarized via a **commutative diagram**.

$$
\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "\rho_V(g)"'] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[d, "\rho_W(g)"] \\
V \arrow[r, "f"'] & W
\end{tikzcd}
$$

Figure 1.2: Commutative diagram, morphisms.

  - Commutative diagrams are very category-theory-esque things.
- That was a very abstract definition. Let's make it concrete.
  - Suppose you have a pair of representations $V = \mathbb{C}^n, W = \mathbb{C}^m$, and we have our map F between them given by an $m \times n$ matrix.
  - Let $\rho_V(g) = A_g$ be an $n \times n$ matrix, and let $\rho_W(g) = B_g$ be an $m \times m$ matrix.
  - Then $FA_g = B_gF$.
- More examples of representations.
  1. An interesting example: Let's look at $S_3 \circlearrowright V_{\text{perm}} = \mathbb{C}^3$, a **permutation representation**.
     - For all $\sigma \in S_3$, $\rho(\sigma) : (x_1, x_2, x_3) \mapsto (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$.
  2. There's also the trivial representation $S_3 \circlearrowright V_{(3)} = \mathbb{C}$ defined by $\rho(\sigma) : x \mapsto x$.
- Are the above 2 representations related?
  - Yes! We can, in fact, find a *morphism* between them.
  - In particular, define $f : V_{(3)} \to V_{\text{perm}}$ by $f(x) = (x, x, x)$.
    - Since permuting 3 of the same thing does nothing, the commutativity of Figure 1.2 holds. Therefore, f is a morphism of G-representations as defined above.

[2]Recall that the object, $\rho_V(g)$ is a linear map! Thus, it can be composed with other linear maps like f.

    - We may also explicitly confirm that f is a morphism as follows.

$$f[\rho_{(3)}(\sigma)(x)] = f(x) = (x,x,x) = \rho_{\text{perm}}(\sigma)((x,x,x)) = \rho_{\text{perm}}(\sigma)[f(x)]$$

  - Is f **reversible**?
    - Is "reversible" the right word??
  - Define $\tilde{f} : V_{\text{perm}} \to V_{(3)}$ by $\tilde{f} : (x_1, x_2, x_3) \mapsto x_1 + x_2 + x_3$.
    - Since addition is commutative, the commutativity of Figure 1.2 holds.
    - More explicitly,

$$\begin{aligned}
f[\rho_{\text{perm}}(\sigma)((x_1,x_2,x_3))] &= f((x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})) \\
&= x_{\sigma(1)} + x_{\sigma(2)} + x_{\sigma(3)} \\
&= x_1 + x_2 + x_3 && \text{Commutativity of addition} \\
&= f((x_1,x_2,x_3)) \\
&= \rho_{(3)}(\sigma)[f((x_1,x_2,x_3))]
\end{aligned}$$

  - Takeaway: The existence of maps between representations is interesting.

- Next question: How do we define an **isomorphism** of two representations?
- **Isomorphism** (of G-representations): A morphism of G-reps that is an isomorphism of vector spaces.
- Category theory helps us again here because it generalizes the concept of an isomorphism!
  - If $f : V \to W$ and $g : W \to V$ are category-theoretic morphisms, then the constraints $f \circ g = \text{id}_W$ and $g \circ f = \text{id}_V$ make f and g into category-theoretic *iso*morphisms, regardless of what V and W might be.
  - Back in the context of representations, let $f : V \to V$ be an isomorphism of vector spaces. Then we do indeed have $\rho_V(g) \circ f = f \circ \rho_V(g)$, as we would hope from category theory!
- Recall the condition $FA_g = B_gF$. Supposing F is an isomorphism (and thus has an inverse), we get $FA_gF^{-1} = B_g$ as our new condition.
  - Essentially, we can do *simultaneous conjugation* of all matrices.
  - As per usual with isomorphisms, we get to *change bases*.
  - Essentially, we can represent the nice permutation representation in a very nasty basis but still have it be valid.
- Many other notions (e.g., direct sum) will not be explained by Rudenko, but we can read about them!
- However, we'll do a few more.
- A representation sitting inside another: a **subrepresentation**.
- **Subrepresentation** (of V): A subspace $W \subset V$ such that for all $w \in W$ and $g \in G$, we have that $\rho_V(g)W \subset W$, where V is a G-representation with $\rho_V : G \to GL(V)$.
  - Many people will just write the critical condition as $gW \subset W$.
- Subrepresentations in category theory: We have another commutative diagram.
- Example: The trivial representation, the standard representation, and (of course) the **zero representation** are subrepresentations of the permutation representation.
- **Zero representation**: The representation $\rho : G \to GL(\{0\})$ sending $g \mapsto 1$ for all $g \in G$. *Denoted by* $\mathbf{(0)}$.

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d, "\rho_V(g)"'] & V \arrow[d, "\rho_V(g)"] \\ W \arrow[r, hook] & V \end{tikzcd}$$

Figure 1.3: Commutative diagram, subrepresentations.

- What about representations that don't have subrepresentations?
- **Simple** (representation): A G-representation V that has only two subrerpesentations: (0) and V. *Also known as* **irreducible**, **irreps**.
- Example irreducible representations: Line in $\mathbb{C}^2$, triangle in $\mathbb{C}^2$, A_5 and dodecahedron in $\mathbb{C}^3$.
- Notion of a direct sum.
- **Direct sum** (of V_1, V_2): The G-rep associated with the space $V_1 \oplus V_2 = \{(v_1, v_2) \mid v_1 \in V_1, v_2 \in V_2\}$, where $\rho_{V_1 \oplus V_2}(g)(v_1, v_2) = (\rho_{V_1}(g)v_1, \rho_{V_2}(g)v_2)$. *Denoted by* $\boldsymbol{V_1 \oplus V_2}$.
  - The matrix of $\rho_{V_1 \oplus V_2}(g)$ is the following block matrix.

$$\rho_{V_1 \oplus V_2}(g) = \left[\begin{array}{c|c} \rho_{V_1}(g) & 0 \\ \hline 0 & \rho_{V_1}(g) \end{array}\right]$$

- Example: $V_{\text{perm}} = V_{(3)} \oplus V_{(2,1)}$, with $\mathbb{C}^3 = \mathbb{C} \oplus \mathbb{C}^2$ where

$$\mathbb{C} \cong \langle (1,1,1) \rangle \qquad \mathbb{C}^2 \cong \langle (x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0 \rangle$$

  - The decomposition is into simple representations.
  - Relate this to the fact that the JCF of any 3×3 permutation matrix has at most a 1-block and a 2-block, if not three 1-blocks. There will always be one 1D subspace on which the permutation matrix is an identity, i.e., $\text{span}(1,1,1)$, and a 2D orthogonal complement!
  - As a fun and simple exercise, prove that there is no line fixed under the standard representation.
- A simple and important theorem to prove next week.
- Theorem: Let G be a finite group and $\mathbb{F} = \mathbb{C}$. Then...
  1. There are finitely many irreps $V_1, \dots, V_s$ up to isomorphism.
     - Later on, we will see that s is equal to the number of conjugacy classes.
  2. For every G-rep V, there exists a unique $n_1, \dots, n_s \geq 0$ such that $V \cong {V_1}^{n_1} \oplus \cdots \oplus {V_s}^{n_s}$.
- This theorem tells us that if we want to study rep theory, we want to study irreps (which can be kind of complicated) because if we understand them, everything breaks down into them.
- Examples.
  1. $G = \mathbb{Z}/2\mathbb{Z} = S_2$.
     - $V_1 = \mathbb{C}e$ with $ge = e$ and $V_{-1} = \mathbb{C}e$ with $ge = -e$.
     - It follows that $V \cong {V_1}^{n_1} \oplus {V_{-1}}^{n_{-1}}$.
     - We get a diagonal matrix with only 1s and -1s, just like the example from last time!
  2. $G = S_3$.
     - $V_{(3)}, V_{(1,1,1)}, V_{(2,1)}$.

    - $GL_5(\mathbb{F}_4)$.
    - Proven in an elementary way in Section 1.3 of Fulton and Harris (2004), which we have to read for the HW; will be useful for later in the course's HW.

- Plan: Next time, we'll talk about some more abstract stuff like the tensor products of vector spaces.
  - Tensor products are something we should read up on now! The definition is hard and abstract.
  - Then he'll prove the above theorem.

1.3 S Chapter 1: Generalities on Linear Representations

From Serre (1977).

10/3:

- Part I (what we'll be covering) is written for quantum chemists, and thus gives proofs "as elementary as possible, using only the definition of a group and the rudiments of linear algebra" (Serre, 1977, p. v).
  - Recall the story about Serre and his wife, the chemist, who needed to explain group theory and rep theory to her students.
- Indeed, although the book seemed very fast when I first looked at it two years ago, it reads much more easily now and has enough context for most anyone who is comfortable with group theory and theoretical linear algebra.

Section 1.1: Definitions

- Definitions of $\boldsymbol{GL(V)}$, **invertible square matrix**, and **finite group**.
- **Linear representation**: See class notes. *Also known as* **group representation**.
  - Serre (1977) will frequenty write ρ_s for $\rho(s)$.
- **Representation space** (of G): The vector space V corresponding to the linear representation $\rho : G \to GL(V)$ of G. *Also known as* **representation**.
  - The latter term is a self-identified "abuse of language" (Serre, 1977, p. 3).
- "For most applications, one is interested in dealing with a *finite number of elements* x_i of V, and can always find a subrepresentation of V... of finite dimension, which contains the x_i; just take the vector subspace generated by the images $\rho_s(x_i)$ of the x_i" (Serre, 1977, p. 4).
- **Degree** (of a representation): The dimension of the representation space of this representation.
- To give a representation **in matrix form** is to give a set of invertible matrices that are isomorphic to the group elements.
- Important converse: Given invertible matrices satisfying the appropriate homomorphism identities, there is a corresponding group that these matrices represent.
- **Similar** (representations of G): Two representations $\rho : G \to GL(V)$ and $\rho' : G \to GL(V')$ of G for which there exists a linear isomorphism $\tau : V \to V'$ such that

$$\tau \circ \rho(s) = \rho'(s) \circ \tau$$

  for all $s \in G$. *Also known as* **isomorphic**.
  - Equivalent definition (in matrix form): There exists T invertible such that $R_s' = TR_sT^{-1}$.
  - Isomorphic representations have the same degree.

Section 1.2: Basic Examples

- Degree 1 representation: A homomorphism $\rho : G \to \mathbb{C}^*$, where $\mathbb{C}^*$ denotes the roots of unity (all $z \in \mathbb{C}$ with $|z| = 1$).
  - The fact that every $s \in G$ has *finite* order by assumption is what permits this representation.
- **Unit representation**: See class notes. *Also known as* **trivial representation**.
- **Regular representation**: The representation $\rho : G \to GL(V)$ defined by $s \mapsto [e_t \mapsto e_{st}]$ for all $s \in G$, where V has basis $(e_t)_{t \in G}$.
  - $\deg \rho = |G|$.
  - $e_s = \rho_s(e_1)$.
    - Implication: The images of e_1 under the ρ_s's form a basis of V, i.e., $\{\rho_s(e_1) \mid s \in G\}$ is a basis of V.
  - Converse of above: If W is a representation of G containing a vector w such that $\{\rho_s(w) \mid s \in G\}$ forms a basis of W, then W is isomorphic to the regular representation V via $\tau : V \to W$ defined by $\tau(e_s) = \rho_s(w)$.
- **Permutation representation** (associated with X): The representation $\rho : G \to GL(V)$ defined by $s \mapsto [e_x \mapsto e_{s \cdot x}]$ for all $s \in G$, where $G \circlearrowright X$ a finite set and V has a basis $(e_x)_{x \in X}$.

Section 1.3: Subrepresentations

- Definition of **subrepresentation**.
  - Example: Trivial representation $\mathbb{C}(x, \dots, x)$ is a subrepresentation of the regular representation.
- Definitions of **direct sum** (of vector spaces) and **kernel** (of a linear map).
- **Complement** (of a subspace $W \leq V$): Any $(n - m)$-dimensional subspace U that...
  1. Satisfies $W \oplus U = V$;
  2. Intersects W trivially;

  where $n = \dim V$ and $\dim W = m \leq n$.
  - This means that a single subspace can have multiple complements!
    - Only one **orthogonal** complement, but many general *complements*.
    - Example: Consider a line through the origin in $\mathbb{R}^2$; any other line through the origin is a complement of it!
  - It follows that there is a bijection between the complements W' of W in V and the projections p of V onto W (since non-orthogonal complements require non-orthogonal projections).
- **Projection** (of V onto W associated with the decomposition $V = W \oplus W'$): The mapping that sends each $x \in V$ to its component $w \in W$. *Denoted by* $\boldsymbol{p}$.
  - Consequence: The two properties defining a p are...
    1. $\text{Im}(p) = W$;
    2. $p(x) = x$ for all $x \in W$.
  - Consequence: These two properties also imply that if p is a projection onto $W \leq V$, then $V = W \oplus \text{Ker}(p)$.
- If a representation has a subrepresentation, then some complement of this subrepresentation is also a subrepresentation.

Theorem 1. *Let $\rho : G \to GL(V)$ be a linear representation of G in V and let W be a vector subspace of V stable under G. Then there exists a complement W^0 of W in V which is stable under G.*

Proof 1 (limited conditions). Let p be the projection of V onto W that corresponds to some arbitrary complement of W in V. To begin, we may legally — albeit with little motivation — form the average p^0 of the conjugates of p by the elements of G:

$$p^0 := \frac{1}{|G|} \sum_{t \in G} \rho_t \cdot p \cdot \rho_t^{-1}$$

We now seek to prove that p^0 is a projection by showing that it satisfies the two properties of a "p." First, notice that by assumption, every ρ_t (and thus ρ_t^{-1}) preserves W. This combined with the fact that $p(V) = W$ implies that $p^0(V) = W$, as desired. Additionally, for any $x \in W$ and $t \in G$, we know by property (2) of a p and the fact that $p_t^{-1}(x) \in W$ that $p \cdot p_t^{-1}(x) = p_t^{-1}(x)$. Applying p_t to both sides of this equation yields $[p_t \cdot p \cdot p_t^{-1}](x) = x$. Hence, $p^0(x) = x$, as desired. Thus, p^0 is a projection of V onto W, associated with some complement W^0 of W.

So that we can make a substitution later, we will now prove that

$$\rho_s \cdot p^0 = p^0 \cdot \rho_s$$

for all $s \in G$. Pick such an s. Then

$$\rho_s \cdot p^0 \cdot \rho_s^{-1} = \frac{1}{|G|} \sum_{t \in G} \rho_s \cdot \rho_t \cdot p \cdot \rho_t^{-1} \cdot \rho_s^{-1} = \frac{1}{|G|} \sum_{st \in G} \rho_{st} \cdot p \cdot \rho_{st}^{-1} = p^0$$

so we can post-compose both sides of the above equation with ρ_s to yield the final result. This line here should make it clear why we needed to form a projection like p^0.

We now have all of the tools we need to prove that W^0 is stable under G. To do so, it will suffice to show that for all $x \in W^0$ and $s \in G$, we have $\rho_s(x) \in W^0$. Let $x \in W^0$ and $s \in G$ be arbitrary. Since $x \in W^0$, $p^0(x) = 0$ by definition. This combined with the above commutativity rule implies that $p^0 \cdot \rho_s(x) = \rho_s \cdot p^0(x) = \rho_s(0) = 0$. But the only way that p^0 could map $\rho_s(x)$ to 0 is if $\rho_s(x) \in W^0$, as desired. $\square$

Proof 2 (orthogonal complement). Let W^0 be the orthogonal complement of W, and endow V with a **scalar product** $(x \mid y)$ to turn it into an inner product space. Replace $(x \mid y)$ with the new inner product $\sum_{t \in G} (\rho_t x \mid \rho_t y)$. Now, if it wasn't already, the inner product is invariant under ρ_s for all s, i.e., for s arbitrary, we have

$$(\rho_s x \mid \rho_s y) = (x \mid y)$$

This means that vectors that were orthogonal before ρ_s is applied to V, stay orthogonal after ρ_s is applied to V. In particular, since ρ_s preserves W by hypothesis, all vectors orthogonal to W (i.e., all vectors in W^0) stay orthogonal to W (i.e., stay in W^0) after ρ_s is applied. Thus, W^0 is stable under ρ_s as well. $\square$

- Note: Since we can define a scalar product that is invariant under ρ_s, if (e_i) is an orthonormal basis of V, then the matrix of ρ_s with respect to this basis is a **unitary** matrix.
- Consequence of the second, stronger proof: The representations W and W^0 determine the representation V.
  - This allows us to rigorously say that the representation $V = W \oplus W^0$.
  - If W, W^0 are given in matrix form by R_s, R_s^0, then $W \oplus W^0$ is given in matrix form by

$$\left(\begin{array}{c|c} R_s & 0 \\ \hline 0 & R_s^0 \end{array}\right)$$

- We can extend this method of directly summing representations to an arbitrary finite number of them.

Section 1.4: Irreducible Representations

- Definition of **irreducible** representation.
- Fact: Each nonabelian group possesses at least one irreducible representation with deg ≥ 2.
  - Proven later.
- Irreducible representations construct all representations via the direct sum.

  Theorem 2. *Every representation is a direct sum of irreducible representations.*

  Proof. We induct on $\dim(V)$.

  Suppose $\dim(V) = 0$. Since 0 is the direct sum of the empty family of irreducible representations, the theorem is vacuously true.

  Suppose $\dim(V) \geq 1$. We divide into two cases (V is irreducible and V is reducible). In the first case, we are done. In the second case, $V = V' \oplus V''$ for some $V' \perp V''$ (see Theorem 1). Since $\dim(V') < \dim(V)$ and $\dim(V'') < \dim(V)$ by definition, the induction hypothesis implies that V' and V'' are direct sums of irreducible representations. Therefore, the same is true of V. $\square$

- Fact: The direct-sum decomposition is not necessarily unique.
  - Counterexample: If $\rho_s = 1$ for all $s \in G$, then there are a plethora of decompositions of a vector space into a direct sum of lines.
- Fact: The number of W_i (in a direct sum decomposition $V = W_1 \oplus \cdots \oplus W_k$) that are isomorphic to a given irreducible representation *does not* depend on the chosen decomposition.
  - Proven later.