2 Introduction to Character Theory

10/13: 1. **More linear algebra**. Let V be a finite-dimensional vector space.

(a) Prove that under the identification of $V \otimes V^*$ with $\operatorname{Hom}_F(V, V)$, **simple** tensors $v \otimes \varphi$ correspond to linear maps of rank 0 or 1.

Proof. Let $v \otimes \varphi$ be an arbitrary simple tensor im $V \otimes V^*$. Recall from the 10/2 lecture that

$$v_1 \otimes \alpha \mapsto [v_2 \mapsto \alpha(v_2)v_1]$$

is a good isomorphism from $V \otimes V^* \cong \operatorname{Hom}_F(V, V)$. It follows that the linear map to which $v \otimes \varphi$ corresponds is the map $L : V \to V$ defined by $L(v') = \varphi(v')v$. Since $\operatorname{Im}\varphi = \mathbb{C}$, we have that $\operatorname{Im}(L) \leq \mathbb{C}v$. Thus, since $\dim(\mathbb{C}v) = 1$, we have that $\operatorname{rank}(L) \leq 1$, as desired. ☐

(b) Consider the vector space $W = \operatorname{Hom}_F(V, V)$. Prove that any linear functional in W^* has the form $L \mapsto \operatorname{tr}(LM)$ for some $M \in W$. Prove that the vector space $\operatorname{Hom}_F(V, V)$ is "canonically" self-dual.

Proof. Let $\varphi \in W^*$ be arbitrary. Also let $n := \dim V$ for convenience. Notice that the n^2 matrices E_{ij} $(i, j = 1, \dots, n)$, which have a 1 in the i^{th} row and j^{th} column and 0s everywhere else, form a basis of W. Thus, φ is fully characterized by its actions on the E_{ij}. Now define

$$M := (\varphi(E_{ij}))^T$$

It follows that if $L = (\ell_{ij})$, then

$$\varphi(L) = \sum_{i=1}^{n}\sum_{j=1}^{n} \ell_{ij}\varphi(E_{ij}) = \operatorname{tr}(LM)$$

as desired. This completes the proof, but to help illustrate it, I'll include the $n = 3$ case:

$$\begin{aligned}
\underbrace{\begin{bmatrix}
\ell_{11} & \ell_{12} & \ell_{13}\\
\ell_{21} & \ell_{22} & \ell_{23}\\
\ell_{31} & \ell_{32} & \ell_{33}
\end{bmatrix}}_{L=(\ell_{ij})} \circ \underbrace{\begin{bmatrix}
\varphi(E_{11}) & \varphi(E_{12}) & \varphi(E_{13})\\
\varphi(E_{21}) & \varphi(E_{22}) & \varphi(E_{23})\\
\varphi(E_{31}) & \varphi(E_{32}) & \varphi(E_{33})
\end{bmatrix}^T}_{M=(\varphi(E_{ij}))^T}
&= \underbrace{\begin{bmatrix}
\sum_{j=1}^3 \ell_{1j}\varphi(E_{1j}) & & \cdots\\
& \sum_{j=1}^3 \ell_{2j}\varphi(E_{2j}) & \\
\cdots & & \sum_{j=1}^3 \ell_{3j}\varphi(E_{3j})
\end{bmatrix}}_{LM}
\end{aligned}$$

To prove that $\operatorname{Hom}_F(V, V) = W$ is canonically self-dual, it will suffice to construct an isomorphism $W^* \cong W$ that does not depend on a choice of basis. Let $\varphi \in W^*$ be arbitrary. As demonstrated above, there exists a unique corresponding $M \in W$ such that $\varphi(L) = \operatorname{tr}(LM)$ for all $L \in W$. Therefore, the map $f : W^* \to W$ defined by $\varphi \mapsto M$ is a bijection. To show that it is linear, too, we add in a basis and compute as follows.

$$\begin{aligned}
f(\varphi_1 + \varphi_2) &= ([\varphi_1 + \varphi_2](E_{ij}))^T\\
&= (\varphi_1(E_{ij}) + \varphi_2(E_{ij}))^T\\
&= (\varphi_1(E_{ij}))^T + (\varphi_2(E_{ij}))^T\\
&= f(\varphi_1) + f(\varphi_2)
\end{aligned}
\qquad
\begin{aligned}
f(\lambda\varphi) &= ([\lambda\varphi](E_{ij}))^T\\
&= (\lambda\varphi(E_{ij}))^T\\
&= \lambda(\varphi(E_{ij}))^T\\
&= \lambda f(\varphi)
\end{aligned}$$

☐

2. **Characters of abelian groups**. Let A be a finite abelian group.

   (a) A **character** of A is a homomorphism $\chi : A \to \mathbb{C}^\times$. Prove that for every $g \in A$, the value $\chi(g)$ is a root of unity. Prove that the product of characters is a character. Prove that characters form an abelian group. This group is called the **dual** of A and is denoted $\widehat{A}$.

Proof. Let $g \in A$ be arbitrary. Since A is finite, $|g|$ is finite. It follows since χ is a homomorphism that

$$1 = \chi(e) = \chi(g^{|g|}) = \chi(g)^{|g|}$$

Therefore, since the only complex numbers having 1 as a power are the roots of unity, $\chi(g)$ is a root of unity, as desired.

Let χ_1, χ_2 be two characters of A. To prove that their product $\chi_1\chi_2$ is a character, it will suffice to show that $\chi_1\chi_2$ is a group homomorphism. To do so, we must confirm that

$$\chi_1\chi_2(e) = 1 \qquad \chi_1\chi_2(g_1g_2) = \chi_1\chi_2(g_1) \cdot \chi_1\chi_2(g_2) \qquad \chi_1\chi_2(g^{-1}) = \chi_1\chi_2(g)^{-1}$$

We can do this using the analogous statements satisfied by χ_1 and χ_2 separately. Specifically,

$$\begin{aligned} \chi_1\chi_2(e) &= \chi_1(e) \cdot \chi_2(e) \\ &= 1 \cdot 1 \\ &= 1 \end{aligned} \qquad \begin{aligned} \chi_1\chi_2(g^{-1}) &= \chi_1(g^{-1}) \cdot \chi_2(g^{-1}) \\ &= \chi_1(g)^{-1} \cdot \chi_2(g)^{-1} \\ &= (\chi_1(g) \cdot \chi_2(g))^{-1} \\ &= \chi_1\chi_2(g)^{-1} \end{aligned}$$

$$\begin{aligned} \chi_1\chi_2(g_1g_2) &= \chi_1(g_1g_2) \cdot \chi_2(g_1g_2) \\ &= \chi_1(g_1) \cdot \chi_1(g_2) \cdot \chi_2(g_1) \cdot \chi_2(g_2) \\ &= \chi_1(g_1) \cdot \chi_2(g_1) \cdot \chi_1(g_2) \cdot \chi_2(g_2) \\ &= \chi_1\chi_2(g_1) \cdot \chi_1\chi_2(g_2) \end{aligned}$$

Let $\widehat{A}$ denote the set of all characters of A. Also let $\cdot$ denote the operation of function multiplication, which was shown in the above proof to be a binary operation on $\widehat{A}$. To prove that $(\widehat{A}, \cdot)$ is an abelian group, it will suffice to show that it has an identity element, inverses, associativity, and commutativity. Let's begin.

Identity: Consider the character χ_e defined by $g \mapsto 1$ for all $g \in A$. Let $\chi \in \widehat{A}$ be arbitrary. Then since we have the following, letting $g \in A$ be arbitrary, we know that $\chi\chi_e = \chi = \chi_e\chi$, as desired.

$$\chi\chi_e(g) = \chi(g) \cdot \chi_e(g) = \chi(g) \cdot 1 = \chi(g) = 1 \cdot \chi(g) = \chi_e(g) \cdot \chi(g) = \chi_e\chi(g)$$

Inverses: Let $\chi \in \widehat{A}$ be arbitrary. Consider the character $\bar{\chi}$ defined by $g \mapsto \overline{\chi(g)}$ for all $g \in A$, where the overbar denotes taking the complex conjugate. Then since we have the following, letting $g \in A$ be arbitrary, we know that $\chi\bar{\chi} = \bar{\chi}\chi = \chi_e$, as desired. Note that the complex conjugates multiply to 1 because we showed above that all $\chi(g)$ are roots of unity (for any $\chi \in \widehat{A}$).

$$\chi\bar{\chi}(g) = \chi(g) \cdot \bar{\chi}(g) = \bar{\chi}(g) \cdot \chi(g) = 1 = \chi_e(g)$$

Associativity: Let $\chi_1, \chi_2, \chi_3 \in \widehat{A}$ be arbitrary. Then since we have the following, letting $g \in A$ be arbitrary, we know that $\chi_1(\chi_2\chi_3) = (\chi_1\chi_2)\chi_3$, as desired.

$$[\chi_1(\chi_2\chi_3)](g) = \chi_1(g) \cdot \chi_2\chi_3(g) = \chi_1(g) \cdot \chi_2(g) \cdot \chi_3(g) = \chi_1\chi_2(g) \cdot \chi_3(g) = [(\chi_1\chi_2)\chi_3](g)$$

Commutativity: Let $\chi_1, \chi_2 \in \widehat{A}$ be arbitrary. Then since we have the following, letting $g \in A$ be arbitrary, we know that $\chi_1\chi_2 = \chi_2\chi_1$, as desired.

$$\chi_1\chi_2(g) = \chi_1(g) \cdot \chi_2(g) = \chi_2(g) \cdot \chi_1(g) = \chi_2\chi_1(g)$$

□

(b) Prove directly that for every nontrivial character $\chi \in \widehat{A}$, the following identity holds.

$$\sum_{g\in A}\chi(g) = 0$$

Proof. Let $\chi \in \widehat{A}$ be a nontrivial character. Since it is nontrivial, there exists $h \in A$ for which $\chi(h) \neq 1$. Additionally, we have by the Sudoku Lemma that

$$\sum_{g\in A}\chi(g) = \sum_{g\in A}\chi(hg)$$

But then since χ is a homomorphism, we have

$$\begin{aligned}\sum_{g\in A}\chi(g) = \sum_{g\in A}\chi(hg) &= \sum_{g\in A}\chi(h)\chi(g) = \chi(h)\sum_{g\in A}\chi(g)\\ (1-\chi(h))\sum_{g\in A}\chi(g) &= 0\end{aligned}$$

Thus, by the zero-product property, either $1-\chi(h) = 0$ or $\sum_{g\in A}\chi(g) = 0$. Since $\chi(h) \neq 1$ as proven above, $1-\chi(h) \neq 0$ so we must have

$$\sum_{g\in A}\chi(g) = 0$$

as desired. $\square$

(c) Prove that characters are the same as the 1-dimensional representations of A; product of characters is the same as a tensor product of representations, and the inverse of the character is the same as the dual representation.

Proof. Let $\rho_1, \rho_2 : A \to GL(\mathbb{C}) = \mathbb{C}^\times$ be two arbitrary 1-dimensional representations of A. Notice that ρ_1, ρ_2 have the same domain and codomain as characters, and are homomorphisms. Additionally, by the definition of the Kronecker product for 1×1 matrices, we have that

$$[\rho_1 \otimes \rho_2](g) = \rho_1(g)\cdot\rho_2(g)$$

for all $g \in A$. Thus, the tensor product of these representations is the same the character product of their values. Lastly, we have that

$$\rho_1^*(g) = \rho_1(g^{-1})^T = \rho_1(g^{-1}) = \rho_1(g)^{-1} = \overline{\rho_1(g)}$$

Thus, the dual representation of this representation is computed using the character inverse. $\square$

(d) Find all characters for $A = \mathbb{Z}/n\mathbb{Z}$. Compute the dual group $\widehat{A}$. Do the same for $A = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

Proof. We treat each group separately.

$\underline{\mathbb{Z}/n\mathbb{Z}}$: Let $\chi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times$ be a character of $\mathbb{Z}/n\mathbb{Z}$. Since χ is a homomorphism and hence $\chi(n) = \chi(1)^n$, the value of $\chi(1)$ completely determines the action of χ. Thus, there is a one-to-one mapping between the characters of $\mathbb{Z}/n\mathbb{Z}$ and the possible values of $\chi(1)$, so let's investigate the latter. We know from part (a) that $\chi(1)$ is a root of unity and that $\chi(1)^n = 1$. Consequently, $\chi(1)$ is an n^{th} root of unity. Any such root of unity will work, so the characters $\chi_0, \dots, \chi_{n-1}$ of $\mathbb{Z}/n\mathbb{Z}$ are defined by

$$\boxed{\widehat{\mathbb{Z}/n\mathbb{Z}} = \{\chi_k \mid k = 0, \dots, n-1;\ \chi_k(1) = \mathrm{e}^{2\pi i k/n}\}}$$

$\underline{K_4}$: The maximum order of any element in this group is 2, so $\chi : K_4 \to \{-1, 1\}$. While there are $2^4 = 16$ such maps, only four are homomorphisms: Those sending $((0,0),(0,1),(1,0),(1,1))$ to...

$$\boxed{\widehat{K}_4 = \{(1,1,1,1),(1,1,-1,-1),(1,-1,1,-1),(1,-1,-1,1)\}}$$

$\square$

(e) Prove that $\widehat{A_1 \times A_2}$ is isomorphic to $\widehat{A}_1 \times \widehat{A}_2$. Prove that groups A and $\widehat{A}$ are isomorphic as abstract groups. Deduce that an abelian group of order n has exactly n characters.

Proof. Define $h : \widehat{A}_1 \times \widehat{A}_2 \to \widehat{A_1 \times A_2}$ by

$$h(\chi_1, \chi_2) = \chi_1 \otimes \chi_2$$

where $[\chi_1 \otimes \chi_2](a_1, a_2) = \chi_1(a_1) \cdot \chi_2(a_2)$. Note that we are borrowing part (c)'s conclusion that characters can be treated like representations and, in particular, can have tensor products. To prove that h is an isomorphism of groups, it will suffice to show that it is a bijective homomorphism of groups. The following suffices to show that it is a homomorphism of groups.

$$\begin{aligned}
h[(\chi_1, \chi_2) \cdot (\chi_3, \chi_4)] &= h(\chi_1\chi_3, \chi_2\chi_4) \\
&= \chi_1\chi_3 \otimes \chi_2\chi_4 \\
&= \chi_1 \otimes \chi_2 \cdot \chi_3 \otimes \chi_4 \\
&= h(\chi_1, \chi_2) \cdot h(\chi_3, \chi_4)
\end{aligned}$$

Note that the transition form the second to the third line above is justified because the equality becomes $\chi_1(a_1) \cdot \chi_3(a_1) \cdot \chi_2(a_2) \cdot \chi_4(a_2) = \chi_1(a_1) \cdot \chi_2(a_2) \cdot \chi_3(a_1) \cdot \chi_4(a_2)$ when applied to (a_1, a_2) and expanded. As to bijectivity, we will prove injectivity then surjectivity. For injectivity, suppose $h(\chi_1, \chi_2) = h(\chi_3, \chi_4)$. Then for all $a_1 \in A_1$,

$$\begin{aligned}
[h(\chi_1, \chi_2)](a_1, e) &= [h(\chi_3, \chi_4)](a_1, e) \\
\chi_1(a_1) \cdot \chi_2(e) &= \chi_3(a_1) \cdot \chi_4(e) \\
\chi_1(a_1) \cdot 1 &= \chi_3(a_1) \cdot 1 \\
\chi_1(a_1) &= \chi_3(a_1)
\end{aligned}$$

A similar statement holds for χ_2 and χ_4, proving that $(\chi_1, \chi_2) = (\chi_3, \chi_4)$, as desired. For surjectivity, let $\chi \in \widehat{A_1 \times A_2}$ be arbitrary. Define χ_1 and χ_2 by

$$\chi_1(a_1) = \chi(a_1, e) \qquad\qquad \chi_2(a_2) = \chi(e, a_2)$$

for all $a_1 \in A_1$ and $a_2 \in A_2$. That χ_1, χ_2 are characters under these definitions instead of just functions follows immediately from the character-like properties of χ: indeed, with these definitions in hand, we can show that

$$[h(\chi_1, \chi_2)](a_1, a_2) = \chi_1(a_1) \cdot \chi_2(a_2) = \chi(a_1, e) \cdot \chi(e, a_2) = \chi[(a_1, e) \cdot (e, a_2)] = \chi(a_1, a_2)$$

as desired.

By the fundamental theorem of finite abelian groups, A is isomorphic to a direct product of cyclic groups of prime power order. Thus, we may let

$$A \cong (\mathbb{Z}/p_1\mathbb{Z})^{n_1} \times \cdots \times (\mathbb{Z}/p_k\mathbb{Z})^{n_k}$$

Borrowing the notation from the first task of part (d) above, define $h : (\mathbb{Z}/p_1\mathbb{Z})^{n_1} \times \cdots \times (\mathbb{Z}/p_k\mathbb{Z})^{n_k} \to (\widehat{\mathbb{Z}/p_1\mathbb{Z}})^{n_1} \times \cdots \times (\widehat{\mathbb{Z}/p_k\mathbb{Z}})^{n_k}$ by

$$h(a_{11}, \dots, a_{kn_k}) = \chi_{a_{11}} \otimes \cdots \otimes \chi_{a_{kn_k}}$$

For the same reasons mentioned in part (d), h is an isomorphism. Additionally, by consecutive applications of the first claim in this part,

$$(\widehat{\mathbb{Z}/p_1\mathbb{Z}})^{n_1} \times \cdots \times (\widehat{\mathbb{Z}/p_k\mathbb{Z}})^{n_k} \cong (\mathbb{Z}/p_1\mathbb{Z})^{n_1} \widehat{\times \cdots \times} (\mathbb{Z}/p_k\mathbb{Z})^{n_k}$$

But since $A \cong (\mathbb{Z}/p_1\mathbb{Z})^{n_1} \times \cdots \times (\mathbb{Z}/p_k\mathbb{Z})^{n_k}$, the group on the right above is isomorphic to $\widehat{A}$. Thus, by chaining together isomorphisms, we can get all the way from A to $\widehat{A}$, as desired.

Since isomorphic groups have the same order, an abelian group of order n has a dual group with order n, i.e., has n characters, as desired. $\square$

3. Consider the permutational representation of S_n. Decompose it into the sum of (two) irreducible representations.

Proof. Let $\rho : S_n \to GL(V)$ be the permutational representation of S_n. As discussed in class, ρ fixes the one-dimensional subspace $\text{span}(1, \dots, 1) \leq V$. Thus, this subspace forms a trivial subrepresentation of V. It follows by the theorem from class that this subspace has a complement; this complement is the standard representation. Thus,

$$V = (3) \oplus (2, 1)$$

As a one-dimensional representation, (3) is clearly irreducible, but it is not immediately evident that $(2, 1)$ is. Fortunately, the following proves that it is. Assuming $n \geq 2$ since the 1D case is trivial. If we take the column vector $(1, -1, 0, \dots, 0) \in (2, 1)$, we can generate from it $n - 1$ other linearly independent column vectors using consecutive applications of $\sigma = (12 \cdots n)$. For example, if $n = 4$, we generate

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

Therefore, there is no subspace of $(2, 1)$ that we can't get into when we're in $(2, 1)$ *except* that which we've already discussed: (3). It follows that $(2, 1)$ is irreducible, as well. $\square$

4. Let G be a finite group.

   (a) Define the **space of invariants** of a representation V by the formula

   $$V^G = \{v \in V \mid gv = v \;\forall\; g \in G\}$$

   Prove that V^G is a subrepresentation of V. Prove that it is isomorphic to a sum of trivial representations.

   Proof. To prove that V^G is a subrepresentation of V, it will suffice to show that it is a subspace of V, and $gV^G \subset V^G$ for all $g \in G$. Since we clearly have $g \cdot 0 = 0$ for all $g \in G$, $g(v_1 + v_2) = v_1 + v_2$ for all v_1, v_2 satisfying $gv_i = v_i$, and $agv = g(av)$, V^G is a subspace of V. Now for closure under the g's, let $v \in V^G$ be arbitrary. But by the definition of V^g, we have $gv = v \in V^g$ for all $g \in G$, as desired.

   Let $e_1, \dots, e_k$ be a basis of V^G. Since $g(\lambda e_i) = \lambda e_i$ for all $\lambda e_i \in \text{span}(e_i)$, $i = 1, \dots, k$, each $\text{span}(e_i)$ is, itself, fixed by all g and hence a trivial subrepresentation of V^G. Therefore,

   $$V^G \cong \underbrace{(3) \oplus \cdots \oplus (3)}_{k \text{ times}}$$

   as desired. $\square$

   (b) Prove that $(\text{Hom}_F(V, W))^G$ is isomorphic to $\text{Hom}_G(V, W)$.

   Proof. To prove the claim, it will suffice to prove the stronger condition that $(\text{Hom}_F(V, W))^G = \text{Hom}_G(V, W)$ as sets. We will proceed via a bidirectional inclusion proof. Let's begin.[1]

   First, let $f \in (\text{Hom}_F(V, W))^G$ be arbitrary. Then by the definition of the space of invariants, $g \cdot f = f$ for all $g \in G$. Additionally, since $G \circlearrowright \text{Hom}_F(V, W)$ via $g \cdot f = gfg^{-1}$, we have that $gfg^{-1} = f$, i.e., $gf = fg$ for all $g \in G$. But this implies that f is a morphism of G-representations, i.e., $f \in \text{Hom}_G(V, W)$, as desired.

   The proof is symmetric in the reverse direction. $\square$

[1]Note: Beware rampant abuses of notation throughout this proof. For example, the statement $gf = fg$ stands in for the much more complex $\rho_V(g) \circ f = f \circ \rho_W(g)$.

5. Let $\rho : G \to GL_n(\mathbb{C})$ be a representation with character χ.

   (a) Prove that $\operatorname{Ker}(\rho) = \{g \in G \mid \chi(g) = n\}$.

   Proof. We proceed via a bidirectional inclusion proof.

   Suppose first that $g \in \operatorname{Ker}(\rho)$. Then $\rho(g) = I_n$. But since $\operatorname{tr}(I_n) = n$ and $\chi(g) = \operatorname{tr}(\rho(g))$, we have by transitivity that $\chi(g) = n$, as desired.

   Now suppose that $\chi(g) = n$. Recall from class that every eigenvalue λ_i of $\rho(g)$ is a root of unity. Additionally, since $\lambda_1 + \cdots + \lambda_n = \chi(g) = n$, we must have $\lambda_i = 1$ $(i = 1, \dots, n)$. But this implies that $\rho(g) = I_n \in \operatorname{Ker}(\rho)$, as desired. □

   (b) Prove that for any $g \in G$, we have $|\chi(g)| \leq n$.

   Proof. As in part (a), recall from class that every eigenvalue λ_i of $\rho(g)$ is a root of unity. Then by the triangle inequality,

   $$|\chi(g)| = |\lambda_1 + \cdots + \lambda_n| \leq |\lambda_1| + \cdots + |\lambda_n| = 1 + \cdots + 1 = n$$

   as desired. □

   (c) Prove that for a given $g \in G$, $|\chi(g)| = n$ if and only if there exists $\lambda \in \mathbb{C}$ such that $\rho(g) = \lambda I$.

   Proof. Suppose first that $|\chi(g)| = n$. Then $|\lambda_1 + \cdots + \lambda_n| = n$, so since $|\lambda_i| = 1$ for $i = 1, \dots, n$, we must have $\lambda_1 = \cdots = \lambda_n$. Define $\lambda := \lambda_i$. Recall that a linear operator with n eigenvalues must have a corresponding $n \times n$ matrix in some basis equal to $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Therefore, in this case, the corresponding matrix of $\rho(g)$ is λI (and is λI in any basis), as desired.

   Now suppose that $\rho(g) = \lambda I$. Then

   $$|\chi(g)| = |\operatorname{tr}(\lambda I)| = |n\lambda| = n \cdot |\lambda| = n \cdot 1 = n$$

   as desired. □