Week 6

Abstract Representation Theory

6.1 The Center of the Group Algebra

10/30: - Plan for this week.
    - Today: Briefly discuss a very important concept called the **center**.
    - Wednesday: Do algebraic numbers.
    - Friday: Burnside's theorem.
- **Center** (of a group): The set of all elements of a group G that commute with every other element in G. *Denoted by* $\boldsymbol{Z(G)}$. *Given by*

$$Z(G) = \{g \in G \mid xgx^{-1} = g \ \forall \ x \in G\}$$

    - Note: $Z(G)$ is a subgroup of G.
- The center is one of the most important concepts in all of representation theory.
    - Example: Let A be an abelian group, such as $Z(G)$. Then all its irreps are 1D.
        - See Section 1.3 of Fulton and Harris (2004) for an explanation.
    - Normally, the center of a group is too small to be interesting.
        - However, $Z(\mathbb{C}[G])$ *is* large enough to be interesting.
- **Center** (of an algebra): The set of all elements of an algebra A that commute with every other element in A. *Denoted by* $\boldsymbol{Z(A)}$. *Given by*

$$Z(A) = \{a \in A \mid xa = ax \ \forall \ x \in A\}$$

- Proposition: If A is an algebra over $\mathbb{C}$, M is an irreducible left A-module, and $\rho : A \to \mathrm{End}(M)$ is a corresponding representation, then $x \in Z(A)$ implies that $\rho(x) = \lambda I$, i.e., $\rho(x)$ is a *scalar matrix.*

  Proof. Let $x \in Z(A)$ be arbitrary. Then for all $a \in A$, we know that $\rho(x)\rho(a) = \rho(a)\rho(x)$. Thus, $\rho(x)$ is a morphism of A-modules. Consequently, since M is irreducible (also known as *simple*), Schur's Lemma for associative algebras implies that $\mathrm{Hom}_A(M, M)$ is a division algebra over $\mathbb{C}$. But since $\mathbb{C}$ is the only division algebra over $\mathbb{C}$, we have that $\mathrm{Hom}_A(M, M) \cong \mathbb{C}$. From here, it readily follows that $\rho(x)$ is equal to some λI. $\square$

- Consequence: If M is reducible, we can reduce it into component scalar representations.
- Consequence: If G is an abelian group, then every irrep V is 1-dimensional.

  - Additionally, $\mathbb{C}[G]$ is commutative and hence $\mathbb{C}[G] = Z(\mathbb{C}[G])$.
  - Then if V is an arbitrary representation, V is equal to the direct sum of one dimensional irreducible representations for all g. Hence, $\rho_V(g) = \lambda I$. Could the λ's not be different for the various irreps??

- We now try to compute $Z(\mathbb{C}[G])$.
  - Facts:
$$Z(A_1 \oplus A_2) = Z(A_1) \oplus Z(A_2) \qquad\qquad Z(M_n(\mathbb{C})) = \text{span}(I) \cong \mathbb{C}$$
  - These facts coupled with the fact that G is a finite group (hence $\mathbb{C}[G] \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$ where k is the number of conjugacy classes in G by the example from last Wednesday's class) yield
$$\begin{aligned} Z(\mathbb{C}[G]) &\cong Z(M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})) \\ &\cong \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{k \text{ times}} \\ &= \mathbb{C}^k \end{aligned}$$

- Let $C_1, \dots, C_k$ be conjugacy classes in G. Then we may define
$$e_i = \sum_{g \in C_i} g$$
for each $i = 1, \dots, k$.
  - Example: In S_3 the three e_i's are $\{e, (12) + (13) + (23), (123) + (132)\}$.

- Claim: $Z(G) = \langle e_1, \dots, e_k \rangle$, that is, the e_i commute with every element of G expressed as $1g \in \mathbb{C}[G]$.

Proof. We will use a bidirectional inclusion proof.
$\underline{\langle e_1, \dots, e_k \rangle \subset Z(G)}$: Let e_i and $x \in G$ be arbitrary. Then
$$\begin{aligned} xe_ix^{-1} &= \sum_{g \in C_i} xgx^{-1} = \sum_{h \in C_i} h = e_i \\ xe_i &= e_ix \end{aligned}$$
This naturally extends to any sums and scalar multiples of the e_i's.
$\underline{Z(G) \subset \langle e_1, \dots, e_k \rangle}$: Let $a \in Z(G)$ be arbitrary. As an element of $\mathbb{C}[G]$, we know that $a = \sum a_g g$ for some $a_g \in \mathbb{C}$. Additionally, since $a \in Z(G)$, we have that $xax^{-1} = a$ for all $x \in G$ (that is, $1x \in A$). Combining these last two results, we have that
$$\sum_{g \in G} a_{x^{-1}gx} g = \sum_{g \in G} a_g xgx^{-1} = xax^{-1} = a = \sum_{g \in G} a_g g$$
Comparing like terms in the above equality, we can learn that for all $x \in G$, we have $a_{x^{-1}gx} = a_g$. In other words, all of the a_g's for g's in the same conjugacy class are equal. Therefore, a is of the form $a = \sum_{i=1}^k a_{g_i} e_i$ for $g_i \in C_i$. ☐

- Thus we get $a_e e + a_{(12)}(12) + a_{(13)}(13) + \cdots$??

- Computing products of the e_i: What if we want to compute $[(12) + (13) + (23)]^2$, for example? We have to multiply *noncommutatively*, so HS formulas are out, but we can still do all nine multiplications and sum them:
$$[(12) + (13) + (23)]^2 = 3e + 3[(123) + (132)]$$

- We now tie this claim back into our discussion of $Z(\mathbb{C}[G])$.
  - $Z(\mathbb{C}[G])$ has basis $e_1, \dots, e_k$[1].
  - Recall that $Z(\mathbb{C}[G]) = \mathbb{C} \oplus \cdots \oplus \mathbb{C}$, with characters $\chi_1, \dots, \chi_k$.
  - Then $f_{\chi_i} = (0, \dots, 0, 1, 0, \dots, 0)$, where the 1 lies in the i^{th} slot.
  - Then we get $f_{\chi_1}, \dots, f_{\chi_k}$ as a basis.
  - It follows that $f_{\chi_i}^2 = f_{\chi_i}$ and $f_{\chi_i} f_{\chi_j} = 0$ for $i \neq j$; this is exactly what it means for a space to be $\mathbb{C} \oplus \cdots \oplus \mathbb{C}$.
  - Both of these spaces (center elements and class functions) have these two interconnected bases, so the spaces are quite similar!
- The center of a group algebra $Z(\mathbb{C}[G])$ can be identified "=" with the space of class functions $\mathbb{C}_{\text{cl}}(G)$ via
$$\sum \varphi(g)g \mapsto [g \to \varphi(g)]$$
where $\varphi(xgx^{-1}) = \varphi(g)$.
  - This isomorphism is an isomorphism of vector spaces, *not* an isomorphism of algebras!
  - However, it still has cool properties.
    - For instance, consider the δ_{C_i}: The functions sending $g \in C_i$ to 1 and $g \notin C_i$ to 0.
    - The isomorphism identifies $e_i \mapsto \delta_{C_i}$.
  - Do we get irreducible characters (our other basis of class functions) when we sum the $\varphi(g)g$'s?
    - We do! What is this??
  - Let's consider another basis χ of irreducibles. The basis is $f_\chi = \frac{d_\chi}{|G|} \sum_{g \in G} \chi(g^{-1})g$, and we send it to χ_{V^*}.
  - Claim:
$$f_{\chi_i} f_{\chi_j} = \begin{cases} f_{\chi_i} & \chi_i = \chi_j \\ 0 & \chi_i \neq \chi_j \end{cases}$$
    - Things that multiply like this are called the **central idempotent**.
  - Thus, general multiplication works as follows.
$$(a_1 f_{\chi_1} + \cdots + a_n f_{\chi_n})(b_1 f_{\chi_1} + \cdots + e_n f_{\chi_n}) = a_1 b_1 f_{\chi_1} + \cdots + a_n b_n f_{\chi_n}$$
  - So if we want to send $a \in Z(G)$ to $\bigoplus^k \mathbb{C}$, we map
$$a = a_1 f_{\chi_1} + \cdots + a_k f_{\chi_k} \mapsto (a_1, \dots, a_k)$$
  - The proof of this claim is really simple because we've already done the computation with the projector on the irrep V_x.
    - So if you want to see $\rho(f_\chi)$, see what it does to the identity: It does $\rho(f_\chi)e = f\chi e = f_\chi$. ρ is regular.
- **Central idempotent**: An element such that $a^2 = a$ and $ax = xa$ for all $x \in A$.
- Two approaches to the same thing: Class functions and the center approach.
  - The great thing about the center: You can understand what it looks like because it is well-defined as a commutative algebra.
  - If something is isomorphic to $\mathbb{C} \oplus \cdots \oplus \mathbb{C}$ as an algebra, then there is another space and basis in which your multiplication looks incredibly simple.

[1] How did we get from the previous claim to here??

- We might get to **Hopf algebras** at the end of the course (very interesting).
  - Let $\mathbb{C}[G]$ be an associative algebra.
  - Let $\mathbb{C}[G]^*$ be the functions on the group.
  - Then $A \otimes A \to A$ sends $a_1 \otimes a_2 \mapsto a_1 a_2$.
  - When we dualize to get $A^* \otimes A^* \to A^*$, everything gets reversed, so we actually get a **comultiplication** $A \to A \otimes A$ given by $g \mapsto g \otimes g$. These two multiplications together are called a **Hopf algebra**.
  - Knowing that there's something that we can define and understand might help us untangle the knot of all the spaces.
  - This is pretty heavy math, though, so we won't go too deep into it if we get at all.
- Today was the last associative algebra class.
- Going forward: Integral elements, algebraic integers, dimension of the representation divides the order or the group, Burnside's theorem.
- Midterm is heavily computational: Tensor products, character tables, etc. A few simple questions about things.
  - Comparably less associative algebra stuff (maybe just 1 exercise).