Week 9

Symmetric Group Representation Characteristics

9.1 Frobenius Reciprocity; The Branching Theorem

11/27: • Announcements.

  - OH on Wednesday at 5:30 PM this week; not Tuesday.
  - There will be extra OH next week pre-exam.
    - Roughly like Monday/Wednesday next week.
  - Midterm will be returned on Wednesday; we can pick them up in-person in his office starting then.
  - There are some grade boundaries: Pass/Fail we can do until Friday, withdrawal we can do until 5:00 PM today.

- Let's finish the conversation about induction/restriction and prove the **branching theorem**.
- Reminder to start.
  - We have two mathematical categories, G-reps and H-reps where $H \leq G$.
  - These catagories are related by functors.
    - $\text{Res}_H^G : G\text{-reps} \to H\text{-reps}$ and vice versa for Ind_H^G.
    - See Figure 8.7.
  - Restrictions are stupidly simple.
  - Inductions, most hands-on, we take copies of W times cosets. Formulaically,

$$\text{Ind}_H^G W = g_1 W \oplus \cdots \oplus g_k W$$

  where $k = (G : H)$ and $G = \bigsqcup_{i=1}^k g_i H$.
    - In more detail, the action of g on $g_i w$ is that of $g_{\sigma(i)} h_i w$.
    - This is a genuinely hard construction.
    - A matrix of this thing will be a block-permutation matrix like

$$\begin{array}{cc}
 & \begin{array}{ccc} g_1W & \quad\quad & g_kW \end{array} \\
\begin{array}{c} g_1W \\ \\ g_kW \end{array} & \left[\begin{array}{c|c|c} \quad\quad & 0 & 0 \\ \hline 0 & 0 & \quad\quad \\ \hline 0 & \quad\quad & 0 \end{array}\right]
\end{array}$$

- As an alternate construction, we have that

$$g_1W \oplus \cdots \oplus g_kW \cong \operatorname{Hom}_H(\mathbb{C}[G], W)$$

  - Recall that elements of the set on the right above are functions $f : G \to W$ such that $f(h(g)) = hf(g)$.
  - We map between the two via $f(g) \mapsto f(gx')$.
- What is nice about induced representations is that $\dim[\operatorname{Ind}_H^G W] = (\dim W)[G : H]$.
- There is a very easy statement of the character of an induced representation, the **Frobenius formula**.
  - Recall that

$$\tilde{\chi}_W(g) = \begin{cases} 0 & g \notin H \\ \chi_W(g) & g \in H \end{cases}$$

  - With this, we average:

$$\chi_{\operatorname{Ind}_H^G W}(g) = \frac{1}{|H|} \sum_{x \in G} \tilde{\chi}_W(xgx^{-1})$$

  - Essentially, we're taking a whole bunch of conjugates, summing them up, and dividing to get rid of overcounting.

- We now move onto **Frobenius reciprocity**, which is a relation between the functors/relations Ind_H^G and Res_H^G.
  - The first point where category theory gets interesting is the notion of **adjoint functors**, which we are about to touch on. It is a very subtle notion.
  - Here's version 1 of the statement of Frobenius reciprocity.
    - Recall that we have a scalar product on the space of class function, given by

$$(\chi_1, \chi_2) = \frac{1}{|G|} \sum_{g \in G} \chi_1(g)\chi_2(g^{-1})$$

      where χ_1, χ_2 are class functions on G.
    - Recall that if $\chi_1 = \chi_V$ and $\chi_2 = \chi_W$, then

$$(\chi_1, \chi_2) = \dim \operatorname{Hom}_G(V, W) = \dim \operatorname{Hom}_G\left(\bigoplus_{i=1}^k V_i^{n_i}, \bigoplus_{i=1}^k V_i^{m_i} \right) = \sum_{i=1}^k n_i m_i$$

    - Then the statement is as follows. If V is a G-rep and W is an H-rep, then

$$(V, \operatorname{Ind}_H^G W)_G = (\operatorname{Res}_H^G V, W)_H$$

      - ➢ This notation denotes a scalar product in G and scalar product in H of the characters of each representation.
    - This is similar to the relation between adjoint maps $V \to W$ and $W^* \to V^*$.
  - Version 2.
    - We have that

$$\operatorname{Hom}_G(V, \operatorname{Ind}_H^G W) \cong \operatorname{Hom}_H(\operatorname{Res}_H^G V, W)$$

      where the isomorphism is canonical.
    - We will not check this last definition; we can tediously do it with definitions, and there's nothing complicated. Rudenko leaves this as an exercise to us.
    - The canonical isomorphism sends a map $v \mapsto [g \mapsto \varphi(gv)]$ to the map $\phi : V \to W$.

- We now prove Version 1.

Proof. We have

$$
\begin{aligned}
(\chi_V, \chi_{\operatorname{Ind}_H^G W})_G &= \frac{1}{|G|} \sum_{g_1 \in G} \chi_V(g_1) \chi_{\operatorname{Ind}_H^G W}(g_1^{-1}) \\
&= \frac{1}{|G|} \sum_{g_1 \in G} \chi_V(g_1) \left(\frac{1}{|H|} \sum_{g_2 \in G} \tilde{\chi}_W(g_2 g_1^{-1} g_2^{-1}) \right) \\
&= \frac{1}{|H| \cdot |G|} \sum_{g_1, g_2 \in G} \chi_V(g_1) \tilde{\chi}_W(g_2 g_1^{-1} g_2^{-1}) \\
&= \frac{1}{|H| \cdot |G|} \sum_{g_1, g_2 \in G} \chi_V(\underbrace{g_2 g_1 g_2^{-1}}_{h}) \tilde{\chi}_W(\underbrace{g_2 g_1^{-1} g_2^{-1}}_{h^{-1}}) \\
&= \frac{1}{|H|} \frac{1}{|G|} \sum_{h \in G} |G| \chi_V(h) \tilde{\chi}_W(h^{-1}) \\
&= (\chi_V|_H, \chi_W)_H \\
&= (\chi_{\operatorname{Res}_H^G V}, \chi_W)_H
\end{aligned}
$$

From line 4 to line 5: Fix h; then $g_2 g_1 g_2^{-1} = h$ iff $g_1 = g_2^{-1} h g_2$, so we have overcounted by $|G|$ times. From line 5 to line 6: $\tilde{\chi}_W$ is zero whenever $h^{-1} \notin H$, so this ostensible sum over all $h \in G$ is *de facto* only a sum over all $h \in H$; this is what allows us to consider χ_V as "restricted to H" in line 6. ☐

- We now come to the branching theorem at long last.
- Example first.
  - Consider $S_n > S_{n-1}$, where S_{n-1} is the subgroup that fixes n. I.e., $S_3 > S_2 = \{e, (12)\}$, and we explicitly omit $(13), (23), (123), (132)$ because they all move 3.
  - Let $\lambda \vdash n$.
  - Let $\mu \leq \lambda$ be a Young diagram of a partition of $n - 1$.
  - Then
    1. We have
       $$\operatorname{Res}_{S_{n-1}}^{S_n} V_\lambda = \bigoplus_{\substack{\mu \leq \lambda \\ |\mu| = n-1}} V_\mu$$
       - Example:
         $$\operatorname{Res}_{S_4}^{S_5} \ydiagram{3,2} = \ydiagram{2,2} \oplus \ydiagram{3,1}$$
    2. We have
       $$\operatorname{Ind}_{S_{n-1}}^{S_n} V_\mu = \bigoplus_{\substack{\mu \leq \lambda \\ |\lambda| = n}} V_\lambda$$
       - Example:
         $$\operatorname{Ind}_{S_5}^{S_6} \ydiagram{3,2} = \ydiagram{3,2,1} \oplus \ydiagram{3,3} \oplus \ydiagram{4,2}$$
- The reason that this theorem is called the branching theorem originates from the diagram in Figure 9.1, which (when continued) encapsulates the main idea of the theorem.
  - This graph helps you understand induction and restriction.
  - Dimensions are the number of paths from the bottom to a final Young diagram.
    - For example, the dimension of $(3, 1)$ is 3 because there are 3 paths to it, listed as follows.

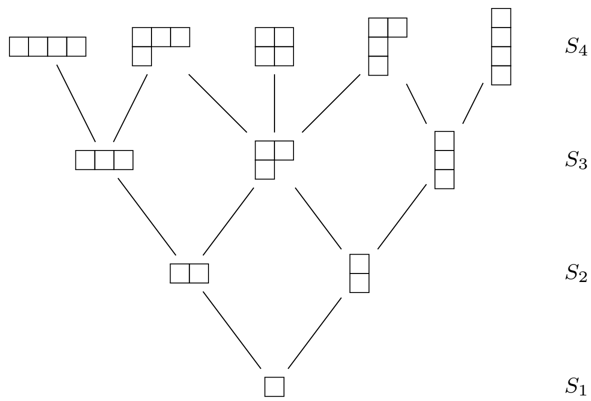


Figure 9.1: The branching theorem.

1. $(1) \to (2) \to (3) \to (3,1)$.
2. $(1) \to (2) \to (2,1) \to (3,1)$.
3. $(1) \to (1,1) \to (2,1) \to (3,1)$.

   – Number of paths is equivalent to number of standard Young tableaux!

- Theorem (Branching): The following two statements are true.

$$\operatorname{Res}^{S_n}_{S_{n-1}} V_\lambda = \bigoplus_{\substack{\mu\leq\lambda \\ |\mu|=n-1}} V_\mu \tag{9.1}$$

$$\operatorname{Ind}^{S_n}_{S_{n-1}} V_\mu = \bigoplus_{\substack{\mu\leq\lambda \\ |\lambda|=n}} V_\lambda \tag{9.2}$$

Proof. We'll talk about the general idea of the proof now, and maybe do the details next time.

<u>(9.1) $\iff$ (9.2)</u>: Suppose first that the left statement above holds true. Then we have that

$$(\operatorname{Res}^{S_n}_{S_{n-1}} V_\lambda, V_\mu) = \begin{cases} 0 \\ 1 & \lambda \geq \mu \end{cases}$$

Thus, by Frobenius reciprocity,

$$(V_\lambda, \operatorname{Ind}^{S_n}_{S_{n-1}} V_\mu) = (\operatorname{Res}^{S_n}_{S_{n-1}} V_\lambda, V_\mu) = \begin{cases} 0 \\ 1 & \lambda \geq \mu \end{cases}$$

Therefore, the second statement holds true. The proof is symmetric in the opposite direction.

<u>(9.1)</u>: Let's look at an example. Consider the Young diagram of S_8 shown in Figure 9.2.

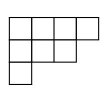
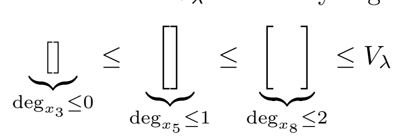


Figure 9.2: Proving the branching theorem.

We want to restrict it down to S_7. Recall that $V_\lambda = \operatorname{span}(S_8 : \Delta(x_1, x_2, x_3)(x_4 - x_5)(x_6 - x_7))$. Now in S_7, we fix x_8. Consider subrepresentations of V_λ filtered by degree as follows.



The proof comes from the fact that if we now take quotients of these subrepresentations, e.g., via

$$\deg = 0, \deg \leq 1/\deg \leq 0, \deg \leq 2/\deg \leq 1, \dots$$

then since x_8 can only appear in three boxes, ... □

- Practice with the above example and think it through.

9.2 The Character of a Symmetric Group Representation

11/29:
- Announcements.
  - OH today at 5:30.
  - Our midterms are graded; we can look at them in his office whenever (I can do this during OH!).
- Today, we'll formulate the main result he wants to prove next time.
- Goal is still to understand representations of S_n.
  - We've constructed all of them using Specht modules, but what else do we want?
  - We have dimension, we want characters, etc.
- The main idea is to look at symmetric polynomials once again.
  - Consider $\mathbb{Q}[x_1,\dots,x_n]^{S_n}$.
  - We have proven the fundamental theorem that $\mathbb{Q}[x_1,\dots,x_n]^{S_n} = \mathbb{Q}[\sigma_1,\dots,\sigma_n]$ where $\sigma_k = \sum_{1\leq i_1\leq\cdots\leq i_k\leq n} x_{i_1}\cdots x_{i_k}$.
  - We also proved in PSet 6, Q6 that these rings are equal to $\mathbb{Q}[p_1,\dots,p_k]$ and $\mathbb{Q}[h_1,\dots,h_k]$ where

$$p_k = \sum_{i=1}^n x_i^k \qquad\qquad h_k = \sum_{1\leq i_1\leq\cdots\leq i_k\leq n} x_1\cdots x_k$$

    - Example: If $n=3$ and $k=2$, then

$$h_2 = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$$

  - Table of bases for n, k.

$k \setminus n$	1	2	3	4
0	1	1	1	1
1	x_1	x_1+x_2	$x_1+x_2+x_3$	$\cdots$
2	x_1^2	$x_1^2+x_2^2$, x_1x_2	σ_1^2, σ_2^2	σ_1^2, σ_2^2
3	x_1^3	...		

Table 9.1: Polynomial bases.

  - Now take

$$\Lambda_k = [\mathbb{C}[x_1,\dots,x_k]]_{\deg=k-1} \cong [\mathbb{C}[x_1,\dots,x_{k+1}]]_{\deg=k-1} \cong \cdots$$

    - Alternatively, we can think of this thing as

$$\Lambda_k = (\mathbb{C}[x_1,\dots])_k$$

      with $\sigma_1^k, \sigma_2\sigma_1^{k-1}, \dots$.

- We call Λ the ring of symmetric functions and define it to be equal to

$$\Lambda = \mathbb{Q}[\sigma_1, \sigma_2, \sigma_3, \dots]$$

- In every complete component, only finitely many of the σ will participate, so we get finite things.
- This is a graded ring! We have

$$\Lambda = \bigoplus_{k\geq 0} \Lambda_k$$

and $\Lambda_k \otimes \Lambda_\ell = \Lambda_{k+\ell}$

- This construction is called the **projective limit**, and we may have encountered it in commutative algebra under the definition

$$\Lambda = \lim_{\rightarrow} \mathbb{C}[x_1, \dots, x_n]^{S_n}$$

- We have identifies such as $p_2 = \sigma_1^2 - 2\sigma_2$. This means that

$$(x_1 + \cdots + x_n)^2 - 2(x_1x_2 + x_1x_3 + \cdots) = x_1^2 + x_2^2 + \cdots$$

- Observation: $\dim_{\mathbb{Q}} \Lambda_n$.

- Now, we need to take a vector space on ring representations; we've done this already with the representation ring.
- Let R_n be the $\mathbb{Q}$-vector space of functions $\chi : S_n \to \mathbb{Q}$ such that $\chi(x\sigma x^{-1}) = \chi(\sigma)$. This is our favorite space of class functions.
- Theorem (Frobenius characteristic map): There is an isomorphism of vector spaces and of rings called the Frobenius characteristic: $\text{ch} : \bigoplus_{n\geq 0} R_n \to \Lambda$.

Proof. Take $\chi_V \in R_k$, and $\chi_W \in R_\ell$. Let V an S_k-rep, and W an S_ℓ-rep. We know that

$$S_k \times S_\ell = S_{k+\ell}$$

So what we can do is induction $\text{Ind}_{S_k\times S_\ell}^{S_{k+\ell}}(V \otimes W)$. Call this operation $\chi_V \boxtimes \chi_W$.

Now we write down the formula:

$$\text{ch}(\chi) = \frac{1}{n!} \sum_{\sigma\in S_n} \chi(\sigma) p_1^{\lambda_1(\sigma)} \cdots p_k^{\lambda_k(\sigma)}$$

where $\lambda_1(\sigma), \lambda_2(\sigma), \dots$ represent the cycle structure of σ; each λ_i is a number of cycles of length $1, 2, \dots$. $\square$

- Examples.

1. S_1.
   - Sends the YD (1) to $p_1 = x_1 + x_2 + x_3 + \cdots$.
2. S_2.
   - Sends (2) to $\frac{1}{2!}(p_1^2 + p_2) = \frac{1}{2}((x_1 + x_2)^2 + x_1^2 + x_2^2) = x_1^2 + x_2^2 + x_1x_2 = h_2$.
   - It also sends $(1, 1)$ to $\frac{1}{2!}(p_1^2 - p_2) = \frac{1}{2}((x_1 + x_2)^2 - x_1^2 - x_2^2) = x_1x_2 = \sigma_2$.
   - Let's check our formula. What is $\text{Ind}_{S_1\times S_1}^{S_2}(1) \otimes (1)$? Since the induction of the trivial representation is the regular representation, which we can decompose, we know that this induction equals $(1, 1) \oplus (2)$. It follows that $p_1^2 = x_1^2 + x_2^2 + x_1x_2 + x_1x_2 = (x_1 + x_2)^2$.
3. S_3.

- Sends (3) to

$$\begin{aligned}\frac{1}{3!}(p_1^3+3p_1p_2+2p_3) &= \frac{1}{6}[(x_1+x_2+x_3)^3+3(x_1+x_2+x_3)(x_1^2+x_2^2+x_3^2)+2(x_1^3+x_2^3+x_3^3)] \\ &= \frac{1}{6}[6(x_1^3+x_2^3+x_3^3)+6(x_1^2x_2+x_1x_2^2+x_1x_3^2+x_1^2x_3+\cdots)+6x_1x_2x_3] \\ &= h_3\end{aligned}$$

- Sends $(1,1,1)$ to

$$\begin{aligned}\frac{1}{3!}(p_1^3-3p_1p_2+2p_3) &= \frac{1}{6}[(x_1+x_2+x_3)^3-3(x_1+x_2+x_3)(x_1^2+x_2^2+x_3^2)+2(x_1^3+x_2^3+x_3^3)] \\ &= x_1x_2x_3 \\ &= \sigma_3\end{aligned}$$

- Sends $(2,1)$ to

$$\begin{aligned}\frac{1}{3!}(2p_1^3-p_3) &= \frac{1}{6}[2(x_1^3+x_2^3+x_3^3)+6(x_1^2x_2+\cdots)+12x_1x_2x_3] \\ &= (x_1^2+\cdots)+2x_1x_2x_3\end{aligned}$$

- Again, we can check that

$$\operatorname{Ind}_{S_2\times S_1}^{S_3}[(1,1)\otimes(1)] = \sigma_1\sigma_2$$

  - We compute $\operatorname{Ind}_{S_2}^{S_3}(1,1)=(1,1,1)\oplus(2,1)$ via the branching formula: There are only two ways to add a box!
  - We have $\sigma_1\sigma_2-\sigma_3=(x_1+x_2+x_3)(x_1x_2+x_1x_3+x_2x_3)-x_1x_2x_3$.

- Do we need to be fluent in the techniques by which you expanded all of the polynomials above??
- Thus, we have two conjectures:

$$\operatorname{ch}[(n)] = h_n \qquad\qquad \operatorname{ch}[\underbrace{(1,\dots,1)}_{n\text{ times}}] = \sigma_n$$

- The theorem is cool because it sends all of representation theory to some symmetric polynomial game!
- How do we compute $\operatorname{ch}(V_\lambda)$?
  - We say it equals S_λ, where S_λ is a Schur polynomial.
  - Take the YT of λ. Recall standard YTs.
  - **Semistandard** (YT): Things strictly increase in columns, but only monotonically increase in rows. *draw picture!*
  - The six semistandard ones give us the Schur polynomial.
  - Relation to RSK correspondence.
- Proving why this stuff is true is not hard.
- To understand *why* this is true, Google the **Schur-Weyl duality**.

9.3 Office Hours

- I got a 68/100 on the midterm: 30, 24, 0, 14.
  - I would have needed to show my work (or at least one example of a calculation) to get full credit for 2, even though it just said "find."
  - Rudenko did not expect that finding conjugacy classes would be so difficult for us; he will adjust for this difficulty on the final.
- Week 3, Lecture 2: You proved that $\langle\chi_V,\chi_W\rangle=\delta_{VW}$. To do so, you used a projection function $p=(1/|G|)\sum_{g\in G}gv$. You began your proof by proving that p is a G-morphism and then never used this result again, as far as I can tell. Did you use it again? See pp. 45-47, 58 (it needs to be a morphism of G-representations to map between the representations V,V^G?).
- Week 3, Lecture 2: Same proof. To prove that $\operatorname{Im}(P)=V^G$, do we need more than $p^2=p$? I think so, but you didn't do it explicitly. See pp. 46-47.
- Week 3, Lecture 2: Same proof. What's up with the trivial special case? See p. 48.
- *Week 3, Lecture 3: Cube thing (see picture from 10/13)?
  - It's just a depiction of two different 3-coordinate bases of the same space. It was drawn to illustrate a possible relation between the orthonormal basis χ_1,χ_2,χ_3 (cube) and the orthogonal basis $\chi_{C_1},\chi_{C_2},\chi_{C_3}$.
- Week 3, Lecture 3: Why did we talk about the infinite-dimensional regular representation here? See p. 50.
- *Week 3, Lecture 3: What is the point of the misc. calculations involved in computing the S_4 character table? See p. 52.
  - Just to check that we were on the right path and shown an example of using the orthogonality relations.
- *Week 3, Lecture 3: Proof of the second orthogonality relation your way? It's in Serre (1977), but I don't think that's the way you proved it. See p. 52.
  - To begin, note that it is a *highly* nontrivial statement that if A,B are matrices such that $AB=I$, then $BA=I$. It seems so simple to us, but think about it! For an arbitrary matrix A,B, AB looks nothing like BA! We have two entirely different systems of equations.
  - However, using this fact, basically it is possible to translate the orthogonality relation for the *columns* into the orthogonality relation about the *rows*.
- *Week 3, Lecture 3: All the talk about the exceptional homomorphisms? See p. 52, 61 (the final representation has something to do with an **involution** of trace 2, and is a representation of a quotient group?).
  - So the representation is $\rho:S_4\twoheadrightarrow S_3\xrightarrow{\tilde{\rho}}GL_n$, where $\tilde{\rho}:S_3\to GL_n$ is the representation of ρ corresponding to the character $(2,0,1)$.
- *Week 4, Lecture 1: Alternate construction of $R(G)$? See p. 63.
- *Week 4, Lecture 1: Extension of scalars with the representation ring? See p. 64.
  - We don't need to know anything about this stuff.
  - What it is though is basically analogous to extending the real numbers into a subset of the complex numbers by treating every $x\in\mathbb{R}$ as $x+0i\in\mathbb{C}$. Very trivial, silly concept.
  - There is also such a thing as a **reduction of scalars**.

- *Week 4, Lecture 1: Does multiplying a column vector in the basis $\{\delta_{C_i}\}$ by the character table put it in the basis $\{\chi_{V_i^*}\}$, or vice versa? See p. 66.
  - Derive it for yourself.
  - Example: Consider the character table for S_3 (Table 3.1) represented as the following matrix.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

  - Denote the conjugacy classes of S_3 by e, (xx), (xxx).
  - Interpretation 1.
    - We see that the standard representation is the class function sending

$$e \mapsto 2 \qquad (xx) \mapsto 0 \qquad (xxx) \mapsto -1$$

    - Additionally, we have

$$\begin{aligned} &e \mapsto 1 && (xx) \mapsto 0 && (xxx) \mapsto 0 && (\delta_e) \\ &e \mapsto 0 && (xx) \mapsto 1 && (xxx) \mapsto 0 && (\delta_{(xx)}) \\ &e \mapsto 0 && (xx) \mapsto 0 && (xxx) \mapsto 1 && (\delta_{(xxx)}) \end{aligned}$$

    - Thus, we can express $\chi_{(2,1)}$ as a linear combination of the δ's via

$$\begin{aligned} \chi_{(2,1)} &= (2)\delta_e + (0)\delta_{(xx)} + (-1)\delta_{(xxx)} \\ &= \chi_{(2,1)}(e)\delta_e + \chi_{(2,1)}(xx)\delta_{(xx)} + \chi_{(2,1)}(xxx)\delta_{(xxx)} \\ &= \sum_{C_i} \chi_{(2,1)}(C_i)\delta_{C_i} \end{aligned}$$

    - It follows in particular that if we represent the δ_{C_i}'s as the standard column vector basis of $\mathbb{C}^3$, then

$$\chi_{(2,1)} = A^T \delta_{(xxx)}$$

  - Interpretation 2.
    - If we multiply A by the column vector equal to each representation weighted by $|C_i|$, then we recover the δ basis:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = A \begin{bmatrix} 1 \cdot 1 \\ 3 \cdot 1 \\ 1 \cdot 1 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = A \begin{bmatrix} 1 \cdot 1 \\ 3 \cdot -1 \\ 1 \cdot 1 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = A \begin{bmatrix} 1 \cdot 2 \\ 3 \cdot 0 \\ 1 \cdot -1 \end{bmatrix}$$

    - This interpretation is also what is expressed by the following formula from Lecture 4.2.

$$\delta_{C_j}(g) = \sum_{V_i} \frac{|C_j|\bar{\chi}_{V_i}(g)}{|G|} \cdot \chi_{V_i}(g)$$

- *Week 4, Lecture 2: Isotypical components example. See p. 68.
- **Week 4, Lecture 3: Proof the $\mathbb{C}$ is the only finite-dimensional division algebra? See p. 71.
  - Let A be an arbitrary finite-dimensional division algebra over $\mathbb{C}$.
  - To prove that $A = \mathbb{C}$, we will use a bidirectional inclusion proof.
  - Naturally, $\mathbb{C} \subset A$.
  - To prove the reverse implication, start by letting $a \in A$ be arbitrary.
  - Define the left-multiplication operator $L_a : A \to A$ by $x \mapsto ax$ for all $x \in A$.

- Recall that A is a complex vector space in addition to being an algebra, the same way a ring is also a group. Thus, L_a is a linear operator on a complex vector space.
- It follows by the theorem of linear algebra that L_a has an eigenvalue $\lambda \in F = \mathbb{C}$ and corresponding eigenvector $b \in A$.
- Consequently, by the cancellation lemma,

$$\begin{aligned} L_a b &= \lambda b \\ ab &= \lambda b \\ a &= \lambda \end{aligned}$$

- Therefore, $a \in A$ implies $a \in \mathbb{C}$, so $A = \mathbb{C}$.

- *Week 6, Lecture 2: Proof that $\sqrt{2}/2$ is not an algebraic integer using Gauss's lemma? See p. 87.
  - Let $\alpha := \sqrt{2}/2$ for the sake of notation.
  - Suppose for the sake of contradiction that α is an algebraic integer.
  - Then there exists a monic polonomial $p(x) \in \mathbb{Z}[x]$ such that $p(\alpha) = 0$.
  - Observe that the minimal polynomial in $\mathbb{Z}[x]$ that annihilates α is $2x^2 - 1$.
  - Thus, by polynomial division,

$$p(x) = q(x) \cdot (2x^2 - 1) + r(x)$$

    for some $q, r \in \mathbb{Q}[x]$ such that $\deg r \leq 2 - 1$.
  - We have that

$$r(\alpha) = p(\alpha) - q(\alpha) \cdot (2\alpha^2 - 1) = 0 - q(\alpha) \cdot 0 = 0$$

  - Additionally, since $r \in \mathbb{Q}[x]$ and $\deg r \leq 1$, we know that $r(x) = ux + v$ for some $u, v \in \mathbb{Q}$.
  - We now prove that $u = v = 0$.
    - Suppose for the sake of contradiction that either u or v was not equal to zero.
    - Combining the previous two claims reveals that

$$\begin{aligned} 0 &= r(\alpha) \\ &= u\alpha + v \\ -\frac{v}{u} &= \alpha \end{aligned}$$

    - If $u = 0$, then α is undefined and we have arrived at a contradiction. Thus, $u \neq 0$.
    - Thus, $\alpha \in \mathbb{Q}$. But since $\alpha \notin \mathbb{Q}$ by definition, we have arrived at a contradiction.
    - Therefore, $u = v = 0$.
  - Having established that $r = 0$, we know that $p = (2x^2 - 1)q$, i.e., $2x^2 - 1$ divides p.
  - Now define N to be the least common multiple of the denominators of the coefficients of q.
  - Consider

$$Np = (Nq)(2x^2 - 1)$$

  - It follows by Gauss's lemma that

$$\begin{aligned} c(Np) &= c[(Nq)(2x^2 - 1)] \\ N &= c(Nq) \cdot c(2x^2 - 1) \\ &= 1 \cdot 1 \\ &= 1 \end{aligned}$$

    where c denotes the **content**.

- But if $N = 1$, then $q \in \mathbb{Z}[x]$, so leading term of p — equal to the product of $2x^2$ and the leading term of q — has a coefficient that is a multiple of 2, i.e., is *not* equal to 1 as is required of a monic polynomial, a contradiction.

- *Week 6, Lecture 3: Questions about Lemma 1 of the proof of Burnside's theorem. See p. 92.
  - The roots $a_1, \dots, a_k$ of the minimal polynomial of the algebraic integer a are known as **conjugate algebraic integers**.
  - The conjugate algebraic integers of a root of unity are also roots of unity.
    - Suppose ε is a root of unity.
    - Then the minimal polynomial of ε is $x^n - 1$ for some $n \in \mathbb{N}$.
    - Naturally, the roots of this polynomial (the conjugate algebraic integers to ε) are all of the other roots of unity of order n.
  - The conjugate algebraic integers of a sum of roots of unity is a sum of roots of unity.
    - It can be shown that the minimal polynomial for $\varepsilon_1 + \varepsilon_2$ is

$$p(x) = \prod_{i,j=1}^{n} (x - \varepsilon_1^i - \varepsilon_2^j)$$

    - Evidently, the above polynomial is symmetric under permutations of $\varepsilon_1^i, \varepsilon_2^j$, and we'd generate the same polynomial with any $\pm\varepsilon_1^i \pm \varepsilon_2^j$ as starting material.
    - Explicit example.
      - ➢ $\pm\sqrt{2}$ are conjugate algebraic integers, as solutions to $x^2 - 2$. Similarly, $\pm\sqrt{3}$ are conjugate algebraic integers as solutions to $x^2 = 3$.
      - ➢ Thus, we expect the minimal polynomial for $\sqrt{2} + \sqrt{3}$ to be

$$p(x) = (x - \sqrt{2} - \sqrt{3})(x - \sqrt{2} + \sqrt{3})(x + \sqrt{2} - \sqrt{3})(x + \sqrt{2} + \sqrt{3})$$

      - ➢ Expanding, we obtain

$$\begin{aligned}
p(x) &= (x^2 - (\sqrt{2} + \sqrt{3})^2)(x^2 - (\sqrt{2} - \sqrt{3})^2) \\
&= x^4 - [(\sqrt{2} + \sqrt{3})^2 + (\sqrt{2} - \sqrt{3})^2]x^2 + (\sqrt{2} + \sqrt{3})^2(\sqrt{2} - \sqrt{3})^2 \\
&= x^4 - 10x^2 + 1
\end{aligned}$$

      - ➢ Indeed, the above polynomial is a monic polynomial
      - ➢ From the definition, this polynomial is evidently also the minimal polynomial for $\sqrt{2} - \sqrt{3}$, $-\sqrt{2} + \sqrt{3}$, and $-\sqrt{2} - \sqrt{3}$.
      - ➢ Thus, the conjugate algebraic integers of $\sqrt{2} + \sqrt{3}$ are the four sums of all individual algebraic integers.
  - How do we extend this argument to the case in the problem?? What about when $\varepsilon_1 = -1$ and $\varepsilon_2 = i$ so that simple powers don't access every combination as the $p(x)$ formula does?
  - We know that $\prod_{i=1}^{n} a_i \in \mathbb{Z}$ because of Vieta's formula.
    - In particular, this tells us that $x_1 \cdots x_n$ is equal to the last coefficient in the minimal polynomial which, by definition, is an integer.
  - Don't worry too much about all this, though: Burnside's theorem will no longer be on the final because Rudenko changed his mind.

- *Week 7, Lecture 2: Symmetric polynomials and roots of symmetric polynomials. See p. 101.

- *Week 7, Lecture 2: Word in blackboard picture? See p. 102.
  - "Remain" to show...

- *Week 7, Lecture 2: What is d in the proof of the alternating polynomials theorem? See p. 103.
  - $d = n - 1$.
- *Week 8, Lecture 2: What is d in the definition on p. 111.
  - Consider the Specht polynomial corresponding to $(2, 2, 1)$.

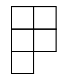


Figure 9.3: Young diagram for $(2, 2, 1)$.

  - Since $(2, 2, 1)' = (3, 2)$, the Specht polynomial is $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \cdot (x_4 - x_5)$.
  - Δ_{123} is of degree $3 = \binom{3}{2}$ because looking at the first column of the YD, which corresponds to $\lambda_1' = 3$, out of the 3 boxes, we must choose 2 for each of the three terms $(x_1-x_2), (x_1-x_3), (x_2-x_3)$. Then we just add this to the 2 choose 2 for the second column of the YD.
- *Week 9, Lecture 2: Do we need to be fluent in the techniques you used to expand and reduce the various polynomial powers? How did you do that again?

9.4 The Frobenius Characteristic Map

12/1:

- Proving the theorem.
- The statement is that there exists a function

$$\text{ch}: \underbrace{\bigoplus_{n\geq 0}\mathbb{Q}_{\text{cl}}(S_n)}_{\{f:S_n\to\mathbb{Q}:f(\sigma=\sigma^{-1})=f(i)\}} \to \bigoplus_{n\geq 0}\Lambda_n$$

where $\Lambda_n = \mathbb{Q}[\sigma_1, \sigma_2, \sigma_3, \dots]_{\deg=n} = [\mathbb{Q}[x_1, \dots, x_n]^{S_n}]_{\deg=n}$. Note that by convention, $\Lambda_0 = \mathbb{Q}$. This function is given by

$$\text{ch}(\chi) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi(\sigma)\cdot p_1^{\lambda_1(\sigma)}\cdot p_n^{\lambda_n(\sigma)}$$

where $p_k = x_1^k + x_2^k + \cdots$ and $\lambda_i(\sigma)$ is the number of cycles in σ of length i. Moreover, ch is an isomorphism of $\mathbb{Q}$-algebras. To create a product of V a S_n-rep and W an S_m-rep, we map

$$V \boxtimes W = \text{Ind}_{S_n\times S_m}^{S_{n+m}}(V\otimes W)$$

Proof. To prove that ch is a ring isomorphism, we need...

1. $\boxtimes$ is associative;
2. $\text{ch}(\chi_1 \boxtimes \chi_2) = \text{ch}(\chi_1)\cdot\text{ch}(\chi_2)$;
3. $\text{ch}((n)) = h_n$.

1,2,3 imply the theorem because 2,3 imply that ch is surjective, Λ_n has a $\mathbb{Q}$-basis $h_{\lambda_1}\cdots h_{\lambda_n}$ for $\lambda_1 \leq \cdots \leq \lambda_n$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = n$. For example, for Λ_5, we have $h_1^5, h_1^3h_2, h_1, h_2^2, h_2h_3, h_1^2h_3, h_1h_4, h_5$. This surjectivity combined with the fact that $\dim\mathbb{Q}_{\text{cl}}[S_n] = \dim\Lambda_n$ implies that ch is an isomorphism of rings.

Last thing: $\text{ch}[(n)] = \frac{1}{n!}\sum p_1^{c_1(\sigma)}\cdots p_n^{c_n(\sigma)}$ where $c_i(\sigma)$ denotes the number of cycles of length i and hence $\sum ic_i = n$. Denote $p_1^{c_1(\sigma)}\cdots p_n^{c_n(\sigma)}$ by $p^{c(\sigma)}$.

Proof. Let

$$\begin{aligned}
\sum h_n t^n &= \sum\Big(\sum_{i_1 \le \cdots \le i_n} x_{i_1} \cdots x_{i_n} t^n \Big) \\
&= \frac{1}{1-x_1 t} \cdot \frac{1}{1-x_2 t} \cdots \frac{1}{1-x_n t} \\
&= \exp\left(\log\left(\prod_{i=1}^{n} \frac{1}{1-x_i t} \right) \right) \\
&= \exp\left(\sum_{i=1}^{n} -\log(1-x_i t) \right) \\
&= \exp\left(x_1 + \frac{x_1^2 t^2}{2} + \frac{x_1^3 t^3}{3} + \cdots + x_2 + \frac{x_2^2 t^2}{2} + \cdots \right) \\
&= \exp\left(p_1 + \frac{p_2 t^2}{2} + \frac{p_3 t^3}{3} + \cdots \right) \\
&= \prod_{m \ge 1} \exp\left(\frac{p_m t^m}{m} \right) \\
&= *
\end{aligned}$$

We get the second equality because each $1/(1-x_i t) = 1 + x_i + x_i^2 t^2 + \cdots$. We need the power series $-\log(1-t) = t + t^2/2 + \cdots$ and $\exp(t) = 1 + t + t^2/2! + \cdots$. Thus, $\exp(\log(1-t)) = 1-t$. Now note that

$$\begin{aligned}
\sum_{n \ge 0} \mathrm{ch}[(n)] \cdot t^n &= \sum_{n \ge 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} p_1^{c_1(\sigma)} \cdots p_n^{c_n(\sigma)} \\
&= \sum_{n \ge 0} \frac{t^n}{n!} \left(\sum_{a_1 + 2a_2 + \cdots + n a_n = n} p_1^{a_1} \cdots p_n^{a_n} \right) \cdot \frac{n!}{1^{a_1} a_1! 2^{a_2} a_2! \cdots n^{a_n} a_n!} \\
&= \sum_{\substack{n \ge 0 \\ a_1, \ldots, a_n : a_1 + 2a_2 + \cdots + n a_n = n}} \frac{1}{a_1!} \left(\frac{p_1}{1} \right)^{a_1} t^{a_1} \frac{1}{a_2!} \left(\frac{p_2}{2} \right)^{a_2} t^{a_2} \cdots \frac{1}{a_n!} \left(\frac{p_n}{n} \right)^{a_n} t^{a_n} \\
&= \prod_{k=1}^{n} \left(\sum_{a_k = 1}^{\infty} \frac{(p_k)^{a_k} t^{a_k k}}{a_k! k a_k} \right) \\
&= \prod_{k \ge 1} \exp\left(\frac{p_k t^k}{k} \right) \\
&= *
\end{aligned}$$

We're overcounting because we can cyclically permute cycles (i.e., (12) = (21)), hence the correction factor in the second line above.

Note: This exponent/logarithm trick is a common computational trick in combinatorics, varieties, etc. □

Now we prove the part 3, i.e., that $\boxtimes$ is associative. We do this by direct computation.

$$\underbrace{\mathrm{Ind}_{S_{n+m} \times S_\ell}^{S_{n+m+\ell}} \left[\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} (\chi_1 \otimes \chi_2) \right] \otimes \chi_3}_{(\chi_1 \boxtimes \chi_2) \boxtimes \chi_3} = \mathrm{Ind}_{s_n \times S_m \times S_\ell}^{S_{n+m+\ell}} (\chi_1 \otimes \chi_2 \otimes \chi_3)$$

...

Then proving 2 (homomorphism bit) is the hardest. We have

$$\begin{aligned}
\operatorname{ch}(\operatorname{Ind}_{S_n\times S_m}^{S_{n+m}}(\chi_1\otimes\chi_2)) &= \frac{1}{n!}\sum_{\sigma\in S_n}(\operatorname{Ind}_{S_n\times S_m}^{S_{n+m}}\chi_1\otimes\chi_2)(\sigma)\underbrace{p_1^{c_1(\sigma)}\cdots p_{n+m}^{c_{n+m}(\sigma)}}_{\psi}\\
&= \left\langle \operatorname{Ind}_{S_n\times S_m}^{S_{n+m}}(\chi_1\otimes\chi_2),\psi\right\rangle_{S_{n+m}}\\
&= \left\langle \chi_1\otimes\chi_2,\operatorname{Res}_{S_n\times S_m}^{S_{n+m}}\psi\right\rangle\\
&= \sum_{\substack{\sigma_1\in S_n\\ \sigma_2\in S_m}}\chi_1(\sigma_1)\chi_2(\sigma_2)p_1^{c_1(\sigma_1)}\cdots p_n^{c_n(\sigma_1)}p_1^{c_1(\sigma_2)}\cdots p_m^{c_m(\sigma_2)}\\
&= \operatorname{ch}(\chi_1)\operatorname{ch}(\chi_2)
\end{aligned}$$

We use Frobenius reciprocity somewhere in here. We also have $\psi : S_n \to \Lambda_n$ and $\psi(\tau\sigma\tau^{-1}) = \psi(\sigma)$. ☐

- After another 10 years of trying to understand the representations of the symmetric group, we'll be here.
- At this point, we can study compact Lie groups.

9.5 Final Review Sheet

1/18:

- Problem-by-problem breakdown.
- (30 points) Question 1: One of the following three prompts. I can memorize answers to these. This is for speed and for credit.
- Define algebraic integers. Prove that they form a ring. Prove that the dimension of an irreducible representation divides the order of the group.

Answer.
Algebraic integers: The set of all $x \in \mathbb{C}$ such that there exist some $a_0,\dots,a_{n-1}\in\mathbb{Z}$ such that

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$$

To prove that the algebraic integers form a ring, it will suffice to show that if $x, y \in \bar{\mathbb{Z}}$, then $x+y, xy \in \bar{\mathbb{Z}}$. Let $x, y \in \bar{\mathbb{Z}}$. Then there exist monic polynomials with integer coefficients such that

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0 \qquad\qquad y^m + b_{m-1}y^{m-1} + \cdots + b_0 = 0$$

To each of these polynomials corresponds a matrix with integer coefficients for which x, y, respectively, is an eigenvalue. Explicitly, these matrices are the Frobenius matrices

$$A = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1\\ -a_0 & -a_1 & \cdots & -a_{n-1}\end{bmatrix} \qquad\qquad B = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1\\ -b_0 & -b_1 & \cdots & -b_{m-1}\end{bmatrix}$$

which have characteristic polynomials equal to the above polynomials and thus have x, y as eigenvalues. To these eigenvalues must correspond some eigenvectors v, w, which we will not explicitly solve for. The important thing is that we now know that

$$Av = xv \qquad\qquad Bw = yw$$

Now, using the Kronecker product, we have that

$$(A \otimes B)(v \otimes w) = Av \otimes Bw = xv \otimes yw = xy(v \otimes w)$$
$$(A \otimes I_m + I_n \otimes B)(v \otimes w) = Av \otimes I_m w + I_n v \otimes Bw = xv \otimes 1w + 1v \otimes yw = (x+y)(v \otimes w)$$

Thus, $x+y, xy$ are eigenvalues of matrices with integer coefficients. Thus, they are zeroes of the characteristic polynomials, which will be monic polynomials with coefficients that are the sums and products of integers, and hence are integers, too. Therefore, $x+y, xy \in \bar{\mathbb{Z}}$.

We now prove that the dimension d_V of an irrep divides the order $|G|$ of a group.

We begin with three definitions: Let $C := \{g_1, \dots, g_s\} \subset G$ be a conjugacy class of the finite group G where naturally $s = |C|$, let $e_C := g_1 + \cdots + g_s \in \mathbb{Z}[G] \subset \mathbb{C}[G]$, and let $\rho : \mathbb{C}[G] \to GL(V)$ be the group homomorphism corresponding to the irreducible representation V.

Recall that the e_C's for all conjugacy classes form a basis of the center of the group algebra $\mathbb{C}[G]$. Thus, $e_C \in Z(\mathbb{C}[G])$. It follows easily that $\rho(e_C)$ satisfies Schur's lemma for associative algebras, and from there that there exists $\lambda \in \mathbb{C}$ such that $\rho(e_C) = \lambda I_{d_V}$. Moreover,

$$\begin{aligned} \rho(g_1 + \cdots + g_s) &= \lambda I_{d_V} \\ \operatorname{tr}(\rho(g_1 + \cdots + g_s)) &= \operatorname{tr}(\lambda I_{d_V}) \\ \operatorname{tr}(\rho(g_1)) + \cdots + \operatorname{tr}(\rho(g_s)) &= \lambda \operatorname{tr}(I_{d_V}) \\ |C|\chi(C) &= \lambda d_V \\ \lambda &= \frac{|C|\chi(C)}{d_V} \end{aligned}$$

Additionally, we can prove that $\lambda \in \bar{\mathbb{Z}}$. Recall that there exist $a_0, \dots, a_{n-1} \in \mathbb{Z}$ such that

$$e_C^n + a_{n-1} e_C^{n-1} + \cdots + a_0 = 0$$

Recall that the essential reason that this is true is that the left multiplication by e_C map from $\mathbb{Z}[G] \to \mathbb{Z}[G]$ will have eigenvalue e_C (by definition) and a matrix of integers (because of its domain and range), so we use the characteristic polynomial as before. Anyway, back to the point: Let $v \in V$ be nonzero. Then

$$\begin{aligned} 0 &= \rho(0)v \\ &= \rho(e_C^n + a_{n-1} e_C^{n-1} + \cdots + a_0)v \\ &= \left[\rho(e_C)^n + a_{n-1}\rho(e_C)^{n-1} + \cdots + a_0\right] v \\ &= \left[\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0\right] I_{d_V} v \\ &= \left[\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0\right] v \end{aligned}$$

It follows by the zero product property that the coefficient in brackets must be zero, giving us the monic, integer coefficient polynomial we need to prove that $\lambda \in \bar{\mathbb{Z}}$.

Additionally, recall that $\chi(C) \in \bar{\mathbb{Z}}$. This is because $g_i^n = e$ for some n, so $\rho(g)$ may be diagonal with roots of unity along the diagonal (i.e., *unitary*), so the character is the sum of n^{th} roots of unity. All of these roots of unity are algebraic integers, meaning that their sum is, too, since we've already proven that the algebraic integers form a ring. Moreover, it follows because complex roots of a polynomial imply that their conjugate is also a root that $\overline{\chi(C)} \in \bar{\mathbb{Z}}$, too. Thus, by the first orthogonality relation,

$$\begin{aligned} |G| &= \sum_C |C|\chi(C)\overline{\chi(C)} \\ \frac{|G|}{d_V} &= \sum_C \frac{|C|\chi(C)}{d_V} \cdot \overline{\chi(C)} \end{aligned}$$

Thus, since $\bar{\mathbb{Z}}$ is a ring, $|G|/d_V \in \bar{\mathbb{Z}}$. Naturally, it is also in $\mathbb{Q}$. It follows that it is in $\mathbb{Q} \cap \bar{\mathbb{Z}} = \mathbb{Z}$. Note that this last equality holds via a somewhat complicated polynomial argument with prime divisors and such. Regardless, since $|G|/d_V$ is an integer, this means that $d_V \mid |G|$, as desired. $\square$

- Explain the construction of irreducible representations of S_n via Specht polynomials. Prove irreducibility of these representations.

Answer.
Roughly speaking, we establish a bijective correspondence between the partitions $\lambda \vdash n$ and the irreps of S_n as follows. Take a partition of n. For the sake of clarity, we will look at a representative example. Consider S_7 and let $\lambda = (4,2,1)$. Then letting $\lambda' = (3,2,1,1)$, we define the Specht polynomial of λ to be

$$\mathrm{Sp}_\lambda(x_1,\dots,x_7) := \Delta(x_1,x_2,x_3)\Delta(x_4,x_5)\Delta(x_6)\Delta(x_7)$$

based off of the inverse partition. Each Δ term denotes a Vandermonde determinant, which we can evaluate to get the final, expanded Specht polynomial as follows:

$$\mathrm{Sp}_\lambda = (x_1-x_2)(x_1-x_3)(x_2-x_3)(x_4-x_5)$$

This is how to construct a Specht polynomial.

Now let's look at a different, simpler example for the full construction of irreducible representations. In particular, let's look at $\lambda = (2,1)$ for S_3. Here,

$$\mathrm{Sp}_\lambda = x_1 - x_2$$

Now we apply all of the elements of S_3 to it to generate a basis of the vector space that will be our irrep. Applying $e,(123),(132),(12),(13),(23)$ in order via the permutation representation, we get

$$\langle x_1-x_2, x_2-x_3, x_3-x_1, x_2-x_1, x_3-x_2, x_1-x_3\rangle$$

Since x_1-x_2 and x_2-x_3 don't even contain the same variables, they're clearly linearly independent in the polynomial space. Let's call them a,b respectively. However, every other polynomial present is linearly dependent on a,b: For example, $x_3-x_1 = -(a+b)$ and $x_2-x_1=-a$. Thus, the true basis of the Specht module, which we will denote V_λ and is our final irrep, is

$$\langle x_1-x_2, x_2-x_3\rangle$$

Alternatively, we can construct a basis directly using standard Young tableaux. The two possible SYTs for $(2,1)$ put x_1,x_2 vertically stacked and x_1,x_3 vertically stacked. This implies a basis $\langle \Delta(x_1,x_2), \Delta(x_1,x_3)\rangle = \langle x_1-x_2, x_1-x_3\rangle$, which works just as well as the one above.

We now prove that Specht modules constitute irreducible representations. Let $\lambda \vdash n$, let V_λ be the corresponding Specht module, let $d := d(\lambda)$ be the degree of Sp_λ, and let R_d be the subset of $\mathbb{C}[x_1,\dots,x_n]$ consisting of polynomials of degree d. By complete reducibility, let $V_\lambda = \bigoplus W_i^{n_i}$ and $R_d = \bigoplus W_i^{m_i}$, where the W_i are the irreps of S_n. Recall that by Schur's lemma,

$$\dim \mathrm{Hom}_{S_n}(V_\lambda, R_d) = \sum_i n_i m_i$$

Moreover, if we actually inspect a homomorphism in this set, we will find that it is a linear map sending polynomials to polynomials. In particular, since the argument is antisymmetric in $x_1,\dots,x_{\lambda_1'}$, so is its image. This pattern continues for all Vandermonde determinants. In fact, what it ends up implying — since any antisymmetric polynomial is equal to the product of a symmetric polynomial and a Vandermonde determinant — is that the image is divisible by all of the Vandermonde determinants composing the argument. Indeed, the image is divisible by the argument. But since both image and argument have the same degree, that means that they are only related by a scalar factor, so the homomorphism must be a homothety. For our purposes, the takeaway is that

$$\dim \mathrm{Hom}_{S_n}(V_\lambda, R_d) = 1$$

It follows by transitivity that there must exist some i for which $n_i = m_i = 1$, and we must have $n_j = m_j = 0$ for all $j \neq i$. But this means that $V_\lambda = W_i$ is irreducible, as desired. $\square$

- Define induction and restriction, state the formula for the character of an induced representation. State and prove Frobenius reciprocity.

Answer. **Restriction** takes a representation $\rho : G \to GL(V)$ and a subgroup $H \leq G$, and returns the restricted representation $\tilde{\rho} : H \to GL(V)$ defined by $\tilde{\rho}(g) := \rho(g)$.

Induction takes a subgroup $H \leq G$ and a representation $\rho : H \to GL(W)$, and returns an induced representation $\tilde{\rho} : G \to GL(V)$. Defining this one is a bit trickier. In a nutshell, let $g_1H, \dots, g_sH$ be all cosets of H in G, where $s = (G : H)$ and we pick $g_1 \in H$. Define vector spaces $g_1W, \dots, g_sW$ identical to W except for the name, and let $V := g_1W \oplus \cdots \oplus g_sW$. Then if $g \in G$ is arbitrary, we say $\tilde{\rho}(g)$ acts on the basis of V as follows: Let g_jw be a basis vector. $gg_j = g_kh$ for some g_k in the system of representatives and $h \in H$. Thus,

$$[\tilde{\rho}(g)](g_jw) = g_k\rho(h)w$$

The formula for the character of an induced representation is as follows.

$$\chi_{\operatorname{Ind}_H^G W}(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} \chi_W(x^{-1}gx)$$

Frobenius reciprocity relates the inner product of the characters of two G-reps to the inner product of two related H-reps: In simplified notation, we have

$$(V, \operatorname{Ind}_H^G W)_G = (\operatorname{Res}_H^G V, W)$$

We can prove this equality through a long transitivity chain as follows.

$$\begin{aligned}
(\chi_V, \chi_{\operatorname{Ind}_H^G W})_G &= \frac{1}{|G|} \sum_{g_1 \in G} \chi_V(g_1)\overline{\chi_{\operatorname{Ind}_H^G W}(g_1)} \\
&= \frac{1}{|G|} \sum_{g_1 \in G} \chi_V(g_1)\chi_{\operatorname{Ind}_H^G W}(g_1^{-1}) \\
&= \frac{1}{|G|} \sum_{g_1 \in G} \chi_V(g_1) \left(\frac{1}{|H|} \sum_{\substack{g_2 \in G \\ g_2^{-1}g_1^{-1}g_2 \in H}} \chi_W(g_2^{-1}g_1^{-1}g_2) \right) \\
&= \frac{1}{|H| \cdot |G|} \sum_{g_1 \in G} \sum_{g_2 \in G} \chi_V(g_1)\chi_W(g_2^{-1}g_1^{-1}g_2) \\
&= \frac{1}{|H| \cdot |G|} \sum_{g_2 \in G} \sum_{g_1 \in G} \chi_V(g_2^{-1}g_1g_2)\chi_W(g_2^{-1}g_1^{-1}g_2) \\
&= \frac{1}{|H| \cdot |G|} \cdot |G| \sum_{g_1 \in G} \chi_V(g_2^{-1}g_1g_2)\chi_W(g_2^{-1}g_1^{-1}g_2) \\
&= \frac{1}{|H|} \sum_{h \in G} \chi_V(h)\chi_W(h^{-1}) \\
&= \frac{1}{|H|} \sum_{h \in H} \chi_V(h)\chi_W(h^{-1}) \\
&= \frac{1}{|H|} \sum_{h \in H} \chi_V(h)\overline{\chi_W(h)} \\
&= (\chi_V|_H, \chi_W)_H \\
&= (\chi_{\operatorname{Res}_H^G V}, \chi_W)_H
\end{aligned}$$

Note that we can expand the second summation to be over all $g_2 \in G$ by defining χ_W to be zero whenever its argument $g_2^{-1}g_1^{-1}g_2 \notin H$. $\square$

- Tips for remembering the answers to each questions/quick generalized outlines.
  1. Algebraic integers.
     - Define: Easy.
     - Steps to prove they form a ring.
       - $x, y \in \bar{\mathbb{Z}}$, then $x + y, xy \in \bar{\mathbb{Z}}$.
       - Apply definition of algebraic integers to x, y.
       - Pull back into Frobenius matrices.
       - Use $A \otimes B$ and $A \otimes I_m + I_n \otimes B$.
       - Pull back to characteristic polynomials.
     - Steps to prove that $d_V \mid |G|$.
       - Three definitions: $C := \{g_1, \dots, g_s\}$, $e_C = g_1 + \cdots + g_s$, $\rho : \mathbb{C}[G] \to GL(V)$.
       - $e_C \in Z(\mathbb{C}[G])$ implies $\rho(e_C) = \lambda I_{d_V}$.
       - Expand above result into $\lambda = |C|\chi(C)/d_V$.
       - Left multiplication map on $\mathbb{Z}[G]$ implies $e_C^n + a_{n-1}e_C^{n-1} + \cdots + a_0 = 0$.
       - Use above to prove λ is an algebraic integer.
       - Use roots of unity and complex roots of a polynomial to prove $\overline{\chi(C)} \in \bar{\mathbb{Z}}$.
       - Use first orthogonality relation and $\mathbb{Q} \cap \bar{\mathbb{Z}} = \mathbb{Z}$ to finish the proof.
  2. Specht modules.
     - Logical way.
       - Take λ.
       - Make a Specht polynomial.
       - Apply everything to it.
       - Choose some elements as a basis.
     - Quick way.
       - Take λ.
       - Apply the hook length formula.
       - Make SYTs.
       - Write the basis in Specht polynomials.
     - Prove irreducibility.
       - Definitions: V_λ, $d = d_\lambda$, R_d (polynomials of degree d).
       - Apply complete reducibility.
       - Look at $\dim \operatorname{Hom}_{S_n}(V_\lambda, R_d)$.
       - Get sum.
       - Get 1. ($P = f(\Delta)$.)
  3. Induction/restriction.
     - Induction: $gg_j = g_k h$.
     - Frobenius recipricity is very much like saying restriction is the *adjoint* (functor) of induction. In fact, that's exactly what the statement says.
- (25 points) Question 2: Given a representation of S_n, write down its basis using standard tableaux, prove that the corresponding polynomials indeed form a basis, and compute the character.
- Let's work some examples.
- Example: $(2, 2)$.

Answer. The hook length formula tells us that we can write

$$\frac{4!}{3\cdot 2\cdot 2\cdot 1} = 2$$

SYTs for $\lambda = (2,2)$. By inspection, these are

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \qquad\qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$$

It follows by taking Vandermonde determinants of the columns that our basis is[1]

$$V_{(2,2)} = \langle \underbrace{(x_1-x_2)(x_3-x_4)}_{a}, \underbrace{(x_1-x_3)(x_2-x_4)}_{b} \rangle$$

We will see as we work toward computing the character that a,b suffice as basis vectors.

We will compute the character directly, that is by finding a matrix for a representative member of each conjugacy class of S_4 and taking the trace. Take as representative members of the five conjugacy classes the elements

$$\{e,(12),(123),(1234),(12)(34)\}$$

To simplify computations, it will be useful to introduce the vector

$$b-a = (x_1-x_4)(x_2-x_3)$$

e will send $a\mapsto a$ and $b\mapsto b$. (12) will send $a\mapsto -a$ and $b\mapsto b-a$. (123) will send $a\mapsto b-a$ and $b\mapsto -a$. (1234) will send $a\mapsto a-b$ and $b\mapsto -b$. $(12)(34)$ will send $a\mapsto a$ and $b\mapsto b$. Thus, the matrices of these five elements are

$$\underbrace{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}_{e} \qquad \underbrace{\begin{bmatrix}-1 & -1\\ 0 & 1\end{bmatrix}}_{(12)} \qquad \underbrace{\begin{bmatrix}-1 & -1\\ 1 & 0\end{bmatrix}}_{(123)} \qquad \underbrace{\begin{bmatrix}1 & 0\\ -1 & -1\end{bmatrix}}_{(1234)} \qquad \underbrace{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}_{(12)(34)}$$

Thus, the character is

$$(2,0,-1,0,2)$$

Notice how all of the representative elements send a,b to linear combinations of themselves verifying $\langle a,b\rangle$ as the basis of $V_{(2,2)}$. $\square$

- Steps.
  1. Hook length formula.
  2. Standard Young tableaux.
  3. Specht module basis.
  4. Explicit character computation.
     - Look for linear combinations that show up often and write them down!
     - Use the procedural method if necessary.
  5. Proof of basis validity falls out.
- (25 points) A problem about characters similar to problems 1 and 2 from the midterm.
- Problem 1 from the midterm tested the following skills.
  - You are given the character table for a group.
  - Fill in holes using the orthogonality relations.

[1] This does, indeed, match with our example from the 11/13 class. See also Figure 8.3.

  - Compute tensor products, wedge/symmetric squares, and/or the permutational representation.
  - Decompose reducible representations into irreducibles.

- Problem 2 from the midterm tested the following skills.
  - You are given a small group (like the quaternion group, the symmetries of a square, A_4, etc.).
  - Compute conjugacy classes.
  - Compute the character table.
  - Decompose reducible representations into isotypical components.
  - Diagonalize an endomorphism.

- Example: Character table for the quaternion group.

  Answer. We are given

$$Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$$

with relations

$$(-1)^2 = 1 \qquad\qquad i^2 = j^2 = k^2 = ijk = -1$$

To compute the conjugacy classes, start by putting the identity (1) by itself, as per usual. Similarly, -1 will commute out of any $g(-1)g^{-1}$, so it's in a class by itself. Then we may observe that $(j)(i)(-j) = -i$, so $i, -i$ are in the same conjugacy class. Similar relations hold for $j, -j$ and $k, -k$. Moreover, there is no way to conjugate i over to $\pm j, \pm k$, so these three classes are disjoint. Thus, the five conjugacy classes are

$$\{1\}, \{-1\}, \{i, -i\}, \{j, -j\}, \{k, -k\}$$

We know from the sum of the squares of the dimensionalities that we're gonna have one 2D representation and four 1D representations. We can immediately construct a trivial representation $(1, 1, 1, 1, 1)$. For the first character to always be 1 and orthogonality to be satisfied, our remaining options are $(1, 1, -1, -1, 1)$, $(1, 1, -1, 1, -1)$, and $(1, 1, 1, -1, -1)$. These correspond to various ways we can assign -1 to i, j for instance and have it multiply with the rest per the rules. We can then recover the 2D one via orthogonality relations. In particular, the second orthogonality relation tells us that it is of the form $(\pm 2, \pm 2, 0, 0, 0)$. Our knowledge that characters of the identity are always positive gets us to $(2, \pm 2, 0, 0, 0)$. And then the first orthogonality relation with the first row gets us to $(2, -2, 0, 0, 0)$. Thus, altogether, the character table is □

	$\{1\}$	$\{-1\}$	$\{i, -i\}$	$\{j, -j\}$	$\{k, -k\}$
V_1	1	1	1	1	1
V_2	1	1	-1	-1	1
V_3	1	1	-1	1	-1
V_4	1	1	1	-1	-1
V_5	2	-2	0	0	0

Table 9.2: Character table for Q_8.

- Review the Midterm 1 review sheet!
- Memorize rotation matrix form:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- Pull together content from the last few weeks.

- **Partition** (of $n \in \mathbb{N}$): An ordered tuple satisfying the following constraints. *Denoted by* $\boldsymbol{\lambda}$, $\boldsymbol{(\lambda_1, \dots, \lambda_k)}$. *Constraints*
  1. $\lambda_i \in \mathbb{N}$ for $i = 1, \dots, k$;
  2. $\lambda_1 \geq \cdots \geq \lambda_k$;
  3. $\lambda_1 + \cdots + \lambda_k = n$.
- **Inverse** (of λ): The partition $(\lambda_1', \dots, \lambda_k')$ defined as follows. *Denoted by* $\boldsymbol{\lambda'}$. *Given by*

$$\lambda_i' = |\{\lambda_j \mid \lambda_j > i\}|$$

  for all $i = 1, \dots, k$.
- **Young diagram** (of λ): A left-aligned set of vertically stacked rows of boxes, with λ_i boxes in the i^{th} row and λ_1' rows total.
- Recall **graded algebras**, though I don't think they're too important.
- **Elementary symmetric polynomial** (of degree k in n): The symmetric polynomial defined as follows. *Denoted by* $\boldsymbol{\sigma_k}$. *Given by*

$$\sigma_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}$$

- Examples:
  - $\sigma_1 = x_1 + \cdots + x_n = \sum_{1 \leq i \leq n} x_i$.
  - $\sigma_2 = x_1 x_2 + \cdots + x_1 x_n + x_2 x_3 + \cdots = \sum_{1 \leq i < j \leq n} x_i x_j$.
  - $\sigma_n = x_1 \cdots x_n$.
- **Lexicographic order** (on monomials): An ordering of monomials based on the following rule. *Denoted by* $\boldsymbol{\succ}$. *Given by*
  $x_1^{a_1} \cdots x_n^{a_n} \succ x_1^{b_1} \cdots x_n^{b_n} \dots$
  1. If $a_1 > b_1$ OR...
  2. If $a_1 = b_1$ and $a_2 > b_2$ OR...
  3. $a_1 = b_1$ and $a_2 = b_2$ and $a_3 > b_3$ OR...
  4. So on and so forth.
- **Largest monomial** (of $P \neq 0$); The monomial in $P(x_1, \dots, x_n) \neq 0$ that is the largest lexicographically. *Denoted by* $\boldsymbol{LM(P)}$.
- $\boldsymbol{C_{LM}(P)}$: The coefficient of $LM(P)$.
- $LM(PQ) = LM(P)LM(Q)$, if $P, Q \neq 0$.
- Lemma: If $P \in \mathbb{Q}[x_1, \dots, x_n]^{S_n}$ and $LM(P) = x_1^{a_1} \cdots x_n^{a_n}$, then $a_1 \geq \cdots \geq a_n$.
- Fundamental theorem of symmetric polynomials:

$$\mathbb{Q}[x_1, \dots, x_n]^{S_n} = \mathbb{Q}[\sigma_1, \dots, \sigma_n]$$

- **Antisymmetric** (polynomial): A polynomial $P(x_1, \dots, x_n)$ such that

$$\sigma P = (-1)^\sigma P$$

- **Vandermonde determinant**:

$$\Delta(x_1,\dots,x_n) = \prod_{1\leq i<j\leq n}(x_j - x_i) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & \vdots & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{bmatrix}$$

- Theorem: If $P \in \mathbb{Q}[x_1,\dots,x_n]^{\text{alt}}$ (i.e., $P \in \mathbb{Q}[x_1,\dots,x_n]$ and P is antisymmetric), then $P = P'\Delta(x_1,\dots,x_n)$ where P' is symmetric (i.e., $P' \in \mathbb{Q}[x_1,\dots,x_n]^{S_n}$).
- Corollary: If P is antisymmetric and $\deg(P) < n(n-1)/2$, then $P = 0$.
- **Specht polynomial** (of λ): The polynomial derived by taking the Young diagram of λ, writing numbers down the columns from the left all the way until n, and taking Vandermonde determinants of the columns. *Denoted by* $\mathbf{Sp}_{\boldsymbol{\lambda}}$. *Given by*

$$\text{Sp}_\lambda = \Delta(x_1,\dots,x_{\lambda_1'})\Delta(x_{\lambda_1'+1},\dots,x_{\lambda_2'})\cdots\Delta(x_{\lambda_1'+\cdots+\lambda_{k-1}'+1},\dots,x_n)$$

  - For example, $\text{Sp}_{(4,2,1)}$ is written from

$$\begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

    and equals

$$\text{Sp}_{(4,2,1)} = \Delta(x_1,x_2,x_3)\Delta(x_4,x_5)\Delta(x_6)\Delta(x_7) = (x_1-x_2)(x_1-x_3)(x_2-x_3)(x_4-x_5)$$

- **Degree** (of a Specht polynomial): The degree of the given polynomial as defined in Lecture 7.1. *Denoted by* $\mathbf{d(\boldsymbol{\lambda})}$. *Given by*

$$d(\lambda) = \sum_{i=1}^{k'}\frac{\lambda_i'(\lambda_i'-1)}{2}$$

- Corollary: If $d' < d$, then $\text{Hom}(V_\lambda, R_{d'}) = 0$.
- Theorem 2: Let λ_1,λ_2 be partitions of n. Then $V_{\lambda_1} \cong V_{\lambda_2}$ iff $\lambda_1 = \lambda_2$.
- Corollary: V_λ's are all irreps of S_n.
- We have that

$$d = \binom{\lambda_1}{2} + \binom{\lambda_2}{2} + \cdots$$

- **Young tableau**: A Young diagram filled with integers. *Also known as* **YT**.
- **Standard** (Young tableau): A YT filled in with numbers $1,\dots,n$, wherein each appears exactly once and the numbers increase in rows and in columns. *Also known as* **SYT**.
- **Specht module** (of λ): The vector space where each basis element is a polynomial written from a standard Young tableau of λ in the same manner that Sp_λ is written from its filled-in Young diagram. *Denoted by* $\boldsymbol{V_\lambda}$.
  - For example, the Specht module of $(2,2)$ is written from

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} \qquad\qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$$

    and equals

$$V_{(2,2)} = \langle (x_1-x_2)(x_3-x_4), (x_1-x_3)(x_2-x_4)\rangle$$

- Theorem: $\dim V_\lambda$ is the number of SYTs of shape λ.

- Corollary: $\dim V_\lambda = \dim V_{\lambda'}$.
- Fact: We have the following identity.
$$V_{\lambda'} = V_\lambda \otimes (\text{sign})$$
- Theorem (RSK): There exists a bijection between permutations in S_n and pairs of SYTs of the same shape (i.e., of **area** n).
  - See Figure 8.5 and the associated discussion.
- **Hook** (of a cell): The set of all cells in a Young diagram directly to the right of or directly beneath the cell in question, including the cell in question.
- **Length** (of a hook): The cardinality of the hook in question.
- **Hook length formula**: The formula given as follows, where n is the number being partitioned. *Given by*
$$\#\,\mathrm{SYT}_\lambda = \frac{n!}{\prod \text{length of all hooks}}$$
- **Restriction** (of V to $H \leq G$): The vector space V viewed as an H-representation, i.e., a straight-up functional restriction of ρ_V. *Denoted by* $\boldsymbol{\mathrm{Res}_H^G(V)}$.
  - Sometimes, we see how the space breaks into subrepresentations under the subgroup.
- **Induction**: *See above.*
- The dimension of an induced representation can be calculated via
$$\dim \mathrm{Ind}_H^G W = (\dim W)(G : H)$$
- Theorem (Frobenius): Let $H < G$, and let W be an H-rep. Then
$$\chi_{\mathrm{Ind}_H^G W}(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} \chi_W(x^{-1}gx)$$
- **Frobenius reciprocity**: The following equality. *Given by*
$$(\chi_V, \chi_{\mathrm{Ind}_H^G W})_G = (\chi_{\mathrm{Res}_H^G V}, \chi_W)_H$$
- Recall the Branching Theorem.
- Recall the Frobenius characteristic map.
- Search for any place I wrote something down about the final!
  - Learn representations of S_4 by heart. See Table 3.2.
  - Final will include some stuff from the last HW, but it will be easier.
  - Rudenko did not expect that finding conjugacy classes would be so difficult for us; he will adjust for this difficulty on the final.
  - No Burnside's theorem on the final.
- Review PSets!

9.6 Final

1. **(30)** Explain the construction of the irreducible representations of S_n via Specht polynomials. Prove irreducibility of these representations.

   Answer. See corresponding answer in Final Review Sheet above. ☐

2. **(25)** Fill in the character of the group S_4. You can use the permutational representation, induction/restriction, orthogonality relations, Specht modules, tensor products, wedge/symmetric powers, etc.

Representation	(1)(2)(3)(4)	(12)(34)	(12)(3)(4)	(1234)	(123)(4)
(4)					
$(1,1,1,1)$					
$(2,2)$					
$(3,1)$					
$(2,1,1)$					

   Answer. $\underline{(4)}$: This is the trivial representation, which always has character

$$(1,1,1,1,1)$$

   $\underline{(1,1,1,1)}$: This is the alternating representation, which has character equal to the sign of each element.
   $e = (12)(12)$, for example, which is an *even* number of transpositions, so sign $= 1$.
   $(12)(34)$ has an even number, so sign $= 1$.
   (12): Odd, sign $= -1$.
   $(1234) = (12)(23)(34)$: Odd, sign $= -1$.
   $(123) = (12)(23)$: Even, sign $= 1$.
   Thus, overall, the character is

$$(1,1,-1,-1,1)$$

   $\underline{(2,2)}$: Use Specht modules, as in the example from the Final Review Sheet above. This yields the character

$$(2,2,0,0,-1)$$

   $\underline{(3,1)}$: Subtract the trivial from the permutational representation. The trivial, we've already found. The permutational is given by the number of elements fixed under each type of permutation, i.e., is $(4,0,2,0,1)$. Subtracting 1 from each of these values yields

$$(3,-1,1,-1,0)$$

   $\underline{(2,1,1)}$: Try $(3,1) \otimes (1,1,1,1)$. This yields the following representation, distinct from the others, that is still irreducible by the irreducibility criterion.

$$(3,-1,-1,1,0)$$

   Note: An alternative, quicker way to solve this question would have been to save $(2,2)$ until last and then fill it in with orthogonality relations. ☐

3. **(25)** Consider a representation $(3,2)$ of the group S_5.

   (a) Write the corresponding Specht polynomial (it has degree 2). Compute its dimension (using any theorems that we discussed). Write a basis for this representation (the natural way to do it is to use standard Young tableaux). Prove that this is indeed a basis.

Answer. Using the construction discussed in Question 1,

$$\mathrm{Sp}_{(3,2)} = \Delta(x_1, x_2)\Delta(x_3, x_4)\Delta(x_5)$$
$$\boxed{\mathrm{Sp}_{(3,2)} = (x_1 - x_2)(x_3 - x_4)}$$

Using the hook length formula, we have that

$$\dim V_{(3,2)} = \frac{5!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1}$$
$$\boxed{\dim V_{(3,2)} = 5}$$

The 5 SYTs we can write are

$$\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}$$

which correspond to

$$\boxed{\begin{gathered} V_{(3,2)} = \Big\langle \overbrace{(x_1 - x_2)(x_3 - x_4)}^{a}, \overbrace{(x_1 - x_3)(x_2 - x_4)}^{b}, \overbrace{(x_1 - x_4)(x_2 - x_5)}^{c} \\ \underbrace{(x_1 - x_2)(x_3 - x_5)}_{d}, \underbrace{(x_1 - x_3)(x_2 - x_5)}_{e} \Big\rangle \end{gathered}}$$

To prove that the above basis is a basis, we need to show that it is spanning and linearly independent. Since there are 5 vectors in it and the hook length formula tells us that we only need 5 vectors, it is spanning. We will prove that the vectors are linearly independent by considering them as column vectors in $R_2 \subset \mathbb{C}[x_1, \dots, x_5]$:

$$\begin{array}{c} \\ x_1x_2 \\ x_1x_3 \\ x_1x_4 \\ x_1x_5 \\ x_2x_3 \\ x_2x_4 \\ x_2x_5 \\ x_3x_4 \\ x_3x_5 \\ x_4x_5 \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ -1 & -1 & 0 & -1 & -1 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Although we could row-reduce this matrix to prove that it is full column rank, we can also observe that b, c, d, e each have a different nonzero entry in the bottom four slots, while a is zero in all four slots, thus proving linear independence. □

(b) Compute the character of this representation.

Answer. We do this directly by looking at where certain representative elements of S_5 map the basis of $V_{(3,2)}$. Filling in the following table, to matrices, to traces finishes the problem.

	e	(12)	(123)	(1234)	(12345)	(12)(34)	(123)(45)
a							
b							
c							
d							
e							

□

4. **(20)** Consider finite groups G and H. Suppose that $V_1, \dots, V_n$ are irreducible representations of G and $W_1, \dots, W_m$ are irreducible representations of H. Define representations of the group $G \times H$ on the vector spaces $V_i \otimes W_j$. Prove that these representations are irreducible and that these are all irreducible representations of $G \times H$.

Answer. Let $\rho_i^G : G \to GL(V_i)$ and $\rho_j^H : H \to GL(W_j)$ denote the group homomorphisms corresponding to the irreps V_i, W_j. Define $\rho_{ij} : G \times H \to GL(V_i \otimes W_j)$ by

$$\rho_{ij}(g,h)(v_i, w_j) = \rho_i^G(g)(v_i) \otimes \rho_j^H(h)(w_j)$$

Then if we suppose that $U \subsetneq V_i \otimes W_j$ is stable under ρ_{ij}, this implies the existence of a proper stable subspace of V_i under ρ_i^G, contradicting the hypothesis that V_i is irreducible. Therefore, the $V_i \otimes W_j$ must be irreducible.

Moreover, these account for all irreps. The number of irreps is equal to the number of conjugacy classes, and since $xgx^{-1} \in C_g$ and $yhy^{-1} \in C_h$ imply that

$$(x,y)(g,h)(x,y)^{-1} \in C_{(g,h)}$$

we can see that the number of conjugacy classes of the product is equal to the product of the number of conjugacy classes of the groups. Specifically, this number is nm since n irreps of G means n conjugacy classes, and similarly for H. But we already have nm irreps in the $V_i \otimes W_j$, so this must be all of them! $\square$