

## Week 7

# Representations of the Symmetric Group

## 7.1 Partitions, Young Diagrams, and Specht Modules

11/6:

- Announcements.
  - Midterm description is on the Canvas page.
  - Review it then PSets. The operator averaging stuff and  $S_4$ ,  $S_5$  examples are most important.
  - New HW will be due next Friday (not this Friday).
- New topic: Representations of  $S_n$ .
  - We will talk about these almost until the end of the course; very hard.
  - Any specialist in rep theory will say that they know some approaches, but nobody understands this stuff completely.
  - We'll explore some phenomena, but if we feel after this course that we still don't understand everything about  $S_n$ , that's typical; if we think we understand everything, we're probably wrong.
- Aside: Representation theory of  $GL_n(\mathbb{F}_{p^k})$  is related but even worse.
  - Same with  $O_n(\mathbb{F}_{p^k})$ .
  - Recently, all this stuff was understood with something called **linguistic theory** (right name??), but that's far beyond us.
- Let's begin. What do we know about  $S_n$ ?
  - $|S_n| = n!$ .
  - The conjugacy classes of  $S_n$  are in bijection with cyclic structures of a permutation.
    - Our good understanding of the conjugacy classes of  $S_n$  is the only thing that makes this problem the slightest bit tractable.
  - These cyclic structures are also in bijection with the **partitions** of a number; recall that we briefly talked about partitions in MATH 25700!
- **Partition** (of  $n \in \mathbb{N}$ ): An ordered tuple satisfying the following constraints. *Denoted by  $\lambda$ ,  $(\lambda_1, \dots, \lambda_k)$ .*  
*Constraints*
  1.  $\lambda_i \in \mathbb{N}$  for  $i = 1, \dots, k$ ;
  2.  $\lambda_1 \geq \dots \geq \lambda_k$ ;
  3.  $\lambda_1 + \dots + \lambda_k = n$ .

- Example: The partitions of the number “4” are (4), (3, 1), (2, 2), (2, 1, 1), and (1, 1, 1, 1).
  - This is the same way we’ve been denoting representations!
- $p(n)$ : The number of possible partitions of  $n$ .
  - Hardy and Ramanujan helped understand the number  $p(n)$  of partitions of  $n$ , but they’re still very hard to understand.
- One way to understand  $p(n)$  is through its encoding in the **generating function**

$$\sum_{n \geq 1} p(n)x^n = 1 + x + 2x^2 + 3x^3 + 5x^4 + \dots$$

- We can think of the above generating function as an actual function of  $x$  if it converges for small  $x$ ; if it doesn’t converge, then we just think of it as a “meaningless” **formal power series**.
- To choose a partition, we need to choose a certain number of 1’s, a certain number of 2’s, a certain number of 3’s, etc. all the way up to  $n$ .
- So let’s look at

$$(1 + x + x^2 + \dots)(1 + x^2 + x^4 + \dots)(1 + x^3 + x^6 + \dots)(1 + x^4 + x^8 + \dots) \dots$$

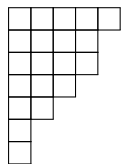
- Formally, this is

$$\prod_{i=1}^{\infty} \left( \sum_{j=0}^{\infty} x^{ij} \right)$$

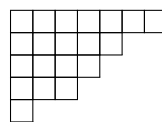
- This equals the generating function! It tells us that to compute  $p(100)x^{100}$ , we need only look at certain terms.
- Recall that we can write  $1 + x + x^2 + \dots = 1/(1 - x)$ . Doing similarly for other terms transforms the above product into

$$\frac{1}{1-x} \frac{1}{1-x^2} \frac{1}{1-x^3} \dots$$

- **Generating function**: An encoding of an infinite sequence of numbers as the coefficients of a formal power series.
- **Formal power series**: An infinite sum of terms of the form  $ax^n$  that is considered independently from any notion of convergence.
- The above discussion of  $p(n)$  as a generating function is only for our fun; Rudenko is not going to use this in any way.
  - It’s just a pretty function.
  - Takeaway: We can write the generating function as something nice and then use it to prove something.
- We can visualize these partitions using something called a **Young diagram**.



(a)  $\lambda = (5, 4, 4, 3, 2, 1, 1)$ .



(b)  $\lambda' = (7, 5, 4, 3, 1)$ .

Figure 7.1: Young diagrams for a partition of 20.

- Suppose we have the following partition of 20:  $(5, 4, 4, 3, 2, 1, 1)$ .
- Then we draw 5 cages for 5 little birds, followed by 4 cages for 4 little birds, etc.
- Thus, the  $i^{\text{th}}$  row of boxes has length  $\lambda_i$ .
- The same way you can denote by  $\lambda$  the whole *partition*, you can denote by  $\lambda$  the whole *diagram*.
- This is just a way to visualize partitions.
- Recall the three partitions of  $S_3$ , corresponding to its representations:  $(3)$ ,  $(2, 1)$ ,  $(1, 1, 1)$ .
- Moreover, these diagrams are actually meaningful!

- **Inverse** (of  $\lambda$ ): The partition  $(\lambda'_1, \dots, \lambda'_k)$  defined as follows. *Denoted by  $\lambda'$ . Given by*

$$\lambda'_i = |\{\lambda_j \mid \lambda_j \geq i\}|$$

for all  $i = 1, \dots, k$ .

- We can see that  $\lambda'_1 \geq \dots \geq \lambda'_k$ .
- We can also see that the sum will still be  $n$ .
- Moreover, if we do this twice, we'll get back to  $\lambda$ , i.e.,  $(\lambda')' = \lambda$ .
- We can prove  $(\lambda')' = \lambda$  combinatorially, too, (that is, without Young diagrams) but that gets pretty complicated. We will do this in the HW.
- Example: If  $\lambda = (5, 4, 4, 3, 2, 1, 1)$  as above, then  $\lambda' = (7, 5, 4, 3, 1)$ .
  - See Figure 7.1b.
  - The Young diagrams are related by a flip, akin to matrix transposition!
  - Notice how the definition of inversion *exactly* specifies this flip in the picture: The number of  $\lambda_j$ 's that have length at least 1 is all the first column of Figure 7.1a, the number of length at least 2 is all the second column, etc.
- Onto the next question, which is the main miracle.
  - Main miracle: There exists a natural (i.e., canonical) bijection between the conjugacy classes and irreducible representations of  $S_n$ .
  - We've explored a duality for general finite groups  $G$ , before, but never a bijection.
    - In  $S_n$ , there *is* this natural bijection.
    - If you understand why intuitively, you will have started to understand the representation theory of  $S_n$ .
- Let  $\lambda \vdash n$ .<sup>[1]</sup> Then there is some irrep  $V_\lambda$  corresponding to  $\lambda$ . We will look at the **Specht module** construction of  $V_\lambda$ .
  - Some of the proofs Rudenko will present, he stole from Etingof et al. (2011), and some of the proofs he invented himself.
  - This is *by far* the best construction, even though it's exceedingly rare in the literature.
- The usual construction.
  - Take  $\mathbb{C}[S_n]$  with coefficients  $a_\lambda, b_\lambda$ , etc. similar over conjugacy classes and do something with it??
  - “Just say NO!” to this construction.
- Here is the better idea.
  - Consider an algebra of polynomials with rational coefficients:  $\mathbb{Q}[x_1, \dots, x_n]$ .

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<sup>1</sup>“Lambda partitions en.”

- We could also do real or complex, but rational is nice.
- For symmetric groups, all representations will be integers, etc.??
- One thing to emphasize about this algebra: It is a **graded** algebra.
  - If represented by  $A$ , then it equals  $A_0 \oplus A_1 \oplus A_2 \oplus \dots$  where

$$A_m = \left\{ \sum_{k_1 + \dots + k_n = m} a_{k_1 \dots k_n} x_1^{k_1} \dots x_n^{k_n} \right\}$$

- I.e.,  $A_m$  is the sum of all polynomials with **degree** equal to  $m$ .
  - Example: If we take  $1 + x_1^2 x_2^3 + x_1 x_2 + x_1^{100} + x_1 x_2^{99}$ , we can then break this polynomial up into polynomials of degree 1, 5, 2, and 100.
- We also have  $A_{m_1} \cdot A_{m_2} \subset A_{m_1+m_2}$ .
  - Example:  $x_1 x_2^2 \cdot (x_1 + x_2) = x_1^2 x_2^2 + x_1 x_2^3$ .
- With this algebra in hand, we may let  $S_n \subset \mathbb{Q}[x_1, \dots, x_n]$  via

$$\sigma P(x_1, \dots, x_n) = P(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$$

- In other words,  $\sigma$  is transposing polynomials.
  - Example:  $(12)(x_1^2 + x_2^3 + x_3) = x_2^2 + x_1^3 + x_3$ .
- Thus, we call as  $A_1$  the representation  $V_{\text{perm}}^*$ .
  - This is because  $A_1 = \text{span}(x_1 + \dots + x_n)$ , and permuting these is much like permuting the basis of a vector space, as the typical permutation representation does.
  - It could technically be the isomorphic representation  $V_{\text{perm}}$ , but the dual fits better here for reasons??
- Then  $A_2 = S^2 V^*$ .
  - So if  $A_1$  had basis  $e^1, \dots, e^m$ ,  $A_2$  has basis  $\{e^i e^j\}$ .
  - Why are we choosing these sets??
- Continuing,  $A_3 = S^3 V^*$ .
- It follows that the representation of the overall thing is

$$\bigoplus_{m \geq 0} (S^m V_{\text{perm}}^*)$$

- This is called the **symmetric algebra**.
- **Graded** (algebra): An algebra for which the underlying additive group is a direct sum of abelian groups  $A_i$  such that  $A_i A_j \subset A_{i+j}$ .
- **Degree** (of a monomial  $P$ ): The sum of the powers of its variables. *Denoted by  $d(P)$* .
  - Example: If  $P = x_1^3 x_2^4$ , then  $d(P) = 2 + 3 = 5$ .
- **Degree** (of a polynomial  $P$ ): The greatest degree of each of its monomials. *Denoted by  $d(P)$* .
- So how do we construct representations?
  - For  $S_2$ ,  $x_1 - x_2$  changes sign when we apply  $S_2$ .
  - For  $S_3 \dots$ 
    - The trivial's polynomial is 1 and  $\square\square\square$ .
    - The standard is  $(2, 1)$ . When we apply  $S_3$  to  $(x_1 - x_2)$ , we get

$$\langle (x_1 - x_2), (x_2 - x_1), (x_1 - x_3), (x_3 - x_1), (x_2 - x_3), (x_3 - x_2) \rangle$$

- If we let  $a = x_1 - x_2$ ,  $b = x_2 - x_3$ , then the third element equals  $a + b$ .
- Similarly, the the second element equals  $-a$ , the fourth element equals  $-a - b$ , and the sixth element equals  $-b$ .
- This is another way to think about the action.
- What about the alternating representation? We have  $(x_1 - x_2)(x_2 - x_3)(x_1 - x_3) = \Delta_{123}$ , which changes sign when we use any element with sign  $-1$  to permute the  $x_i$ !
- For  $S_4 \dots$ 
  - $(4)$  is 1.
  - $(3, 1)$  is  $S_4(x_1 - x_2) = \Delta_{12}$ .
  - $(1, 1, 1, 1)$  is  $(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4) = \Delta_{1234}$ .
  - $(2, 1, 1)$  is  $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$ .
    - We got this polynomial by guessing; the same way  $(x_1 - x_2)$  worked in multiple cases, maybe this one does too! And it does.
    - Something to check is that  $\Delta_{123} - \Delta_{124} - \Delta_{134} - \Delta_{234} = 0$ .
  - $(2, 2)$  is  $(x_1 - x_2)(x_3 - x_4)$ .
    - Something related we can prove is that
$$(x_1 - x_2)(x_3 - x_4) - (x_1 - x_3)(x_2 - x_4) - (x_1 - x_4)(x_2 - x_3) = 0$$
    - This formula appears in **cross ratios**, which we can discuss in Rudenko's algebraic geometry course next quarter.
  - For  $\lambda = (4, 3, 1)$ , we have  $\Delta_{123}\Delta_{45}\Delta_{67}$ , and we act by  $S_8$  upon this! Explicitly, we have  $S_8(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_4 - x_5)(x_6 - x_7)$ .
  - Takeaway: It all depends on column length!
- These polynomials are called **Vandermonde determinants**; those are the little  $\Delta$  things with subscripts. We'll talk about these next times.
- We need to prove reducibility and not pairwise isomorphic to make sure that this construction is valid, but that's easy! We'll do this next Monday.

## 7.2 Symmetric Polynomials; Vandermonde Determinants

11/8:

- Announcements.
  - OH tonight at 6:00 PM.
- Consider  $S_n$ .
  - Recall the symmetric algebra  $R = \mathbb{Q}[x_1, \dots, x_n]$ , which is a graded ring  $\bigoplus_{d \geq 0} R_d$  where  $R_d = S^d V_{\text{perm}}^*$ .
  - The action of  $S_n \subset \mathbb{Q}[x_1, \dots, x_n]$  is  $\sigma P(x_1, \dots, x_n) = P(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$ .
- With the definitions of last class behind us, we can now look at the **space of invariants**  $R^{S_n}$ , isotypical components of which  $\sigma$  acts on trivially.
$$R^{S_n} := \{P(x_1, \dots, x_n) \mid \forall \sigma \in S_n, P(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = P(x_1, \dots, x_n)\}$$
  - This is the ring of symmetric polynomials.
  - Example: If  $n = 3$ , then  $x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3 \in \mathbb{Z}^{S_3}$ .
- We now define some stuff to help us prove a major result: The  $n$  elementary symmetric polynomials.
  - $\sigma_1 = x_1 + \dots + x_n = \sum_{1 \leq i \leq n} x_i$ .

- $\sigma_2 = x_1x_2 + \cdots + x_1x_n + x_2x_3 + \cdots = \sum_{1 \leq i < j \leq n} x_i x_j$ .
- $\sigma_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}$ .
- $\sigma_n = x_1 \cdots x_n$ .

- With these definitions, we can say that

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = (x - x_1) \cdots (x - x_n)$$

where

$$a_{n-1} = -\sigma_1 \qquad a_{n-2} = \sigma_2 \qquad \dots \qquad a_0 = (-1)^n \sigma_n$$

- Intro to the fundamental theorem of symmetric polynomials.
  - Basically: Every symmetric polynomial is a polynomial in the elementary symmetric polynomials.
  - A more precise statement follows.

- Theorem (Fundamental theorem of symmetric polynomials): We have that

$$\mathbb{Q}[x_1, \dots, x_n]^{S_n} = \mathbb{Q}[\sigma_1, \dots, \sigma_n]$$

- Before we prove the fundamental theorem, there are a few points we need to discuss.

- Example:

- Consider the polynomial  $x^2 + px + q = 0$ .
- Let  $x_1, x_2$  denote its roots. Then  $\sigma_1 = x_1 + x_2 = -p$  and  $\sigma_2 = x_1x_2 = q$ .
- We may observe that  $x_1^2 + x_2^2 = \sigma_1^2 - 2\sigma_2 = p^2 - 2q$ .
- We may observe that  $(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2 = \sigma_1^2 - 4\sigma_2 = p^2 - 4q$ , where the rightmost term is the discriminant of the original polynomial.
- In general,  $x_1^n + x_2^n$  has an expression as a polynomial in  $\sigma_1, \sigma_2$ . This will be a homework problem.
- What is going on here?? Is  $x^2 + px + q$  even in  $\mathbb{Q}[x]^{S_n}$ ? If so, why do we factor it into  $\sigma_1, \sigma_2$  instead of just  $\sigma$ ? What are the other examples about?

- **Lexicographic order** (on monomials): An ordering of monomials based on the following rule. *Denoted by  $\succ$ . Given by*

$$x_1^{a_1} \cdots x_n^{a_n} \succ x_1^{b_1} \cdots x_n^{b_n} \dots$$

1. If  $a_1 > b_1$  OR...
2. If  $a_1 = b_1$  and  $a_2 > b_2$  OR...
3.  $a_1 = b_1$  and  $a_2 = b_2$  and  $a_3 > b_3$  OR...
4. So on and so forth.

- Notes on the lexicographic ordering.

- Don't think of this order like an ordering on integers.
- This allows us to define the key notion for a number of proofs we'll see in the coming days.
- Although it may seem counterintuitive, the lexicographic ordering is still determined for polynomials such as  $\sigma_1$ . For example, we may look at

$$\sigma_1 = x_1 + \cdots + x_n$$

and think, "Wait a second — all these terms have the same order: They all have the same exponent of 1." However, we would be discounting the fact that the lexicographic ordering views  $\sigma_1$  as

$$\sigma_1 = x_1^1 x_2^0 \cdots x_n^0 + \cdots + x_1^0 \cdots x_{n-1}^0 x_n^1$$

From here, we can see that  $LM(\sigma_1) = x_1^1 x_2^0 \cdots x_n^0 = x_1$ .

- **Largest monomial** (of  $P \neq 0$ ): The monomial in  $P(x_1, \dots, x_n) \neq 0$  that is the largest lexicographically. Denoted by  $LM(P)$ .
- $C_{LM}(P)$ : The coefficient of  $LM(P)$ .
- Example: Consider the polynomial  $P = x_1^2 + x_1^3 x_2 x_3 - 7x_1^3 x_2 x_3^{100}$ .
  - Then  $LM(P) = x_1^3 x_2 x_3^{100}$  and  $C_{LM}(P) = -7$ .
- Properties.
  1.  $P, Q \neq 0$  implies that  $LM(PQ) = LM(P)LM(Q)$ .
    - Using inductive reasoning, try multiplying the example above by  $Q = x_1^2 + x_2^2 + x_3^2$ !
    - Rudenko will not give rigorous proofs of any of these properties; they will just confuse us. It's better to do everything intuitively here.
- Lemma: If  $P \in \mathbb{Q}[x_1, \dots, x_n]^{S_n}$  and  $LM(P) = x_1^{a_1} \cdots x_n^{a_n}$ , then  $a_1 \geq \cdots \geq a_n$ .

*Proof.* Let  $i < j$ . Suppose for the sake of contradiction that  $a_i < a_j$ . Let  $\sigma = (ij) \in S_n$ . Since  $P$  is symmetric,  $\sigma P = P$ . But then in particular, the monomial  $\sigma LM(P)$  in  $P$  is lexicographically larger than  $LM(P)$ . Thus,  $LM(P)$  is not the lexicographically largest monomial in  $P$ , a contradiction.  $\square$

- Here's a simple example to illustrate the idea behind this proof: Let  $P = x^2 y + xy^2 \in \mathbb{Q}[x, y]^{S_n}$ . Suppose we pick  $LM(P) = xy^2$  (obviously this is the wrong choice, but that's the contradiction we'll see). We observe that  $2 = a_2 > a_1 = 1$  in this case. Let  $\sigma = (12)$ . Then  $\sigma LM(P) = yx^2 = x^2 y \succ xy^2 = LM(P)$ . So  $\sigma LM(P) \succ LM(P)$ . Thus,  $LM(P)$  is not the lexicographically largest monomial in  $P$ , and we have *formally* proven that our initial choice of  $LM(P)$  was incorrect.
- We now have everything we need to prove the fundamental theorem. As such, we will restate and prove it.
- Theorem (Fundamental theorem of symmetric polynomials): We have that

$$\mathbb{Q}[x_1, \dots, x_n]^{S_n} = \mathbb{Q}[\sigma_1, \dots, \sigma_n]$$

*Proof.* We will prove this theorem using the well-ordering principle (every set of natural numbers has a smallest element), which is equivalent to induction. Let's begin.

Suppose for the sake of contradiction that there exists a symmetric polynomial that cannot be expressed via  $\sigma_1, \dots, \sigma_n$ . Given this counterexample, factor out as many terms as we want (successively reducing the degree) until it ceases to be a counterexample, thus yielding the counterexample of smallest degree. Similarly, get to the counterexample with smallest  $LM$ . Call this counterexample  $P(x_1, \dots, x_n)$ . Let

$$P(x_1, \dots, x_n) = C_{LM}(P) \underbrace{x_1^{a_1} \cdots x_n^{a_n}}_{LM(P)} + \text{smaller monomials}$$

Since  $P$  is symmetric and the term above is the lexicographically largest monomial, the Lemma implies that  $a_1 \geq \cdots \geq a_n$ . We now construct a polynomial  $Q$  out of the  $\sigma_i$  such that  $LM(P) = LM(Q)$ . To begin, note that

$$LM(\sigma_1) = x_1 \quad LM(\sigma_2) = x_1 x_2 \quad \cdots \quad LM(\sigma_n) = x_1 \cdots x_n$$

Now consider  $\sigma_n^{a_n}$ . This clearly divides  $LM(P)$  since  $a_n$  is minimal. Now multiply by  $\sigma_{n-1}^{a_{n-1} - a_n}$ . Continuing on, we get

$$Q = \sigma_n^{a_n} \sigma_{n-1}^{a_{n-1} - a_n} \sigma_{n-2}^{a_{n-2} - a_{n-1}} \cdots \sigma_1^{a_1 - a_2}$$

Now it follows that

$$LM(P - C_{LM}(P) \cdot Q) \prec LM(P)$$

Since  $C_{LM}(P) \in \mathbb{Q}$  and  $Q \in \mathbb{Q}[x_1, \dots, x_n]^{S_n}$ , it also follows that  $P - C_{LM}(P) \cdot Q \in \mathbb{Q}[x_1, \dots, x_n]^{S_n}$ . But then by the assumption that  $P$  was the counterexample with smallest  $LM$ , we know that  $P - C_{LM}(P) \cdot Q \in \mathbb{Q}[\sigma_1, \dots, \sigma_n]$ . It follows that  $P \in \mathbb{Q}[\sigma_1, \dots, \sigma_n]$ , a contradiction.  $\square$

- Note: This is an effective proof; we can write an algorithm to do this for us, and it's actually pretty fast and efficient.
- Remain to show:  $\sigma_1, \dots, \sigma_n$  are algebraically independent  $P(\sigma_1, \dots, \sigma_n) = 0$  implies that  $P = 0$ .
  - This will be a homework problem; hint, it's pretty easy.
- Back to representation theory.

- **Antisymmetric** (polynomial): A polynomial  $P(x_1, \dots, x_n)$  such that

$$\sigma P = (-1)^\sigma P$$

- Example.

- $n = 2$ :  $x_1 - x_2$ .
- $n = 3$ :  $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$ .

- These main examples are the **Vandermonde determinant** from last time!
- **Vandermonde determinant**:

$$\Delta(x_1, \dots, x_n) = \prod_{1 \leq i < j \leq n} (x_j - x_i)$$

- Exercise:  $\Delta(x_1, \dots, x_n)$  is antisymmetric.
- One of the nicest definitions of sign comes from these determinants!

$$(-1)^\sigma = \frac{\prod_{i < j} (x_{\sigma(i)} - x_{\sigma(j)})}{\prod_{i < j} (x_i - x_j)}$$

- Theorem: If  $P \in \mathbb{Q}[x_1, \dots, x_n]^{\text{alt}}$  (i.e.,  $P \in \mathbb{Q}[x_1, \dots, x_n]$  and  $P$  is antisymmetric), then  $P = P' \Delta(x_1, \dots, x_n)$  where  $P'$  is symmetric (i.e.,  $P' \in \mathbb{Q}[x_1, \dots, x_n]^{S_n}$ ).
- Corollary: If  $P$  is antisymmetric and  $\deg(P) < n(n-1)/2$ , then  $P = 0$ .
  - We'll use this many times, this fact that "antisymmetric polynomials have a smallest possible degree."
- We now prove the Theorem.

*Proof.* Let  $P$  be antisymmetric. Then  $(12)P = -P$ . It follows that  $P(x_1, \dots, x_n)|_{x_1=x_2} = 0$ . Now, rewrite  $P$  as a polynomial in one variable where all of the coefficients are polynomials in other variables. In particular, let

$$P = P_d(x_1 - x_2)^d + P_{d-1}(x_1 - x_2)^{d-1} + \dots + P_0$$

where each  $P_i \in \mathbb{Q}[x_1, \dots, x_d]$ . What is  $d$ ? (Less than  $n$ , I'm assuming, but any other constraints?) Plugging in  $x_1 = x_2$  once again, we get  $0 = P = P_0$ . But this implies that  $P_0 = 0$ . Thus,  $P$  is divisible by  $x_1 - x_2$ . Similarly, for all  $i < j$ ,  $(x_i - x_j) \mid P$ . But since the  $x_i - x_j$  are irreducible polynomials, we have that  $\prod_{i < j} (x_j - x_i) \mid P$ . This is justified because we are in a unique factorization domain (how is this relevant?). Thus, we have that  $P = P' \cdot \Delta(x_1, \dots, x_n)$ . Lastly, it follows that  $P' \in \mathbb{Q}[x_1, \dots, x_n]^{S_n}$  because under any sign  $-1$  permutation,  $\Delta(x_1, \dots, x_n)$  will flip signs and  $P$  will still be equal, so  $P'$  had better just stay itself under this permutation (i.e., be symmetric).  $\square$

- Remark: Where does the name Vandermonde *determinant* come from?



- We have that

$$\Delta(x_1, \dots, x_n) = \begin{vmatrix} 1 & 1 \\ x_1 & x_n \\ \vdots & \vdots \\ x_1^{n-1} & x_n^{n-1} \end{vmatrix}$$

- Final reminder before the final.
  - Don't forget our awesome central construction!
  - If  $\lambda = (\lambda_1, \dots, \lambda_k)$  is a partition with  $\lambda_1 \geq \dots \geq \lambda_k$ , then we can draw a Young Diagram and construct an associated representation  $V_\lambda \in \mathbb{C}[x_1, \dots, x_n]$ .
  - But what we do is  $V_\lambda = \mathbb{C}[S_n]\Delta_\lambda$ , where

$$\Delta_{\lambda'} = \Delta(x_1, \dots, x_{\lambda'_1})\Delta(x_{\lambda'_1+1}, \dots, x_{\lambda'_1+\lambda'_2}) \cdots$$

- Example: For  $\lambda = (2, 2, 1)$ , we have  $V_{(2,2,1)} = \mathbb{C}[S_n](x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_4 - x_5)$ .
- Next time, we'll prove that  $V_{(2,2,1)}$  is irreducible.
- This Specht construction is in a tiny footnote of Fulton and Harris (2004), but that's about it!

## 7.3 Midterm Review Sheet

11/10:

- The following definitions and results will be useful in solving the midterm problems.
- **Group representation:** A group homomorphism  $\rho : G \rightarrow GL(V)$  for  $G$  a finite group,  $V$  a finite-dimensional vector space over some field  $\mathbb{F}$  with basis  $\{e_1, \dots, e_n\}$ , and  $GL(V)$  the set of isomorphic linear maps  $L : V \rightarrow V$ .
- **Morphism** (of  $G$ -representations): A map  $f : V \rightarrow W$  such that...
  1.  $f$  is linear;
  2. For every  $g \in G$ ,  $\rho_W(g) \circ f = f \circ \rho_V(g)$ .
    - To remember this rule, draw out the commutative diagram!
- Theorem (complete reducibility): Any finite-dimensional representation can be decomposed into a direct sum of irreducible representations via

$$V = V_1^{n_1} \oplus \dots \oplus V_k^{n_k}$$

- Lemma (Schur's Lemma): Let  $G$  be a finite group, let  $V, W$  be irreducible representations over  $\mathbb{C}$ , and let  $f \in \text{Hom}_G(V, W)$ . Then...
  1. If  $V \not\cong W$ , then  $f = 0$ . If  $V \cong W$ , then  $f$  is an isomorphism of  $G$ -representations.
  2. If  $f : V \rightarrow V$ , then  $f(v) = \lambda v$ .
- **Algebraic integer:** A number  $x \in \mathbb{C}$  for which there exist  $a_0, \dots, a_{n-1} \in \mathbb{Z}$  such that

$$x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$$

- **Character** (of  $\rho$ ): The function  $\chi_\rho : G \rightarrow \mathbb{C}$  defined by

$$\chi_\rho(g) = \text{tr}(\rho(g))$$

- **First orthogonality relation:** If  $\chi_1, \chi_2$  are the characters of irreducible representations, then

$$\sum_{g \in G} \chi_1(g) \overline{\chi_2(g)} = \begin{cases} 0 & \chi_1 \neq \chi_2 \\ |G| & \chi_1 = \chi_2 \end{cases}$$

- Follows from the fact that the characters form an orthonormal set within the space of class functions, and the definition of the inner product on this space.

- **Second orthogonality relation:** If  $C_G(g)$  is the number of elements in the conjugacy class of  $g$ , then

$$\sum_{\chi} \chi(g_1) \overline{\chi(g_2)} = \begin{cases} 0 & g_1 \not\sim g_2 \\ \frac{|G|}{|C_G(g_1)|} & g_1 \sim g_2 \end{cases}$$

- **Permutational representation:** The representation  $\rho : S_n \rightarrow \mathbb{C}^n (= V_{\text{perm}})$  defined by

$$\rho(\sigma) : (x_1, \dots, x_n) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

- Character: Compute  $\text{Fix}(\sigma)$  for each type of  $\sigma$ .

- **Class function:** A function that is constant on the conjugacy classes of  $G$ . Explicitly, for all  $s, t \in G$ ,

$$f(tst^{-1}) = f(s)$$

- **Group algebra:** The complex vector space with basis  $\{e_g\}$  corresponding to the elements of group  $G$ , plus the definition  $e_g \cdot e_h = e_{gh}$ .

- **Semisimple module:** A module  $M$  that satisfies any of the following three conditions.

1.  $M = \bigoplus_{i \in I} S_i$ , where each  $S_i$  is a simple module and  $I$  is an indexing set.
2.  $M = \sum_{i \in I} S_i$ .
3. For all submodules  $N \subset M$ , there exists  $N'$  such that  $M = N \oplus N'$ .

- Additional notes on semisimple modules.

- Simple module: A module that is nonzero and has no nonzero proper submodules.

- **Division algebra:** An algebra  $D$  such that for all nonzero  $x \in D$ , there exists a  $y \in D$  such that  $xy = 1$ .

- **Semisimple algebra:** An algebra for which every finite-dimensional  $A$ -module is semisimple.

- **Wedderburn-Artin theorem:** If  $A$  is a finite-dimensional semisimple associative algebra, then

$$A \cong M_{n_1}(D_1) \oplus \dots \oplus M_{n_k}(D_k)$$

- **Schur's Lemma** (over an arbitrary field): Let  $A$  be a finite-dimensional algebra, and let  $M_1, M_2$  be simple  $A$ -modules. Then...

1. If  $f : M_1 \rightarrow M_2$  is a nonzero morphism of  $A$ -modules,  $f$  is isomorphic;
2. If  $M$  is simple,  $\text{Hom}_A(M, M)$  is a division algebra.

- **Center** (of  $A$ ): The following set.

$$Z(A) = \{a \in A \mid xa = ax \ \forall x \in A\}$$

- **Jacobson radical:** The finite-dimensional  $A$ -algebra defined as follows.

$$\text{Rad}(A) = \{a \in A \mid aS = 0 \text{ for any simple module } S\}$$

- Here's an outline of what to remember for the problems.
- Strategies for computing the following things from the character table of a group.

## 1. Tensor products.

- Multiply corresponding characters.

## 2. Wedge/symmetric squares.

- If  $\chi$  is the character of a representation  $\rho : G \rightarrow GL(V)$ , then the characters  $\chi_\sigma^2$  of the symmetric square  $S^2V$  of  $V$  and  $\chi_\alpha^2$  of the alternating square  $\Lambda^2V$  of  $V$  are given by the following for each  $s \in G$ .

$$\chi_\sigma^2(s) = \frac{1}{2} (\chi(s)^2 + \chi(s^2)) \qquad \chi_\alpha^2(s) = \frac{1}{2} (\chi(s)^2 - \chi(s^2))$$

- Note that just like  $V^{\otimes 2} = S^2V + \Lambda^2V$ , we have  $\chi^2 = \chi_\sigma^2 + \chi_\alpha^2$ .

## 3. Decomposing permutational representations into irreducibles.

- If the representation of interest is  $\chi_V$ , we find the coefficients  $n_i$  of its decomposition

$$\chi_V = \sum n_i \chi_{V_i}$$

via the inner product

$$n_i = \langle \chi_V, \chi_{V_i} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \overline{\chi_{V_i}(g)}$$

## 4. More.

- Strategies for computing the following things given a small group (e.g., the quaternion group).

## 1. Conjugacy classes.

- Take an element, conjugate it by everything, round up the products. Then move onto another elements.

## 2. Character table.

- Find the conjugacy classes and put them at the top of the table. This also tells us how many irreps we need to get to.
- Start with the trivial, alternating, and standard representations.
- Tensor products of representations with 1D representations (e.g., the alternating) are often linearly independent.
- We can recover the standard as the difference of the permutation and trivial.
- We can solve for representations as components of the regular representation via

$$V_R = \bigoplus_{i=1}^k V_i^{\dim V_i}$$

- We can calculate the degrees of remaining representations via the sums of the squares of the dimensionalities.
- We can fill in final representations with the orthogonality relations, *especially* the second one.

## 3. Decomposing representations into a sum of isotypical components.

- Use the inner product/decomposition formula from above.

## 4. Diagonalizing an endomorphism.

- Start with its matrix  $A$ .
- Find the characteristic polynomial by computing  $\det(A - \lambda I)$ .
- Solve for the eigenvalues.
- Find, by inspection or by solving systems of equations, elements of the null space of  $A - \lambda I$  for each  $\lambda$ . Beware eigenvalues with multiplicity greater than one!

## 5. More.

- Strategies for solving an abstract problem about characters.
  - *reread notes*
- Strategies for solving an abstract problem about representations.
  - *reread notes*
- Other misc. concepts that are probably good to remember (my own ideas).
- **Left  $A$ -module:** A pair  $(M, \rho)$  where  $(M, +)$  is an abelian group and  $\rho : A \rightarrow \text{End}(M)$  is the ring homomorphism defined as follows, where  $A$  is a ring: For all  $a \in A$ ,  $\rho(a) : M \rightarrow M$  is given by  $\rho(a)v = av$  for all  $v \in M$  and satisfies the following constraints.
  1.  $\rho(a) : M \rightarrow M$  is a group homomorphism on  $(M, +)$ .
  2.  $\rho$  is a ring homomorphism.
    - That is to say,  $\rho(a + b) = \rho(a) + \rho(b)$ ,  $\rho(ab) = \rho(a)\rho(b)$ , and  $\rho(1_A) = 1_{\text{End}(M)}$ .
- Lemma (Gauss's Lemma): If  $f, g \in R[X]$  are both nonzero polynomials with coefficients in the ring  $R$ , then  $c(fg) = c(f)c(g)$ .
  - Note that  $c(f)$  denotes the **content** of  $f$ , which is the gcd of its coefficients.
  - Use: If  $p$  is reducible in a fraction field, then it's reducible in the native UFD.
- $A$  is semisimple iff  $\text{Rad}(A) = 0$ .
- Formulas for the decomposition of the regular representation/misc. formulas from IChem.
  - Sum of the squares of the dimensionalities (from the second orthogonality relation):

$$|G| = \sum_{i=1}^k (\dim V_i)^2$$

- Sum of the squares of an irrep's characters (from the first orthogonality relation):

$$|G| = \sum_{g \in G} \chi(g)^2$$

## 7.4 Midterm

1. **(30)** Here is the character table of the group  $S_4$ .

Representation	(1)(2)(3)(4)	(12)(34)	(12)(3)(4)	(1234)	(123)(4)
(4)	1	1	1	1	1
(1, 1, 1, 1)	1	1	-1	-1	1
(2, 2)	2	2	0	0	?
(3, 1)	3	?	1	-1	0
(2, 1, 1)	3	-1	-1	1	0

- (a) State two orthogonality relations for characters of a general group  $G$ . Apply them to fill in holes in the table above.
  - (b) Compute the character of  $S^2(2, 2)$  and decompose it into irreducibles.
  - (c) Compute the character of  $(3, 1) \otimes (2, 2)$  and decompose it into irreducibles.
2. **(40)** Consider the group  $G$  of symmetries of a square (it has size 8).

- (a) Find the conjugacy classes of  $G$ .
  - (b) The action of  $G$  on the plane (by symmetries of a square) defines a two-dimensional complex representation  $V$  of  $G$ . Find the character of  $V$ . Prove that  $V$  is irreducible.
  - (c) Compute the character table of  $G$ .
  - (d) Consider the action of  $G$  on the set of functions on edges of the square. This defines a four-dimensional representation of  $G$ . Find its characters and decompose it into isotypical components.
3. **(15)** Suppose that for a finite group  $G$  all irreducible representations are one-dimensional. Prove that  $G$  is abelian.
4. **(15)** State Schur's lemma for complex representations of a finite group. Assume that  $V, W$  are two distinct complex-dimensional irreducible representations of a finite group  $G$ . Find the dimension of the space  $\text{Hom}_G(V \oplus W, V \oplus W)$ .