Week 8

Symmetric Group Representation Formulations

8.1 Specht Modules are Irreducible and Well-Defined

11/13:
- Announcements.
  - This week's homework is the next to last one.
- Review.
  - Miracuously, we can understand all representations of S_n.
  - We start with partitions λ that are defined a certain way. We visualize them with Young diagrams.
  - The number of partitions of n is equal to the number of conjugacy classes in S_n is equal to the number of irreps in S_n.
    - It is a special feature of S_n that this is true.
  - How do we construct the irreducible representation V_λ due to λ?
    - Consider $(4,2,1)' = (3,2,1,1)$ as an example (recall the definition of an inverse partition).
    - Take Vandermonde determinants (recall the explicit definition of these, too) of column elements and multiply them together.
    - Then we define V_λ[1] to be the subspace of the polynomial space $\mathbb{C}[x_1,\dots,x_n]$ that is spanned by the Vandermonde determinants polynomial and all actions of S_n on it.
    - In particular, take Vandermonde determinant of variables corresponding to the successive columns to obtain

$$\Delta(x_1,\dots,x_{\lambda_1'})\Delta(x_{\lambda_1'+1},\dots,x_{\lambda_2'})\cdots\Delta(x_{\lambda_{k-1}'+1},\dots,\lambda_k')$$

    - Thus, in our specific example, we let $\mathbb{C}[S_n]$ act on $(x_1-x_2)(x_1-x_3)(x_2-x_3)(x_4-x_5)$.
- One more example.
  - $\lambda=(2,2)$.
  - Let $\mathbb{C}[S_4]$ act on $(x_1-x_2)(x_3-x_4)$.
  - Then

$$V_\lambda = \langle (x_1-x_2)(x_3-x_4),(x_1-x_3)(x_2-x_4),(x_1-x_4)(x_2-x_3)\rangle$$

[1]Something about $V_\lambda = \mathbb{C}[S_n]$?? I though we were "just say NO!"-ing this usual construction, though?

- But we're expecting a 2D representation. Indeed, we get one because if we define the first term above to be a and the second to be b, then the third is $b-a$:

$$\begin{aligned}
b-a &= [(x_1-x_3)(x_2-x_4)] - [(x_1-x_2)(x_3-x_4)] \\
&= [x_1x_2 - x_1x_4 - x_2x_3 + x_3x_4] - [x_1x_3 - x_1x_4 - x_2x_3 + x_2x_4] \\
&= x_1x_2 + x_3x_4 - x_1x_3 - x_2x_4 \\
&= (x_1-x_4)(x_2-x_3)
\end{aligned}$$

- Consequence of the above: There are only two linearly independent polynomials herein.
- Thus, the final Specht module is

$$V_\lambda = \langle \underbrace{(x_1-x_2)(x_3-x_4)}_{a}, \underbrace{(x_1-x_3)(x_2-x_4)}_{b} \rangle$$

- Now we calculate entries in the character table as follows: See how representatives of conjugacy classes like (12) and (123) acts on a, b via matrices, and then calculate traces of these matrices.
  - For example, using the definitions of a, b from above, we can see that

$$\begin{aligned}
(12)\cdot a &= (12)\cdot(x_1-x_2)(x_3-x_4) = (x_2-x_1)(x_3-x_4) = -(x_1-x_2)(x_3-x_4) = -a \\
(12)\cdot b &= (12)\cdot(x_1-x_3)(x_2-x_4) = (x_2-x_3)(x_1-x_4) = (x_1-x_4)(x_2-x_3) = b-a
\end{aligned}$$

  - In matrix form, the above equations become

$$\begin{bmatrix} -a \\ b-a \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}}_{\rho(12)} \begin{bmatrix} a \\ b \end{bmatrix}$$

  - Thus, $\chi(12) = \operatorname{tr}[\rho(12)] = 0$.
  - Similarly, we can calculate that

$$\begin{bmatrix} b-a \\ -a \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}}_{\rho(123)} \begin{bmatrix} a \\ b \end{bmatrix}$$

    so $\chi(123) = -1$.
- One of the HW problems is to do exactly this for S_4 just for practice.

- Today: A theorem that proves the V_λ are irreducible (as our notation would suggest), and a theorem that proves the V_λ are well-defined (i.e., only one V_λ can be constructed from each λ).
- Note: V_λ is a **Specht module** and the original polynomial constructed is the **Specht polynomial**.
- We now build up to the theorems.
- **Degree** (of a Specht polynomial): The degree of the given polynomial as defined in Lecture 7.1. *Denoted by* $\mathbf{d(\lambda)}$. *Given by*

$$d(\lambda) = \sum_{i=1}^{k'} \frac{\lambda_i'(\lambda_i'-1)}{2}$$

- Let R_d denote the subset of $\mathbb{C}[x_1, \dots, x_n]$ consisting of polynomials of degree d.
  - Recall also that as a definition, $R_d = S^d(V_{\text{perm}}^*)$.
  - Note that if $d = d(\lambda)$, then $V_\lambda \subset R_d$.

- Lemma: Let λ be a partition of n, $d = d(\lambda)$, R_d be the subset of $\mathbb{C}[x_1, \dots, x_n]$ consisting of polynomials of degree d, and V_λ be the Specht module of λ. Then

$$\operatorname{Hom}_{S_n}(V_\lambda, R_d) \cong \mathbb{C}$$

Proof. Let $f \in \operatorname{Hom}_{S_n}(V_\lambda, R_d)$ be arbitrary, let $\Delta(x_1, \dots, x_{\lambda_1'})\Delta(x_{\lambda_1'+1}, \dots, x_{\lambda_2'})\dots$ be an arbitrary element of V_λ, and let

$$P(x_1, \dots, x_n) := f(\Delta(x_1, \dots, x_{\lambda_1'})\Delta(x_{\lambda_1'+1}, \dots, x_{\lambda_2'})\dots)$$

By definition, the degree of P is d. Additionally, because f is a morphism of S_n-representations as an element of $\operatorname{Hom}_{S_n}(V_\lambda, R_d)$, we have that f is linear and hence, since the argument of f is antisymmetric in $x_1, \dots, x_{\lambda_1'}$, so P similarly antisymmetric. P is also antisymmetric in $x_{\lambda_1'+1}, \dots, x_{\lambda_2'}$. In fact, P is antisymmetric in all such sets all the way up to $x_{\lambda_{k'-1}'+1}, \dots, x_{\lambda_{k'}'}$. It follows that $P(x_1, \dots, x_n)$ is divisible by $\Delta(x_1, \dots, x_{\lambda_i'})$, etc., i.e., all Vandermonde determinants. Thus, $P(x_1, \dots, x_n)$ is divisible by the product, which is the d-degree Specht polynomial argument of f. It follows that

$$P(x_1, \dots, x_n) = u \cdot \Delta(x_1, \dots, x_{\lambda_1'})\Delta(x_{\lambda_1'+1}, \dots, x_{\lambda_2'})\dots$$

from which it follows that $f = uI$. This implies the claim via the isomorphism $f \mapsto u$. $\square$

- Theorem 1: V_λ is irreducible.

Proof. Let $V_\lambda = \bigoplus W_i^{n_i}$ and $R_d = \bigoplus W_i^{m_i}$, where the W_i are all irreps. From previous classes, we have a nice way to compute a morphism of S_n-representations $V_\lambda \to R_d$: Explicitly, we apply Schur's lemma to find that the only acceptable constituent morphisms are those which send $W_i \to W_i$. Thus, $\dim \operatorname{Hom}_{S_n}(V_\lambda, R_d) = \sum n_i m_i$. (Any transformation from $W_i^{n_i}$ to $W_i^{m_i}$ has the form of a $m_i \times n_i$-blocked matrix, so there are $n_i m_i$ degrees of freedom.) But by the lemma, $\dim \operatorname{Hom}_{S_n}(V_\lambda, R_d) = 1$. Additionally, since we are in a subrepresentation, i.e., $V_\lambda \subset R_d$, we have that $n_i \leq m_i$ for all i. Thus, we must have $n_i = 1, m_i = 1$ for some i and that $n_j, m_j = 0$ for all other j. This means that

$$V_\lambda = W_1^0 \oplus \dots \oplus W_{i-1}^0 \oplus W_i^1 \oplus W_{i+1}^0 \oplus \dots \oplus W_k^0 = W_i$$

Therefore, since it is equal to an irrep, V_λ is irreducible. $\square$

- Corollary: If $d' < d$, then $\operatorname{Hom}(V_\lambda, R_{d'}) = 0$.
- Theorem 2: Let λ_1, λ_2 be partitions of n. Then $V_{\lambda_1} \cong V_{\lambda_2}$ iff $\lambda_1 = \lambda_2$.

Proof. We will prove both directions independently here. Let's begin.

($\Rightarrow$): Suppose that $V_{\lambda_1} \cong V_{\lambda_2}$.

Then $d(\lambda_1) = d(\lambda_2)$. We can see this two ways. First and most obviously, take the columns of each Young diagram and compute the degree of the Specht polynomial. Second and more formally, suppose for the sake of contradiction that $d(\lambda_1) \neq d(\lambda_2)$. WLOG let $d(\lambda_1) < d(\lambda_2)$. Then $V_{\lambda_2} \cong V_{\lambda_1} \hookrightarrow R_{d(\lambda_1)}$. But then by the above corollary, this ostensibly injective embedding is the zero map, a contradiction.

Let $d := d(\lambda_1) = d(\lambda_2)$. At this point, we have $V_{\lambda_1} \hookrightarrow R_d$ and $V_{\lambda_2} \hookrightarrow R_d$. It follows that $V_{\lambda_1} = V_{\lambda_2}$ as subspaces of R_d. Essentially, since we have the isomorphism $V_{\lambda_1} \cong V_{\lambda_2}$, we can construct the second embedding by factoring through the first, but then this second embedding should just give the same image. The factorization would look something like Figure 8.1.

$$\begin{tikzcd} V_{\lambda_1} \arrow[dr, hook] & \\ \rotatebox{90}{$\cong$} & R_d \\ V_{\lambda_2} \arrow[ur, hook] & \end{tikzcd}$$

Figure 8.1: Specht modules are equal as subspaces of R_d.

We now show that the polynomials in $V_{\lambda_1}, V_{\lambda_2}$ (which we can think of as subspaces/explicit polynomials) have no monomials in common[2]. For this, it's enough to understand monomials in one V_{λ_1}. Which monomials appear in V_λ? Here's an example. We will do a representative example instead of a formal proof. Consider $\lambda = (5,4,2,2)$ and S_{13}. $\lambda' = (4,4,2,2,1)$. Our Specht polynomial is

$$\Delta(x_1,x_2,x_3,x_4)\Delta(x_5,x_6,x_7,x_8)\Delta(x_9,x_{10})\Delta(x_{11},x_{12})$$

since $\Delta(x_{13}) = 1$. We also have that

$$\Delta(x_1,x_2,x_3,x_4) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{vmatrix} = \sum_{\sigma \in S_4} (-1)^\sigma x_{\sigma(1)}^0 x_{\sigma(2)}^1 x_{\sigma(3)}^2 x_{\sigma(4)}^3$$

Then for each column, we will have a number of variables in each power from $0,\dots,3$. Now we multiply out the individual Vandermonde determinants and count the number of variables in power $0,\dots,3$ to get 5,4,2,2; that is, every monomial will have 5 variables in power 0, 4 variables in power 1, 2 variables in power 2, and 2 variables in power 3. Thus, from every monomial, we immediately reconstruct λ. It means that we can reconstruct from any monomial this representation, so this implies that we must have $\lambda_1 = \lambda_2$. ☐

- Corollary: V_λ's are all irreps of S_n

  Proof. They are pairwise isomorphic and their number equals n. ☐

8.2 Standard Young Tableaux

11/15: • Recap.

  - Recall S_n and Young diagrams.
  - We've discussed conjugate Young diagrams corresponding to inverses λ' as well.
  - For every λ, we've constructed representations $V_{\lambda'}$.
  - Recall that $V_{\lambda'}$ is some representation inside the space of polynomials. In particular,

$$V_{\lambda'} = \operatorname{span}(\sigma[\Delta(x_1,\dots,x_{\lambda_1})\Delta(x_{\lambda_1+1},\dots,x_{\lambda_2})\cdots] \mid \sigma \in S_n)$$

    - Any $\sigma[\Delta(x_1,\dots,x_{\lambda_1})\Delta(x_{\lambda_1+1},\dots,x_{\lambda_2})\cdots]$ is a Specht polynomial $\mathrm{Sp}_\lambda(x_1,\dots,x_n)$.
    - All of these Specht polynomials together span the irrep given by the corresponding Specht module.
  - Last time, we proved that Specht modules are irreducible.
  - Specht polynomials are polynomials in R_d, where R is the ring of polynomials in $x_1,\dots,x_n$ and

$$d = \binom{\lambda_1}{2} + \binom{\lambda_2}{2} + \cdots$$

    - What is this definition of d??

- So how do we further study these representations?
  - Dimension?
  - Characters?
  - Basis?

[2] What does this mean?? Does it mean that in each polynomial in these spaces, there are no two monomials in the same variables, so no monomials cancel and all monomials have coefficient 1?

- Guiding question for today: Which Specht polynomials $\mathrm{Sp}_\lambda(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ form a basis of $V_{\lambda'}$?
- **Young tableau**: A Young diagram filled with integers. *Also known as* **YT**.
- **Standard** (Young tableaux): A YT filled in with numbers $1, \dots, n$, wherein each appears exactly once and the numbers increase in rows and in columns. *Also known as* **SYT**.
- Example of an SYT.

$$\begin{array}{ccccccc} 1 & < & 3 & < & 4 & < & 8 \\ \wedge & & \wedge & & \wedge & & \\ 2 & < & 5 & < & 6 & & \\ \wedge & & & & & & \\ 7 & & & & & & \end{array}$$

Figure 8.2: Example standard Young tableau.

  - We start with a Young diagram.
  - We need to fill it with 8 numbers.
  - There are relations between the boxes.
  - There are some constraints on what can go where, but multiple fillings are still possible.
  - In total, there are three SYT_8.
  - Denote by T a tableau within this set of three.
- Theorem: $\dim V_\lambda$ is the number of SYTs of shape λ.
- Examples.

$$\begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}$$

(a) $\Delta(1234)$.

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array}$$

(b) $(x_1 - x_2)$. (c) $(x_1 - x_3)$. (d) $(x_1 - x_4)$.

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

(e) $(x_1 - x_3)(x_2 - x_4)$. (f) $(x_1 - x_2)(x_3 - x_4)$.

Figure 8.3: Standard Young tableaux of $m\lambda = 4$.

  1. Only ONE way to fill trivial and alternating Young diagrams.
  2. Three ways to fill $(3, 1)$.
  3. Two ways to fill $(2, 2)$.
- Tip: Learn the representations of S_4 by heart!
  - Good for the final and in general.
- We denote the Specht polynomial written from a standard Young tableau by $\mathrm{Sp}(T)$.
  - Given an SYT T, $\mathrm{Sp}(T)$ is the product of the Vandermonde determinants for each column where the numbers in the column tell you which variables to plug into said determinant.
  - For example, the captions of each subfigure in Figure 8.3 are $\mathrm{Sp}(T)$ for the SYT depicted therein.

- We now build up to proving the theorem.
- Lemma: Fix a symmetric group S_n and a partition $\lambda \vdash n$. Then the collection $\{\text{Sp}(T)\}$ of Specht polynomials written from all $T \in \text{SYT}_\lambda$ (that is, all standard Young tableaux T of shape λ) is linearly independent.

  Proof. The basic reason that this lemma is true is that each $\text{Sp}(T)$ contains a certain monomial that none of the others contain; specifically, this will be the lexicographically smallest monomial SM. To get started, fix $\text{Sp}(T)$, and consider $SM[\text{Sp}(T)]$. Our goal is to reconstruct T from it. In this argument, we will look at a representative example instead of a formal proof. In particular, we will look at the example from Figure 8.2. Let's begin.

  First off, note that we have an analogous lemma to last time, i.e., we have

  $$SM(PQ) = SM(P)SM(Q)$$

  Reading from Figure 8.2, we have

  $$\text{Sp}(T) = \Delta(x_1, x_2, x_7)\Delta(x_3, x_5)\Delta(x_4, x_6)$$

  By considering the determinant interpretation of each Vandermonde determinant, we can determine by inspection that the lexicographically smallest monomial. Essentially, the smallest combinations lie along the diagonal of the matrix: Thus, we have

  $$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_7 \\ x_1^2 & x_2^2 & x_7^2 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ x_3 & x_5 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ x_4 & x_6 \end{vmatrix}$$

  Figure 8.4: Determining the lexicographically smallest monomial.

  $$SM[\text{Sp}(T)] = (1 \cdot x_2 \cdot x_7^2)(1 \cdot x_5)(1 \cdot x_6) = x_2 x_7^2 x_5 x_6$$

  From this monomial, we can reconstruct the SYT by putting x_7 in the bottom by necessity, then we have to put $2, 5, 6$ (other coefficients of SM) above in the certain order to get the right ordering. Then, we have to put the ones that aren't there $(x_1^0 x_3^0 x_4^0 x_8^0)$ in the top row. This gives us our YT back. ☐

- Theorem: $\dim V_\lambda = |\,\text{SYT}_\lambda\,|$.

  Proof. Since the $\text{Sp}(T)$ are linearly independent by the lemma, $\dim V_\lambda \geq |\,\text{SYT}_\lambda\,|$. Additionally, the $\text{Sp}(T)$ span $V_{\lambda'}$ because... (Rudenko will not finish this proof.) ☐

- Corollary: $\dim V_\lambda = \dim V_{\lambda'}$.

  Proof. Any representation of S_n will be self-dual. Essentially, because partition inversion only induces a transposition of the Young tableau and a transposed SYT is still standard, the number of SYTs will remain fixed under inversion, so so will the quantity its equal to by the above theorem, namely $\dim V_\lambda$. ☐

- Fact: We have the following identity.

  $$V_{\lambda'} = V_\lambda \otimes (\text{sign})$$

- Let f_λ be the number of SYTs of shape λ. We have shown that $f_\lambda \leq \dim(V_\lambda)$.

- Theorem (RSK): There exists a bijection between permutations in S_n and pairs of SYTs of the same shape (i.e., of **area** n).
  - RSK stands for Robinson-Schensted-Knuth.
  - Errata??: According to Wikipedia, this theorem (as stated) is the Robinson-Schensted correspondence, a special case of and predecessor to the RSK correspondence that trades under its own name.
- Corollary: $f_\lambda = \dim V_\lambda$.

Proof. $\sum_{\lambda=\text{YT of area } n} f_\lambda^2 = n! = \sum(\dim V_\lambda^2)$. This proves that $f_\lambda \leq \dim V_\lambda$ and $f_\lambda = \dim V_\lambda$. $\square$

  - What's the difference between this and the previous theorem?? And how does this proof work?
- Let's see how the RSK correspondence works through an example.
  - Consider the permutation

$$\sigma = (13)(27654) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 1 & 2 & 4 & 5 & 6 & 8 \end{pmatrix} \in S_8$$

  - We will construct two SYTs of identical shape from σ.
  - Start with a pair of empty YTs.

$$\square \qquad\qquad \square$$

    - We will call the left one the **insertion tableau** and the right one the **recording tableau**.
  - Fill $\sigma(1) = 3$ into the insertion tableau and record that this is the first (1) number inserted in the corresponding box of the recording tableau.

$$\begin{ytableau} 3 \end{ytableau} \qquad\qquad \begin{ytableau} 1 \end{ytableau}$$

  - We now have a new pair of tableaux. How do we insert the next number $\sigma(2) = 7$? Try adding it to the right of 3 in the insertion tableau. Record this addition in the recording tableau by adding to it a new box in the same relative position as the new 7 box and filling it with 2.

$$\begin{ytableau} 3 & 7 \end{ytableau} \qquad\qquad \begin{ytableau} 1 & 3 \end{ytableau}$$

  - How do we insert the next number $\sigma(3) = 1$? In the insertion tableau, push out 3 with 1 and move 3 to the next row. In the recording tableau, add a new square in the corresponding bottom position and fill it with 3.

$$\begin{ytableau} 1 & 7 \\ 3 \end{ytableau} \qquad\qquad \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}$$

  - Next number: 2 pushes out 7 in the insertion tableau. 4 goes in the new box in the recording tableau.

$$\begin{ytableau} 1 & 2 \\ 3 & 7 \end{ytableau} \qquad\qquad \begin{ytableau} 1 & 2 \\ 3 & 4 \end{ytableau}$$

  - 4 gets inserted to the right; 5 fills the new box.

$$\begin{ytableau} 1 & 2 & 4 \\ 3 & 7 \end{ytableau} \qquad\qquad \begin{ytableau} 1 & 2 & 5 \\ 3 & 4 \end{ytableau}$$

  - 5,6,8 go further to the right; 6,7,8 in the second one.

$$\begin{ytableau} 1 & 2 & 4 & 5 & 6 & 8 \\ 3 & 7 \end{ytableau} \qquad\qquad \begin{ytableau} 1 & 2 & 5 & 6 & 7 & 8 \\ 3 & 4 \end{ytableau}$$

- Now we have a pair of standard Young tableaux.

- For every permutation, the above algorithm gives us a pair of SYTs.
- Formally, we are following the Schensted row-insertion algorithm, formalized as follows.

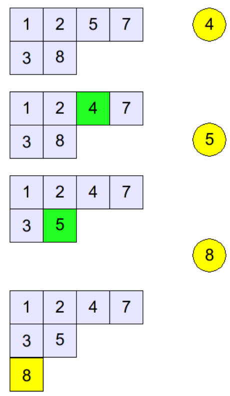

Figure 8.5: The Schensted algorithm.

1. Begin with an insertion tableau and a new number to insert. The new number must not appear anywhere in the insertion tableau.
2. Find the smallest number in row one that is greater than the new number. If no such number exists, append the new number to the end of the row, and we are done. If such a number does exist, replace it with the new number and prepare to insert the number that just got replaced into row two.
3. In the same manner as Step 2, find the smallest number in row two that is greater than the number that just got replaced. Replace it and move onto the third row.
4. Keep repeating until we reach a row where we can append the number from the previous row to the end of the line. If no such row exists, eventually we will reach the bottom of the Young tableau and we may start a new row.

- This algorithm provides a constructive proof of the RSK theorem.
  - In particular, this algorithm takes us between a pair (T, T') of Young tableaux and a permutation σ.
- This map has many interesting properties that are hard to prove. Here's a few.
  1. The map takes $(T', T) \mapsto \sigma^{-1}$.
  2. λ_1 and λ_1' are the length of the longest increasing (resp. decreasing) subsequence of your permutation variables.
- Last word: There is a famous theorem called the **Erdős-Szekeres theorem**.
  - This correspondence is a deep way to understand permutations/sequences of numbers. This is a big tool in CS.
- Next time: Induction and restriction.

8.3 Induction and Restriction

11/17:
- Review: Representations of S_n.
  - λ is a partition.
  - V_λ is the span of $\mathrm{Sp}_\lambda(x_1, \dots, x_n)$.
  - $\dim V_\lambda = \#\,\mathrm{SYT}_\lambda$, i.e., equals the number of standard Young tableaux of shape λ.
- Naturally, it is desirable to find a better way of counting SYT_λ. We will do this with the **hook length formula**.
- **Hook length formula**: The formula given as follows, where n is the number being partitioned. *Given by*

$$\#\,\mathrm{SYT}_\lambda = \frac{n!}{\prod \text{length of all hooks}}$$

  - We should feel free to use this formula, but know that it's quite difficult to prove, so Rudenko will forego such a proof.
- **Hook** (of a cell): The set of all cells in a Young diagram directly to the right of or directly beneath the cell in question, including the cell in question.
- **Length** (of a hook): The cardinality of the hook in question.
- Example.

$$\begin{array}{|c|c|c|c|}
\hline
7 & 4 & 3 & 1 \\ \hline
5 & 2 & 1 \\ \cline{1-3}
2 \\ \cline{1-1}
1 \\ \cline{1-1}
\end{array}$$

Figure 8.6: Hook length formula.

  - The Young diagram in Figure 8.6 corresponds to the partition $9 = (4, 3, 1, 1)$.
  - In each cell of the diagram is the length of the hook corresponding to that cell.
  - Thus, using the hook length formula, the number of standard Young tableaux of shape $(4, 3, 1, 1)$ is

$$\frac{9!}{7 \cdot 4 \cdot 3 \cdot 1 \cdot 5 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = 9 \cdot 8 \cdot 3 = 216$$

- We're headed toward **branching**.
- We'll cover **induction** and **restriction** first.

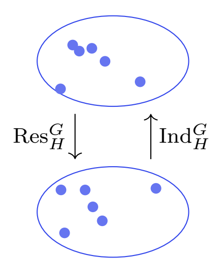

Figure 8.7: Induction and restriction as functors.

  - These are pretty natural things related to group G and subgroup $H \leq G$.
  - Essentially, we have G-reps and H-reps, and we want to interconvert between them.

- Induction allows you to go from H-reps up to G-reps, and vice versa for restrictions.
- In category theory, we call these maps **functors**.
  - Now would be a good time to dive into the definition of functor a bit more deeply!

- The map down is denoted by Res_H^G, and the map up is denoted by Ind_H^G.
- **Restriction** (of V to $H \leq G$): The vector space V viewed as an H-representation, i.e., a straight-up functional restriction of ρ_V. *Denoted by* $\mathbf{Res}_{\boldsymbol{H}}^{\boldsymbol{G}}\boldsymbol{(V)}$.
- Example: Consider $S_2 < S_3$. Find $\text{Res}_{S_2}^{S_3}(2,1)$.
  - Recall that $(2,1) = \langle x_1 - x_2, x_1 - x_3 \rangle$, so this is our answer.
  - However, suppose we want to express $(2,1)$ in a form that tells us a bit more about its status as an S_2-representation. In particular, while $(2,1)$ was an irrep of S_3, it is *not* an irrep of S_2. Indeed, the group S_2 is abelian and hence only has one-dimensional irreps, so we should be able to decompose $(2,1)$ into a sum of two invariant subspaces.
  - So, looking at polynomials fixed and flipped under S_2, we obtain

$$\text{Res}_{S_2}^{S_3}(2,1) = \langle x_1 - x_2 \rangle \oplus \langle x_1 - x_3 + x_2 - x_3 \rangle$$

    - The left polynomial flips under S_2. Specifically, $(12) \cdot (x_1 - x_2) = (x_2 - x_1) = -(x_1 - x_2)$.
    - The right polynomial stays the same under S_2. Specifically, $(12) \cdot (x_1 - x_3 + x_2 - x_3) = (x_2 - x_3 + x_1 - x_3) = (x_1 - x_3 + x_2 - x_3)$.
    - Obviously, neither polynomial changes under e.
    - Note also that adding the right and left polynomials yields $2(x_1 - x_3) \in \text{span}(2,1)$, as expected.
  - Let's highlight a few other features of this decomposition.
  - The two representations in the decomposition are the alternating and trivial — $(1,1)$ and (2) — respectively.
  - While it is fairly obvious that $x_1 - x_2$ — one of the basis vectors of $(2,1)$ — is the basis for the alternating subrepresentation, finding the other one by inspection is trickier. Thus, here's a procedural way to do it.
    - Since the subspaces fixed under (12) are just its eigenspaces, let's compute the eigenvectors of the transformation.
    - Let $a := x_1 - x_2$ and $b := x_1 - x_3$.
    - Observe that

$$\begin{aligned}(12) \cdot a &= x_2 - x_1 = -(x_1 - x_2) = -a \\ (12) \cdot b &= x_2 - x_3 = (x_1 - x_3) - (x_1 - x_2) = b - a\end{aligned}$$

    - Thus,

$$\rho_{(2,1)}(12) = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$$

      - Letting $(1,0) = a$ and $(0,1) = b$, we have

$$\underbrace{\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}}_{\rho_{(2,1)}(12)} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{a} = \underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{-a} \qquad \underbrace{\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}}_{\rho_{(2,1)}(12)} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{a} = \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{b-a}$$

      as expected.
    - Computing the eigenvectors of this matrix, we obtain

$$e_1 = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{a} \qquad e_2 = \underbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{2b-a}$$

  - Plugging in the original definitions of a, b, we obtain

$$\begin{aligned} e_1 &= x_1 - x_2 \\ e_2 &= 2(x_1 - x_3) - (x_1 - x_2) = x_1 - x_3 + x_2 - x_3 \end{aligned}$$

    as expected.

- If we treat x_1, x_2, x_3 as the standard basis of $\mathbb{R}^3$, then $x_1 - x_2$ and $x_1 - x_3$ do span a plane containing $x_1 - x_3 + x_2 - x_3$, as we can prove with vector algebra. Moreover, as we would expect for a direct sum, $x_1 - x_2$ and $x_1 - x_3 + x_2 - x_3$ are orthogonal:

$$\underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{x_1 - x_2} \cdot \underbrace{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}}_{x_1 - x_3 + x_2 - x_3} = (1)(1) + (-1)(1) + (0)(-2) = 0$$

- Moreover, the action of (12) on 3D space (i.e., with standard basis x_1, x_2, x_3) is a reflection along $x_1 - x_2$ and through the plane containing $x_1 - x_3 + x_2 - x_3$; this really shows how (12) fixes both axes while "flipping" one.

- We now move onto induction.
- Induction is highly nontrivial. It is one of those things in math that is just very tricky. All definitions of it are slightly uncomfortable.
- Here's a first attempt at a definition.
  - Let $H \circlearrowright W$ as an H-rep.
  - We want to create a G-rep $\operatorname{Ind}_H^G W$.
  - To start, let's decompose G into left cosets via

$$G = g_1 H \sqcup g_2 H \sqcup \cdots \sqcup g_k H$$

    - Suppose $g_1 \in H$ so that $g_1 H = H$.
  - $\operatorname{Ind}_H^G W$ is defined as the vector space $g_1 W \oplus \cdots \oplus g_k W$. As a vector space, each $g_i W = W$, but there is additional structure as representations.
  - Indeed, we must answer the question, "how does g act on $g_i w$?"
  - Answer: Via

$$g \cdot (g_i w) = g_{\sigma(i)} h_i w = g_{\sigma(i)} (h_i w)$$

    - The first equality comes from defining $\sigma \in S_n$ and $h_i \in H$ so that $g g_i = g_{\sigma(i)} h_i$.
    - We have this equality because $g g_i \in G$, so $g g_i \in \bigsqcup g_i H$, so for some $j = \sigma(i)$, $g g_i \in g_j H = g_{\sigma(i)} H$, i.e., $g g_i = g_{\sigma(i)} h_i$ for some $h_i \in H$.
    - Note that in the last part of the equality, $h_i w$ is the action of h_i on w via the H-rep.
  - So basically, g takes $g_i w$, isolates w, acts on it via h using the original representation to make $h_i w$, and then places this element within the subspace $g_{\sigma(i)} W$.
  - Each element of G acts in a big block-triangular matrix. Inside each block, you will see how g acts on H.
- Let W be the trivial representation of H. Then $\operatorname{Ind}_H^G(\mathrm{id})$ is the permutational representation of G acting on left cosets.
  - Example: $\operatorname{Ind}_{\{e\}}^G(\mathrm{id}) = \mathbb{C}[G]$.
  - See Example 2 from Section 3.3 of Serre (1977).
- There is a correspondence between $H / \operatorname{Stab}(x)$ and G??

  - This is the master construction of representations when you have a subgroup.
- The dimension of an induced representation can be calculated via

$$\dim \operatorname{Ind}_H^G W = (\dim W)(G : H)$$

- A slightly fancier way to think about this stuff, if you're unsatisfied at this point.
  - Take $H < G$, and H-rep W.
  - Let's look at functions on G with values in W, i.e., functions $f : G \to W$. This would be $\operatorname{Hom}_{\mathbb{C}}(\mathbb{C}[G], W)$, once we've linearized G so that we can consider linear maps. These linear maps are the exact same thing as the original functions because the basis of $\mathbb{C}[G]$ is G!
  - Since we know how to calculate the dimension of a space of homomorphisms, we have $\dim \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}[G], W) = (|G|)(\dim W)$.
  - Under this construction, we get to say that

$$\operatorname{Ind}_H^G W = \operatorname{Hom}_H(\mathbb{C}[G], W) = \{f : G \to W \mid f(x \cdot h) = f(xh^{-1}) = hf(x)\}$$

  - We can easily see that this space has the right dimension; such a function is uniquely defined by its values on $g_1, \dots, g_k$.
    - $f(g_1), \dots, f(g_k)$ and $f(g_1 h) = h^{-1} f(g_1)$.
  - What is g acting on in this function? So $[g(f)](x) = f(gx)$...
  - In this case, it's very easy to see that this is a construction with no choices of g_i's, of cosets, etc. Thus fancier.
- Once again, there is no easy way to understand this; we just have to work with it.
- Even fancier construction!
  - Let W be an H-representation. Abstractly, this means that W is a module over $\mathbb{C}[H]$.
  - Take $W \otimes_{\mathbb{C}[H]} \mathbb{C}[G]$.
    - Essentially, this means that if $w \otimes g$, then $hw \otimes g = w \otimes g$.
  - This has something to do with the second representation.
  - This is the most abstractly nice construction because it's much more general.
    - We don't need to use it on groups; we can use it on algebras and modules over them.
    - Indeed, this works in complete generality and has all the same properties.
    - Takeaway: This induced representation is something very, very general, but thinking of it more generally does not help you understand it to start.
- We will do a bunch of computations of such **induced representations** on the homework.
- Theorem (Frobenius): Let $H < G$, and let W be an H-rep. Then

$$\chi_{\operatorname{Ind}_H^G W}(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ g_i^{-1} g g_i \in H}} \chi_W(g_i^{-1} g g_i)$$

- Discussion.
  - Look at the picture from the blackboard.
    - Characters are class functions; they don't change under conjugation.
  - χ_W is a class function in H. An alternate formulation of the formula can be obtained by extending the χ_W (which is a class function in H) to $\tilde{\chi}_W$ (which is a class function in G).

      - We could extend it to $\chi_W : G \to \mathbb{C}$ via

$$\tilde{\chi}_W = \begin{cases} 0 & g \notin H \\ \chi_W(g) & g \in H \end{cases}$$

      - From here, take

$$\chi_{\mathrm{Ind}_H^G W} = \frac{1}{|H|} \sum_{x \in G} \tilde{\chi}_W(xgx^{-1})$$

      - So we're basically just averaging again.

- Proof: See Theorem 12 from Section 3.3 of Serre (1977).
- Next week:
  - If we want to construct $\mathrm{Res}^{S_n}_{S_{n-1}}$, we will take all diagrams inside the Young diagram but one box less.
  - For example,

$$\mathrm{Res}^{S_n}_{S_{n-1}} \ydiagram{3,2,1,1} = \ydiagram{3,2,1} \oplus \ydiagram{3,1,1,1} \oplus \ydiagram{2,2,1,1}$$

  - This is the branching rules.
  - New HW will appear soon and be due Friday of Week 9.
    - Will be a bit harder (some hands-on, some fancy); if you can't do everything there, don't worry.
  - Midterms will be returned early after Thanksgiving.
  - Final will include some stuff from the last HW, but it will be easier.

8.4 S Chapter 3: Subgroups, Products, Induced Representations

From Serre (1977).

Section 3.1: Abelian Subgroups

12/28:
- Definition of **abelian/commutative** (group).
- Irreducible representations of abelian groups.

  Theorem 9. *The following properties are equivalent:*

  (i) *G is abelian.*

  (ii) *All the irreducible representations of G have degree 1.*

  Proof. Let $n_1, \dots, n_h$ be the degrees of the distinct irreducible representations of G. Recall that

$$|G| = \sum_{i=1}^{h} n_i^2$$

  If G is abelian, then there are $|G|$ conjugacy classes and hence $|G|$ irreducible representations. But observe from the above equation that $|G| = h$ iff each $n_i = 1$, which implies the theorem. ☐

  - See Section 1.3 of Fulton and Harris (2004) for an alternate approach to this theorem.

- Irreducible representations of abelian subgroups.

Corollary 1. *Let A be an abelian subgroup of G. Then each irreducible representation V of G has degree*

$$\dim(V) \leq (G : A)$$

Proof. Let $\rho : G \to GL(V)$ be an irreducible representation of G. Through **restriction** to the subgroup A, it defines a representation $\rho_A : A \to GL(V)$ of A. Let $W \subset V$ be an irreducible subrepresentation of ρ_A; by Theorem 9, we have $\dim(W) = 1$. Let

$$V' = \bigcup_{g \in G} \rho(g)W$$

Since V' is a subspace of V that is clearly stable under G by definition, the fact that V is irreducible implies that $V' = V$. Additionally, we have for any $g \in G$ and $a \in A$ that

$$\rho(ga)W = \rho(g)\rho(a)W = \rho(g)W$$

This means that many of the $\rho(g)W$ composing V' are identical; in fact, it is only each *coset* of G with respect to A that can possibly contribute a new dimension to V'. Moreover, since $\dim(W) = 1$, it will only be at most 1 dimension that any $\rho(g)W$ contributes to the union. Essentially, each of the $(G : A)$ cosets contributes at most one dimension to $V' = V$, so $\dim(V)$ cannot exceed $(G : A)$, as desired. $\square$

- Example: A dihedral group contains a cyclic subgroup of index 2; its irreps thus have degree 1 or 2.

Section 3.2: Product of Two Groups

- **Product** (of G_1, G_2): The set of pairs (s_1, s_2) with $s_1 \in G_1$ and $s_2 \in G_2$. *Denoted by* $\boldsymbol{G_1 \times G_2}$.
- **Group product** (of G_1, G_2): The group $(G_1 \times G_2, \cdot)$, where $\cdot$ is the group structure defined by

$$(s_1, s_2) \cdot (t_1, t_2) = (s_1t_1, s_2t_2)$$

  Denoted by $\boldsymbol{G_1 \times G_2}$.
- Properties of $G_1 \times G_2$.
  1. $|G_1 \times G_2| = |G_1| \times |G_2|$.
  2. $G_1 \cong \{(s_1, 1) \mid s_1 \in G_1\} \leq G_1 \times G_2$ and $G_2 \cong \{(1, s_2) \mid s_2 \in G_2\} \leq G_1 \times G_2$.
  3. With the identifications in Property 2, each element of G_1 commutes with each element of G_2.
- These properties characterize $G_1 \times G_2$ completely, as we will see immediately below.
- **Direct product** (of G_1, G_2): The group G containing G_1, G_2 as subgroups and satisfying the conditions...
  (i) Each $s \in G$ can be written uniquely in the form $s = s_1s_2$ with $s_1 \in G_1$ and $s_2 \in G_2$;
  (ii) For $s_1 \in G_1$ and $s_2 \in G_2$, we have $s_1s_2 = s_2s_1$.

  Also known as **product**.
- Equivalence of the direct product with the group product.
  - Because of the two conditions, the product of any two $s, t \in G$ can be written as follows.

$$\begin{aligned} st &= (s_1s_2)(t_1t_2) && \text{Condition (i)} \\ &= s_1(s_2t_1)t_2 \\ &= s_1(t_1s_2)t_2 && \text{Condition (ii)} \\ &= (s_1t_1)(s_2t_2) \end{aligned}$$

- Thus, the group structure of G mirrors that of $G_1 \times G_2$, too.
- This gives us everything we need to define an isomorphism $G_1 \times G_2 \to G$ by

$$(s_1, s_2) \mapsto s_1 s_2$$

- **Tensor product** (of $\rho^1 : G_1 \to GL(V_1), \rho^2 : G_2 : GL(V_2)$): The linear representation of $G_1 \times G_2$ into $V_1 \otimes V_2$ defined as follows. *Denoted by* $\boldsymbol{\rho^1 \otimes \rho^2}$. *Given by*

$$(\rho^1 \otimes \rho^2)(s_1, s_2) = \rho^1(s_1) \otimes \rho^2(s_2)$$

- The characters χ of $\rho^1 \otimes \rho^2$, χ_1 of ρ^1, and χ_2 of ρ^2 are related as follows.

$$\chi(s_1, s_2) = \chi_1(s_1) \cdot \chi_2(s_2)$$

- **Diagonal** (subgroup of $G \times G$): The set of pairs (s, s) for all $s \in G$.
- Note that the representation $\rho^1 \otimes \rho^2$ defined above equals the representation denoted $\rho^1 \otimes \rho^2$ in Section 1.5 of Serre (1977) when $G_1 = G_2$ and when it is restricted to the diagonal subgroup of $G \times G$.
- Irreducible representations of group products.

  Theorem 10.

  (i) *If ρ^1 and ρ^2 are irreducible, $\rho^1 \otimes \rho^2$ is an irreducible representation of $G_1 \times G_2$.*

  Proof. Since ρ^1, ρ^2 are irreducible, Theorem 5 implies that

$$\frac{1}{|G_1|} \sum_{s_1 \in G_1} |\chi_1(s_1)|^2 = 1 \qquad\qquad \frac{1}{|G_2|} \sum_{s_2 \in G_2} |\chi_2(s_2)|^2 = 1$$

  By multiplication, this gives

$$\frac{1}{|G_1 \times G_2|} \sum_{(s_1, s_2) \in G_1 \times G_2} |\chi(s_1, s_2)|^2 = 1$$

  It follows by Theorem 5 that $\rho^1 \otimes \rho^2$ is irreducible. □

  (ii) *Each irreducible representation of $G_1 \times G_2$ is isomorphic to a representation $\rho^1 \otimes \rho^2$, where ρ^i is an irreducible representation of G_i ($i = 1, 2$).*

  Proof. A pair of really elegant proofs are given. □

- Takeaway: "The above theorem completely reduces the study of representations of $G_1 \times G_2$ to that of representations of G_1 and representations of G_2" (Serre, 1977, p. 28).

Section 3.3: Induced Representations

- Definition of a **left coset**, $\boldsymbol{G/H}$, and **index** (of H in G).
- **Congruent modulo** (H elements $s, s' \in G$): Two elements $s, s' \in G$ that belong to the same left coset. *Denoted by* $\boldsymbol{s' \equiv s}$ **(mod $\boldsymbol{H}$)**;
  - Alternative definition: $s^{-1}s' \in H$. ($s = gh_1$ and $s' = gh_2 \Longrightarrow s^{-1}s' = h_1^{-1}g^{-1}gh_2 = h_1^{-1}h_2 \in H$.)
- **System of representatives** (of G/H): A subset $R \subset G$ containing an element from each left coset of H. *Denoted by* $\boldsymbol{R}$.
- Each $s \in G$ can be written uniquely as $s = rt$ for some $r \in R$ and $t \in H$.

- We now build up to defining an **induced** representation. We will construct the definition abstractly first, and then work through a specific, simple example to illustrate the definition. Let's begin.
  - Let $\rho : G \to GL(V)$ be a linear representation of G.
  - Let $\rho_H : H \to GL(V)$ denote the restriction $\rho|_H$ of ρ to H, and let $W \subset V$ be a subrepresentation of ρ_H, i.e., be stable under ρ_t for all $t \in H$.
  - Let $\theta : H \to GL(W)$ be the linear representation that acts on W as ρ_H acts on V.
    - Note that this does *not* mean that $\theta(h) = \rho_H(h) = \rho(h)$; rather, since $\dim W < \dim V$ for a nontrivial case of this construction, $\theta(h)$ will yield a matrix/linear transformation of smaller dimension than $\rho_H(h)$.
    - See the following example for details.
  - Let $s \in G$ be arbitrary. Observe that the vector space $\rho_s W$ depends only on the left coset sH of s; indeed, if we replace s by st where $t \in H$, then we have $\rho_{st}W = \rho_s\rho_t W = \rho_s W$ since $\rho_t W = W$.
  - Define a subspace $W_\sigma \subset V$ for each left coset σ of H by $W_\sigma = \rho_s W$ for some (it does not matter which) $s \in \sigma$.
    - Note that the ρ_s permute the W_σ. Symbolically, if $W_\sigma = \rho_{s_2}W$, then $\rho_{s_1}W_\sigma = \rho_{s_1}\rho_{s_2}W = \rho_{s_1s_2}W = W_{\sigma'}$ for some coset σ' of H to which s_1s_2 belongs.
  - Thus, since each $\rho_s = \rho_{rt}$ moves the W_σ around internally via ρ_t and between each other via ρ_r, the sum $\sum_{\sigma\in G/H} W_\sigma$ is a subrepresentation of V.
- We now have all the definitions and tools we need to formally define an induced representation.
- **Induced** (representation $\rho : G \to GL(V)$ by $\theta : H \to GL(W)$ for $H \leq G$): The representation $\rho : G \to GL(V)$ defined above, if V is equal to the sum of the W_σ ($\sigma \in G/H$) and if this sum is direct (that is, if $V = \bigoplus_{\sigma\in G/H} W_\sigma$).
- Example.
  - Let $G = \mathbb{Z}/4\mathbb{Z}$, and let $\rho : \mathbb{Z}/4\mathbb{Z} \to GL(\mathbb{R}^2)$ send $0, 1, 2, 3$ to the $0°, 90°, 180°, 270°$ rotation matrices. Explicitly, we have

$$\rho_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \rho_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \rho_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \rho_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

  - Let $H = \{0, 2\} \cong \mathbb{Z}/2\mathbb{Z}$, and let $W = \operatorname{span}(1, 0) \cong \mathbb{R}$.
    - Since $\rho_0(1, 0) = (1, 0) \in \operatorname{span}(1, 0)$ and $\rho_2(1, 0) = (-1, 0) \in \operatorname{span}(1, 0)$, W is indeed a subrepresentation that is stable under ρ_H.
  - Let $\theta : \{0, 2\} \to GL[\operatorname{span}(1, 0)]$ be defined as follows. Since ρ_0 maps the basis vector $(1, 0)$ of $\operatorname{span}(1, 0)$ to itself, θ_0 should be the identity as well. Since ρ_2 maps the basis vector $(1, 0)$ of $\operatorname{span}(1, 0)$ to $(-1, 0)$, θ_2 should be the opposite of the identity as well. Altogether,

$$\theta_0 = \begin{bmatrix} 1 \end{bmatrix} \qquad \theta_2 = \begin{bmatrix} -1 \end{bmatrix}$$

  - Denote the two left cosets of H by $\sigma = 0 + H$ and $\tau = 1 + H$. Then

$$W_\sigma = \rho_0 \operatorname{span}(1, 0) = \operatorname{span}(1, 0) \qquad W_\tau = \rho_1 \operatorname{span}(1, 0) = \operatorname{span}(0, 1)$$

    - We have the rightmost equality above because of ρ_1's matrix definition, which implies that its action on the basis vector $(1, 0)$ of $\operatorname{span}(1, 0)$ is $\rho_1(1, 0) = (0, 1)$.
  - Thus, altogether, if ρ is to be the representation induced by θ, we must have

$$V = \bigoplus_{\sigma\in G/H} W_\sigma = W_0 \oplus W_1 = \operatorname{span}(1, 0) \oplus \operatorname{span}(0, 1)$$

- Let's go a bit deeper with this example, and see how we could recover ρ if all we started with was G, H, θ, W.
  - Define $\sigma = 0 + H$ and $\tau = 1 + H$.
  - Then $W_\sigma = \rho_0 W$ and $W_\tau = \rho_1 W$.
  - It follows that $V = W_\sigma \oplus W_\tau$.
  - We can now define ρ's action on the basis $\{W_\sigma, W_\tau\}$ of V.

$$
\begin{aligned}
\rho_0 W_\sigma &= \rho_0(\rho_0 W) = \rho_0(\rho_0 W) = \rho_0(\theta_0 W) = \rho_0 W = W_\sigma \\
\rho_0 W_\tau &= \rho_0(\rho_1 W) = \rho_1(\rho_0 W) = \rho_1(\theta_0 W) = \rho_1 W = W_\tau \\
\rho_1 W_\sigma &= \rho_1(\rho_0 W) = \rho_1(\rho_0 W) = \rho_1(\theta_0 W) = \rho_1 W = W_\tau \\
\rho_1 W_\tau &= \rho_1(\rho_1 W) = \rho_0(\rho_2 W) = \rho_0(\theta_2 W) = -\rho_0 W = -W_\sigma \\
\rho_2 W_\sigma &= \rho_2(\rho_0 W) = \rho_0(\rho_2 W) = \rho_0(\theta_2 W) = -\rho_0 W = -W_\sigma \\
\rho_2 W_\tau &= \rho_2(\rho_1 W) = \rho_1(\rho_2 W) = \rho_1(\theta_2 W) = -\rho_1 W = -W_\tau \\
\rho_3 W_\sigma &= \rho_3(\rho_0 W) = \rho_1(\rho_2 W) = \rho_1(\theta_2 W) = -\rho_1 W = -W_\tau \\
\rho_3 W_\tau &= \rho_3(\rho_1 W) = \rho_0(\rho_0 W) = \rho_0(\theta_0 W) = \rho_0 W = W_\sigma
\end{aligned}
$$

  - If we associate W_σ with $(1,0) \in \mathbb{R}^2 \cong V$ and W_τ with $(0,1) \in \mathbb{R}^2 \cong V$, then the matrices of ρ_s are those given in the original example above.
- Note: To get a nontrivial induced representation in this manner, we must have $W < V$. I.e., in the above example, we could not take $\rho_{\{0,2\}} : G \to GL(\mathbb{R}^2)$ and induce *it* up; rather, we needed to deal with $\rho_{\{0,2\}} : G \to GL(\mathbb{R})$ and induce it.
- Let's now look at a couple of reformulations of the definition of an induced representation.
  1. Each $x \in V$ can be written uniquely as $\sum_{\sigma \in G/H} x_\sigma$, with $x_\sigma \in W_\sigma$ for each σ.
  2. If R is a system of representatives of G/H, the vector space V is the direct sum of the $\rho_r W$ with $r \in R$.
- A consequence of the second formulation above is that

$$\dim(V) = \sum_{r \in R} \dim(\rho_r W) = (G : H) \cdot \dim(W)$$

- Examples.
  1. If $\rho : G \to GL(V)$ is the regular representation of G and W is the subspace with basis $(e_t)_{t \in H}$, then $\theta : H \to GL(W)$ is the regular representation of W and ρ is induced by θ. This is a fairly straightforward case of adding more dimensions to build up the full representation!
  2. The **permutation representation** of G associated with G/H. e_H is invariant under H. The representation of H in the subspace $\mathbb{C}e_H$ is the **unit representation** of H, and this representation induces ρ.
     - This is a more general case of the example presented above, where I chose $G = \mathbb{Z}/4\mathbb{Z}$ and $H = \mathbb{Z}/2\mathbb{Z}$, and restricted $\mathbb{C}e_H$ to $\mathbb{R}e_h$.
  3. If ρ_1 is induced by θ_1 and ρ_2 is induced by θ_2, then $\rho_1 \oplus \rho_2$ is induced by $\theta_1 \oplus \theta_2$.
  4. If (V, ρ) is induced by (W, θ), and if W_1 is a stable subspace of W, the subspace $V_1 = \sum_{r \in R} \rho_r W_1$ of V is stable under G, and the representation of G in V_1 is induced by the representation of H in W_1.
  5. If ρ is induced by θ, if ρ' is a representation of G, and if ρ'_H is the restriction of ρ' to H, then $\rho \otimes \rho'$ is induced by $\theta \otimes \rho'_H$.

- **Permutation representation** (of G associated with G/H): The representation $\rho : G \to GL(V)$, where $V = (e_\sigma)_{\sigma \in G/H}$ and $\rho_s e_\sigma = e_{s\sigma}$.
- We now prove the existence and uniqueness of induced representations.
  - While the above examples are specific, explicitly verifiable cases of induced representations, we have not yet proven that an induced representation (V, ρ) exists for *every* (W, θ).
  - This is our present goal.
- Note: This construction here is related to the intermediately fancy construction of induced representations from Friday's class.
- To begin, we first state and prove a lemma that will later be useful in proving the uniqueness of the induced representation.

  Lemma 1. *Suppose that (V, ρ) is induced by (W, θ). Let $\rho' : G \to GL(V')$ be a linear representation of G, and let $f : W \to V'$ be a linear map such that $f(\theta_t w) = \rho'_t f(w)$ for all $t \in H$ and $w \in W$[3]. Then there exists a unique linear map $F : V \to V'$ which extends f and satisfies $F \circ \rho_s = \rho'_s \circ F$ for all $s \in G$.*

  Proof. We first prove the uniqueness of F so that we can use an aspect of this argument to prove its existence. Let's begin.

  To prove that F is unique, it will suffice to give a formula derived from the given constraints that wholly characterizes it on V. Let $x \in \rho_s W \subset V$ be arbitrary. Then $\rho_s^{-1} x \in W$, hence

  $$F(x) = F(\rho_s \rho_s^{-1}) = \rho'_s F(\rho_s^{-1} x) = \rho'_s f(\rho_s^{-1} x)$$

  as desired.

  To prove that F exists, it will suffice to define it by formula and then show that this formula is well-defined. Let $x \in W_\sigma$ be arbitrary. Define $F(x)$ by $F(x) = \rho'_s f(\rho_s^{-1} x)$ for some $s \in \sigma$, mirroring the above. While it may seem that varying the choice of s could vary the definition of F, it actually does not: Replace s by st ($t \in H$) to see that

  $$\rho'_{st} f(\rho_{st}^{-1} x) = \rho'_s \rho'_t f(\theta_t^{-1} \rho_s^{-1} x) = \rho'_s (\theta_t \theta_t^{-1} \rho_s^{-1} x) = \rho'_s f(\rho_s^{-1} x)$$

  We can then check that $F \circ \rho_s = \rho'_s \circ F$ for all $s \in G$. (How?? Serre (1977) says it's easy but I'm not seeing it.) □

- Now we state and prove the full existence and uniqueness result.

  Theorem 11. *Let (W, θ) be a linear representation of H. There exists a linear representation (V, ρ) of G which is induced by (W, θ), and it is unique up to isomorphism.*

  Proof. We will prove existence and then uniqueness. Let's begin.

  <u>Existence</u>: In view of Example 3, we may assume that θ is irreducible. In this case, θ is isomorphic to a subrepresentation of the regular representation of H, which can be induced to the regular representation of G by Example 1. Then applying Example 4, we conclude that θ, itself, can be inducted.

  <u>Uniqueness</u>: Let $(V, \rho), (V', \rho')$ be two representations induced by (W, θ). Since W is a subspace of V', we may consider the linear injection $f : W \to V'$. As an injection, f is the identity on W, so for any $t \in H$ and $w \in W$, we have $f(\theta_t w) = \theta_t w = \rho'_t w = \rho'_t f(w)$. Thus, applying Lemma 1, we see that there exists a linear map $F : V \to V'$ which is the identity on W and satisfies $F \circ \rho_s = \rho'_s \circ F$ for all $s \in G$. Since F is the identity on W, $\mathrm{Im}(F)$ contains all the $\rho'_s W$ and thus is isomorphic to V'. This combined with the fact that $\dim V' = (G : H) \cdot \dim(W) = \dim V$ proves that F is an isomorphism overall, hence completing the proof. □

[3] Note that this is not quite a morphism of G-representations because only ρ' maps from G — θ maps from H!

- We now discuss the character of an induced representation.
- Motivation: Since (W, θ) determines (V, ρ) up to isomorphism, we should be able to compute χ_ρ from χ_θ.
- Here's how:

  Theorem 12. *Let h be the order of H and let R be a system of representatives of G/H. For each $u \in G$, we have*
  $$\chi_\rho(u) = \sum_{\substack{r \in R \\ r^{-1}ur \in H}} \chi_\theta(r^{-1}ur) = \frac{1}{|H|} \sum_{\substack{s \in G \\ s^{-1}us \in H}} \chi_\theta(s^{-1}us)$$
  In particular, $\chi_\rho(u)$ is a linear combination of the values of χ_θ on the intersection of H with the conjugacy class of u in G.

  Proof. We will proceed from the definition of $\chi_\rho(u)$ as
  $$\chi_\rho(u) = \operatorname{tr}_V(\rho_u)$$
  To begin, consider the matrix of ρ_u. Since ρ_u permutes the $\rho_r W$ composing V, only spaces $\rho_r W$ that ρ_u maps into themselves (i.e., spaces on the block diagonal of the matrix of ρ_u) will affect the trace. More precisely, these are spaces for which $ur = rt$ for some $t \in H$. Observe that this condition can be rewritten $r^{-1}ur \in H$. Thus,
  $$\chi_\rho(u) = \sum_{\substack{r \in R \\ r^{-1}ur \in H}} \operatorname{tr}_{\rho_r W}(\rho_u|_{\rho_r W})$$
  Recall that the origin of the condition $ur = rt$ is that we needed $\rho_u\rho_r = \rho_r\theta_t$ with $t = r^{-1}ur \in H$. More specifically, since we only care about the action of ρ_u on $\rho_R W$ right now, we have $\rho_u|_{\rho_r W}\rho_r = \rho_r\theta_t$. It follows since $\operatorname{tr}(ab) = \operatorname{tr}(ba)$ and hence $\operatorname{tr}(aba^{-1}) = \operatorname{tr}(b)$ that $\operatorname{tr}(\rho_u|_{\rho_r W}) = \operatorname{tr}(\rho_r\theta_t\rho_{r^{-1}}) = \operatorname{tr}(\theta_t) = \chi_\theta(t) = \chi_\theta(r^{-1}ur)$, which yields
  $$\chi_\rho(u) = \sum_{\substack{r \in R \\ r^{-1}ur \in H}} \chi_\theta(r^{-1}ur)$$
  as desired.

  The second formula given for $\chi_\rho(u)$ follows from the first by noting that all elements $s \in G$ in the left coset rH ($r \in R_u$) satisfy $\chi_\theta(s^{-1}us) = \chi_\theta(r^{-1}ur)$. ☐

- Another property of induced representations discussed further in part II: The **Frobenius reciprocity formula**, which is given by
  $$(f_H \mid \chi_\theta)_H = (f \mid \chi_\rho)_G$$

8.5 S Chapter 7: Induced Representations; Mackey's Criterion

From Serre (1977).

Section 7.1: Induction

12/29:

- We are now treating V as a $\mathbb{C}[G]$-module and W as a $\mathbb{C}[H]$-submodule of V.
- Definition of **induced** representation.
- We now reformulate the induction property.
- Let W' (defined as follows) be the $\mathbb{C}[G]$-module obtained from W by **scalar extension** from $\mathbb{C}[H]$ to $\mathbb{C}[G]$.
  $$W' = \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W$$

  - The injection $W \to V$ extends by linearity to a $\mathbb{C}[G]$-homomorphism $i : W' \to V$.
  - There is definitely more for me to understand here in the realm of exactly what a scalar extension is!

- We now relate W' to V.

  Proposition 18. *In order that V be induced by W, it is necessary and sufficient that the homomorphism*

  $$i : \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \to V$$

  be an isomorphism.

  Proof. This follows (somehow??) from the fact that the elements of the elements of a system of left coset representatives for H form a basis of $\mathbb{C}[G]$ considered as a right $\mathbb{C}[H]$-module. □

- Notes.

  1. Update on Theorem 11: Because the tensor product is well-defined, his formulation of the representation induced by W obviously *exists* and *is unique*.
     - At this point, Serre (1977) introduces the $\int_H^G(W)$ notation.
  2. Update on Lemma 1: If V is induced by W and if E is a $\mathbb{C}[G]$-module, we have a canonical isomorphism

     $$\operatorname{Hom}^H(W, E) \cong \operatorname{Hom}^G(V, E)$$

     where $\operatorname{Hom}^G(V, E)$ denotes the vector space of $\mathbb{C}[G]$-homomorphisms of V into E, and $\operatorname{Hom}^H(W, E)$ is defined similarly.
     - This follows from a property of tensor products.
  3. Induction is transitive: If G is a subgroup of a group K, then we have

     $$\operatorname{Ind}_G^K(\operatorname{Ind}_H^G(W)) \cong \operatorname{Ind}_H^K(W)$$

     - Two ways to see this: Directly or via the associativity of the tensor product.

- A criterion for when a subspace and subgroup can induce a representation.

  Proposition 19. *Let V be a $\mathbb{C}[G]$-module which is a direct sum $V = \bigoplus_{i \in I} W_i$ of vector subspaces permuted* ***transitively*** *by G. Let $i_0 \in I$, $W = W_{i_0}$, and let H be the* ***stabilizer*** *of W in G. Then W is stable under the subgroup H and the $\mathbb{C}[G]$-module V is induced by the $\mathbb{C}[H]$-module W.*

  Proof. Obvious, according to Serre (1977). □

- **Transitive** (permutation by G on a set X): A group action for which the orbit of x is X for some (any) $x \in X$.
- **Stabilizer** (of W in G): The set of all $s \in G$ such that $sW = W$.
- Note on the proposition: In order to apply it to an irrep $V = \bigoplus W_i$ of G, it is enough to check that the W_i are permuted among themselves by G; the transitivity condition is automatic because each orbit of G in the set of W_i's defines a subrepresentation of V.
- **Monomial** (representation): A representation V for which the W_i's are of dimension 1.