Week 2

The Structure of Representations

2.1 The Tensor Product

10/2:

- Before lecture, I chatted with a few people about tensor products and the exterior and symmetric powers.
  - Patrick: A **tensor** $v \otimes w$ is just an element of a vector space, indexed differently than in a column.
  - Raman: There is no canonical way to transform tensors into column vectors.
- Course logistics.
  - OH: T 5:30-6:30(+) and W 5:30-6:30(+). We can also meet one-on-one.
  - HW is due Thursdays at midnight.
- Today: Constructing new representations from old.
  - Rudenko will skim through tensor products really quickly.
- Reminder: Last time, we talked about how representation theory is really quite simple. If G is a finite group and $F = \mathbb{C}$, there exist a finite set $V_1, \dots, V_s$ of irreps up to isomorphism, and every finite-dimensional representation $V \cong V_1^{n_1} \oplus \cdots \oplus V_s^{n_s}$.
- If V is a representation of G, then there are loads of things we can do with it.
  - We can construct the dual representation V^*.
  - We can construct the representation $V \otimes V$.
  - We can construct symmetric powers.
  - We can construct wedge powers.
  - There are more, but this is enough for now.
- Even when we take a very simple group and representation, there are some very interesting things that can fall out.
  - Example: If you take the symmetric powers of S_3, as in the homework, you get something really interesting.
- Now, we go to linear algebra.
- Let V, W be vector spaces over a field F. How do we produce a new vector space out of these?
- $\mathrm{Hom}_F(V, W)$ is the vector space of linear maps $F : V \to W$!
  - $\dim = (\dim V)(\dim W)$.

- Can we make $\mathrm{Hom}_F(V,W)$ into a representation of G? Yes!

$$\begin{tikzcd}
V \arrow[r, "L"] \arrow[d, "\rho_V(g)"'] & W \arrow[d, "\rho_W(g)"] \\
V \arrow[r, "gL"'] & W
\end{tikzcd}$$

Figure 2.1: Commutative diagram, linear maps space representation.

  - Suppose that V,W are G-reps, which gives us $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$.
  - Suppose also that we have $L \in \mathrm{Hom}_F(V,W)$.
  - Now infer from the commutative diagram that it will work to define $gL = \rho_W(g) \circ L \circ \rho_V(g)^{-1}$.
  - This is pretty standard.
- Recall that there is a different space $\mathrm{Hom}_G(V,W)$ of morphisms of G-representations (see Figure 1.2 and the associated discussion).
  - This is a very very small subspace of $\mathrm{Hom}_F(V,W)$.
- Special case of the above construction: **Dual representation**.
  - Consider $\mathrm{Hom}_F(V,F)$. This the **dual vector space**.
  - Basic fact 1: Let $e_1,\dots,e_n$ be a basis of V. Then V^* also has a corresponding basis $e^1,\dots,e^n$, known as its **dual basis**.
    - Computing coordinates already depends on a basis, and having bases is super nice.
    - Corollary: $\dim V = \dim V^*$.
    - This is the first time **canonical** comes into linear algebra. Canonical (nobody understands what it means) basically means that something doesn't depend on choices.
    - In particular, V,V^* are isomorphic because they have the same dimension, but for no more natural reason. They can be the same representation, or they can be different.
  - Basic fact 2: If V is finite-dimensional, then $(V^*)^* \cong V$. The formula for this isomorphism is canonical, because it does not depend on a choice of basis. In particular, choose the map $V \to (V^*)^*$ sending v to the map sending $\varphi \in V^*$ to $\varphi(v)$.
  - If V is infinite dimensional, none of this is true and you are in the realm of functional analysis.
  - Ok, so all of this was good information about the dual *space*, but what is the dual *representation*?? Does it matter, and do we need to know for now?
- **Dual vector space** (of V): The vector space defined as follows, given that V is a vector space over F. *Denoted by* $\boldsymbol{V^*}$. *Given by*
$$V^* = \mathrm{Hom}_F(V,F)$$
- **Dual basis** (of V^* to $e_1,\dots,e_n$): The basis defined as follows for $i = 1,\dots,n$, where $e_1,\dots,e_n$ is a basis of V. *Denoted by* $\mathbf{e^1,\dots,e^n}$. *Given by*
$$e^i(x_1e_1 + \cdots + x_ne_n) = x_i$$
- We now move onto the tensor product.
  - The tensor product is very hard to understand. If you learn about it and you feel you don't understand it, that's typical; nobody understands it at first.
  - For now, we'll discuss two ways of thinking about tensor products that won't bring us any comfort.
- Let V,W be two vector spaces over a field F.

- Abstract definition of the tensor product.
  - We have discussed maps from $V \to W$, but there is another related space.
  - Indeed, we can look at the space of bilinear maps from $V \times W \to F$.
    - Example: A map $f : V \times W \to F$ that satisfies the constraints $f(\lambda v, w) = \lambda f(v, w)$, $f(v_1 + v_2, w) = f(v_1, w) + f(v_2, w)$, and likewise for the second index. Recall that this is a **bilinear map**.
  - Let V have basis $e_1, \dots, e_n$ and W have basis $f_1, \dots, f_m$.
  - Notice that every bilinear map f can be defined as a linear combination of the $f(e_i, f_j)$. In other words, the $f(e_i, f_j)$ form the basis of a function space.
    - This "bilinear maps space" has dimension nm.
  - Now, one way to understand a tensor product: Is this "bilinear maps space" actually some other space? It is! It is $(V \otimes W)^*$.
  - Bilinear maps are linear maps from where? From $V \otimes W$!
- **Bilinear** (map): A function $f : V \times W \to Z$ that satisfies the following constraints, where V, W, Z are vector spaces over F, $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, and $\lambda \in F$. *Constraints*

$$f(v_1 + v_2, w) = f(v_1, w) + f(v_2, w) \qquad f(\lambda v, w) = \lambda f(v, w)$$
$$f(v, w_1 + w_2) = f(v, w_1) + f(v, w_2) \qquad f(v, \lambda w) = \lambda f(v, w)$$

- We now look at a much more elementary definition of the tensor product.
- Explicit definition of the tensor product.
  - $V \otimes W$ is equal to a huge vector space with basis consisting of pairs of elements (v, w). Even if V, W are one dimensional, this is like all pairs of real numbers; it's huge. Then, we quotient it by the space of all elements satisfying $\lambda(v, w) = (\lambda v, w) = (v, \lambda w)$, $(v_1 + v_2, w) = (v_1, w) + (v_2, w)$, and the like. This forces these relationships to be true.
    - Clarify this methodology??
    - Essentially, this allows us to treat tensor multiplication much like real multiplication, endowing the operation with distributivity, etc.
    - For example, the rule $(v_1 + v_2, w) = (v_1, w) + (v_2, w)$ becomes, in tensor product notation, $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$.
  - Example: Suppose $V = \mathbb{C}e_1 + \mathbb{C}e_2$. We want to look at $V \otimes V$.
    - A priori[1], it's spanned by $(ae_1 + be_2) \otimes (ce_1 + de_2) = ace_1 \otimes e_1 + ade_1 \otimes e_2 + bce_2 \otimes e_1 + cde_2 \otimes e_2$.
    - Thus, $V_1 \otimes V_2$ has 4-element basis $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$.
- These two definitions constitute a first approximation to what the tensor product is.
- Takeaway: What is true in general is that if V has basis $e_1, \dots, e_n$ and W has basis $f_1, \dots, f_m$, then $V \otimes W$ has basis $e_i \otimes f_j$ ($i = 1, \dots, n$ and $j = 1, \dots, m$).
- Having discussed the tensor product of vector spaces, let's think about the tensor product of *representations*.
  - Suppose $g : V \to V$ and $g : W \to W$.
    - We're starting to make notation sloppy.
  - How does $g : V \otimes W \to V \otimes W$? Well, we just send $v \otimes w \mapsto (gv) \otimes (gw)$.
    - Why is this map well-defined?

[1] I.e., it follows from some logic. In particular, it follows from the logic that any element $v \in V$ is of the form $v = ae_1 + be_2$, so of course all $v \otimes v$ must be of the given form for choices of a, b, c, d.

        - We invoke the **universal property of the tensor product operation**.
        - This guarantees us that given g — which is effectively a map from $V \times W \to V \otimes W$ as defined — there nevertheless exists a complete extension $\tilde{g} : V \otimes W \to V \otimes W$.
    - As a matrix, this map is pretty strange!
        - Example: Let $g : V \to V$ be a 2×2 matrix. What is the matrix of $g : V \otimes V \to V \otimes V$?
        - If
$$\rho_V(g) = g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} =: A$$
          then we have
$$\begin{aligned} g(e_1 \otimes e_1) &= ge_1 \otimes ge_1 \\ &= (ae_1 + ce_2) \otimes (ae_1 + ce_2) \\ &= a^2 e_1 \otimes e_1 + ace_1 \otimes e_2 + ace_2 \otimes e_1 + c^2 e_2 \otimes e_2 \end{aligned}$$
        - Evaluating similarly for all basis vectors, we get a very curious block matrix:
$$\begin{array}{c c} & \begin{array}{cccc} e_1 \otimes e_1 & e_1 \otimes e_2 & e_2 \otimes e_1 & e_2 \otimes e_2 \end{array} \\ \begin{array}{c} e_1 \otimes e_1 \\ e_1 \otimes e_2 \\ e_2 \otimes e_1 \\ e_2 \otimes e_2 \end{array} & \begin{bmatrix} a^2 & ab & ab & b^2 \\ ac & ad & bc & bd \\ ac & bc & ad & bd \\ c^2 & cd & cd & d^2 \end{bmatrix} \end{array} = \left[\begin{array}{c|c} aA & bA \\ \hline cA & dA \end{array}\right]$$
        - Notice how, for example, this takes the tensor $e_1 \otimes e_1$, represented as $(1, 0, 0, 0)$, to the tensor $a^2 e_1 \otimes e_1 + ace_1 \otimes e_2 + ace_2 \otimes e_1 + c^2 e_2 \otimes e_2$, represented as (a^2, ac, ac, c^2).
        - Does this construction imply a canonical way to convert from tensors to column vectors??
    - Classically, this is called the **Kronecker product** of two matrices.
    - People discovered all of this stuff before they unified it as tensor math.

- **Universal property of the tensor product operation**: For every bilinear map $h : V \times W \to Z$, there exists a *unique* linear map $\tilde{h} : V \otimes W \to Z$ such that $h = \tilde{h} \circ \otimes$.

$$\begin{tikzcd} V \times W \arrow[r, "\otimes"] \arrow[dr, "h"'] & V \otimes W \arrow[d, "\tilde{h}"] \\ & Z \end{tikzcd}$$

Figure 2.2: Universal property, tensor product operation.

Proof. See the solid explanation linked here. Write out at a later date, and/or review MATH 25800 notes further. ☐

- **Kronecker product** (of A, B): The matrix product defined as follows. *Denoted by* $\boldsymbol{A} \otimes \boldsymbol{B}$. *Given by*
$$A \otimes B = n \overset{n}{\begin{bmatrix} A \end{bmatrix}} \otimes m \overset{m}{\begin{bmatrix} B \end{bmatrix}} = nm \overset{nm}{\begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix}}$$
    - The Kronecker product is *not* commutative, but the matrices you get are related by conjugacy and by commuting the columns.

- Vector spaces of the same dimension are all alike, but vector space representations are very interesting. By the end of this course, we'll understand what's going on.

- How we understand tensor stuff: Look at the abstract definition, look at the concrete definition, look at 5 examples, and then go in a circle. Repeat again and again until it makes sense.
- He's just trying to tell us all relevant words so that they will fit together later.
- Fact: If V, W finite-dimensional, $\mathrm{Hom}_F(V, W) \cong V \otimes W^*$.
  - Tensor products are very nice to construct maps from.
  - Let's construct a reverse map, then.
  - Take $\alpha \otimes w \in V^* \otimes W$, where $\alpha : V \to F$ by definition. Send $\alpha \otimes w$ to the map $v \mapsto \alpha(v)w$. This is a *canonical* map!! We can show that they span everything.
    - For example, if we want to choose $\alpha \otimes w$ mapping to the matrix with a 1 in the upper left-hand corner and zeroes everywhere else, let $w = e_1 \in W$ and let $\alpha = e^1 \in V^*$.
    - We can do similarly for all other such matrices, mapping this basis of $\mathrm{Hom}_F(V, W)$ to $e^i \otimes e_j$ ($i = 1, \dots, n$ and $j = 1, \dots, m$).
    - Note that this also allows us to define a (noncanonical) inverse map.
  - This inverse map from $\mathrm{Hom}_F(V, W) \to V^* \otimes W$ is clearly a bit harder to work out.
  - Hidden in this story is why trace is invariant under conjugation (see below discussion).
- If we now take $\mathrm{Hom}_F(V, V)$, then this is isomorphic to $V^* \otimes V$. There is a very natural map from these isomorphic spaces to F defined by the trace, and/or $\alpha \otimes v \mapsto \alpha(v)$. We can prove this. And this is canonical, as well. This is why the main property of the trace is that it's invariant under conjugation. This fact is hidden in the story very nicely.
- Tensor products are hard, it will be a pain, we will understand them very well, but it will not be nice for now.
- Symmetric products and wedge powers will be discussed briefly next time.
  - There is a nice description in Serre (1977) that we can use for the homework.
- Extra homework: Please read about tensor products in whatever textbook you like, try some examples, and repeat.

2.2 Office Hours (Rudenko)

10/3:

- Problem 2a:
  - $\Lambda^2 V$ is *exterior powers.*
  - The exact canonical isomorphism we need is briefly discussed on Fulton and Harris (2004, p. 473).
  - I.e., we have to construct isomorphisms between the structures that don't rely on the choice of any basis. Recall the classic example of $V \cong V^{**}$, as explained in the well-written MSE post "basic difference between canonical isomorphism and isomorphims." Recall that the isomorphism from $V \to V^*$ defined by sending each element of the basis of V to the corresponding element of the dual basis of V^* is *not* canonical because *it involves choosing bases.* Definitions of canonical maps are available in MATH20510Notes, p. 2.
  - From a quick look at this, it looks like the proof may be analogous to the classic middle-school algebra identity $(v + w)^2 = v^2 + vw + w^2$.
  - The second exterior power $\Lambda^2 V$ of a finite-dimensional vector space V is the dual space of the vector space of alternating bilinear forms on V. Elements of $\Lambda^2 V$ are called 2-vectors.
- Problem 2b:
  - $S^2 V$ is *symmetric powers.*

- The exact canonical isomorphism we need is briefly discussed on Fulton and Harris (2004, p. 473).

- Problem 3a:
  - This is the determinant of the multiplication table, in relation to that theorem that you showed us at the end of the first class? Yep!
- Problem 3b:
  - So a circulant matrix is a matrix like the multiplication table from (a)? Yep!
  - Is $\zeta = e^{2\pi i/n}$? Sort of. It can be any n^{th} root of unity.
- Problem 4d:
  - We'll cover higher symmetric powers in class tomorrow.
  - However, it basically just means that we're now working with elements of the form $e_1 \otimes e_2 \otimes e_3 \in S^3V$ and on and on.
- Problem 5a:
  - Is $V^\vee = V^*$? Yes. This is "vee check," and is a notation that some people prefer.
- Problem 5b:
  - Is "tr" the trace function of the linear map corresponding to L? Yes.
  - What is L?
    - An element of $V \otimes V^*$ is a linear combination of elements of the form $v \otimes \alpha$, not necessarily just one of these "decomposable" products.
    - There is an isomorphism $V \otimes V^* \cong \mathrm{Hom}(V)$.
    - Consider the matrix
      $$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
      It sends $e_1 \mapsto e_1$ and $e_2 \mapsto 0$. Thus, it is well-matched with $e_1 \otimes e^1$, which also grabs e_1 (with e^1) and sends it to e_1.
    - Consider the matrix
      $$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
      It sends $e_1 \mapsto 0$ and $e_2 \mapsto e_1$. Thus, it is well-matched with $e_1 \otimes e^2$, which also grabs e_2 (with e^2) and sends it to e_1.
    - In full,
      $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ae_1 \otimes e^1 + be_1 \otimes e^2 + ce_2 \otimes e^1 + de_2 \otimes e^2$$
    - This map *is* canonical! This is because the bases must be chosen to even begin talking about matrices.
    - If you change the matrix, the bases change, too??
    - Takeaway: We have to walk backwards from matrix to linear transformation to representation in $V \otimes V^*$ to a scalar in F.
- Problem 5c:
  - So trace of such a map is equal to the dimension of its image? Yes.

2.3 S Chapter 1: Generalities on Linear Representations

From Serre (1977).

Section 1.5: Tensor Product of Two Representations

10/4:

- **Tensor product** (of V_1, V_2): The vector space W that (a) is furnished with a map $V_1 \times V_2 \to W$ sending $(x_1, x_2) \mapsto x_1 \cdot x_2$ and (b) satisfies the following two conditions.
  (i) $x_1 \cdot x_2$ is bilinear.
  (ii) If (e_{i_1}) is a basis of V_1 and (e_{i_2}) is a basis of V_2, the family of products $e_{i_1} \cdot e_{i_2}$ is a basis of W.

  Denoted by $\boldsymbol{V_1 \otimes V_2}$.
  - It can be shown that such a space exists and is unique up to isomorphism (see proof here).
- This definition allows us to say some things quite expediently. For example, (ii) implies that

$$\dim(V_1 \otimes V_2) = \dim(V_1) \cdot \dim(V_2)$$

- **Tensor product** (of ρ^1, ρ^2): The representation $\rho : G \to GL(V_1 \otimes V_2)$ defined as follows for all $s \in G$, $x_1 \in V_1$, and $x_2 \in V_2$, where $\rho^1 : G \to GL(V_1)$ and $\rho^2 : G \to GL(V_2)$ are representations. *Given by*

$$[\rho_s^1 \otimes \rho_s^2](x_1 \cdot x_2) = \rho_s^1(x_1) \cdot \rho_s^2(x_2)$$

- A more formal write up of the matrix translation of this definition.
  - Let (e_{i_1}) be a basis for V_1, and let (e_{i_2}) be a basis for V_2.
  - Let $r_{i_1j_1}(s)$ be the matrix of ρ_s^1 with respect to this basis, and let $r_{i_2j_2}(s)$ be the matrix of ρ_s^2 with respect to this basis.
  - It follows that

$$\rho_s^1(e_{j_1}) = \sum_{i_1} r_{i_1j_1}(s)e_{i_1} \qquad\qquad \rho_s^2(e_{j_2}) = \sum_{i_2} r_{i_2j_2}(s)e_{i_2}$$

  - Therefore,

$$[\rho_s^1 \otimes \rho_s^2](e_{j_1} \cdot e_{j_2}) = \sum_{i_1,i_2} r_{i_1j_1}(s)r_{i_2j_2}(s)e_{i_1} \cdot e_{i_2}$$

    and

$$\mathcal{M}(\rho_s^1 \otimes \rho_s^2) = (r_{i_1j_1}(s)r_{i_2j_2}(s))$$

- Aside on quantum chemistry to come back to later; I can't quite connect the dots yet.

Section 1.6: Symmetric Square and Alternating Square

- Herein, we investigate the tensor product when $V_1 = V_2 = V$.
- Let (e_i) be a basis of V.
- Define the automorphism $\theta : V \otimes V \to V \otimes V$ by

$$\theta(e_i \cdot e_j) = e_j \cdot e_i$$

  for all 2-indices (i, j).
- Properties of θ.
  - Since θ is linear, it follows that

$$\theta(x \cdot y) = y \cdot x$$

    for all $x, y \in V$.
    - Implication: θ is independent of the chosen basis (e_i)!

  - $\theta^2 = 1$, where 1 is the identity map on $V \otimes V$.
- Assertion: $V \otimes V$ decomposes into
$$V \otimes V = S^2(V) \oplus \Lambda^2(V)$$
  - Rudenko: We do not have to worry about proving this... yet, at least.
- **Symmetric square representation**: The subspace of $V \otimes V$ containing all elements z satisfying $\theta(z) = z$. *Denoted by* $\boldsymbol{S^2V}$, $\boldsymbol{S^2(V)}$, $\boldsymbol{\mathbb{S}^2V}$, $\textbf{Sym}^{\boldsymbol{2}}\boldsymbol{(V)}$.
  - Basis: $(e_i \cdot e_j + e_j \cdot e_i)_{i \leq j}$.
    - Rudenko: How do we know everything is linearly independent? Well, when we add two linearly independent vectors out of a set, the sum is still linearly independent from everything else!
    - Example when $\dim V = 2$: The basis of $V \otimes V$ is $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$, where all four of these vectors are linearly independent. So naturally, the basis of the corresponding symmetric square representation — which is $2e_1 \otimes e_1, e_1 \otimes e_2 + e_2 \otimes e_1, 2e_2 \otimes e_2$ — will still be a linearly independent list of vectors.
  - Dimension: If $\dim V = n$, then
$$\dim S^2(V) = \frac{n(n+1)}{2}$$
- **Symmetric square representation**: The subspace of $V \otimes V$ containing all elements z satisfying $\theta(z) = -z$. *Denoted by* $\boldsymbol{S^2V}$, $\boldsymbol{S^2(V)}$, $\boldsymbol{\mathbb{S}^2V}$, $\textbf{Sym}^{\boldsymbol{2}}\boldsymbol{(V)}$.
  - Basis: $(e_i \cdot e_j - e_j \cdot e_i)_{i < j}$.
  - Dimension: If $\dim V = n$, then
$$\dim \Lambda^2(V) = \frac{n(n-1)}{2}$$