MATH 26700 (Introduction to Representation Theory of Finite Groups) Problem Sets

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Contents

1 Introduction to Representations

10/5:

1. Read Section 1.3 in Fulton and Harris (2004).
2. Let V, W be finite-dimensional vector spaces. Construct canonical isomorphisms...
   (a) $\Lambda^2(V \oplus W) \cong (\Lambda^2 V) \oplus (V \otimes W) \oplus (\Lambda^2 W)$;
   (b) $S^2(V \oplus W) \cong (S^2 V) \oplus (V \otimes W) \oplus (S^2 W)$.
3. (a) Factorize the group determinant for $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.
   (b) A **circulant matrix** is a square matrix in which all row vectors are composed of the same elements and each row vector is rotated one element to the right relative to the preceding row vector. Prove that for $\zeta \in \mu_n$, vector $(1, \zeta, \dots, \zeta^{n-1})$ is an eigenvector of any circulant matrix of size n.
   (c) Compute the eigenvalues and the determinant of a circulant matrix. Factorize the group determinant for $G = \mathbb{Z}/n\mathbb{Z}$.
4. **Plethysm for S_3.** Let (3), $(1,1,1)$, and $(2,1)$ be the trivial, alternating, and standard representations of S_3.
   (a) Consider the permutational representation $V \cong (3) \oplus (2,1)$. Decompose $\Lambda^2 V$ into irreducibles.
   (b) Decompose $S^2(2,1)$ and $S^3(2,1)$ into irreducibles.
   (c) Decompose the regular representation R into irreducibles.
   (d) Prove that $S^{k+6}(2,1) \cong S^k(2,1) \oplus R$. Compute $S^k(2,1)$ for all k.
5. Let V be a vector space over F with a basis $e_1, \dots, e_n$; let $e^1, \dots, e^n$ be the dual basis. Prove the following.
   (a) Element $e_1 \otimes e^1 + \cdots + e_n \otimes e^n \in V \otimes V^\vee$ does not depend on the choice of basis.
   (b) Consider a linear map $\mathrm{ev} : V \otimes V^* \to F$ sending $v \otimes \alpha$ to $\alpha(v) \in F$. Prove that $\mathrm{ev}(L) = \mathrm{tr}(L)$.
   (c) A **projector** is a linear map $P : V \to V$ such that $P^2 = P$. Prove that $\mathrm{tr}(P) = \dim(\mathrm{Im}(P))$.
   (d) Let V be a representation of a finite group G. Prove that the representation $V \otimes V^*$ has a trivial subrepresentation.

References

Fulton, W., & Harris, J. (2004). *Representation theory: A first course* (S. Axler, F. W. Gehring, & K. A. Ribet, Eds.). Springer.