Week 9

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9.1 ???

11/27:

- Announcements.
  - OH on Wednesday at 5:30 PM this week; not Tuesday.
  - There will be extra OH next week pre-exam.
    - Roughly like Monday/Wednesday next week.
  - Midterm will be returned on Wednesday; we can pick them up in-person in his office starting then.
  - There are some grade boundaries: Pass/Fail we can do til Friday, withdrawal we can do til 5:00 PM today.
- Let's finish the conversation about induction/restriction and prove the **branching theorem**.
- Reminder to start.
  - We have two mathematical categories, G-reps and H-reps where $H \leq G$.
  - These catagories are related by functors.
  - $\operatorname{Res}_H^G : G\text{-reps} \to H\text{-reps}$ and vice versa for Ind_H^G.
  - Restrictions are stupidly simple.
  - Inductions, most hands-on, we take copies of W times cosets. Formulaically,

$$\operatorname{Ind}_H^G W = g_1 W \oplus \cdots \oplus g_k W$$

    where $k = (G : H)$ and $G = \bigsqcup_{i=1}^k g_i H$.
    - In more detail, the action of g on $g_i w$ is that of $g_{\sigma(i)} h_i w$.
    - This is a genuinely hard construction.
    - A matrix of this thing will be a permutation matrix via

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    - Note that

$$g_1 W \oplus \cdots \oplus g_k W \cong \operatorname{Hom}_H(\mathbb{C}[G], W)$$

      - Recall that elements of the set on the right above are functions $f : G \to W$ such that $f(h(g)) = hf(g)$.
      - We map between the two via $f(g) \mapsto f(gx')$.
    - What is nice about induced representations is that $\dim[\operatorname{Ind}_H^G W] = (\dim W)[G : H]$.

- Moreover, there is a very easy statement, the **Frobenius formula**.
  - Recall that
$$\tilde{\chi}_W(g) = \begin{cases} 0 & g \notin H \\ \chi_W(g) & g \in H \end{cases}$$
  - With this, we average.
$$\chi_{\operatorname{Ind}_H^G W}(g) = \sum_{x \in G} \tilde{\chi}_W(xgx^{-1})$$
  - Essentially, we're taking a whole bunch of conjugates, summing them up, and dividing to get rid of overcounting.

- We now move onto **Frobenius reciprocity**, which is a relation between the functors/relations Ind_H^G and Res_H^G.
  - The first point where category theory gets interesting is the notion of **adjoint functors**, which we are about to touch on. It is a very subtle notion.
  - Here's version 1 of the statement of Frobenius reciprocity.
    - Recall that we have a scalar product on the space of class function, given by
$$(\chi_1, \chi_2) = \frac{1}{|G|} \sum_{g \in G} \chi_1(g)\chi_2(g^{-1})$$
      where χ_1, χ_2 are class functions on G.
    - Recall that if $\chi_1 = \chi_V$ and $\chi_2 = \chi_W$, then
$$(\chi_1, \chi_2) = \dim \operatorname{Hom}_G(V, W) = \dim \operatorname{Hom}_G\left(\bigoplus_{i=1}^k V_i^{n_i}, \bigoplus_{i=1}^k V_i^{m_i}\right) = \sum_{i=1}^k n_i m_i$$
    - Then the statement is as follows. If V is a G-rep and W is an H-rep, then
$$(V, \operatorname{Ind}_H^G W)_G = (\operatorname{Res}_H^G V, W)_H$$
      - ➢ Denoting scalar product in G and scalar product in W of the characters of each representation.
    - This is similar to the relation between adjoint maps $V \to W$ and $W^* \to V^*$.
  - Version 2.
    - We have that
$$\operatorname{Hom}_G(V, \operatorname{Ind}_H^G W) \cong \operatorname{Hom}_H(\operatorname{Res}_H^G V, W)$$
      where the isomorphism is canonical.
    - We will not check this last definition; we can tediously do it with definitions, and there's nothing complicated. Rudenko leaves this as an exercise to us.
  - Constructing...something: Take $v \in V$, $g \in G$, $\varphi : V \to W$. We send $g \mapsto \varphi(gv)$.

- We now prove Version 1.

Proof. We have

$$\begin{aligned}
(\chi_V, \chi_{\operatorname{Ind}_H^G W})_G &= \frac{1}{|G|}\sum_{g_1\in G}\chi_V(g_1)\left(\frac{1}{|H|}\sum_{g_2\in G}\tilde{\chi}_W(g_2g_1^{-1}g_2^{-1})\right)\\
&= \frac{1}{|H|\cdot|G|}\sum_{g_1,g_2\in G}\chi_V(g)\tilde{\chi}_W(g_2g_1^{-1}g_2^{-1})\\
&= \frac{1}{|H|\cdot|G|}\sum_{g_1,g_2\in G}\chi_V(\underbrace{g_2g_1g_2^{-1}}_{h})\tilde{\chi}_W(\underbrace{g_2g_1^{-1}g_2^{-1}}_{h^{-1}})\\
&= \frac{1}{|H|}\frac{1}{|G|}\sum_{h\in G}|G|\chi_V(h)\tilde{\chi}_W(h^{-1})\\
&= (\chi_V|_H, \chi_W)_H\\
&= (\operatorname{Res}_H^G V, \chi_W)_H
\end{aligned}$$

From line 3 to line 4: Fix h; then $g_2g_1g_2^{-1} = h$ iff $g_1 = g_2^{-1}hg_2$, so we have overcounted by $|G|$ times. ☐

- We now come to the branching theorem at long last.
- Example first.
  - Consider $S_n > S_{n-1}$, where S_{n-1} is the subgroup of permutations fixing n. I.e., $S_3 > S_2 = \{e, (12)\}$.
  - Let λ be a partition of n; *there's notation for this*!
  - Let $\mu \leq \lambda$ be a Young diagram of a partition of $n-1$.
  - Then
    1. We have
    $$\operatorname{Res}_{S_{n-1}}^{S_n} V_\lambda = \bigoplus_{\mu\leq\lambda} V_\mu$$
       - Example: *Draw out pictures*
    2. We have
    $$\operatorname{Ind}_{S_{n-1}}^{S_n} V_\mu = \bigoplus_{\mu\leq\lambda} V_\lambda$$
       - Example: *Draw out pictures*
- The reason that this theorem is called the branching theorem originates from the following diagram, which (when continued) encapsulates the main idea of the theorem. *picture*
  - This graph helps you understand induction and restriction.
  - Dimensions are the number of paths from the left to a a final Young diagram.
    - For example, the dimension of $(3,1)$ is 3 because there are 3 paths to it (*list them*).
  - Number of paths formula is equivalent to standard Young tableaux!
- Theorem (Branching): The following two statements are true.

$$\operatorname{Res}_{S_{n-1}}^{S_n} V_\lambda = \bigoplus_{\mu\leq\lambda} V_\mu \tag{9.1}$$

$$\operatorname{Ind}_{S_{n-1}}^{S_n} V_\mu = \bigoplus_{\mu\leq\lambda} V_\lambda \tag{9.2}$$

Proof. We'll talk about the general idea of the proof now, and maybe do the details next time.

(1) $\Longleftrightarrow$ (2): We have that *stuff at bottom of board*

(1): Let's look at an example. Here's a YD of S_8. We want to restrict it down to S_7. Recall that $V_\lambda = \operatorname{span}(S_8 : \Delta(x_1, x_2, x_3)(x_4 - x_5)(x_6 - x_7))$. Now in S_7, we fix x_8. Consider subrepresentations of V_λ filtered by degree as follows.

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The proof comes from the fact that if we now take quotients of these subrepresentations, then since x_8 can only appear in three boxes, ... ☐

- Practice with the above example and think it through.

9.2 ???

11/29:
- Announcements.
  - OH today at 5:30.
  - Our midterms are graded; we can look at them in his office whenever (I can do this during OH!).
- Today, we'll formulate the main result he wants to prove next time.
- Goal is still to understand representations of S_n.
  - We've constructed all of them using Specht modules, but what else do we want?
  - We have dimension, we want characters, etc.
- The main idea is to look at symmetric polynomials once again.
  - Consider $\mathbb{Q}[x_1, \dots, x_n]^{S_n}$.
  - We have proven the fundamental theorem that $\mathbb{Q}[x_1, \dots, x_n]^{S_n} = \mathbb{Q}[\sigma_1, \dots, \sigma_n]$ where $\sigma_k = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} x_{i_1} \cdots x_{i_k}$.
  - We also proved in the homework that these rings are equal to $\mathbb{Q}[p_1, \dots, p_k]$ and $\mathbb{Q}[h_1, \dots, h_k]$ where

$$p_k = \sum x_i^k \qquad h_k = \sum_{i_1 \leq \cdots \leq i_k} x_1 \cdots x_k$$

  - Example: If $n = 3$, then

$$h_2 = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$$

  - Table of bases for n, k.

kn	1	2	3	4
0	1	1	1	1
1	x_1	$x_1 + x_2$	$x_1 + x_2 + x_3$	$\cdots$
2	x_1^2	$x_1^2 + x_2^2$, x_1x_2	σ_1^2, σ_2^2	σ_1^2, σ_2^2
3	x_1^3	...		

Table 9.1: ...

  - Now take

$$\Lambda_k = [\mathbb{C}[x_1, \dots, x_k]]_{\deg = k-1} \cong [\mathbb{C}[x_1, \dots, x_{k+1}]]_{\deg = k-1} \cong \cdots$$

    - ■ Alternatively, we can think of this thing as

$$\Lambda_k = (\mathbb{C}[x_1, \dots])_k$$

with $\sigma_1^k, \sigma_2\sigma_1^{k-1}, \dots$.

  - We call Λ the ring of symmetric functions and define it to be equal to

$$\Lambda = \mathbb{Q}[\sigma_1, \sigma_2, \sigma_3, \dots]$$

  - In every complete component, only finitely many of the σ will participate, so we get finite things.
  - This is a graded ring! We have

$$\Lambda = \bigoplus_{k \geq 0} \Lambda_k$$

and $\Lambda_k \otimes \Lambda_\ell = \Lambda_{k+\ell}$

  - This construction is called the **projective limit**, and we may have encountered it in commutative algebra under the definition

$$\Lambda = \lim_{\rightarrow} \mathbb{C}[x_1, \dots, x_n]^{S_n}$$

  - We have identifies such as $p_2 = \sigma_1^2 - 2\sigma_2$. This means that

$$(x_1 + \dots + x_n)^2 - 2(x_1x_2 + x_1x_3 + \cdots) = x_1^2 + x_2^2 + \cdots$$

  - Observation: $\dim_{\mathbb{Q}} \Lambda_n$.

- Now, we need to take a vector space on ring representations; we've done this already with the representation ring.

- Let R_n be the $\mathbb{Q}$-vector space of functions $\chi : S_n \to \mathbb{Q}$ such that $\chi(x\sigma x^{-1}) = \chi(\sigma)$. This is our favorite space of class functions.

- Theorem (Frobenius characteristic map): There is an isomorphism of vector spaces and of rings called the Frobenius characteristic: $\text{ch} : \bigoplus_{n \geq 0} R_n \to \Lambda$.

  Proof. Take $\chi_V \in R_k$, and $\chi_W \in R_\ell$. Let V an S_k-rep, and W an S_ℓ-rep. We know that

$$S_k \times S_\ell = S_{k+\ell}$$

  So what we can do is induction $\text{Ind}_{S_k \times S_\ell}^{S_{k+\ell}}(V \otimes W)$. Call this operation $\chi_V \boxtimes \chi_W$.

  Now we write down the formula:

$$\text{ch}(\chi) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi(\sigma) p_1^{\lambda_1(\sigma)} \cdots p_k^{\lambda_k(\sigma)}$$

  where $\lambda_1(\sigma), \lambda_2(\sigma), \dots$ represent the cycle structure of σ; each λ_i is a number of cycles of length $1, 2, \dots$. □

- Examples.

  1. S_1.
     - Sends the YD (1) to $p_1 = x_1 + x_2 + x_3 + \cdots$.
  2. S_2.
     - Sends (2) to $\frac{1}{2!}(p_1^2 + p_2) = \frac{1}{2}((x_1 + x_2)^2 + x_1^2 + x_2^2) = x_1^2 + x_2^2 + x_1x_2 = h_2$.
     - It also sends $(1, 1)$ to $\frac{1}{2!}(p_1^2 - p_2) = \frac{1}{2}((x_1 + x_2)^2 - x_1^2 - x_2^2) = x_1x_2 = \sigma_2$.

  - Let's check our formula. What is $\operatorname{Ind}_{S_1 \times S_1}^{S_2}(1) \otimes (1)$? Since the induction of the trivial representation is the regular representation, which we can decompose, we know that this induction equals $(1,1) \oplus (2)$. It follows that $p_1^2 = x_1^2 + x_2^2 + x_1x_2 + x_1x_2 = (x_1+x_2)^2$.

3. S_3.
  - Sends (3) to

$$\begin{aligned}\frac{1}{3!}(p_1^3 + 3p_1p_2 + 2p_3) &= \frac{1}{6}[(x_1+x_2+x_3)^3 + 3(x_1+x_2+x_3)(x_1^2+x_2^2+x_3^2) + 2(x_1^3+x_2^3+x_3^3)] \\ &= \frac{1}{6}[6(x_1^3+x_2^3+x_3^3) + 6(x_1^2x_2 + x_1x_2^2 + x_1x_3^2 + x_1^2x_3 + \cdots) + 6x_1x_2x_3] \\ &= h_3\end{aligned}$$

  - Sends $(1,1,1)$ to

$$\begin{aligned}\frac{1}{3!}(p_1^3 - 3p_1p_2 + 2p_3) &= \frac{1}{6}[(x_1+x_2+x_3)^3 - 3(x_1+x_2+x_3)(x_1^2+x_2^2+x_3^2) + 2(x_1^3+x_2^3+x_3^3)] \\ &= x_1x_2x_3 \\ &= \sigma_3\end{aligned}$$

  - Sends $(2,1)$ to

$$\begin{aligned}\frac{1}{3!}(2p_1^3 - p_3) &= \frac{1}{6}[2(x_1^3+x_2^3+x_3^3) + 6(x_1^2x_2 + \cdots) + 12x_1x_2x_3] \\ &= (x_1^2 + \cdots) + 2x_1x_2x_3\end{aligned}$$

  - Again, we can check that

$$\operatorname{Ind}_{S_2 \times S_1}^{S_3}[(1,1) \otimes (1)] = \sigma_1\sigma_2$$

    - We compute $\operatorname{Ind}_{S_2}^{S_3}(1,1) = (1,1,1) \oplus (2,1)$ via the branching formula: There are only two ways to add a box!
    - We have $\sigma_1\sigma_2 - \sigma_3 = (x_1+x_2+x_3)(x_1x_2 + x_1x_3 + x_2x_3) - x_1x_2x_3$.

- Do we need to be fluent in the techniques by which you expanded all of the polynomials above??
- Thus, we have two conjectures:

$$\operatorname{ch}[(n)] = h_n \qquad\qquad \operatorname{ch}[\underbrace{(1,\dots,1)}_{n \text{ times}}] = \sigma_n$$

- The theorem is cool because it sends all of representation theory to some symmetric polynomial game!
- How do we compute $\operatorname{ch}(V_\lambda)$?
  - We say it equals S_λ, where S_λ is a Schur polynomial.
  - Take the YT of λ. Recall standard YTs.
  - **Semistandard** (YT): Things strictly increase in columns, but only monotonically increase in rows. *draw picture!*
  - The six semistandard ones give us the Schur polynomial.
  - Relation to RSK correspondence.
- Proving why this stuff is true is not hard.
- To understand *why* this is true, Google the **Schur-Weyl duality**.