# Week 9

# ???

## 9.1 ???

11/27:

- Announcements.
  - OH on Wednesday at 5:30 PM this week; not Tuesday.
  - There will be extra OH next week pre-exam.
    - Roughly like Monday/Wednesday next week.
  - Midterm will be returned on Wednesday; we can pick them up in-person in his office starting then.
  - There are some grade boundaries: Pass/Fail we can do til Friday, withdrawal we can do til 5:00 PM today.
- Let's finish the conversation about induction/restriction and prove the **branching theorem**.
- Reminder to start.
  - We have two mathematical categories, $G$-reps and $H$-reps where $H \leq G$.
  - These catagories are related by functors.
  - $\text{Res}_H^G : G\text{-reps} \to H\text{-reps}$ and vice versa for $\text{Ind}_H^G$.
  - Restrictions are stupidly simple.
  - Inductions, most hands-on, we take copies of $W$ times cosets. Formulaically,

    $$\text{Ind}_H^G W = g_1 W \oplus \cdots \oplus g_k W$$

    where $k = (G : H)$ and $G = \bigsqcup_{i=1}^k g_i H$.
    - In more detail, the action of $g$ on $g_i w$ is that of $g_{\sigma(i)} h_i w$.
    - This is a genuinely hard construction.
    - A matrix of this thing will be a permutation matrix via

      $$copyfromboard$$

    - Note that

      $$g_1 W \oplus \cdots \oplus g_k W \cong \text{Hom}_H(\mathbb{C}[G], W)$$

      - Recall that elements of the set on the right above are functions $f : G \to W$ such that $f(h(g)) = hf(g)$.
      - We map between the two via $f(g) \mapsto f(gx')$.
    - What is nice about induced representations is that $\dim[\text{Ind}_H^G W] = (\dim W)[G : H]$.

- Moreover, there is a very easy statement, the **Frobenius formula**.
  - Recall that
$$\tilde{\chi}_W(g) = \begin{cases} 0 & g \notin H \\ \chi_W(g) & g \in H \end{cases}$$
  - With this, we average.
$$\chi_{\operatorname{Ind}_H^G W}(g) = \sum_{x \in G} \tilde{\chi}_W(xgx^{-1})$$
  - Essentially, we're taking a whole bunch of conjugates, summing them up, and dividing to get rid of overcounting.

- We now move onto **Frobenius reciprocity**, which is a relation between the functors/relations $\operatorname{Ind}_H^G$ and $\operatorname{Res}_H^G$.
  - The first point where category theory gets interesting is the notion of **adjoint functors**, which we are about to touch on. It is a very subtle notion.
  - Here's version 1 of the statement of Frobenius reciprocity.
    - Recall that we have a scalar product on the space of class function, given by
$$(\chi_1, \chi_2) = \frac{1}{|G|} \sum_{g \in G} \chi_1(g)\chi_2(g^{-1})$$
where $\chi_1, \chi_2$ are class functions on $G$.
    - Recall that if $\chi_1 = \chi_V$ and $\chi_2 = \chi_W$, then
$$(\chi_1, \chi_2) = \dim \operatorname{Hom}_G(V, W) = \dim \operatorname{Hom}_G\left(\bigoplus_{i=1}^k V_i^{n_i}, \bigoplus_{i=1}^k V_i^{m_i}\right) = \sum_{i=1}^k n_i m_i$$
    - Then the statement is as follows. If $V$ is a $G$-rep and $W$ is an $H$-rep, then
$$(V, \operatorname{Ind}_H^G W)_G = (\operatorname{Res}_H^G V, W)_H$$
      - ➢ Denoting scalar product in $G$ and scalar product in $W$ of the characters of each representation.
    - This is similar to the relation between adjoint maps $V \to W$ and $W^* \to V^*$.
  - Version 2.
    - We have that
$$\operatorname{Hom}_G(V, \operatorname{Ind}_H^G W) \cong \operatorname{Hom}_H(\operatorname{Res}_H^G V, W)$$
where the isomorphism is canonical.
    - We will not check this last definition; we can tediously do it with definitions, and there's nothing complicated. Rudenko leaves this as an exercise to us.
  - Constructing...something: Take $v \in V$, $g \in G$, $\varphi : V \to W$. We send $g \mapsto \varphi(gv)$.

- We now prove Version 1.

*Proof.* We have

$$\begin{aligned}
(\chi_V, \chi_{\operatorname{Ind}_H^G W})_G &= \frac{1}{|G|}\sum_{g_1\in G}\chi_V(g_1)\left(\frac{1}{|H|}\sum_{g_2\in G}\tilde{\chi}_W(g_2g_1^{-1}g_2^{-1})\right)\\
&= \frac{1}{|H|\cdot|G|}\sum_{g_1,g_2\in G}\chi_V(g)\tilde{\chi}_W(g_2g_1^{-1}g_2^{-1})\\
&= \frac{1}{|H|\cdot|G|}\sum_{g_1,g_2\in G}\chi_V(\underbrace{g_2g_1g_2^{-1}}_{h})\tilde{\chi}_W(\underbrace{g_2g_1^{-1}g_2^{-1}}_{h^{-1}})\\
&= \frac{1}{|H|}\frac{1}{|G|}\sum_{h\in G}|G|\chi_V(h)\tilde{\chi}_W(h^{-1})\\
&= (\chi_V|_H, \chi_W)_H\\
&= (\operatorname{Res}_H^G V, \chi_W)_H
\end{aligned}$$

From line 3 to line 4: Fix $h$; then $g_2g_1g_2^{-1} = h$ iff $g_1 = g_2^{-1}hg_2$, so we have overcounted by $|G|$ times. ☐

- We now come to the branching theorem at long last.
- Example first.
  - Consider $S_n > S_{n-1}$, where $S_{n-1}$ is the subgroup of permutations fixing $n$. I.e., $S_3 > S_2 = \{e, (12)\}$.
  - Let $\lambda$ be a partition of $n$; *there's notation for this*!
  - Let $\mu \leq \lambda$ be a Young diagram of a partition of $n-1$.
  - Then
    1. We have
    $$\operatorname{Res}_{S_{n-1}}^{S_n} V_\lambda = \bigoplus_{\mu\leq\lambda} V_\mu$$
       - Example: *Draw out pictures*
    2. We have
    $$\operatorname{Ind}_{S_{n-1}}^{S_n} V_\mu = \bigoplus_{\mu\leq\lambda} V_\lambda$$
       - Example: *Draw out pictures*
- The reason that this theorem is called the branching theorem originates from the following diagram, which (when continued) encapsulates the main idea of the theorem. *picture*
  - This graph helps you understand induction and restriction.
  - Dimensions are the number of paths from the left to a a final Young diagram.
    - For example, the dimension of $(3,1)$ is 3 because there are 3 paths to it (*list them*).
  - Number of paths formula is equivalent to standard Young tableaux!
- Theorem (Branching): The following two statements are true.

$$\operatorname{Res}_{S_{n-1}}^{S_n} V_\lambda = \bigoplus_{\mu\leq\lambda} V_\mu \tag{9.1}$$

$$\operatorname{Ind}_{S_{n-1}}^{S_n} V_\mu = \bigoplus_{\mu\leq\lambda} V_\lambda \tag{9.2}$$

*Proof.* We'll talk about the general idea of the proof now, and maybe do the details next time.

(1) $\iff$ (2): We have that *stuff at bottom of board*

(1): Let's look at an example. Here's a YD of $S_8$. We want to restrict it down to $S_7$. Recall that $V_\lambda = \text{span}(S_8 : \Delta(x_1, x_2, x_3)(x_4 - x_5)(x_6 - x_7))$. Now in $S_7$, we fix $x_8$. Consider subrepresentations of $V_\lambda$ filtered by degree as follows.

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The proof comes from the fact that if we now take quotients of these subrepresentations, then since $x_8$ can only appear in three boxes, ... ☐

- Practice with the above example and think it through.

## 9.2 ???

11/29: - Announcements.
    - OH today at 5:30.
    - Our midterms are graded; we can look at them in his office whenever (I can do this during OH!).
- Today, we'll formulate the main result he wants to prove next time.
- Goal is still to understand representations of $S_n$.
    - We've constructed all of them using Specht modules, but what else do we want?
    - We have dimension, we want characters, etc.
- The main idea is to look at symmetric polynomials once again.
    - Consider $\mathbb{Q}[x_1, \dots, x_n]^{S_n}$.
    - We have proven the fundamental theorem that $\mathbb{Q}[x_1, \dots, x_n]^{S_n} = \mathbb{Q}[\sigma_1, \dots, \sigma_n]$ where $\sigma_k = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq n} x_{i_1} \cdots x_{i_k}$.
    - We also proved in the homework that these rings are equal to $\mathbb{Q}[p_1, \dots, p_k]$ and $\mathbb{Q}[h_1, \dots, h_k]$ where

$$p_k = \sum x_i^k \qquad\qquad h_k = \sum_{i_1 \leq \cdots \leq i_k} x_1 \cdots x_k$$

    - Example: If $n = 3$, then

$$h_2 = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$$

    - Table of bases for $n$, $k$.

| kn | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| 1 | $x_1$ | $x_1 + x_2$ | $x_1 + x_2 + x_3$ | $\cdots$ |
| 2 | $x_1^2$ | $x_1^2 + x_2^2$, $x_1x_2$ | $\sigma_1^2$, $\sigma_2^2$ | $\sigma_1^2$, $\sigma_2^2$ |
| 3 | $x_1^3$ | ... | | |

Table 9.1: ...

    - Now take

$$\Lambda_k = [\mathbb{C}[x_1, \dots, x_k]]_{\deg = k-1} \cong [\mathbb{C}[x_1, \dots, x_{k+1}]]_{\deg = k-1} \cong \cdots$$

        - Alternatively, we can think of this thing as

$$\Lambda_k = (\mathbb{C}[x_1, \dots])_k$$

          with $\sigma_1^k, \sigma_2\sigma_1^{k-1}, \dots$.
    - We call $\Lambda$ the ring of symmetric functions and define it to be equal to

$$\Lambda = \mathbb{Q}[\sigma_1, \sigma_2, \sigma_3, \dots]$$

    - In every complete component, only finitely many of the $\sigma$ will participate, so we get finite things.
    - This is a graded ring! We have

$$\Lambda = \bigoplus_{k \geq 0} \Lambda_k$$

      and $\Lambda_k \otimes \Lambda_\ell = \Lambda_{k+\ell}$
    - This construction is called the **projective limit**, and we may have encountered it in commutative algebra under the definition

$$\Lambda = \lim_{\rightarrow} \mathbb{C}[x_1, \dots, x_n]^{S_n}$$

    - We have identifies such as $p_2 = \sigma_1^2 - 2\sigma_2$. This means that

$$(x_1 + \cdots + x_n)^2 - 2(x_1x_2 + x_1x_3 + \cdots) = x_1^2 + x_2^2 + \cdots$$

    - Observation: $\dim_{\mathbb{Q}} \Lambda_n$.

- Now, we need to take a vector space on ring representations; we've done this already with the representation ring.
- Let $R_n$ be the $\mathbb{Q}$-vector space of functions $\chi : S_n \to \mathbb{Q}$ such that $\chi(x\sigma x^{-1}) = \chi(\sigma)$. This is our favorite space of class functions.
- Theorem (Frobenius characteristic map): There is an isomorphism of vector spaces and of rings called the Frobenius characteristic: $\mathrm{ch} : \bigoplus_{n \geq 0} R_n \to \Lambda$.

  *Proof.* Take $\chi_V \in R_k$, and $\chi_W \in R_\ell$. Let $V$ an $S_k$-rep, and $W$ an $S_\ell$-rep. We know that

$$S_k \times S_\ell = S_{k+\ell}$$

  So what we can do is induction $\mathrm{Ind}_{S_k \times S_\ell}^{S_{k+\ell}}(V \otimes W)$. Call this operation $\chi_V \boxtimes \chi_W$.
  Now we write down the formula:

$$\mathrm{ch}(\chi) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi(\sigma) p_1^{\lambda_1(\sigma)} \cdots p_k^{\lambda_k(\sigma)}$$

  where $\lambda_1(\sigma), \lambda_2(\sigma), \dots$ represent the cycle structure of $\sigma$; each $\lambda_i$ is a number of cycles of length $1, 2, \dots$. $\square$

- Examples.
    1. $S_1$.
        - Sends the YD (1) to $p_1 = x_1 + x_2 + x_3 + \cdots$.
    2. $S_2$.
        - Sends (2) to $\frac{1}{2!}(p_1^2 + p_2) = \frac{1}{2}((x_1 + x_2)^2 + x_1^2 + x_2^2) = x_1^2 + x_2^2 + x_1x_2 = h_2$.
        - It also sends $(1,1)$ to $\frac{1}{2!}(p_1^2 - p_2) = \frac{1}{2}((x_1 + x_2)^2 - x_1^2 - x_2^2) = x_1x_2 = \sigma_2$.

  - Let's check our formula. What is $\mathrm{Ind}_{S_1\times S_1}^{S_2}(1)\otimes(1)$? Since the induction of the trivial representation is the regular representation, which we can decompose, we know that this induction equals $(1,1)\oplus(2)$. It follows that $p_1^2 = x_1^2 + x_2^2 + x_1x_2 + x_1x_2 = (x_1+x_2)^2$.

3. $S_3$.
  - Sends $(3)$ to

$$\begin{aligned}\frac{1}{3!}(p_1^3+3p_1p_2+2p_3) &= \frac{1}{6}[(x_1+x_2+x_3)^3 + 3(x_1+x_2+x_3)(x_1^2+x_2^2+x_3^2) + 2(x_1^3+x_2^3+x_3^3)] \\ &= \frac{1}{6}[6(x_1^3+x_2^3+x_3^3) + 6(x_1^2x_2 + x_1x_2^2 + x_1x_3^2 + x_1^2x_3 + \cdots) + 6x_1x_2x_3] \\ &= h_3\end{aligned}$$

  - Sends $(1,1,1)$ to

$$\begin{aligned}\frac{1}{3!}(p_1^3-3p_1p_2+2p_3) &= \frac{1}{6}[(x_1+x_2+x_3)^3 - 3(x_1+x_2+x_3)(x_1^2+x_2^2+x_3^2) + 2(x_1^3+x_2^3+x_3^3)] \\ &= x_1x_2x_3 \\ &= \sigma_3\end{aligned}$$

  - Sends $(2,1)$ to

$$\begin{aligned}\frac{1}{3!}(2p_1^3-p_3) &= \frac{1}{6}[2(x_1^3+x_2^3+x_3^3) + 6(x_1^2x_2+\cdots) + 12x_1x_2x_3] \\ &= (x_1^2+\cdots) + 2x_1x_2x_3\end{aligned}$$

  - Again, we can check that

$$\mathrm{Ind}_{S_2\times S_1}^{S_3}[(1,1)\otimes(1)] = \sigma_1\sigma_2$$

    - We compute $\mathrm{Ind}_{S_2}^{S_3}(1,1) = (1,1,1)\oplus(2,1)$ via the branching formula: There are only two ways to add a box!
    - We have $\sigma_1\sigma_2 - \sigma_3 = (x_1+x_2+x_3)(x_1x_2+x_1x_3+x_2x_3) - x_1x_2x_3$.

- Do we need to be fluent in the techniques by which you expanded all of the polynomials above??
- Thus, we have two conjectures:

$$\mathrm{ch}[(n)] = h_n \qquad\qquad \mathrm{ch}[\underbrace{(1,\dots,1)}_{n\text{ times}}] = \sigma_n$$

- The theorem is cool because it sends all of representation theory to some symmetric polynomial game!
- How do we compute $\mathrm{ch}(V_\lambda)$?
  - We say it equals $S_\lambda$, where $S_\lambda$ is a Schur polynomial.
  - Take the YT of $\lambda$. Recall standard YTs.
  - **Semistandard** (YT): Things strictly increase in columns, but only monotonically increase in rows. *draw picture!*
  - The six semistandard ones give us the Schur polynomial.
  - Relation to RSK correspondence.
- Proving why this stuff is true is not hard.
- To understand *why* this is true, Google the **Schur-Weyl duality**.

## 9.3 Office Hours

- I got a 68/100 on the midterm: 30, 24, 0, 14.
  - I would have needed to show my work (or at least one example of a calculation) to get full credit for 2, even though it just said "find."
  - Rudenko did not expect that finding conjugacy classes would be so difficult for us; he will adjust for this difficulty on the final.
- Week 3, Lecture 2: You proved that $\langle \chi_V, \chi_W \rangle = \delta_{VW}$. To do so, you used a projection function $p = (1/|G|) \sum_{g \in G} gv$. You began your proof by proving that $p$ is a $G$-morphism and then never used this result again, as far as I can tell. Did you use it again? See pp. 45-47, 58 (it needs to be a morphism of $G$-representations to map between the representations $V, V^G$?).
- Week 3, Lecture 2: Same proof. To prove that $\text{Im}(P) = V^G$, do we need more than $p^2 = p$? I think so, but you didn't do it explicitly. See pp. 46-47.
- Week 3, Lecture 2: Same proof. What's up with the trivial special case? See p. 48.
- *Week 3, Lecture 3: Cube thing (see picture from 10/13)?
  - It's just a depiction of two different 3-coordinate bases of the same space. It was drawn to illustrate a possible relation between the orthonormal basis $\chi_1, \chi_2, \chi_3$ (cube) and the orthogonal basis $\chi_{C_1}, \chi_{C_2}, \chi_{C_3}$.
- Week 3, Lecture 3: Why did we talk about the infinite-dimensional regular representation here? See p. 50.
- *Week 3, Lecture 3: What is the point of the misc. calculations involved in computing the $S_4$ character table? See p. 52.
  - Just to check that we were on the right path and shown an example of using the orthogonality relations.
- *Week 3, Lecture 3: Proof of the second orthogonality relation your way? It's in Serre (1977), but I don't think that's the way you proved it. See p. 52.
  - To begin, note that it is a *highly* nontrivial statement that if $A, B$ are matrices such that $AB = I$, then $BA = I$. It seems so simple to us, but think about it! For an arbitrary matrix $A, B$, $AB$ looks nothing like $BA$! We have two entirely different systems of equations.
  - However, using this fact, basically it is possible to translate the orthogonality relation for the *columns* into the orthogonality relation about the *rows*.
- *Week 3, Lecture 3: All the talk about the exceptional homomorphisms? See p. 52, 61 (the final representation has something to do with an **involution** of trace 2, and is a representation of a quotient group?).
  - So the representation is $\rho : S_4 \twoheadrightarrow S_3 \xrightarrow{\tilde{\rho}} GL_n$, where $\tilde{\rho} : S_3 \to GL_n$ is the representation of $\rho$ corresponding to the character $(2, 0, 1)$.
- *Week 4, Lecture 1: Alternate construction of $R(G)$? See p. 63.
- *Week 4, Lecture 1: Extension of scalars with the representation ring? See p. 64.
  - We don't need to know anything about this stuff.
  - What it is though is basically analogous to extending the real numbers into a subset of the complex numbers by treating every $x \in \mathbb{R}$ as $x + 0i \in \mathbb{C}$. Very trivial, silly concept.
  - There is also such a thing as a **reduction of scalars**.

- *Week 4, Lecture 1: Does multiplying a column vector in the basis $\{\delta_{C_i}\}$ by the character table put it in the basis $\{\chi_{V_i^*}\}$, or vice versa? See p. 66.
  - Derive it for yourself.
- *Week 4, Lecture 2: Isotypical components example. See p. 68.
- **Week 4, Lecture 3: Proof the $\mathbb{C}$ is the only finite-dimensional division algebra? See p. 71.
  - Let $A$ be an arbitrary finite-dimensional division algebra over $\mathbb{C}$.
  - To prove that $A = \mathbb{C}$, we will use a bidirectional inclusion proof.
  - Naturally, $\mathbb{C} \subset A$.
  - To prove the reverse implication, start by letting $a \in A$ be arbitrary.
  - Define the left-multiplication operator $L_a : A \to A$ by $x \mapsto ax$ for all $x \in A$.
  - Recall that $A$ is a complex vector space in addition to being an algebra, the same way a ring is also a group. Thus, $L_a$ is a linear operator on a complex vector space.
  - It follows by the theorem of linear algebra that $L_a$ has an eigenvalue $\lambda \in F = \mathbb{C}$ and corresponding eigenvector $b \in A$.
  - Consequently, by the cancellation lemma,

$$\begin{aligned} L_a b &= \lambda b \\ ab &= \lambda b \\ a &= \lambda \end{aligned}$$

  - Therefore, $a \in A$ implies $a \in \mathbb{C}$, so $A = \mathbb{C}$.
- *Week 6, Lecture 2: Proof that $\sqrt{2}/2$ is not an algebraic integer using Gauss's lemma? See p. 87.
  - Let $\alpha := \sqrt{2}/2$ for the sake of notation.
  - Suppose for the sake of contradiction that $\alpha$ is an algebraic integer.
  - Then there exists a monic polonomial $p(x) \in \mathbb{Z}[x]$ such that $p(\alpha) = 0$.
  - Observe that the minimal polynomial in $\mathbb{Z}[x]$ that annihilates $\alpha$ is $2x^2 - 1$.
  - Thus, by polynomial division,

$$p(x) = q(x) \cdot (2x^2 - 1) + r(x)$$

    for some $q, r \in \mathbb{Q}[x]$ such that $\deg r \leq 2 - 1$.
  - We have that

$$r(\alpha) = p(\alpha) - q(\alpha) \cdot (2\alpha^2 - 1) = 0 - q(\alpha) \cdot 0 = 0$$

  - Additionally, since $r \in \mathbb{Q}[x]$ and $\deg r \leq 1$, we know that $r(x) = ux + v$ for some $u, v \in \mathbb{Q}$.
  - We now prove that $u = v = 0$.
    - Suppose for the sake of contradiction that either $u$ or $v$ was not equal to zero.
    - Combining the previous two claims reveals that

$$\begin{aligned} 0 &= r(\alpha) \\ &= u\alpha + v \\ -\frac{v}{u} &= \alpha \end{aligned}$$

    - If $u = 0$, then $\alpha$ is undefined and we have arrived at a contradiction. Thus, $u \neq 0$.
    - Thus, $\alpha \in \mathbb{Q}$. But since $\alpha \notin \mathbb{Q}$ by definition, we have arrived at a contradiction.

- Therefore, $u = v = 0$.

- Having established that $r = 0$, we know that $p = (2x^2 - 1)q$, i.e., $2x^2 - 1$ divides $p$.
- Now define $N$ to be the least common multiple of the denominators of the coefficients of $q$.
- Consider
$$Np = (Nq)(2x^2 - 1)$$
- It follows by Gauss's lemma that
$$\begin{aligned} c(Np) &= c[(Nq)(2x^2 - 1)] \\ N &= c(Nq) \cdot c(2x^2 - 1) \\ &= 1 \cdot 1 \\ &= 1 \end{aligned}$$
where $c$ denotes the **content**.
- But if $N = 1$, then $q \in \mathbb{Z}[x]$, so leading term of $p$ — equal to the product of $2x^2$ and the leading term of $q$ — has a coefficient that is a multiple of 2, i.e., is *not* equal to 1 as is required of a monic polynomial, a contradiction.

- *Week 6, Lecture 3: Questions about Lemma 1 of the proof of Burnside's theorem. See p. 92.
  - The roots $a_1, \dots, a_k$ of the minimal polynomial of the algebraic integer $a$ are known as **conjugate algebraic integers**.
  - The conjugate algebraic integers of a root of unity are also roots of unity.
    - Suppose $\varepsilon$ is a root of unity.
    - Then the minimal polynomial of $\varepsilon$ is $x^n - 1$ for some $n \in \mathbb{N}$.
    - Naturally, the roots of this polynomial (the conjugate algebraic integers to $\varepsilon$) are all of the other roots of unity of order $n$.
  - The conjugate algebraic integers of a sum of roots of unity is a sum of roots of unity.
    - It can be shown that the minimal polynomial for $\varepsilon_1 + \varepsilon_2$ is
$$p(x) = \prod_{i,j=1}^{n} (x - \varepsilon_1^i - \varepsilon_2^j)$$
    - Evidently, the above polynomial is symmetric under permutations of $\varepsilon_1^i, \varepsilon_2^j$, and we'd generate the same polynomial with any $\pm\varepsilon_1^i \pm \varepsilon_2^j$ as starting material.
    - Explicit example.
      - $\pm\sqrt{2}$ are conjugate algebraic integers, as solutions to $x^2 - 2$. Similarly, $\pm\sqrt{3}$ are conjugate algebraic integers as solutions to $x^2 = 3$.
      - Thus, we expect the minimal polynomial for $\sqrt{2} + \sqrt{3}$ to be
$$p(x) = (x - \sqrt{2} - \sqrt{3})(x - \sqrt{2} + \sqrt{3})(x + \sqrt{2} - \sqrt{3})(x + \sqrt{2} + \sqrt{3})$$
      - Expanding, we obtain
$$\begin{aligned} p(x) &= (x^2 - (\sqrt{2} + \sqrt{3})^2)(x^2 - (\sqrt{2} - \sqrt{3})^2) \\ &= x^4 - [(\sqrt{2} + \sqrt{3})^2 + (\sqrt{2} - \sqrt{3})^2]x^2 + (\sqrt{2} + \sqrt{3})^2(\sqrt{2} - \sqrt{3})^2 \\ &= x^4 - 10x^2 + 1 \end{aligned}$$
      - Indeed, the above polynomial is a monic polynomial
      - From the definition, this polynomial is evidently also the minimal polynomial for $\sqrt{2} - \sqrt{3}$, $-\sqrt{2} + \sqrt{3}$, and $-\sqrt{2} - \sqrt{3}$.

➢ Thus, the conjugate algebraic integers of $\sqrt{2}+\sqrt{3}$ are the four sums of all individual algebraic integers.

- How do we extend this argument to the case in the problem?? What about when $\varepsilon_1 = -1$ and $\varepsilon_2 = i$ so that simple powers don't access every combination as the $p(x)$ formula does?
- We know that $\prod_{i=1}^n a_i \in \mathbb{Z}$ because of Vieta's formula.
  - In particular, this tells us that $x_1 \cdots x_n$ is equal to the last coefficient in the minimal polynomial which, by definition, is an integer.

- *Week 7, Lecture 2: Symmetric polynomials and roots of symmetric polynomials. See p. 101.
- *Week 7, Lecture 2: Word in blackboard picture? See p. 102.
  - "Remain" to show...
- *Week 7, Lecture 2: What is $d$ in the proof of the alternating polynomials theorem? See p. 103.
  - $d = n-1$.
- *Week 8, Lecture 2: What is $d$ in the definition on p. 111.
  - Consider the Specht polynomial corresponding to $(2,2,1)$.

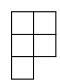

Figure 9.1: Young diagram for $(2,2,1)$.

  - Since $(2,2,1)' = (3,2)$, the Specht polynomial is $(x_1-x_2)(x_1-x_3)(x_2-x_3)\cdot(x_4-x_5)$.
  - $\Delta_{123}$ is of degree $3 = \binom{3}{2}$ because looking at the first column of the YD, which corresponds to $\lambda_1' = 3$, out of the 3 boxes, we must choose 2 for each of the three terms $(x_1-x_2), (x_1-x_3), (x_2-x_3)$. Then we just add this to the 2 choose 2 for the second column of the YD.
- *Week 9, Lecture 2: Do we need to be fluent in the techniques you used to expand and reduce the various polynomial powers? How did you do that again?

## 9.4 ???

12/1:

- Proving the theorem.
- The statement is that there exists a function

$$\mathrm{ch} : \underbrace{\bigoplus_{n\geq 0} \mathbb{Q}_{\mathrm{cl}}(S_n)}_{\{f:S_n\to\mathbb{Q}:f(\sigma=\sigma^{-1})=f(i)\}} \to \bigoplus_{n\geq 0} \Lambda_n$$

where $\Lambda_n = \mathbb{Q}[\sigma_1,\sigma_2,\sigma_3,\dots]_{\deg=n} = [\mathbb{Q}[x_1,\dots,x_n]^{S_n}]_{\deg=n}$. Note that by convention, $\Lambda_0 = \mathbb{Q}$. This function is given by

$$\mathrm{ch}(\chi) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi(\sigma)\cdot p_1^{\lambda_1(\sigma)}\cdot p_n^{\lambda_n(\sigma)}$$

where $p_k = x_1^k + x_2^k + \cdots$ and $\lambda_i(\sigma)$ is the number of cycles in $\sigma$ of length $i$. Moreover, ch is an isomorphism of $\mathbb{Q}$-algebras. To create a product of $V$ a $S_n$-rep and $W$ an $S_m$-rep, we map

$$V \boxtimes W = \mathrm{Ind}_{S_n\times S_m}^{S_{n+m}}(V\otimes W)$$

*Proof.* To prove that ch is a ring isomorphism, we need...

1. $\boxtimes$ is associative;
2. $\mathrm{ch}(\chi_1 \boxtimes \chi_2) = \mathrm{ch}(\chi_1) \cdot \mathrm{ch}(\chi_2)$;
3. $\mathrm{ch}((n)) = h_n$.

1,2,3 imply the theorem because 2,3 imply that ch is surjective, $\Lambda_n$ has a $\mathbb{Q}$-basis $h_{\lambda_1} \cdots h_{\lambda_n}$ for $\lambda_1 \leq \cdots \leq \lambda_n$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = n$. For example, for $\Lambda_5$, we have $h_1^5, h_1^3 h_2, h_1, h_2^2, h_2 h_3, h_1^2 h_3, h_1 h_4, h_5$. This surjectivity combined with the fact that $\dim \mathbb{Q}_{\mathrm{cl}}[S_n] = \dim \Lambda_n$ implies that ch is an isomorphism of rings.

Last thing: $\mathrm{ch}[(n)] = \frac{1}{n!} \sum p_1^{c_1(\sigma)} \cdots p_n^{c_n(\sigma)}$ where $c_i(\sigma)$ denotes the number of cycles of length $i$ and hence $\sum i c_i = n$. Denote $p_1^{c_1(\sigma)} \cdots p_n^{c_n(\sigma)}$ by $p^{c(\sigma)}$.

*Proof.* Let

$$\begin{aligned}
\sum h_n t^n &= \sum \Big( \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n} t^n \Big) \\
&= \frac{1}{1 - x_1 t} \cdot \frac{1}{1 - x_2 t} \cdots \frac{1}{1 - x_n t} \\
&= \exp\left( \log\left( \prod_{i=1}^{n} \frac{1}{1 - x_i t} \right) \right) \\
&= \exp\left( \sum_{i=1}^{n} - \log(1 - x_i t) \right) \\
&= \exp\left( x_1 + \frac{x_1^2 t^2}{2} + \frac{x_1^3 t^3}{3} + \cdots + x_2 + \frac{x_2^2 t^2}{2} + \cdots \right) \\
&= \exp\left( p_1 + \frac{p_2 t^2}{2} + \frac{p_3 t^3}{3} + \cdots \right) \\
&= \prod_{m \geq 1} \exp\left( \frac{p_m t^m}{m} \right) \\
&= *
\end{aligned}$$

We get the second equality because each $1/(1 - x_i t) = 1 + x_i + x_i^2 t^2 + \cdots$. We need the power series $-\log(1-t) = t + t^2/2 + \cdots$ and $\exp(t) = 1 + t + t^2/2! + \cdots$. Thus, $\exp(\log(1-t)) = 1 - t$. Now note that

$$\begin{aligned}
\sum_{n \geq 0} \mathrm{ch}[(n)] \cdot t^n &= \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} p_1^{c_1(\sigma)} \cdots p_n^{c_n(\sigma)} \\
&= \sum_{n \geq 0} \frac{t^n}{n!} \left( \sum_{a_1 + 2a_2 + \cdots + n a_n = n} p_1^{a_1} \cdots p_n^{a_n} \right) \cdot \frac{n!}{1^{a_1} a_1! 2^{a_2} a_2! \cdots n^{a_n} a_n!} \\
&= \sum_{\substack{n \geq 0 \\ a_1, \ldots, a_n : a_1 + 2a_2 + \cdots + n a_n = n}} \frac{1}{a_1!} \left( \frac{p_1}{1} \right)^{a_1} t^{a_1} \frac{1}{a_2!} \left( \frac{p_2}{2} \right)^{a_2} t^{a_2} \cdots \frac{1}{a_n!} \left( \frac{p_n}{n} \right)^{a_n} t^{a_n} \\
&= \prod_{k=1}^{n} \left( \sum_{a_k = 1}^{\infty} \frac{(p_k)^{a_k} t^{a_k k}}{a_k! k a_k} \right) \\
&= \prod_{k \geq 1} \exp\left( \frac{p_k t^k}{k} \right) \\
&= *
\end{aligned}$$

We're overcounting because we can cyclically permute cycles (i.e., (12) = (21)), hence the correction factor in the second line above.

Note: This exponent/logarithm trick is a common computational trick in combinatorics, varieties, etc. □

Now we prove the part 3, i.e., that $\boxtimes$ is associative. We do this by direct computation.

$$\underbrace{\operatorname{Ind}_{S_{n+m}\times S_\ell}^{S_{n+m+\ell}}\left[\operatorname{Ind}_{S_n\times S_m}^{S_{n+m}}(\chi_1\otimes\chi_2)\right]\otimes\chi_3}_{(\chi_1\boxtimes\chi_2)\boxtimes\chi_3} = \operatorname{Ind}_{s_n\times S_m\times S_\ell}^{S_{n+m+\ell}}(\chi_1\otimes\chi_2\otimes\chi_3)$$

...

Then proving 2 (homomorphism bit) is the hardest. We have

$$\begin{aligned}
\operatorname{ch}(\operatorname{Ind}_{S_n\times S_m}^{S_{n+m}}(\chi_1\otimes\chi_2)) &= \frac{1}{n!}\sum_{\sigma\in S_n}(\operatorname{Ind}_{S_n\times S_m}^{S_{n+m}}\chi_1\otimes\chi_2)(\sigma)\underbrace{p_1^{c_1(\sigma)}\cdots p_{n+m}^{c_{n+m}(\sigma)}}_{\psi} \\
&= \left\langle \operatorname{Ind}_{S_n\times S_m}^{S_{n+m}}(\chi_1\otimes\chi_2),\psi\right\rangle_{S_{n+m}} \\
&= \left\langle \chi_1\otimes\chi_2, \operatorname{Res}_{S_n\times S_m}^{S_{n+m}}\psi\right\rangle \\
&= \sum_{\substack{\sigma_1\in S_n\\ \sigma_2\in S_m}}\chi_1(\sigma_1)\chi_2(\sigma_2)p_1^{c_1(\sigma_1)}\cdots p_n^{c_n(\sigma_1)}p_1^{c_1(\sigma_2)}\cdots p_m^{c_m(\sigma_2)} \\
&= \operatorname{ch}(\chi_1)\operatorname{ch}(\chi_2)
\end{aligned}$$

We use Frobenius reciprocity somewhere in here. We also have $\psi: S_n\to\Lambda_n$ and $\psi(\tau\sigma\tau^{-1})=\psi(\sigma)$. □

- After another 10 years of trying to understand the representations of the symmetric group, we'll be here.
- At this point, we can study compact Lie groups.