Week 9

???

9.1 ???

11/27:
- Announcements.
  - OH on Wednesday at 5:30 PM this week; not Tuesday.
  - There will be extra OH next week pre-exam.
    - Roughly like Monday/Wednesday next week.
  - Midterm will be returned on Wednesday; we can pick them up in-person in his office starting then.
  - There are some grade boundaries: Pass/Fail we can do til Friday, withdrawal we can do til 5:00 PM today.
- Let's finish the conversation about induction/restriction and prove the **branching theorem**.
- Reminder to start.
  - We have two mathematical categories, G-reps and H-reps where $H \leq G$.
  - These catagories are related by functors.
  - Res_H^G : G-reps $\to$ H-reps and vice versa for Ind_H^G.
  - Restrictions are stupidly simple.
  - Inductions, most hands-on, we take copies of W times cosets. Formulaically,

    $$\text{Ind}_H^G W = g_1 W \oplus \cdots \oplus g_k W$$

    where $k = (G : H)$ and $G = \bigsqcup_{i=1}^k g_i H$.
    - In more detail, the action of g on $g_i w$ is that of $g_{\sigma(i)} h_i w$.
    - This is a genuinely hard construction.
    - A matrix of this thing will be a permutation matrix via

      $$copyfromboard$$

    - Note that

      $$g_1 W \oplus \cdots \oplus g_k W \cong \text{Hom}_H(\mathbb{C}[G], W)$$

      - Recall that elements of the set on the right above are functions $f : G \to W$ such that $f(h(g)) = hf(g)$.
      - We map between the two via $f(g) \mapsto f(gx')$.
    - What is nice about induced representations is that $\dim[\text{Ind}_H^G W] = (\dim W)[G : H]$.

- Moreover, there is a very easy statement, the **Frobenius formula**.
  - Recall that
$$\tilde{\chi}_W(g) = \begin{cases} 0 & g \notin H \\ \chi_W(g) & g \in H \end{cases}$$
  - With this, we average.
$$\chi_{\operatorname{Ind}_H^G W}(g) = \sum_{x \in G} \tilde{\chi}_W(xgx^{-1})$$
  - Essentially, we're taking a whole bunch of conjugates, summing them up, and dividing to get rid of overcounting.

- We now move onto **Frobenius reciprocity**, which is a relation between the functors/relations Ind_H^G and Res_H^G.
  - The first point where category theory gets interesting is the notion of **adjoint functors**, which we are about to touch on. It is a very subtle notion.
  - Here's version 1 of the statement of Frobenius reciprocity.
    - Recall that we have a scalar product on the space of class function, given by
$$(\chi_1, \chi_2) = \frac{1}{|G|} \sum_{g \in G} \chi_1(g)\chi_2(g^{-1})$$
      where χ_1, χ_2 are class functions on G.
    - Recall that if $\chi_1 = \chi_V$ and $\chi_2 = \chi_W$, then
$$(\chi_1, \chi_2) = \dim \operatorname{Hom}_G(V, W) = \dim \operatorname{Hom}_G\left(\bigoplus_{i=1}^k V_i^{n_i}, \bigoplus_{i=1}^k V_i^{m_i}\right) = \sum_{i=1}^k n_i m_i$$
    - Then the statement is as follows. If V is a G-rep and W is an H-rep, then
$$(V, \operatorname{Ind}_H^G W)_G = (\operatorname{Res}_H^G V, W)_H$$
      - ➢ Denoting scalar product in G and scalar product in W of the characters of each representation.
    - This is similar to the relation between adjoint maps $V \to W$ and $W^* \to V^*$.
  - Version 2.
    - We have that
$$\operatorname{Hom}_G(V, \operatorname{Ind}_H^G W) \cong \operatorname{Hom}_H(\operatorname{Res}_H^G V, W)$$
      where the isomorphism is canonical.
    - We will not check this last definition; we can tediously do it with definitions, and there's nothing complicated. Rudenko leaves this as an exercise to us.
  - Constructing...something: Take $v \in V$, $g \in G$, $\varphi : V \to W$. We send $g \mapsto \varphi(gv)$.

- We now prove Version 1.

Proof. We have

$$
\begin{aligned}
(\chi_V, \chi_{\operatorname{Ind}_H^G W})_G &= \frac{1}{|G|} \sum_{g_1 \in G} \chi_V(g_1) \left(\frac{1}{|H|} \sum_{g_2 \in G} \tilde{\chi}_W(g_2 g_1^{-1} g_2^{-1}) \right) \\
&= \frac{1}{|H| \cdot |G|} \sum_{g_1, g_2 \in G} \chi_V(g) \tilde{\chi}_W(g_2 g_1^{-1} g_2^{-1}) \\
&= \frac{1}{|H| \cdot |G|} \sum_{g_1, g_2 \in G} \chi_V(\underbrace{g_2 g_1 g_2^{-1}}_{h}) \tilde{\chi}_W(\underbrace{g_2 g_1^{-1} g_2^{-1}}_{h^{-1}}) \\
&= \frac{1}{|H|} \frac{1}{|G|} \sum_{h \in G} |G| \chi_V(h) \tilde{\chi}_W(h^{-1}) \\
&= (\chi_V|_H, \chi_W)_H \\
&= (\operatorname{Res}_H^G V, \chi_W)_H
\end{aligned}
$$

From line 3 to line 4: Fix h; then $g_2 g_1 g_2^{-1} = h$ iff $g_1 = g_2^{-1} h g_2$, so we have overcounted by $|G|$ times. ☐

- We now come to the branching theorem at long last.
- Example first.
  - Consider $S_n > S_{n-1}$, where S_{n-1} is the subgroup of permutations fixing n. I.e., $S_3 > S_2 = \{e, (12)\}$.
  - Let λ be a partition of n; *there's notation for this*!
  - Let $\mu \leq \lambda$ be a Young diagram of a partition of $n-1$.
  - Then
    1. We have
    $$\operatorname{Res}_{S_{n-1}}^{S_n} V_\lambda = \bigoplus_{\mu \leq \lambda} V_\mu$$
       - Example: *Draw out pictures*
    2. We have
    $$\operatorname{Ind}_{S_{n-1}}^{S_n} V_\mu = \bigoplus_{\mu \leq \lambda} V_\lambda$$
       - Example: *Draw out pictures*
- The reason that this theorem is called the branching theorem originates from the following diagram, which (when continued) encapsulates the main idea of the theorem. *picture*
  - This graph helps you understand induction and restriction.
  - Dimensions are the number of paths from the left to a a final Young diagram.
    - For example, the dimension of $(3, 1)$ is 3 because there are 3 paths to it (*list them*).
  - Number of paths formula is equivalent to standard Young tableaux!
- Theorem (Branching): The following two statements are true.

$$\operatorname{Res}_{S_{n-1}}^{S_n} V_\lambda = \bigoplus_{\mu \leq \lambda} V_\mu \tag{9.1}$$

$$\operatorname{Ind}_{S_{n-1}}^{S_n} V_\mu = \bigoplus_{\mu \leq \lambda} V_\lambda \tag{9.2}$$

Proof. We'll talk about the general idea of the proof now, and maybe do the details next time.

(1) $\iff$ (2): We have that *stuff at bottom of board*

(1): Let's look at an example. Here's a YD of S_8. We want to restrict it down to S_7. Recall that $V_\lambda = \text{span}(S_8 : \Delta(x_1, x_2, x_3)(x_4 - x_5)(x_6 - x_7))$. Now in S_7, we fix x_8. Consider subrepresentations of V_λ filtered by degree as follows.

$$copy from board$$

The proof comes from the fact that if we now take quotients of these subrepresentations, then since x_8 can only appear in three boxes, ... ☐

- Practice with the above example and think it through.