MATH 26700 (Introduction to Representation Theory of Finite Groups) Notes

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Weeks

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Week 1

Introduction to Representation Theory

1.1 Motivating and Defining Representations

9/27:

- Rudenko would happily approve my final substitution, but it's not his call; it's Boller's.
- HW will be due every week on Wednesday or thereabouts.
  - Submit in paper in a mailbox, location TBA.
  - First HW due next Wednesday.
- Midterm eventually and an in-class final.
- Grading scheme in the syllabus.
- OH not available MW after class (Rudenko has to run to something else), but F after class, we can ask him anything.
  - Regular OH MTh, time TBA.
- There is no specific book for the course.
  - First 8 lectures come from Serre (1977); amazing book but very concise; gets confusing later on. Most lectures are made up by Rudenko.
- Course outline.
  1. Character theory: Beautiful, not too hard.
  2. Non-commutative algebra: More abstract/general approach to the same thing.
  3. Advanced topics, S_n.
- This course's focus: Representations of finite groups in finite dimensions over $\mathbb{C}$.
- This course is for math-inclined people (not quite physics) and lays the foundation for all other Rep Theory.
  - The ideas would be presented in a very different way in Physics Rep Theory.
- We can always ask questions and stop him to correct mistakes during class.
- Why we care about representations.
  - Start with a group G, finite. For example, let $G \equiv S_1$.

  - People started to play with S_4 (permutations of roots of a polynomial of degree 4) in Galois theory.
    - Galois theory primer: Consider a polynomial like $x^4 + 3x + 1 = 0$; the roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ satisfy tons of equations, e.g., $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$ since 1 is the x^0 term.
  - But groups also occur in much more natural places, e.g., isometries of $\mathbb{R}^3$ that preserve a tetrahedron.
  - S_4 is also orientation-preserving isometries of $\mathbb{R}^3$ that preserve a cube.
  - Many things lead to the same group!
  - Theory of abstract groups developed far later than any of these perspectives; was developed to unify them.
- Recall group actions: Take $G, X = \{x_1, \dots, x_n\}$ both finite. We want $G \circlearrowright X$, which is a homomorphism $A : G \to S_n$.
- What can we do now?
  - We can look at orbits, which are smaller pieces.
  - We can look at the stabilizer.
  - We can identify orbits with cosets.
  - If we understand all possible subgroups, we understand all possible actions.
- This story is not boring, but it's simplistic.
- Rudenko doesn't assume we remember everything (phew!).
- Main definition (general to start, then we simplify).
- **Group representation** (of G on V): A group homomorphism $G \to GL(V)$, for G a group, V a finite-dimensional vector space over some field $\mathbb{F}$ with basis $\{e_1, \dots, e_n\}$, and $GL(V)$ the set of isomorphic linear maps $L : V \to V$. *Denoted by* $\boldsymbol{\rho}$.
  - Recall that $GL(V) = GL_n(\mathbb{F})$ is the set of all $n \times n$ invertible matrices.
- For every element $g \in G$, $g \mapsto \rho(g) = A_g$. Essentially, you're mapping to elements that satisfy certain equations.
  - For example, $A_e = E_n$, $A_{g_1 g_2} = A_{g_1} A_{g_2}$, and $A_{g^{-1}} = {A_g}^{-1}$.
  - Thus, representations are a "concrete way to think about groups."
  - If you don't understand abstract group G, let us compare it to a group that we do understand! Like a group can *act* on S_n, we can *represent* a group in a vector space.
- In this course, G is finite, $\mathbb{F} = \mathbb{C}$, and V is finite dimensional.
  - This is the most simple case, but also a very interesting one. The theory is much, much easier, so we can get much more complicated, but this is a good place to start.
  - We could make G compact, but we're not gonna go that far.
- Examples to get an idea of what's going on.
  1. $\dim \rho = 1$ (means $\dim V = 1$). Then $\rho : G \to GL_1(V) = \mathbb{C}^\times$. The codomain is referred to as the **character** of the group.
     - An example group homomorphism $S_n \to \mathbb{C}^\times$ is the sign function $\sigma \to \operatorname{sign}(\sigma) = \{\pm 1\}$.
     - Another example is the **trivial representation**, $G \to \mathbb{C}^\times$ and $g \mapsto 1$.
  2. Smallest one: Let $G = S_3$. The structure is already pretty rich, and this will be part of the homework.

  - **Trivial representation** again.
  - **Alternating representation**.
  - **Standard representation**.
  - **Regular representation**.

- **Trivial representation**: The representation $\rho : G \to GL(V)$ sending $g \mapsto 1$ for all $g \in G$. *Denoted by* $\ydiagram{3}$, $\mathbf{(3)}$.
  - The boxes notation is too much of a detour to explain now.
  - Note that $1 \in GL(V)$ is the identity map on V!
- **Alternating representation**: The representation $\rho : G \to GL(V)$ sending $g \mapsto \operatorname{sign}(g)$ for all $g \in G$. *Denoted by* $\ydiagram{1,1,1}$, $\mathbf{(1,1,1)}$.
- **Standard representation**: The representation $\rho : S_n \to GL(V)$ sending $\sigma \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)})$, where $V = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_1 + \cdots + x_n = 0\}$ is a $(n-1)$-dimensional vector space. *Denoted by* $\ydiagram{2,1}$, $\mathbf{(2,1)}$.
  - A 2D representation like rotating a triangle.
  - This gives something with real numbers.
  - Example: $S_3 \circlearrowright V$ by $\sigma((x_1, x_2, x_3)) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$.
- **Regular representation**: The representation $\rho : G \to \operatorname{Hom}(\mathbb{C}^n)$ defined by $g \mapsto \sigma_g$, where $G = \{g_1, \dots, g_n\}$, $\{e_{g_1}, \dots, e_{g_n}\}$ is a basis of $\mathbb{C}^n$, $\cdot$ is the group action of $\rho(G) \circlearrowright \mathbb{C}^n$ by $\rho(g) \cdot e_g = e_{gg_i}$, and $\sigma_g(e_{g_i}) = \rho(g) \cdot e_g = e_{gg_i}$.
  - This is a permutation of vectors.
  - Thus, for S_3, it will already be 6-dimensional (it's very high dimensional).
- How do we know that representation theory is tractable? Sure, we can define all these things, but how do we know that it will lead anywhere? Here's an example.
  - Let $G = \mathbb{Z}/2\mathbb{Z} = \{e, g\}$, $V = \mathbb{C}^n$, A an $n \times n$ matrix over $\mathbb{C}$, $\rho : G \to GL_n(\mathbb{C})$, and $A := \rho(g)$. Since $g^2 = e$, we know for example that $A^2 = E_n$.
  - But how do we find the matrices A? If we look at eigenvalues of A, there are only two possibilities: ± 1. The structure of A can be very complicated with Jordan normal form and all that, but in fact, these are the **semisimple matrices**, so it's not that bad.
  - Since $A^2 = E$, we know that $(A-E)(A+E) = 0$. Consider $(A-E) : V \to V$. Naturally, it has $\operatorname{Ker}(A-E)$ and $\operatorname{Im}(A-E)$. In this particular case, Rudenko claims that $\operatorname{Ker}(A-E) \cap \operatorname{Im}(A-E) = \{0\}$.

    Proof. Let $v \in \operatorname{Ker}(A-E) \cap \operatorname{Im}(A-E)$ be arbitrary. Since $v \in \operatorname{Im}(A-E)$, there exists $w \in V$ such that $v = (A-E)w = Aw - w$. Since $v \in \operatorname{Ker}(A-E)$, we have $(A-E)v = 0$, so $Av = v$. It follows that $A(Aw - w) = Aw - w$ but also $A(Aw - w) = Ew - Aw = w - Aw$. Thus,

    $$\begin{aligned} Aw - w &= w - Aw \\ 2Aw &= 2w \\ Aw &= w \end{aligned}$$

    But then $w \in \operatorname{Ker}(A-E)$, so $v = (A-E)w = 0$. $\square$

  - This combined with the fact that every vector in a vector space is in either the image or the kernel of a linear map[1] implies that $V = \operatorname{Ker}(A-E) \oplus \operatorname{Im}(A-E)$.

[1]See Theorem 3.6 of Axler (2015).

- Let the kernel have basis $e_1, \dots, e_k$ and the image have basis $e_{k+1}, \dots, n$; then all A are of the following form.

$$\begin{array}{cc}
 & \begin{array}{cccccc} 1 & & k & k+1 & & n \end{array} \\
\begin{array}{c} 1 \\ \\ k \\ k+1 \\ \\ n \end{array} & \left[\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{array}\right]
\end{array}$$

- Next time, we will discuss sums of representations, of which this is an example of the theory.

- The same kind of thing, **simple representations**, happens with all finite groups?? This is where we're going. It's not rocket science; in fact, we'll see it next week.
- Last thing for today: A remarkable story.
  - The story of representation theory started quite different.
  - A beautiful theorem that we can prove now!
  - Frobenius determinant.
  - Think of $G = \{g_1, \dots, g_n\}$. Picture its multiplication table.
  - In every row and column, you see each element once.
  - Let's associate to the multiplication table an actual determinant in the linear algebra sense. Consider elements $x_{g_1}, \dots, x_{g_n}$. Define the $n \times n$ matrix $(x_{g_i g_j})$. Take its determinant. It will be a polynomial in n variables, i.e., an element of the ring $\mathbb{Z}[x_{g_1}, \dots, x_{g_n}]$.
  - Example: Consider

$$\begin{vmatrix} e & g \\ g & e \end{vmatrix}$$

    - The determinant is $x_e^2 - x_g^2 = (x_e - x_g)(x_e + x_g)$.
  - Example: $G = \mathbb{Z}/3\mathbb{Z}$.
    - If the elements are e, g, g^2 and we map these, respectively, to variables a, b, c, we get the matrix

$$\begin{bmatrix} e & g & g^2 \\ g & g^2 & e \\ g^2 & e & g \end{bmatrix} \mapsto \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

    - The determinant is $3abc - a^3 - b^3 - c^3 = (a+b+c)(a^2+b^2+c^2-ab-bc-ac) = (a+b+c)(a+\zeta b+\zeta^2 c)(a+\zeta^2 b+\zeta c)$ where $\zeta^3 = 1$ is a root of unity.
  - Frobenius's theorem: If G is a finite group and we take this Frobenius determinant, then this determinant is equal to ${P_1}^{d_1} \cdots {P_k}^{d_k}$ where $P_1, \dots, P_k$ are irreducible polynomials in $x_g, \dots, x_{g_j}$, then $\deg P_i = d_i$ and k is the number of conjugacy classes.
  - Example: Take S_3; we'll get a polynomial of degree $|S_3| = 6$ but the Frobenius determinant $FD = (x_{g_1} + \cdots + x_{g_k})(x_{g_1} \pm \cdots)(\text{some pol. of deg 2})^2$
  - The proof is remarkable and deep and uses what would become character theory. These polynomials are related to representations and the number of simplest irreducible representations. The theory that came out came as a way to understand this miracle. We'll forget FD's for now, but then come back and prove it later.

1.2 Key Definitions and Category Theory Primer

9/29:

- OH: TW 4:30 or 5:00 most likely; he will confirm later.
- Today: Definitions in greater generality.
- As before, let G be a finite group and V be a finite-dimensional vector space.
- Goal of this course: Understand representations of G, that is...
  - Homomorphisms $\rho : G \to GL(V) = GL_n(\mathbb{C})$;
  - That send $g \mapsto A_g \in GL_n(\mathbb{C})$;
  - And satisfy $A_e = E$, $A_{g_1}A_{g_2} = A_{g_1g_2}$, and $A_{g^{-1}} = {A_g}^{-1}$.
- What are some things we might want to do?
  - Build new representations from old? Investigate and/or classify irreducible representations?
  - Before we can see if any of this works or not, we need a ton of definitions: Sum, equality, etc.
- Rudenko will start today's lecture with some general thoughts on the **category** of representations.
- Categories are things that now permeates math.
- **Category**: A *class* (not a set) of *objects* (some things; you don't know anything about them), and then a bunch of properties.

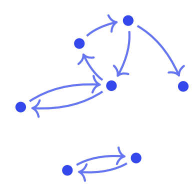

Figure 1.1: The general structure of a category.

  - Objects a, b in category C are denoted by $a, b \in \mathbf{Ob}(\boldsymbol{C})$.
  - There are also **morphisms** between the objects. These are drawn as arrows and lie in $\text{Hom}(a, b)$.
  - There is also composition: $\text{Hom}(a, b) \times \text{Hom}(b, c) \to \text{Hom}(a, c)$.
    - What does this notation mean??
  - Properties.
    1. Associativity.
    2. Existence of a unit: For any object a, there exists $\text{id}_a \in \text{Hom}(a, a)$ such that any morphism pre- or post-composed to this identity yields the same morphism.
       - Example: If $f \in \text{Hom}(a, b)$, then $\text{id}_b \circ f = f = f \circ \text{id}_a$.
  - Rudenko: So a category is basically two pieces of data and a bunch of properties.
- Examples of categories:
  - Category of sets and maps between them.
  - Category of vector spaces over $\mathbb{F}$ where $\text{Ob}(C)$ is the vector spaces and $\text{Hom}(V, W)$ is filled with *linear* maps because you don't just want maps — you want maps that respect the structure.
  - Category of groups where $\text{Hom}(G_1, G_2)$ is the set of group homomorphisms.
  - Category of topological spaces and continuous maps.
  - Category of abelian groups.
  - Trivial category and the identity map; thus, categories need not be chonky.

- Comments on category theory.
  - We'll see some pretty significant category theory at the end of the course.
  - We'll see categories in every course we take; some people try to avoid them. Rudenko doesn't want to go into the material in depth, but he wants to use language from it.
  - Surprisingly, even under the stripped-down of axioms of category theory, you can say quite a lot.
  - Why any of this discussion of category theory matters: If you know the basics of category theory, you can guess the definitions of direct sum, equality, etc. for representations.
- **Category of representations**. *Denoted by* $\mathbf{Rep}_{\boldsymbol{G}}$.
- Take two G-representations V, W; how do we define a map between them?
  - Recall that V, W are vector spaces.
- **Morphism** (of G-representations): A map $f : V \to W$ such that...
  1. f is linear;
  2. f respects the structure of the representations; explicitly, for every $g \in G$, $\rho_V(g) \circ f = f \circ \rho_W(g)$[2].
- On constraint 2, above: This condition is summarized via a **commutative diagram**.

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "\rho_V(g)"'] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[d, "\rho_W(g)"] \\
V \arrow[r, "f"'] & W
\end{tikzcd}$$

Figure 1.2: Commutative diagram, morphisms.

  - Commutative diagrams are very category-theory-esque things.
- That was a very abstract definition; let's make it concrete.
  - Suppose you have a pair of representations $V = \mathbb{C}^n, W = \mathbb{C}^m$, and we have our map f between them given by an $m \times n$ matrix.
  - Let $\rho_V(g) = A_g$ be an $n \times n$ matrix, and let $\rho_W(g) = B_g$ be an $m \times m$ matrix.
  - Then $FA_g = B_gF$.
- Examples.
  1. An interesting example: Let's look at $S_3 \circlearrowright V_{\text{perm}} = \mathbb{C}^3$, a **permutation representation**.
     - For all $\sigma \in S_3$, $\rho(\sigma) : (x_1, x_2, x_3) \mapsto (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$.
  2. There's also the trivial representation $S_3 \circlearrowright V_{(3)} = \mathbb{C}$ defined by $\rho(\sigma) : x \mapsto x$.
- Are the above 2 representations related?
  - Yes! We can, in fact, find a *morphism* between them.
  - In particular, define $f : V_{(3)} \to V_{\text{perm}}$ by $f(x) = (x, x, x)$.
    - Since permuting 3 of the same thing does nothing, the commutativity of Figure 1.2 holds. Therefore, f is a morphism of G-representations as defined above.
    - More explicitly,

$$f[\rho_{(3)}(\sigma)(x)] = f(x) = (x, x, x) = \rho_{\text{perm}}(\sigma)((x, x, x)) = \rho_{\text{perm}}(\sigma)[f(x)]$$

[2]Recall that the object, $\rho_V(g)$ is a linear map! Thus, it can be composed with other linear maps like f.

- Is f **reversible**?
  - Is "reversible" the right word??
- Define $\tilde{f} : V_{\text{perm}} \to V_{(3)}$ by $\tilde{f} : (x_1, x_2, x_3) \mapsto x_1 + x_2 + x_3$.
  - Since addition is commutative, the commutativity of Figure 1.2 holds.
  - More explicitly,

$$\begin{aligned}
f[\rho_{\text{perm}}(\sigma)((x_1, x_2, x_3))] &= f((x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})) \\
&= x_{\sigma(1)} + x_{\sigma(2)} + x_{\sigma(3)} \\
&= x_1 + x_2 + x_3 && \text{Commutativity of addition} \\
&= f((x_1, x_2, x_3)) \\
&= \rho_{(3)}(\sigma)[f((x_1, x_2, x_3))]
\end{aligned}$$

- Takeaway: The existence of maps between representations is interesting.

- Next question: How do we define an **isomorphism** of two representations?
- **Isomorphism** (of G-representations): A morphism of G-reps that is an isomorphism of vector spaces.
- Category theory helps us again here because it generalizes the concepts of an isomorphism!
  - If $f : V \to W$ and $g : W \to V$ are category-theoretic morphisms, then the constraints $f \circ g = \text{id}_W$ and $g \circ f = \text{id}_V$ make f and g into category-theoretic *iso*morphisms, regardless of what V and W might be.
  - Back in the context of representations, let $f : V \to V$ be an isomorphism of vector spaces. Then we do indeed have $\rho_V(g) \circ f = f \circ \rho_V(g)$, as we would hope from category theory!
- Recall the condition $FA_g = B_gF$. Supposing F is an isomorphism (and thus has an inverse), we get $FA_gF^{-1} = B_g$ as our new condition.
  - Essentially, we can do *simultaneous conjugation* of all matrices.
  - As per usual with isomorphisms, we get to *change bases*.
  - Essentially, we can represent the nice permutation representation in a very nasty basis but still have it be valid.
- Many other notions (e.g., direct sum) will not be explained by Rudenko, but we can read about them!
- However, we'll do a few more.
- A representation sitting inside another: a **subrepresentation**.
- **Subrepresentation** (of V): A subspace $W \subset V$ such that for all $w \in W$ and $g \in G$, we have that $\rho_V(g)W \subset W$, where V is a G-representation with $\rho_V : G \to GL(V)$.
  - Many people will just write the critical condition as $gW \subset W$.
- Subrepresentations in category theory: We have another commutative diagram.

$$\begin{tikzcd}
W \arrow[r, hook] \arrow[d, "\rho_V(g)"'] & V \arrow[d, "\rho_V(g)"] \\
W \arrow[r, hook] & V
\end{tikzcd}$$

Figure 1.3: Commutative diagram, subrepresentations.

- Example: The trivial representation, the standard representation, and (of course) the **zero representation** are subrepresentations of the permutation representation.
- **Zero representation**: The representation $\rho : G \to GL(\{0\})$ sending $g \mapsto 1$ for all $g \in G$. *Denoted by* $\mathbf{(0)}$.
- What about representations that don't have subrepresentations?
- **Simple** (representation): A G-representation V that has only two subrerpesentations: (0) and V. *Also known as* **irreducible**, **irreps**.
- Example irreducible representations: Line in $\mathbb{C}^2$, triangle in $\mathbb{C}^2$, A_5 and dodecahedron in $\mathbb{C}^3$.
- Notion of a direct sum.
- **Direct sum** (of V_1, V_2): The G-rep with the space $V_1 \oplus V_2 = \{(v_1, v_2) \mid v_1 \in V_1, v_2 \in V_2\}$ where $\rho_{V_1 \oplus V_2}(g)(v_1, v_2) = (\rho_{V_1}(g)v_1, \rho_{V_2}(g)v_2)$. *Denoted by* $\boldsymbol{V_1 \oplus V_2}$.
  - The matrix of $\rho_{V_1 \oplus V_2}(g)$ is the following block matrix.

$$\rho_{V_1 \oplus V_2}(g) = \left[\begin{array}{c|c} \rho_{V_1}(g) & 0 \\ \hline 0 & \rho_{V_1}(g) \end{array}\right]$$

- Example: $V_{\text{perm}} = V_{(3)} \oplus V_{(2,1)}$, with $\mathbb{C}^3 = \mathbb{C} \oplus \mathbb{C}^2$ where

$$\mathbb{C} = \langle (1, 1, 1) \rangle \qquad\qquad \mathbb{C}^2 = \langle (x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0 \rangle$$

  - The decomposition is into simple representations.
  - Relate this to the fact that the JCF of any 3×3 permutation matrix has at most a 1-block and a 2-block, if not three 1-blocks. There will always be one 1D subspace on which the permutation matrix is an identity, i.e., span$(1, 1, 1)$, and a 2D orthogonal complement!
  - As a fun and simple exercise, prove that there is no line fixed under the standard representation.
- A simple and important theorem to prove next week.
- Theorem: Let G be a finite group and $\mathbb{F} = \mathbb{C}$. Then...
  1. There are finitely many irreps $V_1, \dots, V_s$ up to isomorphism.
     - Later on, we will see that s is equal to the number of conjugacy classes.
  2. For every G-rep V, there exists a unique $n_1, \dots, n_s \geq 0$ such that $V \cong {V_1}^{n_1} \oplus \cdots \oplus {V_s}^{n_s}$.
- This theorem tells us that if we want to study rep theory, we want to study irreps (which can be kind of complicated) because if we understand them, everything breaks down into them.
- Examples.
  1. $G = \mathbb{Z}/2\mathbb{Z} = S_2$.
     - $V_1 = \mathbb{C}e$ with $ge = e$ and $V_{-1} = \mathbb{C}e$ with $ge = -e$.
     - It follows that $V \cong {V_1}^{n_1} \oplus {V_{-1}}^{n_{-1}}$.
     - We get a diagonal matrix with only 1s and -1s.
  2. $G = S_3$.
     - $V_{(3)}, V_{(1,1,1)}, V_{(2,1)}$.
     - $GL_5(\mathbb{F}_4)$.
     - Proven in an elementary way in Section 1.3 of Fulton and Harris (2004), which we have to read for the HW; will be useful for later in the course's HW.
- Plan: Next time, we'll talk about some more abstract stuff; tensor products of vector spaces.
  - Tensor products are something we should read up on now! The definition is hard and abstract.
  - Then he'll prove the above theorem.

1.3 S Chapter 1: Generalities on Linear Representations

From Serre (1977).

10/3:

- Part I (what we'll be covering) is written for quantum chemists, and thus gives proofs "as elementary as possible, using only the definition of a group and the rudiments of linear algebra" (Serre, 1977, p. v).
  - Recall the story about Serre and his wife, the chemist, who needed to explain group theory and rep theory to her students.
- Indeed, although the book seemed very fast when I first looked at it two years ago, it reads much more easily now and has enough context for most anyone who is comfortable with group theory and theoretical linear algebra.

Section 1.1: Definitions

- Definitions of $\boldsymbol{GL(V)}$, **invertible square matrix**, and **finite group**.
- **Linear representation**: See class notes. *Also known as* **group representation**.
  - Serre (1977) will frequenty write ρ_s for $\rho(s)$.
- **Representation space** (of G): The vector space V corresponding to the linear representation $\rho : G \to GL(V)$ of G. *Also known as* **representation**.
  - The latter term is a self-identified "abuse of language" (Serre, 1977, p. 3).
- "For most applications, one is interested in dealing with a *finite number of elements* x_i of V, and can always find a subrepresentation of $V \ldots$ of finite dimension, which contains the x_i; just take the vector subspace generated by the images $\rho_s(x_i)$ of the x_i" (Serre, 1977, p. 4).
- **Degree** (of a representation): The dimension of the representation space of this representation.
- To give a representation **in matrix form** is to give a set of invertible matrices that are isomorphic to the group elements.
- Important converse: Given invertible matrices satisfying the appropriate homomorphism identities, there is a corresponding group that these matrices represent.
- **Similar** (representations of G): Two representations $\rho : G \to GL(V)$ and $\rho' : G \to GL(V')$ of G for which there exists a linear isomorphism $\tau : V \to V'$ such that

  $$\tau \circ \rho(s) = \rho'(s) \circ \tau$$

  for all $s \in G$. *Also known as* **isomorphic**.
  - Equivalent definition (in matrix form): There exists T invertible such that $R_s' = TR_sT^{-1}$.
  - Isomorphic representations have the same degree.

Section 1.2: Basic Examples

- Degree 1 representation: A homomorphism $\rho : G \to \mathbb{C}^*$, where $\mathbb{C}^*$ denotes the roots of unity (all $z \in \mathbb{C}$ with $|z| = 1$).
  - The fact that every $s \in G$ has *finite* order by assumption is what permits this representation.
- **Unit representation**: See class notes. *Also known as* **trivial representation**.
- **Regular representation**: The representation $\rho : G \to GL(V)$ defined by $s \mapsto [e_t \mapsto e_{st}]$ for all $s \in G$, where V has basis $(e_t)_{t \in G}$.

- $\deg \rho = |G|$.
- $e_s = \rho_s(e_1)$.
    - Implication: The images of e_1 under the ρ_s's form a basis of V, i.e., $\{\rho_s(e_1) \mid s \in G\}$ is a basis of V.
- Converse of above: If W is a representation of G containing a vector w such that $\{\rho_s(w) \mid s \in G\}$ forms a basis of W, then W is isomorphic to the regular representation V via $\tau : V \to W$ defined by $\tau(e_s) = \rho_s(w)$.

- **Permutation representation** (associated with X): The representation $\rho : G \to GL(V)$ defined by $s \mapsto [e_x \mapsto e_{s \cdot x}]$ for all $s \in G$, where $G \circlearrowright X$ a finite set and V has a basis $(e_x)_{x \in X}$.

Section 1.3: Subrepresentations

- Definition of **subrepresentation**.
    - Example: Trivial representation $\mathbb{C}(x, \dots, x)$ is a subrepresentation of the regular representation.
- Definitions of **direct sum** of vector spaces and **kernel**.
- **Complement** (of a subspace): Any $(n-m)$-dimensional subspace U that...
    1. Satisfies $W \oplus U = V$;
    2. Intersects W trivially;

  where $\dim V = n$ and $\dim W = m \leq n$.
    - This means that a single subspace can have multiple complements!
        - Only one **orthogonal** complement, but many *complements*.
        - Example: Consider a line through the origin in $\mathbb{R}^2$; any other line through the origin is a complement of it!
    - It follows that there is a bijection between the complements W' of W in V and the projections p of V onto W (since non-orthogonal complements require non-orthogonal projections).
- **Projection** (of V onto W associated with the decomposition $V = W \oplus W'$): The mapping that sends each $x \in V$ to its component $w \in W$. *Denoted by* $\boldsymbol{p}$.
    - Consequence: The two properties defining a p are (1) $\text{Im}(p) = W$ and (2) $p(x) = x$ for all $x \in W$.
    - Consequence: These two properties also imply that a map is a projection and $V = W \oplus \text{Ker}(p)$.
- If a representation has a subrepresentation, then some complement of this subrepresentation is also a subrepresentation.

  Theorem 1. *Let $\rho : G \to GL(V)$ be a linear representation of G in V and let W be a vector subspace of V stable under G. Then there exists a complement W^0 of W in V which is stable under G.*

  Proof 1 (limited conditions). Let p be the projection of V onto W that corresponds to some arbitrary complement of W in V. To begin, we may legally — albeit with little motivation — form the average p^0 of the conjugates of p by the elements of G:

$$p^0 := \frac{1}{|G|} \sum_{t \in G} \rho_t \cdot p \cdot \rho_t^{-1}$$

  We now seek to prove that p^0 is a projection by showing that it satisfies the two properties of a "p." First, notice that by assumption, every ρ_t (and thus ρ_t^{-1}) preserves W. This combined with the fact that $p(V) = W$ implies that $p^0(V) = W$, as desired. Additionally, for any $x \in W$ and $t \in G$, we know by property (2) of a p and the fact that $p_t^{-1}(x) \in W$ that $p \cdot p_t^{-1}(x) = p_t^{-1}(x)$. Applying p_t to both sides of this equation yields $[p_t \cdot p \cdot p_t^{-1}](x) = x$. Hence, $p^0(x) = x$, as desired. Thus, p^0 is a projection of V onto W, associated with some complement W^0 of W.

So that we can make a substitution later, we will now prove that

$$\rho_s \cdot p^0 = p^0 \cdot \rho_s$$

for all $s \in G$. Pick such an s. Then

$$\rho_s \cdot p^0 \cdot \rho_s^{-1} = \frac{1}{|G|} \sum_{t \in G} \rho_s \cdot \rho_t \cdot p \cdot \rho_t^{-1} \cdot \rho_s^{-1} = \frac{1}{|G|} \sum_{t \in G} \rho_{st} \cdot p \cdot \rho_{st}^{-1} = p^0$$

so we can precompose both sides of the above equation with ρ_s to yield the final result. This line here should make it clear why we needed to form a projection like p^0.

We now have all of the tools we need to prove that W^0 is stable under G. To do so, it will suffice to show that for all $x \in W^0$ and $s \in G$, we have $\rho_s(x) \in W^0$. Let $x \in W^0$ and $s \in G$ be arbitrary. Since $x \in W^0$, $p^0(x) = 0$ by definition. This combined with the above commutativity rule implies that $p^0 \cdot \rho_s(x) = \rho_s \cdot p^0(x) = \rho_s(0) = 0$. But the only way that p^0 could map $\rho_s(x)$ to 0 is if $\rho_s(x) \in W^0$, as desired. □

Proof 2 (orthogonal complement). Let W^0 be the orthogonal complement of W, and endow V with a **scalar product** $(x \mid y)$ to turn it into an inner product space. Replace $(x \mid y)$ with the new inner product $\sum_{t \in G} (\rho_t x \mid \rho_t y)$. Now, if it wasn't already, the inner product is invariant under ρ_s for all s, i.e., for s arbitrary, we have

$$(\rho_s x \mid \rho_s y) = (x \mid y)$$

This means that vectors that were orthogonal before ρ_s is applied to V, stay orthogonal after ρ_s is applied to V. In particular, since ρ_s preserves W by hypothesis, all vectors orthogonal to W (i.e., all vectors in W^0) stay orthogonal to W (i.e., stay in W^0) after ρ_s is applied. Thus, W^0 is stable under ρ_s as well. □

- Consequence of the second, stronger proof: The representations W and W^0 determine the representation V.
  - This allows us to rigorously say that the representation $V = W \oplus W^0$.
  - If W, W^0 are given in matrix form by R_s, R_s^0, then $W \oplus W^0$ is given in matrix form by

$$\left(\begin{array}{c|c} R_s & 0 \\ \hline 0 & R_s^0 \end{array}\right)$$

- We can extend this method of directly summing representations to an arbitrary finite number of them.

Section 1.4: Irreducible Representations

- Definition of **irreducible** representation.
- Fact: Each nonabelian group possesses at least one irreducible representation with deg ≥ 2.
  - Proven later.
- Irreducible representations construct all representations via the direct sum.

Theorem 2. *Every representation is a direct sum of irreducible representations.*

Proof. We induct on $\dim(V)$.

Suppose $\dim(V) = 0$. Since 0 is the direct sum of the empty family of irreducible representations, the theorem is vacuously true.

Suppose $\dim(V) \geq 1$. We divide into two cases (V is irreducible and V is reducible). In the first case, we are done. In the second case, $V = V' \oplus V''$ for some $V' \perp V''$ (see Theorem 1). Since $\dim(V') < \dim(V)$ and $\dim(V'') < \dim(V)$ by definition, the induction hypothesis implies that V' and V'' are direct sums of irreducible representations. Therefore, the same is true of V. □

- Fact: The direct-sum decomposition is not necessarily unique.
  - Counterexample: If $\rho_s = 1$ for all $s \in G$, then there are a plethora of decompositions of a vector space into a direct sum of lines.
- Fact: The number of W_i isomorphic to a given irreducible representation *does not* depend on the chosen decomposition.
  - Proven later.

Week 2

The Structure of Representations

2.1 The Tensor Product

10/2:
- Before lecture, I chatted with a few people about tensor products and the exterior and symmetric powers.
  - Patrick: A **tensor** $v \otimes w$ is just an element of a vector space, indexed differently than in a column.
  - Raman: There is no canonical way to transform tensors into column vectors.
- Course logistics.
  - OH: T 5:30-6:30(+) and W 5:30-6:30(+). We can also meet one-on-one.
  - HW is due Thursdays at midnight.
- Today: Constructing new representations from old.
  - Rudenko will skim through tensor products really quickly.
- Reminder: Last time, we talked about how representation theory is really quite simple. If G is a finite group and $F = \mathbb{C}$, there exist a finite set $V_1, \dots, V_s$ of irreps up to isomorphism, and every finite-dimensional representation $V \cong V_1^{n_1} \oplus \cdots \oplus V_s^{n_s}$.
- If V is a representation of G, then there are loads of things we can do with it.
  - We can construct the dual representation V^*.
  - We can construct the representation $V \otimes V$.
  - We can construct symmetric powers.
  - We can construct wedge powers.
  - There are more, but this is enough for now.
- Even when we take a very simple group and representation, there are some very interesting things that can fall out.
  - Example: If you take the symmetric powers of S_3, as in the homework, you get something really interesting.
- Now, we go to linear algebra.
- Let V, W be vector spaces over a field F. How do we produce a new vector space out of these?
- $\text{Hom}_F(V, W)$ is the vector space of linear maps $F : V \to W$!
  - $\dim = (\dim V)(\dim W)$.

- Can we make $\mathrm{Hom}_F(V,W)$ into a representation of G? Yes!

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "\rho_V(g)"'] & W \arrow[d, "\rho_W(g)"] \\ V \arrow[r, "gL"'] & W \end{tikzcd}$$

Figure 2.1: Commutative diagram, linear maps space representation.

  - Suppose that V,W are G-reps, which gives us $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$.
  - Suppose also that we have $L \in \mathrm{Hom}_F(V,W)$.
  - Now infer from the commutative diagram that it will work to define $gL = \rho_W(g) \circ L \circ \rho_V(g)^{-1}$.
  - This is pretty standard.
- Recall that there is a different space $\mathrm{Hom}_G(V,W)$ of morphisms of G-representations (see Figure 1.2 and the associated discussion).
  - This is a very very small subspace of $\mathrm{Hom}_F(V,W)$.
- Special case of the above construction: **Dual representation**.
  - Consider $\mathrm{Hom}_F(V,F)$. This the **dual vector space**.
  - Basic fact 1: Let $e_1,\dots,e_n$ be a basis of V. Then V^* also has a corresponding basis $e^1,\dots,e^n$, known as its **dual basis**.
    - Computing coordinates already depends on a basis, and having bases is super nice.
    - Corollary: $\dim V = \dim V^*$.
    - This is the first time **canonical** comes into linear algebra. Canonical (nobody understands what it means) basically means that something doesn't depend on choices.
    - In particular, V, V^* are isomorphic because they have the same dimension, but for no more natural reason. They can be the same representation, or they can be different.
  - Basic fact 2: If V is finite-dimensional, then $(V^*)^* \cong V$. The formula for this isomorphism is canonical, because it does not depend on a choice of basis. In particular, choose the map $V \to (V^*)^*$ sending v to the map sending $\varphi \in V^*$ to $\varphi(v)$.
  - If V is infinite dimensional, none of this is true and you are in the realm of functional analysis.
  - Ok, so all of this was good information about the dual *space*, but what is the dual *representation*?? Does it matter, and do we need to know for now?
- **Dual vector space** (of V): The vector space defined as follows, given that V is a vector space over F. *Denoted by* $\boldsymbol{V^*}$. *Given by*

$$V^* = \mathrm{Hom}_F(V,F)$$

- **Dual basis** (of V^* to $e_1,\dots,e_n$): The basis defined as follows for $i = 1,\dots,n$, where $e_1,\dots,e_n$ is a basis of V. *Denoted by* $\mathbf{e^1,\dots,e^n}$. *Given by*

$$e^i(x_1e_1 + \cdots + x_ne_n) = x_i$$

- We now move onto the tensor product.
  - The tensor product is very hard to understand. If you learn about it and you feel you don't understand it, that's typical; nobody understands it at first.
  - For now, we'll discuss two ways of thinking about tensor products that won't bring us any comfort.
- Let V,W be two vector spaces over a field F.

- Abstract definition of the tensor product.
  - We have discussed maps from $V \to W$, but there is another related space.
  - Indeed, we can look at the space of bilinear maps from $V \times W \to F$.
    - Example: A map $f : V \times W \to F$ that satisfies the constraints $f(\lambda v, w) = \lambda f(v, w)$, $f(v_1 + v_2, w) = f(v_1, w) + f(v_2, w)$, and likewise for the second index. Recall that this is a **bilinear map**.
  - Let V have basis $e_1, \dots, e_n$ and W have basis $f_1, \dots, f_m$.
  - Notice that every bilinear map f can be defined as a linear combination of the $f(e_i, f_j)$. In other words, the $f(e_i, f_j)$ form the basis of a function space.
    - This "bilinear maps space" has dimension nm.
  - Now, one way to understand a tensor product: Is this "bilinear maps space" actually some other space? It is! It is $(V \otimes W)^*$.
  - Bilinear maps are linear maps from where? From $V \otimes W$!
- **Bilinear** (map): A function $f : V \times W \to Z$ that satisfies the following constraints, where V, W, Z are vector spaces over F, $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, and $\lambda \in F$. *Constraints*

$$
\begin{aligned}
f(v_1 + v_2, w) &= f(v_1, w) + f(v_2, w) & \qquad f(\lambda v, w) &= \lambda f(v, w) \\
f(v, w_1 + w_2) &= f(v, w_1) + f(v, w_2) & \qquad f(v, \lambda w) &= \lambda f(v, w)
\end{aligned}
$$

- We now look at a much more elementary definition of the tensor product.
- Explicit definition of the tensor product.
  - $V \otimes W$ is equal to a huge vector space with basis consisting of pairs of elements (v, w). Even if V, W are one dimensional, this is like all pairs of real numbers; it's huge. Then, we quotient it by the space of all elements satisfying $\lambda(v, w) = (\lambda v, w) = (v, \lambda w)$, $(v_1 + v_2, w) = (v_1, w) + (v_2, w)$, and the like. This forces these relationships to be true.
    - Clarify this methodology??
    - Essentially, this allows us to treat tensor multiplication much like real multiplication, endowing the operation with distributivity, etc.
    - For example, the rule $(v_1 + v_2, w) = (v_1, w) + (v_2, w)$ becomes, in tensor product notation, $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$.
  - Example: Suppose $V = \mathbb{C}e_1 + \mathbb{C}e_2$. We want to look at $V \otimes V$.
    - A priori[1], it's spanned by $(ae_1 + be_2) \otimes (ce_1 + de_2) = ace_1 \otimes e_1 + ade_1 \otimes e_2 + bce_2 \otimes e_1 + cde_2 \otimes e_2$.
    - Thus, $V_1 \otimes V_2$ has 4-element basis $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$.
- These two definitions constitute a first approximation to what the tensor product is.
- Takeaway: What is true in general is that if V has basis $e_1, \dots, e_n$ and W has basis $f_1, \dots, f_m$, then $V \otimes W$ has basis $e_i \otimes f_j$ ($i = 1, \dots, n$ and $j = 1, \dots, m$).
- Having discussed the tensor product of vector spaces, let's think about the tensor product of *representations*.
  - Suppose $g : V \to V$ and $g : W \to W$.
    - We're starting to make notation sloppy.
  - How does $g : V \otimes W \to V \otimes W$? Well, we just send $v \otimes w \mapsto (gv) \otimes (gw)$.
    - Why is this map well-defined?

[1] I.e., it follows from some logic. In particular, it follows from the logic that any element $v \in V$ is of the form $v = ae_1 + be_2$, so of course all $v \otimes v$ must be of the given form for choices of a, b, c, d.

    - We invoke the **universal property of the tensor product operation**.
    - This guarantees us that given g — which is effectively a map from $V \times W \to V \otimes W$ as defined — there nevertheless exists a complete extension $\tilde{g} : V \otimes W \to V \otimes W$.
  - As a matrix, this map is pretty strange!
    - Example: Let $g : V \to V$ be a 2×2 matrix. What is the matrix of $g : V \otimes V \to V \otimes V$?
    - If

$$\rho_V(g) = g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} =: A$$

      then we have

$$\begin{aligned} g(e_1 \otimes e_1) &= ge_1 \otimes ge_1 \\ &= (ae_1 + ce_2) \otimes (ae_1 + ce_2) \\ &= a^2 e_1 \otimes e_1 + ace_1 \otimes e_2 + ace_2 \otimes e_1 + c^2 e_2 \otimes e_2 \end{aligned}$$

    - Evaluating similarly for all basis vectors, we get a very curious block matrix:

$$\begin{array}{cc} & \begin{array}{cccc} e_1 \otimes e_1 & e_1 \otimes e_2 & e_2 \otimes e_1 & e_2 \otimes e_2 \end{array} \\ \begin{array}{c} e_1 \otimes e_1 \\ e_1 \otimes e_2 \\ e_2 \otimes e_1 \\ e_2 \otimes e_2 \end{array} & \begin{bmatrix} a^2 & ab & ab & b^2 \\ ac & ad & bc & bd \\ ac & bc & ad & bd \\ c^2 & cd & cd & d^2 \end{bmatrix} \end{array} = \left[\begin{array}{c|c} aA & bA \\ \hline cA & dA \end{array}\right]$$

    - Notice how, for example, this takes the tensor $e_1 \otimes e_1$, represented as $(1, 0, 0, 0)$, to the tensor $a^2 e_1 \otimes e_1 + ace_1 \otimes e_2 + ace_2 \otimes e_1 + c^2 e_2 \otimes e_2$, represented as (a^2, ac, ac, c^2).
    - Does this construction imply a canonical way to convert from tensors to column vectors??
  - Classically, this is called the **Kronecker product** of two matrices.
  - People discovered all of this stuff before they unified it as tensor math.

- **Universal property of the tensor product operation**: For every bilinear map $h : V \times W \to Z$, there exists a *unique* linear map $\tilde{h} : V \otimes W \to Z$ such that $h = \tilde{h} \circ \otimes$.

$$\begin{tikzcd} V \times W \arrow[r, "\otimes"] \arrow[dr, "h"'] & V \otimes W \arrow[d, "\tilde{h}"] \\ & Z \end{tikzcd}$$

Figure 2.2: Universal property, tensor product operation.

Proof. See the solid explanation linked here. Alternatively, here's my write up.

Let $V = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$, $W = \mathbb{C}f_1 \oplus \cdots \oplus \mathbb{C}f_m$, Z, and $h : V \times W \to Z$ be arbitrary. Define $\tilde{h} : V \otimes W \to Z$ by

$$\tilde{h}(e_i \otimes f_j) := h(e_i, f_j)$$

for $i = 1, \dots, n$ and $j = 1, \dots, m$. Since a linear map is wholly defined by its action on the basis of its domain, this set of equations suffices to define $\tilde{h}$ on all of $V \otimes W$.

Existence: To prove that $\tilde{h}$ satisfies the "universal property," it will suffice to show that $h = \tilde{h} \circ \otimes$. Let

$(v, w) \in V \times W$ be arbitrary, and suppose $v = \sum_{i=1}^n a_i e_i \in V$, and $w = \sum_{i=1}^n b_i f_i \in W$. Then

$$\begin{aligned}
[\tilde{h} \circ \otimes](v, w) &= \tilde{h}(v \otimes w) \\
&= \sum_{i=1}^n \sum_{j=1}^m a_i b_i \tilde{h}(e_i \otimes f_i) \\
&= \sum_{i=1}^n \sum_{j=1}^m a_i b_i h(e_i, f_j) \\
&= h(v, w)
\end{aligned}$$

as desired.

Uniqueness: Now suppose $\tilde{g} : V \otimes W \to Z$ also satisfies the "universal property," that is, $h = \tilde{g} \circ \otimes$. Then by definition,

$$\tilde{h}(e_i \otimes f_j) = h(e_i, f_j) = \tilde{g}(e_i \otimes f_j)$$

for $i = 1, \dots, n$ and $j = 1, \dots, m$. But since a linear map is wholly defined by its action on the basis of its domain, it follows that $\tilde{h} = \tilde{g}$, as desired. $\square$

- **Kronecker product** (of A, B): The matrix product defined as follows. *Denoted by* $\boldsymbol{A} \otimes \boldsymbol{B}$. *Given by*

$$A \otimes B = n \overset{n}{\begin{bmatrix} A \end{bmatrix}} \otimes m \overset{m}{\begin{bmatrix} B \end{bmatrix}} = \begin{array}{cc} & nm \\ nm & \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix} \end{array}$$

  - The Kronecker product is *not* commutative, but the matrices you get are related by conjugacy and by commuting the columns.
- Vector spaces of the same dimension are all alike, but vector space representations are very interesting. By the end of this course, we'll understand what's going on.
- How we understand tensor stuff: Look at the abstract definition, look at the concrete definition, look at 5 examples, and then go in a circle. Repeat again and again until it makes sense.
- He's just trying to tell us all relevant words so that they will fit together later.
- Fact: If V, W finite-dimensional, $\mathrm{Hom}_F(V, W) \cong V \otimes W^*$.
  - Tensor products are very nice to construct maps from.
  - Let's construct a reverse map, then.
  - Take $\alpha \otimes w \in V^* \otimes W$, where $\alpha : V \to F$ by definition. Send $\alpha \otimes w$ to the map $v \mapsto \alpha(v)w$. This is a *canonical* map!! We can show that they span everything.
    - For example, if we want to choose $\alpha \otimes w$ mapping to the matrix with a 1 in the upper left-hand corner and zeroes everywhere else, let $w = e_1 \in W$ and let $\alpha = e^1 \in V^*$.
    - We can do similarly for all other such matrices, mapping this basis of $\mathrm{Hom}_F(V, W)$ to $e^i \otimes e_j$ ($i = 1, \dots, n$ and $j = 1, \dots, m$).
    - Note that this also allows us to define a (noncanonical) inverse map.
  - This inverse map from $\mathrm{Hom}_F(V, W) \to V^* \otimes W$ is clearly a bit harder to work out.
  - Hidden in this story is why trace is invariant under conjugation (see below discussion).
- If we now take $\mathrm{Hom}_F(V, V)$, then this is isomorphic to $V^* \otimes V$. There is a very natural map from these isomorphic spaces to F defined by the trace, and/or $\alpha \otimes v \mapsto \alpha(v)$. We can prove this. And this is canonical, as well. This is why the main property of the trace is that it's invariant under conjugation. This fact is hidden in the story very nicely.

- Tensor products are hard, it will be a pain, we will understand them very well, but it will not be nice for now.
- Symmetric products and wedge powers will be discussed briefly next time.
  - There is a nice description in Serre (1977) that we can use for the homework.
- Extra homework: Please read about tensor products in whatever textbook you like, try some examples, and repeat.

2.2 Office Hours

10/3:

- Problem 2a:
  - $\Lambda^2 V$ is *exterior powers.*
  - The exact canonical isomorphism we need is briefly discussed on Fulton and Harris (2004, p. 473).
  - I.e., we have to construct isomorphisms between the structures that don't rely on the choice of any basis. Recall the classic example of $V \cong V^{**}$, as explained in the well-written MSE post "basic difference between canonical isomorphism and isomorphims." Recall that the isomorphism from $V \to V^*$ defined by sending each element of the basis of V to the corresponding element of the dual basis of V^* is *not* canonical because *it involves choosing bases.* Definitions of canonical maps are available in MATH20510Notes, p. 2.
  - From a quick look at this, it looks like the proof may be analogous to the classic middle-school algebra identity $(v + w)^2 = v^2 + vw + w^2$.
  - The second exterior power $\Lambda^2 V$ of a finite-dimensional vector space V is the dual space of the vector space of alternating bilinear forms on V. Elements of $\Lambda^2 V$ are called 2-vectors.
- Problem 2b:
  - $S^2 V$ is *symmetric powers.*
  - The exact canonical isomorphism we need is briefly discussed on Fulton and Harris (2004, p. 473).
- Problem 3a:
  - This is the determinant of the multiplication table, in relation to that theorem that you showed us at the end of the first class? Yep!
- Problem 3b:
  - So a circulant matrix is a matrix like the multiplication table from (a)? Yep!
  - Is $\zeta = e^{2\pi i/n}$? Sort of. It can be any n^{th} root of unity.
- Problem 4d:
  - We'll cover higher symmetric powers in class tomorrow.
  - However, it basically just means that we're now working with elements of the form $e_1 \otimes e_2 \otimes e_3 \in S^3 V$ and on and on.
- Problem 5a:
  - Is $V^\vee = V^*$? Yes. This is "vee check," and is a notation that some people prefer.
- Problem 5b:
  - Is "tr" the trace function of the linear map corresponding to L? Yes.
  - What is L?

- An element of $V\otimes V^*$ is a linear combination of elements of the form $v\otimes\alpha$, not necessarily just one of these "decomposable" products.
- There is an isomorphism $V\otimes V^*\cong\operatorname{Hom}(V)$.
- Consider the matrix
$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$
It sends $e_1\mapsto e_1$ and $e_2\mapsto 0$. Thus, it is well-matched with $e_1\otimes e^1$, which also grabs e_1 (with e^1) and sends it to e_1.
- Consider the matrix
$$\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$
It sends $e_1\mapsto 0$ and $e_2\mapsto e_1$. Thus, it is well-matched with $e_1\otimes e^2$, which also grabs e_2 (with e^2) and sends it to e_1.
- In full,
$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\mapsto ae_1\otimes e^1+be_1\otimes e^2+ce_2\otimes e^1+de_2\otimes e^2$$
- This map *is* canonical! This is because the bases must be chosen to even begin talking about matrices.
- If you change the matrix, the bases change, too??
- Takeaway: We have to walk backwards from matrix to linear transformation to representation in $V\otimes V^*$ to a scalar in F.

- Problem 5c:
  - So trace of such a map is equal to the dimension of its image? Yes.

2.3 Wedge and Symmetric Powers

10/4:
- OH slightly later today at 5:45-6:45 PM.
- Recap: Last time, we built new reps from old.
  - This stuff can't be learned in 1.5 lectures; he can point us around, but we have to learn it ourselves.
- Tensor product review.
  - Given V,W, make $V\otimes_F W$.
  - This vector space is hard to describe directly, so we more often talk about its dual $(V\otimes W)^*$ because this is actually easier to describe.
  - If you want to work with $V\otimes W$ hands-on, you can do the following.
    - Start with the following easy-to-work-with vector space: The (probably infinite-dimensional) vector space where each $v\otimes w$ is a basis vector for all $v\in V$ and $w\in W$.
    - Then quotient it by relations to force them to hold in the final space.
  - Here's an example of this construction.
    - Let $V=W$ be the one-dimensional vector space over the finite field $F_2=\mathbb{Z}/2\mathbb{Z}$.
    - Thus, the elements of V are $\{0,1\}$ (which is, literally, all linear combinations $a0+b1$ where $a,b\in F_2$ as well; this hearkens back to V's definition as an F_2-module).
    - Then the easy-to-work-with vector space we're talking about is the 4-dimensional **free** vector space $U=\operatorname{span}(0\otimes 0,0\otimes 1,1\otimes 0,1\otimes 1)$.

  - Note that in this space, for example, $(0+1)\otimes 0 \neq 0\otimes 0 + 1\otimes 0$; representing the basis as column vectors, this is equivalent the obvious observation that
$$\begin{bmatrix}0\\0\\1\\0\end{bmatrix} \neq \begin{bmatrix}1\\0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$
  - But we want such relationships to hold true in our conceptual "tensor product space." Thus, we quotient it by the subspace spanning all elements of the form $(a+b)\otimes c - a\otimes c - b\otimes c$.
  - By direct computation, this subspace is $\text{span}(0\otimes 0, 0\otimes 1)$:
$$\begin{aligned}(0+0)\otimes 0 - 0\otimes 0 - 0\otimes 0 &= -0\otimes 0 & \qquad (0+0)\otimes 1 - 0\otimes 1 - 0\otimes 1 &= -0\otimes 1\\ (0+1)\otimes 0 - 0\otimes 0 - 1\otimes 0 &= -0\otimes 0 & (0+1)\otimes 1 - 0\otimes 1 - 1\otimes 1 &= -0\otimes 1\\ (1+1)\otimes 0 - 1\otimes 0 - 1\otimes 0 &= 0\otimes 0 & (1+1)\otimes 1 - 1\otimes 1 - 1\otimes 1 &= 0\otimes 1\end{aligned}$$
Note that once we've considered $(a+b)\otimes c$, we don't need to consider $(b+a)\otimes c$ because of the commutativity of addition in V. That is, it is axiomatic that $a+b=b+a$ for all $a,b\in V$. Additionally, in the last line above, we are using the facts that $1+1=2=0$ in F_2 and $a\otimes b + a\otimes b = 2a\otimes b = 0$ in any F_2-module to simplify the expressions.
  - Similarly, the subspace corresponding to $a\otimes(b+c) - a\otimes b - a\otimes c$ is $\text{span}(0\otimes 0, 1\otimes 0)$. Thus, altogether, we quotient out the subspace $X = \text{span}(0\otimes 0, 0\otimes 1, 1\otimes 0)$. This leaves us with a 1-dimensional $V\otimes V$, as expected for the tensor product of two one-dimensional vector spaces. It is interesting to note that the one vector we didn't quotient out ($1\otimes 1$) is analogous to $e_1\otimes e_1$ since $e_1\in V$ might as well be defined $e_1 := 1$.
  - Now let's see how well this quotienting worked. First off, a bit of notation: let $\pi: U\to V\otimes V$ be the projection $\pi: v\mapsto v+X$, and denote elements $\pi(v_1\otimes v_2)\in V\otimes V$ by $v_1\otimes_\pi v_2$ for now to differentiate them from elements of U.
  - Let $(0+1)\otimes_\pi 0 = (0+1)\otimes 0 + X$ be an element of the quotient space $V\otimes V$. Certainly, the elements $0\otimes_\pi 0$ and $1\otimes_\pi 0$ are also elements of this quotient space. Moreover, we can fairly form the linear combination $(0+1)\otimes_\pi 0 - 0\otimes_\pi 0 - 1\otimes_\pi 0$. However, this element lies in the quotiented-out subspace X. Thus,
$$(0+1)\otimes_\pi 0 - 0\otimes_\pi 0 - 1\otimes_\pi 0 = [(0+1)\otimes 0 - 0\otimes 0 - 1\otimes 0] + X = 0 + X = 0$$
  - But
$$(0+1)\otimes_\pi 0 - 0\otimes_\pi 0 - 1\otimes_\pi 0 = 0 \implies (0+1)\otimes_\pi 0 = 0\otimes_\pi 0 + 1\otimes_\pi 0$$
as desired.
  - Note that this construction also gives us nice things like $0\otimes_\pi 0 = 0$, $0\otimes_\pi 1 = 0$, etc. which were not true in U! It should not be concluded, though, that all we need to quotient out of U for any V is $\text{span}(0\otimes 0, 0\otimes v, v\otimes 0)$ for every $v\in V$; indeed, $V=\mathbb{R}$, for example, will contain
- If V has basis $e_1,\dots,e_n$ and W has basis $f_1,\dots,f_m$, then $e_i\otimes f_j$ is a basis of $V\otimes W$.
- Interesting fact 1: If V,W are finite dimensional, $V^*\otimes W\cong \text{Hom}(V,W)$.
- If we want to work with the tensor product in practice in *rep theory*, the only thing we need to know is the basis of the tensor product space, which can tell us how any map $\rho(g)$ acts on both sides of a $v\otimes w\in V\otimes W$. From here, we recover the Kronecker product of matrices.
- So many things are explained by the concept of tensor products!
- A tensor in *physics* is something with lots of indices that changes in some way.
  - It does come from the math concept.
  - We'll get a huge basis because we have a massive product like $V\otimes\cdots\otimes V\otimes V^*\otimes\cdots\otimes V^*$.

- **Free** (vector space): A vector space that has a basis consisting of linearly independent elements.
  - Example: Think of $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ as a $\mathbb{C}$-module. A free version $F(V)$ of V is infinite dimensional with every $v \in V$ a linearly independent basis vector. Elements of $F(V)$ are of the form $a_1v_1 + \cdots + a_kv_k$ for $a_1, \dots, a_k \in \mathbb{C}$ and $v_1, \dots, v_k \in V$. If $u = v + w$ where $u, v, w \in V$ are all nonzero, then $u \neq v + w$ in $F(V)$ because they are all linearly independent basis vectors.
  - Example: What we formally start with in the example above is $V \times V$, the free F_2-module not the Cartesian product vector space V^2.
  - A terrific explanation of free vector spaces is available here.
- Last 2 useful notions: Wedge powers and symmetric powers.
  - Again, it's much easier to think about the dual space.
- Consider the space $V^{\otimes n}$ (dimension $(\dim V)^n$).
  - $(V^{\otimes n})^*$ are **polylinear** maps $f : V^n \to F$.
    - Note: By contrast, $(V^n)^*$ is the space of all *linear* maps $f : V^n \to F$.
    - This distinction is subtle but important. Note, for instance, that $\dim V^{\otimes n} \neq \dim V^n$ and likewise for the duals.
    - The distinction comes out fully when considering that if, for example, $V = \mathbb{R}^3$, then $V^2 \cong \mathbb{R}^6$ and any map in $(V^2)^*$ is determined by its action on $(e_1, 0), (e_2, 0), (e_3, 0), (0, e_1), (0, e_2), (0, e_3)$. By contrast, any map in $(V^{\otimes 2})^*$ is determined by its action on $(e_1, e_1), (e_1, e_2), (e_1, e_3), (e_2, e_1)$, $(e_2, e_2), (e_2, e_3), (e_3, e_1), (e_3, e_2), (e_3, e_3)$.
    - Important note: What $(V^{\otimes 2})^*$ does is consider these nine elements of V^2 as the basis of another space. This is what it truly means when we say "a bilinear map on V^2 is a linear map on $V^{\otimes 2}$."
    - Takeaway: Polylinearity changes the basis upon which a function $f : V^n \to F$ fundamentally acts.
  - A polylinear map may be **symmetric**, **antisymmetric**, or[2] neither.
  - These maps form vector spaces and the dimension is actually pretty meaningful.
- **Symmetric** (polylinear map): A polylinear map $f : V^n \to F$ that satisfies the following property. *Constraint*
$$f(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = f(v_1, \dots, v_n)$$
- **Antisymmetric** (polylinear map): A polylinear map $f : V^n \to F$ that satisfies the following property. *Constraint*
$$f(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = (-1)^\sigma f(v_1, \dots, v_n)$$
- Suppose you take $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$[3].
  - Consider a symmetric polylinear map $f : V \times V \times V \to \mathbb{C}$.
  - To compute it, we'll need the action of f on the basis of V^3. In particular, we'll need...
$$f(x_1e_1 + y_1e_2, x_2e_1 + y_2e_2, x_3e_1 + y_3e_2) = x_1x_2x_3 f(e_1, e_1, e_1) + x_1x_2y_3 f(e_1, e_1, e_2) + \cdots$$
    - Somewhere in there, you'll also have a $x_1y_2x_3 f(e_1, e_2, e_1)$ term as well.
    - However, because f is symmetric, you know by symmetry that these "bases" are the same, so you don't count them as 2 towards the dimension but as 1.
    - Thus, dim = 4 for symmetric maps.

[2]This is an exclusive "or."

[3]Note that this notation allows you to define a vector space *and* its basis in one go! I.e., the alternative is saying "Let V be a complex vector space with basis e_1, e_2."

- What about antisymmetric maps?
- Suppose $g : V^3 \to \mathbb{C}$ is an antisymmetric polylinear map.
  - Consider $g(e_1, e_1, e_1)$. Suppose you apply (12). Interchanging the first two indices (for instance) obviously won't do anything, so we'll get

$$\begin{aligned} g(e_1,e_1,e_1) &= (-1)^{(12)} g(e_1,e_1,e_1) \\ g(e_1,e_1,e_1) &= -g(e_1,e_1,e_1) \\ 2g(e_1,e_1,e_1) &= 0 \\ g(e_1,e_1,e_1) &= 0 \end{aligned}$$

  - But what about $g(e_1, e_1, e_2)$? We could apply (23) and get $g(e_1, e_2, e_1)$, right? So it appears that we would just be shrinking two options into one. Technically, this is true, but what's more important is that applying (12) again yields the same thing, meaning that $g(e_1, e_1, e_2) = g(e_1, e_2, e_1) = 0$.
  - And thus, since V has dimension 2 but g takes three vectors, any argument submitted to g will always be linearly dependent. Thus, $g = 0$ and, in fact, the space of antisymmetric maps on V^3 has dimension 0.
- Note: It's not always a rule that $V^{\otimes m} \cong S^m V \oplus \Lambda^m V$.

- Mathematically, there's a more natural object to work with than symmetric and antisymmetric maps.
  - Wedge powers and symmetric powers!
  - Given V and $n \in \mathbb{N}$, we can construct $S^n V$ and $\Lambda^n V$. $(S^n V)^*$ is symmetric polylinear maps taking n arguments from V. $(\Lambda^n V)^*$ is antisymmetric polylinear maps taking n arguments from V.
- How about a concrete way to see these? We can relate them to tensor powers.
  - Take a tensor power $V^{\otimes n}$, then look at those tensors which are symmetric and antisymmetric under permutation.
  - Example: Let V be the same as before. Then $V^{\otimes 2}$ has $\dim = 4$.
    - Take as basis elements for $S^2 V$ those that don't change when you change the coordinates.
    - Take as basis elements for $\Lambda^2 V$ those that flip sign when you change the coordinates.
    - In this case, the basis of $V^{\otimes 2}$ is $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$. The basis of $S^2 V$ will be $e_1 \otimes e_1, e_1 \otimes e_2 + e_2 \otimes e_1, e_2 \otimes e_2$. The basis of $\Lambda^2 V$ will be $e_1 \otimes e_2 - e_2 \otimes e_1$. Notice that these bases are identical (up to scaling) with those in Serre (1977) and those produced by applying the **symmetrization** and **alternation** operators to the basis of $V^{\otimes 2}$.
  - $S^2 V$ and $\Lambda^2 V$ direct sum because the dimensions match and they don't intersect, so we're good to go!
  - Everything we're doing is representations, so $g(v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n$.
- Relating this to something we've seen, but that's a little confusing.
  - The product notation is suggestive for symmetric vectors; you can commute $e_1 \cdot e_2 \in S^2 V$, for instance.
  - This allows us to, for example, shrink $e_1 \otimes e_1$ to $2e_1^2$[4], but $e_1 \otimes e_2 + e_2 \otimes e_1$ only to $e_1 \cdot e_2$.
  - Note that $e_1 \wedge e_2 = e_1 \otimes e_2 - e_2 \otimes e_1$ by definition.
  - Fact/exercise: Let V be a vector space of dimension n. V^* is the dual space, but it is also a function space. If $V = \mathbb{R}^k$, its a space of *functions from the blackboard.*
    - Note that $(\Lambda^k V)^* = \Lambda^k V^*$.

[4]Why the 2 coefficient??

  - S^nV^* is homogeneous polynomials of degree n.
  - You can take higher degree polynomials and just keep pushing through.
    - Ask about this??
  - Wedge powers now.
  - By convention, $\Lambda^0V = F$ and $\Lambda^1V = V$. But then you get to Λ^2V and Λ^3V. They grow but then shrink down as the power approaches $\dim V$.
  - Truth: The dimension of wedge powers Λ^iV is $\binom{k}{i}$ for $\dim V = k$. Figuring out why this is the case is another good exercise.
- An interesting connection between wedge powers and the determinant.
  - Let $V = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$.
  - Recall that Λ^nV^* is the space of antisymmetric polylinear functions $V \times \cdots \times V \to F$ taking n arguments from V, and it has a single basis vector $e^1 \wedge \cdots \wedge e^n$.
  - Let $v_1 = \sum a_{i1}e_i$, $v_2 = \sum a_{i2}e_i$, etc.
  - Let $f \in \Lambda^nV^*$, so that f is an alternating polylinear map that takes n arguments.
  - Since f is polylinear, we have that

$$f(v_1,\dots,v_n) = \sum_{i_1,\dots,i_n=1}^{n} a_{i_11}\cdots a_{i_nn}f(e_{i_1},\dots,e_{i_n})$$

  - Because of antisymmetry, we need only look at elements where the indices are all different. Thus, the above equals

$$\sum_{\sigma\in S_n} a_{\sigma(1)1}\cdots a_{\sigma(n)n}f(e_{\sigma(1)},\dots,e_{\sigma(n)})$$

  - Additionally, $f(e_{\sigma(1)},\dots,e_{\sigma(n)}) = (-1)^\sigma f(e_1,\dots,e_n)$ for any $\sigma \in S_n$. Moreover, $f(e_1,\dots,e_n) \in \mathbb{C}$ by definition, so define a constant $\lambda := f(e_1,\dots,e_n)$. Thus, the above equals

$$\lambda \sum_{\sigma\in S_n} a_{\sigma(1)1}\cdots a_{\sigma(n)n}$$

  - But the term following the λ is just the determinant of the $n \times n$ matrix (a_{ij}). Thus, all said,

$$f(v_1,\dots,v_n) = \lambda \det(v_1 \mid \cdots \mid v_n)$$

  - Implication: Wedge powers are something like the determinant.
    - In particular, because Λ^nV^* has only a single basis vector as mentioned above, $f = \lambda e^1 \wedge \cdots \wedge e^n$. It follows that $e^1 \wedge \cdots \wedge e^n = \det$.
  - Takeaway: Wedge powers are something interesting; there's a reason to study them.
- The basis of the wedge powers consists of wedge monomials $e_{j_1} \wedge \cdots \wedge e_{j_i}$. Moreover, no need to have the same list twice, so choose some way of indexing them, e.g., increasing indexes.
  - This is why we do *increasing* bases! There's no particular reason, it's just an arbitrary way of making sure we don't do the same thing twice! We could just as well choose decreasing or any other means of guaranteeing that we don't have duplicates.
- Now let's relate all of this exterior and symmetric product stuff back to representation theory.
  - Let $V = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$.
  - Let $G \circlearrowright V$ via the homomorphism $G \to GL(V) \cong GL_n(\mathbb{C})$.

  - Focusing more on the *matrix* aspect this time, note that under this homomorphism, $g \mapsto A_g$ subject to the homomorphism constraints.
  - Consider the set $\{A_{g_1}, \dots, A_{g_k}\}$ of all matrices in the image of the homomorphism. If we transpose all of them, will they still obey the homomorphism constraints?
    - Nope!
    - Indeed, if we do this, we'll get in trouble. More specifically, transposition is not a representation because $A_{g_1}^T A_{g_2}^T \neq A_{g_1 g_2}^T = A_{g_2}^T A_{g_1}^T$.
  - It's the same story with inverses.
  - *However*, combining the two operations, we get

$$(A_{g_1 g_2}^T)^{-1} = (A_{g_1}^T)^{-1}(A_{g_2}^T)^{-1}$$

    - This is exactly when we take a representation and then go to the dual[5].
  - This will be on next week's homework!
  - Takeaway: This is an application of $\Lambda^j V^*$ to representation theory, $j \neq k, n$.
- Another relation: An application of $\Lambda^n V^*$ to representation theory.
  - Suppose we have a representation $G \circlearrowright V$ that we want to flatten into $G \circlearrowright \mathbb{C}$. How can we turn a relation between a group of matrices into a relation between a group of numbers?
  - Use the determinant!
  - Indeed, we already know that

$$\det(A_e) = 1 \qquad \det(A_{g_1 g_2}) = (\det A_{g_1})(\det A_{g_2}) \qquad \det\big(A_{g^{-1}}\big) = \det(A_g)^{-1}$$

  - In particular, we make formal the transition $G \to GL_j(\mathbb{C}) \to \mathbb{C}$ with the **top wedge power** $\Lambda^n V^*$.
- A last note.
  - Don't think that we're limited to top wedge powers.
  - Recall that we can define tensor products of matrices via the Kronecker product. Well, we can prove that

$$A_{g_1 g_2}^{\otimes 2} = A_{g_1}^{\otimes 2} A_{g_2}^{\otimes 2}$$

    and the like as well!
  - Similarly, we can define Λ^2 of a matrix.
  - We'll get into some weird Kronecker product stuff again, but we can sort through it.
- Plan for Friday and next time.
  - Prove the theorem that every representation is a sum of irreducible representations.
  - He will use projectors.
  - Then a horror story.
  - Then associative algebra.

[5]Relation to MATH 20510 when we discussed dual matrices and pullbacks of matrices.

2.4 Office Hours

10/5:

- Problem 2a:
  - $(V \oplus W) \otimes (V \oplus W) \stackrel{?}{=} V \otimes V \oplus V \otimes W \oplus W \otimes V \oplus W \otimes W$.
  - Check linearity in all terms and then with universal property. Check antisymmetric, linear, injective, surjective; dimensions are the same, so no need to check *both* injectivity and surjectivity (surjectivity is easier to check). We can go to basis to check various properties; we can't use a basis to write the map, but we can use bases to check surjectivity and the like.
- Problem 3a:
  - Bezout and Gauss's lemma is good to learn on my own. Put polynomials in each variable. Throw some stuff about this shit into my answers.
  - Relearn polynomial division.
  - $(1,1,1,1)$, $(1,1,-1,-1)$, $(1,-1,1,-1)$, and $(1,-1,-1,1)$.
  - This is a symmetric matrix.
  - The upper-left and lower-right blocks of this matrix match; so do the lower-left and upper-right.
  - When the eigenvalue is equal to zero, the determinant is equal to zero. So look for eigenvectors to calculate eigenvalues, and then just express the determinant as a product of these.
- Problem 3b:
  - Corresponding eigenvalue is $\sum_{i=1}^n x_i z^{i-1}$.
  - Can I use representation theory to do this? What group has a multiplication table like this? $\mathbb{Z}/n\mathbb{Z}$. The elements of $\mathbb{Z}/n\mathbb{Z}$ are of the form $\{1, \zeta, \dots, \zeta^{n-1}\}$.
  - If that's an eigenvector, then it's a subrepresentation; it is a space that is fixed under the action of the matrix.
  - Other eigenvectors: $(1,1,1)$, $(1,z^2,z)$.
  - We don't need to do induction or anything fancy like that; we can just do dots. As long as your argument is complete and clear, you're good.
- Problem 4a:
  - See FH 1.3. Standard rep, not wedge. Treat τ, σ (generators of the action) on the basis vectors.
  - If both fix, it's the trivial; if one flips, you have alternating; if both flip, you have standard.
  - $(2,1) \oplus (1,1,1)$. Use problem 2.
  - See FH Exercise 1.2??
  - The action of τ on this basis vector can be computed:

$$\tau(\alpha \wedge \beta) = 1\alpha \wedge \beta$$

  - Having obtained an eigenvalue of 1, we can rule out the standard representation.
- Problem 4b:
  - $\{\alpha \otimes \alpha \otimes \alpha, \alpha \otimes \alpha \otimes \beta + \alpha \otimes \beta \otimes \alpha + \beta \otimes \alpha \otimes \alpha, \beta \otimes \beta \otimes \beta\}$
- Problem 5a:
  - Consider an alternate basis $f_1, \dots, f_n$ and dual basis $f^1, \dots, f^n$. Consider the element $f_1 \otimes f^1 + \cdots + f_n \otimes f^n \in V \otimes V^\vee$. We want to prove that it equals the one asked about in the question.

  - Under the isomorphism to $\text{Hom}(V,V)$, we send $e_1 \otimes e^1$ to $[v \mapsto e^1(v)e_1]$. More generally, we end $e_i \otimes e^i$ to $[v \mapsto e^i(v)e_i]$. Adding all these maps together yields the map $[v \mapsto e^1(v)e_1 + \cdots + e^n(v)e_n]$, which is just the identity $1 \in \text{Hom}(V,V)$, regardless of basis.

- Problem 5b:
  - Example:
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to ae_1 \otimes e^1 + be_1 \otimes e^2 + ce_2 \otimes e^1 + de_2 \otimes e^2$$
  - Evaluating this gives
$$e^1(ae_1) + e^2(be_1) + e^1(ce_2) + e^2(de_2) = a + d$$
    since it's only when the indices match (i.e., along the diagonal) that we get a nonzero value.

- Problem 5c:
  - P should have a block-diagonal matrix corresponding to the decomposition $V = W \oplus W^0$. P is the identity on $\text{Im}(P)$. So if our basis is vectors spanning W and then vectors spanning W^0, the matrix should be the identity and then the zero matrix. That should do the trick. How rigorous does this need to be?
  - Let $e_1, \dots, e_k$ be an orthonormal basis of $\text{Im}(P)$. Extend this basis to an orthonormal basis $e_1, \dots, e_n$ of V.

- Problem 5d:
  - Trivial representation: All $g \in G$ get mapped to $1 \in GL(V)$.
  - Part (a) gives us the identity in $\text{Hom}(V,V)$.
  - So we have $\rho : G \to GL(V)$.
  - Is any line acceptable? Span of the identity function? Rudenko: It depends on V. It has *infinitely many* trivial sub representations.
  - Example: $G \circlearrowright \mathbb{C}^2$. with $\rho(g) = I_2$.
  - Dual representation: Defined analogously to the $\text{Hom}_F(V,W)$ representation. We also need an inverse.

- Psets will likely get easier; right now, we have to relearn a lot of old stuff and we are being challenged with harder problems. As the questions become more based on course content and thus will get easier.

- He'll do hard PSets, easy exams, and everythign is curved; he agrees that this is a hard pset, and probably harder than necessary.

2.5 Complete Reducibility

10/6:

- Let G be a finite group.
- We want to study finite dimensional representations over $\mathbb{C}$.
  - Characteristic F, $|G| = 1$.
  - What is this stuff about characteristic??
- Theorem: Any f.d. representation can be decomposed into a sum of irreps via
$$V = V_1^{n_1} \oplus \cdots \oplus V_k^{n_k}$$
  Moreover, this decomposition is unique.
  - See Proposition 1.8.

- Example: We have already seen S_2, S_3 in the homework; now, let $G = \mathbb{Z}/n\mathbb{Z}$.
  - Consider $V_0, \dots, V_{n-1}$.
  - Let V_i be a 1-dimensional rep.
    - We have $\rho : G \to C^\times$ defined by $[k] \mapsto (e^{2\pi i/n})^k$.
  - These are all 1-dimensional representations up to isomorphism.
- Example: Let $G = \mathbb{Z}$. What is a representation of $\mathbb{Z}$? We just need to say what happens to 1.
  - For example, if the map $G \to GL_n(\mathbb{C})$ sends $1 \mapsto A$, then $2 \mapsto A^2$, and on and on.
  - A place where you run into trouble: $n = 2$ and

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

  - The matrix has a fixed subrepresentation (i.e., eigenvector $(1, 1)$).
    - More specifically, $\mathbb{C}(1, 0) \hookrightarrow V$ is a 1D subrepresentation.
  - The theorem basically tells us that $V = \mathbb{C}(1, 0) + \mathbb{C}w$.
  - This is an example of how things can go wrong. How??
- Proving the theorem; we need a miracle!
- Existence: We need a lemma.
- Lemma: Let G be finite, $F = \mathbb{C}$, and V a G-representation. Let $W \leq V$ be a subrepresentation or **invariant subspace**. Then there exists another invariant subspace $W' \subset V$ such that $V = W \oplus W'$.
  - See Theorem 1 (Rudenko replicates all aspects of the "limited conditions" proof).
  - This lemma implies existence.
  - Two proofs: One that only works over complex numbers. He suggests we read about it. Name??
  - He'll do the slightly less intuitive one, which involves **projectors**.
- **Projector**: A linear map $P : V \to V$ such that $P^2 = P$, that is, is **idempotent**.
  - Example: Consider $W := \mathrm{Im}(P) \leq V$. $P|_W$ does nothing; it's the identity.
  - A good mental picture: Things are falling from 3D space onto some smaller space.
  - On the kernel.
    - Importantly, $\mathrm{Ker}\, P \cap \mathrm{Im}\, P = 0$.
    - It follows that $V = \mathrm{Im}(P) \oplus \mathrm{Ker}(P)$.
    - Within the space, $v = (w, w') = w + w'$. What the projector does is $(w, w') \mapsto (w, 0)$.
  - What else can we say about projectors?
    - There is a correspondence between projectors and direct sum decompositions.
- So to prove the lemma, we need a projector $P : V \to V$ with image W and certain properties.
  - More specifically, the goal is to find a projector P (1) with image W and (2) that is a morphism of G-reps.
  - On the second condition, that is, we want $P(gv) = gP(v)$. In this case, $\mathrm{Ker}(P)$ will be a shuffle??
- Strategy.
  - Take any projector $P_0 : V \to W$. And then you can get g-projectors $gP = gPg^{-1}$.

- So define a new projector
$$P=\frac{1}{|G|}\sum_{g\in G}\underbrace{gP_0g^{-1}}_{gP_0}$$
One didn't work, so we hope the average will work, and it will!
- For any $w\in W$, we can prove that the sum thing does fix W's, so it is a projector!
- $P(hv)$ example.
- Note: This computation will be done again later in a different context; this averaging construction is *central* to representation theory.

- Constraints we used in the proof: G is finite (or compact), $|G|$ is invertible.
  - Only when we get into **modular representation theory** is where we get into trouble; this theorem actually kills **extensions**, which are very interesting but are not in finite group rep theory.
- Hermitian inner product isn't common here, but it shows up in physics. Point of this??
- Now for the other part of the original proof: Uniqueness.
- Schur's Lemma: Let G be a finite group, let the field $F=\mathbb{C}$, and let V,W be irreps over F. Let $f\in\operatorname{Hom}_G(V,W)$, which we may recall is the space of morphisms between G-reps V and W, that is, all $h:V\to W$ satisfying $h(gv)=gh(v)$. Then...
  1. $f=0$ if $V\ncong W$. If $V\cong W$, then these maps are isomorphisms.
  2. In particular, if V is an irrep and $f:V\to V$ is such that $f(gv)=gf(v)$, then $f(v)=\lambda v$.

  Altogether, we have that
$$\operatorname{Hom}_G(V,W)\cong\begin{cases}0 & V\ncong W\\ \mathbb{C} & V\cong W\end{cases}$$
  - The statement "$\operatorname{Hom}_G(V,W)=\mathbb{C}$" does not literally mean that the left space is the complex numbers; rather, it is the space of all scalar isomorphisms λI for scalars in $\mathbb{C}$, which happens to be isomorphic to $\mathbb{C}$.
  - Gist: If you want a certain kind of matrix between certain spaces, in some cases, you'll just fail.
- Proof.
  - See Lemma 1.7 (Rudenko replicates all aspects of the "limited conditions" proof).
  - For $f:V\to V$, consider $\operatorname{Ker}(f)$ and $\operatorname{Im}(f)$. The latter two are subrepresentations of V,W, respectively. $\operatorname{Ker}(f)=V$ implies $f=0$; symmetric with $\operatorname{Im}(f)$. If nonzero, then $\operatorname{Ker}=0$ and $\operatorname{Im}=W$, implies f is an isomorphism.
- Schur's Lemma is the easiest step to learn in the whole story.
- Last 2 minutes: Finish the proof of the original theorem.
  - Don't worry if we're confused by this last line; it will be repeated later in a much more powerful way.
  - Analogous to Proposition 1.8.
  - I might have missed some stuff here??
- Plan for next week.
  - Character theory.
  - Serre (1977) is still the best source for tracking lecture content for right now.
- This would have been an interesting but wholly nonessential lecture to pay attention to, since I already did all of the readings.

2.6 S Chapter 1: Generalities on Linear Representations

From Serre (1977).

Section 1.5: Tensor Product of Two Representations

10/4:
- **Tensor product** (of V_1, V_2): The vector space W that (a) is furnished with a map $V_1 \times V_2 \to W$ sending $(x_1, x_2) \mapsto x_1 \cdot x_2$ and (b) satisfies the following two conditions.
  (i) $x_1 \cdot x_2$ is bilinear.
  (ii) If (e_{i_1}) is a basis of V_1 and (e_{i_2}) is a basis of V_2, the family of products $e_{i_1} \cdot e_{i_2}$ is a basis of W.

  Denoted by $\boldsymbol{V_1 \otimes V_2}$.
  - It can be shown that such a space exists and is unique up to isomorphism (see proof here).
- This definition allows us to say some things quite expediently. For example, (ii) implies that
$$\dim(V_1 \otimes V_2) = \dim(V_1) \cdot \dim(V_2)$$
- **Tensor product** (of ρ^1, ρ^2): The representation $\rho : G \to GL(V_1 \otimes V_2)$ defined as follows for all $s \in G$, $x_1 \in V_1$, and $x_2 \in V_2$, where $\rho^1 : G \to GL(V_1)$ and $\rho^2 : G \to GL(V_2)$ are representations. *Given by*
$$[\rho_s^1 \otimes \rho_s^2](x_1 \cdot x_2) = \rho_s^1(x_1) \cdot \rho_s^2(x_2)$$
- A more formal write up of the matrix translation of this definition.
  - Let (e_{i_1}) be a basis for V_1, and let (e_{i_2}) be a basis for V_2.
  - Let $r_{i_1j_1}(s)$ be the matrix of ρ_s^1 with respect to this basis, and let $r_{i_2j_2}(s)$ be the matrix of ρ_s^2 with respect to this basis.
  - It follows that
$$\rho_s^1(e_{j_1}) = \sum_{i_1} r_{i_1j_1}(s)e_{i_1} \qquad\qquad \rho_s^2(e_{j_2}) = \sum_{i_2} r_{i_2j_2}(s)e_{i_2}$$
  - Therefore,
$$[\rho_s^1 \otimes \rho_s^2](e_{j_1} \cdot e_{j_2}) = \sum_{i_1,i_2} r_{i_1j_1}(s)r_{i_2j_2}(s)e_{i_1} \cdot e_{i_2}$$
    and
$$\mathcal{M}(\rho_s^1 \otimes \rho_s^2) = (r_{i_1j_1}(s)r_{i_2j_2}(s))$$
- Aside on quantum chemistry to come back to later; I can't quite connect the dots yet.

Section 1.6: Symmetric Square and Alternating Square

- Herein, we investigate the tensor product when $V_1 = V_2 = V$.
- Let (e_i) be a basis of V.
- Define the automorphism $\theta : V \otimes V \to V \otimes V$ by
$$\theta(e_i \cdot e_j) = e_j \cdot e_i$$
  for all 2-indices (i, j).
- Properties of θ.

- Since θ is linear, it follows that
$$\theta(x \cdot y) = y \cdot x$$
for all $x, y \in V$.
  - Implication: θ is independent of the chosen basis (e_i)!
- $\theta^2 = 1$, where 1 is the identity map on $V \otimes V$.

- Assertion: $V \otimes V$ decomposes into
$$V \otimes V = S^2(V) \oplus \Lambda^2(V)$$
  - Rudenko: We do not have to worry about proving this... yet, at least.
- **Symmetric square representation**: The subspace of $V \otimes V$ containing all elements z satisfying $\theta(z) = z$. *Denoted by* $\boldsymbol{S^2V}$, $\boldsymbol{S^2(V)}$, $\boldsymbol{\mathbb{S}^2V}$, $\textbf{Sym}^{\,2}\boldsymbol{(V)}$.
  - Basis: $(e_i \cdot e_j + e_j \cdot e_i)_{i \leq j}$.
    - Rudenko: How do we know everything is linearly independent? Well, when we add two linearly independent vectors out of a set, the sum is still linearly independent from everything else!
    - Example when $\dim V = 2$: The basis of $V \otimes V$ is $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$, where all four of these vectors are linearly independent. So naturally, the basis of the corresponding symmetric square representation — which is $2e_1 \otimes e_1, e_1 \otimes e_2 + e_2 \otimes e_1, 2e_2 \otimes e_2$ — will still be a linearly independent list of vectors.
  - Dimension: If $\dim V = n$, then
$$\dim S^2(V) = \frac{n(n+1)}{2}$$
- **Alternating square representation**: The subspace of $V \otimes V$ containing all elements z satisfying $\theta(z) = -z$. *Denoted by* $\boldsymbol{\Lambda^2V}$, $\boldsymbol{\Lambda^2(V)}$, $\textbf{Alt}^{\,2}\boldsymbol{(V)}$.
  - Basis: $(e_i \cdot e_j - e_j \cdot e_i)_{i < j}$.
  - Dimension: If $\dim V = n$, then
$$\dim \Lambda^2(V) = \frac{n(n-1)}{2}$$

2.7 FH Appendix B: On Multilinear Algebra

From Fulton and Harris (2004).

Section B.1: Tensor Products

10/5:

- **Tensor product** (of V, W over F): A vector space U equipped with a bilinear map $V \times W \to U$ sending $v \times w \to v \otimes w$ that is universal, i.e., for any bilinear map $\beta : V \times W \to Z$, there is a unique linear map from $U \to Z$ that takes $v \otimes w \mapsto \beta(v, w)$. *Denoted by* $\boldsymbol{V \otimes W}$, $\boldsymbol{V \otimes_F W}$.
  - The so-called *universal property* determines the tensor product up to canonical isomorphism.
- One construction of $V \otimes W$: From the basis $\{e_i \otimes f_j\}$.
  - This construction is **functorial**, implying that linear maps from $f : V \to V'$ and $g : W \to W'$ determine a linear map $f \otimes g : V \otimes W \to V' \otimes W'$, namely that defined by $f \otimes g : v \otimes w \to fv \otimes gw$.
- Definition of the n**-fold tensor product**.
- **Multilinear** (map): A map from a Cartesian product $V_1 \times \cdots \times V_n$ of vector spaces to a vector space U such that when all but one of the factors V_i are fixed, the resulting map from $V_i \to U$ is linear.
- Properties of the tensor product.

1. *Commutativity*:
$$V \otimes W \cong W \otimes V$$
by $v \otimes w \mapsto w \otimes v$.
2. *Distributivity*:
$$(V_1 \oplus V_2) \otimes W \cong (V_1 \otimes W) \oplus (V_2 \otimes W)$$
by $(v_1, v_2) \otimes w \mapsto (v_1 \otimes w, v_2 \otimes w)$.
3. *Associativity*:
$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W) \cong U \otimes V \otimes W$$
by $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w) \mapsto u \otimes v \otimes w$.

- **Tensor power** (of V to n): The tensor product defined as follows. *Denoted by* $\boldsymbol{V^{\otimes n}}$. *Given by*
$$V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}$$
  - Convention: $V^{\otimes 0} = F$.
- Analogous construction of the tensor product for generalized algebras and modules.

Section B.2: Exterior and Symmetric Powers

- **Alternating** (multilinear map): A multilinear map β such that $\beta(v_1, \dots, v_n) = 0$ whenever $v_i = v_j$ for some $i, j \in [n]$.
  - Implication: $\beta(v_1, \dots, v_n)$ changes sign whenever two of the vectors are interchanged.
    - Follows from the definition and the **standard polarization**.
  - Implication:
$$\beta(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = (-1)^\sigma \beta(v_1, \dots, v_n)$$
for all $\sigma \in S_n$.
- **Standard polarization**: The equality
$$\beta(v, w) + \beta(w, v) = \beta(v + w, v + w) - \beta(v, v) - \beta(w, w) = 0 - 0 - 0 = 0$$
- **Exterior powers** (of V): The vector space U equipped with an alternating multilinear map $V \times \cdots \times V \to \Lambda^n V$ sending $v_1 \times \cdots \times v_n \mapsto v_1 \wedge \cdots \wedge v_n$ that is universal, i.e., for any alternating multilinear map $\beta : V^n \to Z$, there is a unique linear map from U to Z that takes $v_1 \wedge \cdots \wedge v_n \mapsto \beta(v_1, \dots, v_n)$. *Denoted by* $\boldsymbol{\Lambda^n V}$.
  - Convention: $\Lambda^0 V = F$.
- Quotient space construction of the exterior powers.
- Projecting from $V^{\otimes n} \to \Lambda^n V$: Define $\pi : V^{\otimes n} \to \Lambda^n V$ by
$$\pi(v_1 \otimes \cdots \otimes v_n) = v_1 \wedge \cdots \wedge v_n$$
- Basis for the exterior powers.
- There is a canonical linear map $\Lambda^a V \otimes \Lambda^b W \to \Lambda^{a+b}(V \oplus W)$, which takes $(v_1 \wedge \cdots \wedge v_a) \otimes (w_1 \wedge \cdots \wedge w_b) \mapsto v_1 \wedge \cdots \wedge v_a \wedge w_1 \wedge \cdots \wedge w_b$.
  - This determines (how??) an isomorphism
$$\Lambda^n(V \oplus W) \cong \bigoplus_{a=0}^{n} \Lambda^a V \otimes \Lambda^{n-a} W$$

- This isomorphism plus induction on n can justify (how??) the basis for $\Lambda^n V$ as the increasing indices.

- **Symmetric** (multilinear map): A multilinear map β such that $\beta(v_1, \dots, v_n)$ is unchanged when any two factors are interchanged, that is

$$\beta(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = \beta(v_1, \dots, v_n)$$

for all $\sigma \in S_n$.

- **Symmetric powers** (of V): The vector space U equipped with a symmetric multilinear map $V \times \cdots \times V \to S^n V$ sending $v_1 \times \cdots \times v_n \mapsto v_1 \cdot \ldots \cdot v_n$ that is universal, i.e., for any symmetric multilinear map $\beta : V^n \to Z$, there is a unique linear map from U to Z that takes $v_1 \cdot \ldots \cdot v_n \mapsto \beta(v_1, \dots, v_n)$. *Denoted by* $\boldsymbol{S^n V}$.
  - Convention: $S^0 V = F$.

- Quotient space construction of the symmetric powers.
  - Quotient out all $v_1 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$, that is, those elements of $V^{\otimes n}$ in which σ permutes two successive factors. How does this work??

- Projecting from $V^{\otimes n} \to S^n V$: Define $\pi : V^{\otimes n} \to S^n V$ by

$$\pi(v_1 \otimes \cdots \otimes v_n) = v_1 \cdot \ldots \cdot v_n$$

- Basis for the symmetric powers.
  - It follows from the basis construction that $S^n V$ can be regarded as the space of homogeneous polynomials of degree n in the variable e_i, since each element is of the form $e_{i_1} \cdot \ldots \cdot e_{i_n}$ and we can add them.

- Canonical isomorphism:

$$S^n(V \oplus W) \cong \bigoplus_{a=0}^{n} S^a V \otimes S^{n-a} W$$

- More on $\Lambda^n V, S^n V$ as subspaces of $V^{\otimes n}$.
  - We inject $\iota : \Lambda^n V \to V^{\otimes n}$ with

$$\iota(v_1 \wedge \cdots \wedge v_n) = \sum_{\sigma \in S_n} (-1)^\sigma v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

    - This relates to Rudenko's note that $v_1 \wedge v_2 = v_1 \otimes v_2 - v_2 \otimes v_2$!
    - There are some more advanced notes on the implications of ι; $[\iota \circ \pi/n!](V^{\otimes n}) = V^{\otimes n}$ is brought up.
  - We inject $\iota : S^n V \to V^{\otimes n}$ with

$$\iota(v_1 \cdot \ldots \cdot v_n) = \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

    - More, related advanced notes; includes the $1/n!$ thing again.

- **Wedge product**: The function $\Lambda^m V \otimes \Lambda^n V \to \Lambda^{m+n} V$ defined as follows. *Denoted by* $\wedge$. *Given by*

$$(v_1 \wedge \cdots \wedge v_m) \otimes (v_{m+1} \wedge \cdots \wedge v_{m+n}) \mapsto v_1 \wedge \cdots \wedge v_m \wedge v_{m+1} \wedge \cdots \wedge v_{m+n}$$

- Properties of the wedge product.

1. *Associativity*:

$$(v_1 \wedge v_2) \wedge v_3 = v_1 \wedge (v_2 \wedge v_3) = v_1 \wedge v_2 \wedge v_3$$

2. *Skew-commutativity*:

$$v_1 \wedge v_2 = -v_2 \wedge v_2$$

- Note that both of the above properties hold in higher-dimensional cases as well.
- Commutativity of the products.

$$\begin{tikzcd}
\Lambda^m V \otimes \Lambda^n V \arrow[r, "\wedge"] \arrow[d, hook, "\iota\otimes\iota"'] & \Lambda^{m+n} V \arrow[d, hook, "\iota"] \\
V^{\otimes m} \otimes V^{\otimes n} \arrow[r, "f_1"'] & V^{\otimes(m+n)}
\end{tikzcd}$$

(a) Wedge product.

$$\begin{tikzcd}
S^m V \otimes S^n V \arrow[r, "\cdot"] \arrow[d, hook, "\iota\otimes\iota"'] & S^{m+n} V \arrow[d, hook, "\iota"] \\
V^{\otimes m} \otimes V^{\otimes n} \arrow[r, "f_2"'] & V^{\otimes(m+n)}
\end{tikzcd}$$

(b) Symmetric product.

Figure 2.3: Commutative diagram, wedge and symmetric products.

  - f_1 is defined by

$$(v_1 \otimes \cdots \otimes v_m) \otimes (v_{m+1} \otimes \cdots \otimes v_{m+n}) \mapsto \sum_{\sigma \in G} (-1)^\sigma v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)} \otimes v_{\sigma(m+1)} \otimes \cdots \otimes v_{\sigma(m+n)}$$

    where G is the subgroup of S_{m+n} preserving the order of the subsets $\{1, \dots, m\}$ and $\{m + 1, \dots, m + n\}$.
  - f_2 is defined analogously.
- The above mappings all commute with linear maps of vector spaces.
  - Example: Our definition $g(v \otimes w) = gv \otimes gw$ could be redrawn as $[g \circ \otimes](v, w) = [\otimes \circ g](v, w)$, where the latter $g : (v, w) \mapsto (gv, gw)$ by abuse of notation.
- Tensor, exterior, and symmetric algebras.

2.8 FH Chapter 1: Representations of Finite Groups

From Fulton and Harris (2004).

10/1: • Starts with a justification for beginning their investigation of rep theory with finite groups.

Section 1.1: Definitions

- Definition of a **representation**.
- ρ "gives V the structure of a G-module!" (Fulton & Harris, 2004, p. 3).
- When there is little ambiguity about ρ, we call V itself a representation of G.
  - This is what Rudenko has been doing in class!
- We also often write $g \cdot v$ for $\rho(g)(v)$, and g for $\rho(g)$.
- **Degree** (of ρ): The dimension of V.
- ***G*-linear** (map): See class notes. *Also known as* **map**, **morphism**.
- The **kernel**, **image**, and **cokernel** of φ are all G-submodules.

- **Kernel** (of a map): The vector subspace containing all $v \in V$ for which $\varphi(v) = 0$. *Denoted by* $\mathbf{Ker}\,\boldsymbol{\varphi}$.
- **Image** (of a map): The vector subspace containing all $w \in W$ for which there exists $v \in V$ such that $\varphi(v) = w$. *Denoted by* $\mathbf{Im}\,\boldsymbol{\varphi}$.
- **Cokernel** (of a map): The quotient space $W/\operatorname{Im}\varphi$.
- Definitions of **subrepresentation**, **irreducible** representation, and **direct sum** of representations.
- **Tensor product** (of V, W): The representation with the space $V \otimes W$ where $g(v \otimes w) = gv \otimes gw$.

10/5:

  - The n^{th} tensor power is also a representation by this rule.
  - The n^{th} exterior and symmetric powers are subrepresentations of the n^{th} tensor power.
- **Natural pairing** (between V^*, V): The pairing defined as follows for all $v^* \in V^*$ and $v \in V$. *Denoted by* $\langle\ ,\ \rangle$. *Given by*
$$\langle v^*, v\rangle = v^*(v) = (v^*)^T v$$
- **Dual representation**: The representation from $G \to GL(V^*)$ defined as follows. *Denoted by* $\boldsymbol{\rho^*}$. *Given by*
$$\rho^*(g) = \rho(g^{-1})^T$$
  - We should — and do — have
$$\langle \rho^*(g)(v^*), \rho(g)(v)\rangle = \langle v^*, v\rangle$$
  - Indeed,
$$\begin{aligned}\langle \rho^*(g)(v^*), \rho(g)(v)\rangle &= \rho^*(g)(v^*)^T \rho(g)v\\ &= [\rho(g^{-1})^T(v^*)]^T \rho(g)v\\ &= (v^*)^T \rho(g^{-1})T\rho(g)v\\ &= (v^*)^T v\\ &= \langle v^*, v\rangle\end{aligned}$$
- $\operatorname{Hom}(V, W)$ is a representation.
  - Definition via the commutative diagram (from class): $g(L)v = [g \circ L \circ g^{-1}]v$.
  - Definition analogously to $V^* \otimes W$: $g(v^* \otimes w) = gv^* \otimes gw = (g^{-1})^T v^* \otimes gw = [(g^{-1})^T v^*](gw) = [(g^{-1})^T v^*]^T gw = (v^*)^T g^{-1} gw = v^*(v)$??
- The rules for normal vector spaces hold for representations as well, e.g.,
$$V \otimes (U \oplus W) = (V \otimes U) \oplus (V \otimes W) \qquad \Lambda^k(V \oplus W) = \bigoplus_{a+b=k} \Lambda^a V \oplus \Lambda^b W \qquad \Lambda^k(V^*) = \lambda^k(V)^*$$
- Definition of **permutation representation** and **regular representation**.

Section 1.2: Complete Reducibility, Schur's Lemma

- **Indecomposable** (representation): See class notes. *Also known as* **irreducible**.
- Proof of Theorem 1 as in Serre (1977).
  - The method is called "integration over the group (with respect to an invariant measure on the group)" (Fulton & Harris, 2004, p. 6).
- **Complete reducibility**: The property that any representation is a direct sum of irreducible representations. *Also known as* **semisimplicity**.

 - Stated here as a corollary; proven as Theorem 2 in Serre (1977).

- The following lemma has several consequences, among which is that it determines how much a representation's direct-sum decomposition is unique.

  Lemma 1.7 (Schur's Lemma)**.** *If V and W are irreducible representations of G and $\varphi : V \to W$ is a G-module homomorphism, then...*

  1. *Either φ is an isomorphism, or $\varphi = 0$;*
  2. *If $V = W$, then $\varphi = \lambda I$ for some $\lambda \in \mathbb{C}$, I being the identity.*

  Proof. Suppose for the sake of contradiction that φ is neither an isomorphism nor zero. Then it has a nontrivial kernel and image, both of which are necessarily invariant under the representation. Therefore, neither V nor W are irreducible representations of G, a contradiction.

  Since $\mathbb{C}$ is algebraically closed, φ must have an eigenvalue λ. Equivalently, for some $\lambda \in \mathbb{C}$, $\varphi - \lambda I$ has nonzero kernel. But then by part (1), we must have $\varphi - \lambda I = 0$, implying that $\varphi = \lambda I$, as desired. $\square$

- Direct sum irreducible decomposition.

  Proposition 1.8. *For any representation V of a finite group G, there is a decomposition*

  $$V = V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}$$

  where the V_i are distinct irreducible representations. The decomposition of V into a direct sum of the k factors is unique, as are the V_i that occur and their multiplicities a_i.

  Proof. Let W be another representation of G, possibly of different dimension. Let $\varphi : V \to W$ be a map of representations. Restrict φ to $V_i^{\oplus a_i}$, a subrepresentation of V. It follows from Schur's Lemma that this restriction either maps into the $W_j^{\oplus b_j}$ satisfying $W_j \cong V_i$, or it does not map it at all.

  Uniqueness for the decomposition of V follows by applying Schur's Lemma to the identity map on V. $\square$

- Goals going forward.

  1. Describe all the irreducible representations of G.
     - We can find all *irreducible* representations of G, then describe *any* representation as a linear combination of these.
  2. Find techniques for giving the direct sum decomposition and the multiplicities of an arbitrary representation.
  3. **Plethysm**: Describe the decompositions, with multiplicities, of representations derived from a given representation V, such as $V \otimes V$, V^*, $\Lambda^k V$, $S^k V$, and $\Lambda^k(\Lambda^1 V)$.
     - Note: If V decomposes into two representations, these representations decompose accordingly, e.g., if $V = U \oplus W$, we may invoke the earlier identity to learn that $\Lambda^k V = \oplus_{i+j=k} \Lambda^i U \otimes \Lambda^j W$.
     - **Clebsch-Gordon problem**: Decompose $V \otimes W$, given two irreducible representations V and W.

Section 1.3: Examples — Abelian Groups, S_3

- Classifying the irreducible representations of abelian groups.
  - *Return to later??*
- Classifying the irreducible representations of S_3.

- There exist two one-dimensional representations of S_3 (and of ever other nontrivial symmetric group).
  - **Trivial representation** (irreducible).
  - **Alternating representation** (irreducible).
- Using the fact that S_3 is a permutation group, we can locate the...
  - Permutation representation (reducible);
  - **Standard represenatation** (irreducible).
- Let W be an arbitrary representation of S_3.
  - Easily done with **character theory**, but we'll only get there later.
- Since the representation theory of finite abelian groups was jut proven to be very simple, we'll start by looking at the action of the finite abelian subgroup $A_3 = \mathbb{Z}/3\mathbb{Z} \subset S_3$ on W.
  - Let τ be a generator of A_3. Explicitly, this means $\tau = (1,2,3)$ or $\tau = (1,3,2)$.
  - Then W is spanned by eigenvectors $v_1, \dots, v_n$ of τ (why?? Schur's Lemma part 2? Relation to classification of abelian groups?) with corresponding eigenvalues $\omega = \mathrm{e}^{2\pi i/3}$.
  - Thus, $W = \bigoplus V_i$ where $V_i = \mathbb{C}v_i$ and $\tau v_i = \omega^{\alpha_i} v_i$.
  - An example representation of A_3 (in the chemistry sense) is

$$\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}, \begin{bmatrix} \mathrm{e}^{2\pi i/3} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathrm{e}^{2\pi i/3} \end{bmatrix}, \begin{bmatrix} \mathrm{e}^{4\pi i/3} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathrm{e}^{4\pi i/3} \end{bmatrix}$$

- Now we want to see how the remainder of S_3 acts on W.
  - Let σ be an arbitrary transposition in S_3.
  - Note: $\{\sigma, \tau\}$ generates S_3.
  - Recall the relationship $\sigma\tau\sigma = \tau^2$.
  - The action of σ on the eigenvectors of τ: Let v be an arbitrary eigenvector of τ, with corresponding eigenvalue ω^j. Notice that

$$\tau(\sigma(v)) = \sigma(\tau^2(v)) = \sigma(\omega^{2j} v) = \omega^{2j}\sigma(v)$$

    Takeaway: v an eigenvector for τ with eigenvalue ω^j implies $\sigma(v)$ an eigenvector for τ with eigenvalue ω^{2j}.
- Exercise 1.10 (A basis for the standard representation of S_3): Verify that with $\sigma = (12)$ and $\tau = (123)$, the standard representation has a basis $\alpha = (\omega, 1, \omega^2), \beta(1, \omega, \omega^2)$, with

$$\tau\alpha = \omega\alpha \qquad \tau\beta = \omega^2\beta \qquad \sigma\alpha = \beta \qquad \sigma\beta = \alpha$$

  - $1 + \omega + \omega^2 = 0$ in the complex plane.
  - We do, indeed, get

$$\begin{aligned} \tau\alpha &= (\omega^2, \omega, 1) & \tau\beta &= (\omega^2, 1, \omega) & \sigma\alpha &= (1, \omega, \omega^2) & \sigma\beta &= (\omega, 1, \omega^2) \\ &= \omega(\omega, 1, \omega^2) & &= \omega^2(1, \omega, \omega^2) & &= \beta & &= \alpha \\ &= \omega\alpha & &= \omega^2\beta \end{aligned}$$

  - We also get — per the aforementioned rule — that

$$\tau\alpha = \omega\alpha$$

    but

$$\tau(\sigma\alpha) = \tau\beta = \omega^2\beta = \omega^2(\sigma\alpha)$$

    for instance, as predicted.

    - Note that both α, β are orthogonal to $(1,1,1)$, but while they are linearly independent, they are not orthogonal to each other. This is fine, because they're computationally simple, but it is noteworthy.
  - The difference in eigenvalues between v and $\sigma(v)$ indicates that these vectors are *not* linearly dependent.
    - Rather, they span a 2D subspace V' that is invariant under S_3!! This is because $v = \sigma(\sigma(v))$ as well.
    - In fact, V' is isomorphic to the standard representation!
  - What if the eigenvalue of v is 1?
    - If $\sigma(v)$ is not linearly independent of v, then the two span a 1D subrepresentation of W, isomorphic to the trivial representation (if $\sigma(v) = v$) and isomorphic to the alternating representation (if $\sigma(v) = -v$).
    - If $\sigma(v)$ is linearly independent of v, then $v + \sigma(v)$ spans a 1D subrepresentation of W isomorphic to the trivial representation and $v - \sigma(v)$ spans a 1D subrepresentation of W isomorphic to the alternating representation.
  - It follows that the only three irreps of S_3 are the trivial, alternating, and standard ones.

- Using the above approach to find the decomposition of the tensor product.
  - Let V be the standard two-dimensional representation. Recall that the basis of V is $\{\alpha, \beta\}$.
  - It follows that the basis of $V \otimes V$ is $\{\alpha \otimes \alpha, \alpha \otimes \beta, \beta \otimes \alpha, \beta \otimes \beta\}$.
  - These are eigenvectors for τ, and we can find their corresponding eigenvalues via direct computation:

$$\begin{aligned} \tau(\alpha \otimes \alpha) &= \tau\alpha \otimes \tau\alpha \\ &= (\omega\alpha) \otimes (\omega\alpha) \\ &= \omega^2 \alpha \otimes \alpha \end{aligned} \qquad \begin{aligned} \tau(\alpha \otimes \beta) &= \tau\alpha \otimes \tau\beta \\ &= (\omega\alpha) \otimes (\omega^2\beta) \\ &= 1\alpha \otimes \beta \end{aligned}$$

$$\begin{aligned} \tau(\beta \otimes \alpha) &= \tau\beta \otimes \tau\alpha \\ &= (\omega^2\beta) \otimes (\omega\alpha) \\ &= 1\beta \otimes \alpha \end{aligned} \qquad \begin{aligned} \tau(\beta \otimes \beta) &= \tau\beta \otimes \tau\beta \\ &= (\omega^2\beta) \otimes (\omega^2\beta) \\ &= \omega\beta \otimes \beta \end{aligned}$$

  - Similarly, we can calculate the effect of σ.

$$\begin{aligned} \sigma(\alpha \otimes \alpha) &= \sigma\alpha \otimes \sigma\alpha \\ &= \beta \otimes \beta \end{aligned} \quad \begin{aligned} \sigma(\alpha \otimes \beta) &= \sigma\alpha \otimes \sigma\beta \\ &= \beta \otimes \alpha \end{aligned} \quad \begin{aligned} \sigma(\beta \otimes \alpha) &= \sigma\beta \otimes \sigma\alpha \\ &= \alpha \otimes \beta \end{aligned} \quad \begin{aligned} \sigma(\beta \otimes \beta) &= \sigma\beta \otimes \sigma\beta \\ &= \alpha \otimes \alpha \end{aligned}$$

  - Because the transformations for $\alpha \otimes \alpha$ and $\beta \otimes \beta$ are directly analogous to the untensored case of α and β, these basis vectors span a subrepresentation isomorphic to the standard representation.
  - Because $\sigma(\alpha \otimes \beta) = \beta \otimes \alpha$ is linearly independent of $\alpha \otimes \beta$ (they are literally different basis vectors), $\alpha \otimes \beta + \beta \otimes \alpha$ spans a trivial representation and $\alpha \otimes \beta - \beta \otimes \alpha$ spans an alternating representation.
  - Altogether, we get that if $V = (2,1)$, then

$$V \otimes V \cong (2,1) \oplus (3) \oplus (1,1,1)$$

Week 3

Character Theory

3.1 Characters

10/9:
- Today, we talk about **characters**, arguably the most important idea in rep theory.
- As per usual, we begin by letting G a finite group.
  - We've been discussing finite dimensional representations of G over $\mathbb{C}$.
  - We've also already talked about irreps, and we know that it's enough to understand those because every rep is a sum of them.
- Goal of characters: Understand the irreps $V_1, \dots, V_k$ of G.
  - Recall the surprising fact about k: It is the number of conjugacy classes of G!
    - We haven't yet proven this, but we will soon!
  - Game plan: Use characters to relate irreps to something that is counted by conjugacy classes.
- Let $V = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$ be a G-rep.
  - Then there exists a homomorphism $\rho : g \mapsto A_g \in GL_n(\mathbb{C})$.
- Motivating question: What doesn't change when we change the basis of V?
  - To isolate the "essence" of the A_g, we want to construct a function $f : GL_n(\mathbb{C}) \to \mathbb{C}$ such that $f(XAX^{-1}) = f(A)$.
- Ideas.
  1. The determinant is a great example of such a function, but it's kind of boring because this rank 1 representation doesn't characterize your product representation.
  2. Trace is the main example of such a function.
- Indeed, you can also take $\operatorname{tr}(A^k)$ for any k.
  - Traces of powers are ubiquitous in physics and math because they contain the same information as the coefficients of the characteristic polynomial. In particular, we can express the determinant in terms of them.
- In fact, we could also take any coefficient of the characteristic polynomial, but others would get complicated.
  - Any characteristic polynomial coefficient can be expressed in terms of traces; this will be an exercise in PSet 3; it's not hard.

- So what do we have at this point?
  - We can associate to ρ a function $\chi_\rho : G \to \mathbb{C}$ defined by $\chi_\rho(g) = \operatorname{tr}(A_g) = \operatorname{tr}(\rho(g))$.
  - This function is invariant under isomorphism.
  - If we know $\operatorname{tr}(A)$, we know $\operatorname{tr}(A^2)$ since $A_g^2 = A_{g^2}$. Thus, if we know all traces, we know all power traces.
    - Something about the following??

$$\sum \lambda_i \lambda_j = \frac{\operatorname{tr}(A)^2 - \operatorname{tr}(A^2)}{2}$$

  - We form a ring of polynomials??
    - Equivalently, χ_ρ has a representation as a polynomial with coefficients in $\mathbb{C}$??
- If V is a G-rep, $\chi_V : G \to \mathbb{C}$ will be our notation for its character.
- Properties.
  1. $\chi_V(xgx^{-1}) = \chi_V(g)$ for any $x, g \in G$.
     - Implication: χ_V is a **class function**.
     - Let $\mathbb{C}[G]$ be the vector space of all functions from $G \to \mathbb{C}$. Its dim $= |G|$.
     - Inside this space, there is the subspace $\mathbb{C}_{\text{cl}}[G]$ of functions $f : G \to \mathbb{C}$ such that $f(xgx^{-1}) = f(g)$ for all $x, g \in G$. These are functions from the sets of conjugacy classes, isomorphic to functions that are constant on conjugacy classes. $\dim \mathbb{C}_{\text{cl}}[G]$ is the number of conjugacy classes.
     - Thus, for every V a G-rep, we get a vector $\chi_V \in \mathbb{C}_{\text{cl}}[G]$. These class functions form a basis of the space; each χ_V for V an irrep forms a linearly independent vector; the set is an *orthogonal* basis. This is the reason for the original theorem holding true!
  2. $\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$.
     - Proof: It's basically tautological (not actually, but it's easy). Let $g \in G$. Compute $\chi_{V_1 \oplus V_2}(g)$. We can compute a basis $e_1, \dots, e_{n+m}$ where the first n vectors form a basis of V_1, and the next m vectors are a basis of V_2. This gives us a block matrix from which we show that the trace of the matrix is the sum of traces.

$$\chi_{V_1 \oplus V_2}(g) = \operatorname{tr} \begin{bmatrix} \rho_{V_1}(g) & 0 \\ 0 & \rho_{V_2}(g) \end{bmatrix} = \operatorname{tr} \rho_{V_1}(g) + \operatorname{tr} \rho_{V_2}(g) = \chi_{V_1}(g) + \chi_{V_2}(g)$$

     - Corollary:

$$\chi_{V_1^{n_1} \oplus \cdots \oplus V_k^{n_k}} = n_1 \chi_{V_1} + \cdots + n_k \chi_{V_k}$$

- We now pause for a fact that will be instrumental in proving the next property, which is a bit more involved.
  - He will explain two ways to prove it; we can also just prove it on our own.
- Fact: A a matrix such that $A^n = 1$. Then A is diagonalizable or "semi-simple."
  - We can prove this with Jordan normal form.
  - It's a slightly surprising statement.
  - Obviously eigenvalues are roots of unity, but still needs some work.
  - This proof is left as an exercise.
- We now resume the list of properties.
  3. $\chi_V(g)$ is a sum of roots of unity.

- Proof: We know that $g^{|G|} = e$. Thus, $A_g^{|G|} = 1$. It follows by the fact above that A_g is diagonalizable with eigenvalues $\lambda_1, \dots, \lambda_n$, each of which satisfies $\lambda_i^{|G|} = 1$.
  - Note: Eigenvalues can repeat in the list $\lambda_1, \dots, \lambda_n$, i.e., we are not asserting n distinct eigenvalues here.
- Therefore, since each λ_i is, individually, a root of unity, we have that $\chi_V(g) = \operatorname{tr} A_g = \lambda_1 + \cdots + \lambda_n$, as desired.

4. $\chi_{V^*} = \bar{\chi}_V$.
   - This property begins to address how characters behave under other operations.
     - Naturally, this is something specific for complex numbers, because the idea of "conjugates" doesn't exist everywhere.
   - Proof: Recall that $\rho_{V^*}(g) = (\rho_V(g)^{-1})^T$.
     - If we know that $\rho_V(G) \sim \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, then we know that $\rho_V^{-1}(g)^T \sim \operatorname{diag}(\lambda_1^{-1}, \dots, \lambda_n^{-1})$.
     - Thus, $\chi_{V^*}(g) = \lambda_1^{-1} + \cdots + \lambda_n^{-1}$.
     - But since we're in the complex plane, $|\lambda_i| = 1$ (equiv. $\lambda_i\bar{\lambda}_i = 1$), so $\lambda_i^{-1} = 1/\lambda_i = \bar{\lambda}_i$.
     - This means that $\chi_{V^*}(g) = \bar{\lambda}_1 + \cdots + \bar{\lambda}_n = \overline{\lambda_1 + \cdots + \lambda_n}$.
   - Note: Every representation we have is **unitary** in certain bases, but unitary representations are not covered in this course.
5. $\chi_{V_1 \otimes V_2} = \chi_{V_1} \cdot \chi_{V_2}$.
   - Proof: We can use a basis or not use a basis.
   - Let's use a basis for now.
     - Let $g \in G$ be arbitrary. Then there exist bases $e_1, \dots, e_n$ of V_1 and $f_1, \dots, f_m$ of V_2 such that $\rho_{V_1}(g)$ and $\rho_{V_2}(g)$ are diagonal.
     - It follows that $\rho_{V_1}(g)e_i = \lambda_i e_i$ $(i = 1, \dots, n)$ and $\rho_{V_2}(g)f_i = \mu_i f_i$ $(i = 1, \dots, m)$.
     - $V_1 \otimes V_2$ thus has basis $e_i \otimes f_j$.
     - But then it follows that $\rho_{V_1 \otimes V_2}(g)e_i \otimes f_j = (\lambda_i e_i) \otimes (\mu_j f_j) = \lambda_i \mu_j (e_i \otimes f_j)$.
     - Thus,

$$\operatorname{tr}(\rho_{V_1 \otimes V_2}(g)) = \sum_{i,j=1}^{n,m} \lambda_i \mu_j = (\lambda_1 + \cdots + \lambda_n)(\mu_1 + \cdots + \mu_m) = \operatorname{tr}(\rho_{V_1}(g)) \cdot \operatorname{tr}(\rho_{V_2}(g))$$

   - Alternate approach.
     - If we don't want to think of eigenvalues, think of tensor product of matrices, the Kronecker product.
     - We get trace is the product of traces once again! *Write this out.*

- **Class function**: A function on a group G that is constant on the conjugacy classes of G.
- Examples.

1. Let A be an abelian group.
   - Then $\chi : A \to \mathbb{C}^\times$.
   - Implication: Character of a character is $\chi_\chi = \chi$.
     - This is horribly repetitive but true.
2. $G = S_3$.
   - The conjugacy classes of this group are $\{e\}$, $\{(12), (13), (23)\}$, and $\{(123), (132)\}$.
   - We construct a **character table** to define all characters.
   - Computing the characters for the trivial representation.
     - We know that ρ sends each g to the matrix (1), which has trace 1.
   - Computing the characters for the sign representation.

	e	(12) (13) (23)	(123) (132)
Trivial	1	1	1
Alternating	1	-1	1
Standard	2	0	-1

Table 3.1: Character table for S_3.

- e and (123) have sign 1 and thus get sent to the matrix (1).
- (12) has sign -1 and thus gets sent to the matrix (-1).

– Computing the characters for the standard representation.
- We can compute these traces via a thought experiment.
- Visualize a triangle in a plane.
- The 2×2 identity matrix (the standard representation of $e \in G$) acts on it by doing nothing, and has trace 2.
- In *some* basis, our matrix fixes one vector and inverts another, so matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and character is 0.
- Last one is rotation by $2\pi/3$, so

$$\begin{pmatrix} \cos(2\pi/3) & \sin(2\pi/3) \\ -\sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}$$

so character is $-1 = 2 \cdot -1/2 = 2 \cdot \cos(2\pi/3)$.

– If V is the standard representation, we can also compute the characters of $V^{\otimes 2}$ for instance. Indeed, by the product rule of characters, they will be the squares of the standard representation's characters, i.e., $(4, 0, 1)$.

– Similarly, since the permutational representation is the direct sum of the standard and trivial representations, we can add their characters to get its characters $(3, 1, 0)$.

3. A very general and very pretty example. Let $G \circlearrowright X$ a finite set.
   – Assign the permutational representation.
   – Let $X = \{x_1, \dots, x_n\}$. Think of these elements as the basis of a vector space; in particular, consider $V = \mathbb{C}e_{x_1} \oplus \cdots \oplus \mathbb{C}e_{x_n}$. Recall that $g(a_1 e_{x_1} + \cdots + a_n e_{x_n}) = a_1 e_{gx_1} + \cdots + a_n e_{gx_n}$. The fact that this is a representation follows immediately from the properties of the group action.
   – Computing the character χ_V of this V: Look at g and write its matrix. In particular, the trace is the number of unmoved/fixed elements, sometimes denoted $\mathrm{Fix}(g)$.
   – This gives us another way of computing V_{perm} from above!

- **Character table**: A table that lists the conjugacy classes across the top, the irreps down the left side, and at each point within it, the value of an irrep's character over that conjugacy class.
  – The character table is a very nice matrix with very nice properties.
  – It is almost orthogonal; not exactly, but very close.
    - Rows aren't orthogonal, but columns are (take direct products)!
    - It is full rank, though.
- The midterm: Take the character table and do fun things with it.

3.2 Office Hours

10/10:
- Problem 1b:
  - Canonically self-dual: $V \cong V^*$ canonically.
- Mathematical methods of quantum mechanics: First few paragraphs of *picture*.
- We should have everything we need to do most of the problem set at this point; maybe not all of 5, but maybe yes, too.
- Problem 3:
  - There is some problem where it decomposes into trivial plus standard, but we still have to prove that standard is irreducible in this case!
  - If you have any vector, you can produce out of this vector something else.
  - If we take any vector and the group acts on it, we'll get a basis. If you hit a vector in the invariant subspace, it will just stay there; if you hit it and it goes everywhere, you get a basis.
  - Now think about a vector when you permute its coordinates.
  - Tomorrow in class, we will learn a quick way to do this problem.
- Problem 5:
  - For some problem, we need to use the fact that $A^n = 1$ proves that $A = I$ in some sense.
  - This is a hard problem!
  - Show that eigenvalues sum to 1; we know that the eigenvalues are roots of unity! Thus, they have to both be 1!
  - When the problem in group theory is harder, that's when you need to go to rep theory.

3.3 Characters are Orthonormal

10/11:
- Announcement: Zoom OH today.
- Recap: The big picture.
  - Representations.
    - We have representations, which are vector spaces on which a group acts.
    - With these representations, we can do a bunch of operations we've discussed: $\oplus, \otimes, V^*, \Lambda^n, S^n$.
    - We'll focus on the first 3 for now, though.
  - Class functions.
    - We also have class functions: Functions $f : G \to \mathbb{C}$ such that for all $g, x \in G$, $f(gxg^{-1}) = f(x)$.
    - The space of class functions forms a ring, since you can add, multiply, and take the complex conjugate of these functions.
    - Moreover, this ring is a vector space and it has dimension equal to the number of conjugacy classes of G.
  - The big idea: These two things (representations and class functions) are closely related!
    - There is a map, called a *character*, that pairs a representation to a class function.
    - Indeed, $V \to \chi_V$.
    - Under this map, operations of representations become operations of functions:

$$\oplus \mapsto + \qquad \otimes \mapsto \cdot \qquad V^* \mapsto \bar{f}$$

    - Additionally, $V_1, \dots, V_s$ become $\chi_{V_1}, \dots, \chi_{V_s}$.

- Theorem we will prove over the next couple of lectures: Irreps become *linearly independent* class functions, and all irreps form a basis of the space of class functions.
  - This theorem is huge! It is our main takeaway for now.
  - For the first part of the course, this is the main thing that we should remember.
- How do we prove that multiple vectors are linearly independent?
  - A strong condition would be to introduce an inner product and prove that the pairwise inner product of the vectors is zero.
- **Orthonormal basis**: A basis for which $\langle e_i, e_j \rangle = \delta_{ij}$.
- Let's begin carrying out this plan by defining an inner product on $\mathbb{C}[G]$. Indeed, let f_1, f_2 be two functions on G and take

$$\langle f_1, f_2 \rangle := \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$$

- Motivation for this definition.
  - Recall the **Hermitian inner product** on $\mathbb{C}^n$.
    - We are essentially mapping f_1, f_2 to $(f_1(g_1), \dots, f_1(g_{|G|})), (f_2(g_1), \dots, f_2(g_{|G|})) \in \mathbb{C}^{|G|}$ and taking the Hermitian inner product there.
    - Thus, we can see that all properties hold for both the Hermitian inner product on $\mathbb{C}^n$ and the one defined above on $\mathbb{C}[G]$.
    - In other words, this kind of construction should inherit its status as a linear, positive definite bilinear form from the Hermitian inner product.
  - Note: The Hermitian product above is ***G*-invariant**.
    - This means that the functions on G from $G \to \mathbb{C}$ in $\mathbb{C}[G]$ form a representation of G.
    - In particular, if $\varphi : G \to \mathbb{C}$, then $g = \rho(g)$ moves it as follows: $g \cdot \varphi = \varphi^g$ where $\varphi^g(h) := \varphi(g^{-1}h)$. Thus, we have an action of G on every φ!
    - Such representations are isomorphic for finite groups??
  - If we have $\langle f_1, f_2 \rangle$, we can ask if

$$\langle f_1, f_2 \rangle \overset{?}{=} \langle f_1^g, f_2^g \rangle$$

    - Left as an exercise that this *is* true!
- **Hermitian inner product** (on $\mathbb{C}^n$): The inner product defined as follows for all $z, w \in \mathbb{C}^n$. *Denoted by* $\langle\ ,\ \rangle$. *Given by*

$$\langle z, w \rangle = \sum_{i=1}^{n} z_i \bar{w}_i$$

  - This inner product gives a complex number $\langle v, w \rangle \in \mathbb{C}$ with the following properties.
    1. $\langle a_1 v_1 + a_2 v_2, w \rangle = a_1 \langle v_1, w \rangle + a_2 \langle v_2, w \rangle$.
    2. $\langle v, b_1 w_1 + b_2 w_2 \rangle = \bar{b}_1 \langle v, w_1 \rangle + \bar{b}_2 \langle v, w_2 \rangle$.
    3. $\langle v, v \rangle \geq 0$, and $\langle v, v \rangle = 0$ implies that $v = 0$.
  - Thus, if $v = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$, then

$$\langle v, w \rangle = \sum z_i \bar{w}_i \qquad\qquad \langle v, v \rangle = \sum |z_i|^2$$

- We now begin tackling today's main theorem: If V_1, V_2 are irreps, then

$$\langle \chi_{V_1}, \chi_{V_2} \rangle = \begin{cases} 0 & V_1 \ncong V_2 \\ 1 & V_1 \cong V_2 \end{cases}$$

- We will prove this theorem in stages.
- The general outline of our approach is to deduce the equality step by step through the transitive property. Some of the equalities we'll eventually end up needing are easier to discuss on their own first, though, so we begin with some lemmas.

- First off, recall the **space of invariants** from PSet 2.
- **Space of invariants** (of a representation V): The vector space defined as follows. *Denoted by* $\boldsymbol{V^G}$. *Given by*

$$V^G = \{v \in V \mid gv = v \; \forall \; g \in G\}$$

- Lemma 1: Let G be a finite group, let $\rho : G \to GL(V)$ be a finite-dimensional representation of it, and let p be defined as above. Then $p \in \mathrm{Hom}_G(V, V)$.

Proof. We can view p as an element of $\mathrm{Hom}(V, V)$. This combined with the fact that for every $h \in G$,

$$p(hv) = \frac{1}{|G|}\sum_{g\in G}(gh)v = \frac{1}{|G|}\sum_{gh\in G}(gh)v = \frac{1}{|G|}h\sum_{g\in G}gv = h(pv)$$

implies that $p \in \mathrm{Hom}_G(V, V)$. In more formal notation,

$$\begin{aligned}
[p \circ \rho_V(h)](v) &= \frac{1}{|G|}\sum_{g\in G}[\rho_V(g) \circ \rho_V(h)](v) \\
&= \frac{1}{|G|}\sum_{g\in G}[\rho_V(gh)](v) \\
&= \frac{1}{|G|}\sum_{gh\in G}[\rho_V(gh)](v) \\
&= \frac{1}{|G|}\sum_{hg\in G}[\rho_V(hg)](v) \\
&= \frac{1}{|G|}\sum_{g\in G}[\rho_V(hg)](v) \\
&= [\rho_V(h)]\left(\frac{1}{|G|}\sum_{g\in G}[\rho_V(g)](v)\right) \\
&= [\rho_V(h) \circ p](v)
\end{aligned}$$

$\square$

- Why do we need this result?? What does it do for the rest of the proof?
- Lemma 2: Let G be a finite group, and let $\rho : G \to GL(V)$ be a finite-dimensional representation of it. Then the map p, defined as follows, is a projector from $V \to V^G$.

$$p = \frac{1}{|G|}\sum_{g\in G} g = \frac{1}{|G|}\sum_{g\in G}\rho_V(g)$$

Proof. To prove that p is a projector, it will suffice to show that $p^2 = p$. To prove that p projects onto V^G, it will suffice to show that $\mathrm{Im}(p) = V^G$. Let's begin.

To show that $p^2 = p$, we have

$$p^2 = \left(\frac{1}{|G|}\sum_{g\in G} g\right)^2 = \frac{1}{|G|^2}\sum_{g_1,g_2\in G} g_1g_2 = \frac{|G|}{|G|^2}\sum_{g\in G} g = p$$

Note that since G is not abelian (i.e., $g_1g_2 \neq g_2g_1$ in all cases), the square of $\sum g$ is as above and cannot be reduced to a smaller sum with a 2 coefficient or something like that. Additionally, note that $\sum_{g_1,g_2\in G} g_1g_2 = |G|\sum g$ since for each g_i, $g_i(g_1+\cdots+g_{|G|}) = g_1+\cdots+g_{|G|}$.

To show that $\mathrm{Im}(p) = V^G$, we will use a bidirectional inclusion proof. To confirm that $\mathrm{Im}(p) \subset V^G$, we have for any $h \in G$ that

$$h\left(\frac{1}{|G|}\sum_{g\in G} gv\right) = \frac{1}{|G|}\sum_{hg\in G} hgv = \frac{1}{|G|}\sum_{g\in G} gv$$

from which it follows that

$$p(v) = \frac{1}{|G|}\sum gv \in V^G$$

as desired. To confirm that $V^G \subset \mathrm{Im}(p)$, let $v \in V^G$. Then $gv = v$. It follows that

$$v = \frac{1}{|G|}\sum_{g\in G} v = \frac{1}{|G|}\sum_{g\in G} gv = p(v) \in \mathrm{Im}(p)$$

as desired. ☐

- You differentiated the first and second parts of the above proof by saying, "this is the algebraic way to prove it; we can also prove it nonalgebraically." Does this mean that $p^2 = p$ somehow *implies* $\mathrm{Im}(p) = V^G$ here, or do we still need to prove that "nonalgebraically," as in Fulton and Harris (2004)??
- Consequence of Lemma 2: There's a very easy way to construct invariant factors.
- We now prove one final lemma using what we have learned about p.
- Lemma 3: Let G be a finite group, and let $\rho : G \to GL(V)$ be a finite-dimensional representation of it. Then $\dim V^G = (1/|G|)\sum_{g\in G}\chi_V(g)$.

  Proof. Define p as above. Then

$$\begin{aligned}
\dim V^G &= \dim(\mathrm{Im}(p)) && \text{Lemma 2}\\
&= \mathrm{tr}(p) && \text{PSet 1, Q5c}\\
&= \mathrm{tr}\left(\frac{1}{|G|}\sum \rho_V(g)\right)\\
&= \frac{1}{|G|}\sum_{g\in G}\mathrm{tr}(\rho_V(g))\\
&= \frac{1}{|G|}\sum_{g\in G}\chi_V(g)
\end{aligned}$$

  as desired. ☐

- We can now prove the main result.
- Theorem: If V, W are irreps, then

$$\langle \chi_V, \chi_W\rangle = \begin{cases} 0 & V \ncong W\\ 1 & V \cong W\end{cases}$$

Proof. We will work towards a formula for the inner product, using various results that we've proven up until now. Let's begin.

$$\begin{aligned}
\langle \chi_V, \chi_W \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \cdot \overline{\chi_W(g)} && \text{Definition} \\
&= \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \cdot \chi_{W^*}(g) && \text{Property 4} \\
&= \frac{1}{|G|} \sum_{g \in G} \chi_{V \otimes W^*}(g) && \text{Property 5} \\
&= \dim[(V \otimes W^*)^G] && \text{Lemma 3} \\
&= \dim([\mathrm{Hom}_F(V, W)]^G) && \text{Lecture 2.1} \\
&= \dim[\mathrm{Hom}_G(V, W)] && \text{PSet 2, Q4b} \\
&= \begin{cases} \dim(\mathrm{span}(I)) & V \cong W \\ \dim(\mathrm{span}(0)) & V \ncong W \end{cases} && \text{Schur's Lemma} \\
&= \begin{cases} 0 & V \ncong W \\ 1 & V \cong W \end{cases}
\end{aligned}$$

□

- In the above proof, Rudenko first surveys the following special case. Why??
  - Then if V is irreducible and trivial, we have

$$\frac{1}{|G|} \sum_{g \in G} \chi_V(g) = 0$$

which happens iff

$$\langle \chi_V, \chi_{\text{triv}} \rangle = 0$$

whereas

$$\langle \chi_{\text{triv}}, \chi_{\text{triv}} \rangle = 1$$

This proves the theorem in a special case, but how do we go from here to all representations? We're very close!

- Corollary: The number of irreps is less than or equal to the number of conjugacy classes.
  - We'll leave it to next time to prove that equality holds.
- Whenever we have a sec, we should try to form a mental picture the whole class function thing.
- Consequence of the theorem: We get an orthogonality relation.
  - If χ_1, χ_2 are characters if irreps, then

$$\sum_{g \in G} \chi_1(g) \overline{\chi_2(g)} = \begin{cases} 0 & \chi_1 \neq \chi_2 \\ |G| & \chi_1 = \chi_2 \end{cases}$$

  - This is related to the character table and IChem!!! Example:
    - Recall Table 3.1, the character table for S_3.
    - Between the trivial and alternating representations, we have

$$(1)(1) + (1)(-1) + (1)(-1) + (1)(-1) + (1)(1) + (1)(1) = 0$$

as expected. Note that we have a term for each element in S_3, so some products get repeated multiple times.

- For the standard representation, we have

$$(2)(2) + (0)(0) + (0)(0) + (0)(0) + (-1)(-1) + (-1)(-1) = 6 = |S_3|$$

as expected.

- Theorem: Characters are equal iff their representations are isomorphic.
- Next time.
  - Prove the theorem.
  - Consequences.
  - Implications for the character table.

3.4 Character Table Properties

10/13: • Announcement: Midterm on November 10.
  - Will mostly involve computing character tables; HW will be good prep.
- Review of the general picture from the first part of the course.

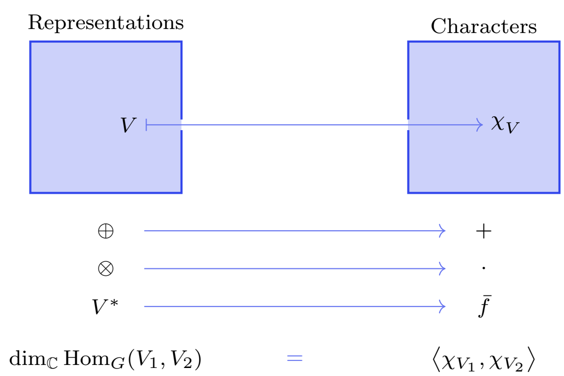

Figure 3.1: The stories of representation theory.

  - Let G be a finite group.
  - First story: We study finite-dimensional representations of G over $\mathbb{C}$; these are vector spaces, so we can direct sum, tensor multiply, and dualize them. We can also look at the morphisms between them.
  - Second story: We study class functions $\mathbb{C}_{\text{cl}}[G] = \{f : G \to \mathbb{C} \mid f(gxg^{-1}) = f(x)\}$; these are elements of a ring, so we can add, multiply, and conjugate them. We can also take the inner product of them.
  - We can map between these two stories: Representations become characters, $\oplus \mapsto +$, $\otimes \mapsto \cdot$, and $V^* \mapsto \bar{f}$.
- Theorem (from last time):

$$\langle \chi_{V_1}, \chi_{V_2} \rangle = \dim \operatorname{Hom}_G(V_1, V_2)$$

- The story in Figure 3.1 tells us stuff about representations.
  - Let $V_1, \dots, V_k$ be irreps. Then the vectors $\chi_{V_1}, \dots, \chi_{V_k}$ are orthonormal.
    - We get this result with the Theorem above and Schur's Lemma.

- Next time, we'll prove that $\chi_{V_1},\dots,\chi_{V_k}$ spans $\mathbb{C}_{\text{cl}}[G]$, i.e., the number of irreps is the number of conjugacy classes.
- *Cube thing??*
- This picture is remarkable because it's so simple.

- We now look at some corollaries to last time's main theorem.
- Corollary 1: If V,W are G-reps, then $\chi_V=\chi_W$ iff $V\cong W$.

Proof. Invoking complete reducibility, we have that $V=\bigoplus V_i^{n_i}$. Thus, to know V, it is enough to know the n_i's. But

$$\chi_V=\sum n_i\chi_{V_i}$$

where

$$n_i=n_i\cdot 1=n_i\left\langle\chi_{V_i},\chi_{V_i}\right\rangle=\left\langle\chi_V,\chi_{V_i}\right\rangle$$

Therefore, since the χ_{V_i} are linearly independent, the only way that $\chi_V=\chi_W$ is if the n_i's match which would mean that $V\cong W$, and vice versa the only way that $V\cong W$ is if the n_i's match which would mean that $\chi_V=\chi_W$. ☐

- Corollary 2: Let V be a G-rep. Then TFAE:
  1. V is irreducible.
  2. $\langle\chi_V,\chi_V\rangle=1$.
  3. $\sum_{g\in G}|\chi_V(g)|^2=|G|$.

Proof. ($1\Rightarrow 2$): We have that

$$\langle\chi_V,\chi_V\rangle=\dim\operatorname{Hom}_G(V,V)=1$$

as desired.

($2\Rightarrow 1$): Complete reducibility implies that $V\cong V_1^{n_1}\oplus\cdots\oplus V_k^{n_k}$, where the V_i's are irreps. This combined with the hypothesis implies that

$$1=\langle\chi_V,\chi_V\rangle=\left\langle\sum_{i=1}^k n_i\chi_{V_i},\sum_{i=1}^k n_i\chi_{V_i}\right\rangle=\sum_{i=1}^k n_i^2$$

But if $\sum n_i^2=1$ where each $n_i\in\mathbb{Z}^+$, then $n_i=1$ for some i and $n_j=0$ for $j\neq i$, from which it follows that $V\cong V_i$.

We can interconvert between 2 and 3 using the definition of the inner product and the property of complex numbers that $zz^*=|z|^2$. ☐

- We now build up to one final corollary.
- We've discussed all of these properties of irreps, but where do we even find them?
  - We might be able to find some by inspection, but here's how we find all of them.
- Review: The regular representation. Here are two different but isomorphic ways to think about it.
  - Think of it as functions on G.
    - Better for infinite groups.
  - Think of it as the permutational representation associated with the action $G\circlearrowright G$.
    - Better for finite groups.
  - Why did we talk about this here??

- Corollary 3: Consider the regular representation V_R. We have that

$$\chi_{V_R}(g) = \begin{cases} 0 & g \neq e \\ |G| & g = e \end{cases}$$

Proof. We can compute its character χ_{V_R} by considering the corresponding permutation matrices. Indeed, the action $\chi_{V_R}(g)$ of this character on g is equal to the number of 1's on the diagonal in the permuttion matrix, which is equal to the number of fixed points of the permutation, i.e., the number of i's such that $gg_i = g_i$. But in a group, $gg_i = g_i$ iff $g = e$, so this number of fixed points is

$$\chi_{V_R}(g) = \mathrm{Fix}(g) = \begin{pmatrix} g_1 & \cdots & g_n \\ gg_1 & \cdots & gg_n \end{pmatrix} = \begin{cases} 0 & g \neq e \\ |G| & g = e \end{cases}$$

as desired. □

- What is the matrix thing??
- We now apply Corollaries 1-3 to the regular representation V_R to obtain some important results.
  - Let V_i be an arbitrary irrep.
  - By complete reducibility, $V_R = \bigoplus_{i=1}^k V_i^{n_i}$ for some set of n_i's.
  - Additionally,

$$\begin{aligned} n_i &= \langle \chi_{V_R}, \chi_{V_i} \rangle && \text{Corollary 1} \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_{V_R}(g) \overline{\chi_{V_i}(g)} \\ &= \frac{1}{|G|} |G| \underbrace{\overline{\chi_{V_i}(e)}}_{\dim V_i} && \text{Corollary 3} \\ &= \dim V_i \end{aligned}$$

  - This implies three remarkable results, all worth remembering.

$$V_R = \bigoplus_{i=1}^k V_i^{\dim V_i} \qquad |G| = \sum_{i=1}^k (\dim V_i)^2 \qquad \#\text{ irreps is finite}$$

    - The first result follows directly by substituting $n_i = \dim V_i$ into complete reducibility.
    - The second result follows because $|G| = \dim(V_R) = \dim(\bigoplus_{i=1}^k V_i^{\dim V_i}) = \sum (\dim V_i)^2$.
    - The third result follows because if there were infinitely many irreps, each with $\dim V_i \geq 1$, then $|G| = \sum_{i=1}^k (\dim V_i)^2 = \infty$, contradicting the hypothesis that $|G|$ is finite.

- We want to investigate S_4, i.e., characterize all irreps of it.

	1	6	8	3	6
	e	(12)	(123)	(12)(34)	(1234)
Trivial	1	1	1	1	1
Sign	1	-1	1	1	-1
V_{std}	3	1	0	-1	-1
Sign $\otimes V_{\text{std}}$	3	-1	0	-1	1
	2	0	-1	2	0

Table 3.2: Character table for S_4.

- We do so by constructing the character table, Table 3.2.
- Initially, this seemed like a very hard problem.
  - However, with all of our theory, it only takes a couple of minutes now!
- We start by inputting the trivial, sign, and standard representations.
  - The trivial is obviously $(1,1,1,1,1)$.
  - The sign can be calculated to be $(1,-1,1,1,-1)$.
  - The standard is found via $V_{\text{std}} = V_{\text{perm}} - V_{\text{triv}} = (4,2,1,0,0) - (1,1,1,1,1) = (3,1,0,-1,-1)$.
- Note that

$$\sum n_i^2 = \langle \chi_{V_{\text{perm}}}, \chi_{V_{\text{perm}}} \rangle = \frac{1}{24}(1 \cdot 4^2 + 6 \cdot 2^2 + 8 \cdot 1^2 + 0 \cdot 0^2 + 0 \cdot 0^2) = 2$$
$$\langle \chi_{V_{\text{perm}}}, \chi_{V_{\text{sign}}} \rangle = \frac{1}{24}(4 - 12 + 8) = 0$$
$$\langle \chi_{V_{\text{perm}}}, \chi_{V_{\text{triv}}} \rangle = 1$$

  - What is the point of these calculations??
- Thus, we can derive representations without having any geometric notion of it using characters!
- To see any of these representations geometrically, look at actions on the tetrahedron in $\mathbb{R}^3$!
- All those computations above used the **first orthogonality relation**; here's the **second orthogonality relation**:

$$\sum_\chi \chi(g_1)\overline{\chi(g_2)} = \begin{cases} 0 & g_1 \nsim g_2 \\ \frac{|G|}{|C_G(g)|} & g_1 \sim g_2 \end{cases}$$

  where $\sim$ denotes conjugacy.
  - Prove the new one with $AB = 1 = BA$. This very simple thing leads to a very powerful statement about systems of equations that we will discuss later. How does this proof work??
  - We can start doing this stuff in the new homework!
  - We constructed the fifth representation using this.
- Where does the fifth irrep come from?
  - Going back to a miracle of group theory: Simple groups.
  - If $f : S_n \to S_n$, we have lots of injective maps, lots of minor actions $S_n \to S_n$ sending $x \mapsto gxg^{-1}$.
  - We have sign : $S_n \to S_2$, $S_4 \twoheadrightarrow S_3$ with kernel equal to K_4.
  - We have the exotic $S_6 \to S_6$.
  - We have $S_5 \hookrightarrow S_6$ that is also exotic.
  - These are called **exceptional homomorphisms**.
  - Since we have $S_4 \to S_3$ and $\rho : S_3 \to GL_n$, we have $S_4 \to GL_n$ with a the same character table. Takeaway: This $(2,0,1)$ thing in the big character table comes from this map, geometrically.
  - Takeaway: The geometry of the fifth irrep comes from S_3.
  - What is going on here??

- Final announcements.
  - Going forward, we'll mostly be following Fulton and Harris (2004).
  - Then we'll get into associative algebra.
  - OH next week will be Zoom, but we should still feel free to meet with him in-person by emailing him for an appointment.

Week 4

???

4.1 Representation Ring; Character Basis

10/16: • Announcements.

  – Reminder: Midterm 11/10.
  – OH this week in-person at normal times.
  – PSet 3 should be fun.

• Today: Finish proving some character things.

• Recall: The main picture.

  – Rudenko redraws Figure 3.1.
  – We have a finite group G and we are studying finite-dimensional G-reps over $\mathbb{C}$.
  – $\mathbb{C}_{\text{cl}}[G]$ is a ring.
  – The map...
    - Respects addition;
    - Sends tensor multiplication to (pointwise) functional multiplication;
    - Sends duality to conjugation;
    - Respects a kind of inner product, whether it be either side of $\dim_{\mathbb{C}} \operatorname{Hom}_G(V, W) = \langle f_1, f_2 \rangle$.

• Today, we will see that $\mathbb{C}_{\text{cl}}[G] \cong \mathbb{C}^k$, where k is the number of conjugacy classes.

  – In other words, we will see that the number of irreps is also exactly equal to k, that there is a bijection $\{V_i\} \to \{\chi_i\}$, and that the $\chi_1, \dots, \chi_k$ form an orthonormal basis of $\mathbb{C}_{\text{cl}}[G]$.

• Visualizing the vector space $\mathbb{C}_{\text{cl}}[G]$.

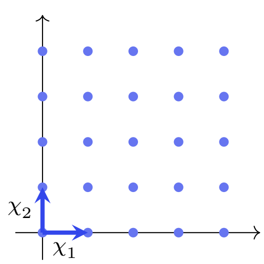

Figure 4.1: Visualizing the space of class functions on G.

- It's a "cone" emanating from the origin with only lattice points.
  - If $\dim \mathbb{C}_{\text{cl}}[G] = 2$, the vector space consists of all the blue points in Figure 4.1.
- Why is it only lattice points instead of a continuous function space?
  - The restrictions on coefficients are inherited from the restrictions on what kinds of spaces you can build of the form $V_1^{n_1} \oplus V_2^{n_2}$.
  - Indeed, if it were continuous, that would imply that there is some meaning to the point $0.3\chi_1 + 2.5\chi_2$, i.e., there is a space $V_1^{0.3} \oplus V_2^{2.5}$. But of course, we cannot define such a space!
- Why is it only *nonnegative* integer coefficients and not *all* integer coefficients?
  - We don't have subtraction to get us to a full ring.
  - Additionally, we can only scale and linearly combine the χ_i's with nonnegative integer coefficients because, as said above, those are the types of reducible rep decompositions we have.

- Let $[V]$ denote the **isomorphism class** of the representation V.
- **Isomorphism class** (of V): The set of all vector spaces W that are isomorphic to V as representations.
- This allows us to define the **representation ring**.
- **Representation ring** (of G): The ring $(R, +, \cdot)$, where R is the free abelian group generated by all isomorphism classes of the representations of G, quotiented by the span of all linear combinations of the form $[V \oplus W] - [V] - [W]$; $+$ is well-defined via the construction of R, which yields $[V]+[W] = [V \oplus W]$ for all $[V], [W]$ in the ring; and $\cdot$ is defined by $[V] \cdot [W] = [V \otimes W]$. *Denoted by* $\boldsymbol{R(G)}$.
  - Basis: $[V_1], \dots, [V_k]$.
  - Thus, structurally,
$$R(G) \cong \mathbb{Z}^k$$
  - Elements are of the form $[V_1] + 2[V_2] - 3[V_3]$.
  - Multiplication is slightly complicated because $V_i \otimes V_j = \bigoplus V_k^{n_{ijk}}$; it follows that
$$[V_i] \cdot [V_j] = \sum n_{ijk}[V_k]$$
- Alternative construction of $R(G)$: Take the subring of the class ring $\mathbb{C}_{\text{cl}}[G]$ that is generated by the characters.
  - To do so, define a map $R(G) \to \mathbb{C}^k$ where the image is linear combinations of characters χ_i with $\mathbb{Z}$-class.
  - Clarify this construction??
- **Virtual representation**: An element of $R(G)$.
  - We need this term because some elements of $R(G)$ — like $-[V]$, for instance — may not correspond to an actual representation.
  - Indeed, note that $-[V]$ is *not* V^*; it is just some thing that when you add it to $[V]$, you get the zero representation.
- Example: Let $G = \mathbb{Z}/2\mathbb{Z} = \{e, x\}$.
  - Then $R(G) = \mathbb{Z}^2 = \mathbb{Z}e \oplus \mathbb{Z}x$ has basis $[1], [-1]$ (corresponding to the trivial and alternating representations) where we define
$$[1]^2 = [1] \qquad [1][-1] = [-1] \qquad [-1]^2 = [1]$$
- One reason people like this $R(G)$ is as follows.

- Initially, understanding this group is not easy because even to get started, you have to find all your characters.
- But, we know that
$$R(G) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}_{\text{cl}}[G]$$
  - So we have a ring that's hard to understand, but if we do something called an **extension of scalars** (shown above) we get an easy ring!
  - Why?? Clarify this construction.
- This is interesting because we can look at the intermediate objects. For example, could we describe $R(G) \otimes \mathbb{R}$ or $R(G) \otimes \mathbb{Q}$. Interestingly, **Artin's theorem** describes $R(G) \otimes \mathbb{Q}$ completely.
- If we try to understand $R(S_n)$, this is still hard work, but if we take $\bigoplus_{n\geq 0} R(S_n)$, we obtain an object that is remarkably, surprisingly simple. That's where we're going. This is why rep theory of finite groups is simultaneously very hard and very simple.

- Lemma: Let G be a finite group, let f be a complex-valued[1] class function, and let V be a G-rep. Then the linear map
$$F = \sum_{g\in G} f(g) \cdot g : V \to V$$
is a morphism of G-representations, that is, $F \in \operatorname{Hom}_G(V, V)$.

Proof. To prove that $F \in \operatorname{Hom}_G(V, V)$, it will suffice to show that $xF = Fx$ for every $x \in G$. Let $x \in G$ be arbitrary. Then
$$F(xv) = \sum_{g\in G} f(g)gxv$$
Since ρ is a group homomorphism, the functions $\rho(g) \in GL(V)$ act just like the elements $g \in G$. *This* is what justifies us to basically move everything around all willy-nilly. Thus, continuing from the above, we have
$$\begin{aligned} &= \sum_{g\in G} f(g)(xx^{-1})gxv \\ &= \sum_{g\in G} f(g)x(x^{-1}gx)v \end{aligned}$$
Since $x = \rho(x)$ is in the general *linear* group, i.e., is a *linear* map, we can factor it out of the sum of functions to get
$$= x \left(\sum_{g\in G} f(g)x^{-1}gx \right) v$$
Since f is a class function by hypothesis, we have $f(g) = f(x^{-1}gx)$, so
$$\begin{aligned} &= x \left(\sum_{g\in G} f(x^{-1}gx)x^{-1}gxv \right) \\ &= x \sum_{g\in G} f(g)gv \\ &= x(Fv) \end{aligned}$$
as desired. ∎

[1] This "complex-valued" hypothesis was not stated in class, but I have to imagine it's true. Is it??

- Recall that previously, we had $(1/|G|)\sum_{g\in G} g : V \to V^G$.
  - He will put something about this being a class function on the midterm?? Review how to prove that this is a class function!
- Another comment: A slightly refined question.
  - Suppose you have a class function f and an irrep V.
  - Then we know that $F = \sum f(g)g : V \to V$ is a G-morphism, so it is a **homothety** by Schur's lemma.
  - So let's find λ.
  - Thinking a big more carefully, we know that F above is

$$\sum_{g\in G} f(g)\rho_V(g) = \lambda I_{d_V}$$

    where d_V denotes the **degree** of V.
  - Now, we will compute λ using the trace. Take the trace of both sides. Then

$$\begin{aligned}
\operatorname{tr}\left(\sum_{g\in G} f(g)\rho_V(g)\right) &= \operatorname{tr}(\lambda I_{d_V}) \\
\sum f(g)\operatorname{tr}(\rho_V(g)) &= \lambda d_V \\
\sum f(g)\chi_V(g) &= \lambda d_V \\
\lambda &= \frac{|G|}{d_V}\frac{1}{|G|}\sum_{g\in G} f(g)\overline{\chi_{V^*}(g)} \\
&= \frac{|G|}{d_V}\langle f, \chi_{V^*}\rangle
\end{aligned}$$

- **Homothety**: A map $F : V \to V$ for which there exists $\lambda \in \mathbb{C}$ such that $Fv = \lambda v$ for all $v \in V$.
  - It just means that we're scaling.
- **Degree** (of V): The dimension of V as a vector space. *Denoted by* $\boldsymbol{d_V}$. *Given by*

$$d_V = \dim V$$

- Now, we can prove the theorem to which we've been building up the whole time.
- Theorem: Let G be a finite group. Then the number of irreps up to isomorphism is equal to the number of conjugacy classes.

  Proof. Let k be the number of conjugacy classes of G, and let $\chi_1, \dots, \chi_s$ be the characters of the irreps. By the theorem from last Wednesday's class, it follows that $\chi_1, \dots, \chi_s$ are orthonormal vectors in $\mathbb{C}_{\text{cl}}[G]$. Thus, by the corollary to the aforementioned theorem, $s \leq k$.

  Now, suppose for the sake of contradiction that $s < k$. Then there exists a nonzero $f \in \mathbb{C}_{\text{cl}}[G]$ such that $\langle f, \chi_{V_i}\rangle = 0$ ($i = 1, \dots, s$). By Gram-Schmidt, we can choose f to be another *orthonormal* vector in the list, extending it to $\chi_1, \dots, \chi_s, f$. We will now build up to proving that $f(g) = 0$ for all $g \in G$ (i.e., $f = 0$), which we will do by using the above lemma to construct a linear independence argument as follows. The first step is to let V_i be an arbitrary irrep of G. Then by the above comment, $F : V_i \to V_i$ may be evaluated on any $v \in V_i$ as follows.

$$F(v) = \lambda I v = \frac{|G|}{d_{V_i}}\langle f, \chi_{V_i^*}\rangle \cdot v = \frac{|G|}{d_{V_i}}\overline{\langle f, \chi_{V_i}\rangle} \cdot v = \frac{|G|}{d_{V_i}}\overline{0} \cdot v = 0$$

It follows that $F = 0$ on *any* representation since by complete reducibility, they're all direct sums of irreps. In particular, $F : V_{\text{reg}} \to V_{\text{reg}}$ is the zero operator, where $V_{\text{reg}} \cong V_1^{d_{V_1}} \oplus \cdots \oplus V_s^{d_{V_s}}$ is the regular representation. Thus, for example, $F(e_e) = 0$. But we also know that

$$F(e_e) = \sum_{g \in G} f(g) \cdot g e_e = \sum_{g \in G} f(g) \cdot e_g$$

Consequently, by transitivity, we have that

$$0 = \sum_{g \in G} f(g) \cdot e_g$$

But since the e_g are all linearly independent by the definition of the regular representation, we have that each $f(g) = 0$, as desired. This means that $f = 0$, contradicting our original supposition. $\square$

- That is the end of this story.
- Here's one consequence of the above theorem.
  - We now know that the space of class functions has an orthonormal basis $\chi_{V_1^*}, \dots, \chi_{V_k^*}$.
  - If we denote the conjugacy classes of G by $C_1, \dots, C_k$, then another obvious basis of $\mathbb{C}_{\text{cl}}[G]$ is $\delta_{C_1}, \dots, \delta_{C_k}$ defined by
$$\delta_{C_i}(g) = \begin{cases} 1 & g \in C_i \\ 0 & g \notin C_i \end{cases}$$
  - This new basis is orthogonal: We have
$$\langle \delta_{C_i}, \delta_{C_j} \rangle = \frac{1}{|G|} \sum_{g \in G} \delta_{C_i}(g) \delta_{C_j}(g) = \begin{cases} 0 & i \neq j \\ \frac{|C_i|}{|G|} & i = j \end{cases}$$
    - Justifying this computation: If $i \neq j$, then at least one of $\delta_{C_i}, \delta_{C_j}$ will be zero; if $i = j$, then they're both nonzero and equal to 1 for all $|C_i|$ elements $g \in C_i$.
  - What is the change of basis matrix between $\{\delta_{C_i}\}$ and $\{\chi_{V_i^*}\}$? It's the character table.
    - The orthogonality condition for characters then just comes from the fact that we're going from one orthogonal basis to another.
    - What are the exact bases we change between??

4.2 Office Hours

10/17:

- **Transitive** (group action): A group action for which the **orbit** of x is equal to X for any $x \in X$.
- **Orbit** (of $x \in X$): The set of $g \cdot x$ for all $g \in G$.
- **Diagonal action** (of G on $X \times X$): The action defined as follows. *Given by*
$$g \cdot (x_1, x_2) = (g \cdot x_1, g \cdot x_2)$$
- Check Etingof et al. (2011) for some things??

4.3 Orthogonality Results

10/18: • Announcements.

- Goal: Finish our discussion of the orthogonality of characters, projection functions, etc.
- Friday: Frobenius determinant.
- Next week: Group algebras, associative algebras, etc.; another perspective on representations.
- After next week: A more advanced part of representation theory related to group theory.

• Describing Figure 3.1 from a different perspective.

- Let G be a finite group, and let k denote the number of conjugacy classes and the number of irreps. Let $C_1,\dots,C_k$ be the conjugacy classes and $V_1,\dots,V_k$ be the irreps.
- There is no natural/canonical bijection between the two sets. For a simple group, there is often a canonical way, and this is where things get interesting.
  - Example: Symmetric group induces canonical bijection, as we'll see later.
- $\mathbb{C}_{\text{cl}}[G]=\mathbb{C}^k$ is a vector space of class functions and a ring.
- We have the Hermitian inner product

$$\langle f_1,f_2\rangle=\frac{1}{|G|}\sum_{g\in G}f_1(g)\overline{f_2(g)}$$

- Recall that $\chi_{V_1},\dots,\chi_{V_k}$ is an orthonormal basis such that

$$\left\langle\chi_{V_i},\chi_{V_j}\right\rangle=\delta_{ij}$$

- We have another basis $\delta_{C_1},\dots,\delta_{C_k}$ defined by

$$\delta_{C_i}(g)=\begin{cases}0 & g\notin C_i\\ 1 & g\in C_i\end{cases}$$

that is orthogonal but not orthonormal:

$$\left\langle\delta_{C_i},\delta_{C_j}\right\rangle=\begin{cases}0 & C_i\neq C_j\\ \frac{|C_i|}{|G|} & C_i=C_j\end{cases}$$

- How do we relate the two bases?
- To begin, fix C_i. Then

$$\delta_{C_j}(g)=\sum_{V_i}\lambda_i\chi_{V_i}(g)$$

- λ_i can be computed immediately using the inner product since the characters are orthonormal:

$$\lambda_i=\left\langle\delta_{C_j},\chi_{V_i}\right\rangle=\frac{1}{|G|}\sum_{g\in G}\delta_{C_j}(g)\overline{\chi_{V_i}(g)}=\frac{|C_j|\bar{\chi}_{V_i}(g)}{|G|}$$

  - You took $\lambda_i=\left\langle\delta_{C_j},\bar{\chi}_{V_i}\right\rangle$; which one is correct??
- But then

$$\delta_{C_j}(g)=\frac{|C_j|}{|G|}\left(\sum_{V_i}\bar{\chi}_{V_i}(C_j)\chi_{V_i}(g)\right)$$

- It follows that we have two bases of $\mathbb{C}_{\text{cl}}[G]$. These are given by

$$\frac{|G|}{|C_j|}\delta_{C_j} \qquad\qquad\qquad \chi_{V_i^*}$$

  where $i,j = 1,\dots,k$.
- How do we convert between these two very natural bases of our space of functions? The change of basis matrix from left to right is the character table.
- Obviously, we have to do some scaling and take some duals, but it's not that bad and it fits the character table really well.
- This gives us some properties of the character table such as orthogonality.
- For example, **orthogonal** matrices convert between orthogonal bases; in the complex domain, such a matrix is **unitary**, i.e., for the character table U, $U\bar{U}^T = E$.

- Orthogonality relations that you can derive.

  1. We can show that
  $$\sum_{g\in G}\chi_1(g)\overline{\chi_2(g)} = \begin{cases} 0 & \chi_1 \neq \chi_2\\ |G| & \chi_1 = \chi_2 \end{cases}$$
     - Use the unitary condition.
  2. We can show that
  $$\sum_{i=1}^{k}\chi_i(g_1)\overline{\chi_i(g_2)} = \begin{cases} 0 & g_1 \neq g_2\\ \frac{|G|}{|C(g_1)|} & g_1 \sim g_2 \end{cases}$$
     - We literally just take the identity defining $\delta_{C_j}(g)$.

- **Isotypical component**: A representation that is equal to the direct sum of isomorphic irreducible representations. *Also known as* **isotypic component**.
  - Illustrative example: For $V = V_1^{n_1}\oplus\cdots\oplus V_k^{n_k}$, each $V_i^{n_i}$ is an isotypical component.

- Examples.

  1. Let $G \circlearrowright \mathbb{C}^2$ by $\rho(g) = E_2$. Thus, we can say that $\mathbb{C}^2 = V_1\oplus V_1$, but we can't say this in any unique, canonical way, i.e., we can choose infinitely many V_1's and have the statement still be true, where V_1 is the trivial rep.
  2. We have $V_1^{n_1} = V^G = \{v\in V \mid gv = v\ \forall\ g\in G\}$. Look at what's invariant under the symmetry group, i.e., define
  $$P = \frac{1}{|G|}\sum g$$
     - All **invariant functions** come from averaging over the group!
     - Then $P^2 = P$ and $\operatorname{Im} P = V^G$.
     - Takeaway: We call each $V_i^{n_i}$ an **isotypical component**.
     - What's going on in this example??
  3. The permutational representation for S_n decomposes into the sum of the trivial and standard reps; there is only one decomposition this way. If we look at $V_1\oplus V_{\text{stand}}^2$, then our decomposition will depend on a choice of a plane.

- Reminder.
  - Last time, we chose an $f\in\mathbb{C}_{\text{cl}}[G]$, a representation V, and then took $\sum f(g)g : V\to V$ so that then $\sum f(g)g\in\operatorname{Hom}_G(V,V)$.
  - Moreover, we proved that if V is irreducible, then this endomorphism is equal to a scalar λ times the identity matrix via Schur's lemma.

  - Computing λ:

$$\lambda = \frac{|G|}{d_V} \langle f, \chi_V^* \rangle$$

    - Hard to remember but easy to derive.
  - Define $V = V_1^{n_1} \oplus \cdots \oplus V_k^{n_k}$ and $P_i : V \to V_i^{n_i}$.
  - In particular, look at

$$P_i = \frac{d_V}{|G|} \sum_{g \in G} \chi_{V_i^*}(g) g$$

    - This averaging operator is consistent with what we had before.
  - P_i acts on V_i by

$$\frac{d_{V_i}}{|G|} \frac{|G|}{d_{V_i}} \left\langle \chi_{V_i^*}, \chi_{V_i^*} \right\rangle = 1$$

  - P_i acts on V_j by

$$\frac{d_{V_i}}{|G|} \frac{|G|}{d_{V_i}} \left\langle \chi_{V_i^*}, \chi_{V_j^*} \right\rangle = 0$$

  - Take $V = V_1^{n_1} \oplus \cdots \oplus V_k^{n_k}$ and apply P_i. It follows by the above that it is exactly the projection on $V_i^{n_i}$.
  - Thus, $P_1 + \cdots + P_k = 1$. $P_i^2 = P_i$. $P_i P_j = 0$. This is called a/the (which one??) **idempotent decompostion**.
  - Example: Let $v \in V$. Then $v = P_1 v + \cdots + P_k v$.
  - Additionally, we can take a function f that is invariant under the group...??

- We're done early.
- We will not start the Frobenius determinant today.
- We will start on next week's content then so we can begin thinking about it.
- **Associative algebra**: A vector space over a field F that is also a (not necessarily commutative) ring, where we have a unit 1 in the ring, addition, and multiplication. Scalar multiplication: $\lambda a = (\lambda \cdot 1) \cdot a$. Associativity condition: $(\lambda a) b = \lambda(ab)$. *Denoted by* $\boldsymbol{A}$.
  - We'll only discuss finite-dimensional algebras in this course.
- Examples:
  1. $\mathbb{R}$, $\mathbb{C}$ (an algebra over $\mathbb{R}$).
  2. $\mathbb{H}$, a 4d algebra over $\mathbb{R}$. The algebra of quaternions. $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$.
     - Hamilton's remarkable discovery: There is a 4D set of numbers that is not commutative but is still associative and helps describe rotation in 3D or 4D space.
     - Multiplication rules:

$$i^2 = j^2 = k^2 = -1 \qquad\qquad ijk = -1$$

     - We should spend part of our weekend reading a history of quaternions!
  3. $M_{n \times n}(F)$, the **matrix algebra**.
  4. $A_1 \oplus \cdots \oplus A_n$, the **direct sum** of algebras.
     - Addition and multiplication are done pairwise.
  5. $A_1 \otimes A_2$.
     - We will not talk about this today, though!

- Let's go back; let G be a finite group and consider $\mathbb{C}[G]$, the set of functions on G.
  - This algebra has some basis $\bigoplus \mathbb{C}e_g$.
  - To get the algebra structure, we just need a rule for multiplying basis elements. In this case, we use $e_{g_1}e_{g_2} = e_{g_1g_2}$.
  - This is the **algebra over $\mathbb{C}$ of dimension G**.
  - Theorem: $\mathbb{C}[G] \cong M_{d_1\times d_1}(\mathbb{C}) \oplus \cdots \oplus M_{d_k\times d_k}(\mathbb{C})$.
    - We can prove this theorem from what we know: Schur's Lemma and complete reducability.
    - We'll discuss it for several consecutive times.
    - A similar result holds for *many* algebras (e.g., semisimple algebra), not just *group* algebras.
- HW1-2 will be graded later this week and handed back on Friday.
- In Etingof et al. (2011), we can find a lot of history of some of this stuff. The comments are interesting and entertaining.

References

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