Week 7

Representations of the Symmetric Group

7.1 Specht Modules

11/6: - Announcements.
    - Midterm description is on the Canvas page.
    - Review what he says to review, and then look at the PSets. The operator averaging stuff and S_4, S_5 examples are most important.
    - New HW will be due next Friday (not this Friday).
- New topic: Representations of S_n.
    - We will talk about these almost until the end of the course.
    - Very hard.
    - Any specialist in rep theory will still say that they know some approaches, but nobody understands this stuff completely.
    - We'll explore some phenomena, but if we feel after this course that we still don't understand everything about S_n, that's typical; if we think we understand everything, we're probably wrong.
- Representation theory of $GL_n(\mathbb{F}_{p^k})$ is related but even worse.
    - Same with $O_n(\mathbb{F}_{p^k})$.
    - Recently, all this stuff was understood with something called linquistic (??) theory, but that's far beyond us.
- $|S_n| = n!$, and the conjugacy classes are in bijection with cyclic structures of a permutation.
    - Our good understanding of the conjugacy classes of S_n is the only thing that makes this problem the slightest bit tractable.
    - Cyclic structures are also in bijection with the **partitions** of a number; recall that we briefly talked about these in MATH 25700!
- **Partition** (of $n \in \mathbb{N}$): An ordered tuple satisfying the following constraints. *Denoted by* $\boldsymbol{\lambda}$, $\boldsymbol{(\lambda_1, \dots, \lambda_k)}$. *Constraints*
    1. $\lambda_i \in \mathbb{N}$ for $i = 1, \dots, k$;
    2. $\lambda_1 \geq \cdots \geq \lambda_k$;
    3. $\lambda_1 + \cdots \lambda_k = n$.

- Example: The partitions of the number "4" are (4), $(3,1)$, $(2,2)$, $(2,1,1)$, and $(1,1,1,1)$.
  - This is the same way we've been denoting representations!
- $\boldsymbol{p(n)}$: The number of possible partitions of n.
  - Hardy and Ramanujan helped understand the number $p(n)$ of partitions of n, but they're still very hard to understand.
- One way to understand $p(n)$ is through its encoding in the **generating function**

$$\sum_{n\geq 1} p(n)x^n = 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots$$

  - We can think of the above generating function as an actual function of x if it converges for small x; of it doesn't converge, then we just think of it as a "meaningless" **formal power series**.
  - To choose a partition, we need to choose a certain number of 1's, a certain number of 2's, a certain number of 3's, etc. all the way up to n.
  - So let's look at

$$(1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)(1 + x^4 + x^8 + \cdots)\cdots$$

    - Formally, this is

$$\prod_{i=1}^{\infty}\left(\sum_{j=0}^{\infty} x^{ij}\right)$$

    - This equals the generating function! It tells us that to compute $p(100)x^{100}$, we need only look at certain terms.
  - Recall that we can write $1 + x + x^2 + \cdots = 1/(1-x)$. Doing similarly for other terms transforms the above product into

$$\frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^3}\cdots$$

- **Generating function**: An encoding of an infinite sequence of numbers as the coefficients of a formal power series.
- **Formal power series**: An infinite sum of terms of the form ax^n that is considered independently from any notion of convergence.
- The above discussion of $p(n)$ as a generating function is only for our fun; Rudenko is not going to use this in any way.
  - It's just a pretty function.
  - Takeaway: We can write the generating function as something nice and then use it to prove something.
- We can visualize these partitions using something called a **Young diagram**.

(a) $\lambda = (5,4,4,3,2,1,1)$. (b) $\lambda' = (7,5,4,3,1)$.

Figure 7.1: Young diagrams for a partition of 20.

- Suppose we have the following partition of 20: $(5, 4, 4, 3, 2, 1, 1)$.
- Then we draw 5 cages for 5 little birds, followed by 4 cages for 4 little birds, etc.
- Thus, the i^{th} row of boxes has length λ_i.
- The same way you can denote by λ the whole *partition*, you can denote by λ the whole *diagram*.
- This is just a way to visualize partitions.
- Recall the three partitions of S_3, corresponding to its representations: (3), $(2,1)$, $(1,1,1)$.
- Moreover, these diagrams are actually meaningful!

- **Inverse** (of λ): The partition $(\lambda'_1, \dots, \lambda'_k)$ defined as follows. *Denoted by* $\boldsymbol{\lambda'}$. *Given by*

$$\lambda'_i = |\{\lambda_j \mid \lambda_j > i\}|$$

for all $i = 1, \dots, k$.

  - We can see that $\lambda'_1 \geq \cdots \geq \lambda'_k$.
  - We can also see that the sum will still be n.
  - Moreover, if we do this twice, we'll get back to λ, i.e., $(\lambda')' = \lambda$.
  - We can prove $(\lambda')' = \lambda$ combinatorially, too, (that is, without Young diagrams) but that gets pretty complicated.

- Example: If $\lambda = (5, 4, 4, 3, 2, 1, 1)$ as above, then $\lambda' = (7, 5, 4, 3, 1)$.
  - See Figure 7.1b.
  - Moreover, the Young diagrams are related by a flip akin to matrix transposition.
  - Notice how the definition of inversion *exactly* specifies this flip in the picture: The number of λ_j's that have length at least 1 is all the first column of Figure 7.1a, the number of length at least 2 is all the second column, etc.

- Onto the next question, which is the main miracle.
  - Main miracle: There exists a natural (i.e., canonical) bijection between the conjugacy classes and irreducible representations of S_n.
  - We've explored a duality for general finite groups G, before, but never a bijection.
    - In S_n, there *is* this natural bijection.
    - If you understand why intuitively, you will have started to understand the representation theory of S_n.

- If we define $\lambda \mapsto n$ (??), then there is some irrep V_λ corresponding to λ. We will look at the **Specht module** construction of V_λ.
  - Some of the proofs Rudenko will present, he stole from Etingof et al. (2011), and some of the proofs he invented himself.
  - This is *by far* the best construction, even though it's exceedingly rare in the literature.

- The usual construction.
  - Take $\mathbb{C}[S_n]$ with coefficients a_λ, b_λ, etc. similar over conjugacy classes and do something with it??
  - "Just say NO!" to this construction.

- Here is the better idea.
  - Consider an algebra of polynomials with rational coefficients: $\mathbb{Q}[x_1, \dots, x_n]$.
    - We could also do real or complex, but rational is nice.

    - For symmetric groups, all representations will be integers, etc.??
    - One thing to emphasize about this algebra: It is a **graded** algebra.
        - If represented by A, then it equals $A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ where

$$A_m = \left\{ \sum_{k_1+\cdots+k_n=m} a_{k_1\cdots k_n} x_1^{k_1} \cdots x_n^{k_n} \right\}$$

            - I.e., A_m is the sum of all polynomials with degree equal to m.
            - Example: If we take $1 + x_1^2x_2^3 + x_1x_2 + x_1^{100} + x_1x_2^{99}$, we can then break this polynomial up into polynomials of degree 1, 5, 2, and 100.
        - We also have $A_{m_1} \cdot A_{m_2} \subset A_{m_1+m_2}$.
            - Exmaple: $x_1x_2^2 \cdot (x_1 + x_2) = x_1^2x_2^2 + x_1x_2^3$.
    - With this algebra in hand, we may let $S_n \circlearrowright \mathbb{Q}[x_1, \dots, x_n]$ via

$$\sigma P(x_1, \dots, x_n) = P(x_{\sigma_{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$$

        - In other words, σ is transposing polynomials.
        - Example: $(12)(x_1^2 + x_2^3 + x_3) = x_2^2 + x_1^3 + x_3$.
    - Thus, we call as A_1 the representation V_{perm}^*.
        - This is because $A_1 = \text{span}(x_1 + \cdots + x_n)$, and permuting these is much like permuting the basis of a vector space, as the typical permutation representation does.
        - It could technically be the isomorphic representation V_{perm}, but the dual fits better here for reasons??
    - Then $A_2 = S^2V^*$.
        - So if A_1 had basis $e^1, \dots, e^m$, A_2 has basis e^ie^j.
        - Why are we choosing these sets??
    - Continuing, $A_3 = S^3V^*$.
    - It follows that the representation of the overall thing is

$$\bigoplus_{m\geq 0} (S^m V_{\text{perm}}^*)$$

        - This is called the **symmetric algebra**.

- **Graded** (algebra): An algebra for which the underlying additive group is a direct sum of abelian groups A_i such that $A_iA_j \subset A_{i+j}$.

- So how do we construct representations?
    - For S_2, $x_1 - x_2$ changes sign when we apply S_2.
    - For S_3...
        - The trivial's polynomial is 1 and *tableaux*.
        - The standard is $(2, 1)$. When we apply S_3 to $(x_1 - x_2)$, we get

$$\langle (x_1 - x_2), (x_2 - x_1), (x_1 - x_3), (x_3 - x_1), (x_2 - x_3), (x_3 - x_2) \rangle$$

            - If we let $a = x_1 - x_2$, $b = x_2 - x_3$, then some elements equal $a + b$. This is another way to think about the action.
        - What about the alternating representation? We have $(x_1 - x_2)(x_2 - x_3)(x_1 - x_3) = \Delta_{123}$, which changes sign when we apply any element with sign -1!
    - For S_4...

      - (4) is 1.
      - $(3, 1)$ is $S_4(x_1 - x_2) = \Delta_{12}$.
      - $(1, 1, 1, 1)$ is $(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4) = \Delta_{1234}$.
      - $(2, 1, 1)$ is $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$.
        - We got this polynomial by guessing; the same way $(x_1 - x_2)$ worked in multiple cases, maybe this one does too! And it does.
        - Something to check is that $\Delta_{123} - \Delta_{124} - \Delta_{134} - \Delta_{234} = 0$.
      - $(2, 2)$ is $(x_1 - x_2)(x_3 - x_4)$.
        - Something related we can prove is that

$$(x_1 - x_2)(x_3 - x_4) - (x_1 - x_3)(x_2 - x_4) - (x_1 - x_4)(x_2 - x_3) = 0$$

        - This formula appears in **cross ratios**, which we can discuss in Rudenko's algebraic geometry course next quarter.
      - For $\lambda = (4, 3, 1)$, we have $\Delta_{123}\Delta_{45}\Delta_{67}$, and we act by S_8 upon this! Explicitly, we have $S_8(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x - 4 - x_5)(x_6 - x_7)$.
      - Takeaway: It all depends on column length!
    - These polynomials are called **Vandermonde determinants**; those are the little Δ things with subscripts. We'll talk about these next times.

- We need to prove reducibility and not pairwise isomorphic to make sure that this construction is valid, but that's easy!