

Week 9

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9.1 ???

11/27:

- Announcements.
 - OH on Wednesday at 5:30 PM this week; not Tuesday.
 - There will be extra OH next week pre-exam.
 - Roughly like Monday/Wednesday next week.
 - Midterm will be returned on Wednesday; we can pick them up in-person in his office starting then.
 - There are some grade boundaries: Pass/Fail we can do til Friday, withdrawal we can do til 5:00 PM today.
- Let's finish the conversation about induction/restriction and prove the **branching theorem**.
- Reminder to start.
 - We have two mathematical categories, G -reps and H -reps where $H \leq G$.
 - These categories are related by functors.
 - $\text{Res}_H^G : G\text{-reps} \rightarrow H\text{-reps}$ and vice versa for Ind_H^G .
 - Restrictions are stupidly simple.
 - Inductions, most hands-on, we take copies of W times cosets. Formulaically,

$$\text{Ind}_H^G W = g_1 W \oplus \cdots \oplus g_k W$$

where $k = (G : H)$ and $G = \bigsqcup_{i=1}^k g_i H$.

- In more detail, the action of g on $g_i w$ is that of $g_{\sigma(i)} h_i w$.
- This is a genuinely hard construction.
- A matrix of this thing will be a permutation matrix via

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- Note that

$$g_1 W \oplus \cdots \oplus g_k W \cong \text{Hom}_H(\mathbb{C}[G], W)$$

- Recall that elements of the set on the right above are functions $f : G \rightarrow W$ such that $f(h(g)) = hf(g)$.
- We map between the two via $f(g) \mapsto f(gx')$.
- What is nice about induced representations is that $\dim[\text{Ind}_H^G W] = (\dim W)[G : H]$.

- Moreover, there is a very easy statement, the **Frobenius formula**.

- Recall that

$$\tilde{\chi}_W(g) = \begin{cases} 0 & g \notin H \\ \chi_W(g) & g \in H \end{cases}$$

- With this, we average.

$$\chi_{\text{Ind}_H^G W}(g) = \sum_{x \in G} \tilde{\chi}_W(xgx^{-1})$$

- Essentially, we're taking a whole bunch of conjugates, summing them up, and dividing to get rid of overcounting.

- We now move onto **Frobenius reciprocity**, which is a relation between the functors/relations Ind_H^G and Res_H^G .

- The first point where category theory gets interesting is the notion of **adjoint functors**, which we are about to touch on. It is a very subtle notion.

- Here's version 1 of the statement of Frobenius reciprocity.

- Recall that we have a scalar product on the space of class function, given by

$$(\chi_1, \chi_2) = \frac{1}{|G|} \sum_{g \in G} \chi_1(g) \chi_2(g^{-1})$$

where χ_1, χ_2 are class functions on G .

- Recall that if $\chi_1 = \chi_V$ and $\chi_2 = \chi_W$, then

$$(\chi_1, \chi_2) = \dim \text{Hom}_G(V, W) = \dim \text{Hom}_G \left(\bigoplus_{i=1}^k V_i^{n_i}, \bigoplus_{i=1}^k V_i^{m_i} \right) = \sum_{i=1}^k n_i m_i$$

- Then the statement is as follows. If V is a G -rep and W is an H -rep, then

$$(V, \text{Ind}_H^G W)_G = (\text{Res}_H^G V, W)_H$$

- Denoting scalar product in G and scalar product in W of the characters of each representation.

- This is similar to the relation between adjoint maps $V \rightarrow W$ and $W^* \rightarrow V^*$.

- Version 2.

- We have that

$$\text{Hom}_G(V, \text{Ind}_H^G W) \cong \text{Hom}_H(\text{Res}_H^G V, W)$$

where the isomorphism is canonical.

- We will not check this last definition; we can tediously do it with definitions, and there's nothing complicated. Rudenko leaves this as an exercise to us.

- Constructing...something: Take $v \in V$, $g \in G$, $\varphi : V \rightarrow W$. We send $g \mapsto \varphi(gv)$.

- We now prove Version 1.

Proof. We have

$$\begin{aligned}
 (\chi_V, \chi_{\text{Ind}_H^G W})_G &= \frac{1}{|G|} \sum_{g_1 \in G} \chi_V(g_1) \left(\frac{1}{|H|} \sum_{g_2 \in G} \tilde{\chi}_W(g_2 g_1^{-1} g_2^{-1}) \right) \\
 &= \frac{1}{|H| \cdot |G|} \sum_{g_1, g_2 \in G} \chi_V(g) \tilde{\chi}_W(g_2 g_1^{-1} g_2^{-1}) \\
 &= \frac{1}{|H| \cdot |G|} \sum_{g_1, g_2 \in G} \chi_V(\underbrace{g_2 g_1 g_2^{-1}}_h) \tilde{\chi}_W(\underbrace{g_2 g_1^{-1} g_2^{-1}}_{h^{-1}}) \\
 &= \frac{1}{|H|} \frac{1}{|G|} \sum_{h \in G} |G| \chi_V(h) \tilde{\chi}_W(h^{-1}) \\
 &= (\chi_V|_H, \chi_W)_H \\
 &= (\text{Res}_H^G V, \chi_W)_H
 \end{aligned}$$

From line 3 to line 4: Fix h ; then $g_2 g_1 g_2^{-1} = h$ iff $g_1 = g_2^{-1} h g_2$, so we have overcounted by $|G|$ times. \square

- We now come to the branching theorem at long last.
- Example first.
 - Consider $S_n > S_{n-1}$, where S_{n-1} is the subgroup of permutations fixing n . I.e., $S_3 > S_2 = \{e, (12)\}$.
 - Let λ be a partition of n ; *there's notation for this!*
 - Let $\mu \leq \lambda$ be a Young diagram of a partition of $n-1$.
 - Then

1. We have

$$\text{Res}_{S_{n-1}}^{S_n} V_\lambda = \bigoplus_{\mu \leq \lambda} V_\mu$$

■ Example: *Draw out pictures*

2. We have

$$\text{Ind}_{S_{n-1}}^{S_n} V_\mu = \bigoplus_{\mu \leq \lambda} V_\lambda$$

■ Example: *Draw out pictures*

- The reason that this theorem is called the branching theorem originates from the following diagram, which (when continued) encapsulates the main idea of the theorem. *picture*
 - This graph helps you understand induction and restriction.
 - Dimensions are the number of paths from the left to a final Young diagram.
 - For example, the dimension of $(3, 1)$ is 3 because there are 3 paths to it (*list them*).
 - Number of paths formula is equivalent to standard Young tableaux!
- Theorem (Branching): The following two statements are true.

$$\text{Res}_{S_{n-1}}^{S_n} V_\lambda = \bigoplus_{\mu \leq \lambda} V_\mu \tag{9.1}$$

$$\text{Ind}_{S_{n-1}}^{S_n} V_\mu = \bigoplus_{\mu \leq \lambda} V_\lambda \tag{9.2}$$

Proof. We'll talk about the general idea of the proof now, and maybe do the details next time.

(1) \iff (2): We have that *stuff at bottom of board*

(1): Let's look at an example. Here's a YD of S_8 . We want to restrict it down to S_7 . Recall that $\overline{V_\lambda} = \text{span}(S_8 : \Delta(x_1, x_2, x_3)(x_4 - x_5)(x_6 - x_7))$. Now in S_7 , we fix x_8 . Consider subrepresentations of V_λ filtered by degree as follows.

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The proof comes from the fact that if we now take quotients of these subrepresentations, then since x_8 can only appear in three boxes, ... \square

- Practice with the above example and think it through.

9.2 ???

11/29:

- Announcements.
 - OH today at 5:30.
 - Our midterms are graded; we can look at them in his office whenever (I can do this during OH!).
- Today, we'll formulate the main result he wants to prove next time.
- Goal is still to understand representations of S_n .
 - We've constructed all of them using Specht modules, but what else do we want?
 - We have dimension, we want characters, etc.
- The main idea is to look at symmetric polynomials once again.
 - Consider $\mathbb{Q}[x_1, \dots, x_n]^{S_n}$.
 - We have proven the fundamental theorem that $\mathbb{Q}[x_1, \dots, x_n]^{S_n} = \mathbb{Q}[\sigma_1, \dots, \sigma_n]$ where $\sigma_k = \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n} x_{i_1} \cdots x_{i_k}$.
 - We also proved in the homework that these rings are equal to $\mathbb{Q}[p_1, \dots, p_k]$ and $\mathbb{Q}[h_1, \dots, h_k]$ where

$$p_k = \sum x_i^k \qquad h_k = \sum_{i_1 \leq \dots \leq i_k} x_{i_1} \cdots x_{i_k}$$

- Example: If $n = 3$, then

$$h_2 = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$$

- Table of bases for n, k .

kn	1	2	3	4
0	1	1	1	1
1	x_1	$x_1 + x_2$	$x_1 + x_2 + x_3$	\dots
2	x_1^2	$x_1^2 + x_2^2, x_1x_2$	σ_1^2, σ_2^2	σ_1^2, σ_2^2
3	x_1^3	\dots		

Table 9.1: ...

- Now take

$$\Lambda_k = [\mathbb{C}[x_1, \dots, x_k]]_{\deg=k-1} \cong [\mathbb{C}[x_1, \dots, x_{k+1}]]_{\deg=k-1} \cong \dots$$

- Alternatively, we can think of this thing as

$$\Lambda_k = (\mathbb{C}[x_1, \dots])_k$$

with $\sigma_1^k, \sigma_2 \sigma_1^{k-1}, \dots$

- We call Λ the ring of symmetric functions and define it to be equal to

$$\Lambda = \mathbb{Q}[\sigma_1, \sigma_2, \sigma_3, \dots]$$

- In every complete component, only finitely many of the σ will participate, so we get finite things.
- This is a graded ring! We have

$$\Lambda = \bigoplus_{k \geq 0} \Lambda_k$$

and $\Lambda_k \otimes \Lambda_\ell = \Lambda_{k+\ell}$

- This construction is called the **projective limit**, and we may have encountered it in commutative algebra under the definition

$$\Lambda = \varinjlim \mathbb{C}[x_1, \dots, x_n]^{S_n}$$

- We have identifies such as $p_2 = \sigma_1^2 - 2\sigma_2$. This means that

$$(x_1 + \dots + x_n)^2 - 2(x_1 x_2 + x_1 x_3 + \dots) = x_1^2 + x_2^2 + \dots$$

- Observation: $\dim_{\mathbb{Q}} \Lambda_n$.

- Now, we need to take a vector space on ring representations; we've done this already with the representation ring.
- Let R_n be the \mathbb{Q} -vector space of functions $\chi : S_n \rightarrow \mathbb{Q}$ such that $\chi(x\sigma x^{-1}) = \chi(\sigma)$. This is our favorite space of class functions.
- Theorem (Frobenius characteristic map): There is an isomorphism of vector spaces and of rings called the Frobenius characteristic: $\text{ch} : \bigoplus_{n \geq 0} R_n \rightarrow \Lambda$.

Proof. Take $\chi_V \in R_k$, and $\chi_W \in R_\ell$. Let V an S_k -rep, and W an S_ℓ -rep. We know that

$$S_k \times S_\ell = S_{k+\ell}$$

So what we can do is induction $\text{Ind}_{S_k \times S_\ell}^{S_{k+\ell}} (V \otimes W)$. Call this operation $\chi_V \boxtimes \chi_W$.

Now we write down the formula:

$$\text{ch}(\chi) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi(\sigma) p_1^{\lambda_1(\sigma)} \dots p_k^{\lambda_k(\sigma)}$$

where $\lambda_1(\sigma), \lambda_2(\sigma), \dots$ represent the cycle structure of σ ; each λ_i is a number of cycles of length $1, 2, \dots$ □

- Examples.

1. S_1 .

- Sends the YD (1) to $p_1 = x_1 + x_2 + x_3 + \dots$.

2. S_2 .

- Sends (2) to $\frac{1}{2!}(p_1^2 + p_2) = \frac{1}{2}((x_1 + x_2)^2 + x_1^2 + x_2^2) = x_1^2 + x_2^2 + x_1 x_2 = h_2$.
- It also sends (1, 1) to $\frac{1}{2!}(p_1^2 - p_2) = \frac{1}{2}((x_1 + x_2)^2 - x_1^2 - x_2^2) = x_1 x_2 = s_2$.

- Let's check our formula. What is $\text{Ind}_{S_1 \times S_1}^{S_2}(1) \otimes (1)$? Since the induction of the trivial representation is the regular representation, which we can decompose, we know that this induction equals $(1, 1) \oplus (2)$. It follows that $p_1^2 = x_1^2 + x_2^2 + x_1x_2 + x_1x_2 = (x_1 + x_2)^2$.

3. S_3 .

- Sends (3) to

$$\begin{aligned} \frac{1}{3!}(p_1^3 + 3p_1p_2 + 2p_3) &= \frac{1}{6}[(x_1 + x_2 + x_3)^3 + 3(x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2) + 2(x_1^3 + x_2^3 + x_3^3)] \\ &= \frac{1}{6}[6(x_1^3 + x_2^3 + x_3^3) + 6(x_1^2x_2 + x_1x_2^2 + x_1x_3^2 + x_1^2x_3 + \cdots) + 6x_1x_2x_3] \\ &= h_3 \end{aligned}$$

- Sends (1, 1, 1) to

$$\begin{aligned} \frac{1}{3!}(p_1^3 - 3p_1p_2 + 2p_3) &= \frac{1}{6}[(x_1 + x_2 + x_3)^3 - 3(x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2) + 2(x_1^3 + x_2^3 + x_3^3)] \\ &= x_1x_2x_3 \\ &= \sigma_3 \end{aligned}$$

- Sends (2, 1) to

$$\begin{aligned} \frac{1}{3!}(2p_1^3 - p_3) &= \frac{1}{6}[2(x_1^3 + x_2^3 + x_3^3) + 6(x_1^2x_2 + \cdots) + 12x_1x_2x_3] \\ &= (x_1^2 + \cdots) + 2x_1x_2x_3 \end{aligned}$$

- Again, we can check that

$$\text{Ind}_{S_2 \times S_1}^{S_3}[(1, 1) \otimes (1)] = \sigma_1\sigma_2$$

- We compute $\text{Ind}_{S_2}^{S_3}(1, 1) = (1, 1, 1) \oplus (2, 1)$ via the branching formula: There are only two ways to add a box!
- We have $\sigma_1\sigma_2 - \sigma_3 = (x_1 + x_2 + x_3)(x_1x_2 + x_1x_3 + x_2x_3) - x_1x_2x_3$.

- Do we need to be fluent in the techniques by which you expanded all of the polynomials above??
- Thus, we have two conjectures:

$$\text{ch}[(n)] = h_n \qquad \text{ch}[\underbrace{(1, \dots, 1)}_{n \text{ times}}] = \sigma_n$$

- The theorem is cool because it sends all of representation theory to some symmetric polynomial game!
- How do we compute $\text{ch}(V_\lambda)$?
 - We say it equals S_λ , where S_λ is a Schur polynomial.
 - Take the YT of λ . Recall standard YTs.
 - **Semistandard** (YT): Things strictly increase in columns, but only monotonically increase in rows. *draw picture!*
 - The six semistandard ones give us the Schur polynomial.
 - Relation to RSK correspondence.
- Proving why this stuff is true is not hard.
- To understand *why* this is true, Google the **Schur-Weyl duality**.

9.3 Office Hours

- I got a 68/100 on the midterm: 30, 24, 0, 14.
 - I would have needed to show my work (or at least one example of a calculation) to get full credit for 2, even though it just said “find.”
 - Rudenko did not expect that finding conjugacy classes would be so difficult for us; he will adjust for this difficulty on the final.
- Week 3, Lecture 2: You proved that $\langle \chi_V, \chi_W \rangle = \delta_{VW}$. To do so, you used a projection function $p = (1/|G|) \sum_{g \in G} gv$. You began your proof by proving that p is a G -morphism and then never used this result again, as far as I can tell. Did you use it again? See pp. 45-47, 58 (it needs to be a morphism of G -representations to map between the representations V, V^G).
- Week 3, Lecture 2: Same proof. To prove that $\text{Im}(P) = V^G$, do we need more than $p^2 = p$? I think so, but you didn't do it explicitly. See pp. 46-47.
- Week 3, Lecture 2: Same proof. What's up with the trivial special case? See p. 48.
- *Week 3, Lecture 3: Cube thing (see picture from 10/13)?
 - It's just a depiction of two different 3-coordinate bases of the same space. It was drawn to illustrate a possible relation between the orthonormal basis χ_1, χ_2, χ_3 (cube) and the orthogonal basis $\chi_{C_1}, \chi_{C_2}, \chi_{C_3}$.
- Week 3, Lecture 3: Why did we talk about the infinite-dimensional regular representation here? See p. 50.
- *Week 3, Lecture 3: What is the point of the misc. calculations involved in computing the S_4 character table? See p. 52.
 - Just to check that we were on the right path and shown an example of using the orthogonality relations.
- *Week 3, Lecture 3: Proof of the second orthogonality relation your way? It's in Serre (1977), but I don't think that's the way you proved it. See p. 52.
 - To begin, note that it is a *highly* nontrivial statement that if A, B are matrices such that $AB = I$, then $BA = I$. It seems so simple to us, but think about it! For an arbitrary matrix A, B , AB looks nothing like BA ! We have two entirely different systems of equations.
 - However, using this fact, basically it is possible to translate the orthogonality relation for the *columns* into the orthogonality relation about the *rows*.
- *Week 3, Lecture 3: All the talk about the exceptional homomorphisms? See p. 52, 61 (the final representation has something to do with an **involution** of trace 2, and is a representation of a quotient group?).
 - So the representation is $\rho : S_4 \twoheadrightarrow S_3 \xrightarrow{\tilde{\rho}} GL_n$, where $\tilde{\rho} : S_3 \rightarrow GL_n$ is the representation of ρ corresponding to the character $(2, 0, 1)$.
- *Week 4, Lecture 1: Alternate construction of $R(G)$? See p. 63.
- *Week 4, Lecture 1: Extension of scalars with the representation ring? See p. 64.
 - We don't need to know anything about this stuff.
 - What it is though is basically analogous to extending the real numbers into a subset of the complex numbers by treating every $x \in \mathbb{R}$ as $x + 0i \in \mathbb{C}$. Very trivial, silly concept.
 - There is also such a thing as a **reduction of scalars**.

- *Week 4, Lecture 1: Does multiplying a column vector in the basis $\{\delta_{C_i}\}$ by the character table put it in the basis $\{\chi_{V_i^*}\}$, or vice versa? See p. 66.
 - Derive it for yourself.
- *Week 4, Lecture 2: Isotypical components example. See p. 68.
- **Week 4, Lecture 3: Proof the \mathbb{C} is the only finite-dimensional division algebra? See p. 71.
 - Let A be an arbitrary finite-dimensional division algebra over \mathbb{C} .
 - To prove that $A = \mathbb{C}$, we will use a bidirectional inclusion proof.
 - Naturally, $\mathbb{C} \subset A$.
 - To prove the reverse implication, start by letting $a \in A$ be arbitrary.
 - Define the left-multiplication operator $L_a : A \rightarrow A$ by $x \mapsto ax$ for all $x \in A$.
 - Recall that A is a complex vector space in addition to being an algebra, the same way a ring is also a group. Thus, L_a is a linear operator on a complex vector space.
 - It follows by the theorem of linear algebra that L_a has an eigenvalue $\lambda \in F = \mathbb{C}$ and corresponding eigenvector $b \in A$.
 - Consequently, by the cancellation lemma,

$$L_a b = \lambda b$$

$$ab = \lambda b$$

$$a = \lambda$$

- Therefore, $a \in A$ implies $a \in \mathbb{C}$, so $A = \mathbb{C}$.
- *Week 6, Lecture 2: Proof that $\sqrt{2}/2$ is not an algebraic integer using Gauss's lemma? See p. 87.
 - Let $\alpha := \sqrt{2}/2$ for the sake of notation.
 - Suppose for the sake of contradiction that α is an algebraic integer.
 - Then there exists a monic polynomial $p(x) \in \mathbb{Z}[x]$ such that $p(\alpha) = 0$.
 - Observe that the minimal polynomial in $\mathbb{Z}[x]$ that annihilates α is $2x^2 - 1$.
 - Thus, by polynomial division,

$$p(x) = q(x) \cdot (2x^2 - 1) + r(x)$$

for some $q, r \in \mathbb{Q}[x]$ such that $\deg r \leq 2 - 1$.

- We have that

$$r(\alpha) = p(\alpha) - q(\alpha) \cdot (2\alpha^2 - 1) = 0 - q(\alpha) \cdot 0 = 0$$

- Additionally, since $r \in \mathbb{Q}[x]$ and $\deg r \leq 1$, we know that $r(x) = ux + v$ for some $u, v \in \mathbb{Q}$.
- We now prove that $u = v = 0$.

- Suppose for the sake of contradiction that either u or v was not equal to zero.
- Combining the previous two claims reveals that

$$0 = r(\alpha)$$

$$= u\alpha + v$$

$$-\frac{v}{u} = \alpha$$

- If $u = 0$, then α is undefined and we have arrived at a contradiction. Thus, $u \neq 0$.
- Thus, $\alpha \in \mathbb{Q}$. But since $\alpha \notin \mathbb{Q}$ by definition, we have arrived at a contradiction.

■ Therefore, $u = v = 0$.

- Having established that $r = 0$, we know that $p = (2x^2 - 1)q$, i.e., $2x^2 - 1$ divides p .
- Now define N to be the least common multiple of the denominators of the coefficients of q .
- Consider

$$Np = (Nq)(2x^2 - 1)$$

- It follows by Gauss's lemma that

$$\begin{aligned} c(Np) &= c[(Nq)(2x^2 - 1)] \\ N &= c(Nq) \cdot c(2x^2 - 1) \\ &= 1 \cdot 1 \\ &= 1 \end{aligned}$$

where c denotes the **content**.

- But if $N = 1$, then $q \in \mathbb{Z}[x]$, so leading term of p — equal to the product of $2x^2$ and the leading term of q — has a coefficient that is a multiple of 2, i.e., is *not* equal to 1 as is required of a monic polynomial, a contradiction.
- *Week 6, Lecture 3: Questions about Lemma 1 of the proof of Burnside's theorem. See p. 92.
 - The roots a_1, \dots, a_k of the minimal polynomial of the algebraic integer a are known as **conjugate algebraic integers**.
 - The conjugate algebraic integers of a root of unity are also roots of unity.
 - Suppose ε is a root of unity.
 - Then the minimal polynomial of ε is $x^n - 1$ for some $n \in \mathbb{N}$.
 - Naturally, the roots of this polynomial (the conjugate algebraic integers to ε) are all of the other roots of unity of order n .
 - The conjugate algebraic integers of a sum of roots of unity is a sum of roots of unity.
 - It can be shown that the minimal polynomial for $\varepsilon_1 + \varepsilon_2$ is

$$p(x) = \prod_{i,j=1}^n (x - \varepsilon_1^i - \varepsilon_2^j)$$

- Evidently, the above polynomial is symmetric under permutations of $\varepsilon_1^i, \varepsilon_2^j$, and we'd generate the same polynomial with any $\pm \varepsilon_1^i \pm \varepsilon_2^j$ as starting material.
- Explicit example.
 - $\pm\sqrt{2}$ are conjugate algebraic integers, as solutions to $x^2 - 2$. Similarly, $\pm\sqrt{3}$ are conjugate algebraic integers as solutions to $x^2 = 3$.
 - Thus, we expect the minimal polynomial for $\sqrt{2} + \sqrt{3}$ to be

$$p(x) = (x - \sqrt{2} - \sqrt{3})(x - \sqrt{2} + \sqrt{3})(x + \sqrt{2} - \sqrt{3})(x + \sqrt{2} + \sqrt{3})$$

- Expanding, we obtain

$$\begin{aligned} p(x) &= (x^2 - (\sqrt{2} + \sqrt{3})^2)(x^2 - (\sqrt{2} - \sqrt{3})^2) \\ &= x^4 - [(\sqrt{2} + \sqrt{3})^2 + (\sqrt{2} - \sqrt{3})^2]x^2 + (\sqrt{2} + \sqrt{3})^2(\sqrt{2} - \sqrt{3})^2 \\ &= x^4 - 10x^2 + 1 \end{aligned}$$

- Indeed, the above polynomial is a monic polynomial
- From the definition, this polynomial is evidently also the minimal polynomial for $\sqrt{2} - \sqrt{3}$, $-\sqrt{2} + \sqrt{3}$, and $-\sqrt{2} - \sqrt{3}$.

- Thus, the conjugate algebraic integers of $\sqrt{2} + \sqrt{3}$ are the four sums of all individual algebraic integers.
- How do we extend this argument to the case in the problem?? What about when $\varepsilon_1 = -1$ and $\varepsilon_2 = i$ so that simple powers don't access every combination as the $p(x)$ formula does?
- We know that $\prod_{i=1}^n a_i \in \mathbb{Z}$ because of Vieta's formula.
 - In particular, this tells us that $x_1 \cdots x_n$ is equal to the last coefficient in the minimal polynomial which, by definition, is an integer.
- *Week 7, Lecture 2: Symmetric polynomials and roots of symmetric polynomials. See p. 101.
- *Week 7, Lecture 2: Word in blackboard picture? See p. 102.
 - “Remain” to show...
- *Week 7, Lecture 2: What is d in the proof of the alternating polynomials theorem? See p. 103.
 - $d = n - 1$.
- *Week 8, Lecture 2: What is d in the definition on p. 111.
 - Consider the Specht polynomial corresponding to $(2, 2, 1)$.

Figure 9.1: Young diagram for $(2, 2, 1)$.

- Since $(2, 2, 1)' = (3, 2)$, the Specht polynomial is $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \cdot (x_4 - x_5)$.
- Δ_{123} is of degree $3 = \binom{3}{2}$ because looking at the first column of the YD, which corresponds to $\lambda'_1 = 3$, out of the 3 boxes, we must choose 2 for each of the three terms $(x_1 - x_2), (x_1 - x_3), (x_2 - x_3)$. Then we just add this to the 2 choose 2 for the second column of the YD.
- *Week 9, Lecture 2: Do we need to be fluent in the techniques you used to expand and reduce the various polynomial powers? How did you do that again?