

## Week 3

# Character Theory

### 3.1 Characters

- 10/9:
- Today, we talk about **characters**, arguably the most important idea in rep theory.
  - As per usual, we begin by letting  $G$  a finite group.
    - We’ve been discussing finite dimensional representations of  $G$  over  $\mathbb{C}$ .
    - We’ve also already talked about irreps, and we know that it’s enough to understand those because every rep is a sum of them.
  - Goal of characters: Understand the irreps  $V_1, \dots, V_k$  of  $G$ .
    - Recall the surprising fact about  $k$ : It is the number of conjugacy classes of  $G$ !
      - We haven’t yet proven this, but we will soon!
    - Game plan: Use characters to relate irreps to something that is counted by conjugacy classes.
  - Let  $V = \mathbb{C}e_1 \oplus \dots \oplus \mathbb{C}e_n$  be a  $G$ -rep.
    - Then there exists a homomorphism  $\rho : g \mapsto A_g \in GL_n(\mathbb{C})$ .
  - Motivating question: What doesn’t change when we change the basis of  $V$ ?
    - To isolate the “essence” of the  $A_g$ , we want to construct a function  $f : GL_n(\mathbb{C}) \rightarrow \mathbb{C}$  such that  $f(XAX^{-1}) = f(A)$ .
  - Ideas.
    1. The determinant is a great example of such a function, but it’s kind of boring because this rank 1 representation doesn’t characterize your product representation.
    2. Trace is the main example of such a function.
  - Indeed, you can also take  $\text{tr}(A^k)$  for any  $k$ .
    - Traces of powers are ubiquitous in physics and math because they contain the same information as the coefficients of the characteristic polynomial. In particular, we can express the determinant in terms of them.
  - In fact, we could also take any coefficient of the characteristic polynomial, but others would get complicated.
    - Any characteristic polynomial coefficient can be expressed in terms of traces; this will be an exercise in PSet 3; it’s not hard.

- So what do we have at this point?
  - We can associate to  $\rho$  a function  $\chi_\rho : G \rightarrow \mathbb{C}$  defined by  $\chi_\rho(g) = \text{tr}(A_g) = \text{tr}(\rho(g))$ .
  - This function is invariant under isomorphism.
  - If we know  $\text{tr}(A)$ , we know  $\text{tr}(A^2)$  since  $A_g^2 = A_{g^2}$ . Thus, if we know all traces, we know all power traces.

■ Something about the following??

$$\sum \lambda_i \lambda_j = \frac{\text{tr}(A)^2 - \text{tr}(A^2)}{2}$$

- We form a ring of polynomials??
  - Equivalently,  $\chi_\rho$  has a representation as a polynomial with coefficients in  $\mathbb{C}$ ??
- If  $V$  is a  $G$ -rep,  $\chi_V : G \rightarrow \mathbb{C}$  will be our notation for its character.
- Properties.

1.  $\chi_V(xgx^{-1}) = \chi_V(g)$  for any  $x, g \in G$ .

– Implication:  $\chi_V$  is a **class function**.

– Let  $\mathbb{C}[G]$  be the vector space of all functions from  $G \rightarrow \mathbb{C}$ . Its  $\dim = |G|$ .

■ Recall that this notation is the same as that for the space of polynomials in  $g_1, \dots, g_n$  with complex coefficients.

■ But since  $G$  is a group, a polynomial in products of  $g_i$ 's can be reduced to a polynomial in the  $g_i$ 's.

➤ For example, if  $G = S_3$ , we know since  $(23)(123)^2 = (12)$  that

$$\begin{aligned} 2 + 3(12) + i(123) + (-3 + 7i)(23)(123)^2 &= 2 + [3 + (-3 + 7i)](12) + i(123) \\ &= 2 + 7i(12) + i(123) \end{aligned}$$

■ Such a polynomial is then easily mapped onto a complex-valued function by sending each  $g_i$  to its coefficient  $a_{g_i}$ .

➤ Continuing with the above example, the corresponding function in  $\mathbb{C}[G]$  would be defined by

$$\begin{array}{lll} e \mapsto 2 & (12) \mapsto 7i & (123) \mapsto i \\ & (13) \mapsto 0 & (132) \mapsto 0 \\ & (23) \mapsto 0 & \end{array}$$

■ Thus,  $\mathbb{C}[G]$  (as a space of polynomials) is canonically isomorphic to  $\mathbb{C}[G]$  (as a space of complex-valued functions on  $G$ ), so the notation is well chosen.

– Inside this space, there is the subspace  $\mathbb{C}_{\text{cl}}[G]$  of functions  $f : G \rightarrow \mathbb{C}$  such that  $f(xgx^{-1}) = f(g)$  for all  $x, g \in G$ . These are functions from the sets of conjugacy classes, isomorphic to functions that are constant on conjugacy classes.  $\dim \mathbb{C}_{\text{cl}}[G]$  is the number of conjugacy classes.

– Thus, for every  $V$  a  $G$ -rep, we get a vector  $\chi_V \in \mathbb{C}_{\text{cl}}[G]$ . These class functions form a basis of the space; each  $\chi_V$  for  $V$  an irrep forms a linearly independent vector; the set is an *orthogonal* basis. This is the reason for the original theorem holding true!

2.  $\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$ .

– Proof: It's basically tautological (not actually, but it's easy). Let  $g \in G$ . Compute  $\chi_{V_1 \oplus V_2}(g)$ . We can compute a basis  $e_1, \dots, e_{n+m}$  where the first  $n$  vectors form a basis of  $V_1$ , and the next  $m$  vectors are a basis of  $V_2$ . This gives us a block matrix from which we show that the trace of the matrix is the sum of traces.

$$\chi_{V_1 \oplus V_2}(g) = \text{tr} \begin{bmatrix} \rho_{V_1}(g) & 0 \\ 0 & \rho_{V_2}(g) \end{bmatrix} = \text{tr} \rho_{V_1}(g) + \text{tr} \rho_{V_2}(g) = \chi_{V_1}(g) + \chi_{V_2}(g)$$

– Corollary:

$$\chi_{V_1^{n_1} \oplus \dots \oplus V_k^{n_k}} = n_1 \chi_{V_1} + \dots + n_k \chi_{V_k}$$

- We now pause for a fact that will be instrumental in proving the next property, which is a bit more involved.

– He will explain two ways to prove it; we can also just prove it on our own.

- Fact:  $A$  a matrix such that  $A^n = 1$ . Then  $A$  is diagonalizable or “semi-simple.”

– We can prove this with Jordan normal form.

– It’s a slightly surprising statement.

– Obviously eigenvalues are roots of unity, but still needs some work.

– This proof is left as an exercise.

- We now resume the list of properties.

3.  $\chi_V(g)$  is a sum of roots of unity.

– Proof: We know that  $g^{|G|} = e$ . Thus,  $A_g^{|G|} = 1$ . It follows by the fact above that  $A_g$  is diagonalizable with eigenvalues  $\lambda_1, \dots, \lambda_n$ , each of which satisfies  $\lambda_i^{|G|} = 1$ .

■ Note: Eigenvalues can repeat in the list  $\lambda_1, \dots, \lambda_n$ , i.e., we are not asserting  $n$  distinct eigenvalues here.

– Therefore, since each  $\lambda_i$  is, individually, a root of unity, we have that  $\chi_V(g) = \text{tr } A_g = \lambda_1 + \dots + \lambda_n$ , as desired.

4.  $\chi_{V^*} = \bar{\chi}_V$ .

– This property begins to address how characters behave under other operations.

■ Naturally, this is something specific for complex numbers, because the idea of “conjugates” doesn’t exist everywhere.

– Proof: Recall that  $\rho_{V^*}(g) = (\rho_V(g)^{-1})^T$ .

■ If we know that  $\rho_V(G) \sim \text{diag}(\lambda_1, \dots, \lambda_n)$ , then we know that  $\rho_V^{-1}(g)^T \sim \text{diag}(\lambda_1^{-1}, \dots, \lambda_n^{-1})$ .

■ Thus,  $\chi_{V^*}(g) = \lambda_1^{-1} + \dots + \lambda_n^{-1}$ .

■ But since we’re in the complex plane,  $|\lambda_i| = 1$  (equiv.  $\lambda_i \bar{\lambda}_i = 1$ ), so  $\lambda_i^{-1} = 1/\lambda_i = \bar{\lambda}_i$ .

■ This means that  $\chi_{V^*}(g) = \bar{\lambda}_1 + \dots + \bar{\lambda}_n = \overline{\lambda_1 + \dots + \lambda_n}$ .

– Note: Every representation we have is **unitary** in certain bases, but unitary representations are not covered in this course.

5.  $\chi_{V_1 \otimes V_2} = \chi_{V_1} \cdot \chi_{V_2}$ .

– Proof: We can use a basis or not use a basis.

– Let’s use a basis for now.

■ Let  $g \in G$  be arbitrary. Then there exist bases  $e_1, \dots, e_n$  of  $V_1$  and  $f_1, \dots, f_m$  of  $V_2$  such that  $\rho_{V_1}(g)$  and  $\rho_{V_2}(g)$  are diagonal.

■ It follows that  $\rho_{V_1}(g)e_i = \lambda_i e_i$  ( $i = 1, \dots, n$ ) and  $\rho_{V_2}(g)f_i = \mu_i f_i$  ( $i = 1, \dots, m$ ).

■  $V_1 \otimes V_2$  thus has basis  $e_i \otimes f_j$ .

■ But then it follows that  $\rho_{V_1 \otimes V_2}(g)e_i \otimes f_j = (\lambda_i e_i) \otimes (\mu_j f_j) = \lambda_i \mu_j (e_i \otimes f_j)$ .

■ Thus,

$$\text{tr}(\rho_{V_1 \otimes V_2}(g)) = \sum_{i,j=1}^{n,m} \lambda_i \mu_j = (\lambda_1 + \dots + \lambda_n)(\mu_1 + \dots + \mu_m) = \text{tr}(\rho_{V_1}(g)) \cdot \text{tr}(\rho_{V_2}(g))$$

– Alternate approach.

■ If we don’t want to think of eigenvalues, think of tensor product of matrices, the Kronecker product.

■ We get trace is the product of traces once again! *Write this out.*

- **Class function:** A function on a group  $G$  that is constant on the conjugacy classes of  $G$ .
- Examples.

1. Let  $A$  be an abelian group.
  - Then  $\chi : A \rightarrow \mathbb{C}^\times$ .
  - Implication: Character of a character is  $\chi_\chi = \chi$ .
    - This is horribly repetitive but true.
2.  $G = S_3$ .

	$e$	$\begin{pmatrix} (12) \\ (13) \\ (23) \end{pmatrix}$	$\begin{pmatrix} (123) \\ (132) \end{pmatrix}$
Trivial	1	1	1
Alternating	1	-1	1
Standard	2	0	-1

Table 3.1: Character table for  $S_3$ .

- The conjugacy classes of this group are  $\{e\}$ ,  $\{(12), (13), (23)\}$ , and  $\{(123), (132)\}$ .
- We construct a **character table** to define all characters.
- Computing the characters for the trivial representation.
  - We know that  $\rho$  sends each  $g$  to the matrix  $(1)$ , which has trace 1.
- Computing the characters for the sign representation.
  - $e$  and  $(123)$  have sign 1 and thus get sent to the matrix  $(1)$ .
  - $(12)$  has sign  $-1$  and thus gets sent to the matrix  $(-1)$ .
- Computing the characters for the standard representation.
  - We can compute these traces via a thought experiment.
  - Visualize a triangle in a plane.
  - The  $2 \times 2$  identity matrix (the standard representation of  $e \in G$ ) acts on it by doing nothing, and has trace 2.
  - In *some* basis, our matrix fixes one vector and inverts another, so matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and character is 0.

- Last one is rotation by  $2\pi/3$ , so

$$\begin{pmatrix} \cos(2\pi/3) & \sin(2\pi/3) \\ -\sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}$$

so character is  $-1 = 2 \cdot -1/2 = 2 \cdot \cos(2\pi/3)$ .

- If  $V$  is the standard representation, we can also compute the characters of  $V^{\otimes 2}$  for instance. Indeed, by the product rule of characters, they will be the squares of the standard representation's characters, i.e.,  $(4, 0, 1)$ .
  - Similarly, since the permutational representation is the direct sum of the standard and trivial representations, we can add their characters to get its characters  $(3, 1, 0)$ .
3. A very general and very pretty example. Let  $G \subset X$  a finite set.
    - Assign the permutational representation.

- Let  $X = \{x_1, \dots, x_n\}$ . Think of these elements as the basis of a vector space; in particular, consider  $V = \mathbb{C}e_{x_1} \oplus \dots \oplus \mathbb{C}e_{x_n}$ . Recall that  $g(a_1e_{x_1} + \dots + a_ne_{x_n}) = a_1e_{gx_1} + \dots + a_ne_{gx_n}$ . The fact that this is a representation follows immediately from the properties of the group action.
- Computing the character  $\chi_V$  of this  $V$ : Look at  $g$  and write its matrix. In particular, the trace is the number of unmoved/fixed elements, sometimes denoted  $\text{Fix}(g)$ .
- This gives us another way of computing  $V_{\text{perm}}$  from above!
- **Character table:** A table that lists the conjugacy classes across the top, the irreps down the left side, and at each point within it, the value of an irrep's character over that conjugacy class.
  - The character table is a very nice matrix with very nice properties.
  - It is almost orthogonal; not exactly, but very close.
    - Rows aren't orthogonal, but columns are (take direct products)!
    - It is full rank, though.
- The midterm: Take the character table and do fun things with it.

## 3.2 Office Hours

10/10:

- Problem 1b:
  - Canonically self-dual:  $V \cong V^*$  canonically.
- Mathematical methods of quantum mechanics: First few paragraphs of *picture*.
- We should have everything we need to do most of the problem set at this point; maybe not all of 5, but maybe yes, too.
- Problem 3:
  - There is some problem where it decomposes into trivial plus standard, but we still have to prove that standard is irreducible in this case!
  - If you have any vector, you can produce out of this vector something else.
  - If we take any vector and the group acts on it, we'll get a basis. If you hit a vector in the invariant subspace, it will just stay there; if you hit it and it goes everywhere, you get a basis.
  - Now think about a vector when you permute its coordinates.
  - Tomorrow in class, we will learn a quick way to do this problem.
- Problem 5:
  - For some problem, we need to use the fact that  $A^n = 1$  proves that  $A = I$  in some sense.
  - This is a hard problem!
  - Show that eigenvalues sum to 1; we know that the eigenvalues are roots of unity! Thus, they have to both be 1!
  - When the problem in group theory is harder, that's when you need to go to rep theory.

### 3.3 Characters are Orthonormal

10/11:

- Announcement: Zoom OH today.
- Recap: The big picture.
  - Representations.
    - We have representations, which are vector spaces on which a group acts.
    - With these representations, we can do a bunch of operations we've discussed:  $\oplus, \otimes, V^*, \Lambda^n, S^n$ .
    - We'll focus on the first 3 for now, though.
  - Class functions.
    - We also have class functions: Functions  $f : G \rightarrow \mathbb{C}$  such that for all  $g, x \in G$ ,  $f(gxg^{-1}) = f(x)$ .
    - The space of class functions forms a ring, since you can add, multiply, and take the complex conjugate of these functions.
    - Moreover, this ring is a vector space and it has dimension equal to the number of conjugacy classes of  $G$ .
  - The big idea: These two things (representations and class functions) are closely related!
    - There is a map, called a *character*, that pairs a representation to a class function.
    - Indeed,  $V \rightarrow \chi_V$ .
    - Under this map, operations of representations become operations of functions:

$$\oplus \mapsto + \qquad \qquad \qquad \otimes \mapsto \cdot \qquad \qquad \qquad V^* \mapsto \bar{f}$$

- Additionally,  $V_1, \dots, V_s$  become  $\chi_{V_1}, \dots, \chi_{V_s}$ .
- Theorem we will prove over the next couple of lectures: Irreps become *linearly independent* class functions, and all irreps form a basis of the space of class functions.
  - This theorem is huge! It is our main takeaway for now.
  - For the first part of the course, this is the main thing that we should remember.
- How do we prove that multiple vectors are linearly independent?
  - A strong condition would be to introduce an inner product and prove that the pairwise inner product of the vectors is zero.
- **Orthonormal basis:** A basis for which  $\langle e_i, e_j \rangle = \delta_{ij}$ .
- Let's begin carrying out this plan by defining an inner product on  $\mathbb{C}[G]$ . Indeed, let  $f_1, f_2$  be two functions on  $G$  and take

$$\langle f_1, f_2 \rangle := \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$$

- Motivation for this definition.
  - Recall the **Hermitian inner product** on  $\mathbb{C}^n$ .
    - We are essentially mapping  $f_1, f_2$  to  $(f_1(g_1), \dots, f_1(g_{|G|})), (f_2(g_1), \dots, f_2(g_{|G|})) \in \mathbb{C}^{|G|}$  and taking the Hermitian inner product there.
    - Thus, we can see that all properties hold for both the Hermitian inner product on  $\mathbb{C}^n$  and the one defined above on  $\mathbb{C}[G]$ .
    - In other words, this kind of construction should inherit its status as a linear, positive definite bilinear form from the Hermitian inner product.
  - Note: The Hermitian product above is **G-invariant**.
    - This means that the functions on  $G$  from  $G \rightarrow \mathbb{C}$  in  $\mathbb{C}[G]$  form a representation of  $G$ .

- In particular, if  $\varphi : G \rightarrow \mathbb{C}$ , then  $g = \rho(g)$  moves it as follows:  $g \cdot \varphi = \varphi^g$  where  $\varphi^g(h) := \varphi(g^{-1}h)$ . Thus, we have an action of  $G$  on every  $\varphi$ !
- Such representations are isomorphic for finite groups??
- If we have  $\langle f_1, f_2 \rangle$ , we can ask if

$$\langle f_1, f_2 \rangle \stackrel{?}{=} \langle f_1^g, f_2^g \rangle$$

- Left as an exercise that this *is* true!

- **Hermitian inner product** (on  $\mathbb{C}^n$ ): The inner product defined as follows for all  $z, w \in \mathbb{C}^n$ . Denoted by  $\langle \cdot, \cdot \rangle$ . Given by

$$\langle z, w \rangle = \sum_{i=1}^n z_i \bar{w}_i$$

- This inner product gives a complex number  $\langle v, w \rangle \in \mathbb{C}$  with the following properties.

1.  $\langle a_1 v_1 + a_2 v_2, w \rangle = a_1 \langle v_1, w \rangle + a_2 \langle v_2, w \rangle$ .
2.  $\langle v, b_1 w_1 + b_2 w_2 \rangle = \bar{b}_1 \langle v, w_1 \rangle + \bar{b}_2 \langle v, w_2 \rangle$ .
3.  $\langle v, v \rangle \geq 0$ , and  $\langle v, v \rangle = 0$  implies that  $v = 0$ .

- Thus, if  $v = (z_1, \dots, z_n)$  and  $w = (w_1, \dots, w_n)$ , then

$$\langle v, w \rangle = \sum z_i \bar{w}_i \qquad \langle v, v \rangle = \sum |z_i|^2$$

- We now begin tackling today's main theorem: If  $V_1, V_2$  are irreps, then

$$\langle \chi_{V_1}, \chi_{V_2} \rangle = \begin{cases} 0 & V_1 \not\cong V_2 \\ 1 & V_1 \cong V_2 \end{cases}$$

- We will prove this theorem in stages.
- The general outline of our approach is to deduce the equality step by step through the transitive property. Some of the equalities we'll eventually end up needing are easier to discuss on their own first, though, so we begin with some lemmas.

- First off, recall the **space of invariants** from PSet 2.

- **Space of invariants** (of a representation  $V$ ): The vector space defined as follows. Denoted by  $V^G$ . Given by

$$V^G = \{v \in V \mid gv = v \ \forall g \in G\}$$

- Lemma 1: Let  $G$  be a finite group, let  $\rho : G \rightarrow GL(V)$  be a finite-dimensional representation of it, and let  $p$  be defined as above. Then  $p \in \text{Hom}_G(V, V)$ .

*Proof.* We can view  $p$  as an element of  $\text{Hom}(V, V)$ . This combined with the fact that for every  $h \in G$ ,

$$p(hv) = \frac{1}{|G|} \sum_{g \in G} (gh)v = \frac{1}{|G|} \sum_{gh \in G} (gh)v = \frac{1}{|G|} h \sum_{g \in G} gv = h(pv)$$

implies that  $p \in \text{Hom}_G(V, V)$ . In more formal notation,

$$\begin{aligned} [p \circ \rho_V(h)](v) &= \frac{1}{|G|} \sum_{g \in G} [\rho_V(g) \circ \rho_V(h)](v) \\ &= \frac{1}{|G|} \sum_{g \in G} [\rho_V(gh)](v) \\ &= \frac{1}{|G|} \sum_{gh \in G} [\rho_V(gh)](v) \\ &= \frac{1}{|G|} \sum_{hg \in G} [\rho_V(hg)](v) \\ &= \frac{1}{|G|} \sum_{g \in G} [\rho_V(hg)](v) \\ &= [\rho_V(h)] \left( \frac{1}{|G|} \sum_{g \in G} [\rho_V(g)](v) \right) \\ &= [\rho_V(h) \circ p](v) \end{aligned}$$

□

- Why do we need this result?? What does it do for the rest of the proof?
- Lemma 2: Let  $G$  be a finite group, and let  $\rho : G \rightarrow GL(V)$  be a finite-dimensional representation of it. Then the map  $p$ , defined as follows, is a projector from  $V \rightarrow V^G$ .

$$p = \frac{1}{|G|} \sum_{g \in G} g = \frac{1}{|G|} \sum_{g \in G} \rho_V(g)$$

*Proof.* To prove that  $p$  is a projector, it will suffice to show that  $p^2 = p$ . To prove that  $p$  projects onto  $V^G$ , it will suffice to show that  $\text{Im}(p) = V^G$ . Let's begin.

To show that  $p^2 = p$ , we have

$$p^2 = \left( \frac{1}{|G|} \sum_{g \in G} g \right)^2 = \frac{1}{|G|^2} \sum_{g_1, g_2 \in G} g_1 g_2 = \frac{|G|}{|G|^2} \sum_{g \in G} g = p$$

Note that since  $G$  is not abelian (i.e.,  $g_1 g_2 \neq g_2 g_1$  in all cases), the square of  $\sum g$  is as above and cannot be reduced to a smaller sum with a 2 coefficient or something like that. Additionally, note that  $\sum_{g_1, g_2 \in G} g_1 g_2 = |G| \sum g$  since for each  $g_i$ ,  $g_i(g_1 + \dots + g_{|G|}) = g_1 + \dots + g_{|G|}$ .

To show that  $\text{Im}(p) = V^G$ , we will use a bidirectional inclusion proof. To confirm that  $\text{Im}(p) \subset V^G$ , we have for any  $h \in G$  that

$$h \left( \frac{1}{|G|} \sum_{g \in G} gv \right) = \frac{1}{|G|} \sum_{hg \in G} hgv = \frac{1}{|G|} \sum_{g \in G} gv$$

from which it follows that

$$p(v) = \frac{1}{|G|} \sum_{g \in G} gv \in V^G$$

as desired. To confirm that  $V^G \subset \text{Im}(p)$ , let  $v \in V^G$ . Then  $gv = v$ . It follows that

$$v = \frac{1}{|G|} \sum_{g \in G} v = \frac{1}{|G|} \sum_{g \in G} gv = p(v) \in \text{Im}(p)$$

as desired.

□



- You differentiated the first and second parts of the above proof by saying, “this is the algebraic way to prove it; we can also prove it nonalgebraically.” Does this mean that  $p^2 = p$  somehow *implies*  $\text{Im}(p) = V^G$  here, or do we still need to prove that “nonalgebraically,” as in Fulton and Harris (2004)??
- Consequence of Lemma 2: There’s a very easy way to construct invariant factors.
- We now prove one final lemma using what we have learned about  $p$ .
- Lemma 3: Let  $G$  be a finite group, and let  $\rho : G \rightarrow GL(V)$  be a finite-dimensional representation of it. Then  $\dim V^G = (1/|G|) \sum_{g \in G} \chi_V(g)$ .

*Proof.* Define  $p$  as above. Then

$$\begin{aligned}
 \dim V^G &= \dim(\text{Im}(p)) && \text{Lemma 2} \\
 &= \text{tr}(p) && \text{PSet 1, Q5c} \\
 &= \text{tr}\left(\frac{1}{|G|} \sum_{g \in G} \rho_V(g)\right) \\
 &= \frac{1}{|G|} \sum_{g \in G} \text{tr}(\rho_V(g)) \\
 &= \frac{1}{|G|} \sum_{g \in G} \chi_V(g)
 \end{aligned}$$

as desired. □

- We can now prove the main result.
- Theorem: If  $V, W$  are irreps, then

$$\langle \chi_V, \chi_W \rangle = \begin{cases} 0 & V \not\cong W \\ 1 & V \cong W \end{cases}$$

*Proof.* We will work towards a formula for the inner product, using various results that we’ve proven up until now. Let’s begin.

$$\begin{aligned}
 \langle \chi_V, \chi_W \rangle &= \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \cdot \overline{\chi_W(g)} && \text{Definition} \\
 &= \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \cdot \chi_{W^*}(g) && \text{Property 4} \\
 &= \frac{1}{|G|} \sum_{g \in G} \chi_{V \otimes W^*}(g) && \text{Property 5} \\
 &= \dim[(V \otimes W^*)^G] && \text{Lemma 3} \\
 &= \dim([\text{Hom}_F(V, W)]^G) && \text{Lecture 2.1} \\
 &= \dim[\text{Hom}_G(V, W)] && \text{PSet 2, Q4b} \\
 &= \begin{cases} \dim(\text{span}(I)) & V \cong W \\ \dim(\text{span}(0)) & V \not\cong W \end{cases} && \text{Schur's Lemma} \\
 &= \begin{cases} 0 & V \not\cong W \\ 1 & V \cong W \end{cases}
 \end{aligned}$$

□

- In the above proof, Rudenko first surveys the following special case. Why??

- Then if  $V$  is irreducible and trivial, we have

$$\frac{1}{|G|} \sum_{g \in G} \chi_V(g) = 0$$

which happens iff

$$\langle \chi_V, \chi_{\text{triv}} \rangle = 0$$

whereas

$$\langle \chi_{\text{triv}}, \chi_{\text{triv}} \rangle = 1$$

This proves the theorem in a special case, but how do we go from here to all representations?  
We're very close!

- Corollary: The number of irreps is less than or equal to the number of conjugacy classes.
  - We'll leave it to next time to prove that equality holds.
- Whenever we have a sec, we should try to form a mental picture the whole class function thing.
- Consequence of the theorem: We get an orthogonality relation.
  - If  $\chi_1, \chi_2$  are characters of irreps, then

$$\sum_{g \in G} \chi_1(g) \overline{\chi_2(g)} = \begin{cases} 0 & \chi_1 \neq \chi_2 \\ |G| & \chi_1 = \chi_2 \end{cases}$$

- This is related to the character table and IChem!!! Example:

- Recall Table 3.1, the character table for  $S_3$ .
- Between the trivial and alternating representations, we have

$$(1)(1) + (1)(-1) + (1)(-1) + (1)(-1) + (1)(1) + (1)(1) = 0$$

as expected. Note that we have a term for each element in  $S_3$ , so some products get repeated multiple times.

- For the standard representation, we have

$$(2)(2) + (0)(0) + (0)(0) + (0)(0) + (-1)(-1) + (-1)(-1) = 6 = |S_3|$$

as expected.

- Theorem: Characters are equal iff their representations are isomorphic.
- Next time.
  - Prove the theorem.
  - Consequences.
  - Implications for the character table.

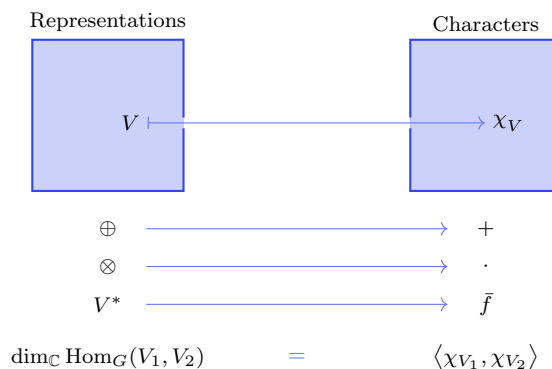


Figure 3.1: The stories of representation theory.

### 3.4 Character Table Properties

10/13:

- Announcement: Midterm on November 10.
  - Will mostly involve computing character tables; HW will be good prep.
- Review of the general picture from the first part of the course.
  - Let  $G$  be a finite group.
  - First story: We study finite-dimensional representations of  $G$  over  $\mathbb{C}$ ; these are vector spaces, so we can direct sum, tensor multiply, and dualize them. We can also look at the morphisms between them.
  - Second story: We study class functions  $\mathbb{C}_{\text{cl}}[G] = \{f : G \rightarrow \mathbb{C} \mid f(gxg^{-1}) = f(x)\}$ ; these are elements of a ring, so we can add, multiply, and conjugate them. We can also take the inner product of them.
  - We can map between these two stories: Representations become characters,  $\oplus \mapsto +$ ,  $\otimes \mapsto \cdot$ , and  $V^* \mapsto \bar{f}$ .
- Theorem (from last time):
 
$$\langle \chi_{V_1}, \chi_{V_2} \rangle = \dim \operatorname{Hom}_G(V_1, V_2)$$
- The story in Figure 3.1 tells us stuff about representations.
  - Let  $V_1, \dots, V_k$  be irreps. Then the vectors  $\chi_{V_1}, \dots, \chi_{V_k}$  are orthonormal.
    - We get this result with the Theorem above and Schur's Lemma.
  - Next time, we'll prove that  $\chi_{V_1}, \dots, \chi_{V_k}$  spans  $\mathbb{C}_{\text{cl}}[G]$ , i.e., the number of irreps is the number of conjugacy classes.
  - *Cube thing??*
  - This picture is remarkable because it's so simple.
- We now look at some corollaries to last time's main theorem.
- Corollary 1: If  $V, W$  are  $G$ -reps, then  $\chi_V = \chi_W$  iff  $V \cong W$ .

*Proof.* Invoking complete reducibility, we have that  $V = \bigoplus V_i^{n_i}$ . Thus, to know  $V$ , it is enough to know the  $n_i$ 's. But

$$\chi_V = \sum n_i \chi_{V_i}$$

where

$$n_i = n_i \cdot 1 = n_i \langle \chi_{V_i}, \chi_{V_i} \rangle = \langle \chi_V, \chi_{V_i} \rangle$$

Therefore, since the  $\chi_{V_i}$  are linearly independent, the only way that  $\chi_V = \chi_W$  is if the  $n_i$ 's match which would mean that  $V \cong W$ , and vice versa the only way that  $V \cong W$  is if the  $n_i$ 's match which would mean that  $\chi_V = \chi_W$ .  $\square$

- Corollary 2: Let  $V$  be a  $G$ -rep. Then TFAE:

1.  $V$  is irreducible.
2.  $\langle \chi_V, \chi_V \rangle = 1$ .
3.  $\sum_{g \in G} |\chi_V(g)|^2 = |G|$ .

*Proof.*  $(1 \Rightarrow 2)$ : We have that

$$\langle \chi_V, \chi_V \rangle = \dim \operatorname{Hom}_G(V, V) = 1$$

as desired.

$(2 \Rightarrow 1)$ : Complete reducibility implies that  $V \cong V_1^{n_1} \oplus \cdots \oplus V_k^{n_k}$ , where the  $V_i$ 's are irreps. This combined with the hypothesis implies that

$$1 = \langle \chi_V, \chi_V \rangle = \left\langle \sum_{i=1}^k n_i \chi_{V_i}, \sum_{i=1}^k n_i \chi_{V_i} \right\rangle = \sum_{i=1}^k n_i^2$$

But if  $\sum n_i^2 = 1$  where each  $n_i \in \mathbb{Z}^+$ , then  $n_i = 1$  for some  $i$  and  $n_j = 0$  for  $j \neq i$ , from which it follows that  $V \cong V_i$ .

We can interconvert between 2 and 3 using the definition of the inner product and the property of complex numbers that  $zz^* = |z|^2$ .  $\square$

- We now build up to one final corollary.
- We've discussed all of these properties of irreps, but where do we even find them?
  - We might be able to find some by inspection, but here's how we find all of them.
- Review: The regular representation. Here are two different but isomorphic ways to think about it.
  - Think of it as functions on  $G$ .
    - Better for infinite groups.
  - Think of it as the permutational representation associated with the action  $G \curvearrowright G$ .
    - Better for finite groups.
  - Why did we talk about this here??
- Corollary 3: Consider the regular representation  $V_R$ . We have that

$$\chi_{V_R}(g) = \begin{cases} 0 & g \neq e \\ |G| & g = e \end{cases}$$

*Proof.* We can compute its character  $\chi_{V_R}$  by considering the corresponding permutation matrices. Indeed, the action  $\chi_{V_R}(g)$  of this character on  $g$  is equal to the number of 1's on the diagonal in the permutation matrix, which is equal to the number of fixed points of the permutation, i.e., the number of  $i$ 's such that  $gi = i$ . But in a group,  $gi = i$  iff  $g = e$ , so this number of fixed points is

$$\chi_{V_R}(g) = \operatorname{Fix}(g) = \begin{pmatrix} g_1 & \cdots & g_n \\ gg_1 & \cdots & gg_n \end{pmatrix} = \begin{cases} 0 & g \neq e \\ |G| & g = e \end{cases}$$

as desired.  $\square$

- What is the matrix thing??
- We now apply Corollaries 1-3 to the regular representation  $V_R$  to obtain some important results.
  - Let  $V_i$  be an arbitrary irrep.
  - By complete reducibility,  $V_R = \bigoplus_{i=1}^k V_i^{n_i}$  for some set of  $n_i$ 's.
  - Additionally,

$$n_i = \langle \chi_{V_R}, \chi_{V_i} \rangle \quad \text{Corollary 1}$$

$$= \frac{1}{|G|} \sum_{g \in G} \chi_{V_R}(g) \overline{\chi_{V_i}(g)}$$

$$= \frac{1}{|G|} |G| \underbrace{\overline{\chi_{V_i}(e)}}_{\dim V_i} \quad \text{Corollary 3}$$

$$= \dim V_i$$

- This implies three remarkable results, all worth remembering.

$$V_R = \bigoplus_{i=1}^k V_i^{\dim V_i} \quad |G| = \sum_{i=1}^k (\dim V_i)^2 \quad \# \text{ irreps is finite}$$

- The first result follows directly by substituting  $n_i = \dim V_i$  into complete reducibility.
- The second result follows because  $|G| = \dim(V_R) = \dim(\bigoplus_{i=1}^k V_i^{\dim V_i}) = \sum (\dim V_i)^2$ .
- The third result follows because if there were infinitely many irreps, each with  $\dim V_i \geq 1$ , then  $|G| = \sum_{i=1}^k (\dim V_i)^2 = \infty$ , contradicting the hypothesis that  $|G|$  is finite.
- We want to investigate  $S_4$ , i.e., characterize all irreps of it.

	1	6	8	3	6
	$e$	(12)	(123)	(12)(34)	(1234)
Trivial	1	1	1	1	1
Sign	1	-1	1	1	-1
$V_{\text{std}}$	3	1	0	-1	-1
$\text{Sign} \otimes V_{\text{std}}$	3	-1	0	-1	1
	2	0	-1	2	0

Table 3.2: Character table for  $S_4$ .

- We do so by constructing the character table, Table 3.2.
- Initially, this seemed like a very hard problem.
  - However, with all of our theory, it only takes a couple of minutes now!
- We start by inputting the trivial, sign, and standard representations.
  - The trivial is obviously  $(1, 1, 1, 1, 1)$ .
  - The sign can be calculated to be  $(1, -1, 1, 1, -1)$ .
  - The standard is found via  $V_{\text{std}} = V_{\text{perm}} - V_{\text{triv}} = (4, 2, 1, 0, 0) - (1, 1, 1, 1, 1) = (3, 1, 0, -1, -1)$ .
- Note that

$$\sum n_i^2 = \langle \chi_{V_{\text{perm}}}, \chi_{V_{\text{perm}}} \rangle = \frac{1}{24} (1 \cdot 4^2 + 6 \cdot 2^2 + 8 \cdot 1^2 + 0 \cdot 0^2 + 0 \cdot 0^2) = 2$$

$$\langle \chi_{V_{\text{perm}}}, \chi_{V_{\text{sign}}} \rangle = \frac{1}{24} (4 - 12 + 8) = 0$$

$$\langle \chi_{V_{\text{perm}}}, \chi_{V_{\text{triv}}} \rangle = 1$$

- What is the point of these calculations??
- Thus, we can derive representations without having any geometric notion of it using characters!
- To see any of these representations geometrically, look at actions on the tetrahedron in  $\mathbb{R}^3$ !
- All those computations above used the **first orthogonality relation**; here's the **second orthogonality relation**:

$$\sum_{\chi} \chi(g_1) \overline{\chi(g_2)} = \begin{cases} 0 & g_1 \not\sim g_2 \\ \frac{|G|}{|C_G(g)|} & g_1 \sim g_2 \end{cases}$$

where  $\sim$  denotes conjugacy.

- Prove the new one with  $AB = 1 = BA$ . This very simple thing leads to a very powerful statement about systems of equations that we will discuss later. How does this proof work??
- We can start doing this stuff in the new homework!
- We constructed the fifth representation using this.
- Where does the fifth irrep come from?
  - Going back to a miracle of group theory: Simple groups.
  - If  $f : S_n \rightarrow S_n$ , we have lots of injective maps, lots of minor actions  $S_n \rightarrow S_n$  sending  $x \mapsto gxg^{-1}$ .
  - We have  $\text{sign} : S_n \rightarrow S_2$ ,  $S_4 \twoheadrightarrow S_3$  with kernel equal to  $K_4$ .
  - We have the exotic  $S_6 \rightarrow S_6$ .
  - We have  $S_5 \hookrightarrow S_6$  that is also exotic.
  - These are called **exceptional homomorphisms**.
  - Since we have  $S_4 \rightarrow S_3$  and  $\rho : S_3 \rightarrow GL_n$ , we have  $S_4 \rightarrow GL_n$  with a the same character table.  
Takeaway: This  $(2, 0, 1)$  thing in the big character table comes from this map, geometrically.
  - Takeaway: The geometry of the fifth irrep comes from  $S_3$ .
  - What is going on here??
- Final announcements.
  - Going forward, we'll mostly be following Fulton and Harris (2004).
  - Then we'll get into associative algebra.
  - OH next week will be Zoom, but we should still feel free to meet with him in-person by emailing him for an appointment.