MATH 26700 (Introduction to Representation Theory of Finite Groups) Problem Sets

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Contents

1 Applications of Linear Algebra to Representation Theory

10/6:

1. Read Section 1.3 in Fulton and Harris (2004).

2. Let V, W be finite-dimensional vector spaces. Construct canonical isomorphisms...

   (a) $\Lambda^2(V \oplus W) \cong (\Lambda^2 V) \oplus (V \otimes W) \oplus (\Lambda^2 W)$;

   Proof. Define the map $\tilde{f}$ on the subset of $\Lambda^2(V \oplus W)$ containing all decomposable antisymmetric tensors by the rule

   $$(v_1, w_1) \wedge (v_2, w_2) \mapsto (v_1 \wedge v_2, v_1 \otimes w_2 - v_2 \otimes w_1, w_1 \wedge w_2)$$

   We now have to prove that this map is bilinear.
   To prove linearity in the first argument, we have

   $$\begin{aligned}\tilde{f}[((v_1, w_1) + (v_1', w_1')) \wedge (v_2, w_2)] &= \tilde{f}[(v_1 + v_1', w_1 + w_1') \wedge (v_2, w_2)] \\ &= ((v_1 + v_1') \wedge v_2, \\ &\qquad (v_1 + v_1') \otimes w_2 - v_2 \otimes (w_1 + w_1'), \\ &\qquad (w_1 + w_1') \wedge w_2) \\ &= (v_1 \wedge v_2 + v_1' \wedge v_2, \\ &\qquad v_1 \otimes w_2 + v_1' \otimes w_2 - v_2 \otimes w_1 - v_2 \otimes w_1', \\ &\qquad w_1 \wedge w_2 + w_1' \wedge w_2) \\ &= (v_1 \wedge v_2, v_1 \otimes w_2 - v_2 \otimes w_1, w_1 \wedge w_2) \\ &\quad + (v_1' \wedge v_2, v_1' \otimes w_2 - v_2 \otimes w_1', w_1' \wedge w_2) \\ &= \tilde{f}[(v_1, w_1) \wedge (v_2, w_2)] + f[(v_1', w_1') \wedge (v_2, w_2)]\end{aligned}$$

   and

   $$\begin{aligned}\tilde{f}[(\lambda(v_1, w_1)) \wedge (v_2, w_2)] &= \tilde{f}[(\lambda v_1, \lambda w_1) \wedge (v_2, w_2)] \\ &= (\lambda v_1 \wedge v_2, \lambda v_1 \otimes w_2 - v_2 \otimes \lambda w_1, \lambda w_1 \wedge w_2) \\ &= (\lambda(v_1 \wedge v_2), \lambda(v_1 \otimes w_2 - v_2 \otimes w_1), \lambda(w_1 \wedge w_2)) \\ &= \lambda(v_1 \wedge v_2, v_1 \otimes w_2 - v_2 \otimes w_1, w_1 \wedge w_2) \\ &= \lambda \tilde{f}[(v_1, w_1) \wedge (v_2, w_2)]\end{aligned}$$

   The proof is symmetric in the second argument.
   To prove that f respects the antisymmetry of the wedge product, it will suffice to show that

   $$f[(v_1, w_1) \wedge (v_2, w_2)] = -f[(v_2, w_2) \wedge (v_1, w_1)]$$

   in general. Let $(v_1, w_1) \wedge (v_2, w_2)$ be an arbitrary decomposable element of $\Lambda^2(V \oplus W)$. Then we have that

   $$\begin{aligned}f[(v_1, w_1) \wedge (v_2, w_2)] &= (v_1 \wedge v_2, v_1 \otimes w_2 - v_2 \otimes w_1, w_1 \wedge w_2) \\ &= (-v_2 \wedge v_1, -(v_2 \otimes w_1 - v_1 \otimes w_2), -w_2 \wedge w_1) \\ &= -(v_2 \wedge v_1, v_2 \otimes w_1 - v_1 \otimes w_2, w_2 \wedge w_1) \\ &= -f[(v_2, w_2) \wedge (v_1, w_1)]\end{aligned}$$

   as desired.

   Since $\tilde{f}$ an alternating bilinear map, the universal property of the exterior powers[1] implies that there exists a map $f : \Lambda^2(V \oplus W) \to (\Lambda^2 V) \oplus (V \otimes W) \oplus (\Lambda^2 W)$.

[1] See Fulton and Harris (2004, p. 472).

Since the domain and codomain have the same dimension, to prove that f is an isomorphism, it will suffice to prove that it's surjective. Let

$$\left(\sum_{i_1,i_2=1}^{r} v_{i_1}\wedge v_{i_2}, \sum_{i_3=1}^{s} v_{i_3}\otimes w_{i_3}, \sum_{i_1,i_2=1}^{t} w_{i_1}\wedge w_{i_2}\right)$$

be an arbitrary element of $(\Lambda^2 V)\oplus(V\otimes W)\oplus(\Lambda^2 W)$. Decompose it into a sum of elements of the form $(a,0,0)$, $(0,a,0)$, or $(0,0,a)$. Explicitly, the above element is equal to

$$\sum_{i_1,i_2=1}^{r}(v_{i_1}\wedge v_{i_2},0,0)+\sum_{i_3=1}^{s}(0,v_{i_3}\otimes w_{i_3},0)+\sum_{i_1,i_2=1}^{t}(0,0,w_{i_1}\wedge w_{i_2})$$

Now, by the definition of f, we know that each of the individual terms in the three sums above satisfy one of

$$\begin{aligned}(v_1\wedge v_2,0,0)&=f[(v_1,0)\wedge(v_2,0)]\\(0,v\otimes w,0)&=f[(v,0)\wedge(0,w)]\\(0,0,w_1\wedge w_2)&=f[(0,w_1)\wedge(0,w_2)]\end{aligned}$$

Therefore, we have that the initial arbitrary element of $(\Lambda^2 V)\oplus(V\otimes W)\oplus(\Lambda^2 W)$ is equal to

$$f\left[\sum_{i_1,i_2=1}^{r}(v_{i_1},0)\wedge(v_{i_2},0)+\sum_{i_3=1}^{s}(v_{i_3},0)\wedge(0,w_{i_3})+\sum_{i_1,i_2=1}^{t}(0,w_{i_1})\wedge(0,w_{i_2})\right]$$

as desired. $\square$

(b) $S^2(V\oplus W)\cong(S^2V)\oplus(V\otimes W)\oplus(S^2W)$.

Proof. Define the map $\tilde{f}$ on the subset of $S^2(V\oplus W)$ containing all decomposable symmetric tensors by the rule

$$(v_1,w_1)\cdot(v_2,w_2)\mapsto(v_1\cdot v_2, v_1\otimes w_2+v_2\otimes w_1, w_1\cdot w_2)$$

The rest of the proof is symmetric to that of part (a). $\square$

3. (a) Factorize the group determinant for $G=\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$.

Proof. The multiplication table for G is

	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,1)	(1,0)	(1,1)
(0,1)	(0,1)	(0,0)	(1,1)	(1,0)
(1,0)	(1,0)	(1,1)	(0,0)	(0,1)
(1,1)	(1,1)	(1,0)	(0,1)	(0,0)

Thus, the group determinant is

$$\begin{vmatrix}a&b&c&d\\b&a&d&c\\c&d&a&b\\d&c&b&a\end{vmatrix}$$

To factorize this group determinant, it will suffice to find the eigenvalues of the matrix it encloses. To find the eigenvalues, we may start by inspecting it for eigenvectors.

First off, recall the Sudoku Lemma from MATH 25700. It implies that every row of the matrix will list each element once, and we can confirm that this is true in this example by looking at it.

It follows that if we propose $(1,1,1,1)$ as an eigenvector, we'll be able to extract an eigenvalue via the commutativity of multiplication as follows.

$$\begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b+c+d \\ b+a+d+c \\ c+d+a+b \\ d+c+b+a \end{bmatrix} = \begin{bmatrix} a+b+c+d \\ a+b+c+d \\ a+b+c+d \\ a+b+c+d \end{bmatrix} = (a+b+c+d) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Second, we may observe that the upper-left and lower-right blocks of this matrix match, as do the lower-left and upper-right blocks. Indeed, this matrix is a block matrix of the form $\left[\begin{smallmatrix} A & C \\ C & A \end{smallmatrix}\right]$. Thus, since the eigenvector of this block matrix that is not $(1,1)$ is $(1,-1)$, the analogous relevant eigenvector of the full matrix is $(1,1,-1,-1)$. Once again, we obtain

$$\begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} a+b-c-d \\ b+a-d-c \\ c+d-a-b \\ d+c-b-a \end{bmatrix} = \begin{bmatrix} (a+b-c-d)\cdot 1 \\ (a+b-c-d)\cdot 1 \\ (a+b-c-d)\cdot -1 \\ (a+b-c-d)\cdot -1 \end{bmatrix} = (a+b-c-d) \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

Third, we may observe that each of the blocks referred to above is also of the form $\left[\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right]$. Thus, we can also apply $(1,-1)$ twice — once to each block — with the eigenvector $(1,-1,1,-1)$. Once again, we obtain

$$\begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a-b+c-d \\ b-a+d-c \\ c-d+a-b \\ d-c+b-a \end{bmatrix} = \begin{bmatrix} (a-b+c-d)\cdot 1 \\ (a-b+c-d)\cdot -1 \\ (a-b+c-d)\cdot 1 \\ (a-b+c-d)\cdot -1 \end{bmatrix} = (a-b+c-d) \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

Lastly, we may observe while the two side columns have a or d in the top and bottom slots and b or c in the middle two slots, it is vice versa for the two middle columns. Thus, we can apply $(1,-1,-1,1)$ to obtain

$$\begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} a-b-c+d \\ b-a-d+c \\ c-d-a+b \\ d-c-b+a \end{bmatrix} = \begin{bmatrix} (a-b-c+d)\cdot 1 \\ (a-b-c+d)\cdot -1 \\ (a-b-c+d)\cdot -1 \\ (a-b-c+d)\cdot 1 \end{bmatrix} = (a-b+c-d) \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Thus, we have found four distinct eigenvectors for a 4×4 matrix. Therefore, we have found all of the eigenvalues. Their product equals the determinant, and is also a factorization of the determinant. In particular, the factorized group determinant for K_4 is

$$\boxed{(a+b+c+d)(a+b-c-d)(a-b+c-d)(a-b-c+d)}$$

$\square$

(b) A **circulant matrix** is a square matrix in which all row vectors are composed of the same elements and each row vector is rotated one element to the right relative to the preceding row vector. Prove that for $\zeta \in \mu_n$, vector $(1, \zeta, \dots, \zeta^{n-1})$ is an eigenvector of any circulant matrix of size n.

Proof. Let

$$\begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_n & x_1 & x_2 & \cdots & x_{n-1} \\ x_{n-1} & x_n & x_1 & \cdots & x_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_2 & x_3 & x_4 & \cdots & x_1 \end{bmatrix}$$

be an arbitrary circulant matrix of size n.

It follows that

$$\begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_n & x_1 & x_2 & \cdots & x_{n-1} \\ x_{n-1} & x_n & x_1 & \cdots & x_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_2 & x_3 & x_4 & \cdots & x_1 \end{bmatrix} \begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \\ \vdots \\ \zeta^{n-1} \end{bmatrix} = \begin{bmatrix} x_1 + x_2\zeta + x_3\zeta^2 + \cdots + x_n\zeta^{n-1} \\ x_n + x_1\zeta + x_2\zeta^2 + \cdots + x_{n-1}\zeta^{n-1} \\ x_{n-1} + x_n\zeta + x_1\zeta^2 + \cdots + x_{n-2}\zeta^{n-1} \\ \vdots \\ x_2 + x_3\zeta + x_4\zeta^2 + \cdots + x_1\zeta^{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} x_1\zeta^0 + x_2\zeta^1 + x_3\zeta^2 + \cdots + x_n\zeta^{n-1} \\ x_n\zeta^n + x_1\zeta^1 + x_2\zeta^2 + \cdots + x_{n-1}\zeta^{n-1} \\ x_{n-1}\zeta^n + x_n\zeta^{n+1} + x_1\zeta^2 + \cdots + x_{n-2}\zeta^{n-1} \\ \vdots \\ x_2\zeta^n + x_3\zeta^{n+1} + x_4\zeta^{n+2} + \cdots + x_1\zeta^{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} x_1\zeta^0 + x_2\zeta^1 + x_3\zeta^2 + \cdots + x_n\zeta^{n-1} \\ x_1\zeta^1 + x_2\zeta^2 + x_3\zeta^3 + \cdots + x_n\zeta^n \\ x_1\zeta^2 + x_2\zeta^3 + x_3\zeta^4 + \cdots + x_n\zeta^{n+1} \\ \vdots \\ x_1\zeta^{n-1} + x_2\zeta^n + x_3\zeta^{n+1} + \cdots + x_n\zeta^{2n-2} \end{bmatrix}$$

$$= \left(\sum_{i=1}^n x_i\zeta^{i-1}\right) \begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \\ \vdots \\ \zeta^{n-1} \end{bmatrix}$$

as desired. ☐

(c) Compute the eigenvalues and the determinant of a circulant matrix. Factorize the group determinant for $G = \mathbb{Z}/n\mathbb{Z}$.

Proof. By the same logic used in part (a), one can find by inspection that the n distinct eigenvectors are of the form $(\zeta^{j(0)}, \dots, \zeta^{j(n-1)})$, $j = 0, \dots, n-1$. It follows by similar logic once again that the corresponding eigenvalues are of the form

$$\boxed{\lambda_j = \sum_{i=1}^n x_i\zeta^{j(i-1)},\ j = 0, \dots, n-1}$$

Moreover, since the determinant is the product of the eigenvalues,

$$\boxed{\det = \lambda_0 \cdots \lambda_{n-1}}$$

where λ_j is defined as above.

Therefore, since the multiplication table of $\mathbb{Z}/n\mathbb{Z}$ converts into an $n \times n$ circulant matrix, the factorization of its group determinant is $\boxed{\text{equal to the above determinant expression.}}$ ☐

4. **Plethysm for S_3.** Let (3), $(1,1,1)$, and $(2,1)$ be the trivial, alternating, and standard representations of S_3.

(a) Consider the permutational representation $V \cong (3) \oplus (2,1)$. Decompose $\Lambda^2 V$ into irreducibles.

Proof. Fix a basis of V equal to $\{(1,1,1), (\omega,1,\omega^2), (1,\omega,\omega^2)\} = \{\gamma, \alpha, \beta\}$. Let $\tau = (123) \in S_3$ and $\sigma = (12) \in S_3$. By Problem 2a,

$$\Lambda^2 V = \Lambda^2((3) \oplus (2,1)) = [\Lambda^2(3)] \oplus [(3) \otimes (2,1)] \oplus [\Lambda^2(2,1)]$$

We now divide into three cases.

Case 1 ($\Lambda^2(3)$): The basis for this vector space is $\{\gamma\wedge\gamma\}=\{0\}$, so

$$\Lambda^2(3)=0$$

Case 2 ($(3)\otimes(2,1)$): The basis for this vector space is $\{\gamma\otimes\alpha,\gamma\otimes\beta\}$. The action of τ on these basis vectors can be computed:

$$\tau(\gamma\otimes\alpha)=\omega\gamma\otimes\alpha \qquad\qquad \tau(\gamma\otimes\beta)=\omega^2\gamma\otimes\beta$$

These equations are directly analogous to the untensored α and β equations, so this is the standard representation. Formally,

$$(3)\otimes(2,1)=(2,1)$$

Case 3 ($\Lambda^2(2,1)$): The basis for this vector space is $\{\alpha\wedge\beta\}$. Thus, $\dim\Lambda^2(2,1)=1$, so $\Lambda^2(2,1)\neq(2,1)$, and we can move on to discriminating between the trivial and alternating representations. In particular, the action of σ on this basis vector is

$$\sigma(\alpha\wedge\beta)=-\alpha\wedge\beta$$

Thus, $\sigma(v)$ is not linearly independent of v. Moreover, $\sigma(v)=-v$. Thus, this is the alternating representation. Formally,

$$\Lambda^2(2,1)=(1,1,1)$$

Putting everything back together, we obtain

$$\begin{aligned}\Lambda^2V&\cong[\Lambda^2(3)]\oplus[(3)\otimes(2,1)]\oplus[\Lambda^2(2,1)]\\&\cong 0\oplus(2,1)\oplus(1,1,1)\\ &\boxed{\Lambda^2V\cong(2,1)\oplus(1,1,1)}\end{aligned}$$

□

(b) Decompose $S^2(2,1)$ and $S^3(2,1)$ into irreducibles.

Proof. We will treat each case separately.

$\underline{S^2(2,1)}$: The basis for this vector space is $\{\alpha\cdot\alpha,\alpha\cdot\beta,\beta\cdot\beta\}$. The action of τ on these basis vectors can be computed:

$$\tau(\alpha\cdot\alpha)=\omega^2\alpha\cdot\alpha \qquad \tau(\alpha\cdot\beta)=1\alpha\cdot\beta \qquad \tau(\beta\cdot\beta)=\omega\beta\cdot\beta$$

The action of σ on these basis vectors can also be computed:

$$\sigma(\alpha\cdot\alpha)=\beta\cdot\beta \qquad \sigma(\alpha\cdot\beta)=\alpha\cdot\beta \qquad \sigma(\beta\cdot\beta)=\alpha\cdot\alpha$$

Thus, $\alpha\cdot\alpha$ and $\beta\cdot\beta$ span a standard representation, and $\alpha\cdot\beta$ spans a trivial representation. Putting everything together, we obtain

$$\boxed{S^2(2,1)\cong(2,1)\oplus(3)}$$

$\underline{S^3(2,1)}$: The basis for this vector space is $\{\alpha\cdot\alpha\cdot\alpha,\alpha\cdot\alpha\cdot\beta,\alpha\cdot\beta\cdot\beta,\beta\cdot\beta\cdot\beta\}$. The action of τ on these basis vectors can be computed:

$$\tau(\alpha\cdot\alpha\cdot\alpha)=1\alpha\cdot\alpha\cdot\alpha \qquad\qquad \tau(\alpha\cdot\alpha\cdot\beta)=\omega\alpha\cdot\alpha\cdot\beta$$

$$\tau(\alpha\cdot\beta\cdot\beta)=\omega^2\alpha\cdot\beta\cdot\beta \qquad\qquad \tau(\beta\cdot\beta\cdot\beta)=1\beta\cdot\beta\cdot\beta$$

The action of σ on these basis vectors can also be computed:

$$\sigma(\alpha\cdot\alpha\cdot\alpha)=\beta\cdot\beta\cdot\beta \qquad \sigma(\alpha\cdot\alpha\cdot\beta)=\alpha\cdot\beta\cdot\beta$$

$$\sigma(\alpha\cdot\beta\cdot\beta)=\alpha\cdot\alpha\cdot\beta \qquad \sigma(\beta\cdot\beta\cdot\beta)=\alpha\cdot\alpha\cdot\alpha$$

Thus, $\alpha\cdot\alpha\cdot\beta$ and $\alpha\cdot\beta\cdot\beta$ span a standard representation, $\alpha\cdot\alpha\cdot\alpha+\beta\cdot\beta\cdot\beta$ spans a trivial representation, and $\alpha\cdot\alpha\cdot\alpha-\beta\cdot\beta\cdot\beta$ spans an alternating representation. Putting everything together, we obtain

$$\boxed{S^3(2,1)\cong(2,1)\oplus(3)\oplus(1,1,1)}$$

□

(c) Decompose the regular representation R into irreducibles.

Proof. Let's start at the beginning of the derivation in Section 1.3 of Fulton and Harris (2004) and go from there. We first consider the action on $\mathbb{C}^6=\operatorname{span}(e_e,e_{(12)},e_{(13)},e_{(23)},e_{(123)},e_{(132)})$ of the abelian subgroup A_3. In particular, if $\tau=(123)$, then

$$\rho_R(\tau)=\begin{bmatrix}0&0&0&0&0&1\\0&0&0&1&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\1&0&0&0&0&0\\0&0&0&0&1&0\end{bmatrix}$$

so that

$$\tau(e_e,e_{(12)},e_{(13)},e_{(23)},e_{(123)},e_{(132)})=(e_{(123)},e_{(13)},e_{(23)},e_{(12)},e_{(132)},e_e)$$

as expected. Letting $\omega=\mathrm{e}^{2\pi i/3}$, the eigenvalues and corresponding eigenvectors of τ are

$$\begin{matrix}\lambda_1=1 & \lambda_2=1 & \lambda_3=\omega^2 & \lambda_4=\omega^2 & \lambda_5=\omega & \lambda_6=\omega\\ v_1=\begin{bmatrix}1\\0\\0\\0\\1\\1\end{bmatrix} & v_2=\begin{bmatrix}0\\1\\1\\1\\0\\0\end{bmatrix} & v_3=\begin{bmatrix}\omega\\0\\0\\0\\\omega^2\\1\end{bmatrix} & v_4=\begin{bmatrix}0\\\omega\\\omega^2\\1\\0\\0\end{bmatrix} & v_5=\begin{bmatrix}\omega^2\\0\\0\\0\\\omega\\1\end{bmatrix} & v_6=\begin{bmatrix}0\\\omega^2\\\omega\\1\\0\\0\end{bmatrix}\end{matrix}$$

Let $\sigma=(12)$. Then computing as before,

$$\rho_R(\sigma)=\begin{bmatrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&1&0&0&0\end{bmatrix}$$

There are four total eigenvectors with eigenvalue not equal to 1. Thus, the subspace they span decomposes into $(2,1)^2$. For the two remaining eigenvectors, we have

$$\sigma(v_1)=v_2 \qquad \sigma(v_2)=v_1$$

Thus, $\sigma(v_1)$ is linearly independent of v_1, so they space they span decomposes into $(3)\oplus(1,1,1)$. Therefore,

$$\boxed{R\cong(2,1)^2\oplus(3)\oplus(1,1,1)}$$

□

(d) Prove that $S^{k+6}(2,1) \cong S^k(2,1) \oplus R$. Compute $S^k(2,1)$ for all k.

Proof. We prove the recursion formula directly. Let's begin.
The basis for $S^{k+6}(2,1)$ is

$$S^{k+6}(2,1) = \left\langle \alpha^{k+6-i} \cdot \beta^i \right\rangle_{i=0}^{k+6}$$

We now partition this basis into the direct sum of two sets, which we will prove are isomorphic to $S^k(2,1)$ and R. Explicitly, we seek to prove that

$$\left\langle \alpha^{k+6-i} \cdot \beta^i \right\rangle_{i=0}^{k+6} = \underbrace{\left\langle \alpha^{k+6-i} \cdot \beta^i \right\rangle_{i=3}^{k+3}}_{\overset{?}{\cong} S^k(2,1)} \oplus \underbrace{\left\langle \alpha^{k+6-i} \cdot \beta^i \right\rangle_{i=0}^{2} \oplus \left\langle \alpha^{k+6-i} \cdot \beta^i \right\rangle_{i=k+4}^{k+6}}_{\overset{?}{\cong} R}$$

Letting

$$S^k(2,1) = \left\langle \alpha^{k-i} \cdot \beta^i \right\rangle_{i=0}^{k}$$

define

$$\begin{aligned} f : \left\langle \alpha^{k+6-i} \cdot \beta^i \right\rangle_{i=3}^{k+3} &\to \left\langle \alpha^{k-i} \cdot \beta^i \right\rangle_{i=0}^{k} \\ \alpha^{k+6-i} \cdot \beta^i &\mapsto \alpha^{k-i+3} \cdot \beta^{i-3} \end{aligned}$$

Observe that for any relevant i, we have

$$\begin{aligned} [\tau \circ f](\alpha^{k+6-i} \cdot \beta^i) &= \tau(\alpha^{k-i+3} \cdot \beta^{i-3}) \\ &= \omega^{i+k-3} \alpha^{k-i+3} \cdot \beta^{i-3} \\ &= \omega^{i+k+6} \alpha^{k-i+3} \cdot \beta^{i-3} \\ &= \omega^{i+k+6} f(\alpha^{k+6-i} \cdot \beta^i) \\ &= [f \circ \tau](\alpha^{k+6-i} \cdot \beta^i) \end{aligned}$$

and

$$\begin{aligned} [\sigma \circ f](\alpha^{k+6-i} \cdot \beta^i) &= \sigma(\alpha^{k-i+3} \cdot \beta^{i-3}) \\ &= \alpha^{i-3} \cdot \beta^{k-i+3} \\ &= f(\alpha^i \cdot \beta^{k+6-i}) \\ &= [f \circ \sigma](\alpha^{k+6-i} \cdot \beta^i) \end{aligned}$$

Hence,

$$\tau \circ f = f \circ \tau \qquad\qquad \sigma \circ f = f \circ \sigma$$

It follows since τ, σ generate S_3 that every element of S_3 commutes with f in this manner. Additionally, since f maps linearly independent elements to linearly independent elements, f is an isomorphism of vector spaces. Therefore, f is an isomorphism of representations and

$$\left\langle \alpha^{k+6-i} \cdot \beta^i \right\rangle_{i=3}^{k+3} \overset{\checkmark}{\cong} S^k(2,1)$$

as desired.
On the other hand, we can show that

$$R \cong (2,1)^2 \oplus (3) \oplus (1,1,1) \cong S^6(2,1) \cong \left\langle \alpha^{k-i} \cdot \beta^i \right\rangle_{\substack{i=0 \\ i \neq 3}}^{6}$$

and define

$$\begin{aligned} f : \left\langle \alpha^{k+6-i} \cdot \beta^i \right\rangle_{i=0}^{2} \oplus \left\langle \alpha^{k+6-i} \cdot \beta^i \right\rangle_{i=k+4}^{k+6} &\to \left\langle \alpha^{k-i} \cdot \beta^i \right\rangle_{\substack{i=0 \\ i \neq 3}}^{6} \\ \alpha^{k+6-i} \cdot \beta^i &\mapsto \alpha^{6-i} \cdot \beta^i && (i \leq 2) \\ \alpha^{k+6-i} \cdot \beta^i &\mapsto \alpha^{k+6-i} \cdot \beta^{i-k} && (i \geq k+4) \end{aligned}$$

From here, the proof is symmetric to the previous case and results in[2]

$$\left\langle \alpha^{k+6-i} \cdot \beta^i \right\rangle_{i=0}^{2} \oplus \left\langle \alpha^{k+6-i} \cdot \beta^i \right\rangle_{i=k+4}^{k+6} \overset{\checkmark}{\cong} R$$

as desired. This concludes the proof of the recursion formula.

As to the second part of the question, we have from Q4b — and by mimicking its methods — that

$$\begin{aligned}
S^0(2,1) &= (3)\\
S^1(2,1) &= (2,1)\\
S^2(2,1) &= (2,1) \oplus (3)\\
S^3(2,1) &= (2,1) \oplus (3) \oplus (1,1,1)\\
S^4(2,1) &= (2,1)^2 \oplus (3)\\
S^5(2,1) &= (2,1)^2 \oplus (3) \oplus (1,1,1)
\end{aligned}$$

Therefore,

$$\boxed{S^k(2,1) = S^{k \bmod 6}(2,1) \oplus R^{\lfloor k/6 \rfloor}}$$

□

5. Let V be a vector space over F with a basis $e_1, \dots, e_n$; let $e^1, \dots, e^n$ be the dual basis. Prove the following.

(a) Element $e_1 \otimes e^1 + \cdots + e_n \otimes e^n \in V \otimes V^\vee$ does not depend on the choice of basis.

Proof. To prove that the given element of $V \otimes V^*$ does not depend on the choice of basis, it will suffice to show that for any choice of basis and associated dual basis, the given element maps to the same place in the isomorphic space $\mathrm{Hom}(V,V)$. Let's begin.

Let $e_1, \dots, e_n$ be an arbitrary basis of V, and let $e^1, \dots, e^n$ be its dual basis. Under the isomorphism constructed in class, the element

$$e_1 \otimes e^1 + \cdots + e_n \otimes e^n \mapsto [v \mapsto e^1(v)e_1] + \cdots + [v \mapsto e^n(v)e_n]$$

where $[v \mapsto e^i(v)e_i]$ denotes the linear map in $\mathrm{Hom}(V,V)$ sending v to its i^{th} component. Importantly, under the usual rules of adding functions, we can see that linear map on the right above is equal to

$$[v \mapsto e^1(v)e_1 + \cdots + e^n(v)e_n] = [v \mapsto v_1e_1 + \cdots + v_ne_n] = [v \mapsto v] = 1$$

where $1 \in \mathrm{Hom}(V,V)$ is the identity map. □

(b) Consider a linear map $\mathrm{ev} : V \otimes V^* \to F$ sending $v \otimes \alpha$ to $\alpha(v) \in F$. Prove that $\mathrm{ev}(L) = \mathrm{tr}(L)$.

Proof. Let $L \in \mathrm{Hom}(V,V)$ be an arbitrary linear map, let it's matrix be $(\ell_{ij}) \in \mathcal{M}(V,V)$ with respect to some basis $e_1, \dots, e_n$, and let $e^1, \dots, e^n$ be dual to this basis. Define a map from $\mathcal{M}(V,V)$ to $V \otimes V^*$ by sending the standard basis (a_{ij}) of $\mathcal{M}(V,V)$ to $e_i \otimes e^j$. It follows that the image of $\mathcal{M}(L)$ under this map is $\sum_{i,j=1}^n \ell_{ij} e_i \otimes e^j \in V \otimes V^*$. It follows by the constructions up to this point and the definition of the trace that

$$\mathrm{ev}(L) = \mathrm{ev}\left(\sum_{i,j=1}^{n} \ell_{ij} e_i \otimes e^j\right)$$

[2] An alternate method of proving this isomorphism *may* be able to be obtained from the converse of the definition of the regular representation stated on Serre (1977, p. 5).

$$\begin{aligned}
&= \sum_{i,j=1}^{n} \ell_{ij} \operatorname{ev}(e_i \otimes e^j) \\
&= \sum_{i,j=1}^{n} \ell_{ij} e^j(e_i) \\
&= \sum_{i,j=1}^{n} \ell_{ij} \delta_{ij} \\
&= \sum_{i=1}^{n} \ell_{ii} \\
&= \operatorname{tr}(L)
\end{aligned}$$

as desired. □

(c) A **projector** is a linear map $P : V \to V$ such that $P^2 = P$. Prove that $\operatorname{tr}(P) = \dim(\operatorname{Im}(P))$.

Proof. Let $v \in \operatorname{Im}(P)$ be arbitrary. Since $v \in \operatorname{Im}(P)$, we have that $v = Pw$ for some $w \in V$. But since $P = P^2$, it follows that

$$Pv = P^2 w = Pw = v$$

Thus, the restriction P_1 of P to $\operatorname{Im}(P)$ is equal to the identity on $\operatorname{Im}(P)$.
Let v be an arbitrary element of the complement of $\operatorname{Im}(P)$. It follows that $v \in \operatorname{Ker}(P)$, so

$$Pv = 0$$

Thus, the restriction P_2 of P to $\operatorname{Ker}(P)$ is equal to the zero map on $\operatorname{Ker}(P)$.
Since $V = \operatorname{Im}(P) \oplus \operatorname{Ker}(P)$, $P = P_1 \oplus P_2$. In particular, there will exist an orthonormal basis in which the first k vectors form a basis of $\operatorname{Im}(P)$ and the next $n-k$ vectors form a basis of $\operatorname{Ker}(P)$. Thus, $\dim(\operatorname{Im}(P)) = k$. Moreover, with respect to this basis, the $n \times n$ matrix of P will have a $k \times k$ block in the upper left-hand corner in which I_k resides, and it will be zeroes everywhere else. Since the trace is invariant under similarity transformations, it can be read off from this matrix as k as well. □

(d) Let V be a representation of a finite group G. Prove that the representation $V \otimes V^*$ has a trivial subrepresentation.

Proof. Consider the element from part (a). When considered as an element of $\operatorname{Hom}(V, V)$, it is the identity map. Additionally, we know from class that a representation on $\operatorname{Hom}(V, V)$ is a homomorphism $\rho : G \to GL[\operatorname{Hom}(V, V)]$ defined by $\rho(g)L = \rho_V(g) \circ L \circ \rho_V(g)^{-1}$. Thus, substituting in I for L, we learn that

$$\rho(g)I = \rho_V(g) \circ I \circ \rho_V(g)^{-1} = \rho_V(g) \circ \rho_V(g)^{-1} = I$$

Thus, regardless of what G, ρ, V are, ρ will preserve I. Therefore, $\operatorname{span}(I)$ is a subrepresentation and, importantly, it is the trivial subrepresentation since all elements of G act as the identity on it. □

2 Introduction to Character Theory

10/13: 1. **More linear algebra**. Let V be a finite-dimensional vector space.

(a) Prove that under the identification of $V \otimes V^*$ with $\text{Hom}_F(V,V)$, **simple** tensors $v \otimes \varphi$ correspond to linear maps of rank 0 or 1.

Proof. Let $v \otimes \varphi$ be an arbitrary simple tensor im $V \otimes V^*$. Recall from the 10/2 lecture that

$$v_1 \otimes \alpha \mapsto [v_2 \mapsto \alpha(v_2)v_1]$$

is a good isomorphism from $V \otimes V^* \cong \text{Hom}_F(V,V)$. It follows that the linear map to which $v \otimes \varphi$ corresponds is the map $L : V \to V$ defined by $L(v') = \varphi(v')v$. Since $\text{Im}\,\varphi = \mathbb{C}$, we have that $\text{Im}(L) \leq \mathbb{C}v$. Thus, since $\dim(\mathbb{C}v) = 1$, we have that $\text{rank}(L) \leq 1$, as desired. $\square$

(b) Consider the vector space $W = \text{Hom}_F(V,V)$. Prove that any linear functional in W^* has the form $L \mapsto \text{tr}(LM)$ for some $M \in W$. Prove that the vector space $\text{Hom}_F(V,V)$ is "canonically" self-dual.

Proof. Let $\varphi \in W^*$ be arbitrary. Also let $n := \dim V$ for convenience. Notice that the n^2 matrices E_{ij} $(i,j = 1,\dots,n)$, which have a 1 in the i^{th} row and j^{th} column and 0s everywhere else, form a basis of W. Thus, φ is fully characterized by its actions on the E_{ij}. Now define

$$M := (\varphi(E_{ij}))^T$$

It follows that if $L = (\ell_{ij})$, then

$$\varphi(L) = \sum_{i=1}^{n}\sum_{j=1}^{n} \ell_{ij}\varphi(E_{ij}) = \text{tr}(LM)$$

as desired. This completes the proof, but to help illustrate it, I'll include the $n = 3$ case:

$$\underbrace{\begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix}}_{L=(\ell_{ij})} \circ \underbrace{\begin{bmatrix} \varphi(E_{11}) & \varphi(E_{12}) & \varphi(E_{13}) \\ \varphi(E_{21}) & \varphi(E_{22}) & \varphi(E_{23}) \\ \varphi(E_{31}) & \varphi(E_{32}) & \varphi(E_{33}) \end{bmatrix}^T}_{M=(\varphi(E_{ij}))^T}$$

$$= \underbrace{\begin{bmatrix} \sum_{j=1}^{3} \ell_{1j}\varphi(E_{1j}) & & \cdots \\ & \sum_{j=1}^{3} \ell_{2j}\varphi(E_{2j}) & \\ \cdots & & \sum_{j=1}^{3} \ell_{3j}\varphi(E_{3j}) \end{bmatrix}}_{LM}$$

To prove that $\text{Hom}_F(V,V) = W$ is canonically self-dual, it will suffice to construct an isomorphism $W^* \cong W$ that does not depend on a choice of basis. Let $\varphi \in W^*$ be arbitrary. As demonstrated above, there exists a unique corresponding $M \in W$ such that $\varphi(L) = \text{tr}(LM)$ for all $L \in W$. Therefore, the map $f : W^* \to W$ defined by $\varphi \mapsto M$ is a bijection. To show that it is linear, too, we add in a basis and compute as follows.

$$\begin{aligned} f(\varphi_1 + \varphi_2) &= ([\varphi_1 + \varphi_2](E_{ij}))^T \\ &= (\varphi_1(E_{ij}) + \varphi_2(E_{ij}))^T \\ &= (\varphi_1(E_{ij}))^T + (\varphi_2(E_{ij}))^T \\ &= f(\varphi_1) + f(\varphi_2) \end{aligned} \qquad \begin{aligned} f(\lambda\varphi) &= ([\lambda\varphi](E_{ij}))^T \\ &= (\lambda\varphi(E_{ij}))^T \\ &= \lambda(\varphi(E_{ij}))^T \\ &= \lambda f(\varphi) \end{aligned}$$

$\square$

2. **Characters of abelian groups**. Let A be a finite abelian group.

   (a) A **character** of A is a homomorphism $\chi : A \to \mathbb{C}^\times$. Prove that for every $g \in A$, the value $\chi(g)$ is a root of unity. Prove that the product of characters is a character. Prove that characters form an abelian group. This group is called the **dual** of A and is denoted $\widehat{A}$.

Proof. Let $g \in A$ be arbitrary. Since A is finite, $|g|$ is finite. It follows since χ is a homomorphism that

$$1 = \chi(e) = \chi(g^{|g|}) = \chi(g)^{|g|}$$

Therefore, since the only complex numbers having 1 as a power are the roots of unity, $\chi(g)$ is a root of unity, as desired.

Let χ_1, χ_2 be two characters of A. To prove that their product $\chi_1\chi_2$ is a character, it will suffice to show that $\chi_1\chi_2$ is a group homomorphism. To do so, we must confirm that

$$\chi_1\chi_2(e) = 1 \qquad \chi_1\chi_2(g_1g_2) = \chi_1\chi_2(g_1) \cdot \chi_1\chi_2(g_2) \qquad \chi_1\chi_2(g^{-1}) = \chi_1\chi_2(g)^{-1}$$

We can do this using the analogous statements satisfied by χ_1 and χ_2 separately. Specifically,

$$\begin{aligned} \chi_1\chi_2(e) &= \chi_1(e) \cdot \chi_2(e) \\ &= 1 \cdot 1 \\ &= 1 \end{aligned} \qquad \begin{aligned} \chi_1\chi_2(g^{-1}) &= \chi_1(g^{-1}) \cdot \chi_2(g^{-1}) \\ &= \chi_1(g)^{-1} \cdot \chi_2(g)^{-1} \\ &= (\chi_1(g) \cdot \chi_2(g))^{-1} \\ &= \chi_1\chi_2(g)^{-1} \end{aligned}$$

$$\begin{aligned} \chi_1\chi_2(g_1g_2) &= \chi_1(g_1g_2) \cdot \chi_2(g_1g_2) \\ &= \chi_1(g_1) \cdot \chi_1(g_2) \cdot \chi_2(g_1) \cdot \chi_2(g_2) \\ &= \chi_1(g_1) \cdot \chi_2(g_1) \cdot \chi_1(g_2) \cdot \chi_2(g_2) \\ &= \chi_1\chi_2(g_1) \cdot \chi_1\chi_2(g_2) \end{aligned}$$

Let $\widehat{A}$ denote the set of all characters of A. Also let $\cdot$ denote the operation of function multiplication, which was shown in the above proof to be a binary operation on $\widehat{A}$. To prove that $(\widehat{A}, \cdot)$ is an abelian group, it will suffice to show that it has an identity element, inverses, associativity, and commutativity. Let's begin.

Identity: Consider the character χ_e defined by $g \mapsto 1$ for all $g \in A$. Let $\chi \in \widehat{A}$ be arbitrary. Then since we have the following, letting $g \in A$ be arbitrary, we know that $\chi\chi_e = \chi = \chi_e\chi$, as desired.

$$\chi\chi_e(g) = \chi(g) \cdot \chi_e(g) = \chi(g) \cdot 1 = \chi(g) = 1 \cdot \chi(g) = \chi_e(g) \cdot \chi(g) = \chi_e\chi(g)$$

Inverses: Let $\chi \in \widehat{A}$ be arbitrary. Consider the character $\bar{\chi}$ defined by $g \mapsto \overline{\chi(g)}$ for all $g \in A$, where the overbar denotes taking the complex conjugate. Then since we have the following, letting $g \in A$ be arbitrary, we know that $\chi\bar{\chi} = \bar{\chi}\chi = \chi_e$, as desired. Note that the complex conjugates multiply to 1 because we showed above that all $\chi(g)$ are roots of unity (for any $\chi \in \widehat{A}$).

$$\chi\bar{\chi}(g) = \chi(g) \cdot \bar{\chi}(g) = \bar{\chi}(g) \cdot \chi(g) = 1 = \chi_e(g)$$

Associativity: Let $\chi_1, \chi_2, \chi_3 \in \widehat{A}$ be arbitrary. Then since we have the following, letting $g \in A$ be arbitrary, we know that $\chi_1(\chi_2\chi_3) = (\chi_1\chi_2)\chi_3$, as desired.

$$[\chi_1(\chi_2\chi_3)](g) = \chi_1(g) \cdot \chi_2\chi_3(g) = \chi_1(g) \cdot \chi_2(g) \cdot \chi_3(g) = \chi_1\chi_2(g) \cdot \chi_3(g) = [(\chi_1\chi_2)\chi_3](g)$$

Commutativity: Let $\chi_1, \chi_2 \in \widehat{A}$ be arbitrary. Then since we have the following, letting $g \in A$ be arbitrary, we know that $\chi_1\chi_2 = \chi_2\chi_1$, as desired.

$$\chi_1\chi_2(g) = \chi_1(g) \cdot \chi_2(g) = \chi_2(g) \cdot \chi_1(g) = \chi_2\chi_1(g)$$

$\square$

(b) Prove directly that for every nontrivial character $\chi \in \widehat{A}$, the following identity holds.

$$\sum_{g\in A} \chi(g) = 0$$

Proof. Let $\chi \in \widehat{A}$ be a nontrivial character. Since it is nontrivial, there exists $h \in A$ for which $\chi(h) \neq 1$. Additionally, we have by the Sudoku Lemma that

$$\sum_{g\in A} \chi(g) = \sum_{g\in A} \chi(hg)$$

But then since χ is a homomorphism, we have

$$\begin{aligned} \sum_{g\in A} \chi(g) &= \sum_{g\in A} \chi(hg) = \sum_{g\in A} \chi(h)\chi(g) = \chi(h) \sum_{g\in A} \chi(g) \\ (1-\chi(h)) \sum_{g\in A} \chi(g) &= 0 \end{aligned}$$

Thus, by the zero-product property, either $1 - \chi(h) = 0$ or $\sum_{g\in A} \chi(g) = 0$. Since $\chi(h) \neq 1$ as proven above, $1 - \chi(h) \neq 0$ so we must have

$$\sum_{g\in A} \chi(g) = 0$$

as desired. □

(c) Prove that characters are the same as the 1-dimensional representations of A; product of characters is the same as a tensor product of representations, and the inverse of the character is the same as the dual representation.

Proof. Let $\rho_1, \rho_2 : A \to GL(\mathbb{C}) = \mathbb{C}^\times$ be two arbitrary 1-dimensional representations of A. Notice that ρ_1, ρ_2 have the same domain and codomain as characters, and are homomorphisms. Additionally, by the definition of the Kronecker product for 1×1 matrices, we have that

$$[\rho_1 \otimes \rho_2](g) = \rho_1(g) \cdot \rho_2(g)$$

for all $g \in A$. Thus, the tensor product of these representations is the same the character product of their values. Lastly, we have that

$$\rho_1^*(g) = \rho_1(g^{-1})^T = \rho_1(g^{-1}) = \rho_1(g)^{-1} = \overline{\rho_1(g)}$$

Thus, the dual representation of this representation is computed using the character inverse. □

(d) Find all characters for $A = \mathbb{Z}/n\mathbb{Z}$. Compute the dual group $\widehat{A}$. Do the same for $A = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

Proof. We treat each group separately.
$\underline{\mathbb{Z}/n\mathbb{Z}}$: Let $\chi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times$ be a character of $\mathbb{Z}/n\mathbb{Z}$. Since χ is a homomorphism and hence $\chi(n) = \chi(1)^n$, the value of $\chi(1)$ completely determines the action of χ. Thus, there is a one-to-one mapping between the characters of $\mathbb{Z}/n\mathbb{Z}$ and the possible values of $\chi(1)$, so let's investigate the latter. We know from part (a) that $\chi(1)$ is a root of unity and that $\chi(1)^n = 1$. Consequently, $\chi(1)$ is an n^{th} root of unity. Any such root of unity will work, so the characters $\chi_0, \dots, \chi_{n-1}$ of $\mathbb{Z}/n\mathbb{Z}$ are defined by

$$\boxed{\widehat{\mathbb{Z}/n\mathbb{Z}} = \{\chi_k \mid k = 0, \dots, n-1;\ \chi_k(1) = \mathrm{e}^{2\pi i k/n}\}}$$

$\underline{K_4}$: The maximum order of any element in this group is 2, so $\chi : K_4 \to \{-1, 1\}$. While there are $2^4 = 16$ such maps, only four are homomorphisms: Those sending $((0,0),(0,1),(1,0),(1,1))$ to...

$$\boxed{\widehat{K}_4 = \{(1,1,1,1),(1,1,-1,-1),(1,-1,1,-1),(1,-1,-1,1)\}}$$

□

(e) Prove that $\widehat{A_1 \times A_2}$ is isomorphic to $\widehat{A}_1 \times \widehat{A}_2$. Prove that groups A and $\widehat{A}$ are isomorphic as abstract groups. Deduce that an abelian group of order n has exactly n characters.

Proof. Define $h : \widehat{A}_1 \times \widehat{A}_2 \to \widehat{A_1 \times A_2}$ by

$$h(\chi_1, \chi_2) = \chi_1 \otimes \chi_2$$

where $[\chi_1 \otimes \chi_2](a_1, a_2) = \chi_1(a_1) \cdot \chi_2(a_2)$. Note that we are borrowing part (c)'s conclusion that characters can be treated like representations and, in particular, can have tensor products. To prove that h is an isomorphism of groups, it will suffice to show that it is a bijective homomorphism of groups. The following suffices to show that it is a homomorphism of groups.

$$\begin{aligned} h[(\chi_1, \chi_2) \cdot (\chi_3, \chi_4)] &= h(\chi_1\chi_3, \chi_2\chi_4) \\ &= \chi_1\chi_3 \otimes \chi_2\chi_4 \\ &= \chi_1 \otimes \chi_2 \cdot \chi_3 \otimes \chi_4 \\ &= h(\chi_1, \chi_2) \cdot h(\chi_3, \chi_4) \end{aligned}$$

Note that the transition form the second to the third line above is justified because the equality becomes $\chi_1(a_1) \cdot \chi_3(a_1) \cdot \chi_2(a_2) \cdot \chi_4(a_2) = \chi_1(a_1) \cdot \chi_2(a_2) \cdot \chi_3(a_1) \cdot \chi_4(a_2)$ when applied to (a_1, a_2) and expanded. As to bijectivity, we will prove injectivity then surjectivity. For injectivity, suppose $h(\chi_1, \chi_2) = h(\chi_3, \chi_4)$. Then for all $a_1 \in A_1$,

$$\begin{aligned} [h(\chi_1, \chi_2)](a_1, e) &= [h(\chi_3, \chi_4)](a_1, e) \\ \chi_1(a_1) \cdot \chi_2(e) &= \chi_3(a_1) \cdot \chi_4(e) \\ \chi_1(a_1) \cdot 1 &= \chi_3(a_1) \cdot 1 \\ \chi_1(a_1) &= \chi_3(a_1) \end{aligned}$$

A similar statement holds for χ_2 and χ_4, proving that $(\chi_1, \chi_2) = (\chi_3, \chi_4)$, as desired. For surjectivity, let $\chi \in \widehat{A_1 \times A_2}$ be arbitrary. Define χ_1 and χ_2 by

$$\chi_1(a_1) = \chi(a_1, e) \qquad\qquad \chi_2(a_2) = \chi(e, a_2)$$

for all $a_1 \in A_1$ and $a_2 \in A_2$. That χ_1, χ_2 are characters under these definitions instead of just functions follows immediately from the character-like properties of χ: indeed, with these definitions in hand, we can show that

$$[h(\chi_1, \chi_2)](a_1, a_2) = \chi_1(a_1) \cdot \chi_2(a_2) = \chi(a_1, e) \cdot \chi(e, a_2) = \chi[(a_1, e) \cdot (e, a_2)] = \chi(a_1, a_2)$$

as desired.

By the fundamental theorem of finite abelian groups, A is isomorphic to a direct product of cyclic groups of prime power order. Thus, we may let

$$A \cong (\mathbb{Z}/p_1\mathbb{Z})^{n_1} \times \cdots \times (\mathbb{Z}/p_k\mathbb{Z})^{n_k}$$

Borrowing the notation from the first task of part (d) above, define $h : (\mathbb{Z}/p_1\mathbb{Z})^{n_1} \times \cdots \times (\mathbb{Z}/p_k\mathbb{Z})^{n_k} \to (\widehat{\mathbb{Z}/p_1\mathbb{Z}})^{n_1} \times \cdots \times (\widehat{\mathbb{Z}/p_k\mathbb{Z}})^{n_k}$ by

$$h(a_{11}, \dots, a_{kn_k}) = \chi_{a_{11}} \otimes \cdots \otimes \chi_{a_{kn_k}}$$

For the same reasons mentioned in part (d), h is an isomorphism. Additionally, by consecutive applications of the first claim in this part,

$$(\widehat{\mathbb{Z}/p_1\mathbb{Z}})^{n_1} \times \cdots \times (\widehat{\mathbb{Z}/p_k\mathbb{Z}})^{n_k} \cong (\mathbb{Z}/p_1\mathbb{Z})^{n_1} \widehat{\times \cdots \times} (\mathbb{Z}/p_k\mathbb{Z})^{n_k}$$

But since $A \cong (\mathbb{Z}/p_1\mathbb{Z})^{n_1} \times \cdots \times (\mathbb{Z}/p_k\mathbb{Z})^{n_k}$, the group on the right above is isomorphic to $\widehat{A}$. Thus, by chaining together isomorphisms, we can get all the way from A to $\widehat{A}$, as desired.

Since isomorphic groups have the same order, an abelian group of order n has a dual group with order n, i.e., has n characters, as desired. □

3. Consider the permutational representation of S_n. Decompose it into the sum of (two) irreducible representations.

Proof. Let $\rho : S_n \to GL(V)$ be the permutational representation of S_n. As discussed in class, ρ fixes the one-dimensional subspace $\text{span}(1, \dots, 1) \leq V$. Thus, this subspace forms a trivial subrepresentation of V. It follows by the theorem from class that this subspace has a complement; this complement is the standard representation. Thus,

$$V = (3) \oplus (2, 1)$$

As a one-dimensional representation, (3) is clearly irreducible, but it is not immediately evident that $(2, 1)$ is. Fortunately, the following proves that it is. Assuming $n \geq 2$ since the 1D case is trivial. If we take the column vector $(1, -1, 0, \dots, 0) \in (2, 1)$, we can generate from it $n - 1$ other linearly independent column vectors using consecutive applications of $\sigma = (12 \cdots n)$. For example, if $n = 4$, we generate

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

Therefore, there is no subspace of $(2, 1)$ that we can't get into when we're in $(2, 1)$ *except* that which we've already discussed: (3). It follows that $(2, 1)$ is irreducible, as well. □

4. Let G be a finite group.

(a) Define the **space of invariants** of a representation V by the formula

$$V^G = \{v \in V \mid gv = v \ \forall \ g \in G\}$$

Prove that V^G is a subrepresentation of V. Prove that it is isomorphic to a sum of trivial representations.

Proof. To prove that V^G is a subrepresentation of V, it will suffice to show that it is a subspace of V, and $gV^G \subset V^G$ for all $g \in G$. Since we clearly have $g \cdot 0 = 0$ for all $g \in G$, $g(v_1 + v_2) = v_1 + v_2$ for all v_1, v_2 satisfying $gv_i = v_i$, and $agv = g(av)$, V^G is a subspace of V. Now for closure under the g's, let $v \in V^G$ be arbitrary. But by the definition of V^g, we have $gv = v \in V^g$ for all $g \in G$, as desired.

Let $e_1, \dots, e_k$ be a basis of V^G. Since $g(\lambda e_i) = \lambda e_i$ for all $\lambda e_i \in \text{span}(e_i)$, $i = 1, \dots, k$, each $\text{span}(e_i)$ is, itself, fixed by all g and hence a trivial subrepresentation of V^G. Therefore,

$$V^G \cong \underbrace{(3) \oplus \cdots \oplus (3)}_{k \text{ times}}$$

as desired. □

(b) Prove that $(\text{Hom}_F(V, W))^G$ is isomorphic to $\text{Hom}_G(V, W)$.

Proof. To prove the claim, it will suffice to prove the stronger condition that $(\text{Hom}_F(V, W))^G = \text{Hom}_G(V, W)$ as sets. We will proceed via a bidirectional inclusion proof. Let's begin.[3]

First, let $f \in (\text{Hom}_F(V, W))^G$ be arbitrary. Then by the definition of the space of invariants, $g \cdot f = f$ for all $g \in G$. Additionally, since $G \circlearrowright \text{Hom}_F(V, W)$ via $g \cdot f = gfg^{-1}$, we have that $gfg^{-1} = f$, i.e., $gf = fg$ for all $g \in G$. But this implies that f is a morphism of G-representations, i.e., $f \in \text{Hom}_G(V, W)$, as desired.

The proof is symmetric in the reverse direction. □

[3]Note: Beware rampant abuses of notation throughout this proof. For example, the statement $gf = fg$ stands in for the much more complex $\rho_V(g) \circ f = f \circ \rho_W(g)$.

5. Let $\rho : G \to GL_n(\mathbb{C})$ be a representation with character χ.

   (a) Prove that $\mathrm{Ker}(\rho) = \{g \in G \mid \chi(g) = n\}$.

   Proof. We proceed via a bidirectional inclusion proof.

   Suppose first that $g \in \mathrm{Ker}(\rho)$. Then $\rho(g) = I_n$. But since $\mathrm{tr}(I_n) = n$ and $\chi(g) = \mathrm{tr}(\rho(g))$, we have by transitivity that $\chi(g) = n$, as desired.

   Now suppose that $\chi(g) = n$. Recall from class that every eigenvalue λ_i of $\rho(g)$ is a root of unity. Additionally, since $\lambda_1 + \cdots + \lambda_n = \chi(g) = n$, we must have $\lambda_i = 1$ $(i = 1, \dots, n)$. But this implies that $\rho(g) = I_n \in \mathrm{Ker}(\rho)$, as desired. □

   (b) Prove that for any $g \in G$, we have $|\chi(g)| \leq n$.

   Proof. As in part (a), recall from class that every eigenvalue λ_i of $\rho(g)$ is a root of unity. Then by the triangle inequality,

   $$|\chi(g)| = |\lambda_1 + \cdots + \lambda_n| \leq |\lambda_1| + \cdots + |\lambda_n| = 1 + \cdots + 1 = n$$

   as desired. □

   (c) Prove that for a given $g \in G$, $|\chi(g)| = n$ if and only if there exists $\lambda \in \mathbb{C}$ such that $\rho(g) = \lambda I$.

   Proof. Suppose first that $|\chi(g)| = n$. Then $|\lambda_1 + \cdots + \lambda_n| = n$, so since $|\lambda_i| = 1$ for $i = 1, \dots, n$, we must have $\lambda_1 = \cdots = \lambda_n$. Define $\lambda := \lambda_i$. Recall that a linear operator with n eigenvalues must have a corresponding $n \times n$ matrix in some basis equal to $\mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Therefore, in this case, the corresponding matrix of $\rho(g)$ is λI (and is λI in any basis), as desired.

   Now suppose that $\rho(g) = \lambda I$. Then

   $$|\chi(g)| = |\,\mathrm{tr}(\lambda I)| = |n\lambda| = n \cdot |\lambda| = n \cdot 1 = n$$

   as desired. □

3 Representation Structure and Characters

10/20: 1. **Permutational representation**. Let X be a finite set on which the group G acts. Let ρ be the corresponding permutational representation with character χ.

(a) Consider an orbit Gx of an element $x \in X$; let c be the number of orbits. Prove that c equals the number of times ρ contains the trivial representation. Deduce that $(\chi, 1) = c$. In particular, if the action is transitive, $\rho = 1 \oplus \theta$ for some representation θ.

Proof. By the proof of Corollary 1 from Lecture 3.3, the number of times ρ contains the trivial representation (i.e., the multiplicity n_1 of the trivial representation) is equal to $(\chi, 1)$. Thus, to prove that the number of times ρ contains the trivial representation is equal to c, we will show that $(\chi, 1) = c$. Indeed, from the Hermitian inner product definition, we have that

$$
\begin{aligned}
(\chi, 1) &= \frac{1}{|G|} \sum_{g \in G} \chi(g) \cdot \bar{1}(g) \\
&= \frac{1}{|G|} \sum_{g \in G} \chi(g) \cdot 1 \\
&= \frac{1}{|G|} \sum_{g \in G} \chi(g) \\
&= \frac{1}{|G|} \sum_{g \in G} \mathrm{Fix}(g) \\
&= \frac{1}{|G|} \sum_{g \in G} |\{x \in X \mid g \cdot x = x\}| = \frac{1}{|G|} \sum_{g \in G} \sum_{x \in X} 1_{g \cdot x = x} \\
&= \frac{1}{|G|} \sum_{x \in X} |\{g \in G \mid g \cdot x = x\}| = \frac{1}{|G|} \sum_{x \in X} \sum_{g \in G} 1_{g \cdot x = x} \\
&= \frac{1}{|G|} \sum_{x \in X} |\mathrm{Stab}(x)| \\
&= \sum_{x \in X} \frac{1}{|G|/|\mathrm{Stab}(x)|} \\
&= \sum_{x \in X} \frac{1}{|Gx|} && \text{Orbit-Stabilizer Theorem} \\
&= c \cdot \sum_{x \in Gx} \frac{1}{|Gx|} \\
&= c \cdot 1 \\
&= c
\end{aligned}
$$

as desired.

If the action is transitive, then $Gx = X$ for any $x \in X$, so there is only one orbit and $(\chi, 1) = 1$. Thus, the multiplicity of the trivial representation in ρ is 1, so by complete reducibility,

$$\rho = 1^1 \oplus \underbrace{V_2^{n_2} \oplus \cdots \oplus V_k^{n_k}}_{\theta}$$

as desired. ☐

(b) Consider a diagonal action of G on $X \times X$. Prove that the character of the corresponding permutational representation is χ^2.

Proof. Since ρ is a permutational representation, we have that $\chi(g) = |\operatorname{Fix}_X(g)|$, where here we let $\operatorname{Fix}_X(g) = \{x \in X \mid g \cdot x = x\}$. It follows via a simple bidirectional inclusion proof that $\operatorname{Fix}_{X\times X}(g) = \operatorname{Fix}_X(g) \times \operatorname{Fix}_X(g)$. Thus,

$$\chi_{X\times X} = |\operatorname{Fix}_{X\times X}(g)| = |\operatorname{Fix}_X(g) \times \operatorname{Fix}_X(g)| = |\operatorname{Fix}_X(g)|^2 = \chi^2$$

as desired. $\square$

(c) Suppose that G acts transitively on X and $|X| \geq 2$. We call this action **doubly transitive** if every pair of distinct elements of X can be sent to any other pair by some element of G. Prove that the following are equivalent.

i. The action is doubly transitive.
ii. The diagonal action on $X \times X$ has exactly two orbits.
iii. $(\chi^2, 1) = 2$.
iv. The representation θ is irreducible.

Proof. We will prove that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). Let's begin.

(i $\Leftrightarrow$ ii): First, suppose $G \circlearrowright X$ is doubly transitive. Since $|X| \geq 2$, we may choose $x, y \in X$ to be distinct, i.e., satisfying $x \neq y$. We will show that the two orbits of the diagonal action of G on $X \times X$ are $G(x, x)$ and $G(x, y)$.
First, we will show that every $(x_1, x_2) \in X \times X$ is in one of these two orbits. Let $(x_1, x_2) \in X \times X$ be arbitrary. We divide into two cases ($x_1 = x_2$ and $x_1 \neq x_2$). If $x_1 = x_2$, then since $G \circlearrowright X$ is transitive, there exists $g \in G$ such that $g \cdot x = x_1$. Thus,

$$g \cdot (x, x) = (g \cdot x, g \cdot x) = (x_1, x_1) = (x_1, x_2)$$

so $(x_1, x_2) \in G(x, x)$, as desired. If $x_1 \neq x_2$, then since $G \circlearrowright X$ is doubly transitive, there exists $g \in G$ such that $g \cdot x = x_1$ and $g \cdot y = x_2$. Thus,

$$g \cdot (x, y) = (g \cdot x, g \cdot y) = (x_1, x_2)$$

so $(x_1, x_2) \in G(x, y)$, as desired.
Now, we will show that $(x, y) \notin G(x, x)$. This follows immediately from the well-definedness of the group action: Suppose for the sake of contradiction that there exists $g \in G$ such that $g \cdot (x, x) = (x, y)$. Then $(g \cdot x, g \cdot x) = (x, y)$, so $x = g \cdot x = y$, contradicting the hypothesis that $x \neq y$.

Second, suppose the diagonal action has exactly two orbits. Since $G \circlearrowright X$ is transitive, by the same reasoning as before, $G(x, x)$ is an orbit. Thus, since there are only two orbits, the other orbit must be $X \times X \setminus G(x, x) = G(x, y)$. The existence of this second orbit implies that any distinct $x, y \in X$ can be mapped to any other pair of elements of X by some $g \in G$, i.e., that the action is doubly transitive.

(ii $\Leftrightarrow$ iii): Suppose the diagonal action on $X \times X$ has exactly two orbits. Then by part (b), the character of the corresponding permutational representation is χ^2. Thus, by part (a), $(\chi^2, 1) = c$, where c is the number of orbits. But by hypothesis (ii), $c = 2$, so $(\chi^2, 1) = 2$, as desired.

Suppose $(\chi^2, 1) = 2$. Then by parts (a) and (b) once again, the diagonal action on $X \times X$ has $2 = c$ orbits.

(iii $\Leftrightarrow$ iv): Suppose $(\chi^2, 1) = 2$. Note that by θ, we mean the θ defined in part (a), *not* the representation θ' defined by $\rho_2 = 1^2 \oplus \theta$ where ρ_2 is the permutational representation corresponding to the diagonal action of G on $X \times X$. Moving on, observe that $(\chi, \chi) = (\chi^2, 1)$, $(1, \chi) = (\chi, 1)$, and $(1, 1) = 1$ by the definition of the inner product. Observe also that $(\chi, 1) = 1$ by part (a) since the action is transitive. Therefore,

$$\begin{aligned}
(\theta,\theta) &= (\chi-1,\chi-1)\\
&= (\chi-1,\chi)-(\chi-1,1)\\
&= [(\chi,\chi)-(1,\chi)]-[(\chi,1)-(1,1)]\\
&= (\chi^2,1)-2(\chi,1)+(1,1)\\
&= 2-2\cdot 1+1\\
&= 1
\end{aligned}$$

so θ is irreducible by Corollary 2 from Lecture 3.3.

Suppose that θ is irreducible. Then $(\theta,\theta)=1$. We still have $(\chi,\chi)=(\chi^2,1)$, $(\chi,1)=(1,\chi)=1$, and $(1,1)=1$ because these claims relied on the definition of the inner product and part (a), not the hypothesis that $(\chi^2,1)=2$. Thus, we have that

$$(\chi^2,1)=(\theta,\theta)+2(\chi,1)-(1,1)=1+2-1=2$$

as desired. ☐

2. Find the character table of the group A_4.

Proof. The conjugacy classes of A_4 in S_4 are $\{e\},\{(xxx)\},\{(xx)(xx)\}$. The true conjugacy classes of A_4 vary slightly, however. e is still in a class by itself

Permutation representation: 4,1,0

	e	(xxx)	$(xx)(xx)$
Trivial	1	1	1
Standard	3	0	-1

☐

3. Consider the space of functions V from the set of faces of a cube to $\mathbb{C}$. This is a representation of S_4.
   (a) Compute the character of V.
   (b) Describe explicitly the decomposition of V into isotypical components.
   (c) Consider a map $A : V \to V$ acting by substituting the value of a function on a face with an average of its values on the adjacent four faces. Prove that A is an automorphism of the corresponding representation. Find its eigenvalues.
4. Consider a finite representation V of a group G with character χ.
   (a) Express the characters of $\Lambda^2 V$ and $S^2 V$ in terms of χ.
   (b) Express the characters of $\Lambda^3 V$ and $S^3 V$ in terms of χ.
   (c) Let $(3,1)$ be the standard representation of S_4. Decompose $\Lambda^2(3,1)$, $\Lambda^3(3,1)$, $S^2(3,1)$, and $S^3(3,1)$ into irreducibles.

4 More Characters and Intro to Associative Algebras

10/27: 1. **Representations of S_5.**

   (a) Prove that there exist only two one-dimensional representations of S_5: The trivial representation (5) and the alternating representation $(1,1,1,1,1)$.

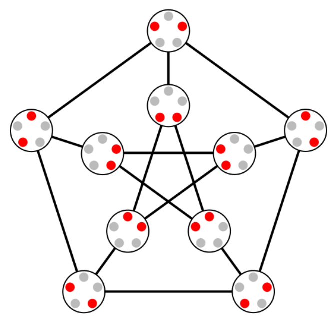
   (b) Compute the character of the standard representation $(4,1)$ by decomposing the permutational representation into irreducibles.

   (c) Prove that the representations $(3,1,1) = \Lambda^2(4,1)$ and $(2,1,1,1) = \Lambda^3(4,1)$ are irreducible. Compute their characters. Prove that $(2,1,1,1) = (1,1,1,1,1) \otimes (4,1)$.

   (d) Find the two remaining irreducible representations of S_5; denote them $(3,2)$ and $(2,2,1)$. Complete the character table.

   (e) Consider an exceptional homomorphism $S_5 \to S_6$. Decompose the corresponding permutational representation into irreducibles.

   (f) The **Petersen graph** is a graph with vertices being 2-element subsets of $\{1,2,3,4,5\}$; two vertices are connected by an edge if the corresponding sets do not intersect (see wiki for a picture).



   Consider a natural action of S_5 on the set of complex-valued functions on the set of vertices of the Peterson graph. Find the character of the corresponding representation V and decompose it into irreducibles.

   (g) Decompose V into isotypical components.

   (h) Consider an endomorphism $A : V \to V$ sending a function to the average of its values on the adjacent vertices. Prove that it is an endomorphism of the corresponding representation. Find the spectrum of A.

2. The character table is a square matrix. Determine the absolute value of its determinant.

3. Prove that for any irreducible character χ of a group G, we have

$$\chi(g)\chi(h) = \frac{d_\chi}{|G|} \sum_{x \in G} \chi(gxhx^{-1})$$

4. Let F be a field.

   (a) Prove that the matrix algebra $M_{n,n}(F)$ is simple, i.e., has no nontrivial ideals.

   (b) Prove that there is a unique simple module over the matrix algebra $M_{n,n}(F)$.

5. Consider the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} \subset \mathbb{H}$ of order 8.

   (a) Find four 1-dimensional representations of Q_8. Find the character of the remaining 2-dimensional representation.

   (b) Prove that $\mathbb{R}[Q_8] \cong \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{H}$ as an algebra.

5 Abstract Representation Theory

11/3:

1. Assume that a finite-dimensional left module M over an associative algebra A is semisimple. Prove that any quotient of M is semisimple.
2. Let F be a field.
   (a) Let D be a finite-dimensional division algebra over F. Prove that D^n is an irreducible module over the matrix algebra $M_n(D)$. Deduce that $M_n(D)$ is semisimple.
   (b) Let $D_1, \dots, D_k$ be finite-dimensional division algebras over F. Prove that an algebra $M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k)$ is semisimple. Prove that it has exactly k irreducible representations.
3. Let A be a finite-dimensional associative algebra over F. Prove that $\text{Rad}(A)$ consists of elements $a \in A$ such that $1 + xa$ has a left inverse for any $x \in A$.
4. Let G be a finite group. Assume that a prime p divides the order of G. Prove that the set $\{\sum a_g g \mid \sum a_g = 0\}$ is an ideal in $\mathbb{F}_p[G]$. Show that it is not a direct summand of $\mathbb{F}_p[G]$. Deduce that $\mathbb{F}_p[G]$ is not semisimple.
5. Prove that under the identification of the group algebra $\mathbb{C}[G]$ with the space of complex-valued functions on G, the product in the group algebra corresponds to the convolution $*$ of functions
$$(f_1 * f_2)(g) = \sum_{x \in G} f_1(x) f_2(x^{-1}g)$$
Deduce that the convolution is associative, and then prove the associativity directly. Which function plays the role of the unit?
6. **Fourier transform**. Let $\rho : G \to GL(V_\rho)$ be a finite-dimensional complex representation of a finite group G. A **Fourier transform** of a function $f : G \to \mathbb{C}$ is an element $\hat{f}(\rho) \in \text{End}_{\mathbb{C}}(V_\rho)$ defined by the formula
$$\hat{f} = \sum_{g \in G} f(g)\rho(g)$$
   (a) Prove that $\widehat{f_1 * f_2} = \widehat{f_1}\widehat{f_2}$.
   (b) Prove the **Fourier inversion formula**, given by
$$f(g) = \frac{1}{|G|} \sum_{\rho} \dim(V_\rho) \operatorname{tr}\left[\rho(g^{-1})\hat{f}(\rho)\right]$$
   where the sum goes over all irreducible representations of G.
   (c) Prove that **Plancherel formula** for functions $f_1, f_2 : G \to \mathbb{C}$, given by
$$\sum_{g \in G} f_1(g^{-1}) f_2(g) = \frac{1}{|G|} \sum_{\rho} \dim(V_\rho) \operatorname{tr}\left[\widehat{f_1}(\rho)\widehat{f_2}(\rho)\right]$$
7. Consider the **Heisenberg group** $H(\mathbb{F}_3)$ consisting of matrices
$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$
for $a, b, c \in \mathbb{F}_3$. Find the character table of this group.
8. The character table is a square matrix. Compute the determinant of the character table.

6 The Symmetric Group and Polynomials

11/17: 1. Let G be a group of symmetries of a cube.

(a) Prove that $|G| = 48$ and that G has a normal subgroup isomorphic to S_4.

Proof. A **symmetry** of a 3D body is a map of the body to itself that sends adjacent vertices to adjacent vertices, and likewise for edges and faces. The cube has 8 vertices. Thus, when mapping it to itself, the first vertex you map can go to any of the 8 vertices, the second vertex you map can go to any of the 3 adjacent vertices, and the third vertex you map can go to one of the remaining two adjacent vertices (to fix the orientation). At this point, the position of all remaining vertices is completely specified by the first three and the constraints on the map. Thus, we have

$$|G| = 8 \cdot 3 \cdot 2$$
$$\boxed{|G| = 48}$$

as desired.

Now consider the group homomorphism $\phi : G \to \{-1, 1\} \cong \mathbb{Z}/2\mathbb{Z}$ that sends an element of G to 1 if the transformation satisfies the following condition: After fixing the first two vertices, the third vertex is sent to the adjacent vertex clockwise from the second looking down the axis connecting the first vertex and the center of the cube. Send the element of G to -1 if the third vertex is "mapped counterclockwise." It can be seen that ϕ is a group homomorphism by working through the four possibilities of $g \circ h$ where $g \mapsto 1$, $h \mapsto 1$; $g \mapsto 1$, $h \mapsto -1$; and so on and so forth. The kernel of ϕ will be a normal subgroup of order 24; moreover, we can see that this normal subgroup is isomorphic to S_4 by looking at its action on the four diagonals of the cube. ☐

(b) Prove that G is isomorphic to a group of signed permutations, i.e., linear maps

$$(x_1, x_2, x_3) \mapsto (\pm x_{\sigma(1)}, \pm x_{\sigma(2)}, \pm x_{\sigma(3)})$$

for $\sigma \in S_3$.

Proof. Let x_1, x_2, x_3 represent vectors pointing to three faces of the cube that surround one vertex. $x_1 \mapsto \pm x_{\sigma(1)}$ sends x_1 to any of the six faces of the cube. Once that is set, $x_2 \mapsto \pm x_{\sigma(2)}$ sends x_2 to any of the four faces *not in the span of* $x_{\sigma(1)}$ since σ is a bijection and thus cannot map 2 to the same place it maps 1. Note that the four faces available to the image of x_2 are all adjacent to $\text{span}(\text{Im}(x_1))$! Finally, $x_3 \mapsto \pm x_{\sigma(3)}$ sends x_3 to any of the two faces not in $\text{span}(\text{Im}(x_1, x_2))$; these are likewise adjacent to the faces of 1 and 2. Altogether, we get $6 \cdot 4 \cdot 2 = 48$ possible maps, once again, so every map is accounted for and there are no duplicates. ☐

(c) Find the character table of G.

Proof. The conjugacy classes of G are all of the maps which are "the same" in a different basis. Categorizing these and calculating characters, we get

1	1	1	1	1	1	1	1	1	1
1	1	−1	−1	1	1	−1	1	1	−1
2	−1	0	0	2	2	0	−1	2	0
3	0	−1	1	−1	3	1	0	−1	−1
3	0	1	−1	−1	3	−1	0	−1	1
1	1	1	1	1	−1	−1	−1	−1	−1
1	1	−1	−1	1	−1	1	−1	−1	1
2	−1	0	0	2	−2	0	1	−2	0
3	0	−1	1	−1	−3	−1	0	1	1
3	0	1	−1	−1	−3	1	0	1	−1

□

2. For a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of n, define the conjugate partition $\lambda' = (\lambda'_1, \dots, \lambda'_{k'})$ by formula $\lambda'_i = \{\#j \mid \lambda_j \geq i\}$. Prove that λ' is also a partition of n and $(\lambda')' = \lambda$ *without* using the geometric picture.

Proof. To prove that λ' is a partition of n, it will suffice to show that (1) each $\lambda'_i \in \mathbb{N}$, (2) $\lambda'_1 \geq \cdots \geq \lambda'_{k'}$, and (3) $\lambda'_1 + \cdots + \lambda'_{k'} = n$. We'll tackle each claim individually.

<u>(1)</u>: As defined, λ'_i is a positive integer. Moreover, it is only equal to zero when $i > \lambda_1 \geq 1$, so we can just truncate the partition at this point and not define such λ'_i.

<u>(2)</u>: Let $1 \leq i_1 < i_2 \leq k'$. If no such i_1, i_2 exist, then the claim is vacuously true. Otherwise, we have that

$$\lambda'_{i_1} = \{\#j \mid \lambda_j \geq i_1\} \qquad\qquad \lambda'_{i_2} = \{\#j \mid \lambda_j \geq i_2\}$$

Since $\lambda_1 \geq \cdots \geq \lambda_k$, the number of j such that $\lambda_j \geq i_1$ will be equal to $|\{\lambda_1, \dots, \lambda_j\}|$. Moreover, since $i_2 > i_1$, the set of λ_i that are greater than i_2 will be a subset of $\{\lambda_1, \dots, \lambda_j\}$, with corresponding cardinality less than or equal to that of this set, as desired.

<u>(3)</u>: To see that $\lambda'_1 + \cdots + \lambda'_{k'} = n$, consider the sum $\lambda_1 + \cdots + \lambda_k = n$. Observe that $\lambda'_1 = k$ counts each number in the partition λ once. λ'_2 counts each number (greater than two) in the partition λ once. Together, λ'_1 and λ'_2 count all of the ones in λ once and all of the twos in λ twice. Continuing on, we see that each λ_i is counted by $\lambda'_1, \dots, \lambda'_{\lambda_i}$. This allows us to split and rearrange the sum $\lambda_1 + \cdots + \lambda_k$ in terms of $\lambda'_1 + \cdots + \lambda'_{k'}$. For example, if $\lambda = (4, 3, 1, 1)$, then

$$\begin{aligned} 9 &= 4 + 3 + 1 + 1 \\ &= (1_1 + 1_2 + 1_3 + 1_4) + (1_1 + 1_2 + 1_3) + (1_1) + (1_1) \\ &= (1_1 + 1_1 + 1_1 + 1_1) + (1_2 + 1_2) + (1_3 + 1_3) + (1_4) \\ &= 4 + 2 + 2 + 1 \end{aligned}$$

where we can calculate that $\lambda' = (4, 2, 2, 1)$ and the subscripts are simply an indexing method. Generalizing this, we obtain

$$n = \sum_{i=1}^{k} \lambda_i = \sum_{i=1}^{k} \sum_{j=1}^{\lambda_i} 1_j = \sum_{j=1}^{\lambda_1} \sum_{i=1}^{\lambda'_j} 1_j = \sum_{j=1}^{k'} \lambda'_j$$

as desired.

We now prove that $(\lambda')' = \lambda$. To do this, we can use an indexing/reindexing method similar to that used in the proof of claim (3) above. Essentially, as before, we split the sum down to a sum of 1's indexed by how many times we count each number. Additionally, however, this time we will also index the 1's by the left to right order in which we count each 1_i. For example,

$$\begin{aligned} 9 &= (1_1 + 1_2 + 1_3 + 1_4) + (1_1 + 1_2 + 1_3) + (1_1) + (1_1) \\ &= (1_1^1 + 1_2 + 1_3 + 1_4) + (1_1^2 + 1_2 + 1_3) + (1_1^3) + (1_1^4) \\ &= (1_1^1 + 1_2^1 + 1_3 + 1_4) + (1_1^2 + 1_2^2 + 1_3) + (1_1^3) + (1_1^4) \\ &= (1_1^1 + 1_2^1 + 1_3^1 + 1_4) + (1_1^2 + 1_2^2 + 1_3^2) + (1_1^3) + (1_1^4) \\ &= (1_1^1 + 1_2^1 + 1_3^1 + 1_4^1) + (1_1^2 + 1_2^2 + 1_3^2) + (1_1^3) + (1_1^4) \end{aligned}$$

What this allows us to see is that rearranging the above sum, as we have done previously, to the following effectively "switches" the top and bottom indices.

$$(1_1^1 + 1_1^2 + 1_1^3 + 1_1^4) + (1_2^1 + 1_2^2) + (1_3^1 + 1_3^2) + (1_4^1)$$

Naturally, then, a further inversion will be an exercise in "switching back" to the original. □

3. Let G be a finite group, $Z(G)$ its center, and V an irreducible representation of it. We proved in class that $\dim(V)$ divides the order of G. Prove the stronger statement that $\dim(V)$ divides the index $(G : Z(G))$.

Proof. Let $n \in \mathbb{N}$ be arbitrary, and let ρ be the homomorphism $G \mapsto GL(V)$ corresponding to V. Let G^n be the n^{th} direct product of G with itself. Then $\rho^n : G^n \to GL(V^{\otimes n})$ defined by

$$(g_1, \dots, g_n) \mapsto \rho(g_1) \otimes \cdots \otimes \rho(g_n)$$

makes $V^{\otimes n}$ into a representation of G^n. Moreover, $V^{\otimes n}$ is an *irreducible* representation of G^n: Since

$$\begin{aligned}
\chi_{V^{\otimes n}}(g_1, \dots, g_n) &= \operatorname{tr}[\rho^n(g_1, \dots, g_n)] \\
&= \operatorname{tr}[\rho(g_1) \otimes \cdots \otimes \rho(g_n)] \\
&= \operatorname{tr}[\rho(g_1)] \times \cdots \times \operatorname{tr}[\rho(g_n)] \\
&= \chi_V(g_1) \times \cdots \times \chi_V(g_n)
\end{aligned}$$

we can use the irreducibility criterion to confirm that

$$\begin{aligned}
\sum_{(g_1,\dots,g_n)\in G^n} \chi_{V^{\otimes n}}(g_1, \dots, g_n)^2 &= \sum_{(g_1,\dots,g_n)\in G^n} \chi_V(g_1)^2 \cdots \chi_V(g_n)^2 \\
&= \prod_{i=1}^n \sum_{g_i \in G} \chi_V(g_i)^2 \\
&= \prod_{i=1}^n |G| \\
&= |G|^n \\
&= |G^n|
\end{aligned}$$

Now consider the subgroup H of G^n defined as follows.

$$H := \{(g_1, \dots, g_n) \in Z(G)^n \mid g_1 \cdots g_n = e\}$$

To begin, H is normal: If $(h_1, \dots, h_n) \in H$ and $(g_1, \dots, g_n) \in G^n$, then

$$\begin{aligned}
(g_1, \dots, g_n)(h_1, \dots, h_n)(g_1, \dots, g_n)^{-1} &= (g_1 h_1 g_1^{-1}, \dots, g_n h_n g_n^{-1}) \\
&= (h_1 g_1 g_1^{-1}, \dots, h_n g_n g_n^{-1}) \\
&= (h_1, \dots, h_n)
\end{aligned}$$

Additionally, by the proposition from the 10/30 lecture, $\rho(g) = \lambda I$ for all $g \in Z(G)$. It follows that elements of H act trivially on $V^{\otimes n}$:

$$\begin{aligned}
\rho^n(h_1, \dots, h_n) V^{\otimes n} &= [\rho(h_1) \otimes \cdots \otimes \rho(h_n)](V \otimes \cdots \otimes V) \\
&= \rho(h_1)V \otimes \cdots \otimes \rho(h_n)V \\
&= \lambda_1 V \otimes \cdots \otimes \lambda_n V \\
&= (\lambda_1 \cdots \lambda_n V) \otimes V^{\otimes(n-1)} \\
&= (\rho(h_1) \cdots \rho(h_n) V) \otimes V^{\otimes(n-1)} \\
&= (\rho(h_1 \cdots h_n) V) \otimes V^{\otimes(n-1)} \\
&= (\rho(e) V) \otimes V^{\otimes(n-1)} \\
&= V^{\otimes n}
\end{aligned}$$

Thus, since H is normal and $H \leq \operatorname{Ker} \rho$, there exists a group homomorphism $\tilde{\rho} : G^n/H \to GL(V^{\otimes n})$ defined by

$$\tilde{\rho}((g_1, \dots, g_n)H) := \rho^n(g_1, \dots, g_n)$$

Moreover, not only does $\tilde{\rho}$ define a representation, but once again, we can confirm that it defines an irreducible representation. To see this, first note that since ρ^n is constant on the cosets of H partitioning G^n. Thus, we can partition the sum of $\chi_{V^{\otimes n}}(g_1, \dots, g_n)$ over all $(g_1, \dots, g_n) \in G^n$ into $|H|$ identical iterations, one for each corresponding member of each coset. For example, if we have group and subgroup S_3 and $\{e, (12)\}$ (with cosets $\{e, (12)\}$, $\{(13), (123)\}$, and $\{(23), (132)\}$), then we know that $\chi(e) = \chi(12)$, $\chi(13) = \chi(123)$, and $\chi(23) = \chi(132)$; thus,

$$\begin{aligned}
\sum_{g \in S_3} \chi(g) &= \chi(e) + \chi(12) + \chi(13) + \chi(123) + \chi(23) + \chi(132) \\
&= (\chi(e) + \chi(13) + \chi(23)) + (\chi(12) + \chi(123) + \chi(132)) \\
&= 2(\chi(e) + \chi(13) + \chi(23)) \\
&= |\{e, (12)\}| \cdot (\chi[e\{e, (12)\}] + \chi[(13)\{e, (12)\}] + \chi[(23)\{e, (12)\}]) \\
\frac{1}{|\{e, (12)\}|} \sum_{g \in S_3} \chi(g) &= \sum_{g\{e,(12)\} \in S_3/\{e,(12)\}} \chi(g\{e, (12)\})
\end{aligned}$$

Consequently, generalizing back to our original quotient group, we have that

$$\begin{aligned}
\sum_{(g_1,\dots,g_n)H \in G^n/H} \chi_{V^{\otimes n}}[(g_1, \dots, g_n)H]^2 &= \frac{1}{|H|} \sum_{(g_1,\dots,g_n) \in G^n} \chi_{V^{\otimes n}}(g_1, \dots, g_n)^2 \\
&= \frac{|G^n|}{|H|} \\
&= |G^n/H|
\end{aligned}$$

Thus, since $V^{\otimes n}$ is an irreducible representation of G^n/H, we have by the Frobenius divisibility theorem that $\dim(V^{\otimes n}) \mid |G^n/H|$. Additionally, the order of H is $|Z(G)|^{n-1}$, since we can pick $g_1, \dots, g_{n-1}$ freely from among $|Z(G)|$ choices, but g_n then must equal $g_{n-1}^{-1} \cdots g_1^{-1}$ to satisfy the constraint. Consequently,

$$\dim(V)^n = \dim(V^{\otimes n}) \mid |G^n/H| = \frac{|G|^n}{|Z(G)|^{n-1}}$$

Now let $|G| = |Z(G)| \cdot x$. Then

$$\dim(V)^n \mid x^n |Z(G)|$$

for all $n \in \mathbb{N}$. In particular, it follows that $\dim(V) \mid x$; we can see this by letting $p_1^{e_1} \cdots p_m^{e_m}$ be the prime factorization of $\dim(V)$ for $e_1, \dots, e_m \geq 1$, $p_1^{f_1} \cdots p_m^{f_m} \cdot c_1$ be the prime factorization of x for $f_1, \dots, f_m \geq 0$ and $(c_1, p_i) = 1$ $(i = 1, \dots, m)$, and $p_1^{g_1} \cdots p_m^{g_m} \cdot c_2$ be the prime factorization of $|Z(G)|$ for $g_1, \dots, g_m \geq 0$ and $(c_2, p_i) = 1$ $(i = 1, \dots, m)$ so that

$$\dim(V)^n \mid x^n |Z(G)| \quad \Longrightarrow \quad p_1^{e_1 n} \cdots p_m^{e_m n} \mid p_1^{f_1 n + g_1} \cdots p_m^{f_m n + g_m} \cdot c_1^n c_2$$

and hence $e_i n \leq f_i n + g_i$ $(i = 1, \dots, m)$; it follows that we always have $-g_i/n \leq f_i - e_i$, so as $n \to \infty$, we obtain $0 \leq f_i - e_i$, or $f_i \geq e_i$, which gives the desired divisibility. Since $\dim(V) \mid x$ and $x = |G|/|Z(G)|$, we have that

$$\dim(V) \mid |G|/|Z(G)| = (G : Z(G))$$

as desired. □

4. Prove the uniqueness part of the fundamental theorem about symmetric polynomials.

   Proof. Suppose $P = Q(\sigma_1, \dots, \sigma_n) = T(\sigma_1, \dots, \sigma_n)$. □

5. Let G be a finite non-abelian simple group. Every 1-dimensional representation of G is trivial (why?).

Proof. Let $\rho : G \to \mathbb{C}^*$ be a one-dimensional representation of G. Since ρ is a group homomorphism, $\operatorname{Ker}\rho \lhd G$. Thus, since G is simple, $\operatorname{Ker}\rho = \{e\}$ or $\operatorname{Ker}\rho = G$. Suppose for the sake of contradiction that $\operatorname{Ker}\rho = \{e\}$. Let $g, h \in G$ be arbitrary. Since the product of two roots of unity (such as $\rho(g)$ and $\rho(h)$) commutes, we have that

$$\begin{aligned} \rho(g) \cdot \rho(h) &= \rho(h) \cdot \rho(g) \\ \rho(gh) &= \rho(hg) \end{aligned}$$

Thus, since ρ is faithful, $gh = hg$. But then G is abelian, a contradiction. Therefore, we must have $\operatorname{Ker}\rho = G$, i.e., ρ is trivial. □

Prove that any 2-dimensional representation of G is trivial as follows. Suppose that $\rho : G \to GL_2(\mathbb{C})$ is nontrivial.

(a) Prove that G has an element x of order 2.

Proof. Only normal subgroups of G are G, $\{e\}$. Let $x \in G$. Suppose no such x exists. Then it's abelian (or not simple). $xgx^{-1} = xg = x^{-1}g$.
We have to deduce this from the supposition?? □

(b) Prove that $\rho(x) = -\operatorname{id}$.

Proof. We know that $\rho(x)^2 = \operatorname{id}$. We also know that $\rho(x)$ is diagonalizable with roots of unity for eigenvalues. Since ρ is faithful, $\rho(x) \neq \operatorname{id}$. What if only one of the diagonal values in the matrix was -1?? □

(c) Prove that $\rho([g, x]) = \operatorname{id}$ for any g and deduce the theorem.

Proof. Let $g \in G$ be arbitrary. Then

$$\begin{aligned} \rho([g, x]) &= \rho(g^{-1}x^{-1}gx) \\ &= \rho(g^{-1}) \circ \rho(x^{-1}) \circ \rho(g) \circ \rho(x) \\ &= \rho(g)^{-1} \circ \rho(x) \circ \rho(g) \circ \rho(x) \\ &= \rho(g)^{-1} \circ (-\operatorname{id}) \circ \rho(g) \circ (-\operatorname{id}) \\ &= \rho(g)^{-1} \circ \rho(g) \circ (-\operatorname{id}) \circ (-\operatorname{id}) \\ &= \operatorname{id} \end{aligned}$$

Thus, $\rho([g, x]) = \rho(e)$. So since ρ is faithful once again, $[g, x] = e$. Since $Z(G) \lhd G$ and G is nonabelian, $Z(G) = \{e\}$. However, we have just shown that $x (\neq e) \in Z(G)$, a contradiction. □

6. Consider the ring of symmetric polynomials $R = \mathbb{Q}[x_1, \dots, x_n]^{S_n}$.

(a) Define the **power-sum symmetric polynomials** as follows.

$$p_k(x_1, \dots, x_n) := x_1^k + \cdots + x_n^k$$

Prove the Newton formulas

$$me_m(x_1, \dots, x_n) = \sum_{i=1}^{m} (-1)^{i-1} e_{m-i}(x_1, \dots, x_n) p_i(x_1, \dots, x_n)$$

Prove that $R = \mathbb{Q}[p_1, \dots, p_n]$.

Proof. Consider a polynomial of degree m with roots $x_1, \dots, x_m$. Then — much like in the characteristic polynomial — we have that

$$\prod_{i=1}^{m} (x - x_i) = \sum_{i=0}^{m} e_i(x_1, \dots, x_n)$$

□

(b) Define the **complete symmetric polynomials** $h_k(x_1, \dots, x_n)$ as a sum of all *distinct* monomials of degree k. For instance,

$$h_3(x_1, x_2) = x_1^3 + x_2^3 + x_1^2 x_2 + x_1 x_2^2 + x_1 x_2 x_3$$

Prove that

$$\sum_{i=0}^{m} (-1)^i e_i(x_1, \dots, x_n) h_{m-i}(x_1, \dots, x_n) = 0$$

Prove that $R = \mathbb{Q}[h_1, \dots, h_n]$.

7. Compute explicitly the characters of all representations V_λ of S_4 using the construction with Specht polynomials. Check that you get the same results as we have obtained before.

Proof. We have

$$\begin{aligned}
V_{(4)} &= \langle 1 \rangle \\
V_{(3,1)} &= \langle \underbrace{(x_1 - x_2)}_{a}, \underbrace{(x_1 - x_3)}_{b}, \underbrace{(x_1 - x_4)}_{c} \rangle \\
V_{(2,2)} &= \langle \underbrace{(x_1 - x_2)(x_3 - x_4)}_{a}, \underbrace{(x_1 - x_3)(x_2 - x_4)}_{b} \rangle \\
V_{(2,1,1)} &= \langle \underbrace{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)}_{a}, \underbrace{(x_1 - x_2)(x_1 - x_4)(x_2 - x_4)}_{b}, \underbrace{(x_1 - x_3)(x_1 - x_4)(x_3 - x_4)}_{c} \rangle \\
V_{(1,1,1,1)} &= \langle (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4) \rangle
\end{aligned}$$

$\underline{V_{(4)}}$: There is nothing to permute, so the characters are

$$(1, 1, 1, 1, 1)$$

$\underline{V_{(3,1)}}$: e will send each element to itself, so $\chi(e) = 3$. The other four, we will bash out. (12) will send $a \mapsto -a$, $b \mapsto b - a$, and $c \mapsto c - a$. (123) will send $a \mapsto b - a$, $b \mapsto -a$, and $c \mapsto c - a$. (1234) will send $a \mapsto b - a$, $b \mapsto c - a$, and $c \mapsto -a$. (12)(34) will send $a \mapsto -a$, $b \mapsto c - a$, and $c \mapsto b - a$. Thus, the matrices of these latter four elements are

$$\underbrace{\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{(12)} \qquad \underbrace{\begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{(123)} \qquad \underbrace{\begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{(1234)} \qquad \underbrace{\begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{(12)(34)}$$

Thus, the characters are

$$(3, 1, 0, -1, -1)$$

$\underline{V_{(2,2)}}$: e will send each element to itself, so $\chi(e) = 2$. The other four, we will bash out. (12) will send $a \mapsto -a$ and $b \mapsto b - a$. (123) will send $a \mapsto b - a$ and $b \mapsto -a$. (1234) will send $a \mapsto a - b$ and $b \mapsto -b$. (12)(34) will send $a \mapsto a$ and $b \mapsto b$. Thus, the matrices of these latter four elements are

$$\underbrace{\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}}_{(12)} \qquad \underbrace{\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}}_{(123)} \qquad \underbrace{\begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}}_{(1234)} \qquad \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{(12)(34)}$$

Thus, the characters are

$$(2, 0, -1, 0, 2)$$

$\underline{V_{(2,1,1)}}$: e will send each element to itself, so $\chi(e) = 2$. The other four, we will bash out. (12) will send $a \mapsto -a$, $b \mapsto -b$, and $c \mapsto c - a$. (123) will send $a \mapsto b - a$, $b \mapsto -a$, and $c \mapsto c - a$. (1234) will send

$a \mapsto b - a$, $b \mapsto c - a$, and $c \mapsto -a$. $(12)(34)$ will send $a \mapsto -a$, $b \mapsto c - a$, and $c \mapsto b - a$. Thus, the matrices of these latter four elements are

$$\underbrace{\begin{bmatrix} -1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{(12)} \qquad \underbrace{\begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{(123)} \qquad \underbrace{\begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{(1234)} \qquad \underbrace{\begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{(12)(34)}$$

Thus, the characters are

$$(3, -1, 0, 1, -1)$$

$\underline{V_{(1,1,1,1)}}$: This is a single antisymmetric polynomial, so any $\sigma \in S_4$ will just map it to $(-1)^\sigma$ times itself. Thus, the characters are

$$(1, -1, 1, -1, 1)$$

Therefore, all characters match previous results! $\square$

7 Classifying Representations of the Symmetric Group

12/1:

1. Prove that the representation $(n-k,1,\dots,1)$ is isomorphic to a wedge power of the standard representation of S_n.
2. Deduce from the hook length formula that if V is an irreducible representation of S_n for $n \geq 5$ and $\dim(V) < n$, then $\dim(V) = 1$ or $\dim(V) = n-1$.
3. Find the pairs of Young tableaux corresponding to the following permutations.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 4 & 3 & 5 & 7 & 6 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 5 & 3 & 6 \end{pmatrix}$$

4. Prove that the character table of S_n is filled with rational numbers.
5. Let $X_i : S_n \to \mathbb{C}$ send each $\sigma \in S_n$ to the number of i-cycles in σ. Clearly X_i is a class function. Moreover, any polynomial $P(X_1,\dots,X_n)$ is a class function called a **character polynomial**.
   (a) Find a basis of class functions consisting of character polynomials for S_2, S_3, and S_4.
   (b) Do the same for S_5.
   (c) Find a character polynomial which equals the character of $(n-1,1)$.
6. Consider a subgroup $H = \{e, (12)(34), (13)(24), (14)(23)\}$ in S_4. Decompose $\mathrm{Ind}_H^G \chi$ into irreducibles for all irreducible characters χ of H.

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