2 Introduction to Character Theory

10/13: 1. **More linear algebra**. Let V be a finite-dimensional vector space.

(a) Prove that under the identification of $V \otimes V^*$ with $\mathrm{Hom}_F(V,V)$, **simple** tensors $v \otimes \varphi$ correspond to linear maps of rank 0 or 1.

(b) Consider the vector space $W = \mathrm{Hom}_F(V,V)$. Prove that any linear functional in W^* has the form $L \mapsto \mathrm{tr}(LM)$ for some $M \in W$. Prove that the vector space $\mathrm{Hom}_F(V,V)$ is "canonically" self-dual.

2. **Characters of abelian groups**. Let A be a finite abelian group.

(a) A **character** of A is a homomorphism $\chi : A \to \mathbb{C}^\times$. Prove that for every $g \in A$, the value $\chi(g)$ is a root of unity. Prove that the product of characters is a character. Prove that characters form an abelian group. This group is called the **dual** of A and is denoted $\widehat{A}$.

(b) Prove directly that for every nontrivial character $\chi \in \widehat{A}$, the following identity holds.

$$\sum_{g \in A} \chi(g) = 0$$

(c) Prove that characters are the same as the 1-dimensional representations of A; product of characters is the same as a tensor product of representations, and the inverse of the character is the same as the dual representation.

(d) Find all characters for $A = \mathbb{Z}/n\mathbb{Z}$. Compute the dual group $\widehat{A}$. Do the same for $A = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

(e) Prove that $\widehat{A_1 \times A_2}$ is isomorphic to $\widehat{A}_1 \times \widehat{A}_2$. Prove that groups A and $\widehat{A}$ are isomorphic as abstract groups. Deduce that an abelian group of order n has exactly n characters.

3. Consider the permutational representation of S_n. Decompose it into the sum of (two) irreducible representations.

4. Let G be a finite group.

(a) Define the **space of invariants** of a representation V by the formula

$$V^G = \{v \in V \mid gv = v \ \forall\ g \in G\}$$

Prove that V^G is a subrepresentation of V. Prove that it is isomorphic to a sum of trivial representations.

(b) Prove that $(\mathrm{Hom}_F(V,W))^G$ is isomorphic to $\mathrm{Hom}_G(V,W)$.

5. Let $\rho : G \to GL_n(\mathbb{C})$ be a representation with character χ.

(a) Prove that $\mathrm{Ker}(\rho) = \{g \in G \mid \chi(g) = n\}$.

(b) Prove that for any $g \in G$, we have $|\chi(g)| \leq n$.

(c) Prove that for a given $g \in G$, $|\chi(g)| = n$ if and only if there exists $\lambda \in \mathbb{C}$ such that $\rho(g) = \lambda I$.