

Week 1

Introduction to Representation Theory

1.1 Motivating and Defining Representations

- 9/27:
- Rudenko would happily approve my final substitution, but it's not his call; it's Boller's.
 - HW will be due every week on Wednesday or thereabouts.
 - Submit in paper in a mailbox, location TBA.
 - First HW due next Wednesday.
 - Midterm eventually and an in-class final.
 - Grading scheme in the syllabus.
 - OH not available MW after class (Rudenko has to run to something else), but F after class, we can ask him anything.
 - Regular OH MTh, time TBA.
 - There is no specific book for the course.
 - First 8 lectures come from Serre (1977); amazing book but very concise; gets confusing later on. Most lectures are made up by Rudenko.
 - Course outline.
 1. Character theory: Beautiful, not too hard.
 2. Non-commutative algebra: More abstract/general approach to the same thing.
 3. Advanced topics, S_n .
 - This course's focus: Representations of finite groups in finite dimensions over \mathbb{C} .
 - This course is for math-inclined people (not quite physics) and lays the foundation for all other Rep Theory.
 - The ideas would be presented in a very different way in Physics Rep Theory.
 - We can always ask questions and stop him to correct mistakes during class.
 - Why we care about representations.
 - Start with a group G , finite. For example, let $G \equiv S_1$.

- People started to play with S_4 (permutations of roots of a polynomial of degree 4) in Galois theory.
 - Galois theory primer: Consider a polynomial like $x^4 + 3x + 1 = 0$; the roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ satisfy tons of equations, e.g., $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$ since 1 is the x^0 term.
- But groups also occur in much more natural places, e.g., isometries of \mathbb{R}^3 that preserve a tetrahedron.
- S_4 is also orientation-preserving isometries of \mathbb{R}^3 that preserve a cube.
- Many things lead to the same group!
- Theory of abstract groups developed far later than any of these perspectives; was developed to unify them.
- Recall group actions: Take $G, X = \{x_1, \dots, x_n\}$ both finite. We want $G \curvearrowright X$, which is a homomorphism $A : G \rightarrow S_n$.
- What can we do now?
 - We can look at orbits, which are smaller pieces.
 - We can look at the stabilizer.
 - We can identify orbits with cosets.
 - If we understand all possible subgroups, we understand all possible actions.
- This story is not boring, but it's simplistic.
- Rudenko doesn't assume we remember everything (phew!).
- Main definition (general to start, then we simplify).
- **Group representation** (of G on V): A group homomorphism $G \rightarrow GL(V)$, for G a group, V a finite-dimensional vector space over some field \mathbb{F} with basis $\{e_1, \dots, e_n\}$, and $GL(V)$ the set of isomorphic linear maps $L : V \rightarrow V$. Denoted by ρ .
 - Recall that $GL(V) = GL_n(\mathbb{F})$ is the set of all $n \times n$ invertible matrices.
- For every element $g \in G$, $g \mapsto \rho(g) = A_g$. Essentially, you're mapping to elements that satisfy certain equations.
 - For example, $A_e = E_n$, $A_{g_1g_2} = A_{g_1}A_{g_2}$, and $A_{g^{-1}} = A_g^{-1}$.
 - Thus, representations are a “concrete way to think about groups.”
 - If you don't understand abstract group G , let us compare it to a group that we do understand! Like a group can *act* on S_n , we can *represent* a group in a vector space.
- In this course, G is finite, $\mathbb{F} = \mathbb{C}$, and V is finite dimensional.
 - This is the most simple case, but also a very interesting one. The theory is much, much easier, so we can get much more complicated, but this is a good place to start.
 - We could make G compact, but we're not gonna go that far.
- Examples to get an idea of what's going on.
 1. $\dim \rho = 1$ (means $\dim V = 1$). Then $\rho : G \rightarrow GL_1(V) = \mathbb{C}^\times$. The codomain is referred to as the **character** of the group.
 - An example group homomorphism $S_n \rightarrow \mathbb{C}^\times$ is the sign function $\sigma \mapsto \text{sign}(\sigma) = \{\pm 1\}$.
 - Another example is the **trivial representation**, $G \rightarrow \mathbb{C}^\times$ and $g \mapsto 1$.
 2. Smallest one: Let $G = S_3$. The structure is already pretty rich, and this will be part of the homework.

- **Trivial representation** again.
 - **Alternating representation.**
 - **Standard representation.**
 - **Regular representation.**
- **Trivial representation:** The representation $\rho : G \rightarrow GL(V)$ sending $g \mapsto 1$ for all $g \in G$. Denoted by $\square\square\square$, **(3)**.
 - The boxes notation is too much of a detour to explain now.
 - Note that $1 \in GL(V)$ is the identity map on V !
 - **Alternating representation:** The representation $\rho : G \rightarrow GL(V)$ sending $g \mapsto \text{sign}(g)$ for all $g \in G$. Denoted by $\begin{smallmatrix} \square \\ \square \end{smallmatrix}$, **(1, 1, 1)**.
 - **Standard representation:** The representation $\rho : S_n \rightarrow GL(V)$ sending $\sigma \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)})$, where $V = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_1 + \dots + x_n = 0\}$ is a $(n-1)$ -dimensional vector space. Denoted by $\begin{smallmatrix} \square & \square \end{smallmatrix}$, **(2, 1)**.
 - A 2D representation like rotating a triangle.
 - This gives something with real numbers.
 - Example: $S_3 \curvearrowright V$ by $\sigma((x_1, x_2, x_3)) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$.
 - **Regular representation:** The representation $\rho : G \rightarrow \text{Hom}(\mathbb{C}^n)$ defined by $g \mapsto \sigma_g$, where $G = \{g_1, \dots, g_n\}$, $\{e_{g_1}, \dots, e_{g_n}\}$ is a basis of \mathbb{C}^n , \cdot is the group action of $\rho(G) \curvearrowright \mathbb{C}^n$ by $\rho(g) \cdot e_g = e_{gg_i}$, and $\sigma_g(e_{g_i}) = \rho(g) \cdot e_g = e_{gg_i}$.
 - This is a permutation of vectors.
 - Thus, for S_3 , it will already be 6-dimensional (it's very high dimensional).
 - How do we know that representation theory is tractable? Sure, we can define all these things, but how do we know that it will lead anywhere? Here's an example.
 - Let $G = \mathbb{Z}/2\mathbb{Z} = \{e, g\}$, $V = \mathbb{C}^n$, A an $n \times n$ matrix over \mathbb{C} , $\rho : G \rightarrow GL_n(\mathbb{C})$, and $A := \rho(g)$. Since $g^2 = e$, we know for example that $A^2 = E_n$.
 - But how do we find the matrices A ? If we look at eigenvalues of A , there are only two possibilities: ± 1 . The structure of A can be very complicated with Jordan normal form and all that, but in fact, these are the **semisimple matrices**, so it's not that bad.
 - Since $A^2 = E$, we know that $(A - E)(A + E) = 0$. Consider $(A - E) : V \rightarrow V$. Naturally, it has $\ker(A - E)$ and $\text{Im}(A - E)$. In this particular case, Rudenko claims that $\ker(A - E) \cap \text{Im}(A - E) = \{0\}$.
 - **Proof:** Let $v \in \ker(A - E) \cap \text{Im}(A - E)$ be arbitrary. Since $v \in \text{Im}(A - E)$, there exists $w \in V$ such that $v = (A - E)w = Aw - w$. Since $v \in \ker(A - E)$, we have $(A - E)v = 0$, so $Av = v$. It follows that $A(Aw - w) = Aw - w$ but also $A(Aw - w) = Ew - Aw = w - Aw$. Thus,

$$\begin{aligned} Aw - w &= w - Aw \\ 2Aw &= 2w \\ Aw &= w \end{aligned}$$
 - But then $w \in \ker(A - E)$, so $v = (A - E)w = 0$.
 - This combined with the fact that every vector in a vector space is in either the image or the kernel of a linear map^[1] implies that $V = \ker(A - E) \oplus \text{Im}(A - E)$.

¹See Theorem 3.6 of Axler (2015).

- Let the kernel have basis e_1, \dots, e_k and the image have basis e_{k+1}, \dots, e_n ; then all A are of the following form.

$$\begin{array}{c} 1 \quad k \quad k+1 \quad n \\ \begin{array}{c} 1 \\ k \\ k+1 \\ n \end{array} \left[\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{array} \right] \end{array}$$

- Next time, we will discuss sums of representations, of which this is an example of the theory.
- The same kind of thing, **simple representations**, happens with all finite groups?? This is where we're going. It's not rocket science; in fact, we'll see it next week.
- Last thing for today: A remarkable story.
 - The story of representation theory started quite different.
 - A beautiful theorem that we can prove now!
 - Frobenius determinant.
 - Think of $G = \{g_1, \dots, g_n\}$. Picture its multiplication table.
 - In every row and column, you see each element once.
 - Let's associate to the multiplication table an actual determinant in the linear algebra sense. Consider elements x_{g_1}, \dots, x_{g_n} . Define the $n \times n$ matrix $(x_{g_i g_j})$. Take its determinant. It will be a polynomial in n variables, i.e., an element of the ring $\mathbb{Z}[x_{g_1}, \dots, x_{g_n}]$.
 - Example: Consider

$$\begin{vmatrix} e & g \\ g & e \end{vmatrix}$$

- The determinant is $x_e^2 - x_g^2 = (x_e - x_g)(x_e + x_g)$.

- Example: $G = \mathbb{Z}/3\mathbb{Z}$.

- If the elements are e, g, g^2 and we map these, respectively, to variables a, b, c , we get the matrix

$$\begin{bmatrix} e & g & g^2 \\ g & g^2 & e \\ g^2 & e & g \end{bmatrix} \mapsto \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

- The determinant is $3abc - a^3 - b^3 - c^3 = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac) = (a + b + c)(a + \zeta b + \zeta^2 c)(a + \zeta^2 b + \zeta c)$ where $\zeta^3 = 1$ is a root of unity.
- Frobenius's theorem: If G is a finite group and we take this Frobenius determinant, then this determinant is equal to $P_1^{d_1} \cdots P_k^{d_k}$ where P_1, \dots, P_k are irreducible polynomials in x_g, \dots, x_{g_j} , then $\deg P_i = d_i$ and k is the number of conjugacy classes.
- Example: Take S_3 ; we'll get a polynomial of degree $|S_3| = 6$ but the Frobenius determinant $FD = (x_{g_1} + \cdots + x_{g_k})(x_{g_1} \pm \cdots)(\text{some pol. of deg } 2)^2$
- The proof is remarkable and deep and uses what would become character theory. These polynomials are related to representations and the number of simplest irreducible representations. The theory that came out came as a way to understand this miracle. We'll forget FD's for now, but then come back and prove it later.

1.2 Key Definitions and Category Theory Primer

- 9/29:
- OH: TW 4:30 or 5:00 most likely; he will confirm later.
 - Today: Definitions in greater generality.
 - As before, let G be a finite group and V be a finite-dimensional vector space.
 - Goal of this course: Understand representations of G , that is...
 - Homomorphisms $\rho : G \rightarrow GL(V) = GL_n(\mathbb{C})$;
 - That send $g \mapsto A_g \in GL_n(\mathbb{C})$;
 - And satisfy $A_e = E$, $A_{g_1}A_{g_2} = A_{g_1g_2}$, and $A_{g^{-1}} = A_g^{-1}$.
 - What are some things we might want to do?
 - Build new representations from old? Investigate and/or classify irreducible representations?
 - Before we can see if any of this works or not, we need a ton of definitions: Sum, equality, etc.
 - Rudenko will start today's lecture with some general thoughts on the **category** of representations.
 - Categories are things that now permeates math.
 - **Category**: A *class* (not a set) of *objects* (some things; you don't know anything about them), and then a bunch of properties.

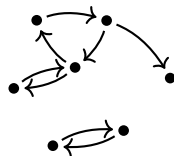


Figure 1.1: The general structure of a category.

- Objects a, b in category C are denoted by $a, b \in \mathbf{Ob}(C)$.
- There are also **morphisms** between the objects. These are drawn as arrows and lie in $\mathrm{Hom}(a, b)$.
- There is also composition: $\mathrm{Hom}(a, b) \times \mathrm{Hom}(b, c) \rightarrow \mathrm{Hom}(a, c)$.
 - What does this notation mean??
- Properties.
 1. Associativity.
 2. Existence of a unit: For any object a , there exists $\mathrm{id}_a \in \mathrm{Hom}(a, a)$ such that any morphism pre- or post-composed to this identity yields the same morphism.
 - Example: If $f \in \mathrm{Hom}(a, b)$, then $\mathrm{id}_b \circ f = f = f \circ \mathrm{id}_a$.
- Rudenko: So a category is basically two pieces of data and a bunch of properties.
- Examples of categories:
 - Category of sets and maps between them.
 - Category of vector spaces over \mathbb{F} where $\mathbf{Ob}(C)$ is the vector spaces and $\mathrm{Hom}(V, W)$ is filled with *linear* maps because you don't just want maps — you want maps that respect the structure.
 - Category of groups where $\mathrm{Hom}(G_1, G_2)$ is the set of group homomorphisms.
 - Category of topological spaces and continuous maps.
 - Category of abelian groups.
 - Trivial category and the identity map; thus, categories need not be chunky.

- Comments on category theory.
 - We'll see some pretty significant category theory at the end of the course.
 - We'll see categories in every course we take; some people try to avoid them. Rudenko doesn't want to go into the material in depth, but he wants to use language from it.
 - Surprisingly, even under the stripped-down of axioms of category theory, you can say quite a lot.
 - Why any of this discussion of category theory matters: If you know the basics of category theory, you can guess the definitions of direct sum, equality, etc. for representations.
- **Category of representations.** Denoted by \mathbf{Rep}_G .
- Take two G -representations V, W ; how do we define a map between them?
 - Recall that V, W are vector spaces.
- **Morphism** (of G -representations): A map $f : V \rightarrow W$ such that...
 1. f is linear;
 2. f respects the structure of the representations; explicitly, for every $g \in G$, $\rho_V(g) \circ f = f \circ \rho_W(g)$ ^[2].
- On constraint 2, above: This condition is summarized via a **commutative diagram**.

$$\begin{array}{ccc}
 V & \xrightarrow{f} & W \\
 \rho_V(g) \downarrow & \circlearrowleft & \downarrow \rho_W(g) \\
 V & \xrightarrow{f} & W
 \end{array}$$

Figure 1.2: Commutative diagram, morphisms.

- Commutative diagrams are very category-theory-esque things.
- That was a very abstract definition; let's make it concrete.
 - Suppose you have a pair of representations $V = \mathbb{C}^n, W = \mathbb{C}^m$, and we have our map f between them given by an $m \times n$ matrix.
 - Let $\rho_V(g) = A_g$ be an $n \times n$ matrix, and let $\rho_W(g) = B_g$ be an $m \times m$ matrix.
 - Then $FA_g = B_gF$.
- Examples.
 1. An interesting example: Let's look at $S_3 \subset V_{\text{perm}} = \mathbb{C}^3$, a **permutation representation**.
 - For all $\sigma \in S_3$, $\rho(\sigma) : (x_1, x_2, x_3) \mapsto (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$.
 2. There's also the trivial representation $S_3 \subset V_{(3)} = \mathbb{C}$ defined by $\rho(\sigma) : x \mapsto x$.
- Are the above 2 representations related?
 - Yes! We can, in fact, find a *morphism* between them.
 - In particular, define $f : V_{(3)} \rightarrow V_{\text{perm}}$ by $f(x) = (x, x, x)$.
 - Since permuting 3 of the same thing does nothing, the commutativity of Figure 1.2 holds. Therefore, f is a morphism of G -representations as defined above.
 - More explicitly,

$$f[\rho_{(3)}(\sigma)(x)] = f(x) = (x, x, x) = \rho_{\text{perm}}(\sigma)((x, x, x)) = \rho_{\text{perm}}(\sigma)[f(x)]$$

²Recall that the object, $\rho_V(g)$ is a linear map! Thus, it can be composed with other linear maps like f .

– Is f **reversible**?

■ Is “reversible” the right word??

– Define $\tilde{f} : V_{\text{perm}} \rightarrow V_{(3)}$ by $\tilde{f} : (x_1, x_2, x_3) \mapsto x_1 + x_2 + x_3$.

■ Since addition is commutative, the commutativity of Figure 1.2 holds.

■ More explicitly,

$$\begin{aligned}
 f[\rho_{\text{perm}}(\sigma)((x_1, x_2, x_3))] &= f((x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})) \\
 &= x_{\sigma(1)} + x_{\sigma(2)} + x_{\sigma(3)} \\
 &= x_1 + x_2 + x_3 && \text{Commutativity of addition} \\
 &= f((x_1, x_2, x_3)) \\
 &= \rho_{(3)}(\sigma)[f((x_1, x_2, x_3))]
 \end{aligned}$$

– Takeaway: The existence of maps between representations is interesting.

- Next question: How do we define an **isomorphism** of two representations?
- **Isomorphism** (of G -representations): A morphism of G -reps that is an isomorphism of vector spaces.
- Category theory helps us again here because it generalizes the concepts of an isomorphism!
 - If $f : V \rightarrow W$ and $g : W \rightarrow V$ are category-theoretic morphisms, then the constraints $f \circ g = \text{id}_W$ and $g \circ f = \text{id}_V$ make f and g into category-theoretic *isomorphisms*, regardless of what V and W might be.
 - Back in the context of representations, let $f : V \rightarrow V$ be an isomorphism of vector spaces. Then we do indeed have $\rho_V(g) \circ f = f \circ \rho_V(g)$, as we would hope from category theory!
- Recall the condition $FA_g = B_gF$. Supposing F is an isomorphism (and thus has an inverse), we get $FA_gF^{-1} = B_g$ as our new condition.
 - Essentially, we can do *simultaneous conjugation* of all matrices.
 - As per usual with isomorphisms, we get to *change bases*.
 - Essentially, we can represent the nice permutation representation in a very nasty basis but still have it be valid.
- Many other notions (e.g., direct sum) will not be explained by Rudenko, but we can read about them!
- However, we’ll do a few more.
- A representation sitting inside another: a **subrepresentation**.
- **Subrepresentation** (of V): A subspace $W \subset V$ such that for all $w \in W$ and $g \in G$, we have that $\rho_V(g)w \in W$, where V is a G -representation with $\rho_V : G \rightarrow GL(V)$.
 - Many people will just write the critical condition as $gW \subset W$.
- Subrepresentations in category theory: We have another commutative diagram.

$$\begin{array}{ccc}
 W & \hookrightarrow & V \\
 \rho_V(g) \downarrow & & \downarrow \rho_V(g) \\
 W & \hookrightarrow & V
 \end{array}$$

Figure 1.3: Commutative diagram, subrepresentations.

- Example: The trivial representation, the standard representation, and (of course) the **zero representation** are subrepresentations of the permutation representation.
- **Zero representation**: The representation $\rho : G \rightarrow GL(\{0\})$ sending $g \mapsto 1$ for all $g \in G$. Denoted by (0) .
- What about representations that don't have subrepresentations?
- **Simple** (representation): A G -representation V that has only two subrepresentations: (0) and V . Also known as **irreducible**, **irreps**.
- Example irreducible representations: Line in \mathbb{C}^2 , triangle in \mathbb{C}^2 , A_5 and dodecahedron in \mathbb{C}^3 .
- Notion of a direct sum.
- **Direct sum** (of V_1, V_2): The G -rep with the space $V_1 \oplus V_2 = \{(v_1, v_2) \mid v_1 \in V_1, v_2 \in V_2\}$ where $\rho_{V_1 \oplus V_2}(g)(v_1, v_2) = (\rho_{V_1}(g)v_1, \rho_{V_2}(g)v_2)$. Denoted by $V_1 \oplus V_2$.
 - The matrix of $\rho_{V_1 \oplus V_2}(g)$ is the following block matrix.

$$\rho_{V_1 \oplus V_2}(g) = \left[\begin{array}{c|c} \rho_{V_1}(g) & 0 \\ \hline 0 & \rho_{V_2}(g) \end{array} \right]$$

- Example: $V_{\text{perm}} = V_{(3)} \oplus V_{(2,1)}$, with $\mathbb{C}^3 = \mathbb{C} \oplus \mathbb{C}^2$ where

$$\mathbb{C} = \langle (1, 1, 1) \rangle \quad \mathbb{C}^2 = \langle (x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0 \rangle$$

- The decomposition is into simple representations.
- Relate this to the fact that the JCF of any 3×3 permutation matrix has at most a 1-block and a 2-block, if not three 1-blocks. There will always be one 1D subspace on which the permutation matrix is an identity, i.e., $\text{span}(1, 1, 1)$, and a 2D orthogonal complement!
- As a fun and simple exercise, prove that there is no line fixed under the standard representation.
- A simple and important theorem to prove next week.
- Theorem: Let G be a finite group and $\mathbb{F} = \mathbb{C}$. Then...
 1. There are finitely many irreps V_1, \dots, V_s up to isomorphism.
 - Later on, we will see that s is equal to the number of conjugacy classes.
 2. For every G -rep V , there exists a unique $n_1, \dots, n_s \geq 0$ such that $V \cong V_1^{n_1} \oplus \dots \oplus V_s^{n_s}$.
- This theorem tells us that if we want to study rep theory, we want to study irreps (which can be kind of complicated) because if we understand them, everything breaks down into them.
- Examples.
 1. $G = \mathbb{Z}/2\mathbb{Z} = S_2$.
 - $V_1 = \mathbb{C}e$ with $ge = e$ and $V_{-1} = \mathbb{C}e$ with $ge = -e$.
 - It follows that $V \cong V_1^{n_1} \oplus V_{-1}^{n_{-1}}$.
 - We get a diagonal matrix with only 1s and -1 s.
 2. $G = S_3$.
 - $V_{(3)}, V_{(1,1,1)}, V_{(2,1)}$.
 - $GL_5(\mathbb{F}_4)$.
 - Proven in an elementary way in Section 1.3 of Fulton and Harris (2004), which we have to read for the HW; will be useful for later in the course's HW.

- Plan: Next time, we'll talk about some more abstract stuff; tensor products of vector spaces.
 - Tensor products are something we should read up on now! The definition is hard and abstract.
 - Then he'll prove the above theorem.