3 Representation Structure and Characters

10/20: 1. **Permutational representation**. Let X be a finite set on which the group G acts. Let ρ be the corresponding permutational representation with character χ.

   (a) Consider an orbit Gx of an element $x \in X$; let c be the number of orbits. Prove that c equals the number of times ρ contains the trivial representation. Deduce that $(\chi, 1) = c$. In particular, if the action is transitive, $\rho = 1 \oplus \theta$ for some representation θ.

   (b) Consider a diagonal action of G on $X \times X$. Prove that the character of the corresponding permutational representation is χ^2.

   (c) Suppose that G acts transitively on X and $|X| \geq 2$. We call this action **doubly transitive** if every pair of distinct elements of X can be sent to any other pair by some element of G. Prove that the following are equivalent.

      i. The action is doubly transitive.
      ii. The diagonal action on $X \times X$ has exactly two orbits.
      iii. $(\chi^2, 1) = 2$.
      iv. The representation θ is irreducible.

2. Find the character table of the group A_4.

3. Consider the space of functions V from the set of faces of a cube to $\mathbb{C}$. This is a representation of S_4.

   (a) Compute the character of V.

   (b) Describe explicitly the decomposition of V into isotypical components.

   (c) Consider a map $A : V \to V$ acting by substituting the value of a function on a face with an average of its values on the adjacent four faces. Prove that A is an automorphism of the corresponding representation. Find its eigenvalues.

4. Consider a finite representation V of a group G with character χ.

   (a) Express the characters of $\Lambda^2 V$ and $S^2 V$ in terms of χ.

   (b) Express the characters of $\Lambda^3 V$ and $S^3 V$ in terms of χ.

   (c) Let $(3, 1)$ be the standard representation of S_4. Decompose $\Lambda^2(3, 1)$, $\Lambda^3(3, 1)$, $S^2(3, 1)$, and $S^3(3, 1)$ into irreducibles.