Week 8

???

8.1 Specht Modules are Irreducible and Well-Defined

11/13:
- Announcements.
  - This week's homework is the next to last one.
- Review.
  - Miracuously, we can understand all representations of S_n.
  - We start with partitions λ that are defined a certain way. We visualize them with Young diagrams.
  - The number of partitions of n is equal to the number of conjugacy classes in S_n is equal to the number of irreps in S_n.
    - It is a special feature of S_n that this is true.
  - How do we construct the irreducible representation V_λ due to λ?
    - Consider $(4,2,1)' = (3,2,1,1)$ as an example (recall the definition of an inverse partition).
    - Take Vandermonde determinants (recall the explicit definition of these, too).
    - Then we define $V_\lambda = \mathbb{C}[S_n]$, take Vandermonde determinant of variables corresponding to the first column, so that $\Delta(x_1,\dots,x_{\lambda_1'})\Delta(x_{\lambda_1'+1},\dots,x_{\lambda_2'})\cdots\Delta(x_{\lambda_{k-1}'+1},\dots,\lambda_k')$.
    - Thus, we let $\mathbb{C}[S_n]$ act on $(x_1-x_2)(x_1-x_3)(x_2-x_3)(x_4-x_5)$.
- One more example.
  - $\lambda = (2,2)$.
  - Let $\mathbb{C}[S_4]$ act on $(x_1-x_2)(x_3-x_4)$.
  - Then
$$V_\lambda = \langle (x_1-x_2)(x_3-x_4),(x_1-x_3)(x_2-x_4),(x_1-x_4)(x_2-x_3)\rangle$$
  - But we're expecting a 2D representation. Indeed, we get one because if we define the first term above to be a and the second to be b, then the third is $b-a$. Thus, there are only two linearly independent polynomials herein.
  - Now we calculate entries in the character table as follows: See how representatives of conjugacy classes like (12) and (123) acts on a,b via matrices, and then calculate traces of these matrices.
    - For example, using the definitions of a,b from above, we can see that
$$\begin{aligned}(12)\cdot a &= (12)\cdot(x_1-x_2)(x_3-x_4) = (x_2-x_2)(x_3-x_4) = -(x_1-x_2)(x_3-x_4) = -a\\ (12)\cdot b &= (12)\cdot(x_1-x_3)(x_2-x_4) = (x_2-x_3)(x_1-x_4) = (x_1-x_4)(x_2-x_3) = b-a\end{aligned}$$

■ In matrix form, the above equations become

$$\begin{bmatrix} -a \\ b-a \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}}_{\rho(12)} \begin{bmatrix} a \\ b \end{bmatrix}$$

■ Thus, $\chi(12) = \mathrm{tr}(\rho(12)) = 0$.

■ Similarly, we can calculate that

$$\begin{bmatrix} b-a \\ -a \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}}_{\rho(123)} \begin{bmatrix} a \\ b \end{bmatrix}$$

so $\chi(123) = -1$.

– One of the HW problems is to do exactly this for S_4 just for practice.

- Today: A theorem that does...??
- Note that V_λ is called a **Specht module** and one of these polynomials on the above blackboard is a **Specht polynomial**.
- Theorem 1: V_λ is irreducible.

Proof. $d(\lambda)$ is the degree of a Specht polynomial and is given by

$$\sum_{i=1}^{k'} \frac{\lambda_i'(\lambda_i'-1)}{2}$$

Let $R_d \subset \mathbb{C}[x_1, \dots, x_n]$, where R_d are just polynomials of degree d. Clearly, by definition, $V_\lambda \subset R_d$. Note that as a definition, $R_d = S^d(V_{\text{perm}}^*)$. We now claim that $\mathrm{Hom}_{S_n}(V_\lambda, R_d) \cong \mathbb{C}$.

This claim implies the theorem because if we assume that $V_\lambda = \bigoplus W_i^{n_i}$ and $R_d = \bigoplus W_i^{m_i}$ where the W_i are all irreps. Then we have a nice way to compute this Hom from previous classes, explicitly, that the only homomorphisms $W_i \to W_i$. Thus, $\dim \mathrm{Hom} = \sum n_i m_i$. What this claim implies is that $\dim \mathrm{Hom} = 1$. Additionally, we are in a subrepresentation, so $n_i \leq m_i$ for all i. Thus, we must have $n_i = 1, m_i = 1$ for some i and that $n_j, m_j = 0$ for all other j. Restated, WLOG let $1 \leq n_i$. Then since $n_i \leq m_i$ and $n_i m_i = 1$, we have $m_i = 1$ and all other $n_j, m_j = 0$. Thus, $V_\lambda = W_i$ is irreducible!

Now we actually have to prove the claim. Let $f \in \mathrm{Hom}_{S_n}(V_\lambda, R_d)$ be arbitrary. Consider

$$f(\Delta(x_1, \dots, x_{\lambda_1'})\Delta(x_{\lambda_1'+1}, \dots, x_{\lambda_2'}) \dots)$$

where the argument is the general Specht polynomial from above. $f(x)$ is a polynomial of degree d; call $f(x)$ by $P(x_1, \dots, x_n)$. It is antisymmetric in $x_1, \dots, x_{\lambda_1'}$. It's also antisymmetric in $x_{\lambda_1'+1}, \dots, x_{\lambda_2'}$. In fact, it's antisymmetric in all such sets all the way up to $x_{\lambda_{k'-1}'+1}, \dots, x_{\lambda_{k'}'}$. It follows that $P(x_1, \dots, x_n)$ is divisible by $\Delta(x_1, \dots, x_{\lambda_i'})$, etc., i.e., all Vandermonde determinants. Thus, $P(x_1, \dots, x_n)$ is divisible by the product, which is the Specht polynomials. It follows that $P(x_1, \dots, x_n) = u \cdot$Specht polynomial, from which it follows that $f = uI$. This implies the claim via the isomorphism $f \mapsto u$! □

- Corollary: If $d' < d$, then $\mathrm{Hom}(V_\lambda, R_d') = 0$.
- Theorem 2: Let λ_1, λ_2 be partitions of n. Then $V_{\lambda_1} \cong V_{\lambda_2}$ iff $\lambda_1 = \lambda_2$.

Proof. Suppose that $V_{\lambda_1} \cong V_{\lambda_2}$.

Then $d(\lambda_1) = d(\lambda_2)$ (take the columns and compute the degree of the Specht polynomial). (If not, WLOG let $d(\lambda_1) > d(\lambda_2)$. Then $V_{\lambda_1} \cong V_{\lambda_2} \hookrightarrow R_{d(\lambda_2)}$. But then by the above corollary, this overall injective embedding is the zero map, a contradiction.)

Let $d := d(\lambda_1) = d(\lambda_2)$. At this point, we have $V_{\lambda_1} \hookrightarrow R_d$ and $V_{\lambda_2} \hookrightarrow R_d$. It follows that $V_{\lambda_1} = V_{\lambda_2}$ as a subspace of R_d. Essentially, since we have the isomorphism $V_{\lambda_1} \cong V_{\lambda_2}$, we can construct the second embedding by factoring through the first; but then this second embedding should just give the same image.

Claim: Polynomials in $V_{\lambda_1}, V_{\lambda_2}$ (which we can think of as subspaces/explicit polynomials) have no monomials in common. For this, it's enough to understand monomials in one V_{λ_1}. Which monomials appear in V_λ? Here's an example. We will do a representative example instead of a formal group. Consider $\lambda = (5, 4, 2, 2)$ and S_{13}. $\lambda' = (4, 4, 2, 2, 1)$. Our Specht polynomial is

$$\Delta(x_1, x_2, x_3, x_4)\Delta(x_5, x_6, x_7, x_8)\Delta(x_9, x_{10})\Delta(x_{11}, x_{12})$$

since $\Delta(x_{13}) = 1$. We have that

$$\Delta(x_1, x_2, x_3, x_4) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{vmatrix} = \sum_{\sigma \in S_4} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)}^0 x_{\sigma(3)}^2 x_{\sigma(4)}^3$$

Then we will have for each column, a number of variables in power each of $0, \dots, 3$; on and on. Now we count the number of variables in power $0, \dots, 3$ to get 5,4,2,2. Thus, every monomial will have 5 variables in power 0, 4 variables in power 1, 2 variables in power 2, and 2 variables in power 3. Thus, from every monomial, we immediately reconstruct λ. It means that we can reconstruct from any monomial this representation, so this implies that we must have $\lambda_1 = \lambda_2$. ☐

- Corollary: V_λ's are all irreps of S_n

  Proof. They are pairwise isomorphic and their number equals n. ☐

8.2 Erdős-Szekeres Theorem

11/15:
- Recap.
  - Recall S_n and Young diagrams.
  - We've discussed conjugate Young diagrams corresponding to inverses λ' as well.
  - For every λ, we've constructed representations $V_{\lambda'}$.
  - Recall that $V_{\lambda'}$ is some representation inside the space of polynomials. In particular,

    $$V_{\lambda'} = \text{span}(\sigma[\Delta(x_1, \dots, x_{\lambda_1})\Delta(x_{\lambda_1+1}, \dots, x_{\lambda_2}) \cdots] \mid \sigma \in S_n)$$

    - Any $\sigma[\Delta(x_1, \dots, x_{\lambda_1})\Delta(x_{\lambda_1+1}, \dots, x_{\lambda_2}) \cdots]$ is a Specht polynomial $\text{Sp}_\lambda(x_1, \dots, x_n)$.
    - All of these Specht polynomials together span the irrep given by the corresponding Specht module.
  - We proved that Specht modules are irreducible last time.
  - Specht polynomials are polynomials in R_d, where R is the ring of polynomials in $x_1, \dots, x_n$ and $d = \binom{\lambda_1}{2} + \binom{\lambda_2}{2} + \cdots$. What is this definition of d??
- So how do we further study these representations?
  - Dimension?
  - Characters?
  - Basis?
- Guiding question for today: Which Specht polynomials $\text{Sp}_\lambda(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ form a baiss of $V_{\lambda'}$?

$$\begin{array}{ccccccc} 1 & < & 3 & < & 4 & < & 8 \\ \wedge & & \wedge & & \wedge & & \\ 2 & < & 5 & < & 6 & & \\ \wedge & & & & & & \\ 7 & & & & & & \end{array}$$

Figure 8.1: Example standard Young tableau.

- **Young tableau**: A Young diagram filled with integers. *Also known as* **YT**.
- **Standard** (Young tableaux): A YT filled in with numbers $1, \dots, n$, wherein each appears exactly once and the numbers increase in rows and in columns. *Also known as* **SYT**.
- Example of an SYT.
  - We start with a Young diagram.
  - We need to fill it with 8 numbers.
  - There are relations between the boxes.
  - Some constraints on what can go where, but there are more of them.
  - So there are three SYT_8. Call a tableaux within here T.
- Theorem: $\dim V_\lambda$ is the number of SYTs of shape λ.
- Examples.

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array}$$

(a) $\Delta(1234)$.

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array}$$

(b) $(x_1 - x_2)$. (c) $(x_1 - x_3)$. (d) $(x_1 - x_4)$.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

(e) $(x_1 - x_3)(x_2 - x_4)$. (f) $(x_1 - x_2)(x_3 - x_4)$.

Figure 8.2: Standard Young tableaux of 4.

  1. Only ONE way to fill trivial and alternating Young diagrams.
  2. Three ways to fill $(3, 1)$.
  3. Two ways to fill $(2, 2)$.
- Learn the representations of S_4 by heart!
- Proving the theorem.

  Proof. How are these tableaux related to the basis of Sp_λ?

  For a SYT, consider the polynomial $\text{Sp}(T)$ (take a particular filling and write down the polynomials; *see how to do so by the corresponding polynomials next to examples above in picture.*).

  Notice that the $\text{Sp}(T)$ form a basis of $V_{\lambda'}$. From here on out, the proof is quite subtle.

  Obvious part of the proof: Lemma: S_n, λ fixed implies that the $\text{Sp}(T)$, where $T \in \text{SYT}_\lambda$ (i.e., T is a SYT of shape λ), are linearly independent. They are each linearly independent because each contains a certain monomial that none of the others contain; specifically, this will be the lexicographically smallest monomial SM.

Proof. Consider the lexicographically smallest monomial in $\mathrm{Sp}(T)$ (the Specht polynomial of tableau T). Our goal is to reconstruct T from it.

Example of how to do this: Note: We have the same lemma as last time: $SM(PQ) = SM(P)SM(Q)$. Starting from the example on the previous blackboard, we have $\Delta(x_1, x_2, x_7)$. Smallest monomial is $x_2x_7^2x_5x_6$. This follows from the determinant interpretation of the vandermonde determinant polynomials by looking at all the smallest combinations we can put together, which goes along the diagonal of the matrix, e.g., $1 \cdot x_2 \cdot x_7^2$. From this monomial, we can reconstruct the SYT by putting x_7 in the bottom by necessity, then we have to put $2, 5, 6$ (other coefficients of SM) above in the certain order to get the right ordering. Then, we have to put the ones that aren't there ($x_1^0x_3^0x_4^0x_8^0$) in the top row. This gives us our YT back.

We will not give a rigorous proof here; the above example is illustrative. ☐

Rudenko will not finish this one. ☐

- Corollary: $\dim V_\lambda = \dim V_{\lambda'}$.

  Proof. Any representation of S_n will be self-dual. ☐

- Fact: We have the following identity.
$$V_{\lambda'} = V_\lambda \otimes (\mathrm{sign})$$
- Let f_λ be the number of SYTs of shape λ. We have shown that $f_\lambda \leq \dim(V_\lambda)$.
- Theorem (RSK): There exists a bijection between permutations in S_n and pairs of SYTs of the same shape (i.e., of **area** n).
  - RSK stands for Robinson-Schensted-Knuth.
- Corollary: $f_\lambda = \dim V_\lambda$.

  Proof. $\sum_{\lambda=\text{YT of area } n} f_\lambda^2 = n! = \sum(\dim V_\lambda^2)$. This is enough to prove that $f_\lambda \leq \dim V_\lambda$ and $f_\lambda = \dim V_\lambda$. ☐

- Let's see how this stuff works through an example.
  - Let's take permutation
$$\sigma = (13)(27654) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 1 & 2 & 4 & 5 & 6 & 8 \end{pmatrix} \in S_8$$
  - Start with a pair of YTs.
  - Start with the number written first (3) and then put 1. You take the empty tableaux and insert [3] inside.
  - We now have a new pair of tableaux. How do we insert the next number (7)? Try adding it to the right, then add 2 to the right of the second one in correspondence.
  - How do we insert the next number (1)? Push out 3 with 1 and move 3 to the next row. In the second one, there's one new square in the bottom position, so we add it there.
  - Next number: 2 pushes out 7. Four goes in the new box.
  - 4 gets inserted to the right; 5 fills the new box.
  - 5,6,8 go further to the right; 6,7,8 in the second one.
  - Now we have a pair of tableaux.
  - This is an algorithm that, for every permutation, gives us a pair of tableaux.

- Why does this work?
  - Everything is increasing in rows.
  - If $y > x$, we need to insert it in the row below but to the left.
  - Perhaps its starting with y in the top row and then it being displaced down and to the left by x.
  - Every time we add something bigger, we add a corner box.
  - This algorithnm proves the theorem.
  - I really need to think about this!!
- This algorithm takes us between a pair (T, T') of young tableaux to a permutation σ.
- Many interesting properties that are hard to prove. Here's a few.
  1. The map takes $(T', T) \mapsto \sigma^{-1}$.
- λ_1 and λ_1' are the length of the longest increasing (resp. decreasing) subsequence of your permutation variables.
- Last word: There is a famous theorem called the **Erdős-Szekeres theorem**.
- This correspondence is a deep way to understand permutations/sequences of numbers. This is a big tool in CS.
- Next time: Induction restriction.