6 The Symmetric Group and Polynomials

11/17: 1. Let G be a group of symmetries of a cube.

   (a) Prove that $|G| = 48$ and that G has a normal subgroup isomorphic to S_4.

   (b) Prove that G is isomorphic to a group of signed permutations, i.e., linear maps

$$(x_1, x_2, x_3) \mapsto (\pm x_{\sigma(1)}, \pm x_{\sigma(2)}, \pm x_{\sigma(3)})$$

   for $\sigma \in S_3$.

   (c) Find the character table of G.

2. For a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of n, define the conjugate partition $\lambda' = (\lambda'_1, \dots, \lambda'_{k'})$ by formula $\lambda'_i = \{\#j \mid \lambda_j \geq i\}$. Prove that λ' is also a partition of n and $(\lambda')' = \lambda$ *without* using the geometric picture.

3. Let G be a finite group, $Z(G)$ its center, and V an irreducible representation of it. We proved in class that $\dim(V)$ divides the order of G. Prove the stronger statement that $\dim(V)$ divides the index $(G : Z(G))$.

4. Prove the uniqueness part of the fundamental theorem about symmetric polynomials.

5. Let G be a finite non-abelian simple group. Every 1-dimensional representation of G is trivial (why?). Prove that any 2-dimensional representation of G is trivial as follows. Suppose that $\rho : G \to GL_2(\mathbb{C})$ is nontrivial.

   (a) Prove that G has an element x of order 2.

   (b) Prove that $\rho(x) = -\mathrm{id}$.

   (c) Prove that $\rho([g, x]) = \mathrm{id}$ for any g and deduce the theorem.

6. Consider the ring of symmetric polynomials $R = \mathbb{Q}[x_1, \dots, x_n]^{S_n}$.

   (a) Define the **power-sum symmetric polynomials** as follows.

$$p_k(x_1, \dots, x_n) := x_1^k + \cdots + x_n^k$$

   Prove the Newton formulas

$$me_m(x_1, \dots, x_n) = \sum_{i=1}^{m} (-1)^{i-1} e_{m-i}(x_1, \dots, x_n) p_i(x_1, \dots, x_n)$$

   Prove that $R = \mathbb{Q}[p_1, \dots, p_n]$.

   (b) Define the **complete symmetric polynomials** $h_k(x_1, \dots, x_n)$ as a sum of all *distinct* monomials of degree k. For instance,

$$h_3(x_1, x_2) = x_1^3 + x_2^3 + x_1^2 x_2 + x_1 x_2^2 + x_1 x_2 x_3$$

   Prove that

$$\sum_{i=0}^{m} (-1)^i e_i(x_1, \dots, x_n) h_{m-i}(x_1, \dots, x_n) = 0$$

   Prove that $R = \mathbb{Q}[h_1, \dots, h_n]$.

7. Compute explicitly the characters of all representations V_λ of S_4 using the construction with Specht polynomials. Check that you get the same results as we have obtained before.