5 Abstract Representation Theory

11/3: 1. Assume that a finite-dimensional left module M over an associative algebra A is semisimple. Prove that any quotient of M is semisimple.

2. Let F be a field.

   (a) Let D be a finite-dimensional division algebra over F. Prove that D^n is an irreducible module over the matrix algebra $M_n(D)$. Deduce that $M_n(D)$ is semisimple.

   (b) Let $D_1, \dots, D_k$ be finite-dimensional division algebras over F. Prove that an algebra $M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k)$ is semisimple. Prove that it has exactly k irreducible representations.

3. Let A be a finite-dimensional associative algebra over F. Prove that $\operatorname{Rad}(A)$ consists of elements $a \in A$ such that $1 + xa$ has a left inverse for any $x \in A$.

4. Let G be a finite group. Assume that a prime p divides the order of G. Prove that the set $\{\sum a_g g \mid \sum a_g = 0\}$ is an ideal in $\mathbb{F}_p[G]$. Show that it is not a direct summand of $\mathbb{F}_p[G]$. Deduce that $\mathbb{F}_p[G]$ is not semisimple.

5. Prove that under the identification of the group algebra $\mathbb{C}[G]$ with the space of complex-valued functions on G, the product in the group algebra corresponds to the convolution $*$ of functions

$$(f_1 * f_2)(g) = \sum_{x \in G} f_1(x) f_2(x^{-1}g)$$

   Deduce that the convolution is associative, and then prove the associativity directly. Which function plays the role of the unit?

   Proof. See MSE post linked in Lecture 3.1!!! □

6. **Fourier transform**. Let $\rho : G \to GL(V_\rho)$ be a finite-dimensional complex representation of a finite group G. A **Fourier transform** of a function $f : G \to \mathbb{C}$ is an element $\hat{f}(\rho) \in \operatorname{End}_{\mathbb{C}}(V_\rho)$ defined by the formula

$$\hat{f} = \sum_{g \in G} f(g)\rho(g)$$

   (a) Prove that $\widehat{f_1 * f_2} = \widehat{f_1}\widehat{f_2}$.

   (b) Prove the **Fourier inversion formula**, given by

$$f(g) = \frac{1}{|G|} \sum_{\rho} \dim(V_\rho) \operatorname{tr}\left[\rho(g^{-1})\hat{f}(\rho)\right]$$

   where the sum goes over all irreducible representations of G.

   (c) Prove that **Plancherel formula** for functions $f_1, f_2 : G \to \mathbb{C}$, given by

$$\sum_{g \in G} f_1(g^{-1}) f_2(g) = \frac{1}{|G|} \sum_{\rho} \dim(V_\rho) \operatorname{tr}\left[\widehat{f_1}(\rho)\widehat{f_2}(\rho)\right]$$

7. Consider the **Heisenberg group** $H(\mathbb{F}_3)$ consisting of matrices

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

   for $a, b, c \in \mathbb{F}_3$. Find the character table of this group.

8. The character table is a square matrix. Compute the determinant of the character table.