Week 4

???

4.1 Representation Ring; Character Basis

10/16:
- Announcements.
  - Reminder: Midterm 11/10.
  - OH this week in-person at normal times.
  - PSet 3 should be fun.
- Today: Finish proving some character things.
- Recall: The main picture.
  - Rudenko redraws Figure 3.1.
  - We have a finite group G and we are studying finite-dimensional G-reps over $\mathbb{C}$.
  - $\mathbb{C}_{\text{cl}}[G]$ is a ring.
  - The map...
    - Respects addition;
    - Sends tensor multiplication to (pointwise) functional multiplication;
    - Sends duality to conjugation;
    - Respects a kind of inner product, whether it be either side of $\dim_{\mathbb{C}} \operatorname{Hom}_G(V, W) = \langle f_1, f_2 \rangle$.
- Today, we will see that $\mathbb{C}_{\text{cl}}[G] \cong \mathbb{C}^k$, where k is the number of conjugacy classes.
  - In other words, we will see that the number of irreps is also exactly equal to k, that there is a bijection $\{V_i\} \to \{\chi_i\}$, and that the $\chi_1, \dots, \chi_k$ form an orthonormal basis of $\mathbb{C}_{\text{cl}}[G]$.
- Visualizing the vector space $\mathbb{C}_{\text{cl}}[G]$.

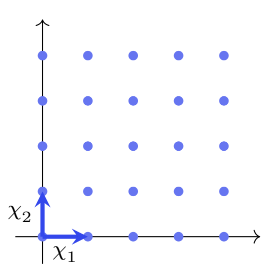

Figure 4.1: Visualizing the space of class functions on G.

  - It's a "cone" emanating from the origin with only lattice points.
    - If $\dim \mathbb{C}_{\text{cl}}[G] = 2$, the vector space consists of all the blue points in Figure 4.1.
  - Why is it only lattice points instead of a continuous function space?
    - The restrictions on coefficients are inherited from the restrictions on what kinds of spaces you can build of the form $V_1^{n_1} \oplus V_2^{n_2}$.
    - Indeed, if it were continuous, that would imply that there is some meaning to the point $0.3\chi_1 + 2.5\chi_2$, i.e., there is a space $V_1^{0.3} \oplus V_2^{2.5}$. But of course, we cannot define such a space!
  - Why is it only *nonnegative* integer coefficients and not *all* integer coefficients?
    - We don't have subtraction to get us to a full ring.
    - Additionally, we can only scale and linearly combine the χ_i's with nonnegative integer coefficients because, as said above, those are the types of reducible rep decompositions we have.

- Let $[V]$ denote the **isomorphism class** of the representation V.
- **Isomorphism class** (of V): The set of all vector spaces W that are isomorphic to V as representations.
- This allows us to define the **representation ring**.
- **Representation ring** (of G): The ring $(R, +, \cdot)$, where R is the free abelian group generated by all isomorphism classes of the representations of G, quotiented by the span of all linear combinations of the form $[V \oplus W] - [V] - [W]$; $+$ is well-defined via the construction of R, which yields $[V] + [W] = [V \oplus W]$ for all $[V], [W]$ in the ring; and $\cdot$ is defined by $[V] \cdot [W] = [V \otimes W]$. *Denoted by* $\boldsymbol{R(G)}$.
  - Basis: $[V_1], \dots, [V_k]$.
  - Thus, structurally,
$$R(G) \cong \mathbb{Z}^k$$
  - Elements are of the form $[V_1] + 2[V_2] - 3[V_3]$.
  - Multiplication is slightly complicated because $V_i \otimes V_j = \bigoplus V_k^{n_{ijk}}$; it follows that
$$[V_i] \cdot [V_j] = \sum n_{ijk} [V_k]$$
- Alternative construction of $R(G)$: Take the subring of the class ring $\mathbb{C}_{\text{cl}}[G]$ that is generated by the characters.
  - To do so, define a map $R(G) \to \mathbb{C}^k$ where the image is linear combinations of characters χ_i with $\mathbb{Z}$-class.
  - Clarify this construction??
- **Virtual representation**: An element of $R(G)$.
  - We need this term because some elements of $R(G)$ — like $-[V]$, for instance — may not correspond to an actual representation.
  - Indeed, note that $-[V]$ is *not* V^*; it is just some thing that when you add it to $[V]$, you get the zero representation.
- Example: Let $G = \mathbb{Z}/2\mathbb{Z} = \{e, x\}$.
  - Then $R(G) = \mathbb{Z}^2 = \mathbb{Z}e \oplus \mathbb{Z}x$ has basis $[1], [-1]$ (corresponding to the trivial and alternating representations) where we define
$$[1]^2 = [1] \qquad [1][-1] = [-1] \qquad [-1]^2 = [1]$$
- One reason people like this $R(G)$ is as follows.

- Initially, understanding this group is not easy because even to get started, you have to find all your characters.
- But, we know that
$$R(G) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}_{\text{cl}}[G]$$
  - So we have a ring that's hard to understand, but if we do something called an **extension of scalars** (shown above) we get an easy ring!
  - Why?? Clarify this construction.
- This is interesting because we can look at the intermediate objects. For example, could we describe $R(G) \otimes \mathbb{R}$ or $R(G) \otimes \mathbb{Q}$. Interestingly, **Artin's theorem** describes $R(G) \otimes \mathbb{Q}$ completely.
- If we try to understand $R(S_n)$, this is still hard work, but if we take $\bigoplus_{n \geq 0} R(S_n)$, we obtain an object that is remarkably, surprisingly simple. That's where we're going. This is why rep theory of finite groups is simultaneously very hard and very simple.

- Lemma: Let G be a finite group, let f be a complex-valued[1] class function, and let V be a G-rep. Then the linear map
$$F = \sum_{g \in G} f(g) \cdot g : V \to V$$
is a morphism of G-representations, that is, $F \in \operatorname{Hom}_G(V, V)$.

Proof. To prove that $F \in \operatorname{Hom}_G(V, V)$, it will suffice to show that $xF = Fx$ for every $x \in G$. Let $x \in G$ be arbitrary. Then
$$F(xv) = \sum_{g \in G} f(g) gxv$$
Since ρ is a group homomorphism, the functions $\rho(g) \in GL(V)$ act just like the elements $g \in G$. *This* is what justifies us to basically move everything around all willy-nilly. Thus, continuing from the above, we have
$$\begin{aligned} &= \sum_{g \in G} f(g)(xx^{-1})gxv \\ &= \sum_{g \in G} f(g)x(x^{-1}gx)v \end{aligned}$$
Since $x = \rho(x)$ is in the general *linear* group, i.e., is a *linear* map, we can factor it out of the sum of functions to get
$$= x \left(\sum_{g \in G} f(g) x^{-1} gx \right) v$$
Since f is a class function by hypothesis, we have $f(g) = f(x^{-1}gx)$, so
$$\begin{aligned} &= x \left(\sum_{g \in G} f(x^{-1}gx) x^{-1} gxv \right) \\ &= x \sum_{g \in G} f(g) gv \\ &= x(Fv) \end{aligned}$$
as desired. ∎

[1] This "complex-valued" hypothesis was not stated in class, but I have to imagine it's true. Is it??

- Recall that previously, we had $(1/|G|)\sum_{g\in G} g : V \to V^G$.
  - He will put something about this being a class function on the midterm?? Review how to prove that this is a class function!
- Another comment: A slightly refined question.
  - Suppose you have a class function f and an irrep V.
  - Then we know that $F = \sum f(g)g : V \to V$ is a G-morphism, so it is a **homothety** by Schur's lemma.
  - So let's find λ.
  - Thinking a big more carefully, we know that F above is
$$\sum_{g\in G} f(g)\rho_V(g) = \lambda I_{d_V}$$
    where d_V denotes the **degree** of V.
  - Now, we will compute λ using the trace. Take the trace of both sides. Then
$$\begin{aligned}
\operatorname{tr}\left(\sum_{g\in G} f(g)\rho_V(g)\right) &= \operatorname{tr}(\lambda I_{d_V})\\
\sum f(g)\operatorname{tr}(\rho_V(g)) &= \lambda d_V\\
\sum f(g)\chi_V(g) &= \lambda d_V\\
\lambda &= \frac{|G|}{d_V}\frac{1}{|G|}\sum_{g\in G} f(g)\overline{\chi_{V^*}(g)}\\
&= \frac{|G|}{d_V}\langle f, \chi_{V^*}\rangle
\end{aligned}$$
- **Homothety**: A map $F : V \to V$ for which there exists $\lambda \in \mathbb{C}$ such that $Fv = \lambda v$ for all $v \in V$.
  - It just means that we're scaling.
- **Degree** (of V): The dimension of V as a vector space. *Denoted by* $\boldsymbol{d_V}$. *Given by*
$$d_V = \dim V$$
- Now, we can prove the theorem to which we've been building up the whole time.
- Theorem: Let G be a finite group. Then the number of irreps up to isomorphism is equal to the number of conjugacy classes.

  Proof. Let k be the number of conjugacy classes of G, and let $\chi_1, \dots, \chi_s$ be the characters of the irreps. By the theorem from last Wednesday's class, it follows that $\chi_1, \dots, \chi_s$ are orthonormal vectors in $\mathbb{C}_{\text{cl}}[G]$. Thus, by the corollary to the aforementioned theorem, $s \leq k$.

  Now, suppose for the sake of contradiction that $s < k$. Then there exists a nonzero $f \in \mathbb{C}_{\text{cl}}[G]$ such that $\langle f, \chi_{V_i}\rangle = 0$ $(i = 1, \dots, s)$. By Gram-Schmidt, we can choose f to be another *orthonormal* vector in the list, extending it to $\chi_1, \dots, \chi_s, f$. We will now build up to proving that $f(g) = 0$ for all $g \in G$ (i.e., $f = 0$), which we will do by using the above lemma to construct a linear independence argument as follows. The first step is to let V_i be an arbitrary irrep of G. Then by the above comment, $F : V_i \to V_i$ may be evaluated on any $v \in V_i$ as follows.
$$F(v) = \lambda I v = \frac{|G|}{d_{V_i}}\langle f, \chi_{V_i^*}\rangle \cdot v = \frac{|G|}{d_{V_i}}\overline{\langle f, \chi_{V_i}\rangle} \cdot v = \frac{|G|}{d_{V_i}}\overline{0} \cdot v = 0$$

It follows that $F = 0$ on *any* representation since by complete reducibility, they're all direct sums of irreps. In particular, $F : V_{\text{reg}} \to V_{\text{reg}}$ is the zero operator, where $V_{\text{reg}} \cong V_1^{d_{V_1}} \oplus \cdots \oplus V_s^{d_{V_s}}$ is the regular representation. Thus, for example, $F(e_e) = 0$. But we also know that

$$F(e_e) = \sum_{g \in G} f(g) \cdot g e_e = \sum_{g \in G} f(g) \cdot e_g$$

Consequently, by transitivity, we have that

$$0 = \sum_{g \in G} f(g) \cdot e_g$$

But since the e_g are all linearly independent by the definition of the regular representation, we have that each $f(g) = 0$, as desired. This means that $f = 0$, contradicting our original supposition. $\square$

- That is the end of this story.
- Here's one consequence of the above theorem.
  - We now know that the space of class functions has an orthonormal basis $\chi_{V_1^*}, \dots, \chi_{V_k^*}$.
  - If we denote the conjugacy classes of G by $C_1, \dots, C_k$, then another obvious basis of $\mathbb{C}_{\text{cl}}[G]$ is $\delta_{C_1}, \dots, \delta_{C_k}$ defined by
  $$\delta_{C_i}(g) = \begin{cases} 1 & g \in C_i \\ 0 & g \notin C_i \end{cases}$$
  - This new basis is orthogonal: We have
  $$\langle \delta_{C_i}, \delta_{C_j} \rangle = \frac{1}{|G|} \sum_{g \in G} \delta_{C_i}(g) \delta_{C_j}(g) = \begin{cases} 0 & i \neq j \\ \frac{|C_i|}{|G|} & i = j \end{cases}$$
    - Justifying this computation: If $i \neq j$, then at least one of $\delta_{C_i}, \delta_{C_j}$ will be zero; if $i = j$, then they're both nonzero and equal to 1 for all $|C_i|$ elements $g \in C_i$.
  - What is the change of basis matrix between $\{\delta_{C_i}\}$ and $\{\chi_{V_i^*}\}$? It's the character table.
    - The orthogonality condition for characters then just comes from the fact that we're going from one orthogonal basis to another.
    - What are the exact bases we change between??