Week 1

???

1.1 Motivating and Defining Representations

9/27:

- Rudenko would happily approve my final substitution, but it's not his call; it's Boller's.
- HW will be due every week on Wednesday or thereabouts.
  - Submit in paper in a mailbox, location TBA.
  - First HW due next Wednesday.
- Midterm eventually and an in-class final.
- Grading scheme in the syllabus.
- OH not available MW after class (Rudenko has to run to something else), but F after class, we can ask him anything.
  - Regular OH MTh, time TBA.
- There is no specific book for the course.
  - First 8 lectures come from Serre (1977); amazing book but very concise; gets confusing later on. Most lectures are made up by Rudenko.
- Course outline.
  1. Character theory: Beautiful, not too hard.
  2. Non-commutative algebra: More abstract/general approach to the same thing.
  3. Advanced topics, S_n.
- This course's focus: Representations of finite groups in finite dimensions over $\mathbb{C}$.
- This course is for math-inclined people (not quite physics) and lays the foundation for all other Rep Theory.
  - The ideas would be presented in a very different way in Physics Rep Theory.
- We can always ask questions and stop him to correct mistakes during class.
- Why we care about representations.
  - Start with a group G, finite. For example, let $G \equiv S_1$.
  - People started to play with S_4 (permutations of roots of a polynomial of degree 4) in Galois theory.

    - Galois theory primer: Consider a polynomial like $x^4 + 3x + 1 = 0$; the roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ satisfy tons of equations, e.g., $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$ since 1 is the x^0 term.
  - But groups also occur in much more natural places, e.g., isometries of $\mathbb{R}^3$ that preserve a tetrahedron.
  - S_4 is also orientation-preserving isometries of $\mathbb{R}^3$ that preserve a cube.
  - Many things lead to the same group!
  - Theory of abstract groups developed far later than any of these perspectives; was developed to unify them.
- Recall group actions: Take $G, X = \{x_1, \dots, x_n\}$ both finite. We want $G \circlearrowright X$, which is a homomorphism $A : G \to S_n$.
- What can we do now?
  - We can look at orbits, which are smaller pieces.
  - We can look at the stabilizer.
  - We can identify orbits with cosets.
  - If we understand all possible subgroups, we understand all possible actions.
- This story is not boring, but it's simplistic.
- Rudenko doesn't assume we remember everything (phew!).
- Main definition (general to start, then we simplify).
- **Group representation** (of G on V): A group homomorphism $G \to GL(V)$, for G a group, V a finite-dimensional vector space over some field $\mathbb{F}$ with basis $\{e_1, \dots, e_n\}$, and $GL(V)$ the set of isomorphic linear maps $L : V \to V$. *Denoted by* $\boldsymbol{\rho}$.
  - Recall that $GL(V) = GL_n(\mathbb{F})$ is the set of all $n \times n$ invertible matrices.
- For every element $g \in G$, $g \mapsto \rho(g) = A_g$. Essentially, you're mapping to elements that satisfy certain equations.
  - For example, $A_e = E_n$, $A_{g_1 g_2} = A_{g_1} A_{g_2}$, and $A_{g^{-1}} = {A_g}^{-1}$.
  - Thus, representations are a "concrete way to think about groups."
  - If you don't understand abstract group G, let us compare it to a group that we do understand! Like a group can *act* on S_n, we can *represent* a group in a vector space.
- In this course, G is finite, $\mathbb{F} = \mathbb{C}$, and V is finite dimensional.
  - This is the most simple case, but also a very interesting one. The theory is much, much easier, so we can get much more complicated, but this is a good place to start.
  - We could make G compact, but we're not gonna go that far.
- Examples to get an idea of what's going on.
  1. $\dim \rho = 1$ (means $\dim V = 1$). Then $\rho : G \to GL_1(V) = \mathbb{C}^\times$. The codomain is referred to as the **character** of the group.
     - An example group homomorphism $S_n \to \mathbb{C}^\times$ is the sign function $\sigma \to \operatorname{sign}(\sigma) = \{\pm 1\}$.
     - Another example is the **trivial representation**, $G \to \mathbb{C}^\times$ and $g \mapsto 1$.
  2. Smallest one: Let $G = S_3$. The structure is already pretty rich, and this will be part of the homework.
     - **Trivial representation** again.
     - **Alternating representation**.

  - **Standard representation**.
  - **Regular representation**.

- **Trivial representation**: The representation $\rho : G \to V$ sending $g \mapsto 1$ for all $g \in G$. *Denoted by* $\begin{array}{|c|c|c|}\hline \; & \; & \; \\ \hline\end{array}$, **(3)**.
  - The boxes notation is too much of a detour to explain now.
- **Alternating representation**: The representation $\rho : G \to V$ sending $g \mapsto \text{sign}(g)$ for all $g \in G$. *Denoted by* $\begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \; \\ \hline\end{array}$, $\mathbf{(1,1,1)}$.
- **Standard representation**: The representation $\rho : S_n \to V$ sending $\sigma \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)})$, where $V = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_1 + \cdots + x_n = 0\}$ is a $(n-1)$-dimensional vector space. *Denoted by* $\begin{array}{|c|c|}\hline \; & \; \\ \hline \; \\ \cline{1-1}\end{array}$, $\mathbf{(2,1)}$.
  - A 2D representation like rotating a triangle.
  - This gives something with real numbers.
  - Example: $S_3 \circlearrowright V$ by $\sigma((x_1, x_2, x_3)) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$.
- **Regular representation**: The representation $\rho : G \to \text{Hom}(\mathbb{C}^n)$ defined by $g \mapsto \sigma_g$, where $G = \{g_1, \dots, g_n\}$, $\{e_{g_1}, \dots, e_{g_n}\}$ is a basis of $\mathbb{C}^n$, $\cdot$ is the group action of $\rho(G) \circlearrowright \mathbb{C}^n$ by $\rho(g) \cdot e_g = e_{gg_i}$, and $\sigma_g(e_{g_i}) = \rho(g) \cdot e_g = e_{gg_i}$.
  - This is a permutation of vectors.
  - Thus, for S_3, it will already be 6-dimensional (it's very high dimensional).
- How do we know that representation theory is tractable? Sure, we can define all these things, but how do we know that it will lead anywhere? Here's an example.
  - Let $G = \mathbb{Z}/2\mathbb{Z} = \{e, g\}$, $V = \mathbb{C}^n$, A an $n \times n$ matrix over $\mathbb{C}$, $\rho : G \to GL_n(\mathbb{C})$, and $A := \rho(g)$. Since $g^2 = e$, we know for example that $A^2 = E_n$.
  - But how do we find the matrices A? If we look at eigenvalues of A, there are only two possibilities: ± 1. The structure of A can be very complicated with Jordan normal form and all that, but in fact, these are the **semisimple matrices**, so it's not that bad.
  - Since $A^2 = E$, we know that $(A - E)(A + E) = 0$. Consider $(A - E) : V \to V$. Naturally, it has $\ker(A - E)$ and $\text{Im}(A - E)$. In this particular case, Rudenko claims that $\ker(A - E) \cap \text{Im}(A - E) = \{0\}$.
    - Proof: Let $v \in \ker(A - E) \cap \text{Im}(A - E)$ be arbitrary. Since $v \in \text{Im}(A - E)$, there exists $w \in V$ such that $v = (A - E)w = Aw - w$. Since $v \in \ker(A - E)$, we have $(A - E)v = 0$, so $Av = v$. It follows that $A(Aw - w) = Aw - w$ but also $A(Aw - w) = Ew - Aw = w - Aw$. Thus,

$$\begin{aligned}
Aw - w &= w - Aw \\
2Aw &= 2w \\
Aw &= w
\end{aligned}$$

      But then $w \in \ker(A - E)$, so $v = (A - E)w = 0$.
  - This combined with the fact that every vector in a vector space is in either the image or the kernel of a linear map[1] implies that $V = \ker(A - E) \oplus \text{Im}(A - E)$.

[1] See Theorem 3.6 of Axler (2015).

- Let the kernel have basis $e_1, \dots, e_k$ and the image have basis $e_{k+1}, \dots, n$; then all A are of the following form.

$$\begin{array}{c} \\ 1 \\ \\ k \\ k+1 \\ \\ n \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & & k & k+1 & & n \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{array}\right] \end{array}$$

- Next time, we will discuss sums of representations, of which this is an example of the theory.

- The same kind of thing, **simple representations**, happens with all finite groups?? This is where we're going. It's not rocket science; in fact, we'll see it next week.

- Last thing for today: A remarkable story.
  - The story of representation theory started quite different.
  - A beautiful theorem that we can prove now!
  - Frobenius determinant.
  - Think of $G = \{g_1, \dots, g_n\}$. Picture its multiplication table.
  - In every row and column, you see each element once.
  - Let's associate to the multiplication table an actual determinant in the linear algebra sense. Consider elements $x_{g_1}, \dots, x_{g_n}$. Define the $n \times n$ matrix $(x_{g_i g_j})$. Take its determinant. It will be a polynomial in n variables, i.e., an element of the ring $\mathbb{Z}[x_{g_1}, \dots, x_{g_n}]$.
  - Example: Consider
  $$\begin{vmatrix} e & g \\ g & e \end{vmatrix}$$
    - The determinant is $x_e^2 - x_g^2 = (x_e - x_g)(x_e + x_g)$.
  - Example: $G = \mathbb{Z}/3\mathbb{Z}$.
    - If the elements are e, g, g^2 and we map these, respectively, to variables a, b, c, we get the matrix
    $$\begin{bmatrix} e & g & g^2 \\ g & g^2 & e \\ g^2 & e & g \end{bmatrix} \mapsto \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$
    - The determinant is $3abc - a^3 - b^3 - c^3 = (a+b+c)(a^2+b^2+c^2-ab-bc-ac) = (a+b+c)(a+\zeta b+\zeta^2 c)(a+\zeta^2 b+\zeta c)$ where $\zeta^3 = 1$ is a root of unity.
  - Frobenius's theorem: If G is a finite group and we take this Frobenius determinant, then this determinant is equal to ${P_1}^{d_1} \cdots {P_k}^{d_k}$ where $P_1, \dots, P_k$ are irreducible polynomials in $x_g, \dots, x_{g_j}$, then $\deg P_i = d_i$ and k is the number of conjugacy classes.
  - Example: Take S_3; we'll get a polynomial of degree $|S_3| = 6$ but the Frobenius determinant $FD = (x_{g_1} + \cdots + x_{g_k})(x_{g_1} \pm \cdots)(\text{some pol. of deg } 2)^2$
  - The proof is remarkable and deep and uses what would become character theory. These polynomials are related to representations and the number of simplest irreducible representations. The theory that came out came as a way to understand this miracle. We'll forget FD's for now, but then come back and prove it later.