Week 7

Representations of the Symmetric Group

7.1 Specht Modules

11/6: • Announcements.

  - Midterm description is on the Canvas page.
  - Review what he says to review, and then look at the PSets. The operator averaging stuff and S_4, S_5 examples are most important.
  - New HW will be due next Friday (not this Friday).

- New topic: Representations of S_n.
  - We will talk about these almost until the end of the course.
  - Very hard.
  - Any specialist in rep theory will still say that they know some approaches, but nobody understands this stuff completely.
  - We'll explore some phenomena, but if we feel after this course that we still don't understand everything about S_n, that's typical; if we think we understand everything, we're probably wrong.
- Representation theory of $GL_n(\mathbb{F}_{p^k})$ is related but even worse.
  - Same with $O_n(\mathbb{F}_{p^k})$.
  - Recently, all this stuff was understood with something called linquistic (??) theory, but that's far beyond us.
- $|S_n| = n!$, and the conjugacy classes are in bijection with cyclic structures of a permutation.
  - Our good understanding of the conjugacy classes of S_n is the only thing that makes this problem the slightest bit tractable.
  - Cyclic structures are also in bijection with the **partitions** of a number; recall that we briefly talked about these in MATH 25700!
- **Partition** (of $n \in \mathbb{N}$): An ordered tuple satisfying the following constraints. *Denoted by* $\boldsymbol{\lambda}$, $(\boldsymbol{\lambda_1, \dots, \lambda_k})$. *Constraints*
  1. $\lambda_i \in \mathbb{N}$ for $i = 1, \dots, k$;
  2. $\lambda_1 \geq \cdots \geq \lambda_k$;
  3. $\lambda_1 + \cdots \lambda_k = n$.

- Example: The partitions of the number "4" are (4), $(3,1)$, $(2,2)$, $(2,1,1)$, and $(1,1,1,1)$.
  - This is the same way we've been denoting representations!
- $\boldsymbol{p(n)}$: The number of possible partitions of n.
  - Hardy and Ramanujan helped understand the number $p(n)$ of partitions of n, but they're still very hard to understand.
- One way to understand $p(n)$ is through its encoding in the **generating function**

$$\sum_{n\geq 1} p(n)x^n = 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots$$

  - We can think of the above generating function as an actual function of x if it converges for small x; of it doesn't converge, then we just think of it as a "meaningless" **formal power series**.
  - To choose a partition, we need to choose a certain number of 1's, a certain number of 2's, a certain number of 3's, etc. all the way up to n.
  - So let's look at

$$(1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)(1 + x^4 + x^8 + \cdots)\cdots$$

    - Formally, this is

$$\prod_{i=1}^{\infty}\left(\sum_{j=0}^{\infty} x^{ij}\right)$$

    - This equals the generating function! It tells us that to compute $p(100)x^{100}$, we need only look at certain terms.
  - Recall that we can write $1 + x + x^2 + \cdots = 1/(1-x)$. Doing similarly for other terms transforms the above product into

$$\frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^3}\cdots$$

- **Generating function**: An encoding of an infinite sequence of numbers as the coefficients of a formal power series.
- **Formal power series**: An infinite sum of terms of the form ax^n that is considered independently from any notion of convergence.
- The above discussion of $p(n)$ as a generating function is only for our fun; Rudenko is not going to use this in any way.
  - It's just a pretty function.
  - Takeaway: We can write the generating function as something nice and then use it to prove something.
- We can visualize these partitions using something called a **Young diagram**.

(a) $\lambda = (5,4,4,3,2,1,1)$. (b) $\lambda' = (7,5,4,3,1)$.

Figure 7.1: Young diagrams for a partition of 20.

  - Suppose we have the following partition of 20: $(5, 4, 4, 3, 2, 1, 1)$.
  - Then we draw 5 cages for 5 little birds, followed by 4 cages for 4 little birds, etc.
  - Thus, the i^{th} row of boxes has length λ_i.
  - The same way you can denote by λ the whole *partition*, you can denote by λ the whole *diagram*.
  - This is just a way to visualize partitions.
  - Recall the three partitions of S_3, corresponding to its representations: (3), $(2, 1)$, $(1, 1, 1)$.
  - Moreover, these diagrams are actually meaningful!
- **Inverse** (of λ): The partition $(\lambda_1', \dots, \lambda_k')$ defined as follows. *Denoted by* $\boldsymbol{\lambda'}$. *Given by*

$$\lambda_i' = |\{\lambda_j \mid \lambda_j > i\}|$$

  for all $i = 1, \dots, k$.
  - We can see that $\lambda_1' \geq \cdots \geq \lambda_k'$.
  - We can also see that the sum will still be n.
  - Moreover, if we do this twice, we'll get back to λ, i.e., $(\lambda')' = \lambda$.
  - We can prove $(\lambda')' = \lambda$ combinatorially, too, (that is, without Young diagrams) but that gets pretty complicated.
- Example: If $\lambda = (5, 4, 4, 3, 2, 1, 1)$ as above, then $\lambda' = (7, 5, 4, 3, 1)$.
  - See Figure 7.1b.
  - Moreover, the Young diagrams are related by a flip akin to matrix transposition.
  - Notice how the definition of inversion *exactly* specifies this flip in the picture: The number of λ_j's that have length at least 1 is all the first column of Figure 7.1a, the number of length at least 2 is all the second column, etc.
- Onto the next question, which is the main miracle.
  - Main miracle: There exists a natural (i.e., canonical) bijection between the conjugacy classes and irreducible representations of S_n.
  - We've explored a duality for general finite groups G, before, but never a bijection.
    - In S_n, there *is* this natural bijection.
    - If you understand why intuitively, you will have started to understand the representation theory of S_n.
- If we define $\lambda \mapsto n$ (??), then there is some irrep V_λ corresponding to λ. We will look at the **Specht module** construction of V_λ.
  - Some of the proofs Rudenko will present, he stole from Etingof et al. (2011), and some of the proofs he invented himself.
  - This is *by far* the best construction, even though it's exceedingly rare in the literature.
- The usual construction.
  - Take $\mathbb{C}[S_n]$ with coefficients a_λ, b_λ, etc. similar over conjugacy classes and do something with it??
  - "Just say NO!" to this construction.
- Here is the better idea.
  - Consider an algebra of polynomials with rational coefficients: $\mathbb{Q}[x_1, \dots, x_n]$.
    - We could also do real or complex, but rational is nice.

- For symmetric groups, all representations will be integers, etc.??
- One thing to emphasize about this algebra: It is a **graded** algebra.
  - If represented by A, then it equals $A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ where

$$A_m = \left\{ \sum_{k_1+\cdots+k_n=m} a_{k_1\cdots k_n} x_1^{k_1} \cdots x_n^{k_n} \right\}$$

    - I.e., A_m is the sum of all polynomials with degree equal to m.
    - Example: If we take $1 + x_1^2x_2^3 + x_1x_2 + x_1^{100} + x_1x_2^{99}$, we can then break this polynomial up into polynomials of degree 1, 5, 2, and 100.
  - We also have $A_{m_1} \cdot A_{m_2} \subset A_{m_1+m_2}$.
    - Exmaple: $x_1x_2^2 \cdot (x_1 + x_2) = x_1^2x_2^2 + x_1x_2^3$.
- With this algebra in hand, we may let $S_n \circlearrowright \mathbb{Q}[x_1, \dots, x_n]$ via

$$\sigma P(x_1, \dots, x_n) = P(x_{\sigma_{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$$

  - In other words, σ is transposing polynomials.
  - Example: $(12)(x_1^2 + x_2^3 + x_3) = x_2^2 + x_1^3 + x_3$.
- Thus, we call as A_1 the representation V_{perm}^*.
  - This is because $A_1 = \text{span}(x_1 + \cdots + x_n)$, and permuting these is much like permuting the basis of a vector space, as the typical permutation representation does.
  - It could technically be the isomorphic representation V_{perm}, but the dual fits better here for reasons??
- Then $A_2 = S^2V^*$.
  - So if A_1 had basis $e^1, \dots, e^m$, A_2 has basis e^ie^j.
  - Why are we choosing these sets??
- Continuing, $A_3 = S^3V^*$.
- It follows that the representation of the overall thing is

$$\bigoplus_{m\geq 0} (S^mV_{\text{perm}}^*)$$

  - This is called the **symmetric algebra**.

- **Graded** (algebra): An algebra for which the underlying additive group is a direct sum of abelian groups A_i such that $A_iA_j \subset A_{i+j}$.
- So how do we construct representations?
  - For S_2, $x_1 - x_2$ changes sign when we apply S_2.
  - For S_3...
    - The trivial's polynomial is 1 and *tableaux*.
    - The standard is $(2,1)$. When we apply S_3 to $(x_1 - x_2)$, we get

$$\langle (x_1 - x_2), (x_2 - x_1), (x_1 - x_3), (x_3 - x_1), (x_2 - x_3), (x_3 - x_2) \rangle$$

      - If we let $a = x_1 - x_2$, $b = x_2 - x_3$, then some elements equal $a + b$. This is another way to think about the action.
    - What about the alternating representation? We have $(x_1 - x_2)(x_2 - x_3)(x_1 - x_3) = \Delta_{123}$, which changes sign when we apply any element with sign -1!
  - For S_4...

- (4) is 1.
- (3, 1) is $S_4(x_1 - x_2) = \Delta_{12}$.
- (1, 1, 1, 1) is $(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4) = \Delta_{1234}$.
- (2, 1, 1) is $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$.
  - We got this polynomial by guessing; the same way $(x_1 - x_2)$ worked in multiple cases, maybe this one does too! And it does.
  - Something to check is that $\Delta_{123} - \Delta_{124} - \Delta_{134} - \Delta_{234} = 0$.
- (2, 2) is $(x_1 - x_2)(x_3 - x_4)$.
  - Something related we can prove is that

$$(x_1 - x_2)(x_3 - x_4) - (x_1 - x_3)(x_2 - x_4) - (x_1 - x_4)(x_2 - x_3) = 0$$

  - This formula appears in **cross ratios**, which we can discuss in Rudenko's algebraic geometry course next quarter.
- For $\lambda = (4, 3, 1)$, we have $\Delta_{123}\Delta_{45}\Delta_{67}$, and we act by S_8 upon this! Explicitly, we have $S_8(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x - 4 - x_5)(x_6 - x_7)$.
- Takeaway: It all depends on column length!

– These polynomials are called **Vandermonde determinants**; those are the little Δ things with subscripts. We'll talk about these next times.

- We need to prove reducibility and not pairwise isomorphic to make sure that this construction is valid, but that's easy!

7.2 Vandermonde Determinants

11/8:
- Announcements.
  - OH tonight at 6:00 PM.
- Consider S_n.
  - Recall the symmetric algebra $R = \mathbb{Q}[x_1, \dots, x_n]$, which is a graded ring $\bigoplus_{d>0} R_d$ where $R_d = S^d V_{\text{perm}}^*$.
  - The action of $S_n \circlearrowright \mathbb{Q}[x_1, \dots, x_n]$ is $\sigma P(x_1, \dots, x_n) = P(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$.
- With the definitions of last class behind us, we can now look at the **space of invariants** R^{S_n}, isotypical components of which σ acts on trivially.

$$R^{S_n} := \{P(x_1, \dots, x_n) \mid \forall\ \sigma \in S_n,\ P(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = P(x_1, \dots, x_n)\}$$

  - This is the ring of symmetric polynomials.
  - Example: If $n = 3$, then $x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3 \in \mathbb{Z}^{S_3}$.
- We now define some stuff to help us prove a major result: The n elementary symmetric polynomials.
  - $\sigma_1 = x_1 + \cdots + x_n = \sum_{1 \leq i \leq n} x_i$.
  - $\sigma_2 = x_1x_2 + \cdots + x_1x_n + x_2x_3 + \cdots = \sum_{1 \leq i < j \leq n} x_ix_j$.
  - $\sigma_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}$.
  - $\sigma_n = x_1 \cdots x_n$.
- With these definitions, we can say that

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = (x - x_1) \cdots (x - x_n)$$

where

$$a_{n-1} = -\sigma_1 \qquad a_{n-2} = \sigma_2 \qquad \dots \qquad a_0 = (-1)^n \sigma_n$$

- Fundamental theorem.
  - Basic statement: Every polynomial is a polynomial in these polynomials.
  - A more precise statement follows.
- Theorem (Fundamental theorem of symmetric polynomials): We have that

$$\mathbb{Q}[x_1,\dots,x_n]^{S_n} = \mathbb{Q}[\sigma_1,\dots,\sigma_n]$$

- Before we prove the fundamental theorem, there are a few points we need to discuss.
- Example:
  - Take $x^2+px+q=0$.
  - If it has two roots x_1,x_2, then $\sigma_1 = x_1+x_2 = -p$ and $\sigma_2 = x_1x_2 = q$.
  - Then $x_1^2+x_2^2 = \sigma_1^2 - 2\sigma_2^2 = p^2-2q$.
  - Thus, if we take $(x_1-x_2)^2 = (x_1+x_2)^2 - 4x_1x_2 = \sigma_1^2 - 4\sigma_2 = p^2-4q$, which is the discriminant.
  - In general, $x_1^n+x_2^n$ has an expression as a polynomial in σ_1,σ_2. This will be a homework problem.
  - What is going on here?? Is x^2+px+q even in $\mathbb{Q}[x]^{S_n}$? If so, why do we factor it into σ_1,σ_2 instead of just σ? What are the other examples about?
- **Lexicographic order** (on monomials): An ordering of monomials based on the following rule. *Denoted by* $\succ$. *Given by*
  $x_1^{a_1}\cdots x_n^{a_n} \succ x_1^{b_1}\cdots x_n^{b_n}\dots$
  1. If $a_1>b_1$ OR...
  2. If $a_1=b_1$ and $a_2>b_2$ OR...
  3. $a_1=b_1$ and $a_2=b_2$ and $a_3>b_3$ OR...
  4. So on and so forth.
- Notes on the lexicographic ordering.
  - Don't think of this order like an ordering on integers.
  - This allows us to define the key notion for a number of proofs we'll see in the coming days.
  - Although it may seem counterintuitive, the lexicographic ordering is still determined for polynomials such as σ_1. For example, we may look at

$$\sigma_1 = x_1+\cdots+x_n$$

    and think, "Wait a second — all these terms have the same order: They all have the same exponent of 1." However, we would be discounting the fact that the lexicographic ordering views σ_1 as

$$\sigma_1 = x_1^1x_2^0\cdots x_n^0 + \cdots + x_1^0\cdots x_{n-1}^0x_n^1$$

    From here, we can see that $LM(\sigma_1) = x_1^1x_2^0\cdots x_n^0 = x_1$.
- **Largest monomial** (of $P\neq 0$); The monomial in $P(x_1,\dots,x_n)\neq 0$ that is the largest lexicographically. *Denoted by* $\boldsymbol{LM(P)}$.
- $\boldsymbol{C_{LM}(P)}$: The coefficient of $LM(P)$.
- Example: Consider the polynomial $P = x_1^2 + x_1^3x_2x_3 - 7x_1^3x_2x_3^{100}$.
  - Then $LM(P) = x_1^3x_2x_3^{100}$ and $C_{LM}(P) = -7$.
- Properties.

1. $P, Q \neq 0$ implies that $LM(PQ) = LM(P)LM(Q)$.
   - Using inductive reasoning, try multiplying the example above by $Q = x_1^2 + x_2^2 + x_3^2$!
   - Rudenko will not give rigorous proofs of any of these properties; they will just confuse us. It's better to do everything intuitively here.

- Lemma: If $P \in \mathbb{Q}[x_1, \dots, x_n]^{S_n}$ and $LM(P) = x_1^{a_1} \cdots x_n^{a_n}$, then $a_1 \geq \cdots \geq a_n$.

  Proof. Let $i < j$. Suppose for the sake of contradiction that $a_j > a_i$. Let $\sigma = (ij) \in S_n$. Since P is symmetric, $\sigma P = P$. But then in particular, the monomial $\sigma LM(P)$ in P is lexicographically larger than $LM(P)$. Thus, $LM(P)$ is not the lexicographically largest monomial in P, a contradiction.

  Here's a simple example to illustrate the idea behind this proof: Let $P = x^2y + xy^2 \in \mathbb{Q}[x, y]^{S_n}$. Suppose we pick $LM(P) = xy^2$ (obviously this is the wrong choice, but that's the contradiction we'll see). We observe that $2 = a_2 > a_1 = 1$ in this case. Let $\sigma = (12)$. Then $\sigma LM(P) = yx^2 = x^2y \succ xy^2 = LM(P)$. So $\sigma LM(P) \succ LM(P)$. Thus, $LM(P)$ is not the lexicographically largest monomial in P, and we have *formally* proven that our initial choice of $LM(P)$ was incorrect. ☐

- We now have everything we need to prove the fundamental theorem. As such, we will restate and prove it.

- Theorem (Fundamental theorem of symmetric polynomials): We have that

$$\mathbb{Q}[x_1, \dots, x_n]^{S_n} = \mathbb{Q}[\sigma_1, \dots, \sigma_n]$$

  Proof. We will prove this theorem using the well-ordering principle (every set of natural numbers has a smallest element), which is equivalent to induction. Let's begin.

  Suppose for the sake of contradiction that there exists a symmetric polynomial that cannot be expressed via $\sigma_1, \dots, \sigma_n$. Given this counterexample, factor out as many terms as we want (successively reducing the degree) until it ceases to be a counterexample, thus yielding the counterexample of smallest degree. Similarly, get to the counterexample with smallest LM. Call this counterexample $P(x_1, \dots, x_n)$. Let

$$P(x_1, \dots, x_n) = C_{LM}(P) \underbrace{x_1^{a_1} \cdots x_n^{a_n}}_{LM(P)} + \text{smaller monomials}$$

  Since P is symmetric and the term above is the lexicographically largest monomial, the Lemma implies that $a_1 \geq \cdots \geq a_n$. We now construct a polynomial Q out of the σ_i such that $LM(P) = LM(Q)$. To begin, note that

$$LM(\sigma_1) = x_1 \qquad LM(\sigma_2) = x_1x_2 \qquad \cdots \qquad LM(\sigma_n) = x_1 \cdots x_n$$

  Now consider $\sigma_n^{a_n}$. This clearly divides $LM(P)$ since a_n is minimal. Now multiply by $\sigma_{n-1}^{a_{n-1}-a_n}$. Continuing on, we get

$$Q = \sigma_n^{a_n} \sigma_{n-1}^{a_{n-1}-a_n} \sigma_{n-2}^{a_{n-2}-a_{n-1}} \cdots \sigma_1^{a_1-a_2}$$

  Now it follows that

$$LM(P - C_{LM}(P) \cdot Q) \prec LM(P)$$

  Since $C_{LM}(P) \in \mathbb{Q}$ and $Q \in \mathbb{Q}[x_1, \dots, x_n]^{S_n}$, it also follows that $P - C_{LM}(P) \cdot Q \in \mathbb{Q}[x_1, \dots, x_n]^{S_n}$. But then by the assumption that P was the counterexample with smallest LM, we know that $P - C_{LM}(P) \cdot Q \in \mathbb{Q}[\sigma_1, \dots, \sigma_n]$. It follows that $P \in \mathbb{Q}[\sigma_1, \dots, \sigma_n]$, a contradiction. ☐

- Note: This is an effective proof; we can write an algorithm to do this for us, and it's actually pretty fast and efficient.

- ?? (*word in blackboard picture*) to show: $\sigma_1, \dots, \sigma_n$ are algebraically independent $P(\sigma_1, \dots, \sigma_n) = 0$ implies that $P = 0$.

- This will be a homework problem; hint, it's pretty easy.

- Back to representation theory.
- **Antisymmetric** (polynomial): A polynomial $P(x_1, \dots, x_n)$ such that

$$\sigma P = (-1)^\sigma P$$

- Example.
  - $n = 2$: $x_1 - x_2$.
  - $n = 3$: $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$.
- These main examples are the **Vandermode determinant** from last time!
- **Vandermonde determinant**:

$$\Delta(x_1, \dots, x_n) = \prod_{1 \leq i < j \leq n} (x_j - x_i)$$

- Exercise: $\Delta(x_1, \dots, x_n)$ is antisymmetric.
- One of the nicest definitions of sign comes from these determinants!

$$(-1)^\sigma = \frac{\prod_{i<j}(x_{\sigma(i)} - x_{\sigma(j)})}{\prod_{i<j}(x_i - x_j)}$$

- Theorem: If $P \in \mathbb{Q}[x_1, \dots, x_n]^{\text{alt}}$ (i.e., $P \in \mathbb{Q}[x_1, \dots, x_n]$ and P is antisymmetric), then $P = P'\Delta(x_1, \dots, x_n)$ where P' is symmetric (i.e., $P' \in \mathbb{Q}[x_1, \dots, x_n]^{S_n}$).
- Corollary: If P is antisymmetric and $\deg(P) < n(n-1)/2$, then $P = 0$.
  - We'll use this many times, this fact that "antisymmetric polynomials have a smallest possible degree."
- We now prove the Theorem.

Proof. Let P be antisymmetric. Then $(12)P = -P$. It follows that $P(x_1, \dots, x_n)\big|_{x_1 = x_2} = 0$. Now, rewrite P as a polynomial in one variable where all of the coefficients are polynomials in other variables. In particular, let

$$P = P_d(x_1 - x_2)^d + P_{d-1}(x_1 - x_2)^{d-1} + \cdots + P_0$$

where each $P_i \in \mathbb{Q}[x_1, \dots, x_d]$. What is d?? (Less than n, I'm assuming, but any other constraints?) Plugging in $x_1 = x_2$ once again, we get $0 = P = P_0$. But this implies that $P_0 = 0$. Thus, P is divisible by $x_1 - x_2$. Similarly, for all $i < j$, $(x_i - x_j) \mid P$. But since the $x_i - x_j$ are irreducible polynomials, we have that $\prod_{i<j}(x_j - x_i) \mid P$. This is justified because we are in a unique factorization domain (how is this relevant??). Thus, we have that $P = P' \cdot \Delta(x_1, \dots, x_n)$. Lastly, it follows that $P' \in \mathbb{Q}[x_1, \dots, x_n]^{S_n}$ because under any sign -1 permutation, $\Delta(x_1, \dots, x_n)$ will flip signs and P will still be equal, so P' had better just stay itself under this permutation (i.e., be symmetric). $\square$

- Remark: Where does the name Vandermonde *determinant* come from?
  - We have that

$$\Delta(x_1, \dots, x_n) = \begin{vmatrix} 1 & & 1 \\ x_1 & & x_n \\ \vdots & & \vdots \\ x_1^{n-1} & & x_n^{n-1} \end{vmatrix}$$

- Final reminder before the final.
  - Don't forget our awesome central construction!
  - If $\lambda = (\lambda_1, \dots, \lambda_k)$ is a partition with $\lambda_1 \geq \cdots \geq \lambda_k$, then we can draw a Young Diagram and construct an associated representation $V_\lambda \in \mathbb{C}[x_1, \dots, x_n]$.
  - But what we do is $V_\lambda = \mathbb{C}[S_n]\Delta_\lambda$, where

$$\Delta_{\lambda'} = \Delta(x_1, \dots, x_{\lambda_1'})\Delta(x_{\lambda_1'+1}, \dots, x_{\lambda_1'+\lambda_2'})\cdots$$

  - Example: For $\lambda = (2,2,1)$, we have $V_{(2,2,1)} = \mathbb{C}[S_n](x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_4 - x_5)$.
  - Next time, we'll prove that $V_{(2,2,1)}$ is irreducible.
- This Specht construction is in a tiny footnote of Fulton and Harris (2004), but that's about it!

7.3 Midterm Review Sheet

11/10:

- The following definitions and results will be useful in solving the midterm problems.
- **Group representation**: A group homomorphism $\rho : G \to GL(V)$ for G a finite group, V a finite-dimensional vector space over some field $\mathbb{F}$ with basis $\{e_1, \dots, e_n\}$, and $GL(V)$ the set of isomorphic linear maps $L : V \to V$.
- **Morphism** (of G-representations): A map $f : V \to W$ such that...
  1. f is linear;
  2. For every $g \in G$, $\rho_W(g) \circ f = f \circ \rho_V(g)$.
     - To remember this rule, draw out the commutative diagram!
- Theorem (complete reducibility): Any finite-dimensional representation can be decomposed into a direct sum of irreducible representations via

$$V = V_1^{n_1} \oplus \cdots \oplus V_k^{n_k}$$

- Lemma (Schur's Lemma): Let G be a finite group, let V, W be irreducible representations over $\mathbb{C}$, and let $f \in \mathrm{Hom}_G(V, W)$. Then...
  1. If $V \ncong W$, then $f = 0$. If $V \cong W$, then f is an isomorphism of G-representations.
  2. If $f : V \to V$, then $f(v) = \lambda v$.
- **Algebraic integer**: A number $x \in \mathbb{C}$ for which there exist $a_0, \dots, a_{n-1} \in \mathbb{Z}$ such that

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

- **Character** (of ρ): The function $\chi_\rho : G \to \mathbb{C}$ defined by

$$\chi_\rho(g) = \mathrm{tr}(\rho(g))$$

- **First orthogonality relation**: If χ_1, χ_2 are the characters of irreducible representations, then

$$\sum_{g \in G} \chi_1(g)\overline{\chi_2(g)} = \begin{cases} 0 & \chi_1 \neq \chi_2 \\ |G| & \chi_1 = \chi_2 \end{cases}$$

  - Follows from the fact that the characters form an orthonormal set within the space of class functions, and the definition of the inner product on this space.

- **Second orthogonality relation**: If $C_G(g)$ is the number of elements in the conjugacy class of g, then

$$\sum_\chi \chi(g_1)\overline{\chi(g_2)} = \begin{cases} 0 & g_1 \nsim g_2 \\ \frac{|G|}{|C_G(g_1)|} & g_1 \sim g_2 \end{cases}$$

- **Permutational representation**: The representation $\rho : S_n \to \mathbb{C}^n (= V_{\text{perm}})$ defined by

$$\rho(\sigma) : (x_1, \dots, x_n) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

  - Character: Compute $\text{Fix}(\sigma)$ for each type of σ.
- **Class function**: A function that is constant on the conjugacy classes of G. Explicitly, for all $s, t \in G$,

$$f(tst^{-1}) = f(s)$$

- **Group algebra**: The complex vector space with basis $\{e_g\}$ corresponding to the elements of group G, plus the definition $e_g \cdot e_h = e_{gh}$.
- **Semisimple module**: A module M that satisfies any of the following three conditions.
  1. $M = \bigoplus_{i\in I} S_i$, where each S_i is a simple module and I is an indexing set.
  2. $M = \sum_{i\in I} S_i$.
  3. For all submodules $N \subset M$, there exists N' such that $M = N \oplus N'$.
- Additional notes on semisimple modules.
  - Simple module: A module that is nonzero and has no nonzero proper submodules.
- **Division algebra**: An algebra D such that for all nonzero $x \in D$, there exists a $y \in D$ such that $xy = 1$.
- **Semisimple algebra**: An algebra for which every finite-dimensional A-module is semisimple.
- **Wedderburn-Artin theorem**: If A is a finite-dimensional semisimple associative algebra, then

$$A \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k)$$

- **Schur's Lemma** (over an arbitrary field): Let A be a finite-dimensional algebra, and let M_1, M_2 be simple A-modules. Then...
  1. If $f : M_1 \to M_2$ is a nonzero morphism of A-modules, f is isomorphic;
  2. If M is simple, $\text{Hom}_A(M, M)$ is a division algebra.
- **Center** (of A): The following set.

$$Z(A) = \{a \in A \mid xa = ax \; \forall \; x \in A\}$$

- **Jacobson radical**: The finite-dimensional A-algebra defined as follows.

$$\text{Rad}(A) = \{a \in A \mid aS = 0 \text{ for any simple module } S\}$$

- Here's an outline of what to remember for the problems.
- Strategies for computing the following things from the character table of a group.
  1. Tensor products.
     - Multiply corresponding characters.

2. Wedge/symmetric squares.
   - If χ is the character of a representation $\rho : G \to GL(V)$, then the characters χ_σ^2 of the symmetric square S^2V of V and χ_α^2 of the alternating square $\Lambda^2 V$ of V are given by the following for each $s \in G$.

$$\chi_\sigma^2(s) = \frac{1}{2}\left(\chi(s)^2 + \chi(s^2)\right) \qquad \chi_\alpha^2(s) = \frac{1}{2}\left(\chi(s)^2 - \chi(s^2)\right)$$

   - Note that just like $V^{\otimes 2} = S^2V + \oplus\Lambda^2 V$, we have $\chi^2 = \chi_\sigma^2 + \chi_\alpha^2$.
3. Decomposing permutational representations into irreducibles.
   - If the representation of interest is χ_V, we find the coefficients n_i of its decomposition

$$\chi_V = \sum n_i \chi_{V_i}$$

     via the inner product

$$n_i = \langle \chi_V, \chi_{V_i} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)\overline{\chi_{V_i}(g)}$$

4. More.

- Strategies for computing the following things given a small group (e.g., the quaternion group).

1. Conjugacy classes.
   - Take an element, conjugate it by everything, round up the products. Then move onto another elements.
2. Character table.
   - Find the conjugacy classes and put them at the top of the table. This also tells us how many irreps we need to get to.
   - Start with the trivial, alternating, and standard representations.
   - Tensor products of representations with 1D representations (e.g., the alternating) are often linearly independent.
   - We can recover the standard as the difference of the permutation and trivial.
   - We can solve for representations as components of the regular representation via

$$V_R = \bigoplus_{i=1}^{k} V_i^{\dim V_i}$$

   - We can calculate the degrees of remaining representations via the sums of the squares of the dimensionalities.
   - We can fill in final representations with the orthogonality relations, *especially* the second one.
3. Decomposing representations into a sum of isotypical components.
   - Use the inner product/decomposition formula from above.
4. Diagonalizing an endomorphism.
   - Start with its matrix A.
   - Find the characteristic polynomial by computing $\det(A - \lambda I)$.
   - Solve for the eigenvalues.
   - Find, by inspection or by solving systems of equations, elements of the null space of $A - \lambda I$ for each λ. Beware eigenvalues with multiplicity greater than one!
5. More.

- Strategies for solving an abstract problem about characters.

  - *reread notes*
- Strategies for solving an abstract problem about representations.
  - *reread notes*
- Other misc. concepts that are probably good to remember (my own ideas).
- **Left *A*-module**: A pair (M,ρ) where $(M,+)$ is an abelian group and $\rho : A \to \text{End}(M)$ is the ring homomorphism defined as follows, where A is a ring: For all $a \in A$, $\rho(a) : M \to M$ is given by $\rho(a)v = av$ for all $v \in M$ and satisfies the following constraints.
  1. $\rho(a) : M \to M$ is a group homomorphism on $(M,+)$.
  2. ρ is a ring homomorphism.
     - That is to say, $\rho(a+b) = \rho(a) + \rho(b)$, $\rho(ab) = \rho(a)\rho(b)$, and $\rho(1_A) = 1_{\text{End}(M)}$.
- Lemma (Gauss's Lemma): If $f,g \in R[X]$ are both nonzero polynomials with coefficients in the ring R, then $c(fg) = c(f)c(g)$.
  - Note that $c(f)$ denotes the **content** of f, which is the gcd of its coefficients.
  - Use: If p is reducible in a fraction field, then it's reducible in the native UFD.
- A is semisimple iff $\text{Rad}(A) = 0$.
- Formulas for the decomposition of the regular representation/misc. formulas from IChem.
  - Sum of the squares of the dimensionalities (from the second orthogonality relation):

$$|G| = \sum_{i=1}^{k} (\dim V_i)^2$$

  - Sum of the squares of an irrep's characters (from the first orthogonality relation):

$$|G| = \sum_{g \in G} \chi(g)^2$$