Week 5

Associative Algebras

5.1 Wedderburn-Artin Theory

10/23: • Share notes with Rudenko at the end of the course!

- Today: Wedderburn-Artin theory.
  - Noncommutative algebra.
  - Noncommutative is a big part of math, partially because of its relation to QMech and partially because of its use in math, itself.
  - There is a textbook: Lang (2002). It's a hard, grad-level textbook but very cleanly written. Not a bad book to have in our mind as we start to encounter category theory.
- So here's what were talking about.
  - Our main object is A, an **associative algebra** over a field F.
- Left vs. right algebras.
  - When A is not commutative, we have to specify which we are dealing with.
  - Let A be an algebra over F.
  - Recall left-modules and right-modules.
    - In a left module, you can multiply $A \times M \to M$ where $(ab)m = a(bm)$.
    - In a right module, $(ab)m = b(am)$. More simply, $m(ab) = (ma)b$.
    - With modules, we get submodules, quotient modules, homomorphisms of modules, etc.
  - Let $I \subset A$ be a left-submodule. Thus, it is a subspace of A such that for all $a \in A$, $aI \subset I$, i.e., a left ideal.
  - In a right-submodule $I \subset A$, we have that for all $b \in A$, $Ib \subset I$, i.e., a right ideal.
  - In a two-sided ideal $I \subset A$, we have for all $a, b \in I$ that $aI \subset I$ and $Ib \subset I$.
  - Example: The matrix algebra is the prototypical noncommutative algebra. Consider $M_{2\times 2}(\mathbb{C})$.
    - Pick $v = (1, 0)$.
    - Look at ideal $I = \{X \in M_{2\times 2} \mid Xv = 0\}$. This is called the **annihilator**, and it is a left ideal. Explicitly, this ideal is the subset of all matrices of the form

$$\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$$

for $a, b \in \mathbb{C}$.

        - An example of a right ideal is all those such that $vX = 0$, i.e., all matrices of the form

$$\begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$$

            - Note that we are treating v as a row vector here.
        - There are *no* two-sided ideals herein, save the trivial one.

- **Simple** (algebra): An algebra for which there are no nontrivial two-sided ideals.
- Every time you go more abstract, it's more boring because you have less things to play with, but we can derive more general rules.
    - We'll only stay so abstract for 2-3 lectures.
- We want to convert left-algebras to right-algebras.
    - To do so, we can construct **opposite algebras**.
- **Opposite algebra** (of A): The algebra with the same vector space structure as A, but with the reversed multiplication such that $a * b$ in this space yields $b * a$ in A. *Denoted by* $\boldsymbol{A^{\text{op}}}$.
    - Left ideals of A become right ideals of A^{op} and vice versa. Two-sided ideals stay the same.
    - In category theory, left-modules over A are equivalent to right-modules over A^{op}.
    - Opposite algebras are briefly defined on Fulton and Harris (2004, p. 308) and are not defined anywhere else in any of the other sources.
- Example: Consider $M_{n\times n}(F)^{\text{op}}$.
    - Claim: This algebra equals regular $M_{n\times n}(F)$.
    - The map between these spaces is $A \mapsto A^T$.
    - There are other maps, such as conjugation and then transpose.
    - Being isomorphic to your opposite is a strange and interesting property!
- Example: $\mathbb{C}[G]^{\text{op}} \cong \mathbb{C}[G]$.
    - Left as an exercise to find the map.
- Let M, N be modules. We now investigate some properties of $\text{Hom}_A(M, N)$, a nice abelian group.
    - Explicitly, it's

$$\text{Hom}_A(M, N) = \{f : M \to N \text{ linear} \mid f(am) = af(m)\ \forall\ a \in A\}$$

    - We have that

$$\text{Hom}_A(M_1 \oplus M_2, N) \cong \text{Hom}_A(M_1, N) \oplus \text{Hom}_A(M_2, N)$$

        - Prove by looking at what happens to vectors of the form $(M_1, 0)$ and $(0, M_2)$.
    - Similarly,

$$\text{Hom}_A(M, N_1 \oplus N_2) \cong \text{Hom}_A(M, N_1) \oplus \text{Hom}_A(M, N_2)$$

- What if we have $\text{Hom}(M_1 \oplus \cdots \oplus M_n, N_1 \oplus \cdots \oplus N_m)$?
    - Then we have by induction from the previous cases that

$$\text{Hom}(M_1 \oplus \cdots \oplus M_n, N_1 \oplus \cdots \oplus N_m) = \bigoplus_{\substack{i=1,\dots,n \\ j=1,\dots,m}} \text{Hom}(M_i, N_j)$$

    - Let $\varphi_{ij} \in \text{Hom}(M_i, N_j)$.

- At this point, it's very natural to write matrices

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} & & \\ & \varphi_{ji} & \\ & & \end{bmatrix} \end{array} \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} \varphi_{11}(m_1)+\cdots+\varphi_{1n}(m_n) \\ \vdots \\ \ \end{pmatrix} = \begin{pmatrix} (\varphi(m)) \\ \vdots \\ \ \end{pmatrix}$$

  - Is it ϕ_{ji} or ϕ_{ij}?? Lang (2002, p. 642) seems to back the latter.
- To make this make sense for ourselves, write out the 2×2 case from $M_1 \oplus M_2 \to M_1 \oplus M_2$.

$$\begin{pmatrix} \varphi_{11} & \varphi_{21} \\ \varphi_{12} & \varphi_{22} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} \quad\quad\quad\quad \\ \ \end{pmatrix}$$

- Matrices made out of maps can seem really confusing when you first start, but in time, it will make sense.

- Recall the result from last time about division algebras.
- The main object we need to understand is a **semisimple algebra**.
- **Semisimple** (module): A module that satisfies any of the conditions in the following theorem.
  - Note that we proved something analogous to condition 3 early on! This was the complements theorem.
  - There is an equivalent for infinite-dimensional algebras; we need **Zorn's lemma** regarding maximal ideals/the axiom of choice here, though.
- Theorem: Let A be an algebra over F, and let M be a left-module. Then TFAE.
  1. $M = \bigoplus_{i \in I} S_i$, where each S_i is a simple module and I is an **indexing set**, not a simple module/ideal.
  2. $M = \sum_{i \in I} S_i$, where the sum is *not* direct.
  3. For all submodules $N \subset M$, there exists N' such that $M = N \oplus N'$.

Proof. This proof only applies for the case that M is finite dimensional; the theorem is more general than that, but we are not interested in the more general case.

($1 \Rightarrow 2$): Very clear; all direct sums are sums.

($2 \Rightarrow 1$): Consider the maximal subset $J \subset I$ (by inclusion, not by indices) of our indexing set such that

$$\sum_{i \in J} S_i = \bigoplus_{i \in J} S_i$$

In other words, J induces the highest-dimension sum of submodules that is a direct sum. Note that we can still find a singleton J in the direct-sum-of-one-thing case, so we're starting from a good base case.

Claim: $\bigoplus_{i \in J} S_i = M$. Suppose not. Then there exists $m \in M$ such that $m \notin \bigoplus_{i \in J} S_i$ and $m = s_{i_1} + \cdots + s_{i_k}$ where each $s_{i_j} \in S_{i_j}$. If all $s_{i_1}, \dots, s_{i_k} \in \bigoplus_{i \in J} S_i$, then we have arrived at a contradiction and we are done. If not, then there exists some s_{i_t} such that $s_{i_t} \notin \bigoplus_{i \in J} S_i$. Now consider $S_{i_t} \cap (\bigoplus_{i \in J} S_i)$. This will be a submodule of S_{i_t}. But since S_{i_t} is simple by hypothesis, this means that $S_{i_t} \cap (\bigoplus_{i \in J} S_i)$ either equals S_{i_t} or 0. However, we know that it can't equal S_{i_t} because above, we found $s_{i_t} \in S_{i_t}$ such that $s_{i_t} \notin \bigoplus_{i \in J} S_i$. Thus, $S_{i_t} \cap (\bigoplus_{i \in J} S_i) = 0$. But this means that $S_{i_t} + \bigoplus_{i \in J} S_i$ is a direct sum, which contradicts the choice of J as maximal.

($1 \Rightarrow 3$): Let's take a submodule $N \subset M$. By 1, $M = \bigoplus_{i \in I} S_i$. Let's look at all subsets J such that

$$N + \sum_{j \in J} S_j = N \oplus \left(\sum S_j \right)$$

Look at the maximal one by inclusion. Then once again, by the same proof strategy as above,

$$N \oplus \underbrace{\left(\sum S_j\right)}_{N'} = M$$

($3 \Rightarrow 1$): We use what we've learned about representations. Let $M = N_1 \oplus N_2$. Then N_2, if nonsimple, has subsets $N_2 \oplus N_3$. We can continue on and on. Because dimensions finitely decrease, we'll eventually have to arrive at a sum $N_1 \oplus \cdots \oplus N_m$ of simples. □

- Now, we have 3 definitions of semisimple modules.
- Corollary: If A is an algebra, M is a semisimple module, and $N \subset M$ is a submodule, then...
  1. N is semisimple.

     Proof. Let L be a submodule of N. We need to find a complement of L inside N. We can find $L' \subset M$ such that $L \oplus L' = M$. Then $L' \cap N \subset N$ is the complement of L in N. Why? Because of the following.
     Claim: $(L' \cap N) \bigoplus L = N$. Not intersecting: $L' \cap N \cap L \subset L' \cap L = 0$. Summing to the whole thing: Let $n \in N$ be arbitrary. Then since $n \in M$, there exists $\ell, \ell' \in L, L'$ such that $n = \ell + \ell'$. But since $n, \ell \in N$, we must have $\ell' \in N$ as well. Therefore, $\ell' \in L' \cap N$. □
  2. M/N is semisimple.
- Takeaway: Submodules and quotient modules of semisimple modules are semisimple modules.
- Lang (2002) has a write-up of the proof from today's class.
  - Funnily enough, it is the only textbook that does! Fulton and Harris (2004) doesn't have it; not even Etingof et al. (2011) has it!

5.2 Semisimple Algebras

10/25:

- More associative algebra today; we'll wrap it up next time.
- Review.
  - Let A be a finite dimensional associative algebra over a field F.
  - We want to understand when this algebra is very close to a *group algebra*.
    - Recall that $A = F[G] = \{a_{g_1}g_1 + \cdots + a_{g_n}g_n \mid a_i \in F\}$ is the group algebra of G a finite group.
  - Recall left modules.
    - These are very similar to representations.
    - Indeed, if we have a left module M, then we have a multiplication map $\rho : A \times M \to M$ with properties such as associativity, etc.
  - Recall right modules.
    - In a group representation, left modules over A are essentially the same thing as right modules over A^{op}.
    - Because there is a bijection between left modules over A and right modules over A^{op}, we sometimes have the case where A doesn't change, i.e., $A \cong A^{\text{op}}$.
  - All of the above motivated the definition of *semisimple*: If A is a finite dimensional algebra and M is a finite-dimensional module, then M is *semisimple* if it satisfies any one of three conditions from last time's theorem.

    - Note: When we describe a module as "finite-dimensional," we mean this in the sense of a vector space, i.e., literally finite-dimensional as opposed to finitely generated or anything like that.
    - Note: "Last time's theorem" refers to the semisimplicity conditions one, which is a part of Wedderburn-Artin theory but is *not* the **Wedderburn-Artin *theorem***. We'll get to this theorem eventually, but that's still in the future.

- Theorem (Maschke's theorem): Let G be a finite group and let F be a field. Suppose $(|G|, \operatorname{char} F) = 1$, i.e., they are coprime. Then every finite-dimensional left module over $F[G]$ is semisimple.

  Proof. We've already basically done this proof as part of last time's theorem. Here's a refresher, though.

  Let M be an arbitrary finite-dimensional left module over $F[G]$. Then there exists a map $F[G] \to \operatorname{End}_{F[G]}(M)$ (left multiplication; the action of elements of this ring on elements of M), or $G \to GL(M)$. Thus, M is a G-representation, which satisfies condition (3) from last time's theorem because of the complements theorem, stated as Theorem 1 from Serre (1977) for instance. $\square$

- Takeaway: The proof actually works for any field under this condition.
  - Rudenko will reprove Maschke's theorem tomorrow a different way.
- In an algebra, we have a multiplication map $\cdot : A \times A \to A$.
  - If we take the perspective that this map defines an action of the left A on the right one, we see that A has the structure of a left A-module.
  - Vice versa for right-modules.
- **Semisimple** (algebra): An algebra for which every finite-dimensional A-module is semisimple. *Also known as* **semi-simple**.
- Theorem: Let A be a finite-dimensional associative algebra. Then TFAE.
  1. A is a semisimple algebra.
  2. A is semisimple as a left-module over A. Equivalently, as an A-module, $A \cong S_1^{n_1} \oplus \cdots \oplus S_k^{n_k}$.
  3. (Wedderburn-Artin theorem) $A \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k)$, where the $D_1, \dots, D_k$ are division algebras. Note that the isomorphism is an isomorphism of algebras.
- We will prove this theorem in just a moment, but there are a few preliminary comments to be made first.
- Let's look at the algebra $\mathbb{H}$.
  - We can create matrices of quaternions, and we can add and multiply these matrices just fine.
  - However, the determinant is weirder: Is it $ad - bc$ or $ad - cb$?
    - There is a theory of determinants of noncommutative fields called **algebraic *k*-theory**, but we will not get into that.
- Example: Proving (3) for $\mathbb{C}[G]$.
  - We have $\mathbb{C}[G]$. There are not many division algebras over complex numbers; only one, in fact: Complex numbers.
  - Let $V_1, \dots, V_k$ be the irreps. Then we want to show that

    $$\mathbb{C}[G] \cong M_{d_1}(\mathbb{C}) \oplus \cdots \oplus M_{d_k}(\mathbb{C})$$

    where $d_i = \deg V_i$.

- Note: Matrices give us a nice way to compute otherwise complicated elements of $\mathbb{C}[G]$.
- Proof: Define a map $F : \mathbb{C}[G] \to M_{d_1}(\mathbb{C}) \oplus \cdots \oplus M_{d_k}(\mathbb{C})$ by

$$x \mapsto (\rho_{V_1}(x), \dots, \rho_{V_k}(x))$$

  - F is injective: $F(x) = 0$ implies that $\rho_{V_i}(x) = 0$ $(i = 1, \dots, k)$, so $xV_i = 0$ $(i = 1, \dots, k)$. In particular, this means that $x = x \cdot 1 = 0$.
  - F is surjective: F is injective and $\dim(\mathbb{C}[G]) = \sum d_i^2 = \dim[M_{d_1}(\mathbb{C}) \oplus \cdots \oplus M_{d_k}(\mathbb{C})]$.
  - F is a homomorphism of algebras: Left as an exercise.
- Note: Remember this theorem very well because it allows you to treat group rings very easily.
- Tomorrow, we'll bring characters into this picture.

- We now state a lemma that will be used to prove $2 \Rightarrow 3$.
- Lemma: Let $\text{End}_A(A)$ denote the set of A-module endomorphisms of A. Then

$$\text{End}_A(A) \cong A^{\text{op}}$$

as algebras.

Proof. To prove the claim, it will suffice to construct an A-algebra isomorphism $F : \text{End}_A(A) \to A^{\text{op}}$. Define F by

$$F(f) := f(1)$$

for all $f \in \text{End}_A(A)$. It should be fairly clear that

$$F(f+g) = F(f) + F(g) \qquad\qquad F(1) = 1$$

Proving that $F(f \circ g) = F(f) * F(g)$ is slightly more involved, but can be done as follows.

$$F(f \circ g) = [f \circ g](1) = f(g(1)) = f(g(1) \cdot 1) = g(1) \cdot f(1) = F(g) \cdot F(f) = F(f) * F(g)$$

Lastly, by plugging $f = a = aI$ and $g = f$ into the above, we can recover

$$F(af) = a * F(f)$$

Thus, F is an A-algebra *homomorphism*. To prove that it is an *isomorphism*, consider the inverse map $G : x \mapsto [a \mapsto ax]$. We can show that $F \circ G = 1_{A^{\text{op}}}$ and $G \circ F = 1_{\text{End}_A(A)}$, thus completing the proof. □

- We now prove the above theorem, which we restate for simplicity.
- Theorem: Let A be a finite-dimensional associative algebra over F. Then TFAE.
  1. A is a semisimple algebra.
  2. A is semisimple as a left-module over A. Equivalently, as an A-module, $A \cong S_1^{n_1} \oplus \cdots \oplus S_k^{n_k}$.
  3. (Wedderburn-Artin theorem) $A \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k)$, where the $D_1, \dots, D_k$ are division algebras. Note that the isomorphism is an isomorphism of algebras.

Proof. One line; very simple, but a little weird conceptually.

($2 \Rightarrow 1$): To prove that A is a semisimple algebra, it will suffice to show that every finite-dimensional A-module is semisimple. Let $M = Ae_1 + \cdots + Ae_n$ be an arbitrary finite-dimensional A-module. To show that it's semisimple, it will suffice to demonstrate that it's equal to the direct sum of simple modules. Define a map $A^n \to M$ by

$$(a_1, \dots, a_n) \mapsto a_1e_1 + \cdots + a_ne_n$$

This should (fairly clearly) be a surjective homomorphism of left A-modules. Moreover, since $A = S_1 \oplus \cdots \oplus S_k$ is semisimple as a left A-module by hypothesis, we have that $A^n = S_1^n \oplus \cdots \oplus S_k^n$. Since the map defined above is also injective, it follows that

$$M \cong A^n = S_1^n \oplus \cdots \oplus S_k^n$$

as desired.

($3 \Rightarrow 2$): Work it out in the HW!

($2 \Rightarrow 3$): Let's take $A = S_1^{n_1} \oplus \cdots \oplus S_k^{n_k}$ a left A-module where each S_i is simple. Then by the lemma,

$$A^{\text{op}} \cong \text{End}_A(A) = \text{Hom}_A(A, A) = \text{Hom}_A(S_1^{n_1} \oplus \cdots \oplus S_k^{n_k}, S_1^{n_1} \oplus \cdots \oplus S_k^{n_k}) = \bigoplus_{i,j=1}^{k} \text{Hom}_A(S_i^{n_i}, S_j^{n_j})$$

By Schur's lemma for associative algebras,

$$\text{Hom}_A(S_i, S_j) = \begin{cases} 0 & i \neq j \\ D_i & i = j \end{cases}$$

where each D_i is a division algebra. Thus, continuing from the above,

$$A^{\text{op}} \cong \bigoplus_{i=1}^{k} \text{Hom}_A(S_i^{n_i}, S_i^{n_i}) = \bigoplus_{i=1}^{k} M_{n_i}(\text{Hom}_A(S_i, S_i)) = \bigoplus_{i=1}^{k} M_{n_i}(D_i)$$

Note that $\text{Hom}_A(S_i^{n_i}, S_i^{n_i}) = M_{n_i}(\text{Hom}_A(S_i, S_i))$ because of the thing about homomorphisms of direct sums of modules equaling matrices of homomorphisms. This was discussed on Monday. We don't include any $\text{Hom}_A(S_i, S_j)$ because all of these are equal to zero; indeed, it appears that these matrices will be strictly diagonal. ☐

- Consequence: It follows because the D_i's are division algebras that

$$A \cong \bigoplus_{i=1}^{k} M_{n_i}(D_i^{\text{op}})$$

  - What was the point of this??

- Note from last time that we forgot to discuss: A quotient module of a semisimple module is semisimple. Proving this will be in the next HW.

- **Radical** (of A): The finite dimensional A-algebra defined as follows. *Also known as* **Jacobson ideal**, **Jacobson radical**. *Denoted by* $\mathbf{Rad(A)}$. *Given by*

$$\text{Rad}(A) = \{a \in A \mid aS = 0 \text{ for any simple module } S\} \subset A$$

  - Immediate fact: $\text{Rad}(A)$ is a two-sided ideal.
  - This is because...
    - $x \in A$ and $a \in \text{Rad}(A) \implies (xa)S = x(aS) = x(0) = 0 \implies xa \in \text{Rad}(A)$;
    - $x \in A$ and $a \in \text{Rad}(A) \implies (ax)S = a(xS) = 0 \implies ax \in \text{Rad}(A)$.
  - Note that xS is simple in the above line because a scaled simple module is still simple.

- Theorem: A is semisimple iff $\text{Rad}(A) = 0$.
  - This will be explained next time.
  - In other words, if there are problematic elements, the algebra is not semisimple.

  - Quotienting algebras by two-sided ideals gives algebras, so if A is not semisimple, we know that $A/\operatorname{Rad}(A)$ *is* semisimple!
- This week: A brief primer on noncommutative algebra that is probably worth studying for the midterm.
- Next week: Number theoretic stuff, integer elements, groups, etc.
- Most people/books don't treat the finite-dimensional case here (so it's not written up anywhere) because they view it as too restrictive; instead, they prefer to use the **Artinian** condition.

5.3 The Jacobson Radical

10/27:

- Review.
  - Let A be an associative algebra over a field F. $\dim_F A = \infty$ (??).
  - A is semisimple if every left A-module is a sum of simple A-modules.
  - Theorem: A is semisimple iff ${}_AA$ is semisimple iff $A \cong M_{n_1\times n_1}(D_1) \oplus \cdots \oplus M_{n_k\times n_k}(D_k)$, where the D_i are division algebras.
    - Note: ${}_AA$ denotes A as a left A-module.
  - The simplest semisimple algebra is a matrix algebra.
- Example: A HW problem solution (PSet 4, Q4b).
  - Let $A = M_{n\times n}(F)$. Then $\dim(A) = n^2$.
  - One linear representation of A that's particularly nice is $S = F^n$, i.e., the set of all column vectors of length n with entries in F.
    - This representation $\rho : A \to GL(S)$ can simply be defined by $\rho(X) = X$.
    - Alternatively, this can be thought of as the map from $A \times S \to S$ sending $(X, v) \mapsto Xv$.
    - This is a simple representation! Using permutation matrices, for instance, we can see that no subspace is fixed under *every* $X \in M_{n\times n}(F)$.
  - The HW problem was to show that F^n is the only simple module over the matrix algebra.
  - Sidebar: To prove that A is semisimple, we can show that ${}_AM_{n\times n}(F) \cong \bigoplus^n S = S^n$.
    - To do so, use the module isomorphism $(v_1 \mid \cdots \mid v_n) \mapsto v_1 \oplus \cdots \oplus v_n$.
  - From here, we can deduce that if T is a simple module, we can construct a homomorphism $A^N = (S^n)^N \twoheadrightarrow T$?? It follows that $S \cong T$? What is this??
- Takeaways.
  - There is a unique simple module over the matrix algebra, i.e., the columns of the matrix.
  - The dimension of every module over a matrix algebra will be a multiple of n. Why?
    - $M_{n,n}(F)$ is semisimple. Thus, any (finite-dimensional) $M_{n,n}(F)$-module M is semisimple. Consequently, $M = \bigoplus_{i\in I} S_i$. But since every $S_i = F^n$, we have $M = (F^n)^{|I|}$ with $\dim(M) = n \cdot |I|$, as desired.
    - Think $n \times 1$ matrices (column vectors; what we just discussed), $n \times 2$ matrices, $n \times 3$ matrices, on and on.
- Moving on.
- We want something more complete about an algebra.
- Recall the radical of A.
- Main theorem: A is semisimple iff $\operatorname{Rad}(A) = 0$.

- Facts.
  1. $\text{Rad}(A)$ is a two-sided ideal.
     - Prove directly by multiplying on both left and right, as at the end of Wednesday's class.
  2. $\text{Rad}(A) = \bigcap L$ where L is a maximal left ideal.
- **Maximal** (left ideal of A): A left ideal L for which there exists no left ideal L' such that $L \subsetneq L' \subsetneq A$.
  - Ideals are subspaces. Maximal means biggest by inclusion, but not necessarily equal to the whole thing.
- We now prove Fact 2.

  Proof. We first establish some facts. Then we do a bidirectional inclusion proof.

  If L is a left ideal, then A/L is a left A-module. If we now assert that L is a *maximal* left ideal, then A/L is a *simple* left A-module. This is because of the following correspondence theorem, a very general fact that's easy to show: Essentially, if you have some modules M, N such that $N \leq M$, then the modules in between $N \subsetneq M$ are in bijection with M/N. This bijection is defined in the forward direction by quotienting modules in between $N \subsetneq M$, and in the reverse direction by taking the preimage of the quotient projection. Thus, maximal left ideals L have nothing in between them and A, so A/L is in bijection with nothing! Moreover, *every* simple module is obtained this way.

  If S is an arbitrary simple module containing $v_0 \neq 0$, then we may define $f : A \to S$ sending $a \mapsto av_0$. Note that $0 \subsetneq \text{Im}(f) \subseteq S$. But since S is simple, we must have $\text{Im}(f) = S$ so f must be surjective. It follows that $S \cong A/L$ for some maximal left ideal L of A.[1]

  If $x \in \text{Rad}(A)$ and L is a maximal left ideal of A, then $x(A/L) = 0$ (since A/L is simple). It follows that $xA \subseteq L$. It follows since $x \in xA$ that $x \in L$. Thus, $\text{Rad}(A) \subset \bigcap L$.

  Now, to show the other inclusion, let $x \in \bigcap L$. Let S be an arbitrary simple module over A. We know that $S \cong A/L$ for some maximal ideal L. To demonstrate that $xS = 0$, it will suffice to confirm that $xv_0 = 0$ for all $v_0 \in S$. Let $0 \neq v_0 \in S$ be arbitrary. Define $f : A \to S$ by $a \mapsto av_0$. Since v_0 is nonzero and hence $\text{Im}(f)$ is nontrivial, the fact that S is simple must mean that $\text{Im}(f) = S$ and hence f is surjective. Thus, $A/\text{Ker}(f) \cong S$. Consequently, $\text{Ker}(f) = L$. It follows since $x \in \bigcap L$ and hence $x \in L$ that $x \in \text{Ker}(f)$. But then $xv_0 = 0$, as desired. □

- Thus, the radical has the equivalent descriptions

$$\text{Rad}(A) = \{a \in A \mid aS = 0 \text{ for any simple module } S\} = \bigcap L$$

- Theorem: A (finite-dimensional) is semisimple iff $\text{Rad}(A) = 0$.

  Proof. We will prove both directions independently here. Let's begin.

  ($\Rightarrow$): Suppose A is semisimple. Then $A = S_1 \oplus \cdots \oplus S_N$. It follows in particular that $1 = s_1 + \cdots + s_N$ for some $s_i \in S_i$ ($i = 1, \dots, n$). Now let $a \in \text{Rad}(A)$ be arbitrary; we hope to show that $a = 0$. Fortunately, we can do this as follows via

$$a = a \cdot 1 = as_1 + \cdots + as_N = 0 + \cdots + 0 = 0$$

  Just to be super clear, $as_i = 0$ because $a \in \text{Rad}(A)$ implies $aS = 0$ for all simple modules S, including S_i of which s_i is an element and is thus annihilated by a.

  ($\Leftarrow$): Suppose $\text{Rad}(A) = 0$. Then $\bigcap L = 0$. This combined with the fact that A is finite dimensional implies that there exists a finite collection $L_1, \dots, L_n$ of maximal ideals such that $\bigcap L = \bigcap^n L_i$. (In particular, $n \leq \dim A$. Essentially, since we're finite dimensional, what we can do is drop dimensions

[1] This seems redundant; perhaps Rudenko meant to say $L = \text{Ker}(f)$ in addition to the other stuff??

from $\dim L_1$ to $\dim L_1 \cap L_2$ to $\dim L_1 \cap L_2 \cap L_3$, so since we're eventually going to hit zero, we're eventually going to have to stop. In other words, choose L_1, then choose L_2 such that $\dim L_1 \cap L_2 < \dim L_1$, then choose L_3 such that $\dim L_1 \cap L_2 \cap L_3 < \dim L_1 \cap L_2$, and continue in this fashion until we have $\dim L_1 \cap \cdots \cap L_n = 0$; because the sequence $\dim L_1 \cap \cdots \cap L_i$ is strictly decreasing and the initial value is finite, the sequence must eventually terminate.) Thus, $\text{Rad}(A) = L_1 \cap \cdots \cap L_n$. One line to finish. View A as a left A-module (denote it ${}_AA$ with left subscript A). Define $f : {}_AA \to A/L_1 \oplus \cdots \oplus A/L_n$ by $f(a) = (\pi_1(a), \dots, \pi_n(a))$, where $\pi_i(a) : A \to A/L_i$ denotes the projection function $a \mapsto a + L_i$. Then

$$\text{Ker}(f) = \bigcap_{i=1}^{n} L_i = \text{Rad}(A) = 0$$

But then f is injective. This combined with the fact that $A/L_1 \oplus \cdots \oplus A/L_n$ is semisimple by definition means that ${}_AA$ is isomorphic to a submodule of a semisimple module. Thus, since the only submodules of a semisimple module are mix-and-match combinations of the semisimple module's constituent simple modules, ${}_AA$ is semisimple itself. Therefore, by the semisimple algebra conditions from Wednesday's class, A is semisimple. □

- **Artinian** (ring): A ring for which every decreasing sequence of ideals has to stabilize.
- Let S_1, S_2 be simple modules, and let M be some module. We get $\text{Hom}_G(S_2, S_1)$, $\text{Ext}^1(S_2, S_1)$, $\text{Ext}^2(S_2, S_1)$, ... This gets very complicated very quickly, and you actually need homological algebra to keep track of everything.
  - Point??
- New HW problem: $A = \mathbb{F}_p[G]$ (p a prime) is never semisimple. This is called **modular representation theory**, it's in our book (where??), and it's hard.
- A very concrete criterion for semisimplicity.
  - Let $F = \mathbb{C}$ and let A be finite dimensional with $\dim_F A = n$.
  - Define a scalar product in A by
    $$\langle x, y\rangle = \text{tr}(L_x L_y)$$
    - $L_x : A \to A$ is the map that sends $a \mapsto xa$.
    - This is a symmetric map; it's got a lot of nice properties actually.
    - Note: $\text{tr}(L_x L_y)$ is colloquially known as $\text{tr}(xy)$.
  - Theorem: Let A be a finite-dimensional algebra over $\mathbb{C}$. Then A is semisimple iff $\text{tr}(x^2)$ is **nondegenerate**, which means that if $\text{tr}(xa) = 0$ for any x, then $a = 0$. We've probably seen this in the context of vector spaces like $V \otimes V \to \mathbb{C}$ or $V \cong V^*$. What is this??
  - Something about $|G|^{|G|}$. What is this??
- **Nondegenerate** (finite-dimensional bilinear form): A bilinear form $f(x, a)$ such that if $f(x, a) = 0$ for any x, then $a = 0$.
- Next week: Number theoretic group theory and then representation theory of symmetric groups.

5.4 L Chapter XVII: Semisimplicity

From Lang (2002).

11/10:
- Sections 1-2 cover a lot of the stuff discussed during Monday's class.
- Rewrite proof of theorem from Monday's class!

12/25:
- Perhaps the reason that we talked about matrices of functions right before semisimplicity is that homomorphisms of semisimple modules, in particular, have this decomposable structure.

- The Wedderburn-Artin theorem is not obviously covered here.
- The radical is talked about in Section 6.