Week 3

Characters

3.1 Defining Characters

10/9:

- Today, we talk about **characters**, arguably the most important idea in rep theory.
- As per usual, we begin by letting G a finite group.
  - We've been discussing finite dimensional representations of G over $\mathbb{C}$.
  - We've also already talked about irreps, and we know that it's enough to understand those because every rep is a sum of them.
- Goal of characters: Understand the irreps $V_1, \dots, V_k$ of G.
  - Recall the surprising fact about k: It is the number of conjugacy classes of G!
    - We haven't yet proven this, but we will soon!
  - Game plan: Use characters to relate irreps to something that is counted by conjugacy classes.
- Let $V = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$ be a G-rep.
  - Then there exists a homomorphism $\rho : g \mapsto A_g \in GL_n(\mathbb{C})$.
- Motivating question: What doesn't change when we change the basis of V?
  - To isolate the "essence" of the A_g, we want to construct a function $f : GL_n(\mathbb{C}) \to \mathbb{C}$ such that $f(XAX^{-1}) = f(A)$, where $X \in GL_n(\mathbb{C})$.
- Ideas.
  1. The determinant is a great example of such a function, but it's kind of boring because this rank 1 representation doesn't characterize your product representation.
  2. Trace is the main example of such a function.
- Indeed, you can also take $\operatorname{tr}(A^k)$ for any k.
  - Traces of powers are ubiquitous in physics and math because they contain the same information as the coefficients of the characteristic polynomial. In particular, we can express the determinant in terms of them.
- In fact, we could also take any coefficient of the characteristic polynomial, but others would get complicated.
  - Any characteristic polynomial coefficient can be expressed in terms of traces; this will be an exercise in PSet 3; it's not hard.

- For example, the second characteristic polynomial coefficient (sum of products of eigenvalues) can be written as follows.

$$\sum \lambda_i \lambda_j = \frac{\operatorname{tr}(A)^2 - \operatorname{tr}\left(A^2\right)}{2}$$

- So what do we have at this point?
  - We can associate to ρ a function $\chi_\rho : G \to \mathbb{C}$ defined by $\chi_\rho(g) = \operatorname{tr}(A_g) = \operatorname{tr}(\rho(g))$.
  - This function is invariant under isomorphism.
  - If we know $\operatorname{tr}(A)$, we know $\operatorname{tr}\left(A^2\right)$ since $A_g^2 = A_{g^2}$. Thus, if we know all traces, we know all power traces.
  - We form a ring of polynomials?? Equivalently, χ_ρ has a representation as a polynomial with coefficients in $\mathbb{C}$?
- If V is a G-rep, $\chi_V : G \to \mathbb{C}$ will be our notation for its character.
- Properties.
  1. $\chi_V(xgx^{-1}) = \chi_V(g)$ for any $x, g \in G$.
     - Implication: χ_V is a **class function**.
     - Let $\mathbb{C}[G]$ be the vector space of all functions from $G \to \mathbb{C}$. Its dim $= |G|$.
       - Recall that this notation is the same as that for the space of polynomials in $g_1, \dots, g_n$ with complex coefficients.
       - But since G is a group, a polynomial in products of g_i's can be reduced to a polynomial in the g_i's.
         - For example, if $G = S_3$, we know since $(23)(123)^2 = (12)$ that

$$\begin{aligned} 2 + 3(12) + i(123) + (-3 + 7i)(23)(123)^2 &= 2 + [3 + (-3 + 7i)](12) + i(123) \\ &= 2e + 7i(12) + i(123) \end{aligned}$$

       - Such a polynomial is then easily mapped onto a complex-valued function by sending each g_i to its coefficient a_{g_i}.
         - Continuing with the above example, the corresponding function in $\mathbb{C}[G]$ would be defined by

$$\begin{aligned} &e \mapsto 2 && (12) \mapsto 7i && (123) \mapsto i \\ & && (13) \mapsto 0 && (132) \mapsto 0 \\ & && (23) \mapsto 0 && \end{aligned}$$

       - Thus, $\mathbb{C}[G]$ (as a space of polynomials) is canonically isomorphic to $\mathbb{C}[G]$ (as a space of complex-valued functions on G), so the notation is well chosen.
       - What do we do for multiplication?? Because functions multiply pointwise but polynomials do not. It appears that we typically go with polynomial multiplication (but not always, potentially; see the claim from Lecture 6.1). MSE also supports polynomial multiplication. Perhaps $\mathbb{C}[G]$ for this function space with pointwise multiplication is just really misleading notation?
     - Inside this space, there is the subspace $\mathbb{C}_{\text{cl}}[G]$ of functions $f : G \to \mathbb{C}$ such that $f(xgx^{-1}) = f(g)$ for all $x, g \in G$. These are functions from the sets of conjugacy classes, isomorphic to functions that are constant on conjugacy classes. $\dim \mathbb{C}_{\text{cl}}[G]$ is the number of conjugacy classes.
     - Thus, for every V a G-rep, we get a vector $\chi_V \in \mathbb{C}_{\text{cl}}[G]$. These class functions form a basis of the space; each χ_V for V an irrep forms a linearly independent vector; the set is an *orthogonal* basis. This is the reason for the original theorem holding true!

2. $\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$.
   - Proof: It's basically tautological (not actually, but it's easy). Let $g \in G$. Compute $\chi_{V_1 \oplus V_2}(g)$. We can compute a basis $e_1, \dots, e_{n+m}$ where the first n vectors form a basis of V_1, and the next m vectors are a basis of V_2. This gives us a block matrix from which we show that the trace of the matrix is the sum of traces.

$$\chi_{V_1 \oplus V_2}(g) = \operatorname{tr} \begin{bmatrix} \rho_{V_1}(g) & 0 \\ 0 & \rho_{V_2}(g) \end{bmatrix} = \operatorname{tr} \rho_{V_1}(g) + \operatorname{tr} \rho_{V_2}(g) = \chi_{V_1}(g) + \chi_{V_2}(g)$$

   - Corollary:

$$\chi_{V_1^{n_1} \oplus \cdots \oplus V_k^{n_k}} = n_1 \chi_{V_1} + \cdots + n_k \chi_{V_k}$$

- We now pause for a fact that will be instrumental in proving the next property, which is a bit more involved.
  - He will explain two ways to prove it; we can also just prove it on our own.
- Fact: If A is a matrix such that $A^n = 1$, then A is diagonalizable or "semi-simple."
  - We can prove this with Jordan normal form.
  - It's a slightly surprising statement.
  - Obviously eigenvalues are roots of unity, but still needs some work.
  - This proof is left as an exercise.
- We now resume the list of properties.

3. $\chi_V(g)$ is a sum of roots of unity.
   - Proof: We know that $g^{|G|} = e$. Thus, $A_g^{|G|} = 1$. It follows by the fact above that A_g is diagonalizable with eigenvalues $\lambda_1, \dots, \lambda_n$, each of which satisfies $\lambda_i^{|G|} = 1$.
     - Note: Eigenvalues can repeat in the list $\lambda_1, \dots, \lambda_n$, i.e., we are not asserting n distinct eigenvalues here.
   - Therefore, since each λ_i is, individually, a root of unity, we have that $\chi_V(g) = \operatorname{tr} A_g = \lambda_1 + \cdots + \lambda_n$, as desired.
4. $\chi_{V^*} = \bar{\chi}_V$.
   - This property begins to address how characters behave under other operations.
     - Naturally, this is something specific for complex numbers, because the idea of "conjugates" doesn't exist everywhere.
   - Proof: Recall that $\rho_{V^*}(g) = (\rho_V(g)^{-1})^T$.
     - If we know that $\rho_V(g) \sim \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, then we know that $\rho_V^{-1}(g)^T \sim \operatorname{diag}(\lambda_1^{-1}, \dots, \lambda_n^{-1})$.
     - Thus, $\chi_{V^*}(g) = \lambda_1^{-1} + \cdots + \lambda_n^{-1}$.
     - But since we're in the complex plane, $|\lambda_i| = 1$ (equiv. $\lambda_i \bar{\lambda}_i = 1$), so $\lambda_i^{-1} = 1/\lambda_i = \bar{\lambda}_i$.
     - This means that $\chi_{V^*}(g) = \bar{\lambda}_1 + \cdots + \bar{\lambda}_n = \overline{\lambda_1 + \cdots + \lambda_n} = \bar{\chi}_V(g)$.
   - Note: Every representation we have is **unitary** in certain bases, but unitary representations are not covered in this course.
5. $\chi_{V_1 \otimes V_2} = \chi_{V_1} \cdot \chi_{V_2}$.
   - Proof: We can use a basis or not use a basis.
   - Let's use a basis for now.
     - Let $g \in G$ be arbitrary. Then there exist bases $e_1, \dots, e_n$ of V_1 and $f_1, \dots, f_m$ of V_2 such that $\rho_{V_1}(g)$ and $\rho_{V_2}(g)$ are diagonal.
     - It follows that $\rho_{V_1}(g)e_i = \lambda_i e_i$ $(i = 1, \dots, n)$ and $\rho_{V_2}(g)f_i = \mu_i f_i$ $(i = 1, \dots, m)$.
     - $V_1 \otimes V_2$ thus has basis $e_i \otimes f_j$.

        - But then it follows that $\rho_{V_1 \otimes V_2}(g) e_i \otimes f_j = (\lambda_i e_i) \otimes (\mu_j f_j) = \lambda_i \mu_j (e_i \otimes f_j)$.
        - Thus,

$$\operatorname{tr}(\rho_{V_1 \otimes V_2}(g)) = \sum_{i,j=1}^{n,m} \lambda_i \mu_j = (\lambda_1 + \cdots + \lambda_n)(\mu_1 + \cdots + \mu_m) = \operatorname{tr}(\rho_{V_1}(g)) \cdot \operatorname{tr}(\rho_{V_2}(g))$$

    - Alternate approach.
        - If we don't want to think of eigenvalues, think of tensor product of matrices, the Kronecker product.
        - Essentially, if we adopt a basis such that our matrices are diagonal, then the block diagonal of the Kronecker product will be $\lambda_1 \rho_{V_2}(g) + \cdots + \lambda_n \rho_{V_2}(g)$, the trace of which will be $\lambda_1(\mu_1 + \cdots + \mu_m) + \cdots + \lambda_n(\mu_1 + \cdots + \mu_m)$.
        - We get trace is the product of traces once again!

- **Class function**: A function on a group G that is constant on the conjugacy classes of G.
- Examples.
    1. Let A be an abelian group.
        - Then $\chi : A \to \mathbb{C}^\times$.
        - Implication: Character of a character is $\chi_\chi = \chi$.
            - This is horribly repetitive but true.
    2. $G = S_3$.

	e	(12) (13) (23)	(123) (132)
Trivial	1	1	1
Alternating	1	-1	1
Standard	2	0	-1

Table 3.1: Character table for S_3.

        - The conjugacy classes of this group are $\{e\}$, $\{(12), (13), (23)\}$, and $\{(123), (132)\}$.
        - We construct a **character table** to define all characters.
        - Computing the characters for the trivial representation.
            - We know that ρ sends each g to the matrix (1), which has trace 1.
        - Computing the characters for the sign representation.
            - e and (123) have sign 1 and thus get sent to the matrix (1).
            - (12) has sign -1 and thus gets sent to the matrix (-1).
        - Computing the characters for the standard representation.
            - We can compute these traces via a thought experiment.
            - Visualize a triangle in a plane.
            - The 2×2 identity matrix (the standard representation of $e \in G$) acts on it by doing nothing, and has trace 2.
            - In *some* basis, our matrix fixes one vector and inverts another, so matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and character is 0.

    - Last one is rotation by $2\pi/3$, so

$$\begin{pmatrix} \cos(2\pi/3) & \sin(2\pi/3) \\ -\sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}$$

so character is $-1 = 2 \cdot -1/2 = 2 \cdot \cos(2\pi/3)$.
  - If V is the standard representation, we can also compute the characters of $V^{\otimes 2}$ for instance. Indeed, by the product rule of characters, they will be the squares of the standard representation's characters, i.e., $(4, 0, 1)$.
  - Similarly, since the permutational representation is the direct sum of the standard and trivial representations, we can add their characters to get its characters $(3, 1, 0)$.

3. A very general and very pretty example. Let $G \circlearrowright X$ a finite set.
  - Assign the permutational representation.
  - Let $X = \{x_1, \dots, x_n\}$. Think of these elements as the basis of a vector space; in particular, consider $V = \mathbb{C}e_{x_1} \oplus \cdots \oplus \mathbb{C}e_{x_n}$. Recall that $g(a_1 e_{x_1} + \cdots + a_n e_{x_n}) = a_1 e_{gx_1} + \cdots + a_n e_{gx_n}$. The fact that this is a representation follows immediately from the properties of the group action.
  - Computing the character χ_V of this V: Look at g and write its matrix. In particular, the trace is the number of unmoved/fixed elements, sometimes denoted $\text{Fix}(g)$.
  - This gives us another way of computing V_{perm} from above!

- **Character table**: A table that lists the conjugacy classes across the top, the irreps down the left side, and at each point within it, the value of an irrep's character over that conjugacy class.
  - The character table is a very nice matrix with very nice properties.
  - It is almost orthogonal; not exactly, but very close.
    - Rows aren't orthogonal, but columns are (take direct products)!
    - It is full rank, though.
- The midterm: Take the character table and do fun things with it.

3.2 Office Hours

10/10:
- Problem 1b:
  - Canonically self-dual: $V \cong V^*$ canonically.
- Mathematical methods of quantum mechanics: First few paragraphs of *picture*.
- We should have everything we need to do most of the problem set at this point; maybe not all of 5, but maybe yes, too.
- Problem 3:
  - There is some problem where it decomposes into trivial plus standard, but we still have to prove that standard is irreducible in this case!
  - If you have any vector, you can produce out of this vector something else.
  - If we take any vector and the group acts on it, we'll get a basis. If you hit a vector in the invariant subspace, it will just stay there; if you hit it and it goes everywhere, you get a basis.
  - Now think about a vector when you permute its coordinates.
  - Tomorrow in class, we will learn a quick way to do this problem.
- Problem 5:

- For some problem, we need to use the fact that $A^n = 1$ proves that $A = I$ in some sense.
- This is a hard problem!
- Show that eigenvalues sum to 1; we know that the eigenvalues are roots of unity! Thus, they have to both be 1!
- When the problem in group theory is harder, that's when you need to go to rep theory.

3.3 Characters are Orthonormal

10/11:

- Announcement: Zoom OH today.
- Recap: The big picture.
  - Representations.
    - We have representations, which are vector spaces on which a group acts.
    - With these representations, we can do a bunch of operations we've discussed: $\oplus, \otimes, V^*, \Lambda^n, S^n$.
    - We'll focus on the first 3 for now, though.
  - Class functions.
    - We also have class functions: Functions $f : G \to \mathbb{C}$ such that for all $g, x \in G$, $f(gxg^{-1}) = f(x)$.
    - The space of class functions forms a ring, since you can add, multiply, and take the complex conjugate of these functions.
    - Moreover, this ring is a vector space and it has dimension equal to the number of conjugacy classes of G.
  - The big idea: These two things (representations and class functions) are closely related!
    - There is a map, called a *character*, that pairs a representation to a class function.
    - Indeed, $V \to \chi_V$.
    - Under this map, operations of representations become operations of functions:

$$\oplus \mapsto + \qquad\qquad \otimes \mapsto \cdot \qquad\qquad V^* \mapsto \bar{f}$$

    - Additionally, $V_1, \dots, V_s$ become $\chi_{V_1}, \dots, \chi_{V_s}$.
- Theorem we will prove over the next couple of lectures: Irreps become *linearly independent* class functions, and all irreps form a basis of the space of class functions.
  - This theorem is huge! It is our main takeaway for now.
  - For the first part of the course, this is the main thing that we should remember.
- How do we prove that multiple vectors are linearly independent?
  - A strong condition would be to introduce an inner product and prove that the pairwise inner product of the vectors is zero.
- **Orthonormal basis**: A basis for which $\langle e_i, e_j \rangle = \delta_{ij}$.
- Let's begin carrying out this plan by defining an inner product on $\mathbb{C}[G]$. Indeed, let f_1, f_2 be two functions on G and take

$$\langle f_1, f_2 \rangle := \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}$$

- Motivation for this definition.
  - Recall the **Hermitian inner product** on $\mathbb{C}^n$.
    - We are essentially mapping f_1, f_2 to $(f_1(g_1), \dots, f_1(g_{|G|})), (f_2(g_1), \dots, f_2(g_{|G|})) \in \mathbb{C}^{|G|}$ and taking the Hermitian inner product there.

- Thus, we can see that all properties hold for both the Hermitian inner product on $\mathbb{C}^n$ and the one defined above on $\mathbb{C}[G]$.
- In other words, this kind of construction should inherit its status as a linear, positive definite bilinear form from the Hermitian inner product.

– Note: The Hermitian product above is ***G*-invariant**.

- This means that the functions on G from $G \to \mathbb{C}$ in $\mathbb{C}[G]$ form a representation of G.
- In particular, if $\varphi : G \to \mathbb{C}$, then $g = \rho(g)$ moves it as follows: $g \cdot \varphi = \varphi^g$ where $\varphi^g(h) := \varphi(g^{-1}h)$. Thus, we have an action of G on every φ!
- Such representations are isomorphic for finite groups??

– If we have $\langle f_1, f_2 \rangle$, we can ask if

$$\langle f_1, f_2 \rangle \stackrel{?}{=} \langle f_1^g, f_2^g \rangle$$

- Left as an exercise that this *is* true!

- **Hermitian inner product** (on $\mathbb{C}^n$): The inner product defined as follows for all $z, w \in \mathbb{C}^n$. *Denoted by* $\langle\ ,\ \rangle$. *Given by*

$$\langle z, w \rangle = \sum_{i=1}^{n} z_i \bar{w}_i$$

– This inner product gives a complex number $\langle v, w \rangle \in \mathbb{C}$ with the following properties.

1. $\langle a_1v_1 + a_2v_2, w \rangle = a_1 \langle v_1, w \rangle + a_2 \langle v_2, w \rangle$.
2. $\langle v, b_1w_1 + b_2w_2 \rangle = \bar{b}_1 \langle v, w_1 \rangle + \bar{b}_2 \langle v, w_2 \rangle$.
3. $\langle v, v \rangle \geq 0$, and $\langle v, v \rangle = 0$ implies that $v = 0$.

– Thus, if $v = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$, then

$$\langle v, w \rangle = \sum z_i \bar{w}_i \qquad\qquad \langle v, v \rangle = \sum |z_i|^2$$

- We now begin tackling today's main theorem: If V_1, V_2 are irreps, then

$$\langle \chi_{V_1}, \chi_{V_2} \rangle = \begin{cases} 0 & V_1 \ncong V_2 \\ 1 & V_1 \cong V_2 \end{cases}$$

– We will prove this theorem in stages.

– The general outline of our approach is to deduce the equality step by step through the transitive property. Some of the equalities we'll eventually end up needing are easier to discuss on their own first, though, so we begin with some lemmas.

- First off, recall the **space of invariants** from PSet 2.

- **Space of invariants** (of a representation V): The vector space defined as follows. *Denoted by* $\boldsymbol{V^G}$. *Given by*

$$V^G = \{v \in V \mid gv = v\ \forall\ g \in G\}$$

- Lemma 1: Let G be a finite group, let $\rho : G \to GL(V)$ be a finite-dimensional representation of it, and let p be defined as above. Then $p \in \mathrm{Hom}_G(V, V)$.

Proof. We can view p as an element of $\mathrm{Hom}(V, V)$. This combined with the fact that for every $h \in G$,

$$p(hv) = \frac{1}{|G|} \sum_{g \in G} (gh)v = \frac{1}{|G|} \sum_{gh \in G} (gh)v = \frac{1}{|G|} h \sum_{g \in G} gv = h(pv)$$

implies that $p \in \mathrm{Hom}_G(V,V)$. In more formal notation,

$$\begin{aligned}
[p \circ \rho_V(h)](v) &= \frac{1}{|G|}\sum_{g\in G}[\rho_V(g)\circ\rho_V(h)](v)\\
&= \frac{1}{|G|}\sum_{g\in G}[\rho_V(gh)](v)\\
&= \frac{1}{|G|}\sum_{gh\in G}[\rho_V(gh)](v)\\
&= \frac{1}{|G|}\sum_{hg\in G}[\rho_V(hg)](v)\\
&= \frac{1}{|G|}\sum_{g\in G}[\rho_V(hg)](v)\\
&= [\rho_V(h)]\left(\frac{1}{|G|}\sum_{g\in G}[\rho_V(g)](v)\right)\\
&= [\rho_V(h)\circ p](v)
\end{aligned}$$

□

- Why do we need this result?? What does it do for the rest of the proof?
- Lemma 2: Let G be a finite group, and let $\rho : G \to GL(V)$ be a finite-dimensional representation of it. Then the map p, defined as follows, is a projector from $V \to V^G$.

$$p = \frac{1}{|G|}\sum_{g\in G} g = \frac{1}{|G|}\sum_{g\in G}\rho_V(g)$$

Proof. To prove that p is a projector, it will suffice to show that $p^2 = p$. To prove that p projects onto V^G, it will suffice to show that $\mathrm{Im}(p) = V^G$. Let's begin.

To show that $p^2 = p$, we have

$$p^2 = \left(\frac{1}{|G|}\sum_{g\in G} g\right)^2 = \frac{1}{|G|^2}\sum_{g_1,g_2\in G} g_1g_2 = \frac{|G|}{|G|^2}\sum_{g\in G} g = p$$

Note that since G is not abelian (i.e., $g_1g_2 \neq g_2g_1$ in all cases), the square of $\sum g$ is as above and cannot be reduced to a smaller sum with a 2 coefficient or something like that. Additionally, note that $\sum_{g_1,g_2\in G} g_1g_2 = |G|\sum g$ since for each g_i, $g_i(g_1 + \cdots + g_{|G|}) = g_1 + \cdots + g_{|G|}$.

To show that $\mathrm{Im}(p) = V^G$, we will use a bidirectional inclusion proof. To confirm that $\mathrm{Im}(p) \subset V^G$, we have for any $h \in G$ that

$$h\left(\frac{1}{|G|}\sum_{g\in G} gv\right) = \frac{1}{|G|}\sum_{hg\in G} hgv = \frac{1}{|G|}\sum_{g\in G} gv$$

from which it follows that

$$p(v) = \frac{1}{|G|}\sum gv \in V^G$$

as desired. To confirm that $V^G \subset \mathrm{Im}(p)$, let $v \in V^G$. Then $gv = v$. It follows that

$$v = \frac{1}{|G|}\sum_{g\in G} v = \frac{1}{|G|}\sum_{g\in G} gv = p(v) \in \mathrm{Im}(p)$$

as desired. □

- You differentiated the first and second parts of the above proof by saying, "this is the algebraic way to prove it; we can also prove it nonalgebraically." Does this mean that $p^2 = p$ somehow *implies* $\operatorname{Im}(p) = V^G$ here, or do we still need to prove that "nonalgebraically," as in Fulton and Harris (2004)??
- Consequence of Lemma 2: There's a very easy way to construct invariant factors.
- We now prove one final lemma using what we have learned about p.
- Lemma 3: Let G be a finite group, and let $\rho : G \to GL(V)$ be a finite-dimensional representation of it. Then $\dim V^G = (1/|G|) \sum_{g \in G} \chi_V(g)$.

  Proof. Define p as above. Then

$$
\begin{aligned}
\dim V^G &= \dim(\operatorname{Im}(p)) && \text{Lemma 2}\\
&= \operatorname{tr}(p) && \text{PSet 1, Q5c}\\
&= \operatorname{tr}\left(\frac{1}{|G|}\sum \rho_V(g)\right)\\
&= \frac{1}{|G|}\sum_{g\in G} \operatorname{tr}(\rho_V(g))\\
&= \frac{1}{|G|}\sum_{g\in G} \chi_V(g)
\end{aligned}
$$

  as desired. □

- We can now prove the main result.
- Theorem: If V, W are irreps, then

$$
\langle \chi_V, \chi_W \rangle = \begin{cases} 0 & V \ncong W\\ 1 & V \cong W \end{cases}
$$

  Proof. We will work towards a formula for the inner product, using various results that we've proven up until now. Let's begin.

$$
\begin{aligned}
\langle \chi_V, \chi_W \rangle &= \frac{1}{|G|}\sum_{g\in G} \chi_V(g) \cdot \overline{\chi_W(g)} && \text{Definition}\\
&= \frac{1}{|G|}\sum_{g\in G} \chi_V(g) \cdot \chi_{W^*}(g) && \text{Property 4}\\
&= \frac{1}{|G|}\sum_{g\in G} \chi_{V\otimes W^*}(g) && \text{Property 5}\\
&= \dim[(V \otimes W^*)^G] && \text{Lemma 3}\\
&= \dim([\operatorname{Hom}_F(V,W)]^G) && \text{Lecture 2.1}\\
&= \dim[\operatorname{Hom}_G(V,W)] && \text{PSet 2, Q4b}\\
&= \begin{cases} \dim(\operatorname{span}(I)) & V \cong W\\ \dim(\operatorname{span}(0)) & V \ncong W \end{cases} && \text{Schur's Lemma}\\
&= \begin{cases} 0 & V \ncong W\\ 1 & V \cong W \end{cases}
\end{aligned}
$$

  □

- In the above proof, Rudenko first surveys the following special case. Why??
  - Then if V is irreducible and trivial, we have
    $$\frac{1}{|G|}\sum_{g\in G}\chi_V(g)=0$$
    which happens iff
    $$\langle\chi_V,\chi_{\text{triv}}\rangle=0$$
    whereas
    $$\langle\chi_{\text{triv}},\chi_{\text{triv}}\rangle=1$$
    This proves the theorem in a special case, but how do we go from here to all representations? We're very close!
- Corollary: The number of irreps is less than or equal to the number of conjugacy classes.
  - We'll leave it to next time to prove that equality holds.
- Whenever we have a sec, we should try to form a mental picture the whole class function thing.
- Consequence of the theorem: We get an orthogonality relation.
  - If χ_1,χ_2 are characters if irreps, then
    $$\sum_{g\in G}\chi_1(g)\overline{\chi_2(g)}=\begin{cases}0 & \chi_1\neq\chi_2\\ |G| & \chi_1=\chi_2\end{cases}$$
  - This is related to the character table and IChem!!! Example:
    - Recall Table 3.1, the character table for S_3.
    - Between the trivial and alternating representations, we have
      $$(1)(1)+(1)(-1)+(1)(-1)+(1)(-1)+(1)(1)+(1)(1)=0$$
      as expected. Note that we have a term for each element in S_3, so some products get repeated multiple times.
    - For the standard representation, we have
      $$(2)(2)+(0)(0)+(0)(0)+(0)(0)+(-1)(-1)+(-1)(-1)=6=|S_3|$$
      as expected.
- Theorem: Characters are equal iff their representations are isomorphic.
- Next time.
  - Prove the theorem.
  - Consequences.
  - Implications for the character table.

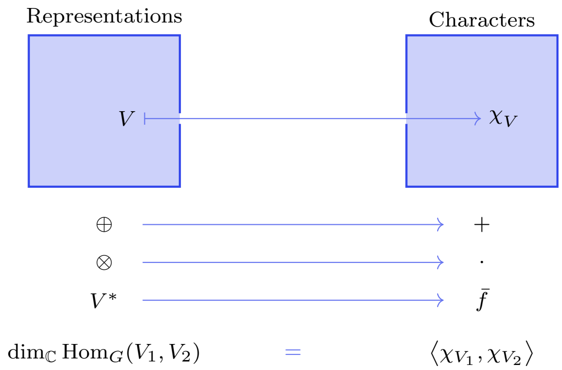

Figure 3.1: The stories of representation theory.

3.4 Character Table Properties

10/13: - Announcement: Midterm on November 10.
    - Will mostly involve computing character tables; HW will be good prep.
- Review of the general picture from the first part of the course.
    - Let G be a finite group.
    - First story: We study finite-dimensional representations of G over $\mathbb{C}$; these are vector spaces, so we can direct sum, tensor multiply, and dualize them. We can also look at the morphisms between them.
    - Second story: We study class functions $\mathbb{C}_{\text{cl}}[G] = \{f : G \to \mathbb{C} \mid f(gxg^{-1}) = f(x)\}$; these are elements of a ring, so we can add, multiply, and conjugate them. We can also take the inner product of them.
    - We can map between these two stories: Representations become characters, $\oplus \mapsto +$, $\otimes \mapsto \cdot$, and $V^* \mapsto \bar{f}$.
- Theorem (from last time):
$$\langle \chi_{V_1}, \chi_{V_2} \rangle = \dim \mathrm{Hom}_G(V_1, V_2)$$
- The story in Figure 3.1 tells us stuff about representations.
    - Let $V_1, \dots, V_k$ be irreps. Then the vectors $\chi_{V_1}, \dots, \chi_{V_k}$ are orthonormal.
        - We get this result with the Theorem above and Schur's Lemma.
    - Next time, we'll prove that $\chi_{V_1}, \dots, \chi_{V_k}$ spans $\mathbb{C}_{\text{cl}}[G]$, i.e., the number of irreps is the number of conjugacy classes.
    - *Cube thing??*
    - This picture is remarkable because it's so simple.
- We now look at some corollaries to last time's main theorem.
- Corollary 1: If V, W are G-reps, then $\chi_V = \chi_W$ iff $V \cong W$.

  Proof. Invoking complete reducibility, we have that $V = \bigoplus V_i^{n_i}$. Thus, to know V, it is enough to know the n_i's. But
$$\chi_V = \sum n_i \chi_{V_i}$$
  where
$$n_i = n_i \cdot 1 = n_i \langle \chi_{V_i}, \chi_{V_i} \rangle = \langle \chi_V, \chi_{V_i} \rangle$$

Therefore, since the χ_{V_i} are linearly independent, the only way that $\chi_V = \chi_W$ is if the n_i's match which would mean that $V \cong W$, and vice versa the only way that $V \cong W$ is if the n_i's match which would mean that $\chi_V = \chi_W$. $\square$

- Corollary 2: Let V be a G-rep. Then TFAE:
  1. V is irreducible.
  2. $\langle \chi_V, \chi_V \rangle = 1$.
  3. $\sum_{g\in G} |\chi_V(g)|^2 = |G|$.

  Proof. ($1 \Rightarrow 2$): We have that
  $$\langle \chi_V, \chi_V \rangle = \dim \mathrm{Hom}_G(V, V) = 1$$
  as desired.

  ($2 \Rightarrow 1$): Complete reducibility implies that $V \cong V_1^{n_1} \oplus \cdots \oplus V_k^{n_k}$, where the V_i's are irreps. This combined with the hypothesis implies that
  $$1 = \langle \chi_V, \chi_V \rangle = \left\langle \sum_{i=1}^k n_i \chi_{V_i}, \sum_{i=1}^k n_i \chi_{V_i} \right\rangle = \sum_{i=1}^k n_i^2$$
  But if $\sum n_i^2 = 1$ where each $n_i \in \mathbb{Z}^+$, then $n_i = 1$ for some i and $n_j = 0$ for $j \neq i$, from which it follows that $V \cong V_i$.

  We can interconvert between 2 and 3 using the definition of the inner product and the property of complex numbers that $zz^* = |z|^2$. $\square$

- We now build up to one final corollary.
- We've discussed all of these properties of irreps, but where do we even find them?
  - We might be able to find some by inspection, but here's how we find all of them.
- Review: The regular representation. Here are two different but isomorphic ways to think about it.
  - Think of it as functions on G.
    - Better for infinite groups.
  - Think of it as the permutational representation associated with the action $G \circlearrowright G$.
    - Better for finite groups.
  - Why did we talk about this here??
- Corollary 3: Consider the regular representation V_R. We have that
  $$\chi_{V_R}(g) = \begin{cases} 0 & g \neq e \\ |G| & g = e \end{cases}$$

  Proof. We can compute its character χ_{V_R} by considering the corresponding permutation matrices. Indeed, the action $\chi_{V_R}(g)$ of this character on g is equal to the number of 1's on the diagonal in the permution matrix, which is equal to the number of fixed points of the permutation, i.e., the number of i's such that $gg_i = g_i$. But in a group, $gg_i = g_i$ iff $g = e$, so this number of fixed points is
  $$\chi_{V_R}(g) = \mathrm{Fix}(g) = \begin{pmatrix} g_1 & \cdots & g_n \\ gg_1 & \cdots & gg_n \end{pmatrix} = \begin{cases} 0 & g \neq e \\ |G| & g = e \end{cases}$$
  as desired. $\square$

- What is the matrix thing?
  - It is most likely a representation of the function $g_i \mapsto gg_i$, and it denotes that $\text{Fix}(g)$ is equal to the number of columns that have the same entry top and bottom.
- We now apply Corollaries 1-3 to the regular representation V_R to obtain some important results.
  - Let V_i be an arbitrary irrep.
  - By complete reducibility, $V_R = \bigoplus_{i=1}^k V_i^{n_i}$ for some set of n_i's.
  - Additionally,

$$\begin{aligned}
n_i &= \langle \chi_{V_R}, \chi_{V_i} \rangle && \text{Corollary 1}\\
&= \frac{1}{|G|} \sum_{g \in G} \chi_{V_R}(g) \overline{\chi_{V_i}(g)}\\
&= \frac{1}{|G|} |G| \underbrace{\overline{\chi_{V_i}(e)}}_{\dim V_i} && \text{Corollary 3}\\
&= \dim V_i
\end{aligned}$$

  - This implies three remarkable results, all worth remembering.

$$V_R = \bigoplus_{i=1}^k V_i^{\dim V_i} \qquad |G| = \sum_{i=1}^k (\dim V_i)^2 \qquad \#\text{ irreps is finite}$$

    - The first result follows directly by substituting $n_i = \dim V_i$ into complete reducibility.
    - The second result follows because $|G| = \dim(V_R) = \dim(\bigoplus_{i=1}^k V_i^{\dim V_i}) = \sum (\dim V_i)^2$.
    - The third result follows because if there were infinitely many irreps, each with $\dim V_i \geq 1$, then $|G| = \sum_{i=1}^k (\dim V_i)^2 = \infty$, contradicting the hypothesis that $|G|$ is finite.
- We want to investigate S_4, i.e., characterize all irreps of it.

	1	6	8	3	6
	e	(12)	(123)	(12)(34)	(1234)
Trivial	1	1	1	1	1
Sign	1	-1	1	1	-1
V_{std}	3	1	0	-1	-1
Sign $\otimes V_{\text{std}}$	3	-1	0	-1	1
	2	0	-1	2	0

Table 3.2: Character table for S_4.

  - We do so by constructing the character table, Table 3.2.
  - Initially, this seemed like a very hard problem.
    - However, with all of our theory, it only takes a couple of minutes now!
  - We start by inputting the trivial, sign, and standard representations.
    - The trivial is obviously $(1, 1, 1, 1, 1)$.
    - The sign can be calculated to be $(1, -1, 1, 1, -1)$.
    - The standard is found via $V_{\text{std}} = V_{\text{perm}} - V_{\text{triv}} = (4, 2, 1, 0, 0) - (1, 1, 1, 1, 1) = (3, 1, 0, -1, -1)$.

- Note that

$$\sum n_i^2 = \left\langle \chi_{V_{\text{perm}}}, \chi_{V_{\text{perm}}} \right\rangle = \frac{1}{24}(1 \cdot 4^2 + 6 \cdot 2^2 + 8 \cdot 1^2 + 0 \cdot 0^2 + 0 \cdot 0^2) = 2$$
$$\left\langle \chi_{V_{\text{perm}}}, \chi_{V_{\text{sign}}} \right\rangle = \frac{1}{24}(4 - 12 + 8) = 0$$
$$\left\langle \chi_{V_{\text{perm}}}, \chi_{V_{\text{triv}}} \right\rangle = 1$$

  - What is the point of these calculations??
- Thus, we can derive representations without having any geometric notion of it using characters!
- To see any of these representations geometrically, look at actions on the tetrahedron in $\mathbb{R}^3$!
- All those computations above used the **first orthogonality relation**; here's the **second orthogonality relation**:

$$\sum_\chi \chi(g_1)\overline{\chi(g_2)} = \begin{cases} 0 & g_1 \nsim g_2 \\ \frac{|G|}{|C_G(g)|} & g_1 \sim g_2 \end{cases}$$

  where $\sim$ denotes conjugacy and $C_G(g)$ is the number of elements in the conjugacy class of g (where we borrow centralizer notation).
  - Prove the new one with $AB = 1 = BA$. This very simple thing leads to a very powerful statement about systems of equations that we will discuss later. How does this proof work??
  - We can start doing this stuff in the new homework!
  - We constructed the fifth representation using this.
- Where does the fifth irrep come from?
  - Going back to a miracle of group theory: Simple groups.
  - If $f : S_n \to S_n$, we have lots of injective maps, lots of minor actions $S_n \to S_n$ sending $x \mapsto gxg^{-1}$.
  - We have sign : $S_n \to S_2$, $S_4 \twoheadrightarrow S_3$ with kernel equal to K_4.
  - We have the exotic $S_6 \to S_6$.
  - We have $S_5 \hookrightarrow S_6$ that is also exotic.
  - These are called **exceptional homomorphisms**.
  - Since we have $S_4 \to S_3$ and $\rho : S_3 \to GL_n$, we have $S_4 \to GL_n$ with a the same character table. Takeaway: This $(2, 0, 1)$ thing in the big character table comes from this map, geometrically.
  - Takeaway: The geometry of the fifth irrep comes from S_3.
  - What is going on here??

- Final announcements.
  - Going forward, we'll mostly be following Fulton and Harris (2004).
  - Then we'll get into associative algebra.
  - OH next week will be Zoom, but we should still feel free to meet with him in-person by emailing him for an appointment.

3.5 S Chapter 2: Character Theory

From Serre (1977).

Section 2.1: The Character of a Representation

11/2:

- Definition of **trace**.
- **Character** (of ρ): The complex-valued function on G defined as follows. *Denoted by* $\boldsymbol{\chi_\rho}$. *Given by*

$$\chi_\rho(s) = \operatorname{tr}(\rho_s)$$

- Notation: Serre (1977) uses both z^* and $\bar{z}$ to denote the complex conjugate of a $z \in \mathbb{C}$.
- Properties of the character as a function.

Proposition 1. *If χ is the character of a representation ρ of degree n, then...*

(i) $\chi(1) = n$;
(ii) $\chi(s^{-1}) = \chi(s)^*$ *for all* $s \in G$;
(iii) $\chi(tst^{-1}) = \chi(s)$ *for all* $s \in G$.

Proof. (i):

$$\chi(1) = \operatorname{tr}(\rho_1) = \operatorname{tr}(1) = n$$

(ii): Recall that there exists a basis with respect to which ρ_s is unitary. Thus, its (not necessarily unique) eigenvalues $\lambda_1, \dots, \lambda_n$ are roots of unity. Consequently, $\lambda_i^{-1} = \lambda_i^*$ ($i = 1, \dots, n$). Therefore,

$$\chi(s^{-1}) = \operatorname{tr}(\rho_{s^{-1}}) = \operatorname{tr}\big(\rho_s^{-1}\big) = \sum \lambda_i^{-1} = \sum \lambda_i^* = \operatorname{tr}(\rho_s)^* = \chi(s)^*$$

(iii): This follows directly from the fact that the trace is invariant under choice of basis. Alternatively (and more explicitly), we may put $u = \rho_t\rho_s$ and $v = \rho_{t^{-1}}$ and use the formula $\operatorname{tr}(ab) = \operatorname{tr}(ba)$[1] as follows.

$$\chi(tst^{-1}) = \operatorname{tr}(\rho_{tst^{-1}}) = \operatorname{tr}(\rho_t\rho_s\rho_{t^{-1}}) = \operatorname{tr}(uv) = \operatorname{tr}(vu) = \operatorname{tr}(\rho_{t^{-1}}\rho_t\rho_s) = \operatorname{tr}(\rho_s) = \chi(s)$$

□

- **Class function**: A function f on G satisfying identity (iii) in Proposition 1.
- Properties of the character with respect to representations.

Proposition 2. *Let $\rho^1 : G \to GL(V_1)$ and $\rho^2 : G \to GL(V_2)$ be two linear representations of G, and let χ_1, χ_2 be their characters. Then...*

(i) *The character χ of the direct sum representation $V_1 \oplus V_2$ is equal to $\chi_1 + \chi_2$;*
(ii) *The character ψ of the tensor product representation $V_1 \otimes V_2$ is equal to $\chi_1 \cdot \chi_2$.*

Proof. As in class. □

- Serre (1977) defines the characters of the symmetric square and alternating square representations.

[1] See Theorem 10.4 of Axler (2015).

Section 2.2: Schur's Lemma; Basic Applications

- Serre (1977) states and proves Schur's Lemma.

  Proposition 4 (Schur's Lemma). *Let $\rho^1 : G \to GL(V_1)$ and $\rho^2 : G \to GL(V_2)$ be two irreducible representations of G, and let f be a linear mapping of V_1 into V_2 such that $\rho_s^2 \circ f = f \circ \rho_s^1$ for all $s \in G$. Then...*

  (i) *If ρ^1 and ρ^2 are not isomorphic, then $f = 0$;*

  (ii) *If $V_1 = V_2$ and $\rho^1 = \rho^2$, then f is a **homothety**.*

  Proof. This proof is identical to the one given in Fulton and Harris (2004). However, I am including it again here because it includes a few key details that Fulton and Harris (2004) leave out. Let's begin.

  (i): To prove the claim, it will suffice to prove the contrapositive. Suppose $f \neq 0$. To prove that $\rho^1 \cong \rho^2$, it will suffice to show that f is an isomorphism of G-representations. Since f is a morphism of G-representations by hypothesis, all that is left is to show that it is an isomorphism of vector spaces. To do so, we will demonstrate that $\text{Ker}(f) = 0$ and $\text{Im}(f) = V_2$, one claim at a time. Let's begin.

  Let $W_1 = \text{Ker}(f)$. Then for any $s \in G$ and $x \in W_1$,

  $$f(\rho_s^1(x)) = [f \circ \rho_s^1](x) = [\rho_s^2 \circ f](x) = \rho_s^2(f(x)) = \rho_s^2(0) = 0$$

  so $\rho_s^1(x) \in W_1$. It follows that W_1 is stable under G. But since V_1 is irreducible, this means that $W_1 = V_1, 0$. This combined with the fact that $f \neq 0$ (hence $W_1 = \text{Ker}(f) \neq V_1$) implies that $W_1 = 0$.

  Let $W_2 = \text{Im}(f)$. Then for any $s \in G$ and $f(x) \in W_2$,

  $$\rho_s^2(f(x)) = [\rho_s^2 \circ f](x) = [f \circ \rho_s^1](x) = f(\rho_s^1(x))$$

  so $\rho_s^2(fx) \in W_2$. It follows that W_2 is stable under G. But since V_2 is irreducible, this means that $W_2 = V_2, 0$. This combined with the fact that $f \neq 0$ (hence $W_2 = \text{Im}(f) \neq 0$) implies that $W_2 = V_2$.

  (ii): Since f is an operator on a finite-dimensional, nonzero, complex vector space, it has[2] an eigenvalue λ. To prove that f is a homothety, it will suffice to show that $f = \lambda I$, which we will do by demonstrating that $f - \lambda I = 0$ using part (i). Indeed, to use part (i) in this manner, we need only show that $f - \lambda I$ *does* satisfy $\rho_s^2 \circ (f - \lambda I) = (f - \lambda I) \circ \rho_s^1$ but *is not* an isomorphism of vector spaces. For the first claim, we have since $\rho^1 = \rho^2$ and $V_1 = V_2$ that

  $$\rho_s^2 \circ (f - \lambda I) = \rho_s^2 \circ f - \rho_s^2 \circ \lambda I = f \circ \rho_s^1 - \lambda I \circ \rho_s^2 = f \circ \rho_s^1 - \lambda I \circ \rho_s^1 = (f - \lambda I) \circ \rho_s^1$$

  For the second claim, we know that the eigenvector corresponding to λ is in $\text{Ker}(f - \lambda I)$, so $f - \lambda I$ has a nontrivial kernel and thus cannot be an isomorphism. ☐

- **Homothety**: A scalar multiple of the identity operator. *Given by*

  $$\lambda I$$

  for some $\lambda \in \mathbb{C}$.

11/8: - Another condition for telling if two representations are isomorphic.

  Corollary 1. *Let h be a linear mapping of V_1 into V_2 and put*

  $$h^0 = \frac{1}{g} \sum_{t \in G} (\rho_t^2)^{-1} h \rho_t^1$$

  with $g = |G|$. Then...

[2]See Theorem 5.5 of Axler (2015).

(i) *If ρ^1 and ρ^2 are not isomorphic, we have $h^0 = 0$;*
(ii) *If $V_1 = V_2$ and $\rho^1 = \rho^2$, h^0 is a homothety of ratio $(1/n)\operatorname{tr}(h)$ with $n = \dim(V_1)$.*

Proof. Given. ☐

- The above corollary in matrix form:

  Corollary 2. *Let $(r_{i_1j_1}(t)) := \rho^1_t$, $(r_{i_2j_2}(t)) := \rho^2_t$, $(x_{i_2i_1}) := h$, and*

  $$x^0_{i_2i_1} := h^0 = \frac{1}{g}\sum_{t,j_1,j_2} r_{i_2j_2}(t^{-1})x_{j_2j_1}r_{j_1i_1}(t)$$

  Then if $(r_{i_1j_1}(t))$ and $(r_{i_2j_2}(t))$ are not isomorphic, we have

  $$\frac{1}{g}\sum_{t,j_1,j_2} r_{i_2j_2}(t^{-1})r_{j_1i_1}(t) = 0$$

  for arbitrary i_1, i_2, j_1, j_2.

  Proof. Described. ☐

  Corollary 3. *Under the same definitions as before, if $(r_{i_1j_1}(t))$ and $(r_{i_2j_2}(t))$ are isomorphic, we have*

  $$\frac{1}{g}\sum_{t\in G} r_{i_2j_2}(t^{-1})r_{j_1i_1}(t) = \frac{1}{n}\delta_{i_2i_1}\delta_{j_2j_1}$$

  Proof. Described. ☐

- Note that under the notation

  $$\langle\phi,\psi\rangle = \frac{1}{g}\sum_{t\in G}\phi(t^{-1})\psi(t) = \frac{1}{g}\sum_{t\in G}\phi(t)\psi(t^{-1})$$

  we have...

  - $\langle\phi,\psi\rangle = \langle\psi,\phi\rangle$;
  - $\langle\phi,\psi\rangle$ is linear in ϕ and ψ;
  - Corollary 2 becomes $\langle r_{i_2j_2}, r_{j_1i_1}\rangle = 0$;
  - Corollary 3 becomes $\langle r_{i_2j_2}, r_{j_1i_1}\rangle = (1/n)\delta_{i_2i_1}\delta_{j_2j_1}$.

- Furthermore, note that if we choose the $(r_{ij}(t))$ to be unitary, then $r_{ij}(t^{-1}) = r_{ji}(t)^*$ and Corollaries 2, 3 are **orthogonality relations** for the scalar product (ϕ,ψ) defined below.

Section 2.3: Orthogonality Relations for Characters

- **Scalar product**: A binary operation $(x \mid y)$, where x, y are elements of some vector space V, that is linear in x, semilinear in y, and satisfies $(x \mid x) > 0$ for all $x \neq 0$.
- Define a scalar product on the space of complex valued functions on G by

  $$(\phi \mid \psi) = \frac{1}{g}\sum_{t\in G}\phi(t)\psi(t)^*$$

- Defining $\tilde{\psi}(t) := \psi(t^{-1})^*$, we have that

  $$(\phi \mid \psi) = \frac{1}{g}\sum_{t\in G}\phi(t)\tilde{\psi}(t^{-1}) = \left\langle\phi,\tilde{\psi}\right\rangle$$

- Special case: By Proposition 1, $\tilde{\chi} = \chi$, so

$$(\phi \mid \chi) = \langle \phi, \chi \rangle$$

- We now prove a theorem analogous to the main theorem from Wednesday's lecture.

Theorem 3.

(i) *If χ is the character of an irreducible representation, we have $(\chi \mid \chi) = 1$ (i.e., χ is "of norm 1").*

(ii) *If χ and χ' are the characters of two nonisomorphic irreducible representations, we have $(\chi \mid \chi') = 0$ (i.e., χ, χ' are orthogonal).*

Proof. Extremely clean argument using Corollaries 2-3. □

- **Irreducible character**: A character of an irreducible representation.
- With this definition in hand, we can see that the irreducible characters form an orthonormal system.
- With the previous result in hand, we can use characters to count how many times an irreducible representation occurs within the direct sum decomposition of a representation.

Theorem 4. *Let V be a linear representation of G, with character ϕ, and suppose V decomposes into a sum of irreducible representations via*

$$V = W_1 \oplus \cdots \oplus W_k$$

Then if W is an irreducible representation with character χ, the number of W_i isomorphic to W is equal to the scalar product $(\phi \mid \chi) = \langle \phi, \chi \rangle$.

Proof. Let χ_i be the character of W_i. Then by Proposition 2, we have

$$\phi = \chi_1 + \cdots + \chi_k$$

Now recall from Theorem 3 that $(\chi_i \mid \chi)$ is 1 or 0 depending on whether W_i is or is not isomorphic to W, respectively. Thus, in the sum

$$(\phi \mid \chi) = (\chi_1 \mid \chi) + \cdots + (\chi_k \mid \chi)$$

all terms for which $\chi_i \neq \chi$ go to zero and all terms for which $\chi_i = \chi$ go to 1, leaving a sum that adds 1 for every occurrence of W, as desired. □

- We now explore some results that immediately follow from this result.
- There is a uniqueness in the decomposition of a representation into irreducible representations.

Corollary 1. *The number of W_i isomorphic to W does not depend on the chosen decomposition.*

- Matching Corollary 1 from Friday's class.

Corollary 2. *Two representations with the same character are isomorphic.*

- Note on this result.
  - This is what reduces the study of representations to the study of characters.
- Serre (1977) reiterates the formula

$$(\phi \mid \phi) = \sum_{i=1}^{h} m_i^2$$

where ϕ is the character of $V = W_1^{m_1} \oplus \cdots \oplus W_h^{m_h}$.

- An **irreducibility criterion** based on Corollary 2 from Friday's class.

Theorem 5. *If ϕ is the character of a representation V, $(\phi \mid \phi)$ is a positive integer and we have $(\phi \mid \phi) = 1$ if and only if V is irreducible.*

Proof. See class. □

Section 2.4: Decomposition of the Regular Representation

- Notation.
  - $\chi_1, \dots, \chi_h$: The irreducible characters of G.
  - $n_1, \dots, n_k$: The degrees of $\chi_1, \dots, \chi_h$; $n_i = \chi_i(1)$.
  - $\rho : G \to GL(R)$: The regular representation of G.
- Matching Corollary 3 from Friday's class: Calculating ρ_s for all $s \in G$.

  Proposition 5. *The character r_G of the regular representation is given by the formulas*

  $$r_G(1) = g \qquad\qquad r_G(s) = 0$$

  where we assume $s \neq 1$ in the right equation above.

  Proof. If $s \neq e$, then $st \neq t$ for all $t \in G$. Thus, the diagonal terms of ρ_s are all 0 in this case, so $\operatorname{tr}(\rho_s) = 0$.

  If $s = e$, then $st = t$ for all $t \in G$. Thus, the diagonal terms of ρ_s are all 1 in this case, so $\operatorname{tr}(\rho_s) = \operatorname{tr}(1) = \dim(R) = g$. ☐

- Matching the leftmost consequence of Corollary 3 from Friday's class.

  Corollary 1. *Every irreducible representation W_i is contained in the regular representation with multiplicity equal to its degree n_i*

- Matching the middle consequence of Corollary 3 from Friday's class (i) and the orthogonality of the first column in a character table with every other column, a special case of the second orthogonality criterion (ii).

  Corollary 2.

  (i) *The degrees n_i satisfy the relation*

  $$\sum_{i=1}^{h} n_i^2 = g$$

  (ii) *If $s \in G$ is different from 1, we have*

  $$\sum_{i=1}^{h} n_i \chi_i(s) = 0$$

  Proof. Corollary 1 says that $r_G(s) = \sum n_i \chi_i(s)$ for all $s \in G$. Taking $s = 1$ gives (i) and $s \neq 1$ gives (ii). ☐

3.6 FH Chapter 2: Characters

From Fulton and Harris (2004).

Section 2.1: Characters

- The example from last time suggests that "knowing all the eigenvalues of each element of G [that is, each $\rho(g)$] should suffice to describe the representation" (Fulton & Harris, 2004, p. 12).
  - We formalize this notion via **character theory**.

  - In particular, note that we do not actually need to specify all eigenvalues of all elements of G; rather, we can opt to specify their sums, since knowing the $\sum \lambda_i^k$ is equivalent to knowing the $\{\lambda_i\}$ of g. This motivates the definition of the character as the trace!
- Definition of a **character** and **class function**.
- Proposition 2.1: Properties of the character as discussed in class.
- Example 2.5: References the character of the permutation representation and $\text{Fix}(g)$.
- An alternate way of deriving the characters of the standard representation.
  - Calculate for the permutation representation $(3, 1, 0)$, which is easy.
  - Note that $V_{\text{perm}} = V_{(2,1)} \oplus V_{(3)}$, so $(3, 1, 0) = (x, y, z) + (1, 1, 1)$, meaning that the character of the standard representation is $(2, 0, -1)$, as desired.
- Using characters to decompose arbitrary representations W of S_3.
  - We take it for granted that $W = V_{(3)}^a \oplus V_{(2,1)}^b \oplus V_{(1,1,1)}^c$.
  - But then $\chi_W = a\chi_{V_{(3)}} + b\chi_{V_{(2,1)}} + c\chi_{V_{(1,1,1)}}$. Moreover, since the three characters are linearly independent, we can solve the system of equations for a, b, c.
  - Takeaway: "W is determined up to isomorphism by its character χ_W" (Fulton & Harris, 2004, p. 14).
- There are some great exercises throughout this section that I could come back to later for more practice!

Section 2.2: The First Projection Formula and its Consequences

- Fulton and Harris (2004) identify a different goal than either Rudenko or Serre (1977) when proving the orthonormality of the irreducible characters, but an interesting one nonetheless! Let's begin.
- We wish to construct an *explicit* formula for a certain projection operator. The operator of interest is the one that projects the vectors in a representation V onto the subspace $V_{\text{triv}}^m \leq V$ consisting of the direct sum of the trivial representations in the decomposition.
- Observe that the subspace $V_{\text{triv}}^m \leq V$ consists of all vectors $v \in V$ such that $gv = v$ for all $g \in G$. We call this subspace "V^G."
- Call our desired projection operator φ. What properties should φ have?
  - If it is to map between representations V and V^G, it should be a morphism of G-representations.
    - How is this related to Lemma 1??
  - It should satisfy $\varphi^2 = \varphi$ and $\text{range}\,\varphi = V^G$.
- Now suppose we want to find m, the number of copies of the trivial representation appearing in the decomposition of V.
  - Consider the matrix of φ in a basis of V such that the first m vectors lie in V^G and the rest of the vectors lie in the complement of V^G.
  - This block diagonal matrix will be the $m \times m$ identity in the upper left and the zero matrix in the bottom right.
  - Thus, $m = \text{tr}(\varphi)$.
  - Expanding, we actually have

$$m = \text{tr}(\varphi) = \frac{1}{|G|} \sum_{g \in G} \text{tr}(g) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)$$

  - Note that both V_{triv} and $V = V_{\text{triv}}^n \oplus \cdots$ make the equality true; it's just that if we want the following consequence, we *need* V in there.

- Important consequence of the above: For an irreducible representation V other than the trivial one, $\sum_{g \in G} \chi_V(g) = 0$.
  - Example: In Table 3.1, notice how $1(1) + 3(-1) + 2(1) = 0$ and $1(2) + 3(0) + 2(-1) = 0$.

- "If V is irreducible, then by Schur's Lemma, $\dim[\operatorname{Hom}(V, W)^G]$ is the multiplicity of V in W; similarly, if W is irreducible, $\dim[\operatorname{Hom}(V, W)^G]$ is the multiplicity of W in V, and in the case where both V and W are irreducible, we have

$$\dim \operatorname{Hom}_G(V, W) = \begin{cases} 1 & V \cong W \\ 0 & V \ncong W \end{cases}$$

  " Fulton and Harris (2004, p. 16).

- Fulton and Harris (2004) finally arrives at the theorem from Wednesday's lecture.

- This theorem gives us a lower bound on the number of conjugacy classes of G.

  Corollary 2.13. *The number of irreducible representations of G is less than or equal to the number of conjugacy classes.*

  Proof. We have that

$$\begin{aligned} \operatorname{span}(\chi_1, \dots, \chi_k) &\leq \mathbb{C}_{\text{cl}}[G] \\ \dim[\operatorname{span}(\chi_1, \dots, \chi_k)] &\leq \dim(\mathbb{C}_{\text{cl}}[G]) \\ \#\text{ irreducible representations} &\leq \#\text{ conjugacy classes} \end{aligned}$$

  as desired. ☐

- Corollary 1 from Friday's class.

  Corollary 2.14. *Any representation is determined by its character.*

- Corollary 2 from Friday's class.

  Corollary 2.15. *A representation V is irreducible iff $(\chi_V, \chi_V) = 1$.*

- A result from the proof of Corollary 1 from Friday's class.

  Corollary 2.16. *The multiplicity a_i of V_i in V is the inner product of χ_V with χ_{V_i}, i.e.,*

$$a_i = (\chi_V, \chi_{V_i})$$

- With respect to the regular representation, Fulton and Harris (2004) states that

$$\chi_R(g) = \begin{cases} 0 & g \neq e \\ |G| & g = e \end{cases}$$

- Matching the leftmost consequence of Corollary 3 from Friday's class.

  Corollary 2.17. *Any irreducible representation V of G appears in the regular representation* $\dim V$ *times.*

- Discusses the middle and rightmost consequences of Corollary 3 from Friday's class.

- Discusses Corollary 2ii from Section 2.4 of Serre (1977).

- Note: These two formulas, copied below for convenience, amount to the **Fourier inversion formula** for finite groups.

$$|G| = \sum_i \dim(V_i)^2 \qquad\qquad 0 = \sum_i (\dim V_i) \cdot \chi_{V_i}(g \neq e)$$

  - Note: If all but one of the characters is known, they given a formula for the unknown character.
- Matching the second orthogonality relation from Friday's class.

Section 2.3: Examples — S_4 and A_4

- Construction of Table 3.2.
  1. List the conjugacy classes in S_4 and the number of elements in each across the top of the table.
     - Since this is a symmetric group S_d, the conjugacy classes correspond to the **partitions** of d via cycle lengths.
     - Thus, our conjugacy classes are...
       - $\{e\}$: $4 = 1 + 1 + 1 + 1$. Number of elements: 1.
       - $\{(xx)\}$: $4 = 2 + 1 + 1$. Number of elements: 6.
       - $\{(xxx)\}$: $4 = 3 + 1$. Number of elements: 8.
       - $\{(xx)(xx)\}$: $4 = 2 + 2$. Number of elements: 3.
       - $\{(xxxx)\}$: $4 = 4$. Number of elements: 6.
  2. Start by listing the trivial, alternating, and standard representations.
     - The character of the trivial is
       $$(1, 1, 1, 1, 1)$$
       by default.
     - The character of the alternating is
       $$((-1)^e, (-1)^{(xx)}, (-1)^{(xxx)}, (-1)^{(xx)(xx)}, (-1)^{(xxxx)}) = (1, -1, 1, 1, -1)$$
       by the definition of the representation.
     - The character of the standard representation is
       $$\chi_{\mathbb{C}^4} - \chi_{(5)} = (4, 2, 1, 0, 0) - (1, 1, 1, 1, 1) = (3, 1, 0, -1, -1)$$
       by the fact that the regular representation always decomposes into the sum of the trivial and standard.
       - We can double check that this representation is irreducible via the irreducibility criterion $|\chi_V| = \sqrt{(\chi_V, \chi_V)} = 1$.
  3. Figure out how many more representations irreducible representations there are. In this case there are two more.
     - We can figure this out by combining two previously used facts.
     - First, we know that the sum of the squares of the dimensions must equal $|S_4| = 24$ (middle consequence of Corollary 3 from Friday's class). Since we're only at $1 + 1 + 9 = 11$ so far, we still have $24 - 11 = 13$ to go. But how is this 13 allocated? To answer this question, we need the second fact.
     - Second, by Corollary 2.13, the number of irreps is less than or equal to the number of conjugacy classes, so we have at most two irreps to go. In fact, since 13 is not the square of any natural number but $13 = 2^2 + 3^2$, we must have *exactly* two irreps to go, of dimensions 2 and 3.
  4. The tensor product of an irrep and a one-dimensional representation is irreducible, so $\text{Sign} \otimes V_{\text{std}} = (3, -1, 0, -1, 1)$ is one of them.

    - Additional check: $|\text{Sign} \otimes V_{\text{std}}| = 1$.
    - Additional check: It is not a scalar multiple *or* linear combination of any of the first three.
  5. Use the second orthogonality relation to solve for each $\chi(g)$ for the final irrep.

- **Partition** (of d): An expression of d as a sum of positive integers $a_1, \dots, a_k$.
- What is an "involution of trace 2" and how is it related to the quotient group (Fulton & Harris, 2004, p. 19)??
  - This is related to the closing comments of Friday's class.
- Description of A_4.