Week 9

Symmetric Group Representation Characteristics

9.1 Frobenius Reciprocity; The Branching Theorem

11/27: • Announcements.

  - OH on Wednesday at 5:30 PM this week; not Tuesday.
  - There will be extra OH next week pre-exam.
    - Roughly like Monday/Wednesday next week.
  - Midterm will be returned on Wednesday; we can pick them up in-person in his office starting then.
  - There are some grade boundaries: Pass/Fail we can do until Friday, withdrawal we can do until 5:00 PM today.

- Let's finish the conversation about induction/restriction and prove the **branching theorem**.
- Reminder to start.
  - We have two mathematical categories, G-reps and H-reps where $H \leq G$.
  - These catagories are related by functors.
    - $\text{Res}_H^G : G\text{-reps} \to H\text{-reps}$ and vice versa for Ind_H^G.
    - See Figure 8.7.
  - Restrictions are stupidly simple.
  - Inductions, most hands-on, we take copies of W times cosets. Formulaically,

$$\text{Ind}_H^G W = g_1 W \oplus \cdots \oplus g_k W$$

    where $k = (G : H)$ and $G = \bigsqcup_{i=1}^k g_i H$.
    - In more detail, the action of g on $g_i w$ is that of $g_{\sigma(i)} h_i w$.
    - This is a genuinely hard construction.
    - A matrix of this thing will be a block-permutation matrix like

$$\begin{array}{c c} & \begin{array}{ccc} g_1W & & g_kW \end{array} \\ \begin{array}{c} g_1W \\ \\ g_kW \end{array} & \left[\begin{array}{c|c|c} & 0 & 0 \\ \hline 0 & 0 & \\ \hline 0 & & 0 \end{array}\right] \end{array}$$

- As an alternate construction, we have that

$$g_1W \oplus \cdots \oplus g_kW \cong \operatorname{Hom}_H(\mathbb{C}[G], W)$$

  - Recall that elements of the set on the right above are functions $f : G \to W$ such that $f(h(g)) = hf(g)$.
  - We map between the two via $f(g) \mapsto f(gx')$.
- What is nice about induced representations is that $\dim[\operatorname{Ind}_H^G W] = (\dim W)[G : H]$.
- There is a very easy statement of the character of an induced representation, the **Frobenius formula**.
  - Recall that

$$\tilde{\chi}_W(g) = \begin{cases} 0 & g \notin H \\ \chi_W(g) & g \in H \end{cases}$$

  - With this, we average:

$$\chi_{\operatorname{Ind}_H^G W}(g) = \frac{1}{|H|} \sum_{x \in G} \tilde{\chi}_W(xgx^{-1})$$

  - Essentially, we're taking a whole bunch of conjugates, summing them up, and dividing to get rid of overcounting.

- We now move onto **Frobenius reciprocity**, which is a relation between the functors/relations Ind_H^G and Res_H^G.
  - The first point where category theory gets interesting is the notion of **adjoint functors**, which we are about to touch on. It is a very subtle notion.
  - Here's version 1 of the statement of Frobenius reciprocity.
    - Recall that we have a scalar product on the space of class function, given by

$$(\chi_1, \chi_2) = \frac{1}{|G|} \sum_{g \in G} \chi_1(g)\chi_2(g^{-1})$$

    where χ_1, χ_2 are class functions on G.
    - Recall that if $\chi_1 = \chi_V$ and $\chi_2 = \chi_W$, then

$$(\chi_1, \chi_2) = \dim \operatorname{Hom}_G(V, W) = \dim \operatorname{Hom}_G \left(\bigoplus_{i=1}^k V_i^{n_i}, \bigoplus_{i=1}^k V_i^{m_i} \right) = \sum_{i=1}^k n_i m_i$$

    - Then the statement is as follows. If V is a G-rep and W is an H-rep, then

$$(V, \operatorname{Ind}_H^G W)_G = (\operatorname{Res}_H^G V, W)_H$$

      - ➢ This notation denotes a scalar product in G and scalar product in H of the characters of each representation.
    - This is similar to the relation between adjoint maps $V \to W$ and $W^* \to V^*$.
  - Version 2.
    - We have that

$$\operatorname{Hom}_G(V, \operatorname{Ind}_H^G W) \cong \operatorname{Hom}_H(\operatorname{Res}_H^G V, W)$$

    where the isomorphism is canonical.
    - We will not check this last definition; we can tediously do it with definitions, and there's nothing complicated. Rudenko leaves this as an exercise to us.
    - The canonical isomorphism sends a map $v \mapsto [g \mapsto \varphi(gv)]$ to the map $\phi : V \to W$.

- We now prove Version 1.

Proof. We have

$$\begin{aligned}
(\chi_V, \chi_{\operatorname{Ind}_H^G W})_G &= \frac{1}{|G|} \sum_{g_1 \in G} \chi_V(g_1)\chi_{\operatorname{Ind}_H^G W}(g_1^{-1}) \\
&= \frac{1}{|G|} \sum_{g_1 \in G} \chi_V(g_1) \left(\frac{1}{|H|} \sum_{g_2 \in G} \tilde{\chi}_W(g_2 g_1^{-1} g_2^{-1}) \right) \\
&= \frac{1}{|H| \cdot |G|} \sum_{g_1, g_2 \in G} \chi_V(g_1)\tilde{\chi}_W(g_2 g_1^{-1} g_2^{-1}) \\
&= \frac{1}{|H| \cdot |G|} \sum_{g_1, g_2 \in G} \chi_V(\underbrace{g_2 g_1 g_2^{-1}}_{h})\tilde{\chi}_W(\underbrace{g_2 g_1^{-1} g_2^{-1}}_{h^{-1}}) \\
&= \frac{1}{|H|} \frac{1}{|G|} \sum_{h \in G} |G| \chi_V(h)\tilde{\chi}_W(h^{-1}) \\
&= (\chi_V|_H, \chi_W)_H \\
&= (\operatorname{Res}_H^G V, \chi_W)_H
\end{aligned}$$

From line 4 to line 5: Fix h; then $g_2 g_1 g_2^{-1} = h$ iff $g_1 = g_2^{-1} h g_2$, so we have overcounted by $|G|$ times. From line 5 to line 6: $\tilde{\chi}_W$ is zero whenever $h^{-1} \notin H$, so this ostensible sum over all $h \in G$ is *de facto* only a sum over all $h \in H$; this is what allows us to consider χ_V as "restricted to H" in line 6. $\square$

- We now come to the branching theorem at long last.
- Example first.
  - Consider $S_n > S_{n-1}$, where S_{n-1} is the subgroup that fixes n. I.e., $S_3 > S_2 = \{e, (12)\}$, and we explicitly omit $(13), (23), (123), (132)$ because they all move 3.
  - Let $\lambda \vdash n$.
  - Let $\mu \leq \lambda$ be a Young diagram of a partition of $n - 1$.
  - Then
    1. We have
    $$\operatorname{Res}_{S_{n-1}}^{S_n} V_\lambda = \bigoplus_{\substack{\mu \leq \lambda \\ |\mu| = n-1}} V_\mu$$
       - Example:
       $$\operatorname{Res}_{S_4}^{S_5} \ydiagram{3,2} = \ydiagram{2,2} \oplus \ydiagram{3,1}$$
    2. We have
    $$\operatorname{Ind}_{S_{n-1}}^{S_n} V_\mu = \bigoplus_{\substack{\mu \leq \lambda \\ |\lambda| = n}} V_\lambda$$
       - Example:
       $$\operatorname{Ind}_{S_5}^{S_6} \ydiagram{3,2} = \ydiagram{3,2,1} \oplus \ydiagram{3,3} \oplus \ydiagram{4,2}$$
- The reason that this theorem is called the branching theorem originates from the diagram in Figure 9.1, which (when continued) encapsulates the main idea of the theorem.
  - This graph helps you understand induction and restriction.
  - Dimensions are the number of paths from the bottom to a final Young diagram.

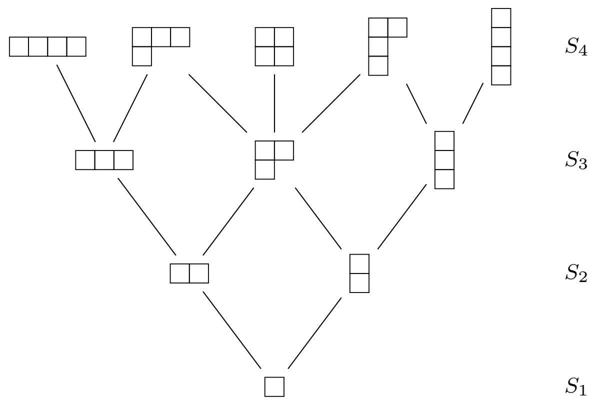

Figure 9.1: The branching theorem.

- ■ For example, the dimension of $(3,1)$ is 3 because there are 3 paths to it, listed as follows.
    1. $(1) \to (2) \to (3) \to (3,1)$.
    2. $(1) \to (2) \to (2,1) \to (3,1)$.
    3. $(1) \to (1,1) \to (2,1) \to (3,1)$.
  - Number of paths is equivalent to number of standard Young tableaux!
- Theorem (Branching): The following two statements are true.

$$\operatorname{Res}^{S_n}_{S_{n-1}} V_\lambda = \bigoplus_{\substack{\mu \leq \lambda \\ |\mu| = n-1}} V_\mu \tag{9.1}$$

$$\operatorname{Ind}^{S_n}_{S_{n-1}} V_\mu = \bigoplus_{\substack{\mu \leq \lambda \\ |\lambda| = n}} V_\lambda \tag{9.2}$$

Proof. We'll talk about the general idea of the proof now, and maybe do the details next time.

(9.1) $\iff$ (9.2): Suppose first that the left statement above holds true. Then we have that

$$(\operatorname{Res}^{S_n}_{S_{n-1}} V_\lambda, V_\mu) = \begin{cases} 0 \\ 1 & \lambda \geq \mu \end{cases}$$

Thus, by Frobenius reciprocity,

$$(V_\lambda, \operatorname{Ind}^{S_n}_{S_{n-1}} V_\mu) = (\operatorname{Res}^{S_n}_{S_{n-1}} V_\lambda, V_\mu) = \begin{cases} 0 \\ 1 & \lambda \geq \mu \end{cases}$$

Therefore, the second statement holds true. The proof is symmetric in the opposite direction.

(9.1): Let's look at an example. Consider the Young diagram of S_8 shown in Figure 9.2.

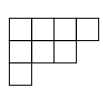

Figure 9.2: Proving the branching theorem.

We want to restrict it down to S_7. Recall that $V_\lambda = \operatorname{span}(S_8 : \Delta(x_1, x_2, x_3)(x_4 - x_5)(x_6 - x_7))$. Now in S_7, we fix x_8. Consider subrepresentations of V_λ filtered by degree as follows.

$$\underbrace{[\,]}_{\deg_{x_3} \leq 0} \leq \underbrace{[\quad]}_{\deg_{x_5} \leq 1} \leq \underbrace{[\qquad]}_{\deg_{x_8} \leq 2} \leq V_\lambda$$

The proof comes from the fact that if we now take quotients of these subrepresentations, e.g., via

$$\deg = 0, \deg \leq 1/\deg \leq 0, \deg \leq 2/\deg \leq 1, \dots$$

then since x_8 can only appear in three boxes, ... $\square$

- Practice with the above example and think it through.

9.2 The Character of a Symmetric Group Representation

11/29:
- Announcements.
  - OH today at 5:30.
  - Our midterms are graded; we can look at them in his office whenever (I can do this during OH!).
- Today, we'll formulate the main result he wants to prove next time.
- Goal is still to understand representations of S_n.
  - We've constructed all of them using Specht modules, but what else do we want?
  - We have dimension, we want characters, etc.
- The main idea is to look at symmetric polynomials once again.
  - Consider $\mathbb{Q}[x_1,\dots,x_n]^{S_n}$.
  - We have proven the fundamental theorem that $\mathbb{Q}[x_1,\dots,x_n]^{S_n} = \mathbb{Q}[\sigma_1,\dots,\sigma_n]$ where $\sigma_k = \sum_{1\leq i_1\leq\cdots\leq i_k\leq n} x_{i_1}\cdots x_{i_k}$.
  - We also proved in PSet 6, Q6 that these rings are equal to $\mathbb{Q}[p_1,\dots,p_k]$ and $\mathbb{Q}[h_1,\dots,h_k]$ where

$$p_k = \sum_{i=1}^n x_i^k \qquad\qquad h_k = \sum_{1\leq i_1\leq\cdots\leq i_k\leq n} x_1\cdots x_k$$

    - Example: If $n=3$ and $k=2$, then

$$h_2 = x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3$$

  - Table of bases for n, k.

$k\setminus n$	1	2	3	4
0	1	1	1	1
1	x_1	x_1+x_2	$x_1+x_2+x_3$	$\cdots$
2	x_1^2	$x_1^2+x_2^2$, x_1x_2	σ_1^2, σ_2^2	σ_1^2, σ_2^2
3	x_1^3	...		

Table 9.1: Polynomial bases.

  - Now take

$$\Lambda_k = [\mathbb{C}[x_1,\dots,x_k]]_{\deg=k-1} \cong [\mathbb{C}[x_1,\dots,x_{k+1}]]_{\deg=k-1} \cong \cdots$$

    - Alternatively, we can think of this thing as

$$\Lambda_k = (\mathbb{C}[x_1,\dots])_k$$

with $\sigma_1^k,\sigma_2\sigma_1^{k-1},\dots$.

  - We call Λ the ring of symmetric functions and define it to be equal to
$$\Lambda = \mathbb{Q}[\sigma_1, \sigma_2, \sigma_3, \dots]$$
  - In every complete component, only finitely many of the σ will participate, so we get finite things.
  - This is a graded ring! We have
$$\Lambda = \bigoplus_{k \geq 0} \Lambda_k$$
and $\Lambda_k \otimes \Lambda_\ell = \Lambda_{k+\ell}$
  - This construction is called the **projective limit**, and we may have encountered it in commutative algebra under the definition
$$\Lambda = \lim_{\to} \mathbb{C}[x_1, \dots, x_n]^{S_n}$$
  - We have identifies such as $p_2 = \sigma_1^2 - 2\sigma_2$. This means that
$$(x_1 + \cdots + x_n)^2 - 2(x_1x_2 + x_1x_3 + \cdots) = x_1^2 + x_2^2 + \cdots$$
  - Observation: $\dim_{\mathbb{Q}} \Lambda_n$.

- Now, we need to take a vector space on ring representations; we've done this already with the representation ring.

- Let R_n be the $\mathbb{Q}$-vector space of functions $\chi : S_n \to \mathbb{Q}$ such that $\chi(x\sigma x^{-1}) = \chi(\sigma)$. This is our favorite space of class functions.

- Theorem (Frobenius characteristic map): There is an isomorphism of vector spaces and of rings called the Frobenius characteristic: $\mathrm{ch} : \bigoplus_{n \geq 0} R_n \to \Lambda$.

Proof. Take $\chi_V \in R_k$, and $\chi_W \in R_\ell$. Let V an S_k-rep, and W an S_ℓ-rep. We know that
$$S_k \times S_\ell = S_{k+\ell}$$
So what we can do is induction $\mathrm{Ind}_{S_k \times S_\ell}^{S_{k+\ell}}(V \otimes W)$. Call this operation $\chi_V \boxtimes \chi_W$.
Now we write down the formula:
$$\mathrm{ch}(\chi) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi(\sigma) p_1^{\lambda_1(\sigma)} \cdots p_k^{\lambda_k(\sigma)}$$
where $\lambda_1(\sigma), \lambda_2(\sigma), \dots$ represent the cycle structure of σ; each λ_i is a number of cycles of length $1, 2, \dots$. ☐

- Examples.

  1. S_1.
     - Sends the YD (1) to $p_1 = x_1 + x_2 + x_3 + \cdots$.
  2. S_2.
     - Sends (2) to $\frac{1}{2!}(p_1^2 + p_2) = \frac{1}{2}((x_1 + x_2)^2 + x_1^2 + x_2^2) = x_1^2 + x_2^2 + x_1x_2 = h_2$.
     - It also sends $(1, 1)$ to $\frac{1}{2!}(p_1^2 - p_2) = \frac{1}{2}((x_1 + x_2)^2 - x_1^2 - x_2^2) = x_1x_2 = \sigma_2$.
     - Let's check our formula. What is $\mathrm{Ind}_{S_1 \times S_1}^{S_2}(1) \otimes (1)$? Since the induction of the trivial representation is the regular representation, which we can decompose, we know that this induction equals $(1, 1) \oplus (2)$. It follows that $p_1^2 = x_1^2 + x_2^2 + x_1x_2 + x_1x_2 = (x_1 + x_2)^2$.
  3. S_3.

- Sends (3) to

$$\begin{aligned}\frac{1}{3!}(p_1^3+3p_1p_2+2p_3) &= \frac{1}{6}[(x_1+x_2+x_3)^3+3(x_1+x_2+x_3)(x_1^2+x_2^2+x_3^2)+2(x_1^3+x_2^3+x_3^3)] \\ &= \frac{1}{6}[6(x_1^3+x_2^3+x_3^3)+6(x_1^2x_2+x_1x_2^2+x_1x_3^2+x_1^2x_3+\cdots)+6x_1x_2x_3] \\ &= h_3\end{aligned}$$

- Sends $(1,1,1)$ to

$$\begin{aligned}\frac{1}{3!}(p_1^3-3p_1p_2+2p_3) &= \frac{1}{6}[(x_1+x_2+x_3)^3-3(x_1+x_2+x_3)(x_1^2+x_2^2+x_3^2)+2(x_1^3+x_2^3+x_3^3)] \\ &= x_1x_2x_3 \\ &= \sigma_3\end{aligned}$$

- Sends $(2,1)$ to

$$\begin{aligned}\frac{1}{3!}(2p_1^3-p_3) &= \frac{1}{6}[2(x_1^3+x_2^3+x_3^3)+6(x_1^2x_2+\cdots)+12x_1x_2x_3] \\ &= (x_1^2+\cdots)+2x_1x_2x_3\end{aligned}$$

- Again, we can check that

$$\operatorname{Ind}_{S_2\times S_1}^{S_3}[(1,1)\otimes(1)] = \sigma_1\sigma_2$$

  - We compute $\operatorname{Ind}_{S_2}^{S_3}(1,1) = (1,1,1)\oplus(2,1)$ via the branching formula: There are only two ways to add a box!
  - We have $\sigma_1\sigma_2-\sigma_3 = (x_1+x_2+x_3)(x_1x_2+x_1x_3+x_2x_3)-x_1x_2x_3$.

- Do we need to be fluent in the techniques by which you expanded all of the polynomials above??
- Thus, we have two conjectures:

$$\operatorname{ch}[(n)] = h_n \qquad\qquad \operatorname{ch}[\underbrace{(1,\dots,1)}_{n\text{ times}}] = \sigma_n$$

- The theorem is cool because it sends all of representation theory to some symmetric polynomial game!
- How do we compute $\operatorname{ch}(V_\lambda)$?
  - We say it equals S_λ, where S_λ is a Schur polynomial.
  - Take the YT of λ. Recall standard YTs.
  - **Semistandard** (YT): Things strictly increase in columns, but only monotonically increase in rows. *draw picture!*
  - The six semistandard ones give us the Schur polynomial.
  - Relation to RSK correspondence.
- Proving why this stuff is true is not hard.
- To understand *why* this is true, Google the **Schur-Weyl duality**.

9.3 Office Hours

- I got a 68/100 on the midterm: 30, 24, 0, 14.
  - I would have needed to show my work (or at least one example of a calculation) to get full credit for 2, even though it just said "find."
  - Rudenko did not expect that finding conjugacy classes would be so difficult for us; he will adjust for this difficulty on the final.
- Week 3, Lecture 2: You proved that $\langle \chi_V, \chi_W \rangle = \delta_{VW}$. To do so, you used a projection function $p = (1/|G|)\sum_{g\in G} gv$. You began your proof by proving that p is a G-morphism and then never used this result again, as far as I can tell. Did you use it again? See pp. 45-47, 58 (it needs to be a morphism of G-representations to map between the representations V, V^G?).
- Week 3, Lecture 2: Same proof. To prove that $\text{Im}(P) = V^G$, do we need more than $p^2 = p$? I think so, but you didn't do it explicitly. See pp. 46-47.
- Week 3, Lecture 2: Same proof. What's up with the trivial special case? See p. 48.
- *Week 3, Lecture 3: Cube thing (see picture from 10/13)?
  - It's just a depiction of two different 3-coordinate bases of the same space. It was drawn to illustrate a possible relation between the orthonormal basis χ_1, χ_2, χ_3 (cube) and the orthogonal basis $\chi_{C_1}, \chi_{C_2}, \chi_{C_3}$.
- Week 3, Lecture 3: Why did we talk about the infinite-dimensional regular representation here? See p. 50.
- *Week 3, Lecture 3: What is the point of the misc. calculations involved in computing the S_4 character table? See p. 52.
  - Just to check that we were on the right path and shown an example of using the orthogonality relations.
- *Week 3, Lecture 3: Proof of the second orthogonality relation your way? It's in Serre (1977), but I don't think that's the way you proved it. See p. 52.
  - To begin, note that it is a *highly* nontrivial statement that if A, B are matrices such that $AB = I$, then $BA = I$. It seems so simple to us, but think about it! For an arbitrary matrix A, B, AB looks nothing like BA! We have two entirely different systems of equations.
  - However, using this fact, basically it is possible to translate the orthogonality relation for the *columns* into the orthogonality relation about the *rows*.
- *Week 3, Lecture 3: All the talk about the exceptional homomorphisms? See p. 52, 61 (the final representation has something to do with an **involution** of trace 2, and is a representation of a quotient group?).
  - So the representation is $\rho : S_4 \twoheadrightarrow S_3 \xrightarrow{\tilde{\rho}} GL_n$, where $\tilde{\rho} : S_3 \to GL_n$ is the representation of ρ corresponding to the character $(2, 0, 1)$.
- *Week 4, Lecture 1: Alternate construction of $R(G)$? See p. 63.
- *Week 4, Lecture 1: Extension of scalars with the representation ring? See p. 64.
  - We don't need to know anything about this stuff.
  - What it is though is basically analogous to extending the real numbers into a subset of the complex numbers by treating every $x \in \mathbb{R}$ as $x + 0i \in \mathbb{C}$. Very trivial, silly concept.
  - There is also such a thing as a **reduction of scalars**.

- *Week 4, Lecture 1: Does multiplying a column vector in the basis $\{\delta_{C_i}\}$ by the character table put it in the basis $\{\chi_{V_i^*}\}$, or vice versa? See p. 66.
  - Derive it for yourself.
  - Example: Consider the character table for S_3 (Table 3.1) represented as the following matrix.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

  - Denote the conjugacy classes of S_3 by e, (xx), (xxx).
  - Interpretation 1.
    - We see that the standard representation is the class function sending

$$e \mapsto 2 \qquad (xx) \mapsto 0 \qquad (xxx) \mapsto -1$$

    - Additionally, we have

$$\begin{aligned} &e \mapsto 1 && (xx) \mapsto 0 && (xxx) \mapsto 0 && (\delta_e) \\ &e \mapsto 0 && (xx) \mapsto 1 && (xxx) \mapsto 0 && (\delta_{(xx)}) \\ &e \mapsto 0 && (xx) \mapsto 0 && (xxx) \mapsto 1 && (\delta_{(xxx)}) \end{aligned}$$

    - Thus, we can express $\chi_{(2,1)}$ as a linear combination of the δ's via

$$\begin{aligned} \chi_{(2,1)} &= (2)\delta_e + (0)\delta_{(xx)} + (-1)\delta_{(xxx)} \\ &= \chi_{(2,1)}(e)\delta_e + \chi_{(2,1)}(xx)\delta_{(xx)} + \chi_{(2,1)}(xxx)\delta_{(xxx)} \\ &= \sum_{C_i} \chi_{(2,1)}(C_i)\delta_{C_i} \end{aligned}$$

    - It follows in particular that if we represent the δ_{C_i}'s as the standard column vector basis of $\mathbb{C}^3$, then

$$\chi_{(2,1)} = A^T \delta_{(xxx)}$$

  - Interpretation 2.
    - If we multiply A by the column vector equal to each representation weighted by $|C_i|$, then we recover the δ basis:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = A \begin{bmatrix} 1 \cdot 1 \\ 3 \cdot 1 \\ 1 \cdot 1 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = A \begin{bmatrix} 1 \cdot 1 \\ 3 \cdot -1 \\ 1 \cdot 1 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = A \begin{bmatrix} 1 \cdot 2 \\ 3 \cdot 0 \\ 1 \cdot -1 \end{bmatrix}$$

    - This interpretation is also what is expressed by the following formula from Lecture 4.2.

$$\delta_{C_j}(g) = \sum_{V_i} \frac{|C_j|\bar{\chi}_{V_i}(g)}{|G|} \cdot \chi_{V_i}(g)$$

- *Week 4, Lecture 2: Isotypical components example. See p. 68.
- **Week 4, Lecture 3: Proof the $\mathbb{C}$ is the only finite-dimensional division algebra? See p. 71.
  - Let A be an arbitrary finite-dimensional division algebra over $\mathbb{C}$.
  - To prove that $A = \mathbb{C}$, we will use a bidirectional inclusion proof.
  - Naturally, $\mathbb{C} \subset A$.
  - To prove the reverse implication, start by letting $a \in A$ be arbitrary.
  - Define the left-multiplication operator $L_a : A \to A$ by $x \mapsto ax$ for all $x \in A$.

- Recall that A is a complex vector space in addition to being an algebra, the same way a ring is also a group. Thus, L_a is a linear operator on a complex vector space.
- It follows by the theorem of linear algebra that L_a has an eigenvalue $\lambda \in F = \mathbb{C}$ and corresponding eigenvector $b \in A$.
- Consequently, by the cancellation lemma,

$$\begin{aligned} L_a b &= \lambda b \\ ab &= \lambda b \\ a &= \lambda \end{aligned}$$

- Therefore, $a \in A$ implies $a \in \mathbb{C}$, so $A = \mathbb{C}$.

- *Week 6, Lecture 2: Proof that $\sqrt{2}/2$ is not an algebraic integer using Gauss's lemma? See p. 87.
  - Let $\alpha := \sqrt{2}/2$ for the sake of notation.
  - Suppose for the sake of contradiction that α is an algebraic integer.
  - Then there exists a monic polonomial $p(x) \in \mathbb{Z}[x]$ such that $p(\alpha) = 0$.
  - Observe that the minimal polynomial in $\mathbb{Z}[x]$ that annihilates α is $2x^2 - 1$.
  - Thus, by polynomial division,

$$p(x) = q(x) \cdot (2x^2 - 1) + r(x)$$

  for some $q, r \in \mathbb{Q}[x]$ such that $\deg r \leq 2 - 1$.
  - We have that

$$r(\alpha) = p(\alpha) - q(\alpha) \cdot (2\alpha^2 - 1) = 0 - q(\alpha) \cdot 0 = 0$$

  - Additionally, since $r \in \mathbb{Q}[x]$ and $\deg r \leq 1$, we know that $r(x) = ux + v$ for some $u, v \in \mathbb{Q}$.
  - We now prove that $u = v = 0$.
    - Suppose for the sake of contradiction that either u or v was not equal to zero.
    - Combining the previous two claims reveals that

$$\begin{aligned} 0 &= r(\alpha) \\ &= u\alpha + v \\ -\frac{v}{u} &= \alpha \end{aligned}$$

    - If $u = 0$, then α is undefined and we have arrived at a contradiction. Thus, $u \neq 0$.
    - Thus, $\alpha \in \mathbb{Q}$. But since $\alpha \notin \mathbb{Q}$ by definition, we have arrived at a contradiction.
    - Therefore, $u = v = 0$.
  - Having established that $r = 0$, we know that $p = (2x^2 - 1)q$, i.e., $2x^2 - 1$ divides p.
  - Now define N to be the least common multiple of the denominators of the coefficients of q.
  - Consider

$$Np = (Nq)(2x^2 - 1)$$

  - It follows by Gauss's lemma that

$$\begin{aligned} c(Np) &= c[(Nq)(2x^2 - 1)] \\ N &= c(Nq) \cdot c(2x^2 - 1) \\ &= 1 \cdot 1 \\ &= 1 \end{aligned}$$

  where c denotes the **content**.

- But if $N = 1$, then $q \in \mathbb{Z}[x]$, so leading term of p — equal to the product of $2x^2$ and the leading term of q — has a coefficient that is a multiple of 2, i.e., is *not* equal to 1 as is required of a monic polynomial, a contradiction.

- *Week 6, Lecture 3: Questions about Lemma 1 of the proof of Burnside's theorem. See p. 92.
  - The roots $a_1, \dots, a_k$ of the minimal polynomial of the algebraic integer a are known as **conjugate algebraic integers**.
  - The conjugate algebraic integers of a root of unity are also roots of unity.
    - Suppose ε is a root of unity.
    - Then the minimal polynomial of ε is $x^n - 1$ for some $n \in \mathbb{N}$.
    - Naturally, the roots of this polynomial (the conjugate algebraic integers to ε) are all of the other roots of unity of order n.
  - The conjugate algebraic integers of a sum of roots of unity is a sum of roots of unity.
    - It can be shown that the minimal polynomial for $\varepsilon_1 + \varepsilon_2$ is

$$p(x) = \prod_{i,j=1}^{n} (x - \varepsilon_1^i - \varepsilon_2^j)$$

    - Evidently, the above polynomial is symmetric under permutations of $\varepsilon_1^i, \varepsilon_2^j$, and we'd generate the same polynomial with any $\pm\varepsilon_1^i \pm \varepsilon_2^j$ as starting material.
    - Explicit example.
      - $\pm\sqrt{2}$ are conjugate algebraic integers, as solutions to $x^2 - 2$. Similarly, $\pm\sqrt{3}$ are conjugate algebraic integers as solutions to $x^2 = 3$.
      - Thus, we expect the minimal polynomial for $\sqrt{2} + \sqrt{3}$ to be

$$p(x) = (x - \sqrt{2} - \sqrt{3})(x - \sqrt{2} + \sqrt{3})(x + \sqrt{2} - \sqrt{3})(x + \sqrt{2} + \sqrt{3})$$

      - Expanding, we obtain

$$\begin{aligned} p(x) &= (x^2 - (\sqrt{2} + \sqrt{3})^2)(x^2 - (\sqrt{2} - \sqrt{3})^2) \\ &= x^4 - [(\sqrt{2} + \sqrt{3})^2 + (\sqrt{2} - \sqrt{3})^2]x^2 + (\sqrt{2} + \sqrt{3})^2(\sqrt{2} - \sqrt{3})^2 \\ &= x^4 - 10x^2 + 1 \end{aligned}$$

      - Indeed, the above polynomial is a monic polynomial
      - From the definition, this polynomial is evidently also the minimal polynomial for $\sqrt{2}-\sqrt{3}$, $-\sqrt{2} + \sqrt{3}$, and $-\sqrt{2} - \sqrt{3}$.
      - Thus, the conjugate algebraic integers of $\sqrt{2} + \sqrt{3}$ are the four sums of all individual algebraic integers.
  - How do we extend this argument to the case in the problem?? What about when $\varepsilon_1 = -1$ and $\varepsilon_2 = i$ so that simple powers don't access every combination as the $p(x)$ formula does?
  - We know that $\prod_{i=1}^{n} a_i \in \mathbb{Z}$ because of Vieta's formula.
    - In particular, this tells us that $x_1 \cdots x_n$ is equal to the last coefficient in the minimal polynomial which, by definition, is an integer.
  - Don't worry too much about all this, though: Burnside's theorem will no longer be on the final because Rudenko changed his mind.

- *Week 7, Lecture 2: Symmetric polynomials and roots of symmetric polynomials. See p. 101.

- *Week 7, Lecture 2: Word in blackboard picture? See p. 102.
  - "Remain" to show...

- *Week 7, Lecture 2: What is d in the proof of the alternating polynomials theorem? See p. 103.
  - $d = n - 1$.
- *Week 8, Lecture 2: What is d in the definition on p. 111.
  - Consider the Specht polynomial corresponding to $(2,2,1)$.

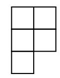

Figure 9.3: Young diagram for $(2,2,1)$.

  - Since $(2,2,1)' = (3,2)$, the Specht polynomial is $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) \cdot (x_4 - x_5)$.
  - Δ_{123} is of degree $3 = \binom{3}{2}$ because looking at the first column of the YD, which corresponds to $\lambda_1' = 3$, out of the 3 boxes, we must choose 2 for each of the three terms $(x_1 - x_2), (x_1 - x_3), (x_2 - x_3)$. Then we just add this to the 2 choose 2 for the second column of the YD.
- *Week 9, Lecture 2: Do we need to be fluent in the techniques you used to expand and reduce the various polynomial powers? How did you do that again?

9.4 The Frobenius Characteristic Map

12/1:
- Proving the theorem.
- The statement is that there exists a function

$$\mathrm{ch} : \underbrace{\bigoplus_{n \geq 0} \mathbb{Q}_{\mathrm{cl}}(S_n)}_{\{f : S_n \to \mathbb{Q} : f(\sigma = \sigma^{-1}) = f(i)\}} \to \bigoplus_{n \geq 0} \Lambda_n$$

where $\Lambda_n = \mathbb{Q}[\sigma_1, \sigma_2, \sigma_3, \dots]_{\deg = n} = [\mathbb{Q}[x_1, \dots, x_n]^{S_n}]_{\deg = n}$. Note that by convention, $\Lambda_0 = \mathbb{Q}$. This function is given by

$$\mathrm{ch}(\chi) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi(\sigma) \cdot p_1^{\lambda_1(\sigma)} \cdot p_n^{\lambda_n(\sigma)}$$

where $p_k = x_1^k + x_2^k + \cdots$ and $\lambda_i(\sigma)$ is the number of cycles in σ of length i. Moreover, ch is an isomorphism of $\mathbb{Q}$-algebras. To create a product of V a S_n-rep and W an S_m-rep, we map

$$V \boxtimes W = \mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} (V \otimes W)$$

Proof. To prove that ch is a ring isomorphism, we need...

1. $\boxtimes$ is associative;
2. $\mathrm{ch}(\chi_1 \boxtimes \chi_2) = \mathrm{ch}(\chi_1) \cdot \mathrm{ch}(\chi_2)$;
3. $\mathrm{ch}((n)) = h_n$.

1,2,3 imply the theorem because 2,3 imply that ch is surjective, Λ_n has a $\mathbb{Q}$-basis $h_{\lambda_1} \cdots h_{\lambda_n}$ for $\lambda_1 \leq \cdots \leq \lambda_n$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = n$. For example, for Λ_5, we have $h_1^5, h_1^3 h_2, h_1, h_2^2, h_2 h_3, h_1^2 h_3, h_1 h_4, h_5$. This surjectivity combined with the fact that $\dim \mathbb{Q}_{\mathrm{cl}}[S_n] = \dim \Lambda_n$ implies that ch is an isomorphism of rings.

Last thing: $\mathrm{ch}[(n)] = \frac{1}{n!} \sum p_1^{c_1(\sigma)} \cdots p_n^{c_n(\sigma)}$ where $c_i(\sigma)$ denotes the number of cycles of length i and hence $\sum i c_i = n$. Denote $p_1^{c_1(\sigma)} \cdots p_n^{c_n(\sigma)}$ by $p^{c(\sigma)}$.

Proof. Let

$$\begin{aligned}
\sum h_n t^n &= \sum\Big(\sum_{i_1\leq\cdots\leq i_n} x_{i_1}\cdots x_{i_n} t^n\Big) \\
&= \frac{1}{1-x_1t}\cdot\frac{1}{1-x_2t}\cdots\frac{1}{1-x_nt} \\
&= \exp\left(\log\left(\prod_{i=1}^n \frac{1}{1-x_it}\right)\right) \\
&= \exp\left(\sum_{i=1}^n -\log(1-x_it)\right) \\
&= \exp\left(x_1+\frac{x_1^2t^2}{2}+\frac{x_1^3t^3}{3}+\cdots+x_2+\frac{x_2^2t^2}{2}+\cdots\right) \\
&= \exp\left(p_1+\frac{p_2t^2}{2}+\frac{p_3t^3}{3}+\cdots\right) \\
&= \prod_{m\geq 1}\exp\left(\frac{p_mt^m}{m}\right) \\
&= *
\end{aligned}$$

We get the second equality because each $1/(1-x_it) = 1+x_i+x_i^2t^2+\cdots$. We need the power series $-\log(1-t) = t+t^2/2+\cdots$ and $\exp(t) = 1+t+t^2/2!+\cdots$. Thus, $\exp(\log(1-t)) = 1-t$. Now note that

$$\begin{aligned}
\sum_{n\geq 0}\mathrm{ch}[(n)]\cdot t^n &= \sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in S_n} p_1^{c_1(\sigma)}\cdots p_n^{c_n(\sigma)} \\
&= \sum_{n\geq 0}\frac{t^n}{n!}\left(\sum_{a_1+2a_2+\cdots+na_n=n} p_1^{a_1}\cdots p_n^{a_n}\right)\cdot\frac{n!}{1^{a_1}a_1!2^{a_2}a_2!\cdots n^{a_n}a_n!} \\
&= \sum_{\substack{n\geq 0\\ a_1,\ldots,a_n:a_1+2a_2+\cdots+na_n=n}} \frac{1}{a_1!}\left(\frac{p_1}{1}\right)^{a_1}t^{a_1}\frac{1}{a_2!}\left(\frac{p_2}{2}\right)^{a_2}t^{a_2}\cdots\frac{1}{a_n!}\left(\frac{p_n}{n}\right)^{a_n}t^{a_n} \\
&= \prod_{k=1}^n\left(\sum_{a_k=1}^\infty\frac{(p_k)^{a_k}t^{a_kk}}{a_k!ka_k}\right) \\
&= \prod_{k\geq 1}\exp\left(\frac{p_kt^k}{k}\right) \\
&= *
\end{aligned}$$

We're overcounting because we can cyclically permute cycles (i.e., (12) = (21)), hence the correction factor in the second line above.

Note: This exponent/logarithm trick is a common computational trick in combinatorics, varieties, etc. □

Now we prove the part 3, i.e., that ⊠ is associative. We do this by direct computation.

$$\underbrace{\mathrm{Ind}_{S_{n+m}\times S_\ell}^{S_{n+m+\ell}}\left[\mathrm{Ind}_{S_n\times S_m}^{S_{n+m}}(\chi_1\otimes\chi_2)\right]\otimes\chi_3}_{(\chi_1\boxtimes\chi_2)\boxtimes\chi_3} = \mathrm{Ind}_{s_n\times S_m\times S_\ell}^{S_{n+m+\ell}}(\chi_1\otimes\chi_2\otimes\chi_3)$$

...

Then proving 2 (homomorphism bit) is the hardest. We have

$$\begin{aligned}
\operatorname{ch}(\operatorname{Ind}_{S_n\times S_m}^{S_{n+m}}(\chi_1\otimes\chi_2)) &= \frac{1}{n!}\sum_{\sigma\in S_n}(\operatorname{Ind}_{S_n\times S_m}^{S_{n+m}}\chi_1\otimes\chi_2)(\sigma)\underbrace{p_1^{c_1(\sigma)}\cdots p_{n+m}^{c_{n+m}(\sigma)}}_{\psi}\\
&= \left\langle \operatorname{Ind}_{S_n\times S_m}^{S_{n+m}}(\chi_1\otimes\chi_2),\psi\right\rangle_{S_{n+m}}\\
&= \left\langle \chi_1\otimes\chi_2,\operatorname{Res}_{S_n\times S_m}^{S_{n+m}}\psi\right\rangle\\
&= \sum_{\substack{\sigma_1\in S_n\\ \sigma_2\in S_m}}\chi_1(\sigma_1)\chi_2(\sigma_2)p_1^{c_1(\sigma_1)}\cdots p_n^{c_n(\sigma_1)}p_1^{c_1(\sigma_2)}\cdots p_m^{c_m(\sigma_2)}\\
&= \operatorname{ch}(\chi_1)\operatorname{ch}(\chi_2)
\end{aligned}$$

We use Frobenius reciprocity somewhere in here. We also have $\psi : S_n \to \Lambda_n$ and $\psi(\tau\sigma\tau^{-1}) = \psi(\sigma)$. $\square$

- After another 10 years of trying to understand the representations of the symmetric group, we'll be here.
- At this point, we can study compact Lie groups.