4 More Characters and Intro to Associative Algebras

10/27: 1. **Representations of S_5.**

   (a) Prove that there exist only two one-dimensional representations of S_5: The trivial representation (5) and the alternating representation $(1,1,1,1,1)$.

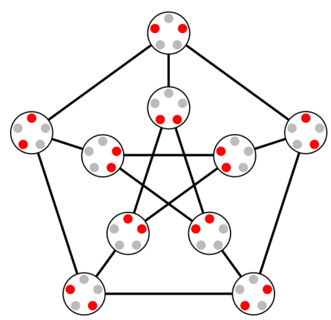
   (b) Compute the character of the standard representation $(4,1)$ by decomposing the permutational representation into irreducibles.

   (c) Prove that the representations $(3,1,1)=\Lambda^2(4,1)$ and $(2,1,1,1)=\Lambda^3(4,1)$ are irreducible. Compute their characters. Prove that $(2,1,1,1)=(1,1,1,1,1)\otimes(4,1)$.

   (d) Find the two remaining irreducible representations of S_5; denote them $(3,2)$ and $(2,2,1)$. Complete the character table.

   (e) Consider an exceptional homomorphism $S_5\to S_6$. Decompose the corresponding permutational representation into irreducibles.

   (f) The **Petersen graph** is a graph with vertices being 2-element subsets of $\{1,2,3,4,5\}$; two vertices are connected by an edge if the corresponding sets do not intersect (see wiki for a picture).



   Consider a natural action of S_5 on the set of complex-valued functions on the set of vertices of the Peterson graph. Find the character of the corresponding representation V and decompose it into irreducibles.

   (g) Decompose V into isotypical components.

   (h) Consider an endomorphism $A:V\to V$ sending a function to the average of its values on the adjacent vertices. Prove that it is an endomorphism of the corresponding representation. Find the spectrum of A.

2. The character table is a square matrix. Determine the absolute value of its determinant.

3. Prove that for any irreducible character χ of a group G, we have

$$\chi(g)\chi(h)=\frac{d_\chi}{|G|}\sum_{x\in G}\chi(gxhx^{-1})$$

4. Let F be a field.

   (a) Prove that the matrix algebra $M_{n,n}(F)$ is simple, i.e., has no nontrivial ideals.

   (b) Prove that there is a unique simple module over the matrix algebra $M_{n,n}(F)$.

5. Consider the quaternion group $Q_8=\{\pm1,\pm i,\pm j,\pm k\}\subset\mathbb{H}$ of order 8.

   (a) Find four 1-dimensional representations of Q_8. Find the character of the remaining 2-dimensional representation.

   (b) Prove that $\mathbb{R}[Q_8]\cong\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{R}\oplus\mathbb{H}$ as an algebra.