Week 3

Character Theory

3.1 Characters

10/9:
- Today, we talk about **characters**, arguably the most important idea in rep theory.
- As per usual, we begin by letting G a finite group.
  - We've been discussing finite dimensional representations of G over $\mathbb{C}$.
  - We've also already talked about irreps, and we know that it's enough to understand those because every rep is a sum of them.
- Goal of characters: Understand the irreps $V_1, \dots, V_k$ of G.
  - Recall the surprising fact about k: It is the number of conjugacy classes of G!
    - We haven't yet proven this, but we will soon!
  - Game plan: Use characters to relate irreps to something that is counted by conjugacy classes.
- Let $V = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$ be a G-rep.
  - Then there exists a homomorphism $\rho : g \mapsto A_g \in GL_n(\mathbb{C})$.
- Motivating question: What doesn't change when we change the basis of V?
  - To isolate the "essence" of the A_g, we want to construct a function $f : GL_n(\mathbb{C}) \to \mathbb{C}$ such that $f(XAX^{-1}) = f(A)$.
- Ideas.
  1. The determinant is a great example of such a function, but it's kind of boring because this rank 1 representation doesn't characterize your product representation.
  2. Trace is the main example of such a function.
- Indeed, you can also take $\operatorname{tr}(A^k)$ for any k.
  - Traces of powers are ubiquitous in physics and math because they contain the same information as the coefficients of the characteristic polynomial. In particular, we can express the determinant in terms of them.
- In fact, we could also take any coefficient of the characteristic polynomial, but others would get complicated.
  - Any characteristic polynomial coefficient can be expressed in terms of traces; this will be an exercise in PSet 3; it's not hard.

- So what do we have at this point?
  - We can associate to ρ a function $\chi_\rho : G \to \mathbb{C}$ defined by $\chi_\rho(g) = \operatorname{tr}(A_g) = \operatorname{tr}(\rho(g))$.
  - This function is invariant under isomorphism.
  - If we know $\operatorname{tr}(A)$, we know $\operatorname{tr}\left(A^2\right)$ since $A_g^2 = A_{g^2}$. Thus, if we know all traces, we know all power traces.
    - Something about the following??

$$\sum \lambda_i \lambda_j = \frac{\operatorname{tr}(A)^2 - \operatorname{tr}\left(A^2\right)}{2}$$

  - We form a ring of polynomials??
    - Equivalently, χ_ρ has a representation as a polynomial with coefficients in $\mathbb{C}$??
- If V is a G-rep, $\chi_V : G \to \mathbb{C}$ will be our notation for its character.
- Properties.
  1. $\chi_V(xgx^{-1}) = \chi_V(g)$ for any $x, g \in G$.
     - Implication: χ_V is a **class function**.
     - Let $\mathbb{C}[G]$ be the vector space of all functions from $G \to \mathbb{C}$. Its dim $= |G|$.
     - Inside this space, there is the subspace $\mathbb{C}_{\text{cl}}[G]$ of functions $f : G \to \mathbb{C}$ such that $f(xgx^{-1}) = f(g)$ for all $x, g \in G$. These are functions from the sets of conjugacy classes, isomorphic to functions that are constant on conjugacy classes. $\dim \mathbb{C}_{\text{cl}}[G]$ is the number of conjugacy classes.
     - Thus, for every V a G-rep, we get a vector $\chi_V \in \mathbb{C}_{\text{cl}}[G]$. These class functions form a basis of the space; each χ_V for V an irrep forms a linearly independent vector; the set is an *orthogonal* basis. This is the reason for the original theorem holding true!
  2. $\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$.
     - Proof: It's basically tautological (not actually, but it's easy). Let $g \in G$. Compute $\chi_{V_1 \oplus V_2}(g)$. We can compute a basis $e_1, \dots, e_{n+m}$ where the first n vectors form a basis of V_1, and the next m vectors are a basis of V_2. This gives us a block matrix from which we show that the trace of the matrix is the sum of traces.

$$\chi_{V_1 \oplus V_2}(g) = \operatorname{tr} \begin{bmatrix} \rho_{V_1}(g) & 0 \\ 0 & \rho_{V_2}(g) \end{bmatrix} = \operatorname{tr} \rho_{V_1}(g) + \operatorname{tr} \rho_{V_2}(g) = \chi_{V_1}(g) + \chi_{V_2}(g)$$

     - Corollary:

$$\chi_{V_1^{n_1} \oplus \cdots \oplus V_k^{n_k}} = n_1 \chi_{V_1} + \cdots + n_k \chi_{V_k}$$

- We now pause for a fact that will be instrumental in proving the next property, which is a bit more involved.
  - He will explain two ways to prove it; we can also just prove it on our own.
- Fact: A a matrix such that $A^n = 1$. Then A is diagonalizable or "semi-simple."
  - We can prove this with Jordan normal form.
  - It's a slightly surprising statement.
  - Obviously eigenvalues are roots of unity, but still needs some work.
  - This proof is left as an exercise.
- We now resume the list of properties.
  3. $\chi_V(g)$ is a sum of roots of unity.

  - Proof: We know that $g^{|G|} = e$. Thus, $A_g^{|G|} = 1$. It follows by the fact above that A_g is diagonalizable with eigenvalues $\lambda_1, \dots, \lambda_n$, each of which satisfies $\lambda_i^{|G|} = 1$.
    - Note: Eigenvalues can repeat in the list $\lambda_1, \dots, \lambda_n$, i.e., we are not asserting n distinct eigenvalues here.
  - Therefore, since each λ_i is, individually, a root of unity, we have that $\chi_V(g) = \operatorname{tr} A_g = \lambda_1 + \cdots + \lambda_n$, as desired.

4. $\chi_{V^*} = \bar{\chi}_V$.
  - This property begins to address how characters behave under other operations.
    - Naturally, this is something specific for complex numbers, because the idea of "conjugates" doesn't exist everywhere.
  - Proof: Recall that $\rho_{V^*}(g) = (\rho_V(g)^{-1})^T$.
    - If we know that $\rho_V(G) \sim \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, then we know that $\rho_V^{-1}(g)^T \sim \operatorname{diag}(\lambda_1^{-1}, \dots, \lambda_n^{-1})$.
    - Thus, $\chi_{V^*}(g) = \lambda_1^{-1} + \cdots + \lambda_n^{-1}$.
    - But since we're in the complex plane, $|\lambda_i| = 1$ (equiv. $\lambda_i\bar{\lambda}_i = 1$), so $\lambda_i^{-1} = 1/\lambda_i = \bar{\lambda}_i$.
    - This means that $\chi_{V^*}(g) = \bar{\lambda}_1 + \cdots + \bar{\lambda}_n = \overline{\lambda_1 + \cdots + \lambda_n}$.
  - Note: Every representation we have is **unitary** in certain bases, but unitary representations are not covered in this course.

5. $\chi_{V_1 \otimes V_2} = \chi_{V_1} \cdot \chi_{V_2}$.
  - Proof: We can use a basis or not use a basis.
  - Let's use a basis for now.
    - Let $g \in G$ be arbitrary. Then there exist bases $e_1, \dots, e_n$ of V_1 and $f_1, \dots, f_m$ of V_2 such that $\rho_{V_1}(g)$ and $\rho_{V_2}(g)$ are diagonal.
    - It follows that $\rho_{V_1}(g)e_i = \lambda_i e_i$ $(i = 1, \dots, n)$ and $\rho_{V_2}(g)f_i = \mu_i f_i$ $(i = 1, \dots, m)$.
    - $V_1 \otimes V_2$ thus has basis $e_i \otimes f_j$.
    - But then it follows that $\rho_{V_1 \otimes V_2}(g)e_i \otimes f_j = (\lambda_i e_i) \otimes (\mu_j f_j) = \lambda_i \mu_j (e_i \otimes f_j)$.
    - Thus,

$$\operatorname{tr}(\rho_{V_1 \otimes V_2}(g)) = \sum_{i,j=1}^{n,m} \lambda_i \mu_j = (\lambda_1 + \cdots + \lambda_n)(\mu_1 + \cdots + \mu_m) = \operatorname{tr}(\rho_{V_1}(g)) \cdot \operatorname{tr}(\rho_{V_2}(g))$$

  - Alternate approach.
    - If we don't want to think of eigenvalues, think of tensor product of matrices, the Kronecker product.
    - We get trace is the product of traces once again! *Write this out.*

- **Class function**: A function on a group G that is constant on the conjugacy classes of G.
- Examples.

1. Let A be an abelian group.
  - Then $\chi : A \to \mathbb{C}^\times$.
  - Implication: Character of a character is $\chi_\chi = \chi$.
    - This is horribly repetitive but true.
2. $G = S_3$.
  - The conjugacy classes of this group are $\{e\}$, $\{(12), (13), (23)\}$, and $\{(123), (132)\}$.
  - We construct a **character table** to define all characters.
  - Computing the characters for the trivial representation.
    - We know that ρ sends each g to the matrix (1), which has trace 1.
  - Computing the characters for the sign representation.

	e	(12) (13) (23)	(123) (132)
Trivial	1	1	1
Sign	1	-1	1
Standard	2	0	-1

Table 3.1: Character table for S_3.

  - e and (123) have sign 1 and thus get sent to the matrix (1).
  - (12) has sign -1 and thus gets sent to the matrix (-1).
- Computing the characters for the standard representation.
  - We can compute these traces via a thought experiment.
  - Visualize a triangle in a plane.
  - The 2×2 identity matrix (the standard representation of $e \in G$) acts on it by doing nothing, and has trace 2.
  - In *some* basis, our matrix fixes one vector and inverts another, so matrix is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

    and character is 0.
  - Last one is rotation by $2\pi/3$, so

$$\begin{pmatrix} \cos(2\pi/3) & \sin(2\pi/3) \\ -\sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}$$

    so character is $-1 = 2 \cdot -1/2 = 2 \cdot \cos(2\pi/3)$.
- If V is the standard representation, we can also compute the characters of $V^{\otimes 2}$ for instance. Indeed, by the product rule of characters, they will be the squares of the standard representation's characters, i.e., $(4, 0, 1)$.
- Similarly, since the permutational representation is the direct sum of the standard and trivial representations, we can add their characters to get its characters $(3, 1, 0)$.

3. A very general and very pretty example. Let $G \circlearrowright X$ a finite set.
   - Assign the permutational representation.
   - Let $X = \{x_1, \dots, x_n\}$. Think of these elements as the basis of a vector space; in particular, consider $V = \mathbb{C}e_{x_1} \oplus \cdots \oplus \mathbb{C}e_{x_n}$. Recall that $g(a_1 e_{x_1} + \cdots + a_n e_{x_n}) = a_1 e_{gx_1} + \cdots + a_n e_{gx_n}$. The fact that this is a representation follows immediately from the properties of the group action.
   - Computing the character χ_V of this V: Look at g and write its matrix. In particular, the trace is the number of unmoved/fixed elements, sometimes denoted $\text{Fix}(g)$.
   - This gives us another way of computing V_{perm} from above!

- **Character table**: A table that lists the conjugacy classes across the top, the irreps down the left side, and at each point within it, the value of an irrep's character over that conjugacy class.
  - The character table is a very nice matrix with very nice properties.
  - It is almost orthogonal; not exactly, but very close.
    - Rows aren't orthogonal, but columns are (take direct products)!
    - It is full rank, though.
- The midterm: Take the character table and do fun things with it.

3.2 Office Hours

10/10:

- Problem 1b:
  - Canonically self-dual: $V \cong V^*$ canonically.
- Mathematical methods of quantum mechanics: First few paragraphs of *picture*.
- We should have everything we need to do most of the problem set at this point; maybe not all of 5, but maybe yes, too.
- Problem 3:
  - There is some problem where it decomposes into trivial plus standard, but we still have to prove that standard is irreducible in this case!
  - If you have any vector, you can produce out of this vector something else.
  - If we take any vector and the group acts on it, we'll get a basis. If you hit a vector in the invariant subspace, it will just stay there; if you hit it and it goes everywhere, you get a basis.
  - Now think about a vector when you permute its coordinates.
  - Tomorrow in class, we will learn a quick way to do this problem.
- Problem 5:
  - For some problem, we need to use the fact that $A^n = 1$ proves that $A = I$ in some sense.
  - This is a hard problem!
  - Show that eigenvalues sum to 1; we know that the eigenvalues are roots of unity! Thus, they have to both be 1!
  - When the problem in group theory is harder, that's when you need to go to rep theory.