MATH 26700 (Introduction to Representation Theory of Finite Groups) Notes

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Weeks

List of Figures

Week 1

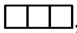
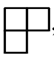
Introduction to Representation Theory

1.1 Motivating and Defining Representations

9/27:
- Rudenko would happily approve my final substitution, but it's not his call; it's Boller's.
- HW will be due every week on Wednesday or thereabouts.
  - Submit in paper in a mailbox, location TBA.
  - First HW due next Wednesday.
- Midterm eventually and an in-class final.
- Grading scheme in the syllabus.
- OH not available MW after class (Rudenko has to run to something else), but F after class, we can ask him anything.
  - Regular OH MTh, time TBA.
- There is no specific book for the course.
  - First 8 lectures come from Serre (1977); amazing book but very concise; gets confusing later on. Most lectures are made up by Rudenko.
- Course outline.
  1. Character theory: Beautiful, not too hard.
  2. Non-commutative algebra: More abstract/general approach to the same thing.
  3. Advanced topics, S_n.
- This course's focus: Representations of finite groups in finite dimensions over $\mathbb{C}$.
- This course is for math-inclined people (not quite physics) and lays the foundation for all other Rep Theory.
  - The ideas would be presented in a very different way in Physics Rep Theory.
- We can always ask questions and stop him to correct mistakes during class.
- Why we care about representations.
  - Start with a group G, finite. For example, let $G \equiv S_1$.

  - People started to play with S_4 (permutations of roots of a polynomial of degree 4) in Galois theory.
    - Galois theory primer: Consider a polynomial like $x^4 + 3x + 1 = 0$; the roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ satisfy tons of equations, e.g., $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$ since 1 is the x^0 term.
  - But groups also occur in much more natural places, e.g., isometries of $\mathbb{R}^3$ that preserve a tetrahedron.
  - S_4 is also orientation-preserving isometries of $\mathbb{R}^3$ that preserve a cube.
  - Many things lead to the same group!
  - Theory of abstract groups developed far later than any of these perspectives; was developed to unify them.
- Recall group actions: Take $G, X = \{x_1, \dots, x_n\}$ both finite. We want $G \circlearrowright X$, which is a homomorphism $A : G \to S_n$.
- What can we do now?
  - We can look at orbits, which are smaller pieces.
  - We can look at the stabilizer.
  - We can identify orbits with cosets.
  - If we understand all possible subgroups, we understand all possible actions.
- This story is not boring, but it's simplistic.
- Rudenko doesn't assume we remember everything (phew!).
- Main definition (general to start, then we simplify).
- **Group representation** (of G on V): A group homomorphism $G \to GL(V)$, for G a group, V a finite-dimensional vector space over some field $\mathbb{F}$ with basis $\{e_1, \dots, e_n\}$, and $GL(V)$ the set of isomorphic linear maps $L : V \to V$. *Denoted by* $\boldsymbol{\rho}$.
  - Recall that $GL(V) = GL_n(\mathbb{F})$ is the set of all $n \times n$ invertible matrices.
- For every element $g \in G$, $g \mapsto \rho(g) = A_g$. Essentially, you're mapping to elements that satisfy certain equations.
  - For example, $A_e = E_n$, $A_{g_1 g_2} = A_{g_1} A_{g_2}$, and $A_{g^{-1}} = {A_g}^{-1}$.
  - Thus, representations are a "concrete way to think about groups."
  - If you don't understand abstract group G, let us compare it to a group that we do understand! Like a group can *act* on S_n, we can *represent* a group in a vector space.
- In this course, G is finite, $\mathbb{F} = \mathbb{C}$, and V is finite dimensional.
  - This is the most simple case, but also a very interesting one. The theory is much, much easier, so we can get much more complicated, but this is a good place to start.
  - We could make G compact, but we're not gonna go that far.
- Examples to get an idea of what's going on.
  1. $\dim \rho = 1$ (means $\dim V = 1$). Then $\rho : G \to GL_1(V) = \mathbb{C}^\times$. The codomain is referred to as the **character** of the group.
     - An example group homomorphism $S_n \to \mathbb{C}^\times$ is the sign function $\sigma \to \text{sign}(\sigma) = \{\pm 1\}$.
     - Another example is the **trivial representation**, $G \to \mathbb{C}^\times$ and $g \mapsto 1$.
  2. Smallest one: Let $G = S_3$. The structure is already pretty rich, and this will be part of the homework.

  - **Trivial representation** again.
  - **Alternating representation**.
  - **Standard representation**.
  - **Regular representation**.

- **Trivial representation**: The representation $\rho : G \to GL(V)$ sending $g \mapsto 1$ for all $g \in G$. *Denoted by* , **(3)**.
  - The boxes notation is too much of a detour to explain now.
  - Note that $1 \in GL(V)$ is the identity map on V!
- **Alternating representation**: The representation $\rho : G \to GL(V)$ sending $g \mapsto \text{sign}(g)$ for all $g \in G$. *Denoted by* , $\mathbf{(1,1,1)}$.
- **Standard representation**: The representation $\rho : S_n \to GL(V)$ sending $\sigma \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)})$, where $V = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_1 + \cdots + x_n = 0\}$ is a $(n-1)$-dimensional vector space. *Denoted by* , $\mathbf{(2,1)}$.
  - A 2D representation like rotating a triangle.
  - This gives something with real numbers.
  - Example: $S_3 \circlearrowright V$ by $\sigma((x_1, x_2, x_3)) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$.
- **Regular representation**: The representation $\rho : G \to \text{Hom}(\mathbb{C}^n)$ defined by $g \mapsto \sigma_g$, where $G = \{g_1, \dots, g_n\}$, $\{e_{g_1}, \dots, e_{g_n}\}$ is a basis of $\mathbb{C}^n$, $\cdot$ is the group action of $\rho(G) \circlearrowright \mathbb{C}^n$ by $\rho(g) \cdot e_g = e_{gg_i}$, and $\sigma_g(e_{g_i}) = \rho(g) \cdot e_g = e_{gg_i}$.
  - This is a permutation of vectors.
  - Thus, for S_3, it will already be 6-dimensional (it's very high dimensional).
- How do we know that representation theory is tractable? Sure, we can define all these things, but how do we know that it will lead anywhere? Here's an example.
  - Let $G = \mathbb{Z}/2\mathbb{Z} = \{e, g\}$, $V = \mathbb{C}^n$, A an $n \times n$ matrix over $\mathbb{C}$, $\rho : G \to GL_n(\mathbb{C})$, and $A := \rho(g)$. Since $g^2 = e$, we know for example that $A^2 = E_n$.
  - But how do we find the matrices A? If we look at eigenvalues of A, there are only two possibilities: ± 1. The structure of A can be very complicated with Jordan normal form and all that, but in fact, these are the **semisimple matrices**, so it's not that bad.
  - Since $A^2 = E$, we know that $(A - E)(A + E) = 0$. Consider $(A - E) : V \to V$. Naturally, it has $\ker(A - E)$ and $\text{Im}(A - E)$. In this particular case, Rudenko claims that $\ker(A - E) \cap \text{Im}(A - E) = \{0\}$.

    Proof. Let $v \in \ker(A - E) \cap \text{Im}(A - E)$ be arbitrary. Since $v \in \text{Im}(A - E)$, there exists $w \in V$ such that $v = (A - E)w = Aw - w$. Since $v \in \ker(A - E)$, we have $(A - E)v = 0$, so $Av = v$. It follows that $A(Aw - w) = Aw - w$ but also $A(Aw - w) = Ew - Aw = w - Aw$. Thus,

$$\begin{aligned} Aw - w &= w - Aw \\ 2Aw &= 2w \\ Aw &= w \end{aligned}$$

    But then $w \in \ker(A - E)$, so $v = (A - E)w = 0$. $\square$

  - This combined with the fact that every vector in a vector space is in either the image or the kernel of a linear map[1] implies that $V = \ker(A - E) \oplus \text{Im}(A - E)$.

[1]See Theorem 3.6 of Axler (2015).

- Let the kernel have basis $e_1, \dots, e_k$ and the image have basis $e_{k+1}, \dots, n$; then all A are of the following form.

$$\begin{array}{c|c} & \begin{matrix} 1 & & k & k+1 & & n \end{matrix} \\ \begin{matrix} 1 \\ \\ k \\ k+1 \\ \\ n \end{matrix} & \left[\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{array}\right] \end{array}$$

- Next time, we will discuss sums of representations, of which this is an example of the theory.

- The same kind of thing, **simple representations**, happens with all finite groups?? This is where we're going. It's not rocket science; in fact, we'll see it next week.
- Last thing for today: A remarkable story.
  - The story of representation theory started quite different.
  - A beautiful theorem that we can prove now!
  - Frobenius determinant.
  - Think of $G = \{g_1, \dots, g_n\}$. Picture its multiplication table.
  - In every row and column, you see each element once.
  - Let's associate to the multiplication table an actual determinant in the linear algebra sense. Consider elements $x_{g_1}, \dots, x_{g_n}$. Define the $n \times n$ matrix $(x_{g_i g_j})$. Take its determinant. It will be a polynomial in n variables, i.e., an element of the ring $\mathbb{Z}[x_{g_1}, \dots, x_{g_n}]$.
  - Example: Consider
$$\begin{vmatrix} e & g \\ g & e \end{vmatrix}$$
    - The determinant is $x_e^2 - x_g^2 = (x_e - x_g)(x_e + x_g)$.
  - Example: $G = \mathbb{Z}/3\mathbb{Z}$.
    - If the elements are e, g, g^2 and we map these, respectively, to variables a, b, c, we get the matrix
$$\begin{bmatrix} e & g & g^2 \\ g & g^2 & e \\ g^2 & e & g \end{bmatrix} \mapsto \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$
    - The determinant is $3abc - a^3 - b^3 - c^3 = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac) = (a + b + c)(a + \zeta b + \zeta^2 c)(a + \zeta^2 b + \zeta c)$ where $\zeta^3 = 1$ is a root of unity.
  - Frobenius's theorem: If G is a finite group and we take this Frobenius determinant, then this determinant is equal to ${P_1}^{d_1} \cdots {P_k}^{d_k}$ where $P_1, \dots, P_k$ are irreducible polynomials in $x_g, \dots, x_{g_j}$, then $\deg P_i = d_i$ and k is the number of conjugacy classes.
  - Example: Take S_3; we'll get a polynomial of degree $|S_3| = 6$ but the Frobenius determinant $FD = (x_{g_1} + \cdots + x_{g_k})(x_{g_1} \pm \cdots)(\text{some pol. of deg 2})^2$
  - The proof is remarkable and deep and uses what would become character theory. These polynomials are related to representations and the number of simplest irreducible representations. The theory that came out came as a way to understand this miracle. We'll forget FD's for now, but then come back and prove it later.

1.2 Key Definitions and Category Theory Primer

9/29:
- OH: TW 4:30 or 5:00 most likely; he will confirm later.
- Today: Definitions in greater generality.
- As before, let G be a finite group and V be a finite-dimensional vector space.
- Goal of this course: Understand representations of G, that is...
  - Homomorphisms $\rho : G \to GL(V) = GL_n(\mathbb{C})$;
  - That send $g \mapsto A_g \in GL_n(\mathbb{C})$;
  - And satisfy $A_e = E$, $A_{g_1}A_{g_2} = A_{g_1g_2}$, and $A_{g^{-1}} = {A_g}^{-1}$.
- What are some things we might want to do?
  - Build new representations from old? Investigate and/or classify irreducible representations?
  - Before we can see if any of this works or not, we need a ton of definitions: Sum, equality, etc.
- Rudenko will start today's lecture with some general thoughts on the **category** of representations.
- Categories are things that now permeates math.
- **Category**: A *class* (not a set) of *objects* (some things; you don't know anything about them), and then a bunch of properties.

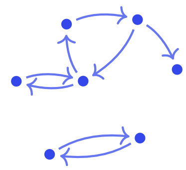


Figure 1.1: The general structure of a category.

  - Objects a, b in category C are denoted by $a, b \in \mathbf{Ob}(\boldsymbol{C})$.
  - There are also **morphisms** between the objects. These are drawn as arrows and lie in $\text{Hom}(a, b)$.
  - There is also composition: $\text{Hom}(a, b) \times \text{Hom}(b, c) \to \text{Hom}(a, c)$.
    - What does this notation mean??
  - Properties.
    1. Associativity.
    2. Existence of a unit: For any object a, there exists $\text{id}_a \in \text{Hom}(a, a)$ such that any morphism pre- or post-composed to this identity yields the same morphism.
       - Example: If $f \in \text{Hom}(a, b)$, then $\text{id}_b \circ f = f = f \circ \text{id}_a$.
  - Rudenko: So a category is basically two pieces of data and a bunch of properties.
- Examples of categories:
  - Category of sets and maps between them.
  - Category of vector spaces over $\mathbb{F}$ where $\text{Ob}(C)$ is the vector spaces and $\text{Hom}(V, W)$ is filled with *linear* maps because you don't just want maps — you want maps that respect the structure.
  - Category of groups where $\text{Hom}(G_1, G_2)$ is the set of group homomorphisms.
  - Category of topological spaces and continuous maps.
  - Category of abelian groups.
  - Trivial category and the identity map; thus, categories need not be chonky.

- Comments on category theory.
  - We'll see some pretty significant category theory at the end of the course.
  - We'll see categories in every course we take; some people try to avoid them. Rudenko doesn't want to go into the material in depth, but he wants to use language from it.
  - Surprisingly, even under the stripped-down of axioms of category theory, you can say quite a lot.
  - Why any of this discussion of category theory matters: If you know the basics of category theory, you can guess the definitions of direct sum, equality, etc. for representations.
- **Category of representations**. *Denoted by* $\mathbf{Rep}_{\boldsymbol{G}}$.
- Take two G-representations V, W; how do we define a map between them?
  - Recall that V, W are vector spaces.
- **Morphism** (of G-representations): A map $f : V \to W$ such that...
  1. f is linear;
  2. f respects the structure of the representations; explicitly, for every $g \in G$, $\rho_V(g) \circ f = f \circ \rho_W(g)$[2].
- On constraint 2, above: This condition is summarized via a **commutative diagram**.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho_V(g)"'] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[d, "\rho_W(g)"] \\ V \arrow[r, "f"'] & W \end{tikzcd}$$

Figure 1.2: Commutative diagram, morphisms.

  - Commutative diagrams are very category-theory-esque things.
- That was a very abstract definition; let's make it concrete.
  - Suppose you have a pair of representations $V = \mathbb{C}^n, W = \mathbb{C}^m$, and we have our map f between them given by an $m \times n$ matrix.
  - Let $\rho_V(g) = A_g$ be an $n \times n$ matrix, and let $\rho_W(g) = B_g$ be an $m \times m$ matrix.
  - Then $FA_g = B_gF$.
- Examples.
  1. An interesting example: Let's look at $S_3 \circlearrowright V_{\text{perm}} = \mathbb{C}^3$, a **permutation representation**.
     - For all $\sigma \in S_3$, $\rho(\sigma) : (x_1, x_2, x_3) \mapsto (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$.
  2. There's also the trivial representation $S_3 \circlearrowright V_{(3)} = \mathbb{C}$ defined by $\rho(\sigma) : x \mapsto x$.
- Are the above 2 representations related?
  - Yes! We can, in fact, find a *morphism* between them.
  - In particular, define $f : V_{(3)} \to V_{\text{perm}}$ by $f(x) = (x, x, x)$.
    - Since permuting 3 of the same thing does nothing, the commutativity of Figure 1.2 holds. Therefore, f is a morphism of G-representations as defined above.
    - More explicitly,

$$f[\rho_{(3)}(\sigma)(x)] = f(x) = (x, x, x) = \rho_{\text{perm}}(\sigma)((x, x, x)) = \rho_{\text{perm}}(\sigma)[f(x)]$$

[2] Recall that the object, $\rho_V(g)$ is a linear map! Thus, it can be composed with other linear maps like f.

- Is f **reversible**?
  - Is "reversible" the right word??
- Define $\tilde{f} : V_{\text{perm}} \to V_{(3)}$ by $\tilde{f} : (x_1, x_2, x_3) \mapsto x_1 + x_2 + x_3$.
  - Since addition is commutative, the commutativity of Figure 1.2 holds.
  - More explicitly,

$$\begin{aligned} f[\rho_{\text{perm}}(\sigma)((x_1, x_2, x_3))] &= f((x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})) \\ &= x_{\sigma(1)} + x_{\sigma(2)} + x_{\sigma(3)} \\ &= x_1 + x_2 + x_3 && \text{Commutativity of addition} \\ &= f((x_1, x_2, x_3)) \\ &= \rho_{(3)}(\sigma)[f((x_1, x_2, x_3))] \end{aligned}$$

- Takeaway: The existence of maps between representations is interesting.

- Next question: How do we define an **isomorphism** of two representations?
- **Isomorphism** (of G-representations): A morphism of G-reps that is an isomorphism of vector spaces.
- Category theory helps us again here because it generalizes the concepts of an isomorphism!
  - If $f : V \to W$ and $g : W \to V$ are category-theoretic morphisms, then the constraints $f \circ g = \text{id}_W$ and $g \circ f = \text{id}_V$ make f and g into category-theoretic *iso*morphisms, regardless of what V and W might be.
  - Back in the context of representations, let $f : V \to V$ be an isomorphism of vector spaces. Then we do indeed have $\rho_V(g) \circ f = f \circ \rho_V(g)$, as we would hope from category theory!
- Recall the condition $FA_g = B_gF$. Supposing F is an isomorphism (and thus has an inverse), we get $FA_gF^{-1} = B_g$ as our new condition.
  - Essentially, we can do *simultaneous conjugation* of all matrices.
  - As per usual with isomorphisms, we get to *change bases*.
  - Essentially, we can represent the nice permutation representation in a very nasty basis but still have it be valid.
- Many other notions (e.g., direct sum) will not be explained by Rudenko, but we can read about them!
- However, we'll do a few more.
- A representation sitting inside another: a **subrepresentation**.
- **Subrepresentation** (of V): A subspace $W \subset V$ such that for all $w \in W$ and $g \in G$, we have that $\rho_V(g)W \subset W$, where V is a G-representation with $\rho_V : G \to GL(V)$.
  - Many people will just write the critical condition as $gW \subset W$.
- Subrepresentations in category theory: We have another commutative diagram.

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d, "\rho_V(g)"'] & V \arrow[d, "\rho_V(g)"] \\ W \arrow[r, hook] & V \end{tikzcd}$$

Figure 1.3: Commutative diagram, subrepresentations.

- Example: The trivial representation, the standard representation, and (of course) the **zero representation** are subrepresentations of the permutation representation.
- **Zero representation**: The representation $\rho : G \to GL(\{0\})$ sending $g \mapsto 1$ for all $g \in G$. *Denoted by* $\mathbf{(0)}$.
- What about representations that don't have subrepresentations?
- **Simple** (representation): A G-representation V that has only two subrerpesentations: (0) and V. *Also known as* **irreducible**, **irreps**.
- Example irreducible representations: Line in $\mathbb{C}^2$, triangle in $\mathbb{C}^2$, A_5 and dodecahedron in $\mathbb{C}^3$.
- Notion of a direct sum.
- **Direct sum** (of V_1, V_2): The G-rep with the space $V_1 \oplus V_2 = \{(v_1, v_2) \mid v_1 \in V_1, v_2 \in V_2\}$ where $\rho_{V_1 \oplus V_2}(g)(v_1, v_2) = (\rho_{V_1}(g)v_1, \rho_{V_2}(g)v_2)$. *Denoted by* $\mathbf{V_1 \oplus V_2}$.
  - The matrix of $\rho_{V_1 \oplus V_2}(g)$ is the following block matrix.

$$\rho_{V_1 \oplus V_2}(g) = \left[\begin{array}{c|c} \rho_{V_1}(g) & 0 \\ \hline 0 & \rho_{V_1}(g) \end{array}\right]$$

- Example: $V_{\text{perm}} = V_{(3)} \oplus V_{(2,1)}$, with $\mathbb{C}^3 = \mathbb{C} \oplus \mathbb{C}^2$ where

$$\mathbb{C} = \langle (1,1,1) \rangle \qquad \mathbb{C}^2 = \langle (x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0 \rangle$$

  - The decomposition is into simple representations.
  - Relate this to the fact that the JCF of any 3×3 permutation matrix has at most a 1-block and a 2-block, if not three 1-blocks. There will always be one 1D subspace on which the permutation matrix is an identity, i.e., span$(1,1,1)$, and a 2D orthogonal complement!
  - As a fun and simple exercise, prove that there is no line fixed under the standard representation.
- A simple and important theorem to prove next week.
- Theorem: Let G be a finite group and $\mathbb{F} = \mathbb{C}$. Then...
  1. There are finitely many irreps $V_1, \dots, V_s$ up to isomorphism.
     - Later on, we will see that s is equal to the number of conjugacy classes.
  2. For every G-rep V, there exists a unique $n_1, \dots, n_s \geq 0$ such that $V \cong {V_1}^{n_1} \oplus \cdots \oplus {V_s}^{n_s}$.
- This theorem tells us that if we want to study rep theory, we want to study irreps (which can be kind of complicated) because if we understand them, everything breaks down into them.
- Examples.
  1. $G = \mathbb{Z}/2\mathbb{Z} = S_2$.
     - $V_1 = \mathbb{C}e$ with $ge = e$ and $V_{-1} = \mathbb{C}e$ with $ge = -e$.
     - It follows that $V \cong {V_1}^{n_1} \oplus {V_{-1}}^{n_{-1}}$.
     - We get a diagonal matrix with only 1s and -1s.
  2. $G = S_3$.
     - $V_{(3)}, V_{(1,1,1)}, V_{(2,1)}$.
     - $GL_5(\mathbb{F}_4)$.
     - Proven in an elementary way in Section 1.3 of Fulton and Harris (2004), which we have to read for the HW; will be useful for later in the course's HW.
- Plan: Next time, we'll talk about some more abstract stuff; tensor products of vector spaces.
  - Tensor products are something we should read up on now! The definition is hard and abstract.
  - Then he'll prove the above theorem.

1.3 S Chapter 1: Generalities on Linear Representations

From Serre (1977).

10/3:
- Part I (what we'll be covering) is written for quantum chemists, and thus gives proofs "as elementary as possible, using only the definition of a group and the rudiments of linear algebra" (Serre, 1977, p. v).
  - Recall the story about Serre and his wife, the chemist, who needed to explain group theory and rep theory to her students.
- Indeed, although the book seemed very fast when I first looked at it two years ago, it reads much more easily now and has enough context for most anyone who is comfortable with group theory and theoretical linear algebra.

Section 1.1: Definitions

- Definitions of $\boldsymbol{GL(V)}$, **invertible square matrix**, and **finite group**.
- **Linear representation**: See class notes. *Also known as* **group representation**.
  - Serre (1977) will frequenty write ρ_s for $\rho(s)$.
- **Representation space** (of G): The vector space V corresponding to the linear representation $\rho : G \to GL(V)$ of G. *Also known as* **representation**.
  - The latter term is a self-identified "abuse of language" (Serre, 1977, p. 3).
- "For most applications, one is interested in dealing with a *finite number of elements* x_i of V, and can always find a subrepresentation of $V\ldots$ of finite dimension, which contains the x_i; just take the vector subspace generated by the images $\rho_s(x_i)$ of the x_i" (Serre, 1977, p. 4).
- **Degree** (of a representation): The dimension of the representation space of this representation.
- To give a representation **in matrix form** is to give a set of invertible matrices that are isomorphic to the group elements.
- Important converse: Given invertible matrices satisfying the appropriate homomorphism identities, there is a corresponding group that these matrices represent.
- **Similar** (representations of G): Two representations $\rho : G \to GL(V)$ and $\rho' : G \to GL(V')$ of G for which there exists a linear isomorphism $\tau : V \to V'$ such that

$$\tau \circ \rho(s) = \rho'(s) \circ \tau$$

  for all $s \in G$. *Also known as* **isomorphic**.
  - Equivalent definition (in matrix form): There exists T invertible such that $R'_s = TR_sT^{-1}$.
  - Isomorphic representations have the same degree.

Section 1.2: Basic Examples

- Degree 1 representation: A homomorphism $\rho : G \to \mathbb{C}^*$, where $\mathbb{C}^*$ denotes the roots of unity (all $z \in \mathbb{C}$ with $|z| = 1$).
  - The fact that every $s \in G$ has *finite* order by assumption is what permits this representation.
- **Unit representation**: See class notes. *Also known as* **trivial representation**.
- **Regular representation**: The representation $\rho : G \to GL(V)$ defined by $s \mapsto [e_t \mapsto e_{st}]$ for all $s \in G$, where V has basis $(e_t)_{t\in G}$.

- $\deg \rho = |G|$.
- $e_s = \rho_s(e_1)$.
  - Implication: The images of e_1 under the ρ_s's form a basis of V, i.e., $\{\rho_s(e_1) \mid s \in G\}$ is a basis of V.
- Converse of above: If W is a representation of G containing a vector w such that $\{\rho_s(w) \mid s \in G\}$ forms a basis of W, then W is isomorphic to the regular representation V via $\tau : V \to W$ defined by $\tau(e_s) = \rho_s(w)$.

- **Permutation representation** (associated with X): The representation $\rho : G \to GL(V)$ defined by $s \mapsto [e_x \mapsto e_{s\cdot x}]$ for all $s \in G$, where $G \circlearrowright X$ a finite set and V has a basis $(e_x)_{x\in X}$.

Section 1.3: Subrepresentations

- Definition of **subrepresentation**.
  - Example: Trivial representation $\mathbb{C}(x, \dots, x)$ is a subrepresentation of the regular representation.
- Definitions of **direct sum** of vector spaces and **kernel**.
- **Complement** (of a subspace): Any $(n-m)$-dimensional subspace U that...
  1. Satisfies $W \oplus U = V$;
  2. Intersects W trivially;

  where $\dim V = n$ and $\dim W = m \leq n$.
  - This means that a single subspace can have multiple complements!
    - Only one **orthogonal** complement, but many *complements*.
    - Example: Consider a line through the origin in $\mathbb{R}^2$; any other line through the origin is a complement of it!
  - It follows that there is a bijection between the complements W' of W in V and the projections p of V onto W (since non-orthogonal complements require non-orthogonal projections).
- **Projection** (of V onto W associated with the decomposition $V = W \oplus W'$): The mapping that sends each $x \in V$ to its component $w \in W$. *Denoted by* $\boldsymbol{p}$.
  - Consequence: The two properties defining a p are (1) $\operatorname{Im}(p) = W$ and (2) $p(x) = x$ for all $x \in W$.
  - Consequence: These two properties also imply that a map is a projection and $V = W \oplus \ker(p)$.
- If a representation has a subrepresentation, then some complement of this subrepresentation is also a subrepresentation.

Theorem 1. *Let $\rho : G \to GL(V)$ be a linear representation of G in V and let W be a vector subspace of V stable under G. Then there exists a complement W^0 of W in V which is stable under G.*

Proof 1 (limited conditions). Let p be the projection of V onto W that corresponds to some arbitrary complement of W in V. To begin, we may legally — albeit with little motivation — form the average p^0 of the conjugates of p by the elements of G:

$$p^0 := \frac{1}{|G|} \sum_{t\in G} \rho_t \cdot p \cdot \rho_t^{-1}$$

We now seek to prove that p^0 is a projection by showing that it satisfies the two properties of a "p." First, notice that by assumption, every ρ_t (and thus ρ_t^{-1}) preserves W. This combined with the fact that $p(V) = W$ implies that $p^0(V) = W$, as desired. Additionally, for any $x \in W$ and $t \in G$, we know by property (2) of a p and the fact that $p_t^{-1}(x) \in W$ that $p \cdot p_t^{-1}(x) = p_t^{-1}(x)$. Applying p_t to both sides of this equation yields $[p_t \cdot p \cdot p_t^{-1}](x) = x$. Hence, $p^0(x) = x$, as desired. Thus, p^0 is a projection of V onto W, associated with some complement W^0 of W.

So that we can make a substitution later, we will now prove that

$$\rho_s \cdot p^0 = p^0 \cdot \rho_s$$

for all $s \in G$. Pick such an s. Then

$$\rho_s \cdot p^0 \cdot \rho_s^{-1} = \frac{1}{|G|} \sum_{t \in G} \rho_s \cdot \rho_t \cdot p \cdot \rho_t^{-1} \cdot \rho_s^{-1} = \frac{1}{|G|} \sum_{t \in G} \rho_{st} \cdot p \cdot \rho_{st}^{-1} = p^0$$

so we can precompose both sides of the above equation with ρ_s to yield the final result. This line here should make it clear why we needed to form a projection like p^0.

We now have all of the tools we need to prove that W^0 is stable under G. To do so, it will suffice to show that for all $x \in W^0$ and $s \in G$, we have $\rho_s(x) \in W^0$. Let $x \in W^0$ and $s \in G$ be arbitrary. Since $x \in W^0$, $p^0(x) = 0$ by definition. This combined with the above commutativity rule implies that $p^0 \cdot \rho_s(x) = \rho_s \cdot p^0(x) = \rho_s(0) = 0$. But the only way that p^0 could map $\rho_s(x)$ to 0 is if $\rho_s(x) \in W^0$, as desired. $\square$

Proof 2 (orthogonal complement). Let W^0 be the orthogonal complement of W, and endow V with a **scalar product** $(x \mid y)$ to turn it into an inner product space. Replace $(x \mid y)$ with the new inner product $\sum_{t \in G} (\rho_t x \mid \rho_t y)$. Now, if it wasn't already, the inner product is invariant under ρ_s for all s, i.e., for s arbitrary, we have

$$(\rho_s x \mid \rho_s y) = (x \mid y)$$

This means that vectors that were orthogonal before ρ_s is applied to V, stay orthogonal after ρ_s is applied to V. In particular, since ρ_s preserves W by hypothesis, all vectors orthogonal to W (i.e., all vectors in W^0) stay orthogonal to W (i.e., stay in W^0) after ρ_s is applied. Thus, W^0 is stable under ρ_s as well. $\square$

- Consequence of the second, stronger proof: The representations W and W^0 determine the representation V.
  - This allows us to rigorously say that the representation $V = W \oplus W^0$.
  - If W, W^0 are given in matrix form by R_s, R_s^0, then $W \oplus W^0$ is given in matrix form by

$$\left(\begin{array}{c|c} R_s & 0 \\ \hline 0 & R_s^0 \end{array}\right)$$

- We can extend this method of directly summing representations to an arbitrary finite number of them.

Section 1.4: Irreducible Representations

- Definition of **irreducible** representation.
- Fact: Each nonabelian group possesses at least one irreducible representation with deg ≥ 2.
  - Proven later.
- Irreducible representations construct all representations via the direct sum.

Theorem 2. *Every representation is a direct sum of irreducible representations.*

Proof. We induct on $\dim(V)$.

Suppose $\dim(V) = 0$. Since 0 is the direct sum of the empty family of irreducible representations, the theorem is vacuously true.

Suppose $\dim(V) \geq 1$. We divide into two cases (V is irreducible and V is reducible). In the first case, we are done. In the second case, $V = V' \oplus V''$ for some $V' \perp V''$ (see Theorem 1). Since $\dim(V') < \dim(V)$ and $\dim(V'') < \dim(V)$ by definition, the induction hypothesis implies that V' and V'' are direct sums of irreducible representations. Therefore, the same is true of V. $\square$

- Fact: The direct-sum decomposition is not necessarily unique.
  - Counterexample: If $\rho_s = 1$ for all $s \in G$, then there are a plethora of decompositions of a vector space into a direct sum of lines.
- Fact: The number of W_i isomorphic to a given irreducible representation *does not* depend on the chosen decomposition.
  - Proven later.

Week 2

The Structure of Representations

2.1 The Tensor Product

10/2:

- Before lecture, I chatted with a few people about tensor products and the exterior and symmetric powers.
  - Patrick: A **tensor** $v \otimes w$ is just an element of a vector space, indexed differently than in a column.
  - Raman: There is no canonical way to transform tensors into column vectors.
- Course logistics.
  - OH: T 5:30-6:30(+) and W 5:30-6:30(+). We can also meet one-on-one.
  - HW is due Thursdays at midnight.
- Today: Constructing new representations from old.
  - Rudenko will skim through tensor products really quickly.
- Reminder: Last time, we talked about how representation theory is really quite simple. If G is a finite group and $F = \mathbb{C}$, there exist a finite set $V_1, \dots, V_s$ of irreps up to isomorphism, and every finite-dimensional representation $V \cong V_1^{n_1} \oplus \cdots \oplus V_s^{n_s}$.
- If V is a representation of G, then there are loads of things we can do with it.
  - We can construct the dual representation V^*.
  - We can construct the representation $V \otimes V$.
  - We can construct symmetric powers.
  - We can construct wedge powers.
  - There are more, but this is enough for now.
- Even when we take a very simple group and representation, there are some very interesting things that can fall out.
  - Example: If you take the symmetric powers of S_3, as in the homework, you get something really interesting.
- Now, we go to linear algebra.
- Let V, W be vector spaces over a field F. How do we produce a new vector space out of these?
- $\operatorname{Hom}_F(V, W)$ is the vector space of linear maps $F : V \to W$!
  - $\dim = (\dim V)(\dim W)$.

- Can we make $\mathrm{Hom}_F(V, W)$ into a representation of G? Yes!

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "\rho_V(g)"'] & W \arrow[d, "\rho_W(g)"] \\ V \arrow[r, "gL"'] & W \end{tikzcd}$$

Figure 2.1: Commutative diagram, linear maps space representation.

  - Suppose that V, W are G-reps, which gives us $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$.
  - Suppose also that we have $L \in \mathrm{Hom}_F(V, W)$.
  - Now infer from the commutative diagram that it will work to define $gL = \rho_W(g) \circ L \circ \rho_V(g)^{-1}$.
  - This is pretty standard.
- Recall that there is a different space $\mathrm{Hom}_G(V, W)$ of morphisms of G-representations (see Figure 1.2 and the associated discussion).
  - This is a very very small subspace of $\mathrm{Hom}_F(V, W)$.
- Special case of the above construction: **Dual representation**.
  - Consider $\mathrm{Hom}_F(V, F)$. This the **dual vector space**.
  - Basic fact 1: Let $e_1, \dots, e_n$ be a basis of V. Then V^* also has a corresponding basis $e^1, \dots, e^n$, known as its **dual basis**.
    - Computing coordinates already depends on a basis, and having bases is super nice.
    - Corollary: $\dim V = \dim V^*$.
    - This is the first time **canonical** comes into linear algebra. Canonical (nobody understands what it means) basically means that something doesn't depend on choices.
    - In particular, V, V^* are isomorphic because they have the same dimension, but for no more natural reason. They can be the same representation, or they can be different.
  - Basic fact 2: If V is finite-dimensional, then $(V^*)^* \cong V$. The formula for this isomorphism is canonical, because it does not depend on a choice of basis. In particular, choose the map $V \to (V^*)^*$ sending v to the map sending $\varphi \in V^*$ to $\varphi(v)$.
  - If V is infinite dimensional, none of this is true and you are in the realm of functional analysis.
  - Ok, so all of this was good information about the dual *space*, but what is the dual *representation*?? Does it matter, and do we need to know for now?
- **Dual vector space** (of V): The vector space defined as follows, given that V is a vector space over F. *Denoted by* $\boldsymbol{V^*}$. *Given by*

$$V^* = \mathrm{Hom}_F(V, F)$$

- **Dual basis** (of V^* to $e_1, \dots, e_n$): The basis defined as follows for $i = 1, \dots, n$, where $e_1, \dots, e_n$ is a basis of V. *Denoted by* $\mathbf{e^1, \dots, e^n}$. *Given by*

$$e^i(x_1 e_1 + \cdots + x_n e_n) = x_i$$

- We now move onto the tensor product.
  - The tensor product is very hard to understand. If you learn about it and you feel you don't understand it, that's typical; nobody understands it at first.
  - For now, we'll discuss two ways of thinking about tensor products that won't bring us any comfort.
- Let V, W be two vector spaces over a field F.

- Abstract definition of the tensor product.
  - We have discussed maps from $V \to W$, but there is another related space.
  - Indeed, we can look at the space of bilinear maps from $V \times W \to F$.
    - Example: A map $f : V \times W \to F$ that satisfies the constraints $f(\lambda v, w) = \lambda f(v, w)$, $f(v_1 + v_2, w) = f(v_1, w) + f(v_2, w)$, and likewise for the second index. Recall that this is a **bilinear map**.
  - Let V have basis $e_1, \dots, e_n$ and W have basis $f_1, \dots, f_m$.
  - Notice that every bilinear map f can be defined as a linear combination of the $f(e_i, f_j)$. In other words, the $f(e_i, f_j)$ form the basis of a function space.
    - This "bilinear maps space" has dimension nm.
  - Now, one way to understand a tensor product: Is this "bilinear maps space" actually some other space? It is! It is $(V \otimes W)^*$.
  - Bilinear maps are linear maps from where? From $V \otimes W$!
- **Bilinear** (map): A function $f : V \times W \to Z$ that satisfies the following constraints, where V, W, Z are vector spaces over F, $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, and $\lambda \in F$. *Constraints*

$$
\begin{aligned}
f(v_1 + v_2, w) &= f(v_1, w) + f(v_2, w) & \qquad f(\lambda v, w) &= \lambda f(v, w) \\
f(v, w_1 + w_2) &= f(v, w_1) + f(v, w_2) & \qquad f(v, \lambda w) &= \lambda f(v, w)
\end{aligned}
$$

- We now look at a much more elementary definition of the tensor product.
- Explicit definition of the tensor product.
  - $V \otimes W$ is equal to a huge vector space with basis consisting of pairs of elements (v, w). Even if V, W are one dimensional, this is like all pairs of real numbers; it's huge. Then, we quotient it by the space of all elements satisfying $\lambda(v, w) = (\lambda v, w) = (v, \lambda w)$, $(v_1 + v_2, w) = (v_1, w) + (v_2, w)$, and the like. This forces these relationships to be true.
    - Clarify this methodology??
    - Essentially, this allows us to treat tensor multiplication much like real multiplication, endowing the operation with distributivity, etc.
    - For example, the rule $(v_1 + v_2, w) = (v_1, w) + (v_2, w)$ becomes, in tensor product notation, $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$.
  - Example: Suppose $V = \mathbb{C}e_1 + \mathbb{C}e_2$. We want to look at $V \otimes V$.
    - A priori[1], it's spanned by $(ae_1 + be_2) \otimes (ce_1 + de_2) = ace_1 \otimes e_1 + ade_1 \otimes e_2 + bce_2 \otimes e_1 + cde_2 \otimes e_2$.
    - Thus, $V_1 \otimes V_2$ has 4-element basis $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$.
- These two definitions constitute a first approximation to what the tensor product is.
- Takeaway: What is true in general is that if V has basis $e_1, \dots, e_n$ and W has basis $f_1, \dots, f_m$, then $V \otimes W$ has basis $e_i \otimes f_j$ ($i = 1, \dots, n$ and $j = 1, \dots, m$).
- Having discussed the tensor product of vector spaces, let's think about the tensor product of *representations*.
  - Suppose $g : V \to V$ and $g : W \to W$.
    - We're starting to make notation sloppy.
  - How does $g : V \otimes W \to V \otimes W$? Well, we just send $v \otimes w \mapsto (gv) \otimes (gw)$.
    - Why is this map well-defined?

[1] I.e., it follows from some logic. In particular, it follows from the logic that any element $v \in V$ is of the form $v = ae_1 + be_2$, so of course all $v \otimes v$ must be of the given form for choices of a, b, c, d.

- We invoke the **universal property of the tensor product operation**.
- This guarantees us that given g — which is effectively a map from $V \times W \to V \otimes W$ as defined — there nevertheless exists a complete extension $\tilde{g} : V \otimes W \to V \otimes W$.

– As a matrix, this map is pretty strange!

- Example: Let $g : V \to V$ be a 2×2 matrix. What is the matrix of $g : V \otimes V \to V \otimes V$?
- If

$$\rho_V(g) = g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} =: A$$

then we have

$$\begin{aligned} g(e_1 \otimes e_1) &= ge_1 \otimes ge_1 \\ &= (ae_1 + ce_2) \otimes (ae_1 + ce_2) \\ &= a^2 e_1 \otimes e_1 + ace_1 \otimes e_2 + ace_2 \otimes e_1 + c^2 e_2 \otimes e_2 \end{aligned}$$

- Evaluating similarly for all basis vectors, we get a very curious block matrix:

$$\begin{array}{c c} & \begin{array}{cccc} e_1 \otimes e_1 & e_1 \otimes e_2 & e_2 \otimes e_1 & e_2 \otimes e_2 \end{array} \\ \begin{array}{c} e_1 \otimes e_1 \\ e_1 \otimes e_2 \\ e_2 \otimes e_1 \\ e_2 \otimes e_2 \end{array} & \begin{bmatrix} a^2 & ab & ab & b^2 \\ ac & ad & bc & bd \\ ac & bc & ad & bd \\ c^2 & cd & cd & d^2 \end{bmatrix} \end{array} = \left[\begin{array}{c|c} aA & bA \\ \hline cA & dA \end{array}\right]$$

- Notice how, for example, this takes the tensor $e_1 \otimes e_1$, represented as $(1, 0, 0, 0)$, to the tensor $a^2 e_1 \otimes e_1 + ace_1 \otimes e_2 + ace_2 \otimes e_1 + c^2 e_2 \otimes e_2$, represented as (a^2, ac, ac, c^2).
- Does this construction imply a canonical way to convert from tensors to column vectors??

– Classically, this is called the **Kronecker product** of two matrices.

– People discovered all of this stuff before they unified it as tensor math.

- **Universal property of the tensor product operation**: For every bilinear map $h : V \times W \to Z$, there exists a *unique* linear map $\tilde{h} : V \otimes W \to Z$ such that $h = \tilde{h} \circ \otimes$.

$$\begin{tikzcd} V \times W \arrow[r, "\otimes"] \arrow[dr, "h"'] & V \otimes W \arrow[d, "\tilde{h}"] \\ & Z \end{tikzcd}$$

Figure 2.2: Universal property, tensor product operation.

Proof. See the solid explanation linked here. Alternatively, here's my write up.

Let $V = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$, $W = \mathbb{C}f_1 \oplus \cdots \oplus \mathbb{C}f_m$, Z, and $h : V \times W \to Z$ be arbitrary. Define $\tilde{h} : V \otimes W \to Z$ by

$$\tilde{h}(e_i \otimes f_j) := h(e_i, f_j)$$

for $i = 1, \dots, n$ and $j = 1, \dots, m$. Since a linear map is wholly defined by its action on the basis of its domain, this set of equations suffices to define $\tilde{h}$ on all of $V \otimes W$.

Existence: To prove that $\tilde{h}$ satisfies the "universal property," it will suffice to show that $h = \tilde{h} \circ \otimes$. Let

$(v, w) \in V \times W$ be arbitrary, and suppose $v = \sum_{i=1}^n a_i e_i \in V$, and $w = \sum_{i=1}^n b_i f_i \in W$. Then

$$\begin{aligned}
[\tilde{h} \circ \otimes](v, w) &= \tilde{h}(v \otimes w) \\
&= \sum_{i=1}^n \sum_{j=1}^m a_i b_i \tilde{h}(e_i \otimes f_i) \\
&= \sum_{i=1}^n \sum_{j=1}^m a_i b_i h(e_i, f_j) \\
&= h(v, w)
\end{aligned}$$

as desired.

Uniqueness: Now suppose $\tilde{g} : V \otimes W \to Z$ also satisfies the "universal property," that is, $h = \tilde{g} \circ \otimes$. Then by definition,

$$\tilde{h}(e_i \otimes f_j) = h(e_i, f_j) = \tilde{g}(e_i \otimes f_j)$$

for $i = 1, \dots, n$ and $j = 1, \dots, m$. But since a linear map is wholly defined by its action on the basis of its domain, it follows that $\tilde{h} = \tilde{g}$, as desired. ☐

- **Kronecker product** (of A, B): The matrix product defined as follows. *Denoted by* $\boldsymbol{A} \otimes \boldsymbol{B}$. *Given by*

$$A \otimes B = \begin{array}{cc} & n \\ n & \begin{bmatrix} A \end{bmatrix} \end{array} \otimes \begin{array}{cc} & m \\ m & \begin{bmatrix} B \end{bmatrix} \end{array} = \begin{array}{cc} & nm \\ nm & \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix} \end{array}$$

  - The Kronecker product is *not* commutative, but the matrices you get are related by conjugacy and by commuting the columns.

- Vector spaces of the same dimension are all alike, but vector space representations are very interesting. By the end of this course, we'll understand what's going on.

- How we understand tensor stuff: Look at the abstract definition, look at the concrete definition, look at 5 examples, and then go in a circle. Repeat again and again until it makes sense.

- He's just trying to tell us all relevant words so that they will fit together later.

- Fact: If V, W finite-dimensional, $\text{Hom}_F(V, W) \cong V \otimes W^*$.
  - Tensor products are very nice to construct maps from.
  - Let's construct a reverse map, then.
  - Take $\alpha \otimes w \in V^* \otimes W$, where $\alpha : V \to F$ by definition. Send $\alpha \otimes w$ to the map $v \mapsto \alpha(v)w$. This is a *canonical* map!! We can show that they span everything.
    - For example, if we want to choose $\alpha \otimes w$ mapping to the matrix with a 1 in the upper left-hand corner and zeroes everywhere else, let $w = e_1 \in W$ and let $\alpha = e^1 \in V^*$.
    - We can do similarly for all other such matrices, mapping this basis of $\text{Hom}_F(V, W)$ to $e^i \otimes e_j$ ($i = 1, \dots, n$ and $j = 1, \dots, m$).
    - Note that this also allows us to define a (noncanonical) inverse map.
  - This inverse map from $\text{Hom}_F(V, W) \to V^* \otimes W$ is clearly a bit harder to work out.
  - Hidden in this story is why trace is invariant under conjugation (see below discussion).

- If we now take $\text{Hom}_F(V, V)$, then this is isomorphic to $V^* \otimes V$. There is a very natural map from these isomorphic spaces to F defined by the trace, and/or $\alpha \otimes v \mapsto \alpha(v)$. We can prove this. And this is canonical, as well. This is why the main property of the trace is that it's invariant under conjugation. This fact is hidden in the story very nicely.

- Tensor products are hard, it will be a pain, we will understand them very well, but it will not be nice for now.
- Symmetric products and wedge powers will be discussed briefly next time.
  - There is a nice description in Serre (1977) that we can use for the homework.
- Extra homework: Please read about tensor products in whatever textbook you like, try some examples, and repeat.

2.2 Office Hours

10/3:

- Problem 2a:
  - $\Lambda^2 V$ is *exterior powers.*
  - The exact canonical isomorphism we need is briefly discussed on Fulton and Harris (2004, p. 473).
  - I.e., we have to construct isomorphisms between the structures that don't rely on the choice of any basis. Recall the classic example of $V \cong V^{**}$, as explained in the well-written MSE post "basic difference between canonical isomorphism and isomorphims." Recall that the isomorphism from $V \to V^*$ defined by sending each element of the basis of V to the corresponding element of the dual basis of V^* is *not* canonical because *it involves choosing bases.* Definitions of canonical maps are available in MATH20510Notes, p. 2.
  - From a quick look at this, it looks like the proof may be analogous to the classic middle-school algebra identity $(v+w)^2 = v^2 + vw + w^2$.
  - The second exterior power $\Lambda^2 V$ of a finite-dimensional vector space V is the dual space of the vector space of alternating bilinear forms on V. Elements of $\Lambda^2 V$ are called 2-vectors.
- Problem 2b:
  - $S^2 V$ is *symmetric powers.*
  - The exact canonical isomorphism we need is briefly discussed on Fulton and Harris (2004, p. 473).
- Problem 3a:
  - This is the determinant of the multiplication table, in relation to that theorem that you showed us at the end of the first class? Yep!
- Problem 3b:
  - So a circulant matrix is a matrix like the multiplication table from (a)? Yep!
  - Is $\zeta = e^{2\pi i/n}$? Sort of. It can be any n^{th} root of unity.
- Problem 4d:
  - We'll cover higher symmetric powers in class tomorrow.
  - However, it basically just means that we're now working with elements of the form $e_1 \otimes e_2 \otimes e_3 \in S^3 V$ and on and on.
- Problem 5a:
  - Is $V^\vee = V^*$? Yes. This is "vee check," and is a notation that some people prefer.
- Problem 5b:
  - Is "tr" the trace function of the linear map corresponding to L? Yes.
  - What is L?

  - An element of $V \otimes V^*$ is a linear combination of elements of the form $v \otimes \alpha$, not necessarily just one of these "decomposable" products.
  - There is an isomorphism $V \otimes V^* \cong \operatorname{Hom}(V)$.
  - Consider the matrix
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
It sends $e_1 \mapsto e_1$ and $e_2 \mapsto 0$. Thus, it is well-matched with $e_1 \otimes e^1$, which also grabs e_1 (with e^1) and sends it to e_1.
  - Consider the matrix
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
It sends $e_1 \mapsto 0$ and $e_2 \mapsto e_1$. Thus, it is well-matched with $e_1 \otimes e^2$, which also grabs e_2 (with e^2) and sends it to e_1.
  - In full,
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ae_1 \otimes e^1 + be_1 \otimes e^2 + ce_2 \otimes e^1 + de_2 \otimes e^2$$
  - This map *is* canonical! This is because the bases must be chosen to even begin talking about matrices.
  - If you change the matrix, the bases change, too??
  - Takeaway: We have to walk backwards from matrix to linear transformation to representation in $V \otimes V^*$ to a scalar in F.

- Problem 5c:
  - So trace of such a map is equal to the dimension of its image? Yes.

2.3 Wedge and Symmetric Powers

10/4:

- OH slightly later today at 5:45-6:45 PM.
- Recap: Last time, we built new reps from old.
  - This stuff can't be learned in 1.5 lectures; he can point us around, but we have to learn it ourselves.
- Tensor product review.
  - Given V, W, make $V \otimes_F W$.
  - This vector space is hard to describe directly, so we more often talk about its dual $(V \otimes W)^*$ because this is actually easier to describe.
  - If you want to work with $V \otimes W$ hands-on, you can do the following.
    - Start with the following easy-to-work-with vector space: The (probably infinite-dimensional) vector space where each $v \otimes w$ is a basis vector for all $v \in V$ and $w \in W$.
    - Then quotient it by relations to force them to hold in the final space.
  - Here's an example of this construction.
    - Let $V = W$ be the one-dimensional vector space over the finite field $F_2 = \mathbb{Z}/2\mathbb{Z}$.
    - Thus, the elements of V are $\{0, 1\}$ (which is, literally, all linear combinations $a0 + b1$ where $a, b \in F_2$ as well; this hearkens back to V's definition as an F_2-module).
    - Then the easy-to-work-with vector space we're talking about is the 4-dimensional **free** vector space $U = \operatorname{span}(0 \otimes 0, 0 \otimes 1, 1 \otimes 0, 1 \otimes 1)$.

- Note that in this space, for example, $(0+1)\otimes 0 \neq 0\otimes 0 + 1\otimes 0$; representing the basis as column vectors, this is equivalent the obvious observation that

$$\begin{bmatrix}0\\0\\1\\0\end{bmatrix} \neq \begin{bmatrix}1\\0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$

- But we want such relationships to hold true in our conceptual "tensor product space." Thus, we quotient it by the subspace spanning all elements of the form $(a+b)\otimes c - a\otimes c - b\otimes c$.
- By direct computation, this subspace is $\operatorname{span}(0\otimes 0, 0\otimes 1)$:

$$\begin{aligned}
(0+0)\otimes 0 - 0\otimes 0 - 0\otimes 0 &= -0\otimes 0 & \qquad (0+0)\otimes 1 - 0\otimes 1 - 0\otimes 1 &= -0\otimes 1\\
(0+1)\otimes 0 - 0\otimes 0 - 1\otimes 0 &= -0\otimes 0 & \qquad (0+1)\otimes 1 - 0\otimes 1 - 1\otimes 1 &= -0\otimes 1\\
(1+1)\otimes 0 - 1\otimes 0 - 1\otimes 0 &= 0\otimes 0 & \qquad (1+1)\otimes 1 - 1\otimes 1 - 1\otimes 1 &= 0\otimes 1
\end{aligned}$$

  Note that once we've considered $(a+b)\otimes c$, we don't need to consider $(b+a)\otimes c$ because of the commutativity of addition in V. That is, it is axiomatic that $a+b=b+a$ for all $a,b\in V$. Additionally, in the last line above, we are using the facts that $1+1=2=0$ in F_2 and $a\otimes b + a\otimes b = 2a\otimes b = 0$ in any F_2-module to simplify the expressions.
- Similarly, the subspace corresponding to $a\otimes(b+c) - a\otimes b - a\otimes c$ is $\operatorname{span}(0\otimes 0, 1\otimes 0)$. Thus, altogether, we quotient out the subspace $X = \operatorname{span}(0\otimes 0, 0\otimes 1, 1\otimes 0)$. This leaves us with a 1-dimensional $V\otimes V$, as expected for the tensor product of two one-dimensional vector spaces. It is interesting to note that the one vector we didn't quotient out $(1\otimes 1)$ is analogous to $e_1\otimes e_1$ since $e_1\in V$ might as well be defined $e_1 := 1$.
- Now let's see how well this quotienting worked. First off, a bit of notation: let $\pi : U \to V\otimes V$ be the projection $\pi : v \mapsto v + X$, and denote elements $\pi(v_1\otimes v_2)\in V\otimes V$ by $v_1\otimes_\pi v_2$ for now to differentiate them from elements of U.
- Let $(0+1)\otimes_\pi 0 = (0+1)\otimes 0 + X$ be an element of the quotient space $V\otimes V$. Certainly, the elements $0\otimes_\pi 0$ and $1\otimes_\pi 0$ are also elements of this quotient space. Moreover, we can fairly form the linear combination $(0+1)\otimes_\pi 0 - 0\otimes_\pi 0 - 1\otimes_\pi 0$. However, this element lies in the quotiented-out subspace X. Thus,

$$(0+1)\otimes_\pi 0 - 0\otimes_\pi 0 - 1\otimes_\pi 0 = [(0+1)\otimes 0 - 0\otimes 0 - 1\otimes 0] + X = 0 + X = 0$$

- But

$$(0+1)\otimes_\pi 0 - 0\otimes_\pi 0 - 1\otimes_\pi 0 = 0 \quad\Longrightarrow\quad (0+1)\otimes_\pi 0 = 0\otimes_\pi 0 + 1\otimes_\pi 0$$

  as desired.
- Note that this construction also gives us nice things like $0\otimes_\pi 0 = 0$, $0\otimes_\pi 1 = 0$, etc. which were not true in U! It should not be concluded, though, that all we need to quotient out of U for any V is $\operatorname{span}(0\otimes 0, 0\otimes v, v\otimes 0)$ for every $v\in V$; indeed, $V=\mathbb{R}$, for example, will contain

- If V has basis $e_1,\dots,e_n$ and W has basis $f_1,\dots,f_m$, then $e_i\otimes f_j$ is a basis of $V\otimes W$.
- Interesting fact 1: If V,W are finite dimensional, $V^*\otimes W \cong \operatorname{Hom}(V,W)$.
- If we want to work with the tensor product in practice in *rep theory*, the only thing we need to know is the basis of the tensor product space, which can tell us how any map $\rho(g)$ acts on both sides of a $v\otimes w\in V\otimes W$. From here, we recover the Kronecker product of matrices.
- So many things are explained by the concept of tensor products!
- A tensor in *physics* is something with lots of indices that changes in some way.
  - It does come from the math concept.
  - We'll get a huge basis because we have a massive product like $V\otimes\cdots\otimes V\otimes V^*\otimes\cdots\otimes V^*$.

- **Free** (vector space): A vector space that has a basis consisting of linearly independent elements.
  - Example: Think of $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ as a $\mathbb{C}$-module. A free version $F(V)$ of V is infinite dimensional with every $v \in V$ a linearly independent basis vector. Elements of $F(V)$ are of the form $a_1v_1 + \cdots + a_kv_k$ for $a_1, \dots, a_k \in \mathbb{C}$ and $v_1, \dots, v_k \in V$. If $u = v + w$ where $u, v, w \in V$ are all nonzero, then $u \neq v + w$ in $F(V)$ because they are all linearly independent basis vectors.
  - Example: What we formally start with in the example above is $V \times V$, the free F_2-module not the Cartesian product vector space V^2.
  - A terrific explanation of free vector spaces is available here.
- Last 2 useful notions: Wedge powers and symmetric powers.
  - Again, it's much easier to think about the dual space.
- Consider the space $V^{\otimes n}$ (dimension $(\dim V)^n$).
  - $(V^{\otimes n})^*$ are **polylinear** maps $f : V^n \to F$.
    - Note: By contrast, $(V^n)^*$ is the space of all *linear* maps $f : V^n \to F$.
    - This distinction is subtle but important. Note, for instance, that $\dim V^{\otimes n} \neq \dim V^n$ and likewise for the duals.
    - The distinction comes out fully when considering that if, for example, $V = \mathbb{R}^3$, then $V^2 \cong \mathbb{R}^6$ and any map in $(V^2)^*$ is determined by its action on $(e_1, 0), (e_2, 0), (e_3, 0), (0, e_1), (0, e_2), (0, e_3)$. By contrast, any map in $(V^{\otimes 2})^*$ is determined by its action on $(e_1, e_1), (e_1, e_2), (e_1, e_3), (e_2, e_1), (e_2, e_2), (e_2, e_3), (e_3, e_1), (e_3, e_2), (e_3, e_3)$.
    - Important note: What $(V^{\otimes 2})^*$ does is consider these nine elements of V^2 as the basis of another space. This is what it truly means when we say "a bilinear map on V^2 is a linear map on $V^{\otimes 2}$."
    - Takeaway: Polylinearity changes the basis upon which a function $f : V^n \to F$ fundamentally acts.
  - A polylinear map may be **symmetric**, **antisymmetric**, or[2] neither.
  - These maps form vector spaces and the dimension is actually pretty meaningful.
- **Symmetric** (polylinear map): A polylinear map $f : V^n \to F$ that satisfies the following property. *Constraint*
$$f(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = f(v_1, \dots, v_n)$$
- **Antisymmetric** (polylinear map): A polylinear map $f : V^n \to F$ that satisfies the following property. *Constraint*
$$f(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = (-1)^\sigma f(v_1, \dots, v_n)$$
- Suppose you take $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$[3].
  - Consider a symmetric polylinear map $f : V \times V \times V \to \mathbb{C}$.
  - To compute it, we'll need the action of f on the basis of V^3. In particular, we'll need...
$$f(x_1e_1 + y_1e_2, x_2e_1 + y_2e_2, x_3e_1 + y_3e_2) = x_1x_2x_3f(e_1, e_1, e_1) + x_1x_2y_3f(e_1, e_1, e_2) + \cdots$$
    - Somewhere in there, you'll also have a $x_1y_2x_3f(e_1, e_2, e_1)$ term as well.
    - However, because f is symmetric, you know by symmetry that these "bases" are the same, so you don't count them as 2 towards the dimension but as 1.
    - Thus, dim = 4 for symmetric maps.

[2]This is an exclusive "or."

[3]Note that this notation allows you to define a vector space *and* its basis in one go! I.e., the alternative is saying "Let V be a complex vector space with basis e_1, e_2."

- What about antisymmetric maps?
- Suppose $g : V^3 \to \mathbb{C}$ is an antisymmetric polylinear map.
  - Consider $g(e_1, e_1, e_1)$. Suppose you apply (12). Interchanging the first two indices (for instance) obviously won't do anything, so we'll get

$$\begin{aligned} g(e_1, e_1, e_1) &= (-1)^{(12)} g(e_1, e_1, e_1) \\ g(e_1, e_1, e_1) &= -g(e_1, e_1, e_1) \\ 2g(e_1, e_1, e_1) &= 0 \\ g(e_1, e_1, e_1) &= 0 \end{aligned}$$

  - But what about $g(e_1, e_1, e_2)$? We could apply (23) and get $g(e_1, e_2, e_1)$, right? So it appears that we would just be shrinking two options into one. Technically, this is true, but what's more important is that applying (12) again yields the same thing, meaning that $g(e_1, e_1, e_2) = g(e_1, e_2, e_1) = 0$.
  - And thus, since V has dimension 2 but g takes three vectors, any argument submitted to g will always be linearly dependent. Thus, $g = 0$ and, in fact, the space of antisymmetric maps on V^3 has dimension 0.
- Note: It's not always a rule that $V^{\otimes m} \cong S^m V \oplus \Lambda^m V$.

• Mathematically, there's a more natural object to work with than symmetric and antisymmetric maps.

- Wedge powers and symmetric powers!
- Given V and $n \in \mathbb{N}$, we can construct $S^n V$ and $\Lambda^n V$. $(S^n V)^*$ is symmetric polylinear maps taking n arguments from V. $(\Lambda^n V)^*$ is antisymmetric polylinear maps taking n arguments from V.

• How about a concrete way to see these? We can relate them to tensor powers.

- Take a tensor power $V^{\otimes n}$, then look at those tensors which are symmetric and antisymmetric under permutation.
- Example: Let V be the same as before. Then $V^{\otimes 2}$ has dim $= 4$.
  - Take as basis elements for $S^2 V$ those that don't change when you change the coordinates.
  - Take as basis elements for $\Lambda^2 V$ those that flip sign when you change the coordinates.
  - In this case, the basis of $V^{\otimes 2}$ is $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$. The basis of $S^2 V$ will be $e_1 \otimes e_1, e_1 \otimes e_2 + e_2 \otimes e_1, e_2 \otimes e_2$. The basis of $\Lambda^2 V$ will be $e_1 \otimes e_2 - e_2 \otimes e_1$. Notice that these bases are identical (up to scaling) with those in Serre (1977) and those produced by applying the **symmetrization** and **alternation** operators to the basis of $V^{\otimes 2}$.
- $S^2 V$ and $\Lambda^2 V$ direct sum because the dimensions match and they don't intersect, so we're good to go!
- Everything we're doing is representations, so $g(v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n$.

• Relating this to something we've seen, but that's a little confusing.

- The product notation is suggestive for symmetric vectors; you can commute $e_1 \cdot e_2 \in S^2 V$, for instance.
- This allows us to, for example, shrink $e_1 \otimes e_1$ to $2e_1^2$[4], but $e_1 \otimes e_2 + e_2 \otimes e_1$ only to $e_1 \cdot e_2$.
- Note that $e_1 \wedge e_2 = e_1 \otimes e_2 - e_2 \otimes e_1$ by definition.
- Fact/exercise: Let V be a vector space of dimension n. V^* is the dual space, but it is also a function space. If $V = \mathbb{R}^k$, its a space of *functions from the blackboard.*
  - Note that $(\Lambda^k V)^* = \Lambda^k V^*$.

[4]Why the 2 coefficient??

  - S^nV^* is homogeneous polynomials of degree n.
  - You can take higher degree polynomials and just keep pushing through.
    - Ask about this??
  - Wedge powers now.
  - By convention, $\Lambda^0 V = F$ and $\Lambda^1 V = V$. But then you get to $\Lambda^2 V$ and $\Lambda^3 V$. They grow but then shrink down as the power approaches $\dim V$.
  - Truth: The dimension of wedge powers $\Lambda^i V$ is $\binom{k}{i}$ for $\dim V = k$. Figuring out why this is the case is another good exercise.
- An interesting connection between wedge powers and the determinant.
  - Let $V = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$.
  - Recall that $\Lambda^n V^*$ is the space of antisymmetric polylinear functions $V \times \cdots \times V \to F$ taking n arguments from V, and it has a single basis vector $e^1 \wedge \cdots \wedge e^n$.
  - Let $v_1 = \sum a_{i1}e_i$, $v_2 = \sum a_{i2}e_i$, etc.
  - Let $f \in \Lambda^n V^*$, so that f is an alternating polylinear map that takes n arguments.
  - Since f is polylinear, we have that
$$f(v_1, \dots, v_n) = \sum_{i_1,\dots,i_n=1}^{n} a_{i_1 1} \cdots a_{i_n n} f(e_{i_1}, \dots, e_{i_n})$$
  - Because of antisymmetry, we need only look at elements where the indices are all different. Thus, the above equals
$$\sum_{\sigma \in S_n} a_{\sigma(1)1} \cdots a_{\sigma(n)n} f(e_{\sigma(1)}, \dots, e_{\sigma(n)})$$
  - Additionally, $f(e_{\sigma(1)}, \dots, e_{\sigma(n)}) = (-1)^\sigma f(e_1, \dots, e_n)$ for any $\sigma \in S_n$. Moreover, $f(e_1, \dots, e_n) \in \mathbb{C}$ by definition, so define a constant $\lambda := f(e_1, \dots, e_n)$. Thus, the above equals
$$\lambda \sum_{\sigma \in S_n} a_{\sigma(1)1} \cdots a_{\sigma(n)n}$$
  - But the term following the λ is just the determinant of the $n \times n$ matrix (a_{ij}). Thus, all said,
$$f(v_1, \dots, v_n) = \lambda \det(v_1 \mid \cdots \mid v_n)$$
  - Implication: Wedge powers are something like the determinant.
    - In particular, because $\Lambda^n V^*$ has only a single basis vector as mentioned above, $f = \lambda e^1 \wedge \cdots \wedge e^n$. It follows that $e^1 \wedge \cdots \wedge e^n = \det$.
  - Takeaway: Wedge powers are something interesting; there's a reason to study them.
- The basis of the wedge powers consists of wedge monomials $e_{j_1} \wedge \cdots \wedge e_{j_i}$. Moreover, no need to have the same list twice, so choose some way of indexing them, e.g., increasing indexes.
  - This is why we do *increasing* bases! There's no particular reason, it's just an arbitrary way of making sure we don't do the same thing twice! We could just as well choose decreasing or any other means of guaranteeing that we don't have duplicates.
- Now let's relate all of this exterior and symmetric product stuff back to representation theory.
  - Let $V = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$.
  - Let $G \circlearrowright V$ via the homomorphism $G \to GL(V) \cong GL_n(\mathbb{C})$.

- Focusing more on the *matrix* aspect this time, note that under this homomorphism, $g \mapsto A_g$ subject to the homomorphism constraints.
- Consider the set $\{A_{g_1}, \dots, A_{g_k}\}$ of all matrices in the image of the homomorphism. If we transpose all of them, will they still obey the homomorphism constraints?
  - Nope!
  - Indeed, if we do this, we'll get in trouble. More specifically, transposition is not a representation because $A_{g_1}^T A_{g_2}^T \neq A_{g_1 g_2}^T = A_{g_2}^T A_{g_1}^T$.
- It's the same story with inverses.
- *However*, combining the two operations, we get

$$(A_{g_1 g_2}^T)^{-1} = (A_{g_1}^T)^{-1}(A_{g_2}^T)^{-1}$$

  - This is exactly when we take a representation and then go to the dual[5].
- This will be on next week's homework!
- Takeaway: This is an application of $\Lambda^j V^*$ to representation theory, $j \neq k, n$.

- Another relation: An application of $\Lambda^n V^*$ to representation theory.
  - Suppose we have a representation $G \circlearrowright V$ that we want to flatten into $G \circlearrowright \mathbb{C}$. How can we turn a relation between a group of matrices into a relation between a group of numbers?
  - Use the determinant!
  - Indeed, we already know that

$$\det(A_e) = 1 \qquad \det(A_{g_1 g_2}) = (\det A_{g_1})(\det A_{g_2}) \qquad \det\left(A_{g^{-1}}\right) = \det(A_g)^{-1}$$

  - In particular, we make formal the transition $G \to GL_j(\mathbb{C}) \to \mathbb{C}$ with the **top wedge power** $\Lambda^n V^*$.

- A last note.
  - Don't think that we're limited to top wedge powers.
  - Recall that we can define tensor products of matrices via the Kronecker product. Well, we can prove that

$$A_{g_1 g_2}^{\otimes 2} = A_{g_1}^{\otimes 2} A_{g_2}^{\otimes 2}$$

    and the like as well!
  - Similarly, we can define Λ^2 of a matrix.
  - We'll get into some weird Kronecker product stuff again, but we can sort through it.

- Plan for Friday and next time.
  - Prove the theorem that every representation is a sum of irreducible representations.
  - He will use projectors.
  - Then a horror story.
  - Then associative algebra.

[5]Relation to MATH 20510 when we discussed dual matrices and pullbacks of matrices.

2.4 Office Hours

10/5:

- Problem 2a:
  - $(V \oplus W) \otimes (V \oplus W) \stackrel{?}{=} V \otimes V \oplus V \otimes W \oplus W \otimes V \oplus W \otimes W$.
  - Check linearity in all terms and then with universal property. Check antisymmetric, linear, injective, surjective; dimensions are the same, so no need to check *both* injectivity and surjectivity (surjectivity is easier to check). We can go to basis to check various properties; we can't use a basis to write the map, but we can use bases to check surjectivity and the like.
- Problem 3a:
  - Bezout and Gauss's lemma is good to learn on my own. Put polynomials in each variable. Throw some stuff about this shit into my answers.
  - Relearn polynomial division.
  - $(1,1,1,1)$, $(1,1,-1,-1)$, $(1,-1,1,-1)$, and $(1,-1,-1,1)$.
  - This is a symmetric matrix.
  - The upper-left and lower-right blocks of this matrix match; so do the lower-left and upper-right.
  - When the eigenvalue is equal to zero, the determinant is equal to zero. So look for eigenvectors to calculate eigenvalues, and then just express the determinant as a product of these.
- Problem 3b:
  - Corresponding eigenvalue is $\sum_{i=1}^n x_i z^{i-1}$.
  - Can I use representation theory to do this? What group has a multiplication table like this? $\mathbb{Z}/n\mathbb{Z}$. The elements of $\mathbb{Z}/n\mathbb{Z}$ are of the form $\{1, \zeta, \dots, \zeta^{n-1}\}$.
  - If that's an eigenvector, then it's a subrepresentation; it is a space that is fixed under the action of the matrix.
  - Other eigenvectors: $(1,1,1)$, $(1, z^2, z)$.
  - We don't need to do induction or anything fancy like that; we can just do dots. As long as your argument is complete and clear, you're good.
- Problem 4a:
  - See FH 1.3. Standard rep, not wedge. Treat τ, σ (generators of the action) on the basis vectors.
  - If both fix, it's the trivial; if one flips, you have alternating; if both flip, you have standard.
  - $(2,1) \oplus (1,1,1)$. Use problem 2.
  - See FH Exercise 1.2??
  - The action of τ on this basis vector can be computed:
    $$\tau(\alpha \wedge \beta) = 1\alpha \wedge \beta$$
  - Having obtained an eigenvalue of 1, we can rule out the standard representation.
- Problem 4b:
  - $\{\alpha \otimes \alpha \otimes \alpha, \alpha \otimes \alpha \otimes \beta + \alpha \otimes \beta \otimes \alpha + \beta \otimes \alpha \otimes \alpha, \beta \otimes \beta \otimes \beta\}$
- Problem 5a:
  - Consider an alternate basis $f_1, \dots, f_n$ and dual basis $f^1, \dots, f^n$. Consider the element $f_1 \otimes f^1 + \cdots + f_n \otimes f^n \in V \otimes V^\vee$. We want to prove that it equals the one asked about in the question.

- Under the isomorphism to $\text{Hom}(V,V)$, we send $e_1 \otimes e^1$ to $[v \mapsto e^1(v)e_1]$. More generally, we end $e_i \otimes e^i$ to $[v \mapsto e^i(v)e_i]$. Adding all these maps together yields the map $[v \mapsto e^1(v)e_1 + \cdots + e^n(v)e_n]$, which is just the identity $1 \in \text{Hom}(V,V)$, regardless of basis.

- Problem 5b:
  - Example:
  $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to ae_1 \otimes e^1 + be_1 \otimes e^2 + ce_2 \otimes e^1 + de_2 \otimes e^2$$
  - Evaluating this gives
  $$e^1(ae_1) + e^2(be_1) + e^1(ce_2) + e^2(de_2) = a + d$$
  since it's only when the indices match (i.e., along the diagonal) that we get a nonzero value.

- Problem 5c:
  - P should have a block-diagonal matrix corresponding to the decomposition $V = W \oplus W^0$. P is the identity on $\text{Im}(P)$. So if our basis is vectors spanning W and then vectors spanning W^0, the matrix should be the identity and then the zero matrix. That should do the trick. How rigorous does this need to be?
  - Let $e_1, \dots, e_k$ be an orthonormal basis of $\text{Im}(P)$. Extend this basis to an orthonormal basis $e_1, \dots, e_n$ of V.

- Problem 5d:
  - Trivial representation: All $g \in G$ get mapped to $1 \in GL(V)$.
  - Part (a) gives us the identity in $\text{Hom}(V,V)$.
  - So we have $\rho : G \to GL(V)$.
  - Is any line acceptable? Span of the identity function? Rudenko: It depends on V. It has *infinitely many* trivial sub representations.
  - Example: $G \circlearrowright \mathbb{C}^2$. with $\rho(g) = I_2$.
  - Dual representation: Defined analogously to the $\text{Hom}_F(V,W)$ representation. We also need an inverse.

- Psets will likely get easier; right now, we have to relearn a lot of old stuff and we are being challenged with harder problems. As the questions become more based on course content and thus will get easier.

- He'll do hard PSets, easy exams, and everythign is curved; he agrees that this is a hard pset, and probably harder than necessary.

2.5 S Chapter 1: Generalities on Linear Representations

From Serre (1977).

Section 1.5: Tensor Product of Two Representations

10/4:

- **Tensor product** (of V_1, V_2): The vector space W that (a) is furnished with a map $V_1 \times V_2 \to W$ sending $(x_1, x_2) \mapsto x_1 \cdot x_2$ and (b) satisfies the following two conditions.
  (i) $x_1 \cdot x_2$ is bilinear.
  (ii) If (e_{i_1}) is a basis of V_1 and (e_{i_2}) is a basis of V_2, the family of products $e_{i_1} \cdot e_{i_2}$ is a basis of W.

  Denoted by $\boldsymbol{V_1 \otimes V_2}$.
  - It can be shown that such a space exists and is unique up to isomorphism (see proof here).

- This definition allows us to say some things quite expediently. For example, (ii) implies that

$$\dim(V_1 \otimes V_2) = \dim(V_1) \cdot \dim(V_2)$$

- **Tensor product** (of ρ^1, ρ^2): The representation $\rho : G \to GL(V_1 \otimes V_2)$ defined as follows for all $s \in G$, $x_1 \in V_1$, and $x_2 \in V_2$, where $\rho^1 : G \to GL(V_1)$ and $\rho^2 : G \to GL(V_2)$ are representations. *Given by*

$$[\rho_s^1 \otimes \rho_s^2](x_1 \cdot x_2) = \rho_s^1(x_1) \cdot \rho_s^2(x_2)$$

- A more formal write up of the matrix translation of this definition.
  - Let (e_{i_1}) be a basis for V_1, and let (e_{i_2}) be a basis for V_2.
  - Let $r_{i_1j_1}(s)$ be the matrix of ρ_s^1 with respect to this basis, and let $r_{i_2j_2}(s)$ be the matrix of ρ_s^2 with respect to this basis.
  - It follows that

$$\rho_s^1(e_{j_1}) = \sum_{i_1} r_{i_1j_1}(s)e_{i_1} \qquad \rho_s^2(e_{j_2}) = \sum_{i_2} r_{i_2j_2}(s)e_{i_2}$$

  - Therefore,

$$[\rho_s^1 \otimes \rho_s^2](e_{j_1} \cdot e_{j_2}) = \sum_{i_1,i_2} r_{i_1j_1}(s)r_{i_2j_2}(s)e_{i_1} \cdot e_{i_2}$$

    and

$$\mathcal{M}(\rho_s^1 \otimes \rho_s^2) = (r_{i_1j_1}(s)r_{i_2j_2}(s))$$

- Aside on quantum chemistry to come back to later; I can't quite connect the dots yet.

Section 1.6: Symmetric Square and Alternating Square

- Herein, we investigate the tensor product when $V_1 = V_2 = V$.
- Let (e_i) be a basis of V.
- Define the automorphism $\theta : V \otimes V \to V \otimes V$ by

$$\theta(e_i \cdot e_j) = e_j \cdot e_i$$

  for all 2-indices (i, j).
- Properties of θ.
  - Since θ is linear, it follows that

$$\theta(x \cdot y) = y \cdot x$$

    for all $x, y \in V$.
    - Implication: θ is independent of the chosen basis (e_i)!
  - $\theta^2 = 1$, where 1 is the identity map on $V \otimes V$.
- Assertion: $V \otimes V$ decomposes into

$$V \otimes V = S^2(V) \oplus \Lambda^2(V)$$

  - Rudenko: We do not have to worry about proving this... yet, at least.
- **Symmetric square representation**: The subspace of $V \otimes V$ containing all elements z satisfying $\theta(z) = z$. *Denoted by* $\boldsymbol{S^2V}$, $\boldsymbol{S^2(V)}$, $\boldsymbol{\mathbb{S}^2V}$, $\textbf{Sym}^{\mathbf{2}}\boldsymbol{(V)}$.
  - Basis: $(e_i \cdot e_j + e_j \cdot e_i)_{i \leq j}$.

  - Rudenko: How do we know everything is linearly independent? Well, when we add two linearly independent vectors out of a set, the sum is still linearly independent from everything else!
  - Example when $\dim V = 2$: The basis of $V \otimes V$ is $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$, where all four of these vectors are linearly independent. So naturally, the basis of the corresponding symmetric square representation — which is $2e_1 \otimes e_1, e_1 \otimes e_2 + e_2 \otimes e_1, 2e_2 \otimes e_2$ — will still be a linearly independent list of vectors.
- Dimension: If $\dim V = n$, then

$$\dim S^2(V) = \frac{n(n+1)}{2}$$

- **Alternating square representation**: The subspace of $V \otimes V$ containing all elements z satisfying $\theta(z) = -z$. *Denoted by* $\boldsymbol{\Lambda^2 V}$, $\boldsymbol{\Lambda^2(V)}$, $\textbf{Alt}\,\boldsymbol{^2(V)}$.
  - Basis: $(e_i \cdot e_j - e_j \cdot e_i)_{i<j}$.
  - Dimension: If $\dim V = n$, then

$$\dim \Lambda^2(V) = \frac{n(n-1)}{2}$$

2.6 FH Appendix B: On Multilinear Algebra

From Fulton and Harris (2004).

Section B.1: Tensor Products

10/5:

- **Tensor product** (of V, W over F): A vector space U equipped with a bilinear map $V \times W \to U$ sending $v \times w \to v \otimes w$ that is universal, i.e., for any bilinear map $\beta : V \times W \to Z$, there is a unique linear map from $U \to Z$ that takes $v \otimes w \mapsto \beta(v, w)$. *Denoted by* $\boldsymbol{V \otimes W}$, $\boldsymbol{V \otimes_F W}$.
  - The so-called *universal property* determines the tensor product up to canonical isomorphism.
- One construction of $V \otimes W$: From the basis $\{e_i \otimes f_j\}$.
  - This construction is **functorial**, implying that linear maps from $f : V \to V'$ and $g : W \to W'$ determine a linear map $f \otimes g : V \otimes W \to V' \otimes W'$, namely that defined by $f \otimes g : v \otimes w \to fv \otimes gw$.
- Definition of the **n-fold tensor product**.
- **Multilinear** (map): A map from a Cartesian product $V_1 \times \cdots \times V_n$ of vector spaces to a vector space U such that when all but one of the factors V_i are fixed, the resulting map from $V_i \to U$ is linear.
- Properties of the tensor product.
  1. *Commutativity*:
$$V \otimes W \cong W \otimes V$$
by $v \otimes w \mapsto w \otimes v$.
  2. *Distributivity*:
$$(V_1 \oplus V_2) \otimes W \cong (V_1 \otimes W) \oplus (V_2 \otimes W)$$
by $(v_1, v_2) \otimes w \mapsto (v_1 \otimes w, v_2 \otimes w)$.
  3. *Associativity*:
$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W) \cong U \otimes V \otimes W$$
by $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w) \mapsto u \otimes v \otimes w$.
- **Tensor power** (of V to n): The tensor product defined as follows. *Denoted by* $\boldsymbol{V^{\otimes n}}$. *Given by*

$$V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}$$

  - Convention: $V^{\otimes 0} = F$.
- Analogous construction of the tensor product for generalized algebras and modules.

Section B.2: Exterior and Symmetric Powers

- **Alternating** (multilinear map): A multilinear map β such that $\beta(v_1, \dots, v_n) = 0$ whenever $v_i = v_j$ for some $i, j \in [n]$.
  - Implication: $\beta(v_1, \dots, v_n)$ changes sign whenever two of the vectors are interchanged.
    - Follows from the definition and the **standard polarization**.
  - Implication:
    $$\beta(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = (-1)^\sigma \beta(v_1, \dots, v_n)$$
    for all $\sigma \in S_n$.
- **Standard polarization**: The equality
  $$\beta(v, w) + \beta(w, v) = \beta(v + w, v + w) - \beta(v, v) - \beta(w, w) = 0 - 0 - 0 = 0$$
- **Exterior powers** (of V): The vector space U equipped with an alternating multilinear map $V \times \cdots \times V \to \Lambda^n V$ sending $v_1 \times \cdots \times v_n \mapsto v_1 \wedge \cdots \wedge v_n$ that is universal, i.e., for any alternating multilinear map $\beta : V^n \to Z$, there is a unique linear map from U to Z that takes $v_1 \wedge \cdots \wedge v_n \mapsto \beta(v_1, \dots, v_n)$. *Denoted by* $\boldsymbol{\Lambda^n V}$.
  - Convention: $\Lambda^0 V = F$.
- Quotient space construction of the exterior powers.
- Projecting from $V^{\otimes n} \to \Lambda^n V$: Define $\pi : V^{\otimes n} \to \Lambda^n V$ by
  $$\pi(v_1 \otimes \cdots \otimes v_n) = v_1 \wedge \cdots \wedge v_n$$
- Basis for the exterior powers.
- There is a canonical linear map $\Lambda^a V \otimes \Lambda^b W \to \Lambda^{a+b}(V \oplus W)$, which takes $(v_1 \wedge \cdots \wedge v_a) \otimes (w_1 \wedge \cdots \wedge w_b) \mapsto v_1 \wedge \cdots \wedge v_a \wedge w_1 \wedge \cdots \wedge w_b$.
  - This determines (how??) an isomorphism
    $$\Lambda^n(V \oplus W) \cong \bigoplus_{a=0}^{n} \Lambda^a V \otimes \Lambda^{n-a} W$$
  - This isomorphism plus induction on n can justify (how??) the basis for $\Lambda^n V$ as the increasing indices.
- **Symmetric** (multilinear map): A multilinear map β such that $\beta(v_1, \dots, v_n)$ is unchanged when any two factors are interchanged, that is
  $$\beta(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = \beta(v_1, \dots, v_n)$$
  for all $\sigma \in S_n$.
- **Symmetric powers** (of V): The vector space U equipped with a symmetric multilinear map $V \times \cdots \times V \to S^n V$ sending $v_1 \times \cdots \times v_n \mapsto v_1 \cdot \ldots \cdot v_n$ that is universal, i.e., for any symmetric multilinear map $\beta : V^n \to Z$, there is a unique linear map from U to Z that takes $v_1 \cdot \ldots \cdot v_n \mapsto \beta(v_1, \dots, v_n)$. *Denoted by* $\boldsymbol{S^n V}$.
  - Convention: $S^0 V = F$.
- Quotient space construction of the symmetric powers.
  - Quotient out all $v_1 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$, that is, those elements of $V^{\otimes n}$ in which σ permutes two successive factors. How does this work??

- Projecting from $V^{\otimes n} \to S^n V$: Define $\pi : V^{\otimes n} \to S^n V$ by

$$\pi(v_1 \otimes \cdots \otimes v_n) = v_1 \cdot \ldots \cdot v_n$$

- Basis for the symmetric powers.
  - It follows from the basis construction that $S^n V$ can be regarded as the space of homogeneous polynomials of degree n in the variable e_i, since each element is of the form $e_{i_1} \cdot \ldots \cdot e_{i_n}$ and we can add them.
- Canonical isomorphism:

$$S^n(V \oplus W) \cong \bigoplus_{a=0}^{n} S^a V \otimes S^{n-a} W$$

- More on $\Lambda^n V, S^n V$ as subspaces of $V^{\otimes n}$.
  - We inject $\iota : \Lambda^n V \to V^{\otimes n}$ with

$$\iota(v_1 \wedge \cdots \wedge v_n) = \sum_{\sigma \in S_n} (-1)^\sigma v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

    - This relates to Rudenko's note that $v_1 \wedge v_2 = v_1 \otimes v_2 - v_2 \otimes v_2$!
    - There are some more advanced notes on the implications of ι; $[\iota \circ \pi/n!](V^{\otimes n}) = V^{\otimes n}$ is brought up.
  - We inject $\iota : S^n V \to V^{\otimes n}$ with

$$\iota(v_1 \cdot \ldots \cdot v_n) = \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

    - More, related advanced notes; includes the $1/n!$ thing again.
- **Wedge product**: The function $\Lambda^m V \otimes \Lambda^n V \to \Lambda^{m+n} V$ defined as follows. *Denoted by* $\wedge$. *Given by*

$$(v_1 \wedge \cdots \wedge v_m) \otimes (v_{m+1} \wedge \cdots \wedge v_{m+n}) \mapsto v_1 \wedge \cdots \wedge v_m \wedge v_{m+1} \wedge \cdots \wedge v_{m+n}$$

- Properties of the wedge product.
  1. *Associativity*:
  $$(v_1 \wedge v_2) \wedge v_3 = v_1 \wedge (v_2 \wedge v_3) = v_1 \wedge v_2 \wedge v_3$$
  2. *Skew-commutativity*:
  $$v_1 \wedge v_2 = -v_2 \wedge v_2$$
- Note that both of the above properties hold in higher-dimensional cases as well.
- Commutativity of the products.

$$\begin{tikzcd} \Lambda^m V \otimes \Lambda^n V \arrow[r, "\wedge"] \arrow[d, hook, "\iota\otimes\iota"'] & \Lambda^{m+n} V \arrow[d, hook, "\iota"] \\ V^{\otimes m} \otimes V^{\otimes n} \arrow[r, "f_1"'] & V^{\otimes(m+n)} \end{tikzcd}$$

(a) Wedge product.

$$\begin{tikzcd} S^m V \otimes S^n V \arrow[r, "\cdot"] \arrow[d, hook, "\iota\otimes\iota"'] & S^{m+n} V \arrow[d, hook, "\iota"] \\ V^{\otimes m} \otimes V^{\otimes n} \arrow[r, "f_2"'] & V^{\otimes(m+n)} \end{tikzcd}$$

(b) Symmetric product.

Figure 2.3: Commutative diagram, wedge and symmetric products.

- f_1 is defined by

$$(v_1 \otimes \cdots \otimes v_m) \otimes (v_{m+1} \otimes \cdots \otimes v_{m+n}) \mapsto \sum_{\sigma \in G} (-1)^\sigma v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)} \otimes v_{\sigma(m+1)} \otimes \cdots \otimes v_{\sigma(m+n)}$$

  where G is the subgroup of S_{m+n} preserving the order of the subsets $\{1, \dots, m\}$ and $\{m+1, \dots, m+n\}$.
- f_2 is defined analogously.

- The above mappings all commute with linear maps of vector spaces.
  - Example: Our definition $g(v \otimes w) = gv \otimes gw$ could be redrawn as $[g \circ \otimes](v, w) = [\otimes \circ g](v, w)$, where the latter $g : (v, w) \mapsto (gv, gw)$ by abuse of notation.
- Tensor, exterior, and symmetric algebras.

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