MATH 26700 (Introduction to Representation Theory of Finite Groups) Notes

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Weeks

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Week 1

Objects of Study and Vocab

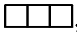
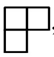
1.1 Motivating and Defining Representations

9/27:

- Rudenko would happily approve my final substitution, but it's not his call; it's Boller's.
- HW will be due every week on Wednesday or thereabouts.
  - Submit in paper in a mailbox, location TBA.
  - First HW due next Wednesday.
- Midterm eventually and an in-class final.
- Grading scheme in the syllabus.
- OH not available MW after class (Rudenko has to run to something else), but F after class, we can ask him anything.
  - Regular OH MTh, time TBA.
- There is no specific book for the course.
  - First 8 lectures come from Serre (1977); amazing book but very concise; gets confusing later on. Most lectures are made up by Rudenko.
- Course outline.
  1. Character theory: Beautiful, not too hard.
  2. Non-commutative algebra: More abstract/general approach to the same thing.
  3. Advanced topics, S_n.
- This course's focus: Representations of finite groups in finite dimensions over $\mathbb{C}$.
- This course is for math-inclined people (not quite physics) and lays the foundation for all other Rep Theory.
  - The ideas would be presented in a very different way in Physics Rep Theory.
- We can always ask questions and stop him to correct mistakes during class.
- Why we care about representations.
  - Start with a group G, finite. For example, let $G \equiv S_1$.
  - People started to play with S_4 (permutations of roots of a polynomial of degree 4) in Galois theory.

        - Galois theory primer: Consider a polynomial like $x^4 + 3x + 1 = 0$; the roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ satisfy tons of equations, e.g., $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$ since 1 is the x^0 term.
    - But groups also occur in much more natural places, e.g., isometries of $\mathbb{R}^3$ that preserve a tetrahedron.
    - S_4 is also orientation-preserving isometries of $\mathbb{R}^3$ that preserve a cube.
    - Many things lead to the same group!
    - Theory of abstract groups developed far later than any of these perspectives; was developed to unify them.
- Recall group actions: Take $G, X = \{x_1, \dots, x_n\}$ both finite. We want $G \circlearrowright X$, which is a homomorphism $A : G \to S_n$.
- What can we do now?
    - We can look at orbits, which are smaller pieces.
    - We can look at the stabilizer.
    - We can identify orbits with cosets.
    - If we understand all possible subgroups, we understand all possible actions.
- This story is not boring, but it's simplistic.
- Rudenko doesn't assume we remember everything (phew!).
- Main definition (general to start, then we simplify).
- **Group representation** (of G on V): A group homomorphism $G \to GL(V)$, for G a group, V a finite-dimensional vector space over some field $\mathbb{F}$ with basis $\{e_1, \dots, e_n\}$, and $GL(V)$ the set of isomorphic linear maps $L : V \to V$. *Denoted by* $\boldsymbol{\rho}$.
    - Recall that $GL(V) \cong GL_n(\mathbb{F})$ is the set of all $n \times n$ invertible matrices.
- For every element $g \in G$, $g \mapsto \rho(g) = A_g$. Essentially, you're mapping to elements that satisfy certain equations.
    - For example, $A_e = E_n$, $A_{g_1 g_2} = A_{g_1} A_{g_2}$, and $A_{g^{-1}} = {A_g}^{-1}$.
    - Thus, representations are a "concrete way to think about groups."
    - If you don't understand abstract group G, let us compare it to a group that we do understand! Like a group can *act* via S_n, we can *represent* a group in a vector space.
- In this course, G is finite, $\mathbb{F} = \mathbb{C}$, and V is finite dimensional.
    - This is the most simple case, but also a very interesting one. The theory is much, much easier, so we can get much more complicated, but this is a good place to start.
    - We could make G compact, but we're not gonna go that far.
- Examples to get an idea of what's going on.
    1. $\deg \rho = 1$ (means $\dim V = 1$). Then $\rho : G \to GL_1(V) \cong \mathbb{C}^\times$. The codomain is referred to as the **character** of the group??
        - An example group homomorphism $S_n \to \mathbb{C}^\times$ is the sign function $\sigma \to \text{sign}(\sigma) = \{\pm 1\}$.
        - Another example is the **trivial representation**, $G \to \mathbb{C}^\times$ and $g \mapsto 1$.
    2. Smallest one: Let $G = S_3$. The structure is already pretty rich, and this will be part of the homework.
        - **Trivial representation** again.
        - **Alternating representation**.

- **Standard representation**.
- **Regular representation**.

- **Trivial representation**: The representation $\rho : G \to GL(V)$ sending $g \mapsto 1$ for all $g \in G$. *Denoted by* , **(3)**.
  - The boxes notation is too much of a detour to explain now.
  - Note that $1 \in GL(V)$ is the identity map on V!
  - Also note that you may define *a* trivial representation on any space V, but *the* trivial *irreducible* representation is necessarily on $\mathbb{C}$. (We'll define **irreducible** and deal much more with this specific trivial representation later.)
- **Alternating representation**: The representation $\rho : G \to GL(V)$ sending $g \mapsto \operatorname{sign}(g)$ for all $g \in G$. *Denoted by* , $\mathbf{(1,1,1)}$.
- **Standard representation**: The representation $\rho : S_n \to GL(V)$ sending $\sigma \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)})$, where $V = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_1 + \cdots + x_n = 0\}$ is a $(n-1)$-dimensional vector space. *Denoted by* , $\mathbf{(2,1)}$.
  - A 2D representation like rotating a triangle.
  - This gives something with real numbers.
  - Example: $S_3 \circlearrowright V$ by $\sigma((x_1, x_2, x_3)) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$.
- **Regular representation**: The representation $\rho : G \to \operatorname{Hom}(\mathbb{C}^n)$ defined by $g \mapsto \sigma_g$, where $G = \{g_1, \dots, g_n\}$, $\{e_{g_1}, \dots, e_{g_n}\}$ is a basis of $\mathbb{C}^n$, $\cdot$ is the group action of $\rho(G) \circlearrowright \mathbb{C}^n$ defined by $\rho(g) \cdot e_g = e_{gg_i}$, and $\sigma_g(e_{g_i}) = \rho(g) \cdot e_g = e_{gg_i}$.
  - This is a permutation of vectors.
  - Thus, for S_3, it will already be 6-dimensional (it's very high dimensional).
- How do we know that representation theory is tractable? Sure, we can define all these things, but how do we know that it will lead anywhere? Here's an example.
  - Let $G = \mathbb{Z}/2\mathbb{Z} = \{e, g\}$, $V = \mathbb{C}^n$, $\rho : G \to GL_n(\mathbb{C}) \cong GL(\mathbb{C}^n)$, and $A = \rho(g)$ an $n \times n$ matrix over $\mathbb{C}$. Since $g^2 = e$, we know for example that $A^2 = E_n$.
  - But how do we find the matrices A? If we look at eigenvalues of A, there are only two possibilities: ± 1. The structure of A can be very complicated with Jordan normal form and all that, but in fact, these are the **semisimple matrices**, so it's not that bad.
  - Since $A^2 = E$, we know that $(A - E)(A + E) = 0$. Consider $(A - E) : V \to V$. Naturally, it has $\operatorname{Ker}(A - E)$ and $\operatorname{Im}(A - E)$. In this case, Rudenko claims that $\operatorname{Ker}(A - E) \cap \operatorname{Im}(A - E) = \{0\}$. Here's the proof to back up that claim.

    Proof. Let $v \in \operatorname{Ker}(A - E) \cap \operatorname{Im}(A - E)$ be arbitrary. Since $v \in \operatorname{Im}(A - E)$, there exists $w \in V$ such that $v = (A - E)w = Aw - w$. Since $v \in \operatorname{Ker}(A - E)$, we have $(A - E)v = 0$, so $Av = v$. It follows that $A(Aw - w) = Aw - w$ but also $A(Aw - w) = Ew - Aw = w - Aw$. Thus,

    $$\begin{aligned} Aw - w &= w - Aw \\ 2Aw &= 2w \\ Aw &= w \end{aligned}$$

    But then $w \in \operatorname{Ker}(A - E)$, so $v = (A - E)w = 0$. ☐
  - This combined with the fact that every vector in a vector space is in either the image or the kernel of a linear map[1] implies that $V = \operatorname{Ker}(A - E) \oplus \operatorname{Im}(A - E)$.

[1]See Theorem 3.6 of Axler (2015).

- Let $\mathrm{Ker}(A - E)$ have basis $e_1, \ldots, e_k$ and let $\mathrm{Im}(A - E)$ have basis $e_{k+1}, \ldots, n$; then all A are of the following form.

$$\begin{array}{cc}
 & \begin{array}{cccccc} 1 & & k & k+1 & & n \end{array} \\
\begin{array}{c} 1 \\ \\ k \\ k+1 \\ \\ n \end{array} & \left[\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{array}\right]
\end{array}$$

  - Next time, we will discuss sums of representations, of which this is an example of the theory.

- The same kind of thing, **simple representations**, happens with all finite groups?? This is where we're going. It's not rocket science; in fact, we'll see it next week.

- Last thing for today: A remarkable story.
  - The story of representation theory started quite different.
  - A beautiful theorem that we can prove now!
  - Frobenius determinant.
  - Think of $G = \{g_1, \ldots, g_n\}$. Picture its multiplication table.
  - In every row and column, you see each element once.
  - Let's associate to the multiplication table an actual determinant in the linear algebra sense. Consider elements $x_{g_1}, \ldots, x_{g_n}$. Define the $n \times n$ matrix $(x_{g_i g_j})$. Take its determinant. It will be a polynomial in n variables, i.e., an element of the ring $\mathbb{Z}[x_{g_1}, \ldots, x_{g_n}]$.
  - Example: Consider

$$\begin{vmatrix} e & g \\ g & e \end{vmatrix}$$

    - The determinant is $x_e^2 - x_g^2 = (x_e - x_g)(x_e + x_g)$.
  - Example: $G = \mathbb{Z}/3\mathbb{Z}$.
    - If the elements are e, g, g^2 and we map these, respectively, to variables a, b, c, we get the matrix

$$\begin{bmatrix} e & g & g^2 \\ g & g^2 & e \\ g^2 & e & g \end{bmatrix} \mapsto \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

    - The determinant is $3abc - a^3 - b^3 - c^3 = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac) = (a + b + c)(a + \zeta b + \zeta^2 c)(a + \zeta^2 b + \zeta c)$ where $\zeta \in \mu_3$ is a third root of unity, i.e., we have that $\zeta^3 = 1$.
  - Frobenius's theorem: If G is a finite group and we take this Frobenius determinant, then this determinant is equal to ${P_1}^{d_1} \cdots {P_k}^{d_k}$, where $P_1, \ldots, P_k$ are irreducible polynomials in $x_g, \ldots, x_{g_j}$, $\deg P_i = d_i$, and k is the number of conjugacy classes.
  - Example: Take S_3; we'll get a polynomial of degree $|S_3| = 6$ but the Frobenius determinant $FD = (x_{g_1} + \cdots + x_{g_k})(x_{g_1} \pm \cdots)(\text{some pol. of deg } 2)^2$
  - The proof is remarkable and deep and uses what would become character theory. These polynomials are related to representations and the number of simplest irreducible representations. The theory that came out came as a way to understand this miracle. We'll forget FD's for now, but then come back and prove it later.

1.2 Key Definitions and Category Theory Primer

9/29:
- OH: TW 4:30 or 5:00 most likely; he will confirm later.
- Today: Definitions in greater generality.
- As before, let G be a finite group and V be a finite-dimensional vector space.
- Goal of this course: Understand representations of G, that is...
  - Homomorphisms $\rho : G \to GL(V) \cong GL_n(\mathbb{C})$;
  - That send $g \mapsto A_g \in GL_n(\mathbb{C})$;
  - And satisfy $A_e = E$, $A_{g_1}A_{g_2} = A_{g_1g_2}$, and $A_{g^{-1}} = {A_g}^{-1}$.
- What are some things we might want to do?
  - Build new representations from old? Investigate and/or classify irreducible representations?
  - Before we can see if any of this works or not, we need a ton of definitions: Sum, equality, etc.
- Rudenko will start today's lecture with some general thoughts on the **category** of representations.
  - Categories are things that now permeate math.
- **Category**: A *class* (not a set) of *objects* (some things; you don't know anything about them), and then a bunch of properties.

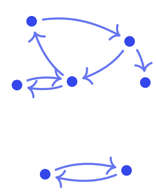


Figure 1.1: The general structure of a category.

  - Objects a, b in category C are denoted by $a, b \in \mathbf{Ob}(\boldsymbol{C})$.
  - There are also **morphisms** between the objects. These are drawn as arrows and lie in $\mathrm{Hom}(a, b)$.
  - There is also composition $\circ : \mathrm{Hom}(a, b) \times \mathrm{Hom}(b, c) \to \mathrm{Hom}(a, c)$.
    - This notation rigorously defines composition, i.e., as a binary operation on functions.
  - Properties.
    1. Associativity.
    2. Existence of a unit: For any object a, there exists $\mathrm{id}_a \in \mathrm{Hom}(a, a)$ such that any morphism pre- or post-composed to this identity yields the same morphism.
       - Example: If $f \in \mathrm{Hom}(a, b)$, then $\mathrm{id}_b \circ f = f = f \circ \mathrm{id}_a$.
  - Rudenko: So a category is basically two pieces of data and a bunch of properties.
- Examples of categories.
  - Category of sets and maps between them.
  - Category of vector spaces over $\mathbb{F}$ where $\mathrm{Ob}(C)$ is the vector spaces and $\mathrm{Hom}(V, W)$ is filled with *linear* maps because you don't just want maps — you want maps that respect the structure.
  - Category of groups where $\mathrm{Hom}(G_1, G_2)$ is the set of group homomorphisms.
  - Category of topological spaces and continuous maps.
  - Category of abelian groups.
  - Trivial category and the identity map; thus, categories need not be chonky.

- Comments on category theory.
  - We'll see some pretty significant category theory at the end of the course.
  - We'll see categories in every course we take; some people try to avoid them. Rudenko doesn't want to go into the material in depth, but he wants to use language from it.
  - Surprisingly, even under the stripped-down of axioms of category theory, you can say quite a lot.
  - Why any of this discussion of category theory matters: If you know the basics of category theory, you can guess the definitions of direct sum, equality, etc. for representations.
- **Category of representations**. *Denoted by* $\mathbf{Rep}_{\boldsymbol{G}}$.
- Take two G-representations V, W; how do we define a map between them?
  - Recall that V, W are vector spaces.
- **Morphism** (of G-representations): A map $f : V \to W$ such that...
  1. f is linear;
  2. f respects the structure of the representations; explicitly, for every $g \in G$, $\rho_W(g) \circ f = f \circ \rho_V(g)$[2].
- $\mathbf{Hom}_{\boldsymbol{G}}\boldsymbol{(V, W)}$: The set of all morphisms of G-representations from V to W.
- On constraint 2, above: This condition is summarized via a **commutative diagram**.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho_V(g)"'] \arrow[dr, phantom, "\circlearrowleft"] & W \arrow[d, "\rho_W(g)"] \\ V \arrow[r, "f"'] & W \end{tikzcd}$$

Figure 1.2: Commutative diagram, morphisms.

  - Commutative diagrams are very category-theory-esque things.
- That was a very abstract definition. Let's make it concrete.
  - Suppose you have a pair of representations $V = \mathbb{C}^n, W = \mathbb{C}^m$, and we have our map F between them given by an $m \times n$ matrix.
  - Let $\rho_V(g) = A_g$ be an $n \times n$ matrix, and let $\rho_W(g) = B_g$ be an $m \times m$ matrix.
  - Then $FA_g = B_gF$.
- More examples of representations.
  1. An interesting example: Let's look at $S_3 \circlearrowright V_{\text{perm}} = \mathbb{C}^3$, a **permutation representation**.
     - For all $\sigma \in S_3$, $\rho(\sigma) : (x_1, x_2, x_3) \mapsto (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$.
  2. There's also the trivial representation $S_3 \circlearrowright V_{(3)} = \mathbb{C}$ defined by $\rho(\sigma) : x \mapsto x$.
- Are the above 2 representations related?
  - Yes! We can, in fact, find a *morphism* between them.
  - In particular, define $f : V_{(3)} \to V_{\text{perm}}$ by $f(x) = (x, x, x)$.
    - Since permuting 3 of the same thing does nothing, the commutativity of Figure 1.2 holds. Therefore, f is a morphism of G-representations as defined above.

[2]Recall that the object, $\rho_V(g)$ is a linear map! Thus, it can be composed with other linear maps like f.

- We may also explicitly confirm that f is a morphism as follows.

$$f[\rho_{(3)}(\sigma)(x)] = f(x) = (x, x, x) = \rho_{\text{perm}}(\sigma)((x, x, x)) = \rho_{\text{perm}}(\sigma)[f(x)]$$

  - Is f **reversible**?
    - Is "reversible" the right word??
  - Define $\tilde{f} : V_{\text{perm}} \to V_{(3)}$ by $\tilde{f} : (x_1, x_2, x_3) \mapsto x_1 + x_2 + x_3$.
    - Since addition is commutative, the commutativity of Figure 1.2 holds.
    - More explicitly,

$$\begin{aligned} f[\rho_{\text{perm}}(\sigma)((x_1, x_2, x_3))] &= f((x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})) \\ &= x_{\sigma(1)} + x_{\sigma(2)} + x_{\sigma(3)} \\ &= x_1 + x_2 + x_3 && \text{Commutativity of addition} \\ &= f((x_1, x_2, x_3)) \\ &= \rho_{(3)}(\sigma)[f((x_1, x_2, x_3))] \end{aligned}$$

  - Takeaway: The existence of maps between representations is interesting.

- Next question: How do we define an **isomorphism** of two representations?
- **Isomorphism** (of G-representations): A morphism of G-reps that is an isomorphism of vector spaces.
- Category theory helps us again here because it generalizes the concept of an isomorphism!
  - If $f : V \to W$ and $g : W \to V$ are category-theoretic morphisms, then the constraints $f \circ g = \text{id}_W$ and $g \circ f = \text{id}_V$ make f and g into category-theoretic *iso*morphisms, regardless of what V and W might be.
  - Back in the context of representations, let $f : V \to V$ be an isomorphism of vector spaces. Then we do indeed have $\rho_V(g) \circ f = f \circ \rho_V(g)$, as we would hope from category theory!
- Recall the condition $FA_g = B_gF$. Supposing F is an isomorphism (and thus has an inverse), we get $FA_gF^{-1} = B_g$ as our new condition.
  - Essentially, we can do *simultaneous conjugation* of all matrices.
  - As per usual with isomorphisms, we get to *change bases*.
  - Essentially, we can represent the nice permutation representation in a very nasty basis but still have it be valid.
- Many other notions (e.g., direct sum) will not be explained by Rudenko, but we can read about them!
- However, we'll do a few more.
- A representation sitting inside another: a **subrepresentation**.
- **Subrepresentation** (of V): A subspace $W \subset V$ such that for all $w \in W$ and $g \in G$, we have that $\rho_V(g)W \subset W$, where V is a G-representation with $\rho_V : G \to GL(V)$.
  - Many people will just write the critical condition as $gW \subset W$.
- Subrepresentations in category theory: We have another commutative diagram.
- Example: The trivial representation, the standard representation, and (of course) the **zero representation** are subrepresentations of the permutation representation.
- **Zero representation**: The representation $\rho : G \to GL(\{0\})$ sending $g \mapsto 1$ for all $g \in G$. *Denoted by* $\mathbf{(0)}$.

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d, "\rho_V(g)"'] & V \arrow[d, "\rho_V(g)"] \\ W \arrow[r, hook] & V \end{tikzcd}$$

Figure 1.3: Commutative diagram, subrepresentations.

- What about representations that don't have subrepresentations?
- **Simple** (representation): A G-representation V that has only two subrerpesentations: (0) and V. *Also known as* **irreducible**, **irreps**.
- Example irreducible representations: Line in $\mathbb{C}^2$, triangle in $\mathbb{C}^2$, A_5 and dodecahedron in $\mathbb{C}^3$.
- Notion of a direct sum.
- **Direct sum** (of V_1, V_2): The G-rep associated with the space $V_1 \oplus V_2 = \{(v_1, v_2) \mid v_1 \in V_1, v_2 \in V_2\}$, where $\rho_{V_1 \oplus V_2}(g)(v_1, v_2) = (\rho_{V_1}(g)v_1, \rho_{V_2}(g)v_2)$. *Denoted by* $\boldsymbol{V_1 \oplus V_2}$.
  - The matrix of $\rho_{V_1 \oplus V_2}(g)$ is the following block matrix.

$$\rho_{V_1 \oplus V_2}(g) = \left[\begin{array}{c|c} \rho_{V_1}(g) & 0 \\ \hline 0 & \rho_{V_1}(g) \end{array}\right]$$

- Example: $V_{\text{perm}} = V_{(3)} \oplus V_{(2,1)}$, with $\mathbb{C}^3 = \mathbb{C} \oplus \mathbb{C}^2$ where

$$\mathbb{C} \cong \langle (1,1,1) \rangle \qquad\qquad \mathbb{C}^2 \cong \langle (x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0 \rangle$$

  - The decomposition is into simple representations.
  - Relate this to the fact that the JCF of any 3×3 permutation matrix has at most a 1-block and a 2-block, if not three 1-blocks. There will always be one 1D subspace on which the permutation matrix is an identity, i.e., $\operatorname{span}(1,1,1)$, and a 2D orthogonal complement!
  - As a fun and simple exercise, prove that there is no line fixed under the standard representation.
- A simple and important theorem to prove next week.
- Theorem: Let G be a finite group and $\mathbb{F} = \mathbb{C}$. Then...
  1. There are finitely many irreps $V_1, \dots, V_s$ up to isomorphism.
     - Later on, we will see that s is equal to the number of conjugacy classes.
  2. For every G-rep V, there exists a unique $n_1, \dots, n_s \geq 0$ such that $V \cong {V_1}^{n_1} \oplus \cdots \oplus {V_s}^{n_s}$.
- This theorem tells us that if we want to study rep theory, we want to study irreps (which can be kind of complicated) because if we understand them, everything breaks down into them.
- Examples.
  1. $G = \mathbb{Z}/2\mathbb{Z} = S_2$.
     - $V_1 = \mathbb{C}e$ with $ge = e$ and $V_{-1} = \mathbb{C}e$ with $ge = -e$.
     - It follows that $V \cong {V_1}^{n_1} \oplus {V_{-1}}^{n_{-1}}$.
     - We get a diagonal matrix with only 1s and -1s, just like the example from last time!
  2. $G = S_3$.
     - $V_{(3)}, V_{(1,1,1)}, V_{(2,1)}$.

  - $GL_5(\mathbb{F}_4)$.
  - Proven in an elementary way in Section 1.3 of Fulton and Harris (2004), which we have to read for the HW; will be useful for later in the course's HW.

- Plan: Next time, we'll talk about some more abstract stuff like the tensor products of vector spaces.
  - Tensor products are something we should read up on now! The definition is hard and abstract.
  - Then he'll prove the above theorem.

1.3 S Chapter 1: Generalities on Linear Representations

From Serre (1977).

10/3:

- Part I (what we'll be covering) is written for quantum chemists, and thus gives proofs "as elementary as possible, using only the definition of a group and the rudiments of linear algebra" (Serre, 1977, p. v).
  - Recall the story about Serre and his wife, the chemist, who needed to explain group theory and rep theory to her students.
- Indeed, although the book seemed very fast when I first looked at it two years ago, it reads much more easily now and has enough context for most anyone who is comfortable with group theory and theoretical linear algebra.

Section 1.1: Definitions

- Definitions of $\boldsymbol{GL(V)}$, **invertible square matrix**, and **finite group**.
- **Linear representation**: See class notes. *Also known as* **group representation**.
  - Serre (1977) will frequenty write ρ_s for $\rho(s)$.
- **Representation space** (of G): The vector space V corresponding to the linear representation $\rho : G \to GL(V)$ of G. *Also known as* **representation**.
  - The latter term is a self-identified "abuse of language" (Serre, 1977, p. 3).
- "For most applications, one is interested in dealing with a *finite number of elements* x_i of V, and can always find a subrepresentation of $V\ldots$ of finite dimension, which contains the x_i; just take the vector subspace generated by the images $\rho_s(x_i)$ of the x_i" (Serre, 1977, p. 4).
- **Degree** (of a representation): The dimension of the representation space of this representation.
- To give a representation **in matrix form** is to give a set of invertible matrices that are isomorphic to the group elements.
- Important converse: Given invertible matrices satisfying the appropriate homomorphism identities, there is a corresponding group that these matrices represent.
- **Similar** (representations of G): Two representations $\rho : G \to GL(V)$ and $\rho' : G \to GL(V')$ of G for which there exists a linear isomorphism $\tau : V \to V'$ such that

$$\tau \circ \rho(s) = \rho'(s) \circ \tau$$

  for all $s \in G$. *Also known as* **isomorphic**.
  - Equivalent definition (in matrix form): There exists T invertible such that $R'_s = TR_sT^{-1}$.
  - Isomorphic representations have the same degree.

Section 1.2: Basic Examples

- Degree 1 representation: A homomorphism $\rho : G \to \mathbb{C}^*$, where $\mathbb{C}^*$ denotes the roots of unity (all $z \in \mathbb{C}$ with $|z| = 1$).
  - The fact that every $s \in G$ has *finite* order by assumption is what permits this representation.
- **Unit representation**: See class notes. *Also known as* **trivial representation**.
- **Regular representation**: The representation $\rho : G \to GL(V)$ defined by $s \mapsto [e_t \mapsto e_{st}]$ for all $s \in G$, where V has basis $(e_t)_{t\in G}$.
  - $\deg \rho = |G|$.
  - $e_s = \rho_s(e_1)$.
    - Implication: The images of e_1 under the ρ_s's form a basis of V, i.e., $\{\rho_s(e_1) \mid s \in G\}$ is a basis of V.
  - Converse of above: If W is a representation of G containing a vector w such that $\{\rho_s(w) \mid s \in G\}$ forms a basis of W, then W is isomorphic to the regular representation V via $\tau : V \to W$ defined by $\tau(e_s) = \rho_s(w)$.
- **Permutation representation** (associated with X): The representation $\rho : G \to GL(V)$ defined by $s \mapsto [e_x \mapsto e_{s\cdot x}]$ for all $s \in G$, where $G \circlearrowright X$ a finite set and V has a basis $(e_x)_{x\in X}$.

Section 1.3: Subrepresentations

- Definition of **subrepresentation**.
  - Example: Trivial representation $\mathbb{C}(x, \dots, x)$ is a subrepresentation of the regular representation.
- Definitions of **direct sum** (of vector spaces) and **kernel** (of a linear map).
- **Complement** (of a subspace $W \leq V$): Any $(n - m)$-dimensional subspace U that...
  1. Satisfies $W \oplus U = V$;
  2. Intersects W trivially;

  where $n = \dim V$ and $\dim W = m \leq n$.
  - This means that a single subspace can have multiple complements!
    - Only one **orthogonal** complement, but many general *complements*.
    - Example: Consider a line through the origin in $\mathbb{R}^2$; any other line through the origin is a complement of it!
  - It follows that there is a bijection between the complements W' of W in V and the projections p of V onto W (since non-orthogonal complements require non-orthogonal projections).
- **Projection** (of V onto W associated with the decomposition $V = W \oplus W'$): The mapping that sends each $x \in V$ to its component $w \in W$. *Denoted by* $\boldsymbol{p}$.
  - Consequence: The two properties defining a p are...
    1. $\text{Im}(p) = W$;
    2. $p(x) = x$ for all $x \in W$.
  - Consequence: These two properties also imply that if p is a projection onto $W \leq V$, then $V = W \oplus \text{Ker}(p)$.
- If a representation has a subrepresentation, then some complement of this subrepresentation is also a subrepresentation.

Theorem 1. *Let $\rho : G \to GL(V)$ be a linear representation of G in V and let W be a vector subspace of V stable under G. Then there exists a complement W^0 of W in V which is stable under G.*

Proof 1 (limited conditions). Let p be the projection of V onto W that corresponds to some arbitrary complement of W in V. To begin, we may legally — albeit with little motivation — form the average p^0 of the conjugates of p by the elements of G:

$$p^0 := \frac{1}{|G|}\sum_{t\in G} \rho_t \cdot p \cdot \rho_t^{-1}$$

We now seek to prove that p^0 is a projection by showing that it satisfies the two properties of a "p." First, notice that by assumption, every ρ_t (and thus ρ_t^{-1}) preserves W. This combined with the fact that $p(V) = W$ implies that $p^0(V) = W$, as desired. Additionally, for any $x \in W$ and $t \in G$, we know by property (2) of a p and the fact that $p_t^{-1}(x) \in W$ that $p \cdot p_t^{-1}(x) = p_t^{-1}(x)$. Applying p_t to both sides of this equation yields $[p_t \cdot p \cdot p_t^{-1}](x) = x$. Hence, $p^0(x) = x$, as desired. Thus, p^0 is a projection of V onto W, associated with some complement W^0 of W.

So that we can make a substitution later, we will now prove that

$$\rho_s \cdot p^0 = p^0 \cdot \rho_s$$

for all $s \in G$. Pick such an s. Then

$$\rho_s \cdot p^0 \cdot \rho_s^{-1} = \frac{1}{|G|}\sum_{t\in G} \rho_s \cdot \rho_t \cdot p \cdot \rho_t^{-1} \cdot \rho_s^{-1} = \frac{1}{|G|}\sum_{st\in G} \rho_{st} \cdot p \cdot \rho_{st}^{-1} = p^0$$

so we can post-compose both sides of the above equation with ρ_s to yield the final result. This line here should make it clear why we needed to form a projection like p^0.

We now have all of the tools we need to prove that W^0 is stable under G. To do so, it will suffice to show that for all $x \in W^0$ and $s \in G$, we have $\rho_s(x) \in W^0$. Let $x \in W^0$ and $s \in G$ be arbitrary. Since $x \in W^0$, $p^0(x) = 0$ by definition. This combined with the above commutativity rule implies that $p^0 \cdot \rho_s(x) = \rho_s \cdot p^0(x) = \rho_s(0) = 0$. But the only way that p^0 could map $\rho_s(x)$ to 0 is if $\rho_s(x) \in W^0$, as desired. □

Proof 2 (orthogonal complement). Let W^0 be the orthogonal complement of W, and endow V with a **scalar product** $(x \mid y)$ to turn it into an inner product space. Replace $(x \mid y)$ with the new inner product $\sum_{t\in G}(\rho_t x \mid \rho_t y)$. Now, if it wasn't already, the inner product is invariant under ρ_s for all s, i.e., for s arbitrary, we have

$$(\rho_s x \mid \rho_s y) = (x \mid y)$$

This means that vectors that were orthogonal before ρ_s is applied to V, stay orthogonal after ρ_s is applied to V. In particular, since ρ_s preserves W by hypothesis, all vectors orthogonal to W (i.e., all vectors in W^0) stay orthogonal to W (i.e., stay in W^0) after ρ_s is applied. Thus, W^0 is stable under ρ_s as well. □

- Note: Since we can define a scalar product that is invariant under ρ_s, if (e_i) is an orthonormal basis of V, then the matrix of ρ_s with respect to this basis is a **unitary** matrix.
- Consequence of the second, stronger proof: The representations W and W^0 determine the representation V.
  - This allows us to rigorously say that the representation $V = W \oplus W^0$.
  - If W, W^0 are given in matrix form by R_s, R_s^0, then $W \oplus W^0$ is given in matrix form by

$$\left(\begin{array}{c|c} R_s & 0 \\ \hline 0 & R_s^0 \end{array}\right)$$

- We can extend this method of directly summing representations to an arbitrary finite number of them.

Section 1.4: Irreducible Representations

- Definition of **irreducible** representation.
- Fact: Each nonabelian group possesses at least one irreducible representation with deg ≥ 2.
  - Proven later.
- Irreducible representations construct all representations via the direct sum.

  Theorem 2. *Every representation is a direct sum of irreducible representations.*

  Proof. We induct on $\dim(V)$.

  Suppose $\dim(V) = 0$. Since 0 is the direct sum of the empty family of irreducible representations, the theorem is vacuously true.

  Suppose $\dim(V) \geq 1$. We divide into two cases (V is irreducible and V is reducible). In the first case, we are done. In the second case, $V = V' \oplus V''$ for some $V' \perp V''$ (see Theorem 1). Since $\dim(V') < \dim(V)$ and $\dim(V'') < \dim(V)$ by definition, the induction hypothesis implies that V' and V'' are direct sums of irreducible representations. Therefore, the same is true of V. ☐

- Fact: The direct-sum decomposition is not necessarily unique.
  - Counterexample: If $\rho_s = 1$ for all $s \in G$, then there are a plethora of decompositions of a vector space into a direct sum of lines.
- Fact: The number of W_i (in a direct sum decomposition $V = W_1 \oplus \cdots \oplus W_k$) that are isomorphic to a given irreducible representation *does not* depend on the chosen decomposition.
  - Proven later.

Week 2

Constructing Representations

2.1 The Tensor Product

10/2:

- Before lecture, I chatted with a few people about tensor products and the exterior and symmetric powers.
  - Patrick: A **tensor** $v \otimes w$ is just an element of a vector space, indexed differently than in a column.
  - Raman: There is no canonical way to transform tensors into column vectors.
- Course logistics.
  - OH: T 5:30-6:30(+) and W 5:30-6:30(+). We can also meet one-on-one.
  - HW is due Thursdays at midnight.
- Today: Constructing new representations from old.
  - Rudenko will skim through tensor products really quickly.
- Reminder: Last time, we talked about how representation theory is really quite simple. If G is a finite group and $F = \mathbb{C}$, there exist a finite set $V_1, \dots, V_s$ of irreps up to isomorphism, and every finite-dimensional representation $V \cong V_1^{n_1} \oplus \cdots \oplus V_s^{n_s}$.
- If V is a representation of G, then there are loads of things we can do with it.
  - We can construct the dual representation V^*.
  - We can construct the representation $V \otimes V$.
  - We can construct symmetric powers.
  - We can construct wedge powers.
  - There are more, but this is enough for now.
- Even when we take a very simple group and representation, there are some very interesting things that can fall out.
  - Example: If you take the symmetric powers of S_3, as in the homework, you get something really interesting.
- Now, we go to linear algebra.
- Let V, W be vector spaces over a field F. How do we produce a new vector space out of these?
- $\mathrm{Hom}_F(V, W)$ is the vector space of linear maps $F : V \to W$!
  - $\dim = (\dim V)(\dim W)$.

- Can we make $\mathrm{Hom}_F(V,W)$ into a representation of G? Yes!

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "\rho_V(g)"'] & W \arrow[d, "\rho_W(g)"] \\ V \arrow[r, "gL"'] & W \end{tikzcd}$$

Figure 2.1: Commutative diagram, linear maps space representation.

  - Suppose that V,W are G-reps, which gives us $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$.
  - Suppose also that we have $L \in \mathrm{Hom}_F(V,W)$.
  - Now infer from the commutative diagram that it will work to define $gL = \rho_W(g) \circ L \circ \rho_V(g)^{-1}$.
  - This is pretty standard.
- Recall that there is a different space $\mathrm{Hom}_G(V,W)$ of morphisms of G-representations (see Figure 1.2 and the associated discussion).
  - This is a very very small subspace of $\mathrm{Hom}_F(V,W)$.
- Special case of the above construction: **Dual representation**.
  - Consider $\mathrm{Hom}_F(V,F)$. This the **dual vector space**.
  - Basic fact 1: Let $e_1,\dots,e_n$ be a basis of V. Then V^* has a corresponding basis $e^1,\dots,e^n$ known as its **dual basis**.
    - Computing coordinates already depends on a basis, and having bases is super nice.
    - Corollary: $\dim V = \dim V^*$.
    - This is the first time **canonical** comes into linear algebra. Canonical (nobody understands what it means) basically means that something doesn't depend on choices.
    - In particular, V,V^* are isomorphic because they have the same dimension, but for no more natural reason. They can be the same representation, or they can be different.
  - Basic fact 2: If V is finite-dimensional, then $(V^*)^* \cong V$. The formula for this isomorphism is canonical, because it does not depend on a choice of basis. In particular, choose the map $V \to (V^*)^*$ sending v to the map sending $\varphi \in V^*$ to $\varphi(v)$.
  - If V is infinite dimensional, none of this is true and you are in the realm of functional analysis.
  - Ok, so all of this was good information about the dual *space*, but what is the dual *representation*?? Does it matter, and do we need to know for now?
    - Defined below in the notes on the reading from Fulton and Harris (2004).
- **Dual vector space** (of V): The vector space defined as follows, given that V is a vector space over F. *Denoted by* $\boldsymbol{V^*}$. *Given by*

$$V^* = \mathrm{Hom}_F(V,F)$$

- **Dual basis** (of V^* to $e_1,\dots,e_n$): The basis defined as follows for $i=1,\dots,n$, where $e_1,\dots,e_n$ is a basis of V. *Denoted by* $\boldsymbol{e^1,\dots,e^n}$. *Given by*

$$e^i(x_1e_1 + \cdots + x_ne_n) = x_i$$

- We now move onto the tensor product.
  - The tensor product is very hard to understand. If you learn about it and you feel you don't understand it, that's typical; nobody understands it at first.
  - For now, we'll discuss two ways of thinking about tensor products that won't bring us any comfort.

- Let V, W be two vector spaces over a field F.
- Abstract definition of the tensor product.
  - We have discussed maps from $V \to W$, but there is another related space.
  - Indeed, we can look at the space of bilinear maps from $V \times W \to F$.
    - Example: A map $f : V \times W \to F$ that satisfies the constraints $f(\lambda v, w) = \lambda f(v, w)$, $f(v_1 + v_2, w) = f(v_1, w) + f(v_2, w)$, and likewise for the second index. Recall that this is a **bilinear map**.
  - Let V have basis $e_1, \dots, e_n$ and W have basis $f_1, \dots, f_m$.
  - Notice that every bilinear map f can be defined as a linear combination of the $f(e_i, f_j)$. In other words, the $f(e_i, f_j)$ form the basis of a function space.
    - This "bilinear maps space" has dimension nm.
  - Now, one way to understand a tensor product: Is this "bilinear maps space" actually some other space? It is! It is $(V \otimes W)^*$.
  - Bilinear maps are linear maps from where? From $V \otimes W$!
- **Bilinear** (map): A function $f : V \times W \to Z$ that satisfies the following constraints, where V, W, Z are vector spaces over F, $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, and $\lambda \in F$. *Constraints*

$$
\begin{aligned}
f(v_1 + v_2, w) &= f(v_1, w) + f(v_2, w) & \qquad f(\lambda v, w) &= \lambda f(v, w) \\
f(v, w_1 + w_2) &= f(v, w_1) + f(v, w_2) & \qquad f(v, \lambda w) &= \lambda f(v, w)
\end{aligned}
$$

- We now look at a much more elementary definition of the tensor product.
- Explicit definition of the tensor product.
  - Start off with the hug, easy-to-work-with vector space with basis consisting of pairs of elements $(v, w) \in V \times W$. For example, even if V, W are one dimensional, this is like all pairs of real numbers $(1, 0)$, $(2, 0)$, (π, e), etc. *as basis vectors*; it's huge.
    - Then, we quotient it by the space of all elements satisfying the relations $\lambda(v, w) = (\lambda v, w) = (v, \lambda w)$, $(v_1 + v_2, w) = (v_1, w) + (v_2, w)$, and the like. These elements will be linear combinations of basis vectors of the following form: $\lambda(v, w) - (\lambda v, w)$, $\lambda(v, w) - (v, \lambda w)$, and $(v_1 + v_2, w) - (v_1, w) - (v_2, w)$.
    - This forces these relationships to be true. For example, in the final quotient space, we can still construct the element $\lambda(v, w) - (\lambda v, w)$. But its inclusion in the quotiented-out subspace will imply that in the quotient space, $\lambda(v, w) - (\lambda v, w) = 0$. It follows from here that $\lambda(v, w) = (\lambda v, w)$, as desired.
  - What do these relations do?
    - Essentially, they allows us to treat tensor multiplication much like real multiplication, endowing the operation with distributivity, etc.
    - For example, the rule $(v_1 + v_2, w) = (v_1, w) + (v_2, w)$ becomes, in tensor product notation, $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$.
  - Here's an example of this construction.
    - Let $V = W$ be the one-dimensional vector space over the finite field $F_2 = \mathbb{Z}/2\mathbb{Z}$.
    - Thus, the elements of V are $\{0, 1\}$ (which is, literally, all linear combinations $a0 + b1$ where $a, b \in F_2$ as well; this hearkens back to V's definition as an F_2-module).
    - Then the easy-to-work-with vector space we're talking about is the 4-dimensional **free** vector space $U = \operatorname{span}(0 \otimes 0, 0 \otimes 1, 1 \otimes 0, 1 \otimes 1)$.

- Note that in this space, for example, $(0+1)\otimes 0 \neq 0\otimes 0 + 1\otimes 0$; representing the basis as column vectors, this is equivalent the obvious observation that

$$\begin{bmatrix}0\\0\\1\\0\end{bmatrix} \neq \begin{bmatrix}1\\0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$

- But we want such relationships to hold true in our conceptual "tensor product space." Thus, we quotient it by the subspace spanning all elements of the form $(a+b)\otimes c - a\otimes c - b\otimes c$.
- By direct computation, this subspace is $\text{span}(0\otimes 0, 0\otimes 1)$:

$$\begin{aligned}
(0+0)\otimes 0 - 0\otimes 0 - 0\otimes 0 &= -0\otimes 0 & (0+0)\otimes 1 - 0\otimes 1 - 0\otimes 1 &= -0\otimes 1\\
(0+1)\otimes 0 - 0\otimes 0 - 1\otimes 0 &= -0\otimes 0 & (0+1)\otimes 1 - 0\otimes 1 - 1\otimes 1 &= -0\otimes 1\\
(1+1)\otimes 0 - 1\otimes 0 - 1\otimes 0 &= 0\otimes 0 & (1+1)\otimes 1 - 1\otimes 1 - 1\otimes 1 &= 0\otimes 1
\end{aligned}$$

  - Note that once we've considered $(a+b)\otimes c$, we don't need to consider $(b+a)\otimes c$ because of the commutativity of addition in V. That is, it is axiomatic that $a+b=b+a$ for all $a,b\in V$.
  - Additionally, in the last line above, we are using the facts that $1+1=2=0$ in F_2 and $a\otimes b + a\otimes b = 2a\otimes b = 0$ in any F_2-module to simplify the expressions.
  - Furthermore, note that since $-1=1\in F_2$, $-0\otimes 0 = 1(0\otimes 0) = 0\otimes 0 \in V\otimes V$.
- Similarly, the subspace corresponding to $a\otimes(b+c) - a\otimes b - a\otimes c$ is $\text{span}(0\otimes 0, 1\otimes 0)$. Thus, altogether, we quotient out the subspace $X = \text{span}(0\otimes 0, 0\otimes 1, 1\otimes 0)$. This leaves us with a 1-dimensional $V\otimes V$, as expected for the tensor product of two one-dimensional vector spaces. It is interesting to note that the one vector we didn't quotient out $(1\otimes 1)$ is analogous to $e_1\otimes e_1$ since $e_1\in V$ might as well be defined $e_1 := 1$.
- Now let's see how well this quotienting worked. First off, a bit of notation: let $\pi : U \to V\otimes V$ be the projection $\pi : v \mapsto v + X$, and denote elements $\pi(v_1\otimes v_2) \in V\otimes V$ by $v_1 \otimes_\pi v_2$ for now to differentiate them from elements of U.
- Let $(0+1)\otimes_\pi 0 = (0+1)\otimes 0 + X$ be an element of the quotient space $V\otimes V$. Certainly, the elements $0\otimes_\pi 0$ and $1\otimes_\pi 0$ are also elements of this quotient space. Moreover, there is no reason we can't form the linear combination $(0+1)\otimes_\pi 0 - 0\otimes_\pi 0 - 1\otimes_\pi 0$. Indeed, when we do, we notice that this element lies in the quotiented-out subspace X. Thus,

$$(0+1)\otimes_\pi 0 - 0\otimes_\pi 0 - 1\otimes_\pi 0 = [(0+1)\otimes 0 - 0\otimes 0 - 1\otimes 0] + X = 0 + X = 0$$

- But

$$(0+1)\otimes_\pi 0 - 0\otimes_\pi 0 - 1\otimes_\pi 0 = 0 \quad\Longrightarrow\quad (0+1)\otimes_\pi 0 = 0\otimes_\pi 0 + 1\otimes_\pi 0$$

  as desired.
- Note that this construction also gives us nice things like $0\otimes_\pi 0 = 0$, $0\otimes_\pi 1 = 0$, etc. which were not true in U!
- It should not be concluded, though, that all we need to quotient out of U for any V is $\text{span}(0\otimes 0, 0\otimes v, v\otimes 0)$ for every $v\in V$; indeed, $V=\mathbb{R}$, for example, will require us to quotient out elements such as $4\otimes 7 - 2\otimes 7 - 2\otimes 7$, which can't even be expressed as a single simple tensor.

- **Free** (vector space): A vector space that has a basis consisting of linearly independent elements.
  - Example: Think of $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ as a $\mathbb{C}$-module. A free version $F(V)$ of V is infinite dimensional with every $v\in V$ a linearly independent basis vector. Elements of $F(V)$ are of the form $a_1v_1 + \cdots + a_kv_k$ for $a_1,\dots,a_k\in\mathbb{C}$ and $v_1,\dots,v_k\in V$. If $u = v+w$ where $u,v,w\in V$ are all nonzero, then $u\neq v+w$ in $F(V)$ because they are all linearly independent basis vectors.

- Example: What we formally start with in the example above is the free F_2-module $V \times V$, *not* the Cartesian product vector space $V \times V$.
- A terrific explanation of free vector spaces is available here.

- These two definitions constitute a first approximation to what the tensor product is.
- Example tensor product space.
  - Suppose $V = \mathbb{C}e_1 + \mathbb{C}e_2$. We want to look at $V \otimes V$.
  - A priori[1], it's spanned by $(ae_1 + be_2) \otimes (ce_1 + de_2) = ace_1 \otimes e_1 + ade_1 \otimes e_2 + bce_2 \otimes e_1 + cde_2 \otimes e_2$.
  - Thus, $V_1 \otimes V_2$ has 4-element basis $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$.
- Takeaway: What is true in general is that if V has basis $e_1, \dots, e_n$ and W has basis $f_1, \dots, f_m$, then $V \otimes W$ has basis $e_i \otimes f_j$ ($i = 1, \dots, n$ and $j = 1, \dots, m$).
- Having discussed the tensor product of vector spaces, let's think about the tensor product of *representations*.
  - Suppose $g : V \to V$ and $g : W \to W$.
    - We're starting to make notation sloppy.
  - How does $g : V \otimes W \to V \otimes W$? Well, we just send $v \otimes w \mapsto (gv) \otimes (gw)$.
    - Why is this map well-defined?
    - We invoke the **universal property of the tensor product operation**.
    - This guarantees us that given g — which is effectively a map from $V \times W \to V \otimes W$ as defined — there nevertheless exists a complete extension $\tilde{g} : V \otimes W \to V \otimes W$.
  - As a matrix, this map is pretty strange!
    - Example: Let $g : V \to V$ be a 2×2 matrix. What is the matrix of $g : V \otimes V \to V \otimes V$?
    - If
      $$\rho_V(g) = g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} =: A$$
      then we have
      $$\begin{aligned} g(e_1 \otimes e_1) &= ge_1 \otimes ge_1 \\ &= (ae_1 + ce_2) \otimes (ae_1 + ce_2) \\ &= a^2 e_1 \otimes e_1 + ace_1 \otimes e_2 + ace_2 \otimes e_1 + c^2 e_2 \otimes e_2 \end{aligned}$$
    - Evaluating similarly for all basis vectors, we get a very curious block matrix:
      $$\begin{array}{c|c} & \begin{array}{cccc} e_1 \otimes e_1 & e_1 \otimes e_2 & e_2 \otimes e_1 & e_2 \otimes e_2 \end{array} \\ \begin{array}{c} e_1 \otimes e_1 \\ e_1 \otimes e_2 \\ e_2 \otimes e_1 \\ e_2 \otimes e_2 \end{array} & \begin{bmatrix} a^2 & ab & ab & b^2 \\ ac & ad & bc & bd \\ ac & bc & ad & bd \\ c^2 & cd & cd & d^2 \end{bmatrix} \end{array} = \left[\begin{array}{c|c} aA & bA \\ \hline cA & dA \end{array}\right]$$
    - Notice how, for example, this takes the tensor $e_1 \otimes e_1$, represented as $(1, 0, 0, 0)$, to the tensor $a^2 e_1 \otimes e_1 + ace_1 \otimes e_2 + ace_2 \otimes e_1 + c^2 e_2 \otimes e_2$, represented as (a^2, ac, ac, c^2).
    - Does this construction imply a canonical way to convert from tensors to column vectors??
  - Classically, this is called the **Kronecker product** of two matrices.
  - People discovered all of this stuff before they unified it as tensor math.
- **Universal property of the tensor product operation**: For every bilinear map $h : V \times W \to Z$, there exists a *unique* linear map $\tilde{h} : V \otimes W \to Z$ such that $h = \tilde{h} \circ \otimes$.

[1] I.e., it follows from some logic. In particular, it follows from the logic that any element $v \in V$ is of the form $v = ae_1 + be_2$, so of course all $v \otimes v$ must be of the given form for choices of a, b, c, d.

$$\begin{tikzcd} V \times W \arrow[r, "\otimes"] \arrow[dr, "h"'] & V \otimes W \arrow[d, "\tilde{h}"] \\ & Z \end{tikzcd}$$

Figure 2.2: Universal property, tensor product operation.

Proof. See the solid explanation here. Alternatively, here's my write up.

Let $V = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$, $W = \mathbb{C}f_1 \oplus \cdots \oplus \mathbb{C}f_m$, Z, and $h : V \times W \to Z$ be arbitrary. Define $\tilde{h} : V \otimes W \to Z$ by

$$\tilde{h}(e_i \otimes f_j) := h(e_i, f_j)$$

for $i = 1, \dots, n$ and $j = 1, \dots, m$. Since a linear map is wholly defined by its action on the basis of its domain, this set of equations suffices to define $\tilde{h}$ on all of $V \otimes W$.

Existence: To prove that $\tilde{h}$ satisfies the "universal property," it will suffice to show that $h = \tilde{h} \circ \otimes$. Let $(v, w) \in V \times W$ be arbitrary, and suppose $v = \sum_{i=1}^n a_i e_i \in V$, and $w = \sum_{i=1}^n b_i f_i \in W$. Then

$$\begin{aligned} [\tilde{h} \circ \otimes](v, w) &= \tilde{h}(v \otimes w) \\ &= \sum_{i=1}^n \sum_{j=1}^m a_i b_i \tilde{h}(e_i \otimes f_i) \\ &= \sum_{i=1}^n \sum_{j=1}^m a_i b_i h(e_i, f_j) \\ &= h(v, w) \end{aligned}$$

as desired.

Uniqueness: Now suppose $\tilde{g} : V \otimes W \to Z$ also satisfies the "universal property," that is, $h = \tilde{g} \circ \otimes$. Then by definition,

$$\tilde{h}(e_i \otimes f_j) = h(e_i, f_j) = \tilde{g}(e_i \otimes f_j)$$

for $i = 1, \dots, n$ and $j = 1, \dots, m$. But since a linear map is wholly defined by its action on the basis of its domain, it follows that $\tilde{h} = \tilde{g}$, as desired. $\square$

- **Kronecker product** (of A, B): The matrix product defined as follows. *Denoted by* $\boldsymbol{A} \otimes \boldsymbol{B}$. *Given by*

$$A \otimes B = n \overset{n}{[A]} \otimes m \overset{m}{[B]} = \begin{array}{cc} & nm \\ nm & \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix} \end{array}$$

  - The Kronecker product is *not* commutative, but the matrices you get are related by conjugacy and by commuting the columns.
- Vector spaces of the same dimension are all alike, but vector space representations are very interesting. By the end of this course, we'll understand what's going on.
- How we understand tensor stuff: Look at the abstract definition, look at the concrete definition, look at 5 examples, and then go in a circle. Repeat again and again until it makes sense.
  - Rudenko is just trying to tell us all relevant words so that they will fit together later.
- Fact: If V, W finite-dimensional, $\mathrm{Hom}_F(V, W) \cong V \otimes W^*$.
  - Tensor products are very nice spaces from which to construct maps.

- Let's construct a reverse map, then.
- Take $\alpha \otimes w \in V^* \otimes W$, where $\alpha : V \to F$ by definition. Send $\alpha \otimes w$ to the map $v \mapsto \alpha(v)w$. This is a *canonical* map!! We can show that they span everything.
  - For example, if we want to choose $\alpha \otimes w$ mapping to the matrix with a 1 in the upper left-hand corner and zeroes everywhere else, let $w = e_1 \in W$ and let $\alpha = e^1 \in V^*$.
  - We can do similarly for all other such matrices, mapping this basis of $\mathrm{Hom}_F(V, W)$ to $e^i \otimes e_j$ ($i = 1, \dots, n$ and $j = 1, \dots, m$).
  - Note that this also allows us to define a (noncanonical) inverse map.
- This inverse map from $\mathrm{Hom}_F(V, W) \to V^* \otimes W$ is clearly a bit harder to work out.
- Hidden in this story is why trace is invariant under matrix conjugation, e.g., why $\mathrm{tr}\left(SAS^{-1}\right) = \mathrm{tr}(A)$.
  - If we take $\mathrm{Hom}_F(V, V)$, then this is isomorphic to $V^* \otimes V$.
  - There is a very natural map from these isomorphic spaces to F.
  - This map is defined by the trace (which sends $\mathrm{Hom}_F(V, V) \to F$), and $\alpha \otimes v \mapsto \alpha(v)$ (which sends $V^* \otimes V \to F$). We can prove this.
  - Moreover, this map is canonical, as well.
  - This is why the main property of the trace is that it's invariant under conjugation, i.e., because SAS^{-1} and A both map to the same element of $V^* \otimes V$. This fact is hidden in the story very nicely.

- Tensor products are hard, it will be a pain, we will understand them very well, but it will not be nice for now.
- Symmetric products and wedge powers will be discussed briefly next time.
  - There is a nice description in Serre (1977) that we can use for the homework.
- Extra homework: Please read about tensor products in whatever textbook you like, try some examples, and repeat.

2.2 Office Hours

10/3:

- Problem 2a:
  - $\Lambda^2 V$ is *exterior powers.*
  - The exact canonical isomorphism we need is briefly discussed on Fulton and Harris (2004, p. 473).
  - I.e., we have to construct isomorphisms between the structures that don't rely on the choice of any basis. Recall the classic example of $V \cong V^{**}$, as explained in the well-written MSE post "basic difference between canonical isomorphism and isomorphims." Recall that the isomorphism from $V \to V^*$ defined by sending each element of the basis of V to the corresponding element of the dual basis of V^* is *not* canonical because *it involves choosing bases.* Definitions of canonical maps are available in MATH20510Notes, p. 2.
  - From a quick look at this, it looks like the proof may be analogous to the classic middle-school algebra identity $(v + w)^2 = v^2 + vw + w^2$.
  - The second exterior power $\Lambda^2 V$ of a finite-dimensional vector space V is the dual space of the vector space of alternating bilinear forms on V. Elements of $\Lambda^2 V$ are called 2-vectors.
- Problem 2b:
  - $S^2 V$ is *symmetric powers.*
  - The exact canonical isomorphism we need is briefly discussed on Fulton and Harris (2004, p. 473).

- Problem 3a:
  - This is the determinant of the multiplication table, in relation to that theorem that you showed us at the end of the first class? Yep!
- Problem 3b:
  - So a circulant matrix is a matrix like the multiplication table from (a)? Yep!
  - Is $\zeta = e^{2\pi i/n}$? Sort of. It can be any n^{th} root of unity.
- Problem 4d:
  - We'll cover higher symmetric powers in class tomorrow.
  - However, it basically just means that we're now working with elements of the form $e_1 \otimes e_2 \otimes e_3 \in S^3V$ and on and on.
- Problem 5a:
  - Is $V^\vee = V^*$? Yes. This is "vee check," and is a notation that some people prefer.
- Problem 5b:
  - Is "tr" the trace function of the linear map corresponding to L? Yes.
  - What is L?
    - An element of $V \otimes V^*$ is a linear combination of elements of the form $v \otimes \alpha$, not necessarily just one of these "decomposable" products.
    - There is an isomorphism $V \otimes V^* \cong \text{Hom}(V)$.
    - Consider the matrix
      $$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
      It sends $e_1 \mapsto e_1$ and $e_2 \mapsto 0$. Thus, it is well-matched with $e_1 \otimes e^1$, which also grabs e_1 (with e^1) and sends it to e_1.
    - Consider the matrix
      $$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
      It sends $e_1 \mapsto 0$ and $e_2 \mapsto e_1$. Thus, it is well-matched with $e_1 \otimes e^2$, which also grabs e_2 (with e^2) and sends it to e_1.
    - In full,
      $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ae_1 \otimes e^1 + be_1 \otimes e^2 + ce_2 \otimes e^1 + de_2 \otimes e^2$$
    - This map *is* canonical! This is because the bases must be chosen to even begin talking about matrices.
    - If you change the matrix, the bases change, too??
    - Takeaway: We have to walk backwards from matrix to linear transformation to representation in $V \otimes V^*$ to a scalar in F.
- Problem 5c:
  - So trace of such a map is equal to the dimension of its image? Yes.

2.3 Wedge and Symmetric Powers

10/4:
- OH slightly later today at 5:45-6:45 PM.
- Recap: Last time, we built new reps from old.
  - This stuff can't be learned in 1.5 lectures; he can point us around, but we have to learn it ourselves.
- Tensor product review.
  - Given V, W, make $V \otimes_F W$.
  - This vector space is hard to describe directly, so we more often talk about its dual $(V \otimes W)^*$ because this is actually easier to describe.
  - If you want to work with $V \otimes W$ hands-on, you can do the following.
    - Start with the following easy-to-work-with vector space: The (probably infinite-dimensional) vector space where each $v \otimes w$ is a basis vector for all $v \in V$ and $w \in W$.
    - Then quotient it by relations to force them to hold in the final space.
  - If V has basis $e_1, \dots, e_n$ and W has basis $f_1, \dots, f_m$, then $e_i \otimes f_j$ is a basis of $V \otimes W$.
  - Interesting fact 1: If V, W are finite dimensional, $V^* \otimes W \cong \text{Hom}(V, W)$.
  - If we want to work with the tensor product in practice in *rep theory*, the only thing we need to know is the basis of the tensor product space, which can tell us how any map $\rho(g)$ acts on both sides of a $v \otimes w \in V \otimes W$. From here, we recover the Kronecker product of matrices.
  - So many things are explained by the concept of tensor products!
  - A tensor in *physics* is something with lots of indices that changes in some way.
    - It does come from the math concept.
    - We'll get a huge basis because we have a massive product like $V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$.
- Last 2 useful notions: Wedge powers and symmetric powers.
  - Again, it's much easier to think about the dual space.
- Consider the space $V^{\otimes n}$ (dimension $(\dim V)^n$).
  - $(V^{\otimes n})^*$ are **polylinear** maps $f : V^n \to F$.
    - Note: By contrast, $(V^n)^*$ is the space of all *linear* maps $f : V^n \to F$.
    - This distinction is subtle but important. Note, for instance, that $\dim V^{\otimes n} \neq \dim V^n$ and likewise for the duals.
    - The distinction comes out fully when considering that if, for example, $V = \mathbb{R}^3$, then $V^2 \cong \mathbb{R}^6$ and any map in $(V^2)^*$ is determined by its action on $(e_1, 0), (e_2, 0), (e_3, 0), (0, e_1), (0, e_2), (0, e_3)$. By contrast, any map in $(V^{\otimes 2})^*$ is determined by its action on $(e_1, e_1), (e_1, e_2), (e_1, e_3), (e_2, e_1)$, $(e_2, e_2), (e_2, e_3), (e_3, e_1), (e_3, e_2), (e_3, e_3)$.
    - Important note: What $(V^{\otimes 2})^*$ does is consider these nine elements of V^2 as the basis of another space. This is what it truly means when we say "a bilinear map on V^2 is a linear map on $V^{\otimes 2}$."
    - Takeaway: Polylinearity changes the basis upon which a function $f : V^n \to F$ fundamentally acts.
  - A polylinear map may be **symmetric**, **antisymmetric**, or[2] neither.
  - These maps form vector spaces and the dimension is actually pretty meaningful.
- **Symmetric** (polylinear map): A polylinear map $f : V^n \to F$ that satisfies the following property. *Constraint*

$$f(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = f(v_1, \dots, v_n)$$

[2]This is an exclusive "or."

- **Antisymmetric** (polylinear map): A polylinear map $f : V^n \to F$ that satisfies the following property. *Constraint*
$$f(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = (-1)^\sigma f(v_1, \dots, v_n)$$
- Suppose you take $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$[3].
  - Consider a symmetric polylinear map $f : V \times V \times V \to \mathbb{C}$.
  - To compute it, we'll need the action of f on the basis of V^3. In particular, we'll need...
$$f(x_1e_1 + y_1e_2, x_2e_1 + y_2e_2, x_3e_1 + y_3e_2) = x_1x_2x_3 f(e_1, e_1, e_1) + x_1x_2y_3 f(e_1, e_1, e_2) + \cdots$$
    - Somewhere in there, you'll also have a $x_1y_2x_3f(e_1, e_2, e_1)$ term as well.
    - However, because f is symmetric, you know by symmetry that these "bases" are the same, so you don't count them as 2 towards the dimension but as 1.
    - Thus, dim $= 4$ for symmetric maps.
  - What about antisymmetric maps?
  - Suppose $g : V^3 \to \mathbb{C}$ is an antisymmetric polylinear map.
    - Consider $g(e_1, e_1, e_1)$. Suppose you apply (12). Interchanging the first two indices (for instance) obviously won't do anything, so we'll get
$$\begin{aligned} g(e_1, e_1, e_1) &= (-1)^{(12)} g(e_1, e_1, e_1) \\ g(e_1, e_1, e_1) &= -g(e_1, e_1, e_1) \\ 2g(e_1, e_1, e_1) &= 0 \\ g(e_1, e_1, e_1) &= 0 \end{aligned}$$
    - But what about $g(e_1, e_1, e_2)$? We could apply (23) and get $g(e_1, e_2, e_1)$, right? So it appears that we would just be shrinking two options into one. Technically, this is true, but what's more important is that applying (12) again yields the same thing, meaning that $g(e_1, e_1, e_2) = g(e_1, e_2, e_1) = 0$.
    - And thus, since V has dimension 2 but g takes three vectors, any argument submitted to g will always be linearly dependent. Thus, $g = 0$ and, in fact, the space of antisymmetric maps on V^3 has dimension 0.
  - Takeaway: It's not always a rule that $V^{\otimes m} \cong S^m V \oplus \Lambda^m V$.
- Mathematically, there's a more natural object to work with than symmetric and antisymmetric maps.
  - Wedge powers and symmetric powers!
  - Given V and $n \in \mathbb{N}$, we can construct $S^n V$ and $\Lambda^n V$. $(S^n V)^*$ is symmetric polylinear maps taking n arguments from V. $(\Lambda^n V)^*$ is antisymmetric polylinear maps taking n arguments from V.
- How about a concrete way to see these? We can relate them to tensor powers.
  - Take a tensor power $V^{\otimes n}$, then look at those tensors which are symmetric and antisymmetric under permutation.
  - Example: Let V be the same as before. Then $V^{\otimes 2}$ has dim $= 4$.
    - Take as basis elements for $S^2 V$ those that don't change when you change the coordinates.
    - Take as basis elements for $\Lambda^2 V$ those that flip sign when you change the coordinates.
    - In this case, the basis of $V^{\otimes 2}$ is $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$. The basis of $S^2 V$ will be $e_1 \otimes e_1, e_1 \otimes e_2 + e_2 \otimes e_1, e_2 \otimes e_2$. The basis of $\Lambda^2 V$ will be $e_1 \otimes e_2 - e_2 \otimes e_1$. Notice that these bases are identical (up to scaling) with those in Serre (1977) and those produced by applying the **symmetrization** and **alternation** operators to the basis of $V^{\otimes 2}$.

[3]Note that this notation allows you to define a vector space *and* its basis in one go! I.e., the alternative is saying "Let V be a complex vector space with basis e_1, e_2."

    - S^2V and Λ^2V can form a *direct* sum because the dimensions match and they don't intersect.
    - Everything we're doing is representations, so $g(v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n$.
- Relating this to something that we've seen but that is a little confusing.
    - The product notation is suggestive for symmetric vectors; you can commute $e_1 \cdot e_2 \in S^2V$, for instance.
    - This allows us to, for example, shrink $e_1 \otimes e_1$ to $2e_1^2$[4], but $e_1 \otimes e_2 + e_2 \otimes e_1$ only to $e_1 \cdot e_2$.
    - Note that $e_1 \wedge e_2 = e_1 \otimes e_2 - e_2 \otimes e_1$ by definition.
    - Fact/exercise: Let V be a vector space of dimension n. V^* is the dual space, but it is also a function space. If $V = \mathbb{R}^k$, its a space of *functions from the blackboard.*
        - Note that $(\Lambda^k V)^* = \Lambda^k V^*$.
    - $S^n V^*$ is homogeneous polynomials of degree n.
    - You can take higher degree polynomials and just keep pushing through.
        - Ask about this??
    - Wedge powers now.
    - By convention, $\Lambda^0 V = F$ and $\Lambda^1 V = V$. But then you get to $\Lambda^2 V$ and $\Lambda^3 V$. They grow but then shrink down as the power approaches $\dim V$.
    - Truth: The dimension of wedge powers $\Lambda^i V$ is $\binom{k}{i}$ for $\dim V = k$. Figuring out why this is the case is another good exercise.
- An interesting connection between wedge powers and the determinant.
    - Let $V = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$.
    - Recall that $\Lambda^n V^*$ is the space of antisymmetric polylinear functions $V \times \cdots \times V \to F$ taking n arguments from V, and it has a single basis vector $e^1 \wedge \cdots \wedge e^n$.
    - Let $v_1 = \sum a_{i1} e_i$, $v_2 = \sum a_{i2} e_i$, etc.
    - Let $f \in \Lambda^n V^*$, so that f is an alternating polylinear map that takes n arguments.
    - Since f is polylinear, we have that

$$f(v_1, \dots, v_n) = \sum_{i_1, \dots, i_n = 1}^{n} a_{i_1 1} \cdots a_{i_n n} f(e_{i_1}, \dots, e_{i_n})$$

    - Because of antisymmetry, we need only look at elements where the indices are all different. Thus, the above equals

$$\sum_{\sigma \in S_n} a_{\sigma(1)1} \cdots a_{\sigma(n)n} f(e_{\sigma(1)}, \dots, e_{\sigma(n)})$$

    - Additionally, $f(e_{\sigma(1)}, \dots, e_{\sigma(n)}) = (-1)^\sigma f(e_1, \dots, e_n)$ for any $\sigma \in S_n$. Moreover, $f(e_1, \dots, e_n) \in \mathbb{C}$ by definition, so define a constant $\lambda := f(e_1, \dots, e_n)$. Thus, the above equals

$$\lambda \sum_{\sigma \in S_n} a_{\sigma(1)1} \cdots a_{\sigma(n)n}$$

    - But the term following the λ is just the determinant of the $n \times n$ matrix (a_{ij}). Thus, all said,

$$f(v_1, \dots, v_n) = \lambda \det(v_1 \mid \cdots \mid v_n)$$

    - Implication: Wedge powers are something like the determinant.

[4]Why the 2 coefficient? Because technically, the symmetrization operator takes $e_1 \otimes e_1 \mapsto e_1 \otimes e_1 + e_1 \otimes e_1 = 2e_1 \cdot e_1 = 2e_1^2$.

  - In particular, because $\Lambda^n V^*$ has only a single basis vector as mentioned above, $f = \lambda e^1 \wedge \cdots \wedge e^n$. It follows that $e^1 \wedge \cdots \wedge e^n = \det$.
- Takeaway: Wedge powers are something interesting; there's a reason to study them.

- The basis of the wedge powers consists of wedge monomials $e_{j_1} \wedge \cdots \wedge e_{j_i}$. Moreover, no need to have the same list twice, so choose some way of indexing them, e.g., increasing indexes.
  - This is why we do *increasing* bases! There's no particular reason, it's just an arbitrary way of making sure we don't do the same thing twice! We could just as well choose decreasing or any other means of guaranteeing that we don't have duplicates.
- Now let's relate all of this exterior and symmetric product stuff back to representation theory.
  - Let $V = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$.
  - Let $G \circlearrowright V$ via the homomorphism $G \to GL(V) \cong GL_n(\mathbb{C})$.
  - Focusing more on the *matrix* aspect this time, note that under this homomorphism, $g \mapsto A_g$ subject to the homomorphism constraints $A_e = E_n$, etc.
  - Consider the set $\{A_{g_1}, \dots, A_{g_k}\}$ of all matrices in the image of the homomorphism. If we transpose all of them, will they still obey the homomorphism constraints?
    - Nope!
    - Indeed, if we do this, we'll get in trouble. More specifically, transposition is not a representation because $A_{g_1}^T A_{g_2}^T \neq A_{g_1 g_2}^T = A_{g_2}^T A_{g_1}^T$.
  - It's the same story with inverses.
  - *However*, combining the two operations, we get

$$(A_{g_1 g_2}^T)^{-1} = (A_{g_1}^T)^{-1} (A_{g_2}^T)^{-1}$$

    - This is exactly when we take a representation and then go to the dual[5].
  - This will be on next week's homework!
  - Takeaway: This is an application of $\Lambda^j V^*$ to representation theory, $j \neq k, n$.
- Another relation: An application of $\Lambda^n V^*$ to representation theory.
  - Suppose we have a representation $G \circlearrowright V$ that we want to flatten into $G \circlearrowright \mathbb{C}$. How can we turn a relation between a group of matrices into a relation between a group of numbers?
  - Use the determinant!
  - Indeed, we already know that

$$\det(A_e) = 1 \qquad \det(A_{g_1 g_2}) = (\det A_{g_1})(\det A_{g_2}) \qquad \det\big(A_{g^{-1}}\big) = \det(A_g)^{-1}$$

  - In particular, we make formal the transition $G \to GL_j(\mathbb{C}) \to \mathbb{C}$ with the **top wedge power** $\Lambda^n V^*$.
- A last note.
  - Don't think that we're limited to top wedge powers.
  - Recall that we can define tensor products of matrices via the Kronecker product. Well, we can prove that

$$A_{g_1 g_2}^{\otimes 2} = A_{g_1}^{\otimes 2} A_{g_2}^{\otimes 2}$$

    and the like as well!
  - Similarly, we can define Λ^2 of a matrix.

[5] Relation to MATH 20510 when we discussed dual matrices and pullbacks of matrices.

- We'll get into some weird Kronecker product stuff again, but we can sort through it.

- Plan for Friday and next time.
  - Prove the theorem that every representation is a sum of irreducible representations.
  - He will use projectors.
  - Then a horror story.
  - Then associative algebra.

2.4 Office Hours

10/5:
- Problem 2a:
  - $(V \oplus W) \otimes (V \oplus W) \stackrel{?}{=} V \otimes V \oplus V \otimes W \oplus W \otimes V \oplus W \otimes W$.
  - Check linearity in all terms and then with universal property. Check antisymmetric, linear, injective, surjective; dimensions are the same, so no need to check *both* injectivity and surjectivity (surjectivity is easier to check). We can go to basis to check various properties; we can't use a basis to write the map, but we can use bases to check surjectivity and the like.

- Problem 3a:
  - Bezout and Gauss's lemma is good to learn on my own. Put polynomials in each variable. Throw some stuff about this shit into my answers.
  - Relearn polynomial division.
  - $(1,1,1,1)$, $(1,1,-1,-1)$, $(1,-1,1,-1)$, and $(1,-1,-1,1)$.
  - This is a symmetric matrix.
  - The upper-left and lower-right blocks of this matrix match; so do the lower-left and upper-right.
  - When the eigenvalue is equal to zero, the determinant is equal to zero. So look for eigenvectors to calculate eigenvalues, and then just express the determinant as a product of these.

- Problem 3b:
  - Corresponding eigenvalue is $\sum_{i=1}^{n} x_i z^{i-1}$.
  - Can I use representation theory to do this? What group has a multiplication table like this? $\mathbb{Z}/n\mathbb{Z}$. The elements of $\mathbb{Z}/n\mathbb{Z}$ are of the form $\{1, \zeta, \dots, \zeta^{n-1}\}$.
  - If that's an eigenvector, then it's a subrepresentation; it is a space that is fixed under the action of the matrix.
  - Other eigenvectors: $(1,1,1)$, $(1, z^2, z)$.
  - We don't need to do induction or anything fancy like that; we can just do dots. As long as your argument is complete and clear, you're good.

- Problem 4a:
  - See FH 1.3. Standard rep, not wedge. Treat τ, σ (generators of the action) on the basis vectors.
  - If both fix, it's the trivial; if one flips, you have alternating; if both flip, you have standard.
  - $(2,1) \oplus (1,1,1)$. Use problem 2.
  - See FH Exercise 1.2??
  - The action of τ on this basis vector can be computed:

$$\tau(\alpha \wedge \beta) = 1\alpha \wedge \beta$$

- Having obtained an eigenvalue of 1, we can rule out the standard representation.

- Problem 4b:
  - $\{\alpha \otimes \alpha \otimes \alpha, \alpha \otimes \alpha \otimes \beta + \alpha \otimes \beta \otimes \alpha + \beta \otimes \alpha \otimes \alpha, \beta \otimes \beta \otimes \beta\}$

- Problem 5a:
  - Consider an alternate basis $f_1, \dots, f_n$ and dual basis $f^1, \dots, f^n$. Consider the element $f_1 \otimes f^1 + \cdots + f_n \otimes f^n \in V \otimes V^\vee$. We want to prove that it equals the one asked about in the question.
  - Under the isomorphism to $\mathrm{Hom}(V, V)$, we send $e_1 \otimes e^1$ to $[v \mapsto e^1(v)e_1]$. More generally, we end $e_i \otimes e^i$ to $[v \mapsto e^i(v)e_i]$. Adding all these maps together yields the map $[v \mapsto e^1(v)e_1 + \cdots + e^n(v)e_n]$, which is just the identity $1 \in \mathrm{Hom}(V, V)$, regardless of basis.

- Problem 5b:
  - Example:
  $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to ae_1 \otimes e^1 + be_1 \otimes e^2 + ce_2 \otimes e^1 + de_2 \otimes e^2$$
  - Evaluating this gives
  $$e^1(ae_1) + e^2(be_1) + e^1(ce_2) + e^2(de_2) = a + d$$
  since it's only when the indices match (i.e., along the diagonal) that we get a nonzero value.

- Problem 5c:
  - P should have a block-diagonal matrix corresponding to the decomposition $V = W \oplus W^0$. P is the identity on $\mathrm{Im}(P)$. So if our basis is vectors spanning W and then vectors spanning W^0, the matrix should be the identity and then the zero matrix. That should do the trick. How rigorous does this need to be?
  - Let $e_1, \dots, e_k$ be an orthonormal basis of $\mathrm{Im}(P)$. Extend this basis to an orthonormal basis $e_1, \dots, e_n$ of V.

- Problem 5d:
  - Trivial representation: All $g \in G$ get mapped to $1 \in GL(V)$.
  - Part (a) gives us the identity in $\mathrm{Hom}(V, V)$.
  - So we have $\rho : G \to GL(V)$.
  - Is any line acceptable? Span of the identity function? Rudenko: It depends on V. It has *infinitely many* trivial sub representations.
  - Example: $G \circlearrowright \mathbb{C}^2$. with $\rho(g) = I_2$.
  - Dual representation: Defined analogously to the $\mathrm{Hom}_F(V, W)$ representation. We also need an inverse.

- Psets will likely get easier; right now, we have to relearn a lot of old stuff and we are being challenged with harder problems. As the questions become more based on course content and thus will get easier.

- He'll do hard PSets, easy exams, and everything is curved; he agrees that this is a hard pset, and probably harder than necessary.

2.5 Complete Reducibility

10/6:

- Let G be a finite group.
- We want to study finite dimensional representations over $\mathbb{C}$.
  - Characteristic F, $|G| = 1$.
  - What is this stuff about characteristic??
- Theorem: Any f.d. representation can be decomposed into a sum of irreps via

$$V = V_1^{n_1} \oplus \cdots \oplus V_k^{n_k}$$

  Moreover, this decomposition is unique.
  - See Proposition 1.8.
- Example: We have already seen S_2, S_3 in the homework; now, let $G = \mathbb{Z}/n\mathbb{Z}$.
  - Consider $V_0, \dots, V_{n-1}$.
  - Let V_i be a 1-dimensional rep.
    - We have $\rho : G \to C^\times$ defined by $[k] \mapsto (\mathrm{e}^{2\pi i/n})^k$.
  - These are all 1-dimensional representations up to isomorphism.
- Example: Let $G = \mathbb{Z}$. What is a representation of $\mathbb{Z}$? We just need to say what happens to 1.
  - For example, if the map $G \to GL_n(\mathbb{C})$ sends $1 \mapsto A$, then $2 \mapsto A^2$, and on and on.
  - A place where you run into trouble: $n = 2$ and

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

  - The matrix has a fixed subrepresentation (i.e., eigenvector $(1, 1)$).
    - More specifically, $\mathbb{C}(1, 0) \hookrightarrow V$ is a 1D subrepresentation.
  - The theorem basically tells us that $V = \mathbb{C}(1, 0) + \mathbb{C}w$.
  - This is an example of how things can go wrong. How??
- Proving the theorem; we need a miracle!
- Existence: We need a lemma.
- Lemma: Let G be finite, $F = \mathbb{C}$, and V a G-representation. Let $W \leq V$ be a subrepresentation or **invariant subspace**. Then there exists another invariant subspace $W' \subset V$ such that $V = W \oplus W'$.
  - See Theorem 1 (Rudenko replicates all aspects of the "limited conditions" proof).
  - This lemma implies existence.
  - Two proofs: One that only works over complex numbers. He suggests we read about it. Name??
  - He'll do the slightly less intuitive one, which involves **projectors**.
- **Projector**: A linear map $P : V \to V$ such that $P^2 = P$, that is, is **idempotent**.
  - Example: Consider $W := \mathrm{Im}(P) \leq V$. $P|_W$ does nothing; it's the identity.
  - A good mental picture: Things are falling from 3D space onto some smaller space.
  - On the kernel.
    - Importantly, $\mathrm{Ker}\, P \cap \mathrm{Im}\, P = 0$.
    - It follows that $V = \mathrm{Im}(P) \oplus \mathrm{Ker}(P)$.

        - Within the space, $v = (w, w') = w + w'$. What the projector does is $(w, w') \mapsto (w, 0)$.
    - What else can we say about projectors?
        - There is a correspondence between projectors and direct sum decompositions.
- So to prove the lemma, we need a projector $P : V \to V$ with image W and certain properties.
    - More specifically, the goal is to find a projector P...
        1. With image W;
        2. That is a morphism of G-reps.
    - On the second condition, that is, we want $P(gv) = gP(v)$. In this case, $\text{Ker}(P)$ will be a shuffle??
- Strategy.
    - Take any projector $P_0 : V \to W$. And then you can get g-projectors $gP = gPg^{-1}$.
    - So define a new projector
      $$P = \frac{1}{|G|} \sum_{g \in G} \underbrace{gP_0g^{-1}}_{gP_0}$$
      One didn't work, so we hope the average will work, and it will!
    - For any $w \in W$, we can prove that the sum thing does fix W's, so it is a projector!
    - $P(hv)$ example.
    - Note: This computation will be done again later in a different context; this averaging construction is *central* to representation theory.
- Constraints we used in the proof: G is finite (or compact), $|G|$ is invertible.
    - Only when we get into **modular representation theory** is where we get into trouble; this theorem actually kills **extensions**, which are very interesting but are not in finite group rep theory.
- Hermitian inner product isn't common here, but it shows up in physics. We will talk a bit more about inner products later, though.
- Now for the other part of the original proof: Uniqueness.
- Schur's Lemma: Let G be a finite group, let the field $F = \mathbb{C}$, and let V, W be irreps over F. Let $f \in \text{Hom}_G(V, W)$, which we may recall is the space of morphisms between G-reps V and W, that is, all $h : V \to W$ satisfying $h(gv) = gh(v)$. Then...
    1. $f = 0$ if $V \ncong W$. If $V \cong W$, then these maps are isomorphisms.
    2. In particular, if V is an irrep and $f : V \to V$ is such that $f(gv) = gf(v)$, then $f(v) = \lambda v$.

  Altogether, we have that
  $$\text{Hom}_G(V, W) \cong \begin{cases} 0 & V \ncong W \\ \mathbb{C} & V \cong W \end{cases}$$
    - The statement "$\text{Hom}_G(V, W) \cong \mathbb{C}$" reflects the fact that the left space is the space of all scalar isomorphisms λI for scalars $\lambda \in \mathbb{C}$, which happens to be isomorphic to $\mathbb{C}$.
    - Gist: If you want a certain kind of matrix between certain spaces, in some cases, you'll just fail.
- Proof.
    - See Lemma 1.7 (Rudenko replicates all aspects of the "limited conditions" proof).
    - For $f : V \to V$, consider $\text{Ker}(f)$ and $\text{Im}(f)$. The latter two are subrepresentations of V, W, respectively. $\text{Ker}(f) = V$ implies $f = 0$; symmetric with $\text{Im}(f)$. If nonzero, then $\text{Ker} = 0$ and $\text{Im} = W$, implies f is an isomorphism.

- Schur's Lemma is the easiest step to learn in the whole story.
- Last 2 minutes: Finish the proof of the original theorem.
  - Don't worry if we're confused by this last line; it will be repeated later in a much more powerful way.
  - Analogous to Proposition 1.8.
  - I might have missed some stuff here??
- Plan for next week.
  - Character theory.
  - Serre (1977) is still the best source for tracking lecture content for right now.
- This would have been an interesting but wholly nonessential lecture to pay attention to, since I already did all of the readings.

2.6 S Chapter 1: Generalities on Linear Representations

From Serre (1977).

Section 1.5: Tensor Product of Two Representations

10/4: - **Tensor product** (of V_1, V_2): The vector space W that (a) is furnished with a map $V_1 \times V_2 \to W$ sending $(x_1, x_2) \mapsto x_1 \cdot x_2$ and (b) satisfies the following two conditions.
  (i) $x_1 \cdot x_2$ is bilinear.
  (ii) If (e_{i_1}) is a basis of V_1 and (e_{i_2}) is a basis of V_2, the family of products $e_{i_1} \cdot e_{i_2}$ is a basis of W.

  Denoted by $\boldsymbol{V_1 \otimes V_2}$.
  - It can be shown that such a space exists and is unique up to isomorphism (see proof here).
- This definition allows us to say some things quite expediently. For example, (ii) implies that
$$\dim(V_1 \otimes V_2) = \dim(V_1) \cdot \dim(V_2)$$
- **Tensor product** (of ρ^1, ρ^2): The representation $\rho : G \to GL(V_1 \otimes V_2)$ defined as follows for all $s \in G$, $x_1 \in V_1$, and $x_2 \in V_2$, where $\rho^1 : G \to GL(V_1)$ and $\rho^2 : G \to GL(V_2)$ are representations. *Given by*
$$[\rho_s^1 \otimes \rho_s^2](x_1 \cdot x_2) = \rho_s^1(x_1) \cdot \rho_s^2(x_2)$$
- A more formal write up of the matrix translation of this definition.
  - Let (e_{i_1}) be a basis for V_1, and let (e_{i_2}) be a basis for V_2.
  - Let $r_{i_1j_1}(s)$ be the matrix of ρ_s^1 with respect to this basis, and let $r_{i_2j_2}(s)$ be the matrix of ρ_s^2 with respect to this basis.
  - It follows that
$$\rho_s^1(e_{j_1}) = \sum_{i_1} r_{i_1j_1}(s) e_{i_1} \qquad \rho_s^2(e_{j_2}) = \sum_{i_2} r_{i_2j_2}(s) e_{i_2}$$
  - Therefore,
$$[\rho_s^1 \otimes \rho_s^2](e_{j_1} \cdot e_{j_2}) = \sum_{i_1,i_2} r_{i_1j_1}(s) r_{i_2j_2}(s) e_{i_1} \cdot e_{i_2}$$
and
$$\mathcal{M}(\rho_s^1 \otimes \rho_s^2) = (r_{i_1j_1}(s) r_{i_2j_2}(s))$$
- Aside on quantum chemistry to come back to later; I can't quite connect the dots yet.

Section 1.6: Symmetric Square and Alternating Square

- Herein, we investigate the tensor product when $V_1 = V_2 = V$.
- Let (e_i) be a basis of V.
- Define the automorphism $\theta : V \otimes V \to V \otimes V$ by
$$\theta(e_i \cdot e_j) = e_j \cdot e_i$$
for all 2-indices (i, j).
- Properties of θ.
  - Since θ is linear, it follows that
  $$\theta(x \cdot y) = y \cdot x$$
  for all $x, y \in V$.
    - Implication: θ is independent of the chosen basis (e_i)!
  - $\theta^2 = 1$, where 1 is the identity map on $V \otimes V$.
- Assertion: $V \otimes V$ decomposes into
$$V \otimes V = S^2(V) \oplus \Lambda^2(V)$$
  - Rudenko: We do not have to worry about proving this... yet, at least.
- **Symmetric square representation**: The subspace of $V \otimes V$ containing all elements z satisfying $\theta(z) = z$. *Denoted by* $\boldsymbol{S^2V}$, $\boldsymbol{S^2(V)}$, $\boldsymbol{\mathbb{S}^2V}$, $\textbf{Sym}^{\,\boldsymbol{2}}\boldsymbol{(V)}$.
  - Basis: $(e_i \cdot e_j + e_j \cdot e_i)_{i \leq j}$.
    - Rudenko: How do we know everything is linearly independent? Well, when we add two linearly independent vectors out of a set, the sum is still linearly independent from everything else!
    - Example when $\dim V = 2$: The basis of $V \otimes V$ is $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$, where all four of these vectors are linearly independent. So naturally, the basis of the corresponding symmetric square representation — which is $2e_1 \otimes e_1, e_1 \otimes e_2 + e_2 \otimes e_1, 2e_2 \otimes e_2$ — will still be a linearly independent list of vectors.
  - Dimension: If $\dim V = n$, then
  $$\dim S^2(V) = \frac{n(n+1)}{2}$$
- **Alternating square representation**: The subspace of $V \otimes V$ containing all elements z satisfying $\theta(z) = -z$. *Denoted by* $\boldsymbol{\Lambda^2V}$, $\boldsymbol{\Lambda^2(V)}$, $\textbf{Alt}^{\,\boldsymbol{2}}\boldsymbol{(V)}$.
  - Basis: $(e_i \cdot e_j - e_j \cdot e_i)_{i<j}$.
  - Dimension: If $\dim V = n$, then
  $$\dim \Lambda^2(V) = \frac{n(n-1)}{2}$$

2.7 FH Appendix B: On Multilinear Algebra

From Fulton and Harris (2004).

Section B.1: Tensor Products

10/5:

- **Tensor product** (of V, W over F): A vector space U equipped with a bilinear map $V \times W \to U$ sending $v \times w \to v \otimes w$ that is universal, i.e., for any bilinear map $\beta : V \times W \to Z$, there is a unique linear map from $U \to Z$ that takes $v \otimes w \mapsto \beta(v, w)$. *Denoted by* $\boldsymbol{V \otimes W}$, $\boldsymbol{V \otimes_F W}$.
  - The so-called *universal property* determines the tensor product up to canonical isomorphism.
- One construction of $V \otimes W$: From the basis $\{e_i \otimes f_j\}$.
  - This construction is **functorial**, implying that linear maps from $f : V \to V'$ and $g : W \to W'$ determine a linear map $f \otimes g : V \otimes W \to V' \otimes W'$, namely that defined by $f \otimes g : v \otimes w \to fv \otimes gw$.
- Definition of the $\boldsymbol{n}$**-fold tensor product**.
- **Multilinear** (map): A map from a Cartesian product $V_1 \times \cdots \times V_n$ of vector spaces to a vector space U such that when all but one of the factors V_i are fixed, the resulting map from $V_i \to U$ is linear.
- Properties of the tensor product.
  1. *Commutativity*:
  $$V \otimes W \cong W \otimes V$$
  by $v \otimes w \mapsto w \otimes v$.
  2. *Distributivity*:
  $$(V_1 \oplus V_2) \otimes W \cong (V_1 \otimes W) \oplus (V_2 \otimes W)$$
  by $(v_1, v_2) \otimes w \mapsto (v_1 \otimes w, v_2 \otimes w)$.
  3. *Associativity*:
  $$(U \otimes V) \otimes W \cong U \otimes (V \otimes W) \cong U \otimes V \otimes W$$
  by $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w) \mapsto u \otimes v \otimes w$.
- **Tensor power** (of V to n): The tensor product defined as follows. *Denoted by* $\boldsymbol{V^{\otimes n}}$. *Given by*
$$V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}$$
  - Convention: $V^{\otimes 0} = F$.
- Analogous construction of the tensor product for generalized algebras and modules.

Section B.2: Exterior and Symmetric Powers

- **Alternating** (multilinear map): A multilinear map β such that $\beta(v_1, \dots, v_n) = 0$ whenever $v_i = v_j$ for some $i, j \in [n]$.
  - Implication: $\beta(v_1, \dots, v_n)$ changes sign whenever two of the vectors are interchanged.
    - Follows from the definition and the **standard polarization**.
  - Implication:
  $$\beta(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = (-1)^\sigma \beta(v_1, \dots, v_n)$$
  for all $\sigma \in S_n$.
- **Standard polarization**: The equality
$$\beta(v, w) + \beta(w, v) = \beta(v + w, v + w) - \beta(v, v) - \beta(w, w) = 0 - 0 - 0 = 0$$

- **Exterior powers** (of V): The vector space U equipped with an alternating multilinear map $V \times \cdots \times V \to \Lambda^n V$ sending $v_1 \times \cdots \times v_n \mapsto v_1 \wedge \cdots \wedge v_n$ that is universal, i.e., for any alternating multilinear map $\beta : V^n \to Z$, there is a unique linear map from U to Z that takes $v_1 \wedge \cdots \wedge v_n \mapsto \beta(v_1, \dots, v_n)$. *Denoted by* $\boldsymbol{\Lambda^n V}$.
  - Convention: $\Lambda^0 V = F$.
- Quotient space construction of the exterior powers.
- Projecting from $V^{\otimes n} \to \Lambda^n V$: Define $\pi : V^{\otimes n} \to \Lambda^n V$ by

$$\pi(v_1 \otimes \cdots \otimes v_n) = v_1 \wedge \cdots \wedge v_n$$

- Basis for the exterior powers.
- There is a canonical linear map $\Lambda^a V \otimes \Lambda^b W \to \Lambda^{a+b}(V \oplus W)$, which takes $(v_1 \wedge \cdots \wedge v_a) \otimes (w_1 \wedge \cdots \wedge w_b) \mapsto v_1 \wedge \cdots \wedge v_a \wedge w_1 \wedge \cdots \wedge w_b$.
  - This determines (how??) an isomorphism

$$\Lambda^n(V \oplus W) \cong \bigoplus_{a=0}^{n} \Lambda^a V \otimes \Lambda^{n-a} W$$

  - This isomorphism plus induction on n can justify (how??) the basis for $\Lambda^n V$ as the increasing indices.
- **Symmetric** (multilinear map): A multilinear map β such that $\beta(v_1, \dots, v_n)$ is unchanged when any two factors are interchanged, that is

$$\beta(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = \beta(v_1, \dots, v_n)$$

  for all $\sigma \in S_n$.
- **Symmetric powers** (of V): The vector space U equipped with a symmetric multilinear map $V \times \cdots \times V \to S^n V$ sending $v_1 \times \cdots \times v_n \mapsto v_1 \cdot \ldots \cdot v_n$ that is universal, i.e., for any symmetric multilinear map $\beta : V^n \to Z$, there is a unique linear map from U to Z that takes $v_1 \cdot \ldots \cdot v_n \mapsto \beta(v_1, \dots, v_n)$. *Denoted by* $\boldsymbol{S^n V}$.
  - Convention: $S^0 V = F$.
- Quotient space construction of the symmetric powers.
  - Quotient out all $v_1 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$, that is, those elements of $V^{\otimes n}$ in which σ permutes two successive factors. How does this work??
- Projecting from $V^{\otimes n} \to S^n V$: Define $\pi : V^{\otimes n} \to S^n V$ by

$$\pi(v_1 \otimes \cdots \otimes v_n) = v_1 \cdot \ldots \cdot v_n$$

- Basis for the symmetric powers.
  - It follows from the basis construction that $S^n V$ can be regarded as the space of homogeneous polynomials of degree n in the variable e_i, since each element is of the form $e_{i_1} \cdot \ldots \cdot e_{i_n}$ and we can add them.
- Canonical isomorphism:

$$S^n(V \oplus W) \cong \bigoplus_{a=0}^{n} S^a V \otimes S^{n-a} W$$

- More on $\Lambda^n V, S^n V$ as subspaces of $V^{\otimes n}$.
  - We inject $\iota : \Lambda^n V \to V^{\otimes n}$ with

$$\iota(v_1 \wedge \cdots \wedge v_n) = \sum_{\sigma \in S_n} (-1)^\sigma v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

    - This relates to Rudenko's note that $v_1 \wedge v_2 = v_1 \otimes v_2 - v_2 \otimes v_2$!
    - There are some more advanced notes on the implications of ι; $[\iota \circ \pi/n!](V^{\otimes n}) = V^{\otimes n}$ is brought up.
  - We inject $\iota : S^n V \to V^{\otimes n}$ with

$$\iota(v_1 \cdot \ldots \cdot v_n) = \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

    - More, related advanced notes; includes the $1/n!$ thing again.
- **Wedge product**: The function $\Lambda^m V \otimes \Lambda^n V \to \Lambda^{m+n} V$ defined as follows. *Denoted by* $\wedge$. *Given by*

$$(v_1 \wedge \cdots \wedge v_m) \otimes (v_{m+1} \wedge \cdots \wedge v_{m+n}) \mapsto v_1 \wedge \cdots \wedge v_m \wedge v_{m+1} \wedge \cdots \wedge v_{m+n}$$

- Properties of the wedge product.
  1. *Associativity*:
$$(v_1 \wedge v_2) \wedge v_3 = v_1 \wedge (v_2 \wedge v_3) = v_1 \wedge v_2 \wedge v_3$$
  2. *Skew-commutativity*:
$$v_1 \wedge v_2 = -v_2 \wedge v_2$$
- Note that both of the above properties hold in higher-dimensional cases as well.
- Commutativity of the products.

$$\begin{tikzcd} \Lambda^m V \otimes \Lambda^n V \arrow[r, "\wedge"] \arrow[d, hook, "\iota\otimes\iota"'] & \Lambda^{m+n} V \arrow[d, hook, "\iota"] \\ V^{\otimes m} \otimes V^{\otimes n} \arrow[r, "f_1"'] & V^{\otimes(m+n)} \end{tikzcd}$$

(a) Wedge product.

$$\begin{tikzcd} S^m V \otimes S^n V \arrow[r, "\cdot"] \arrow[d, hook, "\iota\otimes\iota"'] & S^{m+n} V \arrow[d, hook, "\iota"] \\ V^{\otimes m} \otimes V^{\otimes n} \arrow[r, "f_2"'] & V^{\otimes(m+n)} \end{tikzcd}$$

(b) Symmetric product.

Figure 2.3: Commutative diagram, wedge and symmetric products.

  - f_1 is defined by

$$(v_1 \otimes \cdots \otimes v_m) \otimes (v_{m+1} \otimes \cdots \otimes v_{m+n}) \mapsto \sum_{\sigma \in G} (-1)^\sigma v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)} \otimes v_{\sigma(m+1)} \otimes \cdots \otimes v_{\sigma(m+n)}$$

    where G is the subgroup of S_{m+n} preserving the order of the subsets $\{1, \ldots, m\}$ and $\{m+1, \ldots, m+n\}$.
  - f_2 is defined analogously.
- The above mappings all commute with linear maps of vector spaces.
  - Example: Our definition $g(v \otimes w) = gv \otimes gw$ could be redrawn as $[g \circ \otimes](v, w) = [\otimes \circ g](v, w)$, where the latter $g : (v, w) \mapsto (gv, gw)$ by abuse of notation.
- Tensor, exterior, and symmetric algebras.

2.8 FH Chapter 1: Representations of Finite Groups

From Fulton and Harris (2004).

10/1: • Starts with a justification for beginning their investigation of rep theory with finite groups.

Section 1.1: Definitions

- Definition of a **representation**.
- ρ "gives V the structure of a G-module!" (Fulton & Harris, 2004, p. 3).
- When there is little ambiguity about ρ, we call V itself a representation of G.
  - This is what Rudenko has been doing in class!
- We also often write $g \cdot v$ for $\rho(g)(v)$, and g for $\rho(g)$.
- **Degree** (of ρ): The dimension of V.
- ***G*-linear** (map): See class notes. *Also known as* **map**, **morphism**.
- The **kernel**, **image**, and **cokernel** of φ are all G-submodules.
- **Kernel** (of a map): The vector subspace containing all $v \in V$ for which $\varphi(v) = 0$. *Denoted by* $\mathbf{Ker}\,\boldsymbol{\varphi}$.
- **Image** (of a map): The vector subspace containing all $w \in W$ for which there exists $v \in V$ such that $\varphi(v) = w$. *Denoted by* $\mathbf{Im}\,\boldsymbol{\varphi}$.
- **Cokernel** (of a map): The quotient space $W/\operatorname{Im}\varphi$.
- Definitions of **subrepresentation**, **irreducible** representation, and **direct sum** of representations.
- **Tensor product** (of V, W): The representation with the space $V \otimes W$ where $g(v \otimes w) = gv \otimes gw$.

10/5:
  - The n^{th} tensor power is also a representation by this rule.
  - The n^{th} exterior and symmetric powers are subrepresentations of the n^{th} tensor power.
- **Natural pairing** (between V^*, V): The pairing defined as follows for all $v^* \in V^*$ and $v \in V$. *Denoted by* $\langle\ ,\ \rangle$. *Given by*
$$\langle v^*, v\rangle = v^*(v) = (v^*)^T v$$
- **Dual representation**: The representation from $G \to GL(V^*)$ defined as follows. *Denoted by* $\boldsymbol{\rho^*}$. *Given by*
$$\rho^*(g) = \rho(g^{-1})^T$$
  - We should — and do — have
$$\langle \rho^*(g)(v^*), \rho(g)(v)\rangle = \langle v^*, v\rangle$$
  - Indeed,
$$\begin{aligned}
\langle \rho^*(g)(v^*), \rho(g)(v)\rangle &= \rho^*(g)(v^*)^T \rho(g)v \\
&= [\rho(g^{-1})^T (v^*)]^T \rho(g)v \\
&= (v^*)^T \rho(g^{-1}) T \rho(g)v \\
&= (v^*)^T v \\
&= \langle v^*, v\rangle
\end{aligned}$$
- $\operatorname{Hom}(V, W)$ is a representation.
  - Definition via the commutative diagram (from class): $g(L)v = [g \circ L \circ g^{-1}]v$.

- Definition via the isomorphic space $V^* \otimes W$ and the dual representation:

$$\begin{aligned} g(v^* \otimes w) &= gv^* \otimes gw \\ &= (g^{-1})^T v^* \otimes gw \\ &= [(g^{-1})^T v^*](gw) \\ &= [(g^{-1})^T v^*]^T gw \\ &= (v^*)^T g^{-1} gw \\ &= (v^*)^T w \\ &= v^*(w) \end{aligned}$$

- The rules for normal vector spaces hold for representations as well, e.g.,

$$V \otimes (U \oplus W) = (V \otimes U) \oplus (V \otimes W) \qquad \Lambda^k(V \oplus W) = \bigoplus_{a+b=k} \Lambda^a V \oplus \Lambda^b W \qquad \Lambda^k(V^*) = \Lambda^k(V)^*$$

- Definition of **permutation representation** and **regular representation**.

Section 1.2: Complete Reducibility, Schur's Lemma

- **Indecomposable** (representation): See class notes. *Also known as* **irreducible**.
- Proof of Theorem 1 as in Serre (1977).
  - The method is called "integration over the group (with respect to an invariant measure on the group)" (Fulton & Harris, 2004, p. 6).
- **Complete reducibility**: The property that any representation is a direct sum of irreducible representations. *Also known as* **semisimplicity**.
  - Stated here as a corollary; proven as Theorem 2 in Serre (1977).
- The following lemma has several consequences, among which is that it determines how much a representation's direct-sum decomposition is unique.

  Lemma 1.7 (Schur's Lemma)**.** *If V and W are irreducible representations of G and $\varphi : V \to W$ is a G-module homomorphism, then...*

  1. *Either φ is an isomorphism, or $\varphi = 0$;*
  2. *If $V = W$, then $\varphi = \lambda I$ for some $\lambda \in \mathbb{C}$, I being the identity.*

  Proof. Suppose for the sake of contradiction that φ is neither an isomorphism nor zero. Then it has a nontrivial kernel and image, both of which are necessarily invariant under the representation[6]. Therefore, neither V nor W are irreducible representations of G, a contradiction.

  Since $\mathbb{C}$ is algebraically closed, φ must have an eigenvalue λ. Equivalently, for some $\lambda \in \mathbb{C}$, $\varphi - \lambda I$ has nonzero kernel. But then by part (1), we must have $\varphi - \lambda I = 0$, implying that $\varphi = \lambda I$, as desired. ☐

- Direct sum irreducible decomposition.

  Proposition 1.8. *For any representation V of a finite group G, there is a decomposition*

$$V = V_1^{\oplus a_1} \oplus \cdots \oplus V_k^{\oplus a_k}$$

  where the V_i are distinct irreducible representations. The decomposition of V into a direct sum of the k factors is unique, as are the V_i that occur and their multiplicities a_i.

[6] See the proof of Schur's Lemma in Serre (1977) for an explanation of this fact.

Proof. Let W be another representation of G, possibly of different dimension. Let $\varphi : V \to W$ be a map of representations. Restrict φ to $V_i^{\oplus a_i}$, a subrepresentation of V. It follows from Schur's Lemma that this restriction either maps into the $W_j^{\oplus b_j}$ satisfying $W_j \cong V_i$, or it does not map it at all.

Uniqueness for the decomposition of V follows by applying Schur's Lemma to the identity map on V. ∎

- Goals going forward.
  1. Describe all the irreducible representations of G.
     - We can find all *irreducible* representations of G, then describe *any* representation as a linear combination of these.
  2. Find techniques for giving the direct sum decomposition and the multiplicities of an arbitrary representation.
  3. **Plethysm**: Describe the decompositions, with multiplicities, of representations derived from a given representation V, such as $V \otimes V$, V^*, $\Lambda^k V$, $S^k V$, and $\Lambda^k(\Lambda^1 V)$.
     - Note: If V decomposes into two representations, these representations decompose accordingly, e.g., if $V = U \oplus W$, we may invoke the earlier identity to learn that $\Lambda^k V = \oplus_{i+j=k} \Lambda^i U \otimes \Lambda^j W$.
     - **Clebsch-Gordon problem**: Decompose $V \otimes W$, given two irreducible representations V and W.

Section 1.3: Examples — Abelian Groups, S_3

- Classifying the irreducible representations of abelian groups.
  - Let G be an arbitrary finite abelian group, and let V be an irreducible representation of it.
  - Observe that since $gh = hg$ for all $g, h \in G$, we have

$$\rho_V(gh) = \rho_V(hg)$$
$$\rho_V(g) \circ \rho_V(h) = \rho_V(h) \circ \rho_V(g)$$

  - Thus, each $\rho_V(g)$ is a morphism of G-representations.
  - It follows by Schur's Lemma that each $\rho_V(g) = \lambda_g I$.
  - Consequently, every subspace of V is invariant under $\rho_V(g)$ for all $v \in V$. Therefore, V must be one dimensional, hence isomorphic to $\mathbb{C}$.
- Classifying the irreducible representations of S_3.
  - There exist two one-dimensional representations of S_3 (and of every other nontrivial symmetric group).
    - **Trivial representation** (irreducible).
    - **Alternating representation** (irreducible).
  - Using the fact that S_3 is a permutation group, we can locate the...
    - Permutation representation (reducible);
    - **Standard represenatation** (irreducible).
  - Let W be an arbitrary representation of S_3.
    - Easily done with **character theory**, but we'll only get there later.
  - Since the representation theory of finite abelian groups was jut proven to be very simple, we'll start by looking at the action of the finite abelian subgroup $A_3 = \mathbb{Z}/3\mathbb{Z} \subset S_3$ on W.
    - Let τ be a generator of A_3. Explicitly, this means $\tau = (1, 2, 3)$ or $\tau = (1, 3, 2)$.

- Then W is spanned by eigenvectors $v_1, \dots, v_n$ of τ (why?? Schur's Lemma part 2? Relation to classification of abelian groups?) with corresponding eigenvalues $\omega = \mathrm{e}^{2\pi i/3}$.
- Thus, $W = \bigoplus V_i$ where $V_i = \mathbb{C}v_i$ and $\tau v_i = \omega^{\alpha_i} v_i$.
- An example representation of A_3 (in the chemistry sense) is

$$\begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}, \begin{bmatrix} \mathrm{e}^{2\pi i/3} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathrm{e}^{2\pi i/3} \end{bmatrix}, \begin{bmatrix} \mathrm{e}^{4\pi i/3} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathrm{e}^{4\pi i/3} \end{bmatrix}$$

– Now we want to see how the remainder of S_3 acts on W.
  - Let σ be an arbitrary transposition in S_3.
  - Note: $\{\sigma, \tau\}$ generates S_3.
  - Recall the relationship $\sigma\tau\sigma = \tau^2$.
  - The action of σ on the eigenvectors of τ: Let v be an arbitrary eigenvector of τ, with corresponding eigenvalue ω^j. Notice that

$$\tau(\sigma(v)) = \sigma(\tau^2(v)) = \sigma(\omega^{2j}v) = \omega^{2j}\sigma(v)$$

    Takeaway: v an eigenvector for τ with eigenvalue ω^j implies $\sigma(v)$ an eigenvector for τ with eigenvalue ω^{2j}.

– Exercise 1.10 (A basis for the standard representation of S_3): Verify that with $\sigma = (12)$ and $\tau = (123)$, the standard representation has a basis $\alpha = (\omega, 1, \omega^2), \beta(1, \omega, \omega^2)$, with

$$\tau\alpha = \omega\alpha \qquad \tau\beta = \omega^2\beta \qquad \sigma\alpha = \beta \qquad \sigma\beta = \alpha$$

  - $1 + \omega + \omega^2 = 0$ in the complex plane.
  - We do, indeed, get

$$\begin{aligned} \tau\alpha &= (\omega^2, \omega, 1) & \tau\beta &= (\omega^2, 1, \omega) & \sigma\alpha &= (1, \omega, \omega^2) & \sigma\beta &= (\omega, 1, \omega^2) \\ &= \omega(\omega, 1, \omega^2) & &= \omega^2(1, \omega, \omega^2) & &= \beta & &= \alpha \\ &= \omega\alpha & &= \omega^2\beta \end{aligned}$$

  - We also get — per the aforementioned rule — that

$$\tau\alpha = \omega\alpha$$

    but

$$\tau(\sigma\alpha) = \tau\beta = \omega^2\beta = \omega^2(\sigma\alpha)$$

    for instance, as predicted.
  - Note that both α, β are orthogonal to $(1, 1, 1)$, but while they are linearly independent, they are not orthogonal to each other. This is fine, because they're computationally simple, but it is noteworthy.

– The difference in eigenvalues between v and $\sigma(v)$ indicates that these vectors are *not* linearly dependent.
  - Rather, they span a 2D subspace V' that is invariant under S_3!! This is because $v = \sigma(\sigma(v))$ as well.
  - In fact, V' is isomorphic to the standard representation!

– What if the eigenvalue of v is 1?
  - If $\sigma(v)$ is not linearly independent of v, then the two span a 1D subrepresentation of W, isomorphic to the trivial representation (if $\sigma(v) = v$) and isomorphic to the alternating representation (if $\sigma(v) = -v$).

- If $\sigma(v)$ is linearly independent of v, then $v+\sigma(v)$ spans a 1D subrepresentation of W isomorphic to the trivial representation and $v-\sigma(v)$ spans a 1D subrepresentation of W isomorphic to the alternating representation.

– It follows that the only three irreps of S_3 are the trivial, alternating, and standard ones.

- Using the above approach to find the decomposition of the tensor product.
  - Let V be the standard two-dimensional representation. Recall that the basis of V is $\{\alpha,\beta\}$.
  - It follows that the basis of $V\otimes V$ is $\{\alpha\otimes\alpha,\alpha\otimes\beta,\beta\otimes\alpha,\beta\otimes\beta\}$.
  - These are eigenvectors for τ, and we can find their corresponding eigenvalues via direct computation:

$$\begin{aligned}\tau(\alpha\otimes\alpha)&=\tau\alpha\otimes\tau\alpha & \tau(\alpha\otimes\beta)&=\tau\alpha\otimes\tau\beta\\&=(\omega\alpha)\otimes(\omega\alpha) & &=(\omega\alpha)\otimes(\omega^2\beta)\\&=\omega^2\alpha\otimes\alpha & &=1\alpha\otimes\beta\end{aligned}$$

$$\begin{aligned}\tau(\beta\otimes\alpha)&=\tau\beta\otimes\tau\alpha & \tau(\beta\otimes\beta)&=\tau\beta\otimes\tau\beta\\&=(\omega^2\beta)\otimes(\omega\alpha) & &=(\omega^2\beta)\otimes(\omega^2\beta)\\&=1\beta\otimes\alpha & &=\omega\beta\otimes\beta\end{aligned}$$

  - Similarly, we can calculate the effect of σ.

$$\begin{aligned}\sigma(\alpha\otimes\alpha)&=\sigma\alpha\otimes\sigma\alpha & \sigma(\alpha\otimes\beta)&=\sigma\alpha\otimes\sigma\beta & \sigma(\beta\otimes\alpha)&=\sigma\beta\otimes\sigma\alpha & \sigma(\beta\otimes\beta)&=\sigma\beta\otimes\sigma\beta\\&=\beta\otimes\beta & &=\beta\otimes\alpha & &=\alpha\otimes\beta & &=\alpha\otimes\alpha\end{aligned}$$

  - Because the transformations for $\alpha\otimes\alpha$ and $\beta\otimes\beta$ are directly analogous to the untensored case of α and β, these basis vectors span a subrepresentation isomorphic to the standard representation.
  - Because $\sigma(\alpha\otimes\beta)=\beta\otimes\alpha$ is linearly independent of $\alpha\otimes\beta$ (they are literally different basis vectors), $\alpha\otimes\beta+\beta\otimes\alpha$ spans a trivial representation and $\alpha\otimes\beta-\beta\otimes\alpha$ spans an alternating representation.
  - Altogether, we get that if $V=(2,1)$, then

$$V\otimes V\cong(2,1)\oplus(3)\oplus(1,1,1)$$

Week 3

Characters

3.1 Defining Characters

10/9:

- Today, we talk about **characters**, arguably the most important idea in rep theory.
- As per usual, we begin by letting G a finite group.
  - We've been discussing finite dimensional representations of G over $\mathbb{C}$.
  - We've also already talked about irreps, and we know that it's enough to understand those because every rep is a sum of them.
- Goal of characters: Understand the irreps $V_1, \dots, V_k$ of G.
  - Recall the surprising fact about k: It is the number of conjugacy classes of G!
    - ■ We haven't yet proven this, but we will soon!
  - Game plan: Use characters to relate irreps to something that is counted by conjugacy classes.
- Let $V = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$ be a G-rep.
  - Then there exists a homomorphism $\rho : g \mapsto A_g \in GL_n(\mathbb{C})$.
- Motivating question: What doesn't change when we change the basis of V?
  - To isolate the "essence" of the A_g, we want to construct a function $f : GL_n(\mathbb{C}) \to \mathbb{C}$ such that $f(XAX^{-1}) = f(A)$, where $X \in GL_n(\mathbb{C})$.
- Ideas.
  1. The determinant is a great example of such a function, but it's kind of boring because this rank 1 representation doesn't characterize your product representation.
  2. Trace is the main example of such a function.
- Indeed, you can also take $\operatorname{tr}(A^k)$ for any k.
  - Traces of powers are ubiquitous in physics and math because they contain the same information as the coefficients of the characteristic polynomial. In particular, we can express the determinant in terms of them.
- In fact, we could also take any coefficient of the characteristic polynomial, but others would get complicated.
  - Any characteristic polynomial coefficient can be expressed in terms of traces; this will be an exercise in PSet 3; it's not hard.

- For example, the second characteristic polynomial coefficient (sum of products of eigenvalues) can be written as follows.
$$\sum \lambda_i\lambda_j = \frac{\operatorname{tr}(A)^2 - \operatorname{tr}\left(A^2\right)}{2}$$

- So what do we have at this point?
  - We can associate to ρ a function $\chi_\rho : G \to \mathbb{C}$ defined by $\chi_\rho(g) = \operatorname{tr}(A_g) = \operatorname{tr}(\rho(g))$.
  - This function is invariant under isomorphism.
  - If we know $\operatorname{tr}(A)$, we know $\operatorname{tr}\left(A^2\right)$ since $A_g^2 = A_{g^2}$. Thus, if we know all traces, we know all power traces.
  - We form a ring of polynomials?? Equivalently, χ_ρ has a representation as a polynomial with coefficients in $\mathbb{C}$?
- If V is a G-rep, $\chi_V : G \to \mathbb{C}$ will be our notation for its character.
- Properties.
  1. $\chi_V(xgx^{-1}) = \chi_V(g)$ for any $x, g \in G$.
     - Implication: χ_V is a **class function**.
     - Let $\mathbb{C}[G]$ be the vector space of all functions from $G \to \mathbb{C}$. Its dim $= |G|$.
       - Recall that this notation is the same as that for the space of polynomials in $g_1, \dots, g_n$ with complex coefficients.
       - But since G is a group, a polynomial in products of g_i's can be reduced to a polynomial in the g_i's.
         - For example, if $G = S_3$, we know since $(23)(123)^2 = (12)$ that
$$\begin{aligned}2 + 3(12) + i(123) + (-3+7i)(23)(123)^2 &= 2 + [3 + (-3+7i)](12) + i(123)\\ &= 2e + 7i(12) + i(123)\end{aligned}$$
       - Such a polynomial is then easily mapped onto a complex-valued function by sending each g_i to its coefficient a_{g_i}.
         - Continuing with the above example, the corresponding function in $\mathbb{C}[G]$ would be defined by
$$\begin{aligned} &e \mapsto 2 && (12) \mapsto 7i && (123) \mapsto i\\ & && (13) \mapsto 0 && (132) \mapsto 0\\ & && (23) \mapsto 0 && \end{aligned}$$
       - Thus, $\mathbb{C}[G]$ (as a space of polynomials) is canonically isomorphic to $\mathbb{C}[G]$ (as a space of complex-valued functions on G), so the notation is well chosen.
       - What do we do for multiplication?? Because functions multiply pointwise but polynomials do not. It appears that we typically go with polynomial multiplication (but not always, potentially; see the claim from Lecture 6.1). MSE also supports polynomial multiplication. Perhaps $\mathbb{C}[G]$ for this function space with pointwise multiplication is just really misleading notation?
     - Inside this space, there is the subspace $\mathbb{C}_{\text{cl}}[G]$ of functions $f : G \to \mathbb{C}$ such that $f(xgx^{-1}) = f(g)$ for all $x, g \in G$. These are functions from the sets of conjugacy classes, isomorphic to functions that are constant on conjugacy classes. $\dim \mathbb{C}_{\text{cl}}[G]$ is the number of conjugacy classes.
     - Thus, for every V a G-rep, we get a vector $\chi_V \in \mathbb{C}_{\text{cl}}[G]$. These class functions form a basis of the space; each χ_V for V an irrep forms a linearly independent vector; the set is an *orthogonal* basis. This is the reason for the original theorem holding true!

2. $\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$.
   - Proof: It's basically tautological (not actually, but it's easy). Let $g \in G$. Compute $\chi_{V_1 \oplus V_2}(g)$. We can compute a basis $e_1, \dots, e_{n+m}$ where the first n vectors form a basis of V_1, and the next m vectors are a basis of V_2. This gives us a block matrix from which we show that the trace of the matrix is the sum of traces.
   $$\chi_{V_1 \oplus V_2}(g) = \operatorname{tr} \begin{bmatrix} \rho_{V_1}(g) & 0 \\ 0 & \rho_{V_2}(g) \end{bmatrix} = \operatorname{tr} \rho_{V_1}(g) + \operatorname{tr} \rho_{V_2}(g) = \chi_{V_1}(g) + \chi_{V_2}(g)$$
   - Corollary:
   $$\chi_{V_1^{n_1} \oplus \cdots \oplus V_k^{n_k}} = n_1 \chi_{V_1} + \cdots + n_k \chi_{V_k}$$

- We now pause for a fact that will be instrumental in proving the next property, which is a bit more involved.
  - He will explain two ways to prove it; we can also just prove it on our own.
- Fact: If A is a matrix such that $A^n = 1$, then A is diagonalizable or "semi-simple."
  - We can prove this with Jordan normal form.
  - It's a slightly surprising statement.
  - Obviously eigenvalues are roots of unity, but still needs some work.
  - This proof is left as an exercise.
- We now resume the list of properties.

3. $\chi_V(g)$ is a sum of roots of unity.
   - Proof: We know that $g^{|G|} = e$. Thus, $A_g^{|G|} = 1$. It follows by the fact above that A_g is diagonalizable with eigenvalues $\lambda_1, \dots, \lambda_n$, each of which satisfies $\lambda_i^{|G|} = 1$.
     - Note: Eigenvalues can repeat in the list $\lambda_1, \dots, \lambda_n$, i.e., we are not asserting n distinct eigenvalues here.
   - Therefore, since each λ_i is, individually, a root of unity, we have that $\chi_V(g) = \operatorname{tr} A_g = \lambda_1 + \cdots + \lambda_n$, as desired.
4. $\chi_{V^*} = \bar{\chi}_V$.
   - This property begins to address how characters behave under other operations.
     - Naturally, this is something specific for complex numbers, because the idea of "conjugates" doesn't exist everywhere.
   - Proof: Recall that $\rho_{V^*}(g) = (\rho_V(g)^{-1})^T$.
     - If we know that $\rho_V(g) \sim \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, then we know that $\rho_V^{-1}(g)^T \sim \operatorname{diag}(\lambda_1^{-1}, \dots, \lambda_n^{-1})$.
     - Thus, $\chi_{V^*}(g) = \lambda_1^{-1} + \cdots + \lambda_n^{-1}$.
     - But since we're in the complex plane, $|\lambda_i| = 1$ (equiv. $\lambda_i \bar{\lambda}_i = 1$), so $\lambda_i^{-1} = 1/\lambda_i = \bar{\lambda}_i$.
     - This means that $\chi_{V^*}(g) = \bar{\lambda}_1 + \cdots + \bar{\lambda}_n = \overline{\lambda_1 + \cdots + \lambda_n} = \bar{\chi}_V(g)$.
   - Note: Every representation we have is **unitary** in certain bases, but unitary representations are not covered in this course.
5. $\chi_{V_1 \otimes V_2} = \chi_{V_1} \cdot \chi_{V_2}$.
   - Proof: We can use a basis or not use a basis.
   - Let's use a basis for now.
     - Let $g \in G$ be arbitrary. Then there exist bases $e_1, \dots, e_n$ of V_1 and $f_1, \dots, f_m$ of V_2 such that $\rho_{V_1}(g)$ and $\rho_{V_2}(g)$ are diagonal.
     - It follows that $\rho_{V_1}(g)e_i = \lambda_i e_i$ $(i = 1, \dots, n)$ and $\rho_{V_2}(g)f_i = \mu_i f_i$ $(i = 1, \dots, m)$.
     - $V_1 \otimes V_2$ thus has basis $e_i \otimes f_j$.

- But then it follows that $\rho_{V_1\otimes V_2}(g)e_i\otimes f_j=(\lambda_ie_i)\otimes(\mu_jf_j)=\lambda_i\mu_j(e_i\otimes f_j)$.
- Thus,

$$\operatorname{tr}(\rho_{V_1\otimes V_2}(g))=\sum_{i,j=1}^{n,m}\lambda_i\mu_j=(\lambda_1+\cdots+\lambda_n)(\mu_1+\cdots+\mu_m)=\operatorname{tr}(\rho_{V_1}(g))\cdot\operatorname{tr}(\rho_{V_2}(g))$$

- Alternate approach.
  - If we don't want to think of eigenvalues, think of tensor product of matrices, the Kronecker product.
  - Essentially, if we adopt a basis such that our matrices are diagonal, then the block diagonal of the Kronecker product will be $\lambda_1\rho_{V_2}(g)+\cdots+\lambda_n\rho_{V_2}(g)$, the trace of which will be $\lambda_1(\mu_1+\cdots+\mu_m)+\cdots+\lambda_n(\mu_1+\cdots+\mu_m)$.
  - We get trace is the product of traces once again!

- **Class function**: A function on a group G that is constant on the conjugacy classes of G.
- Examples.
  1. Let A be an abelian group.
     - Then $\chi:A\to\mathbb{C}^\times$.
     - Implication: Character of a character is $\chi_\chi=\chi$.
       - This is horribly repetitive but true.
  2. $G=S_3$.

	e	(12) (13) (23)	(123) (132)
Trivial	1	1	1
Alternating	1	-1	1
Standard	2	0	-1

Table 3.1: Character table for S_3.

- The conjugacy classes of this group are $\{e\}$, $\{(12),(13),(23)\}$, and $\{(123),(132)\}$.
- We construct a **character table** to define all characters.
- Computing the characters for the trivial representation.
  - We know that ρ sends each g to the matrix (1), which has trace 1.
- Computing the characters for the sign representation.
  - e and (123) have sign 1 and thus get sent to the matrix (1).
  - (12) has sign -1 and thus gets sent to the matrix (-1).
- Computing the characters for the standard representation.
  - We can compute these traces via a thought experiment.
  - Visualize a triangle in a plane.
  - The 2×2 identity matrix (the standard representation of $e\in G$) acts on it by doing nothing, and has trace 2.
  - In *some* basis, our matrix fixes one vector and inverts another, so matrix is

$$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

and character is 0.

- Last one is rotation by $2\pi/3$, so

$$\begin{pmatrix} \cos(2\pi/3) & \sin(2\pi/3) \\ -\sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}$$

so character is $-1 = 2 \cdot -1/2 = 2 \cdot \cos(2\pi/3)$.

- If V is the standard representation, we can also compute the characters of $V^{\otimes 2}$ for instance. Indeed, by the product rule of characters, they will be the squares of the standard representation's characters, i.e., $(4, 0, 1)$.
- Similarly, since the permutational representation is the direct sum of the standard and trivial representations, we can add their characters to get its characters $(3, 1, 0)$.

3. A very general and very pretty example. Let $G \circlearrowright X$ a finite set.
   - Assign the permutational representation.
   - Let $X = \{x_1, \dots, x_n\}$. Think of these elements as the basis of a vector space; in particular, consider $V = \mathbb{C}e_{x_1} \oplus \cdots \oplus \mathbb{C}e_{x_n}$. Recall that $g(a_1 e_{x_1} + \cdots + a_n e_{x_n}) = a_1 e_{gx_1} + \cdots + a_n e_{gx_n}$. The fact that this is a representation follows immediately from the properties of the group action.
   - Computing the character χ_V of this V: Look at g and write its matrix. In particular, the trace is the number of unmoved/fixed elements, sometimes denoted $\text{Fix}(g)$.
   - This gives us another way of computing V_{perm} from above!

- **Character table**: A table that lists the conjugacy classes across the top, the irreps down the left side, and at each point within it, the value of an irrep's character over that conjugacy class.
  - The character table is a very nice matrix with very nice properties.
  - It is almost orthogonal; not exactly, but very close.
    - Rows aren't orthogonal, but columns are (take direct products)!
    - It is full rank, though.
- The midterm: Take the character table and do fun things with it.

3.2 Office Hours

10/10:

- Problem 1b:
  - Canonically self-dual: $V \cong V^*$ canonically.
- Mathematical methods of quantum mechanics: First few paragraphs of *picture*.
- We should have everything we need to do most of the problem set at this point; maybe not all of 5, but maybe yes, too.
- Problem 3:
  - There is some problem where it decomposes into trivial plus standard, but we still have to prove that standard is irreducible in this case!
  - If you have any vector, you can produce out of this vector something else.
  - If we take any vector and the group acts on it, we'll get a basis. If you hit a vector in the invariant subspace, it will just stay there; if you hit it and it goes everywhere, you get a basis.
  - Now think about a vector when you permute its coordinates.
  - Tomorrow in class, we will learn a quick way to do this problem.
- Problem 5:

- For some problem, we need to use the fact that $A^n = 1$ proves that $A = I$ in some sense.
- This is a hard problem!
- Show that eigenvalues sum to 1; we know that the eigenvalues are roots of unity! Thus, they have to both be 1!
- When the problem in group theory is harder, that's when you need to go to rep theory.

3.3 Characters are Orthonormal

10/11:

- Announcement: Zoom OH today.
- Recap: The big picture.
  - Representations.
    - We have representations, which are vector spaces on which a group acts.
    - With these representations, we can do a bunch of operations we've discussed: $\oplus, \otimes, V^*, \Lambda^n, S^n$.
    - We'll focus on the first 3 for now, though.
  - Class functions.
    - We also have class functions: Functions $f : G \to \mathbb{C}$ such that for all $g, x \in G$, $f(gxg^{-1}) = f(x)$.
    - The space of class functions forms a ring, since you can add, multiply, and take the complex conjugate of these functions.
    - Moreover, this ring is a vector space and it has dimension equal to the number of conjugacy classes of G.
  - The big idea: These two things (representations and class functions) are closely related!
    - There is a map, called a *character*, that pairs a representation to a class function.
    - Indeed, $V \to \chi_V$.
    - Under this map, operations of representations become operations of functions:

$$\oplus \mapsto + \qquad\qquad \otimes \mapsto \cdot \qquad\qquad V^* \mapsto \bar{f}$$

    - Additionally, $V_1, \dots, V_s$ become $\chi_{V_1}, \dots, \chi_{V_s}$.
- Theorem we will prove over the next couple of lectures: Irreps become *linearly independent* class functions, and all irreps form a basis of the space of class functions.
  - This theorem is huge! It is our main takeaway for now.
  - For the first part of the course, this is the main thing that we should remember.
- How do we prove that multiple vectors are linearly independent?
  - A strong condition would be to introduce an inner product and prove that the pairwise inner product of the vectors is zero.
- **Orthonormal basis**: A basis for which $\langle e_i, e_j \rangle = \delta_{ij}$.
- Let's begin carrying out this plan by defining an inner product on $\mathbb{C}[G]$. Indeed, let f_1, f_2 be two functions on G and take

$$\langle f_1, f_2 \rangle := \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}$$

- Motivation for this definition.
  - Recall the **Hermitian inner product** on $\mathbb{C}^n$.
    - We are essentially mapping f_1, f_2 to $(f_1(g_1), \dots, f_1(g_{|G|})), (f_2(g_1), \dots, f_2(g_{|G|})) \in \mathbb{C}^{|G|}$ and taking the Hermitian inner product there.

- Thus, we can see that all properties hold for both the Hermitian inner product on $\mathbb{C}^n$ and the one defined above on $\mathbb{C}[G]$.
- In other words, this kind of construction should inherit its status as a linear, positive definite bilinear form from the Hermitian inner product.

– Note: The Hermitian product above is ***G*-invariant**.

- This means that the functions on G from $G \to \mathbb{C}$ in $\mathbb{C}[G]$ form a representation of G.
- In particular, if $\varphi : G \to \mathbb{C}$, then $g = \rho(g)$ moves it as follows: $g \cdot \varphi = \varphi^g$ where $\varphi^g(h) := \varphi(g^{-1}h)$. Thus, we have an action of G on every φ!
- Such representations are isomorphic for finite groups??

– If we have $\langle f_1, f_2 \rangle$, we can ask if

$$\langle f_1, f_2 \rangle \stackrel{?}{=} \langle f_1^g, f_2^g \rangle$$

- Left as an exercise that this *is* true!

- **Hermitian inner product** (on $\mathbb{C}^n$): The inner product defined as follows for all $z, w \in \mathbb{C}^n$. *Denoted by* $\langle\ ,\ \rangle$. *Given by*

$$\langle z, w \rangle = \sum_{i=1}^{n} z_i \bar{w}_i$$

– This inner product gives a complex number $\langle v, w \rangle \in \mathbb{C}$ with the following properties.

1. $\langle a_1 v_1 + a_2 v_2, w \rangle = a_1 \langle v_1, w \rangle + a_2 \langle v_2, w \rangle$.
2. $\langle v, b_1 w_1 + b_2 w_2 \rangle = \bar{b}_1 \langle v, w_1 \rangle + \bar{b}_2 \langle v, w_2 \rangle$.
3. $\langle v, v \rangle \geq 0$, and $\langle v, v \rangle = 0$ implies that $v = 0$.

– Thus, if $v = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$, then

$$\langle v, w \rangle = \sum z_i \bar{w}_i \qquad\qquad \langle v, v \rangle = \sum |z_i|^2$$

- We now begin tackling today's main theorem: If V_1, V_2 are irreps, then

$$\langle \chi_{V_1}, \chi_{V_2} \rangle = \begin{cases} 0 & V_1 \ncong V_2 \\ 1 & V_1 \cong V_2 \end{cases}$$

– We will prove this theorem in stages.

– The general outline of our approach is to deduce the equality step by step through the transitive property. Some of the equalities we'll eventually end up needing are easier to discuss on their own first, though, so we begin with some lemmas.

- First off, recall the **space of invariants** from PSet 2.

- **Space of invariants** (of a representation V): The vector space defined as follows. *Denoted by* $\boldsymbol{V^G}$. *Given by*

$$V^G = \{v \in V \mid gv = v \ \forall\ g \in G\}$$

- Lemma 1: Let G be a finite group, let $\rho : G \to GL(V)$ be a finite-dimensional representation of it, and let p be defined as above. Then $p \in \operatorname{Hom}_G(V, V)$.

Proof. We can view p as an element of $\operatorname{Hom}(V, V)$. This combined with the fact that for every $h \in G$,

$$p(hv) = \frac{1}{|G|} \sum_{g \in G} (gh)v = \frac{1}{|G|} \sum_{gh \in G} (gh)v = \frac{1}{|G|} h \sum_{g \in G} gv = h(pv)$$

implies that $p \in \mathrm{Hom}_G(V, V)$. In more formal notation,

$$\begin{aligned}
[p \circ \rho_V(h)](v) &= \frac{1}{|G|}\sum_{g\in G}[\rho_V(g) \circ \rho_V(h)](v) \\
&= \frac{1}{|G|}\sum_{g\in G}[\rho_V(gh)](v) \\
&= \frac{1}{|G|}\sum_{gh\in G}[\rho_V(gh)](v) \\
&= \frac{1}{|G|}\sum_{hg\in G}[\rho_V(hg)](v) \\
&= \frac{1}{|G|}\sum_{g\in G}[\rho_V(hg)](v) \\
&= [\rho_V(h)]\left(\frac{1}{|G|}\sum_{g\in G}[\rho_V(g)](v)\right) \\
&= [\rho_V(h) \circ p](v)
\end{aligned}$$

□

- Why do we need this result?? What does it do for the rest of the proof?
- Lemma 2: Let G be a finite group, and let $\rho : G \to GL(V)$ be a finite-dimensional representation of it. Then the map p, defined as follows, is a projector from $V \to V^G$.

$$p = \frac{1}{|G|}\sum_{g\in G} g = \frac{1}{|G|}\sum_{g\in G}\rho_V(g)$$

Proof. To prove that p is a projector, it will suffice to show that $p^2 = p$. To prove that p projects onto V^G, it will suffice to show that $\mathrm{Im}(p) = V^G$. Let's begin.

To show that $p^2 = p$, we have

$$p^2 = \left(\frac{1}{|G|}\sum_{g\in G} g\right)^2 = \frac{1}{|G|^2}\sum_{g_1,g_2\in G} g_1g_2 = \frac{|G|}{|G|^2}\sum_{g\in G} g = p$$

Note that since G is not abelian (i.e., $g_1g_2 \neq g_2g_1$ in all cases), the square of $\sum g$ is as above and cannot be reduced to a smaller sum with a 2 coefficient or something like that. Additionally, note that $\sum_{g_1,g_2\in G} g_1g_2 = |G|\sum g$ since for each g_i, $g_i(g_1 + \cdots + g_{|G|}) = g_1 + \cdots + g_{|G|}$.

To show that $\mathrm{Im}(p) = V^G$, we will use a bidirectional inclusion proof. To confirm that $\mathrm{Im}(p) \subset V^G$, we have for any $h \in G$ that

$$h\left(\frac{1}{|G|}\sum_{g\in G} gv\right) = \frac{1}{|G|}\sum_{hg\in G} hgv = \frac{1}{|G|}\sum_{g\in G} gv$$

from which it follows that

$$p(v) = \frac{1}{|G|}\sum gv \in V^G$$

as desired. To confirm that $V^G \subset \mathrm{Im}(p)$, let $v \in V^G$. Then $gv = v$. It follows that

$$v = \frac{1}{|G|}\sum_{g\in G} v = \frac{1}{|G|}\sum_{g\in G} gv = p(v) \in \mathrm{Im}(p)$$

as desired. □

- You differentiated the first and second parts of the above proof by saying, "this is the algebraic way to prove it; we can also prove it nonalgebraically." Does this mean that $p^2 = p$ somehow *implies* $\mathrm{Im}(p) = V^G$ here, or do we still need to prove that "nonalgebraically," as in Fulton and Harris (2004)??
- Consequence of Lemma 2: There's a very easy way to construct invariant factors.
- We now prove one final lemma using what we have learned about p.
- Lemma 3: Let G be a finite group, and let $\rho : G \to GL(V)$ be a finite-dimensional representation of it. Then $\dim V^G = (1/|G|) \sum_{g\in G} \chi_V(g)$.

  Proof. Define p as above. Then

$$\begin{aligned}
\dim V^G &= \dim(\mathrm{Im}(p)) && \text{Lemma 2}\\
&= \mathrm{tr}(p) && \text{PSet 1, Q5c}\\
&= \mathrm{tr}\left(\frac{1}{|G|}\sum \rho_V(g)\right)\\
&= \frac{1}{|G|}\sum_{g\in G}\mathrm{tr}(\rho_V(g))\\
&= \frac{1}{|G|}\sum_{g\in G}\chi_V(g)
\end{aligned}$$

  as desired. □

- We can now prove the main result.
- Theorem: If V, W are irreps, then

$$\langle \chi_V, \chi_W\rangle = \begin{cases} 0 & V \ncong W\\ 1 & V \cong W\end{cases}$$

  Proof. We will work towards a formula for the inner product, using various results that we've proven up until now. Let's begin.

$$\begin{aligned}
\langle \chi_V, \chi_W\rangle &= \frac{1}{|G|}\sum_{g\in G}\chi_V(g)\cdot\overline{\chi_W(g)} && \text{Definition}\\
&= \frac{1}{|G|}\sum_{g\in G}\chi_V(g)\cdot\chi_{W^*}(g) && \text{Property 4}\\
&= \frac{1}{|G|}\sum_{g\in G}\chi_{V\otimes W^*}(g) && \text{Property 5}\\
&= \dim[(V\otimes W^*)^G] && \text{Lemma 3}\\
&= \dim([\mathrm{Hom}_F(V,W)]^G) && \text{Lecture 2.1}\\
&= \dim[\mathrm{Hom}_G(V,W)] && \text{PSet 2, Q4b}\\
&= \begin{cases} \dim(\mathrm{span}(I)) & V \cong W\\ \dim(\mathrm{span}(0)) & V \ncong W\end{cases} && \text{Schur's Lemma}\\
&= \begin{cases} 0 & V \ncong W\\ 1 & V \cong W\end{cases}
\end{aligned}$$

  □

- In the above proof, Rudenko first surveys the following special case. Why??
  - Then if V is irreducible and trivial, we have

    $$\frac{1}{|G|}\sum_{g\in G}\chi_V(g)=0$$

    which happens iff

    $$\langle\chi_V,\chi_{\text{triv}}\rangle=0$$

    whereas

    $$\langle\chi_{\text{triv}},\chi_{\text{triv}}\rangle=1$$

    This proves the theorem in a special case, but how do we go from here to all representations? We're very close!
- Corollary: The number of irreps is less than or equal to the number of conjugacy classes.
  - We'll leave it to next time to prove that equality holds.
- Whenever we have a sec, we should try to form a mental picture the whole class function thing.
- Consequence of the theorem: We get an orthogonality relation.
  - If χ_1,χ_2 are characters if irreps, then

    $$\sum_{g\in G}\chi_1(g)\overline{\chi_2(g)}=\begin{cases}0 & \chi_1\neq\chi_2\\ |G| & \chi_1=\chi_2\end{cases}$$

  - This is related to the character table and IChem!!! Example:
    - Recall Table 3.1, the character table for S_3.
    - Between the trivial and alternating representations, we have

      $$(1)(1)+(1)(-1)+(1)(-1)+(1)(-1)+(1)(1)+(1)(1)=0$$

      as expected. Note that we have a term for each element in S_3, so some products get repeated multiple times.
    - For the standard representation, we have

      $$(2)(2)+(0)(0)+(0)(0)+(0)(0)+(-1)(-1)+(-1)(-1)=6=|S_3|$$

      as expected.
- Theorem: Characters are equal iff their representations are isomorphic.
- Next time.
  - Prove the theorem.
  - Consequences.
  - Implications for the character table.

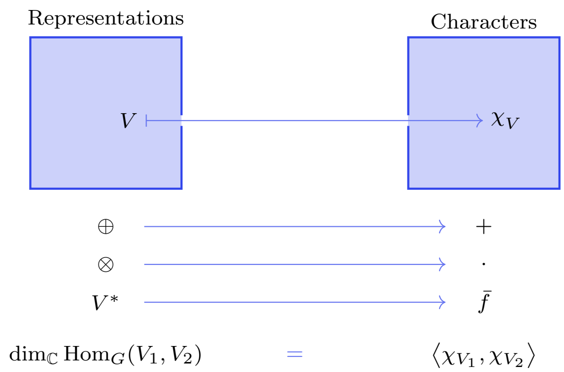


Figure 3.1: The stories of representation theory.

3.4 Character Table Properties

10/13:

- Announcement: Midterm on November 10.
  - Will mostly involve computing character tables; HW will be good prep.
- Review of the general picture from the first part of the course.
  - Let G be a finite group.
  - First story: We study finite-dimensional representations of G over $\mathbb{C}$; these are vector spaces, so we can direct sum, tensor multiply, and dualize them. We can also look at the morphisms between them.
  - Second story: We study class functions $\mathbb{C}_{\text{cl}}[G] = \{f : G \to \mathbb{C} \mid f(gxg^{-1}) = f(x)\}$; these are elements of a ring, so we can add, multiply, and conjugate them. We can also take the inner product of them.
  - We can map between these two stories: Representations become characters, $\oplus \mapsto +$, $\otimes \mapsto \cdot$, and $V^* \mapsto \bar{f}$.
- Theorem (from last time):

$$\langle \chi_{V_1}, \chi_{V_2} \rangle = \dim \operatorname{Hom}_G(V_1, V_2)$$

- The story in Figure 3.1 tells us stuff about representations.
  - Let $V_1, \dots, V_k$ be irreps. Then the vectors $\chi_{V_1}, \dots, \chi_{V_k}$ are orthonormal.
    - We get this result with the Theorem above and Schur's Lemma.
  - Next time, we'll prove that $\chi_{V_1}, \dots, \chi_{V_k}$ spans $\mathbb{C}_{\text{cl}}[G]$, i.e., the number of irreps is the number of conjugacy classes.
  - *Cube thing??*
  - This picture is remarkable because it's so simple.
- We now look at some corollaries to last time's main theorem.
- Corollary 1: If V, W are G-reps, then $\chi_V = \chi_W$ iff $V \cong W$.

Proof. Invoking complete reducibility, we have that $V = \bigoplus V_i^{n_i}$. Thus, to know V, it is enough to know the n_i's. But

$$\chi_V = \sum n_i \chi_{V_i}$$

where

$$n_i = n_i \cdot 1 = n_i \langle \chi_{V_i}, \chi_{V_i} \rangle = \langle \chi_V, \chi_{V_i} \rangle$$

Therefore, since the χ_{V_i} are linearly independent, the only way that $\chi_V = \chi_W$ is if the n_i's match which would mean that $V \cong W$, and vice versa the only way that $V \cong W$ is if the n_i's match which would mean that $\chi_V = \chi_W$. ☐

- Corollary 2: Let V be a G-rep. Then TFAE:
  1. V is irreducible.
  2. $\langle \chi_V, \chi_V \rangle = 1$.
  3. $\sum_{g\in G} |\chi_V(g)|^2 = |G|$.

  Proof. ($1 \Rightarrow 2$): We have that
  $$\langle \chi_V, \chi_V \rangle = \dim \operatorname{Hom}_G(V, V) = 1$$
  as desired.

  ($2 \Rightarrow 1$): Complete reducibility implies that $V \cong V_1^{n_1} \oplus \cdots \oplus V_k^{n_k}$, where the V_i's are irreps. This combined with the hypothesis implies that
  $$1 = \langle \chi_V, \chi_V \rangle = \left\langle \sum_{i=1}^k n_i \chi_{V_i}, \sum_{i=1}^k n_i \chi_{V_i} \right\rangle = \sum_{i=1}^k n_i^2$$
  But if $\sum n_i^2 = 1$ where each $n_i \in \mathbb{Z}^+$, then $n_i = 1$ for some i and $n_j = 0$ for $j \neq i$, from which it follows that $V \cong V_i$.

  We can interconvert between 2 and 3 using the definition of the inner product and the property of complex numbers that $zz^* = |z|^2$. ☐

- We now build up to one final corollary.
- We've discussed all of these properties of irreps, but where do we even find them?
  - We might be able to find some by inspection, but here's how we find all of them.
- Review: The regular representation. Here are two different but isomorphic ways to think about it.
  - Think of it as functions on G.
    - Better for infinite groups.
  - Think of it as the permutational representation associated with the action $G \circlearrowright G$.
    - Better for finite groups.
  - Why did we talk about this here??
- Corollary 3: Consider the regular representation V_R. We have that
  $$\chi_{V_R}(g) = \begin{cases} 0 & g \neq e \\ |G| & g = e \end{cases}$$

  Proof. We can compute its character χ_{V_R} by considering the corresponding permutation matrices. Indeed, the action $\chi_{V_R}(g)$ of this character on g is equal to the number of 1's on the diagonal in the permution matrix, which is equal to the number of fixed points of the permutation, i.e., the number of i's such that $gg_i = g_i$. But in a group, $gg_i = g_i$ iff $g = e$, so this number of fixed points is
  $$\chi_{V_R}(g) = \operatorname{Fix}(g) = \begin{pmatrix} g_1 & \cdots & g_n \\ gg_1 & \cdots & gg_n \end{pmatrix} = \begin{cases} 0 & g \neq e \\ |G| & g = e \end{cases}$$
  as desired. ☐

- What is the matrix thing?
  - It is most likely a representation of the function $g_i \mapsto gg_i$, and it denotes that $\text{Fix}(g)$ is equal to the number of columns that have the same entry top and bottom.
- We now apply Corollaries 1-3 to the regular representation V_R to obtain some important results.
  - Let V_i be an arbitrary irrep.
  - By complete reducibility, $V_R = \bigoplus_{i=1}^k V_i^{n_i}$ for some set of n_i's.
  - Additionally,

$$\begin{aligned}
n_i &= \langle \chi_{V_R}, \chi_{V_i} \rangle && \text{Corollary 1} \\
&= \frac{1}{|G|} \sum_{g \in G} \chi_{V_R}(g) \overline{\chi_{V_i}(g)} \\
&= \frac{1}{|G|} |G| \underbrace{\overline{\chi_{V_i}(e)}}_{\dim V_i} && \text{Corollary 3} \\
&= \dim V_i
\end{aligned}$$

  - This implies three remarkable results, all worth remembering.

$$V_R = \bigoplus_{i=1}^k V_i^{\dim V_i} \qquad |G| = \sum_{i=1}^k (\dim V_i)^2 \qquad \#\text{ irreps is finite}$$

    - The first result follows directly by substituting $n_i = \dim V_i$ into complete reducibility.
    - The second result follows because $|G| = \dim(V_R) = \dim(\bigoplus_{i=1}^k V_i^{\dim V_i}) = \sum (\dim V_i)^2$.
    - The third result follows because if there were infinitely many irreps, each with $\dim V_i \geq 1$, then $|G| = \sum_{i=1}^k (\dim V_i)^2 = \infty$, contradicting the hypothesis that $|G|$ is finite.
- We want to investigate S_4, i.e., characterize all irreps of it.

	1 e	6 (12)	8 (123)	3 (12)(34)	6 (1234)
Trivial	1	1	1	1	1
Sign	1	-1	1	1	-1
V_{std}	3	1	0	-1	-1
Sign $\otimes V_{\text{std}}$	3	-1	0	-1	1
	2	0	-1	2	0

Table 3.2: Character table for S_4.

  - We do so by constructing the character table, Table 3.2.
  - Initially, this seemed like a very hard problem.
    - However, with all of our theory, it only takes a couple of minutes now!
  - We start by inputting the trivial, sign, and standard representations.
    - The trivial is obviously $(1, 1, 1, 1, 1)$.
    - The sign can be calculated to be $(1, -1, 1, 1, -1)$.
    - The standard is found via $V_{\text{std}} = V_{\text{perm}} - V_{\text{triv}} = (4, 2, 1, 0, 0) - (1, 1, 1, 1, 1) = (3, 1, 0, -1, -1)$.

- Note that

$$\sum n_i^2 = \left\langle \chi_{V_{\text{perm}}}, \chi_{V_{\text{perm}}} \right\rangle = \frac{1}{24}(1 \cdot 4^2 + 6 \cdot 2^2 + 8 \cdot 1^2 + 0 \cdot 0^2 + 0 \cdot 0^2) = 2$$
$$\left\langle \chi_{V_{\text{perm}}}, \chi_{V_{\text{sign}}} \right\rangle = \frac{1}{24}(4 - 12 + 8) = 0$$
$$\left\langle \chi_{V_{\text{perm}}}, \chi_{V_{\text{triv}}} \right\rangle = 1$$

  - What is the point of these calculations??
- Thus, we can derive representations without having any geometric notion of it using characters!
- To see any of these representations geometrically, look at actions on the tetrahedron in $\mathbb{R}^3$!
- All those computations above used the **first orthogonality relation**; here's the **second orthogonality relation**:

$$\sum_{\chi} \chi(g_1)\overline{\chi(g_2)} = \begin{cases} 0 & g_1 \nsim g_2 \\ \frac{|G|}{|C_G(g)|} & g_1 \sim g_2 \end{cases}$$

  where $\sim$ denotes conjugacy and $C_G(g)$ is the number of elements in the conjugacy class of g (where we borrow centralizer notation).
  - Prove the new one with $AB = 1 = BA$. This very simple thing leads to a very powerful statement about systems of equations that we will discuss later. How does this proof work??
  - We can start doing this stuff in the new homework!
  - We constructed the fifth representation using this.
- Where does the fifth irrep come from?
  - Going back to a miracle of group theory: Simple groups.
  - If $f : S_n \to S_n$, we have lots of injective maps, lots of minor actions $S_n \to S_n$ sending $x \mapsto gxg^{-1}$.
  - We have sign : $S_n \to S_2$, $S_4 \twoheadrightarrow S_3$ with kernel equal to K_4.
  - We have the exotic $S_6 \to S_6$.
  - We have $S_5 \hookrightarrow S_6$ that is also exotic.
  - These are called **exceptional homomorphisms**.
  - Since we have $S_4 \to S_3$ and $\rho : S_3 \to GL_n$, we have $S_4 \to GL_n$ with a the same character table. Takeaway: This $(2, 0, 1)$ thing in the big character table comes from this map, geometrically.
  - Takeaway: The geometry of the fifth irrep comes from S_3.
  - What is going on here??

- Final announcements.
  - Going forward, we'll mostly be following Fulton and Harris (2004).
  - Then we'll get into associative algebra.
  - OH next week will be Zoom, but we should still feel free to meet with him in-person by emailing him for an appointment.

3.5 S Chapter 2: Character Theory

From Serre (1977).

Section 2.1: The Character of a Representation

11/2: • Definition of **trace**.

- **Character** (of ρ): The complex-valued function on G defined as follows. *Denoted by* $\boldsymbol{\chi_\rho}$. *Given by*

$$\chi_\rho(s) = \operatorname{tr}(\rho_s)$$

- Notation: Serre (1977) uses both z^* and $\bar{z}$ to denote the complex conjugate of a $z \in \mathbb{C}$.
- Properties of the character as a function.

Proposition 1. *If χ is the character of a representation ρ of degree n, then...*

(i) $\chi(1) = n$;

(ii) $\chi(s^{-1}) = \chi(s)^*$ *for all* $s \in G$;

(iii) $\chi(tst^{-1}) = \chi(s)$ *for all* $s \in G$.

Proof. (i):

$$\chi(1) = \operatorname{tr}(\rho_1) = \operatorname{tr}(1) = n$$

(ii): Recall that there exists a basis with respect to which ρ_s is unitary. Thus, its (not necessarily unique) eigenvalues $\lambda_1, \dots, \lambda_n$ are roots of unity. Consequently, $\lambda_i^{-1} = \lambda_i^*$ $(i = 1, \dots, n)$. Therefore,

$$\chi(s^{-1}) = \operatorname{tr}(\rho_{s^{-1}}) = \operatorname{tr}\big(\rho_s^{-1}\big) = \sum \lambda_i^{-1} = \sum \lambda_i^* = \operatorname{tr}(\rho_s)^* = \chi(s)^*$$

(iii): This follows directly from the fact that the trace is invariant under choice of basis. Alternatively (and more explicitly), we may put $u = \rho_t\rho_s$ and $v = \rho_{t^{-1}}$ and use the formula $\operatorname{tr}(ab) = \operatorname{tr}(ba)$[1] as follows.

$$\chi(tst^{-1}) = \operatorname{tr}(\rho_{tst^{-1}}) = \operatorname{tr}(\rho_t\rho_s\rho_{t^{-1}}) = \operatorname{tr}(uv) = \operatorname{tr}(vu) = \operatorname{tr}(\rho_{t^{-1}}\rho_t\rho_s) = \operatorname{tr}(\rho_s) = \chi(s)$$

□

- **Class function**: A function f on G satisfying identity (iii) in Proposition 1.
- Properties of the character with respect to representations.

Proposition 2. *Let $\rho^1 : G \to GL(V_1)$ and $\rho^2 : G \to GL(V_2)$ be two linear representations of G, and let χ_1, χ_2 be their characters. Then...*

(i) *The character χ of the direct sum representation $V_1 \oplus V_2$ is equal to $\chi_1 + \chi_2$;*

(ii) *The character ψ of the tensor product representation $V_1 \otimes V_2$ is equal to $\chi_1 \cdot \chi_2$.*

Proof. As in class. □

- Serre (1977) defines the characters of the symmetric square and alternating square representations.

[1] See Theorem 10.4 of Axler (2015).

Section 2.2: Schur's Lemma; Basic Applications

- Serre (1977) states and proves Schur's Lemma.

  Proposition 4 (Schur's Lemma). *Let $\rho^1 : G \to GL(V_1)$ and $\rho^2 : G \to GL(V_2)$ be two irreducible representations of G, and let f be a linear mapping of V_1 into V_2 such that $\rho_s^2 \circ f = f \circ \rho_s^1$ for all $s \in G$. Then...*

  (i) *If ρ^1 and ρ^2 are not isomorphic, then $f = 0$;*

  (ii) *If $V_1 = V_2$ and $\rho^1 = \rho^2$, then f is a* ***homothety***.

  Proof. This proof is identical to the one given in Fulton and Harris (2004). However, I am including it again here because it includes a few key details that Fulton and Harris (2004) leave out. Let's begin.

  (i): To prove the claim, it will suffice to prove the contrapositive. Suppose $f \neq 0$. To prove that $\rho^1 \cong \rho^2$, it will suffice to show that f is an isomorphism of G-representations. Since f is a morphism of G-representations by hypothesis, all that is left is to show that it is an isomorphism of vector spaces. To do so, we will demonstrate that $\text{Ker}(f) = 0$ and $\text{Im}(f) = V_2$, one claim at a time. Let's begin.

  Let $W_1 = \text{Ker}(f)$. Then for any $s \in G$ and $x \in W_1$,

  $$f(\rho_s^1(x)) = [f \circ \rho_s^1](x) = [\rho_s^2 \circ f](x) = \rho_s^2(f(x)) = \rho_s^2(0) = 0$$

  so $\rho_s^1(x) \in W_1$. It follows that W_1 is stable under G. But since V_1 is irreducible, this means that $W_1 = V_1, 0$. This combined with the fact that $f \neq 0$ (hence $W_1 = \text{Ker}(f) \neq V_1$) implies that $W_1 = 0$.

  Let $W_2 = \text{Im}(f)$. Then for any $s \in G$ and $f(x) \in W_2$,

  $$\rho_s^2(f(x)) = [\rho_s^2 \circ f](x) = [f \circ \rho_s^1](x) = f(\rho_s^1(x))$$

  so $\rho_s^2(fx) \in W_2$. It follows that W_2 is stable under G. But since V_2 is irreducible, this means that $W_2 = V_2, 0$. This combined with the fact that $f \neq 0$ (hence $W_2 = \text{Im}(f) \neq 0$) implies that $W_2 = V_2$.

  (ii): Since f is an operator on a finite-dimensional, nonzero, complex vector space, it has[2] an eigenvalue λ. To prove that f is a homothety, it will suffice to show that $f = \lambda I$, which we will do by demonstrating that $f - \lambda I = 0$ using part (i). Indeed, to use part (i) in this manner, we need only show that $f - \lambda I$ *does* satisfy $\rho_s^2 \circ (f - \lambda I) = (f - \lambda I) \circ \rho_s^1$ but *is not* an isomorphism of vector spaces. For the first claim, we have since $\rho^1 = \rho^2$ and $V_1 = V_2$ that

  $$\rho_s^2 \circ (f - \lambda I) = \rho_s^2 \circ f - \rho_s^2 \circ \lambda I = f \circ \rho_s^1 - \lambda I \circ \rho_s^2 = f \circ \rho_s^1 - \lambda I \circ \rho_s^1 = (f - \lambda I) \circ \rho_s^1$$

  For the second claim, we know that the eigenvector corresponding to λ is in $\text{Ker}(f - \lambda I)$, so $f - \lambda I$ has a nontrivial kernel and thus cannot be an isomorphism. ☐

- **Homothety**: A scalar multiple of the identity operator. *Given by*

  $$\lambda I$$

  for some $\lambda \in \mathbb{C}$.

11/8: - Another condition for telling if two representations are isomorphic.

  Corollary 1. *Let h be a linear mapping of V_1 into V_2 and put*

  $$h^0 = \frac{1}{g} \sum_{t \in G} (\rho_t^2)^{-1} h \rho_t^1$$

  with $g = |G|$. Then...

[2] See Theorem 5.5 of Axler (2015).

(i) *If ρ^1 and ρ^2 are not isomorphic, we have $h^0 = 0$;*
(ii) *If $V_1 = V_2$ and $\rho^1 = \rho^2$, h^0 is a homothety of ratio $(1/n)\operatorname{tr}(h)$ with $n = \dim(V_1)$.*

Proof. Given. □

- The above corollary in matrix form:

  Corollary 2. *Let $(r_{i_1j_1}(t)) := \rho_t^1$, $(r_{i_2j_2}(t)) := \rho_t^2$, $(x_{i_2i_1}) := h$, and*

$$x^0_{i_2i_1} := h^0 = \frac{1}{g}\sum_{t,j_1,j_2} r_{i_2j_2}(t^{-1})x_{j_2j_1}r_{j_1i_1}(t)$$

  Then if $(r_{i_1j_1}(t))$ and $(r_{i_2j_2}(t))$ are not isomorphic, we have

$$\frac{1}{g}\sum_{t,j_1,j_2} r_{i_2j_2}(t^{-1})r_{j_1i_1}(t) = 0$$

  for arbitrary i_1, i_2, j_1, j_2.

  Proof. Described. □

  Corollary 3. *Under the same definitions as before, if $(r_{i_1j_1}(t))$ and $(r_{i_2j_2}(t))$ are isomorphic, we have*

$$\frac{1}{g}\sum_{t\in G} r_{i_2j_2}(t^{-1})r_{j_1i_1}(t) = \frac{1}{n}\delta_{i_2i_1}\delta_{j_2j_1}$$

  Proof. Described. □

- Note that under the notation

$$\langle\phi,\psi\rangle = \frac{1}{g}\sum_{t\in G}\phi(t^{-1})\psi(t) = \frac{1}{g}\sum_{t\in G}\phi(t)\psi(t^{-1})$$

  we have...
  - $\langle\phi,\psi\rangle = \langle\psi,\phi\rangle$;
  - $\langle\phi,\psi\rangle$ is linear in ϕ and ψ;
  - Corollary 2 becomes $\langle r_{i_2j_2}, r_{j_1i_1}\rangle = 0$;
  - Corollary 3 becomes $\langle r_{i_2j_2}, r_{j_1i_1}\rangle = (1/n)\delta_{i_2i_1}\delta_{j_2j_1}$.
- Furthermore, note that if we choose the $(r_{ij}(t))$ to be unitary, then $r_{ij}(t^{-1}) = r_{ji}(t)^*$ and Corollaries 2, 3 are **orthogonality relations** for the scalar product (ϕ, ψ) defined below.

Section 2.3: Orthogonality Relations for Characters

- **Scalar product**: A binary operation $(x \mid y)$, where x, y are elements of some vector space V, that is linear in x, semilinear in y, and satisfies $(x \mid x) > 0$ for all $x \neq 0$.
- Define a scalar product on the space of complex valued functions on G by

$$(\phi \mid \psi) = \frac{1}{g}\sum_{t\in G}\phi(t)\psi(t)^*$$

- Defining $\tilde{\psi}(t) := \psi(t^{-1})^*$, we have that

$$(\phi \mid \psi) = \frac{1}{g}\sum_{t\in G}\phi(t)\tilde{\psi}(t^{-1}) = \left\langle\phi,\tilde{\psi}\right\rangle$$

- Special case: By Proposition 1, $\tilde{\chi} = \chi$, so

$$(\phi \mid \chi) = \langle \phi, \chi \rangle$$

- We now prove a theorem analogous to the main theorem from Wednesday's lecture.

Theorem 3.

(i) *If χ is the character of an irreducible representation, we have $(\chi \mid \chi) = 1$ (i.e., χ is "of norm 1").*
(ii) *If χ and χ' are the characters of two nonisomorphic irreducible representations, we have $(\chi \mid \chi') = 0$ (i.e., χ, χ' are orthogonal).*

Proof. Extremely clean argument using Corollaries 2-3. □

- **Irreducible character**: A character of an irreducible representation.
- With this definition in hand, we can see that the irreducible characters form an orthonormal system.
- With the previous result in hand, we can use characters to count how many times an irreducible representation occurs within the direct sum decomposition of a representation.

Theorem 4. *Let V be a linear representation of G, with character ϕ, and suppose V decomposes into a sum of irreducible representations via*

$$V = W_1 \oplus \cdots \oplus W_k$$

Then if W is an irreducible representation with character χ, the number of W_i isomorphic to W is equal to the scalar product $(\phi \mid \chi) = \langle \phi, \chi \rangle$.

Proof. Let χ_i be the character of W_i. Then by Proposition 2, we have

$$\phi = \chi_1 + \cdots + \chi_k$$

Now recall from Theorem 3 that $(\chi_i \mid \chi)$ is 1 or 0 depending on whether W_i is or is not isomorphic to W, respectively. Thus, in the sum

$$(\phi \mid \chi) = (\chi_1 \mid \chi) + \cdots + (\chi_k \mid \chi)$$

all terms for which $\chi_i \neq \chi$ go to zero and all terms for which $\chi_i = \chi$ go to 1, leaving a sum that adds 1 for every occurrence of W, as desired. □

- We now explore some results that immediately follow from this result.
- There is a uniqueness in the decomposition of a representation into irreducible representations.

Corollary 1. *The number of W_i isomorphic to W does not depend on the chosen decomposition.*

- Matching Corollary 1 from Friday's class.

Corollary 2. *Two representations with the same character are isomorphic.*

- Note on this result.
  - This is what reduces the study of representations to the study of characters.
- Serre (1977) reiterates the formula

$$(\phi \mid \phi) = \sum_{i=1}^{h} m_i^2$$

where ϕ is the character of $V = W_1^{m_1} \oplus \cdots \oplus W_h^{m_h}$.

- An **irreducibility criterion** based on Corollary 2 from Friday's class.

Theorem 5. *If ϕ is the character of a representation V, $(\phi \mid \phi)$ is a positive integer and we have $(\phi \mid \phi) = 1$ if and only if V is irreducible.*

Proof. See class. □

Section 2.4: Decomposition of the Regular Representation

- Notation.
  - $\chi_1, \dots, \chi_h$: The irreducible characters of G.
  - $n_1, \dots, n_k$: The degrees of $\chi_1, \dots, \chi_h$; $n_i = \chi_i(1)$.
  - $\rho : G \to GL(R)$: The regular representation of G.
- Matching Corollary 3 from Friday's class: Calculating ρ_s for all $s \in G$.

  Proposition 5. *The character r_G of the regular representation is given by the formulas*

  $$r_G(1) = g \qquad\qquad r_G(s) = 0$$

  where we assume $s \neq 1$ in the right equation above.

  Proof. If $s \neq e$, then $st \neq t$ for all $t \in G$. Thus, the diagonal terms of ρ_s are all 0 in this case, so $\operatorname{tr}(\rho_s) = 0$.

  If $s = e$, then $st = t$ for all $t \in G$. Thus, the diagonal terms of ρ_s are all 1 in this case, so $\operatorname{tr}(\rho_s) = \operatorname{tr}(1) = \dim(R) = g$. □

- Matching the leftmost consequence of Corollary 3 from Friday's class.

  Corollary 1. *Every irreducible representation W_i is contained in the regular representation with multiplicity equal to its degree n_i*

- Matching the middle consequence of Corollary 3 from Friday's class (i) and the orthogonality of the first column in a character table with every other column, a special case of the second orthogonality criterion (ii).

  Corollary 2.

  (i) *The degrees n_i satisfy the relation*

  $$\sum_{i=1}^{h} n_i^2 = g$$

  (ii) *If $s \in G$ is different from 1, we have*

  $$\sum_{i=1}^{h} n_i \chi_i(s) = 0$$

  Proof. Corollary 1 says that $r_G(s) = \sum n_i \chi_i(s)$ for all $s \in G$. Taking $s = 1$ gives (i) and $s \neq 1$ gives (ii). □

3.6 FH Chapter 2: Characters

From Fulton and Harris (2004).

Section 2.1: Characters

- The example from last time suggests that "knowing all the eigenvalues of each element of G [that is, each $\rho(g)$] should suffice to describe the representation" (Fulton & Harris, 2004, p. 12).
  - We formalize this notion via **character theory**.

  - In particular, note that we do not actually need to specify all eigenvalues of all elements of G; rather, we can opt to specify their sums, since knowing the $\sum \lambda_i^k$ is equivalent to knowing the $\{\lambda_i\}$ of g. This motivates the definition of the character as the trace!
- Definition of a **character** and **class function**.
- Proposition 2.1: Properties of the character as discussed in class.
- Example 2.5: References the character of the permutation representation and $\text{Fix}(g)$.
- An alternate way of deriving the characters of the standard representation.
  - Calculate for the permutation representation $(3,1,0)$, which is easy.
  - Note that $V_{\text{perm}} = V_{(2,1)} \oplus V_{(3)}$, so $(3,1,0) = (x,y,z) + (1,1,1)$, meaning that the character of the standard representation is $(2,0,-1)$, as desired.
- Using characters to decompose arbitrary representations W of S_3.
  - We take it for granted that $W = V_{(3)}^a \oplus V_{(2,1)}^b \oplus V_{(1,1,1)}^c$.
  - But then $\chi_W = a\chi_{V_{(3)}} + b\chi_{V_{(2,1)}} + c\chi_{V_{(1,1,1)}}$. Moreover, since the three characters are linearly independent, we can solve the system of equations for a,b,c.
  - Takeaway: "W is determined up to isomorphism by its character χ_W" (Fulton & Harris, 2004, p. 14).
- There are some great exercises throughout this section that I could come back to later for more practice!

Section 2.2: The First Projection Formula and its Consequences

- Fulton and Harris (2004) identify a different goal than either Rudenko or Serre (1977) when proving the orthonormality of the irreducible characters, but an interesting one nonetheless! Let's begin.
- We wish to construct an *explicit* formula for a certain projection operator. The operator of interest is the one that projects the vectors in a representation V onto the subspace $V_{\text{triv}}^m \leq V$ consisting of the direct sum of the trivial representations in the decomposition.
- Observe that the subspace $V_{\text{triv}}^m \leq V$ consists of all vectors $v \in V$ such that $gv = v$ for all $g \in G$. We call this subspace "V^G."
- Call our desired projection operator φ. What properties should φ have?
  - If it is to map between representations V and V^G, it should be a morphism of G-representations.
    - How is this related to Lemma 1??
  - It should satisfy $\varphi^2 = \varphi$ and $\operatorname{range}\varphi = V^G$.
- Now suppose we want to find m, the number of copies of the trivial representation appearing in the decomposition of V.
  - Consider the matrix of φ in a basis of V such that the first m vectors lie in V^G and the rest of the vectors lie in the complement of V^G.
  - This block diagonal matrix will be the $m \times m$ identity in the upper left and the zero matrix in the bottom right.
  - Thus, $m = \text{tr}(\varphi)$.
  - Expanding, we actually have

$$m = \text{tr}(\varphi) = \frac{1}{|G|}\sum_{g\in G}\text{tr}(g) = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)$$

  - Note that both V_{triv} and $V = V_{\text{triv}}^n \oplus \cdots$ make the equality true; it's just that if we want the following consequence, we *need* V in there.

- Important consequence of the above: For an irreducible representation V other than the trivial one, $\sum_{g\in G}\chi_V(g) = 0$.
  - Example: In Table 3.1, notice how $1(1) + 3(-1) + 2(1) = 0$ and $1(2) + 3(0) + 2(-1) = 0$.

- "If V is irreducible, then by Schur's Lemma, $\dim[\text{Hom}(V,W)^G]$ is the multiplicity of V in W; similarly, if W is irreducible, $\dim[\text{Hom}(V,W)^G]$ is the multiplicity of W in V, and in the case where both V and W are irreducible, we have

$$\dim \text{Hom}_G(V,W) = \begin{cases} 1 & V \cong W \\ 0 & V \ncong W \end{cases}$$

  " Fulton and Harris (2004, p. 16).

- Fulton and Harris (2004) finally arrives at the theorem from Wednesday's lecture.

- This theorem gives us a lower bound on the number of conjugacy classes of G.

  Corollary 2.13. *The number of irreducible representations of G is less than or equal to the number of conjugacy classes.*

  Proof. We have that

$$\begin{aligned} \text{span}(\chi_1,\dots,\chi_k) &\leq \mathbb{C}_{\text{cl}}[G] \\ \dim[\text{span}(\chi_1,\dots,\chi_k)] &\leq \dim(\mathbb{C}_{\text{cl}}[G]) \\ \#\text{ irreducible representations} &\leq \#\text{ conjugacy classes} \end{aligned}$$

  as desired. ☐

- Corollary 1 from Friday's class.

  Corollary 2.14. *Any representation is determined by its character.*

- Corollary 2 from Friday's class.

  Corollary 2.15. *A representation V is irreducible iff $(\chi_V,\chi_V) = 1$.*

- A result from the proof of Corollary 1 from Friday's class.

  Corollary 2.16. *The multiplicity a_i of V_i in V is the inner product of χ_V with χ_{V_i}, i.e.,*

$$a_i = (\chi_V, \chi_{V_i})$$

- With respect to the regular representation, Fulton and Harris (2004) states that

$$\chi_R(g) = \begin{cases} 0 & g \neq e \\ |G| & g = e \end{cases}$$

- Matching the leftmost consequence of Corollary 3 from Friday's class.

  Corollary 2.17. *Any irreducible representation V of G appears in the regular representation* $\dim V$ *times.*

- Discusses the middle and rightmost consequences of Corollary 3 from Friday's class.

- Discusses Corollary 2ii from Section 2.4 of Serre (1977).

- Note: These two formulas, copied below for convenience, amount to the **Fourier inversion formula** for finite groups.

$$|G| = \sum_i \dim(V_i)^2 \qquad\qquad 0 = \sum_i (\dim V_i) \cdot \chi_{V_i}(g \neq e)$$

  - Note: If all but one of the characters is known, they given a formula for the unknown character.
- Matching the second orthogonality relation from Friday's class.

Section 2.3: Examples — S_4 and A_4

- Construction of Table 3.2.
  1. List the conjugacy classes in S_4 and the number of elements in each across the top of the table.
     - Since this is a symmetric group S_d, the conjugacy classes correspond to the **partitions** of d via cycle lengths.
     - Thus, our conjugacy classes are...
       - $\{e\}$: $4 = 1+1+1+1$. Number of elements: 1.
       - $\{(xx)\}$: $4 = 2+1+1$. Number of elements: 6.
       - $\{(xxx)\}$: $4 = 3+1$. Number of elements: 8.
       - $\{(xx)(xx)\}$: $4 = 2+2$. Number of elements: 3.
       - $\{(xxxx)\}$: $4 = 4$. Number of elements: 6.
  2. Start by listing the trivial, alternating, and standard representations.
     - The character of the trivial is
       $$(1, 1, 1, 1, 1)$$
       by default.
     - The character of the alternating is
       $$((-1)^e, (-1)^{(xx)}, (-1)^{(xxx)}, (-1)^{(xx)(xx)}, (-1)^{(xxxx)}) = (1, -1, 1, 1, -1)$$
       by the definition of the representation.
     - The character of the standard representation is
       $$\chi_{\mathbb{C}^4} - \chi_{(5)} = (4, 2, 1, 0, 0) - (1, 1, 1, 1, 1) = (3, 1, 0, -1, -1)$$
       by the fact that the regular representation always decomposes into the sum of the trivial and standard.
       - We can double check that this representation is irreducible via the irreducibility criterion $|\chi_V| = \sqrt{(\chi_V, \chi_V)} = 1$.
  3. Figure out how many more representations irreducible representations there are. In this case there are two more.
     - We can figure this out by combining two previously used facts.
     - First, we know that the sum of the squares of the dimensions must equal $|S_4| = 24$ (middle consequence of Corollary 3 from Friday's class). Since we're only at $1+1+9 = 11$ so far, we still have $24 - 11 = 13$ to go. But how is this 13 allocated? To answer this question, we need the second fact.
     - Second, by Corollary 2.13, the number of irreps is less than or equal to the number of conjugacy classes, so we have at most two irreps to go. In fact, since 13 is not the square of any natural number but $13 = 2^2 + 3^2$, we must have *exactly* two irreps to go, of dimensions 2 and 3.
  4. The tensor product of an irrep and a one-dimensional representation is irreducible, so $\text{Sign} \otimes V_{\text{std}} = (3, -1, 0, -1, 1)$ is one of them.

  - Additional check: $|\text{Sign} \otimes V_{\text{std}}| = 1$.
  - Additional check: It is not a scalar multiple *or* linear combination of any of the first three.

5. Use the second orthogonality relation to solve for each $\chi(g)$ for the final irrep.

- **Partition** (of d): An expression of d as a sum of positive integers $a_1, \dots, a_k$.
- What is an "involution of trace 2" and how is it related to the quotient group (Fulton & Harris, 2004, p. 19)??
  - This is related to the closing comments of Friday's class.
- Description of A_4.

Week 4

Character Theory

4.1 Representation Ring; Character Basis

10/16: • Announcements.

  – Reminder: Midterm 11/10.
  – OH this week in-person at normal times.
  – PSet 3 should be fun.

• Today: Finish proving some character things.

• Recall: The main picture.

  – Rudenko redraws Figure 3.1.
  – We have a finite group G and we are studying finite-dimensional G-reps over $\mathbb{C}$.
  – $\mathbb{C}_{\text{cl}}[G]$ is a ring.
  – The map...
    - Respects addition;
    - Sends tensor multiplication to (pointwise) functional multiplication;
    - Sends duality to conjugation;
    - Respects a kind of inner product, whether it be either side of $\dim_{\mathbb{C}} \operatorname{Hom}_G(V, W) = \langle f_1, f_2 \rangle$.

• Today, we will see that $\mathbb{C}_{\text{cl}}[G] \cong \mathbb{C}^k$, where k is the number of conjugacy classes.

  – In other words, we will see that the number of irreps is also exactly equal to k, that there is a bijection $\{V_i\} \to \{\chi_i\}$, and that the $\chi_1, \dots, \chi_k$ form an orthonormal basis of $\mathbb{C}_{\text{cl}}[G]$.

• Visualizing the vector space $\mathbb{C}_{\text{cl}}[G]$.

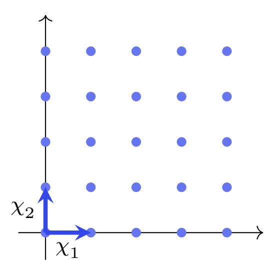


Figure 4.1: Visualizing the space of class functions on G.

- It's a "cone" emanating from the origin with only lattice points.
  - If $\dim \mathbb{C}_{\text{cl}}[G] = 2$, the vector space consists of all the blue points in Figure 4.1.
- Why is it only lattice points instead of a continuous function space?
  - The restrictions on coefficients are inherited from the restrictions on what kinds of spaces you can build of the form $V_1^{n_1} \oplus V_2^{n_2}$.
  - Indeed, if it were continuous, that would imply that there is some meaning to the point $0.3\chi_1 + 2.5\chi_2$, i.e., there is a space $V_1^{0.3} \oplus V_2^{2.5}$. But of course, we cannot define such a space!
- Why is it only *nonnegative* integer coefficients and not *all* integer coefficients?
  - We don't have subtraction to get us to a full ring.
  - Additionally, we can only scale and linearly combine the χ_i's with nonnegative integer coefficients because, as said above, those are the types of reducible rep decompositions we have.

- Let $[V]$ denote the **isomorphism class** of the representation V.
- **Isomorphism class** (of V): The set of all vector spaces W that are isomorphic to V as representations.
- This allows us to define the **representation ring**.
- **Representation ring** (of G): The ring $(R, +, \cdot)$, where R is the free abelian group generated by all isomorphism classes of the representations of G, quotiented by the span of all linear combinations of the form $[V \oplus W] - [V] - [W]$; $+$ is well-defined via the construction of R, which yields $[V] + [W] = [V \oplus W]$ for all $[V], [W]$ in the ring; and $\cdot$ is defined by $[V] \cdot [W] = [V \otimes W]$. *Denoted by* $\boldsymbol{R(G)}$.
  - Basis: $[V_1], \dots, [V_k]$.
  - Thus, structurally,
$$R(G) \cong \mathbb{Z}^k$$
  - Elements are of the form $[V_1] + 2[V_2] - 3[V_3]$.
  - Multiplication is slightly complicated because $V_i \otimes V_j = \bigoplus V_k^{n_{ijk}}$; it follows that
$$[V_i] \cdot [V_j] = \sum n_{ijk}[V_k]$$
- Alternative construction of $R(G)$: Take the subring of the class ring $\mathbb{C}_{\text{cl}}[G]$ that is generated by the characters.
  - To do so, define a map $R(G) \to \mathbb{C}^k$ where the image is linear combinations of characters χ_i with $\mathbb{Z}$-class.
  - Clarify this construction??
- **Virtual representation**: An element of $R(G)$.
  - We need this term because some elements of $R(G)$ — like $-[V]$, for instance — may not correspond to an actual representation.
  - Indeed, note that $-[V]$ is *not* V^*; it is just some thing that when you add it to $[V]$, you get the zero representation.
- Example: Let $G = \mathbb{Z}/2\mathbb{Z} = \{e, x\}$.
  - Then $R(G) = \mathbb{Z}^2 = \mathbb{Z}e \oplus \mathbb{Z}x$ has basis $[1], [-1]$ (corresponding to the trivial and alternating representations) where we define
$$[1]^2 = [1] \qquad [1][-1] = [-1] \qquad [-1]^2 = [1]$$
- One reason people like this $R(G)$ is as follows.

  - Initially, understanding this group is not easy because even to get started, you have to find all your characters.
  - But, we know that
$$R(G)\otimes_{\mathbb{Z}}\mathbb{C}\cong\mathbb{C}_{\text{cl}}[G]$$
    - So we have a ring that's hard to understand, but if we do something called an **extension of scalars** (shown above) we get an easy ring!
    - Why?? Clarify this construction.
  - This is interesting because we can look at the intermediate objects. For example, could we describe $R(G)\otimes\mathbb{R}$ or $R(G)\otimes\mathbb{Q}$. Interestingly, **Artin's theorem** describes $R(G)\otimes\mathbb{Q}$ completely.
  - If we try to understand $R(S_n)$, this is still hard work, but if we take $\bigoplus_{n\geq 0}R(S_n)$, we obtain an object that is remarkably, surprisingly simple. That's where we're going. This is why rep theory of finite groups is simultaneously very hard and very simple.

- Lemma: Let G be a finite group, let f be a complex-valued[1] class function, and let V be a G-rep. Then the linear map
$$F=\sum_{g\in G}f(g)\cdot g:V\to V$$
is a morphism of G-representations, that is, $F\in\text{Hom}_G(V,V)$.

Proof. To prove that $F\in\text{Hom}_G(V,V)$, it will suffice to show that $xF=Fx$ for every $x\in G$. Let $x\in G$ be arbitrary. Then
$$F(xv)=\sum_{g\in G}f(g)gxv$$
Since ρ is a group homomorphism, the functions $\rho(g)\in GL(V)$ act just like the elements $g\in G$. *This* is what justifies us to basically move everything around all willy-nilly. Thus, continuing from the above, we have
$$\begin{aligned}&=\sum_{g\in G}f(g)(xx^{-1})gxv\\&=\sum_{g\in G}f(g)x(x^{-1}gx)v\end{aligned}$$
Since $x=\rho(x)$ is in the general *linear* group, i.e., is a *linear* map, we can factor it out of the sum of functions to get
$$=x\left(\sum_{g\in G}f(g)x^{-1}gx\right)v$$
Since f is a class function by hypothesis, we have $f(g)=f(x^{-1}gx)$, so
$$\begin{aligned}&=x\left(\sum_{g\in G}f(x^{-1}gx)x^{-1}gxv\right)\\&=x\sum_{g\in G}f(g)gv\\&=x(Fv)\end{aligned}$$
as desired. ∎

[1] This "complex-valued" hypothesis was not stated in class, but I have to imagine it's true. Is it??

- Recall that previously, we had $(1/|G|)\sum_{g\in G} g : V \to V^G$.
  - He will put something about this being a class function on the midterm?? Review how to prove that this is a class function!
- Another comment: A slightly refined question.
  - Suppose you have a class function f and an irrep V.
  - Then we know that $F = \sum f(g)g : V \to V$ is a G-morphism, so it is a **homothety** by Schur's lemma.
  - So let's find λ.
  - Thinking a big more carefully, we know that F above is
$$\sum_{g\in G} f(g)\rho_V(g) = \lambda I_{d_V}$$
    where d_V denotes the **degree** of V.
  - Now, we will compute λ using the trace. Take the trace of both sides. Then
$$\begin{aligned}
\operatorname{tr}\left(\sum_{g\in G} f(g)\rho_V(g)\right) &= \operatorname{tr}(\lambda I_{d_V})\\
\sum f(g)\operatorname{tr}(\rho_V(g)) &= \lambda d_V\\
\sum f(g)\chi_V(g) &= \lambda d_V\\
\lambda &= \frac{|G|}{d_V}\frac{1}{|G|}\sum_{g\in G} f(g)\overline{\chi_{V^*}(g)}\\
&= \frac{|G|}{d_V}\langle f, \chi_{V^*}\rangle
\end{aligned}$$
- **Homothety**: A map $F : V \to V$ for which there exists $\lambda \in \mathbb{C}$ such that $Fv = \lambda v$ for all $v \in V$.
  - It just means that we're scaling.
- **Degree** (of V): The dimension of V as a vector space. *Denoted by* $\boldsymbol{d_V}$. *Given by*
$$d_V = \dim V$$
- Now, we can prove the theorem to which we've been building up the whole time.
- Theorem: Let G be a finite group. Then the number of irreps up to isomorphism is equal to the number of conjugacy classes.

  Proof. Let k be the number of conjugacy classes of G, and let $\chi_1, \dots, \chi_s$ be the characters of the irreps. By the theorem from last Wednesday's class, it follows that $\chi_1, \dots, \chi_s$ are orthonormal vectors in $\mathbb{C}_{\text{cl}}[G]$. Thus, by the corollary to the aforementioned theorem, $s \leq k$.

  Now, suppose for the sake of contradiction that $s < k$. Then there exists a nonzero $f \in \mathbb{C}_{\text{cl}}[G]$ such that $\langle f, \chi_{V_i}\rangle = 0$ $(i = 1, \dots, s)$. By Gram-Schmidt, we can choose f to be another *orthonormal* vector in the list, extending it to $\chi_1, \dots, \chi_s, f$. We will now build up to proving that $f(g) = 0$ for all $g \in G$ (i.e., $f = 0$), which we will do by using the above lemma to construct a linear independence argument as follows. The first step is to let V_i be an arbitrary irrep of G. Then by the above comment, $F : V_i \to V_i$ may be evaluated on any $v \in V_i$ as follows.
$$F(v) = \lambda I v = \frac{|G|}{d_{V_i}}\left\langle f, \chi_{V_i^*}\right\rangle \cdot v = \frac{|G|}{d_{V_i}}\overline{\langle f, \chi_{V_i}\rangle} \cdot v = \frac{|G|}{d_{V_i}}\overline{0} \cdot v = 0$$

It follows that $F = 0$ on *any* representation since by complete reducibility, they're all direct sums of irreps. In particular, $F : V_{\text{reg}} \to V_{\text{reg}}$ is the zero operator, where $V_{\text{reg}} \cong V_1^{d_{V_1}} \oplus \cdots \oplus V_s^{d_{V_s}}$ is the regular representation. Thus, for example, $F(e_e) = 0$. But we also know that

$$F(e_e) = \sum_{g \in G} f(g) \cdot g e_e = \sum_{g \in G} f(g) \cdot e_g$$

Consequently, by transitivity, we have that

$$0 = \sum_{g \in G} f(g) \cdot e_g$$

But since the e_g are all linearly independent by the definition of the regular representation, we have that each $f(g) = 0$, as desired. This means that $f = 0$, contradicting our original supposition. $\square$

- That is the end of this story.
- Here's one consequence of the above theorem.
  - We now know that the space of class functions has an orthonormal basis $\chi_{V_1^*}, \dots, \chi_{V_k^*}$.
  - If we denote the conjugacy classes of G by $C_1, \dots, C_k$, then another obvious basis of $\mathbb{C}_{\text{cl}}[G]$ is $\delta_{C_1}, \dots, \delta_{C_k}$ defined by
$$\delta_{C_i}(g) = \begin{cases} 1 & g \in C_i \\ 0 & g \notin C_i \end{cases}$$
  - This new basis is orthogonal: We have
$$\langle \delta_{C_i}, \delta_{C_j} \rangle = \frac{1}{|G|} \sum_{g \in G} \delta_{C_i}(g) \delta_{C_j}(g) = \begin{cases} 0 & i \neq j \\ \frac{|C_i|}{|G|} & i = j \end{cases}$$
    - Justifying this computation: If $i \neq j$, then at least one of $\delta_{C_i}, \delta_{C_j}$ will be zero; if $i = j$, then they're both nonzero and equal to 1 for all $|C_i|$ elements $g \in C_i$.
  - What is the change of basis matrix between $\{\delta_{C_i}\}$ and $\{\chi_{V_i^*}\}$? It's the character table.
    - The orthogonality condition for characters then just comes from the fact that we're going from one orthogonal basis to another.
    - What are the exact bases we change between??

4.2 Office Hours

10/17:

- **Transitive** (group action): A group action for which the **orbit** of x is equal to X for any $x \in X$.
- **Orbit** (of $x \in X$): The set of $g \cdot x$ for all $g \in G$.
- **Diagonal action** (of G on $X \times X$): The action defined as follows. *Given by*
$$g \cdot (x_1, x_2) = (g \cdot x_1, g \cdot x_2)$$
- Check Etingof et al. (2011) for some things??

4.3 Orthogonality Results

10/18:

- Announcements.
  - Goal: Finish our discussion of the orthogonality of characters, projection functions, etc.
  - Friday: Frobenius determinant.
  - Next week: Group algebras, associative algebras, etc.; another perspective on representations.
  - After next week: A more advanced part of representation theory related to group theory.
- Describing Figure 3.1 from a different perspective.
  - Let G be a finite group, and let k denote the number of conjugacy classes and the number of irreps. Let $C_1, \dots, C_k$ be the conjugacy classes and $V_1, \dots, V_k$ be the irreps.
  - There is no natural/canonical bijection between the two sets. For a simple group, there is often a canonical way, and this is where things get interesting.
    - Example: Symmetric group induces canonical bijection, as we'll see later.
  - $\mathbb{C}_{\text{cl}}[G] = \mathbb{C}^k$ is a vector space of class functions and a ring.
  - We have the Hermitian inner product

$$\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$$

  - Recall that $\chi_{V_1}, \dots, \chi_{V_k}$ is an orthonormal basis such that

$$\langle \chi_{V_i}, \chi_{V_j} \rangle = \delta_{ij}$$

  - We have another basis $\delta_{C_1}, \dots, \delta_{C_k}$ defined by

$$\delta_{C_i}(g) = \begin{cases} 0 & g \notin C_i \\ 1 & g \in C_i \end{cases}$$

    that is orthogonal but not orthonormal:

$$\langle \delta_{C_i}, \delta_{C_j} \rangle = \begin{cases} 0 & C_i \neq C_j \\ \frac{|C_i|}{|G|} & C_i = C_j \end{cases}$$

  - How do we relate the two bases?
  - To begin, fix C_i. Then

$$\delta_{C_j}(g) = \sum_{V_i} \lambda_i \chi_{V_i}(g)$$

  - λ_i can be computed immediately using the inner product since the characters are orthonormal:

$$\lambda_i = \langle \delta_{C_j}, \chi_{V_i} \rangle = \frac{1}{|G|} \sum_{g \in G} \delta_{C_j}(g) \overline{\chi_{V_i}(g)} = \frac{|C_j| \bar{\chi}_{V_i}(g)}{|G|}$$

    - You took $\lambda_i = \langle \delta_{C_j}, \bar{\chi}_{V_i} \rangle$; which one is correct??
  - But then

$$\delta_{C_j}(g) = \frac{|C_j|}{|G|} \left(\sum_{V_i} \bar{\chi}_{V_i}(C_j) \chi_{V_i}(g) \right)$$

- It follows that we have two bases of $\mathbb{C}_{\text{cl}}[G]$. These are given by
$$\frac{|G|}{|C_j|}\delta_{C_j} \qquad\qquad \chi_{V_i^*}$$
where $i,j = 1,\dots,k$.
- How do we convert between these two very natural bases of our space of functions? The change of basis matrix from left to right is the character table.
- Obviously, we have to do some scaling and take some duals, but it's not that bad and it fits the character table really well.
- This gives us some properties of the character table such as orthogonality.
- For example, **orthogonal** matrices convert between orthogonal bases; in the complex domain, such a matrix is **unitary**, i.e., for the character table U, $U\bar{U}^T = E$.

- Orthogonality relations that you can derive.
  1. We can show that
  $$\sum_{g\in G}\chi_1(g)\overline{\chi_2(g)} = \begin{cases} 0 & \chi_1 \neq \chi_2 \\ |G| & \chi_1 = \chi_2 \end{cases}$$
     - Use the unitary condition.
  2. We can show that
  $$\sum_{i=1}^{k}\chi_i(g_1)\overline{\chi_i(g_2)} = \begin{cases} 0 & g_1 \neq g_2 \\ \frac{|G|}{|C(g_1)|} & g_1 \sim g_2 \end{cases}$$
     - We literally just take the identity defining $\delta_{C_j}(g)$.

- **Isotypical component**: A representation that is equal to the direct sum of isomorphic irreducible representations. *Also known as* **isotypic component**.
  - Illustrative example: For $V = V_1^{n_1}\oplus\cdots\oplus V_k^{n_k}$, each $V_i^{n_i}$ is an isotypical component.

- Examples.
  1. Let $G \circlearrowright \mathbb{C}^2$ by $\rho(g) = E_2$. Thus, we can say that $\mathbb{C}^2 = V_1\oplus V_1$, but we can't say this in any unique, canonical way, i.e., we can choose infinitely many V_1's and have the statement still be true, where V_1 is the trivial rep.
  2. We have $V_1^{n_1} = V^G = \{v\in V \mid gv = v\ \forall\ g\in G\}$. Look at what's invariant under the symmetry group, i.e., define
  $$P = \frac{1}{|G|}\sum g$$
     - All **invariant functions** come from averaging over the group!
     - Then $P^2 = P$ and $\operatorname{Im} P = V^G$.
     - Takeaway: We call each $V_i^{n_i}$ an **isotypical component**.
     - What's going on in this example??
  3. The permutational representation for S_n decomposes into the sum of the trivial and standard reps; there is only one decomposition this way. If we look at $V_1\oplus V_{\text{stand}}^2$, then our decomposition will depend on a choice of a plane.

- Reminder.
  - Last time, we chose an $f\in\mathbb{C}_{\text{cl}}[G]$, a representation V, and then took $\sum f(g)g : V\to V$ so that then $\sum f(g)g \in \operatorname{Hom}_G(V,V)$.
  - Moreover, we proved that if V is irreducible, then this endomorphism is equal to a scalar λ times the identity matrix via Schur's lemma.

- Computing λ:

$$\lambda = \frac{|G|}{d_V} \langle f, \chi_V^* \rangle$$

  - Hard to remember but easy to derive.
- Define $V = V_1^{n_1} \oplus \cdots \oplus V_k^{n_k}$ and $P_i : V \to V_i^{n_i}$.
- In particular, look at

$$P_i = \frac{d_V}{|G|} \sum_{g \in G} \chi_{V_i^*}(g) g$$

  - This averaging operator is consistent with what we had before.
- P_i acts on V_i by

$$\frac{d_{V_i}}{|G|} \frac{|G|}{d_{V_i}} \left\langle \chi_{V_i^*}, \chi_{V_i^*} \right\rangle = 1$$

- P_i acts on V_j by

$$\frac{d_{V_i}}{|G|} \frac{|G|}{d_{V_i}} \left\langle \chi_{V_i^*}, \chi_{V_j^*} \right\rangle = 0$$

- Take $V = V_1^{n_1} \oplus \cdots \oplus V_k^{n_k}$ and apply P_i. It follows by the above that it is exactly the projection on $V_i^{n_i}$.
- Thus, $P_1 + \cdots + P_k = 1$. $P_i^2 = P_i$. $P_i P_j = 0$. This is called a/the (which one??) **idempotent decompostion**.
- Example: Let $v \in V$. Then $v = P_1 v + \cdots + P_k v$.
- Additionally, we can take a function f that is invariant under the group...??

- We're done early.

- We will not start the Frobenius determinant today.

- We will start on next week's content then so we can begin thinking about it.

- **Associative algebra**: A vector space over a field F that is also a (not necessarily commutative) ring, where we have a unit 1 in the ring, addition, and multiplication. Scalar multiplication: $\lambda a = (\lambda \cdot 1) \cdot a$. Associativity condition: $(\lambda a)b = \lambda(ab)$. *Denoted by* $\boldsymbol{A}$.
  - We'll only discuss finite-dimensional algebras in this course.

- Examples:
  1. $\mathbb{R}$, $\mathbb{C}$ (an algebra over $\mathbb{R}$).
  2. $\mathbb{H}$, a 4d algebra over $\mathbb{R}$. The algebra of quaternions. $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$.
     - Hamilton's remarkable discovery: There is a 4D set of numbers that is not commutative but is still associative and helps describe rotation in 3D or 4D space.
     - Multiplication rules:

$$i^2 = j^2 = k^2 = -1 \qquad\qquad ijk = -1$$

     - We should spend part of our weekend reading a history of quaternions!
  3. $M_{n \times n}(F)$, the **matrix algebra**.
  4. $A_1 \oplus \cdots \oplus A_n$, the **direct sum** of algebras.
     - Addition and multiplication are done pairwise.
  5. $A_1 \otimes A_2$.
     - We will not talk about this today, though!

- Let's go back; let G be a finite group and consider $\mathbb{C}[G]$, the set of functions on G.
  - This algebra has some basis $\bigoplus \mathbb{C}e_g$.
  - To get the algebra structure, we just need a rule for multiplying basis elements. In this case, we use $e_{g_1}e_{g_2} = e_{g_1g_2}$.
  - This is the **algebra over $\mathbb{C}$ of dimension G**.
  - Theorem: $\mathbb{C}[G] \cong M_{d_1\times d_1}(\mathbb{C}) \oplus \cdots \oplus M_{d_k\times d_k}(\mathbb{C})$.
    - We can prove this theorem from what we know: Schur's Lemma and complete reducability.
    - We'll discuss it for several consecutive times.
    - A similar result holds for *many* algebras (e.g., semisimple algebra), not just *group* algebras.
- HW1-2 will be graded later this week and handed back on Friday.
- In Etingof et al. (2011), we can find a lot of history of some of this stuff. The comments are interesting and entertaining.

4.4 Frobenius Determinant; Intro to Associative Algebras

10/20: - Let $G = \{g_1, \dots, g_n\}$.

- **Frobenius determinant**: The polynomial defined as follows. *Denoted by* $\boldsymbol{F(x_{g_1}, \dots, x_{g_n})}$. *Given by*

$$F(x_{g_1}, \dots, x_{g_n}) = \det |x_{g_ig_j}|$$

  - The Frobenius determinant is a homogeneous polynomial with integer coefficients of degree n.
  - $F(x_{g_1}, \dots, x_{g_n}) \in \mathbb{Z}[x_{g_1}, \dots, x_{g_n}]$.

- Theorem: There exist irreducible $P_1, \dots, P_m \in \mathbb{Z}[x_{g_1}, \dots, x_{g_n}]$ such that

$$F = P_1^{\deg P_1} \cdots P_k^{\deg P_k}$$

Moreover, $\chi_i(g) \approx \chi_g^{\deg P_i}$, where χ_g is the coefficient of P_i. (Is this last line correct??)

Proof. Let $\rho : G \to GL_n$ be the regular representation of G, and define $P_\rho = \sum \chi_{g_i}\rho(g_i)$. Then $P_\rho(x_{g_1}, \dots, x_{g_n}) = \pm I(x_{g_1}, \dots, x_{g_n})$.

We have that $P_\rho(e_{g_j}) = \sum x_{g_i}g_ie_{g_j} = \sum x_{g_i}e_{g_ig_j} = \sum x_{g_ig_j^{-1}}e_{g_i}$, so the matrix of P_ρ is $(x_{g_ig_j^{-1}})$ and thus has permuted columns and rows relative to the original matrix of which we took the Frobenius determinant.

Recall that $\mathbb{C}[G] \cong V_1^{d_1} \oplus \cdots \oplus V_k^{d_k}$. Additionally, the matrix of each V_i is $(\sum \chi_g g_i)$.

Understanding this?? ☐

- **Group algebra**: The algebra A over a field F with one basis element e_i for each $g_i \in G$ and the multiplication law $e_i \cdot e_j = \sum_{i=1}^k \lambda_{ij}^k e_k$. *Denoted by* $\boldsymbol{F[G]}$. *Given by*

$$F[G] = \{a_{g_1}g_1 + \cdots + a_{g_n}g_n \mid a_i \in F\}$$

  - $A \cong F^n$.
  - Note that the notation here is well chosen; see the discussion of the notation $\mathbb{C}[G]$ from the 10/9 lecture.

- **Division algebra**: An algebra A such that for all nonzero $x \in A$, there exists a $y \in A$ such that $xy = 1$.

- **Field**: A commutative division algebra.

- Examples.
  1. $\mathbb{C}$ is a 2-dimensional algebra over $\mathbb{R}$.
  2. $\mathbb{H}$ is a 4-dimensional algebra over $\mathbb{R}$.
     - As discussed last time, the elements are of the form $q = a + bi + cj + dk$ where $i^2 = j^2 = k^2 = -1 = ijk$
     - Note that it follows that
$$\bar{q} = a - bi - cj - dk$$
     - Hence,
$$q\bar{q} = a^2 + b^2 + c^2 + d^2$$
     - Thus, we can define
$$q^{-1} = \frac{q}{a^2 + b^2 + c^2 + d^2}$$
- We now prove some results of division algebras.
  1. If $F = \mathbb{C}$, then every finite-dimensional division algebra is $\mathbb{C}$.

     Proof. Let A be an arbitrary finite-dimensional division algebra over $\mathbb{C}$. Let $a \in A$, and let $L_a \in GL_n(A)$ send $a \mapsto [L_a x \mapsto ax]$.
     Then $\mathbb{C} \to M_{2\times 2}(\mathbb{R})$ sends
$$a + bi \mapsto L_{a+bi} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$
     Then $L_a L_b = L_{ab}$ and $L_a + L_b = L_{a^{-1}b}$, so L_a has eigenvalue λ, so $L_a x = ax = \lambda x$, so $a = \lambda \cdot 1$. What is going on here and how does this work?? $\square$
- Example of the above property.
  - $\mathbb{H} \to M_{4\times 4}(\mathbb{R})$ sends $a + bi + cj + dk$ to ?? with determinant $(a^2 + b^2 + c^2 + c^2)^2$.
  - In general, the determinant of $A \to GL_n(A)$.
- Theorem 1: Over $\mathbb{R}$, there are exactly three division algebras: $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$.
- Theorem 2: Over $\mathbb{F}_q$ finite, every finite-dimensional division algebra is a field $\mathbb{F}_{q^n}$.
- **Representation** (of A): A module V over A equipped with a homomorphism of algebras $\rho : A \to M_{n\times n}(V)$.
- Observation:
  - If G is a group and F is a field, then the group algebra is $F[G] = \bigoplus_{i=1}^n Fe_{g_i}$. Herein, we define $e_{g_i} e_{g_j} = e_{g_i g_j}$.
  - Modules over $F[G]$ are equivalent to G reps.
  - $F[G] \to M_{n\times n}(F)$ is equivalent to $G \to GL_n(F)$.
- **Morphism** (of A-modules): A map $f : M \to N$ such that...
  1. f is a module homomorphism;
  2. f respects the structure of the representations; explicitly, for every $g \in G$, $\rho_N(g) \circ f = f \circ \rho_M(g)$.
- $\boldsymbol{\mathrm{Hom}_A(M,N)}$: The set of all morphisms of A-modules from M to N.
- Schur's Lemma for associative algebras: Let A be a finite-dimensional algebra over a field F, and let M_1, M_2 be simple A-modules. Then we have the following statements.
  1. If $f : M_1 \to M_2$ is a nonzero morphism of A-modules, then f is isomorphic.

2. If M is simple, then $\text{Hom}_A(M, M)$ is a division algebra over F.

- Note that this version of Schur's Lemma implies that complete reducibility may fail for associative algebras. (why??)
- Theorem (Complete Reducibility): Let A be a finite-dimensional algebra such that $M_1 \subset M_2$. Then there exists N such that $M_2 = M_1 \oplus N$. Moreover, it follows that

$$A \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k)$$

4.5 S Chapter 2: Character Theory

From Serre (1977).

Section 2.5: Number of Irreducible Representations

11/9:

- We now build up to a proof that the number of irreducible representations is *equal* to the number of conjugacy classes.
- Matching the Lemma and additional comment from Monday's class.

Proposition 6. *Let f be a class function on G, and let $\rho : G \to GL(V)$ be a linear representation of G. Let ρ_f be the linear mapping of V into itself defined by*

$$\rho_f = \sum_{t \in G} f(t)\rho_t$$

If V is irreducible of degree n and character χ, then ρ_f is a homothety of ratio λ given by

$$\lambda = \frac{1}{n} \sum_{t \in G} f(t)\chi(t) = \frac{g}{n}(f \mid \chi^*)$$

Proof. See class. □

- Denote by H the space of class functions on G.
  - Thus, for example, $\chi_1, \dots, \chi_h \in H$.
  - Does this differ somehow from $\mathbb{C}_{\text{cl}}[G]$, which is also used in Serre (1977)??
- Matching the Theorem from Monday's class.

Theorem 6. *The characters $\chi_1, \dots, \chi_h$ form an orthonormal basis of H.*

Proof. See class. □

- **Class**: A subset of G consisting of the elements sts^{-1} for some $g \in G$ and all $s \in G$. *Also known as* **conjugacy class**.
- Matching the final result of the Theorem from Monday's class.

Theorem 7. *The number of irreducible representations of G (up to isomorphism) is equal to the number of classes of G.*

Proof. Essentially, Serre (1977) is just separating out the notion that class functions *are constant on conjugacy classes.* □

- Matching the second orthogonality relation from the class of the Friday of Week 3, and with similarities to the consequence of the Theorem from Monday's class.

  Proposition 7. *Let $s \in G$, and let $c(s)$ be the number of elements in the conjugacy class of s.*

  (i) *We have*
  $$\sum_{i=1}^{h} \chi_i(s)^* \chi_i(s) = \frac{g}{c(s)}$$

  (ii) *For $t \in G$ not conjugate to s, we have*
  $$\sum_{i=1}^{h} \chi_i(s)^* \chi_i(t) = 0$$

  (iii) *For $s = 1$, we recover Corollary 2 to Proposition 5 as a special case of parts (i)-(ii).*

  Proof. For each $s \in G$, define
  $$f_s(t) = \begin{cases} 1 & t \sim s \\ 0 & t \nsim s \end{cases}$$
  Notice that as defined, f_s is a class function. Thus, since the characters form a basis of H,
  $$f_s = \sum_{i=1}^{h} (f_s \mid \chi_i) \chi_i$$
  where
  $$(f_s \mid \chi_i) = \frac{1}{g} \sum_{t \in G} f_s(t) \chi_i(t)^* = \frac{c(s)}{g} \chi_i(s)^*$$
  Combining these two results, we obtain the following equivalent formulation of f_s.
  $$f_s(t) = \frac{c(s)}{g} \sum_{i=1}^{h} \chi_i(s)^* \chi_i(t)$$
  If we set $t = s$ in the above equation, we recover (ii). If we set $t \nsim s$ in the above equation, we recover (ii). □

- Serre (1977) constructs Table 3.1.
  - Postulates trivial and alternating representations.
  - An interesting way to recover the standard θ: From Corollary 1 to Proposition 5, we have that $\chi_1 + \chi_2 + 2\theta = \chi_{\text{reg}} = (6, 0, 0)$.
  - Serre (1977) discusses the geometric foundations of the standard representation.

4.6 FH Chapter 2: Characters

From Fulton and Harris (2004).

Section 2.4: More Projection Formulas; More Consequences

- Again, Fulton and Harris (2004) take an unconventional tack: Instead of looking for an average of endomorphisms this time around, what linear combinations of endomorphisms are G-linear?
- Here's an answer to the question, in a form with which we're quite familiar at this point.

  Proposition 2.28. *Let $\alpha : G \to \mathbb{C}$ be any function on the group G, and for any representation V of G, set*

  $$\varphi_{\alpha,V} = \sum \alpha(g) \cdot g : V \to V$$

  Then $\varphi_{\alpha,V}$ is a homomorphism of G-modules for all V if and only if α is a class function.

  Proof. See class. □

- Matching the Lemma, additional comment, and Theorem from Monday's class.

  Proposition 2.30. *The number of irreducible representations of G is equal to the number of conjugacy classes of G. Equivalently, their characters $\{\chi_V\}$ form an orthonormal basis for $\mathbb{C}_{cl}(G)$.*

  Proof. See class. □

- At this point, we know all there is to know about the characters of a finite group.
- Alternate way to express Proposition 2.30 (and a lot of other things we've learned about representations of a finite group G): Use the **representation ring** of G!
- Definition of **representation ring** and **virtual representation**.
- The character induces a map $\chi : R(G) \to \mathbb{C}_{\text{cl}}[G]$.
  - Analogy of Proposition 2.1: χ is a ring homomorphism.
  - A representation is determined by its characters: χ is injective.
  - Proposition 2.30: χ induces an isomorphism $\chi_{\mathbb{C}} : R(G) \otimes \mathbb{C} \to \mathbb{C}_{\text{cl}}[G]$.
- **Virtual character**: An element of $\text{Im}(\chi)$.
- Goes over Figure 4.1.
- A few more comments.

Week 5

Associative Algebras

5.1 Wedderburn-Artin Theory

10/23:
- Share notes with Rudenko at the end of the course!
- Today: Wedderburn-Artin theory.
  - Noncommutative algebra.
  - Noncommutative is a big part of math, partially because of its relation to QMech and partially because of its use in math, itself.
  - There is a textbook: Lang (2002). It's a hard, grad-level textbook but very cleanly written. Not a bad book to have in our mind as we start to encounter category theory.
- So here's what were talking about.
  - Our main object is A, an **associative algebra** over a field F.
- Left vs. right algebras.
  - When A is not commutative, we have to specify which we are dealing with.
  - Let A be an algebra over F.
  - Recall left-modules and right-modules.
    - In a left module, you can multiply $A \times M \to M$ where $(ab)m = a(bm)$.
    - In a right module, $(ab)m = b(am)$. More simply, $m(ab) = (ma)b$.
    - With modules, we get submodules, quotient modules, homomorphisms of modules, etc.
  - Let $I \subset A$ be a left-submodule. Thus, it is a subspace of A such that for all $a \in A$, $aI \subset I$, i.e., a left ideal.
  - In a right-submodule $I \subset A$, we have that for all $b \in A$, $Ib \subset I$, i.e., a right ideal.
  - In a two-sided ideal $I \subset A$, we have for all $a, b \in I$ that $aI \subset I$ and $Ib \subset I$.
  - Example: The matrix algebra is the prototypical noncommutative algebra. Consider $M_{2\times 2}(\mathbb{C})$.
    - Pick $v = (1, 0)$.
    - Look at ideal $I = \{X \in M_{2\times 2} \mid Xv = 0\}$. This is called the **annihilator**, and it is a left ideal. Explicitly, this ideal is the subset of all matrices of the form

$$\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$$

      for $a, b \in \mathbb{C}$.

    - An example of a right ideal is all those such that $vX = 0$, i.e., all matrices of the form

$$\begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$$

        - Note that we are treating v as a row vector here.
    - There are *no* two-sided ideals herein, save the trivial one.

- **Simple** (algebra): An algebra for which there are no nontrivial two-sided ideals.
- Every time you go more abstract, it's more boring because you have less things to play with, but we can derive more general rules.
    - We'll only stay so abstract for 2-3 lectures.
- We want to convert left-algebras to right-algebras.
    - To do so, we can construct **opposite algebras**.
- **Opposite algebra** (of A): The algebra with the same vector space structure as A, but with the reversed multiplication such that $a * b$ in this space yields $b * a$ in A. *Denoted by* $\boldsymbol{A^{\text{op}}}$.
    - Left ideals of A become right ideals of A^{op} and vice versa. Two-sided ideals stay the same.
    - In category theory, left-modules over A are equivalent to right-modules over A^{op}.
    - Opposite algebras are briefly defined on Fulton and Harris (2004, p. 308) and are not defined anywhere else in any of the other sources.
- Example: Consider $M_{n\times n}(F)^{\text{op}}$.
    - Claim: This algebra equals regular $M_{n\times n}(F)$.
    - The map between these spaces is $A \mapsto A^T$.
    - There are other maps, such as conjugation and then transpose.
    - Being isomorphic to your opposite is a strange and interesting property!
- Example: $\mathbb{C}[G]^{\text{op}} \cong \mathbb{C}[G]$.
    - Left as an exercise to find the map.
- Let M, N be modules. We now investigate some properties of $\text{Hom}_A(M, N)$, a nice abelian group.
    - Explicitly, it's

$$\text{Hom}_A(M, N) = \{f : M \to N \text{ linear} \mid f(am) = af(m)\ \forall\ a \in A\}$$

    - We have that

$$\text{Hom}_A(M_1 \oplus M_2, N) \cong \text{Hom}_A(M_1, N) \oplus \text{Hom}_A(M_2, N)$$

        - Prove by looking at what happens to vectors of the form $(M_1, 0)$ and $(0, M_2)$.
    - Similarly,

$$\text{Hom}_A(M, N_1 \oplus N_2) \cong \text{Hom}_A(M, N_1) \oplus \text{Hom}_A(M, N_2)$$

- What if we have $\text{Hom}(M_1 \oplus \cdots \oplus M_n, N_1 \oplus \cdots \oplus N_m)$?
    - Then we have by induction from the previous cases that

$$\text{Hom}(M_1 \oplus \cdots \oplus M_n, N_1 \oplus \cdots \oplus N_m) = \bigoplus_{\substack{i=1,\dots,n \\ j=1,\dots,m}} \text{Hom}(M_i, N_j)$$

    - Let $\varphi_{ij} \in \text{Hom}(M_i, N_j)$.

- At this point, it's very natural to write matrices

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} & & \\ & \varphi_{ji} & \\ & & \end{bmatrix} \end{array} \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} \varphi_{11}(m_1) + \cdots + \varphi_{1n}(m_n) \\ \vdots \\ \ \end{pmatrix} = \begin{pmatrix} (\varphi(m)) \\ \vdots \\ \ \end{pmatrix}$$

  - Is it ϕ_{ji} or ϕ_{ij}?? Lang (2002, p. 642) seems to back the latter.
- To make this make sense for ourselves, write out the 2×2 case from $M_1 \oplus M_2 \to M_1 \oplus M_2$.

$$\begin{pmatrix} \varphi_{11} & \varphi_{21} \\ \varphi_{12} & \varphi_{22} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} \qquad\qquad \\ \ \end{pmatrix}$$

- Matrices made out of maps can seem really confusing when you first start, but in time, it will make sense.

- Recall the result from last time about division algebras.
- The main object we need to understand is a **semisimple algebra**.
- **Semisimple** (module): A module that satisfies any of the conditions in the following theorem.
  - Note that we proved something analogous to condition 3 early on! This was the complements theorem.
  - There is an equivalent for infinite-dimensional algebras; we need **Zorn's lemma** regarding maximal ideals/the axiom of choice here, though.
- Theorem: Let A be an algebra over F, and let M be a left-module. Then TFAE.
  1. $M = \bigoplus_{i \in I} S_i$, where each S_i is a simple module and I is an **indexing set**, not a simple module/ideal.
  2. $M = \sum_{i \in I} S_i$, where the sum is *not* direct.
  3. For all submodules $N \subset M$, there exists N' such that $M = N \oplus N'$.

Proof. This proof only applies for the case that M is finite dimensional; the theorem is more general than that, but we are not interested in the more general case.

($1 \Rightarrow 2$): Very clear; all direct sums are sums.

($2 \Rightarrow 1$): Consider the maximal subset $J \subset I$ (by inclusion, not by indices) of our indexing set such that

$$\sum_{i \in J} S_i = \bigoplus_{i \in J} S_i$$

In other words, J induces the highest-dimension sum of submodules that is a direct sum. Note that we can still find a singleton J in the direct-sum-of-one-thing case, so we're starting from a good base case.

Claim: $\bigoplus_{i \in J} S_i = M$. Suppose not. Then there exists $m \in M$ such that $m \notin \bigoplus_{i \in J} S_i$ and $m = s_{i_1} + \cdots + s_{i_k}$ where each $s_{i_j} \in S_{i_j}$. If all $s_{i_1}, \dots, s_{i_k} \in \bigoplus_{i \in J} S_i$, then we have arrived at a contradiction and we are done. If not, then there exists some s_{i_t} such that $s_{i_t} \notin \bigoplus_{i \in J} S_i$. Now consider $S_{i_t} \cap (\bigoplus_{i \in J} S_i)$. This will be a submodule of S_{i_t}. But since S_{i_t} is simple by hypothesis, this means that $S_{i_t} \cap (\bigoplus_{i \in J} S_i)$ either equals S_{i_t} or 0. However, we know that it can't equal S_{i_t} because above, we found $s_{i_t} \in S_{i_t}$ such that $s_{i_t} \notin \bigoplus_{i \in J} S_i$. Thus, $S_{i_t} \cap (\bigoplus_{i \in J} S_i) = 0$. But this means that $S_{i_t} + \bigoplus_{i \in J} S_i$ is a direct sum, which contradicts the choice of J as maximal.

($1 \Rightarrow 3$): Let's take a submodule $N \subset M$. By 1, $M = \bigoplus_{i \in I} S_i$. Let's look at all subsets J such that

$$N + \sum_{j \in J} S_j = N \oplus \left(\sum S_j \right)$$

Look at the maximal one by inclusion. Then once again, by the same proof strategy as above,

$$N \oplus \underbrace{\left(\sum S_j\right)}_{N'} = M$$

($3 \Rightarrow 1$): We use what we've learned about representations. Let $M = N_1 \oplus N_2$. Then N_2, if nonsimple, has subsets $N_2 \oplus N_3$. We can continue on and on. Because dimensions finitely decrease, we'll eventually have to arrive at a sum $N_1 \oplus \cdots \oplus N_m$ of simples. ☐

- Now, we have 3 definitions of semisimple modules.
- Corollary: If A is an algebra, M is a semisimple module, and $N \subset M$ is a submodule, then...
  1. N is semisimple.

     Proof. Let L be a submodule of N. We need to find a complement of L inside N. We can find $L' \subset M$ such that $L \oplus L' = M$. Then $L' \cap N \subset N$ is the complement of L in N. Why? Because of the following.
     Claim: $(L' \cap N) \bigoplus L = N$. Not intersecting: $L' \cap N \cap L \subset L' \cap L = 0$. Summing to the whole thing: Let $n \in N$ be arbitrary. Then since $n \in M$, there exists $\ell, \ell' \in L, L'$ such that $n = \ell + \ell'$. But since $n, \ell \in N$, we must have $\ell' \in N$ as well. Therefore, $\ell' \in L' \cap N$. ☐

  2. M/N is semisimple.
- Takeaway: Submodules and quotient modules of semisimple modules are semisimple modules.
- Lang (2002) has a write-up of the proof from today's class.
  - Funnily enough, it is the only textbook that does! Fulton and Harris (2004) doesn't have it; not even Etingof et al. (2011) has it!

5.2 Semisimple Algebras

10/25:

- More associative algebra today; we'll wrap it up next time.
- Review.
  - Let A be a finite dimensional associative algebra over a field F.
  - We want to understand when this algebra is very close to a *group algebra*.
    - Recall that $A = F[G] = \{a_{g_1}g_1 + \cdots + a_{g_n}g_n \mid a_i \in F\}$ is the group algebra of G a finite group.
  - Recall left modules.
    - These are very similar to representations.
    - Indeed, if we have a left module M, then we have a multiplication map $\rho : A \times M \to M$ with properties such as associativity, etc.
  - Recall right modules.
    - In a group representation, left modules over A are essentially the same thing as right modules over A^{op}.
    - Because there is a bijection between left modules over A and right modules over A^{op}, we sometimes have the case where A doesn't change, i.e., $A \cong A^{\text{op}}$.
  - All of the above motivated the definition of *semisimple*: If A is a finite dimensional algebra and M is a finite-dimensional module, then M is *semisimple* if it satisfies any one of three conditions from last time's theorem.

  - Note: When we describe a module as "finite-dimensional," we mean this in the sense of a vector space, i.e., literally finite-dimensional as opposed to finitely generated or anything like that.
  - Note: "Last time's theorem" refers to the semisimplicity conditions one, which is a part of Wedderburn-Artin theory but is *not* the **Wedderburn-Artin *theorem***. We'll get to this theorem eventually, but that's still in the future.

- Theorem (Maschke's theorem): Let G be a finite group and let F be a field. Suppose $(|G|, \operatorname{char} F) = 1$, i.e., they are coprime. Then every finite-dimensional left module over $F[G]$ is semisimple.

  Proof. We've already basically done this proof as part of last time's theorem. Here's a refresher, though.

  Let M be an arbitrary finite-dimensional left module over $F[G]$. Then there exists a map $F[G] \to \operatorname{End}_{F[G]}(M)$ (left multiplication; the action of elements of this ring on elements of M), or $G \to GL(M)$. Thus, M is a G-representation, which satisfies condition (3) from last time's theorem because of the complements theorem, stated as Theorem 1 from Serre (1977) for instance. ☐

- Takeaway: The proof actually works for any field under this condition.
  - Rudenko will reprove Maschke's theorem tomorrow a different way.
- In an algebra, we have a multiplication map $\cdot : A \times A \to A$.
  - If we take the perspective that this map defines an action of the left A on the right one, we see that A has the structure of a left A-module.
  - Vice versa for right-modules.
- **Semisimple** (algebra): An algebra for which every finite-dimensional A-module is semisimple. *Also known as* **semi-simple**.
- Theorem: Let A be a finite-dimensional associative algebra. Then TFAE.
  1. A is a semisimple algebra.
  2. A is semisimple as a left-module over A. Equivalently, as an A-module, $A \cong S_1^{n_1} \oplus \cdots \oplus S_k^{n_k}$.
  3. (Wedderburn-Artin theorem) $A \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k)$, where the $D_1, \dots, D_k$ are division algebras. Note that the isomorphism is an isomorphism of algebras.
- We will prove this theorem in just a moment, but there are a few preliminary comments to be made first.
- Let's look at the algebra $\mathbb{H}$.
  - We can create matrices of quaternions, and we can add and multiply these matrices just fine.
  - However, the determinant is weirder: Is it $ad - bc$ or $ad - cb$?
    - There is a theory of determinants of noncommutative fields called **algebraic *k*-theory**, but we will not get into that.
- Example: Proving (3) for $\mathbb{C}[G]$.
  - We have $\mathbb{C}[G]$. There are not many division algebras over complex numbers; only one, in fact: Complex numbers.
  - Let $V_1, \dots, V_k$ be the irreps. Then we want to show that

$$\mathbb{C}[G] \cong M_{d_1}(\mathbb{C}) \oplus \cdots \oplus M_{d_k}(\mathbb{C})$$

    where $d_i = \deg V_i$.

- Note: Matrices give us a nice way to compute otherwise complicated elements of $\mathbb{C}[G]$.
- Proof: Define a map $F : \mathbb{C}[G] \to M_{d_1}(\mathbb{C}) \oplus \cdots \oplus M_{d_k}(\mathbb{C})$ by

$$x \mapsto (\rho_{V_1}(x), \dots, \rho_{V_k}(x))$$

  - F is injective: $F(x) = 0$ implies that $\rho_{V_i}(x) = 0$ $(i = 1, \dots, k)$, so $xV_i = 0$ $(i = 1, \dots, k)$. In particular, this means that $x = x \cdot 1 = 0$.
  - F is surjective: F is injective and $\dim(\mathbb{C}[G]) = \sum d_i^2 = \dim[M_{d_1}(\mathbb{C}) \oplus \cdots \oplus M_{d_k}(\mathbb{C})]$.
  - F is a homomorphism of algebras: Left as an exercise.
- Note: Remember this theorem very well because it allows you to treat group rings very easily.
- Tomorrow, we'll bring characters into this picture.

- We now state a lemma that will be used to prove $2 \Rightarrow 3$.
- Lemma: Let $\mathrm{End}_A(A)$ denote the set of A-module endomorphisms of A. Then

$$\mathrm{End}_A(A) \cong A^{\mathrm{op}}$$

as algebras.

Proof. To prove the claim, it will suffice to construct an A-algebra isomorphism $F : \mathrm{End}_A(A) \to A^{\mathrm{op}}$. Define F by

$$F(f) := f(1)$$

for all $f \in \mathrm{End}_A(A)$. It should be fairly clear that

$$F(f + g) = F(f) + F(g) \qquad\qquad F(1) = 1$$

Proving that $F(f \circ g) = F(f) * F(g)$ is slightly more involved, but can be done as follows.

$$F(f \circ g) = [f \circ g](1) = f(g(1)) = f(g(1) \cdot 1) = g(1) \cdot f(1) = F(g) \cdot F(f) = F(f) * F(g)$$

Lastly, by plugging $f = a = aI$ and $g = f$ into the above, we can recover

$$F(af) = a * F(f)$$

Thus, F is an A-algebra *homomorphism*. To prove that it is an *isomorphism*, consider the inverse map $G : x \mapsto [a \mapsto ax]$. We can show that $F \circ G = 1_{A^{\mathrm{op}}}$ and $G \circ F = 1_{\mathrm{End}_A(A)}$, thus completing the proof. □

- We now prove the above theorem, which we restate for simplicity.
- Theorem: Let A be a finite-dimensional associative algebra over F. Then TFAE.
  1. A is a semisimple algebra.
  2. A is semisimple as a left-module over A. Equivalently, as an A-module, $A \cong S_1^{n_1} \oplus \cdots \oplus S_k^{n_k}$.
  3. (Wedderburn-Artin theorem) $A \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k)$, where the $D_1, \dots, D_k$ are division algebras. Note that the isomorphism is an isomorphism of algebras.

Proof. One line; very simple, but a little weird conceptually.

($2 \Rightarrow 1$): To prove that A is a semisimple algebra, it will suffice to show that every finite-dimensional A-module is semisimple. Let $M = Ae_1 + \cdots + Ae_n$ be an arbitrary finite-dimensional A-module. To show that it's semisimple, it will suffice to demonstrate that it's equal to the direct sum of simple modules. Define a map $A^n \to M$ by

$$(a_1, \dots, a_n) \mapsto a_1e_1 + \cdots + a_ne_n$$

This should (fairly clearly) be a surjective homomorphism of left A-modules. Moreover, since $A = S_1 \oplus \cdots \oplus S_k$ is semisimple as a left A-module by hypothesis, we have that $A^n = S_1^n \oplus \cdots \oplus S_k^n$. Since the map defined above is also injective, it follows that

$$M \cong A^n = S_1^n \oplus \cdots \oplus S_k^n$$

as desired.

($3 \Rightarrow 2$): Work it out in the HW!

($2 \Rightarrow 3$): Let's take $A = S_1^{n_1} \oplus \cdots \oplus S_k^{n_k}$ a left A-module where each S_i is simple. Then by the lemma,

$$A^{\text{op}} \cong \text{End}_A(A) = \text{Hom}_A(A, A) = \text{Hom}_A(S_1^{n_1} \oplus \cdots \oplus S_k^{n_k}, S_1^{n_1} \oplus \cdots \oplus S_k^{n_k}) = \bigoplus_{i,j=1}^{k} \text{Hom}_A(S_i^{n_i}, S_j^{n_j})$$

By Schur's lemma for associative algebras,

$$\text{Hom}_A(S_i, S_j) = \begin{cases} 0 & i \neq j \\ D_i & i = j \end{cases}$$

where each D_i is a division algebra. Thus, continuing from the above,

$$A^{\text{op}} \cong \bigoplus_{i=1}^{k} \text{Hom}_A(S_i^{n_i}, S_i^{n_i}) = \bigoplus_{i=1}^{k} M_{n_i}(\text{Hom}_A(S_i, S_i)) = \bigoplus_{i=1}^{k} M_{n_i}(D_i)$$

Note that $\text{Hom}_A(S_i^{n_i}, S_i^{n_i}) = M_{n_i}(\text{Hom}_A(S_i, S_i))$ because of the thing about homomorphisms of direct sums of modules equaling matrices of homomorphisms. This was discussed on Monday. We don't include any $\text{Hom}_A(S_i, S_j)$ because all of these are equal to zero; indeed, it appears that these matrices will be strictly diagonal. $\square$

- Consequence: It follows because the D_i's are division algebras that

$$A \cong \bigoplus_{i=1}^{k} M_{n_i}(D_i^{\text{op}})$$

  - What was the point of this??
- Note from last time that we forgot to discuss: A quotient module of a semisimple module is semisimple. Proving this will be in the next HW.
- **Radical** (of A): The finite dimensional A-algebra defined as follows. *Also known as* **Jacobson ideal**, **Jacobson radical**. *Denoted by* $\mathbf{Rad}(\boldsymbol{A})$. *Given by*

$$\text{Rad}(A) = \{a \in A \mid aS = 0 \text{ for any simple module } S\} \subset A$$

  - Immediate fact: $\text{Rad}(A)$ is a two-sided ideal.
  - This is because...
    - $x \in A$ and $a \in \text{Rad}(A) \implies (xa)S = x(aS) = x(0) = 0 \implies xa \in \text{Rad}(A)$;
    - $x \in A$ and $a \in \text{Rad}(A) \implies (ax)S = a(xS) = 0 \implies ax \in \text{Rad}(A)$.
  - Note that xS is simple in the above line because a scaled simple module is still simple.
- Theorem: A is semisimple iff $\text{Rad}(A) = 0$.
  - This will be explained next time.
  - In other words, if there are problematic elements, the algebra is not semisimple.

  - Quotienting algebras by two-sided ideals gives algebras, so if A is not semisimple, we know that $A/\operatorname{Rad}(A)$ *is* semisimple!
- This week: A brief primer on noncommutative algebra that is probably worth studying for the midterm.
- Next week: Number theoretic stuff, integer elements, groups, etc.
- Most people/books don't treat the finite-dimensional case here (so it's not written up anywhere) because they view it as too restrictive; instead, they prefer to use the **Artinian** condition.

5.3 The Jacobson Radical

10/27:
- Review.
  - Let A be an associative algebra over a field F. $\dim_F A = \infty$ (??).
  - A is semisimple if every left A-module is a sum of simple A-modules.
  - Theorem: A is semisimple iff ${}_AA$ is semisimple iff $A \cong M_{n_1\times n_1}(D_1) \oplus \cdots \oplus M_{n_k\times n_k}(D_k)$, where the D_i are division algebras.
    - Note: ${}_AA$ denotes A as a left A-module.
  - The simplest semisimple algebra is a matrix algebra.
- Example: A HW problem solution (PSet 4, Q4b).
  - Let $A = M_{n\times n}(F)$. Then $\dim(A) = n^2$.
  - One linear representation of A that's particularly nice is $S = F^n$, i.e., the set of all column vectors of length n with entries in F.
    - This representation $\rho : A \to GL(S)$ can simply be defined by $\rho(X) = X$.
    - Alternatively, this can be thought of as the map from $A \times S \to S$ sending $(X, v) \mapsto Xv$.
    - This is a simple representation! Using permutation matrices, for instance, we can see that no subspace is fixed under *every* $X \in M_{n\times n}(F)$.
  - The HW problem was to show that F^n is the only simple module over the matrix algebra.
  - Sidebar: To prove that A is semisimple, we can show that ${}_AM_{n\times n}(F) \cong \bigoplus^n S = S^n$.
    - To do so, use the module isomorphism $(v_1 \mid \cdots \mid v_n) \mapsto v_1 \oplus \cdots \oplus v_n$.
  - From here, we can deduce that if T is a simple module, we can construct a homomorphism $A^N = (S^n)^N \twoheadrightarrow T$?? It follows that $S \cong T$? What is this??
- Takeaways.
  - There is a unique simple module over the matrix algebra, i.e., the columns of the matrix.
  - The dimension of every module over a matrix algebra will be a multiple of n. Why?
    - $M_{n,n}(F)$ is semisimple. Thus, any (finite-dimensional) $M_{n,n}(F)$-module M is semisimple. Consequently, $M = \bigoplus_{i\in I} S_i$. But since every $S_i = F^n$, we have $M = (F^n)^{|I|}$ with $\dim(M) = n \cdot |I|$, as desired.
    - Think $n \times 1$ matrices (column vectors; what we just discussed), $n \times 2$ matrices, $n \times 3$ matrices, on and on.
- Moving on.
- We want something more complete about an algebra.
- Recall the radical of A.
- Main theorem: A is semisimple iff $\operatorname{Rad}(A) = 0$.

- Facts.
  1. $\text{Rad}(A)$ is a two-sided ideal.
     - Prove directly by multiplying on both left and right, as at the end of Wednesday's class.
  2. $\text{Rad}(A) = \bigcap L$ where L is a maximal left ideal.
- **Maximal** (left ideal of A): A left ideal L for which there exists no left ideal L' such that $L \subsetneq L' \subsetneq A$.
  - Ideals are subspaces. Maximal means biggest by inclusion, but not necessarily equal to the whole thing.
- We now prove Fact 2.

  Proof. We first establish some facts. Then we do a bidirectional inclusion proof.

  If L is a left ideal, then A/L is a left A-module. If we now assert that L is a *maximal* left ideal, then A/L is a *simple* left A-module. This is because of the following correspondence theorem, a very general fact that's easy to show: Essentially, if you have some modules M, N such that $N \leq M$, then the modules in between $N \subsetneq M$ are in bijection with M/N. This bijection is defined in the forward direction by quotienting modules in between $N \subsetneq M$, and in the reverse direction by taking the preimage of the quotient projection. Thus, maximal left ideals L have nothing in between them and A, so A/L is in bijection with nothing! Moreover, *every* simple module is obtained this way.

  If S is an arbitrary simple module containing $v_0 \neq 0$, then we may define $f : A \to S$ sending $a \mapsto av_0$. Note that $0 \subsetneq \text{Im}(f) \subseteq S$. But since S is simple, we must have $\text{Im}(f) = S$ so f must be surjective. It follows that $S \cong A/L$ for some maximal left ideal L of A.[1]

  If $x \in \text{Rad}(A)$ and L is a maximal left ideal of A, then $x(A/L) = 0$ (since A/L is simple). It follows that $xA \subseteq L$. It follows since $x \in xA$ that $x \in L$. Thus, $\text{Rad}(A) \subset \bigcap L$.

  Now, to show the other inclusion, let $x \in \bigcap L$. Let S be an arbitrary simple module over A. We know that $S \cong A/L$ for some maximal ideal L. To demonstrate that $xS = 0$, it will suffice to confirm that $xv_0 = 0$ for all $v_0 \in S$. Let $0 \neq v_0 \in S$ be arbitrary. Define $f : A \to S$ by $a \mapsto av_0$. Since v_0 is nonzero and hence $\text{Im}(f)$ is nontrivial, the fact that S is simple must mean that $\text{Im}(f) = S$ and hence f is surjective. Thus, $A/\text{Ker}(f) \cong S$. Consequently, $\text{Ker}(f) = L$. It follows since $x \in \bigcap L$ and hence $x \in L$ that $x \in \text{Ker}(f)$. But then $xv_0 = 0$, as desired. ☐

- Thus, the radical has the equivalent descriptions

$$\text{Rad}(A) = \{a \in A \mid aS = 0 \text{ for any simple module } S\} = \bigcap L$$

- Theorem: A (finite-dimensional) is semisimple iff $\text{Rad}(A) = 0$.

  Proof. We will prove both directions independently here. Let's begin.

  ($\Rightarrow$): Suppose A is semisimple. Then $A = S_1 \oplus \cdots \oplus S_N$. It follows in particular that $1 = s_1 + \cdots + s_N$ for some $s_i \in S_i$ ($i = 1, \dots, n$). Now let $a \in \text{Rad}(A)$ be arbitrary; we hope to show that $a = 0$. Fortunately, we can do this as follows via

$$a = a \cdot 1 = as_1 + \cdots + as_N = 0 + \cdots + 0 = 0$$

  Just to be super clear, $as_i = 0$ because $a \in \text{Rad}(A)$ implies $aS = 0$ for all simple modules S, including S_i of which s_i is an element and is thus annihilated by a.

  ($\Leftarrow$): Suppose $\text{Rad}(A) = 0$. Then $\bigcap L = 0$. This combined with the fact that A is finite dimensional implies that there exists a finite collection $L_1, \dots, L_n$ of maximal ideals such that $\bigcap L = \bigcap^n L_i$. (In particular, $n \leq \dim A$. Essentially, since we're finite dimensional, what we can do is drop dimensions

[1] This seems redundant; perhaps Rudenko meant to say $L = \text{Ker}(f)$ in addition to the other stuff??

from $\dim L_1$ to $\dim L_1 \cap L_2$ to $\dim L_1 \cap L_2 \cap L_3$, so since we're eventually going to hit zero, we're eventually going to have to stop. In other words, choose L_1, then choose L_2 such that $\dim L_1 \cap L_2 < \dim L_1$, then choose L_3 such that $\dim L_1 \cap L_2 \cap L_3 < \dim L_1 \cap L_2$, and continue in this fashion until we have $\dim L_1 \cap \cdots \cap L_n = 0$; because the sequence $\dim L_1 \cap \cdots \cap L_i$ is strictly decreasing and the initial value is finite, the sequence must eventually terminate.) Thus, $\text{Rad}(A) = L_1 \cap \cdots \cap L_n$. One line to finish. View A as a left A-module (denote it ${}_AA$ with left subscript A). Define $f : {}_AA \to A/L_1 \oplus \cdots \oplus A/L_n$ by $f(a) = (\pi_1(a), \dots, \pi_n(a))$, where $\pi_i(a) : A \to A/L_i$ denotes the projection function $a \mapsto a + L_i$. Then

$$\text{Ker}(f) = \bigcap_{i=1}^{n} L_i = \text{Rad}(A) = 0$$

But then f is injective. This combined with the fact that $A/L_1 \oplus \cdots \oplus A/L_n$ is semisimple by definition means that ${}_AA$ is isomorphic to a submodule of a semisimple module. Thus, since the only submodules of a semisimple module are mix-and-match combinations of the semisimple module's constituent simple modules, ${}_AA$ is semisimple itself. Therefore, by the semisimple algebra conditions from Wednesday's class, A is semisimple. ☐

- **Artinian** (ring): A ring for which every decreasing sequence of ideals has to stabilize.
- Let S_1, S_2 be simple modules, and let M be some module. We get $\text{Hom}_G(S_2, S_1)$, $\text{Ext}^1(S_2, S_1)$, $\text{Ext}^2(S_2, S_1)$, ... This gets very complicated very quickly, and you actually need homological algebra to keep track of everything.
  - Point??
- New HW problem: $A = \mathbb{F}_p[G]$ (p a prime) is never semisimple. This is called **modular representation theory**, it's in our book (where??), and it's hard.
- A very concrete criterion for semisimplicity.
  - Let $F = \mathbb{C}$ and let A be finite dimensional with $\dim_F A = n$.
  - Define a scalar product in A by
    $$\langle x, y \rangle = \text{tr}(L_x L_y)$$
    - $L_x : A \to A$ is the map that sends $a \mapsto xa$.
    - This is a symmetric map; it's got a lot of nice properties actually.
    - Note: $\text{tr}(L_x L_y)$ is colloquially known as $\text{tr}(xy)$.
  - Theorem: Let A be a finite-dimensional algebra over $\mathbb{C}$. Then A is semisimple iff $\text{tr}(x^2)$ is **nondegenerate**, which means that if $\text{tr}(xa) = 0$ for any x, then $a = 0$. We've probably seen this in the context of vector spaces like $V \otimes V \to \mathbb{C}$ or $V \cong V^*$. What is this??
  - Something about $|G|^{|G|}$. What is this??
- **Nondegenerate** (finite-dimensional bilinear form): A bilinear form $f(x, a)$ such that if $f(x, a) = 0$ for any x, then $a = 0$.
- Next week: Number theoretic group theory and then representation theory of symmetric groups.

5.4 L Chapter XVII: Semisimplicity

From Lang (2002).

11/10:
- Sections 1-2 cover a lot of the stuff discussed during Monday's class.
- Rewrite proof of theorem from Monday's class!

12/25:
- Perhaps the reason that we talked about matrices of functions right before semisimplicity is that homomorphisms of semisimple modules, in particular, have this decomposable structure.

- The Wedderburn-Artin theorem is not obviously covered here.
- The radical is talked about in Section 6.

Week 6

Abstract Representation Theory

6.1 The Center of the Group Algebra

10/30: • Plan for this week.

  - Today: Briefly discuss a very important concept called the **center**.
  - Wednesday: Do algebraic numbers.
  - Friday: Burnside's theorem.

- **Center** (of a group): The set of all elements of a group G that commute with every other element in G. *Denoted by* $\boldsymbol{Z(G)}$. *Given by*

$$Z(G) = \{g \in G \mid xg = gx \ \forall \ x \in G\}$$

  - Note: $Z(G)$ is a subgroup of G.

- The center is one of the most important concepts in all of representation theory.
  - Example: Let A be an abelian group, such as $Z(G)$. Then all its irreps are 1D.
    - See Section 1.3 of Fulton and Harris (2004) for an explanation.
  - Normally, the center of a group is too small to be interesting.
  - However, $Z(\mathbb{C}[G])$ is large enough to be interesting.

- **Center** (of an algebra): The set of all elements of an algebra A that commute with every other element in A. *Denoted by* $\boldsymbol{Z(A)}$. *Given by*

$$Z(A) = \{a \in A \mid xa = ax \ \forall \ x \in A\}$$

- Proposition: If A is an algebra over $\mathbb{C}$, M is an irreducible left A-module, and $\rho : A \to \operatorname{End}(M)$ is a corresponding representation, then $x \in Z(A)$ implies that $\rho(x) = \lambda I$, i.e., $\rho(x)$ is a *scalar matrix.*

  Proof. Let $x \in Z(A)$ be arbitrary. Then for all $a \in A$, we know that $\rho(x)\rho(a) = \rho(a)\rho(x)$. Thus, $\rho(x)$ is a morphism of A-modules. Consequently, since M is irreducible (also known as *simple*), Schur's Lemma for associative algebras implies that $\operatorname{Hom}_A(M, M)$ is a division algebra over $\mathbb{C}$. But since $\mathbb{C}$ is the only division algebra over $\mathbb{C}$, we have that $\operatorname{Hom}_A(M, M) \cong \mathbb{C}$. From here, it readily follows that $\rho(x)$ is equal to some λI. ☐

- Consequence: If M is reducible, we can reduce it into component scalar representations.

- Consequence: If G is an abelian group, then every irrep V is 1-dimensional.

  - Additionally, $\mathbb{C}[G]$ is commutative and hence $\mathbb{C}[G] = Z(\mathbb{C}[G])$.
  - Then if V is an arbitrary representation of $\mathbb{C}[G]$ (i.e., there exists $\rho_V : \mathbb{C}[G] \to \text{End}(V)$), then V is equal to the direct sum of one dimensional irreducible representations. Hence, for all g, we have $\rho_V(g) = \lambda I$.
    - Could we not have different λ's for each irrep, i.e., $\rho_V(g) = \text{diag}(\lambda_1, \dots, \lambda_n)$ instead of only $\rho_V(g) = \text{diag}(\lambda, \dots, \lambda)$??
- We now try to compute $Z(\mathbb{C}[G])$.
  - Facts:

$$Z(A_1 \oplus A_2) = Z(A_1) \oplus Z(A_2) \qquad\qquad Z(M_n(\mathbb{C})) = \text{span}(I) \cong \mathbb{C}$$

  - These facts coupled with the fact that G is a finite group (hence $\mathbb{C}[G] \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$ where k is the number of conjugacy classes in G by the example from last Wednesday's class) yield

$$\begin{aligned} Z(\mathbb{C}[G]) &\cong Z(M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})) \\ &\cong \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{k \text{ times}} \\ &= \mathbb{C}^k \end{aligned}$$

- Let $C_1, \dots, C_k$ be conjugacy classes in G. Then we may define

$$e_i = \sum_{g \in C_i} g$$

  for each $i = 1, \dots, k$.
  - Example: In S_3 the three e_i's are $\{e, (12) + (13) + (23), (123) + (132)\}$.
- We will use "$Z(G)$" to denote $Z(\mathbb{C}[G])$ oftentimes going forward.
  - Let $x \in \mathbb{C}[G]$ be arbitrary. Note that $xg = gx$ for all $g \in G$ if and only if $xy = yx$ for all $y \in \mathbb{C}[G]$.
- Claim: $Z(G) = \langle e_1, \dots, e_k \rangle$.

Proof. We will use a bidirectional inclusion proof.
$\underline{\langle e_1, \dots, e_k \rangle \subset Z(G)}$: Let $e_i \in \mathbb{C}[G]$ and $x \in G$ be arbitrary. As noted above, proving that $e_i x = x e_i$ for any $x \in G$ will suffice to show that $e_i x = x e_i$ for any $x \in \mathbb{C}[G]$. Indeed, we do find that

$$\begin{aligned} x e_i x^{-1} &= \sum_{g \in C_i} x g x^{-1} = \sum_{h \in C_i} h = e_i \\ x e_i &= e_i x \end{aligned}$$

This naturally extends to any sums and scalar multiples of the e_i's.
$\underline{Z(G) \subset \langle e_1, \dots, e_k \rangle}$: Let $a \in Z(G)$ be arbitrary. As an element of $\mathbb{C}[G]$, we know that $a = \sum a_g g$ for some $a_g \in \mathbb{C}$. Additionally, since $a \in Z(G)$, we have that $xax^{-1} = a$ for all $x \in G$. Combining these last two results, we have that

$$\sum_{g \in G} a_{x^{-1}gx} g = \sum_{g \in G} a_g x g x^{-1} = x a x^{-1} = a = \sum_{g \in G} a_g g$$

Comparing like terms in the above equality, we can learn that for all $x \in G$, we have $a_{x^{-1}gx} = a_g$. In other words, all of the a_g's for g's in the same conjugacy class are equal. Therefore, a is of the form $a = \sum_{i=1}^k a_{g_i} e_i$ for $g_i \in C_i$. $\square$

- Thus we get $a_e e + a_{(12)}(12) + a_{(13)}(13) + \cdots$??
- Computing products of the e_i: What if we want to compute $[(12) + (13) + (23)]^2$, for example? We have to multiply *noncommutatively*, so HS formulas are out, but we can still do all nine multiplications and sum them:
$$[(12) + (13) + (23)]^2 = 3e + 3[(123) + (132)]$$
- We now tie this claim back into our discussion of $Z(\mathbb{C}[G])$.
  - As we just claimed and proved, $Z(\mathbb{C}[G])$ has basis $e_1, \dots, e_k$.
  - Recall that $Z(\mathbb{C}[G]) = \mathbb{C} \oplus \cdots \oplus \mathbb{C}$, with characters $\chi_1, \dots, \chi_k$.
  - Then $f_{\chi_i} = (0, \dots, 0, 1, 0, \dots, 0)$, where the 1 lies in the i^{th} slot.
  - Then we get $f_{\chi_1}, \dots, f_{\chi_k}$ as a basis.
  - It follows that $f_{\chi_i}^2 = f_{\chi_i}$ and $f_{\chi_i} f_{\chi_j} = 0$ for $i \neq j$; this is exactly what it means for a space to be $\mathbb{C} \oplus \cdots \oplus \mathbb{C}$.
  - Both of these spaces (center elements and class functions) have these two interconnected bases, so the spaces are quite similar!
- The center of a group algebra $Z(\mathbb{C}[G])$ can be identified "=" with the space of class functions $\mathbb{C}_{\text{cl}}(G)$ via
$$\sum \varphi(g)g \mapsto [g \to \varphi(g)]$$
where $\varphi(xgx^{-1}) = \varphi(g)$.
  - This isomorphism is an isomorphism of vector spaces, *not* an isomorphism of algebras.
    - This is exactly the map briefly described in Lecture 3.1, down to the fact that it maps coefficients to functional outputs but allows for different kinds of multiplication!!!
  - However, it still has cool properties.
    - For instance, consider the δ_{C_i}: The functions sending $g \in C_i$ to 1 and $g \notin C_i$ to 0.
    - The isomorphism identifies $e_i \mapsto \delta_{C_i}$.
  - Do we get irreducible characters (our other basis of class functions) when we sum the $\varphi(g)g$'s?
    - We do! What is this??
  - Let's consider another basis χ of irreducibles. The basis is $f_\chi = \frac{d_\chi}{|G|} \sum_{g \in G} \chi(g^{-1})g$, and we send it to χ_{V^*}.
  - Claim:
$$f_{\chi_i} f_{\chi_j} = \begin{cases} f_{\chi_i} & \chi_i = \chi_j \\ 0 & \chi_i \neq \chi_j \end{cases}$$
    - Things that multiply like this are called the **central idempotent**.
  - Thus, general multiplication works as follows.
$$(a_1 f_{\chi_1} + \cdots + a_n f_{\chi_n})(b_1 f_{\chi_1} + \cdots + e_n f_{\chi_n}) = a_1 b_1 f_{\chi_1} + \cdots + a_n b_n f_{\chi_n}$$
  - So if we want to send $a \in Z(G)$ to $\bigoplus^k \mathbb{C}$, we map
$$a = a_1 f_{\chi_1} + \cdots + a_k f_{\chi_k} \mapsto (a_1, \dots, a_k)$$
  - The proof of this claim is really simple because we've already done the computation with the projector on the irrep V_x.
    - So if you want to see $\rho(f_\chi)$, see what it does to the identity: It does $\rho(f_\chi)e = f\chi e = f_\chi$. ρ is regular.
- **Central idempotent**: An element such that $a^2 = a$ and $ax = xa$ for all $x \in A$.

- Takeaway: Class functions and the center are two approaches to the same thing.
  - The great thing about the center: You can understand what it looks like because it is well-defined as a commutative algebra.
  - If something is isomorphic to $\mathbb{C} \oplus \cdots \oplus \mathbb{C}$ as an algebra, then there is another space and basis in which your multiplication looks incredibly simple.
- We might get to **Hopf algebras** at the end of the course (very interesting).
  - Let $\mathbb{C}[G]$ be an associative algebra.
  - Let $\mathbb{C}[G]^*$ be the functions on the group.
  - Then $A \otimes A \to A$ sends $a_1 \otimes a_2 \mapsto a_1 a_2$.
  - When we dualize to get $A^* \otimes A^* \to A^*$, everything gets reversed, so we actually get a **comultiplication** $A \to A \otimes A$ given by $g \mapsto g \otimes g$. These two multiplications together are called a **Hopf algebra**.
  - Knowing that there's something that we can define and understand might help us untangle the knot of all the spaces.
  - This is pretty heavy math, though, so we won't go too deep into it if we get at all.
- Today was the last associative algebra class.
- Going forward: Integral elements, algebraic integers, dimension of the representation divides the order or the group, Burnside's theorem.
- Midterm is heavily computational: Tensor products, character tables, etc. A few simple questions about things.
  - Comparably less associative algebra stuff (maybe just 1 exercise).

6.2 Algebraic Numbers and the Frobenius Divisibility Theorem

11/1:
- Announcements.
  - OH on Zoom today as well; both OH next week will be in person.
- New topic for the next couple of classes (today and Friday at least, possibly Monday as well).
  - Proving two wonderful theorems.
- Theorem 1 (Frobenius divisibility theorem[1]): Let G be a finite group, and let V be an irreducible representation of G over $\mathbb{C}$. Then the degree of V divides the order of G, i.e.,

$$d_V \mid |G|$$

- Theorem 2 (Burnside): If G is a group and $|G| = p^n q^m$, then G is not simple. In fact, G is **solvable**.
  - Seems completely unrelated to Theorem 1, but the methods are similar.
  - The first statement in this theorem is hard and interesting. We will briefly talk about the second one, but it follows form the first by an easy induction.
- Both proofs are based on number theory.
  - As a warm-up to this branch of mathematics, let's talk about the algebraic integers.

[1] There is no agreed-upon name for this result, but Fulton and Harris (2004) call it the "Frobenius divisibility theorem."

- **Algebraic** (number): A number $x \in \mathbb{C}$ for which there exists $a_0, \dots, a_{n-1} \in \mathbb{Q}$ such that

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

- $\bar{\mathbb{Q}}$: The set of all algebraic numbers.
  - So $\mathbb{Q} \subset \bar{\mathbb{Q}} \subset \mathbb{C}$, where $\bar{\mathbb{Q}}$ is the set of all algebraic numbers.
  - π, e are famous examples of numbers that are *not* algebraic.
- **Algebraic** (integer): An algebraic number for which the corresponding $a_0, \dots, a_{n-1} \in \mathbb{Z}$.
- $\bar{\mathbb{Z}}$: The set of all algebraic integers.
- Examples.
  1. $\sqrt{2} \in \bar{\mathbb{Z}}$.
     - Because $(\sqrt{2})^2 - 2 = 0$.
  2. $\sqrt{3} \in \bar{\mathbb{Z}}$.
  3. $\sqrt{2}/2 \notin \bar{\mathbb{Z}}$.
     - Let $x = \sqrt{2}/2$.
     - We know that $2x^2 - 1 = 0$.
     - Suppose $d(x^n + a_{n-1}x^{n-1} + \cdots + a_0) = (2x^2 - 1)(dx^n + \cdots)$. This is an actual use of Gauss's Lemma from MATH 25800.
     - So $d = 1 \cdot 1$, contradiction.
     - How does this proof work??
- To get a handle on the algebraic integers, we'll prove some basic results (Facts 1-2 below).
- Fact 1: For all $x \in \bar{\mathbb{Q}}$, there exists $d \in \mathbb{N}$ such that $dx \in \bar{\mathbb{Z}}$.

  Proof. Take the polynomial with rational coefficients which is satisfied by x, and then multiply the polynomial by d^n where $d = \operatorname{lcm}(\text{denominators of } a_0, \dots, a_{n-1})$ is the greatest common denominator of all coefficients. This yields the polynomial

$$(dx)^n + da_{n-1}(dx)^{n-1} + \cdots + d^n a_0 = 0$$

  in dx where each coefficient $d^i a_{n-i}$ is, by the definition of d, now an integer. □

- Fact 2: $\mathbb{Q} \cap \bar{\mathbb{Z}} = \mathbb{Z}$.

  Proof. We will use a bidirectional inclusion proof.
  $\underline{\mathbb{Q} \cap \bar{\mathbb{Z}} \subset \mathbb{Z}}$: Let $x \in \mathbb{Q} \cap \bar{\mathbb{Z}}$ be arbitrary. Since $x \in \mathbb{Q}$, there exist $a \in \mathbb{Z}$, $b \in \mathbb{N}$ with $(|a|, |b|) = 1$ (that is, with a, b coprime) such that $x = a/b$. Since $x \in \bar{\mathbb{Z}}$, there exist $a_0, \dots, a_n \in \mathbb{Z}$ such that

$$\begin{aligned} \left(\frac{a}{b}\right)^n + a_{n-1}\left(\frac{a}{b}\right)^{n-1} + a_{n-2}\left(\frac{a}{b}\right)^{n-2} + \cdots + a_0 &= 0 \\ a^n + a_{n-1}a^{n-1}b + a_{n-2}a^{n-2}b^2 + \cdots + a_0 b^n &= 0 \end{aligned}$$

  Now suppose for the sake of contradiction that there exists a prime number p dividing b. Then $b = px$ for some $x \in \mathbb{N}$. Consequently,

$$\begin{aligned} a^n + a_{n-1}a^{n-1}px + a_{n-2}a^{n-2}(px)^2 + \cdots + a_0(px)^n &= 0 \\ a^n + p(a_{n-1}a^{n-1}x + a_{n-2}a^{n-2}px^2 + \cdots + a_0 p^{n-1}x^n) &= 0 \\ p\underbrace{\left(-a_{n-1}a^{n-1}x - a_{n-2}a^{n-2}px^2 - \cdots - a_0 p^{n-1}x^n\right)}_{y} &= a^n \end{aligned}$$

Thus, since $a^n = py$ (where y is an integer as the sum of products of integers), we have that $p \mid a^n$. It follows that $p \mid a$, since p is prime and raising a to a power doesn't introduce any new primes into its factorization. Consequently, since $p > 1$ as a prime number, there exists a number greater than 1 dividing both a and b. Therefore, $(|a|, |b|) > 1$, a contradiction. It follows that no prime number divides b, and hence, we must have $b = 1$ and $x = a \in \mathbb{Z}$, as desired.

$\underline{\mathbb{Z} \subset \mathbb{Q} \cap \bar{\mathbb{Z}}}$: Let $x \in \mathbb{Z}$ be arbitrary. Then $x = x/1 \in \mathbb{Q}$. Additionally, choosing $a_0 = -x$, we have $x + a_0 = 0$. Thus, $x \in \bar{\mathbb{Z}}$. Combining these two results yields $x \in \mathbb{Q} \cap \bar{\mathbb{Z}}$, as desired. □

- We now look at the natural problem to which an algebraic integer is always the solution.
- Fact 3: Let $A \in M_{n\times n}(\mathbb{Z})$. If λ is an eigenvalue of A, then $\lambda \in \bar{\mathbb{Z}}$. More simply, $Av = \lambda v$ implies that $\lambda \in \bar{\mathbb{Z}}$.

  Proof. To prove that $\lambda \in \bar{\mathbb{Z}}$, it will suffice to find a monic polynomial P with integer coefficients such that $P(\lambda) = 0$. Let χ_A be the characteristic polynomial of A. As a characteristic polynomial, χ_A is monic. Additionally, since A is a matrix over the integers, the coefficients of χ_A will all be integers. Lastly, since $Av = \lambda v$, we know that $\chi_A(\lambda) = 0$. □

- Lemma: The converse of Fact 3 is true. That is, if $\lambda \in \bar{\mathbb{Z}}$, then there exists $A \in M_{n\times n}(\mathbb{Z})$ and $v \in \mathbb{C}^n$[2] such that $Av = \lambda v$.

  Proof. $\lambda \in \bar{\mathbb{Z}}$ implies $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0 = 0$. This implies that there exists $A \in M_{n\times n}(\mathbb{Z})$ such that $\chi_A(\lambda) =$ this polynomial $= 0$.

  Rudenko leaves it as an exercise to find this A. The solution is just the Frobenius matrix from MATH 27300, i.e., if we take

$$A = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}$$

  then $\chi_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0 = 0$. □

- We now use the above to give a cryptic proof of an interesting fact.
- Fact 4: $\bar{\mathbb{Z}}$ is a ring. That is, if $x, y \in \bar{\mathbb{Z}}$, then $x + y, xy \in \bar{\mathbb{Z}}$.

  Proof. Since $x, y \in \bar{\mathbb{Z}}$, the lemma implies that there exist A, B, v, w such that

$$Av = xv \qquad\qquad Bw = yw$$

  Note that A can be of dimension $n \times n$ and B of dimension $m \times m$, i.e., they need not be the same dimension. Now how do we find a matrix for which the sum $x + y$ and product xy are eigenvalues? We use the tensor/Kronecker product to start! In particular,

$$(A \otimes B)(v \otimes w) = xy(v \otimes w)$$

  For sum, we take $A \otimes I_m + I_n \otimes B$ so that

$$(A \otimes I_m + I_n \otimes B)(v \otimes w) = xv \otimes w + v \otimes yw = (x + y)v \otimes w$$

  It follows by the two lines above and Fact 3 that $xy, x + y \in \bar{\mathbb{Z}}$, as desired. □

- Notes on the above proof.

[2]Where does v lie?? Is it $\mathbb{Z}^n$ or something, or are there no restrictions as I suspect?

- Types of proofs.
  - This is a nonstandard proof from Etingof et al. (2011).
  - The old proof from the 1800s uses symmetric stuff. It goes something like this:
    - Let $x = x_1, \dots, x_n$ and $y = y_1, \dots, y_m$, and take $\prod_{i,j=1}^{n,m}(t - x_i - y_j)$. Then we observe symmetric polynomials.
    - We'll cover a lot more of this stuff later.
  - There is also one more (more abstract) proof using modules.
- Like algebraic integers form a ring, algebraic numbers form a field.

- So, cool... but why are algebraic integers relevant to us?
  - Observe that if G is a group and χ_V is a character, then for all $g \in G$, we have $\chi_V(g) \in \bar{\mathbb{Z}}$!
  - Why would this be the case?
    - Recall that since $g^n = e$, $\chi(g) = \operatorname{tr}(\rho(g)) = \varepsilon_1 + \cdots + \varepsilon_n$ where the ε_i are n^{th} roots of unity.
    - Each root of unity is an algebraic integer under the polynomial $x^n - 1 = 0$.
    - Thus, by inducting on Fact 4, the sum $\varepsilon_1 + \cdots + \varepsilon_n \in \bar{\mathbb{Z}}$.

- Fact 5: Let $C := \{g_1, \dots, g_s\}$ be a conjugacy class of G, and let $e_C := g_1 + \cdots + g_s \in \mathbb{Z}[G] \subset \mathbb{C}[G]$. Then there exist $a_0, \dots, a_{n-1} \in \mathbb{Z}$ such that

$$e_C^n + a_{n-1}e_C^{n-1} + \cdots + a_0 = 0$$

Proof. Define $L_{e_C} : \mathbb{Z}[G] \to \mathbb{Z}[G]$ by $a \mapsto e_C a$. Thus, L_{e_C} has eigenvalue e_C and matrix representation

$$L_{e_C} = \begin{array}{c} \\ g_1 \\ \vdots \\ g_n \end{array}\begin{array}{c} \begin{array}{ccc} g_1 & \cdots & g_n \end{array} \\ \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \end{array} \in M_{n\times n}(\mathbb{Z})$$

Therefore, by an argument analogous to that used in Fact 3, the desired $a_0, \dots, a_{n-1} \in \mathbb{Z}$ exist. $\square$

- Example to illustrate the above argument: Consider $C = \{(12), (13), (23)\} \subset S_3$.
  - Then $e_C = (12) + (13) + (23)$.
  - Label the elements of S_3 as follows.

$$S_3 = \{\underbrace{e}_{g_1}, \underbrace{(12)}_{g_2}, \underbrace{(13)}_{g_3}, \underbrace{(23)}_{g_4}, \underbrace{(123)}_{g_5}, \underbrace{(132)}_{g_6}\}$$

  - Then the matrix of L_{e_C} is given by the following.

$$L_{e_C} = \begin{array}{c|cccccc} & e & (12) & (13) & (23) & (123) & (132) \\ \hline e & 0 & 1 & 1 & 1 & 0 & 0 \\ (12) & 1 & 0 & 0 & 0 & 1 & 1 \\ (13) & 1 & 0 & 0 & 0 & 1 & 1 \\ (23) & 1 & 0 & 0 & 0 & 1 & 1 \\ (123) & 0 & 1 & 1 & 1 & 0 & 0 \\ (132) & 0 & 1 & 1 & 1 & 0 & 0 \end{array}$$

    - Notice how, for example, representing e as $(1, 0, 0, 0, 0, 0)$ yields

$$L_{e_C}e = (0, 1, 1, 1, 0, 0) = (12) + (13) + (23) = e_C$$

as expected.

- We can then calculate that the characteristic polynomial $\chi_{L_{e_C}}$ of L_{e_C} is

$$\chi_{L_{e_C}}(\lambda) = \det(L_{e_C} - \lambda I) = \lambda^6 - 9\lambda^4$$

- This yields

$$a_0 = 0 \qquad a_1 = 0 \qquad a_2 = 0 \qquad a_3 = 0 \qquad a_4 = -9 \qquad a_5 = 0$$

as the desired coefficients.

- Sanity check: We can confirm that

$$\begin{aligned}
e_C^6 - 9e_C^4 &= e_C^4(e_C^2 - 9) \\
&= (9[e + (123) + (132)])(3[e + (123) + (132)] - 9) \\
&= 27[e + (123) + (132)]^2 - 81[e + (123) + (132)] \\
&= 81[e + (123) + (132)] - 81[e + (123) + (132)] \\
&= 0
\end{aligned}$$

- We will now prove Theorem 1. First, we restate it.
- Theorem 1 (Frobenius divisibility theorem): Let G be a finite group, and let V be an irreducible representation of G over $\mathbb{C}$. Then the degree of V divides the order of G, i.e.,

$$d_V \mid |G|$$

Proof. We begin with four definitions: Let $C := \{g_1, \dots, g_s\} \subset G$ be a conjugacy class of G, let $\mathbb{Z}[G] \subset \mathbb{C}[G]$ be a **group ring**, let $e_C := g_1 + \cdots + g_s \in \mathbb{Z}[G]$, and let $\rho : G \to GL(V)$ be the group homomorphism associated with the irreducible representation V.

With our notation set, let's look at how $\rho(g_1 + \cdots + g_s)$ acts on V. Since $g_1 + \cdots + g_s \in Z(\mathbb{C}[G])$, the proposition from Monday's class implies that

$$\rho(g_1 + \cdots + g_s) = \lambda I_{d_V}$$

Taking the trace of both sides of the above equation, we obtain the following. Note that in the below equations, $\chi(C)$ denotes $\chi(g_i)$ for any $g_i \in C$; all $\chi(g_i)$ are equal because χ is a class function.

$$\begin{aligned}
\operatorname{tr}(\rho(g_1 + \cdots + g_s)) &= \operatorname{tr}(\lambda I_{d_V}) \\
\operatorname{tr}(\rho(g_1)) + \cdots + \operatorname{tr}(\rho(g_s)) &= \lambda \operatorname{tr}(I_{d_V}) \\
\sum_{i=1}^{s} \chi(C) &= \lambda d_V \\
|C|\chi(C) &= \lambda d_V
\end{aligned}$$

It follows by a simple algebraic rearrangement that

$$\frac{|C|\chi(C)}{d_V} = \lambda$$

We can now prove that $\lambda \in \bar{\mathbb{Z}}$ via Fact 4. Let $v \neq 0$. Then

$$\begin{aligned}
0 &= \rho(0)v \\
&= \rho(e_C^n + a_{n-1}e_C^{n-1} + \cdots + a_0)v \\
&= [\rho(e_C)^n + a_{n-1}\rho(e_C)^{n-1} + \cdots + a_0]v \\
&= \underbrace{(\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0)}_{0} v
\end{aligned}$$

Now recall that by the first orthogonality relation, we have that

$$\sum_C |C|\chi(C)\overline{\chi(C)} = |G|$$

It follows by dividing through by d_V that

$$\frac{|G|}{d_V} = \sum_C \frac{|C|\chi(C)}{d_V} \cdot \overline{\chi(C)}$$

But $|C|\chi(C)/d_V = \lambda \in \bar{\mathbb{Z}}$ by the above and $\overline{\chi(C)} \in \bar{\mathbb{Z}}$ by the earlier note about roots of unity, so by Fact 4, the whole sum of products $|G|/d_V \in \bar{\mathbb{Z}}$. Naturally, $|G|/d_V \in \mathbb{Q}$ as well. Consequently, $|G|/d_V \in \bar{\mathbb{Z}} \cap \mathbb{Q}$, so by Fact 2, $|G|/d_V \in \mathbb{Z}$. Therefore, we must have $d_V \mid |G|$. ☐

- Notes on the above proof.
  - In this course, we will not talk to much about integral elements; those will be the focus of Rudenko's next course, Algebraic Geometry.
- Definitely take some time to think through this proof before next class! It's short, but quite subtle. Next class's will be much much harder.
- Rudenko will not be here for next Friday's midterm; someone else will be proctoring, though.
- Next week's HW will be a preparational HW.

6.3 Burnside's Theorem

11/3:
- Announcements.
  - HW2 returned today; HW3 will be returned Monday.
- Today: We have a masterpiece of a theorem.
  - Very short, clean, powerful use of character theory.
  - It's hard to keep the whole proof in your head.
- Theorem (Burnside's theorem): If G is a group and $|G| = p^a q^b$ for p, q prime, then G is not simple (equivalently, G has no normal subgroup N such that $\{e\} \trianglelefteq N \trianglelefteq G$). In fact, G is solvable.
- **Solvable** (group): A group G that has a set of subgroups $G_1, \dots, G_n$ such that...
  1. $\{e\} \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_n = G$;
  2. Each G_i/G_{i-1} is abelian.
- Motivation for solvable groups.
  - Has to do with solving equations in radicals.
  - Equivalent to $|G| = p^a q^b$.
- Before we do the proof, here's a 2-minute Galois theory sprint for a little more context on solvable groups and this theorem as a whole.
  - Galois theory is something we should all learn for fun at some point; nothing is more pleasurable.
    - Artin and Milgram (1944) is a very short (< 100 pages), pleasurable introduction.
  - Let's formulate a result we may not know (very abstract), even if we've taken Galois theory.
    - Phrasing a big part of it in just a few lines.

- First, recall the algebraic numbers $\bar{\mathbb{Q}}$, which contain $\mathbb{Q}$, etc.
  - Using these, we can define the **Galois group**.
  - This group is still very difficult to get a handle on, still an active area of research under the **Langlands Program** (wherein we let $\sigma(x) = \bar{x}$??).
- Now we may state the result.
- Theorem: $G_\mathbb{Q}$ acts on $\bar{\mathbb{Q}}$. Imagine orbits.
  - Then $[\sigma(\sqrt{2})]^2 = \sigma(\sqrt{2}^2) = \sigma(2) = 2$.
  - This theorem acts on orbits and tells you that orbits are in bijection with irreducible polynomials $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ over $\bar{\mathbb{Q}}$, i.e., all $a_i \in \mathbb{Q}$.
  - Then if $\alpha \in \bar{\mathbb{Q}}$ implies $p_A(\alpha) = 0$, $\sigma p(\alpha) = p(\sigma(\alpha)) = 0$.
  - So $\sqrt{2}$ can be sent to $-\sqrt{2}$ and we can do more, too.
  - So it's very hard to construct elements because we need to say what happens in every orbit, and we have infinitely many.
  - What is going on here??
- We won't need much of this, but it's good to talk about.

- **Galois group**: The group defined as follows. *Denoted by* $\boldsymbol{G_\mathbb{Q}}$, $\mathbf{Gal}(\bar{\mathbb{Q}}, \mathbb{Q})$. *Given by*

$$G_\mathbb{Q} = \{\sigma : \bar{\mathbb{Q}} \to \bar{\mathbb{Q}} \mid \sigma(x+y) = \sigma(x) + \sigma(y),\ \sigma(xy) = \sigma(x)\sigma(y),\ \sigma(m/n) = m/n\}$$

  - σ is often thought of as a permutation.

- We now start proving Burnside's theorem in steps, each of which is a theorem in its own right.

- Lemma 1[3]: Suppose $\varepsilon_1, \dots, \varepsilon_n$ are roots of unity such that

$$\frac{\varepsilon_1 + \cdots + \varepsilon_n}{n} \in \bar{\mathbb{Z}}$$

Then either $\varepsilon_1 = \cdots = \varepsilon_n$ or $\varepsilon_1 + \cdots + \varepsilon_n = 0$.

Proof. Assume that it is *not* true that $\varepsilon_1 = \cdots = \varepsilon_n$. Then

$$\left|\frac{\varepsilon_1 + \cdots + \varepsilon_n}{n}\right| < 1$$

Define $a = a_1 := (\varepsilon_1 + \cdots + \varepsilon_n)/n$. Also suppose that $a_1, \dots, a_k$ are roots of the minimal polynomial for a, i.e., $p(x) = (x - a_1) \cdots (x - a_k) \in \mathbb{Z}[X]$ is the polynomial of least degree such that $p(a) = 0$; such a polynomial exists because a is an algebraic integer, per the discussion of roots of unity on Wednesday. Now what can we say about its conjugates? Take an a_i. We claim that a_i is also the sum of roots of unity over n, i.e.,

$$a_i = \frac{\varepsilon_1^i + \cdots + \varepsilon_n^i}{n}$$

So if one coefficient is of this form, they all are! There is a proof of this that follows immediately from Galois theory via[4]

$$\sigma\left(\frac{\varepsilon_1 + \cdots + \varepsilon_n}{n}\right) = \frac{\sigma(\varepsilon_1) + \cdots + \sigma(\varepsilon_n)}{n}$$

So then since each a_i satisfies $|a_i| \leq 1$ and $|a_1| < 1$, we have that

$$\left|\prod_{i=1}^n a_i\right| < 1$$

[3] This is a result of number theory, not one regarding representations. But we are stating and proving it because it is foundational to our argument.

[4] How much do I need to know about this step?? You said there might be something about proving Burnside's theorem on the final??

So $\prod_{i=1}^{n} a_i \in \mathbb{Z}$[5], but since it's an integer with absolute value less than 1, it must be zero.[6] □

- Note: This lemma is basically just a center of mass thing, i.e., the expression $(\varepsilon_1 + \cdots + \varepsilon_n)/n$ essentially gives the center of mass of the roots of unity:

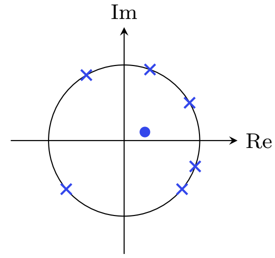


Figure 6.1: Burnside's theorem — roots of unity lemma.

- Stated simply, the first Theorem (below) posits that under relevant constraints, either an element acts as a scalar *or* its character is zero.
- Theorem 1: Let G be a finite group, let V be an irreducible representation of G, and let C be a conjugacy class in G. Assume that $(|C|, \dim V) = 1$[7]. Then for any $g \in C$, either $\chi_V(g) = 0$ or $\rho_V(g) = \lambda I$.

  Proof. Let $g \in C$ be arbitrary. Recall from our proof of the Frobenius divisibility theorem that

$$\frac{|C|\chi_V(g)}{\dim V} \in \bar{\mathbb{Z}}$$

  Recall from number theory that since $(|C|, \dim V) = 1$, there exist $a, b \in \mathbb{Z}$ such that

$$a|C| + b(\dim V) = 1$$

  Multiplying through by $\chi_V(g)/\dim V$ reveals that

$$\frac{\chi_V(g)}{\dim V} = \underbrace{a}_{\in \mathbb{Z}} \cdot \underbrace{\frac{|C|\chi_V(g)}{\dim V}}_{\in \bar{\mathbb{Z}}} + \underbrace{b}_{\in \mathbb{Z}} \cdot \underbrace{\chi_V(g)}_{\in \bar{\mathbb{Z}}} \in \bar{\mathbb{Z}}$$

  Now $\rho_V(g)$ has eigenvalues $\varepsilon_1, \ldots, \varepsilon_d$ that are roots of unity. Thus, substituting into the above, we have that

$$\frac{\varepsilon_1 + \cdots + \varepsilon_d}{d} = \frac{\chi_V(g)}{\dim V} \in \bar{\mathbb{Z}}$$

  Therefore, by Lemma 1, either $\chi_V(g) = \varepsilon_1 + \cdots + \varepsilon_d = 0$ or $\varepsilon_1 = \cdots = \varepsilon_d$ so that $\rho_V(g) = \varepsilon_i I$ for any $i = 1, \ldots, d$, as desired. □

- That was the hard part; it gets easier from here.
- Theorem 2: Let G be a finite group and let C be a conjugacy class of G. Let $|C| = p^k$ for $k > 0$. Then G is not simple.

[5] How do we know that this is an integer??
[6] How does proving that the product of the a_i's is zero prove that $a_1 = 0$??
[7] Both $|C|$ and $\dim V$ divide the order of the group, so they're usually not coprime, but they can be.

Proof. Since $\rho : G \to GL_d(\mathbb{C})$ is a group homomorphism, $\operatorname{Ker}(\rho) \trianglelefteq G$ is a normal subgroup of G. In this proof, we will construct a representation ρ with nontrivial and improper kernel. Let's begin.

Let $g \in C$ be arbitrary. We know that $g \nsim e$: Since p is a prime number, $|C| = p^k > 1$, so $C \neq \{e\}$. It follows by the second orthogonality relation that

$$\sum_{\text{irreps}} \dim(V) \cdot \chi_V(g) = \sum_{V_i} \chi_{V_i}(g)\overline{\chi_{V_i}(e)} = 0$$

The expression on the left above is equal to

$$\underbrace{1}_{V \text{ trivial}} + \sum_{V : p \mid \dim V} \dim(V) \cdot \chi_V(g) + \sum_{V : p \nmid \dim V} \dim(V) \cdot \chi_V(g)$$

We now take a moment to prove that there exists a V such that $p \nmid \dim V$ and $\chi_V(g) \neq 0$. Suppose for the sake of contradiction that no such V exists. Then by the above,

$$1 + \sum_{V : p \mid \dim V} \dim(V) \cdot \chi_V(g) = 0$$

Additionally, since $p \mid \dim(V)$ for all terms in the above sum, we can factor a p out of it to get

$$1 + p \sum_{V : p \mid \dim V} \frac{1}{p} \dim(V) \cdot \chi_V(g) = 0$$

But since $(1/p)\dim(V) \cdot \chi_V(g)$ an integer implies $(1/p)\dim(V) \cdot \chi_V(g)$ an algebraic integer implies $\sum_{V : p \mid \dim V} (1/p)\dim(V) \cdot \chi_V(g)$ an algebraic integer, the above equation implies that $-1/p \in \bar{\mathbb{Z}}$. But since $px + 1 = 0$ is not monic, we cannot have $-1/p \in \bar{\mathbb{Z}}$, and we have a contradiction.

Let's now use this V such that $p \nmid \dim V$ and $\chi_V(g) \neq 0$. Since $p \nmid \dim V$, $|C| = p^k \nmid \dim V$. Thus, $(|C|, \dim V) = 1$. Having proven this fact and $\chi_V(g) \neq 0$ for an *arbitrary* $g \in C$, it follows by Theorem 1 that for all $a \in C$, $\rho_V(a) = \varepsilon \operatorname{diag}(1, \dots, 1) = \varepsilon I$[8]. Thus, if $a_1 \neq a_2 \in C$, $\rho_V(a_1 a_2^{-1}) = I$, so $a_1 a_2^{-1} \in \operatorname{Ker}(\rho_V)$, so the kernel is a normal nontrivial subgroup. The kernel is also not equal to G because elements of it act trivially on V, a nontrivial representation, implying the existence of additional $\rho(g)$'s that act nontrivially. ☐

- Takeaway: If you come up with Theorem 1, you're already almost there.
- We're now ready for Burnside's theorem, which we should be able to prove on our own at this point if we remember the following common trick from group theory.
- Theorem (Burnside's theorem): If G is a group and $|G| = p^a q^b$ for p, q prime, then G is not simple (equivalently, G has no normal subgroup N such that $\{e\} \trianglelefteq N \trianglelefteq G$). In fact, G is solvable.

Proof. Let G have $|G| = p^a q^b$. Suppose for the sake of contradiction that G is simple. We know that $\sum |C| = |G|$. Naturally, we may split the sum as follows.

$$\sum_{|C|=1} |C| + \sum_{|C|>1} |C| = |G|$$

All elements in their own conjugacy class are those in the center! Thus,

$$|Z(G)| + \sum_{|C|>1} |C| = |G|$$

[8]How do we know it's true for all $a \in G$; couldn't they have different ε or couldn't we get different V's??

But if G is simple, then $Z(G) = \{e\}$. So we get

$$1 + \sum_{|C|>1} |C| = |G| = p^a q^b$$

Additionally, recall that as a corollary to the Orbit-Stabilizer theorem, we know that $|C| \mid |G|$. Thus, in this case, $|C| = p^c q^d$ for $c \leq a$ and $d \leq b$. Moreover, by Theorem 2, $c, d \geq 1$. (If one equalled zero and the other did not, $|C|$ would be of prime power order and G could not be simple, a contradiction.) Thus, the order of any conjugacy class in G either equals 1 or is divisible by pq.

It follows that the sum in the above equation is divisible by pq. Thus,

$$\frac{1}{pq} = \left(\underbrace{p^{a-1}q^{b-1}}_{\in\mathbb{Z}} - \underbrace{\frac{1}{pq}\sum_{|C|>1} |C|}_{\in\mathbb{Z}} \right)$$

That is to say, $1/pq$ (which is clearly not an integer) is equal to an integer, a contradiction. ☐

- We will have to prove some parts of Burnside's theorem on the final. This is important because it's so complicated with so much stuff going on that you really have to learn everything by heart before you can understand it.
- Midterm.
  - Computation of character tables, find a character tables, compute wedge powers, tensor powers, symmetric powers, etc. Compute all this elementary stuff and then a few more complicated problems. We'll get an explicit list of topics. Construct character table for symmetries of a square, etc. Make sure you can construct the character table for S_4, etc. Can you sit and from scratch make a character table for S_5? If you can, you'll have no problem on the midterm.
- Next HW is not for submission; it's just a few practice problems.
- Next week: Old works by Specht, which nobody thinks is useful but Rudenko. He thinks it's beautiful, though. Very 19th century feel. Then induction/restriction, and another approach using symmetric polynomials, etc.

6.4 L Chapter XVIII: Representations of Finite Groups

From Lang (2002).

11/11: • Section 4, Proposition 4.1 covers the proof of the first claim from Monday's class.

12/26: **Proposition 4.1.** *An element of $k[G]$ commutes with every element of G if and only if it is a linear combination of conjugacy classes with coefficients in k.*

Proof. Let $\alpha = \sum_{\sigma\in G} a_\sigma \sigma$ and assume $\alpha\tau = \tau\alpha$ for all $\tau \in G$. Then

$$\sum_{\sigma\in G} a_\sigma \tau\sigma\tau^{-1} = \sum_{\sigma\in G} a_\sigma \sigma$$

Hence $a_{\sigma_0} = a_\sigma$ whenever σ is conjugate to σ_0, and this means that we can write

$$\alpha = \sum_\gamma a_\gamma \gamma$$

where the sum is taken over all conjugacy classes γ. ☐

6.5 S Chapter 6: The Group Algebra

From Serre (1977).

11/12:

- Covers some relevant topics from this week.
- Section 6.5, Corollary 2 is the Frobenius divisibility theorem.

6.6 E Chapter 5: Representations of Finite Groups — Further Results

From Etingof et al. (2011).

- Section 5.3 covers Frobenius divisibility.
- Section 5.4 covers Burnside's theorem (very much the way we did it in class!).

Week 7

Representations of the Symmetric Group

7.1 Partitions, Young Diagrams, and Specht Modules

11/6: - Announcements.
    - Midterm description is on the Canvas page.
    - Review it then PSets. The operator averaging stuff and S_4, S_5 examples are most important.
    - New HW will be due next Friday (not this Friday).
- New topic: Representations of S_n.
    - We will talk about these almost until the end of the course; very hard.
    - Any specialist in rep theory will say that they know some approaches, but nobody understands this stuff completely.
    - We'll explore some phenomena, but if we feel after this course that we still don't understand everything about S_n, that's typical; if we think we understand everything, we're probably wrong.
- Aside: Representation theory of $GL_n(\mathbb{F}_{p^k})$ is related but even worse.
    - Same with $O_n(\mathbb{F}_{p^k})$.
    - Recently, all this stuff was understood with something called **linguistic theory** (right name??), but that's far beyond us.
- Let's begin. What do we know about S_n?
    - $|S_n| = n!$.
    - The conjugacy classes of S_n are in bijection with cyclic structures of a permutation.
        - Our good understanding of the conjugacy classes of S_n is the only thing that makes this problem the slightest bit tractable.
    - These cyclic structures are also in bijection with the **partitions** of a number; recall that we briefly talked about partitions in MATH 25700!
- **Partition** (of $n \in \mathbb{N}$): An ordered tuple satisfying the following constraints. *Denoted by* $\boldsymbol{\lambda}$, $\boldsymbol{(\lambda_1, \dots, \lambda_k)}$. *Constraints*
    1. $\lambda_i \in \mathbb{N}$ for $i = 1, \dots, k$;
    2. $\lambda_1 \geq \cdots \geq \lambda_k$;
    3. $\lambda_1 + \cdots + \lambda_k = n$.

- Example: The partitions of the number "4" are (4), $(3,1)$, $(2,2)$, $(2,1,1)$, and $(1,1,1,1)$.
  - This is the same way we've been denoting representations!
- $\boldsymbol{p(n)}$: The number of possible partitions of n.
  - Hardy and Ramanujan helped understand the number $p(n)$ of partitions of n, but they're still very hard to understand.
- One way to understand $p(n)$ is through its encoding in the **generating function**

$$\sum_{n\geq 1} p(n)x^n = 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots$$

  - We can think of the above generating function as an actual function of x if it converges for small x; if it doesn't converge, then we just think of it as a "meaningless" **formal power series**.
  - To choose a partition, we need to choose a certain number of 1's, a certain number of 2's, a certain number of 3's, etc. all the way up to n.
  - So let's look at

$$(1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)(1 + x^4 + x^8 + \cdots)\cdots$$

    - Formally, this is

$$\prod_{i=1}^{\infty}\left(\sum_{j=0}^{\infty} x^{ij}\right)$$

    - This equals the generating function! It tells us that to compute $p(100)x^{100}$, we need only look at certain terms.
  - Recall that we can write $1 + x + x^2 + \cdots = 1/(1-x)$. Doing similarly for other terms transforms the above product into

$$\frac{1}{1-x}\frac{1}{1-x^2}\frac{1}{1-x^3}\cdots$$

- **Generating function**: An encoding of an infinite sequence of numbers as the coefficients of a formal power series.
- **Formal power series**: An infinite sum of terms of the form ax^n that is considered independently from any notion of convergence.
- The above discussion of $p(n)$ as a generating function is only for our fun; Rudenko is not going to use this in any way.
  - It's just a pretty function.
  - Takeaway: We can write the generating function as something nice and then use it to prove something.
- We can visualize these partitions using something called a **Young diagram**.

(a) $\lambda = (5,4,4,3,2,1,1)$. (b) $\lambda' = (7,5,4,3,1)$.

Figure 7.1: Young diagrams for a partition of 20.

- Suppose we have the following partition of 20: $(5,4,4,3,2,1,1)$.
- Then we draw 5 cages for 5 little birds, followed by 4 cages for 4 little birds, etc.
- Thus, the i^{th} row of boxes has length λ_i.
- The same way you can denote by λ the whole *partition*, you can denote by λ the whole *diagram*.
- This is just a way to visualize partitions.
- Recall the three partitions of S_3, corresponding to its representations: (3), $(2,1)$, $(1,1,1)$.
- Moreover, these diagrams are actually meaningful!

- **Inverse** (of λ): The partition $(\lambda_1',\dots,\lambda_k')$ defined as follows. *Denoted by* $\boldsymbol{\lambda'}$. *Given by*

$$\lambda_i' = |\{\lambda_j \mid \lambda_j > i\}|$$

for all $i = 1,\dots,k$.

  - We can see that $\lambda_1' \geq \cdots \geq \lambda_k'$.
  - We can also see that the sum will still be n.
  - Moreover, if we do this twice, we'll get back to λ, i.e., $(\lambda')' = \lambda$.
  - We can prove $(\lambda')' = \lambda$ combinatorially, too, (that is, without Young diagrams) but that gets pretty complicated. We will do this in the HW.

- Example: If $\lambda = (5,4,4,3,2,1,1)$ as above, then $\lambda' = (7,5,4,3,1)$.
  - See Figure 7.1b.
  - The Young diagrams are related by a flip, akin to matrix transposition!
  - Notice how the definition of inversion *exactly* specifies this flip in the picture: The number of λ_j's that have length at least 1 is all the first column of Figure 7.1a, the number of length at least 2 is all the second column, etc.

- Onto the next question, which is the main miracle.
  - Main miracle: There exists a natural (i.e., canonical) bijection between the conjugacy classes and irreducible representations of S_n.
  - We've explored a duality for general finite groups G, before, but never a bijection.
    - In S_n, there *is* this natural bijection.
    - If you understand why intuitively, you will have started to understand the representation theory of S_n.

- Let $\lambda \vdash n$.[1] Then there is some irrep V_λ corresponding to λ. We will look at the **Specht module** construction of V_λ.
  - Some of the proofs Rudenko will present, he stole from Etingof et al. (2011), and some of the proofs he invented himself.
  - This is *by far* the best construction, even though it's exceedingly rare in the literature.

- The usual construction.
  - Take $\mathbb{C}[S_n]$ with coefficients a_λ, b_λ, etc. similar over conjugacy classes and do something with it??
  - "Just say NO!" to this construction.

- Here is the better idea.
  - Consider an algebra of polynomials with rational coefficients: $\mathbb{Q}[x_1,\dots,x_n]$.

[1] "Lambda partitions en."

        - We could also do real or complex, but rational is nice.
    - For symmetric groups, all representations will be integers, etc.??
    - One thing to emphasize about this algebra: It is a **graded** algebra.
        - If represented by A, then it equals $A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ where

$$A_m = \left\{ \sum_{k_1+\cdots+k_n=m} a_{k_1\cdots k_n} x_1^{k_1} \cdots x_n^{k_n} \right\}$$

            - I.e., A_m is the sum of all polynomials with **degree** equal to m.
            - Example: If we take $1 + x_1^2 x_2^3 + x_1 x_2 + x_1^{100} + x_1 x_2^{99}$, we can then break this polynomial up into polynomials of degree 1, 5, 2, and 100.
        - We also have $A_{m_1} \cdot A_{m_2} \subset A_{m_1+m_2}$.
            - Example: $x_1 x_2^2 \cdot (x_1 + x_2) = x_1^2 x_2^2 + x_1 x_2^3$.
    - With this algebra in hand, we may let $S_n \circlearrowright \mathbb{Q}[x_1, \ldots, x_n]$ via

$$\sigma P(x_1, \ldots, x_n) = P(x_{\sigma_{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$$

        - In other words, σ is transposing polynomials.
        - Example: $(12)(x_1^2 + x_2^3 + x_3) = x_2^2 + x_1^3 + x_3$.
    - Thus, we call as A_1 the representation V_{perm}^*.
        - This is because $A_1 = \text{span}(x_1 + \cdots + x_n)$, and permuting these is much like permuting the basis of a vector space, as the typical permutation representation does.
        - It could technically be the isomorphic representation V_{perm}, but the dual fits better here for reasons??
    - Then $A_2 = S^2 V^*$.
        - So if A_1 had basis $e^1, \ldots, e^m$, A_2 has basis $\{e^i e^j\}$.
        - Why are we choosing these sets??
    - Continuing, $A_3 = S^3 V^*$.
    - It follows that the representation of the overall thing is

$$\bigoplus_{m\geq 0} (S^m V_{\text{perm}}^*)$$

        - This is called the **symmetric algebra**.

- **Graded** (algebra): An algebra for which the underlying additive group is a direct sum of abelian groups A_i such that $A_i A_j \subset A_{i+j}$.
- **Degree** (of a monomial P): The sum of the powers of its variables. *Denoted by* $\boldsymbol{d(P)}$.
    - Example: If $P = x_1^3 x_2^4$, then $d(P) = 2 + 3 = 5$.
- **Degree** (of a polynomial P): The greatest degree of each of its monomials. *Denoted by* $\boldsymbol{d(P)}$.
- So how do we construct representations?
    - For S_2, $x_1 - x_2$ changes sign when we apply S_2.
    - For S_3...
        - The trivial's polynomial is 1 and $\square\square\square$.
        - The standard is $(2, 1)$. When we apply S_3 to $(x_1 - x_2)$, we get

$$\langle (x_1 - x_2), (x_2 - x_1), (x_1 - x_3), (x_3 - x_1), (x_2 - x_3), (x_3 - x_2) \rangle$$

        - ➢ If we let $a = x_1 - x_2$, $b = x_2 - x_3$, then the third element equals $a + b$.
        - ➢ Similarly, the the second element equals $-a$, the fourth element equals $-a - b$, and the sixth element equals $-b$.
        - ➢ This is another way to think about the action.
    - ■ What about the alternating representation? We have $(x_1 - x_2)(x_2 - x_3)(x_1 - x_3) = \Delta_{123}$, which changes sign when we use any element with sign -1 to permute the x_i!
  - For S_4...
    - ■ (4) is 1.
    - ■ $(3,1)$ is $S_4(x_1 - x_2) = \Delta_{12}$.
    - ■ $(1,1,1,1)$ is $(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4) = \Delta_{1234}$.
    - ■ $(2,1,1)$ is $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$.
        - ➢ We got this polynomial by guessing; the same way $(x_1 - x_2)$ worked in multiple cases, maybe this one does too! And it does.
        - ➢ Something to check is that $\Delta_{123} - \Delta_{124} - \Delta_{134} - \Delta_{234} = 0$.
    - ■ $(2,2)$ is $(x_1 - x_2)(x_3 - x_4)$.
        - ➢ Something related we can prove is that

$$(x_1 - x_2)(x_3 - x_4) - (x_1 - x_3)(x_2 - x_4) - (x_1 - x_4)(x_2 - x_3) = 0$$

        - ➢ This formula appears in **cross ratios**, which we can discuss in Rudenko's algebraic geometry course next quarter.
    - ■ For $\lambda = (4,3,1)$, we have $\Delta_{123}\Delta_{45}\Delta_{67}$, and we act by S_8 upon this! Explicitly, we have $S_8(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_4 - x_5)(x_6 - x_7)$.
    - ■ Takeaway: It all depends on column length!
  - These polynomials are called **Vandermonde determinants**; those are the little Δ things with subscripts. We'll talk about these next times.

- We need to prove reducibility and not pairwise isomorphic to make sure that this construction is valid, but that's easy! We'll do this next Monday.

7.2 Symmetric Polynomials; Vandermonde Determinants

11/8:
- Announcements.
  - OH tonight at 6:00 PM.
- Consider S_n.
  - Recall the symmetric algebra $R = \mathbb{Q}[x_1, \dots, x_n]$, which is a graded ring $\bigoplus_{d>0} R_d$ where $R_d = S^d V_{\text{perm}}^*$.
  - The action of $S_n \circlearrowright \mathbb{Q}[x_1, \dots, x_n]$ is $\sigma P(x_1, \dots, x_n) = P(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$.
- With the definitions of last class behind us, we can now look at the **space of invariants** R^{S_n}, isotypical components of which σ acts on trivially.

$$R^{S_n} := \{P(x_1, \dots, x_n) \mid \forall\ \sigma \in S_n,\ P(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = P(x_1, \dots, x_n)\}$$

  - This is the ring of symmetric polynomials.
  - Example: If $n = 3$, then $x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3 \in \mathbb{Z}^{S_3}$.
- We now define some stuff to help us prove a major result: The n elementary symmetric polynomials.
  - $\sigma_1 = x_1 + \cdots + x_n = \sum_{1 \le i \le n} x_i$.

  - $\sigma_2 = x_1x_2 + \cdots + x_1x_n + x_2x_3 + \cdots = \sum_{1\leq i<j\leq n} x_ix_j$.
  - $\sigma_k = \sum_{1\leq i_1<\cdots<i_k\leq n} x_{i_1}\cdots x_{i_k}$.
  - $\sigma_n = x_1\cdots x_n$.

- With these definitions, we can say that

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = (x - x_1)\cdots(x - x_n)$$

where

$$a_{n-1} = -\sigma_1 \qquad a_{n-2} = \sigma_2 \qquad \ldots \qquad a_0 = (-1)^n\sigma_n$$

- Intro to the fundamental theorem of symmetric polynomials.
  - Basically: Every symmetric polynomial is a polynomial in the elementary symmetric polynomials.
  - A more precise statement follows.
- Theorem (Fundamental theorem of symmetric polynomials): We have that

$$\mathbb{Q}[x_1,\dots,x_n]^{S_n} = \mathbb{Q}[\sigma_1,\dots,\sigma_n]$$

- Before we prove the fundamental theorem, there are a few points we need to discuss.
- Example:
  - Consider the polynomial $x^2 + px + q = 0$.
  - Let x_1, x_2 denote its roots. Then $\sigma_1 = x_1 + x_2 = -p$ and $\sigma_2 = x_1x_2 = q$.
  - We may observe that $x_1^2 + x_2^2 = \sigma_1^2 - 2\sigma_2^2 = p^2 - 2q$.
  - We may observe that $(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2 = \sigma_1^2 - 4\sigma_2 = p^2 - 4q$, where the rightmost term is the discriminant of the original polynomial.
  - In general, $x_1^n + x_2^n$ has an expression as a polynomial in σ_1, σ_2. This will be a homework problem.
  - What is going on here?? Is $x^2 + px + q$ even in $\mathbb{Q}[x]^{S_n}$? If so, why do we factor it into σ_1, σ_2 instead of just σ? What are the other examples about?
- **Lexicographic order** (on monomials): An ordering of monomials based on the following rule. *Denoted by* $\succ$. *Given by*
$x_1^{a_1}\cdots x_n^{a_n} \succ x_1^{b_1}\cdots x_n^{b_n}\ldots$
  1. If $a_1 > b_1$ OR...
  2. If $a_1 = b_1$ and $a_2 > b_2$ OR...
  3. $a_1 = b_1$ and $a_2 = b_2$ and $a_3 > b_3$ OR...
  4. So on and so forth.
- Notes on the lexicographic ordering.
  - Don't think of this order like an ordering on integers.
  - This allows us to define the key notion for a number of proofs we'll see in the coming days.
  - Although it may seem counterintuitive, the lexicographic ordering is still determined for polynomials such as σ_1. For example, we may look at

$$\sigma_1 = x_1 + \cdots + x_n$$

and think, "Wait a second — all these terms have the same order: They all have the same exponent of 1." However, we would be discounting the fact that the lexicographic ordering views σ_1 as

$$\sigma_1 = x_1^1x_2^0\cdots x_n^0 + \cdots + x_1^0\cdots x_{n-1}^0x_n^1$$

From here, we can see that $LM(\sigma_1) = x_1^1x_2^0\cdots x_n^0 = x_1$.

- **Largest monomial** (of $P \neq 0$); The monomial in $P(x_1, \dots, x_n) \neq 0$ that is the largest lexicographically. *Denoted by* $\boldsymbol{LM(P)}$.
- $\boldsymbol{C_{LM}(P)}$: The coefficient of $LM(P)$.
- Example: Consider the polynomial $P = x_1^2 + x_1^3x_2x_3 - 7x_1^3x_2x_3^{100}$.
  - Then $LM(P) = x_1^3x_2x_3^{100}$ and $C_{LM}(P) = -7$.
- Properties.
  1. $P, Q \neq 0$ implies that $LM(PQ) = LM(P)LM(Q)$.
     - Using inductive reasoning, try multiplying the example above by $Q = x_1^2 + x_2^2 + x_3^2$!
     - Rudenko will not give rigorous proofs of any of these properties; they will just confuse us. It's better to do everything intuitively here.
- Lemma: If $P \in \mathbb{Q}[x_1, \dots, x_n]^{S_n}$ and $LM(P) = x_1^{a_1} \cdots x_n^{a_n}$, then $a_1 \geq \cdots \geq a_n$.

  Proof. Let $i < j$. Suppose for the sake of contradiction that $a_i < a_j$. Let $\sigma = (ij) \in S_n$. Since P is symmetric, $\sigma P = P$. But then in particular, the monomial $\sigma LM(P)$ in P is lexicographically larger than $LM(P)$. Thus, $LM(P)$ is not the lexicographically largest monomial in P, a contradiction. □

  - Here's a simple example to illustrate the idea behind this proof: Let $P = x^2y + xy^2 \in \mathbb{Q}[x, y]^{S_n}$. Suppose we pick $LM(P) = xy^2$ (obviously this is the wrong choice, but that's the contradiction we'll see). We observe that $2 = a_2 > a_1 = 1$ in this case. Let $\sigma = (12)$. Then $\sigma LM(P) = yx^2 = x^2y \succ xy^2 = LM(P)$. So $\sigma LM(P) \succ LM(P)$. Thus, $LM(P)$ is not the lexicographically largest monomial in P, and we have *formally* proven that our initial choice of $LM(P)$ was incorrect.
- We now have everything we need to prove the fundamental theorem. As such, we will restate and prove it.
- Theorem (Fundamental theorem of symmetric polynomials): We have that

$$\mathbb{Q}[x_1, \dots, x_n]^{S_n} = \mathbb{Q}[\sigma_1, \dots, \sigma_n]$$

  Proof. We will prove this theorem using the well-ordering principle (every set of natural numbers has a smallest element), which is equivalent to induction. Let's begin.

  Suppose for the sake of contradiction that there exists a symmetric polynomial that cannot be expressed via $\sigma_1, \dots, \sigma_n$. Given this counterexample, factor out as many terms as we want (successively reducing the degree) until it ceases to be a counterexample, thus yielding the counterexample of smallest degree. Similarly, get to the counterexample with smallest LM. Call this counterexample $P(x_1, \dots, x_n)$. Let

$$P(x_1, \dots, x_n) = C_{LM}(P)\underbrace{x_1^{a_1} \cdots x_n^{a_n}}_{LM(P)} + \text{smaller monomials}$$

  Since P is symmetric and the term above is the lexicographically largest monomial, the Lemma implies that $a_1 \geq \cdots \geq a_n$. We now construct a polynomial Q out of the σ_i such that $LM(P) = LM(Q)$. To begin, note that

$$LM(\sigma_1) = x_1 \qquad LM(\sigma_2) = x_1x_2 \qquad \cdots \qquad LM(\sigma_n) = x_1 \cdots x_n$$

  Now consider $\sigma_n^{a_n}$. This clearly divides $LM(P)$ since a_n is minimal. Now multiply by $\sigma_{n-1}^{a_{n-1}-a_n}$. Continuing on, we get

$$Q = \sigma_n^{a_n}\sigma_{n-1}^{a_{n-1}-a_n}\sigma_{n-2}^{a_{n-2}-a_{n-1}} \cdots \sigma_1^{a_1-a_2}$$

  Now it follows that

$$LM(P - C_{LM}(P) \cdot Q) \prec LM(P)$$

  Since $C_{LM}(P) \in \mathbb{Q}$ and $Q \in \mathbb{Q}[x_1, \dots, x_n]^{S_n}$, it also follows that $P - C_{LM}(P) \cdot Q \in \mathbb{Q}[x_1, \dots, x_n]^{S_n}$. But then by the assumption that P was the counterexample with smallest LM, we know that $P - C_{LM}(P) \cdot Q \in \mathbb{Q}[\sigma_1, \dots, \sigma_n]$. It follows that $P \in \mathbb{Q}[\sigma_1, \dots, \sigma_n]$, a contradiction. □

- Note: This is an effective proof; we can write an algorithm to do this for us, and it's actually pretty fast and efficient.
- Remain to show: $\sigma_1, \dots, \sigma_n$ are algebraically independent $P(\sigma_1, \dots, \sigma_n) = 0$ implies that $P = 0$.
  - This will be a homework problem; hint, it's pretty easy.
- Back to representation theory.
- **Antisymmetric** (polynomial): A polynomial $P(x_1, \dots, x_n)$ such that

$$\sigma P = (-1)^\sigma P$$

- Example.
  - $n = 2$: $x_1 - x_2$.
  - $n = 3$: $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$.
- These main examples are the **Vandermode determinant** from last time!
- **Vandermonde determinant**:

$$\Delta(x_1, \dots, x_n) = \prod_{1 \leq i < j \leq n} (x_j - x_i)$$

- Exercise: $\Delta(x_1, \dots, x_n)$ is antisymmetric.
- One of the nicest definitions of sign comes from these determinants!

$$(-1)^\sigma = \frac{\prod_{i<j}(x_{\sigma(i)} - x_{\sigma(j)})}{\prod_{i<j}(x_i - x_j)}$$

- Theorem: If $P \in \mathbb{Q}[x_1, \dots, x_n]^{\text{alt}}$ (i.e., $P \in \mathbb{Q}[x_1, \dots, x_n]$ and P is antisymmetric), then $P = P'\Delta(x_1, \dots, x_n)$ where P' is symmetric (i.e., $P' \in \mathbb{Q}[x_1, \dots, x_n]^{S_n}$).
- Corollary: If P is antisymmetric and $\deg(P) < n(n-1)/2$, then $P = 0$.
  - We'll use this many times, this fact that "antisymmetric polynomials have a smallest possible degree."
- We now prove the Theorem.

Proof. Let P be antisymmetric. Then $(12)P = -P$. It follows that $P(x_1, \dots, x_n)\big|_{x_1 = x_2} = 0$. Now, rewrite P as a polynomial in one variable where all of the coefficients are polynomials in other variables. In particular, let

$$P = P_d(x_1 - x_2)^d + P_{d-1}(x_1 - x_2)^{d-1} + \cdots + P_0$$

where each $P_i \in \mathbb{Q}[x_1, \dots, x_d]$. What is d?? (Less than n, I'm assuming, but any other constraints?) Plugging in $x_1 = x_2$ once again, we get $0 = P = P_0$. But this implies that $P_0 = 0$. Thus, P is divisible by $x_1 - x_2$. Similarly, for all $i < j$, $(x_i - x_j) \mid P$. But since the $x_i - x_j$ are irreducible polynomials, we have that $\prod_{i<j}(x_j - x_i) \mid P$. This is justified because we are in a unique factorization domain (how is this relevant??). Thus, we have that $P = P' \cdot \Delta(x_1, \dots, x_n)$. Lastly, it follows that $P' \in \mathbb{Q}[x_1, \dots, x_n]^{S_n}$ because under any sign -1 permutation, $\Delta(x_1, \dots, x_n)$ will flip signs and P will still be equal, so P' had better just stay itself under this permutation (i.e., be symmetric). $\square$

- Remark: Where does the name Vandermonde *determinant* come from?

- We have that

$$\Delta(x_1,\dots,x_n) = \begin{vmatrix} 1 & & 1 \\ x_1 & & x_n \\ \vdots & & \vdots \\ x_1^{n-1} & & x_n^{n-1} \end{vmatrix}$$

- Final reminder before the final.
  - Don't forget our awesome central construction!
  - If $\lambda = (\lambda_1,\dots,\lambda_k)$ is a partition with $\lambda_1 \geq \cdots \geq \lambda_k$, then we can draw a Young Diagram and construct an associated representation $V_\lambda \in \mathbb{C}[x_1,\dots,x_n]$.
  - But what we do is $V_\lambda = \mathbb{C}[S_n]\Delta_\lambda$, where

$$\Delta_{\lambda'} = \Delta(x_1,\dots,x_{\lambda_1'})\Delta(x_{\lambda_1'+1},\dots,x_{\lambda_1'+\lambda_2'})\cdots$$

  - Example: For $\lambda = (2,2,1)$, we have $V_{(2,2,1)} = \mathbb{C}[S_n](x_1-x_2)(x_1-x_3)(x_2-x_3)(x_4-x_5)$.
  - Next time, we'll prove that $V_{(2,2,1)}$ is irreducible.
- This Specht construction is in a tiny footnote of Fulton and Harris (2004), but that's about it!

7.3 Midterm Review Sheet

11/10:

- The following definitions and results will be useful in solving the midterm problems.
- **Group representation**: A group homomorphism $\rho : G \to GL(V)$ for G a finite group, V a finite-dimensional vector space over some field $\mathbb{F}$ with basis $\{e_1,\dots,e_n\}$, and $GL(V)$ the set of isomorphic linear maps $L : V \to V$.
- **Morphism** (of G-representations): A map $f : V \to W$ such that...
  1. f is linear;
  2. For every $g \in G$, $\rho_W(g) \circ f = f \circ \rho_V(g)$.
     - To remember this rule, draw out the commutative diagram!
- Theorem (complete reducibility): Any finite-dimensional representation can be decomposed into a direct sum of irreducible representations via

$$V = V_1^{n_1} \oplus \cdots \oplus V_k^{n_k}$$

- Lemma (Schur's Lemma): Let G be a finite group, let V, W be irreducible representations over $\mathbb{C}$, and let $f \in \operatorname{Hom}_G(V,W)$. Then...
  1. If $V \ncong W$, then $f = 0$. If $V \cong W$, then f is an isomorphism of G-representations.
  2. If $f : V \to V$, then $f(v) = \lambda v$.
- **Algebraic integer**: A number $x \in \mathbb{C}$ for which there exist $a_0,\dots,a_{n-1} \in \mathbb{Z}$ such that

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

- **Character** (of ρ): The function $\chi_\rho : G \to \mathbb{C}$ defined by

$$\chi_\rho(g) = \operatorname{tr}(\rho(g))$$

- **First orthogonality relation**: If χ_1, χ_2 are the characters of irreducible representations, then

$$\sum_{g \in G} \chi_1(g)\overline{\chi_2(g)} = \begin{cases} 0 & \chi_1 \neq \chi_2 \\ |G| & \chi_1 = \chi_2 \end{cases}$$

   - Follows from the fact that the characters form an orthonormal set within the space of class functions, and the definition of the inner product on this space.

- **Second orthogonality relation**: If $C_G(g)$ is the number of elements in the conjugacy class of g, then

$$\sum_\chi \chi(g_1)\overline{\chi(g_2)} = \begin{cases} 0 & g_1 \nsim g_2 \\ \frac{|G|}{|C_G(g_1)|} & g_1 \sim g_2 \end{cases}$$

- **Permutational representation**: The representation $\rho : S_n \to \mathbb{C}^n (= V_{\text{perm}})$ defined by

$$\rho(\sigma) : (x_1, \dots, x_n) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

   - Character: Compute $\text{Fix}(\sigma)$ for each type of σ.

- **Class function**: A function that is constant on the conjugacy classes of G. Explicitly, for all $s, t \in G$,

$$f(tst^{-1}) = f(s)$$

- **Group algebra**: The complex vector space with basis $\{e_g\}$ corresponding to the elements of group G, plus the definition $e_g \cdot e_h = e_{gh}$.
- **Semisimple module**: A module M that satisfies any of the following three conditions.
   1. $M = \bigoplus_{i \in I} S_i$, where each S_i is a simple module and I is an indexing set.
   2. $M = \sum_{i \in I} S_i$.
   3. For all submodules $N \subset M$, there exists N' such that $M = N \oplus N'$.
- Additional notes on semisimple modules.
   - Simple module: A module that is nonzero and has no nonzero proper submodules.
- **Division algebra**: An algebra D such that for all nonzero $x \in D$, there exists a $y \in D$ such that $xy = 1$.
- **Semisimple algebra**: An algebra for which every finite-dimensional A-module is semisimple.
- **Wedderburn-Artin theorem**: If A is a finite-dimensional semisimple associative algebra, then

$$A \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k)$$

- **Schur's Lemma** (over an arbitrary field): Let A be a finite-dimensional algebra, and let M_1, M_2 be simple A-modules. Then...
   1. If $f : M_1 \to M_2$ is a nonzero morphism of A-modules, f is isomorphic;
   2. If M is simple, $\text{Hom}_A(M, M)$ is a division algebra.
- **Center** (of A): The following set.

$$Z(A) = \{a \in A \mid xa = ax \ \forall \ x \in A\}$$

- **Jacobson radical**: The finite-dimensional A-algebra defined as follows.

$$\text{Rad}(A) = \{a \in A \mid aS = 0 \text{ for any simple module } S\}$$

- Here's an outline of what to remember for the problems.
- Strategies for computing the following things from the character table of a group.

1. Tensor products.
   - Multiply corresponding characters.
2. Wedge/symmetric squares.
   - If χ is the character of a representation $\rho : G \to GL(V)$, then the characters χ_σ^2 of the symmetric square S^2V of V and χ_α^2 of the alternating square $\Lambda^2 V$ of V are given by the following for each $s \in G$.

$$\chi_\sigma^2(s) = \frac{1}{2}\left(\chi(s)^2 + \chi(s^2)\right) \qquad \chi_\alpha^2(s) = \frac{1}{2}\left(\chi(s)^2 - \chi(s^2)\right)$$

   - Note that just like $V^{\otimes 2} = S^2V + \oplus\Lambda^2 V$, we have $\chi^2 = \chi_\sigma^2 + \chi_\alpha^2$.
3. Decomposing permutational representations into irreducibles.
   - If the representation of interest is χ_V, we find the coefficients n_i of its decomposition

$$\chi_V = \sum n_i \chi_{V_i}$$

     via the inner product

$$n_i = \langle \chi_V, \chi_{V_i} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)\overline{\chi_{V_i}(g)}$$

4. More.

- Strategies for computing the following things given a small group (e.g., the quaternion group).

1. Conjugacy classes.
   - Take an element, conjugate it by everything, round up the products. Then move onto another elements.
2. Character table.
   - Find the conjugacy classes and put them at the top of the table. This also tells us how many irreps we need to get to.
   - Start with the trivial, alternating, and standard representations.
   - Tensor products of representations with 1D representations (e.g., the alternating) are often linearly independent.
   - We can recover the standard as the difference of the permutation and trivial.
   - We can solve for representations as components of the regular representation via

$$V_R = \bigoplus_{i=1}^{k} V_i^{\dim V_i}$$

   - We can calculate the degrees of remaining representations via the sums of the squares of the dimensionalities.
   - We can fill in final representations with the orthogonality relations, *especially* the second one.
3. Decomposing representations into a sum of isotypical components.
   - Use the inner product/decomposition formula from above.
4. Diagonalizing an endomorphism.
   - Start with its matrix A.
   - Find the characteristic polynomial by computing $\det(A - \lambda I)$.
   - Solve for the eigenvalues.
   - Find, by inspection or by solving systems of equations, elements of the null space of $A - \lambda I$ for each λ. Beware eigenvalues with multiplicity greater than one!
5. More.

- Strategies for solving an abstract problem about characters.
  - *reread notes*
- Strategies for solving an abstract problem about representations.
  - *reread notes*
- Other misc. concepts that are probably good to remember (my own ideas).
- **Left A-module**: A pair (M,ρ) where $(M,+)$ is an abelian group and $\rho : A \to \text{End}(M)$ is the ring homomorphism defined as follows, where A is a ring: For all $a \in A$, $\rho(a) : M \to M$ is given by $\rho(a)v = av$ for all $v \in M$ and satisfies the following constraints.
  1. $\rho(a) : M \to M$ is a group homomorphism on $(M,+)$.
  2. ρ is a ring homomorphism.
     - That is to say, $\rho(a+b) = \rho(a) + \rho(b)$, $\rho(ab) = \rho(a)\rho(b)$, and $\rho(1_A) = 1_{\text{End}(M)}$.
- Lemma (Gauss's Lemma): If $f, g \in R[X]$ are both nonzero polynomials with coefficients in the ring R, then $c(fg) = c(f)c(g)$.
  - Note that $c(f)$ denotes the **content** of f, which is the gcd of its coefficients.
  - Use: If p is reducible in a fraction field, then it's reducible in the native UFD.
- A is semisimple iff $\text{Rad}(A) = 0$.
- Formulas for the decomposition of the regular representation/misc. formulas from IChem.
  - Sum of the squares of the dimensionalities (from the second orthogonality relation):

$$|G| = \sum_{i=1}^{k} (\dim V_i)^2$$

  - Sum of the squares of an irrep's characters (from the first orthogonality relation):

$$|G| = \sum_{g \in G} \chi(g)^2$$

7.4 Midterm

1. **(30)** Here is the character table of the group S_4.

Representation	(1)(2)(3)(4)	(12)(34)	(12)(3)(4)	(1234)	(123)(4)
(4)	1	1	1	1	1
$(1,1,1,1)$	1	1	-1	-1	1
$(2,2)$	2	2	0	0	?
$(3,1)$	3	?	1	-1	0
$(2,1,1)$	3	-1	-1	1	0

   (a) State two orthogonality relations for characters of a general group G. Apply them to fill in holes in the table above.

   (b) Compute the character of $S^2(2,2)$ and decompose it into irreducibles.

   (c) Compute the character of $(3,1) \otimes (2,2)$ and decompose it into irreducibles.

2. **(40)** Consider the group G of symmetries of a square (it has size 8).

(a) Find the conjugacy classes of G.

(b) The action of G on the plane (by symmetries of a square) defines a two-dimensional complex representation V of G. Find the character of V. Prove that V is irreducible.

(c) Compute the character table of G.

(d) Consider the action of G on the set of functions on edges of teh square. This defines a four-dimensional representation of G. Find its characters and decompose it into isotypical components.

3. **(15)** Suppose that for a finite group G all irreducible representations are one-dimensional. Prove that G is abelian.

4. **(15)** State Schur's lemma for complex representations of a finite group. Assume that V, W are two distinct complex-dimensional irreducible representations of a finite group G. Find the dimension of the space $\mathrm{Hom}_G(V \oplus W, V \oplus W)$.

Week 8

Symmetric Group Representation Formulations

8.1 Specht Modules are Irreducible and Well-Defined

11/13:

- Announcements.
  - This week's homework is the next to last one.
- Review.
  - Miracuously, we can understand all representations of S_n.
  - We start with partitions λ that are defined a certain way. We visualize them with Young diagrams.
  - The number of partitions of n is equal to the number of conjugacy classes in S_n is equal to the number of irreps in S_n.
    - It is a special feature of S_n that this is true.
  - How do we construct the irreducible representation V_λ due to λ?
    - Consider $(4,2,1)' = (3,2,1,1)$ as an example (recall the definition of an inverse partition).
    - Take Vandermonde determinants (recall the explicit definition of these, too) of column elements and multiply them together.
    - Then we define V_λ[1] to be the subspace of the polynomial space $\mathbb{C}[x_1,\dots,x_n]$ that is spanned by the Vandermonde determinants polynomial and all actions of S_n on it.
    - In particular, take Vandermonde determinant of variables corresponding to the successive columns to obtain

$$\Delta(x_1,\dots,x_{\lambda_1'})\Delta(x_{\lambda_1'+1},\dots,x_{\lambda_2'})\cdots\Delta(x_{\lambda_{k-1}'+1},\dots,\lambda_k')$$

    - Thus, in our specific example, we let $\mathbb{C}[S_n]$ act on $(x_1-x_2)(x_1-x_3)(x_2-x_3)(x_4-x_5)$.
- One more example.
  - $\lambda = (2,2)$.
  - Let $\mathbb{C}[S_4]$ act on $(x_1-x_2)(x_3-x_4)$.
  - Then

$$V_\lambda = \langle (x_1-x_2)(x_3-x_4), (x_1-x_3)(x_2-x_4), (x_1-x_4)(x_2-x_3)\rangle$$

[1]Something about $V_\lambda = \mathbb{C}[S_n]$?? I though we were "just say NO!"-ing this usual construction, though?

- But we're expecting a 2D representation. Indeed, we get one because if we define the first term above to be a and the second to be b, then the third is $b-a$:

$$\begin{aligned}
b-a &= [(x_1-x_3)(x_2-x_4)] - [(x_1-x_2)(x_3-x_4)] \\
&= [x_1x_2 - x_1x_4 - x_2x_3 + x_3x_4] - [x_1x_3 - x_1x_4 - x_2x_3 + x_2x_4] \\
&= x_1x_2 + x_3x_4 - x_1x_3 - x_2x_4 \\
&= (x_1-x_4)(x_2-x_3)
\end{aligned}$$

- Consequence of the above: There are only two linearly independent polynomials herein.
- Thus, the final Specht module is

$$V_\lambda = \langle \underbrace{(x_1-x_2)(x_3-x_4)}_{a}, \underbrace{(x_1-x_3)(x_2-x_4)}_{b} \rangle$$

- Now we calculate entries in the character table as follows: See how representatives of conjugacy classes like (12) and (123) acts on a, b via matrices, and then calculate traces of these matrices.
  - For example, using the definitions of a, b from above, we can see that

$$\begin{aligned}
(12)\cdot a &= (12)\cdot(x_1-x_2)(x_3-x_4) = (x_2-x_1)(x_3-x_4) = -(x_1-x_2)(x_3-x_4) = -a \\
(12)\cdot b &= (12)\cdot(x_1-x_3)(x_2-x_4) = (x_2-x_3)(x_1-x_4) = (x_1-x_4)(x_2-x_3) = b-a
\end{aligned}$$

  - In matrix form, the above equations become

$$\begin{bmatrix} -a \\ b-a \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}}_{\rho(12)} \begin{bmatrix} a \\ b \end{bmatrix}$$

  - Thus, $\chi(12) = \operatorname{tr}[\rho(12)] = 0$.
  - Similarly, we can calculate that

$$\begin{bmatrix} b-a \\ -a \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}}_{\rho(123)} \begin{bmatrix} a \\ b \end{bmatrix}$$

    so $\chi(123) = -1$.
- One of the HW problems is to do exactly this for S_4 just for practice.

- Today: A theorem that proves the V_λ are irreducible (as our notation would suggest), and a theorem that proves the V_λ are well-defined (i.e., only one V_λ can be constructed from each λ).
- Note: V_λ is a **Specht module** and the original polynomial constructed is the **Specht polynomial**.
- We now build up to the theorems.
- **Degree** (of a Specht polynomial): The degree of the given polynomial as defined in Lecture 7.1. *Denoted by* $\mathbf{d(\lambda)}$. *Given by*

$$d(\lambda) = \sum_{i=1}^{k'} \frac{\lambda_i'(\lambda_i'-1)}{2}$$

- Let R_d denote the subset of $\mathbb{C}[x_1, \dots, x_n]$ consisting of polynomials of degree d.
  - Recall also that as a definition, $R_d = S^d(V_{\text{perm}}^*)$.
  - Note that if $d = d(\lambda)$, then $V_\lambda \subset R_d$.

- Lemma: Let λ be a partition of n, $d = d(\lambda)$, R_d be the subset of $\mathbb{C}[x_1, \dots, x_n]$ consisting of polynomials of degree d, and V_λ be the Specht module of λ. Then

$$\operatorname{Hom}_{S_n}(V_\lambda, R_d) \cong \mathbb{C}$$

Proof. Let $f \in \operatorname{Hom}_{S_n}(V_\lambda, R_d)$ be arbitrary, let $\Delta(x_1, \dots, x_{\lambda_1'})\Delta(x_{\lambda_1'+1}, \dots, x_{\lambda_2'})\dots$ be an arbitrary element of V_λ, and let

$$P(x_1, \dots, x_n) := f(\Delta(x_1, \dots, x_{\lambda_1'})\Delta(x_{\lambda_1'+1}, \dots, x_{\lambda_2'})\dots)$$

By definition, the degree of P is d. Additionally, because f is a morphism of S_n-representations as an element of $\operatorname{Hom}_{S_n}(V_\lambda, R_d)$, we have that f is linear and hence, since the argument of f is antisymmetric in $x_1, \dots, x_{\lambda_1'}$, so P similarly antisymmetric. P is also antisymmetric in $x_{\lambda_1'+1}, \dots, x_{\lambda_2'}$. In fact, P is antisymmetric in all such sets all the way up to $x_{\lambda_{k'-1}'+1}, \dots, x_{\lambda_{k'}'}$. It follows that $P(x_1, \dots, x_n)$ is divisible by $\Delta(x_1, \dots, x_{\lambda_i'})$, etc., i.e., all Vandermonde determinants. Thus, $P(x_1, \dots, x_n)$ is divisible by the product, which is the d-degree Specht polynomial argument of f. It follows that

$$P(x_1, \dots, x_n) = u \cdot \Delta(x_1, \dots, x_{\lambda_1'})\Delta(x_{\lambda_1'+1}, \dots, x_{\lambda_2'})\dots$$

from which it follows that $f = uI$. This implies the claim via the isomorphism $f \mapsto u$. $\square$

- Theorem 1: V_λ is irreducible.

Proof. Let $V_\lambda = \bigoplus W_i^{n_i}$ and $R_d = \bigoplus W_i^{m_i}$, where the W_i are all irreps. From previous classes, we have a nice way to compute a morphism of S_n-representations $V_\lambda \to R_d$: Explicitly, we apply Schur's lemma to find that the only acceptable constituent morphisms are those which send $W_i \to W_i$. Thus, $\dim \operatorname{Hom}_{S_n}(V_\lambda, R_d) = \sum n_i m_i$. (Any transformation from $W_i^{n_i}$ to $W_i^{m_i}$ has the form of a $m_i \times n_i$-blocked matrix, so there are $n_i m_i$ degrees of freedom.) But by the lemma, $\dim \operatorname{Hom}_{S_n}(V_\lambda, R_d) = 1$. Additionally, since we are in a subrepresentation, i.e., $V_\lambda \subset R_d$, we have that $n_i \leq m_i$ for all i. Thus, we must have $n_i = 1, m_i = 1$ for some i and that $n_j, m_j = 0$ for all other j. This means that

$$V_\lambda = W_1^0 \oplus \dots \oplus W_{i-1}^0 \oplus W_i^1 \oplus W_{i+1}^0 \oplus \dots \oplus W_k^0 = W_i$$

Therefore, since it is equal to an irrep, V_λ is irreducible. $\square$

- Corollary: If $d' < d$, then $\operatorname{Hom}(V_\lambda, R_{d'}) = 0$.
- Theorem 2: Let λ_1, λ_2 be partitions of n. Then $V_{\lambda_1} \cong V_{\lambda_2}$ iff $\lambda_1 = \lambda_2$.

Proof. We will prove both directions independently here. Let's begin.

($\Rightarrow$): Suppose that $V_{\lambda_1} \cong V_{\lambda_2}$.

Then $d(\lambda_1) = d(\lambda_2)$. We can see this two ways. First and most obviously, take the columns of each Young diagram and compute the degree of the Specht polynomial. Second and more formally, suppose for the sake of contradiction that $d(\lambda_1) \neq d(\lambda_2)$. WLOG let $d(\lambda_1) < d(\lambda_2)$. Then $V_{\lambda_2} \cong V_{\lambda_1} \hookrightarrow R_{d(\lambda_1)}$. But then by the above corollary, this ostensibly injective embedding is the zero map, a contradiction.

Let $d := d(\lambda_1) = d(\lambda_2)$. At this point, we have $V_{\lambda_1} \hookrightarrow R_d$ and $V_{\lambda_2} \hookrightarrow R_d$. It follows that $V_{\lambda_1} = V_{\lambda_2}$ as subspaces of R_d. Essentially, since we have the isomorphism $V_{\lambda_1} \cong V_{\lambda_2}$, we can construct the second embedding by factoring through the first, but then this second embedding should just give the same image. The factorization would look something like Figure 8.1.

$$\begin{tikzcd}
V_{\lambda_1} \arrow[dr, hook] \arrow[dd, phantom, "\cong"] & \\
& R_d \\
V_{\lambda_2} \arrow[ur, hook] &
\end{tikzcd}$$

Figure 8.1: Specht modules are equal as subspaces of R_d.

We now show that the polynomials in $V_{\lambda_1}, V_{\lambda_2}$ (which we can think of as subspaces/explicit polynomials) have no monomials in common[2]. For this, it's enough to understand monomials in one V_{λ_1}. Which monomials appear in V_λ? Here's an example. We will do a representative example instead of a formal proof. Consider $\lambda = (5,4,2,2)$ and S_{13}. $\lambda' = (4,4,2,2,1)$. Our Specht polynomial is

$$\Delta(x_1,x_2,x_3,x_4)\Delta(x_5,x_6,x_7,x_8)\Delta(x_9,x_{10})\Delta(x_{11},x_{12})$$

since $\Delta(x_{13}) = 1$. We also have that

$$\Delta(x_1,x_2,x_3,x_4) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{vmatrix} = \sum_{\sigma\in S_4} (-1)^\sigma x_{\sigma(1)}^0 x_{\sigma(2)}^1 x_{\sigma(3)}^2 x_{\sigma(4)}^3$$

Then for each column, we will have a number of variables in each power from $0,\dots,3$. Now we multiply out the individual Vandermonde determinants and count the number of variables in power $0,\dots,3$ to get 5,4,2,2; that is, every monomial will have 5 variables in power 0, 4 variables in power 1, 2 variables in power 2, and 2 variables in power 3. Thus, from every monomial, we immediately reconstruct λ. It means that we can reconstruct from any monomial this representation, so this implies that we must have $\lambda_1 = \lambda_2$. $\square$

- Corollary: V_λ's are all irreps of S_n

Proof. They are pairwise isomorphic and their number equals n. $\square$

8.2 Standard Young Tableaux

11/15:

- Recap.
  - Recall S_n and Young diagrams.
  - We've discussed conjugate Young diagrams corresponding to inverses λ' as well.
  - For every λ, we've constructed representations $V_{\lambda'}$.
  - Recall that $V_{\lambda'}$ is some representation inside the space of polynomials. In particular,

$$V_{\lambda'} = \operatorname{span}(\sigma[\Delta(x_1,\dots,x_{\lambda_1})\Delta(x_{\lambda_1+1},\dots,x_{\lambda_2})\cdots] \mid \sigma\in S_n)$$

    - Any $\sigma[\Delta(x_1,\dots,x_{\lambda_1})\Delta(x_{\lambda_1+1},\dots,x_{\lambda_2})\cdots]$ is a Specht polynomial $\mathrm{Sp}_\lambda(x_1,\dots,x_n)$.
    - All of these Specht polynomials together span the irrep given by the corresponding Specht module.
  - Last time, we proved that Specht modules are irreducible.
  - Specht polynomials are polynomials in R_d, where R is the ring of polynomials in $x_1,\dots,x_n$ and

$$d = \binom{\lambda_1}{2} + \binom{\lambda_2}{2} + \cdots$$

    - What is this definition of d??
- So how do we further study these representations?
  - Dimension?
  - Characters?
  - Basis?

[2]What does this mean?? Does it mean that in each polynomial in these spaces, there are no two monomials in the same variables, so no monomials cancel and all monomials have coefficient 1?

- Guiding question for today: Which Specht polynomials $\mathrm{Sp}_\lambda(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ form a basis of $V_{\lambda'}$?
- **Young tableau**: A Young diagram filled with integers. *Also known as* **YT**.
- **Standard** (Young tableaux): A YT filled in with numbers $1, \dots, n$, wherein each appears exactly once and the numbers increase in rows and in columns. *Also known as* **SYT**.
- Example of an SYT.

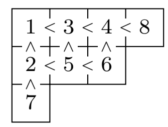


Figure 8.2: Example standard Young tableau.

  - We start with a Young diagram.
  - We need to fill it with 8 numbers.
  - There are relations between the boxes.
  - There are some constraints on what can go where, but multiple fillings are still possible.
  - In total, there are three SYT_8.
  - Denote by T a tableau within this set of three.
- Theorem: $\dim V_\lambda$ is the number of SYTs of shape λ.
- Examples.

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}$$

(a) $\Delta(1234)$.

$$\begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}\qquad\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}$$

(b) $(x_1 - x_2)$. (c) $(x_1 - x_3)$. (d) $(x_1 - x_4)$.

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}\qquad\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}$$

(e) $(x_1 - x_3)(x_2 - x_4)$. (f) $(x_1 - x_2)(x_3 - x_4)$.

Figure 8.3: Standard Young tableaux of $m\lambda = 4$.

  1. Only ONE way to fill trivial and alternating Young diagrams.
  2. Three ways to fill $(3, 1)$.
  3. Two ways to fill $(2, 2)$.
- Tip: Learn the representations of S_4 by heart!
  - Good for the final and in general.
- We denote the Specht polynomial written from a standard Young tableau by $\mathrm{Sp}(T)$.
  - Given an SYT T, $\mathrm{Sp}(T)$ is the product of the Vandermonde determinants for each column where the numbers in the column tell you which variables to plug into said determinant.
  - For example, the captions of each subfigure in Figure 8.3 are $\mathrm{Sp}(T)$ for the SYT depicted therein.

- We now build up to proving the theorem.
- Lemma: Fix a symmetric group S_n and a partition $\lambda \vdash n$. Then the collection $\{\text{Sp}(T)\}$ of Specht polynomials written from all $T \in \text{SYT}_\lambda$ (that is, all standard Young tableaux T of shape λ) is linearly independent.

  Proof. The basic reason that this lemma is true is that each $\text{Sp}(T)$ contains a certain monomial that none of the others contain; specifically, this will be the lexicographically smallest monomial SM. To get started, fix $\text{Sp}(T)$, and consider $SM[\text{Sp}(T)]$. Our goal is to reconstruct T from it. In this argument, we will look at a representative example instead of a formal proof. In particular, we will look at the example from Figure 8.2. Let's begin.

  First off, note that we have an analogous lemma to last time, i.e., we have

  $$SM(PQ) = SM(P)SM(Q)$$

  Reading from Figure 8.2, we have

  $$\text{Sp}(T) = \Delta(x_1, x_2, x_7)\Delta(x_3, x_5)\Delta(x_4, x_6)$$

  By considering the determinant interpretation of each Vandermonde determinant, we can determine by inspection that the lexicographically smallest monomial. Essentially, the smallest combinations lie along the diagonal of the matrix: Thus, we have

  $$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_7 \\ x_1^2 & x_2^2 & x_7^2 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ x_3 & x_5 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ x_4 & x_6 \end{vmatrix}$$

  Figure 8.4: Determining the lexicographically smallest monomial.

  $$SM[\text{Sp}(T)] = (1 \cdot x_2 \cdot x_7^2)(1 \cdot x_5)(1 \cdot x_6) = x_2 x_7^2 x_5 x_6$$

  From this monomial, we can reconstruct the SYT by putting x_7 in the bottom by necessity, then we have to put $2, 5, 6$ (other coefficients of SM) above in the certain order to get the right ordering. Then, we have to put the ones that aren't there $(x_1^0 x_3^0 x_4^0 x_8^0)$ in the top row. This gives us our YT back. ☐

- Theorem: $\dim V_\lambda = |\,\text{SYT}_\lambda\,|$.

  Proof. Since the $\text{Sp}(T)$ are linearly independent by the lemma, $\dim V_\lambda \geq |\,\text{SYT}_\lambda\,|$. Additionally, the $\text{Sp}(T)$ span $V_{\lambda'}$ because... (Rudenko will not finish this proof.) ☐

- Corollary: $\dim V_\lambda = \dim V_{\lambda'}$.

  Proof. Any representation of S_n will be self-dual. Essentially, because partition inversion only induces a transposition of the Young tableau and a transposed SYT is still standard, the number of SYTs will remain fixed under inversion, so so will the quantity its equal to by the above theorem, namely $\dim V_\lambda$. ☐

- Fact: We have the following identity.
  $$V_{\lambda'} = V_\lambda \otimes (\text{sign})$$
- Let f_λ be the number of SYTs of shape λ. We have shown that $f_\lambda \leq \dim(V_\lambda)$.

- Theorem (RSK): There exists a bijection between permutations in S_n and pairs of SYTs of the same shape (i.e., of **area** n).
  - RSK stands for Robinson-Schensted-Knuth.
  - Errata??: According to Wikipedia, this theorem (as stated) is the Robinson-Schensted correspondence, a special case of and predecessor to the RSK correspondence that trades under its own name.
- Corollary: $f_\lambda = \dim V_\lambda$.

Proof. $\sum_{\lambda=\text{YT of area } n} f_\lambda^2 = n! = \sum(\dim V_\lambda^2)$. This proves that $f_\lambda \leq \dim V_\lambda$ and $f_\lambda = \dim V_\lambda$. $\square$

  - What's the difference between this and the previous theorem?? And how does this proof work?
- Let's see how the RSK correspondence works through an example.
  - Consider the permutation

$$\sigma = (13)(27654) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 1 & 2 & 4 & 5 & 6 & 8 \end{pmatrix} \in S_8$$

  - We will construct two SYTs of identical shape from σ.
  - Start with a pair of empty YTs.

$$\begin{ytableau} \none[\ ] \end{ytableau} \qquad\qquad \begin{ytableau} \none[\ ] \end{ytableau}$$

    - We will call the left one the **insertion tableau** and the right one the **recording tableau**.
  - Fill $\sigma(1) = 3$ into the insertion tableau and record that this is the first (1) number inserted in the corresponding box of the recording tableau.

$$\begin{ytableau} 3 \end{ytableau} \qquad\qquad \begin{ytableau} 1 \end{ytableau}$$

  - We now have a new pair of tableaux. How do we insert the next number $\sigma(2) = 7$? Try adding it to the right of 3 in the insertion tableau. Record this addition in the recording tableau by adding to it a new box in the same relative position as the new 7 box and filling it with 2.

$$\begin{ytableau} 3 & 7 \end{ytableau} \qquad\qquad \begin{ytableau} 1 & 3 \end{ytableau}$$

  - How do we insert the next number $\sigma(3) = 1$? In the insertion tableau, push out 3 with 1 and move 3 to the next row. In the recording tableau, add a new square in the corresponding bottom position and fill it with 3.

$$\begin{ytableau} 1 & 7 \\ 3 \end{ytableau} \qquad\qquad \begin{ytableau} 1 & 2 \\ 3 \end{ytableau}$$

  - Next number: 2 pushes out 7 in the insertion tableau. 4 goes in the new box in the recording tableau.

$$\begin{ytableau} 1 & 2 \\ 3 & 7 \end{ytableau} \qquad\qquad \begin{ytableau} 1 & 2 \\ 3 & 4 \end{ytableau}$$

  - 4 gets inserted to the right; 5 fills the new box.

$$\begin{ytableau} 1 & 2 & 4 \\ 3 & 7 \end{ytableau} \qquad\qquad \begin{ytableau} 1 & 2 & 5 \\ 3 & 4 \end{ytableau}$$

  - 5,6,8 go further to the right; 6,7,8 in the second one.

$$\begin{ytableau} 1 & 2 & 4 & 5 & 6 & 8 \\ 3 & 7 \end{ytableau} \qquad\qquad \begin{ytableau} 1 & 2 & 5 & 6 & 7 & 8 \\ 3 & 4 \end{ytableau}$$

- Now we have a pair of standard Young tableaux.

- For every permutation, the above algorithm gives us a pair of SYTs.
- Formally, we are following the Schensted row-insertion algorithm, formalized as follows.

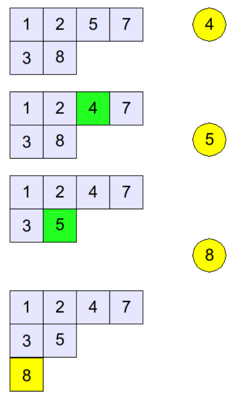


Figure 8.5: The Schensted algorithm.

1. Begin with an insertion tableau and a new number to insert. The new number must not appear anywhere in the insertion tableau.
2. Find the smallest number in row one that is greater than the new number. If no such number exists, append the new number to the end of the row, and we are done. If such a number does exist, replace it with the new number and prepare to insert the number that just got replaced into row two.
3. In the same manner as Step 2, find the smallest number in row two that is greater than the number that just got replaced. Replace it and move onto the third row.
4. Keep repeating until we reach a row where we can append the number from the previous row to the end of the line. If no such row exists, eventually we will reach the bottom of the Young tableau and we may start a new row.

- This algorithm provides a constructive proof of the RSK theorem.
  - In particular, this algorithm takes us between a pair (T, T') of Young tableaux and a permutation σ.
- This map has many interesting properties that are hard to prove. Here's a few.
  1. The map takes $(T', T) \mapsto \sigma^{-1}$.
  2. λ_1 and λ_1' are the length of the longest increasing (resp. decreasing) subsequence of your permutation variables.
- Last word: There is a famous theorem called the **Erdős-Szekeres theorem**.
  - This correspondence is a deep way to understand permutations/sequences of numbers. This is a big tool in CS.
- Next time: Induction and restriction.

8.3 Induction and Restriction

11/17: - Review: Representations of S_n.
    - λ is a partition.
    - V_λ is the span of $\mathrm{Sp}_\lambda(x_1, \dots, x_n)$.
    - $\dim V_\lambda = \#\,\mathrm{SYT}_\lambda$, i.e., equals the number of standard Young tableaux of shape λ.
- Naturally, it is desirable to find a better way of counting SYT_λ. We will do this with the **hook length formula**.
- **Hook length formula**: The formula given as follows, where n is the number being partitioned. *Given by*

$$\#\,\mathrm{SYT}_\lambda = \frac{n!}{\prod \text{length of all hooks}}$$

    - We should feel free to use this formula, but know that it's quite difficult to prove, so Rudenko will forego such a proof.
- **Hook** (of a cell): The set of all cells in a Young diagram directly to the right of or directly beneath the cell in question, including the cell in question.
- **Length** (of a hook): The cardinality of the hook in question.
- Example.

$$\begin{array}{|c|c|c|c|} \hline 7 & 4 & 3 & 1 \\ \hline 5 & 2 & 1 \\ \cline{1-3} 2 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array}$$

Figure 8.6: Hook length formula.

    - The Young diagram in Figure 8.6 corresponds to the partition $9 = (4, 3, 1, 1)$.
    - In each cell of the diagram is the length of the hook corresponding to that cell.
    - Thus, using the hook length formula, the number of standard Young tableaux of shape $(4, 3, 1, 1)$ is

$$\frac{9!}{7 \cdot 4 \cdot 3 \cdot 1 \cdot 5 \cdot 2 \cdot 1 \cdot 2 \cdot 1} = 9 \cdot 8 \cdot 3 = 216$$

- We're headed toward **branching**.
- We'll cover **induction** and **restriction** first.

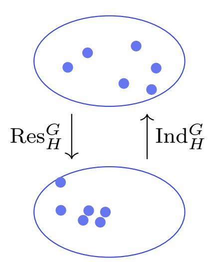


Figure 8.7: Induction and restriction as functors.

    - These are pretty natural things related to group G and subgroup $H \leq G$.
    - Essentially, we have G-reps and H-reps, and we want to interconvert between them.

- Induction allows you to go from H-reps up to G-reps, and vice versa for restrictions.
- In category theory, we call these maps **functors**.
  - Now would be a good time to dive into the definition of functor a bit more deeply!

- The map down is denoted by Res_H^G, and the map up is denoted by Ind_H^G.
- **Restriction** (of V to $H \leq G$): The vector space V viewed as an H-representation, i.e., a straight-up functional restriction of ρ_V. *Denoted by* $\boldsymbol{\text{Res}_H^G(V)}$.
- Example: Consider $S_2 < S_3$. Find $\text{Res}_{S_2}^{S_3}(2,1)$.
  - Recall that $(2,1) = \langle x_1 - x_2, x_1 - x_3 \rangle$, so this is our answer.
  - However, suppose we want to express $(2,1)$ in a form that tells us a bit more about its status as an S_2-representation. In particular, while $(2,1)$ was an irrep of S_3, it is *not* an irrep of S_2. Indeed, the group S_2 is abelian and hence only has one-dimensional irreps, so we should be able to decompose $(2,1)$ into a sum of two invariant subspaces.
  - So, looking at polynomials fixed and flipped under S_2, we obtain

$$\text{Res}_{S_2}^{S_3}(2,1) = \langle x_1 - x_2 \rangle \oplus \langle x_1 - x_3 + x_2 - x_3 \rangle$$

    - The left polynomial flips under S_2. Specifically, $(12) \cdot (x_1 - x_2) = (x_2 - x_1) = -(x_1 - x_2)$.
    - The right polynomial stays the same under S_2. Specifically, $(12) \cdot (x_1 - x_3 + x_2 - x_3) = (x_2 - x_3 + x_1 - x_3) = (x_1 - x_3 + x_2 - x_3)$.
    - Obviously, neither polynomial changes under e.
    - Note also that adding the right and left polynomials yields $2(x_1 - x_3) \in \text{span}(2,1)$, as expected.
  - Let's highlight a few other features of this decomposition.
  - The two representations in the decomposition are the alternating and trivial — $(1,1)$ and (2) — respectively.
  - While it is fairly obvious that $x_1 - x_2$ — one of the basis vectors of $(2,1)$ — is the basis for the alternating subrepresentation, finding the other one by inspection is trickier. Thus, here's a procedural way to do it.
    - Since the subspaces fixed under (12) are just its eigenspaces, let's compute the eigenvectors of the transformation.
    - Let $a := x_1 - x_2$ and $b := x_1 - x_3$.
    - Observe that

$$\begin{aligned}
(12) \cdot a &= x_2 - x_1 = -(x_1 - x_2) = -a \\
(12) \cdot b &= x_2 - x_3 = (x_1 - x_3) - (x_1 - x_2) = b - a
\end{aligned}$$

    - Thus,

$$\rho_{(2,1)}(12) = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$$

      - Letting $(1,0) = a$ and $(0,1) = b$, we have

$$\underbrace{\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}}_{\rho_{(2,1)}(12)} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{a} = \underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{-a} \qquad \underbrace{\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}}_{\rho_{(2,1)}(12)} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{a} = \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{b-a}$$

      as expected.
    - Computing the eigenvectors of this matrix, we obtain

$$e_1 = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{a} \qquad e_2 = \underbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}_{2b-a}$$

  - Plugging in the original definitions of a, b, we obtain

$$\begin{aligned} e_1 &= x_1 - x_2 \\ e_2 &= 2(x_1 - x_3) - (x_1 - x_2) = x_1 - x_3 + x_2 - x_3 \end{aligned}$$

    as expected.

- If we treat x_1, x_2, x_3 as the standard basis of $\mathbb{R}^3$, then $x_1 - x_2$ and $x_1 - x_3$ do span a plane containing $x_1 - x_3 + x_2 - x_3$, as we can prove with vector algebra. Moreover, as we would expect for a direct sum, $x_1 - x_2$ and $x_1 - x_3 + x_2 - x_3$ are orthogonal:

$$\underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{x_1 - x_2} \cdot \underbrace{\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}}_{x_1 - x_3 + x_2 - x_3} = (1)(1) + (-1)(1) + (0)(-2) = 0$$

- Moreover, the action of (12) on 3D space (i.e., with standard basis x_1, x_2, x_3) is a reflection along $x_1 - x_2$ and through the plane containing $x_1 - x_3 + x_2 - x_3$; this really shows how (12) fixes both axes while "flipping" one.

- We now move onto induction.
- Induction is highly nontrivial. It is one of those things in math that is just very tricky. All definitions of it are slightly uncomfortable.
- Here's a first attempt at a definition.
  - Let $H \circlearrowright W$ as an H-rep.
  - We want to create a G-rep $\operatorname{Ind}_H^G W$.
  - To start, let's decompose G into left cosets via

$$G = g_1 H \sqcup g_2 H \sqcup \cdots \sqcup g_k H$$

    - Suppose $g_1 \in H$ so that $g_1 H = H$.
  - $\operatorname{Ind}_H^G W$ is defined as the vector space $g_1 W \oplus \cdots \oplus g_k W$. As a vector space, each $g_i W = W$, but there is additional structure as representations.
  - Indeed, we must answer the question, "how does g act on $g_i w$?"
  - Answer: Via

$$g \cdot (g_i w) = g_{\sigma(i)} h_i w = g_{\sigma(i)}(h_i w)$$

    - The first equality comes from defining $\sigma \in S_n$ and $h_i \in H$ so that $g g_i = g_{\sigma(i)} h_i$.
    - We have this equality because $g g_i \in G$, so $g g_i \in \bigsqcup g_i H$, so for some $j = \sigma(i)$, $g g_i \in g_j H = g_{\sigma(i)} H$, i.e., $g g_i = g_{\sigma(i)} h_i$ for some $h_i \in H$.
    - Note that in the last part of the equality, $h_i w$ is the action of h_i on w via the H-rep.
  - So basically, g takes $g_i w$, isolates w, acts on it via h using the original representation to make $h_i w$, and then places this element within the subspace $g_{\sigma(i)} W$.
  - Each element of G acts in a big block-triangular matrix. Inside each block, you will see how g acts on H.
- Let W be the trivial representation of H. Then $\operatorname{Ind}_H^G(\mathrm{id})$ is the permutational representation of G acting on left cosets.
  - Example: $\operatorname{Ind}_{\{e\}}^G(\mathrm{id}) = \mathbb{C}[G]$.
  - See Example 2 from Section 3.3 of Serre (1977).
- There is a correspondence between $H/\operatorname{Stab}(x)$ and G??

  - This is the master construction of representations when you have a subgroup.
- The dimension of an induced representation can be calculated via

$$\dim \operatorname{Ind}_H^G W = (\dim W)(G : H)$$

- A slightly fancier way to think about this stuff, if you're unsatisfied at this point.
  - Take $H < G$, and H-rep W.
  - Let's look at functions on G with values in W, i.e., functions $f : G \to W$. This would be $\operatorname{Hom}_{\mathbb{C}}(\mathbb{C}[G], W)$, once we've linearized G so that we can consider linear maps. These linear maps are the exact same thing as the original functions because the basis of $\mathbb{C}[G]$ is G!
  - Since we know how to calculate the dimension of a space of homomorphisms, we have $\dim \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}[G], W) = (|G|)(\dim W)$.
  - Under this construction, we get to say that

$$\operatorname{Ind}_H^G W = \operatorname{Hom}_H(\mathbb{C}[G], W) = \{f : G \to W \mid f(x \cdot h) = f(xh^{-1}) = hf(x)\}$$

  - We can easily see that this space has the right dimension; such a function is uniquely defined by its values on $g_1, \dots, g_k$.
    - $f(g_1), \dots, f(g_k)$ and $f(g_1 h) = h^{-1} f(g_1)$.
  - What is g acting on in this function? So $[g(f)](x) = f(gx)$...
  - In this case, it's very easy to see that this is a construction with no choices of g_i's, of cosets, etc. Thus fancier.
- Once again, there is no easy way to understand this; we just have to work with it.
- Even fancier construction!
  - Let W be an H-representation. Abstractly, this means that W is a module over $\mathbb{C}[H]$.
  - Take $W \otimes_{\mathbb{C}[H]} \mathbb{C}[G]$.
    - Essentially, this means that if $w \otimes g$, then $hw \otimes g = w \otimes g$.
  - This has something to do with the second representation.
  - This is the most abstractly nice construction because it's much more general.
    - We don't need to use it on groups; we can use it on algebras and modules over them.
    - Indeed, this works in complete generality and has all the same properties.
    - Takeaway: This induced representation is something very, very general, but thinking of it more generally does not help you understand it to start.
- We will do a bunch of computations of such **induced representations** on the homework.
- Theorem (Frobenius): Let $H < G$, and let W be an H-rep. Then

$$\chi_{\operatorname{Ind}_H^G W}(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ g_i^{-1} g g_i \in H}} \chi_W(g_i^{-1} g g_i)$$

- Discussion.
  - Look at the picture from the blackboard.
    - Characters are class functions; they don't change under conjugation.
  - χ_W is a class function in H. An alternate formulation of the formula can be obtained by extending the χ_W (which is a class function in H) to $\tilde{\chi}_W$ (which is a class function in G).

- We could extend it to $\chi_W : G \to \mathbb{C}$ via
$$\tilde{\chi}_W = \begin{cases} 0 & g \notin H \\ \chi_W(g) & g \in H \end{cases}$$
- From here, take
$$\chi_{\operatorname{Ind}_H^G W} = \frac{1}{|H|} \sum_{x \in G} \tilde{\chi}_W(xgx^{-1})$$
- So we're basically just averaging again.

- Proof: See Theorem 12 from Section 3.3 of Serre (1977).
- Next week:
  - If we want to construct $\operatorname{Res}_{S_{n-1}}^{S_n}$, we will take all diagrams inside the Young diagram but one box less.
  - For example,
$$\operatorname{Res}_{S_{n-1}}^{S_n} \ydiagram{3,2,1,1} = \ydiagram{3,2,1} \oplus \ydiagram{3,1,1,1} \oplus \ydiagram{2,2,1,1}$$
  - This is the branching rules.
  - New HW will appear soon and be due Friday of Week 9.
    - Will be a bit harder (some hands-on, some fancy); if you can't do everything there, don't worry.
  - Midterms will be returned early after Thanksgiving.
  - Final will include some stuff from the last HW, but it will be easier.

8.4 S Chapter 3: Subgroups, Products, Induced Representations

From Serre (1977).

Section 3.1: Abelian Subgroups

12/28:
- Definition of **abelian/commutative** (group).
- Irreducible representations of abelian groups.

Theorem 9. *The following properties are equivalent:*

(i) *G is abelian.*

(ii) *All the irreducible representations of G have degree 1.*

Proof. Let $n_1, \dots, n_h$ be the degrees of the distinct irreducible representations of G. Recall that
$$|G| = \sum_{i=1}^{h} n_i^2$$
If G is abelian, then there are $|G|$ conjugacy classes and hence $|G|$ irreducible representations. But observe from the above equation that $|G| = h$ iff each $n_i = 1$, which implies the theorem. ☐

  - See Section 1.3 of Fulton and Harris (2004) for an alternate approach to this theorem.

- Irreducible representations of abelian subgroups.

Corollary 3. *Let A be an abelian subgroup of G. Then each irreducible representation V of G has degree*

$$\dim(V) \leq (G : A)$$

Proof. Let $\rho : G \to GL(V)$ be an irreducible representation of G. Through **restriction** to the subgroup A, it defines a representation $\rho_A : A \to GL(V)$ of A. Let $W \subset V$ be an irreducible subrepresentation of ρ_A; by Theorem 9, we have $\dim(W) = 1$. Let

$$V' = \bigcup_{g \in G} \rho(g)W$$

Since V' is a subspace of V that is clearly stable under G by definition, the fact that V is irreducible implies that $V' = V$. Additionally, we have for any $g \in G$ and $a \in A$ that

$$\rho(ga)W = \rho(g)\rho(a)W = \rho(g)W$$

This means that many of the $\rho(g)W$ composing V' are identical; in fact, it is only each *coset* of G with respect to A that can possibly contribute a new dimension to V'. Moreover, since $\dim(W) = 1$, it will only be at most 1 dimension that any $\rho(g)W$ contributes to the union. Essentially, each of the $(G : A)$ cosets contributes at most one dimension to $V' = V$, so $\dim(V)$ cannot exceed $(G : A)$, as desired. $\square$

- Example: A dihedral group contains a cyclic subgroup of index 2; its irreps thus have degree 1 or 2.

Section 3.2: Product of Two Groups

- **Product** (of G_1, G_2): The set of pairs (s_1, s_2) with $s_1 \in G_1$ and $s_2 \in G_2$. *Denoted by* $\boldsymbol{G_1 \times G_2}$.
- **Group product** (of G_1, G_2): The group $(G_1 \times G_2, \cdot)$, where $\cdot$ is the group structure defined by

$$(s_1, s_2) \cdot (t_1, t_2) = (s_1t_1, s_2t_2)$$

  Denoted by $\boldsymbol{G_1 \times G_2}$.
- Properties of $G_1 \times G_2$.
  1. $|G_1 \times G_2| = |G_1| \times |G_2|$.
  2. $G_1 \cong \{(s_1, 1) \mid s_1 \in G_1\} \leq G_1 \times G_2$ and $G_2 \cong \{(1, s_2) \mid s_2 \in G_2\} \leq G_1 \times G_2$.
  3. With the identifications in Property 2, each element of G_1 commutes with each element of G_2.
- These properties characterize $G_1 \times G_2$ completely, as we will see immediately below.
- **Direct product** (of G_1, G_2): The group G containing G_1, G_2 as subgroups and satisfying the conditions...
  (i) Each $s \in G$ can be written uniquely in the form $s = s_1s_2$ with $s_1 \in G_1$ and $s_2 \in G_2$;
  (ii) For $s_1 \in G_1$ and $s_2 \in G_2$, we have $s_1s_2 = s_2s_1$.

  Also known as **product**.
- Equivalence of the direct product with the group product.
  - Because of the two conditions, the product of any two $s, t \in G$ can be written as follows.

$$\begin{aligned}
st &= (s_1s_2)(t_1t_2) && \text{Condition (i)}\\
&= s_1(s_2t_1)t_2\\
&= s_1(t_1s_2)t_2 && \text{Condition (ii)}\\
&= (s_1t_1)(s_2t_2)
\end{aligned}$$

- Thus, the group structure of G mirrors that of $G_1 \times G_2$, too.
- This gives us everything we need to define an isomorphism $G_1 \times G_2 \to G$ by

$$(s_1, s_2) \mapsto s_1 s_2$$

- **Tensor product** (of $\rho^1 : G_1 \to GL(V_1), \rho^2 : G_2 : GL(V_2)$): The linear representation of $G_1 \times G_2$ into $V_1 \otimes V_2$ defined as follows. *Denoted by* $\boldsymbol{\rho^1 \otimes \rho^2}$. *Given by*

$$(\rho^1 \otimes \rho^2)(s_1, s_2) = \rho^1(s_1) \otimes \rho^2(s_2)$$

- The characters χ of $\rho^1 \otimes \rho^2$, χ_1 of ρ^1, and χ_2 of ρ^2 are related as follows.

$$\chi(s_1, s_2) = \chi_1(s_1) \cdot \chi_2(s_2)$$

- **Diagonal** (subgroup of $G \times G$): The set of pairs (s, s) for all $s \in G$.
- Note that the representation $\rho^1 \otimes \rho^2$ defined above equals the representation denoted $\rho^1 \otimes \rho^2$ in Section 1.5 of Serre (1977) when $G_1 = G_2$ and when it is restricted to the diagonal subgroup of $G \times G$.
- Irreducible representations of group products.

  Theorem 10.

  (i) *If ρ^1 and ρ^2 are irreducible, $\rho^1 \otimes \rho^2$ is an irreducible representation of $G_1 \times G_2$.*

  Proof. Since ρ^1, ρ^2 are irreducible, Theorem 5 implies that

$$\frac{1}{|G_1|} \sum_{s_1 \in G_1} |\chi_1(s_1)|^2 = 1 \qquad \frac{1}{|G_2|} \sum_{s_2 \in G_2} |\chi_2(s_2)|^2 = 1$$

  By multiplication, this gives

$$\frac{1}{|G_1 \times G_2|} \sum_{(s_1, s_2) \in G_1 \times G_2} |\chi(s_1, s_2)|^2 = 1$$

  It follows by Theorem 5 that $\rho^1 \otimes \rho^2$ is irreducible. □

  (ii) *Each irreducible representation of $G_1 \times G_2$ is isomorphic to a representation $\rho^1 \otimes \rho^2$, where ρ^i is an irreducible representation of G_i ($i = 1, 2$).*

  Proof. A pair of really elegant proofs are given. □

- Takeaway: "The above theorem completely reduces the study of representations of $G_1 \times G_2$ to that of representations of G_1 and representations of G_2" (Serre, 1977, p. 28).

Section 3.3: Induced Representations

- Definition of a **left coset**, $\boldsymbol{G/H}$, and **index** (of H in G).
- **Congruent modulo** (H elements $s, s' \in G$): Two elements $s, s' \in G$ that belong to the same left coset. *Denoted by* $\boldsymbol{s' \equiv s}$ **(mod** $\boldsymbol{H}$**)**;
  - Alternative definition: $s^{-1}s' \in H$. ($s = gh_1$ and $s' = gh_2 \implies s^{-1}s' = h_1^{-1}g^{-1}gh_2 = h_1^{-1}h_2 \in H$.)
- **System of representatives** (of G/H): A subset $R \subset G$ containing an element from each left coset of H. *Denoted by* $\boldsymbol{R}$.
- Each $s \in G$ can be written uniquely as $s = rt$ for some $r \in R$ and $t \in H$.

- We now build up to defining an **induced** representation. We will construct the definition abstractly first, and then work through a specific, simple example to illustrate the definition. Let's begin.
  - Let $\rho : G \to GL(V)$ be a linear representation of G.
  - Let $\rho_H : H \to GL(V)$ denote the restriction $\rho|_H$ of ρ to H, and let $W \subset V$ be a subrepresentation of ρ_H, i.e., be stable under ρ_t for all $t \in H$.
  - Let $\theta : H \to GL(W)$ be the linear representation that acts on W as ρ_H acts on V.
    - Note that this does *not* mean that $\theta(h) = \rho_H(h) = \rho(h)$; rather, since $\dim W < \dim V$ for a nontrivial case of this construction, $\theta(h)$ will yield a matrix/linear transformation of smaller dimension than $\rho_H(h)$.
    - See the following example for details.
  - Let $s \in G$ be arbitrary. Observe that the vector space $\rho_s W$ depends only on the left coset sH of s; indeed, if we replace s by st where $t \in H$, then we have $\rho_{st} W = \rho_s \rho_t W = \rho_s W$ since $\rho_t W = W$.
  - Define a subspace $W_\sigma \subset V$ for each left coset σ of H by $W_\sigma = \rho_s W$ for some (it does not matter which) $s \in \sigma$.
    - Note that the ρ_s permute the W_σ. Symbolically, if $W_\sigma = \rho_{s_2} W$, then $\rho_{s_1} W_\sigma = \rho_{s_1}\rho_{s_2} W = \rho_{s_1 s_2} W = W_{\sigma'}$ for some coset σ' of H to which $s_1 s_2$ belongs.
  - Thus, since each $\rho_s = \rho_{rt}$ moves the W_σ around internally via ρ_t and between each other via ρ_r, the sum $\sum_{\sigma \in G/H} W_\sigma$ is a subrepresentation of V.
- We now have all the definitions and tools we need to formally define an induced representation.
- **Induced** (representation $\rho : G \to GL(V)$ by $\theta : H \to GL(W)$ for $H \leq G$): The representation $\rho : G \to GL(V)$ defined above, if V is equal to the sum of the W_σ ($\sigma \in G/H$) and if this sum is direct (that is, if $V = \bigoplus_{\sigma \in G/H} W_\sigma$).
- Example.
  - Let $G = \mathbb{Z}/4\mathbb{Z}$, and let $\rho : \mathbb{Z}/4\mathbb{Z} \to GL(\mathbb{R}^2)$ send $0, 1, 2, 3$ to the $0°, 90°, 180°, 270°$ rotation matrices. Explicitly, we have

$$\rho_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \rho_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \rho_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \rho_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

  - Let $H = \{0, 2\} \cong \mathbb{Z}/2\mathbb{Z}$, and let $W = \operatorname{span}(1, 0) \cong \mathbb{R}$.
    - Since $\rho_0(1, 0) = (1, 0) \in \operatorname{span}(1, 0)$ and $\rho_2(1, 0) = (-1, 0) \in \operatorname{span}(1, 0)$, W is indeed a subrepresentation that is stable under ρ_H.
  - Let $\theta : \{0, 2\} \to GL[\operatorname{span}(1, 0)]$ be defined as follows. Since ρ_0 maps the basis vector $(1, 0)$ of $\operatorname{span}(1, 0)$ to itself, θ_0 should be the identity as well. Since ρ_2 maps the basis vector $(1, 0)$ of $\operatorname{span}(1, 0)$ to $(-1, 0)$, θ_2 should be the opposite of the identity as well. Altogether,

$$\theta_0 = \begin{bmatrix} 1 \end{bmatrix} \qquad \theta_2 = \begin{bmatrix} -1 \end{bmatrix}$$

  - Denote the two left cosets of H by $\sigma = 0 + H$ and $\tau = 1 + H$. Then

$$W_\sigma = \rho_0 \operatorname{span}(1, 0) = \operatorname{span}(1, 0) \qquad W_\tau = \rho_1 \operatorname{span}(1, 0) = \operatorname{span}(0, 1)$$

    - We have the rightmost equality above because of ρ_1's matrix definition, which implies that its action on the basis vector $(1, 0)$ of $\operatorname{span}(1, 0)$ is $\rho_1(1, 0) = (0, 1)$.
  - Thus, altogether, if ρ is to be the representation induced by θ, we must have

$$V = \bigoplus_{\sigma \in G/H} W_\sigma = W_0 \oplus W_1 = \operatorname{span}(1, 0) \oplus \operatorname{span}(0, 1)$$

- Let's go a bit deeper with this example, and see how we could recover ρ if all we started with was G, H, θ, W.
  - Define $\sigma = 0 + H$ and $\tau = 1 + H$.
  - Then $W_\sigma = \rho_0 W$ and $W_\tau = \rho_1 W$.
  - It follows that $V = W_\sigma \oplus W_\tau$.
  - We can now define ρ's action on the basis $\{W_\sigma, W_\tau\}$ of V.

$$
\begin{aligned}
\rho_0 W_\sigma &= \rho_0(\rho_0 W) = \rho_0(\rho_0 W) = \rho_0(\theta_0 W) = \rho_0 W = W_\sigma \\
\rho_0 W_\tau &= \rho_0(\rho_1 W) = \rho_1(\rho_0 W) = \rho_1(\theta_0 W) = \rho_1 W = W_\tau \\
\rho_1 W_\sigma &= \rho_1(\rho_0 W) = \rho_1(\rho_0 W) = \rho_1(\theta_0 W) = \rho_1 W = W_\tau \\
\rho_1 W_\tau &= \rho_1(\rho_1 W) = \rho_0(\rho_2 W) = \rho_0(\theta_2 W) = -\rho_0 W = -W_\sigma \\
\rho_2 W_\sigma &= \rho_2(\rho_0 W) = \rho_0(\rho_2 W) = \rho_0(\theta_2 W) = -\rho_0 W = -W_\sigma \\
\rho_2 W_\tau &= \rho_2(\rho_1 W) = \rho_1(\rho_2 W) = \rho_1(\theta_2 W) = -\rho_1 W = -W_\tau \\
\rho_3 W_\sigma &= \rho_3(\rho_0 W) = \rho_1(\rho_2 W) = \rho_1(\theta_2 W) = -\rho_1 W = -W_\tau \\
\rho_3 W_\tau &= \rho_3(\rho_1 W) = \rho_0(\rho_0 W) = \rho_0(\theta_0 W) = \rho_0 W = W_\sigma
\end{aligned}
$$

  - If we associate W_σ with $(1,0) \in \mathbb{R}^2 \cong V$ and W_τ with $(0,1) \in \mathbb{R}^2 \cong V$, then the matrices of ρ_s are those given in the original example above.
- Note: To get a nontrivial induced representation in this manner, we must have $W < V$. I.e., in the above example, we could not take $\rho_{\{0,2\}} : G \to GL(\mathbb{R}^2)$ and induce *it* up; rather, we needed to deal with $\rho_{\{0,2\}} : G \to GL(\mathbb{R})$ and induce it.
- Let's now look at a couple of reformulations of the definition of an induced representation.
  1. Each $x \in V$ can be written uniquely as $\sum_{\sigma \in G/H} x_\sigma$, with $x_\sigma \in W_\sigma$ for each σ.
  2. If R is a system of representatives of G/H, the vector space V is the direct sum of the $\rho_r W$ with $r \in R$.
- A consequence of the second formulation above is that

$$\dim(V) = \sum_{r \in R} \dim(\rho_r W) = (G : H) \cdot \dim(W)$$

- Examples.
  1. If $\rho : G \to GL(V)$ is the regular representation of G and W is the subspace with basis $(e_t)_{t \in H}$, then $\theta : H \to GL(W)$ is the regular representation of W and ρ is induced by θ. This is a fairly straightforward case of adding more dimensions to build up the full representation!
  2. The **permutation representation** of G associated with G/H. e_H is invariant under H. The representation of H in the subspace $\mathbb{C}e_H$ is the **unit representation** of H, and this representation induces ρ.
     - This is a more general case of the example presented above, where I chose $G = \mathbb{Z}/4\mathbb{Z}$ and $H = \mathbb{Z}/2\mathbb{Z}$, and restricted $\mathbb{C}e_H$ to $\mathbb{R}e_h$.
  3. If ρ_1 is induced by θ_1 and ρ_2 is induced by θ_2, then $\rho_1 \oplus \rho_2$ is induced by $\theta_1 \oplus \theta_2$.
  4. If (V, ρ) is induced by (W, θ), and if W_1 is a stable subspace of W, the subspace $V_1 = \sum_{r \in R} \rho_r W_1$ of V is stable under G, and the representation of G in V_1 is induced by the representation of H in W_1.
  5. If ρ is induced by θ, if ρ' is a representation of G, and if ρ'_H is the restriction of ρ' to H, then $\rho \otimes \rho'$ is induced by $\theta \otimes \rho'_H$.

- **Permutation representation** (of G associated with G/H): The representation $\rho : G \to GL(V)$, where $V = (e_\sigma)_{\sigma \in G/H}$ and $\rho_s e_\sigma = e_{s\sigma}$.
- We now prove the existence and uniqueness of induced representations.
  - While the above examples are specific, explicitly verifiable cases of induced representations, we have not yet proven that an induced representation (V, ρ) exists for *every* (W, θ).
  - This is our present goal.
- Note: This construction here is related to the intermediately fancy construction of induced representations from Friday's class.
- To begin, we first state and prove a lemma that will later be useful in proving the uniqueness of the induced representation.

  Lemma 1. *Suppose that (V, ρ) is induced by (W, θ). Let $\rho' : G \to GL(V')$ be a linear representation of G, and let $f : W \to V'$ be a linear map such that $f(\theta_t w) = \rho'_t f(w)$ for all $t \in H$ and $w \in W$*[3]*. Then there exists a unique linear map $F : V \to V'$ which extends f and satisfies $F \circ \rho_s = \rho'_s \circ F$ for all $s \in G$.*

  Proof. We first prove the uniqueness of F so that we can use an aspect of this argument to prove its existence. Let's begin.

  To prove that F is unique, it will suffice to give a formula derived from the given constraints that wholly characterizes it on V. Let $x \in \rho_s W \subset V$ be arbitrary. Then $\rho_s^{-1} x \in W$, hence

  $$F(x) = F(\rho_s \rho_s^{-1}) = \rho'_s F(\rho_s^{-1} x) = \rho'_s f(\rho_s^{-1} x)$$

  as desired.

  To prove that F exists, it will suffice to define it by formula and then show that this formula is well-defined. Let $x \in W_\sigma$ be arbitrary. Define $F(x)$ by $F(x) = \rho'_s f(\rho_s^{-1} x)$ for some $s \in \sigma$, mirroring the above. While it may seem that varying the choice of s could vary the definition of F, it actually does not: Replace s by st ($t \in H$) to see that

  $$\rho'_{st} f(\rho_{st}^{-1} x) = \rho'_s \rho'_t f(\theta_t^{-1} \rho_s^{-1} x) = \rho'_s(\theta_t \theta_t^{-1} \rho_s^{-1} x) = \rho'_s f(\rho_s^{-1} x)$$

  We can then check that $F \circ \rho_s = \rho'_s \circ F$ for all $s \in G$. (How?? Serre (1977) says it's easy but I'm not seeing it.) □

- Now we state and prove the full existence and uniqueness result.

  Theorem 11. *Let (W, θ) be a linear representation of H. There exists a linear representation (V, ρ) of G which is induced by (W, θ), and it is unique up to isomorphism.*

  Proof. We will prove existence and then uniqueness. Let's begin.

  Existence: In view of Example 3, we may assume that θ is irreducible. In this case, θ is isomorphic to a subrepresentation of the regular representation of H, which can be induced to the regular representation of G by Example 1. Then applying Example 4, we conclude that θ, itself, can be inducted.

  Uniqueness: Let $(V, \rho), (V', \rho')$ be two representations induced by (W, θ). Since W is a subspace of V', we may consider the linear injection $f : W \to V'$. As an injection, f is the identity on W, so for any $t \in H$ and $w \in W$, we have $f(\theta_t w) = \theta_t w = \rho'_t w = \rho'_t f(w)$. Thus, applying Lemma 1, we see that there exists a linear map $F : V \to V'$ which is the identity on W and satisfies $F \circ \rho_s = \rho'_s \circ F$ for all $s \in G$. Since F is the identity on W, $\mathrm{Im}(F)$ contains all the $\rho'_s W$ and thus is isomorphic to V'. This combined with the fact that $\dim V' = (G : H) \cdot \dim(W) = \dim V$ proves that F is an isomorphism overall, hence completing the proof. □

[3]Note that this is not quite a morphism of G-representations because only ρ' maps from G — θ maps from H!

- We now discuss the character of an induced representation.
- Motivation: Since (W,θ) determines (V,ρ) up to isomorphism, we should be able to compute χ_ρ from χ_θ.
- Here's how:

  Theorem 12. *Let h be the order of H and let R be a system of representatives of G/H. For each $u\in G$, we have*

  $$\chi_\rho(u)=\sum_{\substack{r\in R\\ r^{-1}ur\in H}}\chi_\theta(r^{-1}ur)=\frac{1}{|H|}\sum_{\substack{s\in G\\ s^{-1}us\in H}}\chi_\theta(s^{-1}us)$$

  In particular, $\chi_\rho(u)$ is a linear combination of the values of χ_θ on the intersection of H with the conjugacy class of u in G.

  Proof. We will proceed from the definition of $\chi_\rho(u)$ as

  $$\chi_\rho(u)=\operatorname{tr}_V(\rho_u)$$

  To begin, consider the matrix of ρ_u. Since ρ_u permutes the ρ_rW composing V, only spaces ρ_rW that ρ_u maps into themselves (i.e., spaces on the block diagonal of the matrix of ρ_u) will affect the trace. More precisely, these are spaces for which $ur=rt$ for some $t\in H$. Observe that this condition can be rewritten $r^{-1}ur\in H$. Thus,

  $$\chi_\rho(u)=\sum_{\substack{r\in R\\ r^{-1}ur\in H}}\operatorname{tr}_{\rho_rW}(\rho_u|_{\rho_rW})$$

  Recall that the origin of the condition $ur=rt$ is that we needed $\rho_u\rho_r=\rho_r\theta_t$ with $t=r^{-1}ur\in H$. More specifically, since we only care about the action of ρ_u on ρ_RW right now, we have $\rho_u|_{\rho_rW}\rho_r=\rho_r\theta_t$. It follows since $\operatorname{tr}(ab)=\operatorname{tr}(ba)$ and hence $\operatorname{tr}\big(aba^{-1}\big)=\operatorname{tr}(b)$ that $\operatorname{tr}(\rho_u|_{\rho_rW})=\operatorname{tr}(\rho_r\theta_t\rho_{r^{-1}})=\operatorname{tr}(\theta_t)=\chi_\theta(t)=\chi_\theta(r^{-1}ur)$, which yields

  $$\chi_\rho(u)=\sum_{\substack{r\in R\\ r^{-1}ur\in H}}\chi_\theta(r^{-1}ur)$$

  as desired.

  The second formula given for $\chi_\rho(u)$ follows from the first by noting that all elements $s\in G$ in the left coset rH ($r\in R_u$) satisfy $\chi_\theta(s^{-1}us)=\chi_\theta(r^{-1}ur)$. ☐

- Another property of induced representations discussed further in part II: The **Frobenius reciprocity formula**, which is given by

  $$(f_H\mid\chi_\theta)_H=(f\mid\chi_\rho)_G$$

8.5 S Chapter 7: Induced Representations; Mackey's Criterion

From Serre (1977).

Section 7.1: Induction

12/29:
- We are now treating V as a $\mathbb{C}[G]$-module and W as a $\mathbb{C}[H]$-submodule of V.
- Definition of **induced** representation.
- We now reformulate the induction property.
- Let W' (defined as follows) be the $\mathbb{C}[G]$-module obtained from W by **scalar extension** from $\mathbb{C}[H]$ to $\mathbb{C}[G]$.

  $$W'=\mathbb{C}[G]\otimes_{\mathbb{C}[H]}W$$

- The injection $W \to V$ extends by linearity to a $\mathbb{C}[G]$-homomorphism $i : W' \to V$.
- There is definitely more for me to understand here in the realm of exactly what a scalar extension is!

- We now relate W' to V.

Proposition 18. *In order that V be induced by W, it is necessary and sufficient that the homomorphism*

$$i : \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \to V$$

be an isomorphism.

Proof. This follows (somehow??) from the fact that the elements of the elements of a system of left coset representatives for H form a basis of $\mathbb{C}[G]$ considered as a right $\mathbb{C}[H]$-module. $\square$

- Notes.

1. Update on Theorem 11: Because the tensor product is well-defined, his formulation of the representation induced by W obviously *exists* and *is unique*.
   - At this point, Serre (1977) introduces the $\int_H^G(W)$ notation.
2. Update on Lemma 1: If V is induced by W and if E is a $\mathbb{C}[G]$-module, we have a canonical isomorphism
$$\mathrm{Hom}^H(W, E) \cong \mathrm{Hom}^G(V, E)$$
where $\mathrm{Hom}^G(V, E)$ denotes the vector space of $\mathbb{C}[G]$-homomorphisms of V into E, and $\mathrm{Hom}^H(W, E)$ is defined similarly.
   - This follows from a property of tensor products.
3. Induction is transitive: If G is a subgroup of a group K, then we have
$$\mathrm{Ind}_G^K(\mathrm{Ind}_H^G(W)) \cong \mathrm{Ind}_H^K(W)$$
   - Two ways to see this: Directly or via the associativity of the tensor product.

- A criterion for when a subspace and subgroup can induce a representation.

Proposition 19. *Let V be a $\mathbb{C}[G]$-module which is a direct sum $V = \bigoplus_{i \in I} W_i$ of vector subspaces permuted **transitively** by G. Let $i_0 \in I$, $W = W_{i_0}$, and let H be the **stabilizer** of W in G. Then W is stable under the subgroup H and the $\mathbb{C}[G]$-module V is induced by the $\mathbb{C}[H]$-module W.*

Proof. Obvious, according to Serre (1977). $\square$

- **Transitive** (permutation by G on a set X): A group action for which the orbit of x is X for some (any) $x \in X$.
- **Stabilizer** (of W in G): The set of all $s \in G$ such that $sW = W$.
- Note on the proposition: In order to apply it to an irrep $V = \bigoplus W_i$ of G, it is enough to check that the W_i are permuted among themselves by G; the transitivity condition is automatic because each orbit of G in the set of W_i's defines a subrepresentation of V.
- **Monomial** (representation): A representation V for which the W_i's are of dimension 1.

Week 9

Symmetric Group Representation Characteristics

9.1 Frobenius Reciprocity; The Branching Theorem

11/27: • Announcements.

- OH on Wednesday at 5:30 PM this week; not Tuesday.
- There will be extra OH next week pre-exam.
  - Roughly like Monday/Wednesday next week.
- Midterm will be returned on Wednesday; we can pick them up in-person in his office starting then.
- There are some grade boundaries: Pass/Fail we can do until Friday, withdrawal we can do until 5:00 PM today.

• Let's finish the conversation about induction/restriction and prove the **branching theorem**.

• Reminder to start.

- We have two mathematical categories, G-reps and H-reps where $H \leq G$.
- These catagories are related by functors.
  - $\mathrm{Res}_H^G : G\text{-reps} \to H\text{-reps}$ and vice versa for Ind_H^G.
  - See Figure 8.7.
- Restrictions are stupidly simple.
- Inductions, most hands-on, we take copies of W times cosets. Formulaically,

$$\mathrm{Ind}_H^G W = g_1 W \oplus \cdots \oplus g_k W$$

where $k = (G : H)$ and $G = \bigsqcup_{i=1}^k g_i H$.

  - In more detail, the action of g on $g_i w$ is that of $g_{\sigma(i)} h_i w$.
  - This is a genuinely hard construction.
  - A matrix of this thing will be a block-permutation matrix like

$$\begin{array}{c c}
 & \begin{array}{ccc} g_1 W & & g_k W \end{array} \\
\begin{array}{c} g_1 W \\ \\ g_k W \end{array} & \left[\begin{array}{c|c|c} \text{/////} & 0 & 0 \\ \hline 0 & 0 & \text{/////} \\ \hline 0 & \text{/////} & 0 \end{array}\right]
\end{array}$$

- As an alternate construction, we have that

$$g_1W \oplus \cdots \oplus g_kW \cong \operatorname{Hom}_H(\mathbb{C}[G], W)$$

  - Recall that elements of the set on the right above are functions $f : G \to W$ such that $f(h(g)) = hf(g)$.
  - We map between the two via $f(g) \mapsto f(gx')$.
- What is nice about induced representations is that $\dim[\operatorname{Ind}_H^G W] = (\dim W)[G : H]$.
- There is a very easy statement of the character of an induced representation, the **Frobenius formula**.
  - Recall that

$$\tilde{\chi}_W(g) = \begin{cases} 0 & g \notin H \\ \chi_W(g) & g \in H \end{cases}$$

  - With this, we average:

$$\chi_{\operatorname{Ind}_H^G W}(g) = \frac{1}{|H|} \sum_{x \in G} \tilde{\chi}_W(xgx^{-1})$$

  - Essentially, we're taking a whole bunch of conjugates, summing them up, and dividing to get rid of overcounting.

- We now move onto **Frobenius reciprocity**, which is a relation between the functors/relations Ind_H^G and Res_H^G.
  - The first point where category theory gets interesting is the notion of **adjoint functors**, which we are about to touch on. It is a very subtle notion.
  - Here's version 1 of the statement of Frobenius reciprocity.
    - Recall that we have a scalar product on the space of class function, given by

$$(\chi_1, \chi_2) = \frac{1}{|G|} \sum_{g \in G} \chi_1(g)\chi_2(g^{-1})$$

      where χ_1, χ_2 are class functions on G.
    - Recall that if $\chi_1 = \chi_V$ and $\chi_2 = \chi_W$, then

$$(\chi_1, \chi_2) = \dim \operatorname{Hom}_G(V, W) = \dim \operatorname{Hom}_G\left(\bigoplus_{i=1}^k V_i^{n_i}, \bigoplus_{i=1}^k V_i^{m_i} \right) = \sum_{i=1}^k n_i m_i$$

    - Then the statement is as follows. If V is a G-rep and W is an H-rep, then

$$(V, \operatorname{Ind}_H^G W)_G = (\operatorname{Res}_H^G V, W)_H$$

      - ➢ This notation denotes a scalar product in G and scalar product in H of the characters of each representation.
    - This is similar to the relation between adjoint maps $V \to W$ and $W^* \to V^*$.
  - Version 2.
    - We have that

$$\operatorname{Hom}_G(V, \operatorname{Ind}_H^G W) \cong \operatorname{Hom}_H(\operatorname{Res}_H^G V, W)$$

      where the isomorphism is canonical.
    - We will not check this last definition; we can tediously do it with definitions, and there's nothing complicated. Rudenko leaves this as an exercise to us.
    - The canonical isomorphism sends a map $v \mapsto [g \mapsto \varphi(gv)]$ to the map $\phi : V \to W$.

- We now prove Version 1.

Proof. We have

$$\begin{aligned}
(\chi_V, \chi_{\operatorname{Ind}_H^G W})_G &= \frac{1}{|G|}\sum_{g_1\in G}\chi_V(g_1)\chi_{\operatorname{Ind}_H^G W}(g_1^{-1}) \\
&= \frac{1}{|G|}\sum_{g_1\in G}\chi_V(g_1)\left(\frac{1}{|H|}\sum_{g_2\in G}\tilde{\chi}_W(g_2g_1^{-1}g_2^{-1})\right) \\
&= \frac{1}{|H|\cdot|G|}\sum_{g_1,g_2\in G}\chi_V(g_1)\tilde{\chi}_W(g_2g_1^{-1}g_2^{-1}) \\
&= \frac{1}{|H|\cdot|G|}\sum_{g_1,g_2\in G}\chi_V(\underbrace{g_2g_1g_2^{-1}}_{h})\tilde{\chi}_W(\underbrace{g_2g_1^{-1}g_2^{-1}}_{h^{-1}}) \\
&= \frac{1}{|H|}\frac{1}{|G|}\sum_{h\in G}|G|\chi_V(h)\tilde{\chi}_W(h^{-1}) \\
&= (\chi_V|_H, \chi_W)_H \\
&= (\operatorname{Res}_H^G V, \chi_W)_H
\end{aligned}$$

From line 4 to line 5: Fix h; then $g_2g_1g_2^{-1} = h$ iff $g_1 = g_2^{-1}hg_2$, so we have overcounted by $|G|$ times. From line 5 to line 6: $\tilde{\chi}_W$ is zero whenever $h^{-1}\notin H$, so this ostensible sum over all $h\in G$ is *de facto* only a sum over all $h\in H$; this is what allows us to consider χ_V as "restricted to H" in line 6. ☐

- We now come to the branching theorem at long last.
- Example first.
  - Consider $S_n > S_{n-1}$, where S_{n-1} is the subgroup that fixes n. I.e., $S_3 > S_2 = \{e, (12)\}$, and we explicitly omit $(13), (23), (123), (132)$ because they all move 3.
  - Let $\lambda \vdash n$.
  - Let $\mu \leq \lambda$ be a Young diagram of a partition of $n-1$.
  - Then
    1. We have
       $$\operatorname{Res}_{S_{n-1}}^{S_n} V_\lambda = \bigoplus_{\substack{\mu\leq\lambda \\ |\mu|=n-1}} V_\mu$$
       - Example:
         $$\operatorname{Res}_{S_4}^{S_5} \ydiagram{3,2} = \ydiagram{2,2} \oplus \ydiagram{3,1}$$
    2. We have
       $$\operatorname{Ind}_{S_{n-1}}^{S_n} V_\mu = \bigoplus_{\substack{\mu\leq\lambda \\ |\lambda|=n}} V_\lambda$$
       - Example:
         $$\operatorname{Ind}_{S_5}^{S_6} \ydiagram{3,2} = \ydiagram{3,2,1} \oplus \ydiagram{3,3} \oplus \ydiagram{4,2}$$

- The reason that this theorem is called the branching theorem originates from the diagram in Figure 9.1, which (when continued) encapsulates the main idea of the theorem.
  - This graph helps you understand induction and restriction.
  - Dimensions are the number of paths from the bottom to a final Young diagram.

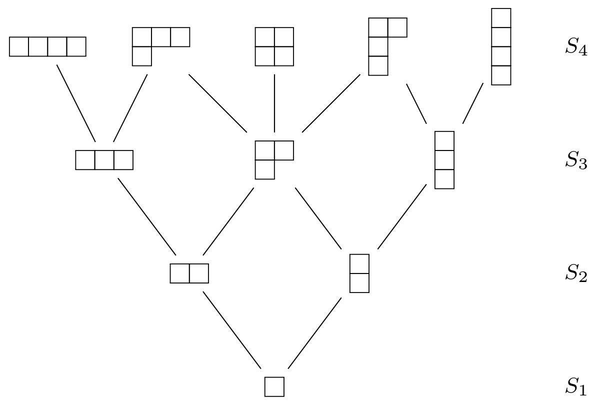


Figure 9.1: The branching theorem.

- For example, the dimension of $(3,1)$ is 3 because there are 3 paths to it, listed as follows.
  1. $(1)\to(2)\to(3)\to(3,1)$.
  2. $(1)\to(2)\to(2,1)\to(3,1)$.
  3. $(1)\to(1,1)\to(2,1)\to(3,1)$.
- Number of paths is equivalent to number of standard Young tableaux!

- Theorem (Branching): The following two statements are true.

$$\operatorname{Res}_{S_{n-1}}^{S_n} V_\lambda = \bigoplus_{\substack{\mu\leq\lambda \\ |\mu|=n-1}} V_\mu \tag{9.1}$$

$$\operatorname{Ind}_{S_{n-1}}^{S_n} V_\mu = \bigoplus_{\substack{\mu\leq\lambda \\ |\lambda|=n}} V_\lambda \tag{9.2}$$

Proof. We'll talk about the general idea of the proof now, and maybe do the details next time.

(9.1) $\Longleftrightarrow$ (9.2): Suppose first that the left statement above holds true. Then we have that

$$(\operatorname{Res}_{S_{n-1}}^{S_n} V_\lambda, V_\mu) = \begin{cases} 0 & \\ 1 & \lambda \geq \mu \end{cases}$$

Thus, by Frobenius reciprocity,

$$(V_\lambda, \operatorname{Ind}_{S_{n-1}}^{S_n} V_\mu) = (\operatorname{Res}_{S_{n-1}}^{S_n} V_\lambda, V_\mu) = \begin{cases} 0 & \\ 1 & \lambda \geq \mu \end{cases}$$

Therefore, the second statement holds true. The proof is symmetric in the opposite direction.

(9.1): Let's look at an example. Consider the Young diagram of S_8 shown in Figure 9.2.

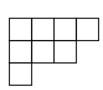


Figure 9.2: Proving the branching theorem.

We want to restrict it down to S_7. Recall that $V_\lambda = \operatorname{span}(S_8 : \Delta(x_1,x_2,x_3)(x_4-x_5)(x_6-x_7))$. Now in S_7, we fix x_8. Consider subrepresentations of V_λ filtered by degree as follows.

$$\underbrace{[\;]}_{\deg_{x_3}\leq 0} \leq \underbrace{\left[\;\right]}_{\deg_{x_5}\leq 1} \leq \underbrace{\left[\quad\right]}_{\deg_{x_8}\leq 2} \leq V_\lambda$$

The proof comes from the fact that if we now take quotients of these subrepresentations, e.g., via

$$\deg = 0, \deg \leq 1/\deg \leq 0, \deg \leq 2/\deg \leq 1, \dots$$

then since x_8 can only appear in three boxes, ... □

- Practice with the above example and think it through.

9.2 The Character of a Symmetric Group Representation

11/29:
- Announcements.
  - OH today at 5:30.
  - Our midterms are graded; we can look at them in his office whenever (I can do this during OH!).
- Today, we'll formulate the main result he wants to prove next time.
- Goal is still to understand representations of S_n.
  - We've constructed all of them using Specht modules, but what else do we want?
  - We have dimension, we want characters, etc.
- The main idea is to look at symmetric polynomials once again.
  - Consider $\mathbb{Q}[x_1,\dots,x_n]^{S_n}$.
  - We have proven the fundamental theorem that $\mathbb{Q}[x_1,\dots,x_n]^{S_n} = \mathbb{Q}[\sigma_1,\dots,\sigma_n]$ where $\sigma_k = \sum_{1\leq i_1\leq\cdots\leq i_k\leq n} x_{i_1}\cdots x_{i_k}$.
  - We also proved in PSet 6, Q6 that these rings are equal to $\mathbb{Q}[p_1,\dots,p_k]$ and $\mathbb{Q}[h_1,\dots,h_k]$ where

$$p_k = \sum_{i=1}^n x_i^k \qquad\qquad h_k = \sum_{1\leq i_1\leq\cdots\leq i_k\leq n} x_1\cdots x_k$$

    - Example: If $n=3$ and $k=2$, then

$$h_2 = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$$

  - Table of bases for n, k.

$k \setminus n$	1	2	3	4
0	1	1	1	1
1	x_1	x_1+x_2	$x_1+x_2+x_3$	$\cdots$
2	x_1^2	$x_1^2+x_2^2$, x_1x_2	σ_1^2, σ_2^2	σ_1^2, σ_2^2
3	x_1^3	...		

Table 9.1: Polynomial bases.

  - Now take

$$\Lambda_k = [\mathbb{C}[x_1,\dots,x_k]]_{\deg=k-1} \cong [\mathbb{C}[x_1,\dots,x_{k+1}]]_{\deg=k-1} \cong \cdots$$

    - Alternatively, we can think of this thing as

$$\Lambda_k = (\mathbb{C}[x_1,\dots])_k$$

      with $\sigma_1^k, \sigma_2\sigma_1^{k-1}, \dots$.

- We call Λ the ring of symmetric functions and define it to be equal to

$$\Lambda = \mathbb{Q}[\sigma_1, \sigma_2, \sigma_3, \dots]$$

- In every complete component, only finitely many of the σ will participate, so we get finite things.
- This is a graded ring! We have

$$\Lambda = \bigoplus_{k \geq 0} \Lambda_k$$

and $\Lambda_k \otimes \Lambda_\ell = \Lambda_{k+\ell}$

- This construction is called the **projective limit**, and we may have encountered it in commutative algebra under the definition

$$\Lambda = \lim_{\rightarrow} \mathbb{C}[x_1, \dots, x_n]^{S_n}$$

- We have identifies such as $p_2 = \sigma_1^2 - 2\sigma_2$. This means that

$$(x_1 + \cdots + x_n)^2 - 2(x_1x_2 + x_1x_3 + \cdots) = x_1^2 + x_2^2 + \cdots$$

- Observation: $\dim_{\mathbb{Q}} \Lambda_n$.

- Now, we need to take a vector space on ring representations; we've done this already with the representation ring.
- Let R_n be the $\mathbb{Q}$-vector space of functions $\chi : S_n \to \mathbb{Q}$ such that $\chi(x\sigma x^{-1}) = \chi(\sigma)$. This is our favorite space of class functions.
- Theorem (Frobenius characteristic map): There is an isomorphism of vector spaces and of rings called the Frobenius characteristic: $\text{ch} : \bigoplus_{n \geq 0} R_n \to \Lambda$.

Proof. Take $\chi_V \in R_k$, and $\chi_W \in R_\ell$. Let V an S_k-rep, and W an S_ℓ-rep. We know that

$$S_k \times S_\ell = S_{k+\ell}$$

So what we can do is induction $\text{Ind}_{S_k \times S_\ell}^{S_{k+\ell}}(V \otimes W)$. Call this operation $\chi_V \boxtimes \chi_W$.

Now we write down the formula:

$$\text{ch}(\chi) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi(\sigma) p_1^{\lambda_1(\sigma)} \cdots p_k^{\lambda_k(\sigma)}$$

where $\lambda_1(\sigma), \lambda_2(\sigma), \dots$ represent the cycle structure of σ; each λ_i is a number of cycles of length $1, 2, \dots$. □

- Examples.

1. S_1.
   - Sends the YD (1) to $p_1 = x_1 + x_2 + x_3 + \cdots$.
2. S_2.
   - Sends (2) to $\frac{1}{2!}(p_1^2 + p_2) = \frac{1}{2}((x_1 + x_2)^2 + x_1^2 + x_2^2) = x_1^2 + x_2^2 + x_1x_2 = h_2$.
   - It also sends $(1, 1)$ to $\frac{1}{2!}(p_1^2 - p_2) = \frac{1}{2}((x_1 + x_2)^2 - x_1^2 - x_2^2) = x_1x_2 = \sigma_2$.
   - Let's check our formula. What is $\text{Ind}_{S_1 \times S_1}^{S_2}(1) \otimes (1)$? Since the induction of the trivial representation is the regular representation, which we can decompose, we know that this induction equals $(1, 1) \oplus (2)$. It follows that $p_1^2 = x_1^2 + x_2^2 + x_1x_2 + x_1x_2 = (x_1 + x_2)^2$.
3. S_3.

- Sends (3) to

$$\begin{aligned}\frac{1}{3!}(p_1^3+3p_1p_2+2p_3) &= \frac{1}{6}[(x_1+x_2+x_3)^3+3(x_1+x_2+x_3)(x_1^2+x_2^2+x_3^2)+2(x_1^3+x_2^3+x_3^3)]\\ &= \frac{1}{6}[6(x_1^3+x_2^3+x_3^3)+6(x_1^2x_2+x_1x_2^2+x_1x_3^2+x_1^2x_3+\cdots)+6x_1x_2x_3]\\ &= h_3\end{aligned}$$

- Sends $(1,1,1)$ to

$$\begin{aligned}\frac{1}{3!}(p_1^3-3p_1p_2+2p_3) &= \frac{1}{6}[(x_1+x_2+x_3)^3-3(x_1+x_2+x_3)(x_1^2+x_2^2+x_3^2)+2(x_1^3+x_2^3+x_3^3)]\\ &= x_1x_2x_3\\ &= \sigma_3\end{aligned}$$

- Sends $(2,1)$ to

$$\begin{aligned}\frac{1}{3!}(2p_1^3-p_3) &= \frac{1}{6}[2(x_1^3+x_2^3+x_3^3)+6(x_1^2x_2+\cdots)+12x_1x_2x_3]\\ &= (x_1^2+\cdots)+2x_1x_2x_3\end{aligned}$$

- Again, we can check that

$$\operatorname{Ind}_{S_2\times S_1}^{S_3}[(1,1)\otimes(1)] = \sigma_1\sigma_2$$

  - We compute $\operatorname{Ind}_{S_2}^{S_3}(1,1) = (1,1,1)\oplus(2,1)$ via the branching formula: There are only two ways to add a box!
  - We have $\sigma_1\sigma_2-\sigma_3 = (x_1+x_2+x_3)(x_1x_2+x_1x_3+x_2x_3)-x_1x_2x_3$.

- Do we need to be fluent in the techniques by which you expanded all of the polynomials above??
- Thus, we have two conjectures:

$$\operatorname{ch}[(n)] = h_n \qquad\qquad \operatorname{ch}[\underbrace{(1,\dots,1)}_{n\text{ times}}] = \sigma_n$$

- The theorem is cool because it sends all of representation theory to some symmetric polynomial game!
- How do we compute $\operatorname{ch}(V_\lambda)$?
  - We say it equals S_λ, where S_λ is a Schur polynomial.
  - Take the YT of λ. Recall standard YTs.
  - **Semistandard** (YT): Things strictly increase in columns, but only monotonically increase in rows. *draw picture!*
  - The six semistandard ones give us the Schur polynomial.
  - Relation to RSK correspondence.
- Proving why this stuff is true is not hard.
- To understand *why* this is true, Google the **Schur-Weyl duality**.

9.3 Office Hours

- I got a 68/100 on the midterm: 30, 24, 0, 14.
  - I would have needed to show my work (or at least one example of a calculation) to get full credit for 2, even though it just said "find."
  - Rudenko did not expect that finding conjugacy classes would be so difficult for us; he will adjust for this difficulty on the final.
- Week 3, Lecture 2: You proved that $\langle \chi_V, \chi_W \rangle = \delta_{VW}$. To do so, you used a projection function $p = (1/|G|)\sum_{g\in G} gv$. You began your proof by proving that p is a G-morphism and then never used this result again, as far as I can tell. Did you use it again? See pp. 45-47, 58 (it needs to be a morphism of G-representations to map between the representations V, V^G?).
- Week 3, Lecture 2: Same proof. To prove that $\text{Im}(P) = V^G$, do we need more than $p^2 = p$? I think so, but you didn't do it explicitly. See pp. 46-47.
- Week 3, Lecture 2: Same proof. What's up with the trivial special case? See p. 48.
- *Week 3, Lecture 3: Cube thing (see picture from 10/13)?
  - It's just a depiction of two different 3-coordinate bases of the same space. It was drawn to illustrate a possible relation between the orthonormal basis χ_1, χ_2, χ_3 (cube) and the orthogonal basis $\chi_{C_1}, \chi_{C_2}, \chi_{C_3}$.
- Week 3, Lecture 3: Why did we talk about the infinite-dimensional regular representation here? See p. 50.
- *Week 3, Lecture 3: What is the point of the misc. calculations involved in computing the S_4 character table? See p. 52.
  - Just to check that we were on the right path and shown an example of using the orthogonality relations.
- *Week 3, Lecture 3: Proof of the second orthogonality relation your way? It's in Serre (1977), but I don't think that's the way you proved it. See p. 52.
  - To begin, note that it is a *highly* nontrivial statement that if A, B are matrices such that $AB = I$, then $BA = I$. It seems so simple to us, but think about it! For an arbitrary matrix A, B, AB looks nothing like BA! We have two entirely different systems of equations.
  - However, using this fact, basically it is possible to translate the orthogonality relation for the *columns* into the orthogonality relation about the *rows*.
- *Week 3, Lecture 3: All the talk about the exceptional homomorphisms? See p. 52, 61 (the final representation has something to do with an **involution** of trace 2, and is a representation of a quotient group?).
  - So the representation is $\rho : S_4 \twoheadrightarrow S_3 \xrightarrow{\tilde{\rho}} GL_n$, where $\tilde{\rho} : S_3 \to GL_n$ is the representation of ρ corresponding to the character $(2, 0, 1)$.
- *Week 4, Lecture 1: Alternate construction of $R(G)$? See p. 63.
- *Week 4, Lecture 1: Extension of scalars with the representation ring? See p. 64.
  - We don't need to know anything about this stuff.
  - What it is though is basically analogous to extending the real numbers into a subset of the complex numbers by treating every $x \in \mathbb{R}$ as $x + 0i \in \mathbb{C}$. Very trivial, silly concept.
  - There is also such a thing as a **reduction of scalars**.

- *Week 4, Lecture 1: Does multiplying a column vector in the basis $\{\delta_{C_i}\}$ by the character table put it in the basis $\{\chi_{V_i^*}\}$, or vice versa? See p. 66.
  - Derive it for yourself.
- *Week 4, Lecture 2: Isotypical components example. See p. 68.
- **Week 4, Lecture 3: Proof the $\mathbb{C}$ is the only finite-dimensional division algebra? See p. 71.
  - Let A be an arbitrary finite-dimensional division algebra over $\mathbb{C}$.
  - To prove that $A = \mathbb{C}$, we will use a bidirectional inclusion proof.
  - Naturally, $\mathbb{C} \subset A$.
  - To prove the reverse implication, start by letting $a \in A$ be arbitrary.
  - Define the left-multiplication operator $L_a : A \to A$ by $x \mapsto ax$ for all $x \in A$.
  - Recall that A is a complex vector space in addition to being an algebra, the same way a ring is also a group. Thus, L_a is a linear operator on a complex vector space.
  - It follows by the theorem of linear algebra that L_a has an eigenvalue $\lambda \in F = \mathbb{C}$ and corresponding eigenvector $b \in A$.
  - Consequently, by the cancellation lemma,
$$\begin{aligned}L_a b &= \lambda b\\ ab &= \lambda b\\ a &= \lambda\end{aligned}$$
  - Therefore, $a \in A$ implies $a \in \mathbb{C}$, so $A = \mathbb{C}$.
- *Week 6, Lecture 2: Proof that $\sqrt{2}/2$ is not an algebraic integer using Gauss's lemma? See p. 87.
  - Let $\alpha := \sqrt{2}/2$ for the sake of notation.
  - Suppose for the sake of contradiction that α is an algebraic integer.
  - Then there exists a monic polonomial $p(x) \in \mathbb{Z}[x]$ such that $p(\alpha) = 0$.
  - Observe that the minimal polynomial in $\mathbb{Z}[x]$ that annihilates α is $2x^2 - 1$.
  - Thus, by polynomial division,
$$p(x) = q(x) \cdot (2x^2 - 1) + r(x)$$
    for some $q, r \in \mathbb{Q}[x]$ such that $\deg r \leq 2 - 1$.
  - We have that
$$r(\alpha) = p(\alpha) - q(\alpha) \cdot (2\alpha^2 - 1) = 0 - q(\alpha) \cdot 0 = 0$$
  - Additionally, since $r \in \mathbb{Q}[x]$ and $\deg r \leq 1$, we know that $r(x) = ux + v$ for some $u, v \in \mathbb{Q}$.
  - We now prove that $u = v = 0$.
    - Suppose for the sake of contradiction that either u or v was not equal to zero.
    - Combining the previous two claims reveals that
$$\begin{aligned}0 &= r(\alpha)\\ &= u\alpha + v\\ -\frac{v}{u} &= \alpha\end{aligned}$$
    - If $u = 0$, then α is undefined and we have arrived at a contradiction. Thus, $u \neq 0$.
    - Thus, $\alpha \in \mathbb{Q}$. But since $\alpha \notin \mathbb{Q}$ by definition, we have arrived at a contradiction.

        - Therefore, $u = v = 0$.
    - Having established that $r = 0$, we know that $p = (2x^2 - 1)q$, i.e., $2x^2 - 1$ divides p.
    - Now define N to be the least common multiple of the denominators of the coefficients of q.
    - Consider
$$Np = (Nq)(2x^2 - 1)$$
    - It follows by Gauss's lemma that
$$\begin{aligned} c(Np) &= c[(Nq)(2x^2 - 1)] \\ N &= c(Nq) \cdot c(2x^2 - 1) \\ &= 1 \cdot 1 \\ &= 1 \end{aligned}$$
      where c denotes the **content**.
    - But if $N = 1$, then $q \in \mathbb{Z}[x]$, so leading term of p — equal to the product of $2x^2$ and the leading term of q — has a coefficient that is a multiple of 2, i.e., is *not* equal to 1 as is required of a monic polynomial, a contradiction.

- *Week 6, Lecture 3: Questions about Lemma 1 of the proof of Burnside's theorem. See p. 92.
    - The roots $a_1, \dots, a_k$ of the minimal polynomial of the algebraic integer a are known as **conjugate algebraic integers**.
    - The conjugate algebraic integers of a root of unity are also roots of unity.
        - Suppose ε is a root of unity.
        - Then the minimal polynomial of ε is $x^n - 1$ for some $n \in \mathbb{N}$.
        - Naturally, the roots of this polynomial (the conjugate algebraic integers to ε) are all of the other roots of unity of order n.
    - The conjugate algebraic integers of a sum of roots of unity is a sum of roots of unity.
        - It can be shown that the minimal polynomial for $\varepsilon_1 + \varepsilon_2$ is
$$p(x) = \prod_{i,j=1}^{n} (x - \varepsilon_1^i - \varepsilon_2^j)$$
        - Evidently, the above polynomial is symmetric under permutations of $\varepsilon_1^i, \varepsilon_2^j$, and we'd generate the same polynomial with any $\pm\varepsilon_1^i \pm \varepsilon_2^j$ as starting material.
        - Explicit example.
            - $\pm\sqrt{2}$ are conjugate algebraic integers, as solutions to $x^2 - 2$. Similarly, $\pm\sqrt{3}$ are conjugate algebraic integers as solutions to $x^2 = 3$.
            - Thus, we expect the minimal polynomial for $\sqrt{2} + \sqrt{3}$ to be
$$p(x) = (x - \sqrt{2} - \sqrt{3})(x - \sqrt{2} + \sqrt{3})(x + \sqrt{2} - \sqrt{3})(x + \sqrt{2} + \sqrt{3})$$
            - Expanding, we obtain
$$\begin{aligned} p(x) &= (x^2 - (\sqrt{2} + \sqrt{3})^2)(x^2 - (\sqrt{2} - \sqrt{3})^2) \\ &= x^4 - [(\sqrt{2} + \sqrt{3})^2 + (\sqrt{2} - \sqrt{3})^2]x^2 + (\sqrt{2} + \sqrt{3})^2(\sqrt{2} - \sqrt{3})^2 \\ &= x^4 - 10x^2 + 1 \end{aligned}$$
            - Indeed, the above polynomial is a monic polynomial
            - From the definition, this polynomial is evidently also the minimal polynomial for $\sqrt{2} - \sqrt{3}$, $-\sqrt{2} + \sqrt{3}$, and $-\sqrt{2} - \sqrt{3}$.

        ➢ Thus, the conjugate algebraic integers of $\sqrt{2}+\sqrt{3}$ are the four sums of all individual algebraic integers.

  - How do we extend this argument to the case in the problem?? What about when $\varepsilon_1 = -1$ and $\varepsilon_2 = i$ so that simple powers don't access every combination as the $p(x)$ formula does?
  - We know that $\prod_{i=1}^{n} a_i \in \mathbb{Z}$ because of Vieta's formula.
    - In particular, this tells us that $x_1 \cdots x_n$ is equal to the last coefficient in the minimal polynomial which, by definition, is an integer.
  - Don't worry too much about all this, though: Burnside's theorem will no longer be on the final because Rudenko changed his mind.

- *Week 7, Lecture 2: Symmetric polynomials and roots of symmetric polynomials. See p. 101.
- *Week 7, Lecture 2: Word in blackboard picture? See p. 102.
  - "Remain" to show...
- *Week 7, Lecture 2: What is d in the proof of the alternating polynomials theorem? See p. 103.
  - $d = n-1$.
- *Week 8, Lecture 2: What is d in the definition on p. 111.
  - Consider the Specht polynomial corresponding to $(2,2,1)$.

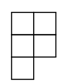


Figure 9.3: Young diagram for $(2,2,1)$.

  - Since $(2,2,1)' = (3,2)$, the Specht polynomial is $(x_1-x_2)(x_1-x_3)(x_2-x_3)\cdot(x_4-x_5)$.
  - Δ_{123} is of degree $3 = \binom{3}{2}$ because looking at the first column of the YD, which corresponds to $\lambda_1' = 3$, out of the 3 boxes, we must choose 2 for each of the three terms $(x_1-x_2),(x_1-x_3),(x_2-x_3)$. Then we just add this to the 2 choose 2 for the second column of the YD.
- *Week 9, Lecture 2: Do we need to be fluent in the techniques you used to expand and reduce the various polynomial powers? How did you do that again?

9.4 The Frobenius Characteristic Map

12/1:
- Proving the theorem.
- The statement is that there exists a function

$$\mathrm{ch}: \underbrace{\bigoplus_{n\geq 0}\mathbb{Q}_{\mathrm{cl}}(S_n)}_{\{f:S_n\to\mathbb{Q}:f(\sigma=\sigma^{-1})=f(i)\}} \to \bigoplus_{n\geq 0}\Lambda_n$$

where $\Lambda_n = \mathbb{Q}[\sigma_1,\sigma_2,\sigma_3,\dots]_{\deg=n} = [\mathbb{Q}[x_1,\dots,x_n]^{S_n}]_{\deg=n}$. Note that by convention, $\Lambda_0 = \mathbb{Q}$. This function is given by

$$\mathrm{ch}(\chi) = \frac{1}{n!}\sum_{\sigma\in S_n}\chi(\sigma)\cdot p_1^{\lambda_1(\sigma)}\cdot p_n^{\lambda_n(\sigma)}$$

where $p_k = x_1^k + x_2^k + \cdots$ and $\lambda_i(\sigma)$ is the number of cycles in σ of length i. Moreover, ch is an isomorphism of $\mathbb{Q}$-algebras. To create a product of V a S_n-rep and W an S_m-rep, we map

$$V\boxtimes W = \mathrm{Ind}_{S_n\times S_m}^{S_{n+m}}(V\otimes W)$$

Proof. To prove that ch is a ring isomorphism, we need...

1. $\boxtimes$ is associative;
2. $\mathrm{ch}(\chi_1 \boxtimes \chi_2) = \mathrm{ch}(\chi_1) \cdot \mathrm{ch}(\chi_2)$;
3. $\mathrm{ch}((n)) = h_n$.

1,2,3 imply the theorem because 2,3 imply that ch is surjective, Λ_n has a $\mathbb{Q}$-basis $h_{\lambda_1} \cdots h_{\lambda_n}$ for $\lambda_1 \leq \cdots \leq \lambda_n$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = n$. For example, for Λ_5, we have $h_1^5, h_1^3 h_2, h_1, h_2^2, h_2 h_3, h_1^2 h_3, h_1 h_4, h_5$. This surjectivity combined with the fact that $\dim \mathbb{Q}_{\mathrm{cl}}[S_n] = \dim \Lambda_n$ implies that ch is an isomorphism of rings.

Last thing: $\mathrm{ch}[(n)] = \frac{1}{n!} \sum p_1^{c_1(\sigma)} \cdots p_n^{c_n(\sigma)}$ where $c_i(\sigma)$ denotes the number of cycles of length i and hence $\sum i c_i = n$. Denote $p_1^{c_1(\sigma)} \cdots p_n^{c_n(\sigma)}$ by $p^{c(\sigma)}$.

Proof. Let

$$
\begin{aligned}
\sum h_n t^n &= \sum \Big(\sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n} t^n \Big) \\
&= \frac{1}{1 - x_1 t} \cdot \frac{1}{1 - x_2 t} \cdots \frac{1}{1 - x_n t} \\
&= \exp\left(\log\left(\prod_{i=1}^n \frac{1}{1 - x_i t} \right) \right) \\
&= \exp\left(\sum_{i=1}^n - \log(1 - x_i t) \right) \\
&= \exp\left(x_1 + \frac{x_1^2 t^2}{2} + \frac{x_1^3 t^3}{3} + \cdots + x_2 + \frac{x_2^2 t^2}{2} + \cdots \right) \\
&= \exp\left(p_1 + \frac{p_2 t^2}{2} + \frac{p_3 t^3}{3} + \cdots \right) \\
&= \prod_{m \geq 1} \exp\left(\frac{p_m t^m}{m} \right) \\
&= *
\end{aligned}
$$

We get the second equality because each $1/(1 - x_i t) = 1 + x_i + x_i^2 t^2 + \cdots$. We need the power series $-\log(1 - t) = t + t^2/2 + \cdots$ and $\exp(t) = 1 + t + t^2/2! + \cdots$. Thus, $\exp(\log(1 - t)) = 1 - t$. Now note that

$$
\begin{aligned}
\sum_{n \geq 0} \mathrm{ch}[(n)] \cdot t^n &= \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in S_n} p_1^{c_1(\sigma)} \cdots p_n^{c_n(\sigma)} \\
&= \sum_{n \geq 0} \frac{t^n}{n!} \left(\sum_{a_1 + 2a_2 + \cdots + n a_n = n} p_1^{a_1} \cdots p_n^{a_n} \right) \cdot \frac{n!}{1^{a_1} a_1! 2^{a_2} a_2! \cdots n^{a_n} a_n!} \\
&= \sum_{\substack{n \geq 0 \\ a_1, \ldots, a_n : a_1 + 2a_2 + \cdots + n a_n = n}} \frac{1}{a_1!} \left(\frac{p_1}{1} \right)^{a_1} t^{a_1} \frac{1}{a_2!} \left(\frac{p_2}{2} \right)^{a_2} t^{a_2} \cdots \frac{1}{a_n!} \left(\frac{p_n}{n} \right)^{a_n} t^{a_n} \\
&= \prod_{k=1}^n \left(\sum_{a_k = 1}^{\infty} \frac{(p_k)^{a_k} t^{a_k k}}{a_k! k a_k} \right) \\
&= \prod_{k \geq 1} \exp\left(\frac{p_k t^k}{k} \right) \\
&= *
\end{aligned}
$$

We're overcounting because we can cyclically permute cycles (i.e., (12) = (21)), hence the correction factor in the second line above.

Note: This exponent/logarithm trick is a common computational trick in combinatorics, varieties, etc. $\square$

Now we prove the part 3, i.e., that $\boxtimes$ is associative. We do this by direct computation.

$$\underbrace{\mathrm{Ind}_{S_{n+m}\times S_\ell}^{S_{n+m+\ell}}\left[\mathrm{Ind}_{S_n\times S_m}^{S_{n+m}}(\chi_1\otimes\chi_2)\right]\otimes\chi_3}_{(\chi_1\boxtimes\chi_2)\boxtimes\chi_3} = \mathrm{Ind}_{s_n\times S_m\times S_\ell}^{S_{n+m+\ell}}(\chi_1\otimes\chi_2\otimes\chi_3)$$

...

Then proving 2 (homomorphism bit) is the hardest. We have

$$\begin{aligned}
\mathrm{ch}(\mathrm{Ind}_{S_n\times S_m}^{S_{n+m}}(\chi_1\otimes\chi_2)) &= \frac{1}{n!}\sum_{\sigma\in S_n}(\mathrm{Ind}_{S_n\times S_m}^{S_{n+m}}\chi_1\otimes\chi_2)(\sigma)\,\underbrace{p_1^{c_1(\sigma)}\cdots p_{n+m}^{c_{n+m}(\sigma)}}_{\psi} \\
&= \left\langle \mathrm{Ind}_{S_n\times S_m}^{S_{n+m}}(\chi_1\otimes\chi_2),\psi\right\rangle_{S_{n+m}} \\
&= \left\langle \chi_1\otimes\chi_2, \mathrm{Res}_{S_n\times S_m}^{S_{n+m}}\psi\right\rangle \\
&= \sum_{\substack{\sigma_1\in S_n\\ \sigma_2\in S_m}}\chi_1(\sigma_1)\chi_2(\sigma_2)p_1^{c_1(\sigma_1)}\cdots p_n^{c_n(\sigma_1)}p_1^{c_1(\sigma_2)}\cdots p_m^{c_m(\sigma_2)} \\
&= \mathrm{ch}(\chi_1)\,\mathrm{ch}(\chi_2)
\end{aligned}$$

We use Frobenius reciprocity somewhere in here. We also have $\psi : S_n \to \Lambda_n$ and $\psi(\tau\sigma\tau^{-1}) = \psi(\sigma)$. $\square$

- After another 10 years of trying to understand the representations of the symmetric group, we'll be here.
- At this point, we can study compact Lie groups.

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