Week 2

The Structure of Representations

2.1 The Tensor Product

10/2:

- Before lecture, I chatted with a few people about tensor products and the exterior and symmetric powers.
  - Patrick: A **tensor** $v \otimes w$ is just an element of a vector space, indexed differently than in a column.
  - Raman: There is no canonical way to transform tensors into column vectors.
- Course logistics.
  - OH: T 5:30-6:30(+) and W 5:30-6:30(+). We can also meet one-on-one.
  - HW is due Thursdays at midnight.
- Today: Constructing new representations from old.
  - Rudenko will skim through tensor products really quickly.
- Reminder: Last time, we talked about how representation theory is really quite simple. If G is a finite group and $F = \mathbb{C}$, there exist a finite set $V_1, \dots, V_s$ of irreps up to isomorphism, and every finite-dimensional representation $V \cong V_1^{n_1} \oplus \cdots \oplus V_s^{n_s}$.
- If V is a representation of G, then there are loads of things we can do with it.
  - We can construct the dual representation V^*.
  - We can construct the representation $V \otimes V$.
  - We can construct symmetric powers.
  - We can construct wedge powers.
  - There are more, but this is enough for now.
- Even when we take a very simple group and representation, there are some very interesting things that can fall out.
  - Example: If you take the symmetric powers of S_3, as in the homework, you get something really interesting.
- Now, we go to linear algebra.
- Let V, W be vector spaces over a field F. How do we produce a new vector space out of these?
- $\mathrm{Hom}_F(V, W)$ is the vector space of linear maps $F : V \to W$!
  - $\dim = (\dim V)(\dim W)$.

- Can we make $\mathrm{Hom}_F(V,W)$ into a representation of G? Yes!

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ {\scriptstyle\rho_V(g)}\downarrow & & \downarrow{\scriptstyle\rho_W(g)} \\ V & \xrightarrow[gL]{} & W \end{array}$$

Figure 2.1: Commutative diagram, linear maps space representation.

  - Suppose that V,W are G-reps, which gives us $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$.
  - Suppose also that we have $L \in \mathrm{Hom}_F(V,W)$.
  - Now infer from the commutative diagram that it will work to define $gL = \rho_W(g) \circ L \circ \rho_V(g)^{-1}$.
  - This is pretty standard.
- Recall that there is a different space $\mathrm{Hom}_G(V,W)$ of morphisms of G-representations (see Figure 1.2 and the associated discussion).
  - This is a very very small subspace of $\mathrm{Hom}_F(V,W)$.
- Special case of the above construction: **Dual representation**.
  - Consider $\mathrm{Hom}_F(V,F)$. This the **dual vector space**.
  - Basic fact 1: Let $e_1,\dots,e_n$ be a basis of V. Then V^* also has a corresponding basis $e^1,\dots,e^n$, known as its **dual basis**.
    - Computing coordinates already depends on a basis, and having bases is super nice.
    - Corollary: $\dim V = \dim V^*$.
    - This is the first time **canonical** comes into linear algebra. Canonical (nobody understands what it means) basically means that something doesn't depend on choices.
    - In particular, V,V^* are isomorphic because they have the same dimension, but for no more natural reason. They can be the same representation, or they can be different.
  - Basic fact 2: If V is finite-dimensional, then $(V^*)^* \cong V$. The formula for this isomorphism is canonical, because it does not depend on a choice of basis. In particular, choose the map $V \to (V^*)^*$ sending v to the map sending $\varphi \in V^*$ to $\varphi(v)$.
  - If V is infinite dimensional, none of this is true and you are in the realm of functional analysis.
  - Ok, so all of this was good information about the dual *space*, but what is the dual *representation*?? Does it matter, and do we need to know for now?
- **Dual vector space** (of V): The vector space defined as follows, given that V is a vector space over F. *Denoted by* $\boldsymbol{V^*}$. *Given by*

$$V^* = \mathrm{Hom}_F(V,F)$$

- **Dual basis** (of V^* to $e_1,\dots,e_n$): The basis defined as follows for $i=1,\dots,n$, where $e_1,\dots,e_n$ is a basis of V. *Denoted by* $\mathbf{e^1,\dots,e^n}$. *Given by*

$$e^i(x_1e_1 + \cdots + x_ne_n) = x_i$$

- We now move onto the tensor product.
  - The tensor product is very hard to understand. If you learn about it and you feel you don't understand it, that's typical; nobody understands it at first.
  - For now, we'll discuss two ways of thinking about tensor products that won't bring us any comfort.
- Let V,W be two vector spaces over a field F.

- Abstract definition of the tensor product.
  - We have discussed maps from $V \to W$, but there is another related space.
  - Indeed, we can look at the space of bilinear maps from $V \times W \to F$.
    - Example: A map $f : V \times W \to F$ that satisfies the constraints $f(\lambda v, w) = \lambda f(v, w)$, $f(v_1 + v_2, w) = f(v_1, w) + f(v_2, w)$, and likewise for the second index. Recall that this is a **bilinear map**.
  - Let V have basis $e_1, \dots, e_n$ and W have basis $f_1, \dots, f_m$.
  - Notice that every bilinear map f can be defined as a linear combination of the $f(e_i, f_j)$. In other words, the $f(e_i, f_j)$ form the basis of a function space.
    - This "bilinear maps space" has dimension nm.
  - Now, one way to understand a tensor product: Is this "bilinear maps space" actually some other space? It is! It is $(V \otimes W)^*$.
  - Bilinear maps are linear maps from where? From $V \otimes W$!
- **Bilinear** (map): A function $f : V \times W \to Z$ that satisfies the following constraints, where V, W, Z are vector spaces over F, $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, and $\lambda \in F$. *Constraints*

$$
\begin{aligned}
f(v_1 + v_2, w) &= f(v_1, w) + f(v_2, w) & \qquad f(\lambda v, w) &= \lambda f(v, w) \\
f(v, w_1 + w_2) &= f(v, w_1) + f(v, w_2) & \qquad f(v, \lambda w) &= \lambda f(v, w)
\end{aligned}
$$

- We now look at a much more elementary definition of the tensor product.
- Explicit definition of the tensor product.
  - $V \otimes W$ is equal to a huge vector space with basis consisting of pairs of elements (v, w). Even if V, W are one dimensional, this is like all pairs of real numbers; it's huge. Then, we quotient it by the space of all elements satisfying $\lambda(v, w) = (\lambda v, w) = (v, \lambda w)$, $(v_1 + v_2, w) = (v_1, w) + (v_2, w)$, and the like. This forces these relationships to be true.
    - Clarify this methodology??
    - Essentially, this allows us to treat tensor multiplication much like real multiplication, endowing the operation with distributivity, etc.
    - For example, the rule $(v_1 + v_2, w) = (v_1, w) + (v_2, w)$ becomes, in tensor product notation, $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$.
  - Example: Suppose $V = \mathbb{C}e_1 + \mathbb{C}e_2$. We want to look at $V \otimes V$.
    - A priori[1], it's spanned by $(ae_1 + be_2) \otimes (ce_1 + de_2) = ace_1 \otimes e_1 + ade_1 \otimes e_2 + bce_2 \otimes e_1 + cde_2 \otimes e_2$.
    - Thus, $V_1 \otimes V_2$ has 4-element basis $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$.
- These two definitions constitute a first approximation to what the tensor product is.
- Takeaway: What is true in general is that if V has basis $e_1, \dots, e_n$ and W has basis $f_1, \dots, f_m$, then $V \otimes W$ has basis $e_i \otimes f_j$ ($i = 1, \dots, n$ and $j = 1, \dots, m$).
- Having discussed the tensor product of vector spaces, let's think about the tensor product of *representations*.
  - Suppose $g : V \to V$ and $g : W \to W$.
    - We're starting to make notation sloppy.
  - How does $g : V \otimes W \to V \otimes W$? Well, we just send $v \otimes w \mapsto (gv) \otimes (gw)$.
    - Why is this map well-defined?

[1] I.e., it follows from some logic. In particular, it follows from the logic that any element $v \in V$ is of the form $v = ae_1 + be_2$, so of course all $v \otimes v$ must be of the given form for choices of a, b, c, d.

  - We invoke the **universal property of the tensor product operation**.
  - This guarantees us that given g — which is effectively a map from $V \times W \to V \otimes W$ as defined — there nevertheless exists a complete extension $\tilde{g} : V \otimes W \to V \otimes W$.
- As a matrix, this map is pretty strange!
  - Example: Let $g : V \to V$ be a 2×2 matrix. What is the matrix of $g : V \otimes V \to V \otimes V$?
  - If
$$\rho_V(g) = g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} =: A$$
then we have
$$\begin{aligned} g(e_1 \otimes e_1) &= ge_1 \otimes ge_1 \\ &= (ae_1 + ce_2) \otimes (ae_1 + ce_2) \\ &= a^2 e_1 \otimes e_1 + ace_1 \otimes e_2 + ace_2 \otimes e_1 + c^2 e_2 \otimes e_2 \end{aligned}$$
  - Evaluating similarly for all basis vectors, we get a very curious block matrix:
$$\begin{array}{c c} & \begin{array}{cccc} e_1 \otimes e_1 & e_1 \otimes e_2 & e_2 \otimes e_1 & e_2 \otimes e_2 \end{array} \\ \begin{array}{c} e_1 \otimes e_1 \\ e_1 \otimes e_2 \\ e_2 \otimes e_1 \\ e_2 \otimes e_2 \end{array} & \begin{bmatrix} a^2 & ab & ab & b^2 \\ ac & ad & bc & bd \\ ac & bc & ad & bd \\ c^2 & cd & cd & d^2 \end{bmatrix} \end{array} = \left[\begin{array}{c|c} aA & bA \\ \hline cA & dA \end{array}\right]$$
  - Notice how, for example, this takes the tensor $e_1 \otimes e_1$, represented as $(1, 0, 0, 0)$, to the tensor $a^2 e_1 \otimes e_1 + ace_1 \otimes e_2 + ace_2 \otimes e_1 + c^2 e_2 \otimes e_2$, represented as (a^2, ac, ac, c^2).
  - Does this construction imply a canonical way to convert from tensors to column vectors??
- Classically, this is called the **Kronecker product** of two matrices.
- People discovered all of this stuff before they unified it as tensor math.

• **Universal property of the tensor product operation**: For every bilinear map $h : V \times W \to Z$, there exists a *unique* linear map $\tilde{h} : V \otimes W \to Z$ such that $h = \tilde{h} \circ \otimes$.

$$\begin{tikzcd} V \times W \arrow[r, "\otimes"] \arrow[dr, "h"'] & V \otimes W \arrow[d, "\tilde{h}"] \\ & Z \end{tikzcd}$$

Figure 2.2: Universal property, tensor product operation.

Proof. See the solid explanation linked here. Alternatively, here's my write up.

Let $V = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$, $W = \mathbb{C}f_1 \oplus \cdots \oplus \mathbb{C}f_m$, Z, and $h : V \times W \to Z$ be arbitrary. Define $\tilde{h} : V \otimes W \to Z$ by
$$\tilde{h}(e_i \otimes f_j) := h(e_i, f_j)$$
for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Since a linear map is wholly defined by its action on the basis of its domain, this set of equations suffices to define $\tilde{h}$ on all of $V \otimes W$.

Existence: To prove that $\tilde{h}$ satisfies the "universal property," it will suffice to show that $h = \tilde{h} \circ \otimes$. Let

$(v,w) \in V \times W$ be arbitrary, and suppose $v = \sum_{i=1}^n a_i e_i \in V$, and $w = \sum_{i=1}^n b_i f_i \in W$. Then

$$\begin{aligned}
[\tilde{h} \circ \otimes](v,w) &= \tilde{h}(v \otimes w) \\
&= \sum_{i=1}^n \sum_{j=1}^m a_i b_i \tilde{h}(e_i \otimes f_i) \\
&= \sum_{i=1}^n \sum_{j=1}^m a_i b_i h(e_i, f_j) \\
&= h(v,w)
\end{aligned}$$

as desired.

Uniqueness: Now suppose $\tilde{g} : V \otimes W \to Z$ also satisfies the "universal property," that is, $h = \tilde{g} \circ \otimes$. Then by definition,

$$\tilde{h}(e_i \otimes f_j) = h(e_i, f_j) = \tilde{g}(e_i \otimes f_j)$$

for $i = 1, \dots, n$ and $j = 1, \dots, m$. But since a linear map is wholly defined by its action on the basis of its domain, it follows that $\tilde{h} = \tilde{g}$, as desired. $\square$

- **Kronecker product** (of A, B): The matrix product defined as follows. *Denoted by* $\boldsymbol{A} \otimes \boldsymbol{B}$. *Given by*

$$A \otimes B = n \overset{n}{\begin{bmatrix} A \end{bmatrix}} \otimes m \overset{m}{\begin{bmatrix} B \end{bmatrix}} = nm \overset{nm}{\begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix}}$$

  - The Kronecker product is *not* commutative, but the matrices you get are related by conjugacy and by commuting the columns.

- Vector spaces of the same dimension are all alike, but vector space representations are very interesting. By the end of this course, we'll understand what's going on.

- How we understand tensor stuff: Look at the abstract definition, look at the concrete definition, look at 5 examples, and then go in a circle. Repeat again and again until it makes sense.

- He's just trying to tell us all relevant words so that they will fit together later.

- Fact: If V, W finite-dimensional, $\text{Hom}_F(V,W) \cong V \otimes W^*$.
  - Tensor products are very nice to construct maps from.
  - Let's construct a reverse map, then.
  - Take $\alpha \otimes w \in V^* \otimes W$, where $\alpha : V \to F$ by definition. Send $\alpha \otimes w$ to the map $v \mapsto \alpha(v)w$. This is a *canonical* map!! We can show that they span everything.
    - For example, if we want to choose $\alpha \otimes w$ mapping to the matrix with a 1 in the upper left-hand corner and zeroes everywhere else, let $w = e_1 \in W$ and let $\alpha = e^1 \in V^*$.
    - We can do similarly for all other such matrices, mapping this basis of $\text{Hom}_F(V,W)$ to $e^i \otimes e_j$ ($i = 1, \dots, n$ and $j = 1, \dots, m$).
    - Note that this also allows us to define a (noncanonical) inverse map.
  - This inverse map from $\text{Hom}_F(V,W) \to V^* \otimes W$ is clearly a bit harder to work out.
  - Hidden in this story is why trace is invariant under conjugation (see below discussion).

- If we now take $\text{Hom}_F(V,V)$, then this is isomorphic to $V^* \otimes V$. There is a very natural map from these isomorphic spaces to F defined by the trace, and/or $\alpha \otimes v \mapsto \alpha(v)$. We can prove this. And this is canonical, as well. This is why the main property of the trace is that it's invariant under conjugation. This fact is hidden in the story very nicely.

- Tensor products are hard, it will be a pain, we will understand them very well, but it will not be nice for now.
- Symmetric products and wedge powers will be discussed briefly next time.
  - There is a nice description in Serre (1977) that we can use for the homework.
- Extra homework: Please read about tensor products in whatever textbook you like, try some examples, and repeat.

2.2 Office Hours (Rudenko)

10/3:
- Problem 2a:
  - $\Lambda^2 V$ is *exterior powers.*
  - The exact canonical isomorphism we need is briefly discussed on Fulton and Harris (2004, p. 473).
  - I.e., we have to construct isomorphisms between the structures that don't rely on the choice of any basis. Recall the classic example of $V \cong V^{**}$, as explained in the well-written MSE post "basic difference between canonical isomorphism and isomorphims." Recall that the isomorphism from $V \to V^*$ defined by sending each element of the basis of V to the corresponding element of the dual basis of V^* is *not* canonical because *it involves choosing bases.* Definitions of canonical maps are available in MATH20510Notes, p. 2.
  - From a quick look at this, it looks like the proof may be analogous to the classic middle-school algebra identity $(v+w)^2 = v^2 + vw + w^2$.
  - The second exterior power $\Lambda^2 V$ of a finite-dimensional vector space V is the dual space of the vector space of alternating bilinear forms on V. Elements of $\Lambda^2 V$ are called 2-vectors.
- Problem 2b:
  - $S^2 V$ is *symmetric powers.*
  - The exact canonical isomorphism we need is briefly discussed on Fulton and Harris (2004, p. 473).
- Problem 3a:
  - This is the determinant of the multiplication table, in relation to that theorem that you showed us at the end of the first class? Yep!
- Problem 3b:
  - So a circulant matrix is a matrix like the multiplication table from (a)? Yep!
  - Is $\zeta = e^{2\pi i/n}$? Sort of. It can be any n^{th} root of unity.
- Problem 4d:
  - We'll cover higher symmetric powers in class tomorrow.
  - However, it basically just means that we're now working with elements of the form $e_1 \otimes e_2 \otimes e_3 \in S^3 V$ and on and on.
- Problem 5a:
  - Is $V^\vee = V^*$? Yes. This is "vee check," and is a notation that some people prefer.
- Problem 5b:
  - Is "tr" the trace function of the linear map corresponding to L? Yes.
  - What is L?

- An element of $V \otimes V^*$ is a linear combination of elements of the form $v \otimes \alpha$, not necessarily just one of these "decomposable" products.
- There is an isomorphism $V \otimes V^* \cong \operatorname{Hom}(V)$.
- Consider the matrix
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
It sends $e_1 \mapsto e_1$ and $e_2 \mapsto 0$. Thus, it is well-matched with $e_1 \otimes e^1$, which also grabs e_1 (with e^1) and sends it to e_1.
- Consider the matrix
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
It sends $e_1 \mapsto 0$ and $e_2 \mapsto e_1$. Thus, it is well-matched with $e_1 \otimes e^2$, which also grabs e_2 (with e^2) and sends it to e_1.
- In full,
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ae_1 \otimes e^1 + be_1 \otimes e^2 + ce_2 \otimes e^1 + de_2 \otimes e^2$$
- This map *is* canonical! This is because the bases must be chosen to even begin talking about matrices.
- If you change the matrix, the bases change, too??
- Takeaway: We have to walk backwards from matrix to linear transformation to representation in $V \otimes V^*$ to a scalar in F.

- Problem 5c:
  - So trace of such a map is equal to the dimension of its image? Yes.

2.3 Wedge and Symmetric Powers

10/4:
- OH slightly later today at 5:45-6:45 PM.
- Recap: Last time, we built new reps from old.
  - This stuff can't be learned in 1.5 lectures; he can point us around, but we have to learn it ourselves.
- Tensor product review.
  - Given V, W, make $V \otimes_F W$.
  - This vector space is hard to describe directly, so we more often talk about its dual $(V \otimes W)^*$ because this is actually easier to describe.
  - If you want to work with $V \otimes W$ hands-on, you can do the following.
    - Start with the following easy-to-work-with vector space: The (probably infinite-dimensional) vector space where each $v \otimes w$ is a basis vector for all $v \in V$ and $w \in W$.
    - Then quotient it by relations to force them to hold in the final space.
  - Here's an example of this construction.
    - Let $V = W$ be the one-dimensional vector space over the finite field $F_2 = \mathbb{Z}/2\mathbb{Z}$.
    - Thus, the elements of V are $\{0, 1\}$ (which is, literally, all linear combinations $a0 + b1$ where $a, b \in F_2$ as well; this hearkens back to V's definition as an F_2-module).
    - Then the easy-to-work-with vector space we're talking about is the 4-dimensional **free** vector space $U = \operatorname{span}(0 \otimes 0, 0 \otimes 1, 1 \otimes 0, 1 \otimes 1)$.

- Note that in this space, for example, $(0+1)\otimes 0 \neq 0\otimes 0 + 1\otimes 0$; representing the basis as column vectors, this is equivalent the obvious observation that

$$\begin{bmatrix}0\\0\\1\\0\end{bmatrix} \neq \begin{bmatrix}1\\0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$

- But we want such relationships to hold true in our conceptual "tensor product space." Thus, we quotient it by the subspace spanning all elements of the form $(a+b)\otimes c - a\otimes c - b\otimes c$.
- By direct computation, this subspace is $\operatorname{span}(0\otimes 0, 0\otimes 1)$:

$$\begin{aligned}
&(0+0)\otimes 0 - 0\otimes 0 - 0\otimes 0 = -0\otimes 0 && (0+0)\otimes 1 - 0\otimes 1 - 0\otimes 1 = -0\otimes 1\\
&(0+1)\otimes 0 - 0\otimes 0 - 1\otimes 0 = -0\otimes 0 && (0+1)\otimes 1 - 0\otimes 1 - 1\otimes 1 = -0\otimes 1\\
&(1+1)\otimes 0 - 1\otimes 0 - 1\otimes 0 = 0\otimes 0 && (1+1)\otimes 1 - 1\otimes 1 - 1\otimes 1 = 0\otimes 1
\end{aligned}$$

  Note that once we've considered $(a+b)\otimes c$, we don't need to consider $(b+a)\otimes c$ because of the commutativity of addition in V. That is, it is axiomatic that $a+b=b+a$ for all $a,b\in V$. Additionally, in the last line above, we are using the facts that $1+1=2=0$ in F_2 and $a\otimes b + a\otimes b = 2a\otimes b = 0$ in any F_2-module to simplify the expressions.
- Similarly, the subspace corresponding to $a\otimes(b+c) - a\otimes b - a\otimes c$ is $\operatorname{span}(0\otimes 0, 1\otimes 0)$. Thus, altogether, we quotient out the subspace $X = \operatorname{span}(0\otimes 0, 0\otimes 1, 1\otimes 0)$. This leaves us with a 1-dimensional $V\otimes V$, as expected for the tensor product of two one-dimensional vector spaces. It is interesting to note that the one vector we didn't quotient out $(1\otimes 1)$ is analogous to $e_1\otimes e_1$ since $e_1\in V$ might as well be defined $e_1 := 1$.
- Now let's see how well this quotienting worked. First off, a bit of notation: let $\pi : U \to V\otimes V$ be the projection $\pi : v \mapsto v + X$, and denote elements $\pi(v_1\otimes v_2)\in V\otimes V$ by $v_1\otimes_\pi v_2$ for now to differentiate them from elements of U.
- Let $(0+1)\otimes_\pi 0 = (0+1)\otimes 0 + X$ be an element of the quotient space $V\otimes V$. Certainly, the elements $0\otimes_\pi 0$ and $1\otimes_\pi 0$ are also elements of this quotient space. Moreover, we can fairly form the linear combination $(0+1)\otimes_\pi 0 - 0\otimes_\pi 0 - 1\otimes_\pi 0$. However, this element lies in the quotiented-out subspace X. Thus,

$$(0+1)\otimes_\pi 0 - 0\otimes_\pi 0 - 1\otimes_\pi 0 = [(0+1)\otimes 0 - 0\otimes 0 - 1\otimes 0] + X = 0 + X = 0$$

- But

$$(0+1)\otimes_\pi 0 - 0\otimes_\pi 0 - 1\otimes_\pi 0 = 0 \quad\Longrightarrow\quad (0+1)\otimes_\pi 0 = 0\otimes_\pi 0 + 1\otimes_\pi 0$$

  as desired.
- Note that this construction also gives us nice things like $0\otimes_\pi 0 = 0$, $0\otimes_\pi 1 = 0$, etc. which were not true in U! It should not be concluded, though, that all we need to quotient out of U for any V is $\operatorname{span}(0\otimes 0, 0\otimes v, v\otimes 0)$ for every $v\in V$; indeed, $V=\mathbb{R}$, for example, will contain

- If V has basis $e_1,\dots,e_n$ and W has basis $f_1,\dots,f_m$, then $e_i\otimes f_j$ is a basis of $V\otimes W$.
- Interesting fact 1: If V,W are finite dimensional, $V^*\otimes W \cong \operatorname{Hom}(V,W)$.
- If we want to work with the tensor product in practice in *rep theory*, the only thing we need to know is the basis of the tensor product space, which can tell us how any map $\rho(g)$ acts on both sides of a $v\otimes w\in V\otimes W$. From here, we recover the Kronecker product of matrices.
- So many things are explained by the concept of tensor products!
- A tensor in *physics* is something with lots of indices that changes in some way.
  - It does come from the math concept.
  - We'll get a huge basis because we have a massive product like $V\otimes\cdots\otimes V\otimes V^*\otimes\cdots\otimes V^*$.

- **Free** (vector space): A vector space that has a basis consisting of linearly independent elements.
  - Example: Think of $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ as a $\mathbb{C}$-module. A free version $F(V)$ of V is infinite dimensional with every $v \in V$ a linearly independent basis vector. Elements of $F(V)$ are of the form $a_1v_1 + \cdots + a_kv_k$ for $a_1, \dots, a_k \in \mathbb{C}$ and $v_1, \dots, v_k \in V$. If $u = v + w$ where $u, v, w \in V$ are all nonzero, then $u \neq v + w$ in $F(V)$ because they are all linearly independent basis vectors.
  - Example: What we formally start with in the example above is $V \times V$, the free F_2-module not the Cartesian product vector space V^2.
  - A terrific explanation of free vector spaces is available here.
- Last 2 useful notions: Wedge powers and symmetric powers.
  - Again, it's much easier to think about the dual space.
- Consider the space $V^{\otimes n}$ (dimension $(\dim V)^n$).
  - $(V^{\otimes n})^*$ are **polylinear** maps $f : V^n \to F$.
    - Note: By contrast, $(V^n)^*$ is the space of all *linear* maps $f : V^n \to F$.
    - This distinction is subtle but important. Note, for instance, that $\dim V^{\otimes n} \neq \dim V^n$ and likewise for the duals.
    - The distinction comes out fully when considering that if, for example, $V = \mathbb{R}^3$, then $V^2 \cong \mathbb{R}^6$ and any map in $(V^2)^*$ is determined by its action on $(e_1, 0), (e_2, 0), (e_3, 0), (0, e_1), (0, e_2), (0, e_3)$. By contrast, any map in $(V^{\otimes 2})^*$ is determined by its action on $(e_1, e_1), (e_1, e_2), (e_1, e_3), (e_2, e_1), (e_2, e_2), (e_2, e_3), (e_3, e_1), (e_3, e_2), (e_3, e_3)$.
    - Important note: What $(V^{\otimes 2})^*$ does is consider these nine elements of V^2 as the basis of another space. This is what it truly means when we say "a bilinear map on V^2 is a linear map on $V^{\otimes 2}$."
    - Takeaway: Polylinearity changes the basis upon which a function $f : V^n \to F$ fundamentally acts.
  - A polylinear map may be **symmetric**, **antisymmetric**, or[2] neither.
  - These maps form vector spaces and the dimension is actually pretty meaningful.
- **Symmetric** (polylinear map): A polylinear map $f : V^n \to F$ that satisfies the following property. *Constraint*

$$f(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = f(v_1, \dots, v_n)$$

- **Antisymmetric** (polylinear map): A polylinear map $f : V^n \to F$ that satisfies the following property. *Constraint*

$$f(v_{\sigma(1)}, \dots, v_{\sigma(n)}) = (-1)^\sigma f(v_1, \dots, v_n)$$

- Suppose you take $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$[3].
  - Consider a symmetric polylinear map $f : V \times V \times V \to \mathbb{C}$.
  - To compute it, we'll need the action of f on the basis of V^3. In particular, we'll need...

$$f(x_1e_1 + y_1e_2, x_2e_1 + y_2e_2, x_3e_1 + y_3e_2) = x_1x_2x_3f(e_1, e_1, e_1) + x_1x_2y_3f(e_1, e_1, e_2) + \cdots$$

    - Somewhere in there, you'll also have a $x_1y_2x_3f(e_1, e_2, e_1)$ term as well.
    - However, because f is symmetric, you know by symmetry that these "bases" are the same, so you don't count them as 2 towards the dimension but as 1.
    - Thus, $\dim = 4$ for symmetric maps.

[2]This is an exclusive "or."

[3]Note that this notation allows you to define a vector space *and* its basis in one go! I.e., the alternative is saying "Let V be a complex vector space with basis e_1, e_2."

- What about antisymmetric maps?
- Suppose $g : V^3 \to \mathbb{C}$ is an antisymmetric polylinear map.
  - Consider $g(e_1, e_1, e_1)$. Suppose you apply (12). Interchanging the first two indices (for instance) obviously won't do anything, so we'll get

$$\begin{aligned} g(e_1, e_1, e_1) &= (-1)^{(12)} g(e_1, e_1, e_1) \\ g(e_1, e_1, e_1) &= -g(e_1, e_1, e_1) \\ 2g(e_1, e_1, e_1) &= 0 \\ g(e_1, e_1, e_1) &= 0 \end{aligned}$$

  - But what about $g(e_1, e_1, e_2)$? We could apply (23) and get $g(e_1, e_2, e_1)$, right? So it appears that we would just be shrinking two options into one. Technically, this is true, but what's more important is that applying (12) again yields the same thing, meaning that $g(e_1, e_1, e_2) = g(e_1, e_2, e_1) = 0$.
  - And thus, since V has dimension 2 but g takes three vectors, any argument submitted to g will always be linearly dependent. Thus, $g = 0$ and, in fact, the space of antisymmetric maps on V^3 has dimension 0.
- Note: It's not always a rule that $V^{\otimes m} \cong S^m V \oplus \Lambda^m V$.

- Mathematically, there's a more natural object to work with than symmetric and antisymmetric maps.
  - Wedge powers and symmetric powers!
  - Given V and $n \in \mathbb{N}$, we can construct $S^n V$ and $\Lambda^n V$. $(S^n V)^*$ is symmetric polylinear maps taking n arguments from V. $(\Lambda^n V)^*$ is antisymmetric polylinear maps taking n arguments from V.
- How about a concrete way to see these? We can relate them to tensor powers.
  - Take a tensor power $V^{\otimes n}$, then look at those tensors which are symmetric and antisymmetric under permutation.
  - Example: Let V be the same as before. Then $V^{\otimes 2}$ has $\dim = 4$.
    - Take as basis elements for $S^2 V$ those that don't change when you change the coordinates.
    - Take as basis elements for $\Lambda^2 V$ those that flip sign when you change the coordinates.
    - In this case, the basis of $V^{\otimes 2}$ is $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$. The basis of $S^2 V$ will be $e_1 \otimes e_1, e_1 \otimes e_2 + e_2 \otimes e_1, e_2 \otimes e_2$. The basis of $\Lambda^2 V$ will be $e_1 \otimes e_2 - e_2 \otimes e_1$. Notice that these bases are identical (up to scaling) with those in Serre (1977) and those produced by applying the **symmetrization** and **alternation** operators to the basis of $V^{\otimes 2}$.
  - $S^2 V$ and $\Lambda^2 V$ direct sum because the dimensions match and they don't intersect, so we're good to go!
  - Everything we're doing is representations, so $g(v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n$.
- Relating this to something we've seen, but that's a little confusing.
  - The product notation is suggestive for symmetric vectors; you can commute $e_1 \cdot e_2 \in S^2 V$, for instance.
  - This allows us to, for example, shrink $e_1 \otimes e_1$ to $2e_1^2$[4], but $e_1 \otimes e_2 + e_2 \otimes e_1$ only to $e_1 \cdot e_2$.
  - Note that $e_1 \wedge e_2 = e_1 \otimes e_2 - e_2 \otimes e_1$ by definition.
  - Fact/exercise: Let V be a vector space of dimension n. V^* is the dual space, but it is also a function space. If $V = \mathbb{R}^k$, its a space of *functions from the blackboard*.
    - Note that $(\Lambda^k V)^* = \Lambda^k V^*$.

[4] Why the 2 coefficient??

- S^nV^* is homogeneous polynomials of degree n.
- You can take higher degree polynomials and just keep pushing through.
  - Ask about this??
- Wedge powers now.
- By convention, $\Lambda^0V = F$ and $\Lambda^1V = V$. But then you get to Λ^2V and Λ^3V. They grow but then shrink down as the power approaches $\dim V$.
- Truth: The dimension of wedge powers Λ^iV is $\binom{k}{i}$ for $\dim V = k$. Figuring out why this is the case is another good exercise.

- An interesting connection between wedge powers and the determinant.
  - Let $V = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$.
  - Recall that Λ^nV^* is the space of antisymmetric polylinear functions $V \times \cdots \times V \to F$ taking n arguments from V, and it has a single basis vector $e^1 \wedge \cdots \wedge e^n$.
  - Let $v_1 = \sum a_{i1}e_i$, $v_2 = \sum a_{i2}e_i$, etc.
  - Let $f \in \Lambda^nV^*$, so that f is an alternating polylinear map that takes n arguments.
  - Since f is polylinear, we have that
$$f(v_1, \dots, v_n) = \sum_{i_1,\dots,i_n=1}^{n} a_{i_1 1} \cdots a_{i_n n} f(e_{i_1}, \dots, e_{i_n})$$
  - Because of antisymmetry, we need only look at elements where the indices are all different. Thus, the above equals
$$\sum_{\sigma \in S_n} a_{\sigma(1)1} \cdots a_{\sigma(n)n} f(e_{\sigma(1)}, \dots, e_{\sigma(n)})$$
  - Additionally, $f(e_{\sigma(1)}, \dots, e_{\sigma(n)}) = (-1)^\sigma f(e_1, \dots, e_n)$ for any $\sigma \in S_n$. Moreover, $f(e_1, \dots, e_n) \in \mathbb{C}$ by definition, so define a constant $\lambda := f(e_1, \dots, e_n)$. Thus, the above equals
$$\lambda \sum_{\sigma \in S_n} a_{\sigma(1)1} \cdots a_{\sigma(n)n}$$
  - But the term following the λ is just the determinant of the $n \times n$ matrix (a_{ij}). Thus, all said,
$$f(v_1, \dots, v_n) = \lambda \det(v_1 \mid \cdots \mid v_n)$$
  - Implication: Wedge powers are something like the determinant.
    - In particular, because Λ^nV^* has only a single basis vector as mentioned above, $f = \lambda e^1 \wedge \cdots \wedge e^n$. It follows that $e^1 \wedge \cdots \wedge e^n = \det$.
  - Takeaway: Wedge powers are something interesting; there's a reason to study them.

- The basis of the wedge powers consists of wedge monomials $e_{j_1} \wedge \cdots \wedge e_{j_i}$. Moreover, no need to have the same list twice, so choose some way of indexing them, e.g., increasing indexes.
  - This is why we do *increasing* bases! There's no particular reason, it's just an arbitrary way of making sure we don't do the same thing twice! We could just as well choose decreasing or any other means of guaranteeing that we don't have duplicates.

- Now let's relate all of this exterior and symmetric product stuff back to representation theory.
  - Let $V = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$.
  - Let $G \circlearrowright V$ via the homomorphism $G \to GL(V) \cong GL_n(\mathbb{C})$.

  - Focusing more on the *matrix* aspect this time, note that under this homomorphism, $g \mapsto A_g$ subject to the homomorphism constraints.
  - Consider the set $\{A_{g_1}, \dots, A_{g_k}\}$ of all matrices in the image of the homomorphism. If we transpose all of them, will they still obey the homomorphism constraints?
    - Nope!
    - Indeed, if we do this, we'll get in trouble. More specifically, transposition is not a representation because $A_{g_1}^T A_{g_2}^T \neq A_{g_1 g_2}^T = A_{g_2}^T A_{g_1}^T$.
  - It's the same story with inverses.
  - *However*, combining the two operations, we get

$$(A_{g_1 g_2}^T)^{-1} = (A_{g_1}^T)^{-1}(A_{g_2}^T)^{-1}$$

    - This is exactly when we take a representation and then go to the dual[5].
  - This will be on next week's homework!
  - Takeaway: This is an application of $\Lambda^j V^*$ to representation theory, $j \neq k, n$.
- Another relation: An application of $\Lambda^n V^*$ to representation theory.
  - Suppose we have a representation $G \circlearrowright V$ that we want to flatten into $G \circlearrowright \mathbb{C}$. How can we turn a relation between a group of matrices into a relation between a group of numbers?
  - Use the determinant!
  - Indeed, we already know that

$$\det(A_e) = 1 \qquad \det(A_{g_1 g_2}) = (\det A_{g_1})(\det A_{g_2}) \qquad \det\left(A_{g^{-1}}\right) = \det(A_g)^{-1}$$

  - In particular, we make formal the transition $G \to GL_j(\mathbb{C}) \to \mathbb{C}$ with the **top wedge power** $\Lambda^n V^*$.
- A last note.
  - Don't think that we're limited to top wedge powers.
  - Recall that we can define tensor products of matrices via the Kronecker product. Well, we can prove that

$$A_{g_1 g_2}^{\otimes 2} = A_{g_1}^{\otimes 2} A_{g_2}^{\otimes 2}$$

    and the like as well!
  - Similarly, we can define Λ^2 of a matrix.
  - We'll get into some weird Kronecker product stuff again, but we can sort through it.
- Plan for Friday and next time.
  - Prove the theorem that every representation is a sum of irreducible representations.
  - He will use projectors.
  - Then a horror story.
  - Then associative algebra.

2.4 S Chapter 1: Generalities on Linear Representations

From Serre (1977).

[5] Relation to MATH 20510 when we discussed dual matrices and pullbacks of matrices.

Section 1.5: Tensor Product of Two Representations

- **Tensor product** (of V_1, V_2): The vector space W that (a) is furnished with a map $V_1 \times V_2 \to W$ sending $(x_1, x_2) \mapsto x_1 \cdot x_2$ and (b) satisfies the following two conditions.
  (i) $x_1 \cdot x_2$ is bilinear.
  (ii) If (e_{i_1}) is a basis of V_1 and (e_{i_2}) is a basis of V_2, the family of products $e_{i_1} \cdot e_{i_2}$ is a basis of W.

  Denoted by $\boldsymbol{V_1 \otimes V_2}$.
  - It can be shown that such a space exists and is unique up to isomorphism (see proof here).
- This definition allows us to say some things quite expediently. For example, (ii) implies that
$$\dim(V_1 \otimes V_2) = \dim(V_1) \cdot \dim(V_2)$$
- **Tensor product** (of ρ^1, ρ^2): The representation $\rho : G \to GL(V_1 \otimes V_2)$ defined as follows for all $s \in G$, $x_1 \in V_1$, and $x_2 \in V_2$, where $\rho^1 : G \to GL(V_1)$ and $\rho^2 : G \to GL(V_2)$ are representations. *Given by*
$$[\rho^1_s \otimes \rho^2_s](x_1 \cdot x_2) = \rho^1_s(x_1) \cdot \rho^2_s(x_2)$$
- A more formal write up of the matrix translation of this definition.
  - Let (e_{i_1}) be a basis for V_1, and let (e_{i_2}) be a basis for V_2.
  - Let $r_{i_1 j_1}(s)$ be the matrix of ρ^1_s with respect to this basis, and let $r_{i_2 j_2}(s)$ be the matrix of ρ^2_s with respect to this basis.
  - It follows that
$$\rho^1_s(e_{j_1}) = \sum_{i_1} r_{i_1 j_1}(s) e_{i_1} \qquad\qquad \rho^2_s(e_{j_2}) = \sum_{i_2} r_{i_2 j_2}(s) e_{i_2}$$
  - Therefore,
$$[\rho^1_s \otimes \rho^2_s](e_{j_1} \cdot e_{j_2}) = \sum_{i_1, i_2} r_{i_1 j_1}(s) r_{i_2 j_2}(s) e_{i_1} \cdot e_{i_2}$$
    and
$$\mathcal{M}(\rho^1_s \otimes \rho^2_s) = (r_{i_1 j_1}(s) r_{i_2 j_2}(s))$$
- Aside on quantum chemistry to come back to later; I can't quite connect the dots yet.

Section 1.6: Symmetric Square and Alternating Square

- Herein, we investigate the tensor product when $V_1 = V_2 = V$.
- Let (e_i) be a basis of V.
- Define the automorphism $\theta : V \otimes V \to V \otimes V$ by
$$\theta(e_i \cdot e_j) = e_j \cdot e_i$$
  for all 2-indices (i, j).
- Properties of θ.
  - Since θ is linear, it follows that
$$\theta(x \cdot y) = y \cdot x$$
    for all $x, y \in V$.
    - Implication: θ is independent of the chosen basis (e_i)!

- $\theta^2 = 1$, where 1 is the identity map on $V \otimes V$.

- Assertion: $V \otimes V$ decomposes into

$$V \otimes V = S^2(V) \oplus \Lambda^2(V)$$

  - Rudenko: We do not have to worry about proving this... yet, at least.

- **Symmetric square representation**: The subspace of $V \otimes V$ containing all elements z satisfying $\theta(z) = z$. *Denoted by* $\boldsymbol{S^2V}$, $\boldsymbol{S^2(V)}$, $\boldsymbol{\mathbb{S}^2V}$, $\textbf{Sym}^{\boldsymbol{2}}\boldsymbol{(V)}$.
  - Basis: $(e_i \cdot e_j + e_j \cdot e_i)_{i \leq j}$.
    - Rudenko: How do we know everything is linearly independent? Well, when we add two linearly independent vectors out of a set, the sum is still linearly independent from everything else!
    - Example when $\dim V = 2$: The basis of $V \otimes V$ is $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$, where all four of these vectors are linearly independent. So naturally, the basis of the corresponding symmetric square representation — which is $2e_1 \otimes e_1, e_1 \otimes e_2 + e_2 \otimes e_1, 2e_2 \otimes e_2$ — will still be a linearly independent list of vectors.
  - Dimension: If $\dim V = n$, then

$$\dim S^2(V) = \frac{n(n+1)}{2}$$

- **Alternating square representation**: The subspace of $V \otimes V$ containing all elements z satisfying $\theta(z) = -z$. *Denoted by* $\boldsymbol{\Lambda^2V}$, $\boldsymbol{\Lambda^2(V)}$, $\textbf{Alt}^{\boldsymbol{2}}\boldsymbol{(V)}$.
  - Basis: $(e_i \cdot e_j - e_j \cdot e_i)_{i < j}$.
  - Dimension: If $\dim V = n$, then

$$\dim \Lambda^2(V) = \frac{n(n-1)}{2}$$