3 Representation Structure and Characters

10/20: 1. **Permutational representation**. Let X be a finite set on which the group G acts. Let ρ be the corresponding permutational representation with character χ.

(a) Consider an orbit Gx of an element $x \in X$; let c be the number of orbits. Prove that c equals the number of times ρ contains the trivial representation. Deduce that $(\chi, 1) = c$. In particular, if the action is transitive, $\rho = 1 \oplus \theta$ for some representation θ.

Proof. By the proof of Corollary 1 from Lecture 3.3, the number of times ρ contains the trivial representation (i.e., the multiplicity n_1 of the trivial representation) is equal to $(\chi, 1)$. Thus, to prove that the number of times ρ contains the trivial representation is equal to c, we will show that $(\chi, 1) = c$. Indeed, from the Hermitian inner product definition, we have that

$$
\begin{aligned}
(\chi, 1) &= \frac{1}{|G|} \sum_{g \in G} \chi(g) \cdot \bar{1}(g) \\
&= \frac{1}{|G|} \sum_{g \in G} \chi(g) \cdot 1 \\
&= \frac{1}{|G|} \sum_{g \in G} \chi(g) \\
&= \frac{1}{|G|} \sum_{g \in G} \operatorname{Fix}(g) \\
&= \frac{1}{|G|} \sum_{g \in G} |\{x \in X \mid g \cdot x = x\}| = \frac{1}{|G|} \sum_{g \in G} \sum_{x \in X} 1_{g \cdot x = x} \\
&= \frac{1}{|G|} \sum_{x \in X} |\{g \in G \mid g \cdot x = x\}| = \frac{1}{|G|} \sum_{x \in X} \sum_{g \in G} 1_{g \cdot x = x} \\
&= \frac{1}{|G|} \sum_{x \in X} |\operatorname{Stab}(x)| \\
&= \sum_{x \in X} \frac{1}{|G| / |\operatorname{Stab}(x)|} \\
&= \sum_{x \in X} \frac{1}{|Gx|} && \text{Orbit-Stabilizer Theorem} \\
&= c \cdot \sum_{x \in Gx} \frac{1}{|Gx|} \\
&= c \cdot 1 \\
&= c
\end{aligned}
$$

as desired.

If the action is transitive, then $Gx = X$ for any $x \in X$, so there is only one orbit and $(\chi, 1) = 1$. Thus, the multiplicity of the trivial representation in ρ is 1, so by complete reducibility,

$$\rho = 1^1 \oplus \underbrace{V_2^{n_2} \oplus \cdots \oplus V_k^{n_k}}_{\theta}$$

as desired. ☐

(b) Consider a diagonal action of G on $X \times X$. Prove that the character of the corresponding permutational representation is χ^2.

Proof. Since ρ is a permutational representation, we have that $\chi(g) = |\operatorname{Fix}_X(g)|$, where here we let $\operatorname{Fix}_X(g) = \{x \in X \mid g \cdot x = x\}$. It follows via a simple bidirectional inclusion proof that $\operatorname{Fix}_{X\times X}(g) = \operatorname{Fix}_X(g) \times \operatorname{Fix}_X(g)$. Thus,

$$\chi_{X\times X} = |\operatorname{Fix}_{X\times X}(g)| = |\operatorname{Fix}_X(g) \times \operatorname{Fix}_X(g)| = |\operatorname{Fix}_X(g)|^2 = \chi^2$$

as desired. ☐

(c) Suppose that G acts transitively on X and $|X| \geq 2$. We call this action **doubly transitive** if every pair of distinct elements of X can be sent to any other pair by some element of G. Prove that the following are equivalent.

i. The action is doubly transitive.
ii. The diagonal action on $X \times X$ has exactly two orbits.
iii. $(\chi^2, 1) = 2$.
iv. The representation θ is irreducible.

Proof. We will prove that (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). Let's begin.

(i $\Leftrightarrow$ ii): First, suppose $G \circlearrowright X$ is doubly transitive. Since $|X| \geq 2$, we may choose $x, y \in X$ to be distinct, i.e., satisfying $x \neq y$. We will show that the two orbits of the diagonal action of G on $X \times X$ are $G(x, x)$ and $G(x, y)$.

First, we will show that every $(x_1, x_2) \in X \times X$ is in one of these two orbits. Let $(x_1, x_2) \in X \times X$ be arbitrary. We divide into two cases ($x_1 = x_2$ and $x_1 \neq x_2$). If $x_1 = x_2$, then since $G \circlearrowright X$ is transitive, there exists $g \in G$ such that $g \cdot x = x_1$. Thus,

$$g \cdot (x, x) = (g \cdot x, g \cdot x) = (x_1, x_1) = (x_1, x_2)$$

so $(x_1, x_2) \in G(x, x)$, as desired. If $x_1 \neq x_2$, then since $G \circlearrowright X$ is doubly transitive, there exists $g \in G$ such that $g \cdot x = x_1$ and $g \cdot y = x_2$. Thus,

$$g \cdot (x, y) = (g \cdot x, g \cdot y) = (x_1, x_2)$$

so $(x_1, x_2) \in G(x, y)$, as desired.

Now, we will show that $(x, y) \notin G(x, x)$. This follows immediately from the well-definedness of the group action: Suppose for the sake of contradiction that there exists $g \in G$ such that $g \cdot (x, x) = (x, y)$. Then $(g \cdot x, g \cdot x) = (x, y)$, so $x = g \cdot x = y$, contradicting the hypothesis that $x \neq y$.

Second, suppose the diagonal action has exactly two orbits. Since $G \circlearrowright X$ is transitive, by the same reasoning as before, $G(x, x)$ is an orbit. Thus, since there are only two orbits, the other orbit must be $X \times X \setminus G(x, x) = G(x, y)$. The existence of this second orbit implies that any distinct $x, y \in X$ can be mapped to any other pair of elements of X by some $g \in G$, i.e., that the action is doubly transitive.

(ii $\Leftrightarrow$ iii): Suppose the diagonal action on $X \times X$ has exactly two orbits. Then by part (b), the character of the corresponding permutational representation is χ^2. Thus, by part (a), $(\chi^2, 1) = c$, where c is the number of orbits. But by hypothesis (ii), $c = 2$, so $(\chi^2, 1) = 2$, as desired.

Suppose $(\chi^2, 1) = 2$. Then by parts (a) and (b) once again, the diagonal action on $X \times X$ has $2 = c$ orbits.

(iii $\Leftrightarrow$ iv): Suppose $(\chi^2, 1) = 2$. Note that by θ, we mean the θ defined in part (a), *not* the representation θ' defined by $\rho_2 = 1^2 \oplus \theta$ where ρ_2 is the permutational representation corresponding to the diagonal action of G on $X \times X$. Moving on, observe that $(\chi, \chi) = (\chi^2, 1)$, $(1, \chi) = (\chi, 1)$, and $(1, 1) = 1$ by the definition of the inner product. Observe also that $(\chi, 1) = 1$ by part (a) since the action is transitive. Therefore,

$$\begin{aligned}(\theta,\theta) &= (\chi-1,\chi-1)\\ &= (\chi-1,\chi)-(\chi-1,1)\\ &= [(\chi,\chi)-(1,\chi)]-[(\chi,1)-(1,1)]\\ &= (\chi^2,1)-2(\chi,1)+(1,1)\\ &= 2-2\cdot 1+1\\ &= 1\end{aligned}$$

so θ is irreducible by Corollary 2 from Lecture 3.3.

Suppose that θ is irreducible. Then $(\theta,\theta)=1$. We still have $(\chi,\chi)=(\chi^2,1)$, $(\chi,1)=(1,\chi)=1$, and $(1,1)=1$ because these claims relied on the definition of the inner product and part (a), not the hypothesis that $(\chi^2,1)=2$. Thus, we have that

$$(\chi^2,1)=(\theta,\theta)+2(\chi,1)-(1,1)=1+2-1=2$$

as desired. □

2. Find the character table of the group A_4.

Proof. The conjugacy classes of A_4 in S_4 are $\{e\},\{(xxx)\},\{(xx)(xx)\}$. The true conjugacy classes of A_4 vary slightly, however. e is still in a class by itself

Permutation representation: 4,1,0

	e	(xxx)	$(xx)(xx)$
Trivial	1	1	1
Standard	3	0	-1

□

3. Consider the space of functions V from the set of faces of a cube to $\mathbb{C}$. This is a representation of S_4.
   (a) Compute the character of V.
   (b) Describe explicitly the decomposition of V into isotypical components.
   (c) Consider a map $A: V\to V$ acting by substituting the value of a function on a face with an average of its values on the adjacent four faces. Prove that A is an automorphism of the corresponding representation. Find its eigenvalues.
4. Consider a finite representation V of a group G with character χ.
   (a) Express the characters of Λ^2V and S^2V in terms of χ.
   (b) Express the characters of Λ^3V and S^3V in terms of χ.
   (c) Let $(3,1)$ be the standard representation of S_4. Decompose $\Lambda^2(3,1)$, $\Lambda^3(3,1)$, $S^2(3,1)$, and $S^3(3,1)$ into irreducibles.