4 More Characters and Intro to Associative Algebras

10/27: 1. **Representations of S_5.**

(a) Prove that there exist only two one-dimensional representations of S_5: The trivial representation (5) and the alternating representation $(1,1,1,1,1)$.

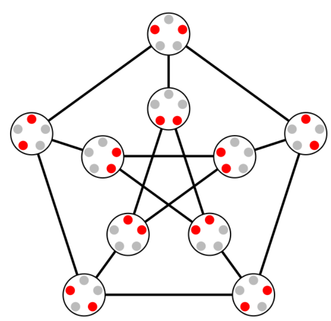
(b) Compute the character of the standard representation $(4,1)$ by decomposing the permutational representation into irreducibles.

(c) Prove that the representations $(3,1,1) = \Lambda^2(4,1)$ and $(2,1,1,1) = \Lambda^3(4,1)$ are irreducible. Compute their characters. Prove that $(2,1,1,1) = (1,1,1,1,1) \otimes (4,1)$.

(d) Find the two remaining irreducible representations of S_5; denote them $(3,2)$ and $(2,2,1)$. Complete the character table.

(e) Consider an exceptional homomorphism $S_5 \to S_6$. Decompose the corresponding permutational representation into irreducibles.

(f) The **Petersen graph** is a graph with vertices being 2-element subsets of $\{1,2,3,4,5\}$; two vertices are connected by an edge if the corresponding sets do not intersect (see wiki for a picture).



Consider a natural action of S_5 on the set of complex-valued functions on the set of vertices of the Peterson graph. Find the character of the corresponding representation V and decompose it into irreducibles.

(g) Decompose V into isotypical components.

(h) Consider an endomorphism $A : V \to V$ sending a function to the average of its values on the adjacent vertices. Prove that it is an endomorphism of the corresponding representation. Find the spectrum of A.

2. The character table is a square matrix. Determine the absolute value of its determinant.

3. Prove that for any irreducible character χ of a group G, we have

$$\chi(g)\chi(h) = \frac{d_\chi}{|G|} \sum_{x \in G} \chi(gxhx^{-1})$$

4. Let F be a field.

(a) Prove that the matrix algebra $M_{n,n}(F)$ is simple, i.e., has no nontrivial ideals.

Proof. To prove that $M_{n,n}(F)$ is simple, it will suffice to show that any nonzero ideal of it contains the identity. Let I be a nonzero ideal of $M_{n,n}(F)$. Then there exists some nonzero $A \in I$. In particular, if A is nonzero, then it contains some entry $a_{ij} \neq 0$. Now, let $E_{ij} \in M_{n,n}(F)$ be the matrix with 1 in the i,j position and 0 everywhere else. Then for each $k = 1, \dots, n$,

$$a_{ij}^{-1} E_{ki} A E_{jk} = E_{kk}$$

so $E_{kk} \in I$. Summing all of the E_{kk}'s within I yields the $n \times n$ identity matrix, as desired. $\square$

(b) Prove that there is a unique simple module over the matrix algebra $M_{n,n}(F)$.

Proof. We will prove that F^n is a simple module over $M_{n,n}(F)$, and then that every simple module over $M_{n,n}(F)$ is isomorphic to F^n. Note that the action of $M_{n,n}(F)$ on F^n is defined to be left matrix multiplication. Let's begin.

To prove that F^n is a simple module, it will suffice to show that any nonzero submodule of F^n equals F^n. Let N be an arbitrary nonzero submodule of F^n. To show that $N = F^n$, we will use a bidirectional inclusion proof. As a submodule of F^n, we immediately have $N \subset F^n$. In the other direction, let $y = (y_1, \dots, y_n) \in F^n$ be arbitrary. Since N is nonzero, it contains some nonzero element $x = (x_1, \dots, x_n)$. Suppose $x_j \neq 0_F$. Let $A \in M_{n,n}(F)$ be zero everywhere except in column j, where we choose $a_{ij} = y_i x_j^{-1}$ $(i = 1, \dots, n)$. Thus, $y = Ax$, so $y \in N$, as desired.

Now let S be an arbitrary simple module over $M_{n,n}(F)$. Let $s \in S$ nonzero be arbitrary. To prove that $S \cong F^n$ as a module, we will show that $S = M_{n,n}(F)s$, use this definition of S to construct a map $f : S \to F^n$, and verify that this map is a module isomorphism. Let's begin.
Consider the set

$$M_{n,n}(F)s = \{As \mid A \in M_{n,n}(F)\}$$

Since $0 \neq s \in M_{n,n}(F)s$, $As + Bs = (A + B)s \in M_{n,n}(F)s$, and $(-A)s \in M_{n,n}(F)s$ for all $As \in M_{n,n}(F)s$, we know that $M_{n,n}(F)s \leq S$. Since, additionally, $B(As) = (BA)s \in M_{n,n}(F)s$ for all $B \in M_{n,n}(F)$ and $As \in M_{n,n}(F)s$, we know that $M_{n,n}(F)s$ is a (nonzero) submodule of S. Consequently, since S is simple, $M_{n,n}(F)s = S$. As a special case, note that F^n being a simple $M_{n,n}(F)$-module implies that $F^n = M_{n,n}(F)e_1$, for instance.
Define $f : S \to F^n$ by

$$As \mapsto Ae_1$$

We have that f is a homomorphism of abelian groups because

$$f(As + Bs) = f[(A + B)s] = (A + B)e_1 = Ae_1 + Be_1 = f(As) + f(Bs)$$

We have that f commutes with scalar multiplication because

$$f(B(As)) = f[(BA)s] = (BA)e_1 = B(Ae_1) = Bf(As)$$

Thus, f is a module homomorphism. Additionally, since $\mathrm{Ker}(f)$ is a submodule of S, S is simple, and f is nontrivial, we have that $\mathrm{Ker}(f) = 0$ and hence f is injective. Similarly, since $\mathrm{Im}(f)$ is a submodule of F^n, F^n is simple, and f is nontrivial, we have that $\mathrm{Im}(f) = F^n$ and hence f is surjective. Thus, as an injective, surjective module homomorphism, f is a module *isomorphism* as well. Therefore, $S \cong F^n$, as desired. $\square$

5. Consider the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} \subset \mathbb{H}$ of order 8.

(a) Find four 1-dimensional representations of Q_8. Find the character of the remaining 2-dimensional representation.

Proof. Take the trivial representation.
Then take the representations sending i, j to (-1), i, k to (-1), and j, k to (-1); everything else gets sent to the identity in each case.
Working out the character table at this point, we can see that the character for the final representation must be given explicitly by the following.

$$+1 \mapsto 2 \qquad -1 \mapsto -2 \qquad +i \mapsto 0 \qquad -i \mapsto 0 \qquad +j \mapsto 0 \qquad -j \mapsto 0 \qquad +k \mapsto 0 \qquad -k \mapsto 0$$

$\square$

(b) Prove that $\mathbb{R}[Q_8] \cong \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{H}$ as an algebra.