MATH 26700 (Introduction to Representation Theory of Finite Groups) Problem Sets

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Contents

1 Applications of Linear Algebra to Representation Theory

10/5: 1. Read Section 1.3 in Fulton and Harris (2004).

2. Let V, W be finite-dimensional vector spaces. Construct canonical isomorphisms...

(a) $\Lambda^2(V \oplus W) \cong (\Lambda^2 V) \oplus (V \otimes W) \oplus (\Lambda^2 W)$;

Proof. Define the map $\tilde{f}$ on the subset of $\Lambda^2(V \oplus W)$ containing all decomposable antisymmetric tensors by the rule

$$(v_1, w_1) \wedge (v_2, w_2) \mapsto (v_1 \wedge v_2, v_1 \otimes w_2 - v_2 \otimes w_1, w_1 \wedge w_2)$$

We now have to prove that this map is bilinear.

To prove linearity in the first argument, we have

$$\begin{aligned}
\tilde{f}[((v_1, w_1) + (v_1', w_1')) \wedge (v_2, w_2)] &= \tilde{f}[(v_1 + v_1', w_1 + w_1') \wedge (v_2, w_2)] \\
&= ((v_1 + v_1') \wedge v_2, \\
&\qquad (v_1 + v_1') \otimes w_2 - v_2 \otimes (w_1 + w_1'), \\
&\qquad (w_1 + w_1') \wedge w_2) \\
&= (v_1 \wedge v_2 + v_1' \wedge v_2, \\
&\qquad v_1 \otimes w_2 + v_1' \otimes w_2 - v_2 \otimes w_1 - v_2 \otimes w_1', \\
&\qquad w_1 \wedge w_2 + w_1' \wedge w_2) \\
&= (v_1 \wedge v_2, v_1 \otimes w_2 - v_2 \otimes w_1, w_1 \wedge w_2) \\
&\quad + (v_1' \wedge v_2, v_1' \otimes w_2 - v_2 \otimes w_1', w_1' \wedge w_2) \\
&= \tilde{f}[(v_1, w_1) \wedge (v_2, w_2)] + f[(v_1', w_1') \wedge (v_2, w_2)]
\end{aligned}$$

and

$$\begin{aligned}
\tilde{f}[(\lambda(v_1, w_1)) \wedge (v_2, w_2)] &= \tilde{f}[(\lambda v_1, \lambda w_1) \wedge (v_2, w_2)] \\
&= (\lambda v_1 \wedge v_2, \lambda v_1 \otimes w_2 - v_2 \otimes \lambda w_1, \lambda w_1 \wedge w_2) \\
&= (\lambda(v_1 \wedge v_2), \lambda(v_1 \otimes w_2 - v_2 \otimes w_1), \lambda(w_1 \wedge w_2)) \\
&= \lambda(v_1 \wedge v_2, v_1 \otimes w_2 - v_2 \otimes w_1, w_1 \wedge w_2) \\
&= \lambda \tilde{f}[(v_1, w_1) \wedge (v_2, w_2)]
\end{aligned}$$

The proof is symmetric in the second argument.

To prove that f respects the antisymmetry of the wedge product, it will suffice to show that

$$f[(v_1, w_1) \wedge (v_2, w_2)] = -f[(v_2, w_2) \wedge (v_1, w_1)]$$

in general. Let $(v_1, w_1) \wedge (v_2, w_2)$ be an arbitrary decomposable element of $\Lambda^2(V \oplus W)$. Then we have that

$$\begin{aligned}
f[(v_1, w_1) \wedge (v_2, w_2)] &= (v_1 \wedge v_2, v_1 \otimes w_2 - v_2 \otimes w_1, w_1 \wedge w_2) \\
&= (-v_2 \wedge v_1, -(v_2 \otimes w_1 - v_1 \otimes w_2), -w_2 \wedge w_1) \\
&= -(v_2 \wedge v_1, v_2 \otimes w_1 - v_1 \otimes w_2, w_2 \wedge w_1) \\
&= -f[(v_2, w_2) \wedge (v_1, w_1)]
\end{aligned}$$

as desired.

Since $\tilde{f}$ an alternating bilinear map, the universal property of the exterior powers[1] implies that there exists a map $f : \Lambda^2(V \oplus W) \to (\Lambda^2 V) \oplus (V \otimes W) \oplus (\Lambda^2 W)$.

[1] See Fulton and Harris (2004, p. 472).

Since the domain and codomain have the same dimension, to prove that f is an isomorphism, it will suffice to prove that it's surjective. Let

$$\left(\sum_{i=1}^{r} v_{i_1} \wedge v_{i_2}, \sum_{i=1}^{s} v_{i_3} \otimes w_{i_3}, \sum_{i=1}^{t} w_{i_1} \wedge w_{i_2}\right)$$

be an arbitrary element of $(\Lambda^2 V) \oplus (V \otimes W) \oplus (\Lambda^2 W)$. Decompose it into a sum of elements of the form $(a, 0, 0)$, $(0, a, 0)$, or $(0, 0, a)$. Explicitly, the above element is equal to

$$\sum_{i=1}^{r}(v_{i_1} \wedge v_{i_2}, 0, 0) + \sum_{i=1}^{s}(0, v_{i_3} \otimes w_{i_3}, 0) + \sum_{i=1}^{t}(0, 0, w_{i_1} \wedge w_{i_2})$$

Now, by the definition of f, we know that each of the individual terms in the three sums above satisfy one of

$$\begin{aligned}
(v_1 \wedge v_2, 0, 0) &= f[(v_1, 0) \wedge (v_2, 0)] \\
(0, v \otimes w, 0) &= f[(v, 0) \wedge (0, w)] \\
(0, 0, w_1 \wedge w_2) &= f[(0, w_1) \wedge (0, w_2)]
\end{aligned}$$

Therefore, we have that the initial arbitrary element of $(\Lambda^2 V) \oplus (V \otimes W) \oplus (\Lambda^2 W)$ is equal to

$$f\left[\sum_{i=1}^{r}(v_{i_1}, 0) \wedge (v_{i_2}, 0) + \sum_{i=1}^{s}(v_{i_3}, 0) \wedge (0, w_{i_3}) + \sum_{i=1}^{t}(0, w_{i_1}) \wedge (0, w_{i_2})\right]$$

as desired. □

(b) $S^2(V \oplus W) \cong (S^2 V) \oplus (V \otimes W) \oplus (S^2 W)$.

Proof. Define the map $\tilde{f}$ on the subset of $S^2(V \oplus W)$ containing all decomposable symmetric tensors by the rule

$$(v_1, w_1) \cdot (v_2, w_2) \mapsto (v_1 \cdot v_2, v_1 \otimes w_2 + v_2 \otimes w_1, w_1 \cdot w_2)$$

The rest of the proof is symmetric to that of part (a). □

3. (a) Factorize the group determinant for $G = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Proof. The multiplication table for G is

	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,1)	(1,0)	(1,1)
(0,1)	(0,1)	(0,0)	(1,1)	(1,0)
(1,0)	(1,0)	(1,1)	(0,0)	(0,1)
(1,1)	(1,1)	(1,0)	(0,1)	(0,0)

Thus, the group determinant is

$$\begin{vmatrix}
a & b & c & d \\
b & a & d & c \\
c & d & a & b \\
d & c & b & a
\end{vmatrix}$$

To factorize this group determinant, it will suffice to find the eigenvalues of the matrix it encloses. To find the eigenvalues, we may start by inspecting it for eigenvectors.

First off, recall the Sudoku Lemma from MATH 25700. It implies that every row of the matrix will list each element once, and we can confirm that this is true in this example by looking at it.

It follows that if we propose $(1,1,1,1)$ as an eigenvector, we'll be able to extract an eigenvalue via the commutativity of multiplication as follows.

$$\begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b+c+d \\ b+a+d+c \\ c+d+a+b \\ d+c+b+a \end{bmatrix} = \begin{bmatrix} a+b+c+d \\ a+b+c+d \\ a+b+c+d \\ a+b+c+d \end{bmatrix} = (a+b+c+d)\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Second, we may observe that the upper-left and lower-right blocks of this matrix match, as do the lower-left and upper-right blocks. Indeed, this matrix is a block matrix of the form $\left[\begin{smallmatrix} A & C \\ C & A \end{smallmatrix}\right]$. Thus, since the eigenvector of this block matrix that is not $(1,1)$ is $(1,-1)$, the analogous relevant eigenvector of the full matrix is $(1,1,-1,-1)$. Once again, we obtain

$$\begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} a+b-c-d \\ b+a-d-c \\ c+d-a-b \\ d+c-b-a \end{bmatrix} = \begin{bmatrix} (a+b-c-d)\cdot 1 \\ (a+b-c-d)\cdot 1 \\ (a+b-c-d)\cdot -1 \\ (a+b-c-d)\cdot -1 \end{bmatrix} = (a+b-c-d)\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

Third, we may observe that each of the blocks referred to above is also of the form $\left[\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right]$. Thus, we can also apply $(1,-1)$ twice — once to each block — with the eigenvector $(1,-1,1,-1)$. Once again, we obtain

$$\begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a-b+c-d \\ b-a+d-c \\ c-d+a-b \\ d-c+b-a \end{bmatrix} = \begin{bmatrix} (a-b+c-d)\cdot 1 \\ (a-b+c-d)\cdot -1 \\ (a-b+c-d)\cdot 1 \\ (a-b+c-d)\cdot -1 \end{bmatrix} = (a-b+c-d)\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

Lastly, we may observe while the two side columns have a or d in the top and bottom slots and b or c in the middle two slots, it is vice versa for the two middle columns. Thus, we can apply $(1,-1,-1,1)$ to obtain

$$\begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} a-b-c+d \\ b-a-d+c \\ c-d-a+b \\ d-c-b+a \end{bmatrix} = \begin{bmatrix} (a-b-c+d)\cdot 1 \\ (a-b-c+d)\cdot -1 \\ (a-b-c+d)\cdot -1 \\ (a-b-c+d)\cdot 1 \end{bmatrix} = (a-b+c-d)\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Thus, we have found four distinct eigenvectors for a 4×4 matrix. Therefore, we have found all of the eigenvalues. Their product equals the determinant, and is also a factorization of the determinant. In particular, the factorized group determinant for K_4 is

$$\boxed{(a+b+c+d)(a+b-c-d)(a-b+c-d)(a-b-c+d)}$$

□

(b) A **circulant matrix** is a square matrix in which all row vectors are composed of the same elements and each row vector is rotated one element to the right relative to the preceding row vector. Prove that for $\zeta \in \mu_n$, vector $(1,\zeta,\dots,\zeta^{n-1})$ is an eigenvector of any circulant matrix of size n.

Proof. Let

$$\begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_n & x_1 & x_2 & \cdots & x_{n-1} \\ x_{n-1} & x_n & x_1 & \cdots & x_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_2 & x_3 & x_4 & \cdots & x_1 \end{bmatrix}$$

be an arbitrary circulant matrix of size n. It follows that

$$\begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_n & x_1 & x_2 & \cdots & x_{n-1} \\ x_{n-1} & x_n & x_1 & \cdots & x_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_2 & x_3 & x_4 & \cdots & x_1 \end{bmatrix} \begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \\ \vdots \\ \zeta^{n-1} \end{bmatrix} = \begin{bmatrix} x_1 + x_2\zeta + x_3\zeta^2 + \cdots + x_n\zeta^{n-1} \\ x_n + x_1\zeta + x_2\zeta^2 + \cdots + x_{n-1}\zeta^{n-1} \\ x_{n-1} + x_n\zeta + x_1\zeta^2 + \cdots + x_{n-2}\zeta^{n-1} \\ \vdots \\ x_2 + x_3\zeta + x_4\zeta^2 + \cdots + x_1\zeta^{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} x_1\zeta^0 + x_2\zeta^1 + x_3\zeta^2 + \cdots + x_n\zeta^{n-1} \\ x_n\zeta^n + x_1\zeta^1 + x_2\zeta^2 + \cdots + x_{n-1}\zeta^{n-1} \\ x_{n-1}\zeta^n + x_n\zeta^{n+1} + x_1\zeta^2 + \cdots + x_{n-2}\zeta^{n-1} \\ \vdots \\ x_2\zeta^n + x_3\zeta^{n+1} + x_4\zeta^{n+2} + \cdots + x_1\zeta^{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} x_1\zeta^0 + x_2\zeta^1 + x_3\zeta^2 + \cdots + x_n\zeta^{n-1} \\ x_1\zeta^1 + x_2\zeta^2 + x_3\zeta^3 + \cdots + x_n\zeta^{n} \\ x_1\zeta^2 + x_2\zeta^3 + x_3\zeta^4 + \cdots + x_n\zeta^{n+1} \\ \vdots \\ x_1\zeta^{n-1} + x_2\zeta^n + x_3\zeta^{n+1} + \cdots + x_n\zeta^{2n-2} \end{bmatrix}$$

$$= \left(\sum_{i=1}^n x_i\zeta^{i-1}\right) \begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \\ \vdots \\ \zeta^{n-1} \end{bmatrix}$$

as desired. ☐

(c) Compute the eigenvalues and the determinant of a circulant matrix. Factorize the group determinant for $G = \mathbb{Z}/n\mathbb{Z}$.

Proof. By the same logic used in part (a), one can find by inspection that the n distinct eigenvectors are of the form $(\zeta^{j(0)}, \dots, \zeta^{j(n-1)})$, $j = 0, \dots, n-1$. It follows by similar logic once again that the corresponding eigenvalues are of the form

$$\boxed{\lambda_j = \sum_{i=1}^n x_i\zeta^{j(i-1)},\ j = 0, \dots, n-1}$$

Moreover, since the determinant is the product of the eigenvalues,

$$\boxed{\det = \lambda_0 \cdots \lambda_{n-1}}$$

where λ_j is defined as above.

Therefore, since the multiplication table of $\mathbb{Z}/n\mathbb{Z}$ converts into an $n \times n$ circulant matrix, the factorization of its group determinant is $\boxed{\text{equal to the above determinant expression.}}$ ☐

4. **Plethysm for S_3.** Let (3), $(1,1,1)$, and $(2,1)$ be the trivial, alternating, and standard representations of S_3.

(a) Consider the permutational representation $V \cong (3) \oplus (2,1)$. Decompose $\Lambda^2 V$ into irreducibles.

Proof. Fix a basis of V equal to $\{(1,1,1), (\omega,1,\omega^2), (1,\omega,\omega^2)\} = \{\gamma, \alpha, \beta\}$. Let $\tau = (123) \in S_3$ and $\sigma = (12) \in S_3$. By Problem 2a,

$$\Lambda^2 V = \Lambda^2((3) \oplus (2,1)) = [\Lambda^2(3)] \oplus [(3) \otimes (2,1)] \oplus [\Lambda^2(2,1)]$$

We now divide into three cases.

Case 1 ($\Lambda^2(3)$): The basis for this vector space is $\{\gamma \wedge \gamma\} = \{0\}$, so

$$\Lambda^2(3) = 0$$

Case 2 ($(3) \otimes (2,1)$): The basis for this vector space is $\{\gamma \otimes \alpha, \gamma \otimes \beta\}$. The action of τ on these basis vectors can be computed:

$$\tau(\gamma \otimes \alpha) = \omega\gamma \otimes \alpha \qquad \tau(\gamma \otimes \beta) = \omega^2\gamma \otimes \beta$$

These equations are directly analogous to the untensored α and β equations, so this is the standard representation. Formally,

$$(3) \otimes (2,1) = (2,1)$$

Case 3 ($\Lambda^2(2,1)$): The basis for this vector space is $\{\alpha \wedge \beta\}$. Thus, $\dim \Lambda^2(2,1) = 1$, so $\Lambda^2(2,1) \neq (2,1)$, and we can more on to discriminating between the trivial and alternating representations. In particular, the action of σ on this basis vector is

$$\sigma(\alpha \wedge \beta) = -\alpha \wedge \beta$$

Thus, $\sigma(v)$ is not linearly independent of v. Moreover, $\sigma(v) = -v$. Thus, this is the alternating representation. Formally,

$$\Lambda^2(2,1) = (1,1,1)$$

Putting everything back together, we obtain

$$\begin{aligned}\Lambda^2 V &\cong [\Lambda^2(3)] \oplus [(3) \otimes (2,1)] \oplus [\Lambda^2(2,1)] \\ &\cong 0 \oplus (2,1) \oplus (1,1,1) \\ &\boxed{\Lambda^2 V \cong (2,1) \oplus (1,1,1)}\end{aligned}$$

$\square$

(b) Decompose $S^2(2,1)$ and $S^3(2,1)$ into irreducibles.

Proof. We will treat each case separately.

$\underline{S^2(2,1)}$: The basis for this vector space is $\{\alpha \cdot \alpha, \alpha \cdot \beta, \beta \cdot \beta\}$. The action of τ on these basis vectors can be computed:

$$\tau(\alpha \cdot \alpha) = \omega^2\alpha \cdot \alpha \qquad \tau(\alpha \cdot \beta) = 1\alpha \cdot \beta \qquad \tau(\beta \cdot \beta) = \omega\beta \cdot \beta$$

The action of σ on these basis vectors can also be computed:

$$\sigma(\alpha \cdot \alpha) = \beta \cdot \beta \qquad \sigma(\alpha \cdot \beta) = \alpha \cdot \beta \qquad \sigma(\beta \cdot \beta) = \alpha \cdot \alpha$$

Thus, $\alpha \cdot \alpha$ and $\beta \cdot \beta$ span a standard representation, and $\alpha \cdot \beta$ spans a trivial representation. Putting everything together, we obtain

$$\boxed{S^2(2,1) \cong (2,1) \oplus (3)}$$

$\underline{S^3(2,1)}$: The basis for this vector space is $\{\alpha \cdot \alpha \cdot \alpha, \alpha \cdot \alpha \cdot \beta, \alpha \cdot \beta \cdot \beta, \beta \cdot \beta \cdot \beta\}$. The action of τ on these basis vectors can be computed:

$$\tau(\alpha \cdot \alpha \cdot \alpha) = 1\alpha \cdot \alpha \cdot \alpha \qquad \tau(\alpha \cdot \alpha \cdot \beta) = \omega\alpha \cdot \alpha \cdot \beta$$

$$\tau(\alpha \cdot \beta \cdot \beta) = \omega^2\alpha \cdot \beta \cdot \beta \qquad \tau(\beta \cdot \beta \cdot \beta) = 1\beta \cdot \beta \cdot \beta$$

The action of σ on these basis vectors can also be computed:

$$\sigma(\alpha\cdot\alpha\cdot\alpha) = \beta\cdot\beta\cdot\beta \qquad \sigma(\alpha\cdot\alpha\cdot\beta) = \alpha\cdot\beta\cdot\beta$$

$$\sigma(\alpha\cdot\beta\cdot\beta) = \alpha\cdot\alpha\cdot\beta \qquad \sigma(\beta\cdot\beta\cdot\beta) = \alpha\cdot\alpha\cdot\alpha$$

Thus, $\alpha\cdot\alpha\cdot\beta$ and $\alpha\cdot\beta\cdot\beta$ span a standard representation, $\alpha\cdot\alpha\cdot\alpha+\beta\cdot\beta\cdot\beta$ spans a trivial representation, and $\alpha\cdot\alpha\cdot\alpha-\beta\cdot\beta\cdot\beta$ spans an alternating representation. Putting everything together, we obtain

$$\boxed{S^3(2,1)\cong(2,1)\oplus(3)\oplus(1,1,1)}$$

□

(c) Decompose the regular representation R into irreducibles.

Proof. The regular representation is 6-dimensional. We're in $\mathbb{C}^6$ with basis $\{e_e, e_{(12)}, e_{(13)}, e_{(23)}, e_{(123)}, e_{(132)}\}$. Let $\tau=(123)$ again. We get $\tau(1,1,1,1,1,1)=1(1,1,1,1,1,1)$. We have $\tau v_1=\omega^{\alpha_1}v_1$ □

(d) Prove that $S^{k+6}(2,1)\cong S^k(2,1)\oplus R$. Compute $S^k(2,1)$ for all k.

5. Let V be a vector space over F with a basis $e_1,\dots,e_n$; let $e^1,\dots,e^n$ be the dual basis. Prove the following.

(a) Element $e_1\otimes e^1+\cdots+e_n\otimes e^n\in V\otimes V^\vee$ does not depend on the choice of basis.

Proof. To prove that the given element of $V\otimes V^*$ does not depend on the choice of basis, it will suffice to show that for any choice of basis and associated dual basis, the given element maps to the same place in the isomorphic space $\operatorname{Hom}(V,V)$. Let's begin.

Let $e_1,\dots,e_n$ be an arbitrary basis of V, and let $e^1,\dots,e^n$ be its dual basis. Under the isomorphism constructed in class, the element

$$e_1\otimes e^1+\cdots+e_n\otimes e^n\mapsto[v\mapsto e^1(v)e_1]+\cdots+[v\mapsto e^n(v)e_n]$$

where $[v\mapsto e^i(v)e_i]$ denotes the linear map in $\operatorname{Hom}(V,V)$ sending v to its i^{th} component. Importantly, under the usual rules of adding functions, we can see that linear map on the right above is equal to

$$[v\mapsto e^1(v)e_1+\cdots+e^n(v)e_n]=[v\mapsto v_1e_1+\cdots+v_ne_n]=[v\mapsto v]=1$$

where $1\in\operatorname{Hom}(V,V)$ is the identity map. □

(b) Consider a linear map $\operatorname{ev}:V\otimes V^*\to F$ sending $v\otimes\alpha$ to $\alpha(v)\in F$. Prove that $\operatorname{ev}(L)=\operatorname{tr}(L)$.

Proof. Let $L\in\operatorname{Hom}(V,V)$ be an arbitrary linear map, let it's matrix be $(\ell_{ij})\in\mathcal{M}(V,V)$ with respect to some basis $e_1,\dots,e_n$, and let $e^1,\dots,e^n$ be dual to this basis. Define a map from $\mathcal{M}(V,V)$ to $V\otimes V^*$ by sending the standard basis (a_{ij}) of $\mathcal{M}(V,V)$ to $e_i\otimes e^j$. It follows that the image of $\mathcal{M}(L)$ under this map is $\sum_{i,j=1}^n\ell_{ij}e_i\otimes e^j\in V\otimes V^*$. It follows by the constructions up to this point and the definition of the trace that

$$\begin{aligned}
\mathrm{ev}(L) &= \mathrm{ev}\left(\sum_{i,j=1}^{n} \ell_{ij} e_i \otimes e^j\right) \\
&= \sum_{i,j=1}^{n} \ell_{ij}\, \mathrm{ev}(e_i \otimes e^j) \\
&= \sum_{i,j=1}^{n} \ell_{ij} e^j(e_i) \\
&= \sum_{i,j=1}^{n} \ell_{ij} \delta_{ij} \\
&= \sum_{i=1}^{n} \ell_{ii} \\
&= \mathrm{tr}(L)
\end{aligned}$$

as desired. $\square$

(c) A **projector** is a linear map $P : V \to V$ such that $P^2 = P$. Prove that $\mathrm{tr}(P) = \dim(\mathrm{Im}(P))$.

Proof. Let $v \in \mathrm{Im}(P)$ be arbitrary. Since $v \in \mathrm{Im}(P)$, we have that $v = Pw$ for some $w \in V$. But since $P = P^2$, it follows that

$$Pv = P^2 w = Pw = v$$

Thus, the restriction P_1 of P to $\mathrm{Im}(P)$ is equal to the identity on $\mathrm{Im}(P)$.
Let $v \in V \setminus \mathrm{Im}(P)$ be arbitrary. Since $v \notin \mathrm{Im}(P)$, $v \in \mathrm{Ker}(P)$ so

$$Pv = 0$$

Thus, the restriction P_2 of P to $\mathrm{Ker}(P)$ is equal to the zero map on $\mathrm{Ker}(P)$.
Since $V = \mathrm{Im}(P) \oplus \mathrm{Ker}(P)$, $P = P_1 \oplus P_2$. In particular, there will exist an orthonormal basis in which the first k vectors form a basis of $\mathrm{Im}(P)$ and the next $n-k$ vectors form a basis of $\mathrm{Ker}(P)$. Thus, $\dim(\mathrm{Im}(P)) = k$. Moreover, with respect to this basis, the $n \times n$ matrix of P will have a $k \times k$ block in the upper left-hand corner in which I_k resides, and it will be zeroes everywhere else. Since the trace is invariant under similarity transformations, it can be read off from this matrix as k as well. $\square$

(d) Let V be a representation of a finite group G. Prove that the representation $V \otimes V^*$ has a trivial subrepresentation.

Proof. Consider the element from part (a). When considered as an element of $\mathrm{Hom}(V, V)$, it is the identity map. Additionally, we know from class that a representation on $\mathrm{Hom}(V, V)$ is a homomorphism $\rho : G \to GL(\mathrm{Hom}(V, V))$ defined by $\rho(g)L = \rho_V(g) \circ L \circ \rho_V(g)^{-1}$. Thus, substituting in I for L, we learn that

$$\rho(g)I = \rho_V(g) \circ I \circ \rho_V(g)^{-1} = \rho_V(g) \circ \rho_V(g)^{-1} = I$$

Thus, regardless of what G, ρ, V are, ρ will preserve I. Therefore, $\mathrm{span}(I)$ is a subrepresentation and, importantly, it is the trivial subrepresentation since all elements of G act as the identity on it. $\square$

2 Introduction to Character Theory

10/13: 1. **More linear algebra**. Let V be a finite-dimensional vector space.

(a) Prove that under the identification of $V \otimes V^*$ with $\mathrm{Hom}_F(V,V)$, **simple** tensors $v \otimes \varphi$ correspond to linear maps of rank 0 or 1.

Proof. Let $v \otimes \varphi$ be an arbitrary simple tensor im $V \otimes V^*$. Recall from the 10/2 lecture that

$$v_1 \otimes \alpha \mapsto [v_2 \mapsto \alpha(v_2)v_1]$$

is a good isomorphism from $V \otimes V^* \cong \mathrm{Hom}_F(V,V)$. It follows that the linear map to which $v \otimes \varphi$ corresponds is the map $L : V \to V$ defined by $L(v') = \varphi(v')v$. Since $\mathrm{Im}\,\varphi = \mathbb{C}$, we have that $\mathrm{Im}(L) \leq \mathbb{C}v$. Thus, since $\dim(\mathbb{C}v) = 1$, we have that $\mathrm{rank}(L) \leq 1$, as desired. ☐

(b) Consider the vector space $W = \mathrm{Hom}_F(V,V)$. Prove that any linear functional in W^* has the form $L \mapsto \mathrm{tr}(LM)$ for some $M \in W$. Prove that the vector space $\mathrm{Hom}_F(V,V)$ is "canonically" self-dual.

Proof. Let $\varphi \in W^*$ be arbitrary. Also let $n := \dim V$ for convenience. Notice that the n^2 matrices E_{ij} $(i,j = 1,\dots,n)$, which have a 1 in the i^{th} row and j^{th} column and 0s everywhere else, form a basis of W. Thus, φ is fully characterized by its actions on the E_{ij}. Now define

$$M := (\varphi(E_{ij}))^T$$

It follows that if $L = (\ell_{ij})$, then

$$\varphi(L) = \sum_{i=1}^{n}\sum_{j=1}^{n} \ell_{ij}\varphi(E_{ij}) = \mathrm{tr}(LM)$$

as desired. This completes the proof, but to help illustrate it, I'll include the $n = 3$ case:

$$\underbrace{\begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix}}_{L=(\ell_{ij})} \circ \underbrace{\begin{bmatrix} \varphi(E_{11}) & \varphi(E_{12}) & \varphi(E_{13}) \\ \varphi(E_{21}) & \varphi(E_{22}) & \varphi(E_{23}) \\ \varphi(E_{31}) & \varphi(E_{32}) & \varphi(E_{33}) \end{bmatrix}^T}_{M=(\varphi(E_{ij}))^T}$$

$$= \underbrace{\begin{bmatrix} \sum_{j=1}^3 \ell_{1j}\varphi(E_{1j}) & & \cdots \\ & \sum_{j=1}^3 \ell_{2j}\varphi(E_{2j}) & \\ \cdots & & \sum_{j=1}^3 \ell_{3j}\varphi(E_{3j}) \end{bmatrix}}_{LM}$$

To prove that $\mathrm{Hom}_F(V,V) = W$ is canonically self-dual, it will suffice to construct an isomorphism $W^* \cong W$ that does not depend on a choice of basis. Let $\varphi \in W^*$ be arbitrary. As demonstrated above, there exists a unique corresponding $M \in W$ such that $\varphi(L) = \mathrm{tr}(LM)$ for all $L \in W$. Therefore, the map $f : W^* \to W$ defined by $\varphi \mapsto M$ is a bijection. To show that it is linear, too, we add in a basis and compute as follows.

$$\begin{aligned} f(\varphi_1 + \varphi_2) &= ([\varphi_1 + \varphi_2](E_{ij}))^T \\ &= (\varphi_1(E_{ij}) + \varphi_2(E_{ij}))^T \\ &= (\varphi_1(E_{ij}))^T + (\varphi_2(E_{ij}))^T \\ &= f(\varphi_1) + f(\varphi_2) \end{aligned} \qquad \begin{aligned} f(\lambda\varphi) &= ([\lambda\varphi](E_{ij}))^T \\ &= (\lambda\varphi(E_{ij}))^T \\ &= \lambda(\varphi(E_{ij}))^T \\ &= \lambda f(\varphi) \end{aligned}$$

☐

2. **Characters of abelian groups**. Let A be a finite abelian group.

(a) A **character** of A is a homomorphism $\chi : A \to \mathbb{C}^\times$. Prove that for every $g \in A$, the value $\chi(g)$ is a root of unity. Prove that the product of characters is a character. Prove that characters form an abelian group. This group is called the **dual** of A and is denoted $\widehat{A}$.

Proof. Let $g \in A$ be arbitrary. Since A is finite, $|g|$ is finite. It follows since χ is a homomorphism that

$$1 = \chi(e) = \chi(g^{|g|}) = \chi(g)^{|g|}$$

Therefore, since the only complex numbers having 1 as a power are the roots of unity, $\chi(g)$ is a root of unity, as desired.

Let χ_1, χ_2 be two characters of A. To prove that their product $\chi_1\chi_2$ is a character, it will suffice to show that $\chi_1\chi_2$ is a group homomorphism. To do so, we must confirm that

$$\chi_1\chi_2(e) = 1 \qquad \chi_1\chi_2(g_1g_2) = \chi_1\chi_2(g_1) \cdot \chi_1\chi_2(g_2) \qquad \chi_1\chi_2(g^{-1}) = \chi_1\chi_2(g)^{-1}$$

We can do this using the analogous statements satisfied by χ_1 and χ_2 separately. Specifically,

$$\begin{aligned} \chi_1\chi_2(e) &= \chi_1(e) \cdot \chi_2(e) \\ &= 1 \cdot 1 \\ &= 1 \end{aligned} \qquad \begin{aligned} \chi_1\chi_2(g^{-1}) &= \chi_1(g^{-1}) \cdot \chi_2(g^{-1}) \\ &= \chi_1(g)^{-1} \cdot \chi_2(g)^{-1} \\ &= (\chi_1(g) \cdot \chi_2(g))^{-1} \\ &= \chi_1\chi_2(g)^{-1} \end{aligned}$$

$$\begin{aligned} \chi_1\chi_2(g_1g_2) &= \chi_1(g_1g_2) \cdot \chi_2(g_1g_2) \\ &= \chi_1(g_1) \cdot \chi_1(g_2) \cdot \chi_2(g_1) \cdot \chi_2(g_2) \\ &= \chi_1(g_1) \cdot \chi_2(g_1) \cdot \chi_1(g_2) \cdot \chi_2(g_2) \\ &= \chi_1\chi_2(g_1) \cdot \chi_1\chi_2(g_2) \end{aligned}$$

Let $\widehat{A}$ denote the set of all characters of A. Also let $\cdot$ denote the operation of function multiplication, which was shown in the above proof to be a binary operation on $\widehat{A}$. To prove that $(\widehat{A}, \cdot)$ is an abelian group, it will suffice to show that it has an identity element, inverses, associativity, and commutativity. Let's begin.

Identity: Consider the character χ_e defined by $g \mapsto 1$ for all $g \in A$. Let $\chi \in \widehat{A}$ be arbitrary. Then since we have the following, letting $g \in A$ be arbitrary, we know that $\chi\chi_e = \chi = \chi_e\chi$, as desired.

$$\chi\chi_e(g) = \chi(g) \cdot \chi_e(g) = \chi(g) \cdot 1 = \chi(g) = 1 \cdot \chi(g) = \chi_e(g) \cdot \chi(g) = \chi_e\chi(g)$$

Inverses: Let $\chi \in \widehat{A}$ be arbitrary. Consider the character $\bar{\chi}$ defined by $g \mapsto \overline{\chi(g)}$ for all $g \in A$, where the overbar denotes taking the complex conjugate. Then since we have the following, letting $g \in A$ be arbitrary, we know that $\chi\bar{\chi} = \bar{\chi}\chi = \chi_e$, as desired. Note that the complex conjugates multiply to 1 because we showed above that all $\chi(g)$ are roots of unity (for any $\chi \in \widehat{A}$).

$$\chi\bar{\chi}(g) = \chi(g) \cdot \bar{\chi}(g) = \bar{\chi}(g) \cdot \chi(g) = 1 = \chi_e(g)$$

Associativity: Let $\chi_1, \chi_2, \chi_3 \in \widehat{A}$ be arbitrary. Then since we have the following, letting $g \in A$ be arbitrary, we know that $\chi_1(\chi_2\chi_3) = (\chi_1\chi_2)\chi_3$, as desired.

$$[\chi_1(\chi_2\chi_3)](g) = \chi_1(g) \cdot \chi_2\chi_3(g) = \chi_1(g) \cdot \chi_2(g) \cdot \chi_3(g) = \chi_1\chi_2(g) \cdot \chi_3(g) = [(\chi_1\chi_2)\chi_3](g)$$

Commutativity: Let $\chi_1, \chi_2 \in \widehat{A}$ be arbitrary. Then since we have the following, letting $g \in A$ be arbitrary, we know that $\chi_1\chi_2 = \chi_2\chi_1$, as desired.

$$\chi_1\chi_2(g) = \chi_1(g) \cdot \chi_2(g) = \chi_2(g) \cdot \chi_1(g) = \chi_2\chi_1(g)$$

∎

(b) Prove directly that for every nontrivial character $\chi \in \widehat{A}$, the following identity holds.

$$\sum_{g\in A}\chi(g) = 0$$

Proof. Let $\chi \in \widehat{A}$ be a nontrivial character. Since it is nontrivial, there exists $h \in A$ for which $\chi(h) \neq 1$. Additionally, we have by the Sudoku Lemma that

$$\sum_{g\in A}\chi(g) = \sum_{g\in A}\chi(hg)$$

But then since χ is a homomorphism, we have

$$\begin{aligned}\sum_{g\in A}\chi(g) &= \sum_{g\in A}\chi(hg) = \sum_{g\in A}\chi(h)\chi(g) = \chi(h)\sum_{g\in A}\chi(g)\\ (1-\chi(h))\sum_{g\in A}\chi(g) &= 0\end{aligned}$$

Thus, by the zero-product property, either $1-\chi(h) = 0$ or $\sum_{g\in A}\chi(g) = 0$. Since $\chi(h) \neq 1$ as proven above, $1-\chi(h)\neq 0$ so we must have

$$\sum_{g\in A}\chi(g) = 0$$

as desired. □

(c) Prove that characters are the same as the 1-dimensional representations of A; product of characters is the same as a tensor product of representations, and the inverse of the character is the same as the dual representation.

Proof. Let $\rho_1,\rho_2 : A \to GL(\mathbb{C}) = \mathbb{C}^\times$ be two arbitrary 1-dimensional representations of A. Notice that ρ_1,ρ_2 have the same domain and codomain as characters, and are homomorphisms. Additionally, by the definition of the Kronecker product for 1×1 matrices, we have that

$$[\rho_1\otimes\rho_2](g) = \rho_1(g)\cdot\rho_2(g)$$

for all $g\in A$. Thus, the tensor product of these representations is the same the character product of their values. Lastly, we have that

$$\rho_1^*(g) = \rho_1(g^{-1})^T = \rho_1(g^{-1}) = \rho_1(g)^{-1} = \overline{\rho_1(g)}$$

Thus, the dual representation of this representation is computed using the character inverse. □

(d) Find all characters for $A = \mathbb{Z}/n\mathbb{Z}$. Compute the dual group $\widehat{A}$. Do the same for $A = (\mathbb{Z}/2\mathbb{Z})\times(\mathbb{Z}/2\mathbb{Z})$.

Proof. We treat each group separately.

<u>$\mathbb{Z}/n\mathbb{Z}$</u>: Let $\chi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^\times$ be a character of $\mathbb{Z}/n\mathbb{Z}$. Since χ is a homomorphism and hence $\chi(n) = \chi(1)^n$, the value of $\chi(1)$ completely determines the action of χ. Thus, there is a one-to-one mapping between the characters of $\mathbb{Z}/n\mathbb{Z}$ and the possible values of $\chi(1)$, so let's investigate the latter. We know from part (a) that $\chi(1)$ is a root of unity and that $\chi(1)^n = 1$. Consequently, $\chi(1)$ is an n^{th} root of unity. Any such root of unity will work, so the characters $\chi_0,\dots,\chi_{n-1}$ of $\mathbb{Z}/n\mathbb{Z}$ are defined by

$$\boxed{\widehat{\mathbb{Z}/n\mathbb{Z}} = \{\chi_k \mid k = 0,\dots,n-1;\ \chi_k(1) = \mathrm{e}^{2\pi ik/n}\}}$$

<u>K_4</u>: The maximum order of any element in this group is 2, so $\chi : K_4 \to \{-1,1\}$. While there are $2^4 = 16$ such maps, only four are homomorphisms: Those sending $((0,0),(0,1),(1,0),(1,1))$ to...

$$\boxed{\widehat{K}_4 = \{(1,1,1,1),(1,1,-1,-1),(1,-1,1,-1),(1,-1,-1,1)\}}$$

□

(e) Prove that $\widehat{A_1 \times A_2}$ is isomorphic to $\widehat{A}_1 \times \widehat{A}_2$. Prove that groups A and $\widehat{A}$ are isomorphic as abstract groups. Deduce that an abelian group of order n has exactly n characters.

Proof. Define $h : \widehat{A}_1 \times \widehat{A}_2 \to \widehat{A_1 \times A_2}$ by

$$h(\chi_1, \chi_2) = \chi_1 \otimes \chi_2$$

where $[\chi_1 \otimes \chi_2](a_1, a_2) = \chi_1(a_1) \cdot \chi_2(a_2)$. Note that we are borrowing part (c)'s conclusion that characters can be treated like representations and, in particular, can have tensor products. To prove that h is an isomorphism of groups, it will suffice to show that it is a bijective homomorphism of groups. The following suffices to show that it is a homomorphism of groups.

$$\begin{aligned} h[(\chi_1, \chi_2) \cdot (\chi_3, \chi_4)] &= h(\chi_1\chi_3, \chi_2\chi_4) \\ &= \chi_1\chi_3 \otimes \chi_2\chi_4 \\ &= \chi_1 \otimes \chi_2 \cdot \chi_3 \otimes \chi_4 \\ &= h(\chi_1, \chi_2) \cdot h(\chi_3, \chi_4) \end{aligned}$$

Note that the transition form the second to the third line above is justified because the equality becomes $\chi_1(a_1) \cdot \chi_3(a_1) \cdot \chi_2(a_2) \cdot \chi_4(a_2) = \chi_1(a_1) \cdot \chi_2(a_2) \cdot \chi_3(a_1) \cdot \chi_4(a_2)$ when applied to (a_1, a_2) and expanded. As to bijectivity, we will prove injectivity then surjectivity. For injectivity, suppose $h(\chi_1, \chi_2) = h(\chi_3, \chi_4)$. Then for all $a_1 \in A_1$,

$$\begin{aligned} [h(\chi_1, \chi_2)](a_1, e) &= [h(\chi_3, \chi_4)](a_1, e) \\ \chi_1(a_1) \cdot \chi_2(e) &= \chi_3(a_1) \cdot \chi_4(e) \\ \chi_1(a_1) \cdot 1 &= \chi_3(a_1) \cdot 1 \\ \chi_1(a_1) &= \chi_3(a_1) \end{aligned}$$

A similar statement holds for χ_2 and χ_4, proving that $(\chi_1, \chi_2) = (\chi_3, \chi_4)$, as desired. For surjectivity, let $\chi \in \widehat{A_1 \times A_2}$ be arbitrary. Define χ_1 and χ_2 by

$$\chi_1(a_1) = \chi(a_1, e) \qquad\qquad \chi_2(a_2) = \chi(e, a_2)$$

for all $a_1 \in A_1$ and $a_2 \in A_2$. That χ_1, χ_2 are characters under these definitions instead of just functions follows immediately from the character-like properties of χ: indeed, with these definitions in hand, we can show that

$$[h(\chi_1, \chi_2)](a_1, a_2) = \chi_1(a_1) \cdot \chi_2(a_2) = \chi(a_1, e) \cdot \chi(e, a_2) = \chi[(a_1, e) \cdot (e, a_2)] = \chi(a_1, a_2)$$

as desired.

By the fundamental theorem of finite abelian groups, A is isomorphic to a direct product of cyclic groups of prime power order. Thus, we may let

$$A \cong (\mathbb{Z}/p_1\mathbb{Z})^{n_1} \times \cdots \times (\mathbb{Z}/p_k\mathbb{Z})^{n_k}$$

Borrowing the notation from the first task of part (d) above, define $h : (\mathbb{Z}/p_1\mathbb{Z})^{n_1} \times \cdots \times (\mathbb{Z}/p_k\mathbb{Z})^{n_k} \to (\widehat{\mathbb{Z}/p_1\mathbb{Z}})^{n_1} \times \cdots \times (\widehat{\mathbb{Z}/p_k\mathbb{Z}})^{n_k}$ by

$$h(a_{11}, \dots, a_{kn_k}) = \chi_{a_{11}} \otimes \cdots \otimes \chi_{a_{kn_k}}$$

For the same reasons mentioned in part (d), h is an isomorphism. Additionally, by consecutive applications of the first claim in this part,

$$(\widehat{\mathbb{Z}/p_1\mathbb{Z}})^{n_1} \times \cdots \times (\widehat{\mathbb{Z}/p_k\mathbb{Z}})^{n_k} \cong (\mathbb{Z}/p_1\mathbb{Z})^{n_1} \widehat{\times \cdots \times} (\mathbb{Z}/p_k\mathbb{Z})^{n_k}$$

But since $A \cong (\mathbb{Z}/p_1\mathbb{Z})^{n_1} \times \cdots \times (\mathbb{Z}/p_k\mathbb{Z})^{n_k}$, the group on the right above is isomorphic to $\widehat{A}$. Thus, by chaining together isomorphisms, we can get all the way from A to $\widehat{A}$, as desired.

Since isomorphic groups have the same order, an abelian group of order n has a dual group with order n, i.e., has n characters, as desired. $\square$

3. Consider the permutational representation of S_n. Decompose it into the sum of (two) irreducible representations.

Proof. Let $\rho : S_n \to GL(V)$ be the permutational representation of S_n. As discussed in class, ρ fixes the one-dimensional subspace $\operatorname{span}(1, \dots, 1) \leq V$. Thus, this subspace forms a trivial subrepresentation of V. It follows by the theorem from class that this subspace has a complement; this complement is the standard representation. Thus,

$$V = (3) \oplus (2, 1)$$

As a one-dimensional representation, (3) is clearly irreducible, but it is not immediately evident that $(2, 1)$ is. Fortunately, the following proves that it is. Assuming $n \geq 2$ since the 1D case is trivial. If we take the column vector $(1, -1, 0, \dots, 0) \in (2, 1)$, we can generate from it $n - 1$ other linearly independent column vectors using consecutive applications of $\sigma = (12 \cdots n)$. For example, if $n = 4$, we generate

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

Therefore, there is no subspace of $(2, 1)$ that we can't get into when we're in $(2, 1)$ *except* that which we've already discussed: (3). It follows that $(2, 1)$ is irreducible, as well. $\square$

4. Let G be a finite group.

(a) Define the **space of invariants** of a representation V by the formula

$$V^G = \{v \in V \mid gv = v \ \forall \ g \in G\}$$

Prove that V^G is a subrepresentation of V. Prove that it is isomorphic to a sum of trivial representations.

Proof. To prove that V^G is a subrepresentation of V, it will suffice to show that it is a subspace of V, and $gV^G \subset V^G$ for all $g \in G$. Since we clearly have $g \cdot 0 = 0$ for all $g \in G$, $g(v_1 + v_2) = v_1 + v_2$ for all v_1, v_2 satisfying $gv_i = v_i$, and $agv = g(av)$, V^G is a subspace of V. Now for closure under the g's, let $v \in V^G$ be arbitrary. But by the definition of V^g, we have $gv = v \in V^g$ for all $g \in G$, as desired.

Let $e_1, \dots, e_k$ be a basis of V^G. Since $g(\lambda e_i) = \lambda e_i$ for all $\lambda e_i \in \operatorname{span}(e_i)$, $i = 1, \dots, k$, each $\operatorname{span}(e_i)$ is, itself, fixed by all g and hence a trivial subrepresentation of V^G. Therefore,

$$V^G \cong \underbrace{(3) \oplus \cdots \oplus (3)}_{k \text{ times}}$$

as desired. $\square$

(b) Prove that $(\operatorname{Hom}_F(V, W))^G$ is isomorphic to $\operatorname{Hom}_G(V, W)$.

Proof. To prove the claim, it will suffice to prove the stronger condition that $(\operatorname{Hom}_F(V, W))^G = \operatorname{Hom}_G(V, W)$ as sets. We will proceed via a bidirectional inclusion proof. Let's begin.[2]

First, let $f \in (\operatorname{Hom}_F(V, W))^G$ be arbitrary. Then by the definition of the space of invariants, $g \cdot f = f$ for all $g \in G$. Additionally, since $G \circlearrowright \operatorname{Hom}_F(V, W)$ via $g \cdot f = gfg^{-1}$, we have that $gfg^{-1} = f$, i.e., $gf = fg$ for all $g \in G$. But this implies that f is a morphism of G-representations, i.e., $f \in \operatorname{Hom}_G(V, W)$, as desired.

The proof is symmetric in the reverse direction. $\square$

[2]Note: Beware rampant abuses of notation throughout this proof. For example, the statement $gf = fg$ stands in for the much more complex $\rho_V(g) \circ f = f \circ \rho_W(g)$.

5. Let $\rho : G \to GL_n(\mathbb{C})$ be a representation with character χ.

   (a) Prove that $\operatorname{Ker}(\rho) = \{g \in G \mid \chi(g) = n\}$.

   Proof. We proceed via a bidirectional inclusion proof.

   Suppose first that $g \in \operatorname{Ker}(\rho)$. Then $\rho(g) = I_n$. But since $\operatorname{tr}(I_n) = n$ and $\chi(g) = \operatorname{tr}(\rho(g))$, we have by transitivity that $\chi(g) = n$, as desired.

   Now suppose that $\chi(g) = n$. Recall from class that every eigenvalue λ_i of $\rho(g)$ is a root of unity. Additionally, since $\lambda_1 + \cdots + \lambda_n = \chi(g) = n$, we must have $\lambda_i = 1$ $(i = 1, \dots, n)$. But this implies that $\rho(g) = I_n \in \operatorname{Ker}(\rho)$, as desired. □

   (b) Prove that for any $g \in G$, we have $|\chi(g)| \leq n$.

   Proof. As in part (a), recall from class that every eigenvalue λ_i of $\rho(g)$ is a root of unity. Then by the triangle inequality,

$$|\chi(g)| = |\lambda_1 + \cdots + \lambda_n| \leq |\lambda_1| + \cdots + |\lambda_n| = 1 + \cdots + 1 = n$$

   as desired. □

   (c) Prove that for a given $g \in G$, $|\chi(g)| = n$ if and only if there exists $\lambda \in \mathbb{C}$ such that $\rho(g) = \lambda I$.

   Proof. Suppose first that $|\chi(g)| = n$. Then $|\lambda_1 + \cdots + \lambda_n| = n$, so since $|\lambda_i| = 1$ for $i = 1, \dots, n$, we must have $\lambda_1 = \cdots = \lambda_n$. Define $\lambda := \lambda_i$. Recall that a linear operator with n eigenvalues must have a corresponding $n \times n$ matrix in some basis equal to $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Therefore, in this case, the corresponding matrix of $\rho(g)$ is λI (and is λI in any basis), as desired.

   Now suppose that $\rho(g) = \lambda I$. Then

$$|\chi(g)| = |\operatorname{tr}(\lambda I)| = |n\lambda| = n \cdot |\lambda| = n \cdot 1 = n$$

   as desired. □

4 Advanced Character Theory and Introduction to Associative Algebras

10/27: 4. Let F be a field.

(a) Prove that the matrix algebra $M_{n,n}(F)$ is simple, i.e., has no nontrivial ideals.

Proof. To prove that $M_{n,n}(F)$ is simple, it will suffice to show that any nonzero ideal of it contains the identity. Let I be a nonzero ideal of $M_{n,n}(F)$. Then there exists some nonzero $A \in I$. In particular, if A is nonzero, then it contains some entry $a_{ij} \neq 0$. Now, let $E_{ij} \in M_{n,n}(F)$ be the matrix with 1 in the i,j position and 0 everywhere else. Then for each $k = 1, \dots, n$,

$$\frac{1}{a_{ij}} E_{ki} A E_{jk} = E_{kk}$$

so $E_{kk} \in I$. Summing all of the E_{kk}'s within I yields the $n \times n$ identity matrix, as desired. $\square$

5. Consider the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\} \subset \mathbb{H}$ of order 8.

(a) Find four 1-dimensional representations of Q_8. Find the character of the remaining 2-dimensional representation.

Proof. Take the trivial representation.

Then take the representations sending i, j to (-1), i, k to (-1), and j, k to (-1); everything else gets sent to the identity in each case.

Working out the character table at this point, we can see that the character for the final representation must be given explicitly by the following.

$$+1 \mapsto 2 \qquad -1 \mapsto -2 \qquad +i \mapsto 0 \qquad -i \mapsto 0 \qquad +j \mapsto 0 \qquad -j \mapsto 0 \qquad +k \mapsto 0 \qquad -k \mapsto 0$$

$\square$

References

Fulton, W., & Harris, J. (2004). *Representation theory: A first course* (S. Axler, F. W. Gehring, & K. A. Ribet, Eds.). Springer.