Week 5

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5.1 Wedderburn-Artin Theory

10/23:

- Share notes with Rudenko at the end of the course!
- Today: Wedderburn-Artin theory.
  - Noncommutative algebra.
  - Noncommutative is a big part of math, partially because of its relation to QMech and partially because of its use in math, itself.
  - There is a textbook: Lang (2002). It's a hard, grad-level textbook but very cleanly written. Not a bad book to have in our mind as we start to encounter category theory.
- So here's what were talking about.
  - Our main object is A, an **associative algebra** over a field F.
- Left vs. right algebras.
  - When A is not commutative, we have to specify which we are dealing with.
  - Let A be an algebra over F.
  - Recall left-modules and right-modules.
    - In a left module, you can multiply $A \times M \to M$ where $(ab)m = a(bm)$.
    - In a right module, $(ab)m = b(am)$. More simply, $m(ab) = (ma)b$.
    - With modules, we get submodules, quotient modules, homomorphisms of modules, etc.
  - Let $I \subset A$ be a left-submodule. Thus, it is a subspace of A such that for all $a \in A$, $aI \subset I$, i.e., a left ideal.
  - In a right-submodule $I \subset A$, we have that for all $b \in A$, $Ib \subset I$, i.e., a right ideal.
  - In a two-sided ideal $I \subset A$, we have for all $a, b \in I$ that $aI \subset I$ and $Ib \subset I$.
  - Example: The matrix algebra is the prototypical noncommutative algebra. Consider $M_{2\times 2}(\mathbb{C})$.
    - Pick $v = (1, 0)$.
    - Look at ideal $I = \{X \in M_{2\times 2} \mid Xv = 0\}$. This is called the **annihilator**, and it is a left ideal. Explicitly, this ideal is the subset of all matrices of the form

$$\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}$$

      for $a, b \in \mathbb{C}$.

  - An example of a right ideal is all those such that $vX = 0$, i.e., all matrices of the form

$$\begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix}$$

  - There are *no* two-sided ideals herein, save the trivial one.

- **Simple** (algebra): An algebra for which there are no nontrivial two-sided ideals.
- Every time you go more abstract, it's more boring because you have less things to play with, but we can derive more general rules.
  - We'll only stay so abstract for 2-3 lectures.
- We want to convert left-algebras to right-algebras.
  - To do so, we can construct **opposite algebras**.
- **Opposite algebra** (of A): The algebra with the same vector space structure as A, but with the reversed multiplication such that $a * b$ in this space yields $b * a$ in A. *Denoted by* $\boldsymbol{A^{\text{op}}}$.
  - Left ideals of A become right ideals of A^{op} and vice versa. Two-sided ideals stay the same.
  - In category theory, left-modules over A are equivalent to right-modules over A^{op}.
  - Opposite algebras are briefly defined on Fulton and Harris (2004, p. 308) and are not defined anywhere else in any of the other sources.
- Example: Consider $M_{n\times n}(F)^{\text{op}}$.
  - Claim: This algebra equals regular $M_{n\times n}(F)$.
  - The map between these spaces is $A \mapsto A^T$.
  - There are other maps, such as conjugation and then transpose.
  - Being isomorphic to your opposite is a strange and interesting property!
- Example: $\mathbb{C}[G]^{\text{op}} \cong \mathbb{C}[G]$.
  - Left as an exercise to find the map.
- Let M, N be modules. We now investigate some properties of $\text{Hom}_A(M, N)$, a nice abelian group.
  - Explicitly, it's

$$\text{Hom}_A(M, N) = \{f : M \to N \text{ linear} \mid f(am) = af(m)\ \forall\ a \in A\}$$

  - We have that

$$\text{Hom}_A(M_1 \oplus M_2, N) \cong \text{Hom}_A(M_1, N) \oplus \text{Hom}_A(M_2, N)$$

    - Prove by looking at what happens to vectors of the form $(M_1, 0)$ and $(0, M_2)$.
  - Similarly,

$$\text{Hom}_A(M, N_1 \oplus N_2) \cong \text{Hom}_A(M, N_1) \oplus \text{Hom}_A(M, N_2)$$

- What if we have $\text{Hom}(M_1 \oplus \cdots \oplus M_n, N_1 \oplus \cdots \oplus N_m)$?
  - Then we have by induction from the previous cases that

$$\text{Hom}(M_1 \oplus \cdots \oplus M_n, N_1 \oplus \cdots \oplus N_m) = \bigoplus_{\substack{i=1,\dots,n \\ j=1,\dots,m}} \text{Hom}(M_i, N_j)$$

  - Let $\varphi_{ij} \in \text{Hom}(M_i, N_j)$.

- At this point, it's very natural to write matrices

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} & & \\ & \varphi_{ji} & \\ & & \end{bmatrix} \end{array} \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} = \begin{pmatrix} \varphi_{11}(m_1)+\cdots+\varphi_{1n}(m_n) \\ \vdots \\ \ \end{pmatrix} = \begin{pmatrix} (\varphi(m)) \\ \vdots \\ \ \end{pmatrix}$$

  - Is it ϕ_{ji} or ϕ_{ij}?? Lang (2002, p. 642) seems to back the latter.

- To make this make sense for ourselves, write out the 2×2 case from $M_1 \oplus M_2 \to M_1 \oplus M_2$.

$$\begin{pmatrix} \varphi_{11} & \varphi_{21} \\ \varphi_{12} & \varphi_{22} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} \qquad\qquad \\ \ \end{pmatrix}$$

- Matrices made out of maps can seem really confusing when you first start, but in time, it will make sense.

- Recall the result from last time about division algebras.
- The main object we need to understand is a **semisimple algebra**.
- **Semisimple** (module): A module that satisfies any of the conditions in the following theorem.
  - Note that we proved something analogous to condition 3 early on! This was the complements theorem.
  - This is equivalent for infinite-dimensional algebras; we need **Zorn's lemma** regarding maximal ideals/the axiom of choice here, though.
- Theorem: Let A be an algebra over F, and let M be a left-module. Then TFAE.
  1. $M = \bigoplus_{i \in I} S_i$, where each S_i is a simple module and I is an **indexing set**, not a simple module/ideal.
  2. $M = \sum_{i \in I} S_i$, where the sum is *not* direct.
  3. For all submodules $N \subset M$, there exists N' such that $M = N \oplus N'$.

Proof. This proof only applies for the case that M is finite dimensional; the theorem is more general than that, but we are not interested in the more general case.

($1 \Rightarrow 2$): Very clear; all direct sums are sums.

($2 \Rightarrow 1$): Consider the maximal subset $J \subset I$ (by inclusion, not by indices) of our indexing set such that

$$\sum_{i \in J} S_i = \bigoplus_{i \in J} S_i$$

In other words, J induces the highest-dimension sum of submodules that is a direct sum. Note that we can still find a singleton J in the direct-sum-of-one-thing case, so we're starting from a good base case.

Claim: $\bigoplus_{i \in J} S_i = M$. Suppose not. Then there exists $m \in M$ such that $m \notin \bigoplus_{i \in J} S_i$ and $m = s_{i_1} + \cdots + s_{i_k}$ where each $s_{i_j} \in S_{i_j}$. If all $s_{i_1}, \dots, s_{i_k} \in \bigoplus_{i \in J} S_i$, then we have arrived at a contradiction and we are done. If not, then there exists some s_{i_t} such that $s_{i_t} \notin \bigoplus_{i \in J} S_i$. Now consider $S_{i_t} \cap (\bigoplus_{i \in J} S_i)$. This will be a submodule of S_{i_t}. But since S_{i_t} is simple by hypothesis, this means that $S_{i_t} \cap (\bigoplus_{i \in J} S_i)$ either equals S_{i_t} or 0. However, we know that it can't equal S_{i_t} because above, we found $s_{i_t} \in S_{i_t}$ such that $s_{i_t} \notin \bigoplus_{i \in J} S_i$. Thus, $S_{i_t} \cap (\bigoplus_{i \in J} S_i) = 0$. But this means that $S_{i_t} + \bigoplus_{i \in J} S_i$ is a direct sum, which contradicts the choice of J as maximal.

($1 \Rightarrow 3$): Let's take a submodule $N \subset M$. By 1, $M = \bigoplus_{i \in I} S_i$. Let's look a tall subsets J such that

$$N + \sum_{j \in J} S_j = N \oplus \left(\sum S_j \right)$$

Look at the maximal one by inclusion. Then once again, by the same proof strategy as above,

$$N \oplus \underbrace{\left(\sum S_j\right)}_{N'} = M$$

($3 \Rightarrow 1$): We use what we've learned about representations. Let $M = N_1 \oplus N_2$. Then N_2, if nonsimple, has subsets $N_2 \oplus N_3$. We can continue on and on. Because dimensions finitely decrease, we'll eventually have to arrive at a sum $N_1 \oplus \cdots \oplus N_m$ of simples. ☐

- Now, we have 3 definitions of semisimple modules.
- Corollary: If A is an algebra, M is a semisimple module, and $N \subset M$ is a submodule, then...
  1. N is semisimple.

     Proof. Let L be a submodule of N. We need to find a complement of L inside N. We can find $L' \subset M$ such that $L \oplus L' = M$. Then $L' \cap N \subset N$ is the complement of L in N. Why? Because of the following.
     Claim: $(L' \cap N) \bigoplus L = N$. Not intersecting: $L' \cap N \cap L \subset L' \cap L = 0$. Summing to the whole thing: Let $n \in N$ be arbitrary. Then since $n \in M$, there exists $\ell, \ell' \in L, L'$ such that $n = \ell + \ell'$. But since $n, \ell \in N$, we must have $\ell' \in N$ as well. Therefore, $\ell' \in L' \cap N$. ☐

  2. M/N is semisimple.
- Takeaway: Submodules and quotient modules of semisimple modules are semisimple modules.
- Lang (2002) has a write-up of the proof from today's class.
  - Funnily enough, it is the only textbook that does! Fulton and Harris (2004) doesn't have it; not even Etingof et al. (2011) has it!