MATH 26700 (Introduction to Representation Theory of Finite Groups) Notes

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Weeks

List of Figures

Week 1

Introduction to Representation Theory

1.1 Motivating and Defining Representations

9/27:
- Rudenko would happily approve my final substitution, but it's not his call; it's Boller's.
- HW will be due every week on Wednesday or thereabouts.
  - Submit in paper in a mailbox, location TBA.
  - First HW due next Wednesday.
- Midterm eventually and an in-class final.
- Grading scheme in the syllabus.
- OH not available MW after class (Rudenko has to run to something else), but F after class, we can ask him anything.
  - Regular OH MTh, time TBA.
- There is no specific book for the course.
  - First 8 lectures come from Serre (1977); amazing book but very concise; gets confusing later on. Most lectures are made up by Rudenko.
- Course outline.
  1. Character theory: Beautiful, not too hard.
  2. Non-commutative algebra: More abstract/general approach to the same thing.
  3. Advanced topics, S_n.
- This course's focus: Representations of finite groups in finite dimensions over $\mathbb{C}$.
- This course is for math-inclined people (not quite physics) and lays the foundation for all other Rep Theory.
  - The ideas would be presented in a very different way in Physics Rep Theory.
- We can always ask questions and stop him to correct mistakes during class.
- Why we care about representations.
  - Start with a group G, finite. For example, let $G \equiv S_1$.

  - People started to play with S_4 (permutations of roots of a polynomial of degree 4) in Galois theory.
    - Galois theory primer: Consider a polynomial like $x^4 + 3x + 1 = 0$; the roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ satisfy tons of equations, e.g., $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$ since 1 is the x^0 term.
  - But groups also occur in much more natural places, e.g., isometries of $\mathbb{R}^3$ that preserve a tetrahedron.
  - S_4 is also orientation-preserving isometries of $\mathbb{R}^3$ that preserve a cube.
  - Many things lead to the same group!
  - Theory of abstract groups developed far later than any of these perspectives; was developed to unify them.

- Recall group actions: Take $G, X = \{x_1, \dots, x_n\}$ both finite. We want $G \circlearrowright X$, which is a homomorphism $A : G \to S_n$.
- What can we do now?
  - We can look at orbits, which are smaller pieces.
  - We can look at the stabilizer.
  - We can identify orbits with cosets.
  - If we understand all possible subgroups, we understand all possible actions.
- This story is not boring, but it's simplistic.
- Rudenko doesn't assume we remember everything (phew!).
- Main definition (general to start, then we simplify).
- **Group representation** (of G on V): A group homomorphism $G \to GL(V)$, for G a group, V a finite-dimensional vector space over some field $\mathbb{F}$ with basis $\{e_1, \dots, e_n\}$, and $GL(V)$ the set of isomorphic linear maps $L : V \to V$. *Denoted by* $\boldsymbol{\rho}$.
  - Recall that $GL(V) = GL_n(\mathbb{F})$ is the set of all $n \times n$ invertible matrices.
- For every element $g \in G$, $g \mapsto \rho(g) = A_g$. Essentially, you're mapping to elements that satisfy certain equations.
  - For example, $A_e = E_n$, $A_{g_1 g_2} = A_{g_1} A_{g_2}$, and $A_{g^{-1}} = {A_g}^{-1}$.
  - Thus, representations are a "concrete way to think about groups."
  - If you don't understand abstract group G, let us compare it to a group that we do understand! Like a group can *act* on S_n, we can *represent* a group in a vector space.
- In this course, G is finite, $\mathbb{F} = \mathbb{C}$, and V is finite dimensional.
  - This is the most simple case, but also a very interesting one. The theory is much, much easier, so we can get much more complicated, but this is a good place to start.
  - We could make G compact, but we're not gonna go that far.
- Examples to get an idea of what's going on.
  1. $\dim \rho = 1$ (means $\dim V = 1$). Then $\rho : G \to GL_1(V) = \mathbb{C}^\times$. The codomain is referred to as the **character** of the group.
     - An example group homomorphism $S_n \to \mathbb{C}^\times$ is the sign function $\sigma \to \text{sign}(\sigma) = \{\pm 1\}$.
     - Another example is the **trivial representation**, $G \to \mathbb{C}^\times$ and $g \mapsto 1$.
  2. Smallest one: Let $G = S_3$. The structure is already pretty rich, and this will be part of the homework.

  - **Trivial representation** again.
  - **Alternating representation**.
  - **Standard representation**.
  - **Regular representation**.

- **Trivial representation**: The representation $\rho : G \to GL(V)$ sending $g \mapsto 1$ for all $g \in G$. *Denoted by* , **(3)**.
  - The boxes notation is too much of a detour to explain now.
  - Note that $1 \in GL(V)$ is the identity map on V!
- **Alternating representation**: The representation $\rho : G \to GL(V)$ sending $g \mapsto \text{sign}(g)$ for all $g \in G$. *Denoted by* , $\mathbf{(1,1,1)}$.
- **Standard representation**: The representation $\rho : S_n \to GL(V)$ sending $\sigma \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)})$, where $V = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_1 + \cdots + x_n = 0\}$ is a $(n-1)$-dimensional vector space. *Denoted by* , $\mathbf{(2,1)}$.
  - A 2D representation like rotating a triangle.
  - This gives something with real numbers.
  - Example: $S_3 \circlearrowright V$ by $\sigma((x_1, x_2, x_3)) = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$.
- **Regular representation**: The representation $\rho : G \to \text{Hom}(\mathbb{C}^n)$ defined by $g \mapsto \sigma_g$, where $G = \{g_1, \dots, g_n\}$, $\{e_{g_1}, \dots, e_{g_n}\}$ is a basis of $\mathbb{C}^n$, $\cdot$ is the group action of $\rho(G) \circlearrowright \mathbb{C}^n$ by $\rho(g) \cdot e_g = e_{gg_i}$, and $\sigma_g(e_{g_i}) = \rho(g) \cdot e_g = e_{gg_i}$.
  - This is a permutation of vectors.
  - Thus, for S_3, it will already be 6-dimensional (it's very high dimensional).
- How do we know that representation theory is tractable? Sure, we can define all these things, but how do we know that it will lead anywhere? Here's an example.
  - Let $G = \mathbb{Z}/2\mathbb{Z} = \{e, g\}$, $V = \mathbb{C}^n$, A an $n \times n$ matrix over $\mathbb{C}$, $\rho : G \to GL_n(\mathbb{C})$, and $A := \rho(g)$. Since $g^2 = e$, we know for example that $A^2 = E_n$.
  - But how do we find the matrices A? If we look at eigenvalues of A, there are only two possibilities: ± 1. The structure of A can be very complicated with Jordan normal form and all that, but in fact, these are the **semisimple matrices**, so it's not that bad.
  - Since $A^2 = E$, we know that $(A - E)(A + E) = 0$. Consider $(A - E) : V \to V$. Naturally, it has $\ker(A - E)$ and $\text{Im}(A - E)$. In this particular case, Rudenko claims that $\ker(A - E) \cap \text{Im}(A - E) = \{0\}$.
    - Proof: Let $v \in \ker(A - E) \cap \text{Im}(A - E)$ be arbitrary. Since $v \in \text{Im}(A - E)$, there exists $w \in V$ such that $v = (A - E)w = Aw - w$. Since $v \in \ker(A - E)$, we have $(A - E)v = 0$, so $Av = v$. It follows that $A(Aw - w) = Aw - w$ but also $A(Aw - w) = Ew - Aw = w - Aw$. Thus,

$$\begin{aligned} Aw - w &= w - Aw \\ 2Aw &= 2w \\ Aw &= w \end{aligned}$$

      But then $w \in \ker(A - E)$, so $v = (A - E)w = 0$.
  - This combined with the fact that every vector in a vector space is in either the image or the kernel of a linear map[1] implies that $V = \ker(A - E) \oplus \text{Im}(A - E)$.

[1] See Theorem 3.6 of Axler (2015).

- Let the kernel have basis $e_1, \dots, e_k$ and the image have basis $e_{k+1}, \dots, n$; then all A are of the following form.

$$\begin{array}{c|c}
 & \begin{array}{cccc} 1 & k & k+1 & n \end{array} \\
\begin{array}{c} 1 \\ \\ k \\ k+1 \\ \\ n \end{array} & \left[\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{array}\right]
\end{array}$$

- Next time, we will discuss sums of representations, of which this is an example of the theory.

- The same kind of thing, **simple representations**, happens with all finite groups?? This is where we're going. It's not rocket science; in fact, we'll see it next week.
- Last thing for today: A remarkable story.
  - The story of representation theory started quite different.
  - A beautiful theorem that we can prove now!
  - Frobenius determinant.
  - Think of $G = \{g_1, \dots, g_n\}$. Picture its multiplication table.
  - In every row and column, you see each element once.
  - Let's associate to the multiplication table an actual determinant in the linear algebra sense. Consider elements $x_{g_1}, \dots, x_{g_n}$. Define the $n \times n$ matrix $(x_{g_i g_j})$. Take its determinant. It will be a polynomial in n variables, i.e., an element of the ring $\mathbb{Z}[x_{g_1}, \dots, x_{g_n}]$.
  - Example: Consider
$$\begin{vmatrix} e & g \\ g & e \end{vmatrix}$$
    - The determinant is $x_e^2 - x_g^2 = (x_e - x_g)(x_e + x_g)$.
  - Example: $G = \mathbb{Z}/3\mathbb{Z}$.
    - If the elements are e, g, g^2 and we map these, respectively, to variables a, b, c, we get the matrix
$$\begin{bmatrix} e & g & g^2 \\ g & g^2 & e \\ g^2 & e & g \end{bmatrix} \mapsto \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$
    - The determinant is $3abc - a^3 - b^3 - c^3 = (a+b+c)(a^2+b^2+c^2-ab-bc-ac) = (a+b+c)(a+\zeta b+\zeta^2 c)(a+\zeta^2 b+\zeta c)$ where $\zeta^3 = 1$ is a root of unity.
  - Frobenius's theorem: If G is a finite group and we take this Frobenius determinant, then this determinant is equal to ${P_1}^{d_1} \cdots {P_k}^{d_k}$ where $P_1, \dots, P_k$ are irreducible polynomials in $x_g, \dots, x_{g_j}$, then $\deg P_i = d_i$ and k is the number of conjugacy classes.
  - Example: Take S_3; we'll get a polynomial of degree $|S_3| = 6$ but the Frobenius determinant $FD = (x_{g_1} + \cdots + x_{g_k})(x_{g_1} \pm \cdots)(\text{some pol. of deg 2})^2$
  - The proof is remarkable and deep and uses what would become character theory. These polynomials are related to representations and the number of simplest irreducible representations. The theory that came out came as a way to understand this miracle. We'll forget FD's for now, but then come back and prove it later.

1.2 Key Definitions and Category Theory Primer

9/29:
- OH: TW 4:30 or 5:00 most likely; he will confirm later.
- Today: Definitions in greater generality.
- As before, let G be a finite group and V be a finite-dimensional vector space.
- Goal of this course: Understand representations of G, that is...
  - Homomorphisms $\rho : G \to GL(V) = GL_n(\mathbb{C})$;
  - That send $g \mapsto A_g \in GL_n(\mathbb{C})$;
  - And satisfy $A_e = E$, $A_{g_1}A_{g_2} = A_{g_1g_2}$, and $A_{g^{-1}} = {A_g}^{-1}$.
- What are some things we might want to do?
  - Build new representations from old? Investigate and/or classify irreducible representations?
  - Before we can see if any of this works or not, we need a ton of definitions: Sum, equality, etc.
- Rudenko will start today's lecture with some general thoughts on the **category** of representations.
- Categories are things that now permeates math.
- **Category**: A *class* (not a set) of *objects* (some things; you don't know anything about them), and then a bunch of properties.

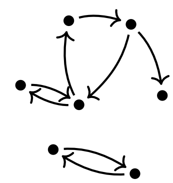

Figure 1.1: The general structure of a category.

  - Objects a, b in category C are denoted by $a, b \in \mathbf{Ob}(C)$.
  - There are also **morphisms** between the objects. These are drawn as arrows and lie in $\mathrm{Hom}(a, b)$.
  - There is also composition: $\mathrm{Hom}(a, b) \times \mathrm{Hom}(b, c) \to \mathrm{Hom}(a, c)$.
    - What does this notation mean??
  - Properties.
    1. Associativity.
    2. Existence of a unit: For any object a, there exists $\mathrm{id}_a \in \mathrm{Hom}(a, a)$ such that any morphism pre- or post-composed to this identity yields the same morphism.
       - Example: If $f \in \mathrm{Hom}(a, b)$, then $\mathrm{id}_b \circ f = f = f \circ \mathrm{id}_a$.
  - Rudenko: So a category is basically two pieces of data and a bunch of properties.
- Examples of categories:
  - Category of sets and maps between them.
  - Category of vector spaces over $\mathbb{F}$ where $\mathrm{Ob}(C)$ is the vector spaces and $\mathrm{Hom}(V, W)$ is filled with *linear* maps because you don't just want maps — you want maps that respect the structure.
  - Category of groups where $\mathrm{Hom}(G_1, G_2)$ is the set of group homomorphisms.
  - Category of topological spaces and continuous maps.
  - Category of abelian groups.
  - Trivial category and the identity map; thus, categories need not be chonky.

- Comments on category theory.
  - We'll see some pretty significant category theory at the end of the course.
  - We'll see categories in every course we take; some people try to avoid them. Rudenko doesn't want to go into the material in depth, but he wants to use language from it.
  - Surprisingly, even under the stripped-down of axioms of category theory, you can say quite a lot.
  - Why any of this discussion of category theory matters: If you know the basics of category theory, you can guess the definitions of direct sum, equality, etc. for representations.
- **Category of representations**. *Denoted by* $\mathbf{Rep}_{\boldsymbol{G}}$.
- Take two G-representations V, W; how do we define a map between them?
  - Recall that V, W are vector spaces.
- **Morphism** (of G-representations): A map $f : V \to W$ such that...
  1. f is linear;
  2. f respects the structure of the representations; explicitly, for every $g \in G$, $\rho_V(g) \circ f = f \circ \rho_W(g)$[2].
- On constraint 2, above: This condition is summarized via a **commutative diagram**.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\rho_V(g)"'] \arrow[dr, phantom, "\circlearrowright"] & W \arrow[d, "\rho_W(g)"] \\ V \arrow[r, "f"'] & W \end{tikzcd}$$

Figure 1.2: Commutative diagram, morphisms.

  - Commutative diagrams are very category-theory-esque things.
- That was a very abstract definition; let's make it concrete.
  - Suppose you have a pair of representations $V = \mathbb{C}^n, W = \mathbb{C}^m$, and we have our map f between them given by an $m \times n$ matrix.
  - Let $\rho_V(g) = A_g$ be an $n \times n$ matrix, and let $\rho_W(g) = B_g$ be an $m \times m$ matrix.
  - Then $FA_g = B_g F$.
- Examples.
  1. An interesting example: Let's look at $S_3 \circlearrowright V_{\text{perm}} = \mathbb{C}^3$, a **permutation representation**.
     - For all $\sigma \in S_3$, $\rho(\sigma) : (x_1, x_2, x_3) \mapsto (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$.
  2. There's also the trivial representation $S_3 \circlearrowright V_{(3)} = \mathbb{C}$ defined by $\rho(\sigma) : x \mapsto x$.
- Are the above 2 representations related?
  - Yes! We can, in fact, find a *morphism* between them.
  - In particular, define $f : V_{(3)} \to V_{\text{perm}}$ by $f(x) = (x, x, x)$.
    - Since permuting 3 of the same thing does nothing, the commutativity of Figure 1.2 holds. Therefore, f is a morphism of G-representations as defined above.
    - More explicitly,

$$f[\rho_{(3)}(\sigma)(x)] = f(x) = (x, x, x) = \rho_{\text{perm}}(\sigma)((x, x, x)) = \rho_{\text{perm}}(\sigma)[f(x)]$$

[2]Recall that the object, $\rho_V(g)$ is a linear map! Thus, it can be composed with other linear maps like f.

- Is f **reversible**?
  - Is "reversible" the right word??
- Define $\tilde{f} : V_{\text{perm}} \to V_{(3)}$ by $\tilde{f} : (x_1, x_2, x_3) \mapsto x_1 + x_2 + x_3$.
  - Since addition is commutative, the commutativity of Figure 1.2 holds.
  - More explicitly,

$$
\begin{aligned}
f[\rho_{\text{perm}}(\sigma)((x_1, x_2, x_3))] &= f((x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})) \\
&= x_{\sigma(1)} + x_{\sigma(2)} + x_{\sigma(3)} \\
&= x_1 + x_2 + x_3 && \text{Commutativity of addition} \\
&= f((x_1, x_2, x_3)) \\
&= \rho_{(3)}(\sigma)[f((x_1, x_2, x_3))]
\end{aligned}
$$

- Takeaway: The existence of maps between representations is interesting.

- Next question: How do we define an **isomorphism** of two representations?
- **Isomorphism** (of G-representations): A morphism of G-reps that is an isomorphism of vector spaces.
- Category theory helps us again here because it generalizes the concepts of an isomorphism!
  - If $f : V \to W$ and $g : W \to V$ are category-theoretic morphisms, then the constraints $f \circ g = \text{id}_W$ and $g \circ f = \text{id}_V$ make f and g into category-theoretic *iso*morphisms, regardless of what V and W might be.
  - Back in the context of representations, let $f : V \to V$ be an isomorphism of vector spaces. Then we do indeed have $\rho_V(g) \circ f = f \circ \rho_V(g)$, as we would hope from category theory!
- Recall the condition $FA_g = B_g F$. Supposing F is an isomorphism (and thus has an inverse), we get $FA_g F^{-1} = B_g$ as our new condition.
  - Essentially, we can do *simultaneous conjugation* of all matrices.
  - As per usual with isomorphisms, we get to *change bases*.
  - Essentially, we can represent the nice permutation representation in a very nasty basis but still have it be valid.
- Many other notions (e.g., direct sum) will not be explained by Rudenko, but we can read about them!
- However, we'll do a few more.
- A representation sitting inside another: a **subrepresentation**.
- **Subrepresentation** (of V): A subspace $W \subset V$ such that for all $w \in W$ and $g \in G$, we have that $\rho_V(g)W \subset W$, where V is a G-representation with $\rho_V : G \to GL(V)$.
  - Many people will just write the critical condition as $gW \subset W$.
- Subrepresentations in category theory: We have another commutative diagram.

$$
\begin{tikzcd}
W \arrow[r, hook] \arrow[d, "\rho_V(g)"'] & V \arrow[d, "\rho_V(g)"] \\
W \arrow[r, hook] & V
\end{tikzcd}
$$

Figure 1.3: Commutative diagram, subrepresentations.

- Example: The trivial representation, the standard representation, and (of course) the **zero representation** are subrepresentations of the permutation representation.
- **Zero representation**: The representation $\rho : G \to GL(\{0\})$ sending $g \mapsto 1$ for all $g \in G$. *Denoted by* $\mathbf{(0)}$.
- What about representations that don't have subrepresentations?
- **Simple** (representation): A G-representation V that has only two subrerpesentations: (0) and V. *Also known as* **irreducible**, **irreps**.
- Example irreducible representations: Line in $\mathbb{C}^2$, triangle in $\mathbb{C}^2$, A_5 and dodecahedron in $\mathbb{C}^3$.
- Notion of a direct sum.
- **Direct sum** (of V_1, V_2): The G-rep with the space $V_1 \oplus V_2 = \{(v_1, v_2) \mid v_1 \in V_1, v_2 \in V_2\}$ where $\rho_{V_1 \oplus V_2}(g)(v_1, v_2) = (\rho_{V_1}(g)v_1, \rho_{V_2}(g)v_2)$. *Denoted by* $\boldsymbol{V_1 \oplus V_2}$.
  - The matrix of $\rho_{V_1 \oplus V_2}(g)$ is the following block matrix.

$$\rho_{V_1 \oplus V_2}(g) = \left[\begin{array}{c|c} \rho_{V_1}(g) & 0 \\ \hline 0 & \rho_{V_1}(g) \end{array}\right]$$

- Example: $V_{\text{perm}} = V_{(3)} \oplus V_{(2,1)}$, with $\mathbb{C}^3 = \mathbb{C} \oplus \mathbb{C}^2$ where

$$\mathbb{C} = \langle (1,1,1) \rangle \qquad\qquad \mathbb{C}^2 = \langle (x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0 \rangle$$

  - The decomposition is into simple representations.
  - Relate this to the fact that the JCF of any 3×3 permutation matrix has at most a 1-block and a 2-block, if not three 1-blocks. There will always be one 1D subspace on which the permutation matrix is an identity, i.e., span$(1,1,1)$, and a 2D orthogonal complement!
  - As a fun and simple exercise, prove that there is no line fixed under the standard representation.
- A simple and important theorem to prove next week.
- Theorem: Let G be a finite group and $\mathbb{F} = \mathbb{C}$. Then...
  1. There are finitely many irreps $V_1, \dots, V_s$ up to isomorphism.
     - Later on, we will see that s is equal to the number of conjugacy classes.
  2. For every G-rep V, there exists a unique $n_1, \dots, n_s \geq 0$ such that $V \cong {V_1}^{n_1} \oplus \cdots \oplus {V_s}^{n_s}$.
- This theorem tells us that if we want to study rep theory, we want to study irreps (which can be kind of complicated) because if we understand them, everything breaks down into them.
- Examples.
  1. $G = \mathbb{Z}/2\mathbb{Z} = S_2$.
     - $V_1 = \mathbb{C}e$ with $ge = e$ and $V_{-1} = \mathbb{C}e$ with $ge = -e$.
     - It follows that $V \cong {V_1}^{n_1} \oplus {V_{-1}}^{n_{-1}}$.
     - We get a diagonal matrix with only 1s and -1s.
  2. $G = S_3$.
     - $V_{(3)}, V_{(1,1,1)}, V_{(2,1)}$.
     - $GL_5(\mathbb{F}_4)$.
     - Proven in an elementary way in Section 1.3 of Fulton and Harris (2004), which we have to read for the HW; will be useful for later in the course's HW.

- Plan: Next time, we'll talk about some more abstract stuff; tensor products of vector spaces.
  - Tensor products are something we should read up on now! The definition is hard and abstract.
  - Then he'll prove the above theorem.

Week 2

The Structure of Representations

2.1 The Tensor Product

10/2:

- Before lecture, I chatted with a few people about tensor products and the exterior and symmetric powers.
  - Patrick: A **tensor** $v \otimes w$ is just an element of a vector space, indexed differently than in a column.
  - Raman: There is no canonical way to transform tensors into column vectors.
- Course logistics.
  - OH: T 5:30-6:30(+) and W 5:30-6:30(+). We can also meet one-on-one.
  - HW is due Thursdays at midnight.
- Today: Constructing new representations from old.
  - Rudenko will skim through tensor products really quickly.
- Reminder: Last time, we talked about how representation theory is really quite simple. If G is a finite group and $F = \mathbb{C}$, there exist a finite set $V_1, \dots, V_s$ of irreps up to isomorphism, and every finite-dimensional representation $V \cong V_1^{n_1} \oplus \cdots \oplus V_s^{n_s}$.
- If V is a representation of G, then there are loads of things we can do with it.
  - We can construct the dual representation V^*.
  - We can construct the representation $V \otimes V$.
  - We can construct symmetric powers.
  - We can construct wedge powers.
  - There are more, but this is enough for now.
- Even when we take a very simple group and representation, there are some very interesting things that can fall out.
  - Example: If you take the symmetric powers of S_3, as in the homework, you get something really interesting.
- Now, we go to linear algebra.
- Let V, W be vector spaces over a field F. How do we produce a new vector space out of these?
- $\mathrm{Hom}_F(V, W)$ is the vector space of linear maps $F : V \to W$!
  - $\dim = (\dim V)(\dim W)$.

- Can we make $\text{Hom}_F(V, W)$ into a representation of G? Yes!

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "\rho_V(g)"'] & W \arrow[d, "\rho_W(g)"] \\ V \arrow[r, "gL"'] & W \end{tikzcd}$$

Figure 2.1: Commutative diagram, linear maps space representation.

  - Suppose that V, W are G-reps, which gives us $\rho_V : G \to GL(V)$ and $\rho_W : G \to GL(W)$.
  - Suppose also that we have $L \in \text{Hom}_F(V, W)$.
  - Now infer from the commutative diagram that it will work to define $gL = \rho_W(g) \circ L \circ \rho_V(g)^{-1}$.
  - This is pretty standard.
- Recall that there is a different space $\text{Hom}_G(V, W)$ of morphisms of G-representations (see Figure 1.2 and the associated discussion).
  - This is a very very small subspace of $\text{Hom}_F(V, W)$.
- Special case of the above construction: **Dual representation**.
  - Consider $\text{Hom}_F(V, F)$. This the **dual vector space**.
  - Basic fact 1: Let $e_1, \dots, e_n$ be a basis of V. Then V^* also has a corresponding basis $e^1, \dots, e^n$, known as its **dual basis**.
    - Computing coordinates already depends on a basis, and having bases is super nice.
    - Corollary: $\dim V = \dim V^*$.
    - This is the first time **canonical** comes into linear algebra. Canonical (nobody understands what it means) basically means that something doesn't depend on choices.
    - In particular, V, V^* are isomorphic because they have the same dimension, but for no more natural reason. They can be the same representation, or they can be different.
  - Basic fact 2: If V is finite-dimensional, then $(V^*)^* \cong V$. The formula for this isomorphism is canonical, because it does not depend on a choice of basis. In particular, choose the map $V \to (V^*)^*$ sending v to the map sending $\varphi \in V^*$ to $\varphi(v)$.
  - If V is infinite dimensional, none of this is true and you are in the realm of functional analysis.
  - Ok, so all of this was good information about the dual *space*, but what is the dual *representation*?? Does it matter, and do we need to know for now?
- **Dual vector space** (of V): The vector space defined as follows, given that V is a vector space over F. *Denoted by* $\boldsymbol{V^*}$. *Given by*

$$V^* = \text{Hom}_F(V, F)$$

- **Dual basis** (of V^* to $e_1, \dots, e_n$): The basis defined as follows for $i = 1, \dots, n$, where $e_1, \dots, e_n$ is a basis of V. *Denoted by* $\mathbf{e^1, \dots, e^n}$. *Given by*

$$e^i(x_1 e_1 + \cdots + x_n e_n) = x_i$$

- We now move onto the tensor product.
  - The tensor product is very hard to understand. If you learn about it and you feel you don't understand it, that's typical; nobody understands it at first.
  - For now, we'll discuss two ways of thinking about tensor products that won't bring us any comfort.
- Let V, W be two vector spaces over a field F.

- Abstract definition of the tensor product.
  - We have discussed maps from $V \to W$, but there is another related space.
  - Indeed, we can look at the space of bilinear maps from $V \times W \to F$.
    - Example: A map $f : V \times W \to F$ that satisfies the constraints $f(\lambda v, w) = \lambda f(v, w)$, $f(v_1 + v_2, w) = f(v_1, w) + f(v_2, w)$, and likewise for the second index. Recall that this is a **bilinear map**.
  - Let V have basis $e_1, \dots, e_n$ and W have basis $f_1, \dots, f_m$.
  - Notice that every bilinear map f can be defined as a linear combination of the $f(e_i, f_j)$. In other words, the $f(e_i, f_j)$ form the basis of a function space.
    - This "bilinear maps space" has dimension nm.
  - Now, one way to understand a tensor product: Is this "bilinear maps space" actually some other space? It is! It is $(V \otimes W)^*$.
  - Bilinear maps are linear maps from where? From $V \otimes W$!
- **Bilinear** (map): A function $f : V \times W \to Z$ that satisfies the following constraints, where V, W, Z are vector spaces over F, $v, v_1, v_2 \in V$, $w, w_1, w_2 \in W$, and $\lambda \in F$. *Constraints*

$$
\begin{aligned}
f(v_1 + v_2, w) &= f(v_1, w) + f(v_2, w) & \qquad f(\lambda v, w) &= \lambda f(v, w) \\
f(v, w_1 + w_2) &= f(v, w_1) + f(v, w_2) & \qquad f(v, \lambda w) &= \lambda f(v, w)
\end{aligned}
$$

- We now look at a much more elementary definition of the tensor product.
- Explicit definition of the tensor product.
  - $V \otimes W$ is equal to a huge vector space with basis consisting of pairs of elements (v, w). Even if V, W are one dimensional, this is like all pairs of real numbers; it's huge. Then, we quotient it by the space of all elements satisfying $\lambda(v, w) = (\lambda v, w) = (v, \lambda w)$, $(v_1 + v_2, w) = (v_1, w) + (v_2, w)$, and the like. This forces these relationships to be true.
    - Clarify this methodology??
    - Essentially, this allows us to treat tensor multiplication much like real multiplication, endowing the operation with distributivity, etc.
    - For example, the rule $(v_1 + v_2, w) = (v_1, w) + (v_2, w)$ becomes, in tensor product notation, $(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$.
  - Example: Suppose $V = \mathbb{C}e_1 + \mathbb{C}e_2$. We want to look at $V \otimes V$.
    - A priori[1], it's spanned by $(ae_1 + be_2) \otimes (ce_1 + de_2) = ace_1 \otimes e_1 + ade_1 \otimes e_2 + bce_2 \otimes e_1 + cde_2 \otimes e_2$.
    - Thus, $V_1 \otimes V_2$ has 4-element basis $e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2$.
- These two definitions constitute a first approximation to what the tensor product is.
- Takeaway: What is true in general is that if V has basis $e_1, \dots, e_n$ and W has basis $f_1, \dots, f_m$, then $V \otimes W$ has basis $e_i \otimes f_j$ ($i = 1, \dots, n$ and $j = 1, \dots, m$).
- Having discussed the tensor product of vector spaces, let's think about the tensor product of *representations*.
  - Suppose $g : V \to V$ and $g : W \to W$.
    - We're starting to make notation sloppy.
  - How does $g : V \otimes W \to V \otimes W$? Well, we just send $v \otimes w \mapsto (gv) \otimes (gw)$.
    - Why is this map well-defined?

[1] I.e., it follows from some logic. In particular, it follows from the logic that any element $v \in V$ is of the form $v = ae_1 + be_2$, so of course all $v \otimes v$ must be of the given form for choices of a, b, c, d.

- We invoke the **universal property of the tensor product operation**.
- This guarantees us that given g — which is effectively a map from $V \times W \to V \otimes W$ as defined — there nevertheless exists a complete extension $\tilde{g} : V \otimes W \to V \otimes W$.

– As a matrix, this map is pretty strange!

- Example: Let $g : V \to V$ be a 2×2 matrix. What is the matrix of $g : V \otimes V \to V \otimes V$?
- If

$$\rho_V(g) = g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} =: A$$

then we have

$$\begin{aligned} g(e_1 \otimes e_1) &= ge_1 \otimes ge_1 \\ &= (ae_1 + ce_2) \otimes (ae_1 + ce_2) \\ &= a^2 e_1 \otimes e_1 + ace_1 \otimes e_2 + ace_2 \otimes e_1 + c^2 e_2 \otimes e_2 \end{aligned}$$

- Evaluating similarly for all basis vectors, we get a very curious block matrix:

$$\begin{array}{cc} & \begin{matrix} e_1 \otimes e_1 & e_1 \otimes e_2 & e_2 \otimes e_1 & e_2 \otimes e_2 \end{matrix} \\ \begin{matrix} e_1 \otimes e_1 \\ e_1 \otimes e_2 \\ e_2 \otimes e_1 \\ e_2 \otimes e_2 \end{matrix} & \begin{bmatrix} a^2 & ab & ab & b^2 \\ ac & ad & bc & bd \\ ac & bc & ad & bd \\ c^2 & cd & cd & d^2 \end{bmatrix} \end{array} = \left[\begin{array}{c|c} aA & bA \\ \hline cA & dA \end{array}\right]$$

- Notice how, for example, this takes the tensor $e_1 \otimes e_1$, represented as $(1, 0, 0, 0)$, to the tensor $a^2 e_1 \otimes e_1 + ace_1 \otimes e_2 + ace_2 \otimes e_1 + c^2 e_2 \otimes e_2$, represented as (a^2, ac, ac, c^2).
- Does this construction imply a canonical way to convert from tensors to column vectors??

– Classically, this is called the **Kronecker product** of two matrices.

– People discovered all of this stuff before they unified it as tensor math.

- **Universal property of the tensor product operation**: For every bilinear map $h : V \times W \to Z$, there exists a *unique* linear map $\tilde{h} : V \otimes W \to Z$ such that $h = \tilde{h} \circ \otimes$.

$$\begin{tikzcd} V \times W \arrow[r, "\otimes"] \arrow[dr, "h"'] & V \otimes W \arrow[d, "\tilde{h}"] \\ & Z \end{tikzcd}$$

Figure 2.2: Universal property, tensor product operation.

Proof. See the solid explanation linked here. Write out at a later date, and/or review MATH 25800 notes further. ☐

- **Kronecker product** (of A, B): The matrix product defined as follows. *Denoted by* $\boldsymbol{A} \otimes \boldsymbol{B}$. *Given by*

$$A \otimes B = n \overset{n}{\begin{bmatrix} A \end{bmatrix}} \otimes m \overset{m}{\begin{bmatrix} B \end{bmatrix}} = nm \overset{nm}{\begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix}}$$

  – The Kronecker product is *not* commutative, but the matrices you get are related by conjugacy and by commuting the columns.

- Vector spaces of the same dimension are all alike, but vector space representations are very interesting. By the end of this course, we'll understand what's going on.

- How we understand tensor stuff: Look at the abstract definition, look at the concrete definition, look at 5 examples, and then go in a circle. Repeat again and again until it makes sense.
- He's just trying to tell us all relevant words so that they will fit together later.
- Fact: If V, W finite-dimensional, $\mathrm{Hom}_F(V, W) \cong V \otimes W^*$.
  - Tensor products are very nice to construct maps from.
  - Let's construct a reverse map, then.
  - Take $\alpha \otimes w \in V^* \otimes W$, where $\alpha : V \to F$ by definition. Send $\alpha \otimes w$ to the map $v \mapsto \alpha(v)w$. This is a *canonical* map!! We can show that they span everything.
    - For example, if we want to choose $\alpha \otimes w$ mapping to the matrix with a 1 in the upper left-hand corner and zeroes everywhere else, let $w = e_1 \in W$ and let $\alpha = e^1 \in V^*$.
    - We can do similarly for all other such matrices, mapping this basis of $\mathrm{Hom}_F(V, W)$ to $e^i \otimes e_j$ ($i = 1, \ldots, n$ and $j = 1, \ldots, m$).
    - Note that this also allows us to define a (noncanonical) inverse map.
  - This inverse map from $\mathrm{Hom}_F(V, W) \to V^* \otimes W$ is clearly a bit harder to work out.
  - Hidden in this story is why trace is invariant under conjugation (see below discussion).
- If we now take $\mathrm{Hom}_F(V, V)$, then this is isomorphic to $V^* \otimes V$. There is a very natural map from these isomorphic spaces to F defined by the trace, and/or $\alpha \otimes v \mapsto \alpha(v)$. We can prove this. And this is canonical, as well. This is why the main property of the trace is that it's invariant under conjugation. This fact is hidden in the story very nicely.
- Tensor products are hard, it will be a pain, we will understand them very well, but it will not be nice for now.
- Symmetric products and wedge powers will be discussed briefly next time.
  - There is a nice description in Serre (1977) that we can use for the homework.
- Extra homework: Please read about tensor products in whatever textbook you like, try some examples, and repeat.

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