Week 4

???

4.1 Representation Ring; Character Basis

10/16: • Announcements.

  - Reminder: Midterm 11/10.
  - OH this week in-person at normal times.
  - PSet 3 should be fun.

- Today: Finish proving some character things.
- Recall: The main picture.
  - Rudenko redraws Figure 3.1.
  - We have a finite group G and we are studying finite-dimensional G-reps over $\mathbb{C}$.
  - $\mathbb{C}_{\text{cl}}[G]$ is a ring.
  - The map...
    - Respects addition;
    - Sends tensor multiplication to (pointwise) functional multiplication;
    - Sends duality to conjugation;
    - Respects a kind of inner product, whether it be either side of $\dim_{\mathbb{C}} \operatorname{Hom}_G(V, W) = \langle f_1, f_2 \rangle$.
- Today, we will see that $\mathbb{C}_{\text{cl}}[G] \cong \mathbb{C}^k$, where k is the number of conjugacy classes.
  - In other words, we will see that the number of irreps is also exactly equal to k, that there is a bijection $\{V_i\} \to \{\chi_i\}$, and that the $\chi_1, \dots, \chi_k$ form an orthonormal basis of $\mathbb{C}_{\text{cl}}[G]$.
- Visualizing the vector space $\mathbb{C}_{\text{cl}}[G]$.

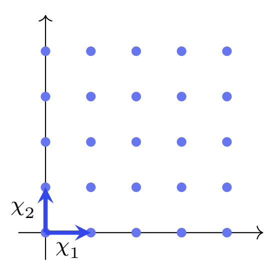

Figure 4.1: Visualizing the space of class functions on G.

  - It's a "cone" emanating from the origin with only lattice points.
    - If $\dim \mathbb{C}_{\text{cl}}[G] = 2$, the vector space consists of all the blue points in Figure 4.1.
  - Why is it only lattice points instead of a continuous function space?
    - The restrictions on coefficients are inherited from the restrictions on what kinds of spaces you can build of the form $V_1^{n_1} \oplus V_2^{n_2}$.
    - Indeed, if it were continuous, that would imply that there is some meaning to the point $0.3\chi_1 + 2.5\chi_2$, i.e., there is a space $V_1^{0.3} \oplus V_2^{2.5}$. But of course, we cannot define such a space!
  - Why is it only *nonnegative* integer coefficients and not *all* integer coefficients?
    - We don't have subtraction to get us to a full ring.
    - Additionally, we can only scale and linearly combine the χ_i's with nonnegative integer coefficients because, as said above, those are the types of reducible rep decompositions we have.

- Let $[V]$ denote the **isomorphism class** of the representation V.
- **Isomorphism class** (of V): The set of all vector spaces W that are isomorphic to V as representations.
- This allows us to define the **representation ring**.
- **Representation ring** (of G): The ring $(R, +, \cdot)$, where R is the free abelian group generated by all isomorphism classes of the representations of G, quotiented by the span of all linear combinations of the form $[V \oplus W] - [V] - [W]$; $+$ is well-defined via the construction of R, which yields $[V] + [W] = [V \oplus W]$ for all $[V], [W]$ in the ring; and $\cdot$ is defined by $[V] \cdot [W] = [V \otimes W]$. *Denoted by* $\boldsymbol{R(G)}$.
  - Basis: $[V_1], \dots, [V_k]$.
  - Thus, structurally,
$$R(G) \cong \mathbb{Z}^k$$
  - Elements are of the form $[V_1] + 2[V_2] - 3[V_3]$.
  - Multiplication is slightly complicated because $V_i \otimes V_j = \bigoplus V_k^{n_{ijk}}$; it follows that
$$[V_i] \cdot [V_j] = \sum n_{ijk} [V_k]$$
- Alternative construction of $R(G)$: Take the subring of the class ring $\mathbb{C}_{\text{cl}}[G]$ that is generated by the characters.
  - To do so, define a map $R(G) \to \mathbb{C}^k$ where the image is linear combinations of characters χ_i with $\mathbb{Z}$-class.
  - Clarify this construction??
- **Virtual representation**: An element of $R(G)$.
  - We need this term because some elements of $R(G)$ — like $-[V]$, for instance — may not correspond to an actual representation.
  - Indeed, note that $-[V]$ is *not* V^*; it is just some thing that when you add it to $[V]$, you get the zero representation.
- Example: Let $G = \mathbb{Z}/2\mathbb{Z} = \{e, x\}$.
  - Then $R(G) = \mathbb{Z}^2 = \mathbb{Z}e \oplus \mathbb{Z}x$ has basis $[1], [-1]$ (corresponding to the trivial and alternating representations) where we define
$$[1]^2 = [1] \qquad [1][-1] = [-1] \qquad [-1]^2 = [1]$$
- One reason people like this $R(G)$ is as follows.

  - Initially, understanding this group is not easy because even to get started, you have to find all your characters.
  - But, we know that
$$R(G) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}_{\text{cl}}[G]$$
    - So we have a ring that's hard to understand, but if we do something called an **extension of scalars** (shown above) we get an easy ring!
    - Why?? Clarify this construction.
  - This is interesting because we can look at the intermediate objects. For example, could we describe $R(G) \otimes \mathbb{R}$ or $R(G) \otimes \mathbb{Q}$. Interestingly, **Artin's theorem** describes $R(G) \otimes \mathbb{Q}$ completely.
  - If we try to understand $R(S_n)$, this is still hard work, but if we take $\bigoplus_{n \geq 0} R(S_n)$, we obtain an object that is remarkably, surprisingly simple. That's where we're going. This is why rep theory of finite groups is simultaneously very hard and very simple.

- Lemma: Let G be a finite group, let f be a complex-valued[1] class function, and let V be a G-rep. Then the linear map
$$F = \sum_{g \in G} f(g) \cdot g : V \to V$$
is a morphism of G-representations, that is, $F \in \text{Hom}_G(V, V)$.

Proof. To prove that $F \in \text{Hom}_G(V, V)$, it will suffice to show that $xF = Fx$ for every $x \in G$. Let $x \in G$ be arbitrary. Then
$$F(xv) = \sum_{g \in G} f(g) gxv$$
Since ρ is a group homomorphism, the functions $\rho(g) \in GL(V)$ act just like the elements $g \in G$. *This* is what justifies us to basically move everything around all willy-nilly. Thus, continuing from the above, we have
$$\begin{aligned} &= \sum_{g \in G} f(g)(xx^{-1}) gxv \\ &= \sum_{g \in G} f(g) x(x^{-1}gx) v \end{aligned}$$
Since $x = \rho(x)$ is in the general *linear* group, i.e., is a *linear* map, we can factor it out of the sum of functions to get
$$= x \left(\sum_{g \in G} f(g) x^{-1} gx \right) v$$
Since f is a class function by hypothesis, we have $f(g) = f(x^{-1}gx)$, so
$$\begin{aligned} &= x \left(\sum_{g \in G} f(x^{-1}gx) x^{-1} gxv \right) \\ &= x \sum_{g \in G} f(g) gv \\ &= x(Fv) \end{aligned}$$
as desired. ∎

[1] This "complex-valued" hypothesis was not stated in class, but I have to imagine it's true. Is it??

- Recall that previously, we had $(1/|G|)\sum_{g\in G} g : V \to V^G$.
    - He will put something about this being a class function on the midterm?? Review how to prove that this is a class function!
- Another comment: A slightly refined question.
    - Suppose you have a class function f and an irrep V.
    - Then we know that $F = \sum f(g)g : V \to V$ is a G-morphism, so it is a **homothety** by Schur's lemma.
    - So let's find λ.
    - Thinking a big more carefully, we know that F above is

      $$\sum_{g\in G} f(g)\rho_V(g) = \lambda I_{d_V}$$

      where d_V denotes the **degree** of V.
    - Now, we will compute λ using the trace. Take the trace of both sides. Then

      $$\begin{aligned}\operatorname{tr}\left(\sum_{g\in G} f(g)\rho_V(g)\right) &= \operatorname{tr}(\lambda I_{d_V})\\ \sum f(g)\operatorname{tr}(\rho_V(g)) &= \lambda d_V\\ \sum f(g)\chi_V(g) &= \lambda d_V\\ \lambda &= \frac{|G|}{d_V}\frac{1}{|G|}\sum_{g\in G} f(g)\overline{\chi_{V^*}(g)}\\ &= \frac{|G|}{d_V}\langle f, \chi_{V^*}\rangle\end{aligned}$$

- **Homothety**: A map $F : V \to V$ for which there exists $\lambda \in \mathbb{C}$ such that $Fv = \lambda v$ for all $v \in V$.
    - It just means that we're scaling.
- **Degree** (of V): The dimension of V as a vector space. *Denoted by* $\boldsymbol{d_V}$. *Given by*

  $$d_V = \dim V$$

- Now, we can prove the theorem to which we've been building up the whole time.
- Theorem: Let G be a finite group. Then the number of irreps up to isomorphism is equal to the number of conjugacy classes.

  Proof. Let k be the number of conjugacy classes of G, and let $\chi_1, \dots, \chi_s$ be the characters of the irreps. By the theorem from last Wednesday's class, it follows that $\chi_1, \dots, \chi_s$ are orthonormal vectors in $\mathbb{C}_{\text{cl}}[G]$. Thus, by the corollary to the aforementioned theorem, $s \leq k$.

  Now, suppose for the sake of contradiction that $s < k$. Then there exists a nonzero $f \in \mathbb{C}_{\text{cl}}[G]$ such that $\langle f, \chi_{V_i}\rangle = 0$ $(i = 1, \dots, s)$. By Gram-Schmidt, we can choose f to be another *orthonormal* vector in the list, extending it to $\chi_1, \dots, \chi_s, f$. We will now build up to proving that $f(g) = 0$ for all $g \in G$ (i.e., $f = 0$), which we will do by using the above lemma to construct a linear independence argument as follows. The first step is to let V_i be an arbitrary irrep of G. Then by the above comment, $F : V_i \to V_i$ may be evaluated on any $v \in V_i$ as follows.

  $$F(v) = \lambda I v = \frac{|G|}{d_{V_i}}\langle f, \chi_{V_i^*}\rangle \cdot v = \frac{|G|}{d_{V_i}}\overline{\langle f, \chi_{V_i}\rangle} \cdot v = \frac{|G|}{d_{V_i}}\overline{0} \cdot v = 0$$

It follows that $F = 0$ on *any* representation since by complete reducibility, they're all direct sums of irreps. In particular, $F : V_{\text{reg}} \to V_{\text{reg}}$ is the zero operator, where $V_{\text{reg}} \cong V_1^{d_{V_1}} \oplus \cdots \oplus V_s^{d_{V_s}}$ is the regular representation. Thus, for example, $F(e_e) = 0$. But we also know that

$$F(e_e) = \sum_{g \in G} f(g) \cdot ge_e = \sum_{g \in G} f(g) \cdot e_g$$

Consequently, by transitivity, we have that

$$0 = \sum_{g \in G} f(g) \cdot e_g$$

But since the e_g are all linearly independent by the definition of the regular representation, we have that each $f(g) = 0$, as desired. This means that $f = 0$, contradicting our original supposition. $\square$

- That is the end of this story.
- Here's one consequence of the above theorem.
  - We now know that the space of class functions has an orthonormal basis $\chi_{V_1^*}, \dots, \chi_{V_k^*}$.
  - If we denote the conjugacy classes of G by $C_1, \dots, C_k$, then another obvious basis of $\mathbb{C}_{\text{cl}}[G]$ is $\delta_{C_1}, \dots, \delta_{C_k}$ defined by

$$\delta_{C_i}(g) = \begin{cases} 1 & g \in C_i \\ 0 & g \notin C_i \end{cases}$$

  - This new basis is orthogonal: We have

$$\langle \delta_{C_i}, \delta_{C_j} \rangle = \frac{1}{|G|} \sum_{g \in G} \delta_{C_i}(g) \delta_{C_j}(g) = \begin{cases} 0 & i \neq j \\ \frac{|C_i|}{|G|} & i = j \end{cases}$$

    - Justifying this computation: If $i \neq j$, then at least one of $\delta_{C_i}, \delta_{C_j}$ will be zero; if $i = j$, then they're both nonzero and equal to 1 for all $|C_i|$ elements $g \in C_i$.
  - What is the change of basis matrix between $\{\delta_{C_i}\}$ and $\{\chi_{V_i^*}\}$? It's the character table.
    - The orthogonality condition for characters then just comes from the fact that we're going from one orthogonal basis to another.
    - What are the exact bases we change between??

4.2 Office Hours

10/17:
- **Transitive** (group action): A group action for which the **orbit** of x is equal to X for any $x \in X$.
- **Orbit** (of $x \in X$): The set of $g \cdot x$ for all $g \in G$.
- **Diagonal action** (of G on $X \times X$): The action defined as follows. *Given by*

$$g \cdot (x_1, x_2) = (g \cdot x_1, g \cdot x_2)$$

- Check Etingof et al. (2011) for some things??

4.3 Orthogonality Results

10/18:

- Announcements.
  - Goal: Finish our discussion of the orthogonality of characters, projection functions, etc.
  - Friday: Frobenius determinant.
  - Next week: Group algebras, associative algebras, etc.; another perspective on representations.
  - After next week: A more advanced part of representation theory related to group theory.
- Describing Figure 3.1 from a different perspective.
  - Let G be a finite group, and let k denote the number of conjugacy classes and the number of irreps. Let $C_1,\dots,C_k$ be the conjugacy classes and $V_1,\dots,V_k$ be the irreps.
  - There is no natural/canonical bijection between the two sets. For a simple group, there is often a canonical way, and this is where things get interesting.
    - Example: Symmetric group induces canonical bijection, as we'll see later.
  - $\mathbb{C}_{\text{cl}}[G]=\mathbb{C}^k$ is a vector space of class functions and a ring.
  - We have the Hermitian inner product

$$\langle f_1,f_2\rangle=\frac{1}{|G|}\sum_{g\in G}f_1(g)\overline{f_2(g)}$$

  - Recall that $\chi_{V_1},\dots,\chi_{V_k}$ is an orthonormal basis such that

$$\left\langle\chi_{V_i},\chi_{V_j}\right\rangle=\delta_{ij}$$

  - We have another basis $\delta_{C_1},\dots,\delta_{C_k}$ defined by

$$\delta_{C_i}(g)=\begin{cases}0 & g\notin C_i\\ 1 & g\in C_i\end{cases}$$

    that is orthogonal but not orthonormal:

$$\left\langle\delta_{C_i},\delta_{C_j}\right\rangle=\begin{cases}0 & C_i\neq C_j\\ \frac{|C_i|}{|G|} & C_i=C_j\end{cases}$$

  - How do we relate the two bases?
  - To begin, fix C_i. Then

$$\delta_{C_j}(g)=\sum_{V_i}\lambda_i\chi_{V_i}(g)$$

  - λ_i can be computed immediately using the inner product since the characters are orthonormal:

$$\lambda_i=\left\langle\delta_{C_j},\chi_{V_i}\right\rangle=\frac{1}{|G|}\sum_{g\in G}\delta_{C_j}(g)\overline{\chi_{V_i}(g)}=\frac{|C_j|\bar{\chi}_{V_i}(g)}{|G|}$$

    - You took $\lambda_i=\left\langle\delta_{C_j},\bar{\chi}_{V_i}\right\rangle$; which one is correct??
  - But then

$$\delta_{C_j}(g)=\frac{|C_j|}{|G|}\left(\sum_{V_i}\bar{\chi}_{V_i}(C_j)\chi_{V_i}(g)\right)$$

  - It follows that we have two bases of $\mathbb{C}_{\text{cl}}[G]$. These are given by

$$\frac{|G|}{|C_j|}\delta_{C_j} \qquad\qquad\qquad \chi_{V_i^*}$$

    where $i,j=1,\dots,k$.
  - How do we convert between these two very natural bases of our space of functions? The change of basis matrix from left to right is the character table.
  - Obviously, we have to do some scaling and take some duals, but it's not that bad and it fits the character table really well.
  - This gives us some properties of the character table such as orthogonality.
  - For example, **orthogonal** matrices convert between orthogonal bases; in the complex domain, such a matrix is **unitary**, i.e., for the character table U, $U\bar{U}^T=E$.

- Orthogonality relations that you can derive.
  1. We can show that
$$\sum_{g\in G}\chi_1(g)\overline{\chi_2(g)}=\begin{cases}0 & \chi_1\neq\chi_2\\ |G| & \chi_1=\chi_2\end{cases}$$
     - Use the unitary condition.
  2. We can show that
$$\sum_{i=1}^k\chi_i(g_1)\overline{\chi_i(g_2)}=\begin{cases}0 & g_1\neq g_2\\ \frac{|G|}{|C(g_1)|} & g_1\sim g_2\end{cases}$$
     - We literally just take the identity defining $\delta_{C_j}(g)$.

- **Isotypical component**: A representation that is equal to the direct sum of isomorphic irreducible representations. *Also known as* **isotypic component**.
  - Illustrative example: For $V=V_1^{n_1}\oplus\cdots\oplus V_k^{n_k}$, each $V_i^{n_i}$ is an isotypical component.

- Examples.
  1. Let $G \circlearrowright \mathbb{C}^2$ by $\rho(g)=E_2$. Thus, we can say that $\mathbb{C}^2=V_1\oplus V_1$, but we can't say this in any unique, canonical way, i.e., we can choose infinitely many V_1's and have the statement still be true, where V_1 is the trivial rep.
  2. We have $V_1^{n_1}=V^G=\{v\in V\mid gv=v\ \forall\ g\in G\}$. Look at what's invariant under the symmetry group, i.e., define
$$P=\frac{1}{|G|}\sum g$$
     - All **invariant functions** come from averaging over the group!
     - Then $P^2=P$ and $\operatorname{Im}P=V^G$.
     - Takeaway: We call each $V_i^{n_i}$ an **isotypical component**.
     - What's going on in this example??
  3. The permutational representation for S_n decomposes into the sum of the trivial and standard reps; there is only one decomposition this way. If we look at $V_1\oplus V_{\text{stand}}^2$, then our decomposition will depend on a choice of a plane.

- Reminder.
  - Last time, we chose an $f\in\mathbb{C}_{\text{cl}}[G]$, a representation V, and then took $\sum f(g)g:V\to V$ so that then $\sum f(g)g\in\operatorname{Hom}_G(V,V)$.
  - Moreover, we proved that if V is irreducible, then this endomorphism is equal to a scalar λ times the identity matrix via Schur's lemma.

- Computing λ:
$$\lambda = \frac{|G|}{d_V} \langle f, \chi_V^* \rangle$$
  - Hard to remember but easy to derive.
- Define $V = V_1^{n_1} \oplus \cdots \oplus V_k^{n_k}$ and $P_i : V \to V_i^{n_i}$.
- In particular, look at
$$P_i = \frac{d_V}{|G|} \sum_{g \in G} \chi_{V_i^*}(g) g$$
  - This averaging operator is consistent with what we had before.
- P_i acts on V_i by
$$\frac{d_{V_i}}{|G|} \frac{|G|}{d_{V_i}} \left\langle \chi_{V_i^*}, \chi_{V_i^*} \right\rangle = 1$$
- P_i acts on V_j by
$$\frac{d_{V_i}}{|G|} \frac{|G|}{d_{V_i}} \left\langle \chi_{V_i^*}, \chi_{V_j^*} \right\rangle = 0$$
- Take $V = V_1^{n_1} \oplus \cdots \oplus V_k^{n_k}$ and apply P_i. It follows by the above that it is exactly the projection on $V_i^{n_i}$.
- Thus, $P_1 + \cdots + P_k = 1$. $P_i^2 = P_i$. $P_i P_j = 0$. This is called a/the (which one??) **idempotent decompostion**.
- Example: Let $v \in V$. Then $v = P_1 v + \cdots + P_k v$.
- Additionally, we can take a function f that is invariant under the group...??

- We're done early.
- We will not start the Frobenius determinant today.
- We will start on next week's content then so we can begin thinking about it.
- **Associative algebra**: A vector space over a field F that is also a (not necessarily commutative) ring, where we have a unit 1 in the ring, addition, and multiplication. Scalar multiplication: $\lambda a = (\lambda \cdot 1) \cdot a$. Associativity condition: $(\lambda a)b = \lambda(ab)$. *Denoted by* $\boldsymbol{A}$.
  - We'll only discuss finite-dimensional algebras in this course.
- Examples:
  1. $\mathbb{R}$, $\mathbb{C}$ (an algebra over $\mathbb{R}$).
  2. $\mathbb{H}$, a 4d algebra over $\mathbb{R}$. The algebra of quaternions. $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$.
     - Hamilton's remarkable discovery: There is a 4D set of numbers that is not commutative but is still associative and helps describe rotation in 3D or 4D space.
     - Multiplication rules:
$$i^2 = j^2 = k^2 = -1 \qquad\qquad ijk = -1$$
     - We should spend part of our weekend reading a history of quaternions!
  3. $M_{n\times n}(F)$, the **matrix algebra**.
  4. $A_1 \oplus \cdots \oplus A_n$, the **direct sum** of algebras.
     - Addition and multiplication are done pairwise.
  5. $A_1 \otimes A_2$.
     - We will not talk about this today, though!

- Let's go back; let G be a finite group and consider $\mathbb{C}[G]$, the set of functions on G.
  - This algebra has some basis $\bigoplus \mathbb{C}e_g$.
  - To get the algebra structure, we just need a rule for multiplying basis elements. In this case, we use $e_{g_1}e_{g_2} = e_{g_1g_2}$.
  - This is the **algebra over $\mathbb{C}$ of dimension G**.
  - Theorem: $\mathbb{C}[G] \cong M_{d_1\times d_1}(\mathbb{C}) \oplus \cdots \oplus M_{d_k\times d_k}(\mathbb{C})$.
    - We can prove this theorem from what we know: Schur's Lemma and complete reducability.
    - We'll discuss it for several consecutive times.
    - A similar result holds for *many* algebras (e.g., semisimple algebra), not just *group* algebras.
- HW1-2 will be graded later this week and handed back on Friday.
- In Etingof et al. (2011), we can find a lot of history of some of this stuff. The comments are interesting and entertaining.