5 Residues

Set A: Graded for Completion

5/17:

1. Fischer and Lieb (2012), QIV.1.3. Show that the image of a simply connected domain under a biholomorphic mapping is simply connected. Is it sufficient to assume that the mapping is locally biholomorphic?

2. Fischer and Lieb (2012), QIV.3.2. Find the principal part of the Laurent expansion of
$$\frac{z-1}{\sin^2 z}$$
in $0<|z|<\pi$ and of
$$\frac{z}{(z^2+b^2)^2}$$
in $0<|z-ib|<2b$.

3. Fischer and Lieb (2012), QIV.4.1. Let $f\in\mathcal{O}(\mathbb{C}\setminus S)$ for some discrete set S of singularities. Show that...
   (a) If f is even, then
   $$\operatorname{res}_{-z} f = -\operatorname{res}_z f$$
   (b) If f is odd, then
   $$\operatorname{res}_{-z} f = \operatorname{res}_z f$$
   (c) If $f(z+\omega)=f(z)$ for some $\omega\in\mathbb{C}$, then
   $$\operatorname{res}_{z+\omega} f = \operatorname{res}_z f$$
   (d) If f is real on $\mathbb{R}$, then
   $$\operatorname{res}_{\bar{z}} f = \overline{\operatorname{res}_z f}$$

4. Fischer and Lieb (2012), QIV.5.1. Prove that
$$\int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\cosh x}=\pi$$
Hint: Integrate over the boundary of the rectangle whose corners are $\pm r$ and $\pm r+i\pi$.

5. Verify the claim from class that the integral of
$$f(z)=\frac{\pi}{z^2\tan(\pi z)}$$
over the square with vertices $[\pm(N+\frac{1}{2}),\pm(N+\frac{1}{2})]$ for $N\in\mathbb{N}$ converges to 0 as $N\to\infty$.

6. For $\alpha\notin\mathbb{Z}$, prove that
$$\sum_{n=-\infty}^{\infty}\frac{1}{(\alpha+n)^2}=\frac{\pi^2}{\sin^2(\pi\alpha)}$$
Hint: Adapt your computation from Problem A.5.

Set B: Graded for Content

1. Fischer and Lieb (2012), QIV.4.3. Let G be a simply connected domain, and let $f \in \mathcal{O}(G \setminus S)$. Show that f has a primitive on $G \setminus S$ if and only if all residues of f vanish.

2. Fischer and Lieb (2012), QIV.5.3. Compute the following.

(a) For $a > 1$,

$$\int_0^\pi \frac{\sin^2 x}{a + \cos x}\, \mathrm{d}x$$

(b) For $a \in \mathbb{C}$ and $|a| \neq 1$,

$$\int_0^{2\pi} \frac{\mathrm{d}t}{1 - 2a\cos t + a^2}$$

Hint: To get started, make the substitution $z = \mathrm{e}^{it}$ as on Fischer and Lieb (2012, p. 127) so that

$$\cos(t) = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \sin(t) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

and so that the integral from $t = 0$ to 2π becomes a contour integral over the unit circle.

3. Fischer and Lieb (2012), QIV.6.3. Let $\lambda > 1$. Show that the equation $\mathrm{e}^{-z} + z = \lambda$ has exactly one solution in the half plane $\operatorname{Re} z > 0$. Show that this solution is real.

4. Show that

$$\int_0^1 \log[\sin(\pi x)]\, \mathrm{d}x = -\log(2)$$

Hint: Consider the contour of integration that goes from ∞ to 0 along the positive imaginary axis, then runs from 0 to 1 along the real axis, then runs from 1 to ∞ along the ray $\{z \mid \operatorname{Re}(z) = 1,\ \operatorname{Im}(z) \geq 0\}$.