1 Holomorphicity

Set A: Graded for Completion

3/29: 1. Fischer and Lieb (2012), QI.2.1. Formulate and prove the chain rule for Wirtinger derivatives. Furthermore, show that

$$\overline{\frac{\partial f}{\partial z}} = \frac{\partial \bar{f}}{\partial \bar{z}}$$

2. Let $\langle (x_1, y_1), (x_2, y_2) \rangle = x_1 x_2 + y_1 y_2$ denote the usual inner product on $\mathbb{R}^2$. We can also define the **Hermitian inner product** on $\mathbb{C}$ via

$$(z, w) = z\bar{w}$$

This term *Hermitian* describes the fact that this product is not symmetric, but satisfies $(w, z) = \overline{(z, w)}$. Show that thinking of z as $x + iy$, we have

$$\langle z, w \rangle = \frac{1}{2}[(z, w) + (w, z)] = \mathrm{Re}(z, w)$$

3. For any integer n, compute the line integral $\int_\gamma z^n \, \mathrm{d}z$ where γ is any circle centered at the origin with counterclockwise orientation. Do not use Cauchy's theorem.

4. Without using Cauchy's theorem, show that for any $|a| < 1 < |b|$,

$$\int_\gamma \frac{1}{(z-a)(z-b)} \, \mathrm{d}z = \frac{2\pi i}{a-b}$$

where γ is the circle of radius 1 centered about the origin, oriented counterclockwise.

5. Determine the image of the following sets under the following conformal mappings. Use level curves to illustrate the geometry of these mappings.

   (a) The unit disk $\mathbb{D} = \{z : |z| < 1\}$ under $z \mapsto 1/z$.

   (b) $\mathbb{D} \setminus \{0\}$ under $z \mapsto z^2$.

   (c) The strip $S = \{z : \mathrm{Im}(z) \in (0, 2\pi)\}$ under $z \mapsto \mathrm{e}^z$.

   (d) The upper half-plane $\mathbb{H} = \{z : \mathrm{Im}(z) > 0\}$ under $z \mapsto z^2$,

   (e) The half disk $\mathbb{D} \cap \mathbb{H}$ under $z \mapsto (1+z)/(1-z)$.

Set B: Graded for Content

1. A prototypical example of a weird function that is differentiable (but not C^1) on all of $\mathbb{R}$ is

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Extend f to a function on $\mathbb{C}$ using the same formula (replacing x's with z's). Is it holomorphic at the origin?

2. Show that if f is holomorphic on a domain $U \subset \mathbb{C}$ and takes only real values, then it is constant.

3. Find a conformal map that takes the upper half-plane onto the "Pac-Man" given by

$$\{z : |z| < 1 \text{ and } \arg(z) \in (\pi/4, 7\pi/4)\}$$

Explain how you obtained this map. *Hint*: Do completion problem 5 first.