Week 6

Special Meromorphic Functions

6.1 Meromorphic Functions and Möbius Transformations

4/23:
- Last time.
  - We defined the Riemann sphere
$$\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$$
  - When is a function $f : \hat{\mathbb{C}} \to \mathbb{C}$ "holomorphic at ∞?"
    - Liouville: $f \in \mathcal{O}(\hat{\mathbb{C}})$ is constant.
  - f is a meromorphic function on $U \subset \hat{\mathbb{C}}$ if and only if f is a holomorphic map to $\hat{\mathbb{C}}$.
  - $f^{-1}(\infty)$ gives the set of poles.
- Note for people doing residues on their final project.
  - Recall that if $f : U \to \hat{\mathbb{C}}$ is meromorphic, then it has a Laurent series around each pole p.
  - Essentially,
$$f(z) = \frac{1}{(z-p)^k} h(z)$$
    - h is holomorphic.
    - $h(p) \neq 0$.
    - k is the order of the pole.
  - Since h is holomorphic, it has a power series expansion
$$h(z) = \sum_{i=0}^{\infty} a_i (z-p)^i$$
    with $a_0 \neq 0$.
  - Thus,
$$\begin{aligned} f(z) &= \sum_{i=0}^{\infty} a_i (z-p)^{i-k} \\ &= \sum_{i=-k}^{\infty} a_i (z-p)^i \end{aligned}$$
  - This allows us to define the **principal part**.
- **Principal part** (of a Laurent series): The sum of the terms with a negative exponent.

- TPS: Suppose you have a pole p in a disk D within the radius of convergence of the Laurent series. Compute
$$\int_{\partial D} f \, \mathrm{d}z$$
  - Because we are in the radius of convergence of the Laurent series, the series converges locally absolutely uniformly, so we can switch the sum and the integral in the following and evaluate.
$$\begin{aligned}\int_{\partial D} f \, \mathrm{d}z &= \int_{\partial D} \sum_{j=-k}^{\infty} a_j (z-p)^j \, \mathrm{d}z = \sum_{j=-k}^{\infty} \int_{\partial D} a_j (z-p)^j \, \mathrm{d}z \\ &= \sum_{\substack{j=-k \\ j\neq -1}}^{\infty} 0 + \int_{\partial D} \frac{a_{-1}}{z-p} \, \mathrm{d}z = a_{-1} \cdot 2\pi i\end{aligned}$$
  - This a_{-1} coefficient is clearly special, so it gets a special name.
- **Residue** (of f at p): The a_{-1} coefficient in the Laurent expansion of f about a pole p. *Denoted by* $\boldsymbol{\mathrm{res}_p(f)}$. *Given by*
$$\mathrm{res}_p(f) := a_{-1}$$
- Corollary: f has a primitive on D (containing a single pole p) iff
$$\mathrm{res}_p(f) = 0$$

  Proof. This goes back to the proposition from the 3/28 class. If the residue is zero, then the closed loop integral is zero, so by homotopy, f has a primitive on the disk?? And if it has a primitive, then it's holomorphic on D and therefore the residue is zero. □

- Theorem: Every meromorphic function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ (by which we mean a meromorphic function on $\mathbb{C}$ that is either bounded or with a pole at ∞) is rational, i.e., $f = p/q$.

  Proof. We will prove this theorem by dividing it into three consecutive claims.

  1. The set of poles of f (including ∞) is finite.
  2. Write f minus the sum of the principal parts of all the poles of f, and call this g.
  3. g is holomorphic on $\hat{\mathbb{C}}$, and therefore is constant.

  Once we have Claim 3, we will know that $f = c + \sum$ principal parts, where all principal parts are rational, meaning that f is rational! Let's begin.

  For claim 1, we will argue that the set of poles of f is discrete and compact because it is a result of set theory that discrete compact sets are finite. To confirm that the set of poles is discrete, observe that $1/f$ has zeroes where f has poles, and we know that these zeroes must be discrete because otherwise $1/f$ would be constant and zero (think power series). To confirm that the set of poles is also compact, observe that it lives in $\hat{\mathbb{C}}$ (which is compact) and is a closed set (since its discrete, it cannot be open).

  For claim 2, it may seem unintuitive that we can subtract off a bunch of partial Laurent series about different points from a function defined more broadly than a Laurent series around a certain pole. However, observe that the principal part of a Laurent expansion around a certain pole is just is a function in its own right! Thus, we can subtract it off just like any other function. After doing this, g is not defined everywhere, but it has an analytic continuation (claim 3) to everywhere, as desired. Note that the principal part of a Laurent expansion about p only has a pole at p, so we *can* subtract it off without introducing any new poles. Additionally, note that we can subtract off a pole at ∞ as follows. Essentially, if f has a pole at ∞, then it is a polynomial. But f having a pole at ∞ is equivalent to $f(1/z)$ having a pole at 0. So just take the principal part of the Laurent expansion of $f(1/z)$ at zero and then switch out all the z's for $1/z$'s again and subtract that. □

- To get a global picture of the poles of f, we need a global picture of the zeroes of $1/f$.
- TPS.
  1. For $k \in \mathbb{N}$, draw $x \mapsto x^k$ on $\mathbb{R}$. For y near 0, how many preimages does y have?
     - If k is odd, y always has one preimage.
     - If k is even, then $y > 0$ has two preimages, $y = 0$ has one preimage, and $y < 0$ has no preimages.
  2. Draw what happens to the sector $\{z \mid z \in r\mathrm{e}^{i\theta},\ \theta \in (0, 2\pi/k]\}$ under $z \mapsto z^k$.
     - Maps to the circle of radius r^k bijectively.
     - Additionally, the interior of the sector goes to the disk minus the slit along the positive real axis.
  3. Do 1, but now for $z \mapsto z^k$ in $\mathbb{C}$.
     - For $y \neq 0$, there are k distinct roots of y.
  4. Draw a "global" picture of $z \mapsto z^k$.
     - For $z \mapsto z^2$, for instance, we cut $\mathbb{C}$ into an upper half plane and lower half plane and wrap them both around into circles.
     - *See class pictures.*
- Last time.
  - Proposition: If X is a Riemann surface and $f : X \to \hat{\mathbb{C}}$ is meromorphic, then f is onto.
- Now we'll do a bit on Möbius transformations.
  - One of Calderon's favorite topics; there will be a bit on the next problem set about these.
- What we just saw is that if $p(z) = z^n$, then $\#p^{-1}(w) = n$[1] for w near zero but nonzero.
  - If $w = 0$, then you still get n preimages if you count with multiplicity.
- If $p(z)$ is a general polynomial of degree n, we also know (by the FTA) that $\#p^{-1}(0) = n$.
  - Moreover, for all $w \in \hat{\mathbb{C}}$, $\#p^{-1}(w) = n$ with multiplicity because $p(z) - w$ has n roots.
- Let $f(z) = p(z)/q(z)$ be a rational function, such as $z/(z-3)^2$.
  - Here, $\#f^{-1}(\infty) = 2$.
  - Here as well, $f^{-1}(0) = 1$ on $\mathbb{C}$ and $= 2$ on $\hat{\mathbb{C}}$.
- **Degree** (of a rational function): The natural number defined as follows, where $f = p/q$ is a rational function representated with p, q coprime. *Denoted by* $\mathbf{deg(f)}$. *Given by*

$$\deg(f) := \max(\deg(p), \deg(q))$$

- Theorem: If $f = p/q$, then for all $w \in \hat{\mathbb{C}}$, $\#f^{-1}(w) = \deg(f)$ when counted with multiplicity.

  Proof. Follows from the FTA. Proof in the notes. □

- Example:
  - If $f(z) = z/(z-3)^2$, then we count 3 twice with multiplicity (giving a pole at ∞). We can also count $0, \infty$ as two distinct poles that give us zero.
  - This just reiterates the previous example.

[1] Recall that # denotes cardinality; in this case, cardinality of the preimage.

- Holomorphic symmetries.
  - What are all biholomorphisms?
  - They are kind of like change of coordinate maps.
  - Can we have entire biholomorphisms? Can we have biholomorphisms $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.
  - $\mathrm{Aut}(\mathbb{C}) = \mathrm{Bihol}(\mathbb{C})$. On Thursday, we'll prove that

$$\mathrm{Bihol}(\mathbb{C}) = \{z \mapsto az + b\}$$

  - For the other one,

$$\mathrm{Bihol}(\hat{\mathbb{C}}) = \left\{ z \mapsto \frac{az+b}{cz+d} \mid a, b, c, d \in \mathbb{C},\ ad - bc \neq 0 \right\}$$

    - The condition $ad - bc \neq 0$ is the same as saying that the map is nonconstant.
    - This is the set of fractional linear transformations.
    - It is isomorphic to

$$\mathrm{PGL}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \det \neq 0 \right\}$$

      - This is the projective linear group.
      - This group acts on the projective linear space $\mathbb{P}^1(\mathbb{C}) = \hat{\mathbb{C}}$ (ask Calderon later??).

  Proof. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a biholomorphism. Thus, it is bijective and hence $\deg(f) = 1$. Additionally, it is meromorphic and hence rational map. But a rational map with degree 1 is just a map of the given kind! □

- A different geometric way of thinking about these **Möbius transformations**: Circle inversion.
  - This is the map $re^{i\theta} \mapsto r^{-1}e^{i\theta}$. In terms of z, this map is $z \mapsto 1/\bar{z}$.
  - This map is not holomorphic. In fact, it is **antiholomorphic** and **anticonformal**.
  - This is the projecting up and down map.
  - It can also be thought of as reflecting over the equator.
  - $\text{Möb}^{\pm}$ (the **extended Möbius group**) is the group generated by circle inversions.
  - Möb is the orientation-preserving subgroup.
- Four examples.
  - Scaling $z \mapsto rz$ where $r \in \mathbb{R}$.
  - Translation.
  - Rotation.
  - One last one.
- Theorem: $\text{Möb} = \mathrm{Bihol}(\hat{\mathbb{C}}) \cong \mathrm{PGL}_2(\mathbb{C})$.

  Proof. $\subset$: Obvious, once you've digested the definitions. This is just because circle inversions are anti-conformal. Two inversions implies conformal, which is equivalent to biholomorphic.

  $\supset$: We have algebraically that

$$\frac{az+b}{cz+d} = \frac{bc-ad}{c^2}\left(z + \frac{d}{c}\right)^{-1} + \frac{a}{c}$$

  where we have two translations, an inversion, and a scaling/rotation. □

6.2 The Complex Logarithm

4/25:

- Review: What do certain complex analysis objects look like?
  - Holomorphic functions look like conformal maps and branching.
  - Meromorphic functions look like conformal maps to $\hat{\mathbb{C}}$ and branching.
  - But what do essential singularities look like?
    - Recall that an essential singularity is a singularity z for which there exist $z_n \to z$ and $w_n \to z$ such that $f(z_n)$ stays bounded and $|f(w_n)| \to \infty$.
- Theorem (Casorati-Weierstrass): Suppose $f \in \mathcal{O}(U \setminus \{z_0\})$ where z_0 is an essential singularity. Then for all $w_0 \in \mathbb{C}$, there exist $z_n \to z_0$ such that $f(z_n) \to w_0$.

  Proof. Suppose for the sake of contradiction that there exists a $w_0 \in \mathbb{C}$ for which every $z_n \to z_0$ is such that $|f(z_n) - w_0| > \varepsilon$. Pictorially, this means that no matter how close the points of the sequence $\{z_n\}$ get to z_0, $f(z_n)$ will always stay outside of a disk (of radius ε) around w_0. Define

  $$g(z) := \frac{1}{f(z) - w_0}$$

  Since $f \in \mathcal{O}(U \setminus \{z_0\})$ and g is a rational function of f, we know that $g \in \mathcal{O}(U \setminus \{z_0\})$. Additionally, let $\{z_n\}$ be an arbitrary sequence converging to z_0. Then

  $$\begin{aligned} |f(z_n) - w_0| &> \varepsilon \\ \frac{1}{\varepsilon} &> \frac{1}{|f(z_n) - w_0|} \\ \frac{1}{\varepsilon} &> \left|\frac{1}{f(z_n) - w_0}\right| \\ \frac{1}{\varepsilon} &> |g(z_n)| \end{aligned}$$

  Thus, we have shown that g is bounded near z_0. This is the last condition we need to invoke Riemann's removable singularity theorem and learn that g can be analytically continued to z_0. But if $\hat{g} \in \mathcal{O}(U)$, then

  $$f = \frac{1}{g} + w_0$$

  has either a removable singularity or a pole at z_0. This contradicts the assumption that f has an essential singularity at z_0. ☐

- Implication: $f(U \setminus \{z_0\})$ is dense in $\mathbb{C}$ and hence f is *very* not injective.
- Example: $z \mapsto \mathrm{e}^z$.

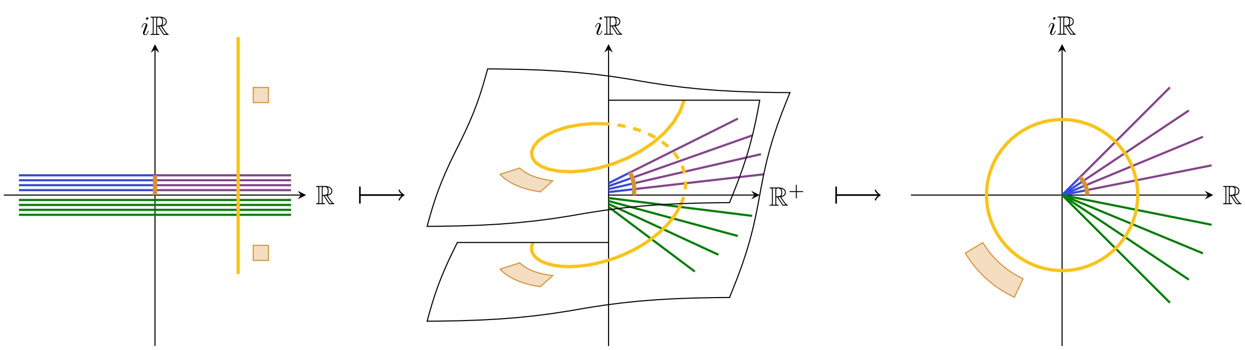


Figure 6.1: The essential singularity of e^z.

- This function has an essential singularity at ∞.
- Say we want to approach the positive real number e^a ($a \in \mathbb{R}$). Then we can take the sequence of points $z_n := a + 2\pi ni$. As $n \to \infty$, $|a + 2\pi ni| \to \infty$. Additionally,
$$e^{a+2\pi ni} = e^a \cdot (e^{2\pi i})^n = e^a \cdot 1^n = e^a$$
- For the negative real number $-e^a$, choose $z_n = a + \pi i + 2\pi ni$. Then
$$e^{a+\pi i+2\pi ni} = e^a \cdot e^{i\pi} \cdot (e^{2\pi i})^n = e^a \cdot -1 \cdot 1^n = -e^a$$
- For 0, choose $z_n = -n$.
- How about the complex number $re^{i\theta}$? Pick $a \in \mathbb{R}$ such that $r = e^a$. Then choose $z_n = a+i\theta+2\pi ni$.
- What is shown in Figure 6.1 is an **∞ branched cover**.

- Theorem (Little Picard): If f is entire and nonconstant, then $f(\mathbb{C})$ equals $\mathbb{C}$, except maybe one point.
- Theorem (Big Picard): For any neighborhood U of an essential singularity z_0, $f(U \setminus \{z_0\})$ is all of $\mathbb{C}$ except maybe one point.
- TPS: Verify that the big Picard theorem holds for exp and ∞.
  - Let U be a neighborhood of ∞, as defined when we discussed the Riemann sphere.
  - This is very much related to what was discussed above!
  - Indeed, let $re^{i\theta} \in \mathbb{C}$ be arbitrary. Choose n big enough so that $a + i\theta + 2\pi ni \in U$. Then $\exp(a + i\theta + 2\pi ni) = re^{i\theta}$, as desired.
  - Here, 0 is our single point exception.
- Corollary (of Casorati-Weierstrass):
$$\text{Bihol}(\mathbb{C}) = \{z \mapsto az + b\}$$

Proof. For the backwards inclusion, let $f(z) = az + b$ be an arbitrary linear map. f is a polynomial (hence holomorphic). Its inverse
$$f^{-1}(z) = a^{-1}z - a^{-1}b$$
is a polynomial (hence holomorphic). And
$$f \circ f^{-1} = f^{-1} \circ f = \text{id}$$
so f is bijective. Therefore, f is biholomorphic by definition.

For the forwards inclusion, suppose first that $f \in \text{Bihol}(\mathbb{C})$ has a pole at infinity. Then $f \in \text{Bihol}(\hat{\mathbb{C}})$. It follows by the result from last time that there exist $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ such that
$$f(z) = \frac{az + b}{cz + d}$$
Additionally, since we know that $f(\infty) = \infty$, we must have $c = 0$ (otherwise, $f(\infty) = a/c$). But if $c = 0$, then
$$f(z) = \frac{a}{d}z + \frac{b}{d} \in \{z \mapsto az + b\}$$
as desired.

Now suppose that $f \in \text{Bihol}(\mathbb{C})$ does not have a pole at infinity. Since $f \notin \text{Bihol}(\hat{\mathbb{C}})$, this also means that f does not have a removable singularity at infinity. Thus, f has an essential singularity at infinity. Consequently, if we let D be an open neighborhood of ∞, then the Casorati-Weierstrass theorem implies that $f(D)$ is dense in $\mathbb{C}$. Additionally, by the open mapping theorem, $f(D)$ is open. These last two results combined imply that $f(D) = \mathbb{C}$. But since $\mathbb{C} \setminus D$ is also in the domain of f (i.e., f must map some points in $\mathbb{C} \setminus D$ to points that it's already covered in D), f is not injective, hence not bijective, hence not biholomorphic. This is a contradiction, and therefore biholomorphic functions can only have a *pole* at infinity, a case in which we have shown they are linear. ☐

- We now finally define a (*not* the) logarithm.
- **Logarithm** (of z): A point $w \in \mathbb{C}$ such that $e^w = z$, where $z \in \mathbb{C}$.
  - Naturally, w is only well-defined up to $\pm 2\pi in$ ($n \in \mathbb{Z}$).
- Log identities don't always hold.
- TPS: Which are always true and which are only true for the appropriate choice of log?
  - $\exp(\log(z)) = z$ is always true.
  - $\log(\exp(z)) = z$ is only true if we happen to choose the correct complex number raised to the exponential.
  - $\log(zw) = \log(z) + \log(w)$ is similarly only true for the appropriate choices of log.
- Setting aside well-definedness for now, let's define what it means to raise one complex number to the power of another.
- $\boldsymbol{z^w}$: The complex number defined as follows. *Given by*

$$z^w := e^{w \log z}$$

  - Such a definition requires a choice of the logarithm.
  - Such a choice is called a **branch** of the logarithm.
- TPS: What do z^n and $z^{1/n}$ give, where $n \in \mathbb{N}$?
  - z^n gives exactly one value.

$$z^n = e^{n \log z} = e^{n(a+bi+2\pi ik)} = e^{na} \cdot e^{bni} \cdot (e^{2\pi in})^k = e^{na} e^{bni}$$

    - In the above, $k \in \mathbb{Z}$.
    - Additionally, it doesn't matter what exact b we choose because if we increase or decrease it by $2\pi ik$, we will just multiply the value by $1^k = 1$. Essentially, e^{na} gives a unique magnitude for the result, and bn gives a unique argument for the result.
  - $z^{1/n}$ gives n values, each of which differs by $e^{2\pi i/n}$ (i.e., by an n^{th} root of 1).

$$z^{1/n} = (re^{i\theta + 2\pi ik})^{1/n} = r^{1/n} e^{i\theta/n} \zeta_k$$

    - In the above, the $\zeta_k = e^{k \cdot 2\pi i/n}$ are the n^{th} roots of unity.
- Example: We can now calculate i^i!

$$i^i = e^{i \log i} = e^{i(\pi i/2 + 2\pi ik)} = e^{-\pi/2} \cdot e^{-2\pi k}$$

- **Principal branch** (of log): The branch that is real for real numbers.
- **Logarithm function** (on U): A continuous inverse to exp, i.e., $e^{\log z} = z$.
- Think of a logarithm function as lifting U to the infinity spiral in the middle of Figure 6.1.
  - Imagine U is the orange domain in the top-down view on the right side of Figure 6.1.
  - Project U up, down, or both onto a connected open subset of the infinity spiral.
    - In the middle of Figure 6.1, we see U projected up infinitely, but with a logarithm, we would have to make a *choice* here.
  - Then invert U back onto the original complex plane on the left side of Figure 6.1.

- Observe that...
  - If $U \ni 0$, then no logarithm function exists on U;
  - If U contains a loop γ winding around U, then no logarithm function exists on U.
- Proposition: If the logarithm makes sense on U, then it is holomorphic and

$$\frac{\mathrm{d}}{\mathrm{d}z}\log(z) = \frac{1}{z}$$

Proof. See Fischer and Lieb (2012). □

- Observe that since the derivative of the logarithm is never zero, the logarithm must be injective. Therefore, it's biholomorphic!
- Corollary: No log exists on $\mathbb{C}^*$.

Proof. Suppose for the sake of contradiction that a logarithm exists on $\mathbb{C}^*$. Then $1/z$ has a primitive on $\mathbb{C}^*$, so by the 3/28 lecture,

$$\int_\gamma \frac{1}{z}\,\mathrm{d}z = 0$$

for all loops γ. But since we know that

$$\int_\gamma \frac{1}{z}\,\mathrm{d}z = 2\pi i \neq 0$$

we have a contradiction. □

- But, there are logs on domains covering $\mathbb{C}^*$.

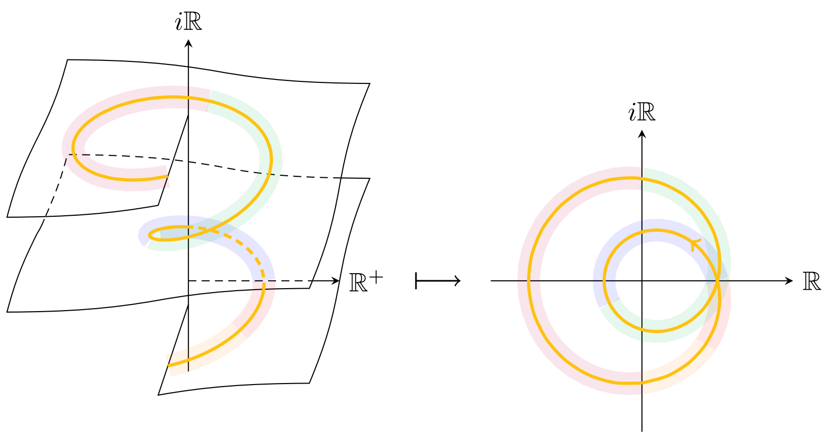


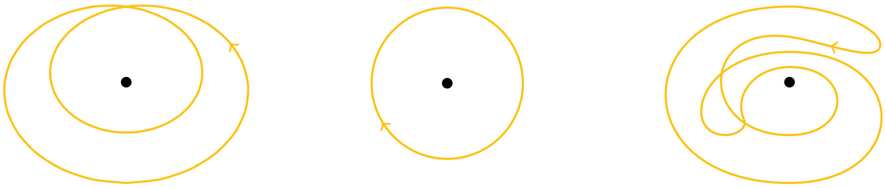
Figure 6.2: Motivating the winding number.

  - If a path γ wraps around 0 more than once, we can break it into segments that individually have logarithms.
  - Then, we find that

$$\int_\gamma \frac{\mathrm{d}z}{z} = \sum \int_{\gamma_i} \frac{\mathrm{d}z}{z} = \text{total change of angle}$$

  - This leads into the following definition.

- **Winding number** (of γ about z_0): The number of times the curve γ wraps around z_0 counterclockwise. *Denoted by* $\mathbf{wn}(\boldsymbol{\gamma}, \boldsymbol{z_0})$. *Given by*
$$\operatorname{wn}(\gamma, z_0) := \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}z}{z - z_0}$$
- Winding number examples.



(a) wn = 2. (b) wn = −1. (c) wn = 0.

Figure 6.3: Winding number examples.

- Reformulation: A branch of the logarithm exists on U if and only if for all $\gamma \subset U$, the winding number of γ about the origin is 0.