3 Cauchy's Integral Formula

Set A: Graded for Completion

4/12: 1. Fischer and Lieb (2012), QII.3.1. Using the Cauchy integral formulas, compute the following integrals.

(a) $\displaystyle\int_{|z+1|=1} \frac{\mathrm{d}z}{(z+1)(z-1)^3}$.

Proof. Let

$$f(z) = \frac{1}{(z-1)^3}$$

Then by the CIF,

$$\begin{aligned} f(-1) &= \frac{1}{2\pi i}\int_{|\zeta+1|=1} \frac{f(\zeta)}{\zeta+1}\mathrm{d}\zeta \\ -\frac{1}{8} &= \frac{1}{2\pi i}\int_{|z+1|=1} \frac{f(z)}{z+1}\,\mathrm{d}z \\ \boxed{-\frac{\pi i}{4} = \int_{|z+1|=1} \frac{\mathrm{d}z}{(z+1)(z-1)^3}} \end{aligned}$$

□

(b) $\displaystyle\int_{|z-i|=3} \frac{\mathrm{d}z}{z^2+\pi^2}$.

Proof. We know that

$$\frac{1}{z^2+\pi^2} = \frac{1}{(z-\pi i)(z+\pi i)}$$

Thus, let

$$f(z) = \frac{1}{z+\pi i}$$

Then by the CIF,

$$\begin{aligned} f(\pi i) &= \frac{1}{2\pi i}\int_{|z-i|=3} \frac{f(z)}{z-\pi i}\,\mathrm{d}z \\ \frac{1}{2\pi i} &= \frac{1}{2\pi i}\int_{|z-i|=3} \frac{1}{(z-\pi i)(z+\pi i)}\,\mathrm{d}z \\ \boxed{1 = \int_{|z-i|=3} \frac{\mathrm{d}z}{z^2+\pi^2}} \end{aligned}$$

□

(c) $\displaystyle\int_{|z|=1/2} \frac{\mathrm{e}^{1-z}}{z^3(1-z)}\,\mathrm{d}z$.

Proof. Let

$$f(z) = \frac{\mathrm{e}^{1-z}}{1-z}$$

Consequently,

$$f'(z) = \frac{(1-z)\cdot -\mathrm{e}^{1-z} - \mathrm{e}^{1-z}\cdot -1}{(1-z)^2} = \frac{z\mathrm{e}^{1-z}}{(1-z)^2}$$

and

$$f''(z) = \frac{(1-z)^2\cdot(\mathrm{e}^{1-z} - z\mathrm{e}^{1-z}) - z\mathrm{e}^{1-z}\cdot 2(1-z)^1\cdot -1}{(1-z)^4} = \frac{(1+z^2)\mathrm{e}^{1-z}}{(1-z)^3}$$

Then by the second derivative of the CIF,

$$\begin{aligned} f''(0) &= \frac{2!}{2\pi i}\int_{|z|=1/2}\frac{f(z)}{(z-0)^{2+1}}\,\mathrm{d}z \\ \mathrm{e} &= \frac{1}{\pi i}\int_{|z|=1/2}\frac{f(z)}{z^3}\,\mathrm{d}z \\ \Aboxed{\pi i\mathrm{e} &= \int_{|z|=1/2}\frac{\mathrm{e}^{1-z}}{z^3(1-z)}\,\mathrm{d}z} \end{aligned}$$

□

(d) $\displaystyle\int_{|z-1|=1}\left(\frac{z}{z-1}\right)^n \mathrm{d}z$ for any $n \geq 1$.

Proof. Let

$$f(z) = z^n$$

Then by the $(n-1)^{\text{th}}$ derivative of the CIF,

$$\begin{aligned} f^{(n-1)}(1) &= \frac{(n-1)!}{2\pi i}\int_{|z-1|=1}\frac{f(z)}{(z-1)^n}\,\mathrm{d}z \\ n! &= \frac{(n-1)!}{2\pi i}\int_{|z-1|=1}\frac{z^n}{(z-1)^n}\,\mathrm{d}z \\ \Aboxed{2\pi n i &= \int_{|z-1|=1}\left(\frac{z}{z-1}\right)^n\mathrm{d}z} \end{aligned}$$

□

2. Fischer and Lieb (2012), QII.4.2. Assume that the power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ converges on $D = D_r(0)$.

(a) Show that if f is real-valued on $\mathbb{R} \cap D$, then all a_k are real.

Proof. Suppose for the sake of contradiction that a_k is complex. By hypothesis, f is a convergent power series in a neighborhood of 0. Thus, by the proposition from the 3/26 class, f is holomorphic in a neighborhood of zero. Consequently, f is C^∞. This means that in particular, f is C^k with

$$f^{(k)}(0) = k! a_k$$

But since a_k is complex, this means that the k^{th} derivative of f is complex, a contradiction for a real-valued function. □

(b) Show that if f is an even (resp. odd) function, then $a_k = 0$ for all odd (resp. even) k.

Proof. Suppose f is even. Then $f(z) = f(-z)$. Thus,

$$\begin{aligned} \sum_{k=0}^{\infty} a_k z^k &= \sum_{k=0}^{\infty} a_k (-z)^k \\ \sum_{k=0}^{\infty} a_{2k} z^{2k} + \sum_{k=0}^{\infty} a_{2k+1} z^{2k+1} &= \sum_{k=0}^{\infty} a_{2k} z^{2k} - \sum_{k=0}^{\infty} a_{2k+1} z^{2k+1} \\ 2\sum_{k=0}^{\infty} a_{2k+1} z^{2k+1} &= 0 \end{aligned}$$

Now the whole power series on the left above cancelling out implies that each term individually goes to zero as well. Therefore, we have proven that all of the odd coefficients go to zero.

The proof is symmetric in the other case. □

(c) Show that if $f(iz) = f(z)$, then a_k can only be nonzero if k is divisible by 4.

Proof. As in part (b), we have that

$$\sum_{k=0}^{\infty} a_k (iz)^k = \sum_{k=0}^{\infty} a_k z^k$$

$$\sum_{k=0}^{\infty} a_{4k} z^{4k} + i \sum_{k=0}^{\infty} a_{4k+1} z^{4k+1} - \sum_{k=0}^{\infty} a_{4k+2} z^{4k+2} - i \sum_{k=0}^{\infty} a_{4k+3} z^{4k+3} = \sum_{j=0}^{3} \sum_{k=0}^{\infty} a_{4k+j} z^{4k+j}$$

$$(i-1) \sum_{k=0}^{\infty} a_{4k+1} z^{4k+1} - 2 \sum_{k=0}^{\infty} a_{4k+2} z^{4k+2} - (i+1) \sum_{k=0}^{\infty} a_{4k+3} z^{4k+3} = 0$$

Indeed, via term-by-term cancellation again, we see that all terms with $k \pmod 4 \not\equiv 0$ go to zero. □

(d) Discuss the equation $f(\rho z) = \mu f(z)$, where $\rho, \mu \in \mathbb{C} \setminus \{0\}$ are given.

Proof. As in part (c), if ρ is a root of unity and $\mu = 1$, then a_k can only be nonzero if k is divisible by the denominator of $\arg \rho$ in reduced form. If ρ has an irrational argument, the equation may not add much of any new information. More generally, power series that satisfy this equation tend to be determined on $\mathbb{C}$ by their values on some subset of $\mathbb{C}$, be it half the plane (as in $f(-z) = f(z)$ or $f(-z) = 2f(z)$), one quadrant (as in $f(iz) = f(z)$), or some other region. □

3. Fischer and Lieb (2012), QII.6.1. Determine the type of singularity that each of the following functions has at z_0. If the singularity is removable, calculate the limit as $z \to z_0$; if the singularity is a pole, find its order and the principal part of f at z_0.

(a) $(1 - \mathrm{e}^z)^{-1}$ at $z_0 = 0$.

Proof. As $z \to z_0 = 0$, $\mathrm{e}^z \to 1$ and hence $1 - \mathrm{e}^z \to 0$. Thus, as $z \to z_0$, $|1/(1 - \mathrm{e}^z)| \to \infty$. Therefore, the singularity is a $\boxed{\text{pole.}}$

If $f(z) = (1 - \mathrm{e}^z)^{-1}$, then $g(z) = 1 - \mathrm{e}^z$ and $g^{(n)}(z) = -\mathrm{e}^z$ $(n \geq 1)$. Thus,

$$g(z) = \sum_{k=1}^{\infty} -\frac{1}{n!} z^n = -z \sum_{k=0}^{\infty} \frac{1}{(n+1)!} z^n$$

so $\boxed{\text{the order is 1.}}$ Moreover, the principal part of f at z_0 is

$$\boxed{-\frac{1}{z}}$$

□

(b) $(z - \sin z)^{-1}$ at $z_0 = 0$.

Proof. As in part (a), the denominator goes to zero as we go to $z_0 = 0$, so the singularity is a $\boxed{\text{pole.}}$

Also as before, we can calculate that

$$g(z) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{z^{2k+1}}{(2k+1)!} = z^3 \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k+3)!}$$

so $\boxed{\text{the order is 3.}}$ Inverting this back again to determine the principal part, we must use the coefficient formulae on Fischer and Lieb (2012, p. 51) to find the Laurent series expansion for f.

In particular,

$$\begin{aligned} b_0 &= a_0^{-1} = 3! = 6 \\ b_1 &= -a_1a_0^{-2} = 0 \\ b_2 &= (a_1^2 - a_0a_2)a_0^{-3} = \frac{3}{10} \end{aligned}$$

so the principal part of f at z_0 is

$$\boxed{6x^{-3} + \frac{3}{10}x^{-1}}$$

□

(c) $ze^{iz}/(z^2+b^2)^2$ at $z_0 = ib$ $(b > 0)$.

Proof. As in parts (a) and (b), the singularity is a $\boxed{\text{pole.}}$
As in part (b), we can calculate that

$$g(z) = 0 + 0z + 4ibe^bz^2 + 24be^bz^3 + \cdots$$

so $\boxed{\text{the order is 2.}}$ Inverting this back again, we learn that

$$b_0 = -\frac{i}{4be^b} \qquad\qquad b_1 = \frac{3}{2be^b}$$

so the principal part of f at z_0 is

$$\boxed{-\frac{i}{4be^b}z^{-2} + \frac{3}{2be^b}z^{-1}}$$

□

(d) $(\sin z + \cos z - 1)^{-2}$ at $z_0 = 0$.

Proof. As in parts (a)-(c), the singularity is a $\boxed{\text{pole.}}$
Once again, we can calculate that

$$g(z) = 0 + 0z + z^2 - z^3$$

so $\boxed{\text{the order is 2.}}$ And once again,

$$b_0 = 1 \qquad\qquad b_1 = 1$$

so we have

$$\boxed{z^{-2} + z^{-1}}$$

□

4. Let $\mathbb{D}$ denote the unit disk and suppose that $f \in \mathcal{O}(\mathbb{D})$.

(a) Prove that for any $R \in (0,1)$ and any z with $|z| < R$, we have that

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\theta})\,\mathrm{Re}\left(\frac{Re^{i\theta}+z}{Re^{i\theta}-z}\right)\mathrm{d}\theta$$

Hint: Observe that setting $w = R^2/\bar{z}$, we have that the integral of $f(\zeta)/(\zeta - w)$ over the circle of radius R centered at the origin is 0.

Proof. Since $f \in \mathcal{O}(\mathbb{D})$, $D := D_R(0) \subset\subset \mathbb{D}$, and $z \in D$, the CIF tells us that

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

Parameterize ∂D by $\zeta = R\mathrm{e}^{i\theta}$ for $\theta \in [0, 2\pi]$. Then, substituting into the above,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_0^{2\pi} f(R\mathrm{e}^{i\theta}) \frac{1}{R\mathrm{e}^{i\theta} - z} \cdot iR\mathrm{e}^{i\theta} \mathrm{d}\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(R\mathrm{e}^{i\theta}) \frac{R\mathrm{e}^{i\theta}}{R\mathrm{e}^{i\theta} - z} \mathrm{d}\theta \end{aligned}$$

□

(b) Compute that

$$\operatorname{Re}\left(\frac{R\mathrm{e}^{i\theta} + r}{R\mathrm{e}^{i\theta} - r}\right) = \frac{R^2 - r^2}{R^2 - 2Rr\cos\theta + r^2}$$

where $R, r, \theta \in \mathbb{R}$.

Proof. We have that

$$\begin{aligned} \operatorname{Re}\left(\frac{R\mathrm{e}^{i\theta} + r}{R\mathrm{e}^{i\theta} - r}\right) &= \frac{1}{2}\left[\frac{R\mathrm{e}^{i\theta} + r}{R\mathrm{e}^{i\theta} - r} + \overline{\frac{R\mathrm{e}^{i\theta} + r}{R\mathrm{e}^{i\theta} - r}}\right] \\ &= \frac{1}{2}\left[\frac{R\mathrm{e}^{i\theta} + r}{R\mathrm{e}^{i\theta} - r} + \frac{R\mathrm{e}^{-i\theta} + r}{R\mathrm{e}^{-i\theta} - r}\right] \\ &= \frac{1}{2}\left[\frac{(R\mathrm{e}^{i\theta} + r)(R\mathrm{e}^{-i\theta} - r) + (R\mathrm{e}^{i\theta} - r)(R\mathrm{e}^{-i\theta} + r)}{(R\mathrm{e}^{i\theta} - r)(R\mathrm{e}^{-i\theta} - r)}\right] \\ &= \frac{1}{2}\left[\frac{(R^2 - Rr\mathrm{e}^{i\theta} + Rr\mathrm{e}^{-i\theta} - r^2) + (R^2 + Rr\mathrm{e}^{i\theta} - Rr\mathrm{e}^{-i\theta} - r^2)}{R^2 - Rr\mathrm{e}^{i\theta} - Rr\mathrm{e}^{-i\theta} + r^2}\right] \\ &= \frac{1}{2}\left[\frac{2R^2 - 2r^2}{R^2 - Rr(\mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta}) + r^2}\right] \\ &= \frac{R^2 - r^2}{R^2 - 2Rr\cos\theta + r^2} \end{aligned}$$

as desired. □

(c) Now suppose that $u = \operatorname{Re}(f)$, so u is a harmonic function. Deduce the **Poisson integral representation formula**: For $z = r\mathrm{e}^{i\theta}$, we have

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - \phi) u(\phi) \mathrm{d}\phi$$

where $P_r(\psi)$ is the **Poisson kernel** for the disk, given by

$$P_r(\psi) = \frac{1 - r^2}{1 - 2r\cos\psi + r^2}$$

Proof.

$$\begin{aligned} f(z) &= \frac{1}{2\pi} \int_0^{2\pi} f(R\mathrm{e}^{i\theta}) \operatorname{Re}\left(\frac{R\mathrm{e}^{i\theta} + z}{R\mathrm{e}^{i\theta} - z}\right) \mathrm{d}\theta \\ f(z) &= \frac{1}{2\pi} \int_0^{2\pi} f(R\mathrm{e}^{i\theta}) \frac{R^2 - r^2}{R^2 - 2Rr\cos\theta + r^2} \mathrm{d}\theta \\ u(z) &= \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - \phi) u(\phi) \mathrm{d}\phi \\ u(z) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2} u(\phi) \mathrm{d}\phi \end{aligned}$$

□

Set B: Graded for Content

1. Fischer and Lieb (2012), QII.4.6.

(a) Suppose the domain G is symmetric with respect to the real axis and that f is holomorphic on G and real-valued on $G \cap \mathbb{R}$. Show that $f(\bar{z}) = \overline{f(z)}$ for all $z \in G$.

Proof. Since f is holomorphic, f has a power series representation

$$\sum_{k=0}^{\infty} a_k z^k$$

on G. It follows by QA.2a that the coefficients a_k of this power series are real. Therefore, we have that

$$\begin{aligned} f(\bar{z}) &= \sum_{k=0}^{\infty} a_k \bar{z}^k \\ &= \sum_{k=0}^{\infty} a_k \overline{z^k} \\ &= \sum_{k=0}^{\infty} \overline{a_k z^k} \\ &= \overline{\sum_{k=0}^{\infty} a_k z^k} \\ &= \overline{f(z)} \end{aligned}$$

as desired. Note that we know that a $\bar{z} \in G$ corresponds to each $z \in G$ because of the hypothesis that G is symmetric with respect to the real axis. □

(b) Suppose $G = D_r(0)$ and f is holomorphic on G and real-valued on $G \cap \mathbb{R}$. Show that if f is even (resp. odd), then the values of f on $G \cap i\mathbb{R}$ are real (resp. imaginary). Prove this without using the power series expansion of f.

Proof. Suppose first that f is even. Let $ib \in G \cap i\mathbb{R}$ be an arbitrary imaginary number. Since f is even,

$$f(ib) = f(-ib)$$

Additionally, by part (a),

$$f(-ib) = \overline{f(ib)}$$

Thus, by transitivity,

$$f(ib) = \overline{f(ib)}$$

But for a complex number to equal its complex conjugate, that complex number must be real, as desired.

Now suppose that f is odd. Then

$$f(ib) = -f(-ib)$$

We still have in addition that $f(-ib) = \overline{f(ib)}$, so by transitivity,

$$\begin{aligned} f(ib) &= -\overline{f(ib)} \\ f(ib) + \overline{f(ib)} &= 0 \\ 2\,\mathrm{Re}[f(ib)] &= 0 \\ \mathrm{Re}[f(ib)] &= 0 \end{aligned}$$

Therefore, $f(ib)$ must be purely imaginary, as desired. □

2. Some setup: Suppose that f is holomorphic on the unit disk $\mathbb{D} = \{|z| < 1\}$. A point w on the circle $\partial D = \{|z| = 1\}$ is **regular** if there is an open neighborhood U of w and an analytic function g on U such that $f = g$ on $U \cap \mathbb{D}$. Notice that f can be analytically continued outside the boundary of $\mathbb{D}$ if and only if there is a point w on ∂D that is regular for f.

Now define the function

$$f(z) = \sum_{k=1}^{\infty} z^{2^k}$$

Show that f converges on $\mathbb{D}$, and that it cannot be analytically continued past $\mathbb{D}$.

Proof. Let z with $|z| < 1$ be arbitrary. By the geometric series test, $\sum_{k=0}^{\infty} z^k$ converges on $\mathbb{D}$, and then f converges via the comparison test.

Now suppose for the sake of contradiction that There is a regular point on $\partial\mathbb{D}$ and corresponding function $g : U \to \mathbb{C}$. Since $f = g$ on $\mathbb{D} \cap U$, the identity theorem tells us that f, g have the same power series (i.e., the one defined above). Now let $z \in U$ with $|z| > 1$. By the ratio test,

$$\lim_{k\to\infty} \left| \frac{z^{2^{k+1}}}{z^{2^k}} \right| = \lim_{k\to\infty} \frac{|z|^{2\cdot 2^k}}{|z|^{2^k}} = \lim_{k\to\infty} |z|^{2^k} = \infty > 1$$

so the series diverges at z, contradicting the existence of g. ☐

3. Suppose that f is an entire function and that, for all sufficiently large z, we have $|f(z)| \leq |z|^n$. Prove that f must be a polynomial.

Proof. Since f is entire (and hence holomorphic), it has a power series. Proving that f is a polynomial is then just a matter of proving that this power series is finite, i.e., truncates somewhere. Combining Cauchy's inequalities with the given condition, we have that

$$\frac{|f^{(m)}(z)|}{m!} \leq \frac{1}{R^m} \max_{\partial D} |f(\zeta)| \leq \frac{1}{R^m} \max_{\partial D} |z|^n = \frac{1}{R^m} \cdot R^n = R^{n-m}$$

for all $m \in \mathbb{N}_0$. Evidently, then, for all $m > n$, we can send $R \to \infty$ and shrink $a_m = |f^{(m)}(z)|/m! \to 0$, as desired. ☐