Week 4

Extrema

4.1 Poles and Maximum Moduli

4/9:

- Announcement.
  - Midterm next week in class.
  - Material up through today, though probably not much on today's content.
- Last time.
  - Cauchy integral formula: If U is a domain, $D \subset\subset U$, and $z \in D$, then
    $$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta$$
  - This implies Riemann's removable singularity theorem, which states that if $f \in \mathcal{O}(U \setminus \{z_0\})$ and f is bounded near z_0, then there exists a $\hat{f} \in \mathcal{O}(U)$ which continues f at z_0.
    - Example: $\sin(z)/z \in \mathcal{O}(\mathbb{C}^*)$ has a continuation to $\mathbb{C}$.
    - In particular, take the Taylor series at zero and evaluate:
      $$\widehat{\frac{\sin(z)}{z}}(0) = 1 - \frac{0^3}{3!} + \frac{0^5}{5!} - \cdots = 1$$
  - Alternatively, if $f \in \mathcal{O}(U \setminus \{z_0\})$ and $|f(z)| \to \infty$ as $z \to z_0$, then z_0 is a **pole** of f.
- Today.
  - Finish up what we couldn't last time.
  - Say something about harmonic functions.
- **Meromorphic** (function): A function $f : U \to \mathbb{C}$ such that $f \in \mathcal{O}(U \setminus P)$ and each $p \in P$ is a pole, where $P \subset U$ is a finite set of points.
- Example: Consider $1/z \in \mathcal{O}(\mathbb{C}^*)$.
  - This has a pole at zero.
  - Thus, $1/z$ is *holomorphic* on the punctured plane $\mathbb{C}^*$, but *meromorphic* on the whole complex plane $\mathbb{C}$.
- Example: The same argument applies to $1/z^k$ ($k \in \mathbb{N}$).
- Example: The function from PSet 2, Q2c:
  $$f(z) = \frac{1}{z(z-1)(z-i)(z-1-i)}$$

- It follows that
$$\{f : f \text{ is holomorphic}\} \subset \{f : f \text{ is meromorphic}\}$$
- Fact: All of the examples kind of look the same.
  - More generally, suppose f has a pole at p and is holomorphic on $U \setminus \{p\}$. Pick a disk $D \ni p$ such that $f \neq 0$ on D. Then $g = 1/f \in \mathcal{O}(D \setminus \{p\})$ and as $z \to p$, $g(z) \to 0$.
  - Thus, we've got a function that's holomorphic and bounded near a point, so by Riemann's removable singularity theorem, it has a unique holomorphic extension $\hat{g} \in \mathcal{O}(D)$.
    - In particular, $g(p) = 0$.
  - Note: We do *not* need to choose D small enough such that it contains only one point in P. However, we will for the time being just to simplify things. The reason we can do this is because singularities — as points of a finite set — are isolated.
- There exists a power series for g about p such that
$$g(z) = \sum_{k=0}^{\infty} a_k (z-p)^k$$
  - We know that $a_0 = 0$ because $g(p) = 0$.
  - It can also happen such that some (or [potentially infinitely] many) of the remaining a_i are zero.
    - Example: if $f = 1/z^3$, then $g = z^3$ and $a_i = 0$ $(i > 3)$.
  - Now let L be the largest natural number such that $a_i = 0$ for all $0 \leq i < L$.
    - Because $a_0 = 0$, $L \geq 1$.
    - Additionally, $a_L \neq 0$.
  - Then we can rewrite the power series as
$$g(z) = (z-p)^L h(z)$$
    where...
    1. $h(z) = \sum_{k=L}^{\infty} a_k (z-p)^{k-L}$;
    2. $h(p) \neq 0$ (and h is nonzero near p).
  - We say that g has a **zero** (of order L at p).
    - Similarly, we say that f has a **pole** (of order L at p).
  - Thus,
$$f(z) = \frac{1}{(z-p)^L} \frac{1}{h(z)}$$
    where, moreover, $1/h \in \mathcal{O}(D')$ for some smaller disk D'.
  - Example: $1/(z^2 + z)$ goes to $z(z+1)$.
  - Takeaway: Near any pole p, f must look like
$$\frac{1}{(z-p)^L} \cdot \phi(z)$$
    where ϕ is holomorphic around p.
    - This implies that there exists a **Laurent series** expansion around any pole.
    - In particular, near p,
$$f(z) = \sum_{k=-L}^{\infty} a_k (z-p)^k$$

- **Zero** (of order L at p): A point p of a holomorphic complex function g such that $g(p) = 0$ and $g(z) = (z-p)^L h(z)$ where $h(p) \neq 0$.
- **Pole** (of order L at p): A point p of a holomorphic complex function f such that $1/f(p) = 0$ and $f(z) = 1/(z-p)^L h(z)$ where $h(p) \neq 0$.
- **Laurent series**: A power series including a finite number of negative coefficients. *Given by*

$$\sum_{k=-L}^{\infty} a_k (z-p)^k$$

- TPS: Consider $\cot(z) = \cos(z)/\sin(z)$, which has a pole at zero. What is the order of the pole? What is the Laurent series?
  - The pole is order 1.
    - One way to see this is to observe how $\tan z$ has a nonzero tangent at 0, so $\tan z = z + \cdots$. Thus, we can only divide one z out of its power series.
    - Alternatively, we have
    $$\cot(z) = \frac{1}{z} \cdot \frac{z}{\sin(z)} \cdot \cos(z)$$
    from which we can observe that $\cos(z) \in \mathcal{O}(\mathbb{C})$, and $\sin(z)/z \in \mathcal{O}(\mathbb{C})$ (at zero, the extension gives 1) so $z/\sin(z)$ is holomorphic near zero. Thus, we can define $\phi(z) = z\cos(z)/\sin(z)$.
      - What if we tried $\tilde{\phi}(z) = z^2 \cos(z)/\sin(z)$? What's different? Well, $\tilde{\phi}$ is still holomorphic, but $\tilde{\phi}(0) = 0$, which is a problem. Notice that $\phi(0) = 1$!
    - As a last way, we could investigate the power series of $\cot(z)^{-1} = \tan(z)$ directly:
    $$\tan z = z + \frac{z^3}{3} + \frac{2z^5}{15}$$
  - The Laurent series was not discussed in class, but here's some comments.
    - It would begin from $k = -1$.
    - We could construct it from the power series for cosine and sine using Calderon's formula above.
    - Figuring out the formula for the power series of an inverted power series is a good exercise!!
- What if $|f(z)| \to \infty$ as $|z| \to \infty$? Then we say that f has a **pole** (at ∞).
  - Otherwise, there exist sequences $z_n \to \infty$ and $w_n \to \infty$ such that $f(z_n) \to \infty$ and $f(w_n)$ stays bounded. This is an **essential singularity** (at ∞).
  - We can mull over this until Thursday when we introduce the solution, the **Riemann sphere**.
  - If $f(z)$ stays bounded, then f has a **removable singularity** (at ∞).
- **Pole** (at ∞): A function f such that $|f(z)| \to \infty$ as $|z| \to \infty$.
- **Essential singularity** (at ∞): A function f for which there exist sequences $z_n \to \infty$ and $w_n \to \infty$ such that $f(z_n) \to \infty$ and $f(w_n)$ stays bounded.
- **Removable singularity** (at ∞): A function f that stays bounded as $|z| \to \infty$.
- We're now going to switch to a completely different topic.
- Suppose $f \in \mathcal{O}(U)$. When does $|f(z)|$ get the biggest? Equivalently, where does $|f(z)|$ take a local max? *Hint*: Look at the Cauchy integral formula!
  - There are no such points, at least on the interior of U!

- Theorem (maximum modulus principle): Let $f \in \mathcal{O}(U)$. If $|f(z)|$ has a local maximum on U, then f is constant.

  Proof. Let z_0 be a local maximum of $|f(z)|$. Pick $D \ni z_0$ small enough such that $|f(z)| \leq |f(z_0)|$ for all $z \in D$. Let r be the radius of D. Now invoking the CIF,

$$
\begin{aligned}
|f(z_0)| &= \left|\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta - z_0}\mathrm{d}\zeta\right| \\
&= \frac{1}{2\pi}\left|\int_{\partial D}\frac{f(\zeta)}{\zeta - z_0}\mathrm{d}\zeta\right| \\
&\leq \frac{1}{2\pi}\int_{\partial D}\left|\frac{f(\zeta)}{\zeta - z_0}\right|\mathrm{d}\zeta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f(z_0 + r\mathrm{e}^{i\theta})}{r\mathrm{e}^{i\theta}}\cdot ir\mathrm{e}^{i\theta}\right|\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}|f(z_0 + r\mathrm{e}^{i\theta})|\mathrm{d}\theta \\
&\leq \frac{1}{2\pi}\cdot 2\pi\cdot\max_{\partial D}|f(\zeta)| \\
&= \max_{\partial D}|f(\zeta)| \\
&\leq |f(z_0)|
\end{aligned}
$$

  But since the above inequality begins and ends with the same value, all $\leq$'s must be $=$'s. Thus, in particular,

$$
\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}|f(z_0 + r\mathrm{e}^{i\theta})|\mathrm{d}\theta &= |f(z_0)| \\
\frac{1}{2\pi}\int_0^{2\pi}\big(|f(z_0 + r\mathrm{e}^{i\theta})| - |f(z_0)|\big)\mathrm{d}\theta &= 0
\end{aligned}
$$

  Combining this with the fact that the above integrand is always ≤ 0 because $f(z_0)$ is a local maximum, we have that

$$
\begin{aligned}
|f(z_0 + r\mathrm{e}^{i\theta})| - |f(z_0)| &= 0 \\
|f(\zeta)| &= |f(z_0)|
\end{aligned}
$$

  on ∂D. Note that this is true for all small ∂D's centered at z_0.

  Now since $|f|$ is constant on ∂D, we must have that $|f|^2 = f \cdot \bar{f}$ is constant on ∂D. Taking the Wirtinger derivative and using its product rule gets us

$$
0 = \frac{\partial}{\partial z}\big(f \cdot \bar{f}\big) = f_z \cdot \bar{f} + f \cdot \bar{f}_z
$$

  Since f is holomorphic (hence satisfies the CR equations) and $f_{\bar{z}} = \bar{f}_z$, we have that

$$
\bar{f}_z = f_{\bar{z}} = 0
$$

  Thus,

$$
0 = f_z \cdot \bar{f} + f \cdot 0 = f_z \cdot \bar{f}
$$

  By the zero-product property, either $f_z = 0$ and $\bar{f} = 0$. In the first case, this means that f is constant, as desired. In the second case, this means that f is zero (and hence constant), as desired.

  At this point, we have shown that f is constant on a small disk. Therefore, we need only invoke the identity theorem, which tells us that since the function is constant for a little bit somewhere, it is constant everywhere. □

- Another way to prove this is by considering the derivative of the Cauchy integral formula and where it's equal to zero.
- Corollary (minimum modulus principle): If $f \in \mathcal{O}(U)$, $f \neq 0$ on U (hence $1/f \in \mathcal{O}(U)$), and $|f(z)|$ takes a minimum in U, then f is constant.
- Application of the maximum modulus principle (the fundamental theorem of algebra): If p is a polynomial of degree d in $\mathbb{C}$, then p has d roots in $\mathbb{C}$ (counted with multiplicity).

  Proof. Suppose inductively that $d \geq 1$.

  Step 1 (show that there exists one root): Suppose for the sake of contradiction that p has no zeros. Since p is a polynomial, we know that $|p(z)| \to \infty$ as $|z| \to \infty$. Thus, there exists $R > 0$ such that for all z with $|z| > R$, $|p(z)| \geq |p(0)|$. Then $|p(z)|$ must take a minimum on $\overline{D_R}$. But to keep p from being constant by the minimum modulus principle, the minimum has to be on ∂D_R. Now take a slightly bigger disk; our global minimum is now in the interior, so p is constant, a contradiction. It follows that p must have a zero in D_R.

  Step 2: Suppose p has a root at z_0. Then power series for p at z_0 is $p(z) = (z - z_0)p_1(z)$. p_1 is a polynomial of degree $d - 1$.

  Step 3: Now iterate to find that p is a product of monomials. $\square$

- Algebraists love to prove this with only algebra, but in reality, the proof is complex analysis.[1]
- We did not get to say something about harmonic functions today, but Calderon will leave the content in his notes in case we want to look at it.
  - The statement: Harmonic functions follow a version of the CIF.
  - There's a related PSet problem.

4.2 Modulus Principles and Harmonic Functions

4/11:

- Last time.
  - Maximum modulus principle: If $f \in \mathcal{O}(U)$, $f(z) \neq 0$ for all $z \in U$, and $|f|$ takes a max inside U, then f is constant.
    - Analogous result: The minimum modulus principle.
  - This result implies the fundamental theorem of algebra.
    - Proof idea: $|f(z)| < |f(\zeta)|$ for $\zeta \in \partial D$, so f must have a zero.
- Another corollary: We have a better understanding of the mapping properties of holomorphic functions.
- Recall that conformal (angle-preserving diffeomorphism) iff biholomorphic (bijective, f, f^{-1} holomorphic).
- In real analysis, we have the **inverse function theorem**.
- **Inverse function theorem**: If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is C^1 and $Df(x) \neq 0$, then f is locally a diffeomorphism from $x \in U$ to $V \ni f(x)$.
- So for $f \in \mathcal{O}(U)$...
  - If f' is never 0 on U, then $f(U)$ is open;
  - If f' is never 0 on U and $f : U \to f(U)$ is a bijection, then f is biholomorphic.

[1] How did this proof work??

- Claim: The "if f' is never 0 on U" condition is actually unnecessary!
- Theorem: Let $f \in \mathcal{O}(U)$.

1. Open mapping theorem: If f is nonconstant, then $f(U)$ is open.

   Proof. To prove that $f(U)$ is open, it will suffice to show that every $w_0 \in f(U)$ is contained in some neighborhood that's a subset of $f(U)$. Let $w_0 = f(z_0)$. Pick a disk $D \subset\subset U$ such that $f(z) - w_0 \neq 0$ on ∂D; this is possible because the zeroes of a nonconstant holomorphic function (like $f - w_0$) must be isolated, or otherwise f would be constant. Thus, we may define the positive number

   $$\delta := \inf_{z \in \partial D} |f(z) - w_0|$$

   Now pick w such that $|w - w_0| < \delta/2$. Then by the triangle inequality, we have that for all $z \in \partial D$,

   $$|f(z) - w| \geq |f(z) - w_0| - |w - w_0| \geq \delta - |w - w_0| > \delta - \frac{\delta}{2} = \frac{\delta}{2}$$

   This means that $|f - w|$ is always greater than the number $\delta/2$ on the boundary of D, but since

   $$|f(z_0) - w| = |w - w_0| < \frac{\delta}{2}$$

   $|f - w|$ does not obtain its minimum on the boundary of D. Thus, since all other hypotheses of the minimum modulus principle are satisfied, there must be a zero of $|f - w|$ on U. This means that there exists a $z \in U$ such that $f(z) = w$, and hence $w \in f(U)$. Therefore, since the choice of $w \in D_{\delta/2}(w_0)$ was arbitrary, we know that $D_{\delta/w}(w_0) \subset f(U)$, as desired. $\square$

2. Complex inverse function theorem: If f is bijective, it's biholomorphic.

   Proof. Define the set

   $$Z := \{z \in U \mid f'(z) = 0\}$$

   of zeroes of f'. To prove that f is biholomorphic, we will quickly show that $f : U \setminus Z \to f(U) \setminus f(Z)$ is biholomorphic and then build up to the point where we can use Riemann's removable singularity theorem to analytically continue this restriction. Let's begin.

   Since $f \in \mathcal{O}(U)$ by hypothesis, $f \in C^\infty \subset C^1$. Additionally, by the definition of z, $Df(x) \neq 0$ at all $x \in U \setminus Z$. Thus, by the real inverse function theorem, f is a diffeomorphism at all $x \in U \setminus Z$. Consequently, $f^{-1} : f(U) \setminus f(Z) \to U \setminus Z$ is differentiable, and hence holomorphic. This combined with the hypothesis that $f : U \setminus Z \to f(U) \setminus f(Z)$ is bijective and holomorphic implies that $f : U \setminus Z \to f(U) \setminus f(Z)$ is biholomorphic.

   Now the first part of the plan is complete. The next step involves building up to the point that we can apply Riemann's removable singularity theorem to $f^{-1} : f(U) \setminus f(Z) \to U \setminus Z$. To do so, we need only verify that $f(U) \setminus f(Z)$ is a domain and f^{-1} is bounded near any $f(z) \in f(Z)$, since $f(z) \in f(Z) \subset f(U)$ by definition and we have just shown that $f^{-1} \in \mathcal{O}(f(U) \setminus f(Z))$.

   First, we verify that $f(U) \setminus f(Z)$ is a domain. To do so, we begin by checking that $f(U)$ is a domain. Since U is a domain (hence connected) and f is holomorphic (hence continuous), Theorem 9.11[2] tells us that $f(U)$ is connected. Additionally, since U is a domain (hence open) and f is bijective (hence nonconstant), the open mapping theorem implies that $f(U)$ is open. But since $f(U)$ is connected and open, it must be a domain, as desired. Next, we check that $f(Z)$ is discrete in $f(U)$. Since f is nonconstant (per the above), f' is nonzero. It follows since f' is holomorphic that Z must be discrete (otherwise, f' holomorphic would be zero on a nondiscrete set, and hence would be zero everywhere, a contradiction). Thus, every $z \in Z$ is contained in an open neighborhood $N_z \subset U$ disjoint from all other $N_{z'}$. It follows by the open mapping theorem that each $f(N_z)$ is an *open* neighborhood of $f(z)$, and by the fact that f is bijective that the set

[2] See MATH 16210 Honors Calculus II notes.

of $f(N_z)$ is pairwise disjoint. Thus, $f(Z)$ is discrete in $f(U)$, as desired. Therefore, $f(U) \setminus f(Z)$ is a (punctured) domain, as desired.

Second, we verify that f^{-1} is bounded near any $f(z) \in f(Z)$. To do so, we begin by checking that $f^{-1} : f(U) \to U$ is continuous. Let $X \subset U$ be open. Since f is bijective, $(f^{-1})^{-1}(X) = f(X)$. By the open mapping theorem, $f(X)$ is open. Thus, by the open-set definition of continuity, f^{-1} is continuous, as desired. But then since f^{-1} is continuous, it maps compact sets to compact sets. Therefore, a closed and bounded neighborhood of $f(z)$ will maps to a closed and bounded neighborhood of z, as desired.

At this point, we may invoke Riemann's removable singularity theorem to analytically continue $f^{-1} : f(U) \setminus f(Z) \to U \setminus Z$ to $f(U)$. Therefore, since $f : U \to f(U)$ is bijective and holomorphic by hypothesis and $f^{-1} : f(U) \to U$ is holomorphic, f is biholomorphic by definition, as desired. $\square$

- Preview: There is also a geometric reason why $f \in \mathcal{O}(U)$ with zeros can't be injective.
- So the maximum modulus principle gets us a lot, and in fact, these kinds of arguments can be used to say even more!
- Example: Where do $\text{Re}(f)$ and $\text{Im}(f)$ take their max?
- Recall that $h : U \to \mathbb{R}$ is *harmonic* if $\Delta h = 0$, where

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \cdot \frac{\partial^2}{\partial z \partial \bar{z}}$$

  - Examples of harmonic functions: $f \in \mathcal{O}(U)$, $\text{Re}(f)$, $\text{Im}(f)$.
  - Nonexample: $|f|$ is not! Take $f(z) = z$; then $\Delta |f| = 1/|z|$.
- Where do harmonic functions take their maxima?
  - This is essentially equivalent to asking about $\text{Re}(f)$ for the following reason.
- A characterization: If $u : U \to \mathbb{R}$ is C^2 and harmonic where U is convex, then there exists $f \in \mathcal{O}(U)$ such that $u = \text{Re}(f)$.

Proof. Since u is harmonic,

$$0 = \Delta u = \frac{\partial}{\partial \bar{z}} \left(\frac{\partial u}{\partial z} \right)$$

This means that u_z is holomorphic! This combined with the fact that U is convex (hence star-shaped) implies by the CIT that $\int_\gamma u_z \, \mathrm{d}z = 0$ for any closed loop $\gamma \subset U$. Thus, by the proposition associated with Figure 2.1, there exists a primitive g for u_z on U. From here, it follows by the rules of complex differentiation that

$$\frac{\partial}{\partial z}(\text{Re}\, g) = \frac{\partial}{\partial z} \left[\frac{1}{2}(g + \bar{g}) \right] = \frac{1}{2} \frac{\partial g}{\partial z} = \frac{1}{2} u_z$$

and

$$\frac{\partial}{\partial \bar{z}}(\text{Re}\, g) = \frac{1}{2} \overline{g_z} = \frac{1}{2} u_{\bar{z}}$$

Therefore, $u = \text{Re}(2g) + C$, as desired. $\square$

- Harmonic functions also satisfy a version of the Cauchy Integral Formula!
  - Let D be a disk centered at z of radius R.

- Then

$$\begin{aligned} u(z) &= \operatorname{Re} f(z) \\ &= \operatorname{Re}\left[\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta\right] \\ &= \frac{1}{2\pi}\operatorname{Im}\left[\int_0^{2\pi} i\cdot f(z+R\mathrm{e}^{i\theta})\mathrm{d}\theta\right] \\ &= \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re} f(z+R\mathrm{e}^{i\theta})\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(z+R\mathrm{e}^{i\theta})\mathrm{d}\theta \end{aligned}$$

- This is called the "mean value property for harmonic functions."
- On the PSet, we'll prove a version for any disk containing z of radius R, namely

$$u(z) = \int_0^{2\pi} u(\zeta)P_R(\zeta, z)\mathrm{d}\theta$$

  - P_R is the Poisson kernel defined by

$$P_R(\zeta, z) = \frac{1}{2\pi}\operatorname{Re}\left(\frac{\zeta + z}{\zeta - z}\right)$$

  - Note that by definition, the Poisson kernel is harmonic!

- Theorem (Maximum modulus principle for harmonic functions): Suppose $h : U \to \mathbb{R}$ is harmonic. If h takes a local maximum (or minimum) at $z_0 \in U$, then h must be *locally constant*, that is, constant in a neighborhood of z_0.

  Proof. We use the same strategy as we did for the holomorphic version.

  In particular, suppose z_0 is a local maximum. Pick a disk $D_R(z_0)$ about z_0 such that $h(z_0) \geq h(z)$ for all $z \in D_R(z_0)$. Using our new CIF, we have that for all $r < R$,

$$h(z_0) = \frac{1}{2\pi}\int_0^{2\pi} h(z_0 + r\mathrm{e}^{i\theta})\mathrm{d}\theta$$

  By a similar integrand argument to before (noting that h is real, so we don't need absolute values), we can conclude that

$$h(z_0) = h(z_0 + r\mathrm{e}^{i\theta})$$

  for all $r < R$. Therefore, h is constant on $D_R(z_0)$, as desired. □

- Corollary: Suppose $f \in \mathcal{O}(U)$. If $\operatorname{Re}(f)$ or $\operatorname{Im}(f)$ take a maximum in U, then f must be everywhere constant.
- Corollary: If U is bounded, then h is either constant or takes its maximum and minimum on ∂U.
- Application: Dirichlet problem (on a disk).
  - Let U be a convex domain, and let g be a function on ∂U. Does there exist a function u such that $u = g$ on ∂D and $\Delta u = 0$ (i.e., u is harmonic)?
  - This is like finding a steady state for the heat equation.
  - If U is a disk, the answer is yes, and the function is unique!
  - Existence.

- Set
$$u(z) := \int_0^{2\pi} g(\zeta) P_R(\zeta, z) \mathrm{d}\theta$$
- Then
$$\Delta_z u = \Delta_z \int_0^{2\pi} g(\zeta) P_R(\zeta, z) \mathrm{d}\theta = \int_0^{2\pi} g(\zeta) \underbrace{\Delta_z P_R(\zeta, z)}_{0} \mathrm{d}\theta = 0$$
- Note that $\Delta_z P_R(\zeta, z) = 0$ because P_R is harmonic, as mentioned earlier.
- The only hard part here is showing that u has a continuous extension to ∂D_R.

– Uniqueness.
  - Suppose that there exist two solutions g_1, g_2. Then $g_1 - g_2$ is harmonic and $g_1 - g_2 = 0$ on ∂D. But then by the maximum (and minimum) modulus principles, $g_1 - g_2 = 0$ on U. Therefore, $g_1 = g_2$ on U, as desired.