Week 5

Consequences of Defining Infinity

5.1 Office Hours

4/15:

- There will not be anything explicit about Thursday's content, but knowing it is helpful for understanding conformal maps.
- The exam is completely closed book.
- Midterm-style questions.
  - Per the mathematical hierarchy of needs (definitions and examples, theorem statements, problems/applying them, proofs of them).
  - He does not want to test our memorization skills but rather our understanding.

5.2 Midterm Review Sheet

4/16:

- Properties of complex numbers.
- **Holomorphic** (f at z_0): A function $f : \mathbb{C} \to \mathbb{C}$ for which the following limit exists. *Also known as* **$\mathbb{C}$-differentiable**. *Constraints*

  $$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} =: f'(z_0) \qquad \iff \qquad f(z) = f(z_0) + \Delta(z)(z - z_0)$$

  where Δ is continuous at z_0 and $\Delta(z_0) = f'(z_0)$.
  - Sum rule, product rule, quotient rule.
  - Chain rule.
  - Holomorphic implies continuous.
- Every $\mathbb{C}$-linear map is just multiplication by a complex number; the matrix must compute with $\mathcal{M}(i)$.
- **Cauchy-Riemann equations**: The following two equations, which identify when a complex function $(x, y) \mapsto (g, h)$ is holomorphic. *Also known as* **CR equations**. *Given by*

  $$\begin{aligned} g_x &= h_y \\ g_y &= -h_x \end{aligned}$$

- **Wirtinger derivatives**: The two differential operators defined as follows. *Denoted by* $\boldsymbol{\partial / \partial z, \partial / \partial \bar{z}}$. *Given by*

  $$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \qquad \qquad \frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

- Theorem: The $\mathbb{R}$-differentiable function $f : U \to \mathbb{C}$ is holomorphic iff $\partial f/\partial\bar{z} = 0$. Moreover, if it is, then
$$f'(z_0) = \left.\frac{\partial f}{\partial z}\right|_{z_0}$$
- **Laplacian**: The differential operator defined as follows. *Denoted by* $\mathbf{\Delta}$. *Given by*
$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$
- **Harmonic** (function): A function $f : \mathbb{R}^2 \to \mathbb{C}$ such that $\Delta f = 0$.
- Corollary: The real and imaginary parts of a C^2 holomorphic function are harmonic.

  Proof. $\Delta(u + iv) = \Delta u + i\Delta v$. □
- **Harmonic conjugates**: Two functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ that satisfy the CR equations.
- Path integration:
$$\int_\gamma f \,\mathrm{d}z = \int_a^b f(\gamma(t)) \cdot \gamma'(t) \,\mathrm{d}t$$
- FTC: Suppose $F' = f$ on $U \subset \mathbb{C}$, and let γ be a **path** inside of U. Then
$$\int_\gamma f \,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$$
- Factoring into rotation and scaling matrices.
$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \qquad (\lambda, \theta \in \mathbb{R})$$
- Lemma: Holomorphic maps preserve angles.

  Proof. Look at the argument at the intersection point and use the chain rule. □
- **Conformal** (map): A function $f : U \to V$, where $U, V \subset \mathbb{C}$, that satisfies the following two constraints. *Constraints*
  1. f is a diffeomorphism.
  2. f preserves angles.
- **Diffeomorphism**: A homeomorphism for which f, f^{-1} are differentiable.
- **Biholomorphic** (map): A function $f : U \to V$ that is bijective, holomorphic, and for which f^{-1} is holomorphic.
- Theorem/observation: Biholomorphic iff conformal.
- Chain rule:
$$\frac{\partial}{\partial t}(f \circ g)(z) = f_z(g(z))g_z(z) + f_{\bar{z}}(g(z))\bar{g}_{\bar{z}}(z)$$
- **Complex linear map**: A map $l : \mathbb{C} \to \mathbb{C}$ characterized by the following. *Constraints*
  1. $l(z + w) = l(z) + l(w)$;
  2. $l(rz) = rl(z)$;

for $z,w,r\in\mathbb{C}$.

- Every complex linear map is of the form

$$w=l(z)=az$$

for a unique $a\in\mathbb{C}$.

- **Real linear map**: A map $l:\mathbb{C}\to\mathbb{C}$ characterized by the following. *Constraints*

1. $l(z+w)=l(z)+l(w)$;
2. $l(rz)=rl(z)$;

for $z,w\in\mathbb{C}$ and $r\in\mathbb{R}$.

- Every real linear map is of the form

$$w=l(z)=az+b\bar{z}=\begin{pmatrix}a & b\end{pmatrix}\begin{pmatrix}z\\ \bar{z}\end{pmatrix}$$

for a unique pair $\begin{pmatrix}a & b\end{pmatrix}\in\mathbb{C}^2$.

- Implication: l is complex linear iff $b=0$.

- **Tangent map** (of f at z_0): The real linear map from $\mathbb{C}\to\mathbb{C}$ determined by the vector $\begin{pmatrix}f_z(z_0) & f_{\bar{z}}(z_0)\end{pmatrix}$.
- Proposition: f is holomorphic at z_0 iff its tangent map at z_0 is complex linear.
- **Exponential function**: The complex function defined as follows. *Denoted by* $\mathbf{e^z}$, $\mathbf{exp(z)}$. *Given by*

$$\mathrm{e}^z=\exp(z):=\sum_{k=0}^{\infty}\frac{z^k}{k!}$$

- **Pointwise** (convergent $\{f_n\}$): A sequence of functions $f_n:\mathbb{C}\to\mathbb{C}$ such that for all $z\in\mathbb{C}$, we have $f_n(z)\to f(z)$.
- **Locally uniformly** (convergent $\{f_n\}$): A sequence of functions $f_n:U\to\mathbb{C}$ and a function $f:U\to\mathbb{C}$ such that for all compact $K\subset U$,

$$\sup_{z\in K}|f_n(z)-f(z)|\to 0$$

- Lemma: If $f_n\to f$ locally uniformly and the f_n are continuous (or integrable; *not* differentiable), then so is f.
- **Taylor's theorem**: If $f:\mathbb{R}\to\mathbb{R}$ is C^{k+1} and $P_\alpha^k(x)$ is the k^{th} Taylor polynomial about $\alpha\in\mathbb{R}$, then for all $\beta\in\mathbb{R}$, there exists some $x\in(\alpha,\beta)$ such that

$$f(\beta)-P_\alpha^k(\beta)=\frac{(\beta-\alpha)^{k+1}}{(k+1)!}f^{(k+1)}(x)$$

- **Analytic** (function): A function $f:\mathbb{R}\to\mathbb{R}$ for which the Taylor polynomials converge (locally uniformly) to f.
- **Absolutely** (locally uniformly convergent power series): A power series $P(z)=\sum_{k=0}^{\infty}a_kz^k$ for which $A_N:\mathbb{C}\to\mathbb{R}$ locally uniformly converges, where

$$A_N(z):=\sum_{k=0}^{N}|a_kz^k|$$

- **Geometric series test**: If $|z| < 1$, then

$$\sum_{k=0}^{\infty} z^k \to \frac{1}{1-z}$$

- Lemma: Let $P(z)$ be a power series about 0. If there exists $z_1 \neq 0$ such that $|a_k z_1^k| \leq M$ for all k, then $P(z) = \sum a_k z^k$ converges on the disk $|z| < |z_1|$.

  Proof. Choice of z_1, z_2, and their ratio. □

- **Disk of convergence**: The largest disk centered at zero on which you converge.
- **Radius of convergence**: The radius of the disk of convergence. *Denoted by* $\boldsymbol{r}$.
- **Cauchy-Hadamard formula**: The radius of convergence is given by

$$r = (\limsup |a_k|^{1/k})^{-1}$$

- Lemma (from real analysis): If $f_n \to f$ locally uniformly and $f_n' \to g$ locally uniformly, then f is differentiable and $f' = g$.
  - Implication: Convergent power series are holomorphic.
- Corollary: Power series representations are unique.
  1. If $P(z) = \sum a_k z^k$ is convergent, then

$$a_k = \frac{1}{k!} P^{(k)}(0)$$

  2. If $P(z) = 0$ in a neighborhood of zero, then $a_k = 0$ for all k.
  3. If $P(z) = Q(z)$ (where $Q(z) = \sum b_k z^k$) in a neighborhood of 0, then $a_k = b_k$ for all k.
- Properties of the complex exponential.
  1. $\exp(z) = [\exp(z)]'$.
     - We obtain this via term-by-term differentiability.
     - This is just our favorite formula $\mathrm{d}/\mathrm{d}t\ (\mathrm{e}^t) = \mathrm{e}^t$ from calculus.
  2. $\overline{\exp(z)} = \exp(\bar{z})$.
  3. $\exp(a+b) = \exp(a) \cdot \exp(b)$.
  4. $|\exp(z)| = \exp[\mathrm{Re}(z)]$.
  5. $\mathrm{e}^{iz} = \cos(z) + i\sin(z)$.
- Complex trigonometric functions.

$$\cos(z) := \frac{1}{2}(\mathrm{e}^{iz} + \mathrm{e}^{-iz}) \qquad \sin(z) := \frac{1}{2i}(\mathrm{e}^{iz} - \mathrm{e}^{-iz})$$
$$\cosh(z) := \cos(iz) \qquad \sinh(z) := i\sin(iz)$$

- **Domain**: A connected, open set $U \subset \mathbb{C}$.
- **Primitive** (of f): A differentiable function whose derivative is equal to the original function f. *Also known as* **antiderivative**, **indefinite integral**. *Denoted by* $\boldsymbol{F}$.

- Corollary to the FTC: If $f = F'$, then for any closed curve γ in U,

$$\int_\gamma f \, \mathrm{d}z = 0$$

- Proposition: If $f : U \to \mathbb{C}$ is continuous and $\int_\gamma f \, \mathrm{d}z = 0$ for every closed loop in U, then f has a primitive on U.

  Proof. Step 1: Choose the integral along arbitrary γ.

  Step 2: Choice of γ doesn't matter (closed loop condition).

  Step 3: Correct derivative; apply FTC along δ and take limit. □

- **Star-shaped** (domain): A domain $U \subset \mathbb{C}$ for which there exists $a \in U$ such that for all $z \in U$, the segment $a \to z$ is in U.
- Lemma: If U is star-shaped and for every triangle with one vertex at a, we have $\int_\triangle f \, \mathrm{d}z = 0$, then f has a primitive in U.
- **Cauchy Integral Theorem**: Suppose U is a star-shaped domain and $f : U \to \mathbb{C}$ is holomorphic. Then $\int_\gamma f \, \mathrm{d}z = 0$ for any closed loop γ in U.

  Proof. Step 1: Prove f has a primitive via lemma & Goursat's lemma.

  Step 2: Apply FTC. □

- **Goursat's lemma**: If f is holomorphic in a neighborhood of a triangle including the interior, then $\int_\triangle f \, \mathrm{d}z = 0$.

  Proof. Subdividing triangles and inequalities. □

- Evaluating integrals using the complex functions and various paths.
- **Ratio test**: For $\sum a_n$, think about

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

- **Root test**: For $\sum a_n$, think about

$$\lim_{n\to\infty} |a_n|^{1/n}$$

- **Majorant test**: If $\sum_{k=0}^{\infty} a_k$ is a convergent series with positive terms and if for almost all k and all $z \in M$ we have $|f_k(z)| \leq a_k$, then $\sum_{k=0}^{\infty} f_k$ is absolutely uniformly convergent on M.
- Exponential mappings.
  - $z = x + iy_0$ maps onto the open ray beginning at 0 and passing through e^{iy_0}.
  - $z = x_0 + iy$ maps onto the circle of radius e^{x_0}.
  - Half-open horizontal strips map bijectively onto $\mathbb{C}^*$.
- **Homotopic** (paths): Two paths $\gamma, \tilde{\gamma} \subset U$ a domain such that $\tilde{\gamma}$ is obtained from γ by modifying γ on a small disk $D \subset U$, keeping the endpoints fixed.
- Claim/TPS: This argument shows that if γ and $\tilde{\gamma}$ are homotopic in U and $f \in \mathcal{O}(U)$, then

$$\int_\gamma f \, \mathrm{d}z = \int_{\tilde{\gamma}} f \, \mathrm{d}z$$

Proof. Each bump is a closed loop for the CIT. ☐

- Corollary: Let U be any domain, D be a disk in U, and $z \in \mathring{D}$. Suppose $f \in \mathcal{O}(U \setminus \{z\})$ and is bounded near z. Then
$$\int_{\partial D} f \,\mathrm{d}z = 0$$

Proof. Homotopy and γ_ε. ☐

- **Cauchy Integral Formula**: Suppose U is any domain, $D \subset U$ is a disk (i.e., $D \subset\subset U$ or $\overline{D} \subset U$), $f \in \mathcal{O}(U)$, and $z \in D$. Then
$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

Proof. Define the helper function
$$g(\zeta) = \begin{cases} \frac{f(\zeta)-f(z)}{\zeta - z} & \zeta \neq z \\ f'(z) & \zeta = z \end{cases}$$
It integrates to zero on ∂D and then splits into the two sides of the CIF. ☐

- Corollary: Holomorphic functions are C^∞.
- Corollary: In general,
$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \mathrm{d}\zeta$$
- **Cauchy's inequalities**:
$$|f^{(n)}(z)| \leq \frac{n!}{R^n} \max_{\partial D} |f(\zeta)|$$
- Liouville's Theorem: Suppose $f \in \mathcal{O}(\mathbb{C})$ (i.e., f is **entire**) and f is bounded. Then it's constant.

Proof. Cauchy's inequalities on a really big disk to limit $|f'|$. ☐

- **Entire** (function): A complex-valued function that is holomorphic on the whole complex plane.
- The Identity Theorem: If two holomorphic functions $f, g \in \mathcal{O}(U)$ agree on an open set in U, then $f = g$.

Proof. True for power series. ☐

  - In fact, more is true: If $z_n \to z_0$ where each z_n is distinct and $f(z_n) = g(z_n)$ for all n, then $f = g$.
- **Analytic continuation** (of f): The function $g \in \mathcal{O}(V)$ where $f \in \mathcal{O}(U)$, $V \supset U$, and $f = g$ on U.
- Morera's Theorem: If U is any domain, $f : U \to \mathbb{C}$ is continuous, and $\int_\triangle f \,\mathrm{d}z = 0$ for all triangles, then f is holomorphic.

Proof. The primitive exists. The primitive is holomorphic. Therefore, $F' = f$ is holomorphic. ☐

- **Riemann's removable singularity theorem**: Suppose U is a domain, $z \in U$, $f \in \mathcal{O}(U \setminus \{z\})$, and f is bounded near z. Then there exists a unique analytic continuation $\hat{f} \in \mathcal{O}(U)$. *Also known as* **Riemann extension theorem**.

Proof. Define a helper function

$$F(\zeta) = \begin{cases} f(\zeta)(\zeta - z) & \zeta \neq z \\ 0 & \zeta = z \end{cases}$$

Use Morera's theorem: F is continuous, triangles in two cases (CIT and γ_ε), and $F' = f$ via the limit definition. ☐

- **Singularity** (of f): A point z_0 such that $f \in \mathcal{O}(U \setminus \{z_0\})$.
- **Removable** (singularity): A singularity of a function that that satisfies the hypotheses of Riemann's removable singularity theorem.
- If a singularity is not removable, then f is not bounded near z_0. This leads to additional definitions.
- **Pole**: A non-removable singularity z_0 of a function f for which $|f(z)| \to \infty$ as $z \to z_0$.
  - So-named because of real analysis where a pole is an asymptote, and asymptotes kind of look like poles!
- **Essential** (singularity): A non-removable singularity that is not a pole; equivalently, a singularity z_0 for which there exist sequences $z_n \to z_0$ and $w_n \to z_0$ such that $|f(z_n)| \to \infty$ and $|f(w_n)|$ stays bounded.
- **Meromorphic** (function): A function $f : U \to \mathbb{C}$ such that $f \in \mathcal{O}(U \setminus P)$ and each $p \in P$ is a pole, where $P \subset U$ is a finite set of points.
- Orders of zeros and poles.
  - Invert the function, find a power series, divide $(z-p)^L$ out, find the power series of h, invert, find the principal part of the **Laurent series**.
- Theorem (maximum modulus principle): Let $f \in \mathcal{O}(U)$. If $|f(z)|$ has a local maximum on U, then f is constant.

  Proof. Step 1: Long inequality through the CIF that becomes equality.

  Step 2: Subtract and get integrand equal to zero; $|f|$ is constant on ∂D.

  Step 3: $|f|^2$ is constant on ∂D, differentiate, casework to f is constant or zero. ☐

- Corollary (minimum modulus principle): If $f \in \mathcal{O}(U)$, $f \neq 0$ on U (hence $1/f \in \mathcal{O}(U)$), and $|f(z)|$ takes a minimum in U, then f is constant.

5.3 Midterm

T/F: 5 points each (1 for answer, 4 for explanation)

Indicate whether each of the following are true or false, and give a complete answer as to why.

1. Any entire function (i.e., any $f \in \mathcal{O}(\mathbb{C})$) is the derivative of another entire function.
2. Let U be a domain, let z be a point in U, and let $f \in \mathcal{O}(U \setminus \{z\})$. Suppose that $\int_\gamma f \,\mathrm{d}z = 0$ for every closed curve γ in $U \setminus \{z\}$. Then $f \in \mathcal{O}(U)$.
3. If $f, g \in \mathcal{O}(\mathbb{C})$ and there are two distinct points $z_1, z_2 \in \mathbb{C}$ such that

   $$f(z_1) - g(z_1) \neq f(z_2) - g(z_2)$$

   then there is a sequence of points $z_n \in \mathbb{C}$ such that $|f(z_n) - g(z_n)| \to \infty$.

4. For any sequence of positive real numbers $\{a_k\}$ and any point $z \in \mathbb{C}$, there is a function f, holomorphic in a neighborhood of z, such that $|f^{(k)}(z)| = a_k$.

5. There is a conformal map that does all of the following.
   - Takes the first quadrant $Q = \{z : \mathrm{Re}(z), \mathrm{Im}(z) > 0\}$ to the strip $S = \{z : \mathrm{Im}(z) \in (-2, 2)\}$.
   - Takes the ray $\{z \in Q : \mathrm{Re} = \mathrm{Im}(z)\}$ to the "sine graph," i.e., $\{z : \mathrm{Im}(z) = \sin[\mathrm{Re}(z)]\}$.
   - Takes the segment $\{z \in Q : \mathrm{Re}(z) + \mathrm{Im}(z) = 1\}$ to $S \cap i\mathbb{R}$.

Problems: 5 points each

In the following, please *fully explain your reasoning* in addition to doing any relevant computations. A correct answer without explanation will receive at most one point on the problem.

1. Show that $u(x + iy) = e^{2x} \sin(2y) + 2x$ is harmonic on $\mathbb{C}$ and find a harmonic conjugate. *Bonus (1pt)*: If v is your harmonic conjugate, express the holomorphic function $f = u + iv$ in terms of z.

2. Suppose that U is a domain, P is a countable set of points in U, and that $f \in \mathcal{O}(U \setminus P)$. Suppose further that f has a pole at each point of P. Prove that P is discrete in U (i.e., P does not have any accumulation points in U).

3. Suppose I tell you that $f(z) = 1/(z - a)$, but I don't tell you what a is. Suppose that you know the real and imaginary parts of a are irrational and that you have an oracle that can compute $\int_\gamma f \, dz$ over any path such that the real and imaginary parts of $\gamma(t)$ are always rational (i.e., any γ that lives in $\mathbb{Q} + i\mathbb{Q}$). How would you go about estimating the value of a?

4. (a) Show that if $\{a_k\}$ is a sequence of non-zero complex numbers, then

$$\lim_{k\to\infty} \frac{|a_k|}{|a_{k-1}|} = L \qquad \Longrightarrow \qquad \lim_{k\to\infty} |a_k|^{1/k} = L$$

(b) Find the radius of convergence of the **order 0 Bessel function**

$$J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} z^{2n}$$

5. Suppose that f is a function on $\mathbb{C}$ such that...

(a) f is $(+i)$ anti-periodic, that is, if f is defined at z then it is defined at $z + i$ and $f(z) = -f(z + i)$;

(b) f is holomorphic in the strip $\{z \mid \mathrm{Im}(z) \in (-0.1, 1.1)\}$ except at the point $z = i/2$;

(c) If γ denotes a *clockwise* circle of radius $1/2$ centered at $i/2$, then $\int_\gamma f \, dz = 17$;

(d) $|f(z)| \to 0$ as $\mathrm{Re}(z) \to \pm\infty$.

Compute $\int_{-\infty}^{\infty} f(x) \, dx$.

5.4 The Riemann Sphere

4/18:
- Midterms are 2/3 graded.
- Project proposal due on Monday by the end of the day.
  - Completion points; Calderon just wants to give us feedback.
- PSets 4 and 5 will be pushed back by 1 week.
  - Get started on your final project in the intervening time!

- Today.
  - More holomorphic and meromorphic functions.
  - Definition of the logarithm.
- Example: Consider
$$f(z) = \frac{1}{z} \in \mathcal{O}(\mathbb{C}^*)$$
How does this look as a conformal map, e.g., where does it send the following two sets?
$$\mathbb{D} \setminus \{0\} \qquad\qquad \{re^{i\theta} \mid \theta \text{ fixed}\}$$
  - $\mathbb{D} \setminus \{0\}$ goes to $\mathbb{C} \setminus \overline{\mathbb{D}}$.
  - $\{re^{i\theta} \mid \theta \text{ fixed}\}$ goes to the line opposite itself at $\theta + 180^\circ$.
    - This is because $1/z$ takes $re^{i\theta}$ to $r^{-1}e^{-i\theta}$.
    - Moreover, as $r \to \infty$ in the initial set, $r^{-1} \to 0$ in the image.
  - Observe that
$$\lim_{z \to 0} |f(z)| = \infty \qquad\qquad \lim_{|z| \to \infty} f(z) = 0$$
  - So can't we just say that $f(0) = \infty$ and $f(\infty) = 0$?
    - Sure! Add in ∞ like with the extended real numbers.
    - We just stated that we can define $i := \sqrt{-1}$ to be a thing, so why not ∞ as well?
  - We now make this definition a bit more rigorous.
- **Riemann sphere**: The S^2-like manifold defined as follows. *Denoted by* $\hat{\mathbb{C}}$. *Given by*
$$\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$$
  - We can thus define a **neighborhood** (of ∞).
- **Neighborhood** (of ∞): The set defined as follows. *Given by*
$$\{|z| > R\} \cup \{\infty\}$$
  - Equivalently, take $\{|1/z| < R\}$.
- Let's visualize this sphere and neighborhood.

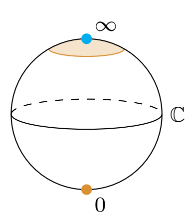

Figure 5.1: The Riemann sphere.

- **Stereographic projection**: A mapping of the sphere (minus the north pole) to the plane. Take the line from ∞ through a point on the sphere and onto the plane (this sets up a one-to-one correspondence between the points of the sphere and those of the plane).

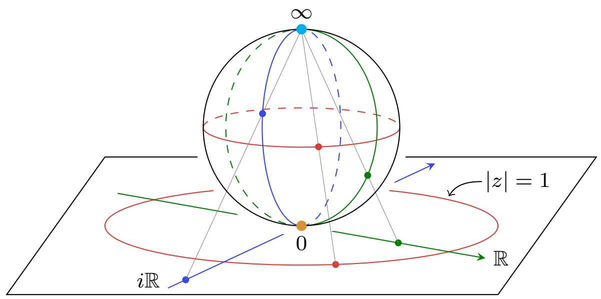

Figure 5.2: Stereographic projection.

  - Put $\{0\}$ at the point of tangency.
  - The equator will get mapped to a circle, so the stereographic projection is actually conformal away from ∞!
    - Moreover, we will identify this circle with the unit circle.
  - The real axis will become a line of longitude/great circle.
  - The imaginary axis will become the complementary line of longitude.
  - So the real and imaginary axes, and the unit circle become three perpendicular great circles on the Riemann sphere.
- This allows us to give an alternate definition of being meromorphic: f on $U \subset \mathbb{C}$ is meromorphic if $f : U \to \hat{\mathbb{C}}$ is such that $f^{-1}(\infty)$ is discrete and $f|_{U \setminus f^{-1}(\infty)}$ is holomorphic.
- As a complex manifold: Let $f : U \ni \infty \to \mathbb{C}$ be a function. We say that f is holomorphic at ∞ if

$$f^*(z) := \begin{cases} f(1/z) & z \neq 0 \\ f(\infty) & z = 0 \end{cases}$$

  is holomorphic at zero.
- Exercise: In Figure 5.2, we projected the Riemann sphere down onto a plane tangent to the sphere at 0. What if we project the Riemann sphere up onto a plane tangent to the sphere at ∞? Clearly these two versions of the complex plane would be parallel, and it actually turns out that the map between them is just $z \mapsto 1/z$!
- TPS: Verify that $1/(z-3)^2$ is holomorphic at ∞.
  - Note: This is a pole of order 2 at $z = 3$.
  - We have that
$$f^*(z) = \begin{cases} \frac{z^2}{(1-3z)^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$$
  - Clearly, $\lim_{z \to 0} f^*(z) = 0$, so we're done.
  - Comments.
    - Saying something *is* holomorphic at ∞ is just saying that it has an analytic continuation at ∞ (via Riemann's removable singularity theorem).
    - f^* has a zero of order 2 at ∞.

- In order to work with the Riemann sphere, you always put a patch of it on the plane and then say, "well, we know complex analysis on the plane."
  - Thus, we have a notion of poles, zeroes, and holomorphisms at ∞.
- The open mapping theorem still applies!
  - If U is a domain and $f : U \subset \hat{C} \to \mathbb{C}$ is holomorphic, then $f(U)$ is open.
- More generally, we should do something about the Riemann sphere.
- **Riemann surface**: A (real) two-manifold (locally like $\mathbb{R}^2$) equipped with an atlas of **charts**. *Denoted by* $\boldsymbol{X}$.
- **Chart**: A map from a domain U on the Riemann surface in question down to the complex plane $\mathbb{C}$ that is locally biholomorphic. *Denoted by* $\boldsymbol{\phi_U}$.
- **Holomorphic** (f at $z \in X$): A function $f : U \to \mathbb{C}$ for which $f \circ \phi_U^{-1}$ is holomorphic at $\phi_U(z)$, where X is a Riemann surface and $z \in U \subset X$.
- Suppose we map the bottom *and* top halves of the Riemann sphere to the unit circle.
  - The interconversion map is once again $1/z$.
- We're now done *defining* the Riemann sphere. We now start doing *analysis* on the Riemann sphere.
- Observation: f is holomorphic at ∞ iff $f(1/z)$ is holomorphic at 0.
  - Thus, $f : U \supset \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is meromorphic iff f is a holomorphic map.
- Theorem: Any holomorphic function $f : X \to \mathbb{C}$ on $\hat{\mathbb{C}}$ (or any compact Riemann surface X) is constant.

  Proof. Since f is holomorphic on X, f is continuous on X. Thus, since continuous functions on compact spaces must take maximums, $|f|$ has to take a maximum on X. Suppose $z \in X$ is such that $|f(z)|$ is max. Take a neighborhood $U \ni z$ and its chart $\phi_U : U \to \mathbb{C}$. Then $\phi_U(z)$ is in the interior of $\phi_U(U)$. It follows that $f \circ \phi_U^{-1}$ is holomorphic on $\phi_U(U) \subset\subset \mathbb{C}$. Thus, by the maximum modulus principle, $f \circ \phi_U^{-1}$ is constant on $\phi_U(U)$. Thus, f is constant on U. Now take overlapping coverings of U to expand the constancy over all of X, i.e., invoke the identity theorem. $\square$

- Note: This method of proof (doing something on a chart and then expanding over) is common.
- Corollary (Liouville's theorem): If $f \in \mathcal{O}(\mathbb{C})$ and bounded, then f is constant.

  Proof. If it's bounded on $\mathbb{C}$, then it's bounded in a neighborhood of ∞. Thus, we do the inversion to f^* which has an analytic continuation to 0, so f is holomorphic on $\hat{\mathbb{C}}$ and hence, by the previous theorem, f is constant. $\square$

- Implication: Holomorphic functions on the Riemann sphere are constant (hence relatively uninteresting).
- More interesting: Let's look at meromorphic functions on $\hat{\mathbb{C}}$.
  - Examples: $1/z$, z, z^2, polynomials, rational functions.
    - This is it!
  - Nonexample: The exponential map is *not* meromorphic on $\hat{\mathbb{C}}$ because it actually has an essential singularity at ∞: $R \to \infty$ goes to ∞, but $R \to -\infty$ goes to 0.
- Claim: Any meromorphic function on the Riemann sphere is a rational function.

Proof. Done next Tuesday. □

- There are lots of meromorphic functions that are called rational functions on other compact Riemann surfaces.
- Example: $zy^2 = x^3 + axz^2 + bz^3$ gives a set of points in $\mathbb{C}^2$ or $\mathbb{R}^2$.
  - Actually defines a genus 1 surface?? Could work for a final project!
  - The function $[x : y : z] \mapsto x/z$ is meromorphic. This map looks like skewering the torus and rotating it like a kebob.
  - What is going on here??
- Proposition: Let $f : X \to \hat{\mathbb{C}}$ be meromorphic and nonconstant, where X is a compact Riemann surface. Then f is onto.

  Proof. We want to show that $f(X) = \hat{\mathbb{C}}$. By the open mapping theorem, $f(X)$ is open. But X is compact, which means that $f(X)$ is compact, which means that $f(X)$ is closed. Thus, $\hat{\mathbb{C}} \setminus f(X)$ is open. But then $f(X)$ and $\hat{\mathbb{C}} \setminus f(X)$ partition $\hat{\mathbb{C}}$ into disjoint open subsets, so since $\hat{\mathbb{C}}$ is connected, one of these sets must be empty. $f(X)$ is nonempty, so $\hat{\mathbb{C}} \setminus f(X)$ must be empty and therefore $f(X) = \hat{\mathbb{C}}$, as desired. □

- Corollary: Fundamental Theorem of Algebra.

  Proof. Given a polynomial, there exists a root. How does this work?? □