# Week 3

# Fundamental Theorems

## 3.1 Cauchy Integral Formula

4/2:

- Last time.
  - Definition of star-shaped.
  - Cauchy integral theorem: $U$ star-shaped, $f \in \mathcal{O}(U)$ implies $\int_\gamma f \, \mathrm{d}z = 0$ for all closed (piecewise $C^1$) loops $\gamma$.
    1. It suffices to prove the theorem for triangles.
    2. Apply Goursat's lemma to treat this triangle case.
  - For Goursat's lemma, apply a clever estimate. Subdivide the big triangle into smaller ones, then

$$\left|\int_{\text{small }\triangle} f \, \mathrm{d}z\right| = \left|\int_a^b f(\gamma(t)) \cdot \gamma'(t) \, \mathrm{d}t\right| \leq \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)| \, \mathrm{d}t \leq \max_{z \in \partial\triangle} |f(z)| \cdot \operatorname{len}(\partial\triangle)$$

- We'll now do a couple exercises to practice applying the concepts we've learned so far.
- TPS: Suppose $f \in \mathcal{O}(\mathbb{C})$. Let $A := \int_0^1 f(x) \, \mathrm{d}x = F(1) - F(0)$, where to be clear we take the integral along the real axis. Let $\gamma$ be the piecewise $C^1$ path in yellow in Figure 3.1. What is $\int_\gamma f \, \mathrm{d}z$?

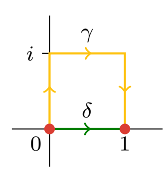



Figure 3.1: Practicing with the Cauchy Integral Theorem (1).

  - Define $\delta$ such that $\int_\delta f \, \mathrm{d}z = \int_0^1 f(x) \, \mathrm{d}x$.
  - Then $\delta^{-1}\gamma$ is a closed loop, so

$$0 = \int_{\delta^{-1}\gamma} f \, \mathrm{d}z$$

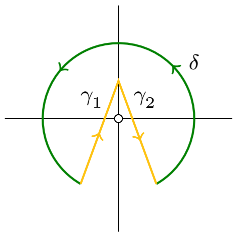
  - Additionally, we have by definition that

$$\int_{\delta^{-1}\gamma} f \, \mathrm{d}z = \int_\gamma f \, \mathrm{d}z - \int_\delta f \, \mathrm{d}z$$

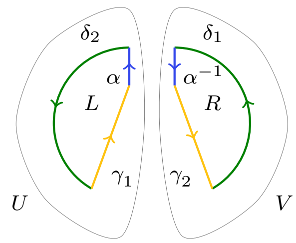
  - Thus, by transitivity and a bit of algebraic rearrangement,

$$\int_\gamma f \, \mathrm{d}z = \int_\delta f \, \mathrm{d}z = A$$

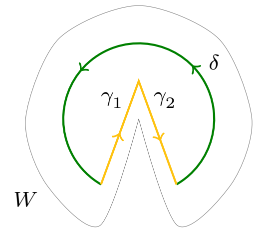
- TPS: Now suppose $f \in \mathcal{O}(\mathbb{C}^*)$, where we must note that $\mathbb{C}^*$ is *not* star-shaped due to the hole at the origin. Suppose we know that $\int_\delta f\,\mathrm{d}z = 0$. What is $\int_\gamma f\,\mathrm{d}z$? The paths $\gamma$ and $\delta$ are visualized in Figure 3.2a. *Hint*: It should be $-\int_\delta f\,\mathrm{d}z$.




(a) Original setup.




(b) Solution: Break in two.




(c) Solution: Pizza pie.

Figure 3.2: Practicing with the Cauchy Integral Theorem (2).

  - There are multiple ways to visualize why the domain does not see the hole/puncture. Here are some examples.
  - Solution 1 (Figure 3.2b): Cut the loop into two loops in star-shaped domains and add them.
    - Draw a straight-line path $\alpha$ from $i/2$ up to $i$.
    - Since $U$ and $V$ are both star-shaped domains, consecutive applications of the Cauchy Integral Theorem imply that

$$\int_{\delta_2\gamma_1\alpha} f\,\mathrm{d}z = \int_L f\,\mathrm{d}z = 0 \qquad\qquad \int_{\delta_1\alpha^{-1}\gamma_2} f\,\mathrm{d}z = \int_R f\,\mathrm{d}z = 0$$

    - Additionally, we know that the sum of the two integrals above is equal to the integral along the entire path in Figure 3.2a because the $\alpha$ and $\alpha^{-1}$ portions cancel. Mathematically,

$$\int_\delta f\,\mathrm{d}z + \int_\gamma f\,\mathrm{d}z = \int_{\delta\gamma} f\,\mathrm{d}z = \underbrace{\int_L f\,\mathrm{d}z}_{0} + \underbrace{\int_R f\,\mathrm{d}z}_{0} = 0$$

    - Therefore,

$$\int_\gamma f\,\mathrm{d}z = -\int_\delta f\,\mathrm{d}z = 0$$

  - Solution 2 (Figure 3.2c): The pizza pie is star-shaped!
    - We can actually draw a star-shaped domain $W$ encapsulating the entire path $\delta\gamma$.
    - Thus, by the Cauchy Integral Theorem,

$$\int_{\delta\gamma} f\,\mathrm{d}z = 0$$

    - From here, we may proceed as before through

$$\begin{aligned}\int_\gamma f\,\mathrm{d}z + \int_\delta f\,\mathrm{d}z &= 0\\ \int_\gamma f\,\mathrm{d}z &= -\int_\delta f\,\mathrm{d}z = 0\end{aligned}$$

- We now investigate a more general principal than the Cauchy integral theorem called **homotopy**.
  - Algebraic topologists would be insulted by the definition of this term that Calderon is about to give, but it will suffice for our purposes.
- **Homotopic** (paths): Two paths $\gamma, \tilde{\gamma} \subset U$ a domain such that $\tilde{\gamma}$ is obtained from $\gamma$ by modifying $\gamma$ on a small disk $D \subset U$, keeping the endpoints fixed.

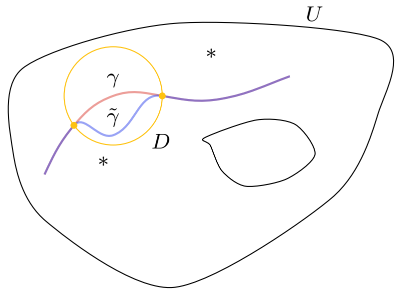
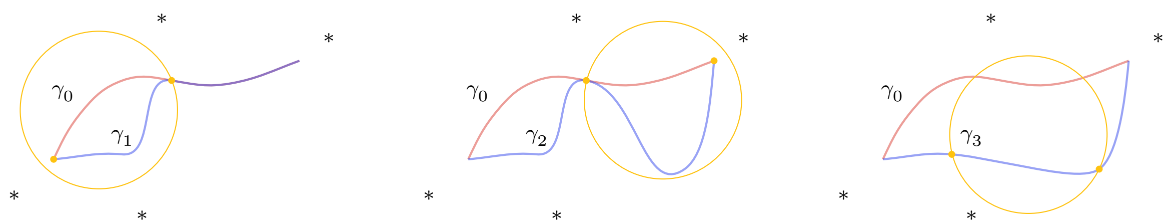



Figure 3.3: Homotopic paths.

- More generally, $\gamma$ and $\tilde{\gamma}$ are **homotopic** if there exists a finite sequence $\gamma = \gamma_0, \gamma_1, \dots, \gamma_n = \tilde{\gamma}$ such that $\gamma_i \to \gamma_{i+1}$ is obtained by modifying on a small ball.




(a) Stage 1. (b) Stage 2. (c) Stage 3.

Figure 3.4: A more general homotopy.

- Claim/TPS: This argument shows that if $\gamma$ and $\tilde{\gamma}$ are homotopic in $U$ and $f \in \mathcal{O}(U)$, then

$$\int_\gamma f \, \mathrm{d}z = \int_{\tilde{\gamma}} f \, \mathrm{d}z$$

  *Hint*: Just go one little bump at a time.

  *Proof.* The start- and endpoints of the bump form a closed loop within a ball (a star-shaped domain), so the bump loop integrates to zero by the CIT. Thus, the integrals within the ball are the same. Additionally, the paths are literally the same outside of the bump, so the integrals there are the same, too. Therefore, the overall integrals are the same, too. □

- Reality check: Let $f \in \mathcal{O}(\mathbb{C}^*)$. As a particular example, consider $f(z) = 1/z$. Now we know that

$$\int_\circ \frac{1}{z} \, \mathrm{d}z = 2\pi i \neq 0$$

  even though we can break the unit circle into the sum of two paths. What's going on?
  - The paths are not homotopic; we can't pull them through the hole in the plane.
  - If we consider the upper hemi-circle and the lower hemi-circle, the two cannot be continuously deformed into each other because we always get stuck at the puncture.

- We now prove a slightly stronger version of the Cauchy integral theorem.
- Corollary: Let $U$ be any domain, $D$ be a disk in $U$, and $z \in \mathring{D}$. Suppose $f \in \mathcal{O}(U \setminus \{z\})$ and is bounded near $z$. Then

$$\int_{\partial D} f \,\mathrm{d}z = 0$$

*Proof.* Step 1: Use homotopy.

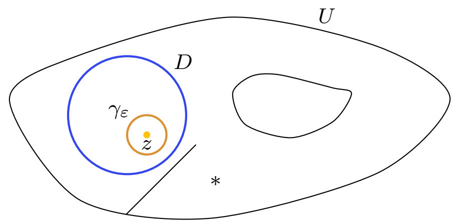



Figure 3.5: Bounded holomorphic functions integrate to zero on disk boundaries.

Via the above claim,

$$\int_{\partial D} f \,\mathrm{d}z = \int_{\gamma_\varepsilon} f \,\mathrm{d}z$$

where $\gamma_\varepsilon$ is a circle around $z$ within the region where $f$ is bounded[1].

Step 2: We have that

$$\left|\int_{\gamma_\varepsilon} f \,\mathrm{d}z\right| \leq \max_{z \in \gamma_\varepsilon} |f(z)| \cdot \operatorname{len}(\gamma_\varepsilon)$$

Since $f$ is bounded near $z$, the maximum is finite. Additionally, the length term is just $2\pi\varepsilon$, so we can send $\varepsilon \to 0$ and thus send the integral to zero. □

- We now look into the **Cauchy Integral Formula**.
- **Cauchy Integral Formula**: Suppose $U$ is any domain, $D \subset U$ is a disk (i.e., $D \subset\subset U$ or $\overline{D} \subset U$), $f \in \mathcal{O}(U)$, and $z \in D$. Then

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

*Proof.* We're going to try to use the corollary and define a function. In particular, define

$$g(\zeta) = \begin{cases} \frac{f(\zeta) - f(z)}{\zeta - z} & \zeta \neq z \\ f'(z) & \zeta = z \end{cases}$$

Because $f$ is holomorphic at $z$, $g$ is continuous at $z$ and hence bounded near $z$. We can also see that since $g$ is a rational function of holomorphic functions on $U \setminus \{z\}$, we have $g \in \mathcal{O}(U \setminus \{z\})$.

Now the corollary says that

$$\int_{\partial D} g \,\mathrm{d}\zeta = 0$$

Additionally, by the definition of $g$, we have that

$$\int_{\partial D} g \,\mathrm{d}\zeta = \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta - \int_{\partial D} \frac{f(z)}{\zeta - z} \,\mathrm{d}\zeta$$

[1] We could also turn the plane into the sum of two star-shaped domains again.

$f(z)$ is just a complex number, so we can pull it out of the rightmost integral above. Additionally, under a change of variables and invoking PSet 1, QA.4, we have that

$$\int_{\partial D} \frac{f(z)}{\zeta - z} \, \mathrm{d}\zeta = f(z) \int_{\partial D} \frac{1}{\zeta - z} \, \mathrm{d}\zeta = \int_{\text{unit circle}} \frac{1}{z - a} \, \mathrm{d}z = 2\pi i f(z)$$

Note: Another way to evaluate this integral is as follows. If $z$ is the center of the disk, then we win and can get $2\pi i$ using PSet 1, QA.4 directly. If $z$ isn't at the center of the disk, we are allowed to slide it. Here's why: Think about the integrand as a function of $z$, so

$$\frac{\partial}{\partial z}\left(\int_{\partial D} \frac{1}{\zeta - z} \, \mathrm{d}\zeta\right) = \int_{\partial D} \frac{\partial}{\partial z}\left(\frac{1}{\zeta - z}\right) \mathrm{d}\zeta = \int_{\partial D} \frac{1}{(\zeta - z)^2} \, \mathrm{d}\zeta = 0$$

Since we're taking the integral and the limit with respect to different things, we can exchange them. Since the second integrand has a primitive, it equals zero. But this means that the integral does not change even as $z$ changes, which is equivalent to saying we can move $z$ around to wherever we want in the disk and the integral will still be $2\pi i$! In other words, if $z$ is somewhere where we can't evaluate the integral directly, we can move $z$ to somewhere where we *can* evaluate the integral directly with no consequence. ☐

- Implication of Cauchy's Integral Theorem: The values of the function are completely determined by the values on the boundary, i.e., holomorphic functions are determined by boundary values.
- Let's now prove another theorem.
- Theorem: Let $U$ be any domain, $f \in \mathcal{O}(U)$. Then $f' \in \mathcal{O}(U)$, $f'' \in \mathcal{O}(U)$, on and on.

  *Proof.* Let's use the Cauchy integral formula. We have that

  $$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

  Now let's take the derivative, which we know exists because $f$ is holomorphic.

  $$\frac{\partial f}{\partial z} = \frac{1}{2\pi i} \int_{\partial D} \frac{\partial}{\partial z}\left(\frac{f(\zeta)}{\zeta - z}\right) \mathrm{d}\zeta = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^2} \mathrm{d}\zeta$$

  Thus, the derivative has a Cauchy integral formula. We can keep taking derivatives on the inside because the integrand is infinitely differentiable. Thus, we can keep taking derivatives on the outside. And that's the proof. ☐

- Corollary: Holomorphic functions are $C^\infty$.
- Corollary: In general,

  $$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \mathrm{d}\zeta$$

- This last result allows us to bound things really easily, giving us **Cauchy's inequalities**.
  - Essentially, let $D$ have radius $R$ and let $z$ be the center of $D$. Then

    $$|f^{(n)}(z)| \leq \frac{n!}{2\pi i} \max_{\partial D} \left|\frac{f(\zeta)}{R^{n+1}}\right| \cdot 2\pi R = \frac{n!}{R^n} \max_{\partial D} |f(\zeta)|$$

- Liouville's Theorem: Suppose $f \in \mathcal{O}(\mathbb{C})$ (i.e., $f$ is **entire**) and $f$ is bounded. Then it's constant.

  *Proof.* Take a point $z \in \mathbb{C}$. Take a huge ball with radius $R$. Cauchy's inequality says that if we take the derivative, then

  $$|f'(z)| \leq \frac{1}{R} \cdot \max_{\partial D} |f(\zeta)|$$

  The maximum is bounded and $R$ is really big, so as $R \to \infty$, the derivative gets arbitrarily small. So if we've got an arbitrary function with zero derivative, then we've got a constant function. ☐

- **Entire** (function): A complex-valued function that is holomorphic on the whole complex plane.

## 3.2 Analytic Continuation and Removable Singularities

4/4:
- Last time.
  - Cauchy integral formula: Let $U$ be any domain, $f \in \mathcal{O}(U)$, $D \subset\subset U$, and $z \in D$. Then
$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$
  - Implies that holomorphic functions are $C^\infty$.
  - Implies Cauchy's inequalities: If $D$ is a disk centered at $z_0$ of radius $R$, then
$$|f^{(n)}(z_0)| \leq \frac{n!}{R^n} \sup_{\partial D} |f(\zeta)|$$
  - Implies Liouville's theorem: Any bounded entire function is constant.
- Our focus today is on results we can get out of power series.
- Observe that if
$$P(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$
is a convergent power series centered at $z_0$, then
$$P^{(n)}(z_0) = n! a_n$$
- Now let $f \in \mathcal{O}(U)$.
  - TPS: What should the power series for $f$ look like?
    - Rearranging the above, we want
$$a_k = \frac{f^{(k)}(z_0)}{k!}$$
    - The following power series formally has the right derivatives.
$$P(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k$$
  - Does this power series converge though, and if so, where?
    - Recall that the Cauchy-Hadamard formula tells us that the radius of convergence $r$ satisfies
$$r = \left( \limsup_{k \to \infty} |a_k|^{1/k} \right)^{-1}$$
    - Pick a $z_0 \in U$ and a disk $D \subset\subset U$ of radius $R$. Then by the Cauchy inequalities,
$$|a_k|^{1/k} = \left| \frac{f^{(k)}(z_0)}{k!} \right|^{1/k} \leq \frac{|\sup_{\partial D} f(\zeta)|^{1/k}}{R} \to \frac{1}{R}$$
    - Thus, returning to the Cauchy-Hadamard formula, the radius of convergence is $\geq R$.
  - So we've got a convergent power series, but why does this power series equal $f(z)$?
    - We know that $P(z)$ and $f(z)$ have all the same derivatives.
    - However, over $\mathbb{R}$, this is not enough! Recall the example of $\mathrm{e}^{-1/x^2}$, which has the same derivatives as its power series at zero but is not equal to it.
    - Over $\mathbb{C}$, however, we claim that having the same derivatives *is* enough.

- Use CIF and expand $1/(\zeta - z)$.
- Note: To keep all of our $z$'s straight, recall that $z_0$ is a point, $\zeta$ lies on $\partial D$ where $D$ is centered at $z_0$, and $z$ is somewhere in $\mathring{D}$.
- Doing this, we obtain

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)} = \frac{1}{\zeta - z_0} \sum_{k=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^k$$

- Thus,

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \int_{\partial D} \sum_{k=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} (z - z_0)^n \mathrm{d}\zeta \\
&= \sum_{k=0}^{\infty} \underbrace{\frac{1}{2\pi i} \int \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \mathrm{d}\zeta}_{f^{(n)}(z_0)/n!} (z - z_0)^n \\
&= P(z)
\end{aligned}$$

- Recall that we can bring the sum outside of the integral because of uniform convergence and our lemma about integrable functions from the 3/26 class.

  - All in all, we've shown that any holomorphic function has a power series representation on any disk that fits within the domain, and the power series representation is the one we think it should be.

- We now discuss an important corollary to this result.
- The Identity Theorem: If two holomorphic functions $f, g \in \mathcal{O}(U)$ agree on an open set in $U$, then $f = g$.

  *Proof.* This is true for power series.

  For every point, there's a power series representation around that series so we can do something with a covering of open sets, though we do not need compactness for $U$. $\square$

  - An analogous result does not hold on the reals. For example, there are plenty of functions that are zero for a while, then bump up to 1 for a while, so they're 0 and 1 on open sets without being either 0 or 1.
  - Implication: "Holomorphic functions are very rigid."

- In fact, more is true: If $z_n \to z_0$ where each $z_n$ is distinct and $f(z_n) = g(z_n)$ for all $n$, then $f = g$.
  - So we don't even need an open set; all we need is an **accumulation point**.
- **Analytic continuation** (of $f$): The function $g \in \mathcal{O}(V)$ where $f \in \mathcal{O}(U)$, $V \supset U$, and $f = g$ on $U$.
  - Note that we get to say "*the* function $g\dots$" because of the identity theorem.
  - Formally, $g_1, g_2$ analytic continuations of $f$ and $g_1 = g_2$ on $U$ open implies $g_1 = g_2$.
- Example: Consider $f(z) = z$ with $f \in \mathcal{O}(\mathbb{C}^*)$. Then $g(z) = z$ with $g \in \mathcal{O}(\mathbb{C})$ is an analytic continuation of $f$.
- What we're essentially doing is taking the power series (which we get via "analytic") and extending them out into $V$.

- Example: The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty} \mathrm{e}^{-s\log n}$$

  Where does this make sense, i.e., where does the series converge?
  - We have $|n^{-s}| = n^{-\operatorname{Re}(s)}$.
  - Thus, the series converges only when $\operatorname{Re}(s) > 1$.
  - The Riemann hypothesis predicts where $\zeta(s) = 0$. We know that it has some zeroes at the negative even integers, and the RH predicts that the rest of them fall on the line $\operatorname{Re}(s) = 1/2$.
  - But $\zeta$ is only defined on a part of the complex plane not including these regions! Thus, to make sense of the RH, we need to analytically continue $\zeta$.
- Given $f \in \mathcal{O}(U)$, what is the "biggest" $V$ on which $f \in \mathcal{O}(V)$? In layman's terms, where should $f$ live?
- Example: $1/z \in \mathcal{O}(\mathbb{C}^*)$ and $1/z \notin \mathcal{O}(\mathbb{C})$.
  - Note that we know the latter statement because if you're holomorphic, the integral around any closed loop in the domain is zero but the integral of this function on the unit circle is $2\pi i$, so it can't be holomorphic on $\mathbb{C}$. Contradiction.
- Example: $\sin(1/z) \in \mathcal{O}(\mathbb{C}^*)$, but is it in $\mathcal{O}(\mathbb{C})$?
  - No; recall from PSet 1 that it's not even *continuous* at 0.
- Example: $\sin(z)/z \in \mathcal{O}(\mathbb{C}^*)$, but is it in $\mathcal{O}(\mathbb{C})$?
- Recall Goursat's lemma: $f \in \mathcal{O}(\mathrm{nbhd}(\triangle))$ implies $\int_\triangle f \,\mathrm{d}z = 0$.
  - If $U$ is star-shaped and $\int_\triangle f \,\mathrm{d}z = 0$ for all triangles, then $f$ has a primitive.
    - Note that we do not need $f$ holomorphic for this result!
  - This latter result has a converse!
- Morera's Theorem: If $U$ is any domain, $f : U \to \mathbb{C}$ is continuous, and $\int_\triangle f \,\mathrm{d}z = 0$ for all triangles, then $f$ is holomorphic.

  *Proof.* Fix a disk $D \subset\subset U$. Disks are star-shaped! This combined with the fact that the integral over all triangles is zero implies that $f$ has a primitive $F \in \mathcal{O}(U)$ by the result a couple lines up. But since $F$ is holomorphic, by our result from last class, $F' = f \in \mathcal{O}(U)$, too. $\square$

- **Riemann's removable singularity theorem**: Suppose $U$ is a domain, $z \in U$, $f \in \mathcal{O}(U \setminus \{z\})$, and $f$ is bounded near $z$. Then there exists a unique analytic continuation $\hat{f} \in \mathcal{O}(U)$. *Also known as* **Riemann extension theorem**.
  - In this case, we call $z$ a **removable singularity**.
  - Note: The contrapositive of this says that if there is not an analytic continuation (i.e., the function is honestly not holomorphic at a point and can't be extended to one, e.g., $1/z$), then $|f|$ has to blow up as you approach $z$ (in some direction).
- **Singularity** (of $f$): A point $z_0$ such that $f \in \mathcal{O}(U \setminus \{z_0\})$.
- **Removable** (singularity): A singularity of a function that that satisfies the hypotheses of Riemann's removable singularity theorem.
- If a singularity is not removable, then $f$ is not bounded near $z_0$. This leads to additional definitions.

- **Pole**: A non-removable singularity $z_0$ of a function $f$ for which $|f(z)| \to \infty$ as $z \to z_0$.
  - So-named because of real analysis where a pole is an asymptote, and asymptotes kind of look like poles!
- **Essential** (singularity): A non-removable singularity that is not a pole; equivalently, a singularity $z_0$ for which there exist sequences $z_n \to z_0$ and $w_n \to z_0$ such that $|f(z_n)| \to \infty$ and $|f(w_n)|$ stays bounded.
- Proving Riemann's removable singularity theorem.

  *Proof.* Set

  $$F(\zeta) = \begin{cases} f(\zeta)(\zeta - z) & \zeta \neq z \\ 0 & \zeta = z \end{cases}$$

  Then $F \in \mathcal{O}(U \setminus \{z\})$ so $F$ is continuous at $z$.

  We want to show that $F$ is holomorphic (using Morera's theorem). To do this, we'll need to show that the integral over all triangles is zero. More specifically, all we need to do is show that $F$ is holomorphic in a little ball $D$ about $z$. Now we need to do some casework.

  Case 1: If $\triangle \not\ni z$, then we can draw a star-shaped domain surrounding the triangle on which $f$ will be holomorphic and invoke the CIT to imply that the integral is zero.

  Case 2: If $\triangle \ni z$, then $\int F \, \mathrm{d}z$ is arbitrarily small. Recall that we get this by using homotopy to replace the integral over the triangle with the integral over some tiny $\gamma_\varepsilon$. Arbitrarily small because $f$ is bounded.

  Morera then tells us that $F \in \mathcal{O}(U)$, so $F' = f \in \mathcal{O}(U)$. Note that $F' = f$ because

  $$F'(z) = \lim_{\zeta \to z} \frac{F(\zeta) - F(z)}{\zeta - z} = \lim_{\zeta \to z} \frac{f(\zeta)(\zeta - z) - 0}{\zeta - z} = \lim_{\zeta \to z} f(\zeta) = f(z)$$

  $\square$

- Go back and add a $z_0$ everywhere and then it should all be ok.
- With the removable singularity theorem, we can now confirm that $\sin(z)/z$ has a removable singularity because although $1/z$ diverges, sine converges faster so this function is bounded near zero.
  - We can prove boundedness with the Taylor series of $\sin(z)/z$.