3 Cauchy's Integral Formula

Set A: Graded for Completion

4/12: 1. Fischer and Lieb (2012), QII.3.1. Using the Cauchy integral formulas, compute the following integrals.

   (a) $\displaystyle\int_{|z+1|=1} \frac{\mathrm{d}z}{(z+1)(z-1)^3}$.

   (b) $\displaystyle\int_{|z-i|=3} \frac{\mathrm{d}z}{z^2+\pi^2}$.

   (c) $\displaystyle\int_{|z|=1/2} \frac{\mathrm{e}^{1-z}}{z^3(1-z)}\,\mathrm{d}z$.

   (d) $\displaystyle\int_{|z-1|=1} \left(\frac{z}{z-1}\right)^n \mathrm{d}z$ for any $n \geq 1$.

2. Fischer and Lieb (2012), QII.4.2. Assume that the power series $f(z) = \sum_{k=0}^{\infty} a_k z^k$ converges on $D = D_r(0)$.

   (a) Show that if f is real-valued on $\mathbb{R} \cap D$, then all a_k are real.

   (b) Show that if f is an even (resp. odd) function, then $a_k = 0$ for all odd (resp. even) k.

   (c) Show that if $f(iz) = f(z)$, then a_k can only be nonzero if k is divisible by 4.

   (d) Discuss the equation $f(\rho z) = \mu f(z)$, where $\rho, \mu \in \mathbb{C} \setminus \{0\}$ are given.

3. Fischer and Lieb (2012), QII.6.1. Determine the type of singularity that each of the following functions has at z_0. If the singularity is removable, calculate the limit as $z \to z_0$; if the singularity is a pole, find its order and the principal part of f at z_0.

   (a) $(1-\mathrm{e}^z)^{-1}$ at $z_0 = 0$.

   (b) $(z - \sin z)^{-1}$ at $z_0 = 0$.

   (c) $z\mathrm{e}^{iz}/(z^2+b^2)^2$ at $z_0 = ib$ $(b > 0)$.

   (d) $(\sin z + \cos z - 1)^{-2}$ at $z_0 = 0$.

4. Let $\mathbb{D}$ denote the unit disk and suppose that $f \in \mathcal{O}(\mathbb{D})$.

   (a) Prove that for any $R \in (0,1)$ and any z with $|z| < R$, we have that

   $$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(R\mathrm{e}^{i\theta})\,\mathrm{Re}\left(\frac{R\mathrm{e}^{i\theta}+z}{R\mathrm{e}^{i\theta}-z}\right)\mathrm{d}\theta$$

   Hint: Observe that setting $w = R^2/\bar{z}$, we have that the integral of $f(\zeta)/(\zeta - w)$ over the circle of radius R centered at the origin is 0.

   (b) Compute that

   $$\mathrm{Re}\left(\frac{R\mathrm{e}^{i\theta}+r}{R\mathrm{e}^{i\theta}-r}\right) = \frac{R^2-r^2}{R^2-2Rr\cos\theta+r^2}$$

   (c) Now suppose that $u = \mathrm{Re}(f)$, so u is a harmonic function. Deduce the **Poisson integral representation formula**: For $z = r\mathrm{e}^{i\theta}$, we have

   $$u(z) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-\phi)u(\phi)\mathrm{d}\phi$$

   where $P_r(\psi)$ is the **Poisson kernel** for the disk, given by

   $$P_r(\psi) = \frac{1-r^2}{1-2r\cos\psi+r^2}$$

Set B: Graded for Content

1. Fischer and Lieb (2012), QII.4.6.

(a) Suppose the domain G is symmetric with respect to the real axis and that f is holomorphic on G and real-valued on $G \cap \mathbb{R}$. Show that $f(\bar{z}) = \overline{f(z)}$ for all $z \in G$.

(b) Suppose $G = D_r(0)$ and f is holomorphic on G and real-valued on $G \cap \mathbb{R}$. Show that if f is even (resp. odd), then the values of f on $G \cap i\mathbb{R}$ are real (resp. imaginary). Prove this without using the power series expansion of f.

2. Some setup: Suppose that f is holomorphic on the unit disk $\mathbb{D} = \{|z| < 1\}$. A point w on the circle $\partial D = \{|z| = 1\}$ is **regular** if there is an open neighborhood U of w and an analytic function g on U such that $f = g$ on $U \cap \mathbb{D}$. Notice that f can be analytically continued outside the boundary of $\mathbb{D}$ if and only if there is a point w on ∂D that is regular for f.

Now define the function

$$f(z) = \sum_{k=1}^{\infty} z^{2^k}$$

Show that f converges on $\mathbb{D}$, and that it cannot be analytically continued past $\mathbb{D}$.

3. Suppose that f is an entire function and that, for all sufficiently large z, we have $|f(z)| \leq |z|^n$. Prove that f must be a polynomial.