Week 3

???

3.1 Cauchy Integral Formula

4/2:

- Last time.
  - Definition of star-shaped.
  - Cauchy integral theorem: U star-shaped, $f \in \mathcal{O}(U)$ implies $\int_\gamma f \, \mathrm{d}z = 0$ for all closed (piecewise C^1) loops γ.
    1. It suffices to prove the theorem for triangles.
    2. Apply Goursat's lemma to treat this triangle case.
  - For Goursat's lemma, apply a clever estimate. Subdivide the big triangle into smaller ones, then

$$\left|\int_{\text{small }\triangle} f \, \mathrm{d}z\right| = \left|\int_a^b f(\gamma(t)) \cdot \gamma'(t) \, \mathrm{d}t\right| \leq \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)| \, \mathrm{d}t \leq \max_{z \in \partial\triangle} |f(z)| \cdot \operatorname{len}(\partial\triangle)$$

- We'll now do a couple exercises to practice applying the concepts we've learned so far.
- TPS: Suppose $f \in \mathcal{O}(\mathbb{C})$. Let $A := \int_0^1 f(x) \, \mathrm{d}x = F(1) - F(0)$, where to be clear we take the integral along the real axis. Let γ be the piecewise C^1 path in yellow in Figure 3.1. What is $\int_\gamma f \, \mathrm{d}z$?

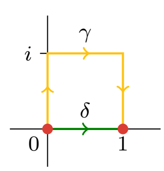


Figure 3.1: Practicing with the Cauchy Integral Theorem (1).

  - Define δ such that $\int_\delta f \, \mathrm{d}z = \int_0^1 f(x) \, \mathrm{d}x$.
  - Then $\delta^{-1}\gamma$ is a closed loop, so

$$0 = \int_{\delta^{-1}\gamma} f \, \mathrm{d}z$$

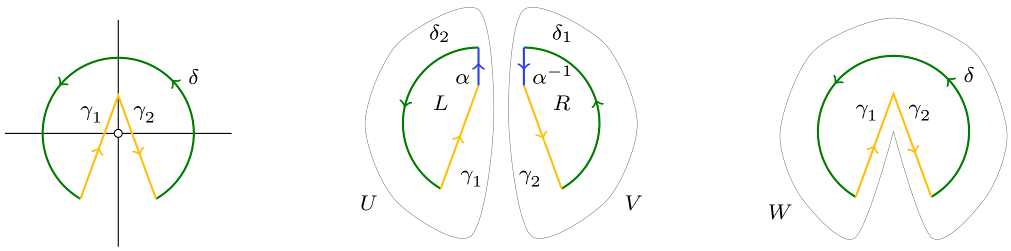
  - Additionally, we have by definition that

$$\int_{\delta^{-1}\gamma} f \, \mathrm{d}z = \int_\gamma f \, \mathrm{d}z - \int_\delta f \, \mathrm{d}z$$

  - Thus, by transitivity and a bit of algebraic rearrangement,

$$\int_\gamma f \, \mathrm{d}z = \int_\delta f \, \mathrm{d}z = A$$

- TPS: Now suppose $f \in \mathcal{O}(\mathbb{C}^*)$, where we must note that $\mathbb{C}^*$ is *not* star-shaped due to the hole at the origin. Suppose we know that $\int_\delta f\,\mathrm{d}z = 0$. What is $\int_\gamma f\,\mathrm{d}z$? The paths γ and δ are visualized in Figure 3.2a. *Hint*: It should be $-\int_\delta f\,\mathrm{d}z$.



(a) Original setup. (b) Solution: Break in two. (c) Solution: Pizza pie.

Figure 3.2: Practicing with the Cauchy Integral Theorem (2).

  - There are multiple ways to visualize why the domain does not see the hole/puncture. Here are some examples.
  - Solution 1 (Figure 3.2b): Cut the loop into two loops in star-shaped domains and add them.
    - Draw a straight-line path α from $i/2$ up to i.
    - Since U and V are both star-shaped domains, consecutive applications of the Cauchy Integral Theorem imply that

$$\int_{\delta_2\gamma_1\alpha} f\,\mathrm{d}z = \int_L f\,\mathrm{d}z = 0 \qquad\qquad \int_{\delta_1\alpha^{-1}\gamma_2} f\,\mathrm{d}z = \int_R f\,\mathrm{d}z = 0$$

    - Additionally, we know that the sum of the two integrals above is equal to the integral along the entire path in Figure 3.2a because the α and α^{-1} portions cancel. Mathematically,

$$\int_\delta f\,\mathrm{d}z + \int_\gamma f\,\mathrm{d}z = \int_{\delta\gamma} f\,\mathrm{d}z = \underbrace{\int_L f\,\mathrm{d}z}_{0} + \underbrace{\int_R f\,\mathrm{d}z}_{0} = 0$$

    - Therefore,

$$\int_\gamma f\,\mathrm{d}z = -\int_\delta f\,\mathrm{d}z = 0$$

  - Solution 2 (Figure 3.2c): The pizza pie is star-shaped!
    - We can actually draw a star-shaped domain W encapsulating the entire path $\delta\gamma$.
    - Thus, by the Cauchy Integral Theorem,

$$\int_{\delta\gamma} f\,\mathrm{d}z = 0$$

    - From here, we may proceed as before through

$$\begin{aligned} \int_\gamma f\,\mathrm{d}z + \int_\delta f\,\mathrm{d}z &= 0 \\ \int_\gamma f\,\mathrm{d}z &= -\int_\delta f\,\mathrm{d}z = 0 \end{aligned}$$

- We now investigate a more general principal than the Cauchy integral theorem called **homotopy**.
  - Algebraic topologists would be insulted by the definition of this term that Calderon is about to give, but it will suffice for our purposes.
- **Homotopic** (paths): Two paths $\gamma, \tilde{\gamma} \subset U$ a domain such that $\tilde{\gamma}$ is obtained from γ by modifying γ on a small disk $D \subset U$, keeping the endpoints fixed.

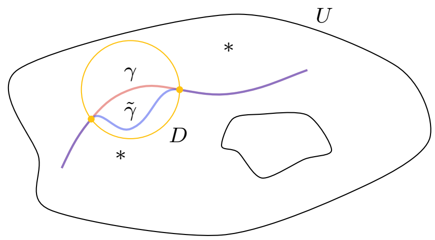
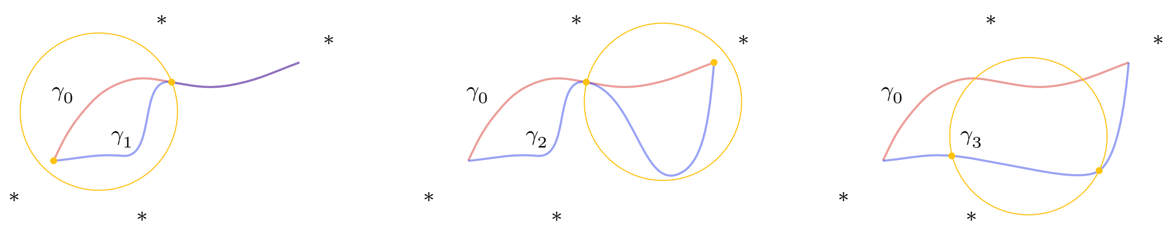


Figure 3.3: Homotopic paths.

- More generally, γ and $\tilde{\gamma}$ are **homotopic** if there exists a finite sequence $\gamma = \gamma_0, \gamma_1, \dots, \gamma_n = \tilde{\gamma}$ such that $\gamma_i \to \gamma_{i+1}$ is obtained by modifying on a small ball.



(a) Stage 1. (b) Stage 2. (c) Stage 3.

Figure 3.4: A more general homotopy.

- Claim/TPS: This argument shows that if γ and $\tilde{\gamma}$ are homotopic in U and $f \in \mathcal{O}(U)$, then

$$\int_\gamma f \, \mathrm{d}z = \int_{\tilde{\gamma}} f \, \mathrm{d}z$$

  Hint: Just go one little bump at a time.

  Proof. The start- and endpoints of the bump form a closed loop within a ball (a star-shaped domain), so the bump loop integrates to zero by the CIT. Thus, the integrals within the ball are the same. Additionally, the paths are literally the same outside of the bump, so the integrals there are the same, too. Therefore, the overall integrals are the same, too. ☐

- Reality check: Let $f \in \mathcal{O}(\mathbb{C}^*)$. As a particular example, consider $f(z) = 1/z$. Now we know that

$$\int_\circ \frac{1}{z} \, \mathrm{d}z = 2\pi i \neq 0$$

  even though we can break the unit circle into the sum of two paths. What's going on?
  - The paths are not homotopic; we can't pull them through the hole in the plane.
  - If we consider the upper hemi-circle and the lower hemi-circle, the two cannot be continuously deformed into each other because we always get stuck at the puncture.

- We now prove a slightly stronger version of the Cauchy integral theorem.
- Corollary: Let U be any domain, D be a disk in U, and $z \in \mathring{D}$. Suppose $f \in \mathcal{O}(U \setminus \{z\})$ and is bounded near z. Then

$$\int_{\partial D} f \, \mathrm{d}z = 0$$

Proof. Step 1: Use homotopy.

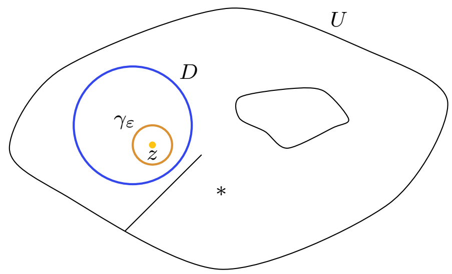


Figure 3.5: Bounded holomorphic functions integrate to zero on disk boundaries.

Via the above claim,

$$\int_{\partial D} f \, \mathrm{d}z = \int_{\gamma_\varepsilon} f \, \mathrm{d}z$$

where γ_ε is a circle around z within the region where f is bounded[1].

Step 2: We have that

$$\left| \int_{\gamma_\varepsilon} f \, \mathrm{d}z \right| \leq \max_{z \in \gamma_\varepsilon} |f(z)| \cdot \operatorname{len}(\gamma_\varepsilon)$$

Since f is bounded near z, the maximum is finite. Additionally, the length term is just $2\pi\varepsilon$, so we can send $\varepsilon \to 0$ and thus send the integral to zero. □

- We now look into the **Cauchy Integral Formula**.
- **Cauchy Integral Formula**: Suppose U is any domain, $D \subset U$ is a disk (i.e., $D \subset\subset U$ or $\overline{D} \subset U$), $f \in \mathcal{O}(U)$, and $z \in D$. Then

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

Proof. We're going to try to use the corollary and define a function. In particular, define

$$g(\zeta) = \begin{cases} \frac{f(\zeta) - f(z)}{\zeta - z} & \zeta \neq z \\ f'(z) & \zeta = z \end{cases}$$

Because f is holomorphic at z, g is continuous at z and hence bounded near z. We can also see that since g is a rational function of holomorphic functions on $U \setminus \{z\}$, we have $g \in \mathcal{O}(U \setminus \{z\})$.

Now the corollary says that

$$\int_{\partial D} g \, \mathrm{d}\zeta = 0$$

Additionally, by the definition of g, we have that

$$\int_{\partial D} g \, \mathrm{d}\zeta = \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta - \int_{\partial D} \frac{f(z)}{\zeta - z} \, \mathrm{d}\zeta$$

[1]We could also turn the plane into the sum of two star-shaped domains again.

$f(z)$ is just a complex number, so we can pull it out of the rightmost integral above. Additionally, under a change of variables and invoking PSet 1, QA.4, we have that

$$\int_{\partial D} \frac{f(z)}{\zeta - z}\,\mathrm{d}\zeta = f(z)\int_{\partial D} \frac{1}{\zeta - z}\,\mathrm{d}\zeta = \int_{\text{unit circle}} \frac{1}{z-a}\,\mathrm{d}z = 2\pi i f(z)$$

Note: Another way to evaluate this integral is as follows. If z is the center of the disk, then we win and can get $2\pi i$ using PSet 1, QA.4 directly. If z isn't at the center of the disk, we are allowed to slide it. Here's why: Think about the integrand as a function of z, so

$$\frac{\partial}{\partial z}\left(\int_{\partial D} \frac{1}{\zeta - z}\,\mathrm{d}\zeta\right) = \int_{\partial D} \frac{\partial}{\partial z}\left(\frac{1}{\zeta - z}\right)\mathrm{d}\zeta = \int_{\partial D} \frac{1}{(\zeta - z)^2}\,\mathrm{d}\zeta = 0$$

Since we're taking the integral and the limit with respect to different things, we can exchange them. Since the second integrand has a primitive, it equals zero. But this means that the integral does not change even as z changes, which is equivalent to saying we can move z around to wherever we want in the disk and the integral will still be $2\pi i$! In other words, if z is somewhere where we can't evaluate the integral directly, we can move z to somewhere where we *can* evaluate the integral directly with no consequence. ☐

- Implication of Cauchy's Integral Theorem: The values of the function are completely determined by the values on the boundary, i.e., holomorphic functions are determined by boundary values.
- Let's now prove another theorem.
- Theorem: Let U be any domain, $f \in \mathcal{O}(U)$. Then $f' \in \mathcal{O}(U)$, $f'' \in \mathcal{O}(U)$, on and on.

  Proof. Let's use the Cauchy integral formula. We have that

  $$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta$$

  Now let's take the derivative, which we know exists because f is holomorphic.

  $$\frac{\partial f}{\partial z} = \frac{1}{2\pi i}\int_{\partial D} \frac{\partial}{\partial z}\left(\frac{f(\zeta)}{\zeta - z}\right)\mathrm{d}\zeta = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^2}\mathrm{d}\zeta$$

  Thus, the derivative has a Cauchy integral formula. We can keep taking derivatives on the inside because the integrand is infinitely differentiable. Thus, we can keep taking derivatives on the outside. And that's the proof. ☐

- Corollary: Holomorphic functions are C^∞.
- Corollary: In general,

  $$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\mathrm{d}\zeta$$

- This last result allows us to bound things really easily, giving us **Cauchy's inequalities**.
  - Essentially, let D have radius R and let z be the center of D. Then

    $$|f^{(n)}(z)| \leq \frac{n!}{2\pi i}\max_{\partial D}\left|\frac{f(\zeta)}{R^{n+1}}\right| \cdot 2\pi R = \frac{n!}{R^n}\max_{\partial D}|f(\zeta)|$$

- Liouville's Theorem: Suppose $f \in \mathcal{O}(\mathbb{C})$ (i.e., f is **entire**) and f is bounded. Then it's constant.

  Proof. Take a point $z \in \mathbb{C}$. Take a huge ball with radius R. Cauchy's inequality says that if we take the derivative, then

  $$|f'(z)| \leq \frac{1}{R} \cdot \max_{\partial D}|f(\zeta)|$$

  The maximum is bounded and R is really big, so as $R \to \infty$, the derivative gets arbitrarily small. So if we've got an arbitrary function with zero derivative, then we've got a constant function. ☐