

MATH 27000 (Basic Complex Variables) Problem Sets

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1 Holomorphicity

Set A: Graded for Completion

- 3/29: 1. Fischer and Lieb (2012), QI.2.1. Formulate and prove the chain rule for Wirtinger derivatives. Furthermore, show that

$$\overline{\frac{\partial f}{\partial z}} = \frac{\partial \bar{f}}{\partial \bar{z}}$$

2. Let $\langle (x_1, y_1), (x_2, y_2) \rangle = x_1 x_2 + y_1 y_2$ denote the usual inner product on \mathbb{R}^2 . We can also define the **Hermitian inner product** on \mathbb{C} via

$$(z, w) = z \bar{w}$$

This term *Hermitian* describes the fact that this product is not symmetric, but satisfies $(w, z) = \overline{(z, w)}$. Show that thinking of z as $x + iy$, we have

$$\langle z, w \rangle = \frac{1}{2}[(z, w) + (w, z)] = \operatorname{Re}(z, w)$$

3. For any integer n , compute the line integral $\int_{\gamma} z^n dz$ where γ is any circle centered at the origin with counterclockwise orientation. Do not use Cauchy's theorem.
4. Without using Cauchy's theorem, show that for any $|a| < 1 < |b|$,

$$\int_{\gamma} \frac{1}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b}$$

where γ is the circle of radius 1 centered about the origin, oriented counterclockwise.

5. Determine the image of the following sets under the following conformal mappings. Use level curves to illustrate the geometry of these mappings.
- (a) The unit disk $\mathbb{D} = \{z : |z| < 1\}$ under $z \mapsto 1/z$.
 - (b) $\mathbb{D} \setminus \{0\}$ under $z \mapsto z^2$.
 - (c) The strip $S = \{z : \operatorname{Im}(z) \in (0, 2\pi)\}$ under $z \mapsto e^z$.
 - (d) The upper half-plane $\mathbb{H} = \{z : \operatorname{Im}(z) > 0\}$ under $z \mapsto z^2$.
 - (e) The half disk $\mathbb{D} \cap \mathbb{H}$ under $z \mapsto (1+z)/(1-z)$.

Set B: Graded for Content

1. A prototypical example of a weird function that is differentiable (but not C^1) on all of \mathbb{R} is

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Extend f to a function on \mathbb{C} using the same formula (replacing x 's with z 's). Is it holomorphic at the origin?

2. Show that if f is holomorphic on a domain $U \subset \mathbb{C}$ and takes only real values, then it is constant.
3. Find a conformal map that takes the upper half-plane onto the “Pac-Man” given by

$$\{z : |z| < 1 \text{ and } \arg(z) \in (\pi/4, 7\pi/4)\}$$

Explain how you obtained this map. *Hint:* Do completion problem 5 first.

References

Fischer, W., & Lieb, I. (2012). *A course in complex analysis: From basic results to advanced topics* (J. Cannizzo, Trans.). Vieweg+Teubner Verlag.