

Week 4

???

4.1 Poles and Maximum Moduli

4/9:

- Announcement.
 - Midterm next week in class.
 - Material up through today, though probably not much on today's content.

- Last time.

- Cauchy integral formula: If U is a domain, $D \subset\subset U$, and $z \in D$, then

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta - z} d\zeta$$

- This implies Riemann's removable singularity theorem, which states that if $f \in \mathcal{O}(U \setminus \{z_0\})$ and f is bounded near z_0 , then there exists a $\hat{f} \in \mathcal{O}(U)$ which continues f at z_0 .

- Example: $\sin(z)/z \in \mathcal{O}(\mathbb{C}^*)$ has a continuation to \mathbb{C} .
- In particular, take the Taylor series at zero and evaluate:

$$\frac{\widehat{\sin(z)}}{z}(0) = 1 - \frac{0^3}{3!} + \frac{0^5}{5!} - \dots = 1$$

- Alternatively, if $f \in \mathcal{O}(U \setminus \{z_0\})$ and $|f(z)| \rightarrow \infty$ as $z \rightarrow z_0$, then z_0 is a **pole** of f .

- Today.

- Finish up what we couldn't last time.
- Say something about harmonic functions.

- **Meromorphic** (function): A function $f : U \rightarrow \mathbb{C}$ such that $f \in \mathcal{O}(U \setminus P)$ and each $p \in P$ is a pole, where $P \subset U$ is a finite set of points.

- Example: Consider $1/z \in \mathcal{O}(\mathbb{C}^*)$.

- This has a pole at zero.
- Thus, $1/z$ is *holomorphic* on the punctured plane \mathbb{C}^* , but *meromorphic* on the whole complex plane \mathbb{C} .

- Example: The same argument applies to $1/z^k$ ($k \in \mathbb{N}$).

- Example: The function from PSet 2, Q2c:

$$f(z) = \frac{1}{z(z-1)(z-i)(z-1-i)}$$

- It follows that

$$\{f : f \text{ is holomorphic}\} \subset \{f : f \text{ is meromorphic}\}$$

- Fact: All of the examples kind of look the same.
 - More generally, suppose f has a pole at p and is holomorphic on $U \setminus \{p\}$. Pick a disk $D \ni p$ such that $f \neq 0$ on D . Then $g = 1/f \in \mathcal{O}(D \setminus \{p\})$ and as $z \rightarrow p$, $g(z) \rightarrow 0$.
 - Thus, we've got a function that's holomorphic and bounded near a point, so by Riemann's removable singularity theorem, it has a unique holomorphic extension $\hat{g} \in \mathcal{O}(D)$.
 - In particular, $g(p) = 0$.
 - Note: We do *not* need to choose D small enough such that it contains only one point in P . However, we will for the time being just to simplify things. The reason we can do this is because singularities — as points of a finite set — are isolated.
- There exists a power series for g about p such that

$$g(z) = \sum_{k=0}^{\infty} a_k (z-p)^k$$

- We know that $a_0 = 0$ because $g(p) = 0$.
- It can also happen such that some (or [potentially infinitely] many) of the remaining a_i are zero.
 - Example: if $f = 1/z^3$, then $g = z^3$ and $a_i = 0$ ($i > 3$).
- Now let L be the largest natural number such that $a_i = 0$ for all $0 \leq i < L$.
 - Because $a_0 = 0$, $L \geq 1$.
 - Additionally, $a_L \neq 0$.
- Then we can rewrite the power series as

$$g(z) = (z-p)^L h(z)$$

where...

1. $h(z) = \sum_{k=L}^{\infty} a_k (z-p)^{k-L}$;
 2. $h(p) \neq 0$ (and h is nonzero near p).
- We say that g has a **zero** (of order L at p).
 - Similarly, we say that f has a **pole** (of order L at p).
 - Thus,

$$f(z) = \frac{1}{(z-p)^L} \frac{1}{h(z)}$$

where, moreover, $1/h \in \mathcal{O}(D')$ for some smaller disk D' .

- Example: $1/(z^2 + z)$ goes to $z(z+1)$.
- Takeaway: Near any pole p , f must look like

$$\frac{1}{(z-p)^L} \cdot \phi(z)$$

where ϕ is holomorphic around p .

- This implies that there exists a **Laurent series** expansion around any pole.
- In particular, near p ,

$$f(z) = \sum_{k=-L}^{\infty} a_k (z-p)^k$$

- **Zero** (of order L at p): A point p of a holomorphic complex function g such that $g(p) = 0$ and $g(z) = (z - p)^L h(z)$ where $h(p) \neq 0$.
- **Pole** (of order L at p): A point p of a holomorphic complex function f such that $1/f(p) = 0$ and $f(z) = 1/(z - p)^L h(z)$ where $h(p) \neq 0$.
- **Laurent series**: A power series including a finite number of negative coefficients. *Given by*

$$\sum_{k=-L}^{\infty} a_k (z - p)^k$$

- TPS: Consider $\cot(z) = \cos(z)/\sin(z)$, which has a pole at zero. What is the order of the pole? What is the Laurent series?

– The pole is order 1.

- One way to see this is to observe how $\tan z$ has a nonzero tangent at 0, so $\tan z = z + \dots$. Thus, we can only divide one z out of its power series.
- Alternatively, we have

$$\cot(z) = \frac{1}{z} \cdot \frac{z}{\sin(z)} \cdot \cos(z)$$

from which we can observe that $\cos(z) \in \mathcal{O}(\mathbb{C})$, and $\sin(z)/z \in \mathcal{O}(\mathbb{C})$ (at zero, the extension gives 1) so $z/\sin(z)$ is holomorphic near zero. Thus, we can define $\phi(z) = z \cos(z)/\sin(z)$.

➤ What if we tried $\tilde{\phi}(z) = z^2 \cos(z)/\sin(z)$? What's different? Well, $\tilde{\phi}$ is still holomorphic, but $\tilde{\phi}(0) = 0$, which is a problem. Notice that $\phi(0) = 1$!

- As a last way, we could investigate the power series of $\cot(z)^{-1} = \tan(z)$ directly:

$$\tan z = z + \frac{z^3}{3} + \frac{2z^5}{15}$$

– The Laurent series was not discussed in class, but here's some comments.

- It would begin from $k = -1$.
- We could construct it from the power series for cosine and sine using Calderon's formula above.
- Figuring out the formula for the power series of an inverted power series is a good exercise!!
- What if $|f(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$? Then we say that f has a **pole** (at ∞).
 - Otherwise, there exist sequences $z_n \rightarrow \infty$ and $w_n \rightarrow \infty$ such that $f(z_n) \rightarrow \infty$ and $f(w_n)$ stays bounded. This is an **essential singularity** (at ∞).
 - We can mull over this until Thursday when we introduce the solution, the **Riemann sphere**.
 - If $f(z)$ stays bounded, then f has a **removable singularity** (at ∞).
- **Pole** (at ∞): A function f such that $|f(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$.
- **Essential singularity** (at ∞): A function f for which there exist sequences $z_n \rightarrow \infty$ and $w_n \rightarrow \infty$ such that $f(z_n) \rightarrow \infty$ and $f(w_n)$ stays bounded.
- **Removable singularity** (at ∞): A function f that stays bounded as $|z| \rightarrow \infty$.
- We're now going to switch to a completely different topic.
- Suppose $f \in \mathcal{O}(U)$. When does $|f(z)|$ get the biggest? Equivalently, where does $|f(z)|$ take a local max? *Hint*: Look at the Cauchy integral formula!
 - There are no such points, at least on the interior of U !

- Theorem (maximum modulus principle): Let $f \in \mathcal{O}(U)$. If $|f(z)|$ has a local maximum on U , then f is constant.

Proof. Let z_0 be a local maximum of $|f(z)|$. Pick $D \ni z_0$ small enough such that $|f(z)| \leq |f(z_0)|$ for all $z \in D$. Let r be the radius of D . Now invoking the CIF,

$$\begin{aligned} |f(z_0)| &= \left| \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z_0} d\zeta \right| \\ &= \frac{1}{2\pi} \left| \int_{\partial D} \frac{f(\zeta)}{\zeta - z_0} d\zeta \right| \\ &\leq \frac{1}{2\pi} \int_{\partial D} \left| \frac{f(\zeta)}{\zeta - z_0} \right| d\zeta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left| \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} \cdot ire^{i\theta} \right| d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta \\ &\leq \frac{1}{2\pi} \cdot 2\pi \cdot \max_{\partial D} |f(\zeta)| \\ &= \max_{\partial D} |f(\zeta)| \\ &\leq |f(z_0)| \end{aligned}$$

But since the above inequality begins and ends with the same value, all \leq 's must be $=$'s. Thus, in particular,

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta &= |f(z_0)| \\ \frac{1}{2\pi} \int_0^{2\pi} (|f(z_0 + re^{i\theta})| - |f(z_0)|) d\theta &= 0 \end{aligned}$$

Combining this with the fact that the above integrand is always ≤ 0 because $f(z_0)$ is a local maximum, we have that

$$\begin{aligned} |f(z_0 + re^{i\theta})| - |f(z_0)| &= 0 \\ |f(\zeta)| &= |f(z_0)| \end{aligned}$$

on ∂D . Note that this is true for all small ∂D 's centered at z_0 .

Now since $|f|$ is constant on ∂D , we must have that $|f|^2 = f \cdot \bar{f}$ is constant on ∂D . Taking the Wirtinger derivative and using its product rule gets us

$$0 = \frac{\partial}{\partial \bar{z}} (f \cdot \bar{f}) = f_z \cdot \bar{f} + f \cdot \bar{f}_z$$

Since f is holomorphic (hence satisfies the CR equations) and $f_z = \bar{f}_z$, we have that

$$\bar{f}_z = f_z = 0$$

Thus,

$$0 = f_z \cdot \bar{f} + f \cdot 0 = f_z \cdot \bar{f}$$

By the zero-product property, either $f_z = 0$ and $\bar{f} = 0$. In the first case, this means that f is constant, as desired. In the second case, this means that f is zero (and hence constant), as desired.

At this point, we have shown that f is constant on a small disk. Therefore, we need only invoke the identity theorem, which tells us that since the function is constant for a little bit somewhere, it is constant everywhere. \square

- Another way to prove this is by considering the derivative of the Cauchy integral formula and where it's equal to zero.
- Corollary (minimum modulus principle): If $f \in \mathcal{O}(U)$, $f \neq 0$ on U (hence $1/f \in \mathcal{O}(U)$), and $|f(z)|$ takes a minimum in U , then f is constant.
- Application of the maximum modulus principle (the fundamental theorem of algebra): If p is a polynomial of degree d in \mathbb{C} , then p has d roots in \mathbb{C} (counted with multiplicity).

Proof. Suppose inductively that $d \geq 1$.

Step 1 (show that there exists one root): Suppose for the sake of contradiction that p has no zeros. Since p is a polynomial, we know that $|p(z)| \rightarrow \infty$ as $|z| \rightarrow \infty$. Thus, there exists $R > 0$ such that for all z with $|z| > R$, $|p(z)| \geq |p(0)|$. Then $|p(z)|$ must take a minimum on $\overline{D_R}$. But to keep p from being constant by the minimum modulus principle, the minimum has to be on ∂D_R . Now take a slightly bigger disk; our global minimum is now in the interior, so p is constant, a contradiction. It follows that p must have a zero in D_R .

Step 2: Suppose p has a root at z_0 . Then power series for p at z_0 is $p(z) = (z - z_0)p_1(z)$. p_1 is a polynomial of degree $d - 1$.

Step 3: Now iterate to find that p is a product of monomials. □

- Algebraists love to prove this with only algebra, but in reality, the proof is complex analysis.^[1]
- We did not get to say something about harmonic functions today, but Calderon will leave the content in his notes in case we want to look at it.
 - The statement: Harmonic functions follow a version of the CIF.
 - There's a related PSet problem.

¹How did this proof work??