Week 1

???

1.1 Holomorphic Functions

3/19:

- We begin by reviewing some properties of the **complex numbers**.
- **Complex numbers**: The field of elements $z = x + iy$ where $x, y \in \mathbb{R}$ and $i^2 = -1$. *Denoted by* $\mathbb{C}$.

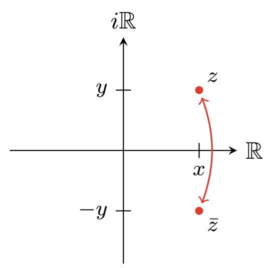

Figure 1.1: The complex plane.

  - Can be visualized as a two-dimensional plane with the number z corresponding to the point (x, y).
- **Real part**: The number x. *Denoted by* $\mathbf{Re}\, \boldsymbol{z}$.
- **Imaginary part**: The number y. *Denoted by* $\mathbf{Im}\, \boldsymbol{z}$.
- **Complex conjugate** (of z): The complex number defined as follows. *Denoted by* $\bar{\boldsymbol{z}}$. *Given by*

$$\bar{z} := x - iy$$

- Now recall the definition of a *real* function that is **differentiable** at a point $x_0 \in \mathbb{R}$.
  - $f'(x_0)(x - x_0)$ is the "best linear approximation" of f near x_0, where $\boldsymbol{f'(x_0)}$ is also defined below.
- **Differentiable** ($f : \mathbb{R} \to \mathbb{R}$ at x_0): A function f for which the following limit exists. *Constraint*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} =: f'(x_0)$$

- We now build up to defining a notion of complex differentiability.
  - Observe that the constraint above is equivalent to the constraint

$$f(x) = f(x_0) + \underbrace{[f'(x_0) + e(x)]}_{\Delta(x)}(x - x_0)$$

    where $e(x) \to 0$ as $x \to x_0$.
  - Note that we are defining a new function $\Delta(x)$ above, with the property that $\Delta(x_0) = f'(x_0)$.

- **Holomorphic** (f at z_0): A function $f : \mathbb{C} \to \mathbb{C}$ for which the following limit exists. *Also known as* **$\mathbb{C}$-differentiable**. *Constraints*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} =: f'(z_0) \qquad \Longleftrightarrow \qquad f(z) = f(z_0) + \Delta(z)(z - z_0)$$

  where Δ is continuous at z_0 and $\Delta(z_0) = f'(z_0)$.
  - Is this the true definition of "holomorphic" / "$\mathbb{C}$-differentiable" function, or is this just a naïve first pass??
- Properties of holomorphic functions: Let $U \subset \mathbb{C}$ be open.
  1. The holomorphic functions on U form a ring $\boldsymbol{\mathcal{O}(U)}$.
     - Equivalently, the $\mathbb{C}$-differentiation operator is $\mathbb{C}$-linear.
     - Equivalently, if f, g are holomorphic, then $f + g$ and fg are holomorphic, too.
     - Equivalently (and most simply), we have the sum rule and the product rule (and the quotient rule if the function in the denominator is nonzero).
  2. We have the chain rule.
  3. Holomorphic implies continuous.
- Examples: Polynomials, rational functions $p(z)/q(z)$ (away from their **poles**).
- Noney[1]: Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$z \mapsto \bar{z}$$

  - TPS[2]: Why?
  - Notice that

$$f(0) = 0 \qquad\qquad f(t) = t \qquad\qquad f(it) = -it$$

  - Thus,

$$\Delta(t) = 1 \qquad\qquad \Delta(it) = -1$$

    for all t.
  - But this means that Δ can't be continuous!
  - Yet f is clearly $\mathbb{R}$-differentiable! What gives?!
  - Note that — viewing f as a mapping of $\mathbb{R}^2 \to \mathbb{R}^2$ — we have

$$Df = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The above example suggests that our definition of complex differentiability may have been to naïve, so we'll do some further investigations now.
- Observe that $\mathbb{C} \cong \mathbb{R}^2$ as $\mathbb{R}$-vector spaces.
- **Differentiable** ($f : \mathbb{R}^2 \to \mathbb{R}^2$ at x_0): A function f for which there exists an $\mathbb{R}$-linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying the following constraint. *Constraint*

$$\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - Ah\|}{\|h\|} = 0$$

  - We also denote A by Df.

[1] What does "Noney" mean??
[2] What does "TPS" mean??

- Example: Consider the function $f : \mathbb{C} \to \mathbb{R}$ defined by
$$x + iy \mapsto x$$
  - Differentiable with total derivative
$$Df = \begin{pmatrix} 1 & 0 \end{pmatrix}$$
- Observation: While $\mathbb{C} \cong \mathbb{R}^2$ as $\mathbb{R}$-vector spaces, as a $\mathbb{C}$-vector space, there is *additional* structure.
  - In particular, all "vectors" should commute with the "multiplication by i" map $J : \mathbb{C} \to \mathbb{C}$ defined by any one of the following three maps.
$$z \mapsto z \qquad\qquad x + iy \mapsto xi - y \qquad\qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
- Exercise: In $(\mathrm{Re}, \mathrm{Im})$ coordinates, write down the matrix for "multiply by w" for any $w \in \mathbb{C}$.
  - Let $w = a + bi$ and let $v = x + iy$. Then
$$\begin{aligned} wv &= (a + bi)(x + iy) = ax - by + i(bx + ay) \\ &= \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix} = \underbrace{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}}_{W} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$
  - The matrix W above is the desired result.
- TPS: Is $f : \mathbb{C} \to \mathbb{C}$ defined as follows a complex linear map? Why not?
$$x + iy \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (x + y) + iy$$
  - Among other properties, a complex linear map should satisfy
$$if(x + iy) = f[i(x + iy)]$$
    for the scalar $i \in \mathbb{C}$.
  - However, we have that
$$if(x + iy) = i[(x + y) + iy] = -y + i(x + y) \neq (x - y) + ix = f(-y + ix) = f[i(x + iy)]$$
- What about the following map?
$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$
  - A complex linear map should satisfy
$$A(v + w) = Av + Aw \qquad\qquad \lambda Av = A(\lambda v)$$
    for all $v, w, \lambda \in \mathbb{C}$.
  - Let $v, w \in \mathbb{C}$ be arbitrary. Then
$$\begin{aligned} A(v + w) &= \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right] = \begin{pmatrix} (v_1 + w_1) + 2(v_2 + w_2) \\ -2(v_1 + w_1) + (v_2 + w_2) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = Av + Aw \end{aligned}$$

- Let $v,\lambda\in\mathbb{C}$. Then

$$\begin{aligned}
\lambda Av &= (\lambda_1+i\lambda_2)\cdot[(v_1+2v_2)+i(-2v_1+v_2)]\\
&= [\lambda_1(v_1+2v_2)-\lambda_2(-2v_1+v_2)]+i[\lambda_2(v_1+2v_2)+\lambda_1(-2v_1+v_2)]\\
&= [(\lambda_1v_1-\lambda_2v_2)+2(\lambda_2v_1+\lambda_1v_2)]+i[-2(\lambda_1v_1-\lambda_2v_2)+(\lambda_2v_1+\lambda_1v_2)]\\
&= A[(\lambda_1v_1-\lambda_2v_2)+i(\lambda_2v_1+\lambda_1v_2)]\\
&= A(\lambda v)
\end{aligned}$$

- Therefore, since A satisfies the two properties, it is complex linear.

- Conclusion: To reiterate from the above, A must commute with J to be complex linear.
- Implication: Every $\mathbb{C}$-linear map of $\mathbb{C}$ is just multiplication by a complex number.
  - This is a special case of the following more general result, which holds for any field K.

$$\operatorname{Hom}_K(K,K)\cong K$$

- Now let's revisit differentiability.
- It turns out that a condition for $\mathbb{C}$-differentiability *equivalent* to the definition of "holomorphic" given above is that there exists a $\mathbb{C}$-linear map $A:\mathbb{C}\to\mathbb{C}$ such that

$$\lim_{h\to 0}\frac{\|f(x_0+h)-f(x_0)-Ah\|}{\|h\|}=0$$

  - From the above discussion, we know that this A is just multiplication by some $w\in\mathbb{C}$.
  - All of the values in the above norms are complex numbers, so *another* equivalent condition is

$$\lim_{z\to z_0}\frac{|f(z)-f(z_0)-w\cdot(z-z_0)|}{|z-z_0|}=0$$

  - This condition is wholly mathematically equivalent to our holomorphic definition,

$$`\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=w$$

- So when is an $\mathbb{R}$-differentiable function actually holomorphic?
  - Let $f:\mathbb{R}^2\to\mathbb{R}^2$ map $(x,y)\mapsto(g,h)$.
  - Let

$$A=Df=\begin{pmatrix}g_x & g_y\\ h_x & h_y\end{pmatrix}$$

    where the subscript notation views g, for instance, as $g(x,y)$ and denotes the partial derivative of g with respect to x.
  - Let J (the "multiply by i") function be defined as above.
  - Then the "commute with i" condition is equivalent to

$$J^{-1}AJ=A$$

  - Expanding the product on the left above in terms of g_x,g_y,h_x,h_y, we obtain

$$\begin{pmatrix}h_y & -h_x\\ -g_y & g_x\end{pmatrix}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}g_x & g_y\\ h_x & h_y\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}=\begin{pmatrix}g_x & g_y\\ h_x & h_y\end{pmatrix}$$

  - This condition is equivalent to A satisfying the **Cauchy-Riemann equations**.

- **Cauchy-Riemann equations**: The following two equations, which identify when a complex function is holomorphic. *Also known as* **CR equations**. *Given by*

$$\begin{aligned} g_x &= h_y \\ g_y &= -h_x \end{aligned}$$

- These equations are satisfied when A is of the form

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

- So at this point, we can differentiate f with respect to z. But what if we want to differentiate it with respect to x and y (of $z = x + iy$)?
  - We will need the following change of basis.
    - Since $z = x + iy$ and $\bar{z} = x - iy$, we have

$$\begin{aligned} 2x &= z + \bar{z} & 2iy &= z - \bar{z} \\ x &= \frac{1}{2}(z + \bar{z}) & y &= -\frac{i}{2}(z - \bar{z}) \end{aligned}$$

    - This tells us that

$$\frac{\partial x}{\partial z} = \frac{1}{2} \qquad\qquad \frac{\partial y}{\partial z} = -\frac{i}{2}$$

  - We can now invoke the multivariable chain rule and simplify the resultant expression.

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}(f_x - if_y)$$

    - Note that once again, the subscript notation "f_x" means $\partial f/\partial x$.
  - Note that we can also similarly work out that

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(f_x + if_y)$$

    - Observe in particular that

$$f_x = g_x + ih_x \qquad\qquad f_y = g_y + ih_y$$

    - Thus, the CR equations ($g_x = h_y$ and $g_y = -h_x$) being satisfied is equivalent to

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(f_x + if_y) = \frac{1}{2}[(g_x + ih_x) + i(g_y + ih_y)] = 0$$

    - Note that $\partial f/\partial \bar{z}$ is not actually a derivative since f depends on z, not $\bar{z}$. Rather, we use "$\partial f/\partial \bar{z}$" to denote the following operator applied to f.

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

- Theorem: The $\mathbb{R}$-differentiable function $f : U \to \mathbb{C}$ is holomorphic iff $\partial f/\partial \bar{z} = 0$. Moreover, if it is, then

$$f'(z_0) = \left.\frac{\partial f}{\partial z}\right|_{z_0}$$

1.2 Chapter I: Analysis in the Complex Plane

From Fischer and Lieb (2012).

- The preface only contains comments and instructions for an instructor planning to use this textbook for a course.
- The chapter begins with two paragraphs.
  - The first discusses topic covered in the chapter.
  - The second gives some historical background on these topics.

Section I.0: Notations and Basic Concepts

- Goal: Reiew the fundamental topological and analytical concepts of real analysis.
- Defines the **complex numbers**, **complex plane**, and **complex conjugate**.
- **Absolute value** (of z): The Euclidean distance of z from zero. *Also known as* **modulus**. *Denoted by* $\boldsymbol{|z|}$. *Given by*
$$|z| := \sqrt{x^2 + y^2}$$
- **Imaginary unit**. *Denoted by* $\boldsymbol{i}$.
- Relating the modulus and complex conjugate.
$$|z| = \sqrt{z\bar{z}}$$
- **Open disk** (of radius ε and center z_0): The set defined as follows. *Also known as* $\boldsymbol{\varepsilon}$**-neighborhood** (of z_0). *Denoted by* $\boldsymbol{D_\varepsilon(z_0)}$, $\boldsymbol{U_\varepsilon(z_0)}$. *Given by*
$$D_\varepsilon(z_0) = U_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$$
- **Unit disk**: The set defined as follows. *Denoted by* $\mathbb{D}$. *Given by*
$$\mathbb{D} := D_1(0)$$
- **Unit circle**: The set defined as follows. *Denoted by* $\mathbb{S}$. *Given by*
$$\mathbb{S} := \{z \in \mathbb{C} : |z| = \varepsilon\}$$
- **Upper half plane**: The set defined as follows. *Denoted by* $\mathbb{H}$. *Given by*
$$\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$
- $\boldsymbol{\mathbb{C}^*}$: The set defined as follows. *Given by*
$$\mathbb{C}^* := \mathbb{C} \setminus \{0\}$$

3/21:

- **Neighborhood** (of z_0): A set U which contains an ε-neighborhood.
- **Open** (set): A set that is a neighborhood of each of its points.
- **Closed** (set): A complement of an open set.
- **Interior** (of M): The largest open set contained in M. *Denoted by* $\mathring{\boldsymbol{M}}$.
- **Closure** (of M): The smallest closed set containing M. *Denoted by* $\overline{\boldsymbol{M}}$.

- **Topological boundary** (of M): The set defined as follows. *Also known as* **boundary**. *Denoted by* $\boldsymbol{\partial M}$. *Given by*
$$\partial M := \overline{M} \setminus \mathring{M}$$
- **Relatively open** (set in M): The intersection of an open set U with an arbitrary set M. *Also known as* **open** (set in M).
- **Relatively closed** (set in M): The intersection of a closed set U with an arbitrary set M. *Also known as* **open** (set in M).