11 Generating Functions and Recursion Formulas

11.1 Hermite Polynomials

5/18:

- In this section, we seek to find the **generating function** for the Hermite polynomials. Besides being interesting in its own right, it will also allow us to derive an integral representation and recursion formula.
- **Exponential generating function** (of $\{p_n\}$): A representation of the infinite sequence $\{p_n\}$ of polynomials as the n^{th} derivatives of a formal power series. *Also known as* **generating function**. *Denoted by* $\boldsymbol{g(x,t)}$. *Given by*

$$g(x,t) := \sum_{n=0}^{\infty} \frac{p_n(x)}{n!} t^n$$

  - Note that "the factorial term $n!$ is merely a counter-term to normalize the derivative operator acting on x^n."[1]
- We now derive the generating function g.
  - Assume g is analytic at and near $t=0$.[2] Then

$$H_n(x) = \frac{\partial^n}{\partial t^n} g(x,t)\bigg|_{t=0}$$

  - Recall from Section 9.4 that

$$H_n(x) = (-1)^n \mathrm{e}^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \mathrm{e}^{-x^2}$$

  - Thus, by transitivity, we need to solve

$$(-1)^n \mathrm{e}^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \mathrm{e}^{-x^2} = \frac{\partial^n}{\partial t^n} g(x,t)\bigg|_{t=0}$$

    for g.
  - Let

$$g(x,t) = \mathrm{e}^{x^2} f(x-t)$$

    for some undetermined function f.
    - Then

$$\frac{\partial^n}{\partial t^n} g(x,t)\bigg|_{t=0} = \mathrm{e}^{x^2} \frac{\partial^n}{\partial t^n} f(x-t)\bigg|_{t=0} = (-1)^n \mathrm{e}^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}u^n} f(u)\bigg|_{t=0}$$

  - It follows by comparison with the Rodrigues formula for $H_n(x)$ that

$$f(u) = \mathrm{e}^{-u^2}$$

  - Therefore, returning the substitution, we have that

$$g(x,t) = \mathrm{e}^{x^2} \mathrm{e}^{-(x-t)^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

[1] Source: https://en.wikipedia.org/wiki/Generating_function#Exponential_generating_function_(EGF).

[2] There is no reason that this must be true, but we are free to assume it and see where it gets us. If it gets us somewhere, we're golden!

- We now derive the first integral representations of the Hermite polynomials listed in Section 10.6.
  - Using the formula for the derivative of the CIF from the 4/2 lecture, another formula for the Taylor series of g about $t=0$ is

$$g(x,t)=\sum_{n=0}^{\infty}\frac{g^{(n)}(x,t)}{n!}t^n=\sum_{n=0}^{\infty}\frac{\partial^n/\partial t^n\ g(x,t)}{n!}t^n=\sum_{n=0}^{\infty}\left(\frac{n!}{2\pi i}\oint_C\frac{g(x,t)}{t^{n+1}}\,\mathrm{d}t\right)\frac{t^n}{n!}$$

    where $C\ni 0$.
  - Thus, by comparing this to the generating function, we learn that

$$H_n(x)=\frac{n!}{2\pi i}\oint_C\frac{g(x,t)}{t^{n+1}}\,\mathrm{d}t=\frac{n!}{2\pi i}\oint_C\frac{\mathrm{e}^{x^2}\mathrm{e}^{-(x-t)^2}}{t^{n+1}}\,\mathrm{d}t=\frac{n!}{2\pi i}\oint_C\frac{\mathrm{e}^{2xt-t^2}}{t^{n+1}}\,\mathrm{d}t$$

    as desired.
- As mentioned in Section 10.6, we now use this integral representation to derive the recursion formula for the Hermite polynomials.
  - We have that

$$H_n'(x)=\frac{n!}{2\pi i}\oint_C\frac{2t\mathrm{e}^{2xt-t^2}}{t^{n+1}}\,\mathrm{d}t=2n\cdot\frac{(n-1)!}{2\pi i}\oint_C\frac{\mathrm{e}^{2xt-t^2}}{t^{(n-1)+1}}\,\mathrm{d}t=2nH_{n-1}(x)$$

  - Differentiating both sides of the above (and using the above), we obtain

$$H_n''(x)=2n\cdot H_{n-1}'(x)=2n\cdot 2(n-1)H_{n-2}(x)=4n(n-1)H_{n-2}(x)$$

  - Now recall that Hermite's equation reads

$$H_n''(x)-2xH_n'(x)+2nH_n(x)=0$$

  - Thus, with our new definitions for $H_n'(x),H_n''(x)$, we obtain

$$\begin{aligned}4n(n-1)H_{n-2}(x)-2x\cdot 2nH_{n-1}(x)+2nH_n(x)&=0\\2(n-1)H_{n-2}(x)-2xH_{n-1}(x)+H_n(x)&=0\\H_n(x)&=2xH_{n-1}(x)-2(n-1)H_{n-2}(x)\end{aligned}$$

  - Redefining the indices $n-1\to n$ in the above yields the final recursion formula

$$H_{n+1}(x)=2xH_n(x)-2nH_{n-1}(x)$$

- **bib:Seaborn<empty citation>** uses the recursion formula along with the initial conditions $H_0(x)=1$ and $H_1(x)=2x$ to compute the first few Hermite polynomials.

11.4 Legendre Polynomials

11.4.1 The Generating Function

- In this section, we will derive an (ordinary) generating function for the Legendre polynomials.
- **Ordinary generating function** (of $\{p_n\}$): A representation of the infinite sequence $\{p_n\}$ of polynomials as the coefficients of a formal power series. *Denoted by* $\boldsymbol{g(x,u)}$. *Given by*

$$g(x,u):=\sum_{n=0}^{\infty}p_n(x)u^n$$

- We now begin the derivation.
  - Like in the previous derivation, another formula for the Taylor series of g about $u=0$ is

$$g(x,u)=\sum_{n=0}^{\infty}\left(\frac{n!}{2\pi i}\oint_C\frac{g(x,u)}{u^{n+1}}\,\mathrm{d}u\right)\frac{u^n}{n!}=\sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\oint_C\frac{g(x,u)}{u^{n+1}}\,\mathrm{d}u\right)u^n$$

  - Consequently, we obtain the following by matching up coefficients.

$$P_n(x)=\frac{1}{2\pi i}\oint_C\frac{g(x,u)}{u^{n+1}}\,\mathrm{d}u$$

  - Additionally, recall the Schläfli integral:

$$P_n(x)=\frac{1}{2^n}\frac{1}{2\pi i}\oint_{C'}\frac{(t^2-1)^n}{(t-x)^{n+1}}\,\mathrm{d}t=\frac{1}{\pi i}\oint_{C'}\left[\frac{(t^2-1)}{2(t-x)}\right]^{n+1}\frac{\mathrm{d}t}{t^2-1}$$

    - Note that C' may equal C, but it need not; it need only enclose 0.
  - Comparing the integrands of the last two equations suggests that a good substitution of variables may be

$$u=\frac{2(t-x)}{t^2-1}$$

    which is equivalent to

$$t=u^{-1}(1\pm\sqrt{1-2xu+u^2})$$