Week 9

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9.1 The Gelfond-Schneider Theorem

5/14: • Today's lecture.
  - By Ben, a postdoc.
  - His choice of topic in complex analysis.
  - Proof that $e^{\sqrt{2}}$ is irrational, pulled from the Math Library's one complex textbook.
  - He chose this topic to illustrate how useful complex analysis is in other areas of math.

- The main theorem we'll use here is the maximum modulus principle, in a slightly modified form.
- Maximum modulus principle (alternate statement): If Ω is a compact domain, $f \in \mathcal{O}(\Omega)$, then

$$|f(z)| \leq \max_{w \in \partial\Omega} |f(w)|$$

  Moreover, if equality holds in any case, then f is constant.

  Proof. For $\Omega = B_p(r)$, this follows from the **mean-value property**.[1] □

- Remark: An entire function with lots of zeroes must grow fast.

  Proof. Let f be the entire function, and suppose it has zeroes at $\{z_i\}$ with multiplicity k_i. Form the new function

$$\frac{f(z)}{\prod_i (z - z_i)^{k_i}}$$

  If we make $|z|$ large, then this function behaves like

$$\frac{f(z)}{\prod_i z^{k_i}}$$

  Since the above function is holomorphic, the MMP says it must obtain its maximum value on the boundary of an arbitrarily large ball around the compact set on which f obtains all its zeroes. But that denominator is growing really fast, so f must grow even faster to compensate. □

- **Strictly ordered** (f by ρ): An entire function f for which there exists $C > 1$ such that

$$|f(z)| \leq C^{R^\rho}$$

  where $R = |z|$.

[1] Ben quickly explains how the mean-value property works.

- Alternatively, we say that "f has *strict order* $\leq \rho$."
- This gives a bound on the growth of the function.
- We will use R to denote $|z|$ throughout lecture today.

- **Algebraically independent** (functions): Two functions f, g for which

$$\sum_{i,j=1}^{N} a_{ij} f^i g^j = 0$$

where $a_{ij} \in \mathbb{C}$ implies that $a_{ij} = 0$ for all i, j.

  - We will apply this to $f(z) = z$ and $g(z) = \mathrm{e}^z$.

- Theorem (Gelfond-Schneider): Let $f_1, \dots, f_n$ be entire functions with strict order less than or equal to ρ a positive number. Assume that at least two of these functions are algebraically independent. Assume $D := \mathrm{d}/\mathrm{d}z$ maps $\mathbb{Q}[f_1, \dots, f_n]$ into itself. Suppose $w_1, \dots, w_N$ are distinct complex numbers such that $f_i(w_j) \in \mathbb{Q}$ for all $1 \leq i \leq n$ and $1 \leq j \leq N$. Then $N \leq 4\rho$.

- Corollary: e^w cannot be rational if $w \in \mathbb{Q}$.

Proof. Apply the Gelfond-Schneider theorem to $\mathbb{Q}[z, \mathrm{e}^z]$. From here, note that if e^w were rational, then $\mathrm{e}^w, \mathrm{e}^{2w}, \mathrm{e}^{3w}, \dots \in \mathbb{Q}$ which would eventually contradict the $N \leq 4\rho$ bound. $\square$

- If we prove the Gelfond-Schneider theorem under the hypothesis that $f_i(w_j) \in \overline{\mathbb{Q}}$, then our corollary may state that e^w cannot be **algebraic**.

- **Algebraic number**: A number that is the zero of a one-variable polynomial.

- Lemma 1 (Siegel): Let

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ &\vdots \\ a_{r1}x_1 + \cdots + a_{rn}x_n &= 0 \end{aligned}$$

be such that (i) $a_{ij} \in \mathbb{Z}$, (ii) $n > r$, and (iii) $|a_{ij}| \leq A$. Then there exists an integral, nonzero solution $(x_1, \dots, x_n)$ to this system of equations with

$$|x_j| \leq 2(2nA)^{\frac{r}{n-r}}$$

Proof. We know that there has to be at least *some* solution by condition (ii) and linear algebra, which confirms sufficient information and a nontrivial kernel.

Let T be the $r \times n$ matrix (a_{ij}). Then T maps $\mathbb{Z}^n(B)$ into $\mathbb{Z}^r(nBA)$, where $\mathbb{Z}^m(s) := B_0(s) \cap \mathbb{Z}^m$.[2] Find $x, y \in \mathbb{Z}^n(B)$ such that $T(x) = T(y)$ and hence $T(x-y) = 0$. Via a pigeonhole principle argument, make B big enough so that $\mathbb{Z}^r(nBA)$ (which is growing slower due to its smaller exponent of r) has cardinality smaller than $\mathbb{Z}^n(B)$; this will mean that two things have to map to the same thing. Then if we do the computation, we get the stated bound.

Essentially, we're relying on the principle that integer balls in higher-dimensional Euclidean spaces have more points in the limit of large radius. $\square$

- **Size** (of a polynomial): The following number, where $P(x_1, \dots, x_n) = \sum_{I=(i_1,\dots,i_n)} a_I x_1^{i_1} \cdots x_n^{i_n}$ is a polynomial. *Denoted by* **size($\boldsymbol{P}$)**. *Given by*

$$\operatorname{size}(P) := \max_I |a_I|$$

[2]Pronounced "the m^{th}-dimensional integer ball of radius s."

- **Denominator** (of $\{a_i\} \subset \mathbb{Q}$): A number d such that $d \cdot a_i \in \mathbb{Z}$ for ever a_i in the subset $\{a_i\} \subset \mathbb{Q}$. *Denoted by* $\mathbf{den(\{a_i\})}$.
- Lemma 2: Let $f_1, \ldots, f_n$ be functions as in the Gelfond-Schneider theorem. Then there exists a constant C_1 such that if $\mathbb{Q}(T_1, \ldots, T_n)$ is a polynomial with rational coefficients and degree less than or equal to r, then
$$P^m(Q(f_1, \ldots, f_n)) = Q_m(f_1, \ldots, f_n)$$
where...
  - i.) $\deg(Q_m) \leq C_1(m + r)$;
  - ii.) $\text{size}(Q_m) \leq \text{size}(Q) m! C_1^{m+r}$;
  - iii.) There exists a denominator for the coefficients of Q_m bounded by $\text{den}(Q) C_1^{m+r}$.
- We are now ready to prove the Gelfond-Schneider theorem.

Proof. By hypothesis, we have common elements $w_1, \ldots, w_N$ of $\mathbb{C}$ such that $f_i(w_j) \in \mathbb{Q}$ and $f_{ij} \in \{f_1, \ldots, f_n\}$ algebraically independent. Let $L \in \mathbb{Z}^+$ be divisible by $2N$, $b_{ij} \in \mathbb{Z}$, and let $F = \sum_{i,j=1}^{L} b_{ij} f^i g^j$ and let $L = 2MN$ be such that

$$D^m F(w_\ell) = 0 \tag{$*$}$$

for $m = 0, \ldots, M-1$ and $\ell = 1, \ldots, N$; we will send both of these constants to infinity eventually.

$(*)$ has L^2 unknowns and MN equations. Multiply the equations in $(*)$ by a common denominator and using Lemma 2 and Siegel's Lemma, we can find b_{ij} such that

$$|b_{ij}| \leq M! C_2^{M+L} \leq M^M C_2^{M+L} \tag{$**$}$$

as $M \to \infty$. Note that in the second inequality, we used Stirling's approximation.

The next observation is that $F \neq 0$ since f and g are algebraically independent. Let s be the smallest integer such that $D^m f(w_i) = 0$ for $m < s$ for all i but $D^s F \neq 0$ at some w_i, which WLOG we will let be w_1.

Let $\alpha := D^s F(w_1)$. Then $\alpha \in \mathbb{Q}$ since $F(W_1) \in \mathbb{Q}$ so all its derivatives will, too. Additionally, $C := \text{den}(\alpha) \leq (C_1)^s$ as $s \to \infty$, this from (i) and (iii) of Lemma 2. Then $C\alpha \in \mathbb{Z}$, which implies that $|C\alpha| \geq 1$ and hence $|\alpha| \geq C^{-1}$. Thus, at this point, we have a lower bound on $|\alpha|$; the next step is to move toward an upper bound and then get what we want.

We upper-bound α using the MMP. Compute

$$D^s F(w_1) = s! \frac{F(w_1)}{(z - w_1)^s}\bigg|_{z=w_1}$$

Estimate

$$H(z) := s! \frac{F(z)}{\prod_{i=1}^{N} (z - w_i)^s} \prod_{i>1}^{N} (w_1 - w_i)^s$$

on the circle of radius $B = s^{1/2\rho}$. Then the MMP tells us that

$$|D^s F(w_1)| = |H(w_1)| \leq \|H\|_R \leq \frac{s^s C^{Ns} \|F\|_R}{R^{Ns}}$$

Then after working this out, we get

$$1 \leq |c\alpha| \leq \frac{s^{2s} C^{Ns}}{e^{Ns \log(s)/2\rho}}$$

which gets to $N \leq 4\rho$. □