Week 6

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6.1 Meromorphic Functions and Möbius Transformations

4/23: • Last time.

  – We defined the Riemann sphere
$$\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$$
  – When is a function $f : \hat{\mathbb{C}} \to \mathbb{C}$ "holomorphic at ∞?"
    ■ Liouville: $f \in \mathcal{O}(\hat{\mathbb{C}})$ is constant.
  – f is a meromorphic function on $U \subset \hat{\mathbb{C}}$ if and only if f is a holomorphic map to $\hat{\mathbb{C}}$.
  – $f^{-1}(\infty)$ gives the set of poles.

• Note for people doing residues on their final project.

  – Recall that if $f : U \to \hat{\mathbb{C}}$ is meromorphic, then it has a Laurent series around each pole p.
  – Essentially,
$$f(z) = \frac{1}{(z-p)^k} h(z)$$
    ■ h is holomorphic.
    ■ $h(p) \neq 0$.
    ■ k is the order of the pole.
  – Since h is holomorphic, it has a power series expansion
$$h(z) = \sum_{i=0}^{\infty} a_i (z-p)^i$$
  with $a_0 \neq 0$.
  – Thus,
$$\begin{aligned} f(z) &= \sum_{i=0}^{\infty} a_i (z-p)^{i-k} \\ &= \sum_{i=-k}^{\infty} a_i (z-p)^i \end{aligned}$$
  – This allows us to define the **principal part**.

• **Principal part** (of a Laurent series): The sum of the terms with a negative exponent.

- TPS: Suppose you have a pole p in a disk D within the radius of convergence of the Laurent series. Compute
$$\int_{\partial D} f \,\mathrm{d}z$$
  - Because we are in the radius of convergence of the Laurent series, the series converges locally absolutely uniformly, so we can switch the sum and the integral in the following and evaluate.
$$\begin{aligned}\int_{\partial D} f \,\mathrm{d}z &= \int_{\partial D} \sum_{j=-k}^{\infty} a_j(z-p)^j \,\mathrm{d}z = \sum_{j=-k}^{\infty} \int_{\partial D} a_j(z-p)^j \,\mathrm{d}z \\ &= \sum_{\substack{j=-k \\ j\neq -1}}^{\infty} 0 + \int_{\partial D} \frac{a_{-1}}{z-p} \,\mathrm{d}z = a_{-1} \cdot 2\pi i\end{aligned}$$
  - This a_{-1} coefficient is clearly special, so it gets a special name.
- **Residue** (of f at p): The a_{-1} coefficient in the Laurent expansion of f about a pole p. *Denoted by* $\mathbf{res}_{\boldsymbol{p}}\boldsymbol{(f)}$. *Given by*
$$\operatorname{res}_p(f) := a_{-1}$$
- Corollary: f has a primitive on D (containing a single pole p) iff
$$\operatorname{res}_p(f) = 0$$

  Proof. This goes back to the proposition from the 3/28 class. If the residue is zero, then the closed loop integral is zero, so by homotopy, f has a primitive on the disk?? And if it has a primitive, then it's holomorphic on D and therefore the residue is zero. □

- Theorem: Every meromorphic function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ (by which we mean a meromorphic function on $\mathbb{C}$ that is either bounded or with a pole at ∞) is rational, i.e., $f = p/q$.

  Proof. We will prove this theorem by dividing it into three consecutive claims.

  1. The set of poles of f (including ∞) is finite.
  2. Write f minus the sum of the principal parts of all the poles of f, and call this g.
  3. g is holomorphic on $\hat{\mathbb{C}}$, and therefore is constant.

  Once we have Claim 3, we will know that $f = c + \sum$ principal parts, where all principal parts are rational, meaning that f is rational! Let's begin.

  For claim 1, we will argue that the set of poles of f is discrete and compact because it is a result of set theory that discrete compact sets are finite. To confirm that the set of poles is discrete, observe that $1/f$ has zeroes where f has poles, and we know that these zeroes must be discrete because otherwise $1/f$ would be constant and zero (think power series). To confirm that the set of poles is also compact, observe that it lives in $\hat{\mathbb{C}}$ (which is compact) and is a closed set (since its discrete, it cannot be open).

  For claim 2, it may seem unintuitive that we can subtract off a bunch of partial Laurent series about different points from a function defined more broadly than a Laurent series around a certain pole. However, observe that the principal part of a Laurent expansion around a certain pole is just is a function in its own right! Thus, we can subtract it off just like any other function. After doing this, g is not defined everywhere, but it has an analytic continuation (claim 3) to everywhere, as desired. Note that the principal part of a Laurent expansion about p only has a pole at p, so we *can* subtract it off without introducing any new poles. Additionally, note that we can subtract off a pole at ∞ as follows. Essentially, if f has a pole at ∞, then it is a polynomial. But f having a pole at ∞ is equivalent to $f(1/z)$ having a pole at 0. So just take the principal part of the Laurent expansion of $f(1/z)$ at zero and then switch out all the z's for $1/z$'s again and subtract that. □

- To get a global picture of the poles of f, we need a global picture of the zeroes of $1/f$.
- TPS.
  1. For $k \in \mathbb{N}$, draw $x \mapsto x^k$ on $\mathbb{R}$. For y near 0, how many preimages does y have?
     - If k is odd, y always has one preimage.
     - If k is even, then $y > 0$ has two preimages, $y = 0$ has one preimage, and $y < 0$ has no preimages.
  2. Draw what happens to the sector $\{z \mid z \in re^{i\theta},\ \theta \in (0, 2\pi/k]\}$ under $z \mapsto z^k$.
     - Maps to the circle of radius r^k bijectively.
     - Additionally, the interior of the sector goes to the disk minus the slit along the positive real axis.
  3. Do 1, but now for $z \mapsto z^k$ in $\mathbb{C}$.
     - For $y \neq 0$, there are k distinct roots of y.
  4. Draw a "global" picture of $z \mapsto z^k$.
     - For $z \mapsto z^2$, for instance, we cut $\mathbb{C}$ into an upper half plane and lower half plane and wrap them both around into circles.
     - *See class pictures.*
- Last time.
  - Proposition: If X is a Riemann surface and $f : X \to \hat{\mathbb{C}}$ is meromorphic, then f is onto.
- Now we'll do a bit on Möbius transformations.
  - One of Calderon's favorite topics; there will be a bit on the next problem set about these.
- What we just saw is that if $p(z) = z^n$, then $\#p^{-1}(w) = n$[1] for w near zero but nonzero.
  - If $w = 0$, then you still get n preimages if you count with multiplicity.
- If $p(z)$ is a general polynomial of degree n, we also know (by the FTA) that $\#p^{-1}(0) = n$.
  - Moreover, for all $w \in \hat{\mathbb{C}}$, $\#p^{-1}(w) = n$ with multiplicity because $p(z) - w$ has n roots.
- Let $f(z) = p(z)/q(z)$ be a rational function, such as $z/(z-3)^2$.
  - Here, $\#f^{-1}(\infty) = 2$.
  - Here as well, $f^{-1}(0) = 1$ on $\mathbb{C}$ and $= 2$ on $\hat{\mathbb{C}}$.
- **Degree** (of a rational function): The natural number defined as follows, where $f = p/q$ is a rational function representated with p, q coprime. *Denoted by* $\mathbf{deg(f)}$. *Given by*

$$\deg(f) := \max(\deg(p), \deg(q))$$

- Theorem: If $f = p/q$, then for all $w \in \hat{\mathbb{C}}$, $\#f^{-1}(w) = \deg(f)$ when counted with multiplicity.

  Proof. Follows from the FTA. Proof in the notes. □

- Example:
  - If $f(z) = z/(z-3)^2$, then we count 3 twice with multiplicity (giving a pole at ∞). We can also count $0, \infty$ as two distinct poles that give us zero.
  - This just reiterates the previous example.

[1] Recall that # denotes cardinality; in this case, cardinality of the preimage.

- Holomorphic symmetries.
  - What are all biholomorphisms?
  - They are kind of like change of coordinate maps.
  - Can we have entire biholomorphisms? Can we have biholomorphisms $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.
  - $\mathrm{Aut}(\mathbb{C}) = \mathrm{Bihol}(\mathbb{C})$. On Thursday, we'll prove that

$$\mathrm{Bihol}(\mathbb{C}) = \{a \mapsto az + b\}$$

  - For the other one,

$$\mathrm{Bihol}(\hat{\mathbb{C}}) = \left\{ z \mapsto \frac{az+b}{cz+d} \mid a, b, c, d \in \mathbb{C},\ ad - bc \neq 0 \right\}$$

    - The condition $ad - bc \neq 0$ is the same as saying that the map is nonconstant.
    - This is the set of fractional linear transformations.
    - It is isomorphic to

$$\mathrm{PGL}_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \det \neq 0 \right\}$$

      - This is the projective linear group.
      - This group acts on the projective linear space $\mathbb{P}^1(\mathbb{C}) = \hat{\mathbb{C}}$ (ask Calderon later??).

  Proof. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a biholomorphism. Thus, it is bijective and hence $\deg(f) = 1$. Additionally, it is meromorphic and hence rational map. But a rational map with degree 1 is just a map of the given kind! □

- A different geometric way of thinking about these **Möbius transformations**: Circle inversion.
  - This is the map $re^{i\theta} \mapsto r^{-1}e^{i\theta}$. In terms of z, this map is $z \mapsto 1/\bar{z}$.
  - This map is not holomorphic. In fact, it is **antiholomorphic** and **anticonformal**.
  - This is the projecting up and down map.
  - It can also be thought of as reflecting over the equator.
  - $\text{Möb}^{\pm}$ (the **extended Möbius group**) is the group generated by circle inversions.
  - Möb is the orientation-preserving subgroup.
- Four examples.
  - Scaling $z \mapsto rz$ where $r \in \mathbb{R}$.
  - Translation.
  - Rotation.
  - One last one.
- Theorem: $\text{Möb} = \mathrm{Bihol}(\hat{\mathbb{C}}) \cong \mathrm{PGL}_2(\mathbb{C})$.

  Proof. $\subset$: Obvious, once you've digested the definitions. This is just because circle inversions are anti-conformal. Two inversions implies conformal, which is equivalent to biholomorphic.

  $\supset$: We have algebraically that

$$\frac{az+b}{cz+d} = \frac{bc - ad}{c^2}\left(z + \frac{d}{c}\right)^{-1} + \frac{a}{c}$$

  where we have two translations, an inversion, and a scaling/rotation. □