Week 1

???

1.1 Holomorphic Functions

3/19: • We begin by reviewing some properties of the **complex numbers**.

- **Complex numbers**: The field of elements $z = x + iy$ where $x, y \in \mathbb{R}$ and $i^2 = -1$. *Denoted by* $\mathbb{C}$.

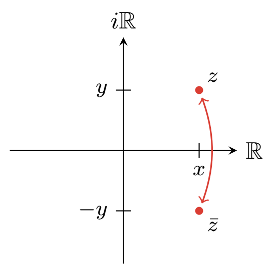

Figure 1.1: The complex plane.

  – Can be visualized as a two-dimensional plane with the number z corresponding to the point (x, y).

- **Real part**: The number x. *Denoted by* $\mathbf{Re}(z)$.
- **Imaginary part**: The number y. *Denoted by* $\mathbf{Im}(z)$.
- **Complex conjugate** (of z): The complex number defined as follows. *Denoted by* $\bar{z}$. *Given by*

$$\bar{z} := x - iy$$

- Now recall the definition of a *real* function that is **differentiable** at a point $x_0 \in \mathbb{R}$.
  – $f'(x_0)(x - x_0)$ is the "best linear approximation" of f near x_0, where $\boldsymbol{f'(x_0)}$ is also defined below.
- **Differentiable** (f at x_0): A function $f : \mathbb{R} \to \mathbb{R}$ for which the following limit exists. *Constraint*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} =: f'(x_0)$$

- We now build up to defining a notion of complex differentiability.
  – Observe that the constraint above is equivalent to the constraint

$$f(x) = f(x_0) + [\underbrace{f'(x_0) + e(x)}_{\Delta(x)}](x - x_0)$$

  where $e(x) \to 0$ as $x \to x_0$.
  – Note that we are defining a new function $\Delta(x)$ above, with the property that $\Delta(x_0) = f'(x_0)$.

- **Holomorphic** (f at z_0): A function $f:\mathbb{C}\to\mathbb{C}$ for which the following limit exists. *Also known as* **$\mathbb{C}$-differentiable**. *Constraints*

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} =: f'(z_0) \qquad\iff\qquad f(z)=f(z_0)+\Delta(z)(z-z_0)$$

  where Δ is continuous at z_0 and $\Delta(z_0)=f'(z_0)$.

  - Is this the true definition of "holomorphic" / "$\mathbb{C}$-differentiable" function, or is this just a naive first pass??

- Properties of holomorphic functions: Let $U\subset\mathbb{C}$ be open.
  1. The holomorphic functions on U form a ring $\boldsymbol{\mathcal{O}(U)}$.
     - Equivalently, the $\mathbb{C}$-differentiation operator is $\mathbb{C}$-linear.
     - Equivalently, if f,g are holomorphic, then $f+g$ and fg are holomorphic, too.
     - Equivalently (and most simply), we have the sum rule and the product rule (and the quotient rule if the function in the denominator is nonzero).
  2. We have the chain rule.
  3. Holomorphic implies continuous.

- Examples: Polynomials, rational functions $p(z)/q(z)$ (away from their **poles**).

- Noney[1]: Consider the function $f:\mathbb{C}\to\mathbb{C}$ defined by

$$z\mapsto\bar{z}$$

  - TPS[2]: Why?
  - Notice that

$$f(0)=0 \qquad\qquad f(t)=t \qquad\qquad f(it)=-it$$

  - Thus,

$$\Delta(t)=1 \qquad\qquad \Delta(it)=-1$$

    for all t.
  - But this means that Δ can't be continuous!
  - Yet f is clearly $\mathbb{R}$-differentiable! What gives?!
  - Note that — viewing f as a mapping of $\mathbb{R}^2\to\mathbb{R}^2$ — we have

$$Df=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

1.2 Chapter I: Analysis in the Complex Plane

From Fischer and Lieb (2012).

- The preface only contains comments and instructions for an instructor planning to use this textbook for a course.

- The chapter begins with two paragraphs.

  - The first discusses topic covered in the chapter.
  - The second gives some historical background on these topics.

[1] What does "Noney" mean??
[2] What does "TPS" mean??

Section I.0: Notations and Basic Concepts

- Goal: Reiew the fundamental topological and analytical concepts of real analysis.
- Defines the **complex numbers**, **complex plane**, and **complex conjugate**.
- **Absolute value** (of z): The Euclidean distance of z from zero. *Also known as* **modulus**. *Denoted by* $\boldsymbol{|z|}$. *Given by*
$$|z| := \sqrt{x^2 + y^2}$$
- **Imaginary unit**. *Denoted by* $\boldsymbol{i}$.
- Relating the modulus and complex conjugate.
$$|z| = \sqrt{z\bar{z}}$$
- **Open disk** (of radius ε and center z_0): The set defined as follows. *Also known as* **ε-neighborhood** (of z_0). *Denoted by* $\boldsymbol{D_\varepsilon(z_0)}$, $\boldsymbol{U_\varepsilon(z_0)}$. *Given by*
$$D_\varepsilon(z_0) = U_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$$
- **Unit disk**: The set defined as follows. *Denoted by* $\mathbb{D}$. *Given by*
$$\mathbb{D} := D_1(0)$$
- **Unit circle**: The set defined as follows. *Denoted by* $\mathbb{S}$. *Given by*
$$\mathbb{S} := \{z \in \mathbb{C} : |z| = \varepsilon\}$$
- **Upper half plane**: The set defined as follows. *Denoted by* $\mathbb{H}$. *Given by*
$$\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$
- $\mathbb{C}^*$: The set defined as follows. *Given by*
$$\mathbb{C}^* := \mathbb{C} \setminus \{0\}$$

3/21:

- **Neighborhood** (of z_0): A set U which contains an ε-neighborhood.
- **Open** (set): A set that is a neighborhood of each of its points.
- **Closed** (set): A complement of an open set.
- **Interior** (of M): The largest open set contained in M. *Denoted by* $\boldsymbol{\mathring{M}}$.
- **Closure** (of M): The smallest closed set containing M. *Denoted by* $\boldsymbol{\overline{M}}$.
- **Topological boundary** (of M): The set defined as follows. *Also known as* **boundary**. *Denoted by* $\boldsymbol{\partial M}$. *Given by*
$$\partial M := \overline{M} \setminus \mathring{M}$$
- **Relatively open** (set in M): The intersection of an open set U with an arbitrary set M. *Also known as* **open** (set in M).
- **Relatively closed** (set in M): The intersection of a closed set U with an arbitrary set M. *Also known as* **open** (set in M).