2 Power Series and Cauchy's Theorem

Set A: Graded for Completion

4/5: 1. Fischer and Lieb (2012), QI.3.2. Prove the Cauchy-Hadamard formula, which states that the radius of convergence r is equal to

$$r = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}}$$

2. Fischer and Lieb (2012), QI.4.4. Show that the function $\tan z$ never takes on the values $\pm i$ and that therefore,

$$\frac{\mathrm{d}}{\mathrm{d}z}(\tan z) \neq 0$$

everywhere. Show that the tangent function maps the strip $S_0 = \{z : -\pi/2 < \operatorname{Re} z < \pi/2\}$ biholomorphically onto $\mathbb{C} \setminus \{it : t \in \mathbb{R},\ t \geq 1\}$.

Also use level sets to illustrate the conformal mapping.

3. Fix $a, b, c \in \mathbb{C}$ so that c is not a negative integer or 0. Show that the **hypergeometric** function

$$F(a, b, c; z) := \sum_{k=0}^{\infty} \frac{a(a+1)\cdots(a+k-1)b(b+1)\cdots(b+k-1)}{c(c+1)\cdots(c+k-1)} \frac{z^k}{k!}$$

converges on the unit disk and satisfies the differential equation

$$z(1-z)F''(z) + [c - (a+b+1)z]F'(z) - abF(z) = 0$$

Set B: Graded for Content

1. Fischer and Lieb (2012), QII.2.3. Compute the **Fresnel integrals**

$$\int_0^\infty \cos(x^2)\,\mathrm{d}x = \sqrt{\frac{\pi}{8}} = \int_0^\infty \sin(x^2)\,\mathrm{d}x$$

Hint: Apply the Cauchy integral theorem to sectors with center 0 and corners given by R and $\mathrm{e}^{i\pi/4}R$, where $R \to \infty$.

2. These problems illustrate the geometric intuition for the radius of convergence of a power series. Do the parts in order.

(a) For each nonzero natural number $n \in \mathbb{N}$, compute the power series expansion for the function $1/z$ around the point $1/n$. What are their radii of convergence?

(b) Describe the set of points $w \in \mathbb{C}$ such that the power series expansion for $1/z$ about w has radius of convergence equal to 1.

(c) Suppose that

$$f(z) = \frac{1}{z(z-1)(z-i)(z-1-i)}$$

Find the unique point w in the unit square $\{\operatorname{Re}(z), \operatorname{Im}(z) \in [0,1]\}$ such that the radius of convergence of the power series for w is maximal. Justify your answer.

3. This problem is to hint at the general formulation of the Cauchy integral theorem. Please solve this problem only using things we have seen in class to this point.

(a) Show that the "L-shaped" domain

$$L = \{z : \operatorname{Re}(z), \operatorname{Im}(z) \in (0,2) \text{ and not both } \operatorname{Re}(z), \operatorname{Im}(z) \in (0,1]\}$$

is star-shaped (hence the Cauchy integral theorem applies).

(b) Show that the "double L-shaped" domain

$$U = \{z : |\operatorname{Re}(z)|, \operatorname{Im}(z) \in (0,2) \text{ and not both } |\operatorname{Re}(z)|, \operatorname{Im}(z) \in (0,1]\}$$

is not star-shaped.

(c) Nevertheless, by breaking up U into two copies of L and using the Cauchy integral theorem for the resultant star-shaped domains, show that for any closed curve γ in U and any $f \in \mathcal{O}(U)$, we have that $\int_\gamma f \,\mathrm{d}z = 0$.

(d) Show that $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ can be written as the union of two star-shaped domains.

(e) Why doesn't your proof for part (c) show that $\int_\gamma f \,\mathrm{d}z = 0$ for any $f \in \mathcal{O}(\mathbb{C}^*)$ and any closed curve γ in $\mathbb{C}^*$?