Week 1

Classifying Complex Functions

1.1 Holomorphic Functions

3/19:

- We begin by reviewing some properties of the **complex numbers**.
- **Complex numbers**: The field of elements $z = x + iy$ where $x, y \in \mathbb{R}$ and $i^2 = -1$. *Denoted by* $\mathbb{C}$.

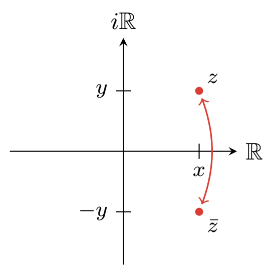


Figure 1.1: The complex plane.

  - Can be visualized as a two-dimensional plane with the number z corresponding to the point (x, y).
- **Real part**: The number x. *Denoted by* $\textbf{Re}\, z$.
- **Imaginary part**: The number y. *Denoted by* $\textbf{Im}\, z$.
- **Complex conjugate** (of z): The complex number defined as follows. *Denoted by* $\bar{z}$. *Given by*

$$\bar{z} := x - iy$$

- Now recall the definition of a *real* function that is **differentiable** at a point $x_0 \in \mathbb{R}$.
  - $f'(x_0)(x - x_0)$ is the "best linear approximation" of f near x_0, where $f'(x_0)$ is also defined below.
- **Differentiable** ($f : \mathbb{R} \to \mathbb{R}$ at x_0): A function f for which the following limit exists. *Constraint*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} =: f'(x_0)$$

- We now build up to defining a notion of complex differentiability.
  - Observe that the constraint above is equivalent to the constraint

$$f(x) = f(x_0) + [\underbrace{f'(x_0) + e(x)}_{\Delta(x)}](x - x_0)$$

    where $e(x) \to 0$ as $x \to x_0$.
  - Note that we are defining a new function $\Delta(x)$ above, with the property that $\Delta(x_0) = f'(x_0)$.

- **Holomorphic** (f at z_0): A function $f : \mathbb{C} \to \mathbb{C}$ for which the following limit exists. *Also known as* **$\mathbb{C}$-differentiable**. *Constraints*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} =: f'(z_0) \qquad \iff \qquad f(z) = f(z_0) + \Delta(z)(z - z_0)$$

  where Δ is continuous at z_0 and $\Delta(z_0) = f'(z_0)$.

  - It will turn out that this is the true definition of "holomorphic" / "$\mathbb{C}$-differentiable" function, not just a naïve first pass.

- Properties of holomorphic functions: Let $U \subset \mathbb{C}$ be open.
  1. The holomorphic functions on U form a ring $\boldsymbol{\mathcal{O}(U)}$.
     - Equivalently, the $\mathbb{C}$-differentiation operator is $\mathbb{C}$-linear.
     - Equivalently, if f, g are holomorphic, then $f + g$ and fg are holomorphic, too.
     - Equivalently (and most simply), we have the sum rule and the product rule (and the quotient rule if the function in the denominator is nonzero).
  2. We have the chain rule.
  3. Holomorphic implies continuous.

- Examples: Polynomials, rational functions $p(z)/q(z)$ (away from their **poles**).

- Non-example: Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$z \mapsto \bar{z}$$

  - Think-Pair-Share (TPS): Why?
  - Notice that

$$f(0) = 0 \qquad\qquad f(t) = t \qquad\qquad f(it) = -it$$

  - Thus,

$$\Delta(t) = 1 \qquad\qquad \Delta(it) = -1$$

    for all t.
  - But this means that Δ can't be continuous!
  - Yet f is clearly $\mathbb{R}$-differentiable! What gives?!
  - Note that — viewing f as a mapping of $\mathbb{R}^2 \to \mathbb{R}^2$ — we have

$$Df = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The above example suggests that our definition of complex differentiability may have been to naïve, so we'll do some further investigations now.

- Observe that $\mathbb{C} \cong \mathbb{R}^2$ as $\mathbb{R}$-vector spaces.

- **Differentiable** ($f : \mathbb{R}^2 \to \mathbb{R}^2$ at x_0): A function f for which there exists an $\mathbb{R}$-linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying the following constraint. *Constraint*

$$\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - Ah\|}{\|h\|} = 0$$

  - We also denote A by Df.

- Example: Consider the function $f : \mathbb{C} \to \mathbb{R}$ defined by

$$x + iy \mapsto x$$

  - Differentiable with total derivative

$$Df = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

- Observation: While $\mathbb{C} \cong \mathbb{R}^2$ as $\mathbb{R}$-vector spaces, as a $\mathbb{C}$-vector space, there is *additional* structure.
  - In particular, all "vectors" should commute with the "multiplication by i" map $J : \mathbb{C} \to \mathbb{C}$ defined by any one of the following three maps.

$$z \mapsto z \qquad\qquad x + iy \mapsto xi - y \qquad\qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

- Exercise: In $(\mathrm{Re}, \mathrm{Im})$ coordinates, write down the matrix for "multiply by w" for any $w \in \mathbb{C}$.
  - Let $w = a + bi$ and let $v = x + iy$. Then

$$\begin{aligned} wv &= (a + bi)(x + iy) = ax - by + i(bx + ay) \\ &= \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix} = \underbrace{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}}_{W} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

  - The matrix W above is the desired result.
- TPS: Is $f : \mathbb{C} \to \mathbb{C}$ defined as follows a complex linear map? Why not?

$$x + iy \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (x + y) + iy$$

  - Among other properties, a complex linear map should satisfy

$$if(x + iy) = f[i(x + iy)]$$

    for the scalar $i \in \mathbb{C}$.
  - However, we have that

$$if(x + iy) = i[(x + y) + iy] = -y + i(x + y) \neq (x - y) + ix = f(-y + ix) = f[i(x + iy)]$$

- What about the following map?

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

  - A complex linear map should satisfy

$$A(v + w) = Av + Aw \qquad\qquad \lambda Av = A(\lambda v)$$

    for all $v, w, \lambda \in \mathbb{C}$.
  - Let $v, w \in \mathbb{C}$ be arbitrary. Then

$$\begin{aligned} A(v + w) &= \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right] = \begin{pmatrix} (v_1 + w_1) + 2(v_2 + w_2) \\ -2(v_1 + w_1) + (v_2 + w_2) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = Av + Aw \end{aligned}$$

- Let $v, \lambda \in \mathbb{C}$. Then

$$\begin{aligned}\lambda A v &= (\lambda_1 + i\lambda_2) \cdot [(v_1 + 2v_2) + i(-2v_1 + v_2)] \\ &= [\lambda_1(v_1 + 2v_2) - \lambda_2(-2v_1 + v_2)] + i[\lambda_2(v_1 + 2v_2) + \lambda_1(-2v_1 + v_2)] \\ &= [(\lambda_1 v_1 - \lambda_2 v_2) + 2(\lambda_2 v_1 + \lambda_1 v_2)] + i[-2(\lambda_1 v_1 - \lambda_2 v_2) + (\lambda_2 v_1 + \lambda_1 v_2)] \\ &= A[(\lambda_1 v_1 - \lambda_2 v_2) + i(\lambda_2 v_1 + \lambda_1 v_2)] \\ &= A(\lambda v)\end{aligned}$$

- Therefore, since A satisfies the two properties, it is complex linear.

- Conclusion: To reiterate from the above, A must commute with J to be complex linear.
- Implication: Every $\mathbb{C}$-linear map of $\mathbb{C}$ is just multiplication by a complex number.
  - This is a special case of the following more general result, which holds for any field K.

$$\operatorname{Hom}_K(K, K) \cong K$$

- Now let's revisit differentiability.
- It turns out that a condition for $\mathbb{C}$-differentiability *equivalent* to the definition of "holomorphic" given above is that there exists a $\mathbb{C}$-linear map $A : \mathbb{C} \to \mathbb{C}$ such that

$$\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - Ah\|}{\|h\|} = 0$$

  - From the above discussion, we know that this A is just multiplication by some $w \in \mathbb{C}$.
  - All of the values in the above norms are complex numbers, so *another* equivalent condition is

$$\lim_{z \to z_0} \frac{|f(z) - f(z_0) - w \cdot (z - z_0)|}{|z - z_0|} = 0$$

  - This condition is wholly mathematically equivalent to our holomorphic definition,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = w$$

- So when is an $\mathbb{R}$-differentiable function actually holomorphic?
  - Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ map $(x, y) \mapsto (g, h)$.
  - Let

$$A = Df = \begin{pmatrix} g_x & g_y \\ h_x & h_y \end{pmatrix}$$

    where the subscript notation views g, for instance, as $g(x, y)$ and denotes the partial derivative of g with respect to x.
  - Let J (the "multiply by i") function be defined as above.
  - Then the "commute with i" condition is equivalent to

$$J^{-1} A J = A$$

  - Expanding the product on the left above in terms of g_x, g_y, h_x, h_y, we obtain

$$\begin{pmatrix} h_y & -h_x \\ -g_y & g_x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} g_x & g_y \\ h_x & h_y \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} g_x & g_y \\ h_x & h_y \end{pmatrix}$$

  - This condition is equivalent to A satisfying the **Cauchy-Riemann equations**.

- **Cauchy-Riemann equations**: The following two equations, which identify when a complex function is holomorphic. *Also known as* **CR equations**. *Given by*

$$g_x = h_y$$
$$g_y = -h_x$$

- These equations are satisfied when A is of the form

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

- So at this point, we can differentiate f with respect to z. But what if we want to differentiate it with respect to x and y (of $z = x + iy$)?
  - We will need the following change of basis.
    - Since $z = x + iy$ and $\bar{z} = x - iy$, we have

$$2x = z + \bar{z} \qquad\qquad 2iy = z - \bar{z}$$
$$x = \frac{1}{2}(z + \bar{z}) \qquad\qquad y = -\frac{i}{2}(z - \bar{z})$$

    - This tells us that

$$\frac{\partial x}{\partial z} = \frac{1}{2} \qquad\qquad \frac{\partial y}{\partial z} = -\frac{i}{2}$$

  - We can now invoke the multivariable chain rule and simplify the resultant expression.

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}(f_x - if_y)$$

    - Note that once again, the subscript notation "f_x" means $\partial f / \partial x$.
  - Note that we can also similarly work out that

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(f_x + if_y)$$

    - Observe in particular that

$$f_x = g_x + ih_x \qquad\qquad f_y = g_y + ih_y$$

    - Thus, the CR equations ($g_x = h_y$ and $g_y = -h_x$) being satisfied is equivalent to

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(f_x + if_y) = \frac{1}{2}[(g_x + ih_x) + i(g_y + ih_y)] = 0$$

    - Note that $\partial f / \partial \bar{z}$ is not actually a derivative since f depends on z, not $\bar{z}$. Rather, we use "$\partial f / \partial \bar{z}$" to denote the following operator applied to f.

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

- **Wirtinger derivatives**: The two differential operators defined as follows. *Denoted by* $\boldsymbol{\partial / \partial z, \partial / \partial \bar{z}}$. *Given by*

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \qquad\qquad \frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

- Theorem: The $\mathbb{R}$-differentiable function $f : U \to \mathbb{C}$ is holomorphic iff $\partial f / \partial \bar{z} = 0$. Moreover, if it is, then

$$f'(z_0) = \left.\frac{\partial f}{\partial z}\right|_{z_0}$$

1.2 Harmonic Functions and Conformal Maps

3/21:
- Tries to remember everyone's name and actually does a pretty good job!
- Has us all turn to our neighbor and meet them! I met Ryan.
- Review.
  - Naïve holomorphic definition: Typical derivative definition.
  - The map $z \mapsto \bar{z}$ is not holomorphic even though it is differential over the reals.
  - The reason this map is not holomorphic is that its matrix derivative is not complex linear. This means that it does not commute with the "multiply by i" matrix, defined by

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

  - Indeed, an *equivalent* definition to the naïve holomorphic one is: $f : \mathbb{C} \cong \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathbb{R}$-differentiable at z_0 with $Df(z)$ is complex linear.
  - Another equivalent one is the Cauchy-Riemann equation definition.
    - Let $f(z) = u(z) + iv(z)$ where $u, v : \mathbb{C} \to \mathbb{R}$.
    - Then f is holomorphic if $u_x = v_y$ and $v_x = -u_y$, or equivalently if $\partial f/\partial\bar{z} = 0$.
  - The above comment motivates the definition of the operators

$$\frac{\partial}{\partial\bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

  - Note: Every $\mathbb{C}$-linear map is "multiply by w" for some $w \in \mathbb{C}$.
  - Note that we have not yet talked about continuity or related things.
- Note: Different books use different conventions.
  - "Holomorphic at a point" and "complex differentiable in a neighborhood of a point" may mean different things.
  - Example: Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$x + iy \mapsto x^2 + iy^2$$

    - Then
$$Df = \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix}$$
    - Evidently, Df is not complex linear in general because we can pick x, y such that $2x \neq 2y$.
    - Thus, this function is not complex differentiable in general.
    - However, it is complex differentiable at zero because here, $Df = 0$.
    - Thus, this function is complex differentiable *at a point*, but not complex differentiable *in a neighborhood*.
  - We will almost always be talking about functions that are complex differentiable *in a neighborhood* in this class.
- Example: $f(x + iy) = x^2\mathbb{I}_{Q(x)} + iy^2\mathbb{I}_{Q(y)}$ is complex differentiable in a neighborhood of the origin, but this is dumb. $\mathbb{I}$ is the **indicator function**.
- Preview (we'll see this next Thursday): Holomorphic implies C^∞.

- Today: Some more things about the Cauchy-Riemann equations and what we can get out of them.
- Let's begin with a consequence of the $\partial/\partial z$ and $\partial/\partial \bar{z}$ operators.
  - Compute (if $f \in C^2$):
$$\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4} \underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)}_{\Delta} f_{xy} = f_{yx}$$
    - Do we need this C^2 condition if holomorphic already implies C^∞?
    - No, but we haven't "learned" this yet. Once we prove this, no more talk of regularity!
  - Solutions to this, the Laplacian Δ (from physics), could be a good final project!
  - Look for solutions to $\Delta f = 0$.
    - Equivalently, look for f such that $f_{xx} + f_{yy} = 0$.
  - Observation: Any f holomorphic implies that $\Delta f = 0$ (since we apply $\partial/\partial \bar{z}$ to f first).
- **Harmonic** (function): A function $f : \mathbb{R}^2 \to \mathbb{C}$ such that $\Delta f = 0$.
- Since the differentiation operator is linear,
$$\Delta(u + iv) = \Delta u + i\Delta v$$
- Corollary: The real and imaginary parts of a C^2 holomorphic function are harmonic.
- So we know that f holomorphic implies u, v real-valued and harmonic. Can we go the other way?
  - We know that these functions have certain properties in terms of their partial derivatives, namely that they satisfy the Cauchy-Riemann equations.
- **Harmonic conjugates**: Two functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ that satisfy the CR equations.
- Fact: Let u, v be two C^∞ harmonic conjugates. Then $f = u + iv$ is holomorphic
  - Easy to prove if you're not concerned about regularity.
- Is $v + iu$ holomorphic?
  - No, partials don't work out. We still get $v_x = -u_y$, but we also get $u_x = -v_y$.
  - However, $v - iu$ is holomorphic!
  - This just means that rotating by i gives us a new holomorphic function since
$$i \cdot (u + iv) = -v + iu$$
- Example: $u = x^2 - y^2$ is harmonic. Find a conjugate and find $f = u + iv$.[1]
  - We have
$$\begin{aligned} v_y &= u_x = 2x \\ v_x &= -u_y = 2y \end{aligned}$$
  - Thus,
$$v = 2xy + C$$
    for some $C \in \mathbb{C}$
  - Then we would have
$$f = u + iv = (x^2 - y^2) + i(2xy + C) = x^2 + 2xyi - y^2 + iC = (x + iy)^2 + iC = z^2 + iC$$

[1] Calderon actually let us work this out in class!

- Let's now talk about integrating functions.
- Let $a, b \in \mathbb{R}$. Consider $f : \mathbb{C} \to \mathbb{C}$, not holomorphic but continuous. How do we take $\int_a^b f \, \mathrm{d}z$?

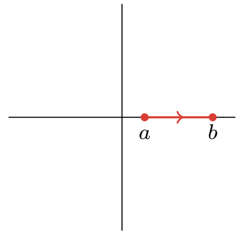


Figure 1.2: Integrating complex functions over real intervals.

  - What we do is just split the integral into real and imaginary parts.

$$\int_a^b f \, \mathrm{d}z = \int_a^b u \, \mathrm{d}t + i \int_a^b v \, \mathrm{d}t$$

  - This is how we integrate between reals in the complex plane.
- How do we integrate over more arbitrary points in the complex plane, e.g., $a\lambda$ and $b\lambda$?

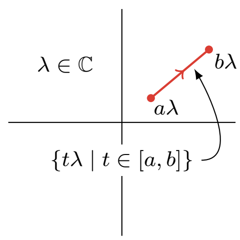


Figure 1.3: Integrating complex functions over line segments in $\mathbb{C}$.

  - We could take any path. Which one?
  - Try over the line segment $\{t\lambda \mid t \in [a, b]\}$.
  - Then we take

$$\int_{a\lambda}^{b\lambda} f(z) \, \mathrm{d}z = \int_a^b f(\lambda t)\lambda \, \mathrm{d}t$$

    via the substitutions $z = t\lambda$ and $\mathrm{d}z = \lambda \, \mathrm{d}t$.
  - This second integral, we can compute in the first way.
- Now what about integrating along an arbitrary curve $\gamma : [a, b] \to \mathbb{C}$, which we will say is piecewise C^1?

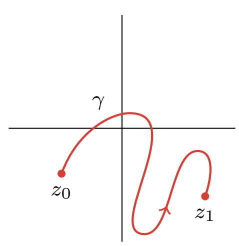
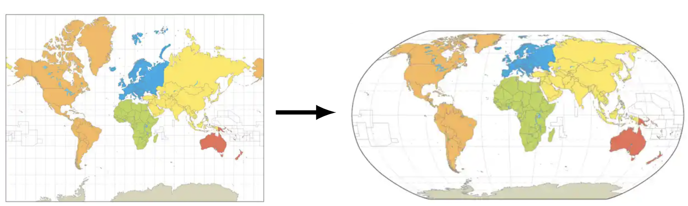


Figure 1.4: Integrating complex functions over arbitrary paths in $\mathbb{C}$.

- Note that $z_0 := \gamma(a)$ and $z_1 := \gamma(b)$.
- Define
$$\int_\gamma f \, \mathrm{d}z = \int_a^b f(\gamma(t)) \cdot \gamma'(t) \, \mathrm{d}t$$
- Where is the γ' coming from? Same place as before! It's just a change of variables: $z = \gamma(t)$ implies $\mathrm{d}z = \gamma'(t) \, \mathrm{d}t$.
- If we know differential forms, $f \, \mathrm{d}z$ is just a complex-valued one-form. And the chain rule is just how we integrate one-forms.
- We'll do lots of basic practice of this in the completion problems on the PSet.

- Note: Whenever we do a path integral, we should ask if the parameterization matters. The parameterization does *not* matter.
- What do we need to compute integrals without having to take the limit of a sum over partitions? We need the fundamental theorem of calculus.
  - The FTC does indeed hold here, too, though we won't prove this.
- FTC: Suppose $F' = f$ on $U \subset \mathbb{C}$, and let γ be a **path** inside of U. Then
$$\int_\gamma f \, \mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$$
  - Again, if we like differential forms, then note that we're just saying in the above equality that
$$\int_\gamma \mathrm{d}F = \int_{\partial\gamma} F$$
- **Path**: A function from an interval of real numbers to a vector space. *Also known as* **contour**. *Denoted by* $\boldsymbol{\gamma}$.
- Gives us a three-minute break from 10:17-10:20 in the middle of the class.
  - The fact that this guy actually teaches in accordance with accepted pedagogical standars is wild.
- How do we want to visualize holomorphic functions?
  - $f : \mathbb{C} \to \mathbb{C}$ is hard to graph because the set of points lives in $\mathbb{R}^4$.
  - So we're out of luck if we want to do graphs.
  - Thus, we'll look at **mappings**.
- Example: Are we looking at the Mercator or Robinson map of the world?



(a) Mercator projection. (b) Robinson projection.

Figure 1.5: Visualizing functions of $\mathbb{R}^2 \to \mathbb{R}^2$.

  - What do these mappings do to the lines of latitude and longitude?
  - This is a mapping of $\mathbb{R}^2 \to \mathbb{R}^2$ that stretches and distorts! By drawing grid lines, we can see what it does to $\mathbb{R}^2$.

- Now recall that f holomorphic implies Df looks like

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

  - What's nice about these matrices is they can always be factored into rotation and scaling matrices.

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \qquad (\lambda, \theta \in \mathbb{R})$$

  - This means that

$$f'(z_0) = w = r\mathrm{e}^{i\theta} \in \mathbb{C}$$

    since multiplication by complex numbers also just rotates and scales!
  - This means that if $z' = r'\mathrm{e}^{i\theta'}$, then $w \cdot z' = r \cdot r'\mathrm{e}^{i(\theta+\theta')}$.
  - This also means that we may have rotation and scaling but no shearing. Formally, we have the following lemma.

- **Argument** (of $z \in \mathbb{C}$): The angle θ such that $z = r\mathrm{e}^{i\theta}$ for some $r \in \mathbb{R}$. *Denoted by* $\boldsymbol{\arg(z)}$.

- Lemma: Suppose two curves γ, δ intersect at a point $z \in \mathbb{C}$. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. Then

$$\angle_z(\gamma, \delta) = \angle_{f(z)}(f(\gamma), f(\delta))$$

  i.e., f preserves angles.

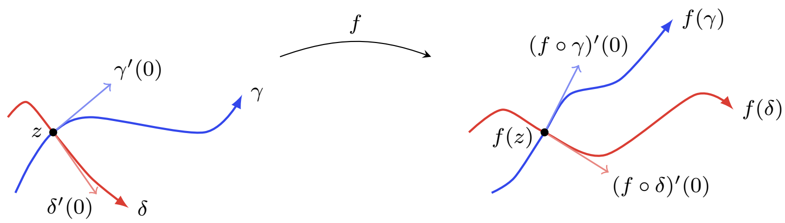


Figure 1.6: Holomorphic maps preserve angles.

Proof. To consider the angle between two curves analytically, let's look at the tangent vectors to the two curves, for example at z. Now while we often think of $\gamma'(0)$ as a *matrix*, remember that we've proven that all of these matrices are equivalent to complex numbers. In particular, since $\gamma : \mathbb{R} \to \mathbb{R}^2 \cong \mathbb{C}$, the total derivative will just be a vector. This vector may easily be represented as a complex number $re^{i\theta}$ in polar coordinates. Similarly, $\delta'(0)$ can be thought of as a complex number $r'\mathrm{e}^{i\theta'}$. Thus, dividing these quantities gives us the angle $\theta - \theta'$ between the two vectors, which we can isolate using the argument function.

Doing the same to the curves at $f(z)$ yields

$$\begin{aligned} \angle_{f(z)}(f(\gamma), f(\delta)) &= \arg\left[\frac{(f \circ \gamma)'(0)}{(f \circ \delta)'(0)}\right] \\ &= \arg\left[\frac{f'(\gamma(0)) \cdot \gamma'(0)}{f'(\gamma(0)) \cdot \delta'(0)}\right] \\ &= \arg\left[\frac{f'(z) \cdot \gamma'(0)}{f'(z) \cdot \delta'(0)}\right] \\ &= \arg\left[\frac{\gamma'(0)}{\delta'(0)}\right] \\ &= \angle_z(\gamma, \delta) \end{aligned}$$

as desired. $\square$

  - Calderon gave us 5 minutes to try to compute this ourselves with the hint: Use the chain rule! $\angle_z(\gamma, \delta) = \arg(\gamma'(0) \cdot [\delta'(0)]^{-1})$.
- **Conformal** (map): A function $f : U \to V$, where $U, V \subset \mathbb{C}$, that satisfies the following two constraints. *Constraints*
  1. f is a diffeomorphism.
  2. f preserves angles.
- **Diffeomorphism**: A homeomorphism for which f, f^{-1} are differentiable.
- **Biholomorphic** (map): A function $f : U \to V$ that is bijective, holomorphic, and for which f^{-1} is holomorphic.
- Theorem/observation: Biholomorphic iff conformal.

  Proof. Follows straight from the definitions and the lemma we just proved. □
- Calderon shows us an applet.
  - We can use the applet to help with the PSet, but we still do have to submit actual proofs.
  - Allows you to visually see the lemma for instance.
  - Example: Under $z \mapsto z^2$, the sector of radius 2 and argument $\pi/4$ goes to the sector of radius $2^2 = 4$ and argument $\pi/2$.

1.3 Chapter I: Analysis in the Complex Plane

From Fischer and Lieb (2012).

3/19:
- The preface only contains comments and instructions for an instructor planning to use this textbook for a course.
- The chapter begins with two paragraphs (as do all the others).
  - The first discusses topic covered in the chapter.
  - The second gives some historical background on these topics.

Section I.0: Notations and Basic Concepts

- Goal: Review the fundamental topological and analytical concepts of real analysis.
- Defines the **complex numbers**, **complex plane**, and **complex conjugate**.
- **Absolute value** (of z): The Euclidean distance of z from zero. *Also known as* **modulus**. *Denoted by* $\boldsymbol{|z|}$. *Given by*
$$|z| := \sqrt{x^2 + y^2}$$
- **Imaginary unit**. *Denoted by* $\boldsymbol{i}$.
- Relating the modulus and complex conjugate.
$$|z| = \sqrt{z\bar{z}}$$
- **Open disk** (of radius ε and center z_0): The set defined as follows. *Also known as* $\boldsymbol{\varepsilon}$**-neighborhood** (of z_0). *Denoted by* $\boldsymbol{D_\varepsilon(z_0)}$, $\boldsymbol{U_\varepsilon(z_0)}$. *Given by*
$$D_\varepsilon(z_0) = U_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$$

- **Unit disk**: The set defined as follows. *Denoted by* $\mathbb{D}$. *Given by*

$$\mathbb{D} := D_1(0)$$

- **Unit circle**: The set defined as follows. *Denoted by* $\mathbb{S}$. *Given by*

$$\mathbb{S} := \{z \in \mathbb{C} : |z| = \varepsilon\}$$

- **Upper half plane**: The set defined as follows. *Denoted by* $\mathbb{H}$. *Given by*

$$\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

- $\mathbb{C}^*$: The set defined as follows. *Given by*

$$\mathbb{C}^* := \mathbb{C} \setminus \{0\}$$

3/21:

- **Neighborhood** (of z_0): A set U which contains an ε-neighborhood.
- **Open** (set): A set that is a neighborhood of each of its points.
- **Closed** (set): A complement of an open set.
- **Interior** (of M): The largest open set contained in M. *Denoted by* $\mathring{M}$.
- **Closure** (of M): The smallest closed set containing M. *Denoted by* $\overline{M}$.
- **Topological boundary** (of M): The set defined as follows. *Also known as* **boundary**. *Denoted by* ∂M. *Given by*

$$\partial M := \overline{M} \setminus \mathring{M}$$

- **Relatively open** (set in M): The intersection of an open set U with an arbitrary set M. *Also known as* **open** (set in M).
- **Relatively closed** (set in M): The intersection of a closed set U with an arbitrary set M. *Also known as* **open** (set in M).

4/4:

- Fischer and Lieb (2012) define **convergent series** and their **limits** in the usual way.
  - Sum, product, and reciprocal rules stated.
- **Accumulation point** (of $M \subset \mathbb{C}$): A point $z_0 \in \mathbb{C}$ for which there is a sequence $\{z_n\} \subset M \setminus \{z_0\}$ with $\lim z_n = z_0$.
- **Bounded** (set): A set K for which $|z| \leq R$ for some R and all $z \in K$.
- **Compact** (set): A closed, bounded set.
  - Each sequence in a compact set contains convergent subsequences with limit in the set.
- **Relatively compact** (set in V): A set U for which $\overline{U}$ is a compact subset of V. *Denoted by* $U \subset\subset V$.
- Definition of a **complex-valued function**.
- **Continuous** (function at z_0): A function $f : U \to \mathbb{C}$ such that for each neighborhood M of $w_0 = f(z_0)$, there is a neighborhood N of z_0 with $f(U \cap N) \subset M$; equivalently, the following holds true for all convergent sequences $\{z_n\} \subset U$.

$$f\left(\lim_{n\to\infty} z_n\right) = \lim_{n\to\infty} f(z_n)$$

- **Real part** (of $f : U \to \mathbb{C}$): The real-valued function g such that $f = g + ih$.

- **Imaginary part** (of $f : U \to \mathbb{C}$): The complex-valued function h such that $f = g + ih$.
- Continuity results.
  - f is continuous iff its real and imaginary parts are.
  - The **composition** of continuous functions is continuous.
  - Example of a continuous function: Any polynomial in $z, \bar{z}$, i.e., any function of the form

$$f(z) = \sum_{j,k=0}^{N} a_{jk} z^j \bar{z}^k$$

- **Path**: A continuous map from a closed finite interval into the complex plane. *Denoted by* $\boldsymbol{\gamma : [a, b] \to \mathbb{C}}$.
  - We say that γ **connects** its **initial point** and **end point**.
- **Trace** (of a path): The image set $\gamma([a, b])$. *Denoted by* $\mathbf{tr}\,\boldsymbol{u}$.
- **Initial point** (of a path): The point $\gamma(a)$.
- **End point** (of a path): The point $\gamma(b)$.
- **Connected** (set): A set U for which any two points of U can be connected by a path whose trace lies in U. *Also known as* **pathwise connected**.
- **Domain**: A connected open set.
  - "An open set U is a domain if and only if no decomposition of U into disjoint nonempty subsets exists" (Fischer & Lieb, 2012, p. 3).
- Images of compact (resp. connected) sets under continuous functions are again compact (resp. connected).
- Images of open (resp. closed) sets under continuous functions are *not necessarily* open (resp. closed).
- Preimages of open (resp. closed) sets under continuous functions are again open (resp. closed).
- Note that the above definitions make sense in higher-dimensional vector spaces upon replacing the absolute value with the **Euclidean norm**.

Section I.1: Holomorphic Functions

4/5:

- Definition of **holomorphic**.
- Examples and nonexamples of holomorphic functions: Constant, identity, and complex conjugate functions.
- Further comments on the complex conjugate function.
  - It's everywhere continuous but nowhere complex differentiable.
  - It's very hard to find real examples of such functions, but we found a complex example like that!
- For the basic properties of complex differentiation (e.g., holomorphic implies continuous, sum and product rules), the proofs are symmetric to the real ones.
- $\mathcal{O}(U)$ is a ring, but it is more technically a $\mathbb{C}$-algebra.
- **Chain rule**: If $f : U \to V$ and $g : V \to \mathbb{C}$ are mappings of open subsets of $\mathbb{C}$ that are holomorphic at $z_0 \in U$ and $f(z_0) = w_0 \in V$, respectively, then $g \circ f : U \to \mathbb{C}$ is holomorphic at z_0 and

$$(g \circ f)'(z_0) = g'(w_0) f'(z_0)$$

- Definition of **biholomorphic**.
- Fischer and Lieb (2012) prove some elementary properties of complex functions in places with nonzero derivatives (e.g., locally one-to-one).
  - This leads into the **complex inverse function theorem**.
  - Preview: This leads to the result that "bijective holomorphic maps are biholomorphic," i.e., if we have a bijective holomorophic map, we don't also need to prove that f^{-1} is holomorphic (Fischer & Lieb, 2012, p. 6).

Section I.2: Real and Complex Differentiability

- Definition of $\mathbb{R}^2$**-differentiable function**.
- Definition of **Wirtinger derivatives**.
  - Fischer and Lieb (2012) arrive here from a slightly different angle than in class.
- Fischer and Lieb (2012) prove the $\partial f/\partial\bar{z} = 0$ condition.
- **Cauchy-Riemann operator**: The differential operator defined as follows. *Denoted by* $\boldsymbol{\partial/\partial\bar{z}}$. *Given by*

$$\frac{\partial}{\partial\bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

- "In requiring that a function be complex differentiable we thus simultaneously require that a certain partial differential equation be satisfied. In other words, *holomorphic functions are the differentiable solutions of the Cauchy-Riemann equation*" given as follows (Fischer & Lieb, 2012, p. 9).

$$\frac{\partial f}{\partial\bar{z}}(z) = 0$$

- **Laplace operator**: The real differential operator defined as follows. *Also known as* **Laplacian**. *Denoted by* $\boldsymbol{\Delta}$. *Given by*

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

  - As a real differential operator,

$$\Delta\bar{f} = \overline{\Delta f}$$

  - More on this operator in Section VII.6.
- Definition of a **harmonic function**, the **Cauchy-Riemann equations** (in real form).
- On the CR equations: "The gradients and, consequently, the equipotential lines of g and h are thus perpendicular to one another at all points where the gradients do not vanish" (Fischer & Lieb, 2012, p. 10).
- We now investigate a special case of the chain rule for the Wirtinger derivatives.
- Lemma 2.4: Let $f : U \to \mathbb{C}$ be a real differentiable function defined on an open set $U \subset \mathbb{C}$ and let $w : [a, b] \to U$ be a differentiable map (i.e., a differentiable path in U). Then for all $t \in [a, b]$,

$$\frac{\partial}{\partial t}(f \circ w)(t) = \begin{pmatrix} f_z(w(t)) & f_{\bar{z}}(w(t)) \end{pmatrix} \begin{pmatrix} \dot{w}(t) \\ \dot{\bar{w}}(t) \end{pmatrix} = f_z(w(t))\dot{w}(t) + f_{\bar{z}}(w(t))\dot{\bar{w}}(t)$$

  Here we denote $\partial w/\partial t$ by $\dot{w}$ and use matrix multiplication.

  Proof. Left to the reader. ☐

- **Complex linear map**: A map $l : \mathbb{C} \to \mathbb{C}$ characterized by the following. *Constraints*
  1. $l(z+w) = l(z) + l(w)$;
  2. $l(rz) = rl(z)$;

  for $z, w, r \in \mathbb{C}$.
  - Every complex linear map is of the form
    $$w = l(z) = az$$
    for a unique $a \in \mathbb{C}$.
- **Real linear map**: A map $l : \mathbb{C} \to \mathbb{C}$ characterized by the following. *Constraints*
  1. $l(z+w) = l(z) + l(w)$;
  2. $l(rz) = rl(z)$;

  for $z, w \in \mathbb{C}$ and $r \in \mathbb{R}$.
  - Every real linear map is of the form
    $$w = l(z) = az + b\bar{z} = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \end{pmatrix}$$
    for a unique pair $\begin{pmatrix} a & b \end{pmatrix} \in \mathbb{C}^2$.
  - Implication: l is complex linear iff $b = 0$.
- **Tangent map** (of f at z_0): The real linear map from $\mathbb{C} \to \mathbb{C}$ determined by the vector $\begin{pmatrix} f_z(z_0) & f_{\bar{z}}(z_0) \end{pmatrix}$.
- Proposition: f is holomorphic at z_0 iff its tangent map at z_0 is complex linear.
- Fischer and Lieb (2012) discuss angle preservation, per Figure 1.6.