4 Modulus Principles, Meromorphicity, and Möbius Transforms

Set A: Graded for Completion

5/3: 1. Fischer and Lieb (2012), QIII.1.1.

(a) Determine the order of the zero of $\sum_{n=1}^{k} b_n(z-z_0)^{-n}$ at ∞.

Proof. Define

$$f^*(z) = f(1/z) = \sum_{n=1}^{k} \frac{b_n}{(\frac{1}{z}-z_0)^n} = \sum_{n=1}^{k} \frac{b_n z^n}{(1-z_0z)^n} = z^1 \underbrace{\sum_{n=1}^{k} \frac{b_n z^{n-1}}{(1-z_0z)^n}}_{h(z)}$$

Therefore, since $h(0) \neq 0$, f has a zero of order $\boxed{1}$ at ∞. □

(b) For the following functions, determine the value $w_0 = f(\infty)$ and its multiplicity.

$$f(z) = \frac{2z^4 - 2z^3 - z^2 - z + 1}{z^4 - z^3 - z + 1} \qquad f(z) = \frac{z^4 + iz^3 + z^2 + 1}{z^4 + iz^3 + z^2 - iz}$$

Proof. Use consecutive applications of L'Hôpital's rule:

$$\lim_{z\to\infty} f(z) = \frac{2\cdot 4!}{4!} = 2$$

Therefore, for the left function above,

$$\boxed{w_0 = f(\infty) = 2}$$

Define

$$g(z) = f(z) - w_0 = \frac{2z^4 - 2z^3 - z^2 - z + 1}{z^4 - z^3 - z + 1} - \frac{2z^4 - 2z^3 - 2z + 2}{z^4 - z^3 - z + 1} = \frac{-z^2 + z - 1}{z^4 - z^3 - z + 1}$$

Then

$$g^*(z) = \frac{-\frac{1}{z^2} + \frac{1}{z} - 1}{\frac{1}{z^4} - \frac{1}{z^3} - \frac{1}{z} + 1} = \frac{-z^2 + z^3 - z^4}{1 - z - z^3 + z^4} = z^2 \cdot \frac{-1 + z - z^2}{1 - z - z^3 + z^4}$$

Thus w_0 has multiplicity $\boxed{2.}$ □

2. Fischer and Lieb (2012), QIII.3.1. Let f and g be entire functions such that $|f| \leq |g|$. Show that $f = cg$ for some constant c.

Proof. If $|f| \leq |g|$, then

$$\left|\frac{f}{g}\right| \leq 1$$

Thus, f/g is bounded and entire, so it must be constant by Liouville's theorem. But if $f/g = c$, then $f = cg$, as desired.

We do need a proof that f/g is entire, which means considering the behavior of f/g at any point that $g(z) = 0$. □

3. Fischer and Lieb (2012), QIII.4.1.

(a) Let $S, T \in$ Möb. Show that a point $z_1 \in \hat{\mathbb{C}}$ is a fixed point of T if and only if Sz_1 is a fixed point of STS^{-1}.

Proof. Suppose z_1 is a fixed point of T. Then $Tz_1 = z_1$. Thus,

$$STS^{-1}(Sz_1) = STz_1 = Sz_1$$

as desired.

Suppose Sz_1 is a fixed point of STS^{-1}. Then

$$\begin{aligned} STS^{-1}(Sz_1) &= Sz_1 \\ STz_1 &= Sz_1 \end{aligned}$$

Applying S^{-1} to both sides of the above yields the desired result. □

(b) Suppose T has exactly one fixed point z_1. Show that there is an $S \in$ Möb such that STS^{-1} is a translation. Moreover, show that for every $z \in \hat{\mathbb{C}}$, we have

$$\lim_{n\to\infty} T^n z = z_1$$

where $T^n = T \circ \cdots \circ T$ denotes the n-fold composition of T with itself.

(c) Suppose T has exactly two fixed points z_1 and z_2. Show that there is an $S \in$ Möb such that STS^{-1} is of the form $z \mapsto az$, where $a \in \mathbb{C}^*$, and that the pair $\{a, a^{-1}\}$ is uniquely determined by T.

(d) Show that if we have $|a| \neq 1$ in part (c), then after a possible renumbering of our fixed points, we have

$$\lim_{n\to\infty} T^n z = z_1$$

for all $z \in \hat{\mathbb{C}} \setminus \{z_2\}$. In the case that $|a| = 1$, show that every point in $\hat{\mathbb{C}} \setminus \{z_1, z_2\}$ lies on a T-invariant Möbius circle.

4. Fischer and Lieb (2012), QIII.5.1. Let f be the branch of the logarithm on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$ that takes the value $-i\pi/2$ at $-i$. Determine...

$$f(i) \qquad f(-\mathrm{e}) \qquad f(-1 - i\sqrt{3}) \qquad f((-1 - i\sqrt{3})^2)$$

Proof.

$$f(i) = -\frac{3\pi i}{2} \qquad f(-\mathrm{e}) = 1 - i\pi$$

□

5. If z_1 and z_2 are related by inversion in a circle C, and z_3 and z_4 are arbitrary (distinct) points of C, show that the cross ratio of the four points has modulus 1.

Set B: Graded for Content

1. Fischer and Lieb (2012), QIII.3.4. Consider the function $f(z) = z + \mathrm{e}^z$. Show that for all $t \in [0, 2\pi]$,

$$\lim_{r\to\infty} f(r\mathrm{e}^{it}) = \infty$$

and that the convergence is uniform with respect to t on the sets $\{t : |t - \pi| \leq \frac{\pi}{2}\}$ and $\{t : |t| \leq \alpha\}$ for every $\alpha < \pi/2$. How does this agree with Proposition 3.4?

2. Fischer and Lieb (2012), QIII.5.2.

(a) Find a maximal domain on which holomorphic functions $\log(1 - z)^2$ and $\sqrt{z + \sqrt{z}}$, respectively, can be defined.

(b) Show that a logarithm of the tangent function exists on the set

$$G = \mathbb{C} \setminus \bigcup_{k\in\mathbb{Z}} [k\pi - \tfrac{\pi}{2}, k\pi]$$

3. Show that a fractional linear transformation

$$z \mapsto \frac{az+b}{cz+d}$$

maps the upper half plane to itself if and only if $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$.

4. Suppose that U is a domain, $f \in \mathcal{O}(U)$ is never zero, and suppose that a holomorphic branch of the logarithm exists on $f(U)$; then the function $\log[f(z)]$ is holomorphic. By considering the real part of $\log f$, show that the maximum modulus principles for harmonic functions and for holomorphic functions are equivalent.