MATH 27000 (Basic Complex Variables) Problem Sets

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Contents

1 Holomorphicity

Set A: Graded for Completion

3/29: 1. Fischer and Lieb (2012), QI.2.1. Formulate and prove the chain rule for Wirtinger derivatives. Furthermore, show that

$$\overline{\frac{\partial f}{\partial z}} = \frac{\partial \bar{f}}{\partial \bar{z}}$$

Proof. Extrapolating from Fischer and Lieb (2012, p. 11), the full chain rule for the Wirtinger derivatives would be

$$\boxed{\frac{\partial}{\partial t}(f \circ g)(z) = f_z(g(z))g_z(z) + f_{\bar{z}}(g(z))\bar{g}_{\bar{z}}(z)}$$

Additionally, since the complex conjugate of the sum or product of two complex numbers is the sum or product of the complex conjugates, we have that

$$\begin{aligned}
\overline{\frac{\partial f}{\partial z}} &= \overline{\frac{1}{2}(f_x + if_y)} \\
&= \frac{1}{2}(\bar{f}_x + \overline{if_y}) \\
&= \frac{1}{2}(\bar{f}_x - i\bar{f}_y) \\
&= \frac{\partial \bar{f}}{\partial \bar{z}}
\end{aligned}$$

□

2. Let $\langle (x_1, y_1), (x_2, y_2) \rangle = x_1x_2 + y_1y_2$ denote the usual inner product on $\mathbb{R}^2$. We can also define the **Hermitian inner product** on $\mathbb{C}$ via

$$(z, w) = z\bar{w}$$

This term *Hermitian* describes the fact that this product is not symmetric, but satisfies $(w, z) = \overline{(z, w)}$. Show that thinking of z as $x + iy$, we have

$$\langle z, w \rangle = \frac{1}{2}[(z, w) + (w, z)] = \operatorname{Re}(z, w)$$

Proof. We have that

$$\begin{aligned}
\frac{1}{2}[(z, w) + (w, z)] &= \frac{1}{2}[z\bar{w} + w\bar{z}] \\
&= \frac{1}{2}[(z_1 + iz_2)(w_1 - iw_2) + (w_1 + iw_2)(z_1 - iz_2)] \\
&= \frac{1}{2}[(z_1w_1 + z_2w_2 + i(z_2w_1 - z_1w_2)) + (w_1z_1 + w_2z_2 + i(w_2z_1 - w_1z_2))] \\
&= z_1w_1 + z_2w_2 \\
&= \langle (z_1, z_2), (w_1, w_2) \rangle \\
&= \langle z, w \rangle
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Re}(z, w) &= \operatorname{Re}(z\bar{w}) \\
&= \operatorname{Re}[(z_1 + iz_2)(w_1 - iw_2)] \\
&= \operatorname{Re}[z_1w_1 + z_2w_2 + i(z_2w_1 - z_1w_2)] \\
&= z_1w_1 + z_2w_2 \\
&= \langle (z_1, z_2), (w_1, w_2) \rangle \\
&= \langle z, w \rangle
\end{aligned}$$

as desired. □

3. For any integer n, compute the line integral $\int_\gamma z^n \,\mathrm{d}z$ where γ is any circle centered at the origin with counterclockwise orientation. Do not use Cauchy's theorem.

Proof. To evaluate such a line integral over a circle centered at the origin with counterclockwise orientation, we may use the parameterization $\gamma : [0, 2\pi) \to \mathbb{C}$ defined by

$$\gamma(t) = a\mathrm{e}^{it}$$

where a is an arbitrary positive real number. Thus, since $\gamma'(t) = ai\mathrm{e}^{it}$, we have the following when $n \neq -1$.

$$\begin{aligned}
\int_\gamma z^n \,\mathrm{d}z &= \int_0^{2\pi} (a\mathrm{e}^{it})^n \cdot ai\mathrm{e}^{it} \,\mathrm{d}t \\
&= a^{n+1} i \int_0^{2\pi} \mathrm{e}^{i(n+1)t} \,\mathrm{d}t \\
&= a^{n+1} i \left[\frac{\mathrm{e}^{i(n+1)t}}{i(n+1)} \right]_0^{2\pi} \\
&= a^{n+1} i \left[\frac{1}{i(n+1)} - \frac{1}{i(n+1)} \right] \\
\boxed{\int_\gamma z^n \,\mathrm{d}z = 0} & \qquad\qquad (n \neq -1)
\end{aligned}$$

When $n = -1$, we have

$$\begin{aligned}
\int_\gamma \frac{1}{z} \,\mathrm{d}z &= \int_0^{2\pi} a^{-1}\mathrm{e}^{-it} \cdot ai\mathrm{e}^{it} \,\mathrm{d}t \\
&= \int_0^{2\pi} i \,\mathrm{d}t \\
\boxed{\int_\gamma z^{-1} \,\mathrm{d}z = 2\pi i} &
\end{aligned}$$

□

4. Without using Cauchy's theorem, show that for any $|a| < 1 < |b|$,

$$\int_\gamma \frac{1}{(z-a)(z-b)} \,\mathrm{d}z = \frac{2\pi i}{a-b}$$

where γ is the circle of radius 1 centered about the origin, oriented counterclockwise.

Proof. Using the method of partial fractions, we set

$$\frac{0z+1}{(z-a)(z-b)} = \frac{A}{z-a} + \frac{B}{z-b} = \frac{A(z-b)+B(z-a)}{(z-a)(z-b)} = \frac{(A+B)z + (-Ab-Ba)}{(z-a)(z-b)}$$

to obtain the two-variable, two-equation system

$$\begin{aligned}
0 &= A + B \\
1 &= -Ab - Ba
\end{aligned}$$

with solution

$$A = \frac{1}{a-b} \qquad\qquad B = \frac{1}{b-a}$$

Thus,

$$\int_\gamma \frac{1}{(z-a)(z-b)}\,\mathrm{d}z = \frac{1}{a-b}\left[\int_\gamma \frac{1}{z-a}\,\mathrm{d}z - \int_\gamma \frac{1}{z-b}\,\mathrm{d}z\right]$$

We will evaluate the left integral first, followed by the right one.

Let's parameterize the circle by $\gamma : [0, 2\pi) \to \mathbb{C}$ defined by $t \mapsto \mathrm{e}^{it}$. Since $|a| < 1$ by hypothesis and $|\gamma(t)| = 1$ for all $t \in [0, 2\pi)$, we know that $|a/z| = |a/\gamma(t)| < 1$; this will allow us to replace a certain expression with the corresponding convergent geometric power series.. Thus, we have that

$$\begin{aligned}\int_\gamma \frac{1}{z-a}\,\mathrm{d}z &= \int_\gamma \frac{1}{z}\frac{1}{1-a/z}\,\mathrm{d}z \\ &= \int_\gamma \frac{1}{z}\sum_{k=0}^\infty \left(\frac{a}{z}\right)^k \mathrm{d}z \\ &= \int_\gamma \sum_{k=0}^\infty \frac{a^k}{z^{k+1}}\,\mathrm{d}z \\ &= \sum_{k=0}^\infty \int_\gamma \frac{a^k}{z^{k+1}}\,\mathrm{d}z \\ &= \sum_{k=0}^\infty a^k \int_\gamma z^{-(k+1)}\,\mathrm{d}z\end{aligned}$$

Note that we are able to exchange the summation and the integral because of the lemma from the 3/26 class regarding convergent series of integrable functions. Additionally, it follows by Problem A.3 that only the $k = 0$ term in the above sum will not evaluate to zero. In particular, this $k = 0$ term will evaluate to $2\pi i$, so overall,

$$\int_\gamma \frac{1}{z-a}\,\mathrm{d}z = 2\pi i$$

For the right integral, we can apply the fundamental theorem of calculus. We have that

$$\int_\gamma \frac{1}{z-b}\,\mathrm{d}z = \int_0^{2\pi} \frac{i\mathrm{e}^{it}}{\mathrm{e}^{it}-b}\,\mathrm{d}t = \left[\ln|\mathrm{e}^{it}-b|\right]_0^{2\pi} = 0$$

Therefore, we have that

$$\begin{aligned}\int_\gamma \frac{1}{(z-a)(z-b)}\,\mathrm{d}z &= \frac{1}{a-b}\left[\int_\gamma \frac{1}{z-a}\,\mathrm{d}z - \int_\gamma \frac{1}{z-b}\,\mathrm{d}z\right] \\ &= \frac{1}{a-b}[2\pi i - 0] \\ &= \frac{2\pi i}{a-b}\end{aligned}$$

as desired. ☐

5. Determine the image of the following sets under the following conformal mappings. Use level curves to illustrate the geometry of these mappings.

(a) The unit disk $\mathbb{D} = \{z : |z| < 1\}$ under $z \mapsto 1/z$.

Proof. This map inverts the modulus of the real part and flips the imaginary part over the real axis. Because of the radial symmetry of the unit disk, the radial symmetry of the final region will be preserved. However, the final region will consist of all points of magnitude $1/r$ ($r < 1$), that is, of magnitude $r > 1$. Thus,

$$\boxed{\mathrm{Im}(\mathbb{D}) = \mathbb{C} \setminus \overline{\mathbb{D}}}$$

Some polar level curves map as follows. ☐

(b) $\mathbb{D} \setminus \{0\}$ under $z \mapsto z^2$.

Proof. This map squares the radius and doubles the argument of a complex number in polar form. Because of the radial symmetry of this disk and the fact that it only contains complex numbers with modulus that shrink when squared, all it will do is map to itself:

$$\boxed{\mathrm{Im}(\mathbb{D} \setminus \{0\}) = \mathbb{D} \setminus \{0\}}$$

$\square$

(c) The strip $S = \{z : \mathrm{Im}(z) \in (0, 2\pi)\}$ under $z \mapsto \mathrm{e}^z$.

Proof. Let $z = x+iy$. Then $\mathrm{e}^z = \mathrm{e}^a \mathrm{e}^{ib}$. By the definition of S, we know that $a \in \mathbb{R}$ and $b \in (0, 2\pi)$. Thus, since the image of the real exponential function is $(0, \infty)$, by picking various values of a, we can reach a complex number of any modulus save zero. Additionally, by picking various values of b, we can reach a complex number of any argument save zero. This means that we can get anywhere in the complex plane except $[0, \infty)$; we can't even access this region by picking $b = \pi$ and a negative modulus because $\mathrm{e}^a > 0$. Therefore,

$$\boxed{\mathrm{Im}(S) = \mathbb{C} \setminus [0, \infty)}$$

$\square$

(d) The upper half-plane $\mathbb{H} = \{z : \mathrm{Im}(z) > 0\}$ under $z \mapsto z^2$,

Proof. As in part (b), we're squaring the modulus and doubling the argument. This means that we can get anywhere except, coincidentally, the same set we miss in part (c). Therefore,

$$\boxed{\mathrm{Im}(\mathbb{H}) = \mathbb{C} \setminus [0, \infty)}$$

$\square$

(e) The half disk $\mathbb{D} \cap \mathbb{H}$ under $z \mapsto (1 + z)/(1 - z)$.

Proof. Via the applet, it appears that

$$\boxed{\mathrm{Im}(\mathbb{D} \cap \mathbb{H}) = \{z \in \mathbb{C}^* : \arg(z) \in (0, \pi/2)\}}$$

$\square$

Set B: Graded for Content

1. A prototypical example of a weird function that is differentiable (but not C^1) on all of $\mathbb{R}$ is

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Extend f to a function on $\mathbb{C}$ using the same formula (replacing x's with z's). Is it holomorphic at the origin?

Proof. As a first step in investigating the complex version of f, let's look at its behavior along the imaginary axis, i.e., for complex numbers ix, $x \in \mathbb{R}$. Since $\sin(ix) = i \sinh(x)$, we have that

$$(ix)^2 \sin\left(\frac{1}{ix}\right) = -x^2 \sin\left(i \cdot -\frac{1}{x}\right) = -ix^2 \sinh\left(-\frac{1}{x}\right)$$

Investigating $x^2 \sinh(-1/x)$, we find that this function is not even continuous at zero, let alone differentiable or holomorphic. Therefore, $\boxed{f \text{ is not holomorphic at the origin.}}$ $\square$

2. Show that if f is holomorphic on a domain $U \subset \mathbb{C}$ and takes only real values, then it is constant.

Proof. Suppose f sends (x, y) to $(g(x, y), h(x, y))$. If f takes only real values, then $h(x, y) = 0$ for all $(x, y) \in U$. Thus,

$$h_x = h_y = 0$$

on U. Additionally, since f is holomorphic on U, it satisfies the Cauchy-Riemann equations. This combined with the above equation implies that

$$g_x = h_y = 0 \qquad\qquad g_y = -h_x = 0$$

Consequently, $f' = 0$ on U, so f must be constant on U. $\square$

3. Find a conformal map that takes the upper half-plane onto the "Pac-Man" given by

$$\{z : |z| < 1 \text{ and } \arg(z) \in (\pi/4, 7\pi/4)\}$$

Explain how you obtained this map. *Hint*: Do completion problem 5 first.

Proof. Define $f, g, h : \mathbb{C} \to \mathbb{C}$ by

$$\begin{aligned} f : r\mathrm{e}^{i\theta} &\mapsto r\mathrm{e}^{i\theta/2} \\ g : z \quad &\mapsto \frac{z-1}{z+1} \\ h : r\mathrm{e}^{i\theta} &\mapsto r\mathrm{e}^{3i\theta/2+\pi/4} \end{aligned}$$

where f, h take $\theta \in [0, 2\pi)$ to avoid ambiguity. Then the desired conformal map is $h \circ g \circ f$. The bijectivity of f, h follows from the bijectivity of the linear manipulations of the arguments, while the bijectivity of g follows from the fact that it is the inverse of the function in Problem A.A.5e. g, g^{-1} are holomorphic as rational functions with no poles in the domain, and f, f^{-1}, h, h^{-1} are holomorphic because their derivatives are rotation maps at every point.

The trickiest part of obtaining this map was figuring out how to get the infinite rectangle into some kind of arc (the job that g does). I toyed around with translating the block up by i and using $1/z$ or something like that to pull it in, but I couldn't work out how to introduce polar-ness. Then, taking the hint, I thought back to Problem A.A.5e and realized that I could hijack this after some pre- and post-transformations. The use of rotation maps was important, too, for f, g because of their smoothness, as opposed to some kind of $x + iy \mapsto |x| + iy$ map for f, for instance. Then it was just a matter of tweaking numbers. $\square$

2 Power Series and Cauchy's Theorem

Set A: Graded for Completion

4/5: 1. Fischer and Lieb (2012), QI.3.2. Prove the Cauchy-Hadamard formula, which states that the radius of convergence r is equal to

$$r = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}}$$

Proof. By definition, the radius of convergence is the largest number $|z_1|$ for which a power series $\sum_{k=0}^{\infty} a_k z^k$ converges locally absolutely uniformly on $D_{|z_1|}(0)$. Equivalently, we have by the Lemma from class on 3/26 that the radius of convergence is the largest number $|z_1|$ for which there exists a positive $M \in \mathbb{R}$ such that $|a_k z_1^k| \leq M$ for all k. Rearranging this latter condition, we learn that

$$|a_k||z_1|^k \leq M$$
$$\sqrt[k]{|a_k|} \leq \frac{\sqrt[k]{M}}{|z_1|}$$

for all k. Now since the limit superior and limit inferior of a sequence of real numbers always exist, we have that

$$\limsup_{k\to\infty} \sqrt[k]{|a_k|} \leq \limsup_{k\to\infty} \frac{\sqrt[k]{M}}{|z_1|} = \frac{1}{|z_1|} \limsup_{k\to\infty} \sqrt[k]{M} = \frac{1}{|z_1|} \cdot 1 = \frac{1}{|z_1|}$$

and

$$\liminf_{k\to\infty} \sqrt[k]{|a_k|} \leq \liminf_{k\to\infty} \frac{\sqrt[k]{M}}{|z_1|} = \frac{1}{|z_1|} \liminf_{k\to\infty} \sqrt[k]{M} = \frac{1}{|z_1|} \cdot 1 = \frac{1}{|z_1|}$$

Consequently, we have

$$|z_1| \leq \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}} \qquad\qquad |z_1| \leq \frac{1}{\liminf_{k\to\infty} \sqrt[k]{|a_k|}}$$

Moreover, since the limit superior is always greater than or equal to the limit inferior of a sequence of real numbers and hence the reciprocal of the limit superior is less than or equal to the limit inferior, the left statement above is the stronger condition. Therefore, we have a well-defined upper bound on $|z_1|$ in the extended real numbers, which should be exactly the radius of convergence by definition. This verifies the Cauchy-Hadamard formula. ☐

2. Fischer and Lieb (2012), QI.4.4. Show that the function $\tan z$ never takes on the values $\pm i$ and that therefore,

$$\frac{\mathrm{d}}{\mathrm{d}z}(\tan z) \neq 0$$

everywhere. Show that the tangent function maps the strip $S_0 = \{z : -\pi/2 < \operatorname{Re} z < \pi/2\}$ biholomorphically onto $\mathbb{C} \setminus \{it : t \in \mathbb{R},\ |t| \geq 1\}$.

Also use level sets to illustrate the conformal mapping.

Proof. Suppose for the sake of contradiction that $z \in \mathbb{C}$ satisfies $\tan z = i$. Then

$$i = \tan z = \frac{\sin z}{\cos z} = \frac{\frac{1}{2i}(\mathrm{e}^{iz} - \mathrm{e}^{-iz})}{\frac{1}{2}(\mathrm{e}^{iz} + \mathrm{e}^{-iz})} = \frac{\mathrm{e}^{iz} - \mathrm{e}^{-iz}}{i(\mathrm{e}^{iz} + \mathrm{e}^{-iz})} = \frac{\mathrm{e}^{2iz} - 1}{i(\mathrm{e}^{2iz} + 1)}$$

so

$$i^2(\mathrm{e}^{2iz} + 1) = \mathrm{e}^{2iz} - 1$$
$$-\mathrm{e}^{2iz} - 1 = \mathrm{e}^{2iz} - 1$$
$$0 = \mathrm{e}^{2iz}$$

Now if the complex number e^{2iz} equals zero, then $|e^{2iz}| = e^{\mathrm{Re}(z)}$ equals zero, too. Thus, $\mathrm{Re}(z) = \log(0)$, but $\log(0)$ is undefined, a contradiction.

Suppose for the sake of contradiction that $z \in \mathbb{C}$ satisfies $\tan z = -i$. Then, similarly to before,

$$-i = \frac{e^{2iz} - 1}{i(e^{2iz} + 1)}$$

so

$$\begin{aligned} -i^2(e^{2iz} + 1) &= e^{2iz} - 1 \\ e^{2iz} + 1 &= e^{2iz} - 1 \\ 2 &= 0 \end{aligned}$$

a contradiction.

Suppose for the sake of contradiction that the derivative of $\tan z$ is everywhere zero. Then $\tan z$ is constant. However, $\tan 0 = 0$ and $\tan(\pi/4) = 1$ for instance, so $\tan z$ is not constant, a contradiction.

To prove that $\tan z$ maps S_0 biholomorphically onto $\mathbb{C} \setminus \{it : t \in \mathbb{R},\ |t| \geq 1\}$, Fischer and Lieb (2012, p. 6) tells us that it will suffice to show that the map is bijective and holomorphic. The holomorphicity condition comes immediately since Fischer and Lieb (2012) states that $\tan z$ is holomorphic everywhere except at the zeroes of $\cos z$ and these zeroes ($\pi/2 \pm \pi n$) are all outside of S_0. Bijectivity, on the other hand, takes a bit more work. To show it, we will decompose $\tan z$ into the composition of four mappings, each of which is individually bijective and the overall composition of which maps the right domain to the right codomain. Explicitly, let $\tan z = \phi(h(g(f(z))))$ where

$$f(z) := 2iz \qquad g(z) := e^z \qquad h(z) := \frac{z-1}{z+1} \qquad \phi(z) := -iz$$

As a "multiply by $w \in \mathbb{C}$" function, f is complex linear and nontrivial, hence bijective. It also maps S_0 to $S_{-\pi} = \{z = x + iy : -\pi < y < \pi\}$. By Fischer and Lieb (2012, p. 21), g maps $S_{-\pi}$ bijectively onto $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. h is bijective because we can derive the explicit formula

$$h^{-1}(z) = -\frac{z+1}{z-1}$$

and confirm that

$$h(h^{-1}(z)) = h^{-1}(h(z)) = z$$

In particular, $h(\mathbb{R}_{\leq 0}) = \{t : t \in \mathbb{R},\ |t| \geq 1\}$. We can see this because via polynomial division,

$$h(x) = \frac{x-1}{x+1} = 1 - \frac{2}{x+1}$$

and this function starts at $-1 = h(0)$, decreases asymptotically toward $-\infty$ as $0 \to -1$, and then decreases asymptotically from ∞ toward 1 as x goes from $-1^- \to -\infty$. Lastly, ϕ is bijective for the same reason as f, and ϕ maps $\{t : t \in \mathbb{R},\ |t| \geq 1\}$ to $\{it : t \in \mathbb{R},\ |t| \geq 1\}$, the final set that we desire to cut out of $\mathbb{C}$. $\square$

3. Fix $a, b, c \in \mathbb{C}$ so that c is not a negative integer or 0. Show that the **hypergeometric** function

$$F(a, b, c; z) := \sum_{k=0}^{\infty} \frac{a(a+1)\cdots(a+k-1)b(b+1)\cdots(b+k-1)}{c(c+1)\cdots(c+k-1)} \frac{z^k}{k!}$$

converges on the unit disk and satisfies the differential equation

$$z(1-z)F''(z) + [c - (a+b+1)z]F'(z) - abF(z) = 0$$

Proof. Before we begin properly, we will introduce the **Pochhammer symbol** $(q)_k$, defined by

$$(q)_k = \begin{cases} 1 & k = 0 \\ q(q+1)\cdots(q+k-1) & k > 0 \end{cases}$$

as shorthand for $a(a+1)\cdots(a+k-1)$, etc. Making use of this notation, we will also preliminarily observe that

$$\begin{aligned} F'(z) &= \sum_{k=1}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{z^{k-1}}{(k-1)!} \\ &= \sum_{k=0}^{\infty} \frac{(a)_{k+1}(b)_{k+1}}{(c)_{k+1}} \frac{z^k}{k!} \\ &= \frac{ab}{c} \sum_{k=0}^{\infty} \frac{(a+1)_k(b+1)_k}{(c+1)_k} \frac{z^k}{k!} \\ &= \frac{(a)_1(b)_1}{(c)_1} F(a+1, b+1, c+1; z) \end{aligned}$$

and

$$\begin{aligned} F''(z) &= \frac{\mathrm{d}}{\mathrm{d}z}[F'(z)] \\ &= \frac{(a)_1(b)_1}{(c)_1} F'(a+1, b+1, c+1; z) \\ &= \frac{(a)_1(b)_1}{(c)_1} \cdot \frac{(a+1)_1(b+1)_1}{(c+1)_1} F(a+2, b+2, c+2; z) \\ &= \frac{(a)_2(b)_2}{(c)_2} F(a+2, b+2, c+2; z) \end{aligned}$$

These formulas will be useful later.

We now begin our argument that the hypergeometric function satisfies the given differential equation in earnest. Upon expanding the given differential equation, we will obtain a sum of terms that can be sorted by the power of z present in the term. In particular, the coefficients a_k of each z^k term must cancel to zero independently, so let's derive a general formula for the coefficient of the k^{th} term.

For the first term, we have

$$\begin{aligned} z(1-z)F''(z) &= z(1-z)\frac{(a)_2(b)_2}{(c)_2} F(a+2, b+2, c+2; z) \\ &= (1-z) \cdot \frac{(a)_2(b)_2}{(c)_2} z \left[1 + \frac{(a+2)(b+2)}{c+2} z + \frac{(a+2)_2(b+2)_2}{(c+2)_2} \frac{z^2}{2} + \cdots\right] \\ &= (1-z)\left[\frac{(a)_2(b)_2}{(c)_2 \cdot 0!} z + \frac{(a)_3(b)_3}{(c)_3 \cdot 1!} z^2 + \frac{(a)_4(b)_4}{(c)_4 \cdot 2!} z^3 + \cdots\right] \\ &= \frac{(a)_2(b)_2}{(c)_2 \cdot 0!} z + \left[\frac{(a)_3(b)_3}{(c)_3 \cdot 1!} - \frac{(a)_2(b)_2}{(c)_2 \cdot 0!}\right] z^2 + \left[\frac{(a)_4(b)_4}{(c)_4 \cdot 2!} - \frac{(a)_3(b)_3}{(c)_3 \cdot 1!}\right] z^3 + \cdots \\ &= 0 + \frac{(a)_2(b)_2}{(c)_2 \cdot 0!} z + \sum_{k=2}^{\infty} \left[\frac{(a)_{k+1}(b)_{k+1}}{(c)_{k+1}(k-1)!} - \frac{(a)_k(b)_k}{(c)_k(k-2)!}\right] z^k \\ &= 0 + \frac{(a)_2(b)_2}{(c)_2 \cdot 0!} z + \sum_{k=2}^{\infty} \frac{(a)_k(b)_k}{(c)_k(k-2)!} \left[\frac{(a+k)(b+k)}{(c+k)(k-1)} - 1\right] z^k \end{aligned}$$

For the second term, we have

$$\begin{aligned}
[c-(a+b+1)z]F'(z) &= [c-(a+b+1)z]\frac{(a)_1(b)_1}{(c)_1}F(a+1,b+1,c+1;z) \\
&= [c-(a+b+1)z]\cdot\frac{(a)_1(b)_1}{(c)_1}\left[1+\frac{(a+1)(b+1)}{c+1}z+\frac{(a+1)_2(b+1)_2}{(c+1)_2}\frac{z^2}{2}+\cdots\right] \\
&= [c-(a+b+1)z]\left[\frac{(a)_1(b)_1}{(c)_1\cdot 0!}+\frac{(a)_2(b)_2}{(c)_2\cdot 1!}z+\frac{(a)_3(b)_3}{(c)_3\cdot 2!}z^2+\cdots\right] \\
&= \frac{(a)_1(b)_1c}{(c)_1\cdot 0!}+\left[\frac{(a)_2(b)_2c}{(c)_2\cdot 1!}-\frac{(a)_1(b)_1(a+b+1)}{(c)_1\cdot 0!}\right]z+\cdots \\
&= \frac{(a)_1(b)_1c}{(c)_1\cdot 0!}+\sum_{k=1}^{\infty}\left[\frac{(a)_{k+1}(b)_{k+1}c}{(c)_{k+1}\cdot k!}-\frac{(a)_k(b)_k(a+b+1)}{(c)_k\cdot(k-1)!}\right]z^k \\
&= \frac{(a)_1(b)_1c}{(c)_1\cdot 0!}+\sum_{k=1}^{\infty}\frac{(a)_k(b)_k}{(c)_k(k-2)!}\left[\frac{(a+k)(b+k)c}{k(k-1)(c+k)}-\frac{a+b+1}{k-1}\right]z^k
\end{aligned}$$

And for the third term, we have

$$\begin{aligned}
-abF(z) &= -abF(a,b,c;z) \\
&= -ab\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}z^k \\
&= \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k(k-2)!}\left[-\frac{ab}{k(k-1)}\right]z^k
\end{aligned}$$

Adding these three infinite series together, we can see that the constant (z^0) term will have the coefficient

$$a_0 = 0+\frac{(a)_1(b)_1c}{(c)_1\cdot 0!}-ab\cdot\frac{(a)_0(b)_0}{(c)_0\cdot 0!}=\frac{abc}{c}-ab\cdot\frac{1}{1}=0$$

as desired. We can also see that the z^1 term will have coefficient

$$\begin{aligned}
a_1 &= \frac{(a)_2(b)_2}{(c)_2\cdot 0!}+\frac{(a)_2(b)_2c}{(c)_2\cdot 1!}-\frac{(a)_1(b)_1(a+b+1)}{(c)_1\cdot 0!}-ab\cdot\frac{(a)_1(b)_1}{(c)_1\cdot 1!} \\
&= \frac{a(a+1)b(b+1)}{c(c+1)}+\frac{a(a+1)b(b+1)c}{c(c+1)}-\frac{ab(a+b+1)}{c}-\frac{a^2b^2}{c} \\
&= \frac{ab(a+1)(b+1)}{c(c+1)}+\frac{abc(a+1)(b+1)}{c(c+1)}-\frac{ab(a+b+1)(c+1)}{c(c+1)}-\frac{a^2b^2(c+1)}{c(c+1)} \\
&= \frac{ab(a+1)(b+1)+abc(a+1)(b+1)-ab(a+b+1)(c+1)-a^2b^2(c+1)}{c(c+1)}
\end{aligned}$$

Now to show that $a_1=0$, it will suffice to show that the numerator of the above fraction equals zero, which we can do as follows.

$$\begin{aligned}
N(a_1) &= ab(a+1)(b+1)+abc(a+1)(b+1)-ab(a+b+1)(c+1)-a^2b^2(c+1) \\
&= ab(ab+a+b+1)+abc(ab+a+b+1) \\
&\quad -abc(a+b+1)-ab(a+b+1)-a^2b^2c-a^2b^2 \\
&= abab+ab(a+b+1)+abcab+abc(a+b+1) \\
&\quad -abc(a+b+1)-ab(a+b+1)-abcab-abab \\
&= 0
\end{aligned}$$

To conclude, we show that the coefficients a_k ($k \geq 2$) are equal to zero all at once, as follows.

$$\begin{aligned}
a_k &\propto \frac{(a+k)(b+k)}{(c+k)(k-1)} - 1 + \frac{(a+k)(b+k)c}{k(k-1)(c+k)} - \frac{a+b+1}{k-1} - \frac{ab}{k(k-1)} \\
&= \frac{k(a+k)(b+k)}{k(k-1)(c+k)} - \frac{k(k-1)(c+k)}{k(k-1)(c+k)} + \frac{(a+k)(b+k)c}{k(k-1)(c+k)} - \frac{k(a+b+1)(c+k)}{k(k-1)(c+k)} - \frac{ab(c+k)}{k(k-1)(c+k)} \\
&= \frac{k(a+k)(b+k) - k(k-1)(c+k) + (a+k)(b+k)c - k(a+b+1)(c+k) - ab(c+k)}{k(k-1)(c+k)}
\end{aligned}$$

And as before, we'll focus on the numerator from here on out.

$$\begin{aligned}
N(a_k) &= k(a+k)(b+k) - k(k-1)(c+k) + (a+k)(b+k)c - k(a+b+1)(c+k) - ab(c+k) \\
&= k(ab+ak+bk+k^2) \\
&\quad - k(ck-c-k+k^2) \\
&\quad + (ab+ak+bk+k^2)c \\
&\quad - k(ac+bc+c+ak+bk+k) \\
&\quad - ab(c+k) \\
&= abk + ak^2 + bk^2 + k^3 \\
&\quad - ck^2 + ck + k^2 - k^3 \\
&\quad + abc + ack + bck + ck^2 \\
&\quad - ack - bck - ck - ak^2 - bk^2 - k^2 \\
&\quad - abc - abk \\
&= 0
\end{aligned}$$

As to convergence on the unit disk, applying the Cauchy-Hadamard formula, we can see that $k! \to \infty$ faster than anything else, so the limit superior will go to zero and the radius of convergence will be ∞. □

Set B: Graded for Content

1. Fischer and Lieb (2012), QII.2.3. Compute the **Fresnel integrals**

$$\int_0^\infty \cos(x^2)\,\mathrm{d}x = \sqrt{\frac{\pi}{8}} = \int_0^\infty \sin(x^2)\,\mathrm{d}x$$

Hint: Apply the Cauchy integral theorem to sectors with center 0 and corners given by R and $\mathrm{e}^{i\pi/4}R$, where $R \to \infty$.

Proof. As in the case of the Dirichlet integral, we will analyze the complex exponential functions composing the complex cosine and then combine our results into the final answer. Let's begin.

Let γ be the sector described in the hint oriented counterclockwise, and let $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ where γ_1 is the segment along the real axis, γ_2 is the curved portion, and γ_3 is the segment between 0 and $R\mathrm{e}^{i\pi/4}$. Then by the Cauchy integral theorem,

$$\int_\gamma \mathrm{e}^{-iz^2}\,\mathrm{d}z = 0$$

Note that we begin our explorations here because this integral closely resembles the Gaussian integral, so we may be able to use that to our advantage. And indeed, for γ_1, $\int_0^R \mathrm{e}^{-it^2}\,\mathrm{d}t$ can be expressed in terms of the Gaussian integral as $R \to \infty$ since the Gaussian distribution is even:

$$\lim_{R\to\infty} \int_0^R \mathrm{e}^{-it^2}\,\mathrm{d}t = \frac{1}{2}\int_{-\infty}^\infty \mathrm{e}^{-it^2}\,\mathrm{d}t = \frac{\sqrt{\pi}}{2}$$

Bounding the integral over γ_2 takes more work, just like in the case of the Dirichlet integral. However, if we first attempt to get a bound on its magnitude, we can end up proving that it converges to zero.

$$\begin{aligned}
\left|\int_{\gamma_2} \mathrm{e}^{-z^2}\,\mathrm{d}z\right| &= \left|\int_0^{\pi/4} \mathrm{e}^{-(R\mathrm{e}^{it})^2} \cdot iR\mathrm{e}^{it}\,\mathrm{d}t\right| \\
&= \left|\int_0^{\pi/4} R\mathrm{e}^{-R^2\mathrm{e}^{i\cdot 2t}} \cdot i\mathrm{e}^{it}\,\mathrm{d}t\right| \\
&= \left|\int_0^{\pi/4} R\mathrm{e}^{-R^2\cos(2t)} \cdot i\mathrm{e}^{i(t-R^2\sin(2t))}\,\mathrm{d}t\right| \\
&\leq \int_0^{\pi/4} \left|R\mathrm{e}^{-R^2\cos(2t)} \cdot i\mathrm{e}^{i(t-R^2\sin(2t))}\right|\mathrm{d}t \\
&= \int_0^{\pi/4} R\mathrm{e}^{-R^2\cos(2t)}\,\mathrm{d}t
\end{aligned}$$

At this point, we'd like to find a way to bound $\cos(2t)$ so that we can evaluate the integral directly, without bounding it so loosely that we lose the convergence. One such way is by noting that $\cos(2t)$ is just slightly greater than the (much simpler) linear function $1 - 4t/\pi$ on the interval $[0, \pi/4]$, and thus the negative exponential of the cosine is slightly less than the negative exponential of the linear. Continuing to evaluate, we obtain an integral that can be computed explicitly:

$$\begin{aligned}
\int_0^{\pi/4} R\mathrm{e}^{-R^2\cos(2t)}\,\mathrm{d}t &\leq \int_0^{\pi/4} R\mathrm{e}^{-R^2(1-4t/\pi)}\,\mathrm{d}t \\
&= -\frac{\pi(\mathrm{e}^{-R^2}-1)}{4R}
\end{aligned}$$

This expression has the form c/∞ in the limit and thus equals zero.

Combining the last several results, we have that

$$\begin{aligned}
0 &= \sum_{k=1}^{3} \int_{\gamma_k} \mathrm{e}^{-z^2}\,\mathrm{d}z \\
\int_{\gamma_3} \mathrm{e}^{-z^2}\,\mathrm{d}z &= -\int_{\gamma_1} \mathrm{e}^{-z^2}\,\mathrm{d}z = -\frac{\sqrt{\pi}}{2}
\end{aligned}$$

This puts us in an interesting and different place from the Dirichlet integral. There, our integral over the real axis was still an unknown, and here, we've already evaluated it. How can we use this situation to our advantage? Well, let's start expanding the γ_3 integral and go from there.

$$\begin{aligned}
\int_{-\gamma_3} \mathrm{e}^{-z^2}\,\mathrm{d}z &= \int_0^{\infty} \mathrm{e}^{-\mathrm{e}^{i\pi/2}t^2} \cdot \mathrm{e}^{i\pi/4}\,\mathrm{d}t \\
&= \int_0^{\infty} \mathrm{e}^{-it^2} \cdot \mathrm{e}^{i\pi/4}\,\mathrm{d}t \\
&= \int_0^{\infty} [\cos(t^2) - i\sin(t^2)] \cdot \frac{\sqrt{2}}{2}(1+i)\,\mathrm{d}t \\
&= \frac{\sqrt{2}}{2}\int_0^{\infty} [\cos(t^2) + \sin(t^2) + i(\cos(t^2) - \sin(t^2))]\,\mathrm{d}t \\
&= \frac{\sqrt{2}}{2}\left[\int_0^{\infty} \cos(t^2)\,\mathrm{d}t + \int_0^{\infty} \sin(t^2)\,\mathrm{d}t + i\left(\int_0^{\infty} \cos(t^2)\,\mathrm{d}t - \int_0^{\infty} \sin(t^2)\,\mathrm{d}t\right)\right]
\end{aligned}$$

What it appears that we have now obtained actually is the two-variable system of equations

$$\frac{\sqrt{2}}{2}\left[\int_0^{\infty} \cos(t^2)\,\mathrm{d}t + \int_0^{\infty} \sin(t^2)\,\mathrm{d}t + i\left(\int_0^{\infty} \cos(t^2)\,\mathrm{d}t - \int_0^{\infty} \sin(t^2)\,\mathrm{d}t\right)\right] = \frac{\sqrt{\pi}}{2} + i(0)$$

or

$$\int_0^\infty \cos(t^2)\,\mathrm{d}t + \int_0^\infty \sin(t^2)\,\mathrm{d}t = \sqrt{\frac{\pi}{2}} \qquad \int_0^\infty \cos(t^2)\,\mathrm{d}t - \int_0^\infty \sin(t^2)\,\mathrm{d}t = 0$$

which we can solve for the desired result. □

2. These problems illustrate the geometric intuition for the radius of convergence of a power series. Do the parts in order.

(a) For each nonzero natural number $n \in \mathbb{N}$, compute the power series expansion for the function $1/z$ around the point $1/n$. What are their radii of convergence?

Proof. The desired power series will be of the form

$$P_n(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(1/n)}{k!}(z - 1/n)^k$$

where $f(z) = 1/z = z^{-1}$. Now by the power rule,

$$f^{(k)}(z) = (-1)^k k! z^{-(k+1)}$$

Therefore, combining the last two results,

$$\boxed{P_n(z) = \sum_{k=0}^{\infty} (-1)^k n^{k+1} (z - 1/n)^k}$$

By the Cauchy-Hadamard formula,

$$\begin{aligned} r &= \frac{1}{\limsup_{k\to\infty} |(-1)^k n^{k+1}|^{1/k}} \\ &= \frac{1}{\limsup_{k\to\infty} n^{k+1/k}} \\ \boxed{r = \frac{1}{n}} \end{aligned}$$

□

(b) Describe the set of points $w \in \mathbb{C}$ such that the power series expansion for $1/z$ about w has radius of convergence equal to 1.

Proof. Working backward in part (a), we start off by finding $n \in \mathbb{C}$ such that $\limsup_{k\to\infty} |(-1)^k n^{k+1}|^{1/k} = 1$. This will happen if $|n| = 1$. Then working backwards, we want $w = 1/n$. But this is still just the set of points of magnitude 1. Therefore, the desired set is

$$\boxed{\{w \in \mathbb{C} : |w| = 1\}}$$

□

(c) Suppose that

$$f(z) = \frac{1}{z(z-1)(z-i)(z-1-i)}$$

Find the unique point w in the unit square $\{\mathrm{Re}(z), \mathrm{Im}(z) \in [0,1]\}$ such that the radius of convergence of the power series for w is maximal. Justify your answer.

Proof. The definition of f singles out the four corners of the unit square as singularities. Thus, the disk of convergence cannot include any of these corners, so we need the point in the unit square that's farthest away from all four corners. This would be

$$\boxed{w = \frac{1}{2}(1+i)}$$

□

3. This problem is to hint at the general formulation of the Cauchy integral theorem. Please solve this problem only using things we have seen in class to this point.

(a) Show that the "L-shaped" domain

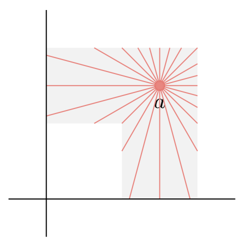
$$L = \{z : \mathrm{Re}(z), \mathrm{Im}(z) \in (0,2) \text{ and not both } \mathrm{Re}(z), \mathrm{Im}(z) \in (0,1]\}$$

is star-shaped (hence the Cauchy integral theorem applies).

Proof. Choose

$$a = \frac{3}{2}(1+i)$$

Then



□

(b) Show that the "double L-shaped" domain

$$U = \{z : |\mathrm{Re}(z)|, \mathrm{Im}(z) \in (0,2) \text{ and not both } |\mathrm{Re}(z)|, \mathrm{Im}(z) \in (0,1]\}$$

is not star-shaped.

Proof. We can do this by casework. If we pick a to be any point in the right "L," straight-line paths to $(-1.5, 0.1)$ will go outside of U and vice versa for the left "L." □

(c) Nevertheless, by breaking up U into two copies of L and using the Cauchy integral theorem for the resultant star-shaped domains, show that for any closed curve γ in U and any $f \in \mathcal{O}(U)$, we have that $\int_\gamma f \, \mathrm{d}z = 0$.

Proof. Any time the curve crosses the imaginary axis once, it will have to cross the imaginary axis at least one more time on the way back since the loop is closed. Thus, when it crosses the imaginary axis, choose the next time it crosses the imaginary axis as we proceed along the path and draw a segment between these two points. Integrate around the loop on the right side and the left side; the integrals along the segment will cancel and the sum will be the original integral. Meanwhile, the one-sided loops in the individual star-shaped domains will evaluate to zero. □

(d) Show that $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ can be written as the union of two star-shaped domains.

Proof. Choose $\mathbb{C}$ without the upper-right quartile and $\mathbb{C}$ without the lower left quartile. □

(e) Why doesn't your proof for part (c) show that $\int_\gamma f \, \mathrm{d}z = 0$ for any $f \in \mathcal{O}(\mathbb{C}^*)$ and any closed curve γ in $\mathbb{C}^*$?

Proof. The two sets in part (d) are not (and cannot be) disjoint. □

References

Fischer, W., & Lieb, I. (2012). *A course in complex analysis: From basic results to advanced topics* (J. Cannizzo, Trans.). Vieweg+Teubner Verlag.