Week 8

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8.1 Counting Zeroes and Laurent Series

5/7: • Announcements.

  - PSet 5 posted, due next Friday.
    - 10 problems (a few more than usual), but heavily computational so shouldn't be that bad.
    - After today's class, we'll have everything we need to do it.
  - Draft of final project report due this Friday.
    - It doesn't need to be a full draft, but it should be a pretty well-fleshed-out outline. The more information we can give him, the better feedback he can give us.
  - Make sure to check out the Canvas post on final presentation scheduling.

• Last time.

  - We deduced the residue theorem from the general CIT.
    - It states: Suppose U is a domain, $S \subset U$ is a discrete (though not necessarily finite) set of singularities, $f \in \mathcal{O}(U \setminus S)$, and $\Gamma = \sum c_i\gamma_i$ is nulhomologous in U (though not necessarily nulhomologous in $U \setminus S$). Then

$$\frac{1}{2\pi i}\int_\Gamma f \, \mathrm{d}z = \sum_{s\in S} \mathrm{wn}(\Gamma, s) \operatorname{res}_s f$$

  - The residue of f about s is

$$\operatorname{res}_s f = \frac{1}{2\pi i}\int_{\partial D} f \, \mathrm{d}z$$

    where D is some little disk about s (and only this $s \in S$).
    - If s is a pole, then $\operatorname{res}_s f$ is also equal to the a_{-1} coefficient of the Laurent expansion.
  - What the residue theorem gets us.
    - Lets us compute integrals over complicated paths.
      - ➢ We get to work with Laurent series instead of integrals, which is easier.
    - Lets us compute sums, as in the Basel problem.
      - ➢ We choose a function, introduce the residue, and express the sum in terms of the integral.

• Today.

  - **The argument principle**: A theoretical (not practical) ramification of the residue theorem.
  - Talking a bit more about essential singularities.

- Consider the function f'/f, where f is either holomorphic or meromorphic on U.
  - Another way to think about this function is as the derivative of $\log f$.
  - We want to investigate the singularities of f'/f.
  - Suppose f has a zero of order k at s.
    - Then locally, $f(z) = (z-s)^k g(z)$ where $g(s) \neq 0$.
    - If f looks like this at s, then $f'(z) = k(z-s)^{k-1}g(z) + (z-s)^k g'(z)$.
    - Thus,
      $$\frac{f'}{f}(z) = \frac{k}{z-s} + \frac{g'(z)}{g(z)}$$
    - Since $g(s) \neq 0$, g'/g is a well-defined number in a disk about s.
    - The other term gives a well-defined pole.
    - Therefore, if f has a zero of order k at s, then f'/f has a simple pole at s with $\operatorname{res}_s(f'/f) = k$.
  - Exercise: If f has a pole of order k at s, then f'/f has a simple pole with $\operatorname{res}_s(f'/f) = -k$.
    - Prove the same way (or see the notes).
- Corollary (the argument principle): Suppose U is a domain, f is meromorphic on U, and γ is an SCC oriented counterclockwise that doesn't hit any poles or zeroes of f. Then
  $$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,\mathrm{d}z = \sum_{s\in S} \operatorname{res}_s(f'/f) = \#\text{ zeroes in }\gamma - \#\text{ poles in }\gamma$$
- **Simple closed curve**: A curve γ that separates the inside from the outside. *Also known as* **SCC**.
  - See Figure 7.2a for two examples.
- TPS: Use the argument principle to compute the number of zeroes of the following polynomial inside the unit disk $\mathbb{D}$.
  $$f(z) = 2z^4 - 5z + 2$$
  - Since f is a polynomial, it has no poles.
  - Thus, by the argument principle, the number of zeroes in $\partial\mathbb{D}$ is
    $$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{f'}{f}\,\mathrm{d}z = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{8z^3-5}{2z^4-5z+2}\,\mathrm{d}z$$
  - The integral on the right is "a pain in the butt" to compute, but we definitely could.
    - We'd just have to do a partial fraction decomposition, substitute in $z = \mathrm{e}^{i\theta}$, and bash it out.
  - A better way to find the number of zeroes uses **Rouché's theorem**, which we'll introduce shortly.
- Understanding the argument principle geometrically. *picture*
  - Suppose you have an SCC γ enclosing a zero or order 1 and a pole of order 1.
  - We'll now investigate f as a mapping of $\mathbb{C} \to \mathbb{C}$ and $\mathbb{C} \to \hat{\mathbb{C}}$.
  - As a mapping into the complex plane, f maps γ to $f(\gamma)$.
    - Note that the curve $f(\gamma)$ is just another mapping of the circle into the complex plane.
  - As a mapping into the Riemann sphere, f maps the zero to 0 and the pole to ∞.
  - Now draw little counterclockwise-oriented curves around the zero and pole.
    - Since the pole has order 1, its little loop maps to a little loop around $\infty \in \hat{\mathbb{C}}$ that goes around 1 time.

        - ➢ If the pole had order 2, for example, then a little loop that goes around it 1 time would map to a loop that goes around ∞ 2 times.
    - ■ Similarly, since the zero has order 1, its little loop maps to a little loop around $0 \in \hat{\mathbb{C}}$ that goes around 1 time.
  - Now pull the two loops down the Riemann sphere to the equator.
    - ■ Observe that their orientations are now inverses, with the orientation of the curve around ∞ having flipped.
        - ➢ This is like the Coriolis effect!
    - ■ Projecting the pulled-down curves into $\mathbb{C}$, we can observe that the one around ∞ is oriented clockwise and encompasses $f(\gamma)$ while the one around 0 is oriented counterclockwise and situated within $f(\gamma)$.
  - ??
  - This all results in the order of zeros minus the order of poles is equal to the winding number of $f(\gamma)$ about zero.

- Here's a computation that justifies all of the handwavey stuff above:

$$\begin{aligned}
\operatorname{wn}(f(\gamma), 0) &= \frac{1}{2\pi i} \int_{f(\gamma)} \frac{1}{z} \, \mathrm{d}z \\
&= \frac{1}{2\pi i} \int_0^1 \frac{1}{f(\gamma(t))} [f(\gamma(t))]' \, \mathrm{d}t \\
&= \frac{1}{2\pi i} \int_0^1 \frac{f'(\gamma(t))}{f(\gamma(t))} \gamma'(t) \, \mathrm{d}t \\
&= \frac{1}{2\pi i} \int_\gamma \frac{f'}{f} \, \mathrm{d}z
\end{aligned}$$

  - This computation is another proof of the argument principle!

- Rouché's theorem: Let U be a domain, $\gamma \subset U$ an SCC, and $f, g \in \mathcal{O}(U)$. Suppose also that $|g| < |f|$ on γ. Then f and $f + g$ have the same number of zeroes.

  Proof. Set $h_\lambda(z) := f + \lambda g$ where $\lambda \in [0, 1]$. Thus, $h_0 = f$ and $h_1 = f + g$. It follows by the argument principle that the number of zeroes of h_λ in γ (which is a discrete set) is

$$\frac{1}{2\pi i} \int_\gamma \frac{h_\lambda'}{h_\lambda} \, \mathrm{d}z = \frac{1}{2\pi i} \int_\gamma \frac{f' + \lambda g'}{f + \lambda g} \, \mathrm{d}z$$

  which is a continuous map in λ. Essentially, we have shown that the "number of zeroes in γ" function is a continuous function from $[0, 1] \to \mathbb{N}_0$. Hence, as a continuous function into a discrete set, it is constant.

  Note that we used the $|g| < |f|$ condition to ensure that $f + \lambda g$ in the denominator of the above integral is never zero, and hence the integral is always well-defined. □

- Example: Solving the TPS from earlier.
  - $g(z) = 2z^4$ has four zeroes inside $\mathbb{D}$.
  - $f(z) = -5z$ has one zero inside $\mathbb{D}$.
  - On $\partial\mathbb{D}$, we have that $|z| = 1$ and hence

$$2 = |g| < |f| = 5$$

    so $2z^4 - 5z$ has the same number of zeroes as $-5z$ (or 1) by Rouché's theorem.

  - Redefine $f(z) = 2z^4 - 5z$ and $g(z) = 2$.
  - On $\partial\mathbb{D}$, we similarly have that

$$|f| = |2z^4 - 5z| \geq \big||2z^4| - |5z|\big| = |2 - 5| = 3 > 2 = |g|$$

  so $2z^4 - 5z + 2$ has the same number of zeroes as $2z^4 - 5z$ (or 1) by Rouché's theorem.

- Takeaway: Whenever you're asked to compute zeroes, Rouché's theorem is probably the way to go.
- Exercise: Prove the FTA using Rouché's theorem.
  - Idea: On big enough circles, eventually the top-degree term dominates.
- Let's now talk a bit more about essential singularities.
- Suppose U is a small disk, $s \in U$, and $f \in \mathcal{O}(U \setminus S)$ (so s is an isolated singularity). Then one of three things can happen.
  1. s is removable.
     - In this case, we can remove it using Riemann's removable singularity theorem and get an analytic continuation $\hat{f} \in \mathcal{O}(U)$.
     - This is really nice, because then we get a power series near s:

$$f(z) = \sum_{k=0}^{\infty} a_k (z - s)^k$$

  2. s is a pole.
     - We get a similar series called a Laurent series with:

$$f(z) = \sum_{k=-N}^{\infty} a_k (z - s)^k$$

  3. s is essential.
     - We get a Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - s)^k$$

- A word on convergence.
  - Before, we used to say the magic words "absolutely locally uniformly" and we'd get convergence. Now we can't do that, and we need the following.
  - Recall that uniform convergence $f_k \to f$ means that

$$\sup_{z \in K} |f_k(z) - f(z)| \to 0$$

  - In addition, recall that local uniform convergence $f_k \to f$ means that

$$\sup_{K \subset U} \sup_{z \in K} |f_k(z) - f(z)| \to 0$$

- Theorem: Suppose $\{f_k\} \in \mathcal{O}(U \setminus S)$ is such that the $f_k \to f$ locally uniformly. Then $f \in \mathcal{O}(U \setminus S)$ and moreover $f_k^{(n)} \to f^{(n)}$ for all n.

Proof. Goursat plus Morera for the first statement. CIF for the second statement. More detail in the notes; Calderon is also happy to talk. ☐

- We now prove that such Laurent expansions exist.
  - Step 1: "Pull off" the singular part.
    - This is a theorem called the **Laurent decomposition**.
  - Step 2: Express f_∞ as
$$\sum_{k=-\infty}^{-1} a_k(z-s)^k$$
- Theorem (Laurent decomposition): There exists a unique $f_\infty \in \mathcal{O}(\mathbb{C}\setminus\{s\})$ with $f = f_0 + f_\infty$ such that $f_0 \in \mathcal{O}(U)$ and $f_\infty \to 0$ as $z \to \infty$.

  Proof. Uniqueness is not interesting; see the book.

  Existence: Let $z \in U$ be arbitrary. Let D_2 be a counterclockwise-oriented curve in U containing 0 and z. Let D_1 be a counterclockwise-oriented curve in U containing just 0. Then $z \in D_2 \setminus D_1$. Altogether, this looks like Figure 8.1.

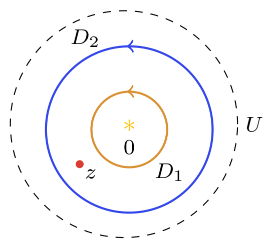

Figure 8.1: Laurent decomposition theorem.

  Additionally, $\partial D_2 - \partial D_1$ is nulhomologous in $U \setminus 0$. Thus, by the general CIF,
$$f(z) = \frac{1}{2\pi i}\int_{\partial D_2 - \partial D_1} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta = \underbrace{\frac{1}{2\pi i}\int_{\partial D_2} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta}_{f_0} - \underbrace{\frac{1}{2\pi i}\int_{\partial D_1} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta}_{-f_\infty}$$
  Finally, $-f_\infty$ — as defined above — has an extension from U to all of $\mathbb{C}\setminus\{0\}$ just like in the proof of the general CIF from the 4/30 lecture. □

- Comments on the Laurent decomposition.
  - Think of f_0 as the nonnegative terms of the Laurent expansion, and f_∞ as everything else (all the negative terms).
- Step 2 proof.

  Proof. For poles, we could just multiply by $(z-s)^k$. Here, we have to do something very clever. WLOG, let $s = 0$. By step 1, f_∞ extends to ∞ and we may set $f_\infty(\infty) = 0$. Thus, let's think of $f_\infty \in \mathcal{O}(\hat{\mathbb{C}}\setminus\{0\})$. Set $g(z) = f_\infty(1/z)$. Now g has a removable singularity at 0. Moreover, $g \in \mathcal{O}(\mathbb{C})$. Then
$$g(z) = \sum_{k=0}^{\infty} b_k(z-0)^k$$
  converges for all z. But therefore
$$f_\infty(z) = g(1/z) = \sum_{k=1}^{\infty} b_k\left(\frac{1}{z}\right)^k = \sum_{k=-\infty}^{-1} a_k z^k$$
  as desired. □