Week 5

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5.1 Office Hours

4/15:
- There will not be anything explicit about Thursday's content, but knowing it is helpful for understanding conformal maps.
- The exam is completely closed book.
- Midterm-style questions.
  - Per the mathematical hierarchy of needs (definitions and examples, theorem statements, problems/applying them, proofs of them).
  - He does not want to test our memorization skills but rather our understanding.

5.2 Midterm Review Sheet

4/16:
- Properties of complex numbers.
- **Holomorphic** (f at z_0): A function $f:\mathbb{C}\to\mathbb{C}$ for which the following limit exists. *Also known as* **$\mathbb{C}$-differentiable**. *Constraints*

  $$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=:f'(z_0)\qquad\iff\qquad f(z)=f(z_0)+\Delta(z)(z-z_0)$$

  where Δ is continuous at z_0 and $\Delta(z_0)=f'(z_0)$.
  - Sum rule, product rule, quotient rule.
  - Chain rule.
  - Holomorphic implies continuous.
- Every $\mathbb{C}$-linear map is just multiplication by a complex number; the matrix must compute with $\mathcal{M}(i)$.
- **Cauchy-Riemann equations**: The following two equations, which identify when a complex function $(x,y)\mapsto(g,h)$ is holomorphic. *Also known as* **CR equations**. *Given by*

  $$\begin{aligned}g_x&=h_y\\g_y&=-h_x\end{aligned}$$

- **Wirtinger derivatives**: The two differential operators defined as follows. *Denoted by* $\boldsymbol{\partial/\partial z,\partial/\partial\bar{z}}$. *Given by*

  $$\frac{\partial}{\partial\bar{z}}:=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)\qquad\qquad\frac{\partial}{\partial z}:=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)$$

- Theorem: The $\mathbb{R}$-differentiable function $f : U \to \mathbb{C}$ is holomorphic iff $\partial f/\partial \bar{z} = 0$. Moreover, if it is, then
$$f'(z_0) = \left.\frac{\partial f}{\partial z}\right|_{z_0}$$
- **Laplacian**: The differential operator defined as follows. *Denoted by* $\boldsymbol{\Delta}$. *Given by*
$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$
- **Harmonic** (function): A function $f : \mathbb{R}^2 \to \mathbb{C}$ such that $\Delta f = 0$.
- Corollary: The real and imaginary parts of a C^2 holomorphic function are harmonic.

  Proof. $\Delta(u + iv) = \Delta u + i\Delta v$. □
- **Harmonic conjugates**: Two functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ that satisfy the CR equations.
- Path integration:
$$\int_\gamma f \,\mathrm{d}z = \int_a^b f(\gamma(t)) \cdot \gamma'(t) \,\mathrm{d}t$$
- FTC: Suppose $F' = f$ on $U \subset \mathbb{C}$, and let γ be a **path** inside of U. Then
$$\int_\gamma f \,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$$
- Factoring into rotation and scaling matrices.
$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \qquad (\lambda, \theta \in \mathbb{R})$$
- Lemma: Holomorphic maps preserve angles.

  Proof. Look at the argument at the intersection point and use the chain rule. □
- **Conformal** (map): A function $f : U \to V$, where $U, V \subset \mathbb{C}$, that satisfies the following two constraints. *Constraints*
  1. f is a diffeomorphism.
  2. f preserves angles.
- **Diffeomorphism**: A homeomorphism for which f, f^{-1} are differentiable.
- **Biholomorphic** (map): A function $f : U \to V$ that is bijective, holomorphic, and for which f^{-1} is holomorphic.
- Theorem/observation: Biholomorphic iff conformal.
- Chain rule:
$$\frac{\partial}{\partial t}(f \circ g)(z) = f_z(g(z))g_z(z) + f_{\bar{z}}(g(z))\bar{g}_{\bar{z}}(z)$$
- **Complex linear map**: A map $l : \mathbb{C} \to \mathbb{C}$ characterized by the following. *Constraints*
  1. $l(z + w) = l(z) + l(w)$;
  2. $l(rz) = rl(z)$;

for $z, w, r \in \mathbb{C}$.

- Every complex linear map is of the form

$$w = l(z) = az$$

for a unique $a \in \mathbb{C}$.

- **Real linear map**: A map $l : \mathbb{C} \to \mathbb{C}$ characterized by the following. *Constraints*

1. $l(z + w) = l(z) + l(w)$;
2. $l(rz) = rl(z)$;

for $z, w \in \mathbb{C}$ and $r \in \mathbb{R}$.

- Every real linear map is of the form

$$w = l(z) = az + b\bar{z} = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \end{pmatrix}$$

for a unique pair $\begin{pmatrix} a & b \end{pmatrix} \in \mathbb{C}^2$.

- Implication: l is complex linear iff $b = 0$.

- **Tangent map** (of f at z_0): The real linear map from $\mathbb{C} \to \mathbb{C}$ determined by the vector $\begin{pmatrix} f_z(z_0) & f_{\bar{z}}(z_0) \end{pmatrix}$.
- Proposition: f is holomorphic at z_0 iff its tangent map at z_0 is complex linear.
- **Exponential function**: The complex function defined as follows. *Denoted by* $\mathbf{e^z}$, $\mathbf{exp(z)}$. *Given by*

$$\mathrm{e}^z = \exp(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

- **Pointwise** (convergent $\{f_n\}$): A sequence of functions $f_n : \mathbb{C} \to \mathbb{C}$ such that for all $z \in \mathbb{C}$, we have $f_n(z) \to f(z)$.
- **Locally uniformly** (convergent $\{f_n\}$): A sequence of functions $f_n : U \to \mathbb{C}$ and a function $f : U \to \mathbb{C}$ such that for all compact $K \subset U$,

$$\sup_{z \in K} |f_n(z) - f(z)| \to 0$$

- Lemma: If $f_n \to f$ locally uniformly and the f_n are continuous (or integrable; *not* differentiable), then so is f.
- **Taylor's theorem**: If $f : \mathbb{R} \to \mathbb{R}$ is C^{k+1} and $P_\alpha^k(x)$ is the k^{th} Taylor polynomial about $\alpha \in \mathbb{R}$, then for all $\beta \in \mathbb{R}$, there exists some $x \in (\alpha, \beta)$ such that

$$f(\beta) - P_\alpha^k(\beta) = \frac{(\beta - \alpha)^{k+1}}{(k+1)!} f^{(k+1)}(x)$$

- **Analytic** (function): A function $f : \mathbb{R} \to \mathbb{R}$ for which the Taylor polynomials converge (locally uniformly) to f.
- **Absolutely** (locally uniformly convergent power series): A power series $P(z) = \sum_{k=0}^{\infty} a_k z^k$ for which $A_N : \mathbb{C} \to \mathbb{R}$ locally uniformly converges, where

$$A_N(z) := \sum_{k=0}^{N} |a_k z^k|$$

- **Geometric series test**: If $|z| < 1$, then

$$\sum_{k=0}^{\infty} z^k \to \frac{1}{1-z}$$

- Lemma: Let $P(z)$ be a power series about 0. If there exists $z_1 \neq 0$ such that $|a_k z_1^k| \leq M$ for all k, then $P(z) = \sum a_k z^k$ converges on the disk $|z| < |z_1|$.

  Proof. Choice of z_1, z_2, and their ratio. □

- **Disk of convergence**: The largest disk centered at zero on which you converge.
- **Radius of convergence**: The radius of the disk of convergence. *Denoted by* $\boldsymbol{r}$.
- **Cauchy-Hadamard formula**: The radius of convergence is given by

$$r = (\limsup |a_k|^{1/k})^{-1}$$

- Lemma (from real analysis): If $f_n \to f$ locally uniformly and $f_n' \to g$ locally uniformly, then f is differentiable and $f' = g$.
  - Implication: Convergent power series are holomorphic.
- Corollary: Power series representations are unique.
  1. If $P(z) = \sum a_k z^k$ is convergent, then

$$a_k = \frac{1}{k!} P^{(k)}(0)$$

  2. If $P(z) = 0$ in a neighborhood of zero, then $a_k = 0$ for all k.
  3. If $P(z) = Q(z)$ (where $Q(z) = \sum b_k z^k$) in a neighborhood of 0, then $a_k = b_k$ for all k.
- Properties of the complex exponential.
  1. $\exp(z) = [\exp(z)]'$.
     - We obtain this via term-by-term differentiability.
     - This is just our favorite formula $\mathrm{d}/\mathrm{d}t\ (\mathrm{e}^t) = \mathrm{e}^t$ from calculus.
  2. $\overline{\exp(z)} = \exp(\bar{z})$.
  3. $\exp(a + b) = \exp(a) \cdot \exp(b)$.
  4. $|\exp(z)| = \exp[\mathrm{Re}(z)]$.
  5. $\mathrm{e}^{iz} = \cos(z) + i\sin(z)$.
- Complex trigonometric functions.

$$\cos(z) := \frac{1}{2}(\mathrm{e}^{iz} + \mathrm{e}^{-iz}) \qquad \sin(z) := \frac{1}{2i}(\mathrm{e}^{iz} - \mathrm{e}^{-iz})$$
$$\cosh(z) := \cos(iz) \qquad \sinh(z) := i\sin(iz)$$

- **Domain**: A connected, open set $U \subset \mathbb{C}$.
- **Primitive** (of f): A differentiable function whose derivative is equal to the original function f. *Also known as* **antiderivative**, **indefinite integral**. *Denoted by* $\boldsymbol{F}$.

- Corollary to the FTC: If $f = F'$, then for any closed curve γ in U,

$$\int_\gamma f \,\mathrm{d}z = 0$$

- Proposition: If $f : U \to \mathbb{C}$ is continuous and $\int_\gamma f \,\mathrm{d}z = 0$ for every closed loop in U, then f has a primitive on U.

  Proof. Step 1: Choose the integral along arbitrary γ.

  Step 2: Choice of γ doesn't matter (closed loop condition).

  Step 3: Correct derivative; apply FTC along δ and take limit. □

- **Star-shaped** (domain): A domain $U \subset \mathbb{C}$ for which there exists $a \in U$ such that for all $z \in U$, the segment $a \to z$ is in U.
- Lemma: If U is star-shaped and for every triangle with one vertex at a, we have $\int_\triangle f \,\mathrm{d}z = 0$, then f has a primitive in U.
- **Cauchy Integral Theorem**: Suppose U is a star-shaped domain and $f : U \to \mathbb{C}$ is holomorphic. Then $\int_\gamma f \,\mathrm{d}z = 0$ for any closed loop γ in U.

  Proof. Step 1: Prove f has a primitive via lemma & Goursat's lemma.

  Step 2: Apply FTC. □

- **Goursat's lemma**: If f is holomorphic in a neighborhood of a triangle including the interior, then $\int_\triangle f \,\mathrm{d}z = 0$.

  Proof. Subdividing triangles and inequalities. □

- Evaluating integrals using the complex functions and various paths.
- **Ratio test**: For $\sum a_n$, think about

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

- **Root test**: For $\sum a_n$, think about

$$\lim_{n\to\infty} |a_n|^{1/n}$$

- **Majorant test**: If $\sum_{k=0}^\infty a_k$ is a convergent series with positive terms and if for almost all k and all $z \in M$ we have $|f_k(z)| \leq a_k$, then $\sum_{k=0}^\infty f_k$ is absolutely uniformly convergent on M.
- Exponential mappings.
  - $z = x + iy_0$ maps onto the open ray beginning at 0 and passing through e^{iy_0}.
  - $z = x_0 + iy$ maps onto the circle of radius e^{x_0}.
  - Half-open horizontal strips map bijectively onto $\mathbb{C}^*$.
- **Homotopic** (paths): Two paths $\gamma, \tilde{\gamma} \subset U$ a domain such that $\tilde{\gamma}$ is obtained from γ by modifying γ on a small disk $D \subset U$, keeping the endpoints fixed.
- Claim/TPS: This argument shows that if γ and $\tilde{\gamma}$ are homotopic in U and $f \in \mathcal{O}(U)$, then

$$\int_\gamma f \,\mathrm{d}z = \int_{\tilde{\gamma}} f \,\mathrm{d}z$$

Proof. Each bump is a closed loop for the CIT. □

- Corollary: Let U be any domain, D be a disk in U, and $z \in \mathring{D}$. Suppose $f \in \mathcal{O}(U \setminus \{z\})$ and is bounded near z. Then
$$\int_{\partial D} f \, \mathrm{d}z = 0$$

Proof. Homotopy and γ_ε. □

- **Cauchy Integral Formula**: Suppose U is any domain, $D \subset U$ is a disk (i.e., $D \subset\subset U$ or $\overline{D} \subset U$), $f \in \mathcal{O}(U)$, and $z \in D$. Then
$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

Proof. Define the helper function
$$g(\zeta) = \begin{cases} \frac{f(\zeta) - f(z)}{\zeta - z} & \zeta \neq z \\ f'(z) & \zeta = z \end{cases}$$
It integrates to zero on ∂D and then splits into the two sides of the CIF. □

- Corollary: Holomorphic functions are C^∞.
- Corollary: In general,
$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \mathrm{d}\zeta$$
- **Cauchy's inequalities**:
$$|f^{(n)}(z)| \leq \frac{n!}{R^n} \max_{\partial D} |f(\zeta)|$$
- Liouville's Theorem: Suppose $f \in \mathcal{O}(\mathbb{C})$ (i.e., f is **entire**) and f is bounded. Then it's constant.

Proof. Cauchy's inequalities on a really big disk to limit $|f'|$. □

- **Entire** (function): A complex-valued function that is holomorphic on the whole complex plane.
- The Identity Theorem: If two holomorphic functions $f, g \in \mathcal{O}(U)$ agree on an open set in U, then $f = g$.

Proof. True for power series. □

  - In fact, more is true: If $z_n \to z_0$ where each z_n is distinct and $f(z_n) = g(z_n)$ for all n, then $f = g$.
- **Analytic continuation** (of f): The function $g \in \mathcal{O}(V)$ where $f \in \mathcal{O}(U)$, $V \supset U$, and $f = g$ on U.
- Morera's Theorem: If U is any domain, $f : U \to \mathbb{C}$ is continuous, and $\int_\triangle f \, \mathrm{d}z = 0$ for all triangles, then f is holomorphic.

Proof. The primitive exists. The primitive is holomorphic. Therefore, $F' = f$ is holomorphic. □

- **Riemann's removable singularity theorem**: Suppose U is a domain, $z \in U$, $f \in \mathcal{O}(U \setminus \{z\})$, and f is bounded near z. Then there exists a unique analytic continuation $\hat{f} \in \mathcal{O}(U)$. *Also known as* **Riemann extension theorem**.

Proof. Define a helper function

$$F(\zeta) = \begin{cases} f(\zeta)(\zeta - z) & \zeta \neq z \\ 0 & \zeta = z \end{cases}$$

Use Morera's theorem: F is continuous, triangles in two cases (CIT and γ_ε), and $F' = f$ via the limit definition. ☐

- **Singularity** (of f): A point z_0 such that $f \in \mathcal{O}(U \setminus \{z_0\})$.
- **Removable** (singularity): A singularity of a function that that satisfies the hypotheses of Riemann's removable singularity theorem.
- If a singularity is not removable, then f is not bounded near z_0. This leads to additional definitions.
- **Pole**: A non-removable singularity z_0 of a function f for which $|f(z)| \to \infty$ as $z \to z_0$.
  - So-named because of real analysis where a pole is an asymptote, and asymptotes kind of look like poles!
- **Essential** (singularity): A non-removable singularity that is not a pole; equivalently, a singularity z_0 for which there exist sequences $z_n \to z_0$ and $w_n \to z_0$ such that $|f(z_n)| \to \infty$ and $|f(w_n)|$ stays bounded.
- **Meromorphic** (function): A function $f : U \to \mathbb{C}$ such that $f \in \mathcal{O}(U \setminus P)$ and each $p \in P$ is a pole, where $P \subset U$ is a finite set of points.
- Orders of zeros and poles.
  - Invert the function, find a power series, divide $(z-p)^L$ out, find the power series of h, invert, find the principal part of the **Laurent series**.
- Theorem (maximum modulus principle): Let $f \in \mathcal{O}(U)$. If $|f(z)|$ has a local maximum on U, then f is constant.

  Proof. Step 1: Long inequality through the CIF that becomes equality.

  Step 2: Subtract and get integrand equal to zero; $|f|$ is constant on ∂D.

  Step 3: $|f|^2$ is constant on ∂D, differentiate, casework to f is constant or zero. ☐

- Corollary (minimum modulus principle): If $f \in \mathcal{O}(U)$, $f \neq 0$ on U (hence $1/f \in \mathcal{O}(U)$), and $|f(z)|$ takes a minimum in U, then f is constant.