Week 2

???

2.1 Office Hours

3/25:

- What exactly are the Wirtinger derivatives?
  - The $\partial/\partial z$ and $\partial/\partial \bar{z}$ operators.
- The initial definition of holomorphic is accurate. It's naïve, but it works out.
- Noney: Non example.
  - As in, we have some examples of holomorphic functions and then we have an example of a function that is *not* holomorphic.
- TPS: Think Pair Share.
- Met Panteleymon and helped him with partial fractions!
- The Δ notation does mean the same Laplacian as $\vec{\nabla}^2$ from Quantum Mechanics.
- Calderon is not related to Calderón; he was Argentinian, Calderon is half-Filipino and has no accent on his name. Both Spanish colonies but that's it.
- We can do all of the problems except Problem 1 at this point.
  - For this, though, we can just look up the definition of the complex sine function.
  - We basically just need to know what $\sin(i)$ is and what sine looks like along the imaginary axis.

2.2 Power Series

3/26:

- Recall: We already know that...
  - Polynomials are elements of $\mathcal{O}(\mathbb{C})$;
  - Rational functions $P(z)/Q(z)$ are elements of $\mathcal{O}(\mathbb{C} \setminus V(Q))$.
- **Affine algebraic set**: The set of solutions in an algebraically closed field K of a system of polynomial equations with coefficients in K. *Also known as* **variety**. *Denoted by* $\boldsymbol{V(f_1, \dots, f_n)}$.
- Today, we want to determine how the other elementary functions behave over the complex numbers.
  - Other functions we want: exp, log, sin, cos.
  - We will do log later, but all the others today.

- **Exponential function**: The complex function defined as follows. *Denoted by* $\mathbf{e^z}$, $\mathbf{exp(z)}$. *Given by*

$$\mathrm{e}^z = \exp(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

- Naïvely, this power series is just be a polynomial $P(z) \in \mathcal{O}(\mathbb{C})$.
- More rigorously, however, we must specify which kind of convergence we mean for the power series.
  - As one example, we could say that for all z,

$$\mathrm{e}^z = P(z) = \lim_{N\to\infty} \sum_{k=0}^{N} \frac{z^k}{k!}$$

    - This would be **pointwise convergence**.
  - But there's an issue: Pointwise convergence of functions doesn't preserve anything, e.g., continuity.
- **Pointwise** (convergent $\{f_n\}$): A sequence of functions $f_n : \mathbb{C} \to \mathbb{C}$ such that for all $z \in \mathbb{C}$, we have $f_n(z) \to f(z)$.
- TPS: Come up with an example of a sequence of continuous functions $\{f_n\}$ that converges pointwise to f, such that the f_n are all...
  1. Continuous but f is not;
     - $f_n(x) = \arctan(nx)$.
     - Converges to the sign function $f(x) = \operatorname{sgn}(x)$.
  2. Odd but f is not;
  3. Differentiable but f is not.
     - These last two cases were not discussed in class.
- We now recall a few definitions and lemmas from real analysis.
- **Locally uniformly** (convergent $\{f_n\}$): A sequence of functions $f_n : U \to \mathbb{C}$ and a function $f : U \to \mathbb{C}$ such that for all compact $K \subset U$,

$$\sup_{z\in K} |f_n(z) - f(z)| \to 0$$

- Lemma: If $f_n \to f$ locally uniformly and the f_n are continuous (or integrable), then so is f.
  - This lemma is *not* true if we sub in "differentiable!"
  - See the Stone-Weierstrass theorem for suitable constraint.
- Thus, to resolve the original question, we mean that $P_N(z) \to \exp(z)$ locally uniformly.
- Aside: Which functions have power series?
  - Remember Taylor polynomials from Calc II? **Taylor's theorem** tells us which ones converge.
- **Taylor's theorem**: If $f : \mathbb{R} \to \mathbb{R}$ is C^{k+1} and $P_\alpha^k(x)$ is the k^{th} Taylor polynomial about $\alpha \in \mathbb{R}$, then for all $\beta \in \mathbb{R}$, there exists some $x \in (\alpha, \beta)$ such that

$$f(\beta) - P_\alpha^k(\beta) = \frac{(\beta-\alpha)^{k+1}}{(k+1)!} f^{(k+1)}(x)$$

  - Essentially a version of the mean value theorem (MVT) for higher-order derivatives.
  - We can use the term of the right side of the equals sign above to get a bound on the error of the Taylor polynomial.

- **Analytic** (function): A function $f : \mathbb{R} \to \mathbb{R}$ for which the Taylor polynomials converge (locally uniformly) to f.
- Non example: The C^∞ function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} \mathrm{e}^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

  - An excellent exercise in real analysis is to check that for all k, the Taylor polynomial about 0 is 0.
  - If we take the Taylor polynomial at some point farther from zero, the polynomial will approximate f well up until zero, but then it will "hit a wall."
    - The point is that f is decaying more rapidly toward 0 than any polynomial possibly could, so the polynomial just thinks it's seeing 0.
- **Absolutely** (locally uniformly convergent power series): A power series $P(z) = \sum_{k=0}^{\infty} a_k z^k$ for which $A_N : \mathbb{C} \to \mathbb{R}$ locally uniformly converges, where

$$A_N(z) := \sum_{k=0}^{N} |a_k z^k|$$

- Absolute local uniform convergence allows you to reorder the terms in the polynomial.
  - It also explains why you cannot reorder the terms in the series $S = 1+1-1+1-1+\cdots$, i.e., why manipulating the order allows you to get any number: This series S does not converge absolutely!
  - Formally, if $\sigma : \mathbb{N} \to \mathbb{N}$ is a permutation and $\sum^\infty a_k$ converges absolutely, then $\sum^\infty a_{\sigma(k)}$ converges.
- Exercise: Show that

$$\sum_{k=0}^{\infty} z^k \to \frac{1}{1-z}$$

converges absolutely locally uniformly on $\mathbb{D} = \{|z| < 1\}$.

Proof. To prove this, we just have to show that $\sum^\infty |z|^k$ converges on $|z| < 1$. But it does so converge because this latter series is just a standard real geometric series. □

- This example generalizes somewhat into the following lemma.
- Lemma: Let $P(z)$ be a power series about 0. If there exists $z_1 \neq 0$ such that $|a_k z_1^k| \leq M$ for all k, then $P(z) = \sum a_k z^k$ converges on the disk $|z| < |z_1|$.

Proof. Uses standard series convergence results from real analysis. May be in Fischer and Lieb (2012)?? □

- **Disk of convergence**: The largest disk centered at zero on which you converge.
- **Radius of convergence**: The radius of the disk of convergence.
- **Cauchy-Hadamard formula**: The radius of convergence is given by

$$\text{rad} = (\limsup |a_k|^{1/k})^{-1}$$

  - We will be using this result on PSet 2.
  - We will also be proving it there!

- What are power series representations good for? Here's an example of how they can be applied to help with PSet 1, QA.4.
  - Question: For $|a| < 1$ and $\gamma(t) = \mathrm{e}^{it}$ a parameterization of a closed loop oriented counterclockwise, compute
$$\int_\gamma \frac{1}{z-a}\,\mathrm{d}z$$
  - Answer:
    - Since $|a| < 1$, we know that on γ, $|a/\gamma(t)| < 1$.
    - Thus, we have that
$$\begin{aligned}\int_\gamma \frac{1}{z-a}\,\mathrm{d}z &= \int_\gamma \frac{1}{z}\frac{1}{1-a/z}\,\mathrm{d}z\\ &= \int_\gamma \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{a}{z}\right)^k \mathrm{d}z\\ &= \int_\gamma \sum_{k=0}^{\infty}\frac{a^k}{z^{k+1}}\,\mathrm{d}z\\ &= \sum_{k=0}^{\infty}\int_\gamma \frac{a^k}{z^{k+1}}\,\mathrm{d}z\\ &= \cdots\\ &= \int_\gamma \frac{1}{z}\,\mathrm{d}z\end{aligned}$$
    - We have the second equality because the power series converges.
    - We have the fourth equality because of the lemma about integrable f_n and the fact that the power series converges.
    - The dots indicate some more steps that we will need to work out for ourselves on PSet 1.
- Lemma (from real analysis): If $f_n \to f$ locally uniformly and $f_n' \to g$ locally uniformly, then f is differentiable and $f' = g$.
  - This is true for both differentiable and holomorphic functions.
- Claim: This lemma implies that convergent power series are holomorphic.

Proof. If
$$f_N = \sum_{k=0}^{N} a_k z^k$$
then
$$f_N' = \sum_{k=0}^{N} k \cdot a_k z^{k-1}$$
We want to show that $\{f_N'\}$ converges (locally absolutely uniformly). Fischer and Lieb (2012) do this by hand. We can also use the Cauchy-Hadamard formula, which we will do presently.

Let's look at $\limsup(k\cdot a_k)^{1/k}$. But this is just equal to
$$\limsup|k\cdot a_k|^{1/k} \leq \limsup(|k|^{1/k})\cdot\limsup(|a_k|^{1/k}) = 1\cdot\limsup(|a_k|^{1/k}) = \limsup|a_k|^{1/k}$$
Moreover, equality holds because that $k^{1/k}$ factor just decays toward 1; think about how k increases linearly and the k^{th} root decays faster. □

- Proposition: Any convergent power series is holomorphic (on its disk) and its derivative is also a power series with the same radius of convergence. It follows that power series are analytic functions and are C^∞.
- Spoiler: Every holomorphic function is analytic.
- Corollary: Power series representations are unique.
  1. If $P(z) = \sum a_k z^k$ is convergent, then
  $$a_k = \frac{1}{k!}P^{(k)}(0)$$
  2. If $P(z) = 0$ in a neighborhood of zero, then $a_k = 0$ for all k.
  3. If $P(z) = Q(z)$ (where $Q(z) = \sum b_k z^k$) in a neighborhood of 0, then $a_k = b_k$ for all k.
- Let's now return to the exponential function, which got this whole discussion started.
- We now know that the definition
  $$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$
  makes sense.
- By manipulating this power series, we can get lots of fun properties.
  1. $\exp(z) = [\exp(z)]'$.
     - We obtain this via term-by-term differentiability.
     - This is just our favorite formula $\mathrm{d}/\mathrm{d}t\ (\mathrm{e}^t) = \mathrm{e}^t$ from calculus.
  2. $\overline{\exp(z)} = \exp(\bar{z})$.
  3. $\exp(a + b) = \exp(a) \cdot \exp(b)$.
  4. $|\exp(z)| = \exp[\mathrm{Re}(z)]$.
- **Complex cosine**: The complex function defined as follows. *Denoted by* $\boldsymbol{\cos(z)}$. *Given by*
  $$\cos(z) := \frac{1}{2}(\mathrm{e}^{iz} + \mathrm{e}^{-iz})$$
- **Complex sine**: The complex function defined as follows. *Denoted by* $\boldsymbol{\sin(z)}$. *Given by*
  $$\sin(z) := \frac{1}{2i}(\mathrm{e}^{iz} - \mathrm{e}^{-iz})$$
- **Complex hyperbolic cosine**: The complex function defined as follows. *Denoted by* $\boldsymbol{\cosh(z)}$. *Given by*
  $$\cosh(z) := \cos(iz)$$
- **Complex hyperbolic sine**: The complex function defined as follows. *Denoted by* $\boldsymbol{\sinh(z)}$. *Given by*
  $$\sinh(z) := i\sin(iz)$$
- We also have
  $$\mathrm{e}^{iz} = \cos(z) + i\sin(z)$$
  - If z is real and in $[0, 2\pi]$, then this simplifies to Euler's formula
  $$\mathrm{e}^{i\theta} = \cos(\theta) + i\sin(\theta)$$
- Calderon draws some mappings of the exponential function but doesn't linger on what's going on.
- These are the preliminaries; now, we'll dive into the meat of the course.