MATH 27000 (Basic Complex Variables) Notes

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Weeks

List of Figures

Week 1

Classifying Complex Functions

1.1 Holomorphic Functions

3/19:
- We begin by reviewing some properties of the **complex numbers**.
- **Complex numbers**: The field of elements $z = x + iy$ where $x, y \in \mathbb{R}$ and $i^2 = -1$. *Denoted by* $\mathbb{C}$.

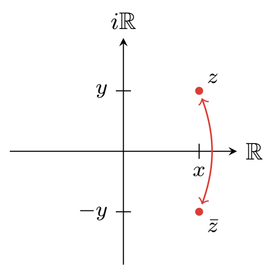


Figure 1.1: The complex plane.

  - Can be visualized as a two-dimensional plane with the number z corresponding to the point (x, y).
- **Real part**: The number x. *Denoted by* $\mathbf{Re}\,z$.
- **Imaginary part**: The number y. *Denoted by* $\mathbf{Im}\,z$.
- **Complex conjugate** (of z): The complex number defined as follows. *Denoted by* $\bar{z}$. *Given by*

$$\bar{z} := x - iy$$

- Now recall the definition of a *real* function that is **differentiable** at a point $x_0 \in \mathbb{R}$.
  - $f'(x_0)(x - x_0)$ is the "best linear approximation" of f near x_0, where $f'(x_0)$ is also defined below.
- **Differentiable** ($f : \mathbb{R} \to \mathbb{R}$ at x_0): A function f for which the following limit exists. *Constraint*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} =: f'(x_0)$$

- We now build up to defining a notion of complex differentiability.
  - Observe that the constraint above is equivalent to the constraint

$$f(x) = f(x_0) + [\underbrace{f'(x_0) + e(x)}_{\Delta(x)}](x - x_0)$$

    where $e(x) \to 0$ as $x \to x_0$.
  - Note that we are defining a new function $\Delta(x)$ above, with the property that $\Delta(x_0) = f'(x_0)$.

- **Holomorphic** (f at z_0): A function $f : \mathbb{C} \to \mathbb{C}$ for which the following limit exists. *Also known as* **$\mathbb{C}$-differentiable**. *Constraints*

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} =: f'(z_0) \qquad \Longleftrightarrow \qquad f(z) = f(z_0) + \Delta(z)(z - z_0)$$

  where Δ is continuous at z_0 and $\Delta(z_0) = f'(z_0)$.
  - It will turn out that this is the true definition of "holomorphic" / "$\mathbb{C}$-differentiable" function, not just a naïve first pass.
- Properties of holomorphic functions: Let $U \subset \mathbb{C}$ be open.
  1. The holomorphic functions on U form a ring $\boldsymbol{\mathcal{O}(U)}$.
     - Equivalently, the $\mathbb{C}$-differentiation operator is $\mathbb{C}$-linear.
     - Equivalently, if f, g are holomorphic, then $f + g$ and fg are holomorphic, too.
     - Equivalently (and most simply), we have the sum rule and the product rule (and the quotient rule if the function in the denominator is nonzero).
  2. We have the chain rule.
  3. Holomorphic implies continuous.
- Examples: Polynomials, rational functions $p(z)/q(z)$ (away from their **poles**).
- Non-example: Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$z \mapsto \bar{z}$$

  - Think-Pair-Share (TPS): Why?
  - Notice that

$$f(0) = 0 \qquad\qquad f(t) = t \qquad\qquad f(it) = -it$$

  - Thus,

$$\Delta(t) = 1 \qquad\qquad \Delta(it) = -1$$

    for all t.
  - But this means that Δ can't be continuous!
  - Yet f is clearly $\mathbb{R}$-differentiable! What gives?!
  - Note that — viewing f as a mapping of $\mathbb{R}^2 \to \mathbb{R}^2$ — we have

$$Df = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The above example suggests that our definition of complex differentiability may have been to naïve, so we'll do some further investigations now.
- Observe that $\mathbb{C} \cong \mathbb{R}^2$ as $\mathbb{R}$-vector spaces.
- **Differentiable** ($f : \mathbb{R}^2 \to \mathbb{R}^2$ at x_0): A function f for which there exists an $\mathbb{R}$-linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying the following constraint. *Constraint*

$$\lim_{h\to 0} \frac{\|f(x_0 + h) - f(x_0) - Ah\|}{\|h\|} = 0$$

  - We also denote A by Df.

- Example: Consider the function $f : \mathbb{C} \to \mathbb{R}$ defined by

$$x + iy \mapsto x$$

  - Differentiable with total derivative

$$Df = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

- Observation: While $\mathbb{C} \cong \mathbb{R}^2$ as $\mathbb{R}$-vector spaces, as a $\mathbb{C}$-vector space, there is *additional* structure.
  - In particular, all "vectors" should commute with the "multiplication by i" map $J : \mathbb{C} \to \mathbb{C}$ defined by any one of the following three maps.

$$z \mapsto z \qquad\qquad x + iy \mapsto xi - y \qquad\qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

- Exercise: In $(\mathrm{Re}, \mathrm{Im})$ coordinates, write down the matrix for "multiply by w" for any $w \in \mathbb{C}$.
  - Let $w = a + bi$ and let $v = x + iy$. Then

$$\begin{aligned} wv &= (a + bi)(x + iy) = ax - by + i(bx + ay) \\ &= \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix} = \underbrace{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}}_{W} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

  - The matrix W above is the desired result.
- TPS: Is $f : \mathbb{C} \to \mathbb{C}$ defined as follows a complex linear map? Why not?

$$x + iy \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (x + y) + iy$$

  - Among other properties, a complex linear map should satisfy

$$if(x + iy) = f[i(x + iy)]$$

    for the scalar $i \in \mathbb{C}$.
  - However, we have that

$$if(x + iy) = i[(x + y) + iy] = -y + i(x + y) \neq (x - y) + ix = f(-y + ix) = f[i(x + iy)]$$

- What about the following map?

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

  - A complex linear map should satisfy

$$A(v + w) = Av + Aw \qquad\qquad \lambda Av = A(\lambda v)$$

    for all $v, w, \lambda \in \mathbb{C}$.
  - Let $v, w \in \mathbb{C}$ be arbitrary. Then

$$\begin{aligned} A(v + w) &= \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right] = \begin{pmatrix} (v_1 + w_1) + 2(v_2 + w_2) \\ -2(v_1 + w_1) + (v_2 + w_2) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = Av + Aw \end{aligned}$$

- Let $v, \lambda \in \mathbb{C}$. Then

$$\begin{aligned}
\lambda Av &= (\lambda_1 + i\lambda_2) \cdot [(v_1 + 2v_2) + i(-2v_1 + v_2)] \\
&= [\lambda_1(v_1 + 2v_2) - \lambda_2(-2v_1 + v_2)] + i[\lambda_2(v_1 + 2v_2) + \lambda_1(-2v_1 + v_2)] \\
&= [(\lambda_1 v_1 - \lambda_2 v_2) + 2(\lambda_2 v_1 + \lambda_1 v_2)] + i[-2(\lambda_1 v_1 - \lambda_2 v_2) + (\lambda_2 v_1 + \lambda_1 v_2)] \\
&= A[(\lambda_1 v_1 - \lambda_2 v_2) + i(\lambda_2 v_1 + \lambda_1 v_2)] \\
&= A(\lambda v)
\end{aligned}$$

- Therefore, since A satisfies the two properties, it is complex linear.

- Conclusion: To reiterate from the above, A must commute with J to be complex linear.
- Implication: Every $\mathbb{C}$-linear map of $\mathbb{C}$ is just multiplication by a complex number.
  - This is a special case of the following more general result, which holds for any field K.

$$\operatorname{Hom}_K(K, K) \cong K$$

- Now let's revisit differentiability.
- It turns out that a condition for $\mathbb{C}$-differentiability *equivalent* to the definition of "holomorphic" given above is that there exists a $\mathbb{C}$-linear map $A : \mathbb{C} \to \mathbb{C}$ such that

$$\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - Ah\|}{\|h\|} = 0$$

  - From the above discussion, we know that this A is just multiplication by some $w \in \mathbb{C}$.
  - All of the values in the above norms are complex numbers, so *another* equivalent condition is

$$\lim_{z \to z_0} \frac{|f(z) - f(z_0) - w \cdot (z - z_0)|}{|z - z_0|} = 0$$

  - This condition is wholly mathematically equivalent to our holomorphic definition,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = w$$

- So when is an $\mathbb{R}$-differentiable function actually holomorphic?
  - Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ map $(x, y) \mapsto (g, h)$.
  - Let

$$A = Df = \begin{pmatrix} g_x & g_y \\ h_x & h_y \end{pmatrix}$$

  where the subscript notation views g, for instance, as $g(x, y)$ and denotes the partial derivative of g with respect to x.
  - Let J (the "multiply by i") function be defined as above.
  - Then the "commute with i" condition is equivalent to

$$J^{-1}AJ = A$$

  - Expanding the product on the left above in terms of g_x, g_y, h_x, h_y, we obtain

$$\begin{pmatrix} h_y & -h_x \\ -g_y & g_x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} g_x & g_y \\ h_x & h_y \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} g_x & g_y \\ h_x & h_y \end{pmatrix}$$

  - This condition is equivalent to A satisfying the **Cauchy-Riemann equations**.

- **Cauchy-Riemann equations**: The following two equations, which identify when a complex function is holomorphic. *Also known as* **CR equations**. *Given by*

$$
\begin{aligned}
g_x &= h_y \\
g_y &= -h_x
\end{aligned}
$$

- These equations are satisfied when A is of the form

$$
A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}
$$

- So at this point, we can differentiate f with respect to z. But what if we want to differentiate it with respect to x and y (of $z = x + iy$)?
  - We will need the following change of basis.
    - Since $z = x + iy$ and $\bar{z} = x - iy$, we have

$$
\begin{aligned}
2x &= z + \bar{z} & 2iy &= z - \bar{z} \\
x &= \frac{1}{2}(z + \bar{z}) & y &= -\frac{i}{2}(z - \bar{z})
\end{aligned}
$$

    - This tells us that

$$
\frac{\partial x}{\partial z} = \frac{1}{2} \qquad\qquad \frac{\partial y}{\partial z} = -\frac{i}{2}
$$

  - We can now invoke the multivariable chain rule and simplify the resultant expression.

$$
\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}(f_x - if_y)
$$

    - Note that once again, the subscript notation "f_x" means $\partial f / \partial x$.
  - Note that we can also similarly work out that

$$
\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(f_x + if_y)
$$

    - Observe in particular that

$$
f_x = g_x + ih_x \qquad\qquad f_y = g_y + ih_y
$$

    - Thus, the CR equations ($g_x = h_y$ and $g_y = -h_x$) being satisfied is equivalent to

$$
\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(f_x + if_y) = \frac{1}{2}[(g_x + ih_x) + i(g_y + ih_y)] = 0
$$

    - Note that $\partial f / \partial \bar{z}$ is not actually a derivative since f depends on z, not $\bar{z}$. Rather, we use "$\partial f / \partial \bar{z}$" to denote the following operator applied to f.

$$
\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)
$$

- **Wirtinger derivatives**: The two differential operators defined as follows. *Denoted by* $\boldsymbol{\partial/\partial z, \partial/\partial \bar{z}}$. *Given by*

$$
\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \qquad\qquad \frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)
$$

- Theorem: The $\mathbb{R}$-differentiable function $f : U \to \mathbb{C}$ is holomorphic iff $\partial f / \partial \bar{z} = 0$. Moreover, if it is, then

$$
f'(z_0) = \left.\frac{\partial f}{\partial z}\right|_{z_0}
$$

1.2 Harmonic Functions and Conformal Maps

3/21:

- Tries to remember everyone's name and actually does a pretty good job!
- Has us all turn to our neighbor and meet them! I met Ryan.
- Review.
  - Naïve holomorphic definition: Typical derivative definition.
  - The map $z \mapsto \bar{z}$ is not holomorphic even though it is differential over the reals.
  - The reason this map is not holomorphic is that its matrix derivative is not complex linear. This means that it does not commute with the "multiply by i" matrix, defined by

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

  - Indeed, an *equivalent* definition to the naïve holomorphic one is: $f : \mathbb{C} \cong \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathbb{R}$-differentiable at z_0 with $Df(z)$ is complex linear.
  - Another equivalent one is the Cauchy-Riemann equation definition.
    - Let $f(z) = u(z) + iv(z)$ where $u, v : \mathbb{C} \to \mathbb{R}$.
    - Then f is holomorphic if $u_x = v_y$ and $v_x = -u_y$, or equivalently if $\partial f / \partial \bar{z} = 0$.
  - The above comment motivates the definition of the operators

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

  - Note: Every $\mathbb{C}$-linear map is "multiply by w" for some $w \in \mathbb{C}$.
  - Note that we have not yet talked about continuity or related things.
- Note: Different books use different conventions.
  - "Holomorphic at a point" and "complex differentiable in a neighborhood of a point" may mean different things.
  - Example: Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$x + iy \mapsto x^2 + iy^2$$

    - Then
$$Df = \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix}$$
    - Evidently, Df is not complex linear in general because we can pick x, y such that $2x \neq 2y$.
    - Thus, this function is not complex differentiable in general.
    - However, it is complex differentiable at zero because here, $Df = 0$.
    - Thus, this function is complex differentiable *at a point*, but not complex differentiable *in a neighborhood*.
  - We will almost always be talking about functions that are complex differentiable *in a neighborhood* in this class.
- Example: $f(x + iy) = x^2 \mathbb{I}_{Q(x)} + iy^2 \mathbb{I}_{Q(y)}$ is complex differentiable in a neighborhood of the origin, but this is dumb. $\mathbb{I}$ is the **indicator function**.
- Preview (we'll see this next Thursday): Holomorphic implies C^∞.

- Today: Some more things about the Cauchy-Riemann equations and what we can get out of them.
- Let's begin with a consequence of the $\partial/\partial z$ and $\partial/\partial \bar{z}$ operators.
  - Compute (if $f \in C^2$):
$$\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4} \underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)}_{\Delta} f_{xy} = f_{yx}$$
    - Do we need this C^2 condition if holomorphic already implies C^∞?
    - No, but we haven't "learned" this yet. Once we prove this, no more talk of regularity!
  - Solutions to this, the Laplacian Δ (from physics), could be a good final project!
  - Look for solutions to $\Delta f = 0$.
    - Equivalently, look for f such that $f_{xx} + f_{yy} = 0$.
  - Observation: Any f holomorphic implies that $\Delta f = 0$ (since we apply $\partial/\partial \bar{z}$ to f first).
- **Harmonic** (function): A function $f : \mathbb{R}^2 \to \mathbb{C}$ such that $\Delta f = 0$.
- Since the differentiation operator is linear,
$$\Delta(u + iv) = \Delta u + i\Delta v$$
- Corollary: The real and imaginary parts of a C^2 holomorphic function are harmonic.
- So we know that f holomorphic implies u, v real-valued and harmonic. Can we go the other way?
  - We know that these functions have certain properties in terms of their partial derivatives, namely that they satisfy the Cauchy-Riemann equations.
- **Harmonic conjugates**: Two functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ that satisfy the CR equations.
- Fact: Let u, v be two C^∞ harmonic conjugates. Then $f = u + iv$ is holomorphic
  - Easy to prove if you're not concerned about regularity.
- Is $v + iu$ holomorphic?
  - No, partials don't work out. We still get $v_x = -u_y$, but we also get $u_x = -v_y$.
  - However, $v - iu$ is holomorphic!
  - This just means that rotating by i gives us a new holomorphic function since
$$i \cdot (u + iv) = -v + iu$$
- Example: $u = x^2 - y^2$ is harmonic. Find a conjugate and find $f = u + iv$.[1]
  - We have
$$\begin{aligned} v_y &= u_x = 2x \\ v_x &= -u_y = 2y \end{aligned}$$
  - Thus,
$$v = 2xy + C$$
    for some $C \in \mathbb{C}$
  - Then we would have
$$f = u + iv = (x^2 - y^2) + i(2xy + C) = x^2 + 2xyi - y^2 + iC = (x + iy)^2 + iC = z^2 + iC$$

[1] Calderon actually let us work this out in class!

- Let's now talk about integrating functions.
- Let $a, b \in \mathbb{R}$. Consider $f : \mathbb{C} \to \mathbb{C}$, not holomorphic but continuous. How do we take $\int_a^b f \, \mathrm{d}z$?

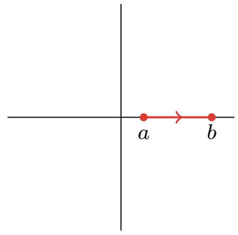


Figure 1.2: Integrating complex functions over real intervals.

  - What we do is just split the integral into real and imaginary parts.

$$\int_a^b f \, \mathrm{d}z = \int_a^b u \, \mathrm{d}t + i \int_a^b v \, \mathrm{d}t$$

  - This is how we integrate between reals in the complex plane.
- How do we integrate over more arbitrary points in the complex plane, e.g., $a\lambda$ and $b\lambda$?

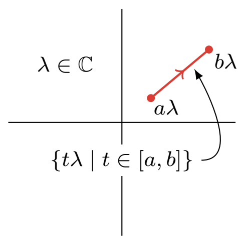


Figure 1.3: Integrating complex functions over line segments in $\mathbb{C}$.

  - We could take any path. Which one?
  - Try over the line segment $\{t\lambda \mid t \in [a, b]\}$.
  - Then we take

$$\int_{a\lambda}^{b\lambda} f(z) \, \mathrm{d}z = \int_a^b f(\lambda t)\lambda \, \mathrm{d}t$$

    via the substitutions $z = t\lambda$ and $\mathrm{d}z = \lambda \, \mathrm{d}t$.
  - This second integral, we can compute in the first way.
- Now what about integrating along an arbitrary curve $\gamma : [a, b] \to \mathbb{C}$, which we will say is piecewise C^1?

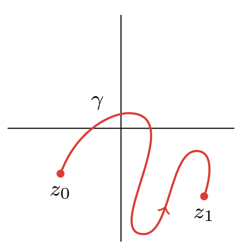
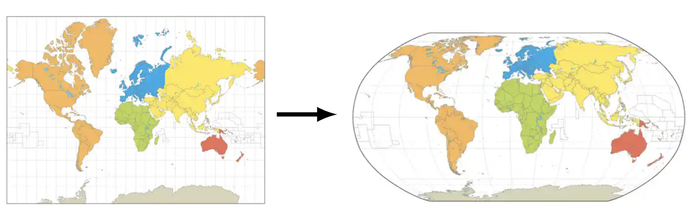


Figure 1.4: Integrating complex functions over arbitrary paths in $\mathbb{C}$.

- Note that $z_0 := \gamma(a)$ and $z_1 := \gamma(b)$.
- Define
$$\int_\gamma f \,\mathrm{d}z = \int_a^b f(\gamma(t)) \cdot \gamma'(t) \,\mathrm{d}t$$
- Where is the γ' coming from? Same place as before! It's just a change of variables: $z = \gamma(t)$ implies $\mathrm{d}z = \gamma'(t)\,\mathrm{d}t$.
- If we know differential forms, $f\,\mathrm{d}z$ is just a complex-valued one-form. And the chain rule is just how we integrate one-forms.
- We'll do lots of basic practice of this in the completion problems on the PSet.

- Note: Whenever we do a path integral, we should ask if the parameterization matters. The parameterization does *not* matter.
- What do we need to compute integrals without having to take the limit of a sum over partitions? We need the fundamental theorem of calculus.
  - The FTC does indeed hold here, too, though we won't prove this.
- FTC: Suppose $F' = f$ on $U \subset \mathbb{C}$, and let γ be a **path** inside of U. Then
$$\int_\gamma f\,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$$
  - Again, if we like differential forms, then note that we're just saying in the above equality that
$$\int_\gamma \mathrm{d}F = \int_{\partial\gamma} F$$
- **Path**: A function from an interval of real numbers to a vector space. *Also known as* **contour**. *Denoted by* $\boldsymbol{\gamma}$.
- Gives us a three-minute break from 10:17-10:20 in the middle of the class.
  - The fact that this guy actually teaches in accordance with accepted pedagogical standars is wild.
- How do we want to visualize holomorphic functions?
  - $f : \mathbb{C} \to \mathbb{C}$ is hard to graph because the set of points lives in $\mathbb{R}^4$.
  - So we're out of luck if we want to do graphs.
  - Thus, we'll look at **mappings**.
- Example: Are we looking at the Mercator or Robinson map of the world?



(a) Mercator projection. (b) Robinson projection.

Figure 1.5: Visualizing functions of $\mathbb{R}^2 \to \mathbb{R}^2$.

  - What do these mappings do to the lines of latitude and longitude?
  - This is a mapping of $\mathbb{R}^2 \to \mathbb{R}^2$ that stretches and distorts! By drawing grid lines, we can see what it does to $\mathbb{R}^2$.

- Now recall that f holomorphic implies Df looks like
$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$
  - What's nice about these matrices is they can always be factored into rotation and scaling matrices.
$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \qquad (\lambda, \theta \in \mathbb{R})$$
  - This means that
$$f'(z_0) = w = re^{i\theta} \in \mathbb{C}$$
since multiplication by complex numbers also just rotates and scales!
  - This means that if $z' = r'e^{i\theta'}$, then $w \cdot z' = r \cdot r'e^{i(\theta+\theta')}$.
  - This also means that we may have rotation and scaling but no shearing. Formally, we have the following lemma.
- **Argument** (of $z \in \mathbb{C}$): The angle θ such that $z = re^{i\theta}$ for some $r \in \mathbb{R}$. *Denoted by* $\mathbf{arg(z)}$.
- Lemma: Suppose two curves γ, δ intersect at a point $z \in \mathbb{C}$. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. Then
$$\angle_z(\gamma, \delta) = \angle_{f(z)}(f(\gamma), f(\delta))$$
i.e., f preserves angles.

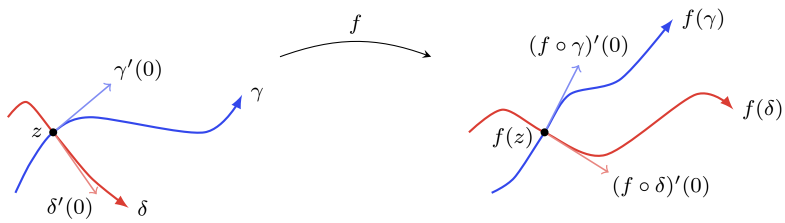


Figure 1.6: Holomorphic maps preserve angles.

Proof. To consider the angle between two curves analytically, let's look at the tangent vectors to the two curves, for example at z. Now while we often think of $\gamma'(0)$ as a *matrix*, remember that we've proven that all of these matrices are equivalent to complex numbers. In particular, since $\gamma : \mathbb{R} \to \mathbb{R}^2 \cong \mathbb{C}$, the total derivative will just be a vector. This vector may easily be represented as a complex number $re^{i\theta}$ in polar coordinates. Similarly, $\delta'(0)$ can be thought of as a complex number $r'e^{i\theta'}$. Thus, dividing these quantities gives us the angle $\theta - \theta'$ between the two vectors, which we can isolate using the argument function.

Doing the same to the curves at $f(z)$ yields
$$\begin{aligned} \angle_{f(z)}(f(\gamma), f(\delta)) &= \arg\left[\frac{(f\circ\gamma)'(0)}{(f\circ\delta)'(0)}\right] \\ &= \arg\left[\frac{f'(\gamma(0))\cdot\gamma'(0)}{f'(\gamma(0))\cdot\delta'(0)}\right] \\ &= \arg\left[\frac{f'(z)\cdot\gamma'(0)}{f'(z)\cdot\delta'(0)}\right] \\ &= \arg\left[\frac{\gamma'(0)}{\delta'(0)}\right] \\ &= \angle_z(\gamma, \delta) \end{aligned}$$
as desired. $\square$

  - Calderon gave us 5 minutes to try to compute this ourselves with the hint: Use the chain rule! $\angle_z(\gamma,\delta) = \arg(\gamma'(0)\cdot[\delta'(0)]^{-1})$.

- **Conformal** (map): A function $f: U \to V$, where $U, V \subset \mathbb{C}$, that satisfies the following two constraints. *Constraints*
  1. f is a diffeomorphism.
  2. f preserves angles.
- **Diffeomorphism**: A homeomorphism for which f, f^{-1} are differentiable.
- **Biholomorphic** (map): A function $f: U \to V$ that is bijective, holomorphic, and for which f^{-1} is holomorphic.
- Theorem/observation: Biholomorphic iff conformal.

  Proof. Follows straight from the definitions and the lemma we just proved. □

- Calderon shows us an applet.
  - We can use the applet to help with the PSet, but we still do have to submit actual proofs.
  - Allows you to visually see the lemma for instance.
  - Example: Under $z \mapsto z^2$, the sector of radius 2 and argument $\pi/4$ goes to the sector of radius $2^2 = 4$ and argument $\pi/2$.

1.3 Chapter I: Analysis in the Complex Plane

From Fischer and Lieb (2012).

3/19:
- The preface only contains comments and instructions for an instructor planning to use this textbook for a course.
- The chapter begins with two paragraphs.
  - The first discusses topic covered in the chapter.
  - The second gives some historical background on these topics.

Section I.0: Notations and Basic Concepts

- Goal: Reiew the fundamental topological and analytical concepts of real analysis.
- Defines the **complex numbers**, **complex plane**, and **complex conjugate**.
- **Absolute value** (of z): The Euclidean distance of z from zero. *Also known as* **modulus**. *Denoted by* $\boldsymbol{|z|}$. *Given by*

$$|z| := \sqrt{x^2 + y^2}$$

- **Imaginary unit**. *Denoted by* $\boldsymbol{i}$.
- Relating the modulus and complex conjugate.

$$|z| = \sqrt{z\bar{z}}$$

- **Open disk** (of radius ε and center z_0): The set defined as follows. *Also known as* **ε-neighborhood** (of z_0). *Denoted by* $\boldsymbol{D_\varepsilon(z_0)}$, $\boldsymbol{U_\varepsilon(z_0)}$. *Given by*

$$D_\varepsilon(z_0) = U_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$$

- **Unit disk**: The set defined as follows. *Denoted by* $\mathbb{D}$. *Given by*

$$\mathbb{D} := D_1(0)$$

- **Unit circle**: The set defined as follows. *Denoted by* $\mathbb{S}$. *Given by*

$$\mathbb{S} := \{z \in \mathbb{C} : |z| = \varepsilon\}$$

- **Upper half plane**: The set defined as follows. *Denoted by* $\mathbb{H}$. *Given by*

$$\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

- $\mathbb{C}^*$: The set defined as follows. *Given by*

$$\mathbb{C}^* := \mathbb{C} \setminus \{0\}$$

3/21:

- **Neighborhood** (of z_0): A set U which contains an ε-neighborhood.
- **Open** (set): A set that is a neighborhood of each of its points.
- **Closed** (set): A complement of an open set.
- **Interior** (of M): The largest open set contained in M. *Denoted by* $\mathring{\boldsymbol{M}}$.
- **Closure** (of M): The smallest closed set containing M. *Denoted by* $\overline{\boldsymbol{M}}$.
- **Topological boundary** (of M): The set defined as follows. *Also known as* **boundary**. *Denoted by* $\boldsymbol{\partial M}$. *Given by*

$$\partial M := \overline{M} \setminus \mathring{M}$$

- **Relatively open** (set in M): The intersection of an open set U with an arbitrary set M. *Also known as* **open** (set in M).
- **Relatively closed** (set in M): The intersection of a closed set U with an arbitrary set M. *Also known as* **open** (set in M).

Week 2

Consequences of Power Series

2.1 Office Hours

3/25:

- What exactly are the Wirtinger derivatives?
  - The $\partial/\partial z$ and $\partial/\partial\bar{z}$ operators.
- The initial definition of holomorphic is accurate. It's naïve, but it works out.
- Noney: Non example.
  - As in, we have some examples of holomorphic functions and then we have an example of a function that is *not* holomorphic.
- TPS: Think Pair Share.
- Met Panteleymon and helped him with partial fractions!
- The Δ notation does mean the same Laplacian as $\vec{\nabla}^2$ from Quantum Mechanics.
- Calderon is not related to Calderón; he was Argentinian, Calderon is half-Filipino and has no accent on his name. Both Spanish colonies but that's it.
- We can do all of the problems except Problem 1 at this point.
  - For this, though, we can just look up the definition of the complex sine function.
  - We basically just need to know what $\sin(i)$ is and what sine looks like along the imaginary axis.

2.2 Power Series

3/26:

- Recall: We already know that...
  - Polynomials are elements of $\mathcal{O}(\mathbb{C})$;
  - Rational functions $P(z)/Q(z)$ are elements of $\mathcal{O}(\mathbb{C}\setminus V(Q))$.
- **Affine algebraic set**: The set of solutions in an algebraically closed field K of a system of polynomial equations with coefficients in K. *Also known as* **variety**. *Denoted by* $\boldsymbol{V(f_1,\dots,f_n)}$.
- Today, we want to determine how the other elementary functions behave over the complex numbers.
  - Other functions we want: exp, log, sin, cos.
  - We will do log later, but all the others today.

- **Exponential function**: The complex function defined as follows. *Denoted by* $\mathbf{e^z}$, $\mathbf{exp(z)}$. *Given by*

$$\mathrm{e}^z = \exp(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

- Naïvely, this power series is just be a polynomial $P(z) \in \mathcal{O}(\mathbb{C})$.
- More rigorously, however, we must specify which kind of convergence we mean for the power series.
  - As one example, we could say that for all z,

$$\mathrm{e}^z = P(z) = \lim_{N\to\infty} \sum_{k=0}^{N} \frac{z^k}{k!}$$

    - This would be **pointwise convergence**.
  - But there's an issue: Pointwise convergence of functions doesn't preserve anything, e.g., continuity.
- **Pointwise** (convergent $\{f_n\}$): A sequence of functions $f_n : \mathbb{C} \to \mathbb{C}$ such that for all $z \in \mathbb{C}$, we have $f_n(z) \to f(z)$.
- TPS: Come up with an example of a sequence of continuous functions $\{f_n\}$ that converges pointwise to f, such that the f_n are all...
  1. Continuous but f is not;
     - $f_n(x) = \arctan(nx)$.
     - Converges to the sign function $f(x) = \operatorname{sgn}(x)$.
  2. Odd but f is not;
  3. Differentiable but f is not.
     - These last two cases were not discussed in class.
- We now recall a few definitions and lemmas from real analysis.
- **Locally uniformly** (convergent $\{f_n\}$): A sequence of functions $f_n : U \to \mathbb{C}$ and a function $f : U \to \mathbb{C}$ such that for all compact $K \subset U$,

$$\sup_{z\in K} |f_n(z) - f(z)| \to 0$$

- Lemma: If $f_n \to f$ locally uniformly and the f_n are continuous (or integrable), then so is f.
  - This lemma is *not* true if we sub in "differentiable!"
  - See the Stone-Weierstrass theorem for suitable constraint.
- Thus, to resolve the original question, we mean that $P_N(z) \to \exp(z)$ locally uniformly.
- Aside: Which functions have power series?
  - Remember Taylor polynomials from Calc II? **Taylor's theorem** tells us which ones converge.
- **Taylor's theorem**: If $f : \mathbb{R} \to \mathbb{R}$ is C^{k+1} and $P_\alpha^k(x)$ is the k^{th} Taylor polynomial about $\alpha \in \mathbb{R}$, then for all $\beta \in \mathbb{R}$, there exists some $x \in (\alpha, \beta)$ such that

$$f(\beta) - P_\alpha^k(\beta) = \frac{(\beta - \alpha)^{k+1}}{(k+1)!} f^{(k+1)}(x)$$

  - Essentially a version of the mean value theorem (MVT) for higher-order derivatives.
  - We can use the term of the right side of the equals sign above to get a bound on the error of the Taylor polynomial.

- **Analytic** (function): A function $f : \mathbb{R} \to \mathbb{R}$ for which the Taylor polynomials converge (locally uniformly) to f.
- Non example: The C^∞ function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} \mathrm{e}^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

  - An excellent exercise in real analysis is to check that for all k, the Taylor polynomial about 0 is 0.
  - If we take the Taylor polynomial at some point farther from zero, the polynomial will approximate f well up until zero, but then it will "hit a wall."
    - The point is that f is decaying more rapidly toward 0 than any polynomial possibly could, so the polynomial just thinks it's seeing 0.
- **Absolutely** (locally uniformly convergent power series): A power series $P(z) = \sum_{k=0}^{\infty} a_k z^k$ for which $A_N : \mathbb{C} \to \mathbb{R}$ locally uniformly converges, where

$$A_N(z) := \sum_{k=0}^{N} |a_k z^k|$$

- Absolute local uniform convergence allows you to reorder the terms in the polynomial.
  - It also explains why you cannot reorder the terms in the series $S = 1+1-1+1-1+\cdots$, i.e., why manipulating the order allows you to get any number: This series S does not converge absolutely!
  - Formally, if $\sigma : \mathbb{N} \to \mathbb{N}$ is a permutation and $\sum^\infty a_k$ converges absolutely, then $\sum^\infty a_{\sigma(k)}$ converges.
- Exercise: Show that

$$\sum_{k=0}^{\infty} z^k \to \frac{1}{1-z}$$

  converges absolutely locally uniformly on $\mathbb{D} = \{|z| < 1\}$.

  Proof. To prove this, we just have to show that $\sum^\infty |z|^k$ converges on $|z| < 1$. But it does so converge because this latter series is just a standard real geometric series. □

- This example generalizes somewhat into the following lemma.
- Lemma: Let $P(z)$ be a power series about 0. If there exists $z_1 \neq 0$ such that $|a_k z_1^k| \leq M$ for all k, then $P(z) = \sum a_k z^k$ converges on the disk $|z| < |z_1|$.

  Proof. Uses standard series convergence results from real analysis. May be in Fischer and Lieb (2012)?? □

- **Disk of convergence**: The largest disk centered at zero on which you converge.
- **Radius of convergence**: The radius of the disk of convergence.
- **Cauchy-Hadamard formula**: The radius of convergence is given by

$$\text{rad} = (\limsup |a_k|^{1/k})^{-1}$$

  - We will be using this result on PSet 2.
  - We will also be proving it there!

- What are power series representations good for? Here's an example of how they can be applied to help with PSet 1, QA.4.
  - Question: For $|a| < 1$ and $\gamma(t) = \mathrm{e}^{it}$ a parameterization of a closed loop oriented counterclockwise, compute
$$\int_\gamma \frac{1}{z-a}\,\mathrm{d}z$$
  - Answer:
    - Since $|a| < 1$, we know that on γ, $|a/\gamma(t)| < 1$.
    - Thus, we have that
$$\begin{aligned}
\int_\gamma \frac{1}{z-a}\,\mathrm{d}z &= \int_\gamma \frac{1}{z}\frac{1}{1-a/z}\,\mathrm{d}z \\
&= \int_\gamma \frac{1}{z}\sum_{k=0}^\infty \left(\frac{a}{z}\right)^k \mathrm{d}z \\
&= \int_\gamma \sum_{k=0}^\infty \frac{a^k}{z^{k+1}}\,\mathrm{d}z \\
&= \sum_{k=0}^\infty \int_\gamma \frac{a^k}{z^{k+1}}\,\mathrm{d}z \\
&= \cdots \\
&= \int_\gamma \frac{1}{z}\,\mathrm{d}z
\end{aligned}$$
    - We have the second equality because the power series converges.
    - We have the fourth equality because of the lemma about integrable f_n and the fact that the power series converges.
    - The dots indicate some more steps that we will need to work out for ourselves on PSet 1.
- Lemma (from real analysis): If $f_n \to f$ locally uniformly and $f_n' \to g$ locally uniformly, then f is differentiable and $f' = g$.
  - This is true for both differentiable and holomorphic functions.
- Claim: This lemma implies that convergent power series are holomorphic.

  Proof. If
$$f_N = \sum_{k=0}^N a_k z^k$$
  then
$$f_N' = \sum_{k=0}^N k \cdot a_k z^{k-1}$$
  We want to show that $\{f_N'\}$ converges (locally absolutely uniformly). Fischer and Lieb (2012) do this by hand. We can also use the Cauchy-Hadamard formula, which we will do presently.

  Let's look at $\limsup(k \cdot a_k)^{1/k}$. But this is just equal to
$$\limsup |k \cdot a_k|^{1/k} \leq \limsup(|k|^{1/k}) \cdot \limsup(|a_k|^{1/k}) = 1 \cdot \limsup(|a_k|^{1/k}) = \limsup |a_k|^{1/k}$$
  Moreover, equality holds because that $k^{1/k}$ factor just decays toward 1; think about how k increases linearly and the k^{th} root decays faster. $\square$

- Proposition: Any convergent power series is holomorphic (on its disk) and its derivative is also a power series with the same radius of convergence. It follows that power series are analytic functions and are C^∞.
- Spoiler: Every holomorphic function is analytic.
- Corollary: Power series representations are unique.
  1. If $P(z) = \sum a_k z^k$ is convergent, then
  $$a_k = \frac{1}{k!} P^{(k)}(0)$$
  2. If $P(z) = 0$ in a neighborhood of zero, then $a_k = 0$ for all k.
  3. If $P(z) = Q(z)$ (where $Q(z) = \sum b_k z^k$) in a neighborhood of 0, then $a_k = b_k$ for all k.
- Let's now return to the exponential function, which got this whole discussion started.
- We now know that the definition
  $$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$
  makes sense.
- By manipulating this power series, we can get lots of fun properties.
  1. $\exp(z) = [\exp(z)]'$.
     - We obtain this via term-by-term differentiability.
     - This is just our favorite formula $\mathrm{d}/\mathrm{d}t\ (\mathrm{e}^t) = \mathrm{e}^t$ from calculus.
  2. $\overline{\exp(z)} = \exp(\bar{z})$.
  3. $\exp(a + b) = \exp(a) \cdot \exp(b)$.
  4. $|\exp(z)| = \exp[\operatorname{Re}(z)]$.
- **Complex cosine**: The complex function defined as follows. *Denoted by* $\boldsymbol{\cos(z)}$. *Given by*
  $$\cos(z) := \frac{1}{2}(\mathrm{e}^{iz} + \mathrm{e}^{-iz})$$
- **Complex sine**: The complex function defined as follows. *Denoted by* $\boldsymbol{\sin(z)}$. *Given by*
  $$\sin(z) := \frac{1}{2i}(\mathrm{e}^{iz} - \mathrm{e}^{-iz})$$
- **Complex hyperbolic cosine**: The complex function defined as follows. *Denoted by* $\boldsymbol{\cosh(z)}$. *Given by*
  $$\cosh(z) := \cos(iz)$$
- **Complex hyperbolic sine**: The complex function defined as follows. *Denoted by* $\boldsymbol{\sinh(z)}$. *Given by*
  $$\sinh(z) := i \sin(iz)$$
- We also have
  $$\mathrm{e}^{iz} = \cos(z) + i \sin(z)$$
  - If z is real and in $[0, 2\pi]$, then this simplifies to Euler's formula
  $$\mathrm{e}^{i\theta} = \cos(\theta) + i \sin(\theta)$$
- Calderon draws some mappings of the exponential function but doesn't linger on what's going on.
- These are the preliminaries; now, we'll dive into the meat of the course.

2.3 Cauchy's Theorem

3/28: • The last three classes have been real analysis with complex numbers; now we get into *complex* analysis.

- **Domain**: A connected, open set $U \subset \mathbb{C}$.
- Recall.
  - $\gamma : [a, b] \to \mathbb{C}$ is a piecewise C^1 curve.
  - $f : \mathbb{C} \to \mathbb{C}$ is continuous.
  - We define
$$\int_\gamma f \,\mathrm{d}z := \int_a^b f(\gamma(t)) \cdot \gamma'(t) \,\mathrm{d}t$$
  - FTC: If $f = F'$ (i.e., F is a **primitive** of f) on a domain $U \subset \mathbb{C}$, then for all paths γ in U,
$$\int_\gamma f \,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$$
- **Primitive** (of f): A differentiable function whose derivative is equal to the original function f. *Also known as* **antiderivative**, **indefinite integral**. *Denoted by* $\boldsymbol{F}$.
- Corollary to the FTC: If $f = F'$, then for any closed curve γ in U,
$$\int_\gamma f \,\mathrm{d}z = 0$$
  - To see why this is true intuitively, look at an example such as $f(z) = 1/z \in \mathcal{O}(\mathbb{C}^*)$, which doesn't have a primitive and
$$\int_\gamma \frac{1}{z} \,\mathrm{d}z \neq 0$$
- Example: Find a primitive of the convergent power series
$$P(z) = \sum_{k=1}^{\infty} a_k z^k$$
  - Via term-by-term integration, we obtain
$$\sum_{k=0}^{\infty} \frac{a_k}{k+1} z^{k+1}$$
- If γ is any closed loop in the disk of convergence,
$$\int_\gamma P(z) \,\mathrm{d}z = 0$$
  - It follows since they are defined in terms of convergent power series that for all closed loops γ,
$$\int_\gamma \mathrm{e}^z \,\mathrm{d}z = \int_\gamma \sin(z) \,\mathrm{d}z = \int_\gamma \cos(z) \,\mathrm{d}z = 0$$
- Question: When is there a primitive?
  - $f : \mathbb{R} \to \mathbb{R}$ continuous always has a primitive by the FTC, specifically that defined by
$$F(x) := \int_a^x f(t) \,\mathrm{d}t$$
    which is differentiable with $F' = f$.

- Proposition: If $f : U \to \mathbb{C}$ is continuous and $\int_\gamma f \, \mathrm{d}z = 0$ for every closed loop in U, then f has a primitive on U.

Proof. Let's try the most naïve thing: The FTC. Consider a domain.

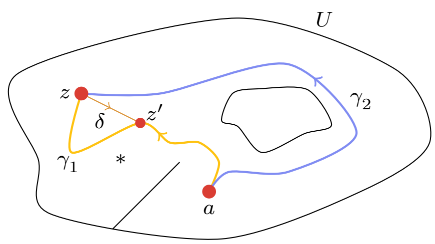


Figure 2.1: Continuous and zero closed-loop integrals implies integrable.

Doesn't have to be simply connected; it can have a **hole**, **slit**, and/or **puncture**. Essentially, to define $F(z)$, choose $a \in U$ and γ connecting a and z and define

$$F(z) = \int_\gamma f \, \mathrm{d}z$$

Claim: This definition is well-defined regardless of the choice of a and γ. In particular, the integral is independent of choice of γ because any two γ can be paired into a closed loop, and we have by hypothesis that the integral over any closed loop is zero.

We now need to show that F is differentiable with $F' = f$. Take z, z' close enough that they can be connected by a straight line path δ. Consider

$$\lim_{z' \to z} \frac{F(z') - F(z)}{z' - z}$$

Now we know that

$$F(z') - F(z) = \int_\delta f \, \mathrm{d}z$$

Let $\gamma : [0, 1] \to \mathbb{C}$ be defined by $t \mapsto tz' + (1 - t)z$; a parameterization we can choose arbitrarily. Then

$$F(z') - F(z) = \int_\delta f \, \mathrm{d}z = \int_0^1 f[tz + (1 - t)z'] \cdot (z' - z) \, \mathrm{d}t$$

so dividing both sides by $z' - z$ and taking the limit yields

$$\begin{aligned}
\lim_{z' \to z} \frac{F(z') - F(z)}{z' - z} &= \lim_{z' \to z} \int_0^1 f[tz + (1 - t)z'] \, \mathrm{d}t \\
&= \int_0^1 \lim_{z' \to z} f(tz + z' - tz') \, \mathrm{d}t \\
&= \int_0^1 f(tz + z - tz) \, \mathrm{d}t \\
&= \int_0^1 f(z) \, \mathrm{d}t \\
&= f(z) \int_0^1 \mathrm{d}t \\
&= f(z)
\end{aligned}$$

and we have everything we wanted. □

- What allows us to interchange the limit and the integral in the final set of equations?
  - Roughly speaking, uniform convergence.
- **Star-shaped** (domain): A domain $U \subset \mathbb{C}$ for which there exists $a \in U$ such that for all $z \in U$, the segment $a \to z$ is in U.

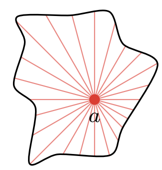


Figure 2.2: Star-shaped domain.

  - There are star-shaped regions that are not **convex**, such as the one in Figure 2.2!
    - Convex implies star-shaped, but not vice versa.
  - Examples of domains that are *not* star-shaped.
    1. The annulus of two circles.
    2. Puncturing the unit disk.
  - Star-shaped implies **simply connected**.
  - Star-shaped is nice because we don't have to check every single curve; see the following lemma.
- Lemma: If U is star-shaped and for every triangle with one vertex at a, we have $\int_\triangle f \, \mathrm{d}z = 0$, then F has a primitive in U.

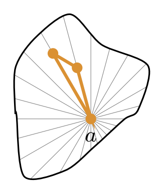


Figure 2.3: A triangle in a star-shaped domain.

Proof. What should be our candidate for $F(z)$? Define

$$F(z) = \int_\gamma f \, \mathrm{d}z$$

where γ is the line segment from $a \to z$ that we know exists because U is star-shaped.

We now have to show that F is holomorphic with $F' = f$, but we just do this as before by constructing a "closed loop," except our closed loop this time will just be a triangle as drawn in Figure 2.3. ☐

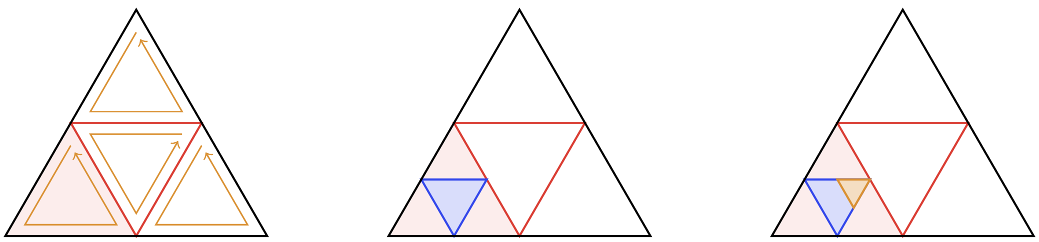
- With these definitions, we now state and prove one of the two main theorems in this class.
- **Cauchy Integral Theorem**: Suppose U is a star-shaped domain and $f : U \to \mathbb{C}$ is holomorphic. Then $\int_\gamma f \, \mathrm{d}z = 0$ for any closed loop γ in U.
  - Whereas the FTC says if you have an *integral*, then the integral around a closed loop is zero. This theorem says that if you have a *derivative*, then the integral around a closed loop is zero.
  - This is Round 1 of the theorem. In round 2, we'll swap the "star-shaped" hypothesis for "simply connected."

- Today we're at least going to prove this, and possibly look at an application, too. If we don't get to the application today, we'll see it next Tuesday.
- Proof idea: Prove that f has a primitive.

  Proof. In order to prove this theorem, we'll use the preceding lemma. Thus, all we need to show is that for every triangle with one vertex on the center of the star, $\int_\triangle f\,\mathrm{d}z = 0$. Since we only have to check this for *triangles*, we can use a really lovely result called **Goursat's lemma**. ☐

- **Goursat's lemma**: If f is holomorphic in a neighborhood of a triangle including the interior, then $\int_\triangle f\,\mathrm{d}z = 0$.

  Proof. Idea: Estimate the integral.



(a) Picking $\triangle_1$. (b) Picking $\triangle_2$. (c) Picking $\triangle_3$.

Figure 2.4: Proving Goursat's lemma.

Fix some $z_0 \in \blacktriangle$ (the exact value will be determined later). We know that f is holomorphic at z_0, which implies that there exists a linear approximation

$$f(z) = \underbrace{f(z_0) + f'(z_0)\cdot(z - z_0)}_{\text{linear}} + E(z)\cdot(z - z_0)$$

where our error function $E(z) \to 0$ as $z \to z_0$. Now the underlined linear portion above is a (stupid) power series, but since it technically *is* a "convergent power series," our previous results imply that it has primitives. In particular, its integral around a closed loop (like a triangle) will be zero. This means that

$$\int_\triangle f\,\mathrm{d}z = \underbrace{\int_\triangle [f(z_0) + f'(z_0)\cdot(z - z_0)]\,\mathrm{d}z}_{0} + \int_\triangle E(z)\cdot(z - z_0)\,\mathrm{d}z = \int_\triangle E(z)\cdot(z - z_0)\,\mathrm{d}z$$

Goursat's idea: Choose a good z_0. To do this, we'll subdivide the original black triangle (see Figure 2.4a) by choosing midpoints and breaking it into four triangles. Keep using the counterclockwise orientation in all cases. All of the red segments get cancelled out from integrating in both directions, so

$$\int_{\triangle_0} f\,\mathrm{d}z = \sum \int_{4 \text{ sub } \triangle\text{'s}} f\,\mathrm{d}z$$

Choose $\triangle_1$ in first stage such that $|\int_{\triangle_1} f\,\mathrm{d}z\,|$ is the greatest among the first stage sub-triangles. Thus,

$$\left|\int_\triangle f\,\mathrm{d}z\right| \le 4\cdot\left|\int_{\triangle_1} f\,\mathrm{d}z\right|$$

Now subdivide $\triangle_1$ and choose $\triangle_2$ the same way (see Figure 2.4b), so that

$$\left|\int_\triangle f\,\mathrm{d}z\right| \le 4\cdot\left|\int_{\triangle_1} f\,\mathrm{d}z\right| \le 4\cdot 4\cdot\left|\int_{\triangle_2} f\,\mathrm{d}z\right|$$

Iterating this process, we obtain

$$\left|\int_{\triangle} f\,\mathrm{d}z\right| \leq 4^n \cdot \left|\int_{\triangle_n} f\,\mathrm{d}z\right|$$

First thing to observe:

$$\operatorname{len}(\triangle_n) = 2^{-n} \cdot \operatorname{len}(\triangle_0) \qquad\qquad \operatorname{diam}(\triangle_n) = 2^{-n} \cdot \operatorname{diam}(\triangle_0)$$

Now fix $\varepsilon > 0$ and take n big enough such that on all of $\triangle_n$,

$$|E(z)| < \frac{\varepsilon}{\operatorname{len}(\triangle_0) \cdot \operatorname{diam}(\triangle_0)}$$

Choose $z_0 \in \bigcap_{n=1}^{\infty} \blacktriangle_n$. Then

$$\begin{aligned}
\left|\int_{\triangle} f\,\mathrm{d}z\right| &\leq 4^n \cdot \left|\int_{\triangle_n} f\,\mathrm{d}z\right| \\
&= 4^n \cdot \left|\int_{\triangle_n} E(z) \cdot (z - z_0)\,\mathrm{d}z\right| \\
&\leq 4^n \cdot \operatorname{len}(\triangle_n) \cdot \max_{\triangle_n} |E(z) \cdot (z - z_0)| \\
&= 4^n \cdot \operatorname{len}(\triangle_n) \cdot \max |E(z)| \cdot \max |z - z_0| \\
&\leq 4^n \cdot \operatorname{len}(\triangle_n) \cdot \operatorname{diam}(\triangle_n) \cdot \max |E(z)| \\
&= 4^n \cdot 2^{-n} \operatorname{len}(\triangle_0) \cdot 2^{-n} \operatorname{diam}(\triangle_0) \cdot \max |E(z)| \\
&= \operatorname{len}(\triangle_0) \cdot \operatorname{diam}(\triangle_0) \cdot \max |E(z)| \\
&< \varepsilon
\end{aligned}$$

Since we can choose ε arbitrarily small, we can thus send the original integral of f over $\triangle$ to zero. $\square$

- We now end class with an example of how complex analysis can be useful, even in calculus!
- Example: Evaluate the following **Dirichlet integral** using complex analysis.

$$\int_0^\infty \frac{\sin(x)}{x}\,\mathrm{d}x$$

  - We will do so via a focused analysis of the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$z \mapsto \frac{\mathrm{e}^{iz}}{z}$$

    - This function is not holomorphic everywhere, but it is on the punctured plane $\mathcal{O}(\mathbb{C}^*)$.
    - However, we only need the upper half $\mathcal{O}(\mathbb{H})$ presently.
  - More specifically, define U to be a domain containing γ as defined as in Figure 2.5.

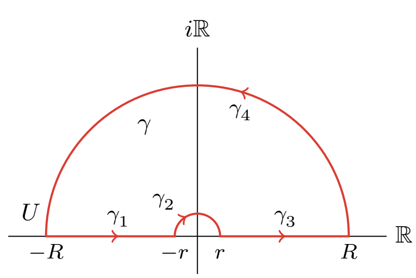


Figure 2.5: Dirichlet integral.

- By the Cauchy integral theorem and our decomposition of γ,

$$0 = \int_\gamma f(z)\,\mathrm{d}z = \sum_{i=1}^{4} \int_{\gamma_i} f\,\mathrm{d}z$$

- We now integrate the segments one at a time.
  - γ_1 and γ_3: Recalling our definition of $\sin(z)$ from last class, we have that

$$\begin{aligned}
\int_{\gamma_1\gamma_3} \frac{\mathrm{e}^{ix}}{x}\,\mathrm{d}x &= \int_{-R}^{-r} \frac{\mathrm{e}^{ix}}{x}\,\mathrm{d}x + \int_r^R \frac{\mathrm{e}^{ix}}{x}\,\mathrm{d}x \\
&= \int_{-r}^{-R} -\frac{\mathrm{e}^{ix}}{x}\,\mathrm{d}x + \int_r^R \frac{\mathrm{e}^{ix}}{x}\,\mathrm{d}x \\
&= \int_{r}^{R} -\frac{\mathrm{e}^{-ix}}{x}\,\mathrm{d}x + \int_r^R \frac{\mathrm{e}^{ix}}{x}\,\mathrm{d}x \\
&= \int_r^R \frac{\mathrm{e}^{ix} - \mathrm{e}^{-ix}}{x}\,\mathrm{d}x \\
&= 2i \int_r^R \frac{\sin(x)}{x}\,\mathrm{d}x
\end{aligned}$$

  - γ_2: We can explicitly compute this integral as $r \to 0$, using the parameterization $\gamma_2 : [0,\pi] \to \mathbb{C}$ defined by $\theta \mapsto r\mathrm{e}^{i(\pi-\theta)}$.

$$\begin{aligned}
\lim_{r\to 0} \int_{\gamma_2} \frac{\mathrm{e}^{iz}}{z}\,\mathrm{d}z &= \lim_{r\to 0} \int_0^\pi \frac{\mathrm{e}^{ir\mathrm{e}^{i(\pi-\theta)}}}{r\mathrm{e}^{i(\pi-\theta)}} \cdot -ir\mathrm{e}^{i(\pi-\theta)}\,\mathrm{d}\theta \\
&= -i \lim_{r\to 0} \int_0^\pi \mathrm{e}^{ir\mathrm{e}^{i(\pi-\theta)}}\,\mathrm{d}\theta \\
&= -i \int_0^\pi \mathrm{e}^0\,\mathrm{d}\theta \\
&= -i\pi
\end{aligned}$$

  - γ_4: We need to bound the $R\mathrm{e}^{i\theta}$ term as $R \to \infty$; see his notes!

$$\int_0^\pi \mathrm{e}^{iR\mathrm{e}^{i\theta}} i\mathrm{d}\theta \to 0$$

- Therefore, by transitivity,

$$\begin{aligned}
2i \int_0^\infty \frac{\sin(x)}{x}\,\mathrm{d}x - i\pi &= 0 \\
\int_0^\infty \frac{\sin(x)}{x}\,\mathrm{d}x &= \frac{\pi}{2}
\end{aligned}$$

2.4 Office Hours

- PSet 1, QA.4: Are a, b real or complex?
  - They can be complex.
  - Hint for this problem: Think about QA.3.
- PSet 1, QB.2: As in, only "takes on" real values, i.e., is a function of the form $f : U \to \mathbb{R}$?
  - Yes.
- We have to give him a heads up before the PSet due date that we want to use a PSet extension.

Week 3

???

3.1 Cauchy Integral Formula

4/2:
- Last time.
  - Definition of star-shaped.
  - Cauchy integral theorem: U star-shaped, $f \in \mathcal{O}(U)$ implies $\int_\gamma f \, \mathrm{d}z = 0$ for all closed (piecewise C^1) loops γ.
    1. It suffices to prove the theorem for triangles.
    2. Apply Goursat's lemma to treat this triangle case.
  - For Goursat's lemma, apply a clever estimate. Subdivide the big triangle into smaller ones, then

$$\left|\int_{\text{small } \triangle} f \, \mathrm{d}z\right| = \left|\int_a^b f(\gamma(t)) \cdot \gamma'(t) \, \mathrm{d}t\right| \leq \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)| \, \mathrm{d}t \leq \max_{z \in \partial\triangle} |f(z)| \cdot \operatorname{len}(\partial\triangle)$$

- We'll now do a couple exercises to practice applying the concepts we've learned so far.
- TPS: Suppose $f \in \mathcal{O}(\mathbb{C})$. Let $A := \int_0^1 f(x) \, \mathrm{d}x = F(1) - F(0)$, where to be clear we take the integral along the real axis. Let γ be the piecewise C^1 path in yellow in Figure 3.1. What is $\int_\gamma f \, \mathrm{d}z$?

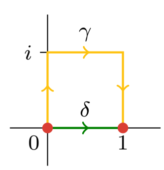


Figure 3.1: Practicing with the Cauchy Integral Theorem (1).

  - Define δ such that $\int_\delta f \, \mathrm{d}z = \int_0^1 f(x) \, \mathrm{d}x$.
  - Then $\delta^{-1}\gamma$ is a closed loop, so

$$0 = \int_{\delta^{-1}\gamma} f \, \mathrm{d}z$$

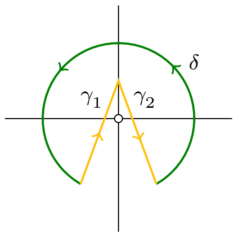
  - Additionally, we have by definition that

$$\int_{\delta^{-1}\gamma} f \, \mathrm{d}z = \int_\gamma f \, \mathrm{d}z - \int_\delta f \, \mathrm{d}z$$

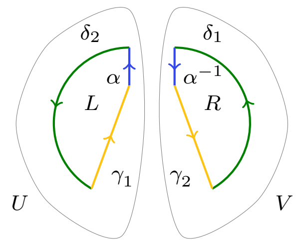
  - Thus, by transitivity and a bit of algebraic rearrangement,

$$\int_\gamma f \, \mathrm{d}z = \int_\delta f \, \mathrm{d}z = A$$

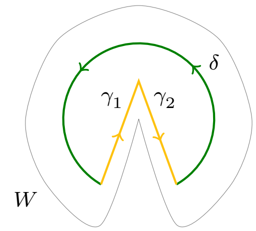
- TPS: Now suppose $f \in \mathcal{O}(\mathbb{C}^*)$, where we must note that $\mathbb{C}^*$ is *not* star-shaped due to the hole at the origin. Suppose we know that $\int_\delta f \, \mathrm{d}z = 0$. What is $\int_\gamma f \, \mathrm{d}z$? The paths γ and δ are visualized in Figure 3.2a. *Hint*: It should be $-\int_\delta f \, \mathrm{d}z$.



(a) Original setup.



(b) Solution: Break in two.



(c) Solution: Pizza pie.

Figure 3.2: Practicing with the Cauchy Integral Theorem (2).

- There are multiple ways to visualize why the domain does not see the hole/puncture. Here are some examples.
- Solution 1 (Figure 3.2b): Cut the loop into two loops in star-shaped domains and add them.
  - Draw a straight-line path α from $i/2$ up to i.
  - Since U and V are both star-shaped domains, consecutive applications of the Cauchy Integral Theorem imply that

$$\int_{\delta_2\gamma_1\alpha} f \, \mathrm{d}z = \int_L f \, \mathrm{d}z = 0 \qquad\qquad \int_{\delta_1\alpha^{-1}\gamma_2} f \, \mathrm{d}z = \int_R f \, \mathrm{d}z = 0$$

  - Additionally, we know that the sum of the two integrals above is equal to the integral along the entire path in Figure 3.2a because the α and α^{-1} portions cancel. Mathematically,

$$\int_\delta f \, \mathrm{d}z + \int_\gamma f \, \mathrm{d}z = \int_{\delta\gamma} f \, \mathrm{d}z = \underbrace{\int_L f \, \mathrm{d}z}_{0} + \underbrace{\int_R f \, \mathrm{d}z}_{0} = 0$$

  - Therefore,

$$\int_\gamma f \, \mathrm{d}z = -\int_\delta f \, \mathrm{d}z = 0$$

- Solution 2 (Figure 3.2c): The pizza pie is star-shaped!
  - We can actually draw a star-shaped domain W encapsulating the entire path $\delta\gamma$.
  - Thus, by the Cauchy Integral Theorem,

$$\int_{\delta\gamma} f \, \mathrm{d}z = 0$$

  - From here, we may proceed as before through

$$\begin{aligned}
\int_\gamma f \, \mathrm{d}z + \int_\delta f \, \mathrm{d}z &= 0 \\
\int_\gamma f \, \mathrm{d}z &= -\int_\delta f \, \mathrm{d}z = 0
\end{aligned}$$

- We now investigate a more general principal than the Cauchy integral theorem called **homotopy**.
  - Algebraic topologists would be insulted by the definition of this term that Calderon is about to give, but it will suffice for our purposes.
- **Homotopic** (paths): Two paths $\gamma, \tilde{\gamma} \subset U$ a domain such that $\tilde{\gamma}$ is obtained from γ by modifying γ on a small disk $D \subset U$, keeping the endpoints fixed.

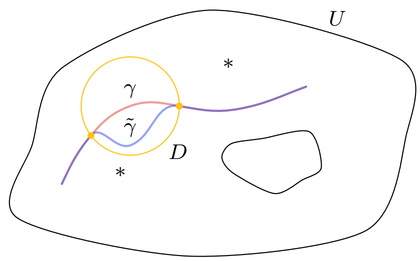
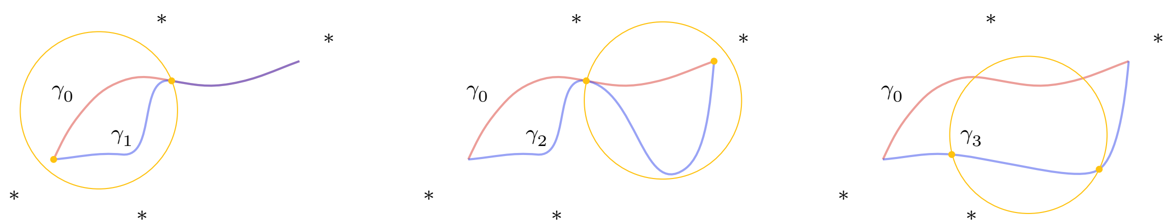


Figure 3.3: Homotopic paths.

- More generally, γ and $\tilde{\gamma}$ are **homotopic** if there exists a finite sequence $\gamma = \gamma_0, \gamma_1, \dots, \gamma_n = \tilde{\gamma}$ such that $\gamma_i \to \gamma_{i+1}$ is obtained by modifying on a small ball.



(a) Stage 1. (b) Stage 2. (c) Stage 3.

Figure 3.4: A more general homotopy.

- Claim/TPS: This argument shows that if γ and $\tilde{\gamma}$ are homotopic in U and $f \in \mathcal{O}(U)$, then

$$\int_\gamma f \,\mathrm{d}z = \int_{\tilde{\gamma}} f \,\mathrm{d}z$$

Hint: Just go one little bump at a time.

Proof. The start- and endpoints of the bump form a closed loop within a ball (a star-shaped domain), so the bump loop integrates to zero by the CIT. Thus, the integrals within the ball are the same. Additionally, the paths are literally the same outside of the bump, so the integrals there are the same, too. Therefore, the overall integrals are the same, too. □

- Reality check: Let $f \in \mathcal{O}(\mathbb{C}^*)$. As a particular example, consider $f(z) = 1/z$. Now we know that

$$\int_\circ \frac{1}{z} \,\mathrm{d}z = 2\pi i \neq 0$$

even though we can break the unit circle into the sum of two paths. What's going on?

  - The paths are not homotopic; we can't pull them through the hole in the plane.
  - If we consider the upper hemi-circle and the lower hemi-circle, the two cannot be continuously deformed into each other because we always get stuck at the puncture.

- We now prove a slightly stronger version of the Cauchy integral theorem.
- Corollary: Let U be any domain, D be a disk in U, and $z \in \mathring{D}$. Suppose $f \in \mathcal{O}(U \setminus \{z\})$ and is bounded near z. Then

$$\int_{\partial D} f \, \mathrm{d}z = 0$$

Proof. Step 1: Use homotopy.

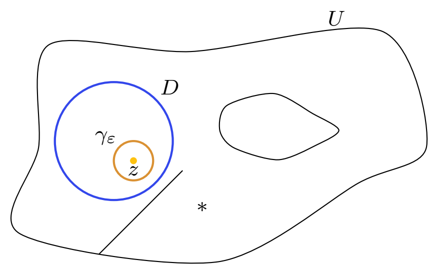


Figure 3.5: Bounded holomorphic functions integrate to zero on disk boundaries.

Via the above claim,

$$\int_{\partial D} f \, \mathrm{d}z = \int_{\gamma_\varepsilon} f \, \mathrm{d}z$$

where γ_ε is a circle around z within the region where f is bounded[1].

Step 2: We have that

$$\left| \int_{\gamma_\varepsilon} f \, \mathrm{d}z \right| \leq \max_{z \in \gamma_\varepsilon} |f(z)| \cdot \mathrm{len}(\gamma_\varepsilon)$$

Since f is bounded near z, the maximum is finite. Additionally, the length term is just $2\pi\varepsilon$, so we can send $\varepsilon \to 0$ and thus send the integral to zero. □

- We now look into the **Cauchy Integral Formula**.
- **Cauchy Integral Formula**: Suppose U is any domain, $D \subset U$ is a disk (i.e., $D \subset\subset U$ or $\overline{D} \subset U$), $f \in \mathcal{O}(U)$, and $z \in D$. Then

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

Proof. We're going to try to use the corollary and define a function. In particular, define

$$g(\zeta) = \begin{cases} \frac{f(\zeta) - f(z)}{\zeta - z} & \zeta \neq z \\ f'(z) & \zeta = z \end{cases}$$

Because f is holomorphic at z, g is continuous at z and hence bounded near z. We can also see that since g is a rational function of holomorphic functions on $U \setminus \{z\}$, we have $g \in \mathcal{O}(U \setminus \{z\})$.

Now the corollary says that

$$\int_{\partial D} g \, \mathrm{d}\zeta = 0$$

Additionally, by the definition of g, we have that

$$\int_{\partial D} g \, \mathrm{d}\zeta = \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta - \int_{\partial D} \frac{f(z)}{\zeta - z} \, \mathrm{d}\zeta$$

[1]We could also turn the plane into the sum of two star-shaped domains again.

$f(z)$ is just a complex number, so we can pull it out of the rightmost integral above. Additionally, under a change of variables and invoking PSet 1, QA.4, we have that

$$\int_{\partial D} \frac{f(z)}{\zeta - z}\,\mathrm{d}\zeta = f(z)\int_{\partial D} \frac{1}{\zeta - z}\,\mathrm{d}\zeta = \int_{\text{unit circle}} \frac{1}{z-a}\,\mathrm{d}z = 2\pi i f(z)$$

Note: Another way to evaluate this integral is as follows. If z is the center of the disk, then we win and can get $2\pi i$ using PSet 1, QA.4 directly. If z isn't at the center of the disk, we are allowed to slide it. Here's why: Think about the integrand as a function of z, so

$$\frac{\partial}{\partial z}\left(\int_{\partial D} \frac{1}{\zeta - z}\,\mathrm{d}\zeta\right) = \int_{\partial D} \frac{\partial}{\partial z}\left(\frac{1}{\zeta - z}\right)\mathrm{d}\zeta = \int_{\partial D} \frac{1}{(\zeta - z)^2}\,\mathrm{d}\zeta = 0$$

Since we're taking the integral and the limit with respect to different things, we can exchange them. Since the second integrand has a primitive, it equals zero. But this means that the integral does not change even as z changes, which is equivalent to saying we can move z around to wherever we want in the disk and the integral will still be $2\pi i$! In other words, if z is somewhere where we can't evaluate the integral directly, we can move z to somewhere where we *can* evaluate the integral directly with no consequence. ☐

- Implication of Cauchy's Integral Theorem: The values of the function are completely determined by the values on the boundary, i.e., holomorphic functions are determined by boundary values.
- Let's now prove another theorem.
- Theorem: Let U be any domain, $f \in \mathcal{O}(U)$. Then $f' \in \mathcal{O}(U)$, $f'' \in \mathcal{O}(U)$, on and on.

  Proof. Let's use the Cauchy integral formula. We have that

  $$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta$$

  Now let's take the derivative, which we know exists because f is holomorphic.

  $$\frac{\partial f}{\partial z} = \frac{1}{2\pi i}\int_{\partial D} \frac{\partial}{\partial z}\left(\frac{f(\zeta)}{\zeta - z}\right)\mathrm{d}\zeta = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^2}\mathrm{d}\zeta$$

  Thus, the derivative has a Cauchy integral formula. We can keep taking derivatives on the inside because the integrand is infinitely differentiable. Thus, we can keep taking derivatives on the outside. And that's the proof. ☐

- Corollary: Holomorphic functions are C^∞.
- Corollary: In general,

  $$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\mathrm{d}\zeta$$

- This last result allows us to bound things really easily, giving us **Cauchy's inequalities**.
  - Essentially, let D have radius R and let z be the center of D. Then

    $$|f^{(n)}(z)| \leq \frac{n!}{2\pi i}\max_{\partial D}\left|\frac{f(\zeta)}{R^{n+1}}\right| \cdot 2\pi R = \frac{n!}{R^n}\max_{\partial D}|f(\zeta)|$$

- Liouville's Theorem: Suppose $f \in \mathcal{O}(\mathbb{C})$ (i.e., f is **entire**) and f is bounded. Then it's constant.

  Proof. Take a point $z \in \mathbb{C}$. Take a huge ball with radius R. Cauchy's inequality says that if we take the derivative, then

  $$|f'(z)| \leq \frac{1}{R}\cdot\max_{\partial D}|f(\zeta)|$$

  The maximum is bounded and R is really big, so as $R \to \infty$, the derivative gets arbitrarily small. So if we've got an arbitrary function with zero derivative, then we've got a constant function. ☐

References

Fischer, W., & Lieb, I. (2012). *A course in complex analysis: From basic results to advanced topics* (J. Cannizzo, Trans.). Vieweg+Teubner Verlag.