MATH 27000 (Basic Complex Variables) Problem Sets

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Contents

1 Holomorphicity

Set A: Graded for Completion

3/29: 1. Fischer and Lieb (2012), QI.2.1. Formulate and prove the chain rule for Wirtinger derivatives. Furthermore, show that

$$\overline{\frac{\partial f}{\partial z}} = \frac{\partial \bar{f}}{\partial \bar{z}}$$

Proof. Extrapolating from Fischer and Lieb (2012, p. 11), the full chain rule for the Wirtinger derivatives would be

$$\boxed{\frac{\partial}{\partial t}(f \circ g)(z) = f_z(g(z))g_z(z) + f_{\bar{z}}(g(z))\bar{g}_{\bar{z}}(z)}$$

Additionally, since the complex conjugate of the sum or product of two complex numbers is the sum or product of the complex conjugates, we have that

$$\begin{aligned}
\overline{\frac{\partial f}{\partial z}} &= \overline{\frac{1}{2}(f_x + if_y)} \\
&= \frac{1}{2}(\bar{f}_x + \overline{if_y}) \\
&= \frac{1}{2}(\bar{f}_x - i\bar{f}_y) \\
&= \frac{\partial \bar{f}}{\partial \bar{z}}
\end{aligned}$$

□

2. Let $\langle (x_1, y_1), (x_2, y_2) \rangle = x_1x_2 + y_1y_2$ denote the usual inner product on $\mathbb{R}^2$. We can also define the **Hermitian inner product** on $\mathbb{C}$ via

$$(z, w) = z\bar{w}$$

This term *Hermitian* describes the fact that this product is not symmetric, but satisfies $(w, z) = \overline{(z, w)}$. Show that thinking of z as $x + iy$, we have

$$\langle z, w \rangle = \frac{1}{2}[(z, w) + (w, z)] = \mathrm{Re}(z, w)$$

Proof. We have that

$$\begin{aligned}
\frac{1}{2}[(z, w) + (w, z)] &= \frac{1}{2}[z\bar{w} + w\bar{z}] \\
&= \frac{1}{2}[(z_1 + iz_2)(w_1 - iw_2) + (w_1 + iw_2)(z_1 - iz_2)] \\
&= \frac{1}{2}[(z_1w_1 + z_2w_2 + i(z_2w_1 - z_1w_2)) + (w_1z_1 + w_2z_2 + i(w_2z_1 - w_1z_2))] \\
&= z_1w_1 + z_2w_2 \\
&= \langle (z_1, z_2), (w_1, w_2) \rangle \\
&= \langle z, w \rangle
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Re}(z, w) &= \mathrm{Re}(z\bar{w}) \\
&= \mathrm{Re}[(z_1 + iz_2)(w_1 - iw_2)] \\
&= \mathrm{Re}[z_1w_1 + z_2w_2 + i(z_2w_1 - z_1w_2)] \\
&= z_1w_1 + z_2w_2 \\
&= \langle (z_1, z_2), (w_1, w_2) \rangle \\
&= \langle z, w \rangle
\end{aligned}$$

as desired. □

3. For any integer n, compute the line integral $\int_\gamma z^n \,\mathrm{d}z$ where γ is any circle centered at the origin with counterclockwise orientation. Do not use Cauchy's theorem.

Proof. To evaluate such a line integral over a circle centered at the origin with counterclockwise orientation, we may use the parameterization $\gamma : [0, 2\pi) \to \mathbb{C}$ defined by

$$\gamma(t) = a\mathrm{e}^{it}$$

where a is an arbitrary positive real number. Thus, since $\gamma'(t) = ai\mathrm{e}^{it}$, we have the following when $n \neq -1$.

$$\begin{aligned}
\int_\gamma z^n \,\mathrm{d}z &= \int_0^{2\pi} (a\mathrm{e}^{it})^n \cdot ai\mathrm{e}^{it} \,\mathrm{d}t \\
&= a^{n+1} i \int_0^{2\pi} \mathrm{e}^{i(n+1)t} \,\mathrm{d}t \\
&= a^{n+1} i \left[\frac{\mathrm{e}^{i(n+1)t}}{i(n+1)} \right]_0^{2\pi} \\
&= a^{n+1} i \left[\frac{1}{i(n+1)} - \frac{1}{i(n+1)} \right] \\
\boxed{\int_\gamma z^n \,\mathrm{d}z = 0} & \qquad\qquad (n \neq -1)
\end{aligned}$$

When $n = -1$, we have

$$\begin{aligned}
\int_\gamma \frac{1}{z} \,\mathrm{d}z &= \int_0^{2\pi} a^{-1}\mathrm{e}^{-it} \cdot ai\mathrm{e}^{it} \,\mathrm{d}t \\
&= \int_0^{2\pi} i \,\mathrm{d}t \\
\boxed{\int_\gamma z^{-1} \,\mathrm{d}z = 2\pi i}
\end{aligned}$$

□

4. Without using Cauchy's theorem, show that for any $|a| < 1 < |b|$,

$$\int_\gamma \frac{1}{(z-a)(z-b)} \,\mathrm{d}z = \frac{2\pi i}{a-b}$$

where γ is the circle of radius 1 centered about the origin, oriented counterclockwise.

Proof. Using the method of partial fractions, we set

$$\frac{0z+1}{(z-a)(z-b)} = \frac{A}{z-a} + \frac{B}{z-b} = \frac{A(z-b)+B(z-a)}{(z-a)(z-b)} = \frac{(A+B)z + (-Ab-Ba)}{(z-a)(z-b)}$$

to obtain the two-variable, two-equation system

$$\begin{aligned}
0 &= A + B \\
1 &= -Ab - Ba
\end{aligned}$$

with solution

$$A = \frac{1}{a-b} \qquad\qquad B = \frac{1}{b-a}$$

Thus,

$$\int_\gamma \frac{1}{(z-a)(z-b)}\,\mathrm{d}z = \frac{1}{a-b}\left[\int_\gamma \frac{1}{z-a}\,\mathrm{d}z - \int_\gamma \frac{1}{z-b}\,\mathrm{d}z\right]$$

We will evaluate the left integral first, followed by the right one.

Let's parameterize the circle by $\gamma : [0, 2\pi) \to \mathbb{C}$ defined by $t \mapsto \mathrm{e}^{it}$. Since $|a| < 1$ by hypothesis and $|\gamma(t)| = 1$ for all $t \in [0, 2\pi)$, we know that $|a/z| = |a/\gamma(t)| < 1$; this will allow us to replace a certain expression with the corresponding convergent geometric power series.. Thus, we have that

$$\begin{aligned}
\int_\gamma \frac{1}{z-a}\,\mathrm{d}z &= \int_\gamma \frac{1}{z}\frac{1}{1-a/z}\,\mathrm{d}z \\
&= \int_\gamma \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{a}{z}\right)^k \mathrm{d}z \\
&= \int_\gamma \sum_{k=0}^{\infty} \frac{a^k}{z^{k+1}}\,\mathrm{d}z \\
&= \sum_{k=0}^{\infty} \int_\gamma \frac{a^k}{z^{k+1}}\,\mathrm{d}z \\
&= \sum_{k=0}^{\infty} a^k \int_\gamma z^{-(k+1)}\,\mathrm{d}z
\end{aligned}$$

Note that we are able to exchange the summation and the integral because of the lemma from the 3/26 class regarding convergent series of integrable functions. Additionally, it follows by Problem A.3 that only the $k = 0$ term in the above sum will not evaluate to zero. In particular, this $k = 0$ term will evaluate to $2\pi i$, so overall,

$$\int_\gamma \frac{1}{z-a}\,\mathrm{d}z = 2\pi i$$

For the right integral, we can apply the fundamental theorem of calculus. We have that

$$\int_\gamma \frac{1}{z-b}\,\mathrm{d}z = \int_0^{2\pi} \frac{i\mathrm{e}^{it}}{\mathrm{e}^{it}-b}\,\mathrm{d}t = \left[\ln|\mathrm{e}^{it}-b|\right]_0^{2\pi} = 0$$

Therefore, we have that

$$\begin{aligned}
\int_\gamma \frac{1}{(z-a)(z-b)}\,\mathrm{d}z &= \frac{1}{a-b}\left[\int_\gamma \frac{1}{z-a}\,\mathrm{d}z - \int_\gamma \frac{1}{z-b}\,\mathrm{d}z\right] \\
&= \frac{1}{a-b}[2\pi i - 0] \\
&= \frac{2\pi i}{a-b}
\end{aligned}$$

as desired. ☐

5. Determine the image of the following sets under the following conformal mappings. Use level curves to illustrate the geometry of these mappings.

   (a) The unit disk $\mathbb{D} = \{z : |z| < 1\}$ under $z \mapsto 1/z$.

   Proof. This map inverts the modulus of the real part and flips the imaginary part over the real axis. Because of the radial symmetry of the unit disk, the radial symmetry of the final region will be preserved. However, the final region will consist of all points of magnitude $1/r$ $(r < 1)$, that is, of magnitude $r > 1$. Thus,

   $$\boxed{\operatorname{Im}(\mathbb{D}) = \mathbb{C} \setminus \overline{\mathbb{D}}}$$

   Some polar level curves map as follows. ☐

(b) $\mathbb{D} \setminus \{0\}$ under $z \mapsto z^2$.

Proof. This map squares the radius and doubles the argument of a complex number in polar form. Because of the radial symmetry of this disk and the fact that it only contains complex numbers with modulus that shrink when squared, all it will do is map to itself:

$$\boxed{\operatorname{Im}(\mathbb{D} \setminus \{0\}) = \mathbb{D} \setminus \{0\}}$$

□

(c) The strip $S = \{z : \operatorname{Im}(z) \in (0, 2\pi)\}$ under $z \mapsto \mathrm{e}^z$.

Proof. Let $z = x+iy$. Then $\mathrm{e}^z = \mathrm{e}^a\mathrm{e}^{ib}$. By the definition of S, we know that $a \in \mathbb{R}$ and $b \in (0, 2\pi)$. Thus, since the image of the real exponential function is $(0, \infty)$, by picking various values of a, we can reach a complex number of any modulus save zero. Additionally, by picking various values of b, we can reach a complex number of any argument save zero. This means that we can get anywhere in the complex plane except $[0, \infty)$; we can't even access this region by picking $b = \pi$ and a negative modulus because $\mathrm{e}^a > 0$. Therefore,

$$\boxed{\operatorname{Im}(S) = \mathbb{C} \setminus [0, \infty)}$$

□

(d) The upper half-plane $\mathbb{H} = \{z : \operatorname{Im}(z) > 0\}$ under $z \mapsto z^2$,

Proof. As in part (b), we're squaring the modulus and doubling the argument. This means that we can get anywhere except, coincidentally, the same set we miss in part (c). Therefore,

$$\boxed{\operatorname{Im}(\mathbb{H}) = \mathbb{C} \setminus [0, \infty)}$$

□

(e) The half disk $\mathbb{D} \cap \mathbb{H}$ under $z \mapsto (1 + z)/(1 - z)$.

Proof. Via the applet, it appears that

$$\boxed{\operatorname{Im}(\mathbb{D} \cap \mathbb{H}) = \{z \in \mathbb{C}^* : \arg(z) \in (0, \pi/2)\}}$$

□

Set B: Graded for Content

1. A prototypical example of a weird function that is differentiable (but not C^1) on all of $\mathbb{R}$ is

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Extend f to a function on $\mathbb{C}$ using the same formula (replacing x's with z's). Is it holomorphic at the origin?

Proof. As a first step in investigating the complex version of f, let's look at its behavior along the imaginary axis, i.e., for complex numbers ix, $x \in \mathbb{R}$. Since $\sin(ix) = i \sinh(x)$, we have that

$$(ix)^2 \sin\left(\frac{1}{ix}\right) = -x^2 \sin\left(i \cdot -\frac{1}{x}\right) = -ix^2 \sinh\left(-\frac{1}{x}\right)$$

Investigating $x^2 \sinh(-1/x)$, we find that this function is not even continuous at zero, let alone differentiable or holomorphic. Therefore, $\boxed{f \text{ is not holomorphic at the origin.}}$ □

2. Show that if f is holomorphic on a domain $U \subset \mathbb{C}$ and takes only real values, then it is constant.

Proof. Suppose f sends (x, y) to $(g(x, y), h(x, y))$. If f takes only real values, then $h(x, y) = 0$ for all $(x, y) \in U$. Thus,

$$h_x = h_y = 0$$

on U. Additionally, since f is holomorphic on U, it satisfies the Cauchy-Riemann equations. This combined with the above equation implies that

$$g_x = h_y = 0 \qquad\qquad g_y = -h_x = 0$$

Consequently, $f' = 0$ on U, so f must be constant on U. □

3. Find a conformal map that takes the upper half-plane onto the "Pac-Man" given by

$$\{z : |z| < 1 \text{ and } \arg(z) \in (\pi/4, 7\pi/4)\}$$

Explain how you obtained this map. *Hint*: Do completion problem 5 first.

Proof. Define $f, g, h : \mathbb{C} \to \mathbb{C}$ by

$$\begin{aligned} f &: r\mathrm{e}^{i\theta} \mapsto r\mathrm{e}^{i\theta/2} \\ g &: z \quad \mapsto \frac{z-1}{z+1} \\ h &: r\mathrm{e}^{i\theta} \mapsto r\mathrm{e}^{3i\theta/2+\pi/4} \end{aligned}$$

where f, h take $\theta \in [0, 2\pi)$ to avoid ambiguity. Then the desired conformal map is $h \circ g \circ f$. The bijectivity of f, h follows from the bijectivity of the linear manipulations of the arguments, while the bijectivity of g follows from the fact that it is the inverse of the function in Problem A.A.5e. g, g^{-1} are holomorphic as rational functions with no poles in the domain, and f, f^{-1}, h, h^{-1} are holomorphic because their derivatives are rotation maps at every point.

The trickiest part of obtaining this map was figuring out how to get the infinite rectangle into some kind of arc (the job that g does). I toyed around with translating the block up by i and using $1/z$ or something like that to pull it in, but I couldn't work out how to introduce polar-ness. Then, taking the hint, I thought back to Problem A.A.5e and realized that I could hijack this after some pre- and post-transformations. The use of rotation maps was important, too, for f, g because of their smoothness, as opposed to some kind of $x + iy \mapsto |x| + iy$ map for f, for instance. Then it was just a matter of tweaking numbers. □

2 Power Series and Cauchy's Theorem

Set A: Graded for Completion

4/5: 1. Fischer and Lieb (2012), QI.3.2. Prove the Cauchy-Hadamard formula, which states that the radius of convergence r is equal to

$$r = \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}}$$

Proof. By definition, the radius of convergence is the largest number $|z_1|$ for which a power series $\sum_{k=0}^{\infty} a_k z^k$ converges locally absolutely uniformly on $D_{|z_1|}(0)$. Equivalently, we have by the Lemma from class on 3/26 that the radius of convergence is the largest number $|z_1|$ for which there exists a positive $M \in \mathbb{R}$ such that $|a_k z_1^k| \leq M$ for all k. Rearranging this latter condition, we learn that

$$\begin{aligned} |a_k||z_1|^k &\leq M \\ \sqrt[k]{|a_k|} &\leq \frac{\sqrt[k]{M}}{|z_1|} \end{aligned}$$

for all k. Now since the limit superior and limit inferior of a sequence of real numbers always exist, we have that

$$\limsup_{k\to\infty} \sqrt[k]{|a_k|} \leq \limsup_{k\to\infty} \frac{\sqrt[k]{M}}{|z_1|} = \frac{1}{|z_1|} \limsup_{k\to\infty} \sqrt[k]{M} = \frac{1}{|z_1|} \cdot 1 = \frac{1}{|z_1|}$$

and

$$\liminf_{k\to\infty} \sqrt[k]{|a_k|} \leq \liminf_{k\to\infty} \frac{\sqrt[k]{M}}{|z_1|} = \frac{1}{|z_1|} \liminf_{k\to\infty} \sqrt[k]{M} = \frac{1}{|z_1|} \cdot 1 = \frac{1}{|z_1|}$$

Consequently, we have

$$|z_1| \leq \frac{1}{\limsup_{k\to\infty} \sqrt[k]{|a_k|}} \qquad\qquad |z_1| \leq \frac{1}{\liminf_{k\to\infty} \sqrt[k]{|a_k|}}$$

Moreover, since the limit superior is always greater than or equal to the limit inferior of a sequence of real numbers and hence the reciprocal of the limit superior is less than or equal to the limit inferior, the left statement above is the stronger condition. Therefore, we have a well-defined upper bound on $|z_1|$ in the extended real numbers, which should be exactly the radius of convergence by definition. This verifies the Cauchy-Hadamard formula. ☐

2. Fischer and Lieb (2012), QI.4.4. Show that the function $\tan z$ never takes on the values $\pm i$ and that therefore,

$$\frac{\mathrm{d}}{\mathrm{d}z}(\tan z) \neq 0$$

everywhere. Show that the tangent function maps the strip $S_0 = \{z : -\pi/2 < \operatorname{Re} z < \pi/2\}$ biholomorphically onto $\mathbb{C} \setminus \{it : t \in \mathbb{R},\ |t| \geq 1\}$.

Also use level sets to illustrate the conformal mapping.

Proof. Suppose for the sake of contradiction that $z \in \mathbb{C}$ satisfies $\tan z = i$. Then

$$i = \tan z = \frac{\sin z}{\cos z} = \frac{\frac{1}{2i}(\mathrm{e}^{iz} - \mathrm{e}^{-iz})}{\frac{1}{2}(\mathrm{e}^{iz} + \mathrm{e}^{-iz})} = \frac{\mathrm{e}^{iz} - \mathrm{e}^{-iz}}{i(\mathrm{e}^{iz} + \mathrm{e}^{-iz})} = \frac{\mathrm{e}^{2iz} - 1}{i(\mathrm{e}^{2iz} + 1)}$$

so

$$\begin{aligned} i^2(\mathrm{e}^{2iz} + 1) &= \mathrm{e}^{2iz} - 1 \\ -\mathrm{e}^{2iz} - 1 &= \mathrm{e}^{2iz} - 1 \\ 0 &= \mathrm{e}^{2iz} \end{aligned}$$

Now if the complex number e^{2iz} equals zero, then $|\mathrm{e}^{2iz}| = \mathrm{e}^{\mathrm{Re}(z)}$ equals zero, too. Thus, $\mathrm{Re}(z) = \log(0)$, but $\log(0)$ is undefined, a contradiction.

Suppose for the sake of contradiction that $z \in \mathbb{C}$ satisfies $\tan z = -i$. Then, similarly to before,

$$-i = \frac{\mathrm{e}^{2iz} - 1}{i(\mathrm{e}^{2iz} + 1)}$$

so

$$\begin{aligned} -i^2(\mathrm{e}^{2iz} + 1) &= \mathrm{e}^{2iz} - 1 \\ \mathrm{e}^{2iz} + 1 &= \mathrm{e}^{2iz} - 1 \\ 2 &= 0 \end{aligned}$$

a contradiction.

Suppose for the sake of contradiction that the derivative of $\tan z$ is everywhere zero. Then $\tan z$ is constant. However, $\tan 0 = 0$ and $\tan(\pi/4) = 1$ for instance, so $\tan z$ is not constant, a contradiction.

To prove that $\tan z$ maps S_0 biholomorphically onto $\mathbb{C} \setminus \{it : t \in \mathbb{R},\ |t| \geq 1\}$, Fischer and Lieb (2012, p. 6) tells us that it will suffice to show that the map is bijective and holomorphic. The holomorphicity condition comes immediately since Fischer and Lieb (2012) states that $\tan z$ is holomorphic everywhere except at the zeroes of $\cos z$ and these zeroes $(\pi/2 \pm \pi n)$ are all outside of S_0. Bijectivity, on the other hand, takes a bit more work. To show it, we will decompose $\tan z$ into the composition of four mappings, each of which is individually bijective and the overall composition of which maps the right domain to the right codomain. Explicitly, let $\tan z = \phi(h(g(f(z))))$ where

$$f(z) := 2iz \qquad g(z) := \mathrm{e}^z \qquad h(z) := \frac{z-1}{z+1} \qquad \phi(z) := -iz$$

As a "multiply by $w \in \mathbb{C}$" function, f is complex linear and nontrivial, hence bijective. It also maps S_0 to $S_{-\pi} = \{z = x + iy : -\pi < y < \pi\}$. By Fischer and Lieb (2012, p. 21), g maps $S_{-\pi}$ bijectively onto $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. h is bijective because we can derive the explicit formula

$$h^{-1}(z) = -\frac{z+1}{z-1}$$

and confirm that

$$h(h^{-1}(z)) = h^{-1}(h(z)) = z$$

In particular, $h(\mathbb{R}_{\leq 0}) = \{t : t \in \mathbb{R},\ |t| \geq 1\}$. We can see this because via polynomial division,

$$h(x) = \frac{x-1}{x+1} = 1 - \frac{2}{x+1}$$

and this function starts at $-1 = h(0)$, decreases asymptotically toward $-\infty$ as $0 \to -1$, and then decreases asymptotically from ∞ toward 1 as x goes from $-1^- \to -\infty$. Lastly, ϕ is bijective for the same reason as f, and ϕ maps $\{t : t \in \mathbb{R},\ |t| \geq 1\}$ to $\{it : t \in \mathbb{R},\ |t| \geq 1\}$, the final set that we desire to cut out of $\mathbb{C}$. $\square$

3. Fix $a, b, c \in \mathbb{C}$ so that c is not a negative integer or 0. Show that the **hypergeometric** function

$$F(a,b,c;z) := \sum_{k=0}^{\infty} \frac{a(a+1)\cdots(a+k-1)b(b+1)\cdots(b+k-1)}{c(c+1)\cdots(c+k-1)} \frac{z^k}{k!}$$

converges on the unit disk and satisfies the differential equation

$$z(1-z)F''(z) + [c - (a+b+1)z]F'(z) - abF(z) = 0$$

Proof. Before we begin properly, we will introduce the **Pochhammer symbol** $(q)_k$, defined by

$$(q)_k = \begin{cases} 1 & k = 0 \\ q(q+1)\cdots(q+k-1) & k > 0 \end{cases}$$

as shorthand for $a(a+1)\cdots(a+k-1)$, etc. Making use of this notation, we will also preliminarily observe that

$$\begin{aligned} F'(z) &= \sum_{k=1}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{z^{k-1}}{(k-1)!} \\ &= \sum_{k=0}^{\infty} \frac{(a)_{k+1}(b)_{k+1}}{(c)_{k+1}} \frac{z^k}{k!} \\ &= \frac{ab}{c} \sum_{k=0}^{\infty} \frac{(a+1)_k(b+1)_k}{(c+1)_k} \frac{z^k}{k!} \\ &= \frac{(a)_1(b)_1}{(c)_1} F(a+1, b+1, c+1; z) \end{aligned}$$

and

$$\begin{aligned} F''(z) &= \frac{\mathrm{d}}{\mathrm{d}z}[F'(z)] \\ &= \frac{(a)_1(b)_1}{(c)_1} F'(a+1, b+1, c+1; z) \\ &= \frac{(a)_1(b)_1}{(c)_1} \cdot \frac{(a+1)_1(b+1)_1}{(c+1)_1} F(a+2, b+2, c+2; z) \\ &= \frac{(a)_2(b)_2}{(c)_2} F(a+2, b+2, c+2; z) \end{aligned}$$

These formulas will be useful later.

We now begin our argument that the hypergeometric function satisfies the given differential equation in earnest. Upon expanding the given differential equation, we will obtain a sum of terms that can be sorted by the power of z present in the term. In particular, the coefficients a_k of each z^k term must cancel to zero independently, so let's derive a general formula for the coefficient of the k^{th} term.

For the first term, we have

$$\begin{aligned} z(1-z)F''(z) &= z(1-z)\frac{(a)_2(b)_2}{(c)_2} F(a+2, b+2, c+2; z) \\ &= (1-z) \cdot \frac{(a)_2(b)_2}{(c)_2} z \left[1 + \frac{(a+2)(b+2)}{c+2} z + \frac{(a+2)_2(b+2)_2}{(c+2)_2} \frac{z^2}{2} + \cdots \right] \\ &= (1-z) \left[\frac{(a)_2(b)_2}{(c)_2 \cdot 0!} z + \frac{(a)_3(b)_3}{(c)_3 \cdot 1!} z^2 + \frac{(a)_4(b)_4}{(c)_4 \cdot 2!} z^3 + \cdots \right] \\ &= \frac{(a)_2(b)_2}{(c)_2 \cdot 0!} z + \left[\frac{(a)_3(b)_3}{(c)_3 \cdot 1!} - \frac{(a)_2(b)_2}{(c)_2 \cdot 0!}\right] z^2 + \left[\frac{(a)_4(b)_4}{(c)_4 \cdot 2!} - \frac{(a)_3(b)_3}{(c)_3 \cdot 1!}\right] z^3 + \cdots \\ &= 0 + \frac{(a)_2(b)_2}{(c)_2 \cdot 0!} z + \sum_{k=2}^{\infty} \left[\frac{(a)_{k+1}(b)_{k+1}}{(c)_{k+1}(k-1)!} - \frac{(a)_k(b)_k}{(c)_k(k-2)!}\right] z^k \\ &= 0 + \frac{(a)_2(b)_2}{(c)_2 \cdot 0!} z + \sum_{k=2}^{\infty} \frac{(a)_k(b)_k}{(c)_k(k-2)!} \left[\frac{(a+k)(b+k)}{(c+k)(k-1)} - 1\right] z^k \end{aligned}$$

For the second term, we have

$$\begin{aligned}
[c-(a+b+1)z]F'(z) &= [c-(a+b+1)z]\frac{(a)_1(b)_1}{(c)_1}F(a+1,b+1,c+1;z) \\
&= [c-(a+b+1)z]\cdot\frac{(a)_1(b)_1}{(c)_1}\left[1+\frac{(a+1)(b+1)}{c+1}z+\frac{(a+1)_2(b+1)_2}{(c+1)_2}\frac{z^2}{2}+\cdots\right] \\
&= [c-(a+b+1)z]\left[\frac{(a)_1(b)_1}{(c)_1\cdot 0!}+\frac{(a)_2(b)_2}{(c)_2\cdot 1!}z+\frac{(a)_3(b)_3}{(c)_3\cdot 2!}z^2+\cdots\right] \\
&= \frac{(a)_1(b)_1c}{(c)_1\cdot 0!}+\left[\frac{(a)_2(b)_2c}{(c)_2\cdot 1!}-\frac{(a)_1(b)_1(a+b+1)}{(c)_1\cdot 0!}\right]z+\cdots \\
&= \frac{(a)_1(b)_1c}{(c)_1\cdot 0!}+\sum_{k=1}^{\infty}\left[\frac{(a)_{k+1}(b)_{k+1}c}{(c)_{k+1}\cdot k!}-\frac{(a)_k(b)_k(a+b+1)}{(c)_k\cdot(k-1)!}\right]z^k \\
&= \frac{(a)_1(b)_1c}{(c)_1\cdot 0!}+\sum_{k=1}^{\infty}\frac{(a)_k(b)_k}{(c)_k(k-2)!}\left[\frac{(a+k)(b+k)c}{k(k-1)(c+k)}-\frac{a+b+1}{k-1}\right]z^k
\end{aligned}$$

And for the third term, we have

$$\begin{aligned}
-abF(z) &= -abF(a,b,c;z) \\
&= -ab\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_kk!}z^k \\
&= \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k(k-2)!}\left[-\frac{ab}{k(k-1)}\right]z^k
\end{aligned}$$

Adding these three infinite series together, we can see that the constant (z^0) term will have the coefficient

$$a_0 = 0+\frac{(a)_1(b)_1c}{(c)_1\cdot 0!}-ab\cdot\frac{(a)_0(b)_0}{(c)_0\cdot 0!}=\frac{abc}{c}-ab\cdot\frac{1}{1}=0$$

as desired. We can also see that the z^1 term will have coefficient

$$\begin{aligned}
a_1 &= \frac{(a)_2(b)_2}{(c)_2\cdot 0!}+\frac{(a)_2(b)_2c}{(c)_2\cdot 1!}-\frac{(a)_1(b)_1(a+b+1)}{(c)_1\cdot 0!}-ab\cdot\frac{(a)_1(b)_1}{(c)_1\cdot 1!} \\
&= \frac{a(a+1)b(b+1)}{c(c+1)}+\frac{a(a+1)b(b+1)c}{c(c+1)}-\frac{ab(a+b+1)}{c}-\frac{a^2b^2}{c} \\
&= \frac{ab(a+1)(b+1)}{c(c+1)}+\frac{abc(a+1)(b+1)}{c(c+1)}-\frac{ab(a+b+1)(c+1)}{c(c+1)}-\frac{a^2b^2(c+1)}{c(c+1)} \\
&= \frac{ab(a+1)(b+1)+abc(a+1)(b+1)-ab(a+b+1)(c+1)-a^2b^2(c+1)}{c(c+1)}
\end{aligned}$$

Now to show that $a_1=0$, it will suffice to show that the numerator of the above fraction equals zero, which we can do as follows.

$$\begin{aligned}
N(a_1) &= ab(a+1)(b+1)+abc(a+1)(b+1)-ab(a+b+1)(c+1)-a^2b^2(c+1) \\
&= ab(ab+a+b+1)+abc(ab+a+b+1) \\
&\quad -abc(a+b+1)-ab(a+b+1)-a^2b^2c-a^2b^2 \\
&= abab+ab(a+b+1)+abcab+abc(a+b+1) \\
&\quad -abc(a+b+1)-ab(a+b+1)-abcab-abab \\
&= 0
\end{aligned}$$

To conclude, we show that the coefficients a_k ($k \geq 2$) are equal to zero all at once, as follows.

$$\begin{aligned}
a_k &\propto \frac{(a+k)(b+k)}{(c+k)(k-1)} - 1 + \frac{(a+k)(b+k)c}{k(k-1)(c+k)} - \frac{a+b+1}{k-1} - \frac{ab}{k(k-1)} \\
&= \frac{k(a+k)(b+k)}{k(k-1)(c+k)} - \frac{k(k-1)(c+k)}{k(k-1)(c+k)} + \frac{(a+k)(b+k)c}{k(k-1)(c+k)} - \frac{k(a+b+1)(c+k)}{k(k-1)(c+k)} - \frac{ab(c+k)}{k(k-1)(c+k)} \\
&= \frac{k(a+k)(b+k) - k(k-1)(c+k) + (a+k)(b+k)c - k(a+b+1)(c+k) - ab(c+k)}{k(k-1)(c+k)}
\end{aligned}$$

And as before, we'll focus on the numerator from here on out.

$$\begin{aligned}
N(a_k) &= k(a+k)(b+k) - k(k-1)(c+k) + (a+k)(b+k)c - k(a+b+1)(c+k) - ab(c+k) \\
&= k(ab+ak+bk+k^2) \\
&\quad - k(ck-c-k+k^2) \\
&\quad + (ab+ak+bk+k^2)c \\
&\quad - k(ac+bc+c+ak+bk+k) \\
&\quad - ab(c+k) \\
&= abk + ak^2 + bk^2 + k^3 \\
&\quad - ck^2 + ck + k^2 - k^3 \\
&\quad + abc + ack + bck + ck^2 \\
&\quad - ack - bck - ck - ak^2 - bk^2 - k^2 \\
&\quad - abc - abk \\
&= 0
\end{aligned}$$

As to convergence on the unit disk, applying the Cauchy-Hadamard formula, we can see that $k! \to \infty$ faster than anything else, so the limit superior will go to zero and the radius of convergence will be ∞. □

Set B: Graded for Content

1. Fischer and Lieb (2012), QII.2.3. Compute the **Fresnel integrals**

$$\int_0^\infty \cos(x^2)\,\mathrm{d}x = \sqrt{\frac{\pi}{8}} = \int_0^\infty \sin(x^2)\,\mathrm{d}x$$

Hint: Apply the Cauchy integral theorem to sectors with center 0 and corners given by R and $\mathrm{e}^{i\pi/4}R$, where $R \to \infty$.

Proof. As in the case of the Dirichlet integral, we will analyze the complex exponential functions composing the complex cosine and then combine our results into the final answer. Let's begin.

Let γ be the sector described in the hint oriented counterclockwise, and let $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ where γ_1 is the segment along the real axis, γ_2 is the curved portion, and γ_3 is the segment between 0 and $R\mathrm{e}^{i\pi/4}$. Then by the Cauchy integral theorem,

$$\int_\gamma \mathrm{e}^{-iz^2}\,\mathrm{d}z = 0$$

Note that we begin our explorations here because this integral closely resembles the Gaussian integral, so we may be able to use that to our advantage. And indeed, for γ_1, $\int_0^R \mathrm{e}^{-it^2}\,\mathrm{d}t$ can be expressed in terms of the Gaussian integral as $R \to \infty$ since the Gaussian distribution is even:

$$\lim_{R\to\infty} \int_0^R \mathrm{e}^{-it^2}\,\mathrm{d}t = \frac{1}{2}\int_{-\infty}^\infty \mathrm{e}^{-it^2}\,\mathrm{d}t = \frac{\sqrt{\pi}}{2}$$

Bounding the integral over γ_2 takes more work, just like in the case of the Dirichlet integral. However, if we first attempt to get a bound on its magnitude, we can end up proving that it converges to zero.

$$\begin{aligned}
\left|\int_{\gamma_2} \mathrm{e}^{-z^2}\,\mathrm{d}z\right| &= \left|\int_0^{\pi/4} \mathrm{e}^{-(R\mathrm{e}^{it})^2}\cdot iR\mathrm{e}^{it}\,\mathrm{d}t\right| \\
&= \left|\int_0^{\pi/4} R\mathrm{e}^{-R^2\mathrm{e}^{i\cdot 2t}}\cdot i\mathrm{e}^{it}\,\mathrm{d}t\right| \\
&= \left|\int_0^{\pi/4} R\mathrm{e}^{-R^2\cos(2t)}\cdot i\mathrm{e}^{i(t-R^2\sin(2t))}\,\mathrm{d}t\right| \\
&\leq \int_0^{\pi/4} \left|R\mathrm{e}^{-R^2\cos(2t)}\cdot i\mathrm{e}^{i(t-R^2\sin(2t))}\right|\mathrm{d}t \\
&= \int_0^{\pi/4} R\mathrm{e}^{-R^2\cos(2t)}\,\mathrm{d}t
\end{aligned}$$

At this point, we'd like to find a way to bound $\cos(2t)$ so that we can evaluate the integral directly, without bounding it so loosely that we lose the convergence. One such way is by noting that $\cos(2t)$ is just slightly greater than the (much simpler) linear function $1-4t/\pi$ on the interval $[0,\pi/4]$, and thus the negative exponential of the cosine is slightly less than the negative exponential of the linear. Continuing to evaluate, we obtain an integral that can be computed explicitly:

$$\begin{aligned}
\int_0^{\pi/4} R\mathrm{e}^{-R^2\cos(2t)}\,\mathrm{d}t &\leq \int_0^{\pi/4} R\mathrm{e}^{-R^2(1-4t/\pi)}\,\mathrm{d}t \\
&= -\frac{\pi(\mathrm{e}^{-R^2}-1)}{4R}
\end{aligned}$$

This expression has the form c/∞ in the limit and thus equals zero.

Combining the last several results, we have that

$$0 = \sum_{k=1}^{3}\int_{\gamma_k} \mathrm{e}^{-z^2}\,\mathrm{d}z$$
$$\int_{\gamma_3} \mathrm{e}^{-z^2}\,\mathrm{d}z = -\int_{\gamma_1} \mathrm{e}^{-z^2}\,\mathrm{d}z = -\frac{\sqrt{\pi}}{2}$$

This puts us in an interesting and different place from the Dirichlet integral. There, our integral over the real axis was still an unknown, and here, we've already evaluated it. How can we use this situation to our advantage? Well, let's start expanding the γ_3 integral and go from there.

$$\begin{aligned}
\int_{-\gamma_3} \mathrm{e}^{-z^2}\,\mathrm{d}z &= \int_0^\infty \mathrm{e}^{-\mathrm{e}^{i\pi/2}t^2}\cdot \mathrm{e}^{i\pi/4}\,\mathrm{d}t \\
&= \int_0^\infty \mathrm{e}^{-it^2}\cdot \mathrm{e}^{i\pi/4}\,\mathrm{d}t \\
&= \int_0^\infty \left[\cos\left(t^2\right) - i\sin\left(t^2\right)\right]\cdot\frac{\sqrt{2}}{2}(1+i)\,\mathrm{d}t \\
&= \frac{\sqrt{2}}{2}\int_0^\infty \left[\cos\left(t^2\right)+\sin\left(t^2\right)+i\left(\cos\left(t^2\right)-\sin\left(t^2\right)\right)\right]\mathrm{d}t \\
&= \frac{\sqrt{2}}{2}\left[\int_0^\infty \cos\left(t^2\right)\mathrm{d}t + \int_0^\infty \sin\left(t^2\right)\mathrm{d}t + i\left(\int_0^\infty \cos\left(t^2\right)\mathrm{d}t - \int_0^\infty \sin\left(t^2\right)\mathrm{d}t\right)\right]
\end{aligned}$$

What it appears that we have now obtained actually is the two-variable system of equations

$$\frac{\sqrt{2}}{2}\left[\int_0^\infty \cos\left(t^2\right)\mathrm{d}t + \int_0^\infty \sin\left(t^2\right)\mathrm{d}t + i\left(\int_0^\infty \cos\left(t^2\right)\mathrm{d}t - \int_0^\infty \sin\left(t^2\right)\mathrm{d}t\right)\right] = \frac{\sqrt{\pi}}{2} + i(0)$$

or

$$\int_0^\infty \cos(t^2)\,\mathrm{d}t + \int_0^\infty \sin(t^2)\,\mathrm{d}t = \sqrt{\frac{\pi}{2}} \qquad\qquad \int_0^\infty \cos(t^2)\,\mathrm{d}t - \int_0^\infty \sin(t^2)\,\mathrm{d}t = 0$$

which we can solve for the desired result. □

2. These problems illustrate the geometric intuition for the radius of convergence of a power series. Do the parts in order.

(a) For each nonzero natural number $n \in \mathbb{N}$, compute the power series expansion for the function $1/z$ around the point $1/n$. What are their radii of convergence?

Proof. The desired power series will be of the form

$$P_n(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(1/n)}{k!}(z - 1/n)^k$$

where $f(z) = 1/z = z^{-1}$. Now by the power rule,

$$f^{(k)}(z) = (-1)^k k! z^{-(k+1)}$$

Therefore, combining the last two results,

$$\boxed{P_n(z) = \sum_{k=0}^{\infty} (-1)^k n^{k+1} (z - 1/n)^k}$$

By the Cauchy-Hadamard formula,

$$\begin{aligned} r &= \frac{1}{\limsup_{k\to\infty} |(-1)^k n^{k+1}|^{1/k}} \\ &= \frac{1}{\limsup_{k\to\infty} n^{k+1/k}} \\ \boxed{r = \frac{1}{n}} \end{aligned}$$

□

(b) Describe the set of points $w \in \mathbb{C}$ such that the power series expansion for $1/z$ about w has radius of convergence equal to 1.

Proof. Working backward in part (a), we start off by finding $n \in \mathbb{C}$ such that $\limsup_{k\to\infty} |(-1)^k n^{k+1}|^{1/k} = 1$. This will happen if $|n| = 1$. Then working backwards, we want $w = 1/n$. But this is still just the set of points of magnitude 1. Therefore, the desired set is

$$\boxed{\{w \in \mathbb{C} : |w| = 1\}}$$

□

(c) Suppose that

$$f(z) = \frac{1}{z(z-1)(z-i)(z-1-i)}$$

Find the unique point w in the unit square $\{\mathrm{Re}(z), \mathrm{Im}(z) \in [0,1]\}$ such that the radius of convergence of the power series for w is maximal. Justify your answer.

Proof. The definition of f singles out the four corners of the unit square as singularities. Thus, the disk of convergence cannot include any of these corners, so we need the point in the unit square that's farthest away from all four corners. This would be

$$\boxed{w = \frac{1}{2}(1+i)}$$

$\square$

3. This problem is to hint at the general formulation of the Cauchy integral theorem. Please solve this problem only using things we have seen in class to this point.

(a) Show that the "L-shaped" domain

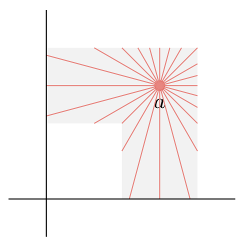
$$L = \{z : \mathrm{Re}(z), \mathrm{Im}(z) \in (0,2) \text{ and not both } \mathrm{Re}(z), \mathrm{Im}(z) \in (0,1]\}$$

is star-shaped (hence the Cauchy integral theorem applies).

Proof. Choose

$$a = \frac{3}{2}(1+i)$$

Then



$\square$

(b) Show that the "double L-shaped" domain

$$U = \{z : |\mathrm{Re}(z)|, \mathrm{Im}(z) \in (0,2) \text{ and not both } |\mathrm{Re}(z)|, \mathrm{Im}(z) \in (0,1]\}$$

is not star-shaped.

Proof. We can do this by casework. If we pick a to be any point in the right "L," straight-line paths to $(-1.5, 0.1)$ will go outside of U and vice versa for the left "L." $\square$

(c) Nevertheless, by breaking up U into two copies of L and using the Cauchy integral theorem for the resultant star-shaped domains, show that for any closed curve γ in U and any $f \in \mathcal{O}(U)$, we have that $\int_\gamma f \, \mathrm{d}z = 0$.

Proof. Any time the curve crosses the imaginary axis once, it will have to cross the imaginary axis at least one more time on the way back since the loop is closed. Thus, when it crosses the imaginary axis, choose the next time it crosses the imaginary axis as we proceed along the path and draw a segment between these two points. Integrate around the loop on the right side and the left side; the integrals along the segment will cancel and the sum will be the original integral. Meanwhile, the one-sided loops in the individual star-shaped domains will evaluate to zero. $\square$

(d) Show that $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ can be written as the union of two star-shaped domains.

Proof. Choose $\mathbb{C}$ without the upper-right quartile and $\mathbb{C}$ without the lower left quartile. $\square$

(e) Why doesn't your proof for part (c) show that $\int_\gamma f \, \mathrm{d}z = 0$ for any $f \in \mathcal{O}(\mathbb{C}^*)$ and any closed curve γ in $\mathbb{C}^*$?

Proof. The two sets in part (d) are not (and cannot be) disjoint. $\square$

3 Cauchy's Integral Formula

Set A: Graded for Completion

4/12: 1. Fischer and Lieb (2012), QII.3.1. Using the Cauchy integral formulas, compute the following integrals.

(a) $\displaystyle\int_{|z+1|=1} \frac{\mathrm{d}z}{(z+1)(z-1)^3}$.

Proof. Let

$$f(z) = \frac{1}{(z-1)^3}$$

Then by the CIF,

$$\begin{aligned} f(-1) &= \frac{1}{2\pi i}\int_{|\zeta+1|=1} \frac{f(\zeta)}{\zeta+1}\mathrm{d}\zeta \\ -\frac{1}{8} &= \frac{1}{2\pi i}\int_{|z+1|=1} \frac{f(z)}{z+1}\,\mathrm{d}z \\ \boxed{-\frac{\pi i}{4} = \int_{|z+1|=1} \frac{\mathrm{d}z}{(z+1)(z-1)^3}} \end{aligned}$$

$\square$

(b) $\displaystyle\int_{|z-i|=3} \frac{\mathrm{d}z}{z^2+\pi^2}$.

Proof. We know that

$$\frac{1}{z^2+\pi^2} = \frac{1}{(z-\pi i)(z+\pi i)}$$

Thus, let

$$f(z) = \frac{1}{z+\pi i}$$

Then by the CIF,

$$\begin{aligned} f(\pi i) &= \frac{1}{2\pi i}\int_{|z-i|=3} \frac{f(z)}{z-\pi i}\,\mathrm{d}z \\ \frac{1}{2\pi i} &= \frac{1}{2\pi i}\int_{|z-i|=3} \frac{1}{(z-\pi i)(z+\pi i)}\,\mathrm{d}z \\ \boxed{1 = \int_{|z-i|=3} \frac{\mathrm{d}z}{z^2+\pi^2}} \end{aligned}$$

$\square$

(c) $\displaystyle\int_{|z|=1/2} \frac{\mathrm{e}^{1-z}}{z^3(1-z)}\,\mathrm{d}z$.

Proof. Let

$$f(z) = \frac{\mathrm{e}^{1-z}}{1-z}$$

Consequently,

$$f'(z) = \frac{(1-z)\cdot -\mathrm{e}^{1-z} - \mathrm{e}^{1-z}\cdot -1}{(1-z)^2} = \frac{z\mathrm{e}^{1-z}}{(1-z)^2}$$

and

$$f''(z) = \frac{(1-z)^2\cdot(\mathrm{e}^{1-z} - z\mathrm{e}^{1-z}) - z\mathrm{e}^{1-z}\cdot 2(1-z)^1\cdot -1}{(1-z)^4} = \frac{(1+z^2)\mathrm{e}^{1-z}}{(1-z)^3}$$

Then by the second derivative of the CIF,

$$\begin{aligned}
f''(0) &= \frac{2!}{2\pi i}\int_{|z|=1/2}\frac{f(z)}{(z-0)^{2+1}}\,\mathrm{d}z \\
\mathrm{e} &= \frac{1}{\pi i}\int_{|z|=1/2}\frac{f(z)}{z^3}\,\mathrm{d}z \\
\Aboxed{\pi i\mathrm{e} &= \int_{|z|=1/2}\frac{\mathrm{e}^{1-z}}{z^3(1-z)}\,\mathrm{d}z}
\end{aligned}$$

□

(d) $\int_{|z-1|=1}\left(\frac{z}{z-1}\right)^n\mathrm{d}z$ for any $n\geq 1$.

Proof. Let

$$f(z)=z^n$$

Then by the $(n-1)^{\text{th}}$ derivative of the CIF,

$$\begin{aligned}
f^{(n-1)}(1) &= \frac{(n-1)!}{2\pi i}\int_{|z-1|=1}\frac{f(z)}{(z-1)^n}\,\mathrm{d}z \\
n! &= \frac{(n-1)!}{2\pi i}\int_{|z-1|=1}\frac{z^n}{(z-1)^n}\,\mathrm{d}z \\
\Aboxed{2\pi n i &= \int_{|z-1|=1}\left(\frac{z}{z-1}\right)^n\mathrm{d}z}
\end{aligned}$$

□

2. Fischer and Lieb (2012), QII.4.2. Assume that the power series $f(z)=\sum_{k=0}^{\infty}a_kz^k$ converges on $D=D_r(0)$.

(a) Show that if f is real-valued on $\mathbb{R}\cap D$, then all a_k are real.

Proof. Suppose for the sake of contradiction that a_k is complex. By hypothesis, f is a convergent power series in a neighborhood of 0. Thus, by the proposition from the 3/26 class, f is holomorphic in a neighborhood of zero. Consequently, f is C^∞. This means that in particular, f is C^k with

$$f^{(k)}(0)=k!a_k$$

But since a_k is complex, this means that the k^{th} derivative of f is complex, a contradiction for a real-valued function. □

(b) Show that if f is an even (resp. odd) function, then $a_k=0$ for all odd (resp. even) k.

Proof. Suppose f is even. Then $f(z)=f(-z)$. Thus,

$$\begin{aligned}
\sum_{k=0}^{\infty}a_kz^k &= \sum_{k=0}^{\infty}a_k(-z)^k \\
\sum_{k=0}^{\infty}a_{2k}z^{2k}+\sum_{k=0}^{\infty}a_{2k+1}z^{2k+1} &= \sum_{k=0}^{\infty}a_{2k}z^{2k}-\sum_{k=0}^{\infty}a_{2k+1}z^{2k+1} \\
2\sum_{k=0}^{\infty}a_{2k+1}z^{2k+1} &= 0
\end{aligned}$$

Now the whole power series on the left above cancelling out implies that each term individually goes to zero as well. Therefore, we have proven that all of the odd coefficients go to zero.
The proof is symmetric in the other case. □

(c) Show that if $f(iz) = f(z)$, then a_k can only be nonzero if k is divisible by 4.

Proof. As in part (b), we have that

$$\begin{aligned} \sum_{k=0}^{\infty} a_k (iz)^k &= \sum_{k=0}^{\infty} a_k z^k \\ \sum_{k=0}^{\infty} a_{4k} z^{4k} + i \sum_{k=0}^{\infty} a_{4k+1} z^{4k+1} - \sum_{k=0}^{\infty} a_{4k+2} z^{4k+2} - i \sum_{k=0}^{\infty} a_{4k+3} z^{4k+3} &= \sum_{j=0}^{3} \sum_{k=0}^{\infty} a_{4k+j} z^{4k+j} \\ (i-1) \sum_{k=0}^{\infty} a_{4k+1} z^{4k+1} - 2 \sum_{k=0}^{\infty} a_{4k+2} z^{4k+2} - (i+1) \sum_{k=0}^{\infty} a_{4k+3} z^{4k+3} &= 0 \end{aligned}$$

Indeed, via term-by-term cancellation again, we see that all terms with $k \pmod 4 \not\equiv 0$ go to zero. □

(d) Discuss the equation $f(\rho z) = \mu f(z)$, where $\rho, \mu \in \mathbb{C} \setminus \{0\}$ are given.

Proof. As in part (c), if ρ is a root of unity and $\mu = 1$, then a_k can only be nonzero if k is divisible by the denominator of $\arg \rho$ in reduced form. If ρ has an irrational argument, the equation may not add much of any new information. More generally, power series that satisfy this equation tend to be determined on $\mathbb{C}$ by their values on some subset of $\mathbb{C}$, be it half the plane (as in $f(-z) = f(z)$ or $f(-z) = 2f(z)$), one quadrant (as in $f(iz) = f(z)$), or some other region. □

3. Fischer and Lieb (2012), QII.6.1. Determine the type of singularity that each of the following functions has at z_0. If the singularity is removable, calculate the limit as $z \to z_0$; if the singularity is a pole, find its order and the principal part of f at z_0.

(a) $(1 - \mathrm{e}^z)^{-1}$ at $z_0 = 0$.

Proof. As $z \to z_0 = 0$, $\mathrm{e}^z \to 1$ and hence $1 - \mathrm{e}^z \to 0$. Thus, as $z \to z_0$, $|1/(1 - \mathrm{e}^z)| \to \infty$. Therefore, the singularity is a $\boxed{\text{pole.}}$

If $f(z) = (1 - \mathrm{e}^z)^{-1}$, then $g(z) = 1 - \mathrm{e}^z$ and $g^{(n)}(z) = -\mathrm{e}^z$ $(n \geq 1)$. Thus,

$$g(z) = \sum_{k=1}^{\infty} -\frac{1}{n!} z^n = -z \sum_{k=0}^{\infty} \frac{1}{(n+1)!} z^n$$

so $\boxed{\text{the order is 1.}}$ Moreover, the principal part of f at z_0 is

$$\boxed{-\frac{1}{z}}$$

□

(b) $(z - \sin z)^{-1}$ at $z_0 = 0$.

Proof. As in part (a), the denominator goes to zero as we go to $z_0 = 0$, so the singularity is a $\boxed{\text{pole.}}$

Also as before, we can calculate that

$$g(z) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{z^{2k+1}}{(2k+1)!} = z^3 \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k+3)!}$$

so $\boxed{\text{the order is 3.}}$ Inverting this back again to determine the principal part, we must use the coefficient formulae on Fischer and Lieb (2012, p. 51) to find the Laurent series expansion for f.

In particular,

$$\begin{aligned} b_0 &= a_0^{-1} = 3! = 6 \\ b_1 &= -a_1 a_0^{-2} = 0 \\ b_2 &= (a_1^2 - a_0 a_2) a_0^{-3} = \frac{3}{10} \end{aligned}$$

so the principal part of f at z_0 is

$$\boxed{6x^{-3} + \frac{3}{10}x^{-1}}$$

□

(c) $ze^{iz}/(z^2+b^2)^2$ at $z_0 = ib$ ($b > 0$).

Proof. As in parts (a) and (b), the singularity is a $\boxed{\text{pole.}}$
As in part (b), we can calculate that

$$g(z) = 0 + 0z + 4ibe^b z^2 + 24be^b z^3 + \cdots$$

so $\boxed{\text{the order is 2.}}$ Inverting this back again, we learn that

$$b_0 = -\frac{i}{4be^b} \qquad\qquad b_1 = \frac{3}{2be^b}$$

so the principal part of f at z_0 is

$$\boxed{-\frac{i}{4be^b}z^{-2} + \frac{3}{2be^b}z^{-1}}$$

□

(d) $(\sin z + \cos z - 1)^{-2}$ at $z_0 = 0$.

Proof. As in parts (a)-(c), the singularity is a $\boxed{\text{pole.}}$
Once again, we can calculate that

$$g(z) = 0 + 0z + z^2 - z^3$$

so $\boxed{\text{the order is 2.}}$ And once again,

$$b_0 = 1 \qquad\qquad b_1 = 1$$

so we have

$$\boxed{z^{-2} + z^{-1}}$$

□

4. Let $\mathbb{D}$ denote the unit disk and suppose that $f \in \mathcal{O}(\mathbb{D})$.

(a) Prove that for any $R \in (0,1)$ and any z with $|z| < R$, we have that

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\theta})\operatorname{Re}\left(\frac{Re^{i\theta}+z}{Re^{i\theta}-z}\right)\mathrm{d}\theta$$

Hint: Observe that setting $w = R^2/\bar{z}$, we have that the integral of $f(\zeta)/(\zeta - w)$ over the circle of radius R centered at the origin is 0.

Proof. Since $f \in \mathcal{O}(\mathbb{D})$, $D := D_R(0) \subset\subset \mathbb{D}$, and $z \in D$, the CIF tells us that

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

Parameterize ∂D by $\zeta = R\mathrm{e}^{i\theta}$ for $\theta \in [0, 2\pi]$. Then, substituting into the above,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_0^{2\pi} f(R\mathrm{e}^{i\theta}) \frac{1}{R\mathrm{e}^{i\theta} - z} \cdot iR\mathrm{e}^{i\theta} \mathrm{d}\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(R\mathrm{e}^{i\theta}) \frac{R\mathrm{e}^{i\theta}}{R\mathrm{e}^{i\theta} - z} \mathrm{d}\theta \end{aligned}$$

□

(b) Compute that

$$\operatorname{Re}\left(\frac{R\mathrm{e}^{i\theta} + r}{R\mathrm{e}^{i\theta} - r}\right) = \frac{R^2 - r^2}{R^2 - 2Rr\cos\theta + r^2}$$

where $R, r, \theta \in \mathbb{R}$.

Proof. We have that

$$\begin{aligned} \operatorname{Re}\left(\frac{R\mathrm{e}^{i\theta} + r}{R\mathrm{e}^{i\theta} - r}\right) &= \frac{1}{2}\left[\frac{R\mathrm{e}^{i\theta} + r}{R\mathrm{e}^{i\theta} - r} + \overline{\frac{R\mathrm{e}^{i\theta} + r}{R\mathrm{e}^{i\theta} - r}}\right] \\ &= \frac{1}{2}\left[\frac{R\mathrm{e}^{i\theta} + r}{R\mathrm{e}^{i\theta} - r} + \frac{R\mathrm{e}^{-i\theta} + r}{R\mathrm{e}^{-i\theta} - r}\right] \\ &= \frac{1}{2}\left[\frac{(R\mathrm{e}^{i\theta} + r)(R\mathrm{e}^{-i\theta} - r) + (R\mathrm{e}^{i\theta} - r)(R\mathrm{e}^{-i\theta} + r)}{(R\mathrm{e}^{i\theta} - r)(R\mathrm{e}^{-i\theta} - r)}\right] \\ &= \frac{1}{2}\left[\frac{(R^2 - Rr\mathrm{e}^{i\theta} + Rr\mathrm{e}^{-i\theta} - r^2) + (R^2 + Rr\mathrm{e}^{i\theta} - Rr\mathrm{e}^{-i\theta} - r^2)}{R^2 - Rr\mathrm{e}^{i\theta} - Rr\mathrm{e}^{-i\theta} + r^2}\right] \\ &= \frac{1}{2}\left[\frac{2R^2 - 2r^2}{R^2 - Rr(\mathrm{e}^{i\theta} + \mathrm{e}^{-i\theta}) + r^2}\right] \\ &= \frac{R^2 - r^2}{R^2 - 2Rr\cos\theta + r^2} \end{aligned}$$

as desired. □

(c) Now suppose that $u = \operatorname{Re}(f)$, so u is a harmonic function. Deduce the **Poisson integral representation formula**: For $z = r\mathrm{e}^{i\theta}$, we have

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - \phi) u(\phi) \mathrm{d}\phi$$

where $P_r(\psi)$ is the **Poisson kernel** for the disk, given by

$$P_r(\psi) = \frac{1 - r^2}{1 - 2r\cos\psi + r^2}$$

Proof.

$$\begin{aligned} f(z) &= \frac{1}{2\pi} \int_0^{2\pi} f(R\mathrm{e}^{i\theta}) \operatorname{Re}\left(\frac{R\mathrm{e}^{i\theta} + z}{R\mathrm{e}^{i\theta} - z}\right) \mathrm{d}\theta \\ f(z) &= \frac{1}{2\pi} \int_0^{2\pi} f(R\mathrm{e}^{i\theta}) \frac{R^2 - r^2}{R^2 - 2Rr\cos\theta + r^2} \mathrm{d}\theta \\ u(z) &= \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - \phi) u(\phi) \mathrm{d}\phi \\ u(z) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2} u(\phi) \mathrm{d}\phi \end{aligned}$$

□

Set B: Graded for Content

1. Fischer and Lieb (2012), QII.4.6.

(a) Suppose the domain G is symmetric with respect to the real axis and that f is holomorphic on G and real-valued on $G \cap \mathbb{R}$. Show that $f(\bar{z}) = \overline{f(z)}$ for all $z \in G$.

Proof. Since f is holomorphic, f has a power series representation

$$\sum_{k=0}^{\infty} a_k z^k$$

on G. It follows by QA.2a that the coefficients a_k of this power series are real. Therefore, we have that

$$\begin{aligned} f(\bar{z}) &= \sum_{k=0}^{\infty} a_k \bar{z}^k \\ &= \sum_{k=0}^{\infty} a_k \overline{z^k} \\ &= \sum_{k=0}^{\infty} \overline{a_k z^k} \\ &= \overline{\sum_{k=0}^{\infty} a_k z^k} \\ &= \overline{f(z)} \end{aligned}$$

as desired. Note that we know that a $\bar{z} \in G$ corresponds to each $z \in G$ because of the hypothesis that G is symmetric with respect to the real axis. $\square$

(b) Suppose $G = D_r(0)$ and f is holomorphic on G and real-valued on $G \cap \mathbb{R}$. Show that if f is even (resp. odd), then the values of f on $G \cap i\mathbb{R}$ are real (resp. imaginary). Prove this without using the power series expansion of f.

Proof. Suppose first that f is even. Let $ib \in G \cap i\mathbb{R}$ be an arbitrary imaginary number. Since f is even,

$$f(ib) = f(-ib)$$

Additionally, by part (a),

$$f(-ib) = \overline{f(ib)}$$

Thus, by transitivity,

$$f(ib) = \overline{f(ib)}$$

But for a complex number to equal its complex conjugate, that complex number must be real, as desired.

Now suppose that f is odd. Then

$$f(ib) = -f(-ib)$$

We still have in addition that $f(-ib) = \overline{f(ib)}$, so by transitivity,

$$\begin{aligned} f(ib) &= -\overline{f(ib)} \\ f(ib) + \overline{f(ib)} &= 0 \\ 2\,\mathrm{Re}[f(ib)] &= 0 \\ \mathrm{Re}[f(ib)] &= 0 \end{aligned}$$

Therefore, $f(ib)$ must be purely imaginary, as desired. $\square$

2. Some setup: Suppose that f is holomorphic on the unit disk $\mathbb{D} = \{|z| < 1\}$. A point w on the circle $\partial D = \{|z| = 1\}$ is **regular** if there is an open neighborhood U of w and an analytic function g on U such that $f = g$ on $U \cap \mathbb{D}$. Notice that f can be analytically continued outside the boundary of $\mathbb{D}$ if and only if there is a point w on ∂D that is regular for f.

Now define the function

$$f(z) = \sum_{k=1}^{\infty} z^{2^k}$$

Show that f converges on $\mathbb{D}$, and that it cannot be analytically continued past $\mathbb{D}$.

Proof. Let z with $|z| < 1$ be arbitrary. By the geometric series test, $\sum_{k=0}^{\infty} z^k$ converges on $\mathbb{D}$, and then f converges via the comparison test.

Now suppose for the sake of contradiction that There is a regular point on $\partial\mathbb{D}$ and corresponding function $g : U \to \mathbb{C}$. Since $f = g$ on $\mathbb{D} \cap U$, the identity theorem tells us that f, g have the same power series (i.e., the one defined above). Now let $z \in U$ with $|z| > 1$. By the ratio test,

$$\lim_{k\to\infty} \left| \frac{z^{2^{k+1}}}{z^{2^k}} \right| = \lim_{k\to\infty} \frac{|z|^{2\cdot 2^k}}{|z|^{2^k}} = \lim_{k\to\infty} |z|^{2^k} = \infty > 1$$

so the series diverges at z, contradicting the existence of g. □

3. Suppose that f is an entire function and that, for all sufficiently large z, we have $|f(z)| \leq |z|^n$. Prove that f must be a polynomial.

Proof. Since f is entire (and hence holomorphic), it has a power series. Proving that f is a polynomial is then just a matter of proving that this power series is finite, i.e., truncates somewhere. Combining Cauchy's inequalities with the given condition, we have that

$$\frac{|f^{(m)}(z)|}{m!} \leq \frac{1}{R^m} \max_{\partial D} |f(\zeta)| \leq \frac{1}{R^m} \max_{\partial D} |z|^n = \frac{1}{R^m} \cdot R^n = R^{n-m}$$

for all $m \in \mathbb{N}_0$. Evidently, then, for all $m > n$, we can send $R \to \infty$ and shrink $a_m = |f^{(m)}(z)|/m! \to 0$, as desired. □

4 Modulus Principles, Meromorphicity, and Möbius Transforms

Set A: Graded for Completion

5/3: 1. Fischer and Lieb (2012), QIII.1.1.

   (a) Determine the order of the zero of $\sum_{n=1}^{k} b_n(z-z_0)^{-n}$ at ∞.

   (b) For the following functions, determine the value $w_0 = f(\infty)$ and its multiplicity.

$$f(z) = \frac{2z^4 - 2z^3 - z^2 - z + 1}{z^4 - z^3 - z + 1} \qquad\qquad f(z) = \frac{z^4 + iz^3 + z^2 + 1}{z^4 + iz^3 + z^2 - iz}$$

2. Fischer and Lieb (2012), QIII.3.1. Let f and g be entire functions such that $|f| \leq |g|$. Show that $f = cg$ for some constant c.

3. Fischer and Lieb (2012), QIII.4.1.

   (a) Let $S, T \in$ Möb. Show that a point $z_1 \in \hat{\mathbb{C}}$ is a fixed point of T if and only if Sz_1 is a fixed point of STS^{-1}.

   (b) Suppose T has exactly one fixed point z_1. Show that there is an $S \in$ Möb such that STS^{-1} is a translation. Moreover, show that for every $z \in \hat{\mathbb{C}}$, we have

$$\lim_{n\to\infty} T^n z = z_1$$

   where $T^n = T \circ \cdots \circ T$ denotes the n-fold composition of T with itself.

   (c) Suppose T has exactly two fixed points z_1 and z_2. Show that there is an $S \in$ Möb such that STS^{-1} is of the form $z \mapsto az$, where $a \in \mathbb{C}^*$, and that the pair $\{a, a^{-1}\}$ is uniquely determined by T.

   (d) Show that if we have $|a| \neq 1$ in part (c), then after a possible renumbering of our fixed points, we have

$$\lim_{n\to\infty} T^n z = z_1$$

   for all $z \in \hat{\mathbb{C}} \setminus \{z_2\}$. In the case that $|a| = 1$, show that every point in $\hat{\mathbb{C}} \setminus \{z_1, z_2\}$ lies on a T-invariant Möbius circle.

4. Fischer and Lieb (2012), QIII.5.1. Let f be the branch of the logarithm on $\mathbb{C} \setminus \mathbb{R}_{\geq 0}$ that takes the value $-i\pi/2$ at $-i$. Determine...

$$f(i) \qquad f(-\mathrm{e}) \qquad f(-1-i\sqrt{3}) \qquad f((-1-i\sqrt{3})^2)$$

5. If z_1 and z_2 are related by inversion in a circle C, and z_3 and z_4 are arbitrary (distinct) points of C, show that the cross ratio of the four points has modulus 1.

Set B: Graded for Content

1. Fischer and Lieb (2012), QIII.3.4. Consider the function $f(z) = z + \mathrm{e}^z$. Show that for all $t \in [0, 2\pi]$,

$$\lim_{r\to\infty} f(r\mathrm{e}^{it}) = \infty$$

and that the convergence is uniform with respect to t on the sets $\{t : |t - \pi| \leq \frac{\pi}{2}\}$ and $\{t : |t| \leq \alpha\}$ for every $\alpha < \pi/2$. How does this agree with Proposition 3.4?

2. Fischer and Lieb (2012), QIII.5.2.

   (a) Find a maximal domain on which holomorphic functions $\log(1-z)^2$ and $\sqrt{z + \sqrt{z}}$, respectively, can be defined.

(b) Show that a logarithm of the tangent function exists on the set

$$G = \mathbb{C} \setminus \bigcup_{k \in \mathbb{Z}} [k\pi - \tfrac{\pi}{2}, k\pi]$$

3. Show that a fractional linear transformation

$$z \mapsto \frac{az + b}{cz + d}$$

maps the upper half plane to itself if and only if $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$.

4. Suppose that U is a domain, $f \in \mathcal{O}(U)$ is never zero, and suppose that a holomorphic branch of the logarithm exists on $f(U)$; then the function $\log[f(z)]$ is holomorphic. By considering the real part of $\log f$, show that the maximum modulus principles for harmonic functions and for holomorphic functions are equivalent.

5 Residues

Set A: Graded for Completion

5/17: 1. Fischer and Lieb (2012), QIV.1.3. Show that the image of a simply connected domain under a biholomorphic mapping is simply connected. Is it sufficient to assume that the mapping is locally biholomorphic?

Proof. Let U be a simply connected domain, f be a biholomorphic mapping, $f(\gamma) \subset f(U)$, and $w_0 \in \mathbb{C} \setminus f(U)$ be arbitrary. We have that

$$\begin{aligned}
\mathrm{wn}(f(\gamma), w_0) &= \frac{1}{2\pi i} \int_{f(\gamma)} \frac{1}{w - w_0} \, \mathrm{d}w \\
&= \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z) - w_0} \, \mathrm{d}z \\
&= \frac{1}{2\pi i} \cdot 0 && \text{CIT 2} \\
&= 0
\end{aligned}$$

Note that it is the fact that f is biholomorphic that allows us to perform a complex change of variables. Also note that $f(z) - w_0$ never equals zero because $w_0 \notin f(U)$, hence why we can call the integrand of the second line holomorphic and apply the Cauchy Integral Theorem. Therefore, by the winding-number definition of a simply connected domain, $f(U)$ is simply connected.

$\boxed{\text{Yes}}$, it is sufficient to assume the mapping is locally biholomorphic. Assuming that "locally biholomorphic" means that f is biholomorphic on some $D_r(x) \cap U$ for all $x \in U$, we know that each $f(D_r(x) \cap U)$ will be individually simply connected and since they overlap, we can build out the image simply connected region with overlapping patches like in the identity theorem and analytic continuation. □

2. Fischer and Lieb (2012), QIV.3.2. Find the principal part of the Laurent expansion of

$$\frac{z-1}{\sin^2 z}$$

in $0 < |z| < \pi$ and of

$$\frac{z}{(z^2 + b^2)^2}$$

in $0 < |z - ib| < 2b$.

Proof. Using the power series expansion of $\sin z$, the Cauchy product, the power series inversion formulas from Fischer and Lieb (2012, p. 51), and some more power series manipulations, we have that

$$\begin{aligned}
\frac{z-1}{\sin^2 z} &= \frac{z-1}{\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}\right)^2} \\
&= \frac{z-1}{z^2} \cdot \frac{1}{\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n}\right)^2} \\
&= \frac{z-1}{z^2} \cdot \frac{1}{\sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{(-1)^m}{(2m+1)!} \cdot \frac{(-1)^{n-m}}{(2(n-m)+1)!}\right) z^{2n}} \\
&= \frac{z-1}{z^2} \cdot \frac{1}{\sum_{n=0}^{\infty} (-1)^n \left(\sum_{m=0}^{n} \frac{1}{(2m+1)! \cdot (2(n-m)+1)!}\right) z^{2n}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{z-1}{z^2} \cdot \left[1 + \frac{2}{3!}z^2 + \left(\frac{3}{3!^2} - \frac{2}{5!}\right) z^4 + \cdots\right] \\
&= (z-1) \cdot \left[\frac{1}{z^2} + \frac{2}{3!} + \left(\frac{3}{3!^2} - \frac{2}{5!}\right) z^2 + \cdots\right] \\
&= -\frac{1}{z^2} + \frac{1}{z} - \frac{2}{3!} + \frac{2}{3!}z - \left(\frac{3}{3!^2} - \frac{2}{5!}\right) z^2 + \left(\frac{3}{3!^2} - \frac{2}{5!}\right) z^3 + \cdots
\end{aligned}
$$

Therefore, the principal part of the Laurent expansion of $(z-1)/\sin^2 z$ in the punctured disk $0 < |z| < \pi$ is

$$\boxed{-\frac{1}{z^2} + \frac{1}{z}}$$

For the other function in question, observe that

$$\frac{z}{(z^2+b^2)^2} = \frac{z}{(z+ib)^2(z-ib)^2} = (z-ib)^{-2} \cdot \frac{z}{(z+ib)^2}$$

This decomposition is important because we have — as in the 4/9 lecture — rewritten the function as the product of a $z - ib$ term and a function that is holomorphic on the entire punctured disk $0 < |z - ib| < 2b$. Let's call the right term above g. Since $g \in \mathcal{O}(D_{2b}(ib))$, we can compute its power series about ib in $D_{2b}(ib)$. In particular, we need only compute the first two terms of its power series because since $(z - ib)$ has a -2 exponent above, any higher order terms in the computed power series will contribute to the regular part of the Laurent expansion, not the principal part. Carrying out this computation, we have

$$g(ib) = \frac{ib}{(2ib)^2} = \frac{1}{4ib}$$

and

$$g'(ib) = \left.\frac{(z+ib)^2 \cdot 1 - z \cdot 2(z+ib)}{(z+ib)^4}\right|_{z=ib} = \left.\frac{ib-z}{(z+ib)^3}\right|_{z=ib} = 0$$

Thus,

$$\frac{z}{(z+ib)^2} = g(z) = \frac{1}{4ib} + 0(z-ib) + \cdots$$

It follows that

$$\frac{z}{(z^2+b^2)^2} = (z-ib)^{-2} \cdot \left(\frac{1}{4ib} + 0(z-ib) + \cdots\right) = \boxed{\frac{1}{4ib(z-ib)^2}} + \cdots$$

as desired. ∎

3. Fischer and Lieb (2012), QIV.4.1. Let $f \in \mathcal{O}(\mathbb{C} \setminus S)$ for some discrete set S of singularities. Show that...

   (a) If f is even, then

   $$\operatorname{res}_{-z} f = -\operatorname{res}_z f$$

   Proof. Let

   $$w = g(\zeta) := -\zeta$$

   Additionally, let D be a small disk around z containing at most one singularity and containing one singularity iff that singularity is z. Then $g(D)$ is a small disk around $-z$ with the same singularity conditions. Therefore,

   $$
   \begin{aligned}
   \operatorname{res}_{-z} f &= \frac{1}{2\pi i} \int_{g(\partial D)} f(w)\,\mathrm{d}w \\
   &= \frac{1}{2\pi i} \int_{\partial D} f(g(\zeta)) \cdot -1 \mathrm{d}\zeta
   \end{aligned}
   $$

$$
\begin{aligned}
&= -\frac{1}{2\pi i}\int_{\partial D} f(-\zeta)\mathrm{d}\zeta \\
&= -\frac{1}{2\pi i}\int_{\partial D} f(\zeta)\mathrm{d}\zeta \\
&= -\operatorname{res}_z f
\end{aligned}
$$

as desired. Note that we can substitute $f(-\zeta) = f(\zeta)$ from the third to the fourth line, above, because f is even by hypothesis. $\square$

(b) If f is odd, then

$$\operatorname{res}_{-z} f = \operatorname{res}_z f$$

Proof. Using the same terminology as in part (a), we have that

$$
\begin{aligned}
\operatorname{res}_{-z} f &= \frac{1}{2\pi i}\int_{g(\partial D)} f(w)\,\mathrm{d}w \\
&= \frac{1}{2\pi i}\int_{\partial D} f(g(\zeta)) \cdot -1\mathrm{d}\zeta \\
&= \frac{1}{2\pi i}\int_{\partial D} -f(-\zeta)\mathrm{d}\zeta \\
&= \frac{1}{2\pi i}\int_{\partial D} f(\zeta)\mathrm{d}\zeta \\
&= \operatorname{res}_z f
\end{aligned}
$$

as desired. $\square$

(c) If $f(z+\omega) = f(z)$ for some $\omega \in \mathbb{C}$, then

$$\operatorname{res}_{z+\omega} f = \operatorname{res}_z f$$

Proof. Let

$$w = g(\zeta) := \zeta + \omega$$

Additionally, let D be a small disk around z with the same singularity conditions as in part (a). Then $g(D)$ is a small disk around $z+\omega$ with the same singularity conditions. Therefore,

$$
\begin{aligned}
\operatorname{res}_{z+\omega} f &= \frac{1}{2\pi i}\int_{g(\partial D)} f(w)\,\mathrm{d}w \\
&= \frac{1}{2\pi i}\int_{\partial D} f(g(\zeta))\mathrm{d}\zeta \\
&= \frac{1}{2\pi i}\int_{\partial D} f(\zeta+\omega)\mathrm{d}\zeta \\
&= \frac{1}{2\pi i}\int_{\partial D} f(\zeta)\mathrm{d}\zeta \\
&= \operatorname{res}_z f
\end{aligned}
$$

as desired. $\square$

(d) If f is real on $\mathbb{R}$, then

$$\operatorname{res}_{\bar{z}} f = \overline{\operatorname{res}_z f}$$

Proof. Let

$$w = g(\zeta) := \bar{\zeta}$$

Additionally, let D be a small disk around z with the same singularity conditions as in part (a). Then $g(D)$ is a small disk around $\bar{z}$ with the same singularity conditions. Furthermore, if f is real on $\mathbb{R}$, then by PSet 3, QA.2a, every coefficient in the Laurent series is real. Thus, $f(\bar{z}) = \overline{f(z)}$. Therefore,

$$\begin{aligned}
\operatorname{res}_{\bar{z}} f &= \frac{1}{2\pi i}\int_{g(\partial D)} f(w)\,\mathrm{d}w \\
&= \frac{1}{2\pi i}\int_{\partial D} f(g(\zeta))\,\mathrm{d}w \\
&= \frac{1}{2\pi i}\int_{\partial D} f(\bar{\zeta})\mathrm{d}\bar{\zeta} \\
&= \frac{1}{2\pi i}\int_{\partial D} \overline{f(\zeta)\mathrm{d}\zeta} \\
&= \frac{1}{2\pi i}\overline{\int_{\partial D} f(\zeta)\mathrm{d}\zeta} \\
&= \overline{\frac{1}{2\pi i}\int_{\partial D} f(\zeta)\mathrm{d}\zeta} \\
&= \overline{\operatorname{res}_z f}
\end{aligned}$$

as desired. Note that from the second to the third line above, we have just renamed $\mathrm{d}w$ to $\mathrm{d}\bar{z}$; we have not taken a derivative and done an infinitesimal substitution as in parts (a)-(c). $\square$

4. Fischer and Lieb (2012), QIV.5.1. Prove that

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{\cosh x} = \pi$$

Hint: Integrate over the boundary of the rectangle whose corners are $\pm r$ and $\pm r + i\pi$.

Proof. Taking the hint, we will analytically continue $1/\cosh x$ to the meromorphic function $1/\cosh z$ and integrate it over the described path, which we will denote by $\square$. Introducing some additional notation, let γ_2 denote the right side of the rectangle (oriented from r to $r + i\pi$), let γ_3 denote the top of the rectangle (oriented from $r + i\pi$ to $-r + i\pi$), and let γ_4 denote the left side of the rectangle (oriented from $-r + i\pi$ to $-r$). Then

$$\int_{\square} \frac{\mathrm{d}z}{\cosh z} = \int_{-r}^{r} \frac{\mathrm{d}x}{\cosh x} + \int_{\gamma_3} \frac{\mathrm{d}z}{\cosh z} + \int_{\gamma_2+\gamma_4} \frac{\mathrm{d}z}{\cosh z}$$

The reason for writing the integrals in this order will soon become apparent. From here, we seek to rearrange the above expression to get the integral over the real line ($\int_{-r}^{r} \mathrm{d}x / \cosh x$) by itself and take the limit as $r \to \infty$. We begin this process by simplifying the above equation.

Since $1/\cosh z$ has only one pole inside $\square$ (namely, at $i\pi/2$), we have by the residue theorem that

$$\int_{\square} \frac{\mathrm{d}z}{\cosh z} = 2\pi i \cdot \operatorname{wn}(\square, \tfrac{i\pi}{2}) \cdot \operatorname{res}_{i\pi/2}\left(\frac{1}{\cosh z}\right)$$

We know just by how $\square$ is defined that it has a winding number of 1 around $i\pi/2$. To evaluate the residue, begin by computing the power series expansion of $\cosh z$ about $i\pi/2$.

$$\begin{aligned}
\cosh z &= \sum_{k=0}^{\infty} \frac{\cosh^{(k)}(\frac{i\pi}{2})}{k!}(z - \tfrac{i\pi}{2})^k \\
&= i \cdot (z - \tfrac{i\pi}{2}) + \frac{i}{3!}(z - \tfrac{i\pi}{2})^3 + \cdots \\
&= (z - \tfrac{i\pi}{2}) \cdot \left[i + \frac{i}{3!}(z - \tfrac{i\pi}{2})^2 + \cdots\right]
\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{\cosh z} &= \frac{1}{z-\frac{i\pi}{2}} \cdot \left[-i + \frac{i}{3!}(z-\tfrac{i\pi}{2})^2 + \cdots\right] \\ &= \frac{-i}{z-\frac{i\pi}{2}} + \frac{i}{3!}(z-\tfrac{i\pi}{2}) + \cdots\end{aligned}$$

This tells us that the desired residue is $-i$. Putting the last several results together, we learn that

$$\int_{\square} \frac{\mathrm{d}z}{\cosh z} = 2\pi i \cdot 1 \cdot -i = 2\pi$$

Additionally, since $1/\cosh z =: f(z)$ is antiperiodic — that is, $-f(z) = f(z+i\pi)$ — we have that

$$\int_{\gamma_3} \frac{\mathrm{d}z}{\cosh z} = \int_r^{-r} \frac{\mathrm{d}x}{\cosh(x+i\pi)} = \int_r^{-r} \frac{\mathrm{d}x}{-\cosh x} = \int_{-r}^{r} \frac{\mathrm{d}x}{\cosh x}$$

Lastly, since $1/\cosh(x+iy) \to 0$ as $|x| \to \infty$, we have that

$$\lim_{r\to\infty} \int_{\gamma_2+\gamma_4} \frac{\mathrm{d}z}{\cosh z} = 0$$

Therefore, the original equation rearranges to

$$\begin{aligned}2\pi &= 2\int_{-r}^{r} \frac{\mathrm{d}x}{\cosh x} \\ \pi &= \int_{-r}^{r} \frac{\mathrm{d}x}{\cosh x}\end{aligned}$$

Taking the limit as $r \to \infty$ yields the desired result. $\square$

5. Verify the claim from class that the integral of

$$f(z) = \frac{\pi}{z^2 \tan(\pi z)}$$

over the square with vertices $[\pm(N+\frac{1}{2}), \pm(N+\frac{1}{2})]$ for $N \in \mathbb{N}$ converges to 0 as $N \to \infty$.

Proof. Let γ_N be the aforementioned square. We will compute the integral by first bounding the integrand f and then showing that said integrand converges to zero as the path grows larger. It will follow that in the limit, the integrand $f = 0$ and hence the integral must be zero.

To begin, let's bound the reciprocal of $\tan(\pi z)$, which is

$$\cot(\pi z) = i \cdot \frac{\mathrm{e}^{\pi i z} + \mathrm{e}^{-\pi i z}}{\mathrm{e}^{\pi i z} - \mathrm{e}^{-\pi i z}}$$

We will do this one side at a time for all four sides of γ_N.

<u>The right side</u>: When $\mathrm{Re}(z) = N + 1/2$ (for some $N \in \mathbb{Z}$) and $y = \mathrm{Im}(z) \in \mathbb{R}$, we have that

$$\mathrm{e}^{\pi i z} = \mathrm{e}^{\pi i(N+1/2+yi)} = \mathrm{e}^{N\pi i} \cdot \mathrm{e}^{\pi i/2} \cdot \mathrm{e}^{-\pi y} = \pm 1 \cdot i \cdot \mathrm{e}^{-\pi y} = \pm i\mathrm{e}^{-\pi y}$$

and

$$\mathrm{e}^{-\pi i z} = \mathrm{e}^{-\pi i(N+1/2+yi)} = \mathrm{e}^{-N\pi i} \cdot \mathrm{e}^{-\pi i/2} \cdot \mathrm{e}^{\pi y} = \pm 1 \cdot -i \cdot \mathrm{e}^{\pi y} = \mp i\mathrm{e}^{\pi y}$$

so hence,

$$|\cot(\pi z)| = \left|\frac{\mathrm{e}^{\pi i z} + \mathrm{e}^{-\pi i z}}{\mathrm{e}^{\pi i z} - \mathrm{e}^{-\pi i z}}\right| = \left|\frac{\pm i\mathrm{e}^{-\pi y} + \mp i\mathrm{e}^{\pi y}}{\pm i\mathrm{e}^{-\pi y} - \mp i\mathrm{e}^{\pi y}}\right| = \left|\frac{\pm\mathrm{e}^{-\pi y} - \pm\mathrm{e}^{\pi y}}{\pm\mathrm{e}^{-\pi y} + \pm\mathrm{e}^{\pi y}}\right| = \left|\frac{\mathrm{e}^{-\pi y} - \mathrm{e}^{\pi y}}{\mathrm{e}^{-\pi y} + \mathrm{e}^{\pi y}}\right| \leq 1$$

The left side: The argument is symmetric to that used for the right side.

The top side: In this case, $\mathrm{Im}(z) = N + 1/2$ and $x = \mathrm{Re}(z) \in \mathbb{R}$. First off, observe that as $\mathrm{Im}(z) \to \infty$, we have that

$$\cot(\pi z) \to -i$$

Thus, if we want to keep $\cot(\pi z)$ bounded, it will suffice to keep it near $-i$. By our observation, to achieve this, we need only require that $\mathrm{Im}(z) = N + 1/2$ is greater than a certain threshold. This threshold can be determined using the applet from the 3/21 lecture, which shows us that if $\mathrm{Im}(z) \geq 0 + 1/2 = 1/2$, then we already have

$$|\cot(\pi z) + i| \ll 1$$

But if $\cot(\pi z)$ is so close to $-i$, then certainly

$$|\cot(\pi z)| \leq 2$$

along the top side.[1]

The bottom side: An analogous argument to the top side holds based on the fact that as $\mathrm{Im}(z) \to -\infty$, $\cot(\pi z) \to i$.

Thus, since $|\cot(\pi z)| \leq 1$ on the right and left sides of γ_N and $|\cot(\pi z) \leq 2$ on the top and bottom of γ_n, we have that $|\cot(\pi z)| \leq 2$ for all $z \in \mathrm{Im}(\gamma_N)$ and $N \in \mathbb{N}$.

Consequently, as $N \to \infty$,

$$|f(z)| = \left|\frac{\pi}{z^2}\cot(\pi z)\right| \leq \left|\frac{2\pi}{z^2}\right| \to 0$$

for all $z \in \mathrm{Im}(\gamma_N)$. Thus, the integral of f over γ_N goes to zero, too. In a statement,

$$\lim_{N\to\infty}\int_{\gamma_N} f\,\mathrm{d}z = 0$$

as desired. ☐

6. For $\alpha \notin \mathbb{Z}$, prove that

$$\sum_{n=-\infty}^{\infty}\frac{1}{(\alpha+n)^2} = \frac{\pi^2}{\sin^2(\pi\alpha)}$$

Hint: Adapt your computation from Problem A.5.

Proof. Taking the hint, we will choose to work with the analogous helper function[2]

$$f(z) = \frac{\pi}{(\alpha+z)^2\tan(\pi z)}$$

Similar to the Basel problem, f is meromorphic on $\mathbb{C}$ with poles of order 1 at every $n \in \mathbb{Z}$ (because tangent is periodic) and an additional pole of order 2 at $-\alpha$. Let's compute the residue of f about

[1] Note that for the sake of bounding the $\cot(\pi z)$ component of $\pi/z^2\tan(\pi z)$, we need not continue to move the top and bottom of γ_N up and down to $\pm\infty$ along with the right and left sides of the box; rather, they could stay at $\mathrm{Im}(z) = \pm 1/2$ and we'd be totally fine on bounding $\cot(\pi z)$. However, we *do* have the top and bottom diverge so that the z^2 term in the denominator of $\pi/z^2\tan(\pi z)$ becomes large at *all* points along γ_N as $N \to \infty$. This fact will be used shortly when we compute $\int_{\gamma_N} f\,\mathrm{d}z$.

[2] A good way to find this function is to first look at analogous functions that have infinitely many residues of the form $(\alpha+n)^{-1}$, of which there are a few. From here, we narrow it down to a function that has an "easily" computable final residue. One pattern that emerges is that computability is improved when we leave the argument of the tangent function alone and only modify the z^2 in the denominator. Another one is that it is good to separate the poles caused by tangent from the pole caused by z^2 so that we can take a positive and negative infinite sum without having to treat the "zero case" separately as we did in the Basel problem; instead, we introduce a new pole whose residue we will compute at the end. These criteria are what leads to the chosen function which, as we are about to see, works quite nicely. So there's an element of "because it works" mathematics at work here, but a bit of strategy, too.

the tangent-caused poles. In these cases, the denominator has a simple zero (because α is not equal to any n by hypothesis) and the numerator is holomorphic, so we may apply "Property 4" from the 5/2 lecture to learn that

$$\operatorname{res}_n f = \frac{\pi\big|_n}{\mathrm{d}/\mathrm{d}z\ [(\alpha+z)^2\tan(\pi z)]\big|_n} = \frac{\pi}{2(\alpha+n)\underbrace{\tan(\pi n)}_{0}+\pi(\alpha+n)^2\underbrace{\sec^2(\pi n)}_{1}} = \frac{\pi}{\pi(\alpha+n)^2} = \frac{1}{(\alpha+n)^2}$$

Defining γ_N as in Problem A.5, we have via the residue theorem (since γ_N has a winding number of 1 around all the poles it encloses) that

$$\frac{1}{2\pi i}\int_{\gamma_N} f(z)\,\mathrm{d}z = \operatorname{res}_{-\alpha} f + \sum_{n=-N}^{N}\operatorname{res}_n f$$

for $N > |\alpha|$. Now analogously to what we did in Problem A.5, we can compute the above integral to be zero in the limit that $N\to\infty$. Consequently, combining the last two results, we have that

$$\begin{aligned}
\frac{1}{2\pi i}\lim_{N\to\infty}\int_{\gamma_N} f(z)\,\mathrm{d}z &= \lim_{N\to\infty}\left(\operatorname{res}_{-\alpha} f + \sum_{n=-N}^{N}\operatorname{res}_n f\right)\\
\frac{1}{2\pi i}\cdot 0 &= \operatorname{res}_{-\alpha} f + \sum_{n=-\infty}^{\infty}\operatorname{res}_n f\\
\sum_{n=-\infty}^{\infty}\frac{1}{(\alpha+n)^2} &= -\operatorname{res}_{-\alpha} f
\end{aligned}$$

Evidently, we must now compute $\operatorname{res}_{-\alpha} f$. Rewrite f as follows.

$$f(z) = \frac{1}{(\alpha+z)^2}\cdot\underbrace{\pi\cot(\pi z)}_{g(z)}$$

Observe that g is holomorphic in a neighborhood of $-\alpha$ (again, since $\alpha\notin\mathbb{Z}$). Thus, g has a power series expansion about $-\alpha$ given by

$$g(z) = \sum_{k=0}^{\infty}\frac{g^{(k)}(-\alpha)}{k!}(z+\alpha)^k$$

Consequently, the Laurent expansion of f about $-\alpha$ is

$$f(z) = \sum_{k=-2}^{\infty}\frac{g^{(k+2)}(-\alpha)}{(k+2)!}(z+\alpha)^k$$

It follows by the definition of the residue as the a_{-1} coefficient that

$$\operatorname{res}_{-\alpha} f = \frac{g'(-\alpha)}{1!} = \pi\cdot -\csc^2(\pi z)\cdot\pi\Big|_{-\alpha} = -\frac{\pi^2}{\sin^2(-\pi\alpha)} = -\frac{\pi^2}{(-1)^2\sin^2(\pi\alpha)} = -\frac{\pi^2}{\sin^2(\pi\alpha)}$$

Therefore,

$$\sum_{n=-\infty}^{\infty}\frac{1}{(\alpha+n)^2} = -\left(-\frac{\pi^2}{\sin^2(\pi\alpha)}\right) = \frac{\pi^2}{\sin^2(\pi\alpha)}$$

as desired. ☐

Set B: Graded for Content

1. Fischer and Lieb (2012), QIV.4.3. Let G be a simply connected domain, and let $f \in \mathcal{O}(G \setminus S)$. Show that f has a primitive on $G \setminus S$ if and only if all residues of f vanish.

Proof. Suppose first that f has a primitive F on $G \setminus S$. Then by the corollary to the fundamental theorem of calculus from the 3/28 lecture,

$$\int_{\partial G} f \, \mathrm{d}z = 0$$

It follows by the residue theorem that

$$\underbrace{\frac{1}{2\pi i}\int_{\partial G} f \, \mathrm{d}z}_{0} = \sum_{s \in S} \underbrace{\mathrm{wn}(\partial G, s)}_{1} \cdot \mathrm{res}_s f$$
$$0 = \sum_{s \in S} \mathrm{res}_s f$$

Therefore, the residues of f vanish, as desired.

Now suppose that all residues of f vanish. Since G is simply connected, it is bounded by a nulhomologous multicurve ∂G. Thus, by the residue theorem,

$$\frac{1}{2\pi i}\int_{\partial G} f \, \mathrm{d}z = \sum_{s \in S} \mathrm{wn}(\partial G, s) \cdot \underbrace{\mathrm{res}_s f}_{0}$$
$$\int_{\partial G} f \, \mathrm{d}z = 0$$

It follows that the integral over any closed loop in G (which is necessarily homotopic to ∂G) must be zero. This combined with the fact that f is holomorphic and hence continuous implies by the proposition from the 3/28 class that f has a primitive on $G \setminus S$. ☐

2. Fischer and Lieb (2012), QIV.5.3. Compute the following. *Hint*: To get started, make the substitution $z = \mathrm{e}^{it}$ as on Fischer and Lieb (2012, p. 127) so that

$$\cos(t) = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \sin(t) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

and so that the integral from $t = 0$ to 2π becomes a contour integral over the unit circle.

(a) For $a > 1$,

$$\int_0^{\pi} \frac{\sin^2 x}{a + \cos x} \, \mathrm{d}x$$

Proof. Let γ denote the upper half circle $\partial\mathbb{D} \cap \overline{\mathbb{H}}$ oriented counterclockwise. Then taking the hint, we have that

$$\int_0^{\pi} \frac{\sin^2 x}{a + \cos x} \, \mathrm{d}x = \int_\gamma \frac{\left[\frac{1}{2i}\left(z - \frac{1}{z}\right)\right]^2}{a + \frac{1}{2}\left(z + \frac{1}{z}\right)} \frac{\mathrm{d}z}{iz} = \int_\gamma \frac{-z^4 + 2z^2 - 1}{4aiz^3 + 2iz^4 + 2iz^2} \, \mathrm{d}z$$

☐

(b) For $a \in \mathbb{C}$ and $|a| \neq 1$,

$$\int_0^{2\pi} \frac{\mathrm{d}t}{1 - 2a\cos t + a^2}$$

Proof. Taking the hint, we have that

$$\int_0^{2\pi} \frac{\mathrm{d}t}{1 - 2a\cos t + a^2} = \int_{\partial\mathbb{D}} \frac{1}{1 - a(z + \frac{1}{z}) + a^2} \frac{\mathrm{d}z}{iz} = \int_{\partial\mathbb{D}} \frac{i}{az^2 - (a^2+1)z + a} \,\mathrm{d}z$$

The integrand has poles at

$$\begin{aligned} z_1 &= \frac{a^2+1}{2a} + \frac{\sqrt{(a^2+1)^2 - 4a^2}}{2a} \\ &= \frac{a^2+1}{2a} + \frac{a^2-1}{2a} \\ &= a \end{aligned} \qquad \begin{aligned} z_2 &= \frac{a^2+1}{2a} - \frac{\sqrt{(a^2+1)^2 - 4a^2}}{2a} \\ &= \frac{a^2+1}{2a} - \frac{a^2-1}{2a} \\ &= \frac{1}{a} \end{aligned}$$

Since $|a| \neq 1$ by hypothesis, exactly one of these poles will lie in $\mathbb{D}$. Thus, by the residue theorem,

$$\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{i}{az^2 - (a^2+1)z + a} \,\mathrm{d}z = \mathrm{res}_{z_i} \left(\frac{i}{a(z-a)(z-a^{-1})} \right)$$

We now divide into two cases: $|a| < 1$ and $|a| > 1$. If $|a| < 1$, then we evaluate the following using the heuristic from class on 5/2 about shrinking the loop around the singularity so that the multiplied terms approach being constant.

$$\mathrm{res}_{z_1} \left(\frac{i}{a(z-a)(z-a^{-1})} \right) = \frac{i}{a(a-a^{-1})}$$

Similarly, if $|a| > 1$, then we evaluate

$$\mathrm{res}_{z_2} \left(\frac{i}{a(z-a)(z-a^{-1})} \right) = \frac{i}{a(a^{-1}-a)}$$

Therefore, the desired integral equals

$$\boxed{\pm \frac{2\pi}{a(a-a^{-1})}}$$

$\square$

3. Fischer and Lieb (2012), QIV.6.3. Let $\lambda > 1$. Show that the equation $\mathrm{e}^{-z} + z = \lambda$ has exactly one solution in the half plane $\mathrm{Re}\, z > 0$. Show that this solution is real.

Proof. Let U be the half plane described in the question. Consider the function

$$f(z) := \mathrm{e}^{-z} + z - \lambda$$

Apply Rouché's theorem. $\square$

4. Show that

$$\int_0^1 \log[\sin(\pi x)] \,\mathrm{d}x = -\log(2)$$

Hint: Consider the contour of integration that goes from ∞ to 0 along the positive imaginary axis, then runs from 0 to 1 along the real axis, then runs from 1 to ∞ along the ray $\{z \mid \mathrm{Re}(z) = 1,\ \mathrm{Im}(z) \geq 0\}$.

Proof. Call the contour described in the hint γ, and call the three segments of it $\gamma_1, \gamma_2, \gamma_3$ in the order they're introduced.

$\sin(\pi z)$ has an essential singularity at ∞. On the Riemann sphere, γ is an SCC.

$$\begin{aligned}
\int_\gamma \log[\sin(\pi z)]\,\mathrm{d}z &= \int_0^1 \log[\sin(\pi x)]\,\mathrm{d}x + \int_{\gamma_1} \log[\sin(\pi z)]\,\mathrm{d}z + \int_{\gamma_3} \log[\sin(\pi z)]\,\mathrm{d}z \\
&= \int_0^1 \log[\sin(\pi x)]\,\mathrm{d}x + \lim_{r\to\infty}\left(\int_{ir}^0 \log[\sin(\pi z)]\,\mathrm{d}z + \int_1^{1+ir} \log[\sin(\pi z)]\,\mathrm{d}z\right) \\
&= \int_0^1 \log[\sin(\pi x)]\,\mathrm{d}x + \lim_{r\to\infty}\left(\int_{ir}^0 \log[\sin(\pi z)]\,\mathrm{d}z + \int_0^{ir} -\log[\sin(\pi z)]\,\mathrm{d}z\right) \\
&= \int_0^1 \log[\sin(\pi x)]\,\mathrm{d}x - 2\lim_{r\to\infty}\left(\int_0^{ir} \log[\sin(\pi z)]\,\mathrm{d}z\right) \\
&= \int_0^1 \log[\sin(\pi x)]\,\mathrm{d}x + 2\int_{\gamma_1} \log[\sin(\pi z)]\,\mathrm{d}z
\end{aligned}$$

$\square$

References

Fischer, W., & Lieb, I. (2012). *A course in complex analysis: From basic results to advanced topics* (J. Cannizzo, Trans.). Vieweg+Teubner Verlag.