

Week 3

Fundamental Theorems

3.1 Cauchy Integral Formula

4/2:

- Last time.
 - Definition of star-shaped.
 - Cauchy integral theorem: U star-shaped, $f \in \mathcal{O}(U)$ implies $\int_{\gamma} f dz = 0$ for all closed (piecewise C^1) loops γ .
 1. It suffices to prove the theorem for triangles.
 2. Apply Goursat's lemma to treat this triangle case.
 - For Goursat's lemma, apply a clever estimate. Subdivide the big triangle into smaller ones, then

$$\left| \int_{\text{small } \triangle} f dz \right| = \left| \int_a^b f(\gamma(t)) \cdot \gamma'(t) dt \right| \leq \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)| dt \leq \max_{z \in \partial \triangle} |f(z)| \cdot \text{len}(\partial \triangle)$$

- We'll now do a couple exercises to practice applying the concepts we've learned so far.
- TPS: Suppose $f \in \mathcal{O}(\mathbb{C})$. Let $A := \int_0^1 f(x) dx = F(1) - F(0)$, where to be clear we take the integral along the real axis. Let γ be the piecewise C^1 path in yellow in Figure 3.1. What is $\int_{\gamma} f dz$?

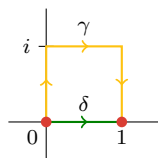


Figure 3.1: Practicing with the Cauchy Integral Theorem (1).

- Define δ such that $\int_{\delta} f dz = \int_0^1 f(x) dx$.
- Then $\delta^{-1}\gamma$ is a closed loop, so

$$0 = \int_{\delta^{-1}\gamma} f dz$$

- Additionally, we have by definition that

$$\int_{\delta^{-1}\gamma} f dz = \int_{\gamma} f dz - \int_{\delta} f dz$$

- Thus, by transitivity and a bit of algebraic rearrangement,

$$\int_{\gamma} f dz = \int_{\delta} f dz = A$$

- TPS: Now suppose $f \in \mathcal{O}(\mathbb{C}^*)$, where we must note that \mathbb{C}^* is *not* star-shaped due to the hole at the origin. Suppose we know that $\int_{\delta} f dz = 0$. What is $\int_{\gamma} f dz$? The paths γ and δ are visualized in Figure 3.2a. *Hint*: It should be $-\int_{\delta} f dz$.

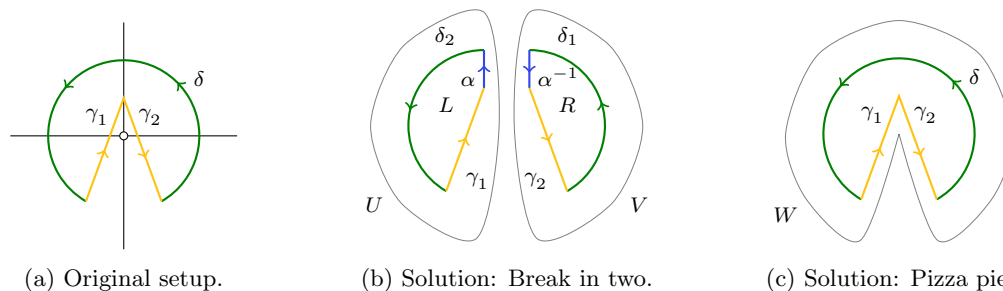


Figure 3.2: Practicing with the Cauchy Integral Theorem (2).

- There are multiple ways to visualize why the domain does not see the hole/puncture. Here are some examples.
- Solution 1 (Figure 3.2b): Cut the loop into two loops in star-shaped domains and add them.
 - Draw a straight-line path α from $i/2$ up to i .
 - Since U and V are both star-shaped domains, consecutive applications of the Cauchy Integral Theorem imply that

$$\int_{\delta_2 \gamma_1 \alpha} f dz = \int_L f dz = 0 \qquad \int_{\delta_1 \alpha^{-1} \gamma_2} f dz = \int_R f dz = 0$$

- Additionally, we know that the sum of the two integrals above is equal to the integral along the entire path in Figure 3.2a because the α and α^{-1} portions cancel. Mathematically,

$$\int_{\delta} f dz + \int_{\gamma} f dz = \int_{\delta \gamma} f dz = \underbrace{\int_L f dz}_0 + \underbrace{\int_R f dz}_0 = 0$$

- Therefore,

$$\int_{\gamma} f dz = - \int_{\delta} f dz = 0$$

- Solution 2 (Figure 3.2c): The pizza pie is star-shaped!
 - We can actually draw a star-shaped domain W encapsulating the entire path $\delta \gamma$.
 - Thus, by the Cauchy Integral Theorem,

$$\int_{\delta \gamma} f dz = 0$$

- From here, we may proceed as before through

$$\begin{aligned} \int_{\gamma} f dz + \int_{\delta} f dz &= 0 \\ \int_{\gamma} f dz &= - \int_{\delta} f dz = 0 \end{aligned}$$

- We now investigate a more general principle than the Cauchy integral theorem called **homotopy**.
 - Algebraic topologists would be insulted by the definition of this term that Calderon is about to give, but it will suffice for our purposes.
- **Homotopic** (paths): Two paths $\gamma, \tilde{\gamma} \subset U$ a domain such that $\tilde{\gamma}$ is obtained from γ by modifying γ on a small disk $D \subset U$, keeping the endpoints fixed.

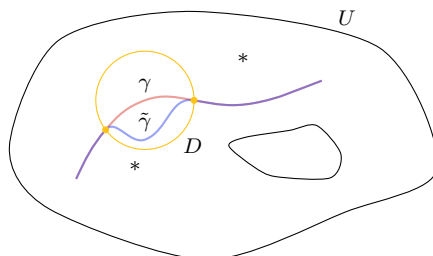


Figure 3.3: Homotopic paths.

- More generally, γ and $\tilde{\gamma}$ are **homotopic** if there exists a finite sequence $\gamma = \gamma_0, \gamma_1, \dots, \gamma_n = \tilde{\gamma}$ such that $\gamma_i \rightarrow \gamma_{i+1}$ is obtained by modifying on a small ball.

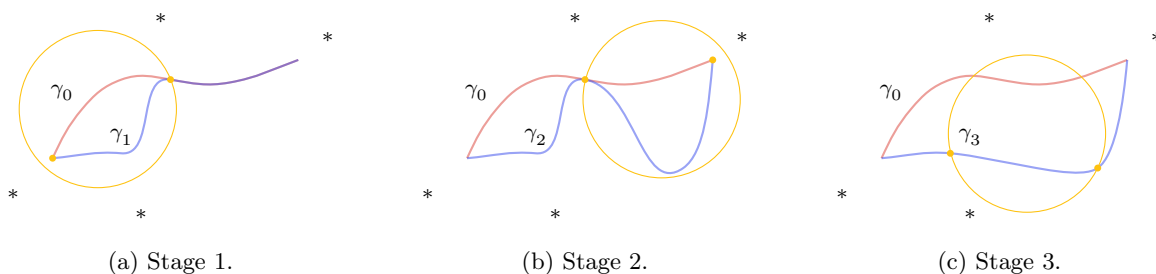


Figure 3.4: A more general homotopy.

- Claim/TPS: This argument shows that if γ and $\tilde{\gamma}$ are homotopic in U and $f \in \mathcal{O}(U)$, then

$$\int_{\gamma} f \, dz = \int_{\tilde{\gamma}} f \, dz$$

Hint: Just go one little bump at a time.

Proof. The start- and endpoints of the bump form a closed loop within a ball (a star-shaped domain), so the bump loop integrates to zero by the CIT. Thus, the integrals within the ball are the same. Additionally, the paths are literally the same outside of the bump, so the integrals there are the same, too. Therefore, the overall integrals are the same, too. \square

- Reality check: Let $f \in \mathcal{O}(\mathbb{C}^*)$. As a particular example, consider $f(z) = 1/z$. Now we know that

$$\int_{\circ} \frac{1}{z} \, dz = 2\pi i \neq 0$$

even though we can break the unit circle into the sum of two paths. What's going on?

- The paths are not homotopic; we can't pull them through the hole in the plane.
- If we consider the upper hemi-circle and the lower hemi-circle, the two cannot be continuously deformed into each other because we always get stuck at the puncture.

- We now prove a slightly stronger version of the Cauchy integral theorem.
- **Corollary:** Let U be any domain, D be a disk in U , and $z \in \overset{\circ}{D}$. Suppose $f \in \mathcal{O}(U \setminus \{z\})$ and is bounded near z . Then

$$\int_{\partial D} f \, dz = 0$$

Proof. Step 1: Use homotopy.

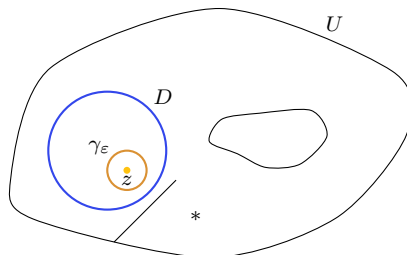


Figure 3.5: Bounded holomorphic functions integrate to zero on disk boundaries.

Via the above claim,

$$\int_{\partial D} f \, dz = \int_{\gamma_\varepsilon} f \, dz$$

where γ_ε is a circle around z within the region where f is bounded^[1].

Step 2: We have that

$$\left| \int_{\gamma_\varepsilon} f \, dz \right| \leq \max_{z \in \gamma_\varepsilon} |f(z)| \cdot \text{len}(\gamma_\varepsilon)$$

Since f is bounded near z , the maximum is finite. Additionally, the length term is just $2\pi\varepsilon$, so we can send $\varepsilon \rightarrow 0$ and thus send the integral to zero. \square

- We now look into the **Cauchy Integral Formula**.
- **Cauchy Integral Formula:** Suppose U is any domain, $D \subset U$ is a disk (i.e., $D \subset\subset U$ or $\overline{D} \subset U$), $f \in \mathcal{O}(U)$, and $z \in D$. Then

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta$$

Proof. We're going to define a function and apply the corollary to it. In particular, define

$$g(\zeta) = \begin{cases} \frac{f(\zeta) - f(z)}{\zeta - z} & \zeta \neq z \\ f'(z) & \zeta = z \end{cases}$$

The fact that f is holomorphic at z implies that g is continuous at z and, hence, bounded near z . Additionally, since g is a rational function of holomorphic functions on $U \setminus \{z\}$, we have $g \in \mathcal{O}(U \setminus \{z\})$. Thus, the corollary says that

$$\int_{\partial D} g \, d\zeta = 0$$

Additionally, by the definition of g , we have that

$$\int_{\partial D} g \, d\zeta = \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta - \int_{\partial D} \frac{f(z)}{\zeta - z} d\zeta$$

¹We could also turn the plane into the sum of two star-shaped domains again.

$f(z)$ is just a complex number, so we can pull it out of the rightmost integral above. Additionally, under a change of variables and invoking PSet 1, QA.4, we have that

$$\int_{\partial D} \frac{f(z)}{\zeta - z} d\zeta = f(z) \int_{\partial D} \frac{1}{\zeta - z} d\zeta = \int_{\text{unit circle}} \frac{1}{z - a} dz = 2\pi i f(z)$$

Note: Another way to evaluate this integral is as follows. If z is the center of the disk, then we win and can get $2\pi i$ using PSet 1, QA.4 directly. If z isn't at the center of the disk, we are allowed to slide it. Here's why: Think about the integrand as a function of z , so

$$\frac{\partial}{\partial z} \left(\int_{\partial D} \frac{1}{\zeta - z} d\zeta \right) = \int_{\partial D} \frac{\partial}{\partial z} \left(\frac{1}{\zeta - z} \right) d\zeta = \int_{\partial D} \frac{1}{(\zeta - z)^2} d\zeta = 0$$

Since we're taking the integral and the limit with respect to different things, we can exchange them. Since the second integrand has a primitive, it equals zero. But this means that the integral does not change even as z changes, which is equivalent to saying we can move z around to wherever we want in the disk and the integral will still be $2\pi i$! In other words, if z is somewhere where we can't evaluate the integral directly, we can move z to somewhere where we *can* evaluate the integral directly with no consequence. \square

- Implication of Cauchy's Integral Formula: The values of the function are completely determined by the values on the boundary, i.e., holomorphic functions are determined by boundary values.
- Let's now prove another theorem.
- Theorem: Let U be any domain, $f \in \mathcal{O}(U)$. Then $f' \in \mathcal{O}(U)$, $f'' \in \mathcal{O}(U)$, on and on.

Proof. Let's use the Cauchy integral formula. We have that

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta$$

Now let's take the derivative, which we know exists because f is holomorphic.

$$\frac{\partial f}{\partial z} = \frac{1}{2\pi i} \int_{\partial D} \frac{\partial}{\partial z} \left(\frac{f(\zeta)}{\zeta - z} \right) d\zeta = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$$

Thus, the derivative has a Cauchy integral formula. We can keep taking derivatives on the inside because the integrand is infinitely differentiable. Thus, we can keep taking derivatives on the outside. And that's the proof. \square

- Corollary: Holomorphic functions are C^∞ .
- Corollary: In general,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta$$

- This last result allows us to bound things really easily, giving us **Cauchy's inequalities**.
 - Essentially, let D have radius R and let z be the center of D . Then

$$|f^{(n)}(z)| \leq \frac{n!}{2\pi i} \max_{\partial D} \left| \frac{f(\zeta)}{R^{n+1}} \right| \cdot 2\pi R = \frac{n!}{R^n} \max_{\partial D} |f(\zeta)|$$

- Liouville's Theorem: Suppose $f \in \mathcal{O}(\mathbb{C})$ (i.e., f is **entire**) and f is bounded. Then it's constant.

Proof. Take a point $z \in \mathbb{C}$. Take a huge ball with radius R . Cauchy's inequality says that if we take the derivative, then

$$|f'(z)| \leq \frac{1}{R} \cdot \max_{\partial D} |f(\zeta)|$$

The maximum is bounded and R is really big, so as $R \rightarrow \infty$, the derivative gets arbitrarily small. So if we've got an arbitrary function with zero derivative, then we've got a constant function. \square

- **Entire** (function): A complex-valued function that is holomorphic on the whole complex plane.

3.2 Analytic Continuation and Removable Singularities

4/4: • Last time.

- Cauchy integral formula: Let U be any domain, $f \in \mathcal{O}(U)$, $D \subset\subset U$, and $z \in D$. Then

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} d\zeta$$

- Implies that holomorphic functions are C^∞ .
- Implies Cauchy's inequalities: If D is a disk centered at z_0 of radius R , then

$$|f^{(n)}(z_0)| \leq \frac{n!}{R^n} \sup_{\partial D} |f(\zeta)|$$

- Implies Liouville's theorem: Any bounded entire function is constant.
- Our focus today is on results we can get out of power series.
- Observe that if

$$P(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

is a convergent power series centered at z_0 , then

$$P^{(n)}(z_0) = n!a_n$$

- Now let $f \in \mathcal{O}(U)$.
- TPS: What should the power series for f look like?

- Rearranging the above, we want

$$a_k = \frac{f^{(k)}(z_0)}{k!}$$

- The following power series formally has the right derivatives.

$$P(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k$$

- Does this power series converge though, and if so, where?
- Recall that the Cauchy-Hadamard formula tells us that the radius of convergence r satisfies

$$r = \left(\limsup_{k \rightarrow \infty} |a_k|^{1/k} \right)^{-1}$$

- Pick a $z_0 \in U$ and a disk $D \subset\subset U$ of radius R . Then by the Cauchy inequalities,

$$|a_k|^{1/k} = \left| \frac{f^{(k)}(z_0)}{k!} \right|^{1/k} \leq \frac{|\sup_{\partial D} f(\zeta)|^{1/k}}{R} \rightarrow \frac{1}{R}$$

- Thus, returning to the Cauchy-Hadamard formula, the radius of convergence is $\geq R$.
- So we've got a convergent power series, but why does this power series equal $f(z)$?
 - We know that $P(z)$ and $f(z)$ have all the same derivatives.
 - However, over \mathbb{R} , this is not enough! Recall the example of e^{-1/x^2} , which has the same derivatives as its power series at zero but is not equal to it.
 - Over \mathbb{C} , however, we claim that having the same derivatives *is* enough.

- Use CIF and expand $1/(\zeta - z)$.
- Note: To keep all of our z 's straight, recall that z_0 is a point, ζ lies on ∂D where D is centered at z_0 , and z is somewhere in \mathring{D} .
- Doing this, we obtain

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)} = \frac{1}{\zeta - z_0} \sum_{k=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^k$$

- Thus,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial D} \sum_{k=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} (z - z_0)^n d\zeta \\ &= \sum_{k=0}^{\infty} \frac{1}{2\pi i} \underbrace{\int \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta}_{f^{(n)}(z_0)/n!} (z - z_0)^n \\ &= P(z) \end{aligned}$$

- Recall that we can bring the sum outside of the integral because of uniform convergence and our lemma about integrable functions from the 3/26 class.
- All in all, we've shown that any holomorphic function has a power series representation on any disk that fits within the domain, and the power series representation is the one we think it should be.
- We now discuss an important corollary to this result.
- The Identity Theorem: If two holomorphic functions $f, g \in \mathcal{O}(U)$ agree on an open set in U , then $f = g$.

Proof. This is true for power series.

For every point, there's a power series representation around that series so we can do something with a covering of open sets, though we do not need compactness for U . \square

- An analogous result does not hold on the reals. For example, there are plenty of functions that are zero for a while, then bump up to 1 for a while, so they're 0 and 1 on open sets without being either 0 or 1.
- Implication: "Holomorphic functions are very rigid."
- In fact, more is true: If $z_n \rightarrow z_0$ where each z_n is distinct and $f(z_n) = g(z_n)$ for all n , then $f = g$.
 - So we don't even need an open set; all we need is an **accumulation point**.
- **Analytic continuation** (of f): The function $g \in \mathcal{O}(V)$ where $f \in \mathcal{O}(U)$, $V \supset U$, and $f = g$ on U .
 - Note that we get to say "the function $g \dots$ " because of the identity theorem.
 - Formally, g_1, g_2 analytic continuations of f and $g_1 = g_2$ on U open implies $g_1 = g_2$.
- Example: Consider $f(z) = z$ with $f \in \mathcal{O}(\mathbb{C}^*)$. Then $g(z) = z$ with $g \in \mathcal{O}(\mathbb{C})$ is an analytic continuation of f .
- What we're essentially doing is taking the power series (which we get via "analytic") and extending them out into V .

- Example: The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty} e^{-s \log n}$$

Where does this make sense, i.e., where does the series converge?

- We have $|n^{-s}| = n^{-\operatorname{Re}(s)}$.
- Thus, the series converges only when $\operatorname{Re}(s) > 1$.
- The Riemann hypothesis predicts where $\zeta(s) = 0$. We know that it has some zeroes at the negative even integers, and the RH predicts that the rest of them fall on the line $\operatorname{Re}(s) = 1/2$.
- But ζ is only defined on a part of the complex plane not including these regions! Thus, to make sense of the RH, we need to analytically continue ζ .
- Given $f \in \mathcal{O}(U)$, what is the “biggest” V on which $f \in \mathcal{O}(V)$? In layman’s terms, where should f live?
- Example: $1/z \in \mathcal{O}(\mathbb{C}^*)$ and $1/z \notin \mathcal{O}(\mathbb{C})$.
 - Note that we know the latter statement because if you’re holomorphic, the integral around any closed loop in the domain is zero but the integral of this function on the unit circle is $2\pi i$, so it can’t be holomorphic on \mathbb{C} . Contradiction.
- Example: $\sin(1/z) \in \mathcal{O}(\mathbb{C}^*)$, but is it in $\mathcal{O}(\mathbb{C})$?
 - No; recall from PSet 1 that it’s not even *continuous* at 0.
- Example: $\sin(z)/z \in \mathcal{O}(\mathbb{C}^*)$, but is it in $\mathcal{O}(\mathbb{C})$?
- Recall Goursat’s lemma: $f \in \mathcal{O}(\operatorname{nbhd}(\Delta))$ implies $\int_{\Delta} f \, dz = 0$.
- Recall the lemma from 3/28: If U is star-shaped and $\int_{\Delta} f \, dz = 0$ for all triangles, then f has a primitive.
 - Note that we do not need f holomorphic for this result.
 - This result has a converse!
- Morera’s Theorem: If U is any domain, $f : U \rightarrow \mathbb{C}$ is continuous, and $\int_{\Delta} f \, dz = 0$ for all triangles, then f is holomorphic.

Proof. Fix a disk $D \subset\subset U$. Disks are star-shaped! This combined with the fact that the integral over all triangles is zero implies that f has a primitive $F \in \mathcal{O}(U)$ by the result a couple lines up. But since F is holomorphic, by our result from last class, $F' = f \in \mathcal{O}(U)$, too. \square

- **Riemann’s removable singularity theorem:** Suppose U is a domain, $z \in U$, $f \in \mathcal{O}(U \setminus \{z\})$, and f is bounded near z . Then there exists a unique analytic continuation $\hat{f} \in \mathcal{O}(U)$. *Also known as Riemann extension theorem.*
 - In this case, we call z a **removable singularity**.
 - Note: The contrapositive of this says that if there is not an analytic continuation (i.e., the function is honestly not holomorphic at a point and can’t be extended to one, e.g., $1/z$), then $|f|$ has to blow up as you approach z (in some direction).
- **Singularity** (of f): A point z_0 such that $f \in \mathcal{O}(U \setminus \{z_0\})$.
- **Removable** (singularity): A singularity of a function that satisfies the hypotheses of Riemann’s removable singularity theorem.

- If a singularity is not removable, then f is not bounded near z_0 . This leads to additional definitions.
- **Pole:** A non-removable singularity z_0 of a function f for which $|f(z)| \rightarrow \infty$ as $z \rightarrow z_0$.
 - So-named because of real analysis where a pole is an asymptote, and asymptotes kind of look like poles!
- **Essential** (singularity): A non-removable singularity that is not a pole; equivalently, a singularity z_0 for which there exist sequences $z_n \rightarrow z_0$ and $w_n \rightarrow z_0$ such that $|f(z_n)| \rightarrow \infty$ and $|f(w_n)|$ stays bounded.
- Proving Riemann's removable singularity theorem.

Proof. Set

$$F(\zeta) = \begin{cases} f(\zeta)(\zeta - z) & \zeta \neq z \\ 0 & \zeta = z \end{cases}$$

Then $F \in \mathcal{O}(U \setminus \{z\})$ so F is continuous at z .

We want to show that F is holomorphic (using Morera's theorem). To do this, we'll need to show that the integral over all triangles is zero. More specifically, all we need to do is show that F is holomorphic in a little ball D about z . Now we need to do some casework.

Case 1: If $\triangle \not\ni z$, then we can draw a star-shaped domain surrounding the triangle on which f will be holomorphic and invoke the CIT to imply that the integral is zero.

Case 2: If $\triangle \ni z$, then $\int F dz$ is arbitrarily small. Recall that we get this by using homotopy to replace the integral over the triangle with the integral over some tiny γ_ε . Arbitrarily small because f is bounded.

Morera then tells us that $F \in \mathcal{O}(U)$, so $F' = f \in \mathcal{O}(U)$. Note that $F' = f$ because

$$F'(z) = \lim_{\zeta \rightarrow z} \frac{F(\zeta) - F(z)}{\zeta - z} = \lim_{\zeta \rightarrow z} \frac{f(\zeta)(\zeta - z) - 0}{\zeta - z} = \lim_{\zeta \rightarrow z} f(\zeta) = f(z)$$

□

- Go back and add a z_0 everywhere and then it should all be ok.
- With the removable singularity theorem, we can now confirm that $\sin(z)/z$ has a removable singularity because although $1/z$ diverges, sine converges faster so this function is bounded near zero.
 - We can prove boundedness with the Taylor series of $\sin(z)/z$.

3.3 Chapter II: The Fundamental Theorems of Complex Analysis

From Fischer and Lieb (2012).

Section II.3: The Cauchy Integral Formula

4/13:

- Statement and proof of the CIF.
 - Fischer and Lieb (2012) come at it in a slightly more complicated way than class.
- Derivatives of the CIF.
- Holomorphic implies C^∞ .
- Morera's theorem and proof.
- The Riemann extension theorem and proof.

Section II.4: Power Series Expansions of Holomorphic Functions

- Proof of the typical power series formula.
- Power series examples.
 1. $(z - a)^{-1}$.
 - Has a standard power series.
 - $(z - a)^{-n}$ is computed via term-by-term differentiation of this base power series and corresponding adjustments.
 - More complicated rational functions are handled via partial fraction decomposition and then the above method.
 2. Products of holomorphic functions.
 - Use **Leibniz's rule**.
 3. Inverse of a power series.
 - Suppose $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ in a neighborhood of z_0 , and assume that $f(z_0) = a_0 \neq 0$.
 - Then $1/f$ is holomorphic near z_0 .
 - Moreover, the coefficients of the expansion

$$\frac{1}{f(z)} = \sum_{n=0}^{\infty} b_n(z - z_0)^n$$

can be determined.

- To do so, compare coefficients in

$$1 = \frac{1}{f(z)} \cdot f(z) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^n b_m a_{n-m} \right) (z - z_0)^n$$

- This yields $b_0 a_0 = 1$ and hence $b_0 = a_0^{-1}$. Continuing, $b_0 a_1 + b_1 a_0 = 0$, so $b_1 = -b_0 a_1 a_0^{-1} = -a_1 a_0^{-2}$. We can continue this computation as far as we like.
- 4. $\tan z$.
 - Observe that $\tan z$ is odd, and hence the even-powered terms disappear.
 - Substitute the power series for cosine, sine, and the undetermined one for tangent into the equation $\cos z \cdot \tan z = \sin z$ and compare coefficients.
 - We could also do $\sin z / \cos z$ using the method of Examples 2-3 above.
- **Leibniz's rule:** The derivative of the product fg of two functions f, g that are both holomorphic in a neighborhood of z_0 is given by

$$\frac{(fg)^{(n)}(z_0)}{n!} = \sum_{m=0}^n \frac{f^{(m)}(z_0)}{m!} \cdot \frac{g^{(n-m)}(z_0)}{(n-m)!}$$

- This is just the Cauchy product of the two formal power series!
- “In order to determine the derivatives $f^{(n)}(z_0)$ of a function f that is holomorphic at z_0 , one only needs to know the values $f(z)$ on, say, a segment $(z_0 - \delta, z_0 + \delta)$ parallel to the real axis” (Fischer & Lieb, 2012, p. 52).
- Build up to, statement of, and proof of the identity theorem.
- Consequence of the identity theorem: Holomorphic functions on U are completely determined by their values on any **nondiscrete** set in U .

- “Properties that can be expressed via identities between holomorphic functions on $[G]$ thus only need to be verified on a nondiscrete set in $[G]$ ” (Fischer & Lieb, 2012, p. 53).
 - Example: $\cot(\pi z)$ and $\cot(\pi z + \pi)$ coincide on the nondiscrete set $\mathbb{C} \setminus \mathbb{Z}$, and thus are equal; therefore, the periodicity of cotangent on the real numbers implies it on the complex numbers.
 - This also means that the exponential function and those derived from it (e.g., sine and cosine) can only be extended from the real to the complex numbers in one way since \mathbb{R} is nondiscrete in \mathbb{C} .
- **Nondiscrete** (set in U): A set $M \subset U$ that contains an accumulation point of M .
 - **Discrete** (set in U): A set $M \subset U$ that is not nondiscrete.
 - Defines a **zero** (of order L), tapping into next week’s content.
 - Since constancy on a nondiscrete set would imply constancy everywhere (for a holomorphic function), the set $f^{-1}(w)$ of points at which a nonconstant holomorphic function takes on the same value $w \in \mathbb{C}$ is discrete.
 - Characterizing holomorphicity: Let $f : U \rightarrow \mathbb{C}$ be a function defined on an open set $U \subset \mathbb{C}$. The following are equivalent:
 - i. f is holomorphic.
 - ii. f is real differentiable and satisfies the Cauchy-Riemann equations.
 - iii. f admits a power series expansion about every point in U .
 - iv. f has local primitives.
 - v. f is continuous, and for every closed triangle $\Delta \subset U$, $\int_{\partial\Delta} f(z) dz = 0$.