Week 7

The Winding Number

7.1 Generalized Cauchy Theorems

4/30: • Questions.

  - PSet 4, QA.4: III.5.1 instead of II.5.1?
    - Yep, should be Chapter 3, not Chapter 2.
  - PSet 4, QB.4: "a holomorphic branch of the logarithm exists on U" or on $f(U)$?
    - Yep, should be $f(U)$.
    - "Which one works, Steven?"

• Recall.

  - The winding number of a curve γ about a point $z_0 \in \mathbb{C}$ is

$$\operatorname{wn}(\gamma, z_0) := \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} \, \mathrm{d}z$$

  - We can also compute the winding number geometrically (see Figure 6.3).

• Additional properties of the winding number.

  - The winding number is invariant under homotopies of γ.
  - Compute by counting how many times you pass a ray from z_0 going counterclockwise!
    - Example: "I'm pointing in this direction, then I rotate, and eventually I point in this direction again, then I rotate, and eventually I'm back where I started so it's winding number 2."
  - We can also think of jumping to a higher plane on the infinity spiral every time we pass the ray.

• TPS: Compute the winding number of γ about the points in Figure 7.1.

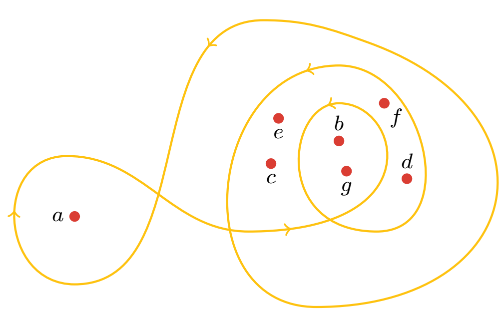


Figure 7.1: Winding number regions.

- We get

$$\mathrm{wn}(\gamma, a) = -1 \qquad \mathrm{wn}(\gamma, b) = 3 \qquad \mathrm{wn}(\gamma, c) = 2$$

$$\mathrm{wn}(\gamma, d) = 2 \qquad \mathrm{wn}(\gamma, e) = 2 \qquad \mathrm{wn}(\gamma, f) = 2 \qquad \mathrm{wn}(\gamma, g) = 3$$

- Do we notice any patterns?
  - Connected regions of the plane appear to yield the same winding number!
  - We formalize this notion via the following lemma.

- Lemma: $\mathrm{wn}(\gamma, z_0)$ is constant on components of $\mathbb{C} \setminus \mathrm{Im}(\gamma)$. It is also 0 on the unbounded component.

  Proof. We address the two claims sequentially.

  Claim 1: Treating z_0 as an argument, $\mathrm{wn}(\gamma, z_0)$ is a function from $\mathbb{C} \setminus \mathrm{Im}(\gamma)$ to $\mathbb{Z}$ defined by

  $$z_0 \mapsto \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{z - z_0}$$

  This is a continuous function into a discrete space and therefore is constant.

  Claim 2: Let z_0 get very big. Then we can make

  $$\left| \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{z - z_0} \right|$$

  arbitrarily small. But an integer that can be made arbitrarily small is just zero. □

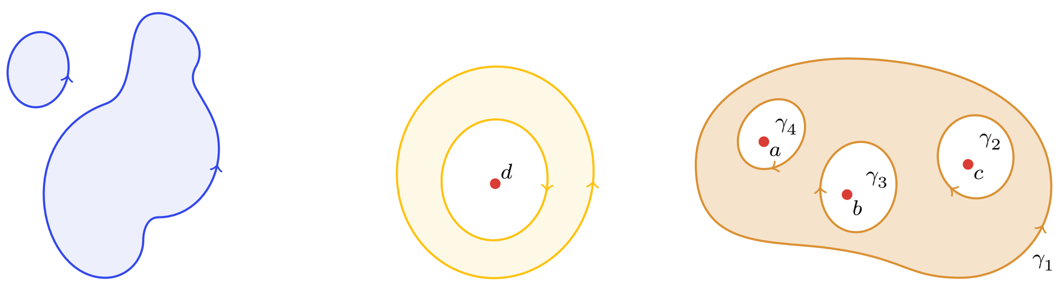
  - This is a complex analytic proof of a topological claim.
  - Justifying that the codomain of the winding number function is the integers: We've done this heuristically using homotopy, but we could formalize it, too.

- We now move onto today's main topic: The proof of the (very) general Cauchy Integral Theorem.
- First, we need a definition.
- **Simply connected** (domain): A domain $U \subset \mathbb{C}$ such that $\mathrm{wn}(\gamma, z_0) = 0$ for all $\gamma \subset U$ and $z \notin U$.
  - There are many other definitions, too.
    - Topology: The **fundamental group** of U is zero.
    - Removing any **arc** (line segment across the domain) from U turns it into a disconnected set.
    - For all arcs δ_1, δ_2 with the same endpoints, δ_1 and δ_2 are homotopic.
  - The last definition above will be particularly useful for our purposes, as we'll see shortly.
  - But these are all formal definitions; what can we think about intuitively?
    - A good first thing to think about is a blob in the plane.
    - But the interior of a fractal domain would also count.
    - A square minus a slit at $1, 1/2, 1/3, \dots$ is also simply connected (though not path connected).
- Jordan curve theorem: Suppose $\gamma : S^1 \to \mathbb{C}$ is a continuous injection. Then γ bounds a disk.
  - Consequence: A domain that is simply connected is homeomorphic to a disk.
  - This appears stupidly obvious, but it was only rigorously proved in the early 1910s.
    - The issue is that we don't really know what *continuous* means.
    - If γ is C^1, this is easy.

- The two generalizations and their proofs.
  - The proof of generalization 1 is very simple, straightforward, and clever.
  - The proof of generalization 2 is much more general and uses almost everything we've done.
- We are now ready to state and prove a first generalization of the CIT.
- Cauchy Integral Theorem: Suppose that U is simply connected and $f \in \mathcal{O}(U)$. Then $\int_\gamma f\,\mathrm{d}z = 0$ for any closed loop γ in U.

  Proof. Let γ be an arbitrary closed loop in U. Because any two arcs with the same endpoints are homotopic, γ is homotopic to the constant path $\tilde{\gamma} : [0,1] \to \{\gamma(0)\}$. This constant path has the property that
  $$\int_{\tilde{\gamma}} f\,\mathrm{d}z = \int_0^1 f(\tilde{\gamma}(t))\tilde{\gamma}'(t)\,\mathrm{d}t = \int_0^1 f(\tilde{\gamma}(t)) \cdot 0\,\mathrm{d}t = 0$$
  Since integrals are the same for homotopic paths, it follows that
  $$\int_\gamma f\,\mathrm{d}z = \int_{\tilde{\gamma}} f\,\mathrm{d}z = 0$$
  as desired. $\square$

- We now build up to an even more general version of the CIT.



(a) Separate oriented domains. (b) Nested curves. (c) Multiple nested curves.

Figure 7.2: Nulhomologous multicurve examples.

  - Suppose $D \subset \mathbb{C}$ is a bounded domain, and ∂D is a union of disjoint simple closed curves (SCCs).
  - Let $\partial\vec{D}$ be the union of the boundaries, oriented so that D is on the left.
    - This is similar to how we orient curves when we're applying Stokes' Theorem.
    - Here as well, the outer one goes counterclockwise and the inner one(s) goes clockwise.
  - More generally, we define a the concept of a **multicurve**.
  - Using this definition, we define the **integral** of f over a multicurve.
    - This definition allows us to compute the winding number of Γ about z_0.
  - Lastly, we define a special kind of multicurve called a **nulhomologous** multicurve.
    - In Figure 7.2c, $\Gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ is nulhomologous.
- **Multicurve**: A formal sum of SCCs γ_i multiplied by coefficients $c_i \in \mathbb{C}$. *Denoted by* $\mathbf{\Gamma}$. *Given by*
  $$\Gamma = \sum c_i \gamma_i$$

- **Integral** (of f over Γ): The path integral defined as follows. *Denoted by* $\boldsymbol{\int_\Gamma f \, \mathrm{d}z}$. *Given by*

$$\int_\Gamma f \, \mathrm{d}z := \sum_{i=0}^{n} c_i \int_{\gamma_i} f \, \mathrm{d}z$$

- **Nulhomologous** (Γ in U): A multicurve Γ in a domain U for which $\Gamma = \partial \vec{D}$ for D as in Figure 7.2. *Also known as* **homologous** (Γ in U to 0).
- TPS: Compute $\operatorname{wn}(\partial \vec{D}, z_0)$ for all $z_0 \notin D$ for each of the domains D in Figure 7.2.
  - $\operatorname{wn}(\partial \vec{D}, z_0) = 0$ because we always get either nothing or a $+1$ and -1 and some zeroes.
- Lemma: If Γ is nulhomologous in U, then for all $z \notin U$, $\operatorname{wn}(\Gamma, z) = 0$.
  - The converse is not true!
    - Example: If $U = \mathbb{C}^*$ and γ_1, γ_2 are intersecting closed curves (e.g., the unit circle and the unit circle translated half a unit to the right), then $\gamma_1 + \gamma_2$ is still nulhomologous even though it doesn't bound a domain.
  - The condition "for all $z \notin U$, $\operatorname{wn}(\Gamma, z) = 0$" is our general definition of nulhomologous in U; what we said earlier was just a precursor definition.
- Example.

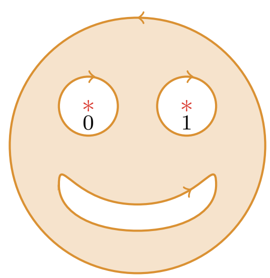


Figure 7.3: Nulhomologous multicurve in a punctured domain.

  - Let

$$f(z) = \frac{\sin(1/z)}{z - 1}$$

  - Then $f \in \mathcal{O}(\mathbb{C} \setminus \{0, 1\})$.
  - An example of a nulhomologous multicurve over which we could integrate f is as follows.
- We are now ready for the statement and proof of the most general version of the CIT and CIF we'll see in this course.
- Suppose U is any domain, $\Gamma \subset U$ is nulhomologous, and $f \in \mathcal{O}(U)$. Then:
  1. General CIT: We have that

$$\int_\Gamma f \, \mathrm{d}z = 0$$

  2. General CIF: For all $z \in U$ and not in $\operatorname{Im}(\Gamma)$,

$$\operatorname{wn}(\Gamma, z) \cdot f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

- Discussion of the proof.
  - We'll sketch the proof today.
  - Think back to the proof for star-shaped domains.
  - We proved the CIT by saying, "if it's true for triangles, then we win."
    - Using triangles, we built a primitive and then invoked Goursat's Lemma.
  - We proved the CIF by first defining the helper function

$$g(\zeta) = \begin{cases} \frac{f(\zeta)-f(z)}{\zeta-z} & \zeta \neq z \\ f'(z) & \zeta = z \end{cases}$$

    - Then we invoked the CIT to say

$$\int_{\partial D} g \, \mathrm{d}z = 0$$

    - The CIF then followed from this and the fact that

$$\int_{\partial D} g \mathrm{d}\zeta = \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta - f(z) \underbrace{\int_{\partial D} \frac{1}{\zeta - z} \mathrm{d}\zeta}_{2\pi i}$$

  - So can't we just replace all the ∂D's with Γ's in the above lines and call it a day?
    - No, because there's no analogy for the CIT. In other words, there may not be a primitive.
    - Thus, we need to fix $\int_{\partial D} g \, \mathrm{d}z = 0$.
  - In sum, the idea of this proof is to prove the CIF and then simply get the CIT.
  - We'll have time to prove the CIF today, but probably will not to get to the CIT.
- We are now ready to sketch the full proof in broad strokes.

Proof. Define $h : U \to \mathbb{C}$ by

$$h(z) := \int_\Gamma g(\zeta, z) \mathrm{d}\zeta$$

We want to show that $h(z) = 0$. We can't do anything as nice as showing that it's a continuous map into a discrete space, but there is still a clever idea. First off, we can see that $h(z) \to 0$ as $z \to \infty$ in U. Essentially, as before, the denominator $\zeta - z$ gets really big so the first term gets really small and the second term has that $\mathrm{wn}(\Gamma, z)$ term which goes to 0. What we now need to show is that h extends to an entire function so that we can make the denominator *arbitrarily* large. This is where we use the assumption that Γ is nulhomologous.

First, we will show that h is continuous. We know that g is continuous in (ζ, z) together. We have holomorphic in ζ for a fixed z.[1]

Next, we need to show that h is holomorphic on U. We know that h is holomorphic as long as $z \neq \zeta$. On the other hand, what if $\zeta = z$? We will invoke Morera's theorem.[2]

Last, we show that h can be analytically continued outside of U. We know that on U,

$$h(z) = \int \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta - f(z) \cdot 2\pi i \, \mathrm{wn}(\Gamma, z)$$

Outside of U, the second term disappears because Γ is nulhomologous. Define

$$h(z) := \int \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

outside of U. Thus, we have two functions that agree on a patch, so we get analytic continuation.

From here, we have an entire function that converges to 0 at ∞ (hence is bounded), so is constant by Liouville's theorem with value that converges to zero (hence is zero).[3] $\square$

[1] There's a bit more detail in the notes, but not much.
[2] There's a bit about the triangle integral condition in the notes.
[3] There is a bit on the CIT in the notes.

7.2 The Residue Theorem

5/2:

- Reminder: PSet 4 due tomorrow.
- Recall.
  - Cauchy integral theorem for star-shaped/simply connected domains: Given such a domain U and $f \in \mathcal{O}(U)$, f has a primitive on U. It follows that for all $\gamma \subset U$, $\int_\gamma f \, \mathrm{d}z = 0$.
  - A cycle $\Gamma = \sum c_i \gamma_i$ in a domain U is **nulhomologous** if for all $z \notin U$,

$$\operatorname{wn}(\Gamma, z) := \sum c_i \operatorname{wn}(\gamma_i, z) = 0$$

  - CIT/CIF in general: Let U be a domain, $\Gamma \subset U$ nulhomologous in U, and $f \in \mathcal{O}(U)$.
    1. We have
$$\int_\Gamma f \, \mathrm{d}z = \sum c_i \int_{\gamma_i} f \, \mathrm{d}z = 0$$
    2. For all $z \notin \operatorname{Im}(\Gamma)$,
$$\operatorname{wn}(\Gamma, z) \cdot f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$
  - This is the most general version of the most important theorem(s) in this class.
  - Comparing and contrasting the old and the new Cauchy theorems.
    - In the first one, we put a restriction on our domain and no restriction on our curve. In the new one, we put a restriction on our curve and no restriction on our domain.
    - In the old one, we worked to construct a primitive for f on U. In the new one, f need not have a primitive on U.
- Further multicurve examples.

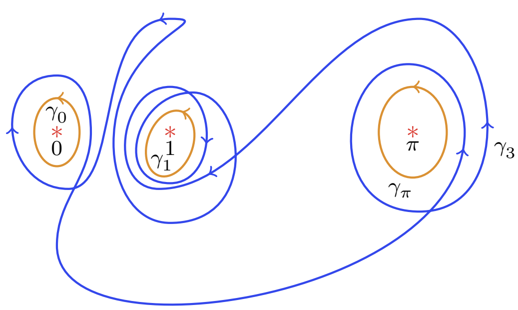


Figure 7.4: Using coefficients to make a multicurve nulhomologous.

1. Let $U = \mathbb{C} \setminus \{p_1, \dots, p_k\}$. Enclose all points p_i in one big loop, and then enclose each individual point in a smaller loop (see Figures 7.2c and 7.3).
   - Then the (unweighted) sum of all $k+1$ curves is nulhomologous.
2. Let $U = \mathbb{C} \setminus \{0, 1, \pi\}$. γ_3, as drawn in Figure 7.4, is not nulhomologous.
   - This is because the winding number of γ_3 about π is 2, about 1 is -2, and about 0 is -1.
   - However, we can make this curve nulhomologous by introducing a counterclockwise-oriented curve about each point and taking

$$\Gamma = \gamma_3 + \gamma_0 + 2\gamma_1 - 2\gamma_\pi$$

- The points in the above examples should be thought of as singularities.

- **Homologous** (cycles in U): Two cycles $\Gamma_1, \Gamma_2 \subset U$ for which $\Gamma_1 - \Gamma_2$ is nulhomologous.
  - Helpful analogy: Think of Γ_1, Γ_2 as elements of a vector space and this is us saying they're equivalent/homologous if $\Gamma_1 - \Gamma_2 = 0$ (i.e., $\Gamma_1 = \Gamma_2$).
  - Example: γ_3 and $2\gamma_\pi - 2\gamma_1 - \gamma_0$ are homologous in $\mathbb{C} \setminus \{0, 1, \pi\}$.
- Corollary: If Γ_1 and Γ_2 are homologous, then for all $f \in \mathcal{O}(U)$,

$$\int_{\Gamma_1} f \,\mathrm{d}z = \int_{\Gamma_2} f \,\mathrm{d}z$$

Proof. We have that

$$\begin{aligned}\int_{\Gamma_1} f \,\mathrm{d}z - \int_{\Gamma_2} f \,\mathrm{d}z &= \int_{\Gamma_1 - \Gamma_2} f \,\mathrm{d}z \\ &= 0 && \text{CIT}\end{aligned}$$

Thus, adding $\int_{\Gamma_2} f \,\mathrm{d}z$ to both sides of the above equation, we obtain

$$\int_{\Gamma_1} f \,\mathrm{d}z = \int_{\Gamma_2} f \,\mathrm{d}z$$

as desired. □

- This is the most important thing we get from the new CIT/CIF.
  - It means that to compute integrals over complicated paths, we can just replace the contour with a homologous ones that is easier to compute!
- TPS: Integrate the following function over γ_3 from Figure 7.4.

$$f(z) = \frac{1}{z(z-1)(z-\pi)}$$

  - By the corollary,

$$\int_{\gamma_3} f \,\mathrm{d}z = 2\int_{\gamma_\pi} f \,\mathrm{d}z - 2\int_{\gamma_1} f \,\mathrm{d}z - \int_{\gamma_0} f \,\mathrm{d}z$$

  - One way to evaluate each of these integrals is with a partial fraction decomposition.
  - Another way is by observing the following.
    - Take the loop around a pole, say π, to be very small.
    - Then on this loop, z is close to π, $z - 1$ is close to $\pi - 1$, and only $z - \pi$ is meaningfully changing.
    - Thus, as the radius of the loop approaches zero, the $1/z$ and $1/(z-1)$ terms approach $1/\pi$ and $1/(\pi - 1)$, respectively. These "constants" can then be factored out, and the remaining integral of $1/(z - \pi)$ evaluated to $2\pi i$.
    - This suggests that

$$\int_{\gamma_\pi} f \,\mathrm{d}z = \frac{1}{\pi(\pi - 1)} \cdot 2\pi i$$

  - We now give a more rigorous justification for the above heuristic.
    - Observe that $\int_{\gamma_\pi} f \,\mathrm{d}z$ is equal to $2\pi i$ times the **residue** of f at π, which we may recall is just the a_{-1} coefficient in the Laurent expansion at π.
    - Let's compute this Laurent expansion!

- We have that
$$f(z) = \frac{1}{z-\pi} \cdot \frac{1}{z(z-1)}$$
where the rightmost term must be holomorphic at π by our previous inquiry.
- Thus, let
$$g(z) := \frac{1}{z(z-1)}$$
- The Taylor expansion of g about π is
$$g(z) = g(\pi) + g'(\pi)(z-\pi) + \frac{g''(z)}{2}(z-\pi)^2 + \cdots$$
- Now recall how we motivated the residue:
$$\int \sum_{k=-n}^{\infty} a_k(z-\pi)^k \, \mathrm{d}z = \int a_{-1}(z-\pi)^{-1} \, \mathrm{d}z = 2\pi i \cdot a_{-1}$$
- Thus, around π,
$$f(z) = \frac{g(\pi)}{z-\pi} + g'(\pi) + \frac{g''(\pi)}{2}(z-\pi) + \cdots$$
- This means that $g(\pi)$ is the a_{-1} coefficient! Thus,
$$\operatorname{res}_\pi(f) = a_{-1} = g(\pi) = \frac{1}{\pi(\pi-1)}$$

- Computing the other two integrals similarly, we obtain in total that
$$\int_{\gamma_\pi} f \, \mathrm{d}z = \frac{2i}{\pi-1} \qquad \int_{\gamma_1} f \, \mathrm{d}z = \frac{2\pi i}{1-\pi} \qquad \int_{\gamma_0} f \, \mathrm{d}z = 2i$$
- Thus, in total,
$$\int_{\gamma_3} f \, \mathrm{d}z = \frac{2i(\pi+3)}{\pi-1}$$
- The point is not that we get a nice final answer. The point is that we can compute complicated integrals in a much simpler way, e.g., just by fiddling with power series.

- **Residue** (of f at p): If p is an isolated singularity and D is a small disk whose only singularity is p, then the residue is defined as follows. *Denoted by* $\boldsymbol{\operatorname{res}_p f}$. *Given by*
$$\operatorname{res}_p f := \frac{1}{2\pi i} \int_{\partial D} f \, \mathrm{d}z$$
- **Residue theorem**: Suppose U is a domain, S (think singularities) is a discrete set in U, $f \in \mathcal{O}(U \setminus S)$, and Γ is nulhomologous in U. Then
$$\frac{1}{2\pi i} \int_\Gamma f \, \mathrm{d}z = \sum_{s \in S} \operatorname{wn}(\Gamma, s) \cdot \operatorname{res}_s(f)$$

Proof. For all $s \in S$, let γ_s be a small loop about s oriented counterclockwise. Define
$$\Gamma' := \Gamma - \sum_{s \in S} \operatorname{wn}(\Gamma, s)\gamma_s$$
Γ' is nulhomologous. Thus, by the CIT,
$$\int_{\Gamma'} f \, \mathrm{d}z = 0$$

Now for all $s \in S$,

$$\operatorname{wn}(\Gamma', s) = \operatorname{wn}(\Gamma, s) - \operatorname{wn}(\Gamma, s) \cdot \underbrace{\operatorname{wn}(\gamma_s, s)}_{1} = 0$$

Therefore, we have that

$$\begin{aligned}
\int_\Gamma f \, \mathrm{d}z &= \int_{\Gamma'} f \, \mathrm{d}z + \int_{\sum_{s \in S} \operatorname{wn}(\Gamma, s)\gamma_s} f \, \mathrm{d}z \\
&= 0 + \sum_{s \in S} \operatorname{wn}(\Gamma, s) \int_{\gamma_s} f \, \mathrm{d}z \\
&= 2\pi i \sum \operatorname{wn}(\Gamma, s) \operatorname{res}_s f
\end{aligned}$$

as desired. ☐

- Comments on the residue theorem.
  - Example: $U = \mathbb{C}$, $S = \{0, 1, \pi\}$, and $\Gamma = \gamma_3$.
  - The residue theorem will be very important, hint hint.
  - The proof is just exactly what we did in the example!
  - Calderon's definition of math: "You do an example, you see something interesting, and then you make a theorem that says, 'This happens always.'"
- We now list some properties of the residue.
  - These properties are true in general, but we'll prove them in the specific case that the functions are meromorphic because this allows us to use Laurent expansions.
- Properties: Let $a \in \mathbb{C}$ and $f, g \in \mathcal{O}(U \setminus S)$.
  1. $\operatorname{res}_s(af + g) = a \operatorname{res}_s f + \operatorname{res}_s g$.

     Proof. Using the linearity of the integral in the definition, we have

     $$\begin{aligned}
     \operatorname{res}_s(af + g) &= \frac{1}{2\pi i} \int_{\partial D} (af + g) \, \mathrm{d}z \\
     &= a \cdot \frac{1}{2\pi i} \int_{\partial D} f \, \mathrm{d}z + \frac{1}{2\pi i} \int_{\partial D} g \, \mathrm{d}z \\
     &= a \operatorname{res}_s f + \operatorname{res}_s g
     \end{aligned}$$

     as desired. ☐
  2. If f has a **simple pole** at s, then

     $$\operatorname{res}_s f = \lim_{z \to s} f(z)(z - s)$$

     Proof. Taking the Laurent expansion, let

     $$f(z) = \sum_{k=-1}^{\infty} a_k (z - s)^k$$

     Then

     $$\begin{aligned}
     \lim_{z \to s} f(z)(z - s) &= \lim_{z \to s} \sum_{k=-1}^{\infty} a_k (z - s)^{k+1} \\
     &= a_{-1} \cdot 0^0 + \sum_{k=0}^{\infty} a_k \cdot 0^{k+1} \\
     &= a_{-1} \\
     &= \operatorname{res}_s f
     \end{aligned}$$

     as desired. ☐

3. If f has a **simple zero** at s and g is holomorphic at s, then

$$\operatorname{res}_s(g \cdot f) = g(s)\operatorname{res}_s f$$

Proof. Taking the Laurent expansions, let

$$f(z) = \sum_{k=-1}^{\infty} a_k (z-s)^k \qquad\qquad g(z) = \sum_{k=0}^{\infty} \frac{g^{(k)}(s)}{k!}(z-s)^k$$

Then taking the Cauchy product, we obtain

$$(f \cdot g)(z) = a_{-1}b_0(z-s)^{-1} + \cdots$$

Therefore,

$$\operatorname{res}_s(g \cdot f) = b_0 a_{-1} = g(s)\operatorname{res}_s f$$

as desired. □

- **Simple** (pole): A pole of order 1.
- **Simple** (zero): A zero of order 1.
- Properties 2 and 3 imply that if f has a simple zero at s and g is holomorphic, then

$$\operatorname{res}_s(g/f) = \frac{g(s)}{f'(s)}$$

- Using residues to compute the real integral

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^n}\,\mathrm{d}x$$

which gets complicated as $n \in \mathbb{N}$ gets big.

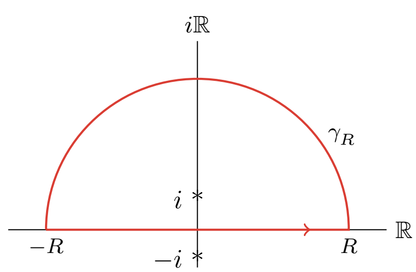


Figure 7.5: Using residues to compute contour integrals.

- We will use contour integration over γ_R, as shown in Figure 7.5.
- First, analytically continue the integrand to

$$f(z) := \frac{1}{(z^2+1)^n}$$

  - Observe that f is meromorphic on $\mathbb{C}$ and, specifically, $f \in \mathcal{O}(\mathbb{C} \setminus \{\pm i\})$.
- Since integrals over homotopic paths are the same, the integral over the boundary of a small disk around i would be the same as the integral over γ_R. Thus,

$$\operatorname{res}_i f = \int_{\gamma_R} f\,\mathrm{d}z$$

- Additionally, we can split

$$\int_{\gamma_R} f \, \mathrm{d}z = \int_{-R}^{R} f(x) \, \mathrm{d}x + \int_{\gamma_R \setminus [-R,R]} f \, \mathrm{d}z$$

- Now as $R \to \infty$, the magnitude of the denominator of f will similarly diverge along $\gamma_R \setminus [-R, R]$. Consequently, the rightmost integral above goes to zero as $R \to \infty$, and we are left with

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^n} \, \mathrm{d}x = \lim_{R\to\infty} \int_{-R}^{R} f(x) \, \mathrm{d}x = \lim_{R\to\infty} \operatorname{res}_i f = \operatorname{res}_i f$$

- Thus, all we need to solve this problem is to compute the Laurent expansion of f about i.
- After more manipulations (see the notes), the final answer is

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^n} \, \mathrm{d}x = \frac{\pi(2n-2)!}{2^{2n-2}[(n-1)!]^2}$$

- Application of complex analysis to number theory: The Basel problem.
  - This was an open problem for over 100 years.
  - It asked for the value of the infinite sum

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2} =: \zeta(2)$$

  - Euler — who was from Basel — proved (with some not-very-rigorous power series manipulations that were only rigorously justified later) that

$$\zeta(2) = \frac{\pi^2}{6}$$

    - Calderon: "That's the good thing about being a pioneer: You don't have to do all the proofs."
  - But what about $\zeta(3)$, $\zeta(4)$, $\zeta(5)$, etc.?
  - We know how to compute $\zeta(2n)$.
  - Apery's theorem (1978): $\zeta(3)$ is proven to be irrational.
    - The argument is supposedly quite complex, and Calderon has never studied it.
    - Is $\zeta(3)$ transcendental like $\zeta(2n)$? Still an open question!
  - For $\zeta(5)$ and any other odd numbers, we're still out of luck.
- A computation of $\zeta(2)$ using residues.

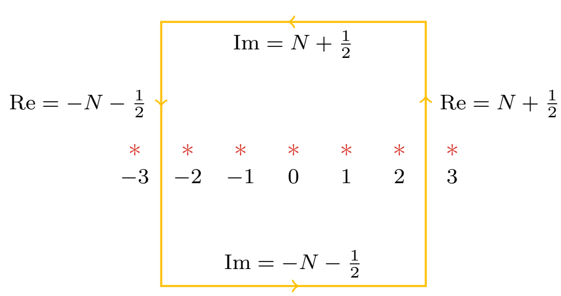


Figure 7.6: Basel problem solution using residues.

- Let's investigate the helper function

$$f(z) = \frac{\pi}{z^2 \tan(\pi z)}$$

- Observe that f is meromorphic on $\mathbb{C}$. In particular...
  - f has a pole of order 3 at $z = 0$;
  - f has a pole of order 1 at ever nonzero $n \in \mathbb{Z}$ (because tangent is periodic).
- Thus, $f \in \mathcal{O}(\mathbb{C} \setminus \mathbb{Z})$.
- Let's compute the residue of f about the nonzero poles.
  - In these cases, the denominator has a simple zero and the numerator is holomorphic, so we can apply "Property 4" to learn that

$$\operatorname{res}_n f = \frac{\pi\big|_n}{\mathrm{d}/\mathrm{d}z\,[z^2\tan(\pi z)]\big|_n} = \frac{\pi}{2n\underbrace{\tan(\pi n)}_{0} + \pi n^2 \underbrace{\sec^2(\pi n)}_{1}} = \frac{\pi}{\pi n^2} = \frac{1}{n^2}$$

- Define γ_N to be the curve in Figure 7.6. Then since γ_N has a winding number of 1 around all of the poles it encloses, the residue theorem tells us that

$$\frac{1}{2\pi i}\int_{\gamma_N} f\,\mathrm{d}z = \sum_{n=-N}^{N} \operatorname{res}_n f$$

  - It follows from the above that

$$\frac{1}{2\pi i}\int_{\gamma_N} f\,\mathrm{d}z = \sum_{n=-N}^{-1} \operatorname{res}_n f + \operatorname{res}_0 f + \sum_{n=1}^{N} \operatorname{res}_n f = \operatorname{res}_0 f + 2\sum_{n=1}^{N} \frac{1}{n^2}$$

- We now compute the above integral.
  - We will do this by bounding it and showing that it converges to zero.
  - To begin, let's bound the reciprocal of $\tan(\pi z)$, which is

$$\cot(\pi z) = i \cdot \frac{\mathrm{e}^{\pi i z} + \mathrm{e}^{-\pi i z}}{\mathrm{e}^{\pi i z} - \mathrm{e}^{-\pi i z}}$$

    - ➢ When $\operatorname{Re}(z) = N + 1/2$ (for some $N \in \mathbb{Z}$) and $y = \operatorname{Im}(z) \in \mathbb{R}$, we have that

$$\mathrm{e}^{\pi i z} = \mathrm{e}^{\pi i (N+1/2+yi)} = \mathrm{e}^{N\pi i} \cdot \mathrm{e}^{\pi i/2} \cdot \mathrm{e}^{-\pi y} = \pm 1 \cdot i \cdot \mathrm{e}^{-\pi y} = \pm i \mathrm{e}^{-\pi y}$$

      and

$$\mathrm{e}^{-\pi i z} = \mathrm{e}^{-\pi i (N+1/2+yi)} = \mathrm{e}^{-N\pi i} \cdot \mathrm{e}^{-\pi i/2} \cdot \mathrm{e}^{\pi y} = \pm 1 \cdot -i \cdot \mathrm{e}^{\pi y} = \mp i \mathrm{e}^{\pi y}$$

      so hence,

$$|\cot(\pi z)| = \left|\frac{\mathrm{e}^{\pi i z} + \mathrm{e}^{-\pi i z}}{\mathrm{e}^{\pi i z} - \mathrm{e}^{-\pi i z}}\right| = \left|\frac{\pm i\mathrm{e}^{-\pi y} + \mp i\mathrm{e}^{\pi y}}{\pm i\mathrm{e}^{-\pi y} - \mp i\mathrm{e}^{\pi y}}\right| = \left|\frac{\pm\mathrm{e}^{-\pi y} - \pm\mathrm{e}^{\pi y}}{\pm\mathrm{e}^{-\pi y} + \pm\mathrm{e}^{\pi y}}\right| = \left|\frac{\mathrm{e}^{-\pi y} - \mathrm{e}^{\pi y}}{\mathrm{e}^{-\pi y} + \mathrm{e}^{\pi y}}\right| \leq 1$$

    - ➢ On the other hand, notice that as $\operatorname{Im}(z) \to \infty$, $\cot(z) \to 1$. Thus, if we want to keep $\cot(z)$ near 1 (and hence bounded in general), we need only require that $\operatorname{Im}(z) = N + 1/2$ is greater than some threshold. In fact, as we can see in the applet from the 3/21 lecture, if $\operatorname{Im}(z) \geq 1/2$, then $|\cot(z) - 1|$ is already less than $1/2$ and hence $|\cot(z)| \leq 2$.[4] An analogous argument holds based on the fact that as $\operatorname{Im}(z) \to -\infty$, $\cot(z) \to -1$.

[4] Note that for the sake of bounding the $\cot(\pi z)$, we need not make the top and bottom of γ_N diverge along with the right and left sides; they could stay at $\operatorname{Im}(z) = \pm 1/2$ and we'd be totally fine on boundedness. However, we do have the top and bottom diverge so that the z^2 term in the denominator of $f(z)$ becomes large at *all* points along γ_N as $N \to \infty$; this fact will be used shortly when we compute $\int_{\gamma_N} f\,\mathrm{d}z$.

    - ➢ Thus, $|\cot(z)| \leq 2$ for all $z \in \mathrm{Im}(\gamma_N)$ and $N \in \mathbb{N}$.
  - ■ Consequently, as $N \to \infty$, $f(z) \to 0$ for all $z \in \mathrm{Im}(\gamma_N)$. Thus, the integral of f over γ_N goes to zero, too. In a statement,

$$\lim_{N\to\infty} \int_{\gamma_N} f \, \mathrm{d}z = 0$$

- – Combining the above two results, we have that

$$\begin{aligned}
\frac{1}{2\pi i} \lim_{N\to\infty} \int_{\gamma_N} f \, \mathrm{d}z &= \lim_{N\to\infty} \left(\operatorname{res}_0 f + 2 \sum_{n=1}^{N} \frac{1}{n^2} \right) \\
\frac{1}{2\pi i} \cdot 0 &= \operatorname{res}_0 f + 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \\
\sum_{n=1}^{\infty} \frac{1}{n^2} &= -\frac{1}{2} \operatorname{res}_0 f
\end{aligned}$$

- – Evidently, we must now compute $\operatorname{res}_0 f$.
  - ■ The Laurent series for $\cot(z)$ about 0 is

$$\cot(z) = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2z^5}{945} - \cdots$$

  - ■ Thus, near zero,[5]

$$\begin{aligned}
f(z) &= \frac{\pi}{z^2} \cdot \left(\frac{1}{\pi z} - \frac{\pi z}{3} - \frac{\pi^3 z^3}{45} - \frac{2\pi^5 z^5}{945} - \cdots \right) \\
&= \frac{1}{z^3} - \frac{\pi^2}{3z} - \frac{\pi^4}{45} z - \frac{2\pi^6}{945} z^4 - \cdots
\end{aligned}$$

  - ■ Consequently,

$$\operatorname{res}_0 f = a_{-1} = -\frac{\pi^2}{3}$$

- – Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{1}{2} \cdot -\frac{\pi^2}{3} = \frac{\pi^2}{6}$$

  as desired.

- This same argument easily extends to the even natural number values of the Riemann zeta function.

[5]Notice that this Laurent series reflects the fact that f has a zero of order 3 at $z = 0$!