MATH 27000 (Basic Complex Variables) Notes

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Weeks

List of Figures

Week 1

Classifying Complex Functions

1.1 Holomorphic Functions

3/19:

- We begin by reviewing some properties of the **complex numbers**.
- **Complex numbers**: The field of elements $z = x + iy$ where $x, y \in \mathbb{R}$ and $i^2 = -1$. *Denoted by* $\mathbb{C}$.

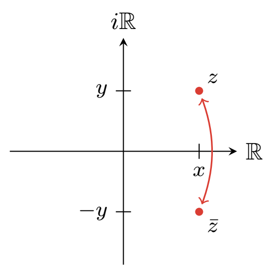


Figure 1.1: The complex plane.

  - Can be visualized as a two-dimensional plane with the number z corresponding to the point (x, y).
- **Real part**: The number x. *Denoted by* $\mathbf{Re}\, \boldsymbol{z}$.
- **Imaginary part**: The number y. *Denoted by* $\mathbf{Im}\, \boldsymbol{z}$.
- **Complex conjugate** (of z): The complex number defined as follows. *Denoted by* $\bar{\boldsymbol{z}}$. *Given by*

$$\bar{z} := x - iy$$

- Now recall the definition of a *real* function that is **differentiable** at a point $x_0 \in \mathbb{R}$.
  - $f'(x_0)(x - x_0)$ is the "best linear approximation" of f near x_0, where $\boldsymbol{f'(x_0)}$ is also defined below.
- **Differentiable** ($f : \mathbb{R} \to \mathbb{R}$ at x_0): A function f for which the following limit exists. *Constraint*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} =: f'(x_0)$$

- We now build up to defining a notion of complex differentiability.
  - Observe that the constraint above is equivalent to the constraint

$$f(x) = f(x_0) + \underbrace{[f'(x_0) + e(x)]}_{\Delta(x)}(x - x_0)$$

    where $e(x) \to 0$ as $x \to x_0$.
  - Note that we are defining a new function $\Delta(x)$ above, with the property that $\Delta(x_0) = f'(x_0)$.

- **Holomorphic** (f at z_0): A function $f : \mathbb{C} \to \mathbb{C}$ for which the following limit exists. *Also known as* **$\mathbb{C}$-differentiable**. *Constraints*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} =: f'(z_0) \qquad \Longleftrightarrow \qquad f(z) = f(z_0) + \Delta(z)(z - z_0)$$

  where Δ is continuous at z_0 and $\Delta(z_0) = f'(z_0)$.

  - It will turn out that this is the true definition of "holomorphic" / "$\mathbb{C}$-differentiable" function, not just a naïve first pass.

- Properties of holomorphic functions: Let $U \subset \mathbb{C}$ be open.

  1. The holomorphic functions on U form a ring $\boldsymbol{\mathcal{O}(U)}$.
     - Equivalently, the $\mathbb{C}$-differentiation operator is $\mathbb{C}$-linear.
     - Equivalently, if f, g are holomorphic, then $f + g$ and fg are holomorphic, too.
     - Equivalently (and most simply), we have the sum rule and the product rule (and the quotient rule if the function in the denominator is nonzero).
  2. We have the chain rule.
  3. Holomorphic implies continuous.

- Examples: Polynomials, rational functions $p(z)/q(z)$ (away from their **poles**).

- Non-example: Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$z \mapsto \bar{z}$$

  - Think-Pair-Share (TPS): Why?
  - Notice that

$$f(0) = 0 \qquad\qquad f(t) = t \qquad\qquad f(it) = -it$$

  - Thus,

$$\Delta(t) = 1 \qquad\qquad \Delta(it) = -1$$

    for all t.
  - But this means that Δ can't be continuous!
  - Yet f is clearly $\mathbb{R}$-differentiable! What gives?!
  - Note that — viewing f as a mapping of $\mathbb{R}^2 \to \mathbb{R}^2$ — we have

$$Df = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The above example suggests that our definition of complex differentiability may have been to naïve, so we'll do some further investigations now.

- Observe that $\mathbb{C} \cong \mathbb{R}^2$ as $\mathbb{R}$-vector spaces.

- **Differentiable** ($f : \mathbb{R}^2 \to \mathbb{R}^2$ at x_0): A function f for which there exists an $\mathbb{R}$-linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying the following constraint. *Constraint*

$$\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - Ah\|}{\|h\|} = 0$$

  - We also denote A by Df.

- Example: Consider the function $f : \mathbb{C} \to \mathbb{R}$ defined by

$$x + iy \mapsto x$$

  - Differentiable with total derivative
$$Df = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

- Observation: While $\mathbb{C} \cong \mathbb{R}^2$ as $\mathbb{R}$-vector spaces, as a $\mathbb{C}$-vector space, there is *additional* structure.
  - In particular, all "vectors" should commute with the "multiplication by i" map $J : \mathbb{C} \to \mathbb{C}$ defined by any one of the following three maps.

$$z \mapsto z \qquad\qquad x + iy \mapsto xi - y \qquad\qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

- Exercise: In (Re, Im) coordinates, write down the matrix for "multiply by w" for any $w \in \mathbb{C}$.
  - Let $w = a + bi$ and let $v = x + iy$. Then

$$\begin{aligned} wv &= (a + bi)(x + iy) = ax - by + i(bx + ay) \\ &= \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix} = \underbrace{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}}_{W} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

  - The matrix W above is the desired result.
- TPS: Is $f : \mathbb{C} \to \mathbb{C}$ defined as follows a complex linear map? Why not?

$$x + iy \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (x + y) + iy$$

  - Among other properties, a complex linear map should satisfy

$$if(x + iy) = f[i(x + iy)]$$

    for the scalar $i \in \mathbb{C}$.
  - However, we have that

$$if(x + iy) = i[(x + y) + iy] = -y + i(x + y) \neq (x - y) + ix = f(-y + ix) = f[i(x + iy)]$$

- What about the following map?
$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$
  - A complex linear map should satisfy

$$A(v + w) = Av + Aw \qquad\qquad \lambda Av = A(\lambda v)$$

    for all $v, w, \lambda \in \mathbb{C}$.
  - Let $v, w \in \mathbb{C}$ be arbitrary. Then

$$\begin{aligned} A(v + w) &= \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \left[\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right] = \begin{pmatrix} (v_1 + w_1) + 2(v_2 + w_2) \\ -2(v_1 + w_1) + (v_2 + w_2) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = Av + Aw \end{aligned}$$

- Let $v, \lambda \in \mathbb{C}$. Then

$$\begin{aligned}
\lambda A v &= (\lambda_1 + i\lambda_2) \cdot [(v_1 + 2v_2) + i(-2v_1 + v_2)] \\
&= [\lambda_1(v_1 + 2v_2) - \lambda_2(-2v_1 + v_2)] + i[\lambda_2(v_1 + 2v_2) + \lambda_1(-2v_1 + v_2)] \\
&= [(\lambda_1 v_1 - \lambda_2 v_2) + 2(\lambda_2 v_1 + \lambda_1 v_2)] + i[-2(\lambda_1 v_1 - \lambda_2 v_2) + (\lambda_2 v_1 + \lambda_1 v_2)] \\
&= A[(\lambda_1 v_1 - \lambda_2 v_2) + i(\lambda_2 v_1 + \lambda_1 v_2)] \\
&= A(\lambda v)
\end{aligned}$$

- Therefore, since A satisfies the two properties, it is complex linear.

- Conclusion: To reiterate from the above, A must commute with J to be complex linear.
- Implication: Every $\mathbb{C}$-linear map of $\mathbb{C}$ is just multiplication by a complex number.
  - This is a special case of the following more general result, which holds for any field K.

$$\mathrm{Hom}_K(K, K) \cong K$$

- Now let's revisit differentiability.
- It turns out that a condition for $\mathbb{C}$-differentiability *equivalent* to the definition of "holomorphic" given above is that there exists a $\mathbb{C}$-linear map $A : \mathbb{C} \to \mathbb{C}$ such that

$$\lim_{h \to 0} \frac{\|f(x_0 + h) - f(x_0) - Ah\|}{\|h\|} = 0$$

  - From the above discussion, we know that this A is just multiplication by some $w \in \mathbb{C}$.
  - All of the values in the above norms are complex numbers, so *another* equivalent condition is

$$\lim_{z \to z_0} \frac{|f(z) - f(z_0) - w \cdot (z - z_0)|}{|z - z_0|} = 0$$

  - This condition is wholly mathematically equivalent to our holomorphic definition,

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = w$$

- So when is an $\mathbb{R}$-differentiable function actually holomorphic?
  - Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ map $(x, y) \mapsto (g, h)$.
  - Let

$$A = Df = \begin{pmatrix} g_x & g_y \\ h_x & h_y \end{pmatrix}$$

    where the subscript notation views g, for instance, as $g(x, y)$ and denotes the partial derivative of g with respect to x.
  - Let J (the "multiply by i") function be defined as above.
  - Then the "commute with i" condition is equivalent to

$$J^{-1} A J = A$$

  - Expanding the product on the left above in terms of g_x, g_y, h_x, h_y, we obtain

$$\begin{pmatrix} h_y & -h_x \\ -g_y & g_x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} g_x & g_y \\ h_x & h_y \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} g_x & g_y \\ h_x & h_y \end{pmatrix}$$

  - This condition is equivalent to A satisfying the **Cauchy-Riemann equations**.

- **Cauchy-Riemann equations**: The following two equations, which identify when a complex function is holomorphic. *Also known as* **CR equations**. *Given by*

$$\begin{aligned} g_x &= h_y \\ g_y &= -h_x \end{aligned}$$

- These equations are satisfied when A is of the form

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

- So at this point, we can differentiate f with respect to z. But what if we want to differentiate it with respect to x and y (of $z = x + iy$)?
  - We will need the following change of basis.
    - Since $z = x + iy$ and $\bar{z} = x - iy$, we have

$$\begin{aligned} 2x &= z + \bar{z} & \qquad 2iy &= z - \bar{z} \\ x &= \frac{1}{2}(z + \bar{z}) & y &= -\frac{i}{2}(z - \bar{z}) \end{aligned}$$

    - This tells us that

$$\frac{\partial x}{\partial z} = \frac{1}{2} \qquad\qquad \frac{\partial y}{\partial z} = -\frac{i}{2}$$

  - We can now invoke the multivariable chain rule and simplify the resultant expression.

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}(f_x - if_y)$$

    - Note that once again, the subscript notation "f_x" means $\partial f/\partial x$.
  - Note that we can also similarly work out that

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(f_x + if_y)$$

    - Observe in particular that

$$f_x = g_x + ih_x \qquad\qquad f_y = g_y + ih_y$$

    - Thus, the CR equations ($g_x = h_y$ and $g_y = -h_x$) being satisfied is equivalent to

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}(f_x + if_y) = \frac{1}{2}[(g_x + ih_x) + i(g_y + ih_y)] = 0$$

    - Note that $\partial f/\partial \bar{z}$ is not actually a derivative since f depends on z, not $\bar{z}$. Rather, we use "$\partial f/\partial \bar{z}$" to denote the following operator applied to f.

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

- **Wirtinger derivatives**: The two differential operators defined as follows. *Denoted by* $\boldsymbol{\partial/\partial z, \partial/\partial \bar{z}}$. *Given by*

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \qquad\qquad \frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

- Theorem: The $\mathbb{R}$-differentiable function $f : U \to \mathbb{C}$ is holomorphic iff $\partial f/\partial \bar{z} = 0$. Moreover, if it is, then

$$f'(z_0) = \left.\frac{\partial f}{\partial z}\right|_{z_0}$$

1.2 Harmonic Functions and Conformal Maps

3/21:

- Tries to remember everyone's name and actually does a pretty good job!
- Has us all turn to our neighbor and meet them! I met Ryan.
- Review.
  - Naïve holomorphic definition: Typical derivative definition.
  - The map $z \mapsto \bar{z}$ is not holomorphic even though it is differential over the reals.
  - The reason this map is not holomorphic is that its matrix derivative is not complex linear. This means that it does not commute with the "multiply by i" matrix, defined by

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

  - Indeed, an *equivalent* definition to the naïve holomorphic one is: $f : \mathbb{C} \cong \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathbb{R}$-differentiable at z_0 with $Df(z)$ is complex linear.
  - Another equivalent one is the Cauchy-Riemann equation definition.
    - Let $f(z) = u(z) + iv(z)$ where $u, v : \mathbb{C} \to \mathbb{R}$.
    - Then f is holomorphic if $u_x = v_y$ and $v_x = -u_y$, or equivalently if $\partial f / \partial \bar{z} = 0$.
  - The above comment motivates the definition of the operators

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

  - Note: Every $\mathbb{C}$-linear map is "multiply by w" for some $w \in \mathbb{C}$.
  - Note that we have not yet talked about continuity or related things.
- Note: Different books use different conventions.
  - "Holomorphic at a point" and "complex differentiable in a neighborhood of a point" may mean different things.
  - Example: Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$x + iy \mapsto x^2 + iy^2$$

    - Then

$$Df = \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix}$$

    - Evidently, Df is not complex linear in general because we can pick x, y such that $2x \neq 2y$.
    - Thus, this function is not complex differentiable in general.
    - However, it is complex differentiable at zero because here, $Df = 0$.
    - Thus, this function is complex differentiable *at a point*, but not complex differentiable *in a neighborhood*.
  - We will almost always be talking about functions that are complex differentiable *in a neighborhood* in this class.
- Example: $f(x + iy) = x^2\mathbb{I}_{Q(x)} + iy^2\mathbb{I}_{Q(y)}$ is complex differentiable in a neighborhood of the origin, but this is dumb. $\mathbb{I}$ is the **indicator function**.
- Preview (we'll see this next Thursday): Holomorphic implies C^∞.

- Today: Some more things about the Cauchy-Riemann equations and what we can get out of them.
- Let's begin with a consequence of the $\partial/\partial z$ and $\partial/\partial\bar{z}$ operators.
  - Compute (if $f \in C^2$):
$$\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{4}\underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)}_{\Delta} f_{xy} = f_{yx}$$
    - Do we need this C^2 condition if holomorphic already implies C^∞?
    - No, but we haven't "learned" this yet. Once we prove this, no more talk of regularity!
  - Solutions to this, the Laplacian Δ (from physics), could be a good final project!
  - Look for solutions to $\Delta f = 0$.
    - Equivalently, look for f such that $f_{xx} + f_{yy} = 0$.
  - Observation: Any f holomorphic implies that $\Delta f = 0$ (since we apply $\partial/\partial\bar{z}$ to f first).
- **Harmonic** (function): A function $f : \mathbb{R}^2 \to \mathbb{C}$ such that $\Delta f = 0$.
- Since the differentiation operator is linear,
$$\Delta(u + iv) = \Delta u + i\Delta v$$
- Corollary: The real and imaginary parts of a C^2 holomorphic function are harmonic.
- So we know that f holomorphic implies u, v real-valued and harmonic. Can we go the other way?
  - We know that these functions have certain properties in terms of their partial derivatives, namely that they satisfy the Cauchy-Riemann equations.
- **Harmonic conjugates**: Two functions $u, v : \mathbb{R}^2 \to \mathbb{R}$ that satisfy the CR equations.
- Fact: Let u, v be two C^∞ harmonic conjugates. Then $f = u + iv$ is holomorphic
  - Easy to prove if you're not concerned about regularity.
- Is $v + iu$ holomorphic?
  - No, partials don't work out. We still get $v_x = -u_y$, but we also get $u_x = -v_y$.
  - However, $v - iu$ is holomorphic!
  - This just means that rotating by i gives us a new holomorphic function since
$$i \cdot (u + iv) = -v + iu$$
- Example: $u = x^2 - y^2$ is harmonic. Find a conjugate and find $f = u + iv$.[1]
  - We have
$$\begin{aligned} v_y &= u_x = 2x \\ v_x &= -u_y = 2y \end{aligned}$$
  - Thus,
$$v = 2xy + C$$
    for some $C \in \mathbb{C}$
  - Then we would have
$$f = u + iv = (x^2 - y^2) + i(2xy + C) = x^2 + 2xyi - y^2 + iC = (x + iy)^2 + iC = z^2 + iC$$

[1] Calderon actually let us work this out in class!

- Let's now talk about integrating functions.
- Let $a, b \in \mathbb{R}$. Consider $f : \mathbb{C} \to \mathbb{C}$, not holomorphic but continuous. How do we take $\int_a^b f\,\mathrm{d}z$?

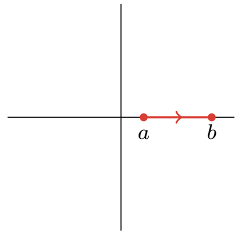


Figure 1.2: Integrating complex functions over real intervals.

  - What we do is just split the integral into real and imaginary parts.

$$\int_a^b f\,\mathrm{d}z = \int_a^b u\,\mathrm{d}t + i\int_a^b v\,\mathrm{d}t$$

  - This is how we integrate between reals in the complex plane.
- How do we integrate over more arbitrary points in the complex plane, e.g., $a\lambda$ and $b\lambda$?

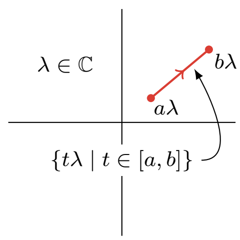


Figure 1.3: Integrating complex functions over line segments in $\mathbb{C}$.

  - We could take any path. Which one?
  - Try over the line segment $\{t\lambda \mid t \in [a, b]\}$.
  - Then we take

$$\int_{a\lambda}^{b\lambda} f(z)\,\mathrm{d}z = \int_a^b f(\lambda t)\lambda\,\mathrm{d}t$$

    via the substitutions $z = t\lambda$ and $\mathrm{d}z = \lambda\,\mathrm{d}t$.
  - This second integral, we can compute in the first way.
- Now what about integrating along an arbitrary curve $\gamma : [a, b] \to \mathbb{C}$, which we will say is piecewise C^1?

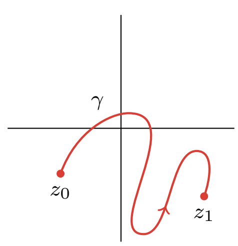
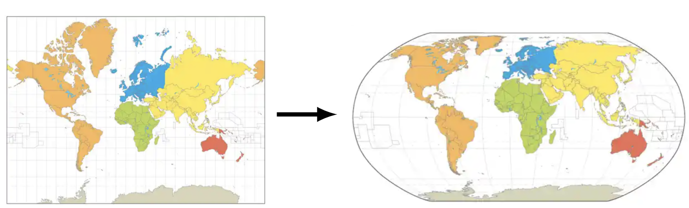


Figure 1.4: Integrating complex functions over arbitrary paths in $\mathbb{C}$.

- Note that $z_0 := \gamma(a)$ and $z_1 := \gamma(b)$.
- Define
$$\int_\gamma f \, \mathrm{d}z = \int_a^b f(\gamma(t)) \cdot \gamma'(t) \, \mathrm{d}t$$
- Where is the γ' coming from? Same place as before! It's just a change of variables: $z = \gamma(t)$ implies $\mathrm{d}z = \gamma'(t) \, \mathrm{d}t$.
- If we know differential forms, $f \, \mathrm{d}z$ is just a complex-valued one-form. And the chain rule is just how we integrate one-forms.
- We'll do lots of basic practice of this in the completion problems on the PSet.

- Note: Whenever we do a path integral, we should ask if the parameterization matters. The parameterization does *not* matter.
- What do we need to compute integrals without having to take the limit of a sum over partitions? We need the fundamental theorem of calculus.
  - The FTC does indeed hold here, too, though we won't prove this.
- FTC: Suppose $F' = f$ on $U \subset \mathbb{C}$, and let γ be a **path** inside of U. Then
$$\int_\gamma f \, \mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$$
  - Again, if we like differential forms, then note that we're just saying in the above equality that
$$\int_\gamma \mathrm{d}F = \int_{\partial\gamma} F$$
- **Path**: A function from an interval of real numbers to a vector space. *Also known as* **contour**. *Denoted by* $\boldsymbol{\gamma}$.
- Gives us a three-minute break from 10:17-10:20 in the middle of the class.
  - The fact that this guy actually teaches in accordance with accepted pedagogical standars is wild.
- How do we want to visualize holomorphic functions?
  - $f : \mathbb{C} \to \mathbb{C}$ is hard to graph because the set of points lives in $\mathbb{R}^4$.
  - So we're out of luck if we want to do graphs.
  - Thus, we'll look at **mappings**.
- Example: Are we looking at the Mercator or Robinson map of the world?



(a) Mercator projection. (b) Robinson projection.

Figure 1.5: Visualizing functions of $\mathbb{R}^2 \to \mathbb{R}^2$.

  - What do these mappings do to the lines of latitude and longitude?
  - This is a mapping of $\mathbb{R}^2 \to \mathbb{R}^2$ that stretches and distorts! By drawing grid lines, we can see what it does to $\mathbb{R}^2$.

- Now recall that f holomorphic implies Df looks like
$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$
  - What's nice about these matrices is they can always be factored into rotation and scaling matrices.
$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \qquad (\lambda, \theta \in \mathbb{R})$$
  - This means that
$$f'(z_0) = w = re^{i\theta} \in \mathbb{C}$$
    since multiplication by complex numbers also just rotates and scales!
  - This means that if $z' = r'e^{i\theta'}$, then $w \cdot z' = r \cdot r'e^{i(\theta+\theta')}$.
  - This also means that we may have rotation and scaling but no shearing. Formally, we have the following lemma.

- **Argument** (of $z \in \mathbb{C}$): The angle θ such that $z = re^{i\theta}$ for some $r \in \mathbb{R}$. *Denoted by* $\boldsymbol{\arg(z)}$.

- Lemma: Suppose two curves γ, δ intersect at a point $z \in \mathbb{C}$. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. Then
$$\angle_z(\gamma, \delta) = \angle_{f(z)}(f(\gamma), f(\delta))$$
  i.e., f preserves angles.

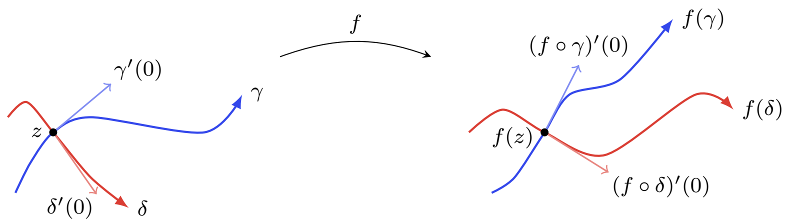


Figure 1.6: Holomorphic maps preserve angles.

Proof. To consider the angle between two curves analytically, let's look at the tangent vectors to the two curves, for example at z. Now while we often think of $\gamma'(0)$ as a *matrix*, remember that we've proven that all of these matrices are equivalent to complex numbers. In particular, since $\gamma : \mathbb{R} \to \mathbb{R}^2 \cong \mathbb{C}$, the total derivative will just be a vector. This vector may easily be represented as a complex number $re^{i\theta}$ in polar coordinates. Similarly, $\delta'(0)$ can be thought of as a complex number $r'e^{i\theta'}$. Thus, dividing these quantities gives us the angle $\theta - \theta'$ between the two vectors, which we can isolate using the argument function.

Doing the same to the curves at $f(z)$ yields
$$\begin{aligned}
\angle_{f(z)}(f(\gamma), f(\delta)) &= \arg\left[\frac{(f\circ\gamma)'(0)}{(f\circ\delta)'(0)}\right] \\
&= \arg\left[\frac{f'(\gamma(0)) \cdot \gamma'(0)}{f'(\gamma(0)) \cdot \delta'(0)}\right] \\
&= \arg\left[\frac{f'(z) \cdot \gamma'(0)}{f'(z) \cdot \delta'(0)}\right] \\
&= \arg\left[\frac{\gamma'(0)}{\delta'(0)}\right] \\
&= \angle_z(\gamma, \delta)
\end{aligned}$$
as desired. $\square$

- Calderon gave us 5 minutes to try to compute this ourselves with the hint: Use the chain rule! $\angle_z(\gamma,\delta) = \arg(\gamma'(0)\cdot[\delta'(0)]^{-1})$.

- **Conformal** (map): A function $f : U \to V$, where $U, V \subset \mathbb{C}$, that satisfies the following two constraints.
  Constraints
  1. f is a diffeomorphism.
  2. f preserves angles.

- **Diffeomorphism**: A homeomorphism for which f, f^{-1} are differentiable.

- **Biholomorphic** (map): A function $f : U \to V$ that is bijective, holomorphic, and for which f^{-1} is holomorphic.

- Theorem/observation: Biholomorphic iff conformal.

  Proof. Follows straight from the definitions and the lemma we just proved. □

- Calderon shows us an applet.
  - We can use the applet to help with the PSet, but we still do have to submit actual proofs.
  - Allows you to visually see the lemma for instance.
  - Example: Under $z \mapsto z^2$, the sector of radius 2 and argument $\pi/4$ goes to the sector of radius $2^2 = 4$ and argument $\pi/2$.

1.3 Chapter I: Analysis in the Complex Plane

From Fischer and Lieb (2012).

3/19:
- The preface only contains comments and instructions for an instructor planning to use this textbook for a course.

- The chapter begins with two paragraphs (as do all the others).
  - The first discusses topic covered in the chapter.
  - The second gives some historical background on these topics.

Section I.0: Notations and Basic Concepts

- Goal: Reiew the fundamental topological and analytical concepts of real analysis.

- Defines the **complex numbers**, **complex plane**, and **complex conjugate**.

- **Absolute value** (of z): The Euclidean distance of z from zero. *Also known as* **modulus**. *Denoted by* $\boldsymbol{|z|}$. *Given by*

$$|z| := \sqrt{x^2 + y^2}$$

- **Imaginary unit**. *Denoted by* $\boldsymbol{i}$.

- Relating the modulus and complex conjugate.

$$|z| = \sqrt{z\bar{z}}$$

- **Open disk** (of radius ε and center z_0): The set defined as follows. *Also known as* **ε-neighborhood** (of z_0). *Denoted by* $\boldsymbol{D_\varepsilon(z_0)}$, $\boldsymbol{U_\varepsilon(z_0)}$. *Given by*

$$D_\varepsilon(z_0) = U_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$$

- **Unit disk**: The set defined as follows. *Denoted by* $\mathbb{D}$. *Given by*

$$\mathbb{D} := D_1(0)$$

- **Unit circle**: The set defined as follows. *Denoted by* $\mathbb{S}$. *Given by*

$$\mathbb{S} := \{z \in \mathbb{C} : |z| = \varepsilon\}$$

- **Upper half plane**: The set defined as follows. *Denoted by* $\mathbb{H}$. *Given by*

$$\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

- $\mathbb{C}^*$: The set defined as follows. *Given by*

$$\mathbb{C}^* := \mathbb{C} \setminus \{0\}$$

3/21:

- **Neighborhood** (of z_0): A set U which contains an ε-neighborhood.
- **Open** (set): A set that is a neighborhood of each of its points.
- **Closed** (set): A complement of an open set.
- **Interior** (of M): The largest open set contained in M. *Denoted by* $\mathring{\boldsymbol{M}}$.
- **Closure** (of M): The smallest closed set containing M. *Denoted by* $\overline{\boldsymbol{M}}$.
- **Topological boundary** (of M): The set defined as follows. *Also known as* **boundary**. *Denoted by* $\boldsymbol{\partial M}$. *Given by*

$$\partial M := \overline{M} \setminus \mathring{M}$$

- **Relatively open** (set in M): The intersection of an open set U with an arbitrary set M. *Also known as* **open** (set in M).
- **Relatively closed** (set in M): The intersection of a closed set U with an arbitrary set M. *Also known as* **open** (set in M).

4/4:

- Fischer and Lieb (2012) define **convergent series** and their **limits** in the usual way.
  - Sum, product, and reciprocal rules stated.
- **Accumulation point** (of $M \subset \mathbb{C}$): A point $z_0 \in \mathbb{C}$ for which there is a sequence $\{z_n\} \subset M \setminus \{z_0\}$ with $\lim z_n = z_0$.
- **Bounded** (set): A set K for which $|z| \leq R$ for some R and all $z \in K$.
- **Compact** (set): A closed, bounded set.
  - Each sequence in a compact set contains convergent subsequences with limit in the set.
- **Relatively compact** (set in V): A set U for which $\overline{U}$ is a compact subset of V. *Denoted by* $\boldsymbol{U \subset\subset V}$.
- Definition of a **complex-valued function**.
- **Continuous** (function at z_0): A function $f : U \to \mathbb{C}$ such that for each neighborhood M of $w_0 = f(z_0)$, there is a neighborhood N of z_0 with $f(U \cap N) \subset M$; equivalently, the following holds true for all convergent sequences $\{z_n\} \subset U$.

$$f\left(\lim_{n\to\infty} z_n\right) = \lim_{n\to\infty} f(z_n)$$

- **Real part** (of $f : U \to \mathbb{C}$): The real-valued function g such that $f = g + ih$.

- **Imaginary part** (of $f : U \to \mathbb{C}$): The complex-valued function h such that $f = g + ih$.
- Continuity results.
  - f is continuous iff its real and imaginary parts are.
  - The **composition** of continuous functions is continuous.
  - Example of a continuous function: Any polynomial in $z, \bar{z}$, i.e., any function of the form

$$f(z) = \sum_{j,k=0}^{N} a_{jk} z^j \bar{z}^k$$

- **Path**: A continuous map from a closed finite interval into the complex plane. *Denoted by* $\boldsymbol{\gamma : [a,b] \to \mathbb{C}}$.
  - We say that γ **connects** its **initial point** and **end point**.
- **Trace** (of a path): The image set $\gamma([a,b])$. *Denoted by* $\mathbf{tr}\,\boldsymbol{u}$.
- **Initial point** (of a path): The point $\gamma(a)$.
- **End point** (of a path): The point $\gamma(b)$.
- **Connected** (set): A set U for which any two points of U can be connected by a path whose trace lies in U. *Also known as* **pathwise connected**.
- **Domain**: A connected open set.
  - "An open set U is a domain if and only if no decomposition of U into disjoint nonempty subsets exists" (Fischer & Lieb, 2012, p. 3).
- Images of compact (resp. connected) sets under continuous functions are again compact (resp. connected).
- Images of open (resp. closed) sets under continuous functions are *not necessarily* open (resp. closed).
- Preimages of open (resp. closed) sets under continuous functions are again open (resp. closed).
- Note that the above definitions make sense in higher-dimensional vector spaces upon replacing the absolute value with the **Euclidean norm**.

Section I.1: Holomorphic Functions

4/5:

- Definition of **holomorphic**.
- Examples and nonexamples of holomorphic functions: Constant, identity, and complex conjugate functions.
- Further comments on the complex conjugate function.
  - It's everywhere continuous but nowhere complex differentiable.
  - It's very hard to find real examples of such functions, but we found a complex example like that!
- For the basic properties of complex differentiation (e.g., holomorphic implies continuous, sum and product rules), the proofs are symmetric to the real ones.
- $\mathcal{O}(U)$ is a ring, but it is more technically a $\mathbb{C}$-algebra.
- **Chain rule**: If $f : U \to V$ and $g : V \to \mathbb{C}$ are mappings of open subsets of $\mathbb{C}$ that are holomorphic at $z_0 \in U$ and $f(z_0) = w_0 \in V$, respectively, then $g \circ f : U \to \mathbb{C}$ is holomorphic at z_0 and

$$(g \circ f)'(z_0) = g'(w_0) f'(z_0)$$

- Definition of **biholomorphic**.
- Fischer and Lieb (2012) prove some elementary properties of complex functions in places with nonzero derivatives (e.g., locally one-to-one).
  - This leads into the **complex inverse function theorem**.
  - Preview: This leads to the result that "bijective holomorphic maps are biholomorphic," i.e., if we have a bijective holomorophic map, we don't also need to prove that f^{-1} is holomorphic (Fischer & Lieb, 2012, p. 6).

Section I.2: Real and Complex Differentiability

- Definition of **$\mathbb{R}^2$-differentiable function**.
- Definition of **Wirtinger derivatives**.
  - Fischer and Lieb (2012) arrive here from a slightly different angle than in class.
- Fischer and Lieb (2012) prove the $\partial f/\partial \bar{z} = 0$ condition.
- **Cauchy-Riemann operator**: The differential operator defined as follows. *Denoted by* $\boldsymbol{\partial/\partial\bar{z}}$. *Given by*

$$\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

- "In requiring that a function be complex differentiable we thus simultaneously require that a certain partial differential equation be satisfied. In other words, *holomorphic functions are the differentiable solutions of the Cauchy-Riemann equation*" given as follows (Fischer & Lieb, 2012, p. 9).

$$\frac{\partial f}{\partial \bar{z}}(z) = 0$$

- **Laplace operator**: The real differential operator defined as follows. *Also known as* **Laplacian**. *Denoted by* $\boldsymbol{\Delta}$. *Given by*

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

  - As a real differential operator,

$$\Delta\bar{f} = \overline{\Delta f}$$

  - More on this operator in Section VII.6.
- Definition of a **harmonic function**, the **Cauchy-Riemann equations** (in real form).
- On the CR equations: "The gradients and, consequently, the equipotential lines of g and h are thus perpendicular to one another at all points where the gradients do not vanish" (Fischer & Lieb, 2012, p. 10).
- We now investigate a special case of the chain rule for the Wirtinger derivatives.
- Lemma 2.4: Let $f : U \to \mathbb{C}$ be a real differentiable function defined on an open set $U \subset \mathbb{C}$ and let $w : [a, b] \to U$ be a differentiable map (i.e., a differentiable path in U). Then for all $t \in [a, b]$,

$$\frac{\partial}{\partial t}(f \circ w)(t) = \begin{pmatrix} f_z(w(t)) & f_{\bar{z}}(w(t)) \end{pmatrix} \begin{pmatrix} \dot{w}(t) \\ \dot{\bar{w}}(t) \end{pmatrix} = f_z(w(t))\dot{w}(t) + f_{\bar{z}}(w(t))\dot{\bar{w}}(t)$$

  Here we denote $\partial w/\partial t$ by $\dot{w}$ and use matrix multiplication.

  Proof. Left to the reader. □

- **Complex linear map**: A map $l : \mathbb{C} \to \mathbb{C}$ characterized by the following. *Constraints*
  1. $l(z+w) = l(z) + l(w)$;
  2. $l(rz) = rl(z)$;

  for $z, w, r \in \mathbb{C}$.
  - Every complex linear map is of the form
  $$w = l(z) = az$$
  for a unique $a \in \mathbb{C}$.
- **Real linear map**: A map $l : \mathbb{C} \to \mathbb{C}$ characterized by the following. *Constraints*
  1. $l(z+w) = l(z) + l(w)$;
  2. $l(rz) = rl(z)$;

  for $z, w \in \mathbb{C}$ and $r \in \mathbb{R}$.
  - Every real linear map is of the form
  $$w = l(z) = az + b\bar{z} = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \end{pmatrix}$$
  for a unique pair $\begin{pmatrix} a & b \end{pmatrix} \in \mathbb{C}^2$.
  - Implication: l is complex linear iff $b = 0$.
- **Tangent map** (of f at z_0): The real linear map from $\mathbb{C} \to \mathbb{C}$ determined by the vector $\begin{pmatrix} f_z(z_0) & f_{\bar{z}}(z_0) \end{pmatrix}$.
- Proposition: f is holomorphic at z_0 iff its tangent map at z_0 is complex linear.
- Fischer and Lieb (2012) discuss angle preservation, per Figure 1.6.

Week 2

Consequences of Power Series

2.1 Office Hours

3/25:
- What exactly are the Wirtinger derivatives?
  - The $\partial/\partial z$ and $\partial/\partial\bar{z}$ operators.
- The initial definition of holomorphic is accurate. It's naïve, but it works out.
- Noney: Non example.
  - As in, we have some examples of holomorphic functions and then we have an example of a function that is *not* holomorphic.
- TPS: Think Pair Share.
- Met Panteleymon and helped him with partial fractions!
- The Δ notation does mean the same Laplacian as $\vec{\nabla}^2$ from Quantum Mechanics.
- Calderon is not related to Calderón; he was Argentinian, Calderon is half-Filipino and has no accent on his name. Both Spanish colonies but that's it.
- We can do all of the problems except Problem 1 at this point.
  - For this, though, we can just look up the definition of the complex sine function.
  - We basically just need to know what $\sin(i)$ is and what sine looks like along the imaginary axis.

2.2 Power Series

3/26:
- Recall: We already know that...
  - Polynomials are elements of $\mathcal{O}(\mathbb{C})$;
  - Rational functions $P(z)/Q(z)$ are elements of $\mathcal{O}(\mathbb{C}\setminus V(Q))$.
- **Affine algebraic set**: The set of solutions in an algebraically closed field K of a system of polynomial equations with coefficients in K. *Also known as* **variety**. *Denoted by* $\boldsymbol{V(f_1,\dots,f_n)}$.
- Today, we want to determine how the other elementary functions behave over the complex numbers.
  - Other functions we want: exp, log, sin, cos.
  - We will do log later, but all the others today.

- **Exponential function**: The complex function defined as follows. *Denoted by* $\mathbf{e}^{\boldsymbol{z}}$, $\mathbf{exp}(\boldsymbol{z})$. *Given by*

$$\mathrm{e}^z = \exp(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

- Naïvely, this power series is just be a polynomial $P(z) \in \mathcal{O}(\mathbb{C})$.
- More rigorously, however, we must specify which kind of convergence we mean for the power series.
  - As one example, we could say that for all z,

$$\mathrm{e}^z = P(z) = \lim_{N\to\infty} \sum_{k=0}^{N} \frac{z^k}{k!}$$

    - This would be **pointwise convergence**.
  - But there's an issue: Pointwise convergence of functions doesn't preserve anything, e.g., continuity.
- **Pointwise** (convergent $\{f_n\}$): A sequence of functions $f_n : \mathbb{C} \to \mathbb{C}$ such that for all $z \in \mathbb{C}$, we have $f_n(z) \to f(z)$.
- TPS: Come up with an example of a sequence of continuous functions $\{f_n\}$ that converges pointwise to f, such that the f_n are all...
  1. Continuous but f is not;
     - $f_n(x) = \arctan(nx)$.
     - Converges to the sign function $f(x) = \operatorname{sgn}(x)$.
  2. Odd but f is not;
  3. Differentiable but f is not.
     - These last two cases were not discussed in class.
- We now recall a few definitions and lemmas from real analysis.
- **Locally uniformly** (convergent $\{f_n\}$): A sequence of functions $f_n : U \to \mathbb{C}$ and a function $f : U \to \mathbb{C}$ such that for all compact $K \subset U$,

$$\sup_{z\in K} |f_n(z) - f(z)| \to 0$$

- Lemma: If $f_n \to f$ locally uniformly and the f_n are continuous (or integrable), then so is f.
  - This lemma is *not* true if we sub in "differentiable!"
  - See the Stone-Weierstrass theorem for suitable constraint.
- Thus, to resolve the original question, we mean that $P_N(z) \to \exp(z)$ locally uniformly.
- Aside: Which functions have power series?
  - Remember Taylor polynomials from Calc II? **Taylor's theorem** tells us which ones converge.
- **Taylor's theorem**: If $f : \mathbb{R} \to \mathbb{R}$ is C^{k+1} and $P_\alpha^k(x)$ is the k^{th} Taylor polynomial about $\alpha \in \mathbb{R}$, then for all $\beta \in \mathbb{R}$, there exists some $x \in (\alpha, \beta)$ such that

$$f(\beta) - P_\alpha^k(\beta) = \frac{(\beta-\alpha)^{k+1}}{(k+1)!} f^{(k+1)}(x)$$

  - Essentially a version of the mean value theorem (MVT) for higher-order derivatives.
  - We can use the term of the right side of the equals sign above to get a bound on the error of the Taylor polynomial.

- **Analytic** (function): A function $f : \mathbb{R} \to \mathbb{R}$ for which the Taylor polynomials converge (locally uniformly) to f.
- Non example: The C^∞ function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

  - An excellent exercise in real analysis is to check that for all k, the Taylor polynomial about 0 is 0.
  - If we take the Taylor polynomial at some point farther from zero, the polynomial will approximate f well up until zero, but then it will "hit a wall."
    - The point is that f is decaying more rapidly toward 0 than any polynomial possibly could, so the polynomial just thinks it's seeing 0.
- **Absolutely** (locally uniformly convergent power series): A power series $P(z) = \sum_{k=0}^{\infty} a_k z^k$ for which $A_N : \mathbb{C} \to \mathbb{R}$ locally uniformly converges, where

$$A_N(z) := \sum_{k=0}^{N} |a_k z^k|$$

- Absolute local uniform convergence allows you to reorder the terms in the polynomial.
  - It also explains why you cannot reorder the terms in the series $S = 1+1-1+1-1+\cdots$, i.e., why manipulating the order allows you to get any number: This series S does not converge absolutely!
  - Formally, if $\sigma : \mathbb{N} \to \mathbb{N}$ is a permutation and $\sum^\infty a_k$ converges absolutely, then $\sum^\infty a_{\sigma(k)}$ converges.
- Exercise: Show that

$$\sum_{k=0}^{\infty} z^k \to \frac{1}{1-z}$$

  converges absolutely locally uniformly on $\mathbb{D} = \{|z| < 1\}$.

  Proof. To prove this, we just have to show that $\sum^\infty |z|^k$ converges on $|z| < 1$. But it does so converge because this latter series is just a standard real geometric series. ☐

- This example generalizes somewhat into the following lemma.
- Lemma: Let $P(z)$ be a power series about 0. If there exists $z_1 \neq 0$ such that $|a_k z_1^k| \leq M$ for all k, then $P(z) = \sum a_k z^k$ converges on the disk $|z| < |z_1|$.

  Proof. Uses standard series convergence results from real analysis. See Fischer and Lieb (2012, pp. 15–16). ☐

- **Disk of convergence**: The largest disk centered at zero on which you converge.
- **Radius of convergence**: The radius of the disk of convergence. *Denoted by* $\boldsymbol{r}$.
- **Cauchy-Hadamard formula**: The radius of convergence is given by

$$r = (\limsup |a_k|^{1/k})^{-1}$$

  - We will be using this result on PSet 2.
  - We will also be proving it there!

- What are power series representations good for? Here's an example of how they can be applied to help with PSet 1, QA.4.
  - Question: For $|a| < 1$ and $\gamma(t) = \mathrm{e}^{it}$ a parameterization of a closed loop oriented counterclockwise, compute
$$\int_\gamma \frac{1}{z-a}\,\mathrm{d}z$$
  - Answer:
    - Since $|a| < 1$, we know that on γ, $|a/\gamma(t)| < 1$.
    - Thus, we have that
$$\begin{aligned}\int_\gamma \frac{1}{z-a}\,\mathrm{d}z &= \int_\gamma \frac{1}{z}\frac{1}{1-a/z}\,\mathrm{d}z\\ &= \int_\gamma \frac{1}{z}\sum_{k=0}^\infty \left(\frac{a}{z}\right)^k \mathrm{d}z\\ &= \int_\gamma \sum_{k=0}^\infty \frac{a^k}{z^{k+1}}\,\mathrm{d}z\\ &= \sum_{k=0}^\infty \int_\gamma \frac{a^k}{z^{k+1}}\,\mathrm{d}z\\ &= \cdots\\ &= \int_\gamma \frac{1}{z}\,\mathrm{d}z\end{aligned}$$
    - We have the second equality because the power series converges.
    - We have the fourth equality because of the lemma about integrable f_n and the fact that the power series converges.
    - The dots indicate some more steps that we will need to work out for ourselves on PSet 1.
- Lemma (from real analysis): If $f_n \to f$ locally uniformly and $f_n' \to g$ locally uniformly, then f is differentiable and $f' = g$.
  - This is true for both differentiable and holomorphic functions.
- Claim: This lemma implies that convergent power series are holomorphic.

Proof. If
$$f_N = \sum_{k=0}^N a_k z^k$$
then
$$f_N' = \sum_{k=0}^N k \cdot a_k z^{k-1}$$
We want to show that $\{f_N'\}$ converges (locally absolutely uniformly). Fischer and Lieb (2012) do this by hand. We can also use the Cauchy-Hadamard formula, which we will do presently.

Let's look at $\limsup(k \cdot a_k)^{1/k}$. But this is just equal to
$$\limsup |k \cdot a_k|^{1/k} \leq \limsup(|k|^{1/k}) \cdot \limsup(|a_k|^{1/k}) = 1 \cdot \limsup(|a_k|^{1/k}) = \limsup |a_k|^{1/k}$$
Moreover, equality holds because that $k^{1/k}$ factor just decays toward 1; think about how k increases linearly and the k^{th} root decays faster. $\square$

- Proposition: Any convergent power series is holomorphic (on its disk) and its derivative is also a power series with the same radius of convergence. It follows that power series are analytic functions and are C^∞.
- Spoiler: Every holomorphic function is analytic.
- Corollary: Power series representations are unique.
  1. If $P(z) = \sum a_k z^k$ is convergent, then
  $$a_k = \frac{1}{k!} P^{(k)}(0)$$
  2. If $P(z) = 0$ in a neighborhood of zero, then $a_k = 0$ for all k.
  3. If $P(z) = Q(z)$ (where $Q(z) = \sum b_k z^k$) in a neighborhood of 0, then $a_k = b_k$ for all k.
- Let's now return to the exponential function, which got this whole discussion started.
- We now know that the definition
  $$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$
  makes sense.
- By manipulating this power series, we can get lots of fun properties.
  1. $\exp(z) = [\exp(z)]'$.
     - We obtain this via term-by-term differentiability.
     - This is just our favorite formula $\mathrm{d}/\mathrm{d}t\ (\mathrm{e}^t) = \mathrm{e}^t$ from calculus.
  2. $\overline{\exp(z)} = \exp(\bar{z})$.
  3. $\exp(a + b) = \exp(a) \cdot \exp(b)$.
  4. $|\exp(z)| = \exp[\operatorname{Re}(z)]$.
- **Complex cosine**: The complex function defined as follows. *Denoted by* $\boldsymbol{\cos(z)}$. *Given by*
  $$\cos(z) := \frac{1}{2}(\mathrm{e}^{iz} + \mathrm{e}^{-iz})$$
- **Complex sine**: The complex function defined as follows. *Denoted by* $\boldsymbol{\sin(z)}$. *Given by*
  $$\sin(z) := \frac{1}{2i}(\mathrm{e}^{iz} - \mathrm{e}^{-iz})$$
- **Complex hyperbolic cosine**: The complex function defined as follows. *Denoted by* $\boldsymbol{\cosh(z)}$. *Given by*
  $$\cosh(z) := \cos(iz)$$
- **Complex hyperbolic sine**: The complex function defined as follows. *Denoted by* $\boldsymbol{\sinh(z)}$. *Given by*
  $$\sinh(z) := i \sin(iz)$$
- We also have
  $$\mathrm{e}^{iz} = \cos(z) + i \sin(z)$$
  - If z is real and in $[0, 2\pi]$, then this simplifies to Euler's formula
  $$\mathrm{e}^{i\theta} = \cos(\theta) + i \sin(\theta)$$
- Calderon draws some mappings of the exponential function but doesn't linger on what's going on.
- These are the preliminaries; now, we'll dive into the meat of the course.

2.3 Cauchy's Theorem

3/28:
- The last three classes have been real analysis with complex numbers; now we get into *complex* analysis.
- **Domain**: A connected, open set $U \subset \mathbb{C}$.
- Recall.
  - $\gamma : [a, b] \to \mathbb{C}$ is a piecewise C^1 curve.
  - $f : \mathbb{C} \to \mathbb{C}$ is continuous.
  - We define
$$\int_\gamma f \, \mathrm{d}z := \int_a^b f(\gamma(t)) \cdot \gamma'(t) \, \mathrm{d}t$$
  - FTC: If $f = F'$ (i.e., F is a **primitive** of f) on a domain $U \subset \mathbb{C}$, then for all paths γ in U,
$$\int_\gamma f \, \mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$$
- **Primitive** (of f): A differentiable function whose derivative is equal to the original function f. *Also known as* **antiderivative**, **indefinite integral**. *Denoted by* $\boldsymbol{F}$.
- Corollary to the FTC: If $f = F'$, then for any closed curve γ in U,
$$\int_\gamma f \, \mathrm{d}z = 0$$
  - To see why this is true intuitively, look at an example such as $f(z) = 1/z \in \mathcal{O}(\mathbb{C}^*)$, which doesn't have a primitive and
$$\int_\gamma \frac{1}{z} \, \mathrm{d}z \neq 0$$
- Example: Find a primitive of the convergent power series
$$P(z) = \sum_{k=1}^{\infty} a_k z^k$$
  - Via term-by-term integration, we obtain
$$\sum_{k=0}^{\infty} \frac{a_k}{k+1} z^{k+1}$$
- If γ is any closed loop in the disk of convergence,
$$\int_\gamma P(z) \, \mathrm{d}z = 0$$
  - It follows since they are defined in terms of convergent power series that for all closed loops γ,
$$\int_\gamma \mathrm{e}^z \, \mathrm{d}z = \int_\gamma \sin(z) \, \mathrm{d}z = \int_\gamma \cos(z) \, \mathrm{d}z = 0$$
- Question: When is there a primitive?
  - $f : \mathbb{R} \to \mathbb{R}$ continuous always has a primitive by the FTC, specifically that defined by
$$F(x) := \int_a^x f(t) \, \mathrm{d}t$$
which is differentiable with $F' = f$.

- Proposition: If $f : U \to \mathbb{C}$ is continuous and $\int_\gamma f \, \mathrm{d}z = 0$ for every closed loop in U, then f has a primitive on U.

  Proof. Let's try the most naïve thing: The FTC. Consider a domain.

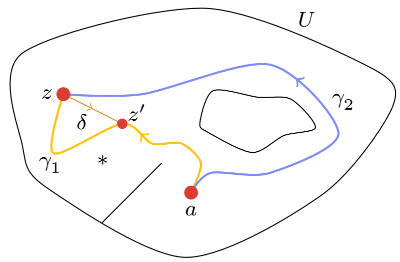
  

  Figure 2.1: Continuous and zero closed-loop integrals implies integrable.

  Doesn't have to be simply connected; it can have a **hole**, **slit**, and/or **puncture**. Essentially, to define $F(z)$, choose $a \in U$ and γ connecting a and z and define

  $$F(z) = \int_\gamma f \, \mathrm{d}z$$

  Claim: This definition is well-defined regardless of the choice of a and γ. In particular, the integral is independent of choice of γ because any two γ can be paired into a closed loop, and we have by hypothesis that the integral over any closed loop is zero.

  We now need to show that F is differentiable with $F' = f$. Take z, z' close enough that they can be connected by a straight line path δ. Consider

  $$\lim_{z' \to z} \frac{F(z') - F(z)}{z' - z}$$

  Now we know that

  $$F(z') - F(z) = \int_\delta f \, \mathrm{d}z$$

  Let $\gamma : [0, 1] \to \mathbb{C}$ be defined by $t \mapsto tz' + (1 - t)z$; a parameterization we can choose arbitrarily. Then

  $$F(z') - F(z) = \int_\delta f \, \mathrm{d}z = \int_0^1 f[tz + (1 - t)z'] \cdot (z' - z) \, \mathrm{d}t$$

  so dividing both sides by $z' - z$ and taking the limit yields

  $$\begin{aligned}
  \lim_{z' \to z} \frac{F(z') - F(z)}{z' - z} &= \lim_{z' \to z} \int_0^1 f[tz + (1 - t)z'] \, \mathrm{d}t \\
  &= \int_0^1 \lim_{z' \to z} f(tz + z' - tz') \, \mathrm{d}t \\
  &= \int_0^1 f(tz + z - tz) \, \mathrm{d}t \\
  &= \int_0^1 f(z) \, \mathrm{d}t \\
  &= f(z) \int_0^1 \mathrm{d}t \\
  &= f(z)
  \end{aligned}$$

  and we have everything we wanted. □

- What allows us to interchange the limit and the integral in the final set of equations?
  - Roughly speaking, uniform convergence.
- **Star-shaped** (domain): A domain $U \subset \mathbb{C}$ for which there exists $a \in U$ such that for all $z \in U$, the segment $a \to z$ is in U.

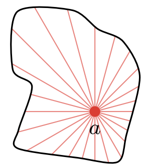


Figure 2.2: Star-shaped domain.

  - There are star-shaped regions that are not **convex**, such as the one in Figure 2.2!
    - Convex implies star-shaped, but not vice versa.
  - Examples of domains that are *not* star-shaped.
    1. The annulus of two circles.
    2. Puncturing the unit disk.
  - Star-shaped implies **simply connected**.
  - Star-shaped is nice because we don't have to check every single curve; see the following lemma.
- Lemma: If U is star-shaped and for every triangle with one vertex at a, we have $\int_{\triangle} f \, \mathrm{d}z = 0$, then F has a primitive in U.

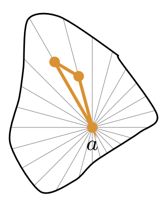


Figure 2.3: A triangle in a star-shaped domain.

Proof. What should be our candidate for $F(z)$? Define

$$F(z) = \int_{\gamma} f \, \mathrm{d}z$$

where γ is the line segment from $a \to z$ that we know exists because U is star-shaped.

We now have to show that F is holomorphic with $F' = f$, but we just do this as before by constructing a "closed loop," except our closed loop this time will just be a triangle as drawn in Figure 2.3. □

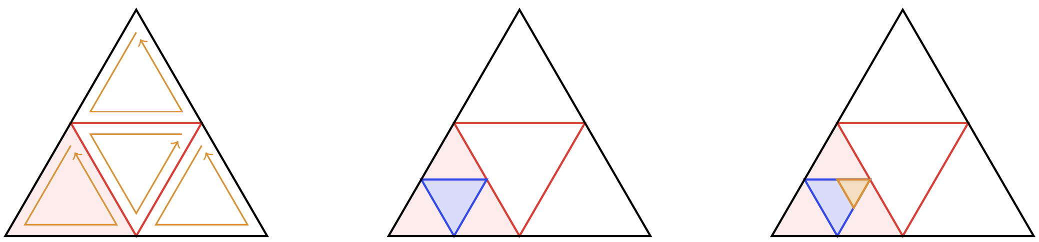
- With these definitions, we now state and prove one of the two main theorems in this class.
- **Cauchy Integral Theorem**: Suppose U is a star-shaped domain and $f : U \to \mathbb{C}$ is holomorphic. Then $\int_{\gamma} f \, \mathrm{d}z = 0$ for any closed loop γ in U.
  - Whereas the FTC says if you have an *integral*, then the integral around a closed loop is zero. This theorem says that if you have a *derivative*, then the integral around a closed loop is zero.
  - This is Round 1 of the theorem. In round 2, we'll swap the "star-shaped" hypothesis for "simply connected."

- Today we're at least going to prove this, and possibly look at an application, too. If we don't get to the application today, we'll see it next Tuesday.
- Proof idea: Prove that f has a primitive.

  Proof. In order to prove this theorem, we'll use the preceding lemma. Thus, all we need to show is that for every triangle with one vertex on the center of the star, $\int_\triangle f\,\mathrm{d}z = 0$. Since we only have to check this for *triangles*, we can use a really lovely result called **Goursat's lemma**. □

- **Goursat's lemma**: If f is holomorphic in a neighborhood of a triangle including the interior, then $\int_\triangle f\,\mathrm{d}z = 0$.

  Proof. Idea: Estimate the integral.



(a) Picking $\triangle_1$. (b) Picking $\triangle_2$. (c) Picking $\triangle_3$.

Figure 2.4: Proving Goursat's lemma.

Fix some $z_0 \in \blacktriangle$ (the exact value will be determined later). We know that f is holomorphic at z_0, which implies that there exists a linear approximation

$$f(z) = \underbrace{f(z_0) + f'(z_0)\cdot(z - z_0)}_{\text{linear}} + E(z)\cdot(z - z_0)$$

where our error function $E(z) \to 0$ as $z \to z_0$. Now the underlined linear portion above is a (stupid) power series, but since it technically *is* a "convergent power series," our previous results imply that it has primitives. In particular, its integral around a closed loop (like a triangle) will be zero. This means that

$$\int_\triangle f\,\mathrm{d}z = \underbrace{\int_\triangle [f(z_0) + f'(z_0)\cdot(z - z_0)]\,\mathrm{d}z}_{0} + \int_\triangle E(z)\cdot(z - z_0)\,\mathrm{d}z = \int_\triangle E(z)\cdot(z - z_0)\,\mathrm{d}z$$

Goursat's idea: Choose a good z_0. To do this, we'll subdivide the original black triangle (see Figure 2.4a) by choosing midpoints and breaking it into four triangles. Keep using the counterclockwise orientation in all cases. All of the red segments get cancelled out from integrating in both directions, so

$$\int_{\triangle_0} f\,\mathrm{d}z = \sum \int_{4\text{ sub }\triangle\text{'s}} f\,\mathrm{d}z$$

Choose $\triangle_1$ in first stage such that $|\int_{\triangle_1} f\,\mathrm{d}z\,|$ is the greatest among the first stage sub-triangles. Thus,

$$\left|\int_\triangle f\,\mathrm{d}z\right| \leq 4\cdot\left|\int_{\triangle_1} f\,\mathrm{d}z\right|$$

Now subdivide $\triangle_1$ and choose $\triangle_2$ the same way (see Figure 2.4b), so that

$$\left|\int_\triangle f\,\mathrm{d}z\right| \leq 4\cdot\left|\int_{\triangle_1} f\,\mathrm{d}z\right| \leq 4\cdot 4\cdot\left|\int_{\triangle_2} f\,\mathrm{d}z\right|$$

Iterating this process, we obtain

$$\left|\int_{\triangle} f\,\mathrm{d}z\right| \leq 4^n \cdot \left|\int_{\triangle_n} f\,\mathrm{d}z\right|$$

First thing to observe:

$$\operatorname{len}(\triangle_n) = 2^{-n} \cdot \operatorname{len}(\triangle_0) \qquad\qquad \operatorname{diam}(\triangle_n) = 2^{-n} \cdot \operatorname{diam}(\triangle_0)$$

Now fix $\varepsilon > 0$ and take n big enough such that on all of $\triangle_n$,

$$|E(z)| < \frac{\varepsilon}{\operatorname{len}(\triangle_0) \cdot \operatorname{diam}(\triangle_0)}$$

Choose $z_0 \in \bigcap_{n=1}^{\infty} \blacktriangle_n$. Then

$$\begin{aligned}
\left|\int_{\triangle} f\,\mathrm{d}z\right| &\leq 4^n \cdot \left|\int_{\triangle_n} f\,\mathrm{d}z\right| \\
&= 4^n \cdot \left|\int_{\triangle_n} E(z) \cdot (z - z_0)\,\mathrm{d}z\right| \\
&\leq 4^n \cdot \operatorname{len}(\triangle_n) \cdot \max_{\triangle_n} |E(z) \cdot (z - z_0)| \\
&= 4^n \cdot \operatorname{len}(\triangle_n) \cdot \max |E(z)| \cdot \max |z - z_0| \\
&\leq 4^n \cdot \operatorname{len}(\triangle_n) \cdot \operatorname{diam}(\triangle_n) \cdot \max |E(z)| \\
&= 4^n \cdot 2^{-n} \operatorname{len}(\triangle_0) \cdot 2^{-n} \operatorname{diam}(\triangle_0) \cdot \max |E(z)| \\
&= \operatorname{len}(\triangle_0) \cdot \operatorname{diam}(\triangle_0) \cdot \max |E(z)| \\
&< \varepsilon
\end{aligned}$$

Since we can choose ε arbitrarily small, we can thus send the original integral of f over $\triangle$ to zero. $\square$

- We now end class with an example of how complex analysis can be useful, even in calculus!
- Example: Evaluate the following **Dirichlet integral** using complex analysis.

$$\int_0^\infty \frac{\sin(x)}{x}\,\mathrm{d}x$$

  - We will do so via a focused analysis of the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$z \mapsto \frac{\mathrm{e}^{iz}}{z}$$

    - This function is not holomorphic everywhere, but it is on the punctured plane $\mathcal{O}(\mathbb{C}^*)$.
    - However, we only need the upper half $\mathcal{O}(\mathbb{H})$ presently.
  - More specifically, define U to be a domain containing γ as defined as in Figure 2.5.

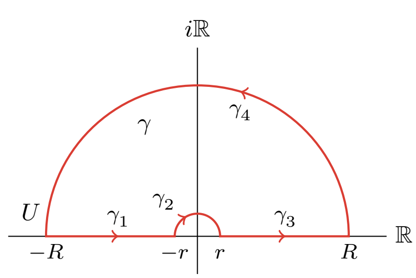


Figure 2.5: Dirichlet integral.

- By the Cauchy integral theorem and our decomposition of γ,

$$0 = \int_\gamma f(z)\,\mathrm{d}z = \sum_{i=1}^{4} \int_{\gamma_i} f\,\mathrm{d}z$$

- We now integrate the segments one at a time.
  - γ_1 and γ_3: Recalling our definition of $\sin(z)$ from last class, we have that

$$\begin{aligned}
\int_{\gamma_1\gamma_3} \frac{\mathrm{e}^{ix}}{x}\,\mathrm{d}x &= \int_{-R}^{-r} \frac{\mathrm{e}^{ix}}{x}\,\mathrm{d}x + \int_r^R \frac{\mathrm{e}^{ix}}{x}\,\mathrm{d}x \\
&= \int_{-r}^{-R} -\frac{\mathrm{e}^{ix}}{x}\,\mathrm{d}x + \int_r^R \frac{\mathrm{e}^{ix}}{x}\,\mathrm{d}x \\
&= \int_{r}^{R} -\frac{\mathrm{e}^{-ix}}{x}\,\mathrm{d}x + \int_r^R \frac{\mathrm{e}^{ix}}{x}\,\mathrm{d}x \\
&= \int_r^R \frac{\mathrm{e}^{ix} - \mathrm{e}^{-ix}}{x}\,\mathrm{d}x \\
&= 2i \int_r^R \frac{\sin(x)}{x}\,\mathrm{d}x
\end{aligned}$$

  - γ_2: We can explicitly compute this integral as $r \to 0$, using the parameterization $\gamma_2 : [0,\pi] \to \mathbb{C}$ defined by $\theta \mapsto r\mathrm{e}^{i(\pi-\theta)}$.

$$\begin{aligned}
\lim_{r\to 0} \int_{\gamma_2} \frac{\mathrm{e}^{iz}}{z}\,\mathrm{d}z &= \lim_{r\to 0} \int_0^\pi \frac{\mathrm{e}^{ir\mathrm{e}^{i(\pi-\theta)}}}{r\mathrm{e}^{i(\pi-\theta)}} \cdot -ir\mathrm{e}^{i(\pi-\theta)}\,\mathrm{d}\theta \\
&= -i \lim_{r\to 0} \int_0^\pi \mathrm{e}^{ir\mathrm{e}^{i(\pi-\theta)}}\,\mathrm{d}\theta \\
&= -i \int_0^\pi \mathrm{e}^0\,\mathrm{d}\theta \\
&= -i\pi
\end{aligned}$$

  - γ_4: We need to bound the $R\mathrm{e}^{i\theta}$ term as $R \to \infty$; see his notes!

$$\int_0^\pi \mathrm{e}^{iR\mathrm{e}^{i\theta}} i\mathrm{d}\theta \to 0$$

- Therefore, by transitivity,

$$\begin{aligned}
2i \int_0^\infty \frac{\sin(x)}{x}\,\mathrm{d}x - i\pi &= 0 \\
\int_0^\infty \frac{\sin(x)}{x}\,\mathrm{d}x &= \frac{\pi}{2}
\end{aligned}$$

2.4 Office Hours

- PSet 1, QA.4: Are a, b real or complex?
  - They can be complex.
  - Hint for this problem: Think about QA.3.
- PSet 1, QB.2: As in, only "takes on" real values, i.e., is a function of the form $f : U \to \mathbb{R}$?
  - Yes.
- We have to give him a heads up before the PSet due date that we want to use a PSet extension.

2.5 Chapter I: Analysis in the Complex Plane

From Fischer and Lieb (2012).

Section I.3: Uniform Convergence and Power Series

4/5:

- Definition of **convergent**, **absolutely convergent**, **comparison test**, **ratio test**, **root test**, and **geometric series test**.
- **Unconditionally** (convergent $\{f_n\}$): A series that will remain convergent (with the same sum) upon any reordering of its terms.
  - Absolutely convergent implies unconditionally convergent.
- Hint at hypergeometric series here?
- Definition of **pointwise convergent**, **uniformly convergent**, and **compactly convergent**.
- Covers implications of uniform convergence from class.
- Introduction of the **Cauchy convergence test**.
- **Majorant test**: If $\sum_{k=0}^{\infty} a_k$ is a convergent series with positive terms and if for almost all k and all $z \in M$ we have $|f_k(z)| \leq a_k$, then $\sum_{k=0}^{\infty} f_k$ is absolutely uniformly convergent on M.
- **Power series** (with base point z_0 and coefficients a_k): An infinite series of the following form, where $a_k \in \mathbb{C}$ ($k = 0, \dots, \infty$) and $z_0 \in \mathbb{C}$. *Denoted by* $\boldsymbol{P(z - z_0)}$. *Given by*

$$P(z - z_0) := \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

- Definition of the **radius of convergence** and **disk of convergence**.
- **Nowhere convergent** (power series): A power series with $r = 0$.
- **Everywhere convergent** (power series): A power series with $r = \infty$.
- Proposition 3.5: Assume that for some point $z_1 \neq 0$, the terms of the power series $\sum_{k=0}^{\infty} a_k z^k$ are bounded, that is $|a_k z_1^k| \leq M$ independently of k. Then the series converges absolutely locally uniformly in the disk $D_{|z_1|}(0) = \{z : |z| < |z_1|\}$.

  Proof. By hypothesis,

$$|a_k||z_1|^k \leq M$$

  for all k. Pick z_2 such that $0 < |z_2| < |z_1|$. Then for all $|z| \leq |z_2|$, we have

$$|a_k||z|^k \leq |a_k||z_2|^k = |a_k||z_1|^k \left|\frac{z_2}{z_1}\right|^k \leq Mq^k$$

  where $q = |z_2/z_1| < 1$. By the geometric series test, $\sum_{k=0}^{\infty} Mq^k$ converges. Thus, by the majorant test, $\sum_{k=0}^{\infty} a_k z^k$ converges absolutely uniformly on $D_{|z_2|}(0)$. Progressively increasing $|z_2|$ guarantees local absolute uniform convergence over all of $D_{|z_1|}(0)$. ☐

- Definition of the **Cauchy-Hadamard formula**.
- Fischer and Lieb (2012) prove that power series are holomorphic.
- Fischer and Lieb (2012) prove the identity theorem (for power series).
- Compute the closed form of the derivatives of power series by differentiating the geometric series.

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)z^{n-k} = \frac{\mathrm{d}^k}{\mathrm{d}z^k} \sum_{n=0}^{\infty} z^n = \frac{\mathrm{d}^k}{\mathrm{d}z^k} \frac{1}{1-z} = \frac{k!}{(1-z)^{k+1}}$$

- Concludes with a few questions that will be answered in the coming sections.

Section I.4: Elementary Functions

- **Elementary functions**: The exponential functions, trigonometric functions, and hyperbolic functions.
- Complex analysis unifies the elementary functions as special cases of the complex exponential function, making clear that they do have more to do with each other than real analysis would suggest.
- Fischer and Lieb (2012) define $\exp z$ and beautifully build its properties almost solely from this axiom.
- Summary of the properties of the exponential function (Fischer & Lieb, 2012, p. 20).
  - "The exponential function is a group homomorphism from $\mathbb{C}$ into $\mathbb{C}^*$ that maps the subgroups $\mathbb{R}$ into $\mathbb{R}_{>0}$ and the subgroup $i\mathbb{R}$ into $\mathbb{S}$."
  - "The group operation on $\mathbb{C}$, $\mathbb{R}$, and $i\mathbb{R}$ is addition and the group operation on $\mathbb{C}^*$, $\mathbb{R}_{>0}$, and $\mathbb{S}$ is multiplication."
  - "The three homomorphisms $\exp : \mathbb{C} \to \mathbb{C}^*$, $\exp : \mathbb{R} \to \mathbb{R}_{>0}$, and $\exp : i\mathbb{R} \to \mathbb{S}$ are surjective."
- Notable new(ish) properties:
  - Since $|\mathrm{e}^z| = \mathrm{e}^{\operatorname{Re} z}$, we can see that "the exponential function... maps vertical lines $\operatorname{Re} z = c$ to circles" (Fischer & Lieb, 2012, p. 20).
  - In particular, $\exp(i\mathbb{R}) = \partial\mathbb{D} = \mathbb{S}$.
- Investigation of the **kernel** of $\exp : \mathbb{C} \to \mathbb{C}^*$.
  - Notably, Fischer and Lieb (2012) prove that it contains only the multiples of $2\pi i$.
- Fischer and Lieb (2012) defines π via the complex exponential function and Euler's identity lol!
- Proposition 4.4:
  i. The exponential function is periodic with period $2\pi i$.
  ii. If a domain contains at most one member of each congruence class modulo $2\pi i$, then the exponential function maps it biholomorphically onto its image in $\mathbb{C}^*$.
- The key mapping properties of $\exp z$.
  1. The mapping $t \mapsto \mathrm{e}^t$ maps $\mathbb{R}$ onto $\mathbb{R}_{>0}$ bijectively.
  2. The mapping $t \mapsto \mathrm{e}^{it}$ maps $[0, 2\pi)$ onto $\mathbb{S}$ bijectively.
- Implications.
  - A horizontal line $z = x + iy_0$ is mapped bijectively onto the open ray L_{y_0} that begins at 0 and passes through the point e^{iy_0} on the unit circle.
  - A vertical line $z = x_0 + iy$ is mapped onto the circle centered at 0 and of radius e^{x_0}.
    - An interval of length less than 2π on this line is mapped injectively into this circle.
  - Every half-open horizontal strip

    $$S_{y_0} = \{z = x + iy : y_0 \leq y < y_0 + 2\pi\}$$

    is mapped bijectively onto $\mathbb{C}^*$.
    - The line $z = x + iy_0$ is mapped to the ray L_{y_0}.
    - The remaining open strip (i.e., S_{y_0} minus its lower boundary) is mapped biholomorphically onto the "slit" plane $\mathbb{C}^* \setminus L_{y_0}$.
- Definition of the **complex cosine**, **complex sine**, **complex hyperbolic cosine**, and **complex hyperbolic sine** functions.

- Proposition 4.5:

   i. The functions $\cos z$ and $\sin z$ are periodic with period 2π; $\cosh z$ and $\sinh z$ are periodic with period $2\pi i$.

   ii.
$$\frac{\mathrm{d}}{\mathrm{d}z}(\sin z) = \cos z \qquad \frac{\mathrm{d}}{\mathrm{d}z}(\cos z) = -\sin z$$
$$\frac{\mathrm{d}}{\mathrm{d}z}(\sinh z) = \cosh z \qquad \frac{\mathrm{d}}{\mathrm{d}z}(\cosh z) = \sinh z$$

   iii.
$$\begin{aligned} \sin(z+w) &= \sin z \cos w + \cos z \sin w \\ \cos(z+w) &= \cos z \cos w - \sin z \sin w \\ \sinh(z+w) &= \sinh z \cosh w + \cosh z \sinh w \\ \cosh(z+w) &= \cosh z \cosh w + \sinh z \sinh w \end{aligned}$$

   iv. For each of the above functions, $f(\bar{z}) = \overline{f(z)}$.

   v.
$$\cos z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} \qquad \sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$$
$$\cosh z = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} \qquad \sinh z = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}$$

   vi. $\mathrm{e}^{iz} = \cos z + i \sin z$.

   vii.
$$\sin^2 z + \cos^2 z = 1 \qquad \cosh^2 z - \sinh^2 z = 1$$

- Decomposition of e^z into its real and imaginary parts:
$$\mathrm{e}^z = \mathrm{e}^{x+iy} = \mathrm{e}^x(\cos y + i \sin y)$$

- The zeroes of the complex sine are just those of the real sine; $\sin z \neq 0$ for any nonreal z.

- Introduction of the **roots of unity**.

- The remaining trigonometric and hyperbolic functions are defined in the usual way, are holomorphic everywhere except at the zeros of their denominators, and have periods π or πi (in the hyperbolic case).
$$\tan z = \frac{\sin z}{\cos z} \qquad \cot z = \frac{\cos z}{\sin z}$$
$$\tanh z = \frac{\sinh z}{\cosh z} \qquad \coth z = \frac{\cosh z}{\sinh z}$$

- Preview: We will show later that this extension of the real analytic functions to holomorphic functions on the complex plane is unique, not arbitrary.

Section I.5: Integration

4/6:

- We can define integration on *generally* continuous functions, i.e., recall that a function need only be **piecewise continuous** in order to be integrable.
- Allusion to **Praffian forms**, e.g., referring to $f(z)\,\mathrm{d}z$ as a "one-form."
- Integration in the mold of Figure 1.2.
- **Piecewise continuous** (function): A function $f:[a,b]\to\mathbb{C}$ for which there exists a partition $a=t_0<\cdots<t_n=b$ of the interval $[a,b]$ into subintervals $[t_{k-1},t_k]$ such that the restriction of f to (t_{k-1},t_k)...
  1. Is continuous for all $k=1,\dots,n$;
  2. Admits a continuous extension to the endpoints t_{k-1} and t_k.
- Define the **functional**
  $$I(f)=\int_a^b f(t)\,\mathrm{d}t$$
  - This functional is complex linear.
  - It also satisfies
    $$I(\bar{f})=\overline{I(f)}$$
  - Therefore, we have that
    $$I(\operatorname{Re}f)=\operatorname{Re}I(f)\qquad\qquad I(\operatorname{Im}f)=\operatorname{Im}I(f)$$
- $|\int_a^b f(t)\,\mathrm{d}t\,|\leq\int_a^b|f(t)|\,\mathrm{d}t$.
- The central theorems of integral calculus (that is, the FTC and u-substitution) still hold:
  - *FTC*: Let f be continuously differentiable on $[a,b]$. Then
    $$\int_a^b f'(t)\,\mathrm{d}t=f(b)-f(a)$$
  - *u-substitution*: Let f be piecewise continuous on $[a,b]$ and let $h:[c,d]\to[a,b]$ be a continuous, nondecreasing, and piecewise continuously differentiable bijection. Then
    $$\int_a^b f(s)\,\mathrm{d}s=\int_c^d f(h(t))h'(t)\,\mathrm{d}t$$
- **Path of integration**: A continuous and piecewise continuously differentiable map $\gamma:[a,b]\to U$.
- **Parameter interval**: The functional domain of a path of integration.
- Proposition 5.3: Paths of integration are rectifiable, and their length is given by
  $$\operatorname{len}(\gamma)=\int_a^b|\gamma'(t)|\,\mathrm{d}t$$
- Integration in the mold of Figure 1.4.
- **Segment** (joining a to b): The path of integration defined as follows, where $a,b\in\mathbb{C}$ and $\gamma:[0,1]\to\mathbb{C}$ is the path defined by $\gamma(t)=a+t(b-a)$. *Denoted by* $\boldsymbol{[a,b]}$. *Given by*
  $$[a,b]:=\gamma$$

- **Circle** (with center z_0 and radius r): The path of integration defined as follows, where $z_0 \in \mathbb{C}$, $r > 0$, and $-\pi \leq t \leq \pi$. *Also known as* **positively oriented circle**. *Denoted by* $\boldsymbol{\kappa(r; z_0)}$. *Given by*

$$\kappa(r; z_0)(t) := z_0 + r\mathrm{e}^{it}$$

- $\boldsymbol{\partial D_r(z_0)}$: The trace of the positively oriented circle with center z_0 and radius r.
- Fischer and Lieb (2012) uses Proposition 5.3 to prove that the length of a circle is $2\pi r$, thus connecting the abstractly defined π of Section I.4 with the definition with which we're familiar.
- An important integral.

$$\int_{\kappa(r;z_0)} \frac{\mathrm{d}z}{z - z_0} = 2\pi i$$

- Discussion of a **parameter transformation** and **reparameterization**.
  - Simply, these concepts describe the same path using two different functions with on different intervals $[a, b]$.
  - Fischer and Lieb (2012) use these concepts to prove that integrals are invariant under reparameterization.
  - "In light of this, we will identify two paths of integration if they are reparametrizations of each other. To be pedantic: a path of integration is an equivalence class of (parametrized) paths of integration with respect to the aforementioned equivalence relation" (Fischer & Lieb, 2012, p. 28).
- Sum rule along consecutive paths.
- You can **subdivide** paths with partitions and consider the **reverse** path.
  - The integral over the reverse path is the negative integral over the path.
- **Constant** paths have zero integrals.
- **Closed** (path): A path for which the initial and end points coincide.
- More common integrals.

$$\int_{\kappa(1;0)} z^n \,\mathrm{d}z = i\int_{-\pi}^{\pi} \mathrm{e}^{i(n+1)t} \,\mathrm{d}t = 0 \qquad (n \neq -1)$$
$$\int_{\kappa(1;0)} \bar{z} \,\mathrm{d}z = \int_{\kappa(1;0)} \frac{\mathrm{d}z}{z} = 2\pi i$$

- The standard estimate: Let f be continuous on the trace of γ. Then

$$\left| \int_\gamma f(z) \,\mathrm{d}z \right| \leq \max_{z \in \operatorname{tr}\gamma} |f(z)| \cdot \operatorname{len}(\gamma)$$

  Proof. Given. □

- Limits and integrals are interchangeable for uniformly convergent sequences of functions.

  Proof. Given. □

- Fischer and Lieb (2012) state some multivariable carryovers from real analysis, including **Fubini's theorem**.

Section I.6: Several Complex Variables

- The concepts given elsewhere in this chapter are briefly extended to the multivariable case.

2.6 Chapter II: The Fundamental Theorems of Complex Analysis

From Fischer and Lieb (2012).

Section II.1: Primitive Functions

- Definition of **primitive**.
- Complex FTC.
  - Implication: If f has a primitive, its integral depends only on the endpoints of the path of integration, not the path in between.
- In $\mathbb{R}$, every function that's continuous on an interval has a primitive; in $\mathbb{C}$, this is not true.
  - Instead, we need f continuous on U *and* $\int_\gamma f(z)\,\mathrm{d}z = 0$ for every closed path of integration $\gamma \subset U$.
  - Proof given as in class (see Figure 2.1).
- Definition of **star-shaped**.
- Zero integral over triangles implies primitive.

  Proof. Given. □

Section II.2: The Cauchy Integral Theorem

- Goursat's lemma and proof.
- The Cauchy Integral Theorem (for star-shaped domains).
  - "This is the 'fundamental theorem of complex analysis'" (Fischer & Lieb, 2012, p. 43).

Week 3

Fundamental Theorems

3.1 Cauchy Integral Formula

4/2: • Last time.

  – Definition of star-shaped.
  – Cauchy integral theorem: U star-shaped, $f \in \mathcal{O}(U)$ implies $\int_\gamma f \, \mathrm{d}z = 0$ for all closed (piecewise C^1) loops γ.
    1. It suffices to prove the theorem for triangles.
    2. Apply Goursat's lemma to treat this triangle case.
  – For Goursat's lemma, apply a clever estimate. Subdivide the big triangle into smaller ones, then

$$\left|\int_{\text{small } \triangle} f \, \mathrm{d}z\right| = \left|\int_a^b f(\gamma(t)) \cdot \gamma'(t) \, \mathrm{d}t\right| \leq \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)| \, \mathrm{d}t \leq \max_{z \in \partial\triangle} |f(z)| \cdot \operatorname{len}(\partial\triangle)$$

• We'll now do a couple exercises to practice applying the concepts we've learned so far.

• TPS: Suppose $f \in \mathcal{O}(\mathbb{C})$. Let $A := \int_0^1 f(x) \, \mathrm{d}x = F(1) - F(0)$, where to be clear we take the integral along the real axis. Let γ be the piecewise C^1 path in yellow in Figure 3.1. What is $\int_\gamma f \, \mathrm{d}z$?

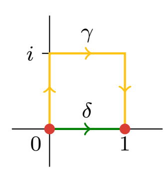


Figure 3.1: Practicing with the Cauchy Integral Theorem (1).

  – Define δ such that $\int_\delta f \, \mathrm{d}z = \int_0^1 f(x) \, \mathrm{d}x$.
  – Then $\delta^{-1}\gamma$ is a closed loop, so

$$0 = \int_{\delta^{-1}\gamma} f \, \mathrm{d}z$$

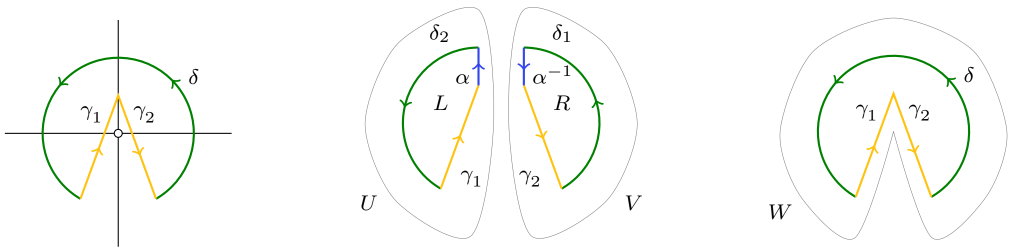
  – Additionally, we have by definition that

$$\int_{\delta^{-1}\gamma} f \, \mathrm{d}z = \int_\gamma f \, \mathrm{d}z - \int_\delta f \, \mathrm{d}z$$

  – Thus, by transitivity and a bit of algebraic rearrangement,

$$\int_\gamma f \, \mathrm{d}z = \int_\delta f \, \mathrm{d}z = A$$

- TPS: Now suppose $f \in \mathcal{O}(\mathbb{C}^*)$, where we must note that $\mathbb{C}^*$ is *not* star-shaped due to the hole at the origin. Suppose we know that $\int_\delta f \, \mathrm{d}z = 0$. What is $\int_\gamma f \, \mathrm{d}z$? The paths γ and δ are visualized in Figure 3.2a. *Hint*: It should be $-\int_\delta f \, \mathrm{d}z$.



(a) Original setup. (b) Solution: Break in two. (c) Solution: Pizza pie.

Figure 3.2: Practicing with the Cauchy Integral Theorem (2).

 - There are multiple ways to visualize why the domain does not see the hole/puncture. Here are some examples.
 - Solution 1 (Figure 3.2b): Cut the loop into two loops in star-shaped domains and add them.
   - Draw a straight-line path α from $i/2$ up to i.
   - Since U and V are both star-shaped domains, consecutive applications of the Cauchy Integral Theorem imply that

$$\int_{\delta_2\gamma_1\alpha} f \, \mathrm{d}z = \int_L f \, \mathrm{d}z = 0 \qquad \int_{\delta_1\alpha^{-1}\gamma_2} f \, \mathrm{d}z = \int_R f \, \mathrm{d}z = 0$$

   - Additionally, we know that the sum of the two integrals above is equal to the integral along the entire path in Figure 3.2a because the α and α^{-1} portions cancel. Mathematically,

$$\int_\delta f \, \mathrm{d}z + \int_\gamma f \, \mathrm{d}z = \int_{\delta\gamma} f \, \mathrm{d}z = \underbrace{\int_L f \, \mathrm{d}z}_{0} + \underbrace{\int_R f \, \mathrm{d}z}_{0} = 0$$

   - Therefore,

$$\int_\gamma f \, \mathrm{d}z = -\int_\delta f \, \mathrm{d}z = 0$$

 - Solution 2 (Figure 3.2c): The pizza pie is star-shaped!
   - We can actually draw a star-shaped domain W encapsulating the entire path $\delta\gamma$.
   - Thus, by the Cauchy Integral Theorem,

$$\int_{\delta\gamma} f \, \mathrm{d}z = 0$$

   - From here, we may proceed as before through

$$\begin{aligned}\int_\gamma f \, \mathrm{d}z + \int_\delta f \, \mathrm{d}z &= 0\\ \int_\gamma f \, \mathrm{d}z &= -\int_\delta f \, \mathrm{d}z = 0\end{aligned}$$

- We now investigate a more general principal than the Cauchy integral theorem called **homotopy**.
  - Algebraic topologists would be insulted by the definition of this term that Calderon is about to give, but it will suffice for our purposes.
- **Homotopic** (paths): Two paths $\gamma, \tilde{\gamma} \subset U$ a domain such that $\tilde{\gamma}$ is obtained from γ by modifying γ on a small disk $D \subset U$, keeping the endpoints fixed.

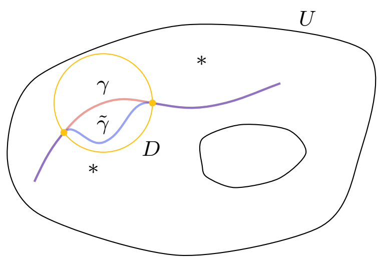
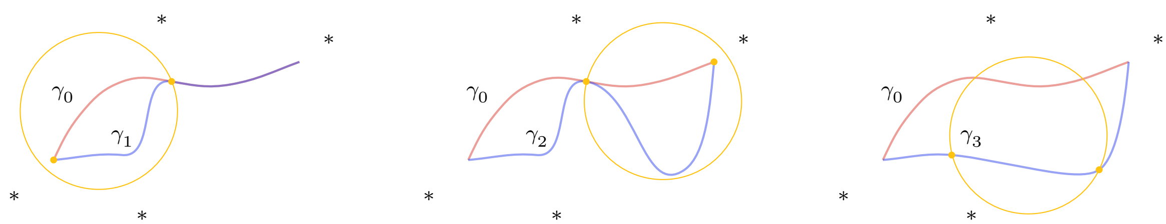


Figure 3.3: Homotopic paths.

- More generally, γ and $\tilde{\gamma}$ are **homotopic** if there exists a finite sequence $\gamma = \gamma_0, \gamma_1, \dots, \gamma_n = \tilde{\gamma}$ such that $\gamma_i \to \gamma_{i+1}$ is obtained by modifying on a small ball.



(a) Stage 1. (b) Stage 2. (c) Stage 3.

Figure 3.4: A more general homotopy.

- Claim/TPS: This argument shows that if γ and $\tilde{\gamma}$ are homotopic in U and $f \in \mathcal{O}(U)$, then

$$\int_\gamma f \, \mathrm{d}z = \int_{\tilde{\gamma}} f \, \mathrm{d}z$$

Hint: Just go one little bump at a time.

Proof. The start- and endpoints of the bump form a closed loop within a ball (a star-shaped domain), so the bump loop integrates to zero by the CIT. Thus, the integrals within the ball are the same. Additionally, the paths are literally the same outside of the bump, so the integrals there are the same, too. Therefore, the overall integrals are the same, too. □

- Reality check: Let $f \in \mathcal{O}(\mathbb{C}^*)$. As a particular example, consider $f(z) = 1/z$. Now we know that

$$\int_\circ \frac{1}{z} \, \mathrm{d}z = 2\pi i \neq 0$$

even though we can break the unit circle into the sum of two paths. What's going on?

  - The paths are not homotopic; we can't pull them through the hole in the plane.
  - If we consider the upper hemi-circle and the lower hemi-circle, the two cannot be continuously deformed into each other because we always get stuck at the puncture.

- We now prove a slightly stronger version of the Cauchy integral theorem.
- Corollary: Let U be any domain, D be a disk in U, and $z \in \mathring{D}$. Suppose $f \in \mathcal{O}(U \setminus \{z\})$ and is bounded near z. Then

$$\int_{\partial D} f \, \mathrm{d}z = 0$$

Proof. Step 1: Use homotopy.

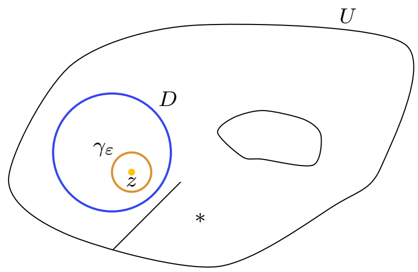


Figure 3.5: Bounded holomorphic functions integrate to zero on disk boundaries.

Via the above claim,

$$\int_{\partial D} f \, \mathrm{d}z = \int_{\gamma_\varepsilon} f \, \mathrm{d}z$$

where γ_ε is a circle around z within the region where f is bounded[1].

Step 2: We have that

$$\left| \int_{\gamma_\varepsilon} f \, \mathrm{d}z \right| \leq \max_{z \in \gamma_\varepsilon} |f(z)| \cdot \operatorname{len}(\gamma_\varepsilon)$$

Since f is bounded near z, the maximum is finite. Additionally, the length term is just $2\pi\varepsilon$, so we can send $\varepsilon \to 0$ and thus send the integral to zero. ☐

- We now look into the **Cauchy Integral Formula**.
- **Cauchy Integral Formula**: Suppose U is any domain, $D \subset U$ is a disk (i.e., $D \subset\subset U$ or $\overline{D} \subset U$), $f \in \mathcal{O}(U)$, and $z \in D$. Then

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

Proof. We're going to try to use the corollary and define a function. In particular, define

$$g(\zeta) = \begin{cases} \frac{f(\zeta) - f(z)}{\zeta - z} & \zeta \neq z \\ f'(z) & \zeta = z \end{cases}$$

Because f is holomorphic at z, g is continuous at z and hence bounded near z. We can also see that since g is a rational function of holomorphic functions on $U \setminus \{z\}$, we have $g \in \mathcal{O}(U \setminus \{z\})$.

Now the corollary says that

$$\int_{\partial D} g \, \mathrm{d}\zeta = 0$$

Additionally, by the definition of g, we have that

$$\int_{\partial D} g \, \mathrm{d}\zeta = \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta - \int_{\partial D} \frac{f(z)}{\zeta - z} \, \mathrm{d}\zeta$$

[1] We could also turn the plane into the sum of two star-shaped domains again.

$f(z)$ is just a complex number, so we can pull it out of the rightmost integral above. Additionally, under a change of variables and invoking PSet 1, QA.4, we have that

$$\int_{\partial D} \frac{f(z)}{\zeta - z}\, \mathrm{d}\zeta = f(z) \int_{\partial D} \frac{1}{\zeta - z}\, \mathrm{d}\zeta = \int_{\text{unit circle}} \frac{1}{z-a}\, \mathrm{d}z = 2\pi i f(z)$$

Note: Another way to evaluate this integral is as follows. If z is the center of the disk, then we win and can get $2\pi i$ using PSet 1, QA.4 directly. If z isn't at the center of the disk, we are allowed to slide it. Here's why: Think about the integrand as a function of z, so

$$\frac{\partial}{\partial z}\left(\int_{\partial D} \frac{1}{\zeta - z}\, \mathrm{d}\zeta\right) = \int_{\partial D} \frac{\partial}{\partial z}\left(\frac{1}{\zeta - z}\right) \mathrm{d}\zeta = \int_{\partial D} \frac{1}{(\zeta - z)^2}\, \mathrm{d}\zeta = 0$$

Since we're taking the integral and the limit with respect to different things, we can exchange them. Since the second integrand has a primitive, it equals zero. But this means that the integral does not change even as z changes, which is equivalent to saying we can move z around to wherever we want in the disk and the integral will still be $2\pi i$! In other words, if z is somewhere where we can't evaluate the integral directly, we can move z to somewhere where we *can* evaluate the integral directly with no consequence. □

- Implication of Cauchy's Integral Theorem: The values of the function are completely determined by the values on the boundary, i.e., holomorphic functions are determined by boundary values.
- Let's now prove another theorem.
- Theorem: Let U be any domain, $f \in \mathcal{O}(U)$. Then $f' \in \mathcal{O}(U)$, $f'' \in \mathcal{O}(U)$, on and on.

  Proof. Let's use the Cauchy integral formula. We have that

  $$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

  Now let's take the derivative, which we know exists because f is holomorphic.

  $$\frac{\partial f}{\partial z} = \frac{1}{2\pi i} \int_{\partial D} \frac{\partial}{\partial z}\left(\frac{f(\zeta)}{\zeta - z}\right) \mathrm{d}\zeta = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^2} \mathrm{d}\zeta$$

  Thus, the derivative has a Cauchy integral formula. We can keep taking derivatives on the inside because the integrand is infinitely differentiable. Thus, we can keep taking derivatives on the outside. And that's the proof. □

- Corollary: Holomorphic functions are C^∞.
- Corollary: In general,

  $$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \mathrm{d}\zeta$$

- This last result allows us to bound things really easily, giving us **Cauchy's inequalities**.
  - Essentially, let D have radius R and let z be the center of D. Then

    $$|f^{(n)}(z)| \leq \frac{n!}{2\pi i} \max_{\partial D} \left| \frac{f(\zeta)}{R^{n+1}} \right| \cdot 2\pi R = \frac{n!}{R^n} \max_{\partial D} |f(\zeta)|$$

- Liouville's Theorem: Suppose $f \in \mathcal{O}(\mathbb{C})$ (i.e., f is **entire**) and f is bounded. Then it's constant.

  Proof. Take a point $z \in \mathbb{C}$. Take a huge ball with radius R. Cauchy's inequality says that if we take the derivative, then

  $$|f'(z)| \leq \frac{1}{R} \cdot \max_{\partial D} |f(\zeta)|$$

  The maximum is bounded and R is really big, so as $R \to \infty$, the derivative gets arbitrarily small. So if we've got an arbitrary function with zero derivative, then we've got a constant function. □

- **Entire** (function): A complex-valued function that is holomorphic on the whole complex plane.

3.2 Analytic Continuation and Removable Singularities

4/4:
- Last time.
  - Cauchy integral formula: Let U be any domain, $f \in \mathcal{O}(U)$, $D \subset\subset U$, and $z \in D$. Then
$$f(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta$$
  - Implies that holomorphic functions are C^∞.
  - Implies Cauchy's inequalities: If D is a disk centered at z_0 of radius R, then
$$|f^{(n)}(z_0)| \leq \frac{n!}{R^n}\sup_{\partial D}|f(\zeta)|$$
  - Implies Liouville's theorem: Any bounded entire function is constant.
- Our focus today is on results we can get out of power series.
- Observe that if
$$P(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$$
is a convergent power series centered at z_0, then
$$P^{(n)}(z_0) = n!a_n$$
- Now let $f \in \mathcal{O}(U)$.
  - TPS: What should the power series for f look like?
    - Rearranging the above, we want
$$a_k = \frac{f^{(k)}(z_0)}{k!}$$
    - The following power series formally has the right derivatives.
$$P(z) = \sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$
  - Does this power series converge though, and if so, where?
    - Recall that the Cauchy-Hadamard formula tells us that the radius of convergence r satisfies
$$r = \left(\limsup_{k\to\infty}|a_k|^{1/k}\right)^{-1}$$
    - Pick a $z_0 \in U$ and a disk $D \subset\subset U$ of radius R. Then by the Cauchy inequalities,
$$|a_k|^{1/k} = \left|\frac{f^{(k)}(z_0)}{k!}\right|^{1/k} \leq \frac{|\sup_{\partial D} f(\zeta)|^{1/k}}{R} \to \frac{1}{R}$$
    - Thus, returning to the Cauchy-Hadamard formula, the radius of convergence is $\geq R$.
  - So we've got a convergent power series, but why does this power series equal $f(z)$?
    - We know that $P(z)$ and $f(z)$ have all the same derivatives.
    - However, over $\mathbb{R}$, this is not enough! Recall the example of e^{-1/x^2}, which has the same derivatives as its power series at zero but is not equal to it.
    - Over $\mathbb{C}$, however, we claim that having the same derivatives *is* enough.

- Use CIF and expand $1/(\zeta - z)$.
- Note: To keep all of our z's straight, recall that z_0 is a point, ζ lies on ∂D where D is centered at z_0, and z is somewhere in $\mathring{D}$.
- Doing this, we obtain

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \left(\frac{z-z_0}{\zeta - z_0}\right)} = \frac{1}{\zeta - z_0} \sum_{k=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^k$$

- Thus,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{\partial D} \sum_{k=0}^{\infty} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} (z - z_0)^n \mathrm{d}\zeta \\ &= \sum_{k=0}^{\infty} \underbrace{\frac{1}{2\pi i} \int \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \mathrm{d}\zeta}_{f^{(n)}(z_0)/n!} (z - z_0)^n \\ &= P(z) \end{aligned}$$

- Recall that we can bring the sum outside of the integral because of uniform convergence and our lemma about integrable functions from the 3/26 class.

  - All in all, we've shown that any holomorphic function has a power series representation on any disk that fits within the domain, and the power series representation is the one we think it should be.

- We now discuss an important corollary to this result.
- The Identity Theorem: If two holomorphic functions $f, g \in \mathcal{O}(U)$ agree on an open set in U, then $f = g$.

  Proof. This is true for power series.

  For every point, there's a power series representation around that series so we can do something with a covering of open sets, though we do not need compactness for U. □

  - An analogous result does not hold on the reals. For example, there are plenty of functions that are zero for a while, then bump up to 1 for a while, so they're 0 and 1 on open sets without being either 0 or 1.
  - Implication: "Holomorphic functions are very rigid."

- In fact, more is true: If $z_n \to z_0$ where each z_n is distinct and $f(z_n) = g(z_n)$ for all n, then $f = g$.
  - So we don't even need an open set; all we need is an **accumulation point**.
- **Analytic continuation** (of f): The function $g \in \mathcal{O}(V)$ where $f \in \mathcal{O}(U)$, $V \supset U$, and $f = g$ on U.
  - Note that we get to say "*the* function g..." because of the identity theorem.
  - Formally, g_1, g_2 analytic continuations of f and $g_1 = g_2$ on U open implies $g_1 = g_2$.
- Example: Consider $f(z) = z$ with $f \in \mathcal{O}(\mathbb{C}^*)$. Then $g(z) = z$ with $g \in \mathcal{O}(\mathbb{C})$ is an analytic continuation of f.
- What we're essentially doing is taking the power series (which we get via "analytic") and extending them out into V.

- Example: The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \sum_{n=1}^{\infty} \mathrm{e}^{-s\log n}$$

  Where does this make sense, i.e., where does the series converge?
  - We have $|n^{-s}| = n^{-\operatorname{Re}(s)}$.
  - Thus, the series converges only when $\operatorname{Re}(s) > 1$.
  - The Riemann hypothesis predicts where $\zeta(s) = 0$. We know that it has some zeroes at the negative even integers, and the RH predicts that the rest of them fall on the line $\operatorname{Re}(s) = 1/2$.
  - But ζ is only defined on a part of the complex plane not including these regions! Thus, to make sense of the RH, we need to analytically continue ζ.
- Given $f \in \mathcal{O}(U)$, what is the "biggest" V on which $f \in \mathcal{O}(V)$? In layman's terms, where should f live?
- Example: $1/z \in \mathcal{O}(\mathbb{C}^*)$ and $1/z \notin \mathcal{O}(\mathbb{C})$.
  - Note that we know the latter statement because if you're holomorphic, the integral around any closed loop in the domain is zero but the integral of this function on the unit circle is $2\pi i$, so it can't be holomorphic on $\mathbb{C}$. Contradiction.
- Example: $\sin(1/z) \in \mathcal{O}(\mathbb{C}^*)$, but is it in $\mathcal{O}(\mathbb{C})$?
  - No; recall from PSet 1 that it's not even *continuous* at 0.
- Example: $\sin(z)/z \in \mathcal{O}(\mathbb{C}^*)$, but is it in $\mathcal{O}(\mathbb{C})$?
- Recall Goursat's lemma: $f \in \mathcal{O}(\operatorname{nbhd}(\triangle))$ implies $\int_\triangle f\,\mathrm{d}z = 0$.
  - If U is star-shaped and $\int_\triangle f\,\mathrm{d}z = 0$ for all triangles, then f has a primitive.
    - Note that we do not need f holomorphic for this result!
  - This latter result has a converse!
- Morera's Theorem: If U is any domain, $f : U \to \mathbb{C}$ is continuous, and $\int_\triangle f\,\mathrm{d}z = 0$ for all triangles, then f is holomorphic.

  Proof. Fix a disk $D \subset\subset U$. Disks are star-shaped! This combined with the fact that the integral over all triangles is zero implies that f has a primitive $F \in \mathcal{O}(U)$ by the result a couple lines up. But since F is holomorphic, by our result from last class, $F' = f \in \mathcal{O}(U)$, too. ☐

- **Riemann's removable singularity theorem**: Suppose U is a domain, $z \in U$, $f \in \mathcal{O}(U \setminus \{z\})$, and f is bounded near z. Then there exists a unique analytic continuation $\hat{f} \in \mathcal{O}(U)$. *Also known as* **Riemann extension theorem**.
  - In this case, we call z a **removable singularity**.
  - Note: The contrapositive of this says that if there is not an analytic continuation (i.e., the function is honestly not holomorphic at a point and can't be extended to one, e.g., $1/z$), then $|f|$ has to blow up as you approach z (in some direction).
- **Singularity** (of f): A point z_0 such that $f \in \mathcal{O}(U \setminus \{z_0\})$.
- **Removable** (singularity): A singularity of a function that that satisfies the hypotheses of Riemann's removable singularity theorem.
- If a singularity is not removable, then f is not bounded near z_0. This leads to additional definitions.

- **Pole**: A non-removable singularity z_0 of a function f for which $|f(z)| \to \infty$ as $z \to z_0$.
  - So-named because of real analysis where a pole is an asymptote, and asymptotes kind of look like poles!
- **Essential** (singularity): A non-removable singularity that is not a pole; equivalently, a singularity z_0 for which there exist sequences $z_n \to z_0$ and $w_n \to z_0$ such that $|f(z_n)| \to \infty$ and $|f(w_n)|$ stays bounded.
- Proving Riemann's removable singularity theorem.

  Proof. Set

  $$F(\zeta) = \begin{cases} f(\zeta)(\zeta - z) & \zeta \neq z \\ 0 & \zeta = z \end{cases}$$

  Then $F \in \mathcal{O}(U \setminus \{z\})$ so F is continuous at z.

  We want to show that F is holomorphic (using Morera's theorem). To do this, we'll need to show that the integral over all triangles is zero. More specifically, all we need to do is show that F is holomorphic in a little ball D about z. Now we need to do some casework.

  Case 1: If $\triangle \not\ni z$, then we can draw a star-shaped domain surrounding the triangle on which f will be holomorphic and invoke the CIT to imply that the integral is zero.

  Case 2: If $\triangle \ni z$, then $\int F \, \mathrm{d}z$ is arbitrarily small. Recall that we get this by using homotopy to replace the integral over the triangle with the integral over some tiny γ_ε. Arbitrarily small because f is bounded.

  Morera then tells us that $F \in \mathcal{O}(U)$, so $F' = f \in \mathcal{O}(U)$. Note that $F' = f$ because

  $$F'(z) = \lim_{\zeta \to z} \frac{F(\zeta) - F(z)}{\zeta - z} = \lim_{\zeta \to z} \frac{f(\zeta)(\zeta - z) - 0}{\zeta - z} = \lim_{\zeta \to z} f(\zeta) = f(z)$$

  $\square$

- Go back and add a z_0 everywhere and then it should all be ok.
- With the removable singularity theorem, we can now confirm that $\sin(z)/z$ has a removable singularity because although $1/z$ diverges, sine converges faster so this function is bounded near zero.
  - We can prove boundedness with the Taylor series of $\sin(z)/z$.

3.3 Chapter II: The Fundamental Theorems of Complex Analysis

From Fischer and Lieb (2012).

Section II.3: The Cauchy Integral Formula

4/13:
- Statement and proof of the CIF.
  - Fischer and Lieb (2012) come at it in a slightly more complicated way than class.
- Derivatives of the CIF.
- Holomorphic implies C^∞.
- Morera's theorem and proof.
- The Riemann extension theorem and proof.

Section II.4: Power Series Expansions of Holomorphic Functions

- Proof of the typical power series formula.
- Power series examples.
  1. $(z-a)^{-1}$.
     - Has a standard power series.
     - $(z-a)^{-n}$ is computed via term-by-term differentiation of this base power series and corresponding adjustments.
     - More complicated rational functions are handled via partial fraction decomposition and then the above method.
  2. Products of holomorphic functions.
     - Use **Leibniz's rule**.
  3. Inverse of a power series.
     - Suppose $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ in a neighborhood of z_0, and assume that $f(z_0)=a_0\neq 0$.
     - Then $1/f$ is holomorphic near z_0.
     - Moreover, the coefficients of the expansion

       $$\frac{1}{f(z)}=\sum_{n=0}^{\infty} b_n(z-z_0)^n$$

       can be determined.
     - To do so, compare coefficients in

       $$1=\frac{1}{f(z)}\cdot f(z)=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} b_m a_{n-m}\right)(z-z_0)^n$$

     - This yields $b_0a_0=1$ and hence $b_0=a_0^{-1}$. Continuing, $b_0a_1+b_1a_0=0$, so $b_1=-b_0a_1a_0^{-1}=-a_1a_0^{-2}$. We can continue this computation as far as we like.
  4. $\tan z$.
     - Observe that $\tan z$ is odd, and hence the even-powered terms disappear.
     - Substitute the power series for cosine, sine, and the undetermined one for tangent into the equation $\cos z\cdot\tan z=\sin z$ and compare coefficients.
     - We could also do $\sin z/\tan z$ using the method of Examples 2-3 above.
- **Leibniz's rule**: The derivative of the product fg of two functions f,g that are both holomorphic in a neighborhood of z_0 is given by

  $$\frac{(fg)^{(n)}(z_0)}{n!}=\sum_{m=0}^{n}\frac{f^{(m)}(z_0)}{m!}\cdot\frac{g^{(n-m)}(z_0)}{(n-m)!}$$

  - This is just the Cauchy product of the two formal power series!
- "In order to determine the derivatives $f^{(n)}(z_0)$ of a function f that is holomorphic at z_0, one only needs to know the values $f(z)$ on, say, a segment $(z_0-\delta,z_0+\delta)$ parallel to the real axis" (Fischer & Lieb, 2012, p. 52).
- Build up to, statement of, and proof of the identity theorem.
- Consequence of the identity theorem: Holomorphic functions on U are completely determined by their values on any **nondiscrete** set in U.

  - "Properties that can be expressed via identities between holomorphic functions on [G] thus only need to be verified on a nondiscrete set in [G]" (Fischer & Lieb, 2012, p. 53).
  - Example: $\cot(\pi z)$ and $\cot(\pi z + \pi)$ coincide on the nondiscrete set $\mathbb{C} \setminus \mathbb{Z}$, and thus are equal; therefore, the periodicity of cotangent on the real numbers implies it on the complex numbers.
  - This also means that the exponential function and those derived from it (e.g., sine and cosine) can only be extended from the real to the complex numbers in one way since $\mathbb{R}$ is nondiscrete in $\mathbb{C}$.
- **Nondiscrete** (set in U): A set $M \subset U$ that contains an accumulation point of M.
- **Discrete** (set in U): A set $M \subset U$ that is not nondiscrete.
- Defines a **zero** (of order L), tapping into next week's content.
- Since constancy on a nondiscrete set would imply constancy everywhere (for a holomorphic function), the set $f^{-1}(w)$ of points at which a nonconstant holomorphic function takes on the same value $w \in \mathbb{C}$ is discrete.
- Characterizing holomorphicity: Let $f : U \to \mathbb{C}$ be a function defined on an open set $U \subset \mathbb{C}$. The following are equivalent:
  i. f is holomorphic.
  ii. f is real differentiable and satisfies the Cauchy-Riemann equations.
  iii. f admits a power series expansion about every point in U.
  iv. f has local primitives.
  v. f is continuous, and for every closed triangle $\triangle \subset U$, $\int_{\partial\triangle} f(z)\,\mathrm{d}z = 0$.

Week 4

Extrema

4.1 Poles and Maximum Moduli

4/9:

- Announcement.
  - Midterm next week in class.
  - Material up through today, though probably not much on today's content.
- Last time.
  - Cauchy integral formula: If U is a domain, $D \subset\subset U$, and $z \in D$, then

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, \mathrm{d}\zeta$$

  - This implies Riemann's removable singularity theorem, which states that if $f \in \mathcal{O}(U \setminus \{z_0\})$ and f is bounded near z_0, then there exists a $\hat{f} \in \mathcal{O}(U)$ which continues f at z_0.
    - Example: $\sin(z)/z \in \mathcal{O}(\mathbb{C}^*)$ has a continuation to $\mathbb{C}$.
    - In particular, take the Taylor series at zero and evaluate:

$$\widehat{\frac{\sin(z)}{z}}(0) = 1 - \frac{0^3}{3!} + \frac{0^5}{5!} - \cdots = 1$$

  - Alternatively, if $f \in \mathcal{O}(U \setminus \{z_0\})$ and $|f(z)| \to \infty$ as $z \to z_0$, then z_0 is a **pole** of f.
- Today.
  - Finish up what we couldn't last time.
  - Say something about harmonic functions.
- **Meromorphic** (function): A function $f : U \to \mathbb{C}$ such that $f \in \mathcal{O}(U \setminus P)$ and each $p \in P$ is a pole, where $P \subset U$ is a finite set of points.
- Example: Consider $1/z \in \mathcal{O}(\mathbb{C}^*)$.
  - This has a pole at zero.
  - Thus, $1/z$ is *holomorphic* on the punctured plane $\mathbb{C}^*$, but *meromorphic* on the whole complex plane $\mathbb{C}$.
- Example: The same argument applies to $1/z^k$ ($k \in \mathbb{N}$).
- Example: The function from PSet 2, Q2c:

$$f(z) = \frac{1}{z(z-1)(z-i)(z-1-i)}$$

- It follows that
$$\{f : f \text{ is holomorphic}\} \subset \{f : f \text{ is meromorphic}\}$$
- Fact: All of the examples kind of look the same.
  - More generally, suppose f has a pole at p and is holomorphic on $U \setminus \{p\}$. Pick a disk $D \ni p$ such that $f \neq 0$ on D. Then $g = 1/f \in \mathcal{O}(D \setminus \{p\})$ and as $z \to p$, $g(z) \to 0$.
  - Thus, we've got a function that's holomorphic and bounded near a point, so by Riemann's removable singularity theorem, it has a unique holomorphic extension $\hat{g} \in \mathcal{O}(D)$.
    - In particular, $g(p) = 0$.
  - Note: We do *not* need to choose D small enough such that it contains only one point in P. However, we will for the time being just to simplify things. The reason we can do this is because singularities — as points of a finite set — are isolated.
- There exists a power series for g about p such that
$$g(z) = \sum_{k=0}^{\infty} a_k (z-p)^k$$
  - We know that $a_0 = 0$ because $g(p) = 0$.
  - It can also happen such that some (or [potentially infinitely] many) of the remaining a_i are zero.
    - Example: if $f = 1/z^3$, then $g = z^3$ and $a_i = 0$ $(i > 3)$.
  - Now let L be the largest natural number such that $a_i = 0$ for all $0 \leq i < L$.
    - Because $a_0 = 0$, $L \geq 1$.
    - Additionally, $a_L \neq 0$.
  - Then we can rewrite the power series as
$$g(z) = (z-p)^L h(z)$$
    where...
    1. $h(z) = \sum_{k=L}^{\infty} a_k (z-p)^{k-L}$;
    2. $h(p) \neq 0$ (and h is nonzero near p).
  - We say that g has a **zero** (of order L at p).
    - Similarly, we say that f has a **pole** (of order L at p).
  - Thus,
$$f(z) = \frac{1}{(z-p)^L} \frac{1}{h(z)}$$
    where, moreover, $1/h \in \mathcal{O}(D')$ for some smaller disk D'.
  - Example: $1/(z^2 + z)$ goes to $z(z+1)$.
  - Takeaway: Near any pole p, f must look like
$$\frac{1}{(z-p)^L} \cdot \phi(z)$$
    where ϕ is holomorphic around p.
    - This implies that there exists a **Laurent series** expansion around any pole.
    - In particular, near p,
$$f(z) = \sum_{k=-L}^{\infty} a_k (z-p)^k$$

- **Zero** (of order L at p): A point p of a holomorphic complex function g such that $g(p) = 0$ and $g(z) = (z-p)^L h(z)$ where $h(p) \neq 0$.
- **Pole** (of order L at p): A point p of a holomorphic complex function f such that $1/f(p) = 0$ and $f(z) = 1/(z-p)^L h(z)$ where $h(p) \neq 0$.
- **Laurent series**: A power series including a finite number of negative coefficients. *Given by*

$$\sum_{k=-L}^{\infty} a_k (z-p)^k$$

- TPS: Consider $\cot(z) = \cos(z)/\sin(z)$, which has a pole at zero. What is the order of the pole? What is the Laurent series?
  - The pole is order 1.
    - One way to see this is to observe how $\tan z$ has a nonzero tangent at 0, so $\tan z = z + \cdots$. Thus, we can only divide one z out of its power series.
    - Alternatively, we have
      $$\cot(z) = \frac{1}{z} \cdot \frac{z}{\sin(z)} \cdot \cos(z)$$
      from which we can observe that $\cos(z) \in \mathcal{O}(\mathbb{C})$, and $\sin(z)/z \in \mathcal{O}(\mathbb{C})$ (at zero, the extension gives 1) so $z/\sin(z)$ is holomorphic near zero. Thus, we can define $\phi(z) = z\cos(z)/\sin(z)$.
      - ➢ What if we tried $\tilde{\phi}(z) = z^2 \cos(z)/\sin(z)$? What's different? Well, $\tilde{\phi}$ is still holomorphic, but $\tilde{\phi}(0) = 0$, which is a problem. Notice that $\phi(0) = 1$!
    - As a last way, we could investigate the power series of $\cot(z)^{-}1 = \tan(z)$ directly:
      $$\tan z = z + \frac{z^3}{3} + \frac{2z^5}{15}$$
  - The Laurent series was not discussed in class, but here's some comments.
    - It would begin from $k = -1$.
    - We could construct it from the power series for cosine and sine using Calderon's formula above.
    - Figuring out the formula for the power series of an inverted power series is a good exercise!!
- What if $|f(z)| \to \infty$ as $|z| \to \infty$? Then we say that f has a **pole** (at ∞).
  - Otherwise, there exist sequences $z_n \to \infty$ and $w_n \to \infty$ such that $f(z_n) \to \infty$ and $f(w_n)$ stays bounded. This is an **essential singularity** (at ∞).
  - We can mull over this until Thursday when we introduce the solution, the **Riemann sphere**.
  - If $f(z)$ stays bounded, then f has a **removable singularity** (at ∞).
- **Pole** (at ∞): A function f such that $|f(z)| \to \infty$ as $|z| \to \infty$.
- **Essential singularity** (at ∞): A function f for which there exist sequences $z_n \to \infty$ and $w_n \to \infty$ such that $f(z_n) \to \infty$ and $f(w_n)$ stays bounded.
- **Removable singularity** (at ∞): A function f that stays bounded as $|z| \to \infty$.
- We're now going to switch to a completely different topic.
- Suppose $f \in \mathcal{O}(U)$. When does $|f(z)|$ get the biggest? Equivalently, where does $|f(z)|$ take a local max? *Hint*: Look at the Cauchy integral formula!
  - There are no such points, at least on the interior of U!

- Theorem (maximum modulus principle): Let $f \in \mathcal{O}(U)$. If $|f(z)|$ has a local maximum on U, then f is constant.

Proof. Let z_0 be a local maximum of $|f(z)|$. Pick $D \ni z_0$ small enough such that $|f(z)| \leq |f(z_0)|$ for all $z \in D$. Let r be the radius of D. Now invoking the CIF,

$$
\begin{aligned}
|f(z_0)| &= \left|\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta - z_0}\mathrm{d}\zeta\right| \\
&= \frac{1}{2\pi}\left|\int_{\partial D}\frac{f(\zeta)}{\zeta - z_0}\mathrm{d}\zeta\right| \\
&\leq \frac{1}{2\pi}\int_{\partial D}\left|\frac{f(\zeta)}{\zeta - z_0}\right|\mathrm{d}\zeta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f(z_0 + r\mathrm{e}^{i\theta})}{r\mathrm{e}^{i\theta}}\cdot ir\mathrm{e}^{i\theta}\right|\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}|f(z_0 + r\mathrm{e}^{i\theta})|\mathrm{d}\theta \\
&\leq \frac{1}{2\pi}\cdot 2\pi\cdot\max_{\partial D}|f(\zeta)| \\
&= \max_{\partial D}|f(\zeta)| \\
&\leq |f(z_0)|
\end{aligned}
$$

But since the above inequality begins and ends with the same value, all $\leq$'s must be $=$'s. Thus, in particular,

$$
\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}|f(z_0 + r\mathrm{e}^{i\theta})|\mathrm{d}\theta &= |f(z_0)| \\
\frac{1}{2\pi}\int_0^{2\pi}\big(|f(z_0 + r\mathrm{e}^{i\theta})| - |f(z_0)|\big)\mathrm{d}\theta &= 0
\end{aligned}
$$

Combining this with the fact that the above integrand is always ≤ 0 because $f(z_0)$ is a local maximum, we have that

$$
\begin{aligned}
|f(z_0 + r\mathrm{e}^{i\theta})| - |f(z_0)| &= 0 \\
|f(\zeta)| &= |f(z_0)|
\end{aligned}
$$

on ∂D. Note that this is true for all small ∂D's centered at z_0.

Now since $|f|$ is constant on ∂D, we must have that $|f|^2 = f \cdot \bar{f}$ is constant on ∂D. Taking the Wirtinger derivative and using its product rule gets us

$$
0 = \frac{\partial}{\partial z}\big(f \cdot \bar{f}\big) = f_z \cdot \bar{f} + f \cdot \bar{f}_z
$$

Since f is holomorphic (hence satisfies the CR equations) and $f_{\bar{z}} = \bar{f}_z$, we have that

$$
\bar{f}_z = f_{\bar{z}} = 0
$$

Thus,

$$
0 = f_z \cdot \bar{f} + f \cdot 0 = f_z \cdot \bar{f}
$$

By the zero-product property, either $f_z = 0$ and $\bar{f} = 0$. In the first case, this means that f is constant, as desired. In the second case, this means that f is zero (and hence constant), as desired.

At this point, we have shown that f is constant on a small disk. Therefore, we need only invoke the identity theorem, which tells us that since the function is constant for a little bit somewhere, it is constant everywhere. ☐

- Another way to prove this is by considering the derivative of the Cauchy integral formula and where it's equal to zero.
- Corollary (minimum modulus principle): If $f \in \mathcal{O}(U)$, $f \neq 0$ on U (hence $1/f \in \mathcal{O}(U)$), and $|f(z)|$ takes a minimum in U, then f is constant.
- Application of the maximum modulus principle (the fundamental theorem of algebra): If p is a polynomial of degree d in $\mathbb{C}$, then p has d roots in $\mathbb{C}$ (counted with multiplicity).

  Proof. Suppose inductively that $d \geq 1$.

  Step 1 (show that there exists one root): Suppose for the sake of contradiction that p has no zeros. Since p is a polynomial, we know that $|p(z)| \to \infty$ as $|z| \to \infty$. Thus, there exists $R > 0$ such that for all z with $|z| > R$, $|p(z)| \geq |p(0)|$. Then $|p(z)|$ must take a minimum on $\overline{D_R}$. But to keep p from being constant by the minimum modulus principle, the minimum has to be on ∂D_R. Now take a slightly bigger disk; our global minimum is now in the interior, so p is constant, a contradiction. It follows that p must have a zero in D_R.

  Step 2: Suppose p has a root at z_0. Then power series for p at z_0 is $p(z) = (z - z_0)p_1(z)$. p_1 is a polynomial of degree $d - 1$.

  Step 3: Now iterate to find that p is a product of monomials. □

- Algebraists love to prove this with only algebra, but in reality, the proof is complex analysis.[1]
- We did not get to say something about harmonic functions today, but Calderon will leave the content in his notes in case we want to look at it.
  - The statement: Harmonic functions follow a version of the CIF.
  - There's a related PSet problem.

4.2 Modulus Principles and Harmonic Functions

4/11:

- Last time.
  - Maximum modulus principle: If $f \in \mathcal{O}(U)$, $f(z) \neq 0$ for all $z \in U$, and $|f|$ takes a max inside U, then f is constant.
    - Analogous result: The minimum modulus principle.
  - This result implies the fundamental theorem of algebra.
    - Proof idea: $|f(z)| < |f(\zeta)|$ for $\zeta \in \partial D$, so f must have a zero.
- Another corollary: We have a better understanding of the mapping properties of holomorphic functions.
- Recall that conformal (angle-preserving diffeomorphism) iff biholomorphic (bijective, f, f^{-1} holomorphic).
- In real analysis, we have the **inverse function theorem**.
- **Inverse function theorem**: If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is C^1 and $Df(x) \neq 0$, then f is locally a diffeomorphism from $x \in U$ to $V \ni f(x)$.
- So for $f \in \mathcal{O}(U)$...
  - If f' is never 0 on U, then $f(U)$ is open;
  - If f' is never 0 on U and $f : U \to f(U)$ is a bijection, then f is biholomorphic.

[1] How did this proof work??

- Claim: The "if f' is never 0 on U" condition is actually unnecessary!
- Theorem: Let $f \in \mathcal{O}(U)$.
  1. Open mapping theorem: If f is nonconstant, then $f(U)$ is open.

     Proof. To prove that $f(U)$ is open, it will suffice to show that every $w_0 \in f(U)$ is contained in some neighborhood that's a subset of $f(U)$. Let $w_0 = f(z_0)$. Pick a disk $D \subset\subset U$ such that $f(z) - w_0 \neq 0$ on ∂D; this is possible because the zeroes of a nonconstant holomorphic function (like $f - w_0$) must be isolated, or otherwise f would be constant. Thus, we may define the positive number
     $$\delta := \inf_{z \in \partial D} |f(z) - w_0|$$
     Now pick w such that $|w - w_0| < \delta/2$. Then by the triangle inequality, we have that for all $z \in \partial D$,
     $$|f(z) - w| \geq |f(z) - w_0| - |w - w_0| \geq \delta - |w - w_0| > \delta - \frac{\delta}{2} = \frac{\delta}{2}$$
     This means that $|f - w|$ is always greater than the number $\delta/2$ on the boundary of D, but since
     $$|f(z_0) - w| = |w - w_0| < \frac{\delta}{2}$$
     $|f - w|$ does not obtain its minimum on the boundary of D. Thus, since all other hypotheses of the minimum modulus principle are satisfied, there must be a zero of $|f - w|$ on U. This means that there exists a $z \in U$ such that $f(z) = w$, and hence $w \in f(U)$. Therefore, since the choice of $w \in D_{\delta/2}(w_0)$ was arbitrary, we know that $D_{\delta/w}(w_0) \subset f(U)$, as desired. □

  2. Complex inverse function theorem: If f is bijective, it's biholomorphic.

     Proof. Define the set
     $$Z := \{z \in U \mid f'(z) = 0\}$$
     of zeroes of f'. To prove that f is biholomorphic, we will quickly show that $f : U \setminus Z \to f(U) \setminus f(Z)$ is biholomorphic and then build up to the point where we can use Riemann's removable singularity theorem to analytically continue this restriction. Let's begin.

     Since $f \in \mathcal{O}(U)$ by hypothesis, $f \in C^\infty \subset C^1$. Additionally, by the definition of z, $Df(x) \neq 0$ at all $x \in U \setminus Z$. Thus, by the real inverse function theorem, f is a diffeomorphism at all $x \in U \setminus Z$. Consequently, $f^{-1} : f(U) \setminus f(Z) \to U \setminus Z$ is differentiable, and hence holomorphic. This combined with the hypothesis that $f : U \setminus Z \to f(U) \setminus f(Z)$ is bijective and holomorphic implies that $f : U \setminus Z \to f(U) \setminus f(Z)$ is biholomorphic.

     Now the first part of the plan is complete. The next step involves building up to the point that we can apply Riemann's removable singularity theorem to $f^{-1} : f(U) \setminus f(Z) \to U \setminus Z$. To do so, we need only verify that $f(U) \setminus f(Z)$ is a domain and f^{-1} is bounded near any $f(z) \in f(Z)$, since $f(z) \in f(Z) \subset f(U)$ by definition and we have just shown that $f^{-1} \in \mathcal{O}(f(U) \setminus f(Z))$.

     First, we verify that $f(U) \setminus f(Z)$ is a domain. To do so, we begin by checking that $f(U)$ is a domain. Since U is a domain (hence connected) and f is holomorphic (hence continuous), Theorem 9.11[2] tells us that $f(U)$ is connected. Additionally, since U is a domain (hence open) and f is bijective (hence nonconstant), the open mapping theorem implies that $f(U)$ is open. But since $f(U)$ is connected and open, it must be a domain, as desired. Next, we check that $f(Z)$ is discrete in $f(U)$. Since f is nonconstant (per the above), f' is nonzero. It follows since f' is holomorphic that Z must be discrete (otherwise, f' holomorphic would be zero on a nondiscrete set, and hence would be zero everywhere, a contradiction). Thus, every $z \in Z$ is contained in an open neighborhood $N_z \subset U$ disjoint from all other $N_{z'}$. It follows by the open mapping theorem that each $f(N_z)$ is an *open* neighborhood of $f(z)$, and by the fact that f is bijective that the set

[2] See MATH 16210 Honors Calculus II notes.

of $f(N_z)$ is pairwise disjoint. Thus, $f(Z)$ is discrete in $f(U)$, as desired. Therefore, $f(U) \setminus f(Z)$ is a (punctured) domain, as desired.

Second, we verify that f^{-1} is bounded near any $f(z) \in f(Z)$. To do so, we begin by checking that $f^{-1} : f(U) \to U$ is continuous. Let $X \subset U$ be open. Since f is bijective, $(f^{-1})^{-1}(X) = f(X)$. By the open mapping theorem, $f(X)$ is open. Thus, by the open-set definition of continuity, f^{-1} is continuous, as desired. But then since f^{-1} is continuous, it maps compact sets to compact sets. Therefore, a closed and bounded neighborhood of $f(z)$ will maps to a closed and bounded neighborhood of z, as desired.

At this point, we may invoke Riemann's removable singularity theorem to analytically continue $f^{-1} : f(U) \setminus f(Z) \to U \setminus Z$ to $f(U)$. Therefore, since $f : U \to f(U)$ is bijective and holomorphic by hypothesis and $f^{-1} : f(U) \to U$ is holomorphic, f is biholomorphic by definition, as desired. □

- Preview: There is also a geometric reason why $f \in \mathcal{O}(U)$ with zeros can't be injective.
- So the maximum modulus principle gets us a lot, and in fact, these kinds of arguments can be used to say even more!
- Example: Where do $\text{Re}(f)$ and $\text{Im}(f)$ take their max?
- Recall that $h : U \to \mathbb{R}$ is *harmonic* if $\Delta h = 0$, where

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \cdot \frac{\partial^2}{\partial z \partial \bar{z}}$$

  - Examples of harmonic functions: $f \in \mathcal{O}(U)$, $\text{Re}(f)$, $\text{Im}(f)$.
  - Nonexample: $|f|$ is not! Take $f(z) = z$; then $\Delta |f| = 1/|z|$.
- Where do harmonic functions take their maxima?
  - This is essentially equivalent to asking about $\text{Re}(f)$ for the following reason.
- A characterization: If $u : U \to \mathbb{R}$ is C^2 and harmonic where U is convex, then there exists $f \in \mathcal{O}(U)$ such that $u = \text{Re}(f)$.

Proof. Since u is harmonic,

$$0 = \Delta u = \frac{\partial}{\partial \bar{z}} \left(\frac{\partial u}{\partial z} \right)$$

This means that u_z is holomorphic! This combined with the fact that U is convex (hence star-shaped) implies by the CIT that $\int_\gamma u_z \, \mathrm{d}z = 0$ for any closed loop $\gamma \subset U$. Thus, by the proposition associated with Figure 2.1, there exists a primitive g for u_z on U. From here, it follows by the rules of complex differentiation that

$$\frac{\partial}{\partial z}(\text{Re}\, g) = \frac{\partial}{\partial z} \left[\frac{1}{2}(g + \bar{g}) \right] = \frac{1}{2} \frac{\partial g}{\partial z} = \frac{1}{2} u_z$$

and

$$\frac{\partial}{\partial \bar{z}}(\text{Re}\, g) = \frac{1}{2} \overline{g_z} = \frac{1}{2} u_{\bar{z}}$$

Therefore, $u = \text{Re}(2g) + C$, as desired. □

- Harmonic functions also satisfy a version of the Cauchy Integral Formula!
  - Let D be a disk centered at z of radius R.

- Then

$$
\begin{aligned}
u(z) &= \operatorname{Re} f(z) \\
&= \operatorname{Re}\left[\frac{1}{2\pi i}\int_{\partial D}\frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta\right] \\
&= \frac{1}{2\pi}\operatorname{Im}\left[\int_0^{2\pi} i\cdot f(z+R\mathrm{e}^{i\theta})\mathrm{d}\theta\right] \\
&= \frac{1}{2\pi}\int_0^{2\pi}\operatorname{Re} f(z+R\mathrm{e}^{i\theta})\mathrm{d}\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} u(z+R\mathrm{e}^{i\theta})\mathrm{d}\theta
\end{aligned}
$$

- This is called the "mean value property for harmonic functions."
- On the PSet, we'll prove a version for any disk containing z of radius R, namely

$$
u(z) = \int_0^{2\pi} u(\zeta)P_R(\zeta, z)\mathrm{d}\theta
$$

  - P_R is the Poisson kernel defined by

$$
P_R(\zeta, z) = \frac{1}{2\pi}\operatorname{Re}\left(\frac{\zeta+z}{\zeta-z}\right)
$$

  - Note that by definition, the Poisson kernel is harmonic!

- Theorem (Maximum modulus principle for harmonic functions): Suppose $h : U \to \mathbb{R}$ is harmonic. If h takes a local maximum (or minimum) at $z_0 \in U$, then h must be *locally constant*, that is, constant in a neighborhood of z_0.

  Proof. We use the same strategy as we did for the holomorphic version.

  In particular, suppose z_0 is a local maximum. Pick a disk $D_R(z_0)$ about z_0 such that $h(z_0) \geq h(z)$ for all $z \in D_R(z_0)$. Using our new CIF, we have that for all $r < R$,

$$
h(z_0) = \frac{1}{2\pi}\int_0^{2\pi} h(z_0 + r\mathrm{e}^{i\theta})\mathrm{d}\theta
$$

  By a similar integrand argument to before (noting that h is real, so we don't need absolute values), we can conclude that

$$
h(z_0) = h(z_0 + r\mathrm{e}^{i\theta})
$$

  for all $r < R$. Therefore, h is constant on $D_R(z_0)$, as desired. $\square$

- Corollary: Suppose $f \in \mathcal{O}(U)$. If $\operatorname{Re}(f)$ or $\operatorname{Im}(f)$ take a maximum in U, then f must be everywhere constant.
- Corollary: If U is bounded, then h is either constant or takes its maximum and minimum on ∂U.
- Application: Dirichlet problem (on a disk).
  - Let U be a convex domain, and let g be a function on ∂U. Does there exist a function u such that $u = g$ on ∂D and $\Delta u = 0$ (i.e., u is harmonic)?
  - This is like finding a steady state for the heat equation.
  - If U is a disk, the answer is yes, and the function is unique!
  - Existence.

- Set
$$u(z) := \int_0^{2\pi} g(\zeta)P_R(\zeta,z)\mathrm{d}\theta$$
- Then
$$\Delta_z u = \Delta_z \int_0^{2\pi} g(\zeta)P_R(\zeta,z)\mathrm{d}\theta = \int_0^{2\pi} g(\zeta)\underbrace{\Delta_z P_R(\zeta,z)}_{0}\mathrm{d}\theta = 0$$
- Note that $\Delta_z P_R(\zeta,z) = 0$ because P_R is harmonic, as mentioned earlier.
- The only hard part here is showing that u has a continuous extension to ∂D_R.

– Uniqueness.
  - Suppose that there exist two solutions g_1, g_2. Then $g_1 - g_2$ is harmonic and $g_1 - g_2 = 0$ on ∂D. But then by the maximum (and minimum) modulus principles, $g_1 - g_2 = 0$ on U. Therefore, $g_1 = g_2$ on U, as desired.

References

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