Week 9

Advanced Topics and Applications

9.1 The Gelfond-Schneider Theorem

5/14: • Today's lecture.
  - By Ben, a postdoc.
  - His choice of topic in complex analysis.
  - Proof that $e^{\sqrt{2}}$ is irrational, pulled from the Math Library's one complex textbook.
  - He chose this topic to illustrate how useful complex analysis is in other areas of math.

- The main theorem we'll use here is the maximum modulus principle, in a slightly modified form.
- Maximum modulus principle (alternate statement): If Ω is a compact domain, $f \in \mathcal{O}(\Omega)$, then

$$|f(z)| \leq \max_{w \in \partial\Omega} |f(w)|$$

  Moreover, if equality holds in any case, then f is constant.

  Proof. For $\Omega = B_p(r)$, this follows from the **mean-value property**.[1] $\square$

- Remark: An entire function with lots of zeroes must grow fast.

  Proof. Let f be the entire function, and suppose it has zeroes at $\{z_i\}$ with multiplicity k_i. Form the new function

$$\frac{f(z)}{\prod_i (z - z_i)^{k_i}}$$

  If we make $|z|$ large, then this function behaves like

$$\frac{f(z)}{\prod_i z^{k_i}}$$

  Since the above function is holomorphic, the MMP says it must obtain its maximum value on the boundary of an arbitrarily large ball around the compact set on which f obtains all its zeroes. But that denominator is growing really fast, so f must grow even faster to compensate. $\square$

- **Strictly ordered** (f by ρ): An entire function f for which there exists $C > 1$ such that

$$|f(z)| \leq C^{R^\rho}$$

  where $R = |z|$.

[1] Ben quickly explains how the mean-value property works.

- Alternatively, we say that "f has *strict order* $\leq \rho$."
- This gives a bound on the growth of the function.
- We will use R to denote $|z|$ throughout lecture today.

- **Algebraically independent** (functions): Two functions f, g for which

$$\sum_{i,j=1}^{N} a_{ij} f^i g^j = 0$$

where $a_{ij} \in \mathbb{C}$ implies that $a_{ij} = 0$ for all i, j.

  - We will apply this to $f(z) = z$ and $g(z) = \mathrm{e}^z$.

- Theorem (Gelfond-Schneider): Let $f_1, \dots, f_n$ be entire functions with strict order less than or equal to ρ a positive number. Assume that at least two of these functions are algebraically independent. Assume $D := \mathrm{d}/\mathrm{d}z$ maps $\mathbb{Q}[f_1, \dots, f_n]$ into itself. Suppose $w_1, \dots, w_N$ are distinct complex numbers such that $f_i(w_j) \in \mathbb{Q}$ for all $1 \leq i \leq n$ and $1 \leq j \leq N$. Then $N \leq 4\rho$.

- Corollary: e^w cannot be rational if $w \in \mathbb{Q}$.

  Proof. Apply the Gelfond-Schneider theorem to $\mathbb{Q}[z, \mathrm{e}^z]$. From here, note that if e^w were rational, then $\mathrm{e}^w, \mathrm{e}^{2w}, \mathrm{e}^{3w}, \dots \in \mathbb{Q}$ which would eventually contradict the $N \leq 4\rho$ bound. ☐

- If we prove the Gelfond-Schneider theorem under the hypothesis that $f_i(w_j) \in \overline{\mathbb{Q}}$, then our corollary may state that e^w cannot be **algebraic**.

- **Algebraic number**: A number that is the zero of a one-variable polynomial.

- Lemma 1 (Siegel): Let

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= 0 \\ &\vdots \\ a_{r1}x_1 + \cdots + a_{rn}x_n &= 0 \end{aligned}$$

be such that (i) $a_{ij} \in \mathbb{Z}$, (ii) $n > r$, and (iii) $|a_{ij}| \leq A$. Then there exists an integral, nonzero solution $(x_1, \dots, x_n)$ to this system of equations with

$$|x_j| \leq 2(2nA)^{\frac{r}{n-r}}$$

Proof. We know that there has to be at least *some* solution by condition (ii) and linear algebra, which confirms sufficient information and a nontrivial kernel.

Let T be the $r \times n$ matrix (a_{ij}). Then T maps $\mathbb{Z}^n(B)$ into $\mathbb{Z}^r(nBA)$, where $\mathbb{Z}^m(s) := B_0(s) \cap \mathbb{Z}^m$.[2] Find $x, y \in \mathbb{Z}^n(B)$ such that $T(x) = T(y)$ and hence $T(x-y) = 0$. Via a pigeonhole principle argument, make B big enough so that $\mathbb{Z}^r(nBA)$ (which is growing slower due to its smaller exponent of r) has cardinality smaller than $\mathbb{Z}^n(B)$; this will mean that two things have to map to the same thing. Then if we do the computation, we get the stated bound.

Essentially, we're relying on the principle that integer balls in higher-dimensional Euclidean spaces have more points in the limit of large radius. ☐

- **Size** (of a polynomial): The following number, where $P(x_1, \dots, x_n) = \sum_{I=(i_1,\dots,i_n)} a_I x_1^{i_1} \cdots x_n^{i_n}$ is a polynomial. *Denoted by* $\boldsymbol{\mathrm{size}(P)}$. *Given by*

$$\mathrm{size}(P) := \max_I |a_I|$$

[2] Pronounced "the m^{th}-dimensional integer ball of radius s."

- **Denominator** (of $\{a_i\} \subset \mathbb{Q}$): A number d such that $d \cdot a_i \in \mathbb{Z}$ for ever a_i in the subset $\{a_i\} \subset \mathbb{Q}$. *Denoted by* $\mathbf{den(\{a_i\})}$.
- Lemma 2: Let $f_1, \ldots, f_n$ be functions as in the Gelfond-Schneider theorem. Then there exists a constant C_1 such that if $\mathbb{Q}(T_1, \ldots, T_n)$ is a polynomial with rational coefficients and degree less than or equal to r, then
$$P^m(Q(f_1, \ldots, f_n)) = Q_m(f_1, \ldots, f_n)$$
where...
  i.) $\deg(Q_m) \leq C_1(m+r)$;
  ii.) $\operatorname{size}(Q_m) \leq \operatorname{size}(Q) m! C_1^{m+r}$;
  iii.) There exists a denominator for the coefficients of Q_m bounded by $\operatorname{den}(Q) C_1^{m+r}$.
- We are now ready to prove the Gelfond-Schneider theorem.

Proof. By hypothesis, we have common elements $w_1, \ldots, w_N$ of $\mathbb{C}$ such that $f_i(w_j) \in \mathbb{Q}$ and $f_{ij} \in \{f_1, \ldots, f_n\}$ algebraically independent. Let $L \in \mathbb{Z}^+$ be divisible by $2N$, $b_{ij} \in \mathbb{Z}$, and let $F = \sum_{i,j=1}^{L} b_{ij} f^i g^j$ and let $L = 2MN$ be such that
$$D^m F(w_\ell) = 0 \tag{$*$}$$
for $m = 0, \ldots, M-1$ and $\ell = 1, \ldots, N$; we will send both of these constants to infinity eventually.

$(*)$ has L^2 unknowns and MN equations. Multiply the equations in $(*)$ by a common denominator and using Lemma 2 and Siegel's Lemma, we can find b_{ij} such that
$$|b_{ij}| \leq M! C_2^{M+L} \leq M^M C_2^{M+L} \tag{$**$}$$
as $M \to \infty$. Note that in the second inequality, we used Stirling's approximation.

The next observation is that $F \neq 0$ since f and g are algebraically independent. Let s be the smallest integer such that $D^m f(w_i) = 0$ for $m < s$ for all i but $D^s F \neq 0$ at some w_i, which WLOG we will let be w_1.

Let $\alpha := D^s F(w_1)$. Then $\alpha \in \mathbb{Q}$ since $F(W_1) \in \mathbb{Q}$ so all its derivatives will, too. Additionally, $C := \operatorname{den}(\alpha) \leq (C_1)^s$ as $s \to \infty$, this from (i) and (iii) of Lemma 2. Then $C\alpha \in \mathbb{Z}$, which implies that $|C\alpha| \geq 1$ and hence $|\alpha| \geq C^{-1}$. Thus, at this point, we have a lower bound on $|\alpha|$; the next step is to move toward an upper bound and then get what we want.

We upper-bound α using the MMP. Compute
$$D^s F(w_1) = s! \frac{F(w_1)}{(z - w_1)^s}\bigg|_{z = w_1}$$
Estimate
$$H(z) := s! \frac{F(z)}{\prod_{i=1}^N (z - w_i)^s} \prod_{i>1}^{N} (w_1 - w_i)^s$$
on the circle of radius $B = s^{1/2\rho}$. Then the MMP tells us that
$$|D^s F(w_1)| = |H(w_1)| \leq \|H\|_R \leq \frac{s^s C^{Ns} \|F\|_R}{R^{Ns}}$$
Then after working this out, we get
$$1 \leq |c\alpha| \leq \frac{s^{2s} C^{Ns}}{\mathrm{e}^{Ns \log(s)/2\rho}}$$
which gets to $N \leq 4\rho$. $\square$

9.2 Moduli Spaces of Elliptic Curves

5/16: - Announcements.
    - PSet 5 due tomorrow.
    - Final Tuesday.
    - Project due end of day Tuesday.
    - Final presentations in my office (E313) unless you hear otherwise.
        - We can show up in his office to watch other people's questions.
    - Stop rescheduling!
- No notes will be posted for today; it's like three weeks worth of content.
    - Nothing from Week 9 will be on the final exam!
- Today: Moduli spaces of elliptic curves.
    - A topic near and dear to Calderon's heart that uses complex analysis heavily.
    - Calderon is first and foremost a topologist/geometer.
- Theorem (Topological classification of surfaces): Consider a 2-manifold space locally homeomorphic to $\mathbb{R}^2$ that is compact without boundary (e.g., closed). All of the closed, orientable 2-manifolds are classified by their number of holes, i.e., homeomorphic go a genus n surface.
- Let's equip our surface with a complex structure. Essentially, instead of charting pieces to $\mathbb{R}^2$, we'll chart them to $\mathbb{C}$!
    - An elliptic curve E is just a complex torus.
    - What are holomorphic functions on Riemann surfaces?
        - Recall that $f \in \mathcal{O}(U)$ iff $f \circ \phi_U^{-1}$ is holomorphic on $\phi_U(U)$.
        - They are constant on $\hat{\mathbb{C}}$! This is just Liouville's theorem again.
        - They are also constant on E.
    - If $U \subset E$ is a nice open set, it maps to a domain.
- There are many Riemann surface structures.
    - For example, transition maps are translations.
    - A projective plane curve is another.
- **Complex projective space** (of dimension n): *Denoted by* $\mathbb{CP}^n$. *Given by*

$$\mathbb{CP}^n := \left(\mathbb{C}^{n+1} \setminus \{(0, \dots, 0)\}\right) /\text{scaling}$$

    - This is the set of lines through the origin, e.g., $(1,1,1) = (2,2,2)$.
- Example: $\mathbb{RP}^1 = \mathbb{R}^2 \setminus 0/\text{scaling}$.
    - We get rays and lines.
- TPS: $\mathbb{CP}^1 \cong \hat{\mathbb{C}}$, where $\cong$ denotes a biholomorphic equivalence.
    - Let $\{(x,y)\}/\text{scaling} \mapsto x/y$.
    - This does preserve scaling: $(2x, 2y) \mapsto x/y$, as well!
    - It also is onto: $(x,1) \mapsto x \in \hat{\mathbb{C}}$, and any point $(x,0) \mapsto \infty$.
    - Then to prove biholomorphic-ness, on charts to $\mathbb{C}$:

    - If $y \neq 0$, send $x, y \mapsto x/y$.
    - If $x \neq 0$, send $x, y \mapsto y/x$.
    - The transition map between these two is $1/z$.

- We can homogenize...

- Uniformization: Every complex torus looks like $\mathbb{C}/\Lambda$, where

$$\Lambda = \{n_1 w_1 + n_2 w_2 \mid n_i \in \mathbb{Z},\ w_i \text{ being } \mathbb{R}\text{-linearly independent}\}$$

  - Every complex torus also has a representation as $\{y^2 z = x^3 + axz^2 + bz^3\}$.
    - Apply the Weierstrass $\wp$-function.

- Question: How many ways are there to do this and get different tori?

  - If $c \in \mathbb{C}$, then $c\omega_2$ and $c\omega_1$ gives the same torus up to biholomorphism.
    - You can scale the torus on the plane.

- Up to scaling, assume $\omega_1 = 1$.

  - We'll now start calling ω_2 by τ.

- Discussion of the Gaussian integers.

- The set of complex tori is equal to $\mathbb{H}$ with $SL_2\mathbb{Z}$ modded out. In particular, $SL_2\mathbb{Z} \circlearrowright \mathbb{Z}^2$ by changing basis.

- In fact, we're interested in $PSL_2\mathbb{Z}$, which lives inside $PSL_2\mathbb{R}$.

- Something with matrices and getting a tesselation of the upper half plane with circular arcs.

- In conclusion, the space of complex tori is called a modular curve.