Week 8

Applying the Complex Logarithm

8.1 Counting Zeroes and Laurent Series

5/7: • Announcements.

  - PSet 5 posted, due next Friday.
    - 10 problems (a few more than usual), but heavily computational so shouldn't be that bad.
    - After today's class, we'll have everything we need to do it.
  - Draft of final project report due this Friday.
    - It doesn't need to be a full draft, but it should be a pretty well-fleshed-out outline. The more information we can give him, the better feedback he can give us.
  - Make sure to check out the Canvas post on final presentation scheduling.

- Last time.
  - We deduced the residue theorem from the general CIT.
    - It states: Suppose U is a domain, $S \subset U$ is a discrete (though not necessarily finite) set of singularities, $f \in \mathcal{O}(U \setminus S)$, and $\Gamma = \sum c_i \gamma_i$ is nulhomologous in U (though not necessarily nulhomologous in $U \setminus S$). Then

$$\frac{1}{2\pi i} \int_\Gamma f \, \mathrm{d}z = \sum_{s \in S} \operatorname{wn}(\Gamma, s) \operatorname{res}_s f$$

  - The residue of f about s is

$$\operatorname{res}_s f = \frac{1}{2\pi i} \int_{\partial D} f \, \mathrm{d}z$$

    where D is some little disk about s (and only this $s \in S$).
    - If s is a pole, then $\operatorname{res}_s f$ is also equal to the a_{-1} coefficient of the Laurent expansion.
  - What the residue theorem gets us.
    - Lets us compute integrals over complicated paths.
      - ➢ We get to work with Laurent series instead of integrals, which is easier.
    - Lets us compute sums, as in the Basel problem.
      - ➢ We choose a function, introduce the residue, and express the sum in terms of the integral.

- Today.
  - **The argument principle**: A theoretical (not practical) ramification of the residue theorem.
  - Talking a bit more about essential singularities.

- Consider the function f'/f, where f is either holomorphic or meromorphic on U.
  - Another way to think about this function is as the derivative of $\log f$.
  - We want to investigate the singularities of f'/f.
  - Suppose f has a zero of order k at s.
    - Then locally, $f(z) = (z-s)^k g(z)$ where $g(s) \neq 0$.
    - If f looks like this at s, then $f'(z) = k(z-s)^{k-1}g(z) + (z-s)^k g'(z)$.
    - Thus,
$$\frac{f'}{f}(z) = \frac{k}{z-s} + \frac{g'(z)}{g(z)}$$
    - Since $g(s) \neq 0$, g'/g is a well-defined number in a disk about s.
    - The other term gives a well-defined pole.
    - Therefore, if f has a zero of order k at s, then f'/f has a simple pole at s with $\operatorname{res}_s(f'/f) = k$.
  - Exercise: If f has a pole of order k at s, then f'/f has a simple pole with $\operatorname{res}_s(f'/f) = -k$.
    - Prove the same way (or see the notes).
- Corollary (the argument principle): Suppose U is a domain, f is meromorphic on U, and γ is an SCC oriented counterclockwise that doesn't hit any poles or zeroes of f. Then
$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,\mathrm{d}z = \sum_{s\in S}\operatorname{res}_s(f'/f) = \#\text{ zeroes in }\gamma - \#\text{ poles in }\gamma$$
- **Simple closed curve**: A curve γ that separates the inside from the outside. *Also known as* **SCC**.
  - See Figure 7.2a for two examples.
- TPS: Use the argument principle to compute the number of zeroes of the following polynomial inside the unit disk $\mathbb{D}$.
$$f(z) = 2z^4 - 5z + 2$$
  - Since f is a polynomial, it has no poles.
  - Thus, by the argument principle, the number of zeroes in $\partial\mathbb{D}$ is
$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{f'}{f}\,\mathrm{d}z = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} \frac{8z^3-5}{2z^4-5z+2}\,\mathrm{d}z$$
  - The integral on the right is "a pain in the butt" to compute, but we definitely could.
    - We'd just have to do a partial fraction decomposition, substitute in $z = \mathrm{e}^{i\theta}$, and bash it out.
  - A better way to find the number of zeroes uses **Rouché's theorem**, which we'll introduce shortly.
- Understanding the argument principle geometrically.
  - See `IMG_9306.JPG` and `IMG_9364.JPG` for some context.
  - Suppose you have an SCC γ enclosing a zero or order 2 and a pole of order 1.
  - We'll now investigate f as a mapping of $\mathbb{C}\to\mathbb{C}$ and $\mathbb{C}\to\hat{\mathbb{C}}$.
  - As a mapping into the complex plane, f maps γ to $f(\gamma)$.
    - Note that the curve $f(\gamma)$ is just another mapping of the circle into the complex plane.
  - As a mapping into the Riemann sphere, f maps the zero to 0 and the pole to ∞.
  - Now draw little counterclockwise-oriented curves around the zero and pole in the domain.
    - Since the pole has order 1, its little loop maps to a little loop around $\infty \in \hat{\mathbb{C}}$ that goes around 1 time.

        - Since the zero has order 2, its little loop maps to a little loop around $0 \in \hat{\mathbb{C}}$ that goes around 2 times.
            - This is because locally, the area near an order 2 zero looks like z^2 and the complex function z^2 rotates the complex plane around on top of itself twice. Thus, a single loop will get spun around twice, a double loop will get spun around 4 times, and so on.
    - Now pull the two loops down the Riemann sphere to the equator.
        - Observe that their orientations are now inverses, with the orientation of the curve around ∞ having flipped.
            - This is like the Coriolis effect!
        - Projecting the pulled-down curves into $\mathbb{C}$, we can observe that the one around ∞ is oriented clockwise and encompasses $f(\gamma)$ while the one around 0 is oriented counterclockwise and situated within $f(\gamma)$.
    - This all results in the order of zeros minus the order of poles is equal to the winding number of $f(\gamma)$ about zero.
    - See Week 9 office hours for more.

- Here's a computation that justifies all of the handwavey stuff above:

$$\begin{aligned}
\mathrm{wn}(f(\gamma),0) &= \frac{1}{2\pi i}\int_{f(\gamma)} \frac{1}{z}\,\mathrm{d}z \\
&= \frac{1}{2\pi i}\int_0^1 \frac{1}{f(\gamma(t))}[f(\gamma(t))]'\,\mathrm{d}t \\
&= \frac{1}{2\pi i}\int_0^1 \frac{f'(\gamma(t))}{f(\gamma(t))}\gamma'(t)\,\mathrm{d}t \\
&= \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}\,\mathrm{d}z
\end{aligned}$$

    - This computation is another proof of the argument principle!

- Rouché's theorem: Let U be a domain, $\gamma \subset U$ an SCC, and $f, g \in \mathcal{O}(U)$. Suppose also that $|g| < |f|$ on γ. Then f and $f+g$ have the same number of zeroes.

  Proof. Set $h_\lambda(z) := f + \lambda g$ where $\lambda \in [0,1]$. Thus, $h_0 = f$ and $h_1 = f+g$. It follows by the argument principle that the number of zeroes of h_λ in γ (which is a discrete set) is

$$\frac{1}{2\pi i}\int_\gamma \frac{h_\lambda'}{h_\lambda}\,\mathrm{d}z = \frac{1}{2\pi i}\int_\gamma \frac{f'+\lambda g'}{f+\lambda g}\,\mathrm{d}z$$

  which is a continuous map in λ. Essentially, we have shown that the "number of zeroes in γ" function is a continuous function from $[0,1] \to \mathbb{N}_0$. Hence, as a continuous function into a discrete set, it is constant.

  Note that we used the $|g| < |f|$ condition to ensure that $f + \lambda g$ in the denominator of the above integral is never zero, and hence the integral is always well-defined. □

- Example: Solving the TPS from earlier.
    - $g(z) = 2z^4$ has four zeroes inside $\mathbb{D}$.
    - $f(z) = -5z$ has one zero inside $\mathbb{D}$.
    - On $\partial\mathbb{D}$, we have that $|z| = 1$ and hence

$$2 = |g| < |f| = 5$$

      so $2z^4 - 5z$ has the same number of zeroes as $-5z$ (or 1) by Rouché's theorem.

- Redefine $f(z) = 2z^4 - 5z$ and $g(z) = 2$.
- On $\partial\mathbb{D}$, we similarly have that

$$|f| = |2z^4 - 5z| \geq \left||2z^4| - |5z|\right| = |2 - 5| = 3 > 2 = |g|$$

so $2z^4 - 5z + 2$ has the same number of zeroes as $2z^4 - 5z$ (or 1) by Rouché's theorem.

- Takeaway: Whenever you're asked to compute zeroes, Rouché's theorem is probably the way to go.
- Exercise: Prove the FTA using Rouché's theorem.
  - Idea: On big enough circles, eventually the top-degree term dominates.
- Let's now talk a bit more about essential singularities.
- Suppose U is a small disk, $s \in U$, and $f \in \mathcal{O}(U \setminus S)$ (so s is an isolated singularity). Then one of three things can happen.
  1. s is removable.
     - In this case, we can remove it using Riemann's removable singularity theorem and get an analytic continuation $\hat{f} \in \mathcal{O}(U)$.
     - This is really nice, because then we get a power series near s:

$$f(z) = \sum_{k=0}^{\infty} a_k (z - s)^k$$

  2. s is a pole.
     - We get a similar series called a Laurent series with:

$$f(z) = \sum_{k=-N}^{\infty} a_k (z - s)^k$$

  3. s is essential.
     - We get a Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - s)^k$$

- A word on convergence.
  - Before, we used to say the magic words "absolutely locally uniformly" and we'd get convergence. Now we can't do that, and we need the following.
  - Recall that uniform convergence $f_k \to f$ means that

$$\sup_{z \in K} |f_k(z) - f(z)| \to 0$$

  - In addition, recall that local uniform convergence $f_k \to f$ means that

$$\sup_{K \subset U} \sup_{z \in K} |f_k(z) - f(z)| \to 0$$

- Theorem: Suppose $\{f_k\} \in \mathcal{O}(U \setminus S)$ is such that the $f_k \to f$ locally uniformly. Then $f \in \mathcal{O}(U \setminus S)$ and moreover $f_k^{(n)} \to f^{(n)}$ for all n.

Proof. Goursat plus Morera for the first statement. CIF for the second statement. More detail in the notes; Calderon is also happy to talk. □

- We now prove that such Laurent expansions exist.
  - Step 1: "Pull off" the singular part.
    - This is a theorem called the **Laurent decomposition**.
  - Step 2: Express f_∞ as

$$\sum_{k=-\infty}^{-1} a_k(z-s)^k$$

- Theorem (Laurent decomposition): There exists a unique $f_\infty \in \mathcal{O}(\mathbb{C} \setminus \{s\})$ with $f = f_0 + f_\infty$ such that $f_0 \in \mathcal{O}(U)$ and $f_\infty \to 0$ as $z \to \infty$.

Proof. Uniqueness is not interesting; see the book.

Existence: Let $z \in U$ be arbitrary. Let D_2 be a counterclockwise-oriented curve in U containing 0 and z. Let D_1 be a counterclockwise-oriented curve in U containing just 0. Then $z \in D_2 \setminus D_1$. Altogether, this looks like Figure 8.1.

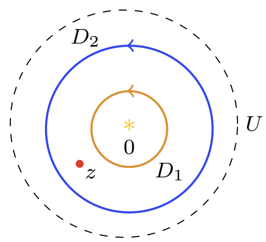

Figure 8.1: Laurent decomposition theorem.

Additionally, $\partial D_2 - \partial D_1$ is nulhomologous in $U \setminus 0$. Thus, by the general CIF,

$$f(z) = \frac{1}{2\pi i}\int_{\partial D_2 - \partial D_1} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta = \underbrace{\frac{1}{2\pi i}\int_{\partial D_2} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta}_{f_0} - \underbrace{\frac{1}{2\pi i}\int_{\partial D_1} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta}_{-f_\infty}$$

Finally, $-f_\infty$ — as defined above — has an extension from U to all of $\mathbb{C} \setminus \{0\}$ just like in the proof of the general CIF from the 4/30 lecture. ☐

- Comments on the Laurent decomposition.
  - Think of f_0 as the nonnegative terms of the Laurent expansion, and f_∞ as everything else (all the negative terms).
- Step 2 proof.

Proof. For poles, we could just multiply by $(z-s)^k$. Here, we have to do something very clever. WLOG, let $s = 0$. By step 1, f_∞ extends to ∞ and we may set $f_\infty(\infty) = 0$. Thus, let's think of $f_\infty \in \mathcal{O}(\hat{\mathbb{C}} \setminus \{0\})$. Set $g(z) = f_\infty(1/z)$. Now g has a removable singularity at 0. Moreover, $g \in \mathcal{O}(\mathbb{C})$. Then

$$g(z) = \sum_{k=0}^{\infty} b_k(z-0)^k$$

converges for all z. But therefore

$$f_\infty(z) = g(1/z) = \sum_{k=1}^{\infty} b_k \left(\frac{1}{z}\right)^k = \sum_{k=-\infty}^{-1} a_k z^k$$

as desired. ☐

8.2 Riemann Mapping Theorem

5/9:

- Announcements.
  - Sign up for a presentation time.
  - Draft due tomorrow.
  - Feedback forms due next week.
  - Next Monday's OH's are TBD.
  - Substitute teacher (postdoc) next Tuesday because Calderon will be out of town.
- Last time: Using the logarithm to find some things out about zeroes.
- Today: Using the logarithm to find some things out about conformal maps.
- Recall.
  - $f : U \to \mathbb{C}$ conformal means angle-preserving diffeomorphism.
  - $f : U \to \mathbb{C}$ biholomorphic means bijective, holomorphic, and f^{-1} holomorphic.
  - Conformal iff biholomorphic.
  - If $f \in \mathcal{O}(U)$ is bijective and f' is nonzero on U, then f^{-1} is holomorphic and hence $f \in \text{Bihol}(U)$.
    - In fact, using the $f'(z) \neq 0$ for all $z \in U$ and the open mapping theorem to imply that f^{-1} is holomorphic was unnecessary! The complex inverse function theorem proved that all we need for biholomorphism is a bijectivity and a holomorphicity condition.
  - Important note: Above, we used the bijectivity condition to imply a "nonzeroness" to the derivative. Suppose $f'(z_0) = 0$ and $f(z_0) = 0$. Then f can't be bijective for the following reason.
    - This combined with the hypothesis that $f \in \mathcal{O}(U)$ and hence f has a Taylor series expansion means that

$$f(z) = \underbrace{f(z_0)}_{0} + \underbrace{f'(z_0)}_{0}(z - z_0) + \sum_{k=2}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k = \sum_{k=2}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z - z_0)^k$$

    - Hence, we can factor $(z - z_0)^2$ out of f, so f locally looks like z^2 and therefore is not injective. This is branching behavior like in Figure 6.1; z^2 cuts a slit in the complex plane at the positive real axis and spins it around and then back to where it started.
- We wish to investigate conformal, bounded maps on bounded, simply connected domains. Before we begin in earnest, we will show that we can simplify any function in this class to an analogous function in a more restricted class that will be easier to work with.
- Let U be a bounded, simply connected domain. Suppose $f \in \mathcal{O}(U)$ is conformal and bounded.

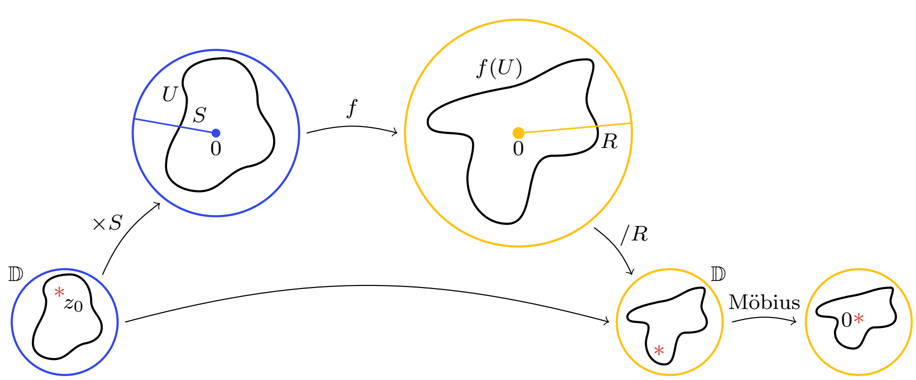

Figure 8.2: Constructing an analogous, simpler class of bounded conformal maps.

- Intuitively, these constraints imply that f maps a blob to a blob.
  - Conformal maps send simply connected domains to simply connected domains.
  - Bounded maps send bounded domains to bounded domains, i.e., the image blob must lie in some disk.
- By the maximum modulus principle, if $|f(z)| = R$, then $z \in \partial U$.
  - This is valid because a conformal map must be nonconstant.
- Now here comes the critical step: We can rescale the problem of a map from a blob of radius S to a blob of radius R to the problem of a map from a blob in $\mathbb{D}$ to another blob in $\mathbb{D}$.
  - We do this by prescaling the initial $\mathbb{D}$ up by S and postscaling the image disk down by R.
- Therefore, via rescaling, we can reduce this problem to understanding maps from $U \subset \mathbb{D} \to \mathbb{D}$.
- Now fix $z_0 \in U$. Recall from PSet 4 that we can use a Möbius transformation to take $f(z_0) \mapsto 0$.
  - Fact/Exercise: A Möbius transformation $\mathbb{D} \to \mathbb{D}$ of the form $z \mapsto (az+b)/(cz+d)$ for $a, b, c, d \in \mathbb{R}$ acts transitively on points in $\mathbb{D}$ (i.e., any point can be taken to zero).
  - It's kind of obvious if you do this with the upper half plane instead of the disk and then bring it to the disk.
- Therefore, via rescaling *and* Möbius transformations, we can reduce this problem to understanding maps from $U \subset \mathbb{D} \to \mathbb{D}$ that take $z_0 \in U \mapsto 0$.

- Let's begin investigating these maps.
- The first (and easiest) case we'll tackle is when $U = \mathbb{D}$. We'll tackle it with the **Schwarz Lemma**.
- Schwarz Lemma: Suppose $f : \mathbb{D} \to \mathbb{D}$ (not necessarily surjective) is conformal and $f(0) = 0$. Then...

1. $|f(z)| \leq |z|$;

   Proof. Consider the function defined by the following ratio.

$$\frac{f(z)}{z}$$

This function is holomorphic for all $z \neq 0$. But what happens at zero? We know that $f(0) = 0$ by hypothesis. Thus, f has a power series expansion at zero. In particular, since $a_0 = f(0) = 0$,

$$f(z) = a_1 z + a_2 z^2 + \cdots$$

so

$$\frac{f(z)}{z} = a_1 + a_2 z + \cdots$$

This means that $f(z)/z$ has a removable singularity which we can fill in via Riemann's removable singularity theorem to get

$$\frac{f(0)}{0} = f'(0)$$

Now pick an $r \in (0, 1)$ and let $z \in \partial D_r(0)$ be arbitrary. Then

$$\left|\frac{f(z)}{z}\right| = \frac{|f(z)|}{|z|} = \frac{|f(z)|}{r} \leq \frac{1}{r}$$

Essentially, we have bounded the value of $f(z)/z$ on the boundary of a disk. Moreover, it follows by the maximum modulus principle that this bound must hold for all $z \in D_r(0)$. (Otherwise, we would get a local maximum inside $D_r(0)$; this would imply by the MMP that f is constant, which we can't have because the function is conformal [hence nonconstant] by hypothesis.) Thus, we have shown that

$$|f(z)| \leq \frac{1}{r} \cdot |z|$$

for all $z \in D_r(0)$. To finish it off, we can send $r \to 1$, thereby including all $z \in \mathbb{D}$ in our argument and transforming the above inequality into the desired one. $\square$

2. If $|f(z)| \leq |z|$ is an equality for any $z \in \mathbb{D}$, then f is a rotation;

   Proof. Let $z_0 \in \mathbb{D}$ be the point at which $|f(z_0)| = |z_0|$. It follows that
   $$\left|\frac{f(z_0)}{z_0}\right| = 1$$
   Additionally, we have by (1) that for all $z \in \mathbb{D}$,
   $$\left|\frac{f(z)}{z}\right| \leq 1$$
   Thus, $|f(z)/z|$ takes a maximum inside $\mathbb{D}$ (in particular, it takes its max at z_0). It follows by the maximum modulus principle that $f(z)/z$ is constant on $\mathbb{D}$.
   Let c be this constant value. Naturally, it follows that in particular, $c = f(z_0)/z_0$ and hence
   $$|c| = \left|\frac{f(z_0)}{z_0}\right| = 1$$
   But via algebraic rearrangement this means that $f(z) = cz$ where $c \in \mathbb{C}$ has $|c| = 1$. Therefore, f is a rotation, as desired. □

3. If $|f'(0)| = 1$, then f is a rotation.

   Proof. In the proof of part (1), we showed that the power series expansion of $f(z)/z$ is
   $$\frac{f(z)}{z} = a_1 + a_2 z + \cdots = f'(0) + \frac{f''(0)}{2!}z + \cdots$$
   Thus, since $|f'(0)| = 1$, we have that
   $$\left|\widehat{\frac{f(0)}{0}}\right| = \left|f'(0) + \frac{f''(0)}{2!}\cdot 0 + \cdots\right| = |f'(0)| = 1$$
   Thus, as in the proof of part (2), we have shown that there exists a point in $\mathbb{D}$ at which $|f(z)/z| = 1$. Therefore, f is a rotation by a symmetric argument, as desired. □

- Note that we did not use conformality anywhere in our proof of the Schwarz Lemma!
  - All we actually needed was a holomorphicity condition.
  - We'll need full conformality, however, to prove the following sort-of converse.
- Lemma (a "converse" to the Schwarz Lemma): Suppose $f : \mathbb{D} \to \mathbb{D}$ is conformal and not onto with $f(0) = 0$. Then there exists $F : \mathbb{D} \to \mathbb{D}$ (still conformal) such that $F(0) = 0$ and $|F'(0)| > |f'(0)|$.

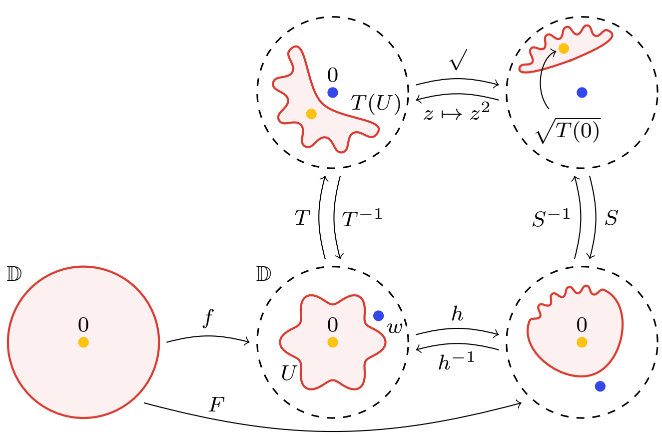

Figure 8.3: A "converse" to the Schwarz Lemma.

Proof. We will prove this claim via *constructive* means, actually building a function F as opposed to just proving its existence in the abstract. In particular...

1. We will construct a function h, and prove that it has some special properties;[1]
2. We will define $F = h \circ f$;
3. We will use the special properties of h to prove that F, as defined, has the desired properties.

Let's begin.

Let $U := f(\mathbb{D})$. Since f is conformal and $\mathbb{D}$ is simply connected, $f(\mathbb{D})$ is simply connected.[2] We now construct the aforementioned function h.

To construct h, we will build three other functions — T, $\sqrt{}$, and S — and let h be their composition.

1. Since f is not onto by hypothesis, there exists $w \in \mathbb{D}$ such that $w \notin U$. Let T be a Möbius transformation sending $w \mapsto 0$.
2. Since T is a Möbius transformation (hence biholomorphic and hence conformal) and U is simply connected, $T(U)$ is simply connected. Thus, there exists a logarithm function on $T(U)$. It follows that there exists a square root function $\sqrt{} : z \mapsto z^{1/2} = \mathrm{e}^{0.5 \log z}$ on U.
3. Lastly, let S be a Möbius transformation sending $\sqrt{T(0)} \mapsto 0$.

At this point, we can finally define $h : U \to \mathbb{D}$ by

$$h(z) := (S \circ \sqrt{} \circ T)(z)$$

We will now prove two special properties of h: $h(0) = 0$ and $|h'(0)| > 1$. By the construction of T, $\sqrt{}$, S, we immediately obtain $h(0) = 0$. As to the other property, begin with defining $h^{-1} : h(U) \to U$ by

$$h^{-1}(z) := (T^{-1} \circ [z \mapsto z^2] \circ S^{-1})(z)$$

Since h^{-1} is a composition of holomorphic and bijective maps, h^{-1} is conformal. This combined with the facts that $h^{-1}(0) = 0$ and h^{-1} is not a rotation (clearly) implies by the contrapositive of the Schwarz Lemma that $|(h^{-1})'(0)| < 1$. Therefore, by the inverse function theorem, $|h'(0)| > 1$.

At this point, we may define a function F by

$$F := h \circ f$$

We now check F's four properties. Since $f : \mathbb{D} \to U$, $h : U \to \mathbb{D}$, and $F = h \circ f$, we have that $F : \mathbb{D} \to \mathbb{D}$ via function composition, as desired. Since T, $\sqrt{}$, and S are conformal, we have that F is conformal, as desired. Since $f(0) = 0$ by hypothesis, $h(0) = 0$ by the above, and $F = h \circ f$, we have that

$$F(0) = h(f(0)) = h(0) = 0$$

as desired. And since $|h'(0)| > 1$, we conclude by observing the following.

$$|F'(0)| = |h'(f(0)) \cdot f'(0)| = |h'(0)| \cdot |f'(0)| > 1 \cdot |f'(0)| = |f'(0)|$$

∎

- Takeaway: The Schwarz Lemma tells us that conformal maps from $\mathbb{D} \to \mathbb{D}$ have derivative ≤ 1, and this lemma tells us that maps from $U \subsetneq \mathbb{D} \to \mathbb{D}$ can be made to have a larger derivative.
- Note that the behavior in the Schwarz and this lemma is analogous to that of *entire* biholomorphims, where f conformal implies f biholomorphic implies $f(z) = az + b$, this plus $f(0) = 0$ implies $f(z) = az$, and $f : \mathbb{D} \to \mathbb{D}$ implies $|a| \leq 1$.

[1] We can also prove a more general version of this claim in which we let $z_0 \in U$ be arbitrary and construct h with $h(z_0) = 0$ and $|h'(z_0)| > 1$ by adding in an additional Möbius transformation in the beginning sending $z_0 \mapsto 0$. This is what we did in class, but it it not actually necessary for the proof, so I have left out the argument here. My raw, unedited notes from class are still available in the .tex file, for future reference.

[2] Some argument about how conformal maps preserve the winding number of $\gamma \subset U$ about an exterior point being zero??

- We now prove our main result for the day.
- Riemann mapping theorem: Let U be any simply connected domain in $\mathbb{C}$ (but not $\mathbb{C}$ itself). Then there exists a conformal map $f : U \to \mathbb{D}$.
- Comments on the RMT.
  - So every simply connected domain is either $\mathbb{C}$ or the unit disk.
  - Exercise: If we stipulate also that some $z_0 \in U \mapsto 0$ and $f'(z_0) \in \mathbb{R}_{>0}$, then f is unique.
  - Why we delayed proving this masterpiece theorem until the end of the class: We just developed the tool we need.
- Three-step proof outline.
  1. Map U into $\mathbb{D}$.

     Proof Sketch. Suppose U misses some open set in $\hat{\mathbb{C}}$. (If it misses some open set, it misses a ball.) Now, just use a Möbius transformation, specifically the one that takes this ball to the entire northern hemisphere, thereby making U something else that lives in the southern hemisphere. Make sure that this map also sends some point $z_0 \in U$ to $0 \in \hat{\mathbb{C}}$! Then remember that the southern hemisphere is just the unit disk (see Figure 5.2.) Great pictures on the board!

     If U doesn't miss an open set (e.g., $U = \mathbb{C} \setminus \mathbb{R}_{\leq 0}$), then use a log, which we would be able to do because the set is simply connected. □
  2. "Maximize" among all such maps.

     Proof Sketch. If $f : U \to \mathbb{D}$ is conformal and not onto, then use the Converse Lemma to find F with a bigger derivative at z_0.

     Thus, "just" take an f maximizing $|f'(z_0)|$. We can't find something with a bigger derivative at this point, so the function must be onto.

     We now pull a rabbit out of the hat in the form of the real analytical/metric space form of the **Arzelà-Ascoli theorem**. □
  3. Take a limit of this maximizing sequence (this is where the real magic happens, using the argument principle and such).

     Proof Sketch. Set $\mathcal{F} = \{f : U \to \mathbb{D} \mid \text{conformal}, f(z_0) = 0\}$. This family is uniformly bounded (e.g., by the unit disk). Let
     $$\alpha := \sup_{f \in \mathcal{F}} |f'(z_0)|$$
     The set of all f_k such that $|f_k'(z_0)| \to \alpha$ has a uniformly convergent subsequence. If $f_k \to f$ locally uniformly, then by the theorem from last class, f is holomorphic. Additionally, f is also conformal because it is holomorphic and bijective (see the following proposition), hence biholomorphic. □
- **Arzelà-Ascoli theorem**: If $\{f_k\} : X \to Y$ (where X, Y are metric spaces) is a family of (locally) uniformly bounded and (locally) equicontinuous functions, then there exists a (locally) uniformly convergent subsequence.
- **Equicontinuity**: Same ε's and δ's work for every f_k.
- Proposition: If f_k are all conformal, then f is either conformal or constant.

Proof. Suppose that f is not bijective. Hence, f is not injective. Thus, there are two points $z_1 \neq z_2$ such that $f(z_1) = w = f(z_2)$. Pick disks D_i about z_i such that on each D_i, only $z_i \mapsto w$.

Thus, $f(z) = w$ (i.e., $f(z) - w = 0$) twice on $D_1 \cup D_2$. Thus, by the argument principle,
$$2 = \int_{\partial D_1 + \partial D_2} \frac{f'(z)}{f(z) - w} \, \mathrm{d}z = \lim_{k \to \infty} \int_{\partial D_1 + \partial D_2} \frac{f_k'(z)}{f_k(z) - w} \, \mathrm{d}z \leq 1$$
which is a contradiction since the f_k are injective, so f is bijective. □

- With just a bit of work, we can get from here to Montel's theorem.
- Montel's theorem: If $\{f_k\}$ is a family of (locally) uniformly bounded holomorphic functions, then there exists a (locally) uniformly convergent subsequence.

  Proof. (Locally) uniformly bounded plus holomorphic implies (locally) equicontinuous. $\square$

- Montel's theorem allows us to extract limits.