2 Differential Equations and Special Functions

2.1 Infinite Series

5/10:
- If we're going to be extensively working with infinite series, we might as well review their properties.
- Defines the $\boldsymbol{n}^{\textbf{th}}$ **partial sum**, **convergence**, **absolute** (convergence), and **uniform** (convergence).
- Properties of uniformly convergent infinite series.
  1. $\{u_k\}$ continuous $\Longrightarrow u$ continuous.
  2. $\{u_k\}$ continuous $\Longrightarrow u$ integrable term by term.
  3. $\{u_k\} \subset C^1$ and $u_k' \to u'$ uniformly $\Longrightarrow u$ differentiable term by term.
- Assume that all of these properties hold for every series in Seaborn (1991) unless explicitly stated otherwise.

2.2 Analytic Functions

- **Real analytic** (function in (a,b)): A function f such that for each point $x_0 \in (a,b)$, $f(x)$ can be written as a power series

$$f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

  where the numbers c_n are independent of x.
  - The functions we encounter in physics and applied mathematics are generally analytic.
  - The functions we encounter in this book certainly will be.
  - Any function f that is analytic in the interval (a,b) may be represented by its **Taylor series** expanded about any point x_0 in the interval.
- **Radius of convergence**. *Denoted by* $\boldsymbol{R}$.
- Seaborn (1991) motivates the **Pochhammer symbol** by using it to rewrite the Taylor series for $f(z) = (1-z)^s$.
- **Pochhammer symbol**: The number defined inductively as follows, where $a \in \mathbb{C}$ and $n \in \mathbb{N}_0$. *Denoted by* $\boldsymbol{(a)_n}$. *Given by*

$$\begin{aligned}
(a)_0 &:= 1 \\
(a)_n &:= a(a+1)(a+2)(a+3)\cdots(a+n-1) && (n = 1,2,3,\dots)
\end{aligned}$$

- Definition of the **geometric series**.
- Identities involving Pochhammer symbols.
  1. $n! = (n-m)!(n-m+1)_m$.
  2. $(c-m+1)_m = (-1)^m(-c)_m$.
  3. $(n+m)! = n!(n+1)_m$.
  4. $n! = m!(m+1)_{n-m}$.
  5. $(2n-2m)! = 2^{2n-2m}(n-m)!(\frac{1}{2})_{n-m}$.
  6. $(c)_{n+m} = (c)_n(c+n)_m$.
  7. $(c)_n = (-1)^m(c)_{n-m}(-c-n+1)_m$.
  8. $(c)_n = (-1)^{n-m}(c)_m(-c-n+1)_{n-m}$.
  9. $(-n)_{m-k} = (-n)_{m-n}(m-2n)_{n-k}$.

2.2.1 Series Expansion with Remainder

- $\boldsymbol{n}^{\textbf{th}}$ **remainder** (of f analytic): The difference between f and the first n terms of its Taylor series. *Denoted by* $\boldsymbol{R_n}$. *Given by*

$$R_n(x) := f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

- Seaborn (1991) derives the Lagrange error bound.
  - He also provides an integral formula and works an example.

2.2.2 Integration of Infinite Series

- "Infinite series that converge uniformly can be integrated term by term" (Seaborn, 1991, p. 22).
- This allows us to find the Taylor series for certain functions.
  - Example: We already know the Taylor series for $(1+x^2)^{-1}$ by extrapolating from the geometric series, and this function is just the derivative of $\tan^{-1}$!

2.2.3 Inversion of Series

- Same as Section II.4 of Fischer and Lieb (2012), but with more terms given.

2.3 Linear Second-Order Differential Equations

- A clever method of solution for any linear, second-order, homogeneous differential equation.
  - Such equations can be written in the form

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}u(z) + P(z)\frac{\mathrm{d}}{\mathrm{d}z}u(z) + Q(z)u(z) = 0$$

  - Rewrite the above as

$$u''(z) = f(z, u, u')$$

  - Suppose u, u' are defined at z_0.
  - Then the above gives $u''(z_0)$.
  - It follows by differentiating to

$$u^{(3)}(z) = \frac{\mathrm{d}f}{\mathrm{d}z} = f'(z, u, u')$$

    that $u^{(3)}$ can also be evaluated at z_0.
  - Assuming u is analytic, all higher derivatives of the above exist as well, so by evaluating these at z_0 and adding on the two given ones, we can construct the following Taylor series for u.

$$u(z) = \sum_{n=0}^{\infty} \frac{u^{(n)}(z_0)}{n!}(z-z_0)^n$$

  - If this series has a nonzero radius of convergence, then the solution exists.

2.3.1 Singularities of a Differential Equation

- **Ordinary point** (of an ODE): A point z_0 for which u, u' can be assigned arbitrary values and the solution still exists.
  - Example: In the harmonic oscillator, all times t_0 are ordinary points of the Newton's second law ODE because we can pick $x(t_0), x'(t_0) = v(t_0)$ arbitrarily and still solve the ODE for a trajectory.
- **Singular point** (of an ODE): A point z_0 for which u, u' cannot be assigned arbitrary values without the solution failing to exist somewhere. *Also known as* **singularity** (at z_0).
  - Example: The ODE
  $$z^2 u''(z) + azu'(z) + bu(z) = 0$$
  has a singularity at 0. Indeed, if $u(0)$ has any value other than 0, the above equation will not hold unless either $u'(0)$ or $u''(0)$ are infinite.
- The above two definitions are often alternatively stated as follows: If both P, Q are analytic at z_0, then z_0 is an ordinary point. Otherwise, the point is singular.

2.3.2 Singularities of a Function

- **Regular** (f at z_0): A point z_0 at which f is analytic.
- **Irregular** (f at z_0): A point z_0 at which f is not analytic.
- Definition of **pole** and **essential singularity**.
  - In Chapter 7, we'll learn about **branch points**, an additional type of singularity.

2.3.3 Regular and Irregular Singularities of a Differential Equation

- **Regular** (singularity of an ODE): A singular point z_0 of an ODE for which $(z - z_0)P(z)$ and $(z - z_0)^2 Q(z)$ are analytic at z_0.
- **Irregular** (singularity of an ODE): A singular point z_0 of an ODE that is not regular.

2.4 The Hypergeometric Function

- Definition of **rational** (function).
- All ODEs encountered in this book have at most three singularities.
  - A differential equation with at most three singularities has P, Q rational.
- A change of variables can convert such an ODE into Gauss's **hypergeometric equation**.
- **Hypergeometric equation**: The differential equation given as follows, where $a, b, c \in \mathbb{C}$ are constants independent of z. *Given by*
  $$z(1-z)\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} + [c - (a+b+1)z]\frac{\mathrm{d}u}{\mathrm{d}z} - abu = 0$$
  - This ODE has its singularities at $0, 1, \infty$.
- Since every ODE we will encounter for the rest of the book can be transformed into the hypergeometric equation, we need only solve it once. After that, we can express solutions to other ODEs in terms of this solution.
- Solving the hypergeometric equation.

- Use the ansatz

$$u(z)=\sum_{n=0}^{\infty}a_nz^{n+s}$$

- Substituting in, collecting terms, and setting each coefficient equal to zero gives the recursion relations

$$s(s+c-1)a_0=0 \qquad\qquad a_{n+1}=\frac{(n+s)(n+s+a+b)+ab}{(n+s+1)(n+s+c)}a_n$$

- We now divide into cases ($a_0=0$, $s=0$, and $s=1-c$).
  - $\underline{a_0=0}$: Implies that $a_n=0$ for all n, and hence $u(z)=0$ is the only solution.
  - $\underline{s=0}$: The recursion relation simplifies to

$$a_{n+1}=\frac{(a+n)(b+n)}{(n+1)(c+n)}a_n$$

    which yields the coefficients of the **hypergeometric function**.
  - $\underline{s=1-c}$: Discussed shortly.

- **Hypergeometric function**: The function defined as follows, which solves the hypergeometric equation in one case. *Denoted by* $\boldsymbol{F(a,b;c;z)}$. *Given by*

$$F(a,b;c;z):=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(c)_n}z^n$$

2.4.1 Examples

- We have

$$\sum_{n=0}^{\infty}z^n=F(1,b;b;z) \qquad\qquad (1-z)^s=F(-s,b;b;z)$$

  - Note that the left equation above is the geometric series!

2.4.2 Linearly Independent Solutions

- Miscellaneous observations, based on the form of the hypergeometric function.
  - If a or b is in $\mathbb{Z}_{\leq 0}$, then the series terminates (i.e., it is a polynomial).
  - The case where c is a negative integer or zero will be discussed shortly.
- Since the hypergeometric equation is a homogeneous, linear, second-order differential equation, its general solution is a linear combination of two **linearly independent** solutions u_1,u_2.
- **Linearly independent** (functions): Two functions u_1,u_2 such that $c_1u_1+c_2u_2=0$ iff $c_1=c_2=0$.
- $u_1(z)=F(a,b;c;z)$ is one solution.
- The other one may be obtained as follows from the $s=1-c$ case.
  - Substituting in and rearranging the original recursion relation yields

$$\begin{aligned}a_{n+1}&=\frac{[n+(2-c)-1][n+(2-c)-1+a+b]+ab}{[n+(2-c)](n+1)}a_n\\&=\frac{(n+c'-1+a)(n+c'-1+b)}{(n+1)(n+c')}a_n\\&=\frac{(a'+n)(b'+n)}{(n+1)(c'+n)}a_n\end{aligned}$$

- Thus, returning the substitutions,

$$u_2(z) = z^{1-c}F(1+a-c, 1+b-c; 2-c; z)$$

- Therefore, the general solution of the hypergeometric equation is

$$u(z) = AF(a,b;c;z) + Bz^{1-c}F(1+a-c, 1+b-c; 2-c; z)$$

2.4.3 If c is an Integer

- If $c = 1$, then $u_2(z)$ is not a new solution.
- If $c \geq 2$, then
$$(2-c)_k = (2-c)(3-c)\cdots(-1)\cdot 0\cdot(-n+k+1)!$$
  - Thus, the denominator vanishes in higher order terms and u_2 is not a valid solution.
- If $c \leq 0$, then
$$(c)_k = (-n)(-n+1)\cdots(-1)\cdot 0\cdot(-n+k-1)!$$
  - Similarly, the denominator vanishes in higher order terms and u_2 is not a valid solution.
- If $c \in \mathbb{Z}$ and a or b is an integer, too, then it may be possible to have solutions given by both series.
  - Example given.

2.5 The Simple Pendulum

- Seaborn (1991) uses the hypergeometric function and elliptic integrals to solve the simple pendulum of classical mechanics *exactly*, i.e., without resorting to the small angle approximation.
- Excellent to see! Come back to if I have time.

2.6 The Generalized Hypergeometric Function

- **Generalized hypergeometric function**: The function defined as follows. *Denoted by* $_p\boldsymbol{F}_q$. *Given by*

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z) := \sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{n!(b_1)_n\cdots(b_q)_n}z^n$$

2.7 Vandermonde's Theorem

- See `AIMEPrep.pdf`.
- **Vandermonde's theorem**: The following powerful relation useful in manipulating sums involving Pochhammer symbols. *Given by*

$$\sum_{m=0}^{n}\frac{(a)_m}{m!}\frac{(b)_{n-m}}{(n-m)!} = \frac{(a+b)_n}{n!}$$

Proof. Given. □

- Definition of the **Cauchy product**.

2.8 Leibniz's Theorem

- **Leibniz's theorem**: The following formula for the m^{th} derivative of the product of two analytic functions u,v. *Given by*

$$\frac{\mathrm{d}^m}{\mathrm{d}x^m}[u(x)v(x)] = \sum_{k=0}^{m} \frac{(m-k+1)_k}{k!} \left[\frac{\mathrm{d}^k}{\mathrm{d}x^k}u(x)\right] \left[\frac{\mathrm{d}^{m-k}}{\mathrm{d}x^{m-k}}v(x)\right]$$

Proof. Given; follows from Vandermonde's theorem. □