10 Integral Representations of Special Functions

10.1 The Gamma Function

5/18:
- Has some good background for why we need integral representations, as well as some other useful forms of Γ that more clearly illustrate its properties (such as poles at the nonpositive integers).
  - In particular, contour integrals provide "recursion formulas, asymptotic forms, and analytic continuations of the special functions" (Seaborn, 1991, p. 171).
- Main results.
  - We get an analytic continuation of Γ to the left half plane.
  - Then its the aforementioned poles.

10.4 Legendre Polynomials

- In this section, we will derive two contour integral representations of the Legendre polynomials. The first one will be much more useful, so we will spend more time on it.
- Deriving the first contour integral representation.
  - Recall Rodrigues's formula, which is

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^2-1\right)^n$$

  - Also recall the n^{th} derivative of the CIF either from class or Seaborn (1991), which states that given a complex function f and C a curve surrounding z such that no singularities of f lie within it, we have

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}f(z) = \frac{n!}{2\pi i}\oint_C \frac{f(t)}{(t-z)^{n+1}}\,\mathrm{d}t$$

  - Analytically continue $(x^2-1)^n$ to $(z^2-1)^n \in \mathcal{O}(\mathbb{C})$ so that by the above,

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(z^2-1\right)^n = \frac{n!}{2\pi i}\oint_C \frac{(t^2-1)^n}{(t-z)^{n+1}}\,\mathrm{d}t$$

  - Then **Schläfli's integral** for the $P_n(z)$ follows by transitivity.
- **Schläfli's integral** (for the $P_n(z)$): The integral formula for the Legendre polynomials given as follows. *Given by*

$$P_n(z) = \frac{1}{2^n}\frac{1}{2\pi i}\oint_C \frac{(t^2-1)^n}{(t-z)^{n+1}}\,\mathrm{d}t$$

- Schläfli's integral is not particularly useful for direct computations, but it gets us both a recursion formula for the Legendre polynomials and, later, the generating function.
- Using Schläfli's integral to find a recursion formula for the Legendre polynomials.
  - First, we write P_n' in a certain form.

$$\begin{aligned}
P_n'(z) &= \frac{1}{2^n}\frac{1}{2\pi i}\oint_C (t^2-1)^n \frac{\mathrm{d}}{\mathrm{d}z}[(t-z)^{-(n+1)}]\,\mathrm{d}t\\
&= \frac{1}{2^n}\frac{1}{2\pi i}\oint_C (t^2-1)^n \cdot -(n+1)(t-z)^{-(n+2)} \cdot -1\,\mathrm{d}t\\
&= (n+1)\frac{1}{2^n}\frac{1}{2\pi i}\oint_C \frac{(t^2-1)^n}{(t-z)^{n+2}}\,\mathrm{d}t
\end{aligned}$$

- Separately, we rewrite Schläfli's integral in a form that lines up with the new integral above.

$$
\begin{aligned}
P_n(z) &= \frac{1}{2^n}\frac{1}{2\pi i}\oint_C \frac{(t-z)(t^2-1)^n}{(t-z)^{n+2}}\,\mathrm{d}t \\
P_n(z) &= \frac{1}{2^n}\frac{1}{2\pi i}\left[\oint_C \frac{t(t^2-1)^n}{(t-z)^{n+2}}\,\mathrm{d}t - z\oint_C \frac{(t^2-1)^n}{(t-z)^{n+2}}\,\mathrm{d}t\right] \\
z\cdot\frac{1}{2^n}\frac{1}{2\pi i}\oint_C \frac{(t^2-1)^n}{(t-z)^{n+2}}\,\mathrm{d}t &= \frac{1}{2^n}\frac{1}{2\pi i}\oint_C \frac{t(t^2-1)^n}{(t-z)^{n+2}}\,\mathrm{d}t - P_n(z) \\
\frac{1}{2^n}\frac{1}{2\pi i}\oint_C \frac{(t^2-1)^n}{(t-z)^{n+2}}\,\mathrm{d}t &= \frac{1}{2^n z}\frac{1}{2\pi i}\oint_C \frac{t(t^2-1)^n}{(t-z)^{n+2}}\,\mathrm{d}t - \frac{1}{z}P_n(z)
\end{aligned}
$$

- Combining the last two results, we obtain

$$
P_n'(z) = \frac{n+1}{2^n z}\frac{1}{2\pi i}\oint_C \frac{t(t^2-1)^n}{(t-z)^{n+2}}\,\mathrm{d}t - \frac{n+1}{z}P_n(z)
$$

- We now start working on simplifying the left term above. Observe that

$$
\begin{aligned}
0 &= \oint_C \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{(t^2-1)^{n+1}}{(t-z)^{n+2}}\right]\mathrm{d}t \\
&= \oint_C \frac{(t-z)^{n+2}\cdot(n+1)(t^2-1)^n\cdot 2t - (t^2-1)^{n+1}\cdot(n+2)(t-z)^{n+1}}{(t-z)^{2n+4}}\,\mathrm{d}t \\
&= 2(n+1)\oint_C \frac{t(t^2-1)^n}{(t-z)^{n+2}}\,\mathrm{d}t - \underbrace{(n+2)\oint_C \frac{(t^2-1)^{n+1}}{(t-z)^{n+3}}\,\mathrm{d}t}_{2\pi i 2^{n+1}P_{n+1}'(z)}
\end{aligned}
$$

  - Note that the integral in the first line, above, is zero because the integrand is an exact differential integrated around a closed loop. Essentially, we are applying the fact (from the 3/28 lecture) that the integrand has a primitive, so we can apply the FTC to a path with the same start and end points.
  - Additionally, it follows by rearranging the above expression that

$$
(n+1)\oint_C \frac{t(t^2-1)^n}{(t-z)^{n+2}}\,\mathrm{d}t = 2\pi i 2^n P_{n+1}'(z)
$$

- Substituting this back into the above expression for $P_n'(z)$ yields

$$
P_n'(z) = \frac{1}{z}P_{n+1}'(z) - \frac{n+1}{z}P_n(z)
$$

- This equation rearranges into the final recursion formula

$$
zP_n'(z) + (n+1)P_n(z) - P_{n+1}'(z) = 0
$$

- Seaborn (1991) — as mentioned — now derives one additional contour integral representation of the Legendre polynomials.
- **Laplace's integral representation** (for $P_n(z)$): The integral formula for the Legendre polynomials given as follows. *Given by*

$$
P_n(z) = \frac{1}{\pi}\int_0^\pi (z+\sqrt{z^2-1}\cos\phi)^n\mathrm{d}\phi
$$

  - Note that despite the integral being taken between two real numbers, this is still a complex contour integral since ϕ feeds into a cosine function that wraps into a contour.

10.6 Hermite Polynomials

- Here are two integral representations that will be derived in later chapters.
  - If C encloses the origin, then
$$H_n(x) = \frac{n!}{2\pi i} \oint_C \frac{\mathrm{e}^{2xt-t^2}}{t^{n+1}} \, \mathrm{d}t$$
    - The recursion formula for the Hermite polynomials will be derived from this contour integral.
  - Additionally, we have
$$H_n(x) = \frac{i^n}{2\sqrt{\pi}} \int_{-\infty}^{\infty} t^n \mathrm{e}^{-(t+2ix)^2/4} \, \mathrm{d}t$$

10.7 The Hypergeometric Function

- Integral representations for both the hypergeometric function and confluent hypergeometric function are derived using properties of Γ.

10.8 Asymptotic Expansions

- **Asymptotic series** (of f): The infinite series $\sum_{k=0}^{\infty} a_k z^{-k}$ satisfying the following constraint. *Also known as* **asymptotic expansion**. *Constraint*
$$\lim_{|z|\to\infty} z^n \left[f(z) - \sum_{k=0}^{n} a_k z^{-k} \right] = 0 \qquad (n > 0)$$
  - This means that for a given n, if $|z|$ is large enough, then the partial sum approximates $f(z)$.
- Asymptotic expansions are useful in quantum mechanics when we want to talk about the behavior of a given wave function at points far from the source of the field to which the quantum particle is subject.
- Seaborn (1991) uses the integral representation of the confluent hypergeometric function to derive its asymptotic series.
  - Come back (for funsies) if I have time!!
- Discussion of **Stokes's phenomenon**.