Week 7

???

7.1 Generalized Cauchy Theorems

4/30:

- Questions.
  - PSet 4, QA.4: III.5.1 instead of II.5.1?
    - Yep, should be Chapter 3, not Chapter 2.
  - PSet 4, QB.4: "a holomorphic branch of the logarithm exists on U" or on $f(U)$?
    - Yep, should be $f(U)$.
    - "Which one works, Steven?"
- Recall.
  - The winding number of a curve γ about a point $z_0 \in \mathbb{C}$ is

$$\operatorname{wn}(\gamma, z_0) := \frac{1}{2\pi i} \int_\gamma \frac{1}{z - z_0} \, \mathrm{d}z$$

  - We can also compute the winding number geometrically (see Figure 6.3).
- Additional properties of the winding number.
  - The winding number is invariant under homotopies of γ.
  - Compute by counting how many times you pass a ray from z_0 going counterclockwise!
    - Example: "I'm pointing in this direction, then I rotate, and eventually I point in this direction again, then I rotate, and eventually I'm back where I started so it's winding number 2."
  - We can also think of jumping to a higher plane on the infinity spiral every time we pass the ray.
- TPS: Compute the winding number of γ about the points in Figure 7.1.

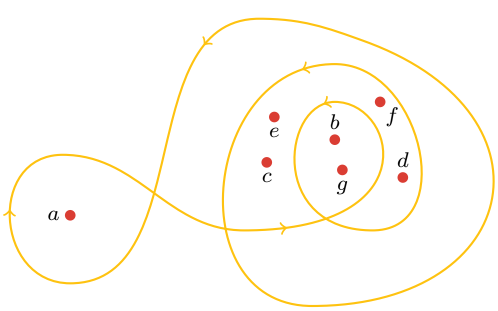


Figure 7.1: Winding number regions.

  - We get

$$\mathrm{wn}(\gamma, a) = -1 \qquad \mathrm{wn}(\gamma, b) = 3 \qquad \mathrm{wn}(\gamma, c) = 2$$

$$\mathrm{wn}(\gamma, d) = 2 \qquad \mathrm{wn}(\gamma, e) = 2 \qquad \mathrm{wn}(\gamma, f) = 2 \qquad \mathrm{wn}(\gamma, g) = 3$$

  - Do we notice any patterns?
    - Connected regions of the plane appear to yield the same winding number!
    - We formalize this notion via the following lemma.

- Lemma: $\mathrm{wn}(\gamma, z_0)$ is constant on components of $\mathbb{C} \setminus \mathrm{Im}(\gamma)$. It is also 0 on the unbounded component.

  Proof. We address the two claims sequentially.

  Claim 1: Treating z_0 as an argument, $\mathrm{wn}(\gamma, z_0)$ is a function from $\mathbb{C} \setminus \mathrm{Im}(\gamma)$ to $\mathbb{Z}$ defined by

$$z_0 \mapsto \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{z - z_0}$$

  This is a continuous function into a discrete space and therefore is constant.

  Claim 2: Let z_0 get very big. Then we can make

$$\left| \frac{1}{2\pi i} \int_\gamma \frac{\mathrm{d}z}{z - z_0} \right|$$

  arbitrarily small. But an integer that can be made arbitrarily small is just zero. □

  - This is a complex analytic proof of a topological claim.
  - Justifying that the codomain of the winding number function is the integers: We've done this heuristically using homotopy, but we could formalize it, too.

- We now move onto today's main topic: The proof of the (very) general Cauchy Integral Theorem.

- First, we need a definition.

- **Simply connected** (domain): A domain $U \subset \mathbb{C}$ such that $\mathrm{wn}(\gamma, z_0) = 0$ for all $\gamma \subset U$ and $z \notin U$.
  - There are many other definitions, too.
    - Topology: The **fundamental group** of U is zero.
    - Removing any **arc** (line segment across the domain) from U turns it into a disconnected set.
    - For all arcs δ_1, δ_2 with the same endpoints, δ_1 and δ_2 are homotopic.
  - The last definition above will be particularly useful for our purposes, as we'll see shortly.
  - But these are all formal definitions; what can we think about intuitively?
    - A good first thing to think about is a blob in the plane.
    - But the interior of a fractal domain would also count.
    - A square minus a slit at $1, 1/2, 1/3, \dots$ is also simply connected (though not path connected).

- Jordan curve theorem: Suppose $\gamma : S^1 \to \mathbb{C}$ is a continuous injection. Then γ bounds a disk.
  - Consequence: A domain that is simply connected is homeomorphic to a disk.
  - This appears stupidly obvious, but it was only rigorously proved in the early 1910s.
    - The issue is that we don't really know what *continuous* means.
    - If γ is C^1, this is easy.

- The two generalizations and their proofs.
  - The proof of generalization 1 is very simple, straightforward, and clever.
  - The proof of generalization 2 is much more general and uses almost everything we've done.
- We are now ready to state and prove a first generalization of the CIT.
- Cauchy Integral Theorem: Suppose that U is simply connected and $f \in \mathcal{O}(U)$. Then $\int_\gamma f \, \mathrm{d}z = 0$ for any closed loop γ in U.

  Proof. Let γ be an arbitrary closed loop in U. Because any two arcs with the same endpoints are homotopic, γ is homotopic to the constant path $\tilde{\gamma} : [0,1] \to \{\gamma(0)\}$. This constant path has the property that

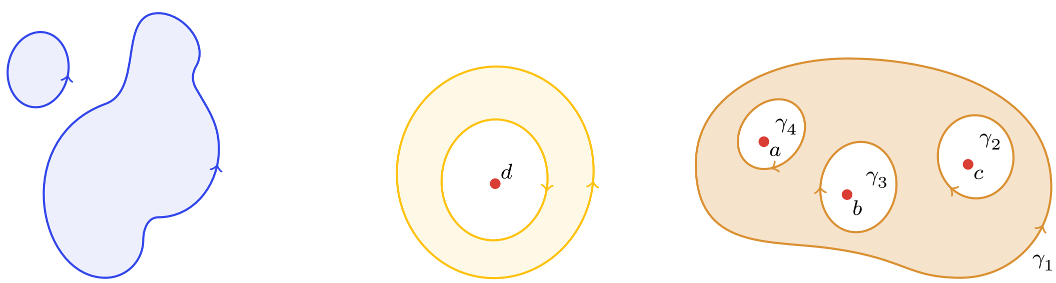
  $$\int_{\tilde{\gamma}} f \, \mathrm{d}z = \int_0^1 f(\tilde{\gamma}(t))\tilde{\gamma}'(t) \, \mathrm{d}t = \int_0^1 f(\tilde{\gamma}(t)) \cdot 0 \, \mathrm{d}t = 0$$

  Since integrals are the same for homotopic paths, it follows that

  $$\int_\gamma f \, \mathrm{d}z = \int_{\tilde{\gamma}} f \, \mathrm{d}z = 0$$

  as desired. ☐

- We now build up to an even more general version of the CIT.



(a) Separate oriented domains. (b) Nested curves. (c) Multiple nested curves.

Figure 7.2: Nulhomologous multicurve examples.

  - Suppose $D \subset \mathbb{C}$ is a bounded domain, and ∂D is a union of disjoint simple closed curves (SCCs).
  - Let $\partial \vec{D}$ be the union of the boundaries, oriented so that D is on the left.
    - This is similar to how we orient curves when we're applying Stokes' Theorem.
    - Here as well, the outer one goes counterclockwise and the inner one(s) goes clockwise.
  - More generally, we define a the concept of a **multicurve**.
  - Using this definition, we define the **integral** of f over a multicurve.
    - This definition allows us to compute the winding number of Γ about z_0.
  - Lastly, we define a special kind of multicurve called a **nulhomologous** multicurve.
    - In Figure 7.2c, $\Gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ is nulhomologous.
- **Multicurve**: A formal sum of SCCs γ_i multiplied by coefficients $c_i \in \mathbb{C}$. *Denoted by* $\boldsymbol{\Gamma}$. *Given by*

  $$\Gamma = \sum c_i \gamma_i$$

- **Integral** (of f over Γ): The path integral defined as follows. *Denoted by* $\int_\Gamma f\,\mathrm{d}z$. *Given by*

$$\int_\Gamma f\,\mathrm{d}z := \sum_{i=0}^{n} c_i \int_{\gamma_i} f\,\mathrm{d}z$$

- **Nulhomologous** (Γ in U): A multicurve Γ in a domain U for which $\Gamma = \partial\vec{D}$ for D as in Figure 7.2. *Also known as* **homologous** (Γ in U to 0).
- TPS: Compute $\text{wn}(\partial\vec{D}, z_0)$ for all $z_0 \notin D$ for each of the domains D in Figure 7.2.
  - $\text{wn}(\partial\vec{D}, z_0) = 0$ because we always get either nothing or a $+1$ and -1 and some zeroes.
- Lemma: If Γ is nulhomologous in U, then for all $z \notin U$, $\text{wn}(\Gamma, z) = 0$.
  - The converse is not true!
    - Example: If $U = \mathbb{C}^*$ and γ_1, γ_2 are intersecting closed curves (e.g., the unit circle and the unit circle translated half a unit to the right), then $\gamma_1 + \gamma_2$ is still nulhomologous even though it doesn't bound a domain.
  - The condition "for all $z \notin U$, $\text{wn}(\Gamma, z) = 0$" is our general definition of nulhomologous in U; what we said earlier was just a precursor definition.
- Example.

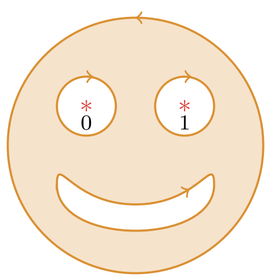


Figure 7.3: Nulhomologous multicurve in a punctured domain.

  - Let
$$f(z) = \frac{\sin(1/z)}{z-1}$$
  - Then $f \in \mathcal{O}(\mathbb{C} \setminus \{0,1\})$.
  - An example of a nulhomologous multicurve over which we could integrate f is as follows.
- We are now ready for the statement and proof of the most general version of the CIT and CIF we'll see in this course.
- Suppose U is any domain, $\Gamma \subset U$ is nulhomologous, and $f \in \mathcal{O}(U)$. Then:
  1. General CIT: We have that
$$\int_\Gamma f\,\mathrm{d}z = 0$$
  2. General CIF: For all $z \in U$ and not in $\text{Im}(\Gamma)$,
$$\text{wn}(\Gamma, z) \cdot f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

- Discussion of the proof.
  - We'll sketch the proof today.
  - Think back to the proof for star-shaped domains.
  - We proved the CIT by saying, "if it's true for triangles, then we win."
    - Using triangles, we built a primitive and then invoked Goursat's Lemma.
  - We proved the CIF by first defining the helper function

$$g(\zeta) = \begin{cases} \frac{f(\zeta)-f(z)}{\zeta - z} & \zeta \neq z \\ f'(z) & \zeta = z \end{cases}$$

    - Then we invoked the CIT to say

$$\int_{\partial D} g \,\mathrm{d}z = 0$$

    - The CIF then followed from this and the fact that

$$\int_{\partial D} g\mathrm{d}\zeta = \int_{\partial D} \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta - f(z) \underbrace{\int_{\partial D} \frac{1}{\zeta - z}\mathrm{d}\zeta}_{2\pi i}$$

  - So can't we just replace all the ∂D's with Γ's in the above lines and call it a day?
    - No, because there's no analogy for the CIT. In other words, there may not be a primitive.
    - Thus, we need to fix $\int_{\partial D} g \,\mathrm{d}z = 0$.
  - In sum, the idea of this proof is to prove the CIF and then simply get the CIT.
  - We'll have time to prove the CIF today, but probably will not to get to the CIT.

- We are now ready to sketch the full proof in broad strokes.

Proof. Define $h : U \to \mathbb{C}$ by

$$h(z) := \int_\Gamma g(\zeta, z)\mathrm{d}\zeta$$

We want to show that $h(z) = 0$. We can't do anything as nice as showing that it's a continuous map into a discrete space, but there is still a clever idea. First off, we can see that $h(z) \to 0$ as $z \to \infty$ in U. Essentially, as before, the denominator $\zeta - z$ gets really big so the first term gets really small and the second term has that $\mathrm{wn}(\Gamma, z)$ term which goes to 0. What we now need to show is that h extends to an entire function so that we can make the denominator *arbitrarily* large. This is where we use the assumption that Γ is nulhomologous.

First, we will show that h is continuous. We know that g is continuous in (ζ, z) together. We have holomorphic in ζ for a fixed z.[1]

Next, we need to show that h is holomorphic on U. We know that h is holomorphic as long as $z \neq \zeta$. On the other hand, what if $\zeta = z$? We will invoke Morera's theorem.[2]

Last, we show that h can be analytically continued outside of U. We know that on U,

$$h(z) = \int \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta - f(z) \cdot 2\pi i \,\mathrm{wn}(\Gamma, z)$$

Outside of U, the second term disappears because Γ is nulhomologous. Define

$$h(z) := \int \frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta$$

outside of U. Thus, we have two functions that agree on a patch, so we get analytic continuation.

From here, we have an entire function that converges to 0 at ∞ (hence is bounded), so is constant by Liouville's theorem with value that converges to zero (hence is zero).[3] □

[1] There's a bit more detail in the notes, but not much.
[2] There's a bit about the triangle integral condition in the notes.
[3] There is a bit on the CIT in the notes.