Week 2

Consequences of Power Series

2.1 Office Hours

3/25:

- What exactly are the Wirtinger derivatives?
  - The $\partial/\partial z$ and $\partial/\partial\bar{z}$ operators.
- The initial definition of holomorphic is accurate. It's naïve, but it works out.
- Noney: Non example.
  - As in, we have some examples of holomorphic functions and then we have an example of a function that is *not* holomorphic.
- TPS: Think Pair Share.
- Met Panteleymon and helped him with partial fractions!
- The Δ notation does mean the same Laplacian as $\vec{\nabla}^2$ from Quantum Mechanics.
- Calderon is not related to Calderón; he was Argentinian, Calderon is half-Filipino and has no accent on his name. Both Spanish colonies but that's it.
- We can do all of the problems except Problem 1 at this point.
  - For this, though, we can just look up the definition of the complex sine function.
  - We basically just need to know what $\sin(i)$ is and what sine looks like along the imaginary axis.

2.2 Power Series

3/26:

- Recall: We already know that...
  - Polynomials are elements of $\mathcal{O}(\mathbb{C})$;
  - Rational functions $P(z)/Q(z)$ are elements of $\mathcal{O}(\mathbb{C}\setminus V(Q))$.
- **Affine algebraic set**: The set of solutions in an algebraically closed field K of a system of polynomial equations with coefficients in K. *Also known as* **variety**. *Denoted by* $\boldsymbol{V(f_1,\dots,f_n)}$.
- Today, we want to determine how the other elementary functions behave over the complex numbers.
  - Other functions we want: exp, log, sin, cos.
  - We will do log later, but all the others today.

- **Exponential function**: The complex function defined as follows. *Denoted by* $\mathbf{e^z}$, $\mathbf{exp(z)}$. *Given by*

$$e^z = \exp(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

- Naïvely, this power series is just be a polynomial $P(z) \in \mathcal{O}(\mathbb{C})$.
- More rigorously, however, we must specify which kind of convergence we mean for the power series.
  - As one example, we could say that for all z,

$$e^z = P(z) = \lim_{N \to \infty} \sum_{k=0}^{N} \frac{z^k}{k!}$$

    - This would be **pointwise convergence**.
  - But there's an issue: Pointwise convergence of functions doesn't preserve anything, e.g., continuity.
- **Pointwise** (convergent $\{f_n\}$): A sequence of functions $f_n : \mathbb{C} \to \mathbb{C}$ such that for all $z \in \mathbb{C}$, we have $f_n(z) \to f(z)$.
- TPS: Come up with an example of a sequence of continuous functions $\{f_n\}$ that converges pointwise to f, such that the f_n are all...
  1. Continuous but f is not;
     - $f_n(x) = \arctan(nx)$.
     - Converges to the sign function $f(x) = \operatorname{sgn}(x)$.
  2. Odd but f is not;
  3. Differentiable but f is not.
     - These last two cases were not discussed in class.
- We now recall a few definitions and lemmas from real analysis.
- **Locally uniformly** (convergent $\{f_n\}$): A sequence of functions $f_n : U \to \mathbb{C}$ and a function $f : U \to \mathbb{C}$ such that for all compact $K \subset U$,

$$\sup_{z \in K} |f_n(z) - f(z)| \to 0$$

- Lemma: If $f_n \to f$ locally uniformly and the f_n are continuous (or integrable), then so is f.
  - This lemma is *not* true if we sub in "differentiable!"
  - See the Stone-Weierstrass theorem for suitable constraint.
- Thus, to resolve the original question, we mean that $P_N(z) \to \exp(z)$ locally uniformly.
- Aside: Which functions have power series?
  - Remember Taylor polynomials from Calc II? **Taylor's theorem** tells us which ones converge.
- **Taylor's theorem**: If $f : \mathbb{R} \to \mathbb{R}$ is C^{k+1} and $P_\alpha^k(x)$ is the k^{th} Taylor polynomial about $\alpha \in \mathbb{R}$, then for all $\beta \in \mathbb{R}$, there exists some $x \in (\alpha, \beta)$ such that

$$f(\beta) - P_\alpha^k(\beta) = \frac{(\beta - \alpha)^{k+1}}{(k+1)!} f^{(k+1)}(x)$$

  - Essentially a version of the mean value theorem (MVT) for higher-order derivatives.
  - We can use the term of the right side of the equals sign above to get a bound on the error of the Taylor polynomial.

- **Analytic** (function): A function $f : \mathbb{R} \to \mathbb{R}$ for which the Taylor polynomials converge (locally uniformly) to f.
- Non example: The C^∞ function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} \mathrm{e}^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

  - An excellent exercise in real analysis is to check that for all k, the Taylor polynomial about 0 is 0.
  - If we take the Taylor polynomial at some point farther from zero, the polynomial will approximate f well up until zero, but then it will "hit a wall."
    - The point is that f is decaying more rapidly toward 0 than any polynomial possibly could, so the polynomial just thinks it's seeing 0.
- **Absolutely** (locally uniformly convergent power series): A power series $P(z) = \sum_{k=0}^{\infty} a_k z^k$ for which $A_N : \mathbb{C} \to \mathbb{R}$ locally uniformly converges, where

$$A_N(z) := \sum_{k=0}^{N} |a_k z^k|$$

- Absolute local uniform convergence allows you to reorder the terms in the polynomial.
  - It also explains why you cannot reorder the terms in the series $S = 1+1-1+1-1+\cdots$, i.e., why manipulating the order allows you to get any number: This series S does not converge absolutely!
  - Formally, if $\sigma : \mathbb{N} \to \mathbb{N}$ is a permutation and $\sum^{\infty} a_k$ converges absolutely, then $\sum^{\infty} a_{\sigma(k)}$ converges.
- Exercise: Show that

$$\sum_{k=0}^{\infty} z^k \to \frac{1}{1-z}$$

  converges absolutely locally uniformly on $\mathbb{D} = \{|z| < 1\}$.

  Proof. To prove this, we just have to show that $\sum^{\infty} |z|^k$ converges on $|z| < 1$. But it does so converge because this latter series is just a standard real geometric series. □

- This example generalizes somewhat into the following lemma.
- Lemma: Let $P(z)$ be a power series about 0. If there exists $z_1 \neq 0$ such that $|a_k z_1^k| \leq M$ for all k, then $P(z) = \sum a_k z^k$ converges on the disk $|z| < |z_1|$.

  Proof. Uses standard series convergence results from real analysis. See Fischer and Lieb (2012, pp. 15–16). □

- **Disk of convergence**: The largest disk centered at zero on which you converge.
- **Radius of convergence**: The radius of the disk of convergence. *Denoted by* $\boldsymbol{r}$.
- **Cauchy-Hadamard formula**: The radius of convergence is given by

$$r = (\limsup |a_k|^{1/k})^{-1}$$

  - We will be using this result on PSet 2.
  - We will also be proving it there!

- What are power series representations good for? Here's an example of how they can be applied to help with PSet 1, QA.4.
  - Question: For $|a| < 1$ and $\gamma(t) = \mathrm{e}^{it}$ a parameterization of a closed loop oriented counterclockwise, compute
$$\int_\gamma \frac{1}{z-a}\,\mathrm{d}z$$
  - Answer:
    - Since $|a| < 1$, we know that on γ, $|a/\gamma(t)| < 1$.
    - Thus, we have that
$$\begin{aligned}\int_\gamma \frac{1}{z-a}\,\mathrm{d}z &= \int_\gamma \frac{1}{z}\frac{1}{1-a/z}\,\mathrm{d}z\\ &= \int_\gamma \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{a}{z}\right)^k \mathrm{d}z\\ &= \int_\gamma \sum_{k=0}^{\infty}\frac{a^k}{z^{k+1}}\,\mathrm{d}z\\ &= \sum_{k=0}^{\infty}\int_\gamma \frac{a^k}{z^{k+1}}\,\mathrm{d}z\\ &= \cdots\\ &= \int_\gamma \frac{1}{z}\,\mathrm{d}z\end{aligned}$$
    - We have the second equality because the power series converges.
    - We have the fourth equality because of the lemma about integrable f_n and the fact that the power series converges.
    - The dots indicate some more steps that we will need to work out for ourselves on PSet 1.
- Lemma (from real analysis): If $f_n \to f$ locally uniformly and $f_n' \to g$ locally uniformly, then f is differentiable and $f' = g$.
  - This is true for both differentiable and holomorphic functions.
- Claim: This lemma implies that convergent power series are holomorphic.

Proof. If
$$f_N = \sum_{k=0}^{N} a_k z^k$$
then
$$f_N' = \sum_{k=0}^{N} k\cdot a_k z^{k-1}$$
We want to show that $\{f_N'\}$ converges (locally absolutely uniformly). Fischer and Lieb (2012) do this by hand. We can also use the Cauchy-Hadamard formula, which we will do presently.

Let's look at $\limsup(k\cdot a_k)^{1/k}$. But this is just equal to
$$\limsup|k\cdot a_k|^{1/k} \le \limsup(|k|^{1/k})\cdot\limsup(|a_k|^{1/k}) = 1\cdot\limsup(|a_k|^{1/k}) = \limsup|a_k|^{1/k}$$
Moreover, equality holds because that $k^{1/k}$ factor just decays toward 1; think about how k increases linearly and the k^{th} root decays faster. ☐

- Proposition: Any convergent power series is holomorphic (on its disk) and its derivative is also a power series with the same radius of convergence. It follows that power series are analytic functions and are C^∞.
- Spoiler: Every holomorphic function is analytic.
- Corollary: Power series representations are unique.
  1. If $P(z) = \sum a_k z^k$ is convergent, then
  $$a_k = \frac{1}{k!} P^{(k)}(0)$$
  2. If $P(z) = 0$ in a neighborhood of zero, then $a_k = 0$ for all k.
  3. If $P(z) = Q(z)$ (where $Q(z) = \sum b_k z^k$) in a neighborhood of 0, then $a_k = b_k$ for all k.
- Let's now return to the exponential function, which got this whole discussion started.
- We now know that the definition
  $$\exp(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$
  makes sense.
- By manipulating this power series, we can get lots of fun properties.
  1. $\exp(z) = [\exp(z)]'$.
     - We obtain this via term-by-term differentiability.
     - This is just our favorite formula $\mathrm{d}/\mathrm{d}t\ (\mathrm{e}^t) = \mathrm{e}^t$ from calculus.
  2. $\overline{\exp(z)} = \exp(\bar{z})$.
  3. $\exp(a + b) = \exp(a) \cdot \exp(b)$.
  4. $|\exp(z)| = \exp[\mathrm{Re}(z)]$.
- **Complex cosine**: The complex function defined as follows. *Denoted by* $\boldsymbol{\cos(z)}$. *Given by*
  $$\cos(z) := \frac{1}{2}(\mathrm{e}^{iz} + \mathrm{e}^{-iz})$$
- **Complex sine**: The complex function defined as follows. *Denoted by* $\boldsymbol{\sin(z)}$. *Given by*
  $$\sin(z) := \frac{1}{2i}(\mathrm{e}^{iz} - \mathrm{e}^{-iz})$$
- **Complex hyperbolic cosine**: The complex function defined as follows. *Denoted by* $\boldsymbol{\cosh(z)}$. *Given by*
  $$\cosh(z) := \cos(iz)$$
- **Complex hyperbolic sine**: The complex function defined as follows. *Denoted by* $\boldsymbol{\sinh(z)}$. *Given by*
  $$\sinh(z) := i \sin(iz)$$
- We also have
  $$\mathrm{e}^{iz} = \cos(z) + i \sin(z)$$
  - If z is real and in $[0, 2\pi]$, then this simplifies to Euler's formula
  $$\mathrm{e}^{i\theta} = \cos(\theta) + i \sin(\theta)$$
- Calderon draws some mappings of the exponential function but doesn't linger on what's going on.
- These are the preliminaries; now, we'll dive into the meat of the course.

2.3 Cauchy's Theorem

3/28:

- The last three classes have been real analysis with complex numbers; now we get into *complex* analysis.
- **Domain**: A connected, open set $U \subset \mathbb{C}$.
- Recall.
  - $\gamma : [a,b] \to \mathbb{C}$ is a piecewise C^1 curve.
  - $f : \mathbb{C} \to \mathbb{C}$ is continuous.
  - We define
$$\int_\gamma f \,\mathrm{d}z := \int_a^b f(\gamma(t)) \cdot \gamma'(t) \,\mathrm{d}t$$
  - FTC: If $f = F'$ (i.e., F is a **primitive** of f) on a domain $U \subset \mathbb{C}$, then for all paths γ in U,
$$\int_\gamma f \,\mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$$
- **Primitive** (of f): A differentiable function whose derivative is equal to the original function f. *Also known as* **antiderivative**, **indefinite integral**. *Denoted by* $\boldsymbol{F}$.
- Corollary to the FTC: If $f = F'$, then for any closed curve γ in U,
$$\int_\gamma f \,\mathrm{d}z = 0$$
  - To see why this is true intuitively, look at an example such as $f(z) = 1/z \in \mathcal{O}(\mathbb{C}^*)$, which doesn't have a primitive and
$$\int_\gamma \frac{1}{z} \,\mathrm{d}z \neq 0$$
- Example: Find a primitive of the convergent power series
$$P(z) = \sum_{k=1}^{\infty} a_k z^k$$
  - Via term-by-term integration, we obtain
$$\sum_{k=0}^{\infty} \frac{a_k}{k+1} z^{k+1}$$
- If γ is any closed loop in the disk of convergence,
$$\int_\gamma P(z) \,\mathrm{d}z = 0$$
  - It follows since they are defined in terms of convergent power series that for all closed loops γ,
$$\int_\gamma \mathrm{e}^z \,\mathrm{d}z = \int_\gamma \sin(z) \,\mathrm{d}z = \int_\gamma \cos(z) \,\mathrm{d}z = 0$$
- Question: When is there a primitive?
  - $f : \mathbb{R} \to \mathbb{R}$ continuous always has a primitive by the FTC, specifically that defined by
$$F(x) := \int_a^x f(t) \,\mathrm{d}t$$
    which is differentiable with $F' = f$.

- Proposition: If $f : U \to \mathbb{C}$ is continuous and $\int_\gamma f \, \mathrm{d}z = 0$ for every closed loop in U, then f has a primitive on U.

Proof. Let's try the most naïve thing: The FTC. Consider a domain.

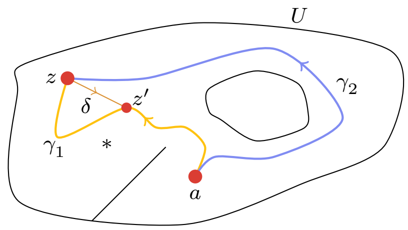


Figure 2.1: Continuous and zero closed-loop integrals implies integrable.

Doesn't have to be simply connected; it can have a **hole**, **slit**, and/or **puncture**. Essentially, to define $F(z)$, choose $a \in U$ and γ connecting a and z and define

$$F(z) = \int_\gamma f \, \mathrm{d}z$$

Claim: This definition is well-defined regardless of the choice of a and γ. In particular, the integral is independent of choice of γ because any two γ can be paired into a closed loop, and we have by hypothesis that the integral over any closed loop is zero.

We now need to show that F is differentiable with $F' = f$. Take z, z' close enough that they can be connected by a straight line path δ. Consider

$$\lim_{z' \to z} \frac{F(z') - F(z)}{z' - z}$$

Now we know that

$$F(z') - F(z) = \int_\delta f \, \mathrm{d}z$$

Let $\gamma : [0,1] \to \mathbb{C}$ be defined by $t \mapsto tz' + (1-t)z$; a parameterization we can choose arbitrarily. Then

$$F(z') - F(z) = \int_\delta f \, \mathrm{d}z = \int_0^1 f[tz + (1-t)z'] \cdot (z' - z) \, \mathrm{d}t$$

so dividing both sides by $z' - z$ and taking the limit yields

$$\begin{aligned}
\lim_{z' \to z} \frac{F(z') - F(z)}{z' - z} &= \lim_{z' \to z} \int_0^1 f[tz + (1-t)z'] \, \mathrm{d}t \\
&= \int_0^1 \lim_{z' \to z} f(tz + z' - tz') \, \mathrm{d}t \\
&= \int_0^1 f(tz + z - tz) \, \mathrm{d}t \\
&= \int_0^1 f(z) \, \mathrm{d}t \\
&= f(z) \int_0^1 \mathrm{d}t \\
&= f(z)
\end{aligned}$$

and we have everything we wanted. $\square$

- What allows us to interchange the limit and the integral in the final set of equations?
  - Roughly speaking, uniform convergence.
- **Star-shaped** (domain): A domain $U \subset \mathbb{C}$ for which there exists $a \in U$ such that for all $z \in U$, the segment $a \to z$ is in U.

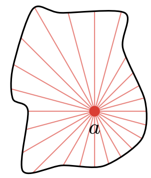


Figure 2.2: Star-shaped domain.

  - There are star-shaped regions that are not **convex**, such as the one in Figure 2.2!
    - Convex implies star-shaped, but not vice versa.
  - Examples of domains that are *not* star-shaped.
    1. The annulus of two circles.
    2. Puncturing the unit disk.
  - Star-shaped implies **simply connected**.
  - Star-shaped is nice because we don't have to check every single curve; see the following lemma.
- Lemma: If U is star-shaped and for every triangle with one vertex at a, we have $\int_\triangle f\,\mathrm{d}z = 0$, then F has a primitive in U.

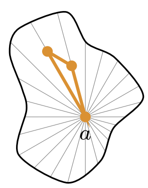


Figure 2.3: A triangle in a star-shaped domain.

Proof. What should be our candidate for $F(z)$? Define

$$F(z) = \int_\gamma f\,\mathrm{d}z$$

where γ is the line segment from $a \to z$ that we know exists because U is star-shaped.

We now have to show that F is holomorphic with $F' = f$, but we just do this as before by constructing a "closed loop," except our closed loop this time will just be a triangle as drawn in Figure 2.3. □

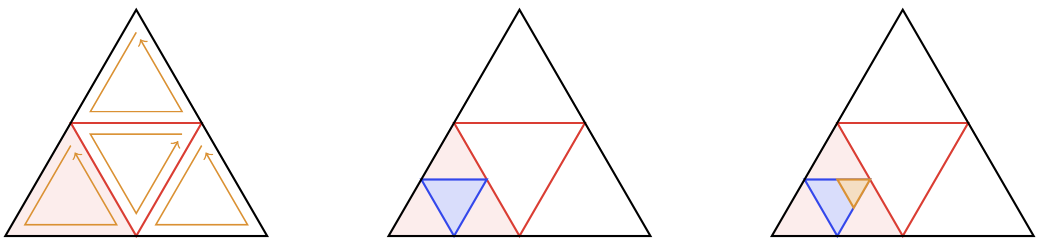
- With these definitions, we now state and prove one of the two main theorems in this class.
- **Cauchy Integral Theorem**: Suppose U is a star-shaped domain and $f : U \to \mathbb{C}$ is holomorphic. Then $\int_\gamma f\,\mathrm{d}z = 0$ for any closed loop γ in U.
  - Whereas the FTC says if you have an *integral*, then the integral around a closed loop is zero. This theorem says that if you have a *derivative*, then the integral around a closed loop is zero.
  - This is Round 1 of the theorem. In round 2, we'll swap the "star-shaped" hypothesis for "simply connected."

- Today we're at least going to prove this, and possibly look at an application, too. If we don't get to the application today, we'll see it next Tuesday.
- Proof idea: Prove that f has a primitive.

  Proof. In order to prove this theorem, we'll use the preceding lemma. Thus, all we need to show is that for every triangle with one vertex on the center of the star, $\int_\triangle f \, \mathrm{d}z = 0$. Since we only have to check this for *triangles*, we can use a really lovely result called **Goursat's lemma**. ☐

- **Goursat's lemma**: If f is holomorphic in a neighborhood of a triangle including the interior, then $\int_\triangle f \, \mathrm{d}z = 0$.

  Proof. Idea: Estimate the integral.



(a) Picking $\triangle_1$. (b) Picking $\triangle_2$. (c) Picking $\triangle_3$.

Figure 2.4: Proving Goursat's lemma.

Fix some $z_0 \in \blacktriangle$ (the exact value will be determined later). We know that f is holomorphic at z_0, which implies that there exists a linear approximation

$$f(z) = \underbrace{f(z_0) + f'(z_0) \cdot (z - z_0)}_{\text{linear}} + E(z) \cdot (z - z_0)$$

where our error function $E(z) \to 0$ as $z \to z_0$. Now the underlined linear portion above is a (stupid) power series, but since it technically *is* a "convergent power series," our previous results imply that it has primitives. In particular, its integral around a closed loop (like a triangle) will be zero. This means that

$$\int_\triangle f \, \mathrm{d}z = \underbrace{\int_\triangle [f(z_0) + f'(z_0) \cdot (z - z_0)] \, \mathrm{d}z}_{0} + \int_\triangle E(z) \cdot (z - z_0) \, \mathrm{d}z = \int_\triangle E(z) \cdot (z - z_0) \, \mathrm{d}z$$

Goursat's idea: Choose a good z_0. To do this, we'll subdivide the original black triangle (see Figure 2.4a) by choosing midpoints and breaking it into four triangles. Keep using the counterclockwise orientation in all cases. All of the red segments get cancelled out from integrating in both directions, so

$$\int_{\triangle_0} f \, \mathrm{d}z = \sum \int_{4 \text{ sub } \triangle\text{'s}} f \, \mathrm{d}z$$

Choose $\triangle_1$ in first stage such that $\left| \int_{\triangle_1} f \, \mathrm{d}z \right|$ is the greatest among the first stage sub-triangles. Thus,

$$\left| \int_\triangle f \, \mathrm{d}z \right| \leq 4 \cdot \left| \int_{\triangle_1} f \, \mathrm{d}z \right|$$

Now subdivide $\triangle_1$ and choose $\triangle_2$ the same way (see Figure 2.4b), so that

$$\left| \int_\triangle f \, \mathrm{d}z \right| \leq 4 \cdot \left| \int_{\triangle_1} f \, \mathrm{d}z \right| \leq 4 \cdot 4 \cdot \left| \int_{\triangle_2} f \, \mathrm{d}z \right|$$

Iterating this process, we obtain

$$\left|\int_{\triangle} f \, \mathrm{d}z\right| \leq 4^n \cdot \left|\int_{\triangle_n} f \, \mathrm{d}z\right|$$

First thing to observe:

$$\operatorname{len}(\triangle_n) = 2^{-n} \cdot \operatorname{len}(\triangle_0) \qquad \operatorname{diam}(\triangle_n) = 2^{-n} \cdot \operatorname{diam}(\triangle_0)$$

Now fix $\varepsilon > 0$ and take n big enough such that on all of $\triangle_n$,

$$|E(z)| < \frac{\varepsilon}{\operatorname{len}(\triangle_0) \cdot \operatorname{diam}(\triangle_0)}$$

Choose $z_0 \in \bigcap_{n=1}^{\infty} \blacktriangle_n$. Then

$$\begin{aligned}
\left|\int_{\triangle} f \, \mathrm{d}z\right| &\leq 4^n \cdot \left|\int_{\triangle_n} f \, \mathrm{d}z\right| \\
&= 4^n \cdot \left|\int_{\triangle_n} E(z) \cdot (z - z_0) \, \mathrm{d}z\right| \\
&\leq 4^n \cdot \operatorname{len}(\triangle_n) \cdot \max_{\triangle_n} |E(z) \cdot (z - z_0)| \\
&= 4^n \cdot \operatorname{len}(\triangle_n) \cdot \max |E(z)| \cdot \max |z - z_0| \\
&\leq 4^n \cdot \operatorname{len}(\triangle_n) \cdot \operatorname{diam}(\triangle_n) \cdot \max |E(z)| \\
&= 4^n \cdot 2^{-n} \operatorname{len}(\triangle_0) \cdot 2^{-n} \operatorname{diam}(\triangle_0) \cdot \max |E(z)| \\
&= \operatorname{len}(\triangle_0) \cdot \operatorname{diam}(\triangle_0) \cdot \max |E(z)| \\
&< \varepsilon
\end{aligned}$$

Since we can choose ε arbitrarily small, we can thus send the original integral of f over $\triangle$ to zero. $\square$

- We now end class with an example of how complex analysis can be useful, even in calculus!
- Example: Evaluate the following **Dirichlet integral** using complex analysis.

$$\int_0^{\infty} \frac{\sin(x)}{x} \, \mathrm{d}x$$

  - We will do so via a focused analysis of the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$z \mapsto \frac{\mathrm{e}^{iz}}{z}$$

    - This function is not holomorphic everywhere, but it is on the punctured plane $\mathcal{O}(\mathbb{C}^*)$.
    - However, we only need the upper half $\mathcal{O}(\mathbb{H})$ presently.
  - More specifically, define U to be a domain containing γ as defined as in Figure 2.5.

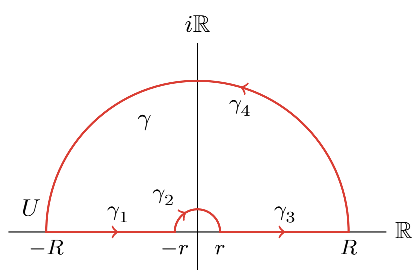


Figure 2.5: Dirichlet integral.

- By the Cauchy integral theorem and our decomposition of γ,

$$0 = \int_\gamma f(z)\,\mathrm{d}z = \sum_{i=1}^4 \int_{\gamma_i} f\,\mathrm{d}z$$

- We now integrate the segments one at a time.
  - γ_1 and γ_3: Recalling our definition of $\sin(z)$ from last class, we have that

$$\begin{aligned}
\int_{\gamma_1\gamma_3} \frac{\mathrm{e}^{ix}}{x}\,\mathrm{d}x &= \int_{-R}^{-r} \frac{\mathrm{e}^{ix}}{x}\,\mathrm{d}x + \int_r^R \frac{\mathrm{e}^{ix}}{x}\,\mathrm{d}x \\
&= \int_{-r}^{-R} -\frac{\mathrm{e}^{ix}}{x}\,\mathrm{d}x + \int_r^R \frac{\mathrm{e}^{ix}}{x}\,\mathrm{d}x \\
&= \int_r^R -\frac{\mathrm{e}^{-ix}}{x}\,\mathrm{d}x + \int_r^R \frac{\mathrm{e}^{ix}}{x}\,\mathrm{d}x \\
&= \int_r^R \frac{\mathrm{e}^{ix} - \mathrm{e}^{-ix}}{x}\,\mathrm{d}x \\
&= 2i \int_r^R \frac{\sin(x)}{x}\,\mathrm{d}x
\end{aligned}$$

  - γ_2: We can explicitly compute this integral as $r \to 0$, using the parameterization $\gamma_2 : [0,\pi] \to \mathbb{C}$ defined by $\theta \mapsto r\mathrm{e}^{i(\pi-\theta)}$.

$$\begin{aligned}
\lim_{r\to 0} \int_{\gamma_2} \frac{\mathrm{e}^{iz}}{z}\,\mathrm{d}z &= \lim_{r\to 0} \int_0^\pi \frac{\mathrm{e}^{ir\mathrm{e}^{i(\pi-\theta)}}}{r\mathrm{e}^{i(\pi-\theta)}} \cdot -ir\mathrm{e}^{i(\pi-\theta)}\,\mathrm{d}\theta \\
&= -i \lim_{r\to 0} \int_0^\pi \mathrm{e}^{ir\mathrm{e}^{i(\pi-\theta)}}\,\mathrm{d}\theta \\
&= -i \int_0^\pi \mathrm{e}^0\,\mathrm{d}\theta \\
&= -i\pi
\end{aligned}$$

  - γ_4: We need to bound the $R\mathrm{e}^{i\theta}$ term as $R \to \infty$; see his notes!

$$\int_0^\pi \mathrm{e}^{iR\mathrm{e}^{i\theta}} i\mathrm{d}\theta \to 0$$

- Therefore, by transitivity,

$$\begin{aligned}
2i \int_0^\infty \frac{\sin(x)}{x}\,\mathrm{d}x - i\pi &= 0 \\
\int_0^\infty \frac{\sin(x)}{x}\,\mathrm{d}x &= \frac{\pi}{2}
\end{aligned}$$

2.4 Office Hours

- PSet 1, QA.4: Are a, b real or complex?
  - They can be complex.
  - Hint for this problem: Think about QA.3.
- PSet 1, QB.2: As in, only "takes on" real values, i.e., is a function of the form $f : U \to \mathbb{R}$?
  - Yes.
- We have to give him a heads up before the PSet due date that we want to use a PSet extension.

2.5 Chapter I: Analysis in the Complex Plane

From Fischer and Lieb (2012).

Section I.3: Uniform Convergence and Power Series

4/5:

- Definition of **convergent**, **absolutely convergent**, **comparison test**, **ratio test**, **root test**, and **geometric series test**.
- **Unconditionally** (convergent $\{f_n\}$): A series that will remain convergent (with the same sum) upon any reordering of its terms.
  - Absolutely convergent implies unconditionally convergent.
- Hint at hypergeometric series here?
- Definition of **pointwise convergent**, **uniformly convergent**, and **compactly convergent**.
- Covers implications of uniform convergence from class.
- Introduction of the **Cauchy convergence test**.
- **Majorant test**: If $\sum_{k=0}^{\infty} a_k$ is a convergent series with positive terms and if for almost all k and all $z \in M$ we have $|f_k(z)| \leq a_k$, then $\sum_{k=0}^{\infty} f_k$ is absolutely uniformly convergent on M.
- **Power series** (with base point z_0 and coefficients a_k): An infinite series of the following form, where $a_k \in \mathbb{C}$ $(k = 0, \dots, \infty)$ and $z_0 \in \mathbb{C}$. *Denoted by* $\boldsymbol{P(z - z_0)}$. *Given by*

$$P(z - z_0) := \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

- Definition of the **radius of convergence** and **disk of convergence**.
- **Nowhere convergent** (power series): A power series with $r = 0$.
- **Everywhere convergent** (power series): A power series with $r = \infty$.
- Proposition 3.5: Assume that for some point $z_1 \neq 0$, the terms of the power series $\sum_{k=0}^{\infty} a_k z^k$ are bounded, that is $|a_k z_1^k| \leq M$ independently of k. Then the series converges absolutely locally uniformly in the disk $D_{|z_1|}(0) = \{z : |z| < |z_1|\}$.

  Proof. By hypothesis,

$$|a_k||z_1|^k \leq M$$

  for all k. Pick z_2 such that $0 < |z_2| < |z_1|$. Then for all $|z| \leq |z_2|$, we have

$$|a_k||z|^k \leq |a_k||z_2|^k = |a_k||z_1|^k \left|\frac{z_2}{z_1}\right|^k \leq Mq^k$$

  where $q = |z_2/z_1| < 1$. By the geometric series test, $\sum_{k=0}^{\infty} Mq^k$ converges. Thus, by the majorant test, $\sum_{k=0}^{\infty} a_k z^k$ converges absolutely uniformly on $D_{|z_2|}(0)$. Progressively increasing $|z_2|$ guarantees local absolute uniform convergence over all of $D_{|z_1|}(0)$. ☐

- Definition of the **Cauchy-Hadamard formula**.
- Fischer and Lieb (2012) prove that power series are holomorphic.
- Fischer and Lieb (2012) prove the identity theorem (for power series).
- Compute the closed form of the derivatives of power series by differentiating the geometric series.

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)z^{n-k} = \frac{\mathrm{d}^k}{\mathrm{d}z^k} \sum_{n=0}^{\infty} z^n = \frac{\mathrm{d}^k}{\mathrm{d}z^k} \frac{1}{1-z} = \frac{k!}{(1-z)^{k+1}}$$

- Concludes with a few questions that will be answered in the coming sections.

Section I.4: Elementary Functions

- **Elementary functions**: The exponential functions, trigonometric functions, and hyperbolic functions.
- Complex analysis unifies the elementary functions as special cases of the complex exponential function, making clear that they do have more to do with each other than real analysis would suggest.
- Fischer and Lieb (2012) define $\exp z$ and beautifully build its properties almost solely from this axiom.
- Summary of the properties of the exponential function (Fischer & Lieb, 2012, p. 20).
  - "The exponential function is a group homomorphism from $\mathbb{C}$ into $\mathbb{C}^*$ that maps the subgroups $\mathbb{R}$ into $\mathbb{R}_{>0}$ and the subgroup $i\mathbb{R}$ into $\mathbb{S}$."
  - "The group operation on $\mathbb{C}$, $\mathbb{R}$, and $i\mathbb{R}$ is addition and the group operation on $\mathbb{C}^*$, $\mathbb{R}_{>0}$, and $\mathbb{S}$ is multiplication."
  - "The three homomorphisms $\exp : \mathbb{C} \to \mathbb{C}^*$, $\exp : \mathbb{R} \to \mathbb{R}_{>0}$, and $\exp : i\mathbb{R} \to \mathbb{S}$ are surjective."
- Notable new(ish) properties:
  - Since $|\mathrm{e}^z| = \mathrm{e}^{\operatorname{Re} z}$, we can see that "the exponential function... maps vertical lines $\operatorname{Re} z = c$ to circles" (Fischer & Lieb, 2012, p. 20).
  - In particular, $\exp(i\mathbb{R}) = \partial\mathbb{D} = \mathbb{S}$.
- Investigation of the **kernel** of $\exp : \mathbb{C} \to \mathbb{C}^*$.
  - Notably, Fischer and Lieb (2012) prove that it contains only the multiples of $2\pi i$.
- Fischer and Lieb (2012) defines π via the complex exponential function and Euler's identity lol!
- Proposition 4.4:
  i. The exponential function is periodic with period $2\pi i$.
  ii. If a domain contains at most one member of each congruence class modulo $2\pi i$, then the exponential function maps it biholomorphically onto its image in $\mathbb{C}^*$.
- The key mapping properties of $\exp z$.
  1. The mapping $t \mapsto \mathrm{e}^t$ maps $\mathbb{R}$ onto $\mathbb{R}_{>0}$ bijectively.
  2. The mapping $t \mapsto \mathrm{e}^{it}$ maps $[0, 2\pi)$ onto $\mathbb{S}$ bijectively.
- Implications.
  - A horizontal line $z = x + iy_0$ is mapped bijectively onto the open ray L_{y_0} that begins at 0 and passes through the point e^{iy_0} on the unit circle.
  - A vertical line $z = x_0 + iy$ is mapped onto the circle centered at 0 and of radius e^{x_0}.
    - An interval of length less than 2π on this line is mapped injectively into this circle.
  - Every half-open horizontal strip
$$S_{y_0} = \{z = x + iy : y_0 \leq y < y_0 + 2\pi\}$$
    is mapped bijectively onto $\mathbb{C}^*$.
    - The line $z = x + iy_0$ is mapped to the ray L_{y_0}.
    - The remaining open strip (i.e., S_{y_0} minus its lower boundary) is mapped biholomorphically onto the "slit" plane $\mathbb{C}^* \setminus L_{y_0}$.
- Definition of the **complex cosine**, **complex sine**, **complex hyperbolic cosine**, and **complex hyperbolic sine** functions.

- Proposition 4.5:

  i. The functions $\cos z$ and $\sin z$ are periodic with period 2π; $\cosh z$ and $\sinh z$ are periodic with period $2\pi i$.

  ii.
$$\frac{\mathrm{d}}{\mathrm{d}z}(\sin z) = \cos z \qquad \frac{\mathrm{d}}{\mathrm{d}z}(\cos z) = -\sin z$$
$$\frac{\mathrm{d}}{\mathrm{d}z}(\sinh z) = \cosh z \qquad \frac{\mathrm{d}}{\mathrm{d}z}(\cosh z) = \sinh z$$

  iii.
$$\begin{aligned}\sin(z+w) &= \sin z \cos w + \cos z \sin w\\ \cos(z+w) &= \cos z \cos w - \sin z \sin w\\ \sinh(z+w) &= \sinh z \cosh w + \cosh z \sinh w\\ \cosh(z+w) &= \cosh z \cosh w + \sinh z \sinh w\end{aligned}$$

  iv. For each of the above functions, $f(\bar{z}) = \overline{f(z)}$.

  v.
$$\cos z = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k}}{(2k)!} \qquad \sin z = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+1}}{(2k+1)!}$$
$$\cosh z = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} \qquad \sinh z = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}$$

  vi. $\mathrm{e}^{iz} = \cos z + i \sin z$.

  vii.
$$\sin^2 z + \cos^2 z = 1 \qquad \cosh^2 z - \sinh^2 z = 1$$

- Decomposition of e^z into its real and imaginary parts:
$$\mathrm{e}^z = \mathrm{e}^{x+iy} = \mathrm{e}^x(\cos y + i \sin y)$$

- The zeroes of the complex sine are just those of the real sine; $\sin z \neq 0$ for any nonreal z.

- Introduction of the **roots of unity**.

- The remaining trigonometric and hyperbolic functions are defined in the usual way, are holomorphic everywhere except at the zeros of their denominators, and have periods π or πi (in the hyperbolic case).
$$\tan z = \frac{\sin z}{\cos z} \qquad \cot z = \frac{\cos z}{\sin z}$$
$$\tanh z = \frac{\sinh z}{\cosh z} \qquad \coth z = \frac{\cosh z}{\sinh z}$$

- Preview: We will show later that this extension of the real analytic functions to holomorphic functions on the complex plane is unique, not arbitrary.