Week 1

???

1.1 Holomorphic Functions

3/19:
- We begin by reviewing some properties of the **complex numbers**.
- **Complex numbers**: The field of elements $z = x + iy$ where $x, y \in \mathbb{R}$ and $i^2 = -1$. *Denoted by* $\mathbb{C}$.

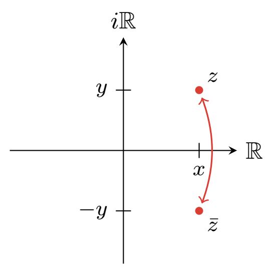

Figure 1.1: The complex plane.

  - Can be visualized as a two-dimensional plane with the number z corresponding to the point (x, y).
- **Real part**: The number x. *Denoted by* $\mathbf{Re}(z)$.
- **Imaginary part**: The number y. *Denoted by* $\mathbf{Im}(z)$.
- **Complex conjugate** (of z): The complex number defined as follows. *Denoted by* $\bar{z}$. *Given by*

$$\bar{z} := x - iy$$

- Now recall the definition of a *real* function that is **differentiable** at a point $x_0 \in \mathbb{R}$.
  - $f'(x_0)(x - x_0)$ is the "best linear approximation" of f near x_0, where $f'(x_0)$ is also defined below.
- **Differentiable** (f at x_0): A function $f : \mathbb{R} \to \mathbb{R}$ for which the following limit exists. *Constraint*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} =: f'(x_0)$$

- We now build up to defining a notion of complex differentiability.
  - Observe that the constraint above is equivalent to the constraint

$$f(x) = f(x_0) + [\underbrace{f'(x_0) + e(x)}_{\Delta(x)}](x - x_0)$$

    where $e(x) \to 0$ as $x \to x_0$.
  - Note that we are defining a new function $\Delta(x)$ above, with the property that $\Delta(x_0) = f'(x_0)$.

- **Holomorphic** (f at z_0): A function $f : \mathbb{C} \to \mathbb{C}$ for which the following limit exists. *Also known as* **$\mathbb{C}$-differentiable**. *Constraints*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} =: f'(z_0) \qquad \iff \qquad f(z) = f(z_0) + \Delta(z)(z - z_0)$$

  where Δ is continuous at z_0 and $\Delta(z_0) = f'(z_0)$.

  - Is this the true definition of "holomorphic" / "$\mathbb{C}$-differentiable" function, or is this just a naive first pass??

- Properties of holomorphic functions: Let $U \subset \mathbb{C}$ be open.
  1. The holomorphic functions on U form a ring $\boldsymbol{\mathcal{O}(U)}$.
     - Equivalently, the $\mathbb{C}$-differentiation operator is $\mathbb{C}$-linear.
     - Equivalently, if f, g are holomorphic, then $f + g$ and fg are holomorphic, too.
     - Equivalently (and most simply), we have the sum rule and the product rule (and the quotient rule if the function in the denominator is nonzero).
  2. We have the chain rule.
  3. Holomorphic implies continuous.

- Examples: Polynomials, rational functions $p(z)/q(z)$ (away from their **poles**).

- Noney[1]: Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$z \mapsto \bar{z}$$

  - TPS[2]: Why?
  - Notice that

$$f(0) = 0 \qquad\qquad f(t) = t \qquad\qquad f(it) = -it$$

  - Thus,

$$\Delta(t) = 1 \qquad\qquad \Delta(it) = -1$$

    for all t.
  - But this means that Δ can't be continuous!
  - Yet f is clearly $\mathbb{R}$-differentiable! What gives?!
  - Note that — viewing f as a mapping of $\mathbb{R}^2 \to \mathbb{R}^2$ — we have

$$Df = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

[1] What does "Noney" mean??
[2] What does "TPS" mean??