8 Applications of Contour Integrals

5/13:

- **Multiply connected** (region): A simply connected region with several holes or places where f is not analytic.

8.1 The Cauchy Residue Theorem

- **Cauchy residue theorem**: If C is a curve that encloses N isolated singularities of f, the k^{th} one being at z_k, then we have

$$\begin{aligned}\oint_C f(z)\,\mathrm{d}z &= \sum_{k=1}^N \oint_{C_k} f(z)\,\mathrm{d}z = 2\pi i\sum_{k=1}^N a_{-1}(z_k)\\ &= 2\pi i[\text{sum of residues}]\end{aligned}$$

Proof. For the sake of this argument, we will discuss a region with two holes/singularities, but the argument easily generalizes. Draw curves C_1, C_2 around these holes/singularities oriented counterclockwise as well. Make cut lines from C to C_1 and from C to C_2. Thus, the single continuous contour

$$C' := C + (-C_1) + (-C_2) + \text{cut lines}$$

encloses a simply connected region. Thus, by the definition of integrating over multiple curves,

$$\oint_{C'} f(z)\,\mathrm{d}z = \oint_C f(z)\,\mathrm{d}z + \oint_{-C_1} f(z)\,\mathrm{d}z + \oint_{-C_2} f(z)\,\mathrm{d}z$$

By the CIT, the left-hand side of the above vanishes. Thus, rearranging, we obtain

$$\oint_C f(z)\,\mathrm{d}z = \oint_{C_1} f(z)\,\mathrm{d}z + \oint_{C_2} f(z)\,\mathrm{d}z$$

At this point, we can evaluate each of the integrals on the RHS above via the definition of the residue to get the desired result. □

- Takeaway: Applications to evaluating definite integrals on closed contours.
- Goes over the residue properties from the 5/2 lecture.

8.2 Evaluation of Definite Integrals by Contour Integration

- General strategy: Choose a contour C such that part of it (which we'll call C_1) lies along the real axis and such that the integral along the remaining part C_2 is either zero or simple to evaluate. Then

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x &= \lim_{R\to\infty}\int_{-R}^{R} f(z)\,\mathrm{d}z\\ &= \lim_{R\to\infty}\left[\oint_C f(z)\,\mathrm{d}z - \int_{C_2} f(z)\,\mathrm{d}z\right]\\ &= 2\pi i[\text{sum of residues in } C] - \lim_{R\to\infty}\int_{C_2} f(z)\,\mathrm{d}z\end{aligned}$$

- Does $f(z) = (z^2+1)^{-1}$ as an example.
- Goes through some more examples, including higher-order poles and trigonometric functions.

8.2.1 Jordan's Lemma

- Solves a certain integral two ways to motivate and build **Jordan's lemma**.
- Introduces and rigorously proves the following bound in the process.

$$\sin\theta \geq \frac{2\theta}{\pi}$$

- **Jordan's lemma**: Given a function of the form $\mathrm{e}^{iaz}f(z)$, where $a>0$, if we have $|f(R\mathrm{e}^{i\theta})| \leq g(R)$ for all $\theta \in [0,\pi]$, where $g:\mathbb{R}\to\mathbb{R}$, then

$$\left|\int_{C_2} \mathrm{e}^{iaz}f(z)\,\mathrm{d}z\right| \leq \frac{\pi}{a}g(R)(1-\mathrm{e}^{-aR})$$

  If, in addition, $g(R)\to 0$ as $R\to\infty$, then

$$\int_{-\infty}^{\infty} \mathrm{e}^{iax}f(x)\,\mathrm{d}x = 2\pi i[\text{sum of residues of } \mathrm{e}^{iaz}f(z) \text{ in } \mathbb{H}]$$

- An analogous results exists for the lower half plane.
- More examples.

8.2.2 Cauchy Principal Value

- **Cauchy principal value** (of a compact integral over a pole): The number defined as follows, where $f(z)$ is a function with a simple pole on the real axis at $z=x_0$ and $x_0\in(a,b)$. *Denoted by* $\boldsymbol{P\int_a^b f(x)\,\mathrm{d}x}$. *Given by*

$$P\int_a^b f(x)\,\mathrm{d}x = \lim_{r\to 0}\left[\int_a^{x_0-r} f(x)\,\mathrm{d}x + \int_{x_0+r}^b f(x)\,\mathrm{d}x\right]$$

- We define the Cauchy principal value because for such functions, $\int_a^b f(x)\,\mathrm{d}x$ does not strictly exist.
- Evaluating over the contour in Figure 2.5 from the class notes, we obtain

$$P\int_a^b f(x)\,\mathrm{d}x = \pi i\operatorname{res}_{x_0} f + 2\pi i[\text{sum of residues of } f \text{ enclosed by } C] - \int_{C_2=\gamma_4} f(z)\,\mathrm{d}z$$

- Example given.

8.2.3 A Branch Point

- Example given.
  - Looks like you take cut lines along the branch cut.