Week 5

End Quantitative and Intro to Qualitative

5.1 Planar Autonomous Linear Systems

10/24:

- Review of vector fields.
- **Phase diagram**: A diagram that shows the qualitative behavior of an autonomous ordinary differential equation. *Also known as* **phase portrait**.

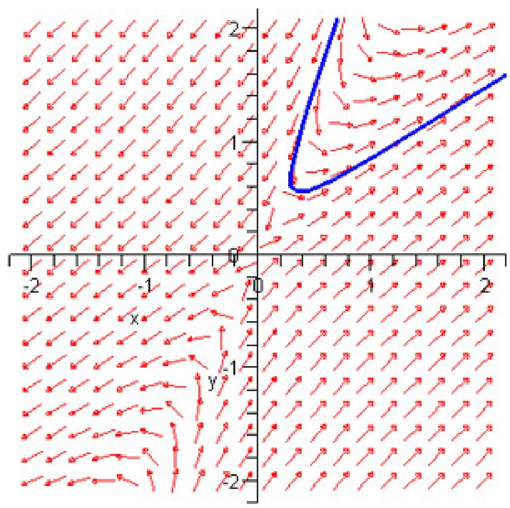
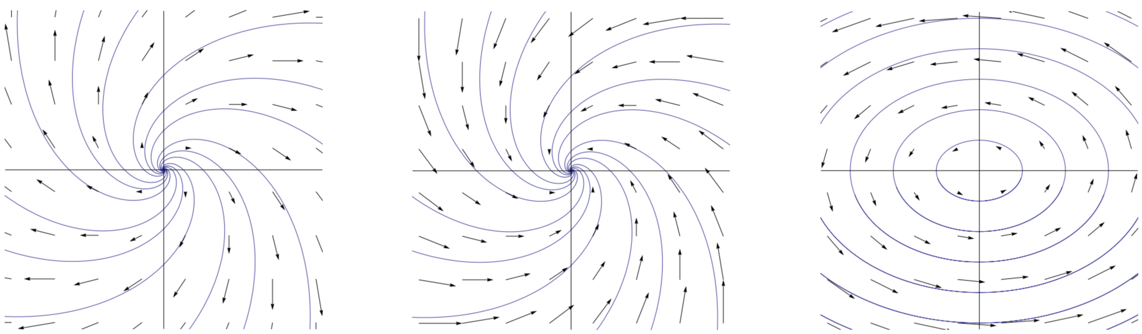


Figure 5.1: Phase diagram example.

  - Consists of a selection of arrows describing, to some extent, a vector field and is often paired with integral curves.
- Suppose $\Omega \subset \mathbb{R}^n$ is open.
- **Vector field** (on Ω): A mapping from $\Omega \to \mathbb{R}^n$. *Denoted by* $\boldsymbol{X}$.
  - Essentially, a vector field assigns to every point of some region a vector; the definition just formalizes this notion.
- **Flow**: A formalization of the idea of the motion of particles in a fluid.
  - The solution to the IVP $\frac{\mathrm{d}y}{\mathrm{d}t} = X(y)$, $y(0) = x$.

- If X is C^1, then for all $x \in \Omega$, there exists a unique solution y to the above IVP.
- **Orbit** (of x under X): The trajectory $y(t, x)$.
  - Recall that the tangent vector to any trajectory at any point coincides with the vector to which X maps that point.
- **Fixed point**: A point $x_0 \in \Omega$ such that $X(x_0) = \bar{0}$.
  - If x_0 is a fixed point, then the trajectory is $y(t) = x_0$.
- Today: We will consider flows on vector fields where the dimension is two and our vector field is linear. In particular...
- Let A be a 2×2 real matrix, and let $X(x) = Ax$.
  - In this case, $x_0 = 0$ is the only fixed point.
  - The flow is given by the linear differential equation $y' = Ay$, $y(0) = x$. The solution is $y(t) = \mathrm{e}^{tA}x$.
- Case 1: A has no real eigenvalues.



(a) $\sigma > 0$. (b) $\sigma < 0$. (c) $\sigma = 0$.

Figure 5.2: Phase diagrams for a planar system with no real eigenvalues.

  - We know that $\chi_A(z)$ is a real polynomial: $\chi_A(z) = z^2 + (\operatorname{trace} A)z + \det A$, and since A is real, both $\operatorname{trace} A$ and $\det A$ are real.
  - Thus, the eigenvalues appear as conjugate pair, i.e., we may write $\lambda = \sigma + i\beta$ and $\bar{\lambda} = \sigma - i\beta$.
    - $\alpha = \gamma = 1$ for both eigenvalues.
    - The eigenvectors must also be complex conjugates.
  - Distinct eigenvalues imply that A is diagonalizable.
  - However, this is not what we want because if we use the complex form, then

$$\mathrm{e}^{tA} = Q \begin{pmatrix} \mathrm{e}^{t\lambda} & 0 \\ 0 & \mathrm{e}^{t\bar{\lambda}} \end{pmatrix} Q^{-1}$$

  - Indeed, we want to get a real matrix out of $Q, \mathrm{e}^{t\Lambda}, Q^{-1}$ all complex. We have

$$\begin{aligned} \mathrm{e}^{tA}x &= Q \begin{pmatrix} \mathrm{e}^{t(\sigma+i\beta)} & 0 \\ 0 & \mathrm{e}^{t(\sigma-i\beta)} \end{pmatrix} \underbrace{Q^{-1}x}_{z} \\ &= Q \begin{pmatrix} \mathrm{e}^{t(\sigma+i\beta)} z^1 \\ \mathrm{e}^{t(\sigma-i\beta)} z^2 \end{pmatrix} \\ &= z^1 \mathrm{e}^{t(\sigma+i\beta)} v + z^2 \mathrm{e}^{t(\sigma-i\beta)} \bar{v} \end{aligned}$$

- Since $y(0) = x = z^1 v + z^2 \bar{v} \in \mathbb{R}^2$ (i.e., $z^1 v + z^2 v$ is *real*), we know that it is equal to its complex conjugate. This tells us that

$$\begin{aligned} z^1 v + z^2 \bar{v} &= \bar{z^1}\bar{v} + \bar{z^2} v \\ z^1 &= \bar{z^2} \end{aligned}$$

- It follows that

$$\begin{aligned} y(t) &= \mathrm{e}^{tA} x \\ &= z^1 \mathrm{e}^{t(\sigma + i\beta)} v + \bar{z^1} \mathrm{e}^{t(\sigma - i\beta)} \bar{v} \\ &= z^1 \mathrm{e}^{t(\sigma + i\beta)} v + \overline{z^1 \mathrm{e}^{t(\sigma + i\beta)} v} \\ &= 2\,\mathrm{Re}(z^1 \mathrm{e}^{t(\sigma + i\beta)} v) \\ &= 2\,\mathrm{Re}(z^1 \mathrm{e}^{\sigma t}(\cos(\beta t) + i\sin(\beta t))(v_1 + iv_2)) \\ &= 2\,\mathrm{Re}(z^1 \mathrm{e}^{\sigma t}(\cos(\beta t) v_1 + i\cos(\beta t) v_2 + i\sin(\beta t) v_1 - \sin(\beta t) v_2)) \\ &= 2\mathrm{e}^{\sigma t}\cos(\beta t)\cdot \mathrm{Re}(z^1 v) - 2\mathrm{e}^{\sigma t}\sin(\beta t)\cdot \mathrm{Im}(z^1 v) \end{aligned}$$

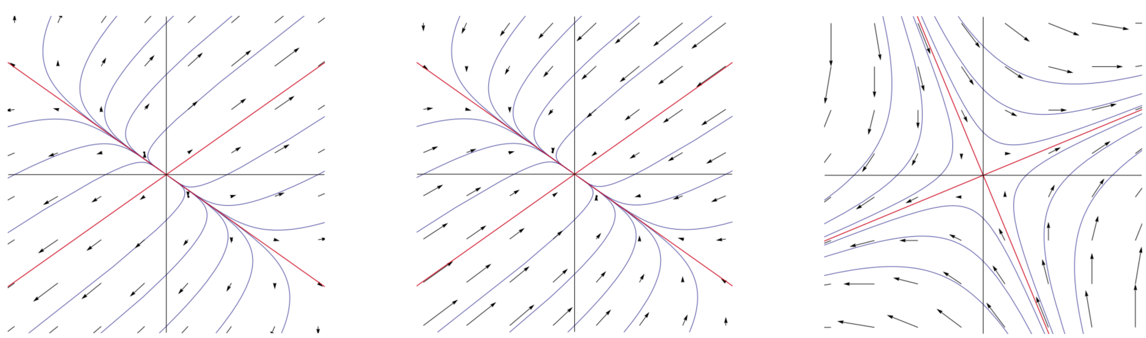
- How do we get from the second-to-last line to the last line above??
- Suppose $\sigma \neq 0$. Then

$$x \mapsto \begin{pmatrix} \mathrm{Re}(z^1 v) \\ \mathrm{Im}(z^1 v) \end{pmatrix}$$

  is a real linear transformation on $\mathbb{R}^2$.
- It follows that the trajectories are just spirals in the complex plane.
- If $\sigma > 0$, then the spiral repels from the origin. If $\sigma < 0$, then the spiral attracts to the origin. If $\sigma = 0$, we get an ellipse.
- Therefore, we have completely classified equations of the form

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix}' = \begin{pmatrix} y^2 \\ -\omega^2 y^1 \end{pmatrix}$$

- Case 2: A has real eigenvalues and *is* diagonalizable.



(a) $\lambda_1, \lambda_2 > 0$. (b) $\lambda_1, \lambda_2 < 0$. (c) $\lambda_1 < 0 < \lambda_2$ (WLOG).

Figure 5.3: Phase diagrams for a diagonalizable planar system with real eigenvalues.

- Suppose $\lambda_1, \lambda_2 \in \mathbb{R}$ have corresponding linearly independent eigenvectors v_1, v_2.
- If we choose v_1, v_2 to be our basis, then

$$\mathrm{e}^{tA} = Q \begin{pmatrix} \mathrm{e}^{t\lambda_1} & 0 \\ 0 & \mathrm{e}^{t\lambda_2} \end{pmatrix} Q^{-1}$$

  where $Q = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$.

- Thus, as before, the solution may be expressed in the following form, where $z = Q^{-1}x$.

$$y(t) = \mathrm{e}^{tA}x = \mathrm{e}^{\lambda_1 t}z^1 v_1 + \mathrm{e}^{\lambda_2 t}z^2 v_2$$

- Moving forward, it will be convenient to work in the v_1, v_2 basis. We divide into three subcases ($\lambda_1, \lambda_2 > 0$ [Figure 5.3a], $\lambda_1, \lambda_2 < 0$ [Figure 5.3b], and WLOG $\lambda_1 < 0 < \lambda_2$ [Figure 5.3c]).
  1. Notice that
  $$\mathrm{e}^{\lambda_2 t} = \mathrm{e}^{(\lambda_2/\lambda_1)(\lambda_1 t)}$$
  i.e., $\mathrm{e}^{\lambda_2 t}$ is a power of $\mathrm{e}^{\lambda_1 t}$. Thus, when the signs are the same, we get a power function $v_2 = v_1^{\lambda_2/\lambda_1}$.
     - Both subspaces v_1, v_2 are unstable here.
  2. If $\lambda_1, \lambda_2 < 0$, then we have the same trajectories, but they're all attracted to the origin instead of repelled.
     - Both subspaces v_1, v_2 are stable here.
  3. When both eigenvalues have different signs, we are considering power functions of a negative power.
     - The stable subspace is v_2 and the unstable subspace is v_1 here.

- Case 3: A has real eigenvalues and *is not* diagonalizable.

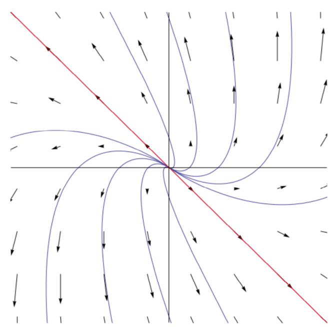


Figure 5.4: Phase diagrams for a nondiagonalizable planar system with real eigenvalues.

- In this case, the matrix exponential is given by

$$\mathrm{e}^{tA} = Q \begin{pmatrix} \mathrm{e}^{t\lambda} & t\mathrm{e}^{t\lambda} \\ 0 & \mathrm{e}^{t\lambda} \end{pmatrix} Q^{-1}$$

- The solution is given by
$$\mathrm{e}^{tA}x = (z^1\mathrm{e}^{t\lambda} + z^2 t\mathrm{e}^{t\lambda})v + z^2\mathrm{e}^{t\lambda}u$$
where $Q^{-1}x = z$ again.
- In graphing, note that here we have (a distorted version of) the form $y = x \pm x \log x$:

$$y = (z^1\mathrm{e}^{t\lambda} + z^2 t\mathrm{e}^{t\lambda})\hat{\imath} + z^2\mathrm{e}^{t\lambda}\hat{\jmath}$$

Define $x := \mathrm{e}^{t\lambda}$. Then $t = \lambda^{-1}\ln x$. Substituting, we have

$$\begin{aligned} &= (z^1 x + z^2(\lambda^{-1}\ln x)x)\hat{\imath} + z^2 x\hat{\jmath} \\ &= (z^1 x + z^2\lambda^{-1}x\ln x)\hat{\imath} + z^2 x\hat{\jmath} \end{aligned}$$

- When $\lambda > 0$, the whole space is unstable. Thus, the phase diagram is tangent to the origin.
- When $\lambda < 0$, the trajectories take the same form but this time are attracted to zero. In this case, the whole space is stable.

- We can take x_1 to x_2 iff they are in the same orbit. Conclusion: Orbits never cross.
- Takeaway: You should be able to compute the eigenvalues and eigenvectors and sketch these graphs.
- Shao will post lecture notes after today's lecture!
- Next lecture: The final explicitly solveable case, which is the driven harmonic oscillator.

5.2 Driven Harmonic Oscillator and Resonance

10/26:
- We are interested in the 2nd order constant coefficient equation

$$x'' + \mu x' + \omega_0^2 x = H_0 \mathrm{e}^{i\omega t}$$

  where $\mu \geq 0$ and $\omega_0, \omega > 0$.

- Two cases where this ODE arises:

(a) A driven harmonic oscillator. (b) An RLC circuit.

Figure 5.5: Origins of the driven harmonic oscillator equation.

1. The driven harmonic oscillator.
   - Consider a mass on a spring.
   - The extent of friction between the mass point and the surface is described by μ.
   - The oscillation is periodically driven by a force of magnitude $H_0 \cos \omega t$.
2. RLC circuit.
   - R is resistance, L is inductance, C is capacitance.
   - We have the laws

$$LI_L'(t) = V_L \qquad CV_C'(t) = I_C \qquad I_R(t) = V_R(t)/R$$

     - Left: Self-inductance.
     - Right: Ohm's law.
   - Combining them with Kirchhoff's laws

$$I(t) = I_R = I_C = I_L \qquad V(t) = V_R + V_L + V_C$$

     we get the RLC circuit equation

$$LI'' + RI' + \frac{I}{C} = V'(t)$$

   - The most interesting cases is when we have a source of alternating current of frequency ω. In this case, $V(t) = V_0 \cos \omega t$ or, in the complex case, $V(t) = V_0 \mathrm{e}^{i\omega t}$. This yields the complex equation

$$I'' + \frac{R}{L} I' + \frac{1}{LC} I = \frac{i\omega V_0}{L} \mathrm{e}^{i\omega t}$$

   - Here, the friction coefficient $\mu = R/L$ and the frequency is $\omega_0 = \sqrt{1/LC}$.

- Recall that we want to solve the following ODE.

$$x'' + \mu x' + \omega_0^2 x = H_0 \mathrm{e}^{i\omega t}$$

  - The homogeneous linear equation $x'' + \mu x' + \omega_0^2 x = 0$ is well-understood, i.e., we can find all of the *homogeneous* solutions to the above equation.
  - Thus, to solve the above inhomogeneous equation, we just have to find a particular solution.
- WLOG let $\omega > 0$.
- From the homework, a particular solution $x_p(t)$ with initial condition $x_p(0) = x_p'(0) = \mu = 0$ can be obtained from the Duhamel formula as follows.

$$x_p(t) = H_0 \int_0^t \frac{\sin \omega_0 (t - \tau)}{\omega_0} \mathrm{e}^{i\omega\tau} \mathrm{d}\tau$$

  - Substituting

$$\sin\theta = \frac{\mathrm{e}^{i\theta} - \mathrm{e}^{-i\theta}}{2i}$$

    into the above allows us to evaluate it.
  - In particular, it follows that

$$x_p(t) = \begin{cases} \dfrac{H_0}{\omega_0^2 - \omega^2} \left(\mathrm{e}^{i\omega t} - \cos\omega_0 t - \dfrac{i\omega}{\omega_0} \sin\omega_0 t \right) & \omega \neq \omega_0 \\ -\dfrac{iH_0}{2\omega_0} \left(t\mathrm{e}^{i\omega_0 t} - \dfrac{\sin\omega_0 t}{\omega_0} \right) & \omega = \omega_0 \end{cases}$$

    - We compute the $\omega = \omega_0$ case using L'Hôpital's rule to analyze the $\omega \neq \omega_0$ case as $\omega \to \omega_0$.
  - If we pump in energy at the same point that we have deviation ($\omega = \omega_0$), then the amplitude of oscillation goes to ∞.
    - Practically, when $\omega \approx \omega_0$, the long-time behavior of the driven oscillator will be very much like a growing oscillator.
    - Eventually, the amplitude will be approximately $(\omega - \omega_0)^{-1}$.
- **Resonance catastrophe**: Inputing energy into a system at its natural frequency, causing the total energy to grow until a mechanical failure occurs.
  - This is what happened at the Millenium Bridge in London; synchronized footsteps caused the bridge to shake really wildly.
- If $\mu > 0$, there will be a particular solution of the form

$$x_p(t) = A(\omega) H_0 \mathrm{e}^{i\omega t}$$

  - From HW1, we have three cases when $\mu > 0$: $0 < \mu < 2\omega_0$, $\mu = 2\omega_0$, and $\mu > 2\omega_0$. These are just the three cases of the characteristic polynomial??
  - Substituting the proposed form of the particular solution into the differential equation, we get

$$\begin{aligned} x_p'' + \mu x_p' + \omega_0^2 x_p &= H_0 \mathrm{e}^{i\omega t} \\ (-\omega^2 + i\omega\mu + \omega_0^2) H_0 A(\omega) \mathrm{e}^{i\omega t} &= H_0 \mathrm{e}^{i\omega t} \\ (-\omega^2 + i\omega\mu + \omega_0^2) A(\omega) &= 1 \\ A(\omega) &= \frac{1}{\omega_0^2 - \omega^2 + i\mu\omega} \end{aligned}$$

  - In theory, we avoid the resonance catastrophe in this case. In practice, however, when $\omega \to 0$, we still run into issues.

- For mass point:
$$|H_0 A(\omega)| = \frac{|H_0|}{\sqrt{(\omega^2 - \omega_0^2)^2 + \mu^2\omega^2}}$$
  - The norm $|H_0 A(\omega)|$ is maximized when $\omega_r = \sqrt{\omega_0^2 + \mu^2/2}$.
  - $\omega_r \to \omega_0$ implies $\mu \to 0$??
- As for the argument/angle,
$$\arg(H_0 A(\omega)) = \arg H_0 + \arg A(\omega)$$
- We consider $\omega : 0 \to \omega_0 \to +\infty$.
  - When $\omega = 0$, the complex amplitude is $1/\omega_0^2$ so it's a real number in the complex plane.
  - If ω is increased a bit, we get the reciprocal of a complex number. Norm is reciprocal, argument is negative.
  - For $\omega = \omega_0$, we have a purely imaginary number.
  - As $\omega \to \infty$, the argument approaches $-\pi$??
  - Showing the shape of the norm and the argument with respect to ω. This allows us to completely describe the resonance phenomena.

- For the RLC circuit, the discussion is a bit different.
  - The external voltage $V(t) = V_0 e^{i\omega t}$. Thus, $V'(t) = iV_0\omega e^{i\omega t}$.
  - Here,
$$x_p(t) = \frac{iV_0\omega e^{i\omega t}}{\omega_0^2 - \omega^2 + iR\omega/L}$$
  - Look at the complex amplitude.
    - Multiply the numerator and denominator by the inductance L to get
$$x_p(t) = \frac{iV_0\omega L e^{i\omega t}}{L\omega_0^2 - L\omega^2 + iR\omega}$$
    - Then,
$$\text{Norm} = \frac{V_0 L}{\sqrt{R^2 + \left(\frac{1}{C\omega} - \frac{\omega}{L}\right)^2}}$$
    - For an RLC circuit, the resistance does not affect the resonance frequency.
$$\omega_r = \sqrt{\frac{1}{LC}} = \omega_0$$
    - If you have an external source of voltage, then you can vary the capacity of your circuit to ensure that the voltage will be maximized at a given frequency. We can tune our circuit to a very specific resonance frequency (this is used to filter our radio stations). The RLC circuit is only observable when the resonance coincides with the external resonance.

- There will be a bonus problem which is a PDE describing the vibration of a string.
  - Suppose we have a string with fixed endpoints, and suppose it is undergoing a small vibration.
  - Deviation from the equilibrium is described by a function $u(x,t)$.
  - The simplest equation we can derive is the 1D linear wave equation
$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial x^2} = f(x,t)$$
    - c is the speed of the wave.

    - $f(x,t)$ is the given external force.
  - We can show that when $f(x,t) = 0$, then the vibration of the string is the linear supposition of infinitely many standing waves.

$$u(x,t) = \sum_{k=1}^{\infty} a_k e^{\frac{\pi k t}{\ell}} \sin \frac{k\pi}{\ell} x$$

    - There are $k-1$ nodes in the string. These are called standing waves.
  - If you drive it with frequency

$$f(x,t) = \cos \omega t \sin \frac{k\pi}{\ell} x$$

    you encounter the resonance catastrophe.

- We are interested in the driven harmonic oscillator because it describes the vibrations, even of PDEs.
- This concludes our discussion of explicitly solvable differential equations.
- Those that are solvable by power series require complex analysis.
- Starting this Friday, we will talk about the qualitative theory of differential equations.
- Cauchy-Lipschitz this Friday.
- Next week: Continuous dependence on initial values and differentiation with respect to the parameter of this equation.
- After this, we will be able to compute classical examples in the theory of perturbations.
- We will be able to solve the procession of Mercury problem (which was the first experimental verification of general relativity).

5.3 Qualitative Theory of ODEs

10/28:

- First issue: Uniqueness — we want to be able to talk about *the* solution to the IVP.
- We will be considering the IVP $y'(t) = f(t,y)$, $y(t_0) = y_0$ for $y(t)$ an $\mathbb{R}^n$-valued function.
- To embed our rough outline of the Cauchy-Lipschitz theorem into analysis, we start with metric spaces.
- **Metric space**: A set and a metric. *Denoted by* $\boldsymbol{(X, d)}$.
- **Metric**: A function from $X \times X \to [0, +\infty)$ that satisfies the following three axioms. *Denoted by* $\boldsymbol{d}$.
  1. $d(x,y) = d(y,x)$.
  2. $d(x,y) \geq 0$ and $d(x,y) = 0$ iff $x = y$.
  3. Triangle inequality: $d(x,z) \leq d(x,y) + d(y,z)$.
- Examples:
  1. $\mathbb{R}^n$ with $d(x,y) = \|x - y\| = \sqrt{\sum_{j=1}^{n} |x^j - y^j|^2}$.
  2. Continuous functions $y : [a,b] \to \mathbb{R}^n$ with $d(y_1, y_2) = \|y_1 - y_2\| = \sup_{t \in [a,b]} |y_1(t) - y_2(t)|$.
- In Euclidean spaces, we have **completeness**.
- **Cauchy** (sequence): A sequence $\{x_n\} \subset X$ such that for all $\varepsilon > 0$, there exists $N \geq 0$ such that $d(x_m, x_n) < \varepsilon$ for all $m, n > N$.

- **Convergent** (sequence): A sequence $\{x_n\} \subset X$ for which there exists $x \in X$ such that
$$\lim_{n\to\infty} d(x, x_n) = 0$$
- **Complete** (metric space): A metric space (X, d) such that every Cauchy sequence is convergent.
- Theorem (Banach fixed point theorem): Let (X, d) be a complete metric space and let $\Phi : X \to X$ be a function for which there exists $q \in (0, 1)$ such that for all $x, y \in X$,
$$d(\Phi(x), \Phi(y)) \le q \cdot d(x, y)$$
Then there exists a unique $x \in X$ such that $x = \Phi(x)$.

Proof. We first construct the desired fixed point x.

Pick any $x_0 \in X$. Inductively define $\{x_n\}$ by $x_{n+1} = \Phi(x_n)$, starting from $n = 0$. We will now show that $\{x_n\}$ is a Cauchy sequence. As a lemma, we will prove by induction that
$$d(x_j, x_{j+1}) \le q^j \cdot d(x_0, x_1)$$
for all $j \in \mathbb{N}_0$. For the base case $j = 0$, equality evidently holds. Now suppose inductively that we have proven that $d(x_j, x_{j+1}) \le q^j \cdot d(x_0, x_1)$; we want to prove the claim for $j + 1$. But we have that
$$\begin{aligned} d(x_{j+1}, x_{j+2}) &= d(\Phi(x_j), \Phi(x_{j+1})) \\ &\le q \cdot d(x_j, x_{j+1}) \\ &\le q \cdot q^j \cdot d(x_0, x_1) \\ &= q^{j+1} \cdot d(x_0, x_1) \end{aligned}$$
as desired.

It follows that
$$\begin{aligned} d(x_n, x_{n+m}) &\le \sum_{k=0}^{m-1} d(x_{n+k}, x_{n+k+1}) && \text{Triangle inequality} \\ &\le \sum_{k=0}^{m-1} q^{n+k} \cdot d(x_0, x_1) && \text{Lemma} \\ &= q^n(1 + q + \cdots + q^{m-1}) \cdot d(x_0, x_1) \\ &< q^n(1 + q + \cdots + q^{m-1} + \cdots) \cdot d(x_0, x_1) \\ &= \frac{q^n}{1-q} \cdot d(x_0, x_1) \end{aligned}$$
It follows that the above term will converge to zero as $n \to \infty$, so $\{x_n\}$ is a Cauchy sequence and there exists an x such that $x_n \xrightarrow{d} x$.

We now prove that x is a fixed point of Φ, i.e., that $\Phi(x) = x$. We have that
$$\begin{aligned} d(x, \Phi(x)) &\le d(x, x_n) + d(x_n, \Phi(x_n)) + d(\Phi(x_n), \Phi(x)) \\ &\le d(x, x_n) + d(x_n, x_{n+1}) + q \cdot d(x_n, x) \\ &= (1 + q) \cdot d(x, x_n) + d(x_n, x_{n+1}) \end{aligned}$$
where the first term converges since $\{x_n\}$ is convergent and the second term converges since $\{x_n\}$ is Cauchy. Thus, $d(x, \Phi(x)) \to 0$ as $n \to \infty$, so $x = \Phi(x)$, as desired.

Lastly, we prove that x is unique. Suppose that there exists $y \in X$ such that $y = \Phi(y)$. Then
$$d(x, y) = d(\Phi(x), \Phi(y)) \le q \cdot d(x, y)$$
It follows that $d(x, y) \le q^n \cdot d(x, y)$, i.e., that $d(x, y) \to 0$ as $n \to \infty$. Therefore, we must have that $d(x, y) = 0$, from which it follows that $x = y$, as desired. □

- Notes on the Banach fixed point theorem.
  - Φ is a **contraction**.
  - Shao gives the example of crumpling a sheet of paper (more specifically, dropping a map of a park in that park; a point coincides).
- Example: Fixed point of the cosine function.
  - Define $\{x_n\}$ by $x_{n+1} = \cos x_n$. If $x_0 \in \mathbb{R}$, then $x_1 \in [-1, 1]$ and $x_2 \in [\cos 1, 1]$.
  - Thus, while cosine is not a contraction on the real numbers ($\cos'(-\pi/2) = 1$, for example), we can show that $\cos : [\cos 1, 1] \to [\cos 1, 1]$ is a contraction: If $x, y \in [\cos 1, 1]$, then

$$\begin{aligned}|\cos x - \cos y| &= \left|\int_y^x -\sin t \, \mathrm{d}t\right| \\ &\leq |x - y| \sup_{t \in [\cos 1, 1]} |\sin t| \\ &\leq (\sin 1)|x - y|\end{aligned}$$

  - Thus, cosine has a fixed point at the intersection of $y = \cos x$ and $y = x$ of approximate value $0.739\ldots$
  - Overall, this is a pretty bad example, though.
- Theorem: Let $y_k : [a, b] \to \mathbb{R}^n$ be a Cauchy sequence of continuous functions under the sup norm. Then the limit exists and is continuous.
  - The proof is based on uniform convergence, which we've encountered before in analysis.
  - It follows that $C[a, b]$ (the metric space of all continuous functions on $[a, b]$) is complete.
  - If $\{y_k\} \subset \bar{B}(0, M)$[1], then the limit y is in $\bar{B}(0, M)$.
- Let's return to our ODE $y'(t) = f(t, y(t))$, $y(t_0) = y_0 \in \mathbb{R}^n$.
- We now have the tools to prove the Cauchy-Lipschitz theorem, and we will presently build up to that.
- Although we do not typically think of it this way, f is still a function with a domain and range. In particular, its domain is the set of ordered pairs where the first entry is a real number and the second entry is an element of the range of y, i.e., an element of $\mathbb{R}^n$. Thus, to begin, we are allowed to impose the following conditions on f.
  - Let $f(t, z)$ be defined on $[t_0, t_0 + a] \times \bar{B}(y_0, b)$ for some $a, b \in \mathbb{R}_+$ (we will put further constraints on the values of a, b later).
  - On this domain, suppose $|f|$ is bounded by some $M \in \mathbb{R}$, i.e., $|f(t, z)| \leq M$ for all t, z in the above set.
  - Let f be Lipschitz continuous in the second argument. In particular, there exists $L > 0$ such that $|f(t, z_1) - f(t, z_2)| \leq L|z_1 - z_2|$ for any $z_1, z_2 \in \mathbb{R}^n$.
- We usually consider a given ODE in differential form. However, there's no reason we can't consider the equivalent integral equation

$$y(t) = y_0 + \int_{t_0}^t f(\tau, y(\tau)) \mathrm{d}\tau$$

  - The reason for this change of perspective will become apparent shortly.

[1] $\bar{B}(0, M)$ denotes the set of all functions $y : [a, b] \to \mathbb{R}^n$ with sup norm at most M; topologically, it is the closed ball of radius M centered at the origin in $C[a, b]$.

- Let $\Phi : C[t_0, t_0 + a] \to C[t_0, t_0 + a]$ map functions to functions. Specifically, let it send

$$y(t) \mapsto y_0 + \int_{t_0}^{t} f(\tau, y(\tau))\mathrm{d}\tau$$

  - We denote this by writing $\Phi[y](t) = y_0 + \int_{t_0}^{t} f(\tau, y(\tau))\mathrm{d}\tau$.
  - Notice that the solution of our IVP is exactly the point of $C[t_0, t_0 + a]$ fixed by Φ because

$$y(t) = y_0 + \int_{t_0}^{t} f(\tau, y(\tau))\mathrm{d}\tau \quad\iff\quad y = \Phi[y]$$

  - *This* motivates all steps taken thus far.
- All that remains is to show that Φ is a contraction on some subset of $C[t_0, t_0 + a]$. Then we can apply the Banach fixed point theorem.
- We first identify this subset. Let

$$X_b = \{y : [t_0, t_0 + a] \to \bar{B}(y_0, b)\}$$

  - By the previous theorem, this is a complete metric space.
  - We now want to relate a and b so that $\Phi(X_b) \subset X_b$ and Φ is a contraction.
  - For $\Phi(X_b) \subset X_b$, we need

$$\|\Phi[y] - y_0\| \leq \int_{t_0}^{t_0+a} |f(\tau, y(\tau))|\mathrm{d}\tau \leq a \cdot M \leq b$$

    so we want $a < b/M$.
  - Moreover, if Φ is to be a contraction, then since

$$\begin{aligned}\|\Phi[y_1] - \Phi[y_2]\| &\leq \int_{t_0}^{t+a} |f(\tau, y_1(\tau)) - f(\tau, y_2(\tau))|\mathrm{d}\tau\\ &\leq La \cdot \|y_1 - y_2\|\end{aligned}$$

    we want $La \in (0, 1)$. We can achieve this by requiring $a < 1/2L$.
  - Thus, choosing

$$a < \min\left(\frac{1}{2L}, \frac{b}{M}\right)$$

    accomplishes all of our goals.
- Therefore, by the Banach fixed point theorem, there exists a unique y such that $y = \Phi[y]$.
  - As we have already remarked, this fixed point is exactly the aforementioned solution to the IVP.
- Conclusion:
- Theorem (Cauchy-Lipschitz theorem): Let $f(t, z)$ be defined on an open subset $\Omega \subset \mathbb{R}_t \times \mathbb{R}_z^n$, $(t_0, y_0) \in \Omega$, such that f is Lipschitz continuous wrt. t, z in some neighborhood of (t_0, y_0). Then the IVP $y'(t) = f(t, y(t))$, $y(t_0) = y_0$ has a unique solution for some $T > 0$ on $[t_0, t_0 + T]$ such that $y(t)$ does not escape that neighborhood.
- If $f \in C_1$, then

$$|f(t, z_1) - f(t, z_2)| \leq \sup_{z \in \bar{B}(y_0, r)} \left\|\frac{\partial f}{\partial z}(t, z)\right\| \cdot |z_1 - z_2|$$

- We use the finite increment theorem of differential calculus to prove that f is Lipschitz continuous if it's continuously differentiable.
- The norm on the RHS above is the matrix norm.
- We have $y(t) = \int_{t_0}^{t} f(\tau, y(\tau))\mathrm{d}\tau + y_0$ and we use the Banach fixed point theorem (which is proved constructively).
- We have $y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(\tau, y_n(\tau))\mathrm{d}\tau$.
  - Thus, the Picard iteration is justified by the Banach fixed point theorem.
- We do not use the algorithm from the proof (the Picard iteration) computationally; we use the polygon algorithm. This algorithm is only of theoretical significance.
- **Interval of existence**: The union of intervals containing the interval $[t_0, t_0 + T]$ on which the IVP has a solution.
  - The interval of existence is always open. If $t_0 \in I$ such that $y(t_0) = y$, then $y'(t) = f(t, y(t))$, $y(t_0) = y$.
  - Note that the theorem does not predict when singularity can occur.
  - Example: The interval of existence will always be $x' = 1 + x^2$, $x(t_0) = x_0$. Then $x(t) = \tan(t - t_0 + \arctan(x_0))$. The length of existence is always π.
- Interval of existence: If you consider the IVP $y'(t) = f(t, y(t))$, $y(t_0) = y_0$, then $[t_0, t_0 + T_1], [t_0 + T_1, t_0 + T_2]$. The first is of length T_1, and the second of length $T_2 - T_1$. Continuing on, we get $T_n - T_1$ so that $T_n \to \infty$ or T_n is bounded. This gives us the maximal solution/interval of existence.
- The motherfucker (Shao) made us stay 10 minutes late.

5.4 Chapter 1: Introduction

From Teschl (2012).

Section 1.5: Qualitative Analysis of First-Order Equations

11/15:

- Only a few ODEs are explicitly solvable. However, in many situations, only certain qualitative aspects of the solution (e.g., whether or not it stays within a certain region, what it looks like for large t, etc.) are of interest.
  - "Moreover, even in situations where an exact solution can be obtained, a qualitative analysis can give a better overview of the behavior than the formula for the solution" (Teschl, 2012, p. 20).
- Example: Qualitative analysis of a model of logistic growth.

$$\dot{x}(t) = (1 - x(t))x(t) - h$$

  - Plot the parabola $f(x) = (1 - x)x - h$.
  - The sign of f tells us what direction the solution will move.
  - We divide into three cases ($0 < h < 1/4$, $h = 1/4$, and $h > 1/4$)
  - $0 < h < 1/4$:
    - There are two unique zeroes of the parabola, namely $x_{1,2} = 0.5(1 \pm \sqrt{1 - 4h})$ where we choose $x_1 < x_2$.
    - If $x_0 < x_1$, $f(x_0) < 0$, and the solution will decrease and converge to $-\infty$.
    - If $x_0 = x_1$ (resp., x_2), then the solution will stay fixed for all t, i.e., $x(t) = x_0$ is the solution.

  - If $x_1 < x_0 < x_2$, the solution will increase and converge to x_2.
  - If $x_0 > x_2$, the solution will decrease and converge to x_2.
  - Note how this mirrors our understanding of the logistic growth model, as well as the discussion from Lecture 2.1.

- What we have seen in the above example motivates the following lemma.
- Lemma 1.1: Consider the first-order autonomous initial value problem

$$\dot{x} = f(x), \quad x(0) = x_0$$

  where $f \in C(\mathbb{R})$ is such that the solutions are unique.

  1. If $f(x_0) = 0$, then $x(t) = x_0$ for all t.
  2. If $f(x_0) < 0$, then $x(t)$ converges to the first zero left of x_0. If there is no such zero, the solution converges to $-\infty$.
  3. If $f(x_0) > 0$, then $x(t)$ converges to the first zero right of x_0. If there is no such zero, the solution converges to $+\infty$.

- Teschl (2012) qualitatively analyzes a Riccati type ODE; I would need to study the previous subsection to understand this in full depth.
- Observations from a Mathematica-facilitated numerical analysis:
  - Mathematica complains about the step size getting too small on one side of the interval, so the solution only exists for finite time.
  - There is symmetry with respect to the transformation $(t, x) \mapsto (-t, -x)$; thus, we only need to consider $t \geq 0$.
  - For different sets of initial conditions, solutions never cross; this is a consequence of uniqueness (see Lecture 5.1).
  - There seem to be two cases: either the solution escapes to $+\infty$ in finite time or it converges to the line $x = -t$.
  - We need further proof, however, to verify that these observations are true and not just numerical glitches. In other words, Mathematica can guide our intuition, but we must still do the hard math ourselves.
- A more in-depth qualitative sign analysis of the same equation follows to verify the above claims.
- The extent to which this qualitative analysis is covered, though, goes well-beyond anything mentioned in class.
- The final theorem could be useful, though again, it is never covered.

Section 1.6: Qualitative Analysis of First-Order Periodic Equations

- Again, largely beyond the scope of class.
- **Periodic** (ODE): An ODE for which $f(t+1, x) = f(t, x)$, where we take the period to be 1 WLOG.

5.5 Chapter 2: Initial Value Problems

From Teschl (2012).

Section 2.1: Fixed Point Theorems

- Starting now, we are going to follow a brief tangent into linear algebra and mathematical analysis. The purpose of this is to facilitate "an easy and transparent proof of our basic existence and uniqueness theorem" (Teschl, 2012, p. 35).
- **Vector space**: *Definition assumed. Denoted by* $\boldsymbol{X}$.
- **Norm** (on X): A map from $X \to [0, \infty)$ satisfying the following requirements. *Denoted by* $||\cdot||$.
  1. $\|0\| = 0$, $\|x\| > 0$ for $x \in X \setminus \{0\}$.
  2. $\|\alpha x\| = |\alpha|\|x\|$ for all $\alpha \in \mathbb{R}$ and $x \in X$.
  3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (**triangle inequality**).
- **Inverse triangle inequality**: The following inequality, which holds for all $x, y \in X$. *Given by*

$$|\|x\| - \|y\|| \leq \|x - y\|$$

- **Normed vector space**: A vector space X along with a norm on X. *Also known as* **normed space**. *Denoted by* $\boldsymbol{(X, ||\cdot||)}$.
- **Convergent** (sequence of vectors): A sequence of vectors $\{f_n\}$ such that

$$\lim_{n \to \infty} \|f_n - f\| = 0$$

  for some $f \in X$. *Denoted by* $\boldsymbol{f_n \to f}$.
- **Limit** (of a convergent sequence of vectors): The vector f in the above definition.
- **Continuous** (mapping between two normed spaces): A function $F : X \to Y$, where X, Y are normed spaces, such that $f_n \to f$ implies $F(f_n) \to F(f)$.
- Example: The norm, vector addition, and scalar multiplication are all continuous under the above definition.
  - Problem 2.2 walks us through this.
- **Cauchy** (sequence): Defined as in class.
- **Complete** (vector space): A vector space for which every Cauchy sequence has a limit.
- **Banach space**: A complete normed space.
- Example: $\mathbb{R}^n$ and $\mathbb{C}^n$ are Banach spaces under the usual **Euclidean norm**.
- **Euclidean norm**: The following norm, applicable to $\mathbb{R}^n$ and $\mathbb{C}^n$. *Given by*

$$|x| = \sqrt{\sum_{j=1}^{n} |x_j|^2}$$

- We will mainly be interested in the following example of a Banach space.
- Example: The set of continuous function $C(I)$ on $I \subset \mathbb{R}$ a compact interval.
  - A vector space if addition and scalar multiplication are defined pointwise on the functions.
  - A normed space under

$$\|x\| = \sup_{t \in I} |x(t)|$$

    - Problem 2.3 walks us through verifying that the above norm satisfies the three axioms.

- A complete vector space because...
  - $\{x_n\}$ Cauchy implies that $x_n \to x$ pointwise. But is $x \in C(I)$?
  - Well, $x_n \to x$ implies

$$\lim_{n\to\infty} \sup_{t\in I} |x_n(t) - x(t)| = \lim_{n\to\infty} \|x_n - x\| = 0$$

  - The above equation implies that x_n converges **uniformly** to x.
  - Since $x_n \to x$ uniformly and each x_n is continuous, we have by the Uniform Limit Theorem[2] that x is continuous, i.e., $x \in C(I)$ as desired.

- **Uniformly convergent** (sequence of functions): A convergent sequence of functions $\{x_n\}$ such that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then

$$|x_n(t) - x(t)| < \varepsilon$$

for all $t \in I$.

- **Endofunction**: A function from some set to the same set. *Denoted by* $\boldsymbol{K : C \to C}$.
  - Note that we do not require that K be bijective, only that it map a set to (some part of) itself.
- **Fixed point** (of an endofunction): An element $x \in C$ such that $K(x) = x$.
- **Contraction**: An endofunction K for which there exists a constant $\theta \in [0, 1)$ such that

$$\|K(x) - K(y)\| \leq \theta \|x - y\|$$

for all $x, y \in C$.

- Denoting function composition.
  - We inductively define $K^0(x) = \mathrm{id}(x) = x$, $K^n(x) = K(K^{n-1}(x))$.
- Theorem 2.1 (Contraction principle): Let C be a (nonempty) closed subset of a Banach space X and let $K : C \to C$ be a contraction. Then K has a unique fixed point $\bar{x} \in C$ such that

$$\|K^n(x) - \bar{x}\| \leq \frac{\theta^n}{1-\theta} \|K(x) - x\|$$

for all $x \in C$.

Proof. As in class.

Note that we use the closed-ness hypothesis to guarantee that C is still complete, i.e., that $\bar{x} \in C$ (we could very well define K on $C \setminus \{\bar{x}\}$, and it would still be a contraction). ☐

- Note that the contraction principle and the Banach fixed point theorem from class are the same statement; the contraction principle just states the inequality used in the proof of the Banach fixed point theorem as an additional result.

[2]Theorem 17.6 from Honors Calculus IBL.

Section 2.2: The Basic Existence and Uniqueness Result

- We now prove the basic existence and uniqueness result, using the theory from the previous section.
- **Initial value problem**: A mathematical question for which the desired solution is a function that satisfies both a differential equation and maps a specific point of its domain to a specific point of its codomain. *Also known as* **IVP**. *Given by*

$$\dot{x} = f(t,x), \quad x(t_0) = x_0$$

- For the purposes of our study, we will suppose that $f \in C(U, \mathbb{R}^n)$, where $U \subset \mathbb{R}^{n+1}$ is open, and that $(t_0, x_0) \in U$.
- To begin, integrate the ODE as follows.

$$\begin{aligned}
\int_{x_0}^{x} \mathrm{d}\chi &= \int_{t_0}^{t} f(s, x(s)) \, \mathrm{d}s \\
x(t) - x_0 &= \int_{t_0}^{t} f(s, x(s)) \, \mathrm{d}s \\
x(t) &= x_0 + \int_{t_0}^{t} f(s, x(s)) \, \mathrm{d}s
\end{aligned}$$

- **Integral equation**: A form of an ODE that reexpresses the equation in terms of (the) integral(s) of a function instead of its deriavative(s).
  - The last line above is the integral equation corresponding to the IVP ODE.
- How changing to the integral equation helps.
  - Notice that $x_0(t) = x_0$ is an approximate solution for t close to t_0.
  - Plugging $x_0(t)$ into the integral gives another approximate solution

$$x_1(t) = x_0 + \int_{t_0}^{t} f(s, x_0(s)) \, \mathrm{d}s$$

  - In fact, as we will soon show, $x_1(t)$ is a "better" approximation than $x_0(t)$. To this end, iterating the procedure infinitely many times will eventually get you to $x(t)$. Indeed, if we recursively define a sequence of functions
$$x_m(t) = K^m(x_0)(t)$$
where
$$K(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s)) \, \mathrm{d}s$$
we can show that the limit is the desired solution.
  - Since the limit will necessarily satisfy $K(x) = x$, the tack we take is proving that K is a contraction on some subset of $C(I)$ (which we have proven is a Banach space) and invoking the contraction principle.
  - At this point, the previous section should be fully motivated.
- Going forward, we will take $t_0 = 0$ and only consider the case where $t \geq 0$; the other cases can be handled analogously (but require considerably more absolute value symbols).
- Take
$$X = C([0, T], \mathbb{R}^n)$$
to be our Banach space for some suitable T (we will put constraints on the value of T as we build the rest of our argument).

- We need some $C \subset X$ on which to define K as an endofunction. Let's try

$$C = \overline{B_\delta(x_0)}$$

  - In words, we let C be the closed ball of radius δ surrounding the constant function $x_0(t)$ in $X = C([0,T], \mathbb{R}^n)$.
    - Geometrically, C is the set of all functions $f : [0,T] \to \mathbb{R}^n$ such that the vector $f(t) - x_0(t) \in \mathbb{R}^n$ is within δ of the origin for all $t \in [0,T]$.
    - Notice how the definition of the sup norm on $C(I)$ motivates the above geometric picture.
  - This will end up working, as long as we have a suitable relation (to be derived) between T and δ.
- Constraints on f.
  - By hypothesis, $(0, x_0) \in U$, where we recall that U is the domain of f.
  - We additionally require that U contains $V = [0,T] \times \overline{B_\delta(x_0)}$.
  - Since V is compact and f is continuous, we have by the Extreme Value Theorem in $\mathbb{R}^n$[3] that f has a maximum M on V. In particular, there exists

$$M = \max_{(t,x)\in V} |f(t,x)|$$

  - When we later seek to prove that K is a contraction, it will be useful to know that f is **locally Lipschitz continuous** in the second argument and **uniformly Lipschitz continuous** in the first argument. In particular, for every compact $V_0 \subset U$, the **Lipschitz constant** (which depends on V_0) is finite.
- **Lipschitz continuous** (function): A function $f : X \to Y$, where X, Y are normed spaces, such that the following number is finite.

$$L = \sup_{x_1 \neq x_2 \in X} \frac{|f(x_1) - f(x_2)}{|x_1 - x_2|}$$

Also known as **uniformly Lipschitz continuous**.

  - Lipschitz continuity is a **strong** form of **uniform continuity**.
- **Lipschitz constant** (of a Lipschitz continuous function): The following value, where f is Lipschitz continuous. *Denoted by* $\boldsymbol{L}$. *Given by*

$$L = \sup_{x_1 \neq x_2 \in X} \frac{|f(x_1) - f(x_2)}{|x_1 - x_2|}$$

  - For functions in $C(I)$, L corresponds to the smallest bound on the slope of f.
- **Uniformly continuous** (function): A function $f : X \to Y$, where X, Y are normed spaces, such that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x_1, x_2 \in X$, $|x_1 - x_2| < \delta$ implies

$$|f(y) - f(x)| < \varepsilon$$

- **Strong** (condition): A condition that implies some other condition that does not, in turn, imply the original condition.
  - For example, all Lipschitz continuous functions are uniformly continuous, but not all uniformly continuous functions are Lipschitz continuous.
- **Locally Lipschitz continuous** (function): A function $f : X \to Y$, where X, Y are normed spaces, such that for every compact $X_0 \subset X$, $f : X_0 \to Y$ is Lipschitz continuous.

[3]Theorem 18.44 from Honors Calculus IBL.

- **Graph** (of $x \in C$): The set defined as follows. *Denoted by* $\boldsymbol{G(x)}$. *Given by*

$$G(x) = \{(t, x(t)) \mid t \in [0, T]\}$$

- Proving that $K : C \to C$.
  - At this point, we can compute

$$\begin{aligned}
|K(x)(t) - x_0| &= \left| \int_0^t f(s, x(s)) \, \mathrm{d}s \right| \\
&\leq \int_0^t |f(s, x(s))| \, \mathrm{d}s && \text{Theorem 13.26}^{[4]} \\
&\leq tM && \text{Theorem 13.27}^{[4]}
\end{aligned}$$

    for any x satisfying $G(x) \subset V$.
  - Thus, if we take

$$T \leq \frac{\delta}{M}$$

    then $|K(x)(t) - x_0| \leq TM \leq \delta$ for all $t \in [0, T]$.
  - It follows that under this definition of T, $\|K(x) - x_0\| \leq \delta$, so $K(x) \in \overline{B_\delta(x_0)}$ for all x with graph in V.
  - In the special case $M = 0$ (which would imply $T = \infty$), we may take T to be some arbitrary positive real number.

- Proving that K is a contraction.
  - To estimate $|K(x)(t) - K(y)(t)|$, we finally invoke the Lipschitz continuity constraint on f. In particular, we have

$$\begin{aligned}
|K(x)(t) - K(y)(t)| &\leq \int_0^t |f(s, x(s)) - f(s, y(s))| \, \mathrm{d}s \\
&\leq L \int_0^t |x(s) - y(s)| \, \mathrm{d}s \\
&\leq Lt \sup_{0 \leq s \leq t} |x(s) - y(s)| \\
&\leq LT \|x - y\|
\end{aligned}$$

  - Thus, if we take

$$T < \frac{1}{L}$$

    we have that K is a contraction.

- Having two constraints on the value of T, we may now formally take

$$T = \min\left(\frac{\delta}{M}, \frac{1}{2L}\right)$$

  - Note that this definition satisfies $T \leq \delta/M$ and $T < L^{-1}$.
  - If either of $M, L = 0$, we understand that that constraint no longer matters and we need only consider the other one. If $M = L = 0$, then we may take T to be any positive real number.

- Having defined a contraction $K : C \to C$, the existence and uniqueness of our ODE follows from the contraction principle:

[4] From Honors Calculus IBL.

- Theorem 2.2 (Picard-Lindelöf): Suppose $f \in C(U, \mathbb{R}^n)$, where $U \subset \mathbb{R}^{n+1}$ is open and $(t_0, x_0) \in U$. If f is locally Lipschitz continuous in the second argument, uniformly with respect to the first, then there exists a unique local solution $\bar{x}(t) \in C^1(I)$ of the IVP

$$\dot{x} = f(t, x), \quad x(t_0) = x_0$$

  where I is some interval around t_0.

  More specifically, if $V = [t_0, t_0 + T] \times \overline{B_\delta(x_0)} \subset U$ and M denotes the maximum of $|f|$ on V, then the solution exists for at least $t \in [t_0, t_0 + T]$ and remains in $\overline{B_\delta(x_0)}$, where

$$T = \frac{\delta}{M}$$ [5]

  The analogous result holds for the interval $[t_0 - T, t_0]$.

- **Picard iteration**: The procedure for finding the solution, involving calculating successive elements of the sequence $\{x_m\}$ as defined above by $x_m(t) = K^m(x_0)(t)$.
  - The Picard iteration is useful for proofs, such as the above one, but it is not suitable for actually finding the solution since the integrals are not computable in general except by tedious numerical methods.
  - If $f(t, x)$ is **analytic**, however, then $x_m(t)$ equals the Taylor expansion of the solution $\bar{x}(t)$ about t_0 up to order m. This can be used for numerical computations.
    - Problem 4.4 walks us through this.
- **Analytic** (function): A function that is locally given by a convergent power series.
- $f \in C^1(U, \mathbb{R}^n)$ implies that f is locally Lipschitz continuous in the second argument, uniformly with respect to the first.
  - Problem 2.5 walks us through this.
  - It follows from this statement that if $f \in C^1(U, \mathbb{R}^n)$, then the corresponding IVP $\dot{x} = f(t, x)$, $x(t_0) = x_0$ has a unique solution as per the Picard-Lindelöf theorem.
- This observation can be slightly extended via the following lemma.
- Lemma 2.3: Suppose $f \in C^k(U, \mathbb{R}^n)$ for some $k \geq 1$, where $U \subset \mathbb{R}^{n+1}$ is open and $(t_0, x_0) \in U$. Then the local solution $\bar{x}$ of the corresponding IVP is $C^{k+1}(I)$.

  Proof. We induct on k. For the base case $k = 1$, $\bar{x}(t) \in C^1$ by the Picard-Lindelöf theorem. Additionally, since $f \in C^1$, we have that $\dot{\bar{x}}(t) = f(t, \bar{x}(t)) \in C^1$. Thus, since the derivative of $\bar{x}$ is continuously differentiable, $\bar{x}(t)$ must be *twice* continuously differentiable, or C^2. The inductive step is very similar. $\square$

Section 2.3: Some Extensions

- Extensions of the Picard-Lindelöf theorem that weren't covered in class.

[5] We are not missing the Lipschitz constraint here; indeed, it is superfluous, as will be shown in the next section.