2 Linear Algebra

Required Problems

10/19: **1.** This question helps to complete the computations omitted in class. In deriving the Kepler orbits for the two-body problem, we have successfully reduced the differential equation satisfied by the curve $r = r(\varphi)$ to

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 + r^2 = \frac{2GMr^3}{l_0^2} + \frac{2Er^4}{ml_0^2}$$

Show that the function $\mu = 1/r$ satisfies the differential equation

$$\left(\frac{\mathrm{d}\mu}{\mathrm{d}\varphi}\right)^2 + \mu^2 = \frac{2GM\mu}{l_0^2} + \frac{2E}{ml_0^2}$$

By differentiating with respect to φ again, this reduces to either $\mathrm{d}\mu/\mathrm{d}\varphi = 0$ or

$$\frac{\mathrm{d}^2\mu}{\mathrm{d}\varphi^2} + \mu - \frac{GM}{l_0^2} = 0$$

Find the general solution of the latter, hence conclude that $r = r(\varphi)$ represents a conic section. *Hint*: There is a very obvious particular solution.

2. The general formula for the inverse of an $n \times n$ invertible matrix is very lengthy. However, for a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfying $ad - bc \neq 0$, there is a very simple formula. Try to find it; this could be very helpful if you can remember it.

3. Compute the determinant of the following matrices. Determine whether they are invertible or not.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 2 & 3 & 6 \\ 1 & 3 & 4 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} -1 & 2 & 1 \\ 3 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$$

4. Determine whether the following linear systems admit solution(s); if they do, write down the solution (or the formula for the general solution).

(1)

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

(2)

$$\begin{pmatrix} -1 & 2 & 1 \\ 3 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

(3)

$$\begin{pmatrix} -1 & 2 & 1 \\ 3 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

5. Find the connecting matrix from the basis $\begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix}$ to the new basis $\begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}$, where

$$\begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix} \qquad \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

That is, represent q_1, q_2, q_3 as linear combinations of p_1, p_2, p_3.

6. Let $\theta \in [0, 2\pi)$. The rotation through angle θ in the plane is represented by the matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Compute its determinant, characteristic polynomial, and eigenvalues. Compute its eigenvectors in $\mathbb{C}^2$. You need to use the Euler formula $\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$. For two angles θ, φ, compute the product $R(\theta)R(\varphi)$ and represent it in terms of $\theta + \varphi$. What is the geometric meaning of this equality?

8. Find the algebraic and geometric multiplicities of the eigenvalues of the following matrices.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

9. Compute the Jordan normal form of the following 2×2 matrices.

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}$$

Notice that you not only need to find all the Jordan blocks, but also need to find the Jordan basis matrix Q such that $Q^{-1}AQ$ is in Jordan normal form.

10. Compute the Jordan normal form of the following 3×3 matrices.

$$A = \begin{pmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -1 & -1 \\ 2 & -1 & -2 \\ -1 & 1 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

Notice that you not only need to find all the Jordan blocks, but also need to find the Jordan basis matrix Q such that $Q^{-1}AQ$ is in Jordan normal form. *Hint*: These three matrices represent three different possibilities of nondiagonalizable Jordan normal forms of a 3×3 matrix: A reduces to $(2 \times 2) \oplus (1 \times 1)$ Jordan blocks with different eigenvalues, B reduces to $(2 \times 2) \oplus (1 \times 1)$ Jordan blocks with the same eigenvalue, and C reduces to a 3×3 Jordan block.

Bonus Problems

1. You may find the characteristic root method for the second-order equation $y'' + ay' + b = 0$ quite abrupt. This problem helps you see where it comes from. The origin of this method is in fact a comparison with the linear recursive relation

$$y_{n+2} + ay_{n+1} + by_n = 0$$

where a, b are given complex numbers.

(1) The linear recursive relation $y_{n+1} + ay_n = 0$ gives rise to a geometric sequence

$$y_0, y_0(-a), y_0(-a)^2, \ldots$$

We now want to try to reduce the second-order recursive relation $y_{n+2} + ay_{n+1} + by_n = 0$ to a first-order relation. Thus, we look for complex numbers λ, μ such that

$$(y_{n+2} - \lambda y_{n+1}) - \mu(y_{n+1} - \lambda y_n) = 0$$

Then λ, μ should be the roots of the characteristic polynomial

$$X^2 + aX + b$$

Taking λ, μ as known quantities, find the general formula for y_n, regarding y_0, y_1 as known quantities. *Hint*: $y_{n+1} - \lambda y_n$ is a geometric sequence with ratio μ. You should also discuss $\mu \neq \lambda$ and $\mu = \lambda$ separately.

(2) Use the method of part (1) to find the general formula for the linear discursive relation

$$y_{n+2} - 2y_{n+1} + y_n = 0$$

Use the same method to find the general formula for the Fibonacci sequence

$$F_{n+2} = F_{n+1} + F_n$$

2. In this exercise, we aim to prove an important theorem in linear algebra:

Complex Hermitian matrices are always diagonalizable.

Here the term "Hermitian" means that the matrix equals its conjugate transpose. In terms of entries, this means that in general, $a_{ij} = \bar{a}_{ji}$. For example,

$$\begin{pmatrix} 2 & 1 & -i \\ 1 & 3 & -2i \\ i & 2i & 1 \end{pmatrix}$$

is Hermitian.

(1) Let $\langle \cdot, \cdot \rangle$ be the standard Hermitian inner product, that is, for $x, y \in \mathbb{C}^n$,

$$\langle x, y \rangle = \sum_{j=1}^{n} x^j \bar{y}^j$$

Show that for any $n \times n$ real matrix,

$$\langle Ax, y \rangle = \langle x, A^* y \rangle$$

for any $x, y \in \mathbb{C}^n$, where A^* denotes the conjugate transpose of A. For example,

$$A = \begin{pmatrix} 1 & 1 & 2i \\ 0 & 3+i & 3 \\ 2 & 0 & 1 \end{pmatrix} \quad \Longleftrightarrow \quad A^* = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 3-i & 0 \\ -2i & 3 & 1 \end{pmatrix}$$

(2) Suppose now that A is Hermitian. Use part (1) to show that any eigenvalue of A must be a real number. Show further that if x, y are eigenvectors corresponding to different eigenvalues, then $\langle x, y \rangle = 0$, that is, x is orthogonal to y.

(3) Prove that every Hermitian matrix A is diagonalizable. *Hint*: Take any eigenvector v_1 of A. Decompose $\mathbb{C}^n$ into the direct sum of $\operatorname{span}(v_1)$ and its orthogonal complement. Show that the orthogonal complement is an invariant subspace for A.