Week 1

Introduction to ODEs

1.1 Definitions and Scope

9/28:

- Questions:
  - When will the PDFs be made available?
- Office: Eckhart 309.
  - Office hours: MWF 3:00-4:00.
- Reader: Walker Lewis. His contact info is in the syllabus.
- Final grade is based on...
  - 2 midterms (15 pts. each; weeks 4 and 8).
  - Final exam (35 pts.).
  - HW (35 pts.).
  - Bonus problems (15 pts).
- Total points for the quarter is 115. The bonus problems usually arise from advanced math and incorporate more advanced knowledge, and we are encouraged to seek out all relevant resources as long as we write up our own solutions.
- **Ordinary differential equation**: An equation that involves an unknown function of a single variable; an equation that takes the form $F(t,y,y',\dots,y^{(n)})=0$. *Also known as* **ODE**.
  - F is a known function.
  - t is an argument (time). x is also used (when space is involved).
  - $y=y(t)$ is an unknown function.
- **Order n** (ODE): An ODE for which the n^{th} derivative of y is the highest-order derivative involved (and is involved).
- ODEs are of the form $y'=f(t,y)$ or, more generally, $y^{(n)}=F(t,y,y',\dots,y^{(n-1)})$.
  - We can transform this second form into the first form via

$$Y=\begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix} \qquad\qquad f(t,y)=\begin{pmatrix} Y^2 \\ Y^3 \\ \vdots \\ F(t,Y^1,Y^2,\dots,Y^{n-1}) \end{pmatrix}$$

This makes $Y' = f(t, Y)$ equal to the system of equations

$$\begin{aligned}(Y^1)' &= Y^2\\(Y^2)' &= Y^3\\&\vdots\\(Y^{n-1})' &= F(t, Y^1, Y^2, \dots, Y^{n-1})\end{aligned}$$

  - Think about this conversion more.

- Thus, we mainly focus on equations of the form $y' = f(t, y)$ (where y may be a scalar or vector function), because that's general enough.

- **Linear** (ODE): Any ODE that can be written in the form

$$y' = A(t)y + f(t)$$

- Because of the above, this naturally includes equations of the form

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_0(t)y = b(t)$$

  - Indeed, if we define $Y = (y, y', \dots, y^{(n-1)})$, then we may express this equation in the form

$$\begin{aligned}\underbrace{\begin{pmatrix}Y^1\\Y^2\\\vdots\\Y^n\end{pmatrix}'}_{Y'} &= \underbrace{\begin{pmatrix}Y^2\\Y^3\\\vdots\\b(t) - a_0(t)Y^1 - \cdots - a_{n-1}(t)Y^{n-1}\end{pmatrix}}_{g(t,y)}\\&= \begin{pmatrix}Y^2\\Y^3\\\vdots\\-a_0(t)Y^1 - \cdots - a_{n-1}(t)Y^{n-1}\end{pmatrix} + \underbrace{\begin{pmatrix}0\\0\\\vdots\\b(t)\end{pmatrix}}_{f(t)}\\&= \underbrace{\begin{pmatrix}0 & 1 & 0 & \cdots & 0\\0 & 0 & 1 & \cdots & 0\\\vdots & \vdots & \vdots & \ddots & \vdots\\0 & 0 & 0 & \cdots & 1\\-a_0(t) & -a_1(t) & -a_2(t) & \cdots & -a_{n-1}(t)\end{pmatrix}}_{A(t)}\underbrace{\begin{pmatrix}Y^1\\Y^2\\\vdots\\Y^n\end{pmatrix}}_{Y} + \underbrace{\begin{pmatrix}0\\0\\\vdots\\b(t)\end{pmatrix}}_{f(t)}\end{aligned}$$

  - This conversion and its implications is covered in more depth in Lecture 4.1.

- **Nonlinear** (ODE): An ODE that is not linear.

- **Autonomous** (ODE): An ODE that can be written in the form

$$y' = f(y)$$

  - Remember that y can be a scalar or a vector function.
  - Solutions to autonomous ODEs can start at *any* time t and still be valid.
    - For example, take the scalar ODE $y' = y$. It's general solution is $y(t) = a\mathrm{e}^{t-t_0}$ for some $a \in \mathbb{R}$ and t_0 being the start time. Importantly, notice that we can make t_0 take any value we want and $y(t)$ will still solve $y' = y$.

- **Nonautonomous** (ODE): An ODE that is not autonomous.
  - We will not investigate these in this course.
- **Initial value problem**: A problem of the form, "find $y(t)$ such that the following holds."

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$$

  Also known as **IVP**, **Cauchy problem**.
- Locally well-posed (LWP) conditions:
  1. Existence (local in time).
  2. Uniqueness (you cannot have multiple solutions).
  3. Local stability (if you perturb your initial value or equation a little bit, you do not expect your solution to vary crazily [esp. locally]).
- Example of a nonunique ODE:
  - $y' = \sqrt{y}$, $y(0) = 0$ has solutions $y_1(t) = 0$ ($t \geq 0$) and $y_2(t) = t^2/4$ ($t \geq 0$).
  - We will investigate the reason later.
- Preview of the reason: **Cauchy-Lipschitz Theorem** or **Picard-Lindelöf Theorem**.
  - As long as the ODE is **Lipschitz continuous**, it's locally stable.
- **Lipschitz continuous** (function): A function f such that

$$|f(t,y_1) - f(t,y_2)| \leq L|y_1 - y_2|$$

  - But in the counterexample above, the slope of the chord from 0 to $y(t)$ approaches infinity as $t \to 0$.
- **Peano Existence Theorem**: Under certain conditions, there exists a solution to a given IVP.
- **Dynamical system**: A law under which a particle evolves over time. $y' = f(t,y)$, IVP is LWP.
- If the IVP $y' = f(t,y)$, $y(t_0) = y_0$ is locally well-posed, then the map $\Phi(t,x)$ which solves

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,x) = f(t,\Phi(t,x)) \\ \Phi(0,x) = x \end{cases}$$

  is well-defined and satisfies the property

$$\Phi(t_2, \Phi(t_1,x)) = \Phi(t_1 + t_2, x)$$

  - Φ is very related to y, though how exactly is still a bit of a mystery?? Perhaps it's

$$\Phi(t,x) = y(t)$$

    where y is the solution to the IVP $y' = f(t,y)$, $y(0) = x$.
  - It appears that $\Phi(t,x)$ is related to $f_t(x)$ from Guillemin and Haine (2018), i.e., we are picking a point x and traveling along its integral curve for time t.
  - Think about $y(t) = a\mathrm{e}^{t-t_0}$ as an integral curve of the one-dimensional vector field $X(x) = x$.
  - The final property appears to express the notion that if you have a system and evolve it by time t_1 and then time t_2, that's equivalent to evolving it by time $t_1 + t_2$.

- **Steady flow**: A vector field on a manifold contained in $\mathbb{R}^2$ or $\mathbb{R}^3$ that does not vary with time.
- Let X be a vector field.
  - Trajectory of a particle: At $x \in \Omega$, the velocity of the particle should coincide with $X(x)$.
  - The differential equation $\dot{x} = X(x)$ is what we're interested in.
  - A solid shape gets shifted and deformed (imagine a chunk of water falling out of the end of a pipe). This is the **local group of transformation**.
  - Differential geometry is the purview of such things.
- Newton's law of motion $F = m \cdot a$ applied to n particles is nothing but the system of equations
$$m_i x_i'' = F_i(x_1, \dots, x_n)$$
for $i = 1, \dots, n$.
  - Many well-known examples.
  - The best known one perhaps is that of uniform acceleration of a single particle. In this case,
$$m_0 x'' = f_0$$
    - The solution is
$$x(t) = \frac{f_0}{2m_0} t^2 + v_0 t + x_0$$
where $x_0 = x(0)$ and $v_0 = x'(0)$ are the initial conditions.
  - A simple example is downwards motion due to gravity. Then
$$x(t) = \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} t^2 + v_0 t + x_0$$
    - The trajectory in general is a parabola.
  - Another example: The mathematical pendulum.
    - The radial directions balance ($mg \cos\theta$).
    - The tangential directions do not ($mg \sin\theta$). Thus, our ODE is
$$l \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = g \sin\theta$$
  - One last set of examples from ecology:
    - Imagine an petri dish of infinite nutrition. The population growth of the bacteria will obey the exponential growth law
$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky$$
    - Suppose we have a system capacity M. Then we obey the logistic growth law
$$\frac{\mathrm{d}y}{\mathrm{d}t} = k(M - y)$$
    - Lotka-Volterra prey-predator model: Wolf population (W) and rabbit population (R). We have
$$R' = k_1 R - aWR$$
$$W' = -k_2 W + bWR$$
    - We can also introduce more species and capacities and et cetera, et cetera.
- Conclusion: Dynamical systems are everywhere, especially in physics, chemistry, and ecology.
- We can also consider long-term behavior.
  - We can have chaos, but chaos can be reasoned with using oscillation, systems that converge to oscillation, etc. We will mostly be focusing on the regular aspect of the long-term behavior.

1.2 Origin of ODEs: Boundary Value Problems

9/30:
- Textbook PDFs will be posted today.
- Note: Equations of order n generally require n parameters to solve.
- Today, we will consider boundary value problems, which are separate from dynamical systems but not entirely unrelated.
- **Boundary Value Problem**: A problem in which we are solving for a y that has fixed values at the boundaries $x = a, b$. *Also known as* **BVP**.
- The **Brachistochrone problem** is an example of a BVP.
- **Brachistochrone problem**: Suppose you have a frictionless track from $(0,0)$ to (a, y_0) and release a particle from $(0,0)$. Which path allows the particle to get to (a, y_0) in the shortest amount of time? *Etymology* brákhistos "shortest" + khrónos "time."

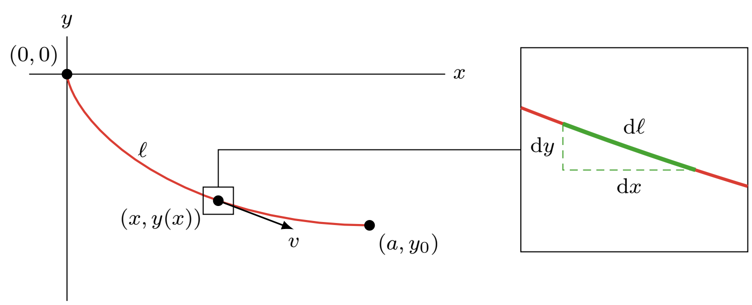

Figure 1.1: Brachistochrone problem.

  - Throughout this derivation, we will make several assumptions. We will do our best to note these assumptions as we go in footnotes. Note that while all of these assumptions are justified in the case of solving this problem, they may not be justified in every related variational problem. Let's begin.
  - Since the track is frictionless, the mechanical energy should be conserved.
  - At a given point along the curve, the particle has a velocity v and is vertical distance y from where it started. We know from physics that

$$\begin{aligned} \frac{1}{2}mv^2 &= mgy \\ v &= \sqrt{2gy} \end{aligned}$$

  - Since $v = \mathrm{d}\ell/\mathrm{d}t$, the time $\mathrm{d}t$ it takes for the particle to traverse an infinitesimal section of track of arc length $\mathrm{d}\ell$ is $\mathrm{d}t = \mathrm{d}\ell/v$.
  - The track should be given by $y = y(x)$[1].
  - Let ℓ denote the arc length of the whole track. Then

$$\mathrm{d}\ell = \sqrt{1 + (y'(x))^2}\,\mathrm{d}x$$

  - Thus, the total time for the particle to traverse the curve is

$$t(y) = \int_0^t \mathrm{d}\tau = \int_0^a \frac{\mathrm{d}\ell}{v} = \int_0^a \frac{\sqrt{1 + (y'(x))^2}\,\mathrm{d}x}{\sqrt{2gy(x)}}$$

[1] There are paths that connect $(0,0)$ and (a, y_0) that are not functions of x. We are taking those out of consideration.

- We also have $y(0) = 0$ and $y(a) = y_0$.
- We want to find y such that the above integral is minimized. Thus, we define the following **functional**, which is used to solve general fixed-endpoint variational problems (the Brachistochrone problem is a problem of this type).
- Let $J[y] = \int_a^b F(x, y(x), y'(x))\,\mathrm{d}x$.
- The space of functions we're considering is C^1 (the set of all continuously differentiable functions)[2].
- Take a function h, vanishing at a, b.
- Let $f(t) = J[y + th]$. Then

$$f(t) = \int_a^b F(x, \underbrace{y(x) + th(x)}_{z(x,t)}, \underbrace{y'(x) + th'(x)}_{w(x,t)})\,\mathrm{d}x$$

- We know that[3]

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b F\,\mathrm{d}x &= \int_a^b \frac{\mathrm{d}F}{\mathrm{d}t}\,\mathrm{d}x \\ &= \int_a^b \left(\frac{\partial F}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial F}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial F}{\partial w}\frac{\mathrm{d}w}{\mathrm{d}t}\right)\mathrm{d}x \\ &= \int_a^b \left(\frac{\partial F}{\partial x}\cdot 0 + \frac{\partial F}{\partial z}\cdot h(x) + \frac{\partial F}{\partial w}\cdot h'(x)\right)\mathrm{d}x \\ &= \int_a^b \left(\frac{\partial F}{\partial z}\cdot h(x) + \frac{\partial F}{\partial w}\cdot h'(x)\right)\mathrm{d}x\end{aligned}$$

  - The last term in the above equation may be integrated by parts as follows. Note that we make use of the hypothesis $h(a) = h(b) = 0$ in eliminating the $[uv]_a^b$ term.

$$\begin{aligned}\int_a^b \frac{\partial F}{\partial w}h'(x)\,\mathrm{d}x &= \left[\frac{\partial F}{\partial w}h(x)\right]_{x=a}^b - \int_a^b h(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial w}\right)\mathrm{d}x \\ &= \left[\left.\frac{\partial F}{\partial w}\right|_b \cdot 0 - \left.\frac{\partial F}{\partial w}\right|_a \cdot 0\right] - \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial w}\right)h(x)\,\mathrm{d}x \\ &= -\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial w}\right)h(x)\,\mathrm{d}x\end{aligned}$$

  - Substituting back into the original equation, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b F\,\mathrm{d}x &= \int_a^b \left[\frac{\partial F}{\partial z}\cdot h(x) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial w}\right)h(x)\right]\mathrm{d}x \\ &= \int_a^b \left[\frac{\partial F}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial w}\right)\right]h(x)\,\mathrm{d}x\end{aligned}$$

  - Therefore,

$$f'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b F\,\mathrm{d}x = \int_a^b \left\{\frac{\partial F}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial F}{\partial w}\right]\right\}h(x)\,\mathrm{d}x$$

- Thus,

$$f'(0) = \int_a^b \left\{\frac{\partial F}{\partial z}(x, y(x), y'(x)) - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial F}{\partial w}(x, y(x), y'(x))\right]\right\}h(x)\,\mathrm{d}x = 0$$

  for all h.

[2] This also eliminates some possible paths from consideration.

[3] We must assume sufficient regularity of F here. In particular, we must assume that the derivative of the integral of F is equal to the integral of the derivative of F.

- Now suppose y is the solution. Then y minimizes $J[y]$. But if this is true, then any variation th will cause $J[y+th] > J[y]$. It follows that for every h, $f(t)$ has a minium at $t=0$. But if f has a minimum at 0 for all h, then $f'(0) = 0$ for all h.

– Lemma: Let ϕ be continuous on (a,b). If for every $h \in C^1([a,b])$ vanishing on a,b we have that

$$\int_a^b \phi(x)h(x)\,\mathrm{d}x = 0$$

then $\phi(x) = 0$.

Proof. Suppose for the sake of contradiction that (WLOG) $\phi(x_0) > 0$. Then within some neighborhood $N_\delta(x)$ of x_0, $\phi(x) > 0$ for all $x \in N_\delta(x)$. Now choose h to be a bump function on that interval. Then $\int_a^b \phi(x)h(x)\,\mathrm{d}x > 0$, a contradiction. □

– It follows that

$$\frac{\partial F}{\partial z}(x, y(x), y'(x)) - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial F}{\partial w}(x, y(x), y'(x))\right] = 0$$

- This is a second-order differential equation, specifically the **Euler-Lagrange equation**.
- It is a necessary condition for y to be an extrema.
- Euler-Lagrange equations are not easy to solve in general. However, we're lucky here.

– In our example,

$$F(x,z,w) = \sqrt{\frac{1+w^2}{2gz}}$$

- What's nice here is that $F(x,z,w) = F(z,w)$, i.e., there is no dependence on x. This is crucial.

– With this observation in mind, notice that

$$\begin{aligned}\frac{\mathrm{d}F}{\mathrm{d}x} &= \frac{\partial F}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{\partial F}{\partial w}\frac{\mathrm{d}w}{\mathrm{d}x}\\ &= \frac{\partial F}{\partial z}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial F}{\partial w}\frac{\mathrm{d}y'}{\mathrm{d}x}\\ &= \frac{\partial F}{\partial z}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial F}{\partial w}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\end{aligned}$$

– We now rearrange the E-L equation and multiply through by $\mathrm{d}y/\mathrm{d}x$.

$$\begin{aligned}\frac{\partial F}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial w}\right) &= 0\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial w}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\partial F}{\partial z}\frac{\mathrm{d}y}{\mathrm{d}x}\end{aligned}$$

– Subtracting the last two results yields

$$\begin{aligned}\frac{\mathrm{d}F}{\mathrm{d}x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}F}{\mathrm{d}w}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\partial F}{\partial w}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\\ \frac{\mathrm{d}F}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}F}{\mathrm{d}w}\right)\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial F}{\partial w}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}F}{\mathrm{d}w}\frac{\mathrm{d}y}{\mathrm{d}x}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(F - \frac{\mathrm{d}F}{\mathrm{d}w}\frac{\mathrm{d}y}{\mathrm{d}x}\right) &= 0\\ F - \frac{\mathrm{d}F}{\mathrm{d}w}\frac{\mathrm{d}y}{\mathrm{d}x} &= A\end{aligned}$$

where $A \in \mathbb{R}$ depends on the initial conditions.

- From the definition of F, we can calculate

$$\frac{\partial F}{\partial w} = \frac{w}{\sqrt{1+w^2}} \cdot \frac{1}{\sqrt{2gz}} = \frac{y'}{\sqrt{1+(y')^2}} \cdot \frac{1}{\sqrt{2gy}}$$

- It follows that our solution function y satisfies the separable differential equation

$$\begin{aligned}
\sqrt{\frac{1+(y')^2}{2gy}} - \frac{y'}{\sqrt{1+(y')^2}} \cdot \frac{1}{\sqrt{2gy}} \cdot y' &= A \\
\frac{1+(y')^2}{\sqrt{1+(y')^2}\sqrt{2gy}} - \frac{(y')^2}{\sqrt{1+(y')^2}\sqrt{2gy}} &= A \\
\frac{1}{\sqrt{2gy(1+(y')^2)}} &= A \\
(y')^2 &= \frac{1/2A^2g - y}{y}
\end{aligned}$$

- The solution, as we can determine using methods from Calculus I-II, is the **cycloid**

$$\begin{cases}
x = a(\theta - \sin\theta) \\
y = a(1 - \cos\theta)
\end{cases}$$

where the specific parameters come from the boundary values.

- **Functional**: A map from a function space to a set of numbers.
- **Sturm-Liouville problems**: Boundary value problems concerning the integral

$$\int_a^b \left[p(x)(y'(x))^2 + q(x)(y(x))^2\right] \mathrm{d}x$$

  - The most basic BVP is a vibrating string. In finding the eigenmode of the vibration, you need to solve the above differential equation.
  - Very important in physics.
  - If time permits at the end of the course, Shao will return to the following topic in detail.
- Next several weeks: *Solvable* differential equations.

1.3 Chapter 1: Introduction

From Teschl (2012).

Section 1.1: Newton's Equations

11/11:

- Before we begin defining abstract terms, let's look at an example in which many of these terms arise.
- Investigation: Describing the motion of a particle using classical mechanics.
  - The location, velocity, and acceleration of a particle are typically given by the related functions[4]

$$x : \mathbb{R} \to \mathbb{R}^3 \qquad v = \dot{x} : \mathbb{R} \to \mathbb{R}^3 \qquad a = \dot{v} : \mathbb{R} \to \mathbb{R}^3$$

[4]Newton's notation for derivatives uses a number of dots above the dependent variable to indicate the order of the derivative to which we are referring. For example, $\dot{x}$ denotes the first derivative of x with respect to it's independent variable, and $\ddot{x}$ denotes the second derivative of x with respect to it's independent variable.

- The particle does not move randomly, though; its motion is governed by an external force field $F : \mathbb{R}^3 \to \mathbb{R}^3$ which exerts a vector force $F(x)$ on the particle when it is at $x \in \mathbb{R}^3$.
- Additionally, Newton's second law of motion asserts that at every $x \in \mathbb{R}^3$, the force acting on the particle must equal the acceleration of the particle times its mass, that is,

$$m\ddot{x}(t) = F(x(t))$$

for all $t \in \mathbb{R}$.
- Given a force field F, physicists often seek to determine how bodies evolve under F over time. Mathematically, they seek functions $x(t)$ that satisfy $m\ddot{x}(t) = F(x(t))$ for a given F.
- Consider the example of a stone falling toward the Earth under gravity from class.
  - In the vicinity of the surface of Earth, the gravitational force is approximately constant and given by

$$F(x) = -mg \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

where g is the positive **gravitational constant** and the $+x_3$ direction is taken to be normal to the Earth's surface.
  - Hence, the system of differential equations reads

$$m\ddot{x}_1 = 0 \qquad m\ddot{x}_2 = 0 \qquad m\ddot{x}_3 = -mg$$

  - The first equation can be integrated with respect to t twice, yielding $x_1(t) = C_2 + C_1 t$. Computing $x_1(0)$ and $\dot{x}_1(0)$ shows that $C_2 = x_1(0)$ and $C_1 = v_1(0)$. An analogous result holds for the second equation. For the third equation, we get $x_3(t) = C_2 + C_1 t - \frac{1}{2}mgt^2$ where $C_2 = x_3(0)$ and $C_1 = v_1(0)$, again. Thus, the full solution reads

$$x(t) = x(0) + v(0)t - \frac{g}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} t^2$$

- **Differential equation**: A relation between a function $x(t)$ and its derivatives.
  - The equation $m\ddot{x}(t) = F(x(t))$, above, is an example of a differential equation.
- **Second-order** (differential equation): A differential equation in which the highest derivative is of second degree.
  - The equation $m\ddot{x}(t) = F(x(t))$, above, is an example of a second-order differential equation.
- $\boldsymbol{n^{\text{th}}}$**-order** (differential equation): A differential equation in which the highest derivative is of degree n.
- **System** (of differential equations): A finite set of differential equations.
  - Systems of differential equations vary in how related they are, that is, they may or may not share variables.
  - Technically, $m\ddot{x}(t) = F(x(t))$ is a system of differential equations since $x = (x_1, x_2, x_3)$ gives rise to three independent differential equations, one for each Cartesian dimension, as follows.

$$m\ddot{x}_1 = F(x_1(t)) \qquad m\ddot{x}_2 = F(x_2(t)) \qquad m\ddot{x}_3 = F(x_3(t))$$

- **Dependent** (variable): A variable whose value depends on that of another.
  - In our example, x, v, and a are all dependent variables.

- **Independent** (variable): A variable whose value does not depend on that of another.
  - In our example, t is the independent variable.
- **First-order** (system): A system of differential equations in which the differential equation of highest order is first order.
  - One possible rewrite of the system $m\ddot{x}(t) = F(x(t))$ works by increasing the number of dependent variables from $x \in \mathbb{R}^3$ to $(x, v) \in \mathbb{R}^6$. Indeed, we can rewrite the second-order differential equation $m\ddot{x}(t) = F(x(t))$ as the following first-order system.

$$\begin{aligned} \dot{x}(t) &= v(t) \\ \dot{v}(t) &= \frac{1}{m}F(x(t)) \end{aligned}$$

  - We will find that the above form is often better suited to theoretical investigations.
- ***n*th-order** (system): A system of differential equations in which the differential equation of highest order is n^{th} order.
- One conclusion of our investigation of gravity's effect on a particle is that the entire fate (past and future) of our particle's position, velocity, and acceleration (under simple gravity) is uniquely determined by its initial location $x(0)$ and velocity $v(0)$.
  - While we could use simple integration to solve this system, we cannot always do this (not even under Newtonian universal gravitation).

Problems

11/24: **1.1.** Consider the case of a stone dropped from a height h above the Earth's surface. Denote by r the distance of the stone from the Earth's surface. The initial condition reads $r(0) = h$, $\dot{r}(0) = 0$. The equation of motion reads

$$\ddot{r} = -\frac{\gamma M}{(R+r)^2}$$

for the exact model and

$$\ddot{r} = -g$$

for the approximate model, where R is the radius of the Earth, M is the mass of the Earth, and $g = \gamma M/R^2$. Note that $\ddot{r}$ is acceleration, not force, and hence we do not see the mass of the stone in the RHS's above.

(i) Transform both equations into a first-order system.

Proof. Define $v = \dot{r}$. Then the first-order system corresponding to the exact model is

$$\boxed{\begin{aligned} \dot{r} &= v \\ \dot{v} &= -\frac{\gamma M}{(R+r)^2} \end{aligned}}$$

and the first-order system corresponding to the approximate model is

$$\boxed{\begin{aligned} \dot{r} &= v \\ \dot{v} &= -g \end{aligned}}$$

∎

(ii) Compute the solution to the approximate system corresponding to the given initial condition. Compute the time it takes for the stone to hit the surface ($r = 0$).

Proof. We can integrate $\dot{v} = -g$ to determine that $v(t) = -gt + C_1$ for some $C_1 \in \mathbb{R}$ and then integrate $\dot{r} = v$ to determine that

$$r(t) = -\frac{1}{2}gt^2 + C_1 t + C_2$$

for some additional $C_2 \in \mathbb{R}$. Using the initial conditions, we can determine that

$$h = r(0) = C_2 \qquad\qquad 0 = \dot{r}(0) = v(0) = C_1$$

Thus, the solution to the approximate system corresponding to the given initial conditions is

$$\boxed{r(t) = -\frac{1}{2}gt^2 + h}$$

We can solve the equation $r(t) = 0$ for t as follows.

$$\begin{aligned}
0 &= r(t) \\
&= -\frac{1}{2}gt^2 + h \\
t &= \pm\sqrt{\frac{2h}{g}}
\end{aligned}$$

Knowing that $t \geq 0$ by definition, we choose

$$\boxed{t = \sqrt{\frac{2h}{g}}}$$

□

(iii) Assume that the exact equation also has a unique solution corresponding to the given initial condition. What can you say about the time it takes for the stone to hit the surface in comparison to the approximate model? Will it be longer or shorter? Estimate the difference between the solutions in the exact and in the approximate case. *Hints*: You should not compute the solution to the exact equation! Look at the minimum and maximum of the force.

Proof. It will take the stone $\boxed{\text{longer}}$ to hit the surface in the exact model than in the approximate model. How do we *estimate* the difference?? □

(iv) Grab your physics book from high school and give numerical values for the case $h = 10\,\text{m}$.

Proof. For the approximate model,

$$\boxed{t \approx 1.43\,\text{s}}$$

and for the exact model, ... □

1.2. Consider again the exact model from the previous problem and write

$$\ddot{r} = -\frac{\gamma M \varepsilon^2}{(1 + \varepsilon r)^2}$$

where $\varepsilon = 1/R$. It can be shown that the solution $r(t) = r(t, \varepsilon)$ to the above with the given initial conditions is C^∞ with respect to both t, ε. Show that

$$r(t) = h - g\left(1 - \frac{2h}{R}\right)\frac{t^2}{2} + O\left(\frac{1}{R^4}\right)$$

where $g = \gamma M/R^2$. *Hint*: Insert $r(t, \varepsilon) = r_0(t) + r_1(t)\varepsilon + r_2(t)\varepsilon^2 + r_3(t)\varepsilon^3 + O(\varepsilon^4)$ into the differential equation and collect powers of ε. Then solve the corresponding differential equations for $r_0(t), r_1(t), \ldots$ and note that the initial conditions follow from $r(0, \varepsilon) = h$ and $\dot{r}(0, \varepsilon) = 0$. A rigorous justification for this procedure will be given in Section 2.5.

Proof. Taking the hint, we get that

$$r_0''(t) + r_1''(t)\varepsilon + r_2''(t)\varepsilon^2 + r_3''(t)\varepsilon^3 + O(\varepsilon^4) = -\frac{\gamma M\varepsilon^2}{(1+\varepsilon(r_0(t)+r_1(t)\varepsilon+r_2(t)\varepsilon^2+r_3(t)\varepsilon^3+O(\varepsilon^4)))^2}$$

We can rewrite the denominator d as follows.

$$\begin{aligned}
d &= (1 + r_0(t)\varepsilon + r_1(t)\varepsilon^2 + r_2(t)\varepsilon^3 + O(\varepsilon^4))^2 \\
&= 1 + r_0(t)\varepsilon + r_1(t)\varepsilon^2 + r_2(t)\varepsilon^3 + r_0(t)\varepsilon + r_0(t)^2\varepsilon^2 + r_0(t)r_1(t)\varepsilon^3 \\
&\quad + r_1(t)\varepsilon^2 + r_0(t)r_1(t)\varepsilon^3 + r_2(t)\varepsilon^3 + O(\varepsilon^4) \\
&= 1 + 2r_0(t)\varepsilon + (r_0(t)^2 + 2r_1(t))\varepsilon^2 + 2(r_0(t)r_1(t) + r_2(t))\varepsilon^3 + O(\varepsilon^4)
\end{aligned}$$

Multiplying both sides of the original equation by d yields

$$\begin{aligned}
-\gamma M\varepsilon^2 &= r_0'' + 2r_0r_0''\varepsilon + (r_0^2 + 2r_1)r_0''\varepsilon^2 + 2(r_0r_1 + r_2)r_0''\varepsilon^3 \\
&\quad + r_1''\varepsilon + 2r_0r_1''\varepsilon^2 + (r_0^2 + 2r_1)r_1''\varepsilon^3 \\
&\quad + r_2''\varepsilon^2 + 2r_0r_2''\varepsilon^3 \\
&\quad + r_3''\varepsilon^3 + O(\varepsilon^4) \\
0 + 0\varepsilon + 0\varepsilon^2 + 0\varepsilon^3 &= r_0'' \\
&\quad + (2r_0r_0'' + r_1'')\varepsilon \\
&\quad + [\gamma M + (r_0^2 + 2r_1)r_0'' + 2r_0r_1'' + r_2'']\varepsilon^2 \\
&\quad + [2(r_0r_1 + r_2)r_0'' + (r_0^2 + 2r_1)r_1'' + 2r_0r_2'' + r_3'']\varepsilon^3
\end{aligned}$$

Thus, by collecting powers of ε and comparing, we have the system of differential equations

$$\begin{aligned}
r_0''(t) &= 0 \\
r_1''(t) &= -2r_0(t)r_0''(t) \\
r_2''(t) &= -\gamma M - (r_0(t)^2 + 2r_1(t))r_0''(t) - 2r_0(t)r_1''(t) \\
r_3''(t) &= -2(r_0(t)r_1(t) + r_2(t))r_0''(t) - (r_0(t)^2 + 2r_1(t))r_1''(t) - 2r_0(t)r_2''(t)
\end{aligned}$$

which we can sequentially solve as follows.

Before we tackle the differential equations, however, a quick note on the initial conditions. We need

$$\begin{aligned}
h &= r(0,\varepsilon) & 0 &= \dot{r}(0,\varepsilon) \\
&= r_0(0) + r_1(0)\varepsilon + r_2(0)\varepsilon^2 + r_3(0)\varepsilon^3 & &= r_0'(0) + r_1'(0)\varepsilon + r_2'(0)\varepsilon^2 + r_3'(0)\varepsilon^3
\end{aligned}$$

But the only way these equations can be satisfied for all ε is if

$$r_0(0) = h \qquad r_1(0) = \cdots = r_3(0) = r_0'(0) = \cdots = r_3'(0) = 0$$

in agreement with the unjustified method from class!

The first differential equation has general solution

$$r_0(t) = C_1t + C_2$$

Applying the initial conditions, we can learn that $C_1 = 0$ and $C_2 = h$. This yields the solution

$$r_0(t) = h$$

We can now move onto the second differential equation, which in light of the formula for r_0 can be rewritten as

$$r_1''(t) = -2 \cdot h \cdot 0 = 0$$

Solving and applying the initial conditions here yields

$$r_1(t) = 0$$

Moving onto the third differential in the same fashion, we have

$$r_2''(t) = -\gamma M - (h^2 + 2 \cdot 0) \cdot 0 - 2 \cdot h \cdot 0 = -\gamma M$$

Thus,

$$r_2(t) = -\frac{\gamma M}{2} t^2$$

For the fourth and final differential equation, we have

$$r_3''(t) = -2 \left(h \cdot 0 - \frac{\gamma M}{2} t^2 \right) \cdot 0 - (h^2 + 2 \cdot 0) \cdot 0 - 2 \cdot h \cdot -\gamma M = 2h\gamma M$$

Thus,

$$r_3(t) = h\gamma M t^2$$

Combining the last four results, we have that

$$\begin{aligned}
r(t) &= r(t, \varepsilon) \\
&= r_0(t) + r_1(t)\varepsilon + r_2(t)\varepsilon^2 + r_3(t)\varepsilon^3 + O(\varepsilon^4) \\
&= h + 0 \cdot \varepsilon + \left(-\frac{\gamma M}{2} t^2 \right) \cdot \varepsilon^2 + h\gamma M t^2 \varepsilon^3 + O(\varepsilon^4) \\
&= h - \gamma M \varepsilon^2 (1 - 2h\varepsilon) \frac{t^2}{2} + O(\varepsilon^4) \\
&= h - \underbrace{\frac{\gamma M}{R^2}}_{g} \left(1 - \frac{2h}{R} \right) \frac{t^2}{2} + O\left(\frac{1}{R^4} \right)
\end{aligned}$$

as desired. □

Section 1.2: Classification of Differential Equations

11/11:

- $\boldsymbol{C^k(U,V)}$: The set of functions $f : U \to V$ having continuous derivatives up to order k, where $U \subseteq \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$, and $k \in \mathbb{N}_0$.
- $\boldsymbol{C(U,V)}$: The set of continuous functions $f : U \to V$. *Given by*

$$C(U,V) = C^0(U,V)$$

- $\boldsymbol{C^\infty(U,V)}$: The set of smooth functions $f : U \to V$. *Given by*

$$C^\infty(U,V) = \bigcap_{k \in \mathbb{N}} C^k(U,V)$$

- $\boldsymbol{C^k(U)}$: The set of real functions $f : U \to \mathbb{R}$ having continuous derivatives up to order k. *Given by*

$$C^k(U) = C^k(U, \mathbb{R})$$

- **Ordinary differential equation**: An equation of the form

$$F(t, x, x^{(1)}, \dots, x^{(k)}) = 0$$

  where $F \in C(U)$ ($U \subseteq \mathbb{R}^{k+2}$ open) relates the unknown function $x \in C^k(J)$ ($J \subseteq \mathbb{R}$), its independent variable t, and its first k derivatives;

$$x^{(j)} := \frac{\mathrm{d}^j x}{\mathrm{d}t^j}$$

  for $j \in \mathbb{N}_0$. *Also known as* **ODE**.

- **Order** (of an ODE): The highest derivative appearing in the argument of F.
- **Solution** (of an ODE): A function $\phi \in C^k(I)$ ($I \subseteq J$ an interval) satisfying the equation
$$F(t, \phi(t), \phi^{(1)}(t), \dots, \phi^{(k)}(t)) = 0$$
for all $t \in I$.
- Very little can be said about completely general ODEs.
- Thus, we begin our investigation with the subclass of ODEs that can solved for their highest order derivative, that is, with ODEs of the form
$$x^{(k)} = f(t, x, x^{(1)}, \dots, x^{(k-1)})$$
  - Relation between these ODEs and general ODEs: These ODEs are the ones for which F has nonzero partial derivative with respect to y_k. Indeed, if F satisfies this condition locally near $(x, y_k) \in U$, then the implicit function theorem permits the above rearrangement.
- **System** (of ODEs): A finite set of ODEs. *Given by*
$$\begin{aligned} x_1^{(k)} &= f_1(t, x, x^{(1)}, \dots, x^{(k-1)}) \\ &\vdots \\ x_n^{(k)} &= f_n(t, x, x^{(1)}, \dots, x^{(k-1)}) \end{aligned}$$
  - Note that the use of an unsubscripted $x^{(j)}$ in the argument of each f_i reflects the fact that every f_i is a function of *all* of the components of of x and their derivatives (not just x_i and its derivatives). Symbolically, each f_i is a function of $x_l^{(j)}$ ($l = 1, \dots, n$ and $j = 0, \dots, k-1$), and
$$f_i : \mathbb{R} \times \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k \text{ times}} \to \mathbb{R}$$
- **Linear** (system): A system of n ordinary differential equations for which each $x_i^{(k)}$ is of the form
$$x_i^{(k)} = g_i(t) + \sum_{l=1}^{n} \sum_{j=0}^{k-1} f_{i,j,l}(t) x_l^{(j)}$$
  - The summations are over the n components of x, and each of their k derivatives up from the function itself ($l = 0$) to $l = k - 1$. In other words, if a (derivative of) a component of x appears in a linear system, the only modification to it from itself should be a functional coefficient in the independent variable.
- **Homogeneous** (linear system): A linear system for which $g_i(t) = 0$.
- **Inhomogeneous** (linear system): A linear system for which $g_i(t) = 0$.
- Teschl (2012) goes over the conversion of a system to a first-order system, as covered in class.
  - Also noted: We can include t as a dependent variables by taking $z = (t, y)$ and making $\dot{z}_1 = 1$, $\dot{z}_i = z_{i+1}$ ($i = 2, \dots, k$), and $\dot{z}_{k+1} = f(z)$.
- **Autonomous** (system): A system in which f does not depend on t.
- We will often limit our studies to autonomous first-order ODEs since, as per the past two results, these encapsulate all higher order, potentially non-autonomous systems as well.
- **Partial differential equation**: A differential equation for which $t \in \mathbb{R}^m$. *Also known as* **PDE**.
  - Name justification: $t \in \mathbb{R}^m$ necessitates the use of partial derivatives.
- Complex values will not be considered until later.