Week 7

???

7.1   Peano Existence Theorem

- Today: Peano Existence Theorem.

- For an IVP of a first-order differential system, as long as the RHS is continuous, we get at least one solution.

- The proof provides an algorithm that can be really useful in computing the solution provided that uniqueness exists.

- We will need a theorem from analysis to start.

- Theorem (Arzelà-Ascoli[1]): Let $h_k : [a, b] \to \mathbb{R}^n$ be a sequence of functions that is uniformly bounded and uniformly Lipschitz continuous wrt. L. Then $\{h_k\}$ contains a uniformly convergent subsequence and the limit has the same bound and Lipschitz constant.

  Proof. Recall the property of sequential compactness[2], i.e., that every bounded sequence of numbers contains a convergent subsequence. We want to prove this for a sequence of functions. To do so, we will need the Cantor diagonalization technique.

  $\mathbb{Q}$ is countable. Thus, we can enumerate the rationals in $[a, b]$ by $r_1, r_2, r_3, \dots$. Since $\{h_k(r_1)\}$ is a bounded sequence of numbers, we have by the above that there is a subsequence C_1 — say $h_1^{(1)}, h_2^{(1)}, h_3^{(1)}, \dots$ — such that $C_1 = \{h_k^{(1)}(r_1)\}$ is a convergent subsequence in $\mathbb{R}^n$ of the original sequence. Now C_1 is still a bounded sequence, so we can obtain a subsequence C_2 of *it* — say $h_1^{(2)}, h_2^{(2)}, h_3^{(2)}, \dots$ — such that $C_2 = \{h_k^{(2)}(r_2)\}$ is a convergent subsequence in $\mathbb{R}^n$ at r_2 (and, by inductive hypothesis, at r_1!). Inductively, we can obtain $C_\ell = \{h_k^{(\ell)}\}_{\ell,k=1}^{\infty}$ convergent at $r_1, r_2, \dots, r_\ell$. We then write down the elements of the sequences as a table. (For example, the kth row of the table is a sequence that converges at $r_1, \dots, r_k$.)

$$
\begin{array}{cccc}
h_1^{(1)} & h_2^{(1)} & h_3^{(1)} & \cdots \\
h_1^{(2)} & h_2^{(2)} & h_3^{(2)} & \cdots \\
h_1^{(3)} & h_2^{(3)} & h_3^{(3)} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{array}
$$

  Consider the diagonal sequence $\{f_\ell\}_{\ell=1}^{\infty}$ where $f_\ell = h_\ell^{(\ell)}$. By definition, it converges at all rational points. We now seek to prove that it converges uniformly at *all* points.

---

[1]This is not the full Arzelà-Ascoli theorem, but a special case. The proof is similar, regardless, though. See Honors Analysis in $\mathbb{R}^n$ I Notes.

[2]The Bolzano-Weierstrass Theorem/Theorem 15.18 from Honors Calculus IBL.

To prove that $\{f_\ell\}$ is a uniformly convergent sequence of functions, it will suffice to show that for all $\varepsilon > 0$, there exists N such that if $k, \ell > N$, then $|f_k(t) - f_\ell(t)| < \varepsilon$ for all $t \in [a, b]$. Let $\varepsilon > 0$ be arbitrary. Divide $[a, b]$ into m congruent subintervals I_α $(\alpha = 1, \ldots, m)$ such that $|I_\alpha| \leq \varepsilon/3L$ for all α. This guarantees that the oscillation of each f_k on any I_α is $\leq \varepsilon/3$ since if $x, y \in I_\alpha$ for some α, then

$$|f_\ell(x) - f_\ell(y)| \leq L|x - y| \leq L \cdot \frac{\varepsilon}{3L} = \frac{\varepsilon}{3}$$

Using the fact that $\{f_\ell\}$ is convergent and hence Cauchy on the rationals, pick N large enough so that $r_\alpha \in I_\alpha$ implies $|f_k(r_\alpha) - f_\ell(r_\alpha)| < \varepsilon/3$ for $k, \ell > N$. We will choose this N to be our N. Now let $t \in [a, b]$ be arbitrary. By their definition, we know $t \in I_\alpha$ for some α. Therefore,

$$\begin{aligned} |f_k(t) - f_\ell(t)| &\leq |f_k(t) - f_k(r_\alpha)| + |f_k(r_\alpha) - f_\ell(r_\alpha)| + |f_\ell(r_\alpha) - f_\ell(t)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

as desired.

Lastly, we can prove that the limit function f of $\{f_\ell\}$ is L-Lipschitz as follows. Let $t, t' \in [a, b]$ be arbitrary. Then

$$\left| \frac{f(t) - f(t')}{t - t'} \right| = \lim_{k \to \infty} \left| \frac{f_k(t) - f_k(t')}{t - t'} \right| \leq \lim_{k \to \infty} \left| \frac{L|t - t'|}{t - t'} \right| = \lim_{k \to \infty} |L| = L$$

as desired. $\square$

- Now we come to the proof of the Peano Existence Theorem.

- Theorem (Peano Existence Theorem): Let $f : [t_0, t_0 + a] \times \bar{B}(y_0, b) \to \mathbb{R}^n$ be bounded $(|f(t, z)| \leq M)$ and continuous. Then the IVP

$$y'(t) = f(t, y(t)), \quad y(t_0) = b$$

has at least one solution for $t \in [t_0, t_0 + T]$ where $T = \min(a, b/M)$.

Proof. Since there is no Lipschitz condition, we use another strategy to find approximate solutions.

picture Fix $T = \min(a, b/M)$. We divide $[t_0, t_0 + T]$ into m congruent closed subintervals I_α $(\alpha = 0, \ldots, m - 1)$, each of length $h_m = T/m$. Define a continuous function $y_m(t)$ as follows: The values at the nodes t_α (the intersection points of adjacent congruent subintervals) are defined inductively via

$$y_m(t_{\alpha+1}) = y_m(t_\alpha) + f(t_\alpha, y_m(t_\alpha))h_m$$

for $\alpha = 0, \ldots, m - 1$, and y_m is taken to be linear between the nodes[3]. The idea is that we replace the derivative $y'(t)$ by the difference quotient $[y(t + h) - y(t)]/h$. It follows by the construction that every function in the set $\{y_k(t) : [t_0, t_0 + T] \to \bar{B}(y_0, b)\}$ is piecewise linear (hence continuous), uniformly bounded, and uniformly M-Lipschitz continuous. Therefore, by the Arzelá-Ascoli theorem, $\{y_k\}$ contains a uniformly convergent subsequence $y_{m_k} \to y$.

It remains to verify that y is a solution to the integral equation

$$y(t) = y_0 + \int_{t_0}^t f(\tau, y(\tau)) \mathrm{d}\tau$$

Observe that the domain of f is a closed and bounded subset of the real numbers. Thus, it is compact by the Heine-Borel theorem[4]. Moreover, since f is a continuous function on a compact domain, we

---

[3]Note that this construction is quite similar to that employed in Euler's method.
[4]Theorem 10.16 of Honors Calculus IBL.

have by the Heine-Cantor theorem[5] that f is uniformly continuous. Thus, for any $\varepsilon > 0$, there exists N such that if $m > N$, then

$$|f(t, y_m(t)) - f(t_\alpha, y_m(t_\alpha))| < \frac{\varepsilon}{T}$$

for all $\alpha = 0, \ldots, m-1$ and $t \in I_\alpha$. Additionally, observe that

$$y_{m_k}(t) = y_0 + \sum_{\alpha=0}^{m-1} \int_{t_\alpha}^{t_{\alpha+1}} \chi_t(\tau) f(t_\alpha, y_{m_k}(t_\alpha)) \mathrm{d}\tau$$

where $\chi_t(\tau)$ denotes the **characteristic function** of $[t_0, t]$. To see this, compare with the original inductive definition of $y_m(t_{\alpha+1})$. *picture* We thus see that y_0 in the above equation corresponds to $y_m(t_0) = y(t_0)$, as we would expect. We see that we are summing a series of side-by-side integrals so that in the end, we integrate over all of $[t_0, t_0 + T]$. We see that the characteristic function restricts us to integrating over the ODE only up until t, as we would want for an approximation $y_{m_k}(t)$ at t using Euler's method. And we see that since $f(t_\alpha, y_{m_k}(t_\alpha))$ is constant and $h_m = t_{\alpha+1} - t_\alpha$, the integral does take on the expected value $f(t_\alpha, y_m(t_\alpha)) h_m$. Moving right along, we see that

$$\left| y_{m_k}(t) - y_0 - \int_{t_0}^{t} f(\tau, y_{m_k}(\tau)) \mathrm{d}\tau \right| \leq \sum_{\alpha=0}^{m-1} \int_{t_\alpha}^{t_{\alpha+1}} \chi_t(\tau) |f(t_\alpha, y_{m_k}(t_\alpha)) - f(\tau, y_{m_k}(\tau))| \mathrm{d}\tau$$

$$< \int_{t_0}^{t_0+T} \chi_t(\tau) \cdot \frac{\varepsilon}{T} \mathrm{d}\tau$$

$$= \int_{t_0}^{t} \frac{\varepsilon}{T} \mathrm{d}\tau$$

$$= \varepsilon \cdot \frac{t - t_0}{T}$$

$$\leq \varepsilon$$

Thus, by uniform convergence, $\int_{t_0}^{t} f(\tau, y_{m_k}(\tau)) \mathrm{d}\tau \to \int_{t_0}^{t} f(\tau, y(\tau)) \mathrm{d}\tau$ uniformly, so y does satisfy the integral equation, as desired.   □

- **Characteristic function** (of $[a, b]$): The function defined as follows. *Denoted by* $\boldsymbol{\chi_{[a,b]}}$. *Given by*

$$\chi_{[a,b]}(t) = \begin{cases} 1 & x \in [a, b] \\ 0 & x \notin [a, b] \end{cases}$$

- Utility of the Peano Existence Theorem: Proves the *existence* of a solution, but the proof is not constructive; it does not give an algorithm for finding the desired sequence. Nor does the PET make any statement on uniqueness.

- We now look to use a related method to define a sequence of functions that will converge to the desired solution of the ODE.

  - While the PET does not require it, in practice, most f we would be interested in will satisfy an additional Lipschitz condition.

  - Define the integral operator

$$\Phi[u] = y_0 + \int_{t_0}^{t} f(\tau, u(\tau)) \mathrm{d}\tau$$

  We will prove that Φ is a contraction on the function space. This will imply that $\Phi^N[u]$ converges across the entire interval $[t_0, t_0 + T]$ to the solution y for any $u : [t_0, t_0 + T] \to \bar{B}(y_0, b)$, giving us our desired computational strategy. Let's begin.

---

[5]Theorem 13.6 of Honors Calculus IBL.

– To prove that Φ is a contraction, it will suffice to show that $\left\|\Phi^j[u_1] - \Phi^j[u_2]\right\| \to 0$ as $j \to \infty$. Thus, we wish to put a bound on $\left\|\Phi^j[u_1] - \Phi^j[u_2]\right\|$ that decreases as j increases. To that end, we will prove that

$$\left\|\Phi^j[u_1] - \Phi^j[u_2]\right\| \leq \frac{(LT)^j}{j!} \cdot \|u_1 - u_2\|$$

for all j.

■ We induct on j. For the base case $j = 1$, we have that

$$
\begin{aligned}
|\Phi[u_1](t) - \Phi[u_2](t)| &\leq \int_{t_0}^t L|u_1(\tau) - u_2(\tau)|\mathrm{d}\tau \\
&\leq L(t - t_0)\|u_1 - u_2\| \\
&\leq LT\|u_1 - u_2\| \\
&= \frac{(LT)^1}{1!} \cdot \|u_1 - u_2\|
\end{aligned}
$$

for all t.

■ Now suppose inductively that $\left\|\Phi^j[u_1] - \Phi^j[u_2]\right\| \leq (LT)^j/j! \cdot \|u_1 - u_2\|$. Then we have that

$$
\begin{aligned}
|\Phi^{j+1}[u_1](t) - \Phi^{j+1}[u_2](t)| &\leq \int_{t_0}^t L|\Phi^j[u_1](\tau) - \Phi^j[u_2](\tau)|\mathrm{d}\tau \\
&\leq \int_{t_0}^t L \cdot \frac{(LT)^j}{j!} \cdot \|u_1 - u_2\|\mathrm{d}\tau \\
&= \cdots \\
&\leq \frac{(LT)^{j+1}}{j!} \cdot \|u_1 - u_2\|
\end{aligned}
$$

for all t, implying the desired result.

– We now estimate the error between y_m and y in terms of y_m, alone. Indeed, we have from the above that

$$
\begin{aligned}
\left\|y_m - \Phi^N[y_m]\right\| &\leq \sum_{j=0}^{N-1} \left\|\Phi^j[y_m] - \Phi^{j+1}[y_m]\right\| \\
&\leq \|y_m - \Phi[y_m]\| \sum_{j=0}^{N-1} \frac{(TL)^j}{j!} \\
\|y_m - y\| &\leq \|y_m - \Phi[y_m]\|\mathrm{e}^{TL}
\end{aligned}
$$

where we get from the second to the third line by letting $N \to \infty$.

■ The proof of the PET guarantees that $\|y_m - \Phi[y_m]\|$ is small when m is large, no matter whether y_m itself converges or not.

■ In fact, when $f \in C^1$, the error is estimated as

$$\|y_m - y\| \leq \frac{LT\mathrm{e}^{TL}}{m}$$

for $L = \|f\|_{C^1}$.

• Takeaway: This polygon method gives rise to an algorithm to solve ODEs. Theoretically, it converges much slower than the Picard iteration, but in practice, it has the advantage that we do not need to do any numerical integration. Indeed, to obtain the desired precision using the Picard iteration, the numerical integration will need more and more steps and the total accumulated error will not be less than this polygon method.

• Better difference methods include Runge-Kutta or Heun, but please refer to monographs on numerical ODEs for these.