MATH 27300 (Basic Theory of Ordinary Differential Equations) Problem Sets

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Contents

1 IVP Examples and Physical Problems

Required Problems

10/12: **1.** Classify the following ordinary differential equations (systems) by indicating the order, if they are linear, and if they are autonomous.

(1) $y'(x) + y(x) = 0$.

Answer.

Order	Linear?	Autonomous?
1	Yes	Yes

□

(2) $y''(t) = t\sin(y(t))$.

Answer.

Order	Linear?	Autonomous?
2	No	No

□

(3) $x' = -y$, $y' = 2x$.

Answer.

Order	Linear?	Autonomous?
1	Yes	Yes

□

(4) $y'(t) = y(t)\sin(t) + \cos(y(t))$.

Answer.

Order	Linear?	Autonomous?
1	No	No

□

2. Transform the following differential equations to first-order systems.

(1) $y^{(3)} + 2y'' - y' + y = 0$.

Proof. Let

$$x = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}$$

Then

$$x' = \begin{pmatrix} y' \\ y'' \\ y^{(3)} \end{pmatrix}$$

so, by comparing components between the above two vectors and then using the original linear equation to define the last entry (with substitutions), we obtain

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -2x_3 + x_2 - x_1 \end{aligned}}$$

□

(2) $x'' - t\sin x' = x$.

Proof. In an analogous manner to the above, we can determine that

$$\boxed{\begin{aligned} y_1' &= y_2 \\ y_2' &= y_1 + t\sin y_2 \end{aligned}}$$

□

3. Solve the following differential equations with initial value $x(0) = x_0$. Also identify the set of x_0 for which these solutions are extendable to the whole of $t \geq 0$. When a solution cannot be extended to the whole of $t \geq 0$, determine its lifespan in terms of x_0.

Example: Solve $x' = x^2$ with $x(0) = x_0$. By separation of variables, the solution reads

$$\int_{x_0}^{x} \frac{\mathrm{d}w}{w^2} = \int_0^t \mathrm{d}\tau$$

where the integral on the left-hand side cannot pass through $w = 0$. The result is

$$-\frac{1}{x} + \frac{1}{x_0} = t \quad \Longleftrightarrow \quad x(t) = \frac{x_0}{1 - x_0 t}$$

When $x_0 \leq 0$, the solution exists throughout $t \geq 0$. When $x_0 > 0$, the solution only exists in $[0, 1/x_0)$[1].

(1) $x' = x\sin t$.

Proof. By separation of variables, the solution reads

$$\int_{x_0}^{x} \frac{\mathrm{d}w}{w} = \int_0^t \sin\tau\,\mathrm{d}\tau$$

The result is

$$\ln\frac{x}{x_0} = 1 - \cos t \quad \Longleftrightarrow \quad \boxed{x(t) = x_0\mathrm{e}^{1-\cos t}}$$

The set of x_0 for which this solution is extendable to the whole of $t \geq 0$ is $\boxed{\mathbb{R}}$. □

(2) $x' = t^2\tan x$.

Proof. By separation of variables, the solution reads

$$\int_{x_0}^{x} \cot w\,\mathrm{d}w = \int_0^t \tau^2\mathrm{d}\tau$$

where the integral on the left-hand side cannot pass through $x = \pi n$ for any $n \in \mathbb{Z}$. The result is

$$\ln\left|\frac{\sin x}{\sin x_0}\right| = \frac{t^3}{3} \quad \Longleftrightarrow \quad \boxed{x(t) = \arcsin\left(\mathrm{e}^{t^3/3}\sin x_0\right)}$$

Based on the above, the solution is extendable to the whole of $t \geq 0$ only when $\mathrm{e}^{t^3/3}\sin x_0$ remains in the domain of arcsine, i.e., $[-1, 1]$ for all t. Based on what we know about exponential functions, this happens iff $\sin x_0 = 0$, i.e., iff

$$\boxed{x_0 = \pi n, \quad n \in \mathbb{Z}}$$

When $x_0 \neq \pi n$ for some $n \in \mathbb{Z}$, the lifespan is

$$\boxed{\sqrt[3]{3\ln\left|\frac{1}{\sin(x_0)}\right|}}$$

□

[1]This is an interval of existence. The problem asks for the lifespan. Thus, this example is wrong. It is probably what led most people to give an interval of existence in their answer instead of a lifespan.

(3) $x' = 1 + x^2$.

Proof. By separation of variables, the solution reads

$$\int_{x_0}^{x} \frac{1}{1+w^2}\,\mathrm{d}w = \int_0^t \mathrm{d}\tau$$

The result is

$$\arctan(x) - \arctan(x_0) = t \quad\Longleftrightarrow\quad \boxed{x(t) = \tan(t + \arctan(x_0))}$$

Since the tangent function blows up periodically, the set of x_0 for which the solution is extendable to the whole of $t \geq 0$ is $\boxed{\emptyset}$. With respect to lifespan, we use a similar approach to the above. In particular, the solution exists for $t \geq 0$ such that $t + \arctan(x_0) \in (-\pi/2, \pi/2)$, i.e., for t in $[0, \pi/2 - \arctan(x_0))$. Therefore, the lifespan is

$$\boxed{\frac{\pi}{2} - \arctan(x_0)}$$

□

(4) $x' = \mathrm{e}^x \sin t$.

Proof. By separation of variables, the solution reads

$$\int_{x_0}^{x} \mathrm{e}^{-w}\,\mathrm{d}w = \int_0^t \sin\tau\,\mathrm{d}\tau$$

The result is

$$-\mathrm{e}^{-x} + \mathrm{e}^{-x_0} = 1 - \cos t \quad\Longleftrightarrow\quad \boxed{x(t) = -\ln\left(\mathrm{e}^{-x_0} - 1 + \cos t\right)}$$

The set of x_0 for which the solution is extendable to the whole of $t \geq 0$ is

$$\boxed{\{x_0 \in \mathbb{R} \mid x_0 < \ln(1/2)\}}$$

When $x_0 \geq \ln(1/2)$, the lifespan is

$$\boxed{\arccos\left(1 - \mathrm{e}^{-x_0}\right)}$$

□

4. Consider the harmonic oscillator equation, as mentioned in class:

$$x'' + \mu x' + \omega^2 x = 0$$

Here, the initial data $x(0) = x_0$ and $x'(0) = x_1$ are real numbers.

(1) Derive two linearly independent *real* solutions when $\mu > 0$. (Hint: You should consider the cases $\mu < 2\omega$ and $\mu > 2\omega$ separately.)

Proof. From class, we know that the form of the two linearly independent solutions to such an equation depends on whether or not the roots of the characteristic polynomial

$$r^2 + \mu r + \omega^2 = 0$$

are equal. At this point, we will take the hint and divide into three cases ($\mu > |2\omega|$, $\mu = 2\omega$, and $\mu < |2\omega|$).

$\underline{\mu > |2\omega|}$: In this case, the two roots

$$r_1 = \frac{-\mu + \sqrt{\mu^2 - 4\omega^2}}{2} \qquad\qquad r_2 = \frac{-\mu - \sqrt{\mu^2 - 4\omega^2}}{2}$$

are not equal. Thus, the two linearly independent solutions are

$$\boxed{\mathrm{e}^{r_1 t}, \mathrm{e}^{r_2 t}}$$

Moreover, since r_1, r_2 are both real, the above constitute two linearly independent *real* solutions, as desired.

$\underline{\mu = 2\omega}$: In this case, the two roots are equal, i.e., $r_1 = r_2$. Specifically, both roots coincide at $-\mu/2$. Thus, the two linearly independent solutions are

$$\boxed{\mathrm{e}^{-\mu t/2}, t\mathrm{e}^{-\mu t/2}}$$

For the same reason as in the previous case, the above are also real solutions, as desired.

$\underline{\mu < |2\omega|}$: In this case, the two roots are once again unequal. Thus, the two linearly independent solutions are

$$\mathrm{e}^{r_1 t}, \mathrm{e}^{r_2 t}$$

However, this time, r_1, r_2 are *not* real, so the above solutions are not real either. However, we can still obtain real solutions from these using Euler's formula. Notice that

$$r_1 = -\frac{\mu}{2} + i\sqrt{\omega^2 - \frac{\mu^2}{4}} \qquad\qquad r_2 = -\frac{\mu}{2} - i\sqrt{\omega^2 - \frac{\mu^2}{4}}$$

where both $-\mu/2$ and $\sqrt{\omega^2 - \mu^2/4}$ are real numbers, which we may call α and β, respectively. Thus, since any linear combination of $\mathrm{e}^{r_1 t}, \mathrm{e}^{r_2 t}$ is another solution, we know in particular that the functions

$$\begin{aligned}
\frac{1}{2}(\mathrm{e}^{r_1 t} + \mathrm{e}^{r_2 t}) &= \frac{1}{2}(\mathrm{e}^{\alpha t + it\beta} + \mathrm{e}^{\alpha t - it\beta}) \\
&= \frac{1}{2}\mathrm{e}^{\alpha t}[(\cos(t\beta) + i\sin(t\beta)) + (\cos(t\beta) - i\sin(t\beta))] \\
&= \mathrm{e}^{\alpha t}\cos(t\beta) \\
&= \mathrm{e}^{-\mu t/2}\cos\left(t\sqrt{\omega^2 - \frac{\mu^2}{4}}\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2i}(\mathrm{e}^{r_1 t} - \mathrm{e}^{r_2 t}) &= \frac{1}{2i}(\mathrm{e}^{\alpha t + it\beta} - \mathrm{e}^{\alpha t - it\beta}) \\
&= \frac{1}{2i}\mathrm{e}^{\alpha t}[(\cos(t\beta) + i\sin(t\beta)) - (\cos(t\beta) - i\sin(t\beta))] \\
&= \mathrm{e}^{\alpha t}\sin(t\beta) \\
&= \mathrm{e}^{-\mu t/2}\sin\left(t\sqrt{\omega^2 - \frac{\mu^2}{4}}\right)
\end{aligned}$$

are solutions. These solutions are linearly independent. Additionally, they are real. Therefore, we may take them, restated as

$$\boxed{\mathrm{e}^{-\mu t/2}\cos\left(t\sqrt{\omega^2 - \frac{\mu^2}{4}}\right), \mathrm{e}^{-\mu t/2}\sin\left(t\sqrt{\omega^2 - \frac{\mu^2}{4}}\right)}$$

to be our desired linearly independent real solutions in this case. $\square$

(2) Recall that $\mu = b/m$ and $\omega^2 = k/m$. Recall also that the mechanical energy for the oscillator reads

$$E = \frac{1}{2}m|x'|^2 + \frac{1}{2}kx^2$$

Compute the time derivative of E and conclude that E is exponentially decaying for $b > 0$, i.e., the mechanical energy is not conserved in this case. Does this violate the law of conservation of mechanical energy?

Proof. Applying the chain rule, we have that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = mx'x'' + kxx'$$

It follows that

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= mx'(-\mu x' - \omega^2 x) + kxx' \\ &= x'(-bx' - kx) + kxx' \\ &= -b(x')^2\end{aligned}$$

Now $x' \neq 0$ (as an exponential function). Hence, $(x')^2 > 0$. This and $b > 0$ show that $\frac{\mathrm{d}E}{\mathrm{d}t}$ is always equal to a negative value. But this is characteristic of exponential decay, as desired.
Mechanical energy is conserved; it is dispersed from system to surroundings by the drag b. □

5. Use the transformation $y = tw$ to convert

$$y' = f(y/t)$$

to an ODE in w. Write down this equation for w. Use this transformation to solve

$$tyy' + 4t^2 + y^2 = 0, \quad y(2) = -7$$

Determine the lifespan (you can use a calculator for an approximate value).

Proof. If $y = tw$, then

$$\frac{\mathrm{d}y}{\mathrm{d}t} = w + t\frac{\mathrm{d}w}{\mathrm{d}t}$$

Thus, the ODE in terms of w is

$$\boxed{\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{f(w) - w}{t}}$$

which is a separable differential equation.

We have that

$$tyy' + 4t^2 + y^2 = 0 \quad \Longleftrightarrow \quad y' = -4\left(\frac{y}{t}\right)^{-1} - \frac{y}{t}$$

Using the above transformation yields

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{(-4w^{-1} - w) - w}{t}$$

Transforming the initial condition as well gives

$$w(2) = \frac{y(2)}{2} = -\frac{7}{2}$$

We can simplify and solve the above as follows.

$$
\begin{aligned}
\frac{\mathrm{d}w}{-4w^{-1}-2w} &= \frac{\mathrm{d}t}{t}\\
-\frac{1}{4}\int_{-7/2}^{w}\frac{2v\,\mathrm{d}v}{v^2+2} &= \int_2^t \frac{\mathrm{d}\tau}{\tau}\\
-\frac{1}{4}\left[\ln\!\left(w^2+2\right)-\ln(14.25)\right] &= \ln\!\left(\frac{t}{2}\right)\\
w &= \pm\frac{1}{t^2}\sqrt{228-2t^4}\\
\Aboxed{y(t) &= -\frac{1}{t}\sqrt{228-2t^4}}
\end{aligned}
$$

Note that we pick the negative in the final step to fit the initial condition.

The initial value of t is 2 by hypothesis. The final value of t can be determined by calculating when $228-2t^4=0$. This occurs such that the interval of existence is approximately $[2,3.27]$ and the lifespan is thus approximately $\boxed{1.27}$. $\square$

6. Use the transformation $w=y^{1-\alpha}$ to convert Bernoulli's equation

$$y'+p(t)y=q(t)y^{\alpha},\quad \alpha\neq 0,1$$

to an ODE in w. Write down this equation for w. Use this transformation to solve

$$6y'-2y=ty^4,\quad y(0)=-2$$

Determine the lifespan (you can use a calculator for an approximate value).

Proof. If $w=y^{1-\alpha}$, then

$$y=w^{1/(1-\alpha)}\qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}t}=\frac{w^{\alpha/(1-\alpha)}}{1-\alpha}\frac{\mathrm{d}w}{\mathrm{d}t}$$

Thus, the ODE in terms of w is

$$\boxed{\frac{w^{\alpha/(1-\alpha)}}{1-\alpha}\frac{\mathrm{d}w}{\mathrm{d}t}+p(t)w^{1/(1-\alpha)}=q(t)w^{\alpha/(1-\alpha)}}$$

which is an exact differential equation.

We have that

$$6y'-2y=ty^4\quad\Longleftrightarrow\quad y'+\left(-\frac{1}{3}\right)y=\left(\frac{t}{6}\right)y^4$$

Using the above transformation yields

$$-\frac{w^{-4/3}}{3}\frac{\mathrm{d}w}{\mathrm{d}t}-\frac{w^{-1/3}}{3}=\frac{tw^{-4/3}}{6}$$

We can simplify and evaluate the above as follows.

$$
\begin{aligned}
\frac{1}{3}w^{-4/3}\frac{\mathrm{d}w}{\mathrm{d}t}+\frac{1}{3}w^{-1/3} &= -\frac{t}{6}w^{-4/3}\\
\frac{\mathrm{d}w}{\mathrm{d}t}+w &= -\frac{t}{2}\\
\mathrm{e}^t\frac{\mathrm{d}w}{\mathrm{d}t}+\mathrm{e}^tw &= -\frac{t}{2}\mathrm{e}^t\\
\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^tw) &= -\frac{t}{2}\mathrm{e}^t\\
\mathrm{e}^tw &= -\frac{1}{2}\int t\mathrm{e}^t\,\mathrm{d}t\\
&= -\frac{1}{2}\mathrm{e}^t(t-1)+C\\
w &= -\frac{1}{2}(t-1)+C\mathrm{e}^{-t}\\
y^{-3} &= -\frac{1}{2}(t-1)+C\mathrm{e}^{-t}\\
y &= \left[-\frac{1}{2}(t-1)+C\mathrm{e}^{-t}\right]^{-1/3}
\end{aligned}
$$

We now apply the initial condition.

$$
\begin{aligned}
\left[-\frac{1}{2}(0-1)+C\mathrm{e}^{-0}\right]^{-1/3} &= y(0)\\
\left[\frac{1}{2}+C\right]^{-1/3} &= -2\\
C &= -\frac{5}{8}
\end{aligned}
$$

Therefore, the solution to the ODE in question is

$$\boxed{y(t)=\left[-\frac{1}{2}(t-1)-\frac{5}{8}\mathrm{e}^{-t}\right]^{-1/3}}$$

The equation $\boxed{\text{does not have finite lifespan}}$. □

7. Show that

$$(4bxy+3x+5)y'+3x^2+8ax+2by^2+3y=0$$

is an exact equation, no matter what value a,b take. Find the implicit relation satisfied by the solution $y(x)$ and x.

Proof. To show that an equation of the form $g\ \mathrm{d}y/\mathrm{d}x+f=0$ is exact, it will suffice to confirm that

$$\frac{\partial g}{\partial x}=\frac{\partial f}{\partial y}$$

Since the equation in question is of this form, we may evaluate directly:

$$\frac{\partial g}{\partial x}=4by+3 \qquad\qquad \frac{\partial f}{\partial y}=4by+3$$

By transitivity, we have the desired result.

Having confirmed that the given equation is exact as written (i.e., we do *not* need integrating factors to make it so), we now want to find F such that $\partial F/\partial x = f$ and $\partial F/\partial y = g$. Starting with the former constraint, we can determine that

$$\begin{aligned}F(x,y) &= \int (3x^2 + 8ax + 2by^2 + 3y)\,\mathrm{d}x \\ &= x^3 + 4ax^2 + 2bxy^2 + 3xy + h(y)\end{aligned}$$

where $h(y)$ is a function of integration (as opposed to a constant of integration). We now differentiate with respect to y.

$$\frac{\partial F}{\partial y} = 4bxy + 3x + \frac{\mathrm{d}h}{\mathrm{d}y}$$

Knowing that $\partial F/\partial y = g$, we can use the above equation to solve for h as follows.

$$\begin{aligned}4bxy + 3x + 5 &= 4bxy + 3x + \frac{\mathrm{d}h}{\mathrm{d}y} \\ \frac{\mathrm{d}h}{\mathrm{d}y} &= 5 \\ h(y) &= 5y\end{aligned}$$

Therefore, we know that

$$F(x,y) = x^3 + 4ax^2 + 2bxy^2 + 3xy + 5y$$

Since F is constant along any solution by construction, the desired implicit relation is given by setting the above equal to some constant $C \in \mathbb{R}$, as follows.

$$\boxed{x^3 + 4ax^2 + 2bxy^2 + 3xy + 5y = C}$$

$\square$

8. Let a, b be constants. For Euler's equation

$$t^2y'' + aty' + by = f(t)$$

consider the transformation $w(\tau) = y(\mathrm{e}^\tau)$. What is the differential equation satisfied by $w(\tau)$? Use this transformation to solve

$$2t^2y'' + 3ty' - 15y = 0, \quad y(1) = 0, \quad y'(1) = 1$$

Proof. The given transformation can be split into the following two transformations (one for the dependent variable and one for the independent variable).

$$w = y \qquad\qquad \begin{aligned} t &= \mathrm{e}^\tau \\ \tau &= \ln(t)\end{aligned}$$

Thus, differentiating our new dependent variable with respect to our new independent variable, we get

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}\tau} &= \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} \\ w'(\tau) &= y'(t)\mathrm{e}^\tau \\ y'(t) &= \frac{w'(\tau)}{\mathrm{e}^\tau}\end{aligned}$$

Differentiating again (and substituting in the above in a later step), we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}w}{\mathrm{d}\tau}\right) &= \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)\cdot\frac{\mathrm{d}t}{\mathrm{d}\tau} + \frac{\mathrm{d}y}{\mathrm{d}t}\cdot\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right) \\
\frac{\mathrm{d}^2 w}{\mathrm{d}\tau^2} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau}\cdot\frac{\mathrm{d}t}{\mathrm{d}\tau} + \frac{\mathrm{d}y}{\mathrm{d}t}\cdot\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} \\
&= \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} \\
w''(\tau) &= y''(t)[\mathrm{e}^\tau]^2 + y'(t)\mathrm{e}^\tau \\
&= y''(t)\mathrm{e}^{2\tau} + w'(\tau) \\
y''(t) &= \frac{w''(\tau) - w'(\tau)}{\mathrm{e}^{2\tau}}
\end{aligned}$$

We can now substitute our transformed definitions of y, y', y'', t back into the original Euler's equation.

$$\begin{aligned}
t^2 y'' + aty' + by &= f(t) \\
[\mathrm{e}^\tau]^2\cdot\frac{w''(\tau)-w'(\tau)}{\mathrm{e}^{2\tau}} + a\cdot\mathrm{e}^\tau\cdot\frac{w'(\tau)}{\mathrm{e}^\tau} + b\cdot w(\tau) &= f(\mathrm{e}^\tau) \\
[w''(\tau) - w'(\tau)] + aw'(\tau) + bw(\tau) &= f(\mathrm{e}^\tau) \\
\boxed{w''(\tau) + (a-1)w'(\tau) + bw(\tau) = f(\mathrm{e}^\tau)}
\end{aligned}$$

This is an inhomogeneous second-order ODE with constant coefficients; we know how to solve these.

We now consider the given example. We can transform it into the general form by dividing through by 2 to yield

$$t^2 y'' + \frac{3}{2}ty' - \frac{15}{2}y = 0$$

Applying the transformation, we get

$$w'' + \frac{1}{2}w' - \frac{15}{2}w = 0$$

as our new differential equation,

$$\tau = \ln(1) = 0$$

as our new initial time, and

$$w(0) = y(1) = 0 \qquad\qquad w'(0) = y'(1)\mathrm{e}^0 = 1$$

as our new initial conditions. We can solve the characteristic polynomial for this differential equation as follows[2].

$$\begin{aligned}
0 &= z^2 + \frac{1}{2}z - \frac{15}{2} \\
&= (z - 5/2)(z+3) \\
z &= \frac{5}{2}, -3
\end{aligned}$$

Therefore, the general solution is

$$w(\tau) = A\mathrm{e}^{5\tau/2} + B\mathrm{e}^{-3\tau}$$

where $A, B \in \mathbb{R}$. We can use the initial conditions to solve for specific values of A, B. In particular, A, B are the solutions to the system of equations

$$0 = w(0) = A + B \qquad\qquad 1 = w'(0) = \frac{5}{2}A - 3B$$

[2]Refer to Theorem 3.7 in Teschl (n.d.) for a better justification of the solution. Alternatively, we can transform it into a 2D planar system and solve it using JNF.

or

$$A = \frac{2}{11} \qquad\qquad B = -\frac{2}{11}$$

Thus, the solution to the transformed IVP is

$$w(\tau) = \frac{2}{11}\mathrm{e}^{5\tau/2} - \frac{2}{11}\mathrm{e}^{-3\tau}$$

Using our inverse coordinate transforms, we can determine that the solution to the original IVP is

$$\begin{aligned}
y(t) &= w(\ln(t)) \\
&= \frac{2}{11}\mathrm{e}^{5\ln(t)/2} - \frac{2}{11}\mathrm{e}^{-3\ln(t)} \\
&= \frac{2}{11}\left(\mathrm{e}^{\ln(t)}\right)^{5/2} - \frac{2}{11}\left(\mathrm{e}^{\ln(t)}\right)^{-3} \\
\Aboxed{y(t) &= \frac{2}{11}t^{5/2} - \frac{2}{11}t^{-3}}
\end{aligned}$$

□

9. Suppose there is a capacitor with capacitance C being charged by a battery of fixed voltage V_0. Suppose there is a resistor R connected to C. Then the charge $Q(t)$ of the capacitor satisfies the differential equation

$$RQ'(t) + \frac{Q(t)}{C} = V_0$$

This is the equation for an RC charging circuit.

Find the explicit solution of this equation with $Q(0) = 0$. Explain why the product RC is important in determining the charging time. For $R = 10^3\,\Omega$, $V_0 = 1\,\mathrm{V}$, $C = 1\,\mu\mathrm{F}$, how much time does it take for the capacitor to be charged to 98%? (You may use a calculator.)

Proof. We can evaluate the ODE as follows.

$$\begin{aligned}
\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{1}{RC}Q &= V_0 \\
\mathrm{e}^{t/RC}\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{1}{RC}\mathrm{e}^{t/RC}Q &= \mathrm{e}^{t/RC}V_0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(Q\mathrm{e}^{t/RC}\right) &= \mathrm{e}^{t/RC}V_0 \\
Q\mathrm{e}^{t/RC} &= RCV_0\mathrm{e}^{t/RC} + C_1 \\
Q(t) &= RCV_0 + C_1\mathrm{e}^{-t/RC}
\end{aligned}$$

We now apply the initial condition.

$$\begin{aligned}
0 &= Q(0) \\
&= RCV_0 + C_1 \\
C_1 &= -RCV_0
\end{aligned}$$

Therefore, the solution to the ODE in question is

$$\boxed{Q(t) = RCV_0\left(1 - \mathrm{e}^{-t/RC}\right)}$$

The product RC (technically referred to as the time constant) is important in determining charging time because it is directly proportional to the rate of exponential charging. Indeed, if RC doubles, the capacitor will take twice as long to charge (and vice versa, for example, if RC halves).

The amount of time it takes for the capacitor to charge to 98% under the given conditions ($R = 10^3\,\Omega$ and $C = 10^{-6}\,\mathrm{F}$) may be determined as follows.

$$\begin{aligned}
0.98 &= 1 - \mathrm{e}^{-t/RC} \\
t &= -RC\ln(0.02) \\
\Aboxed{t &= 3.9 \times 10^{-3}\,\mathrm{s}}
\end{aligned}$$

□

10. A parachutist is falling from a plane. Suppose the parachute is opened at height H, when the falling velocity is v_0. Suppose that the air resistance exerted on the parachute is proportional to the square of the velocity with ratio η. Let the gravitational constant be g, and suppose that the total mass of the parachutist and the parachute is m. Write down the differential equation satisfied by the shift x, together with the initial conditions. Solve this IVP. What is the velocity as $t \to +\infty$? Can you derive the final velocity based on physical considerations?

Proof. For the sake of simplicity, we will write a one-dimensional differential equation corresponding to vertical displacement. Let's begin.

When the parachutist is falling freely, there is only one (idealized) force acting on them: gravity (F_g). As soon as the parachute is opened, another force is added to the mix: drag (F_d). By Newton's second law, the net force is equal to the parachutist/parachute's mass times their acceleration. Taking a convention of upwards displacement being positive, we can thus write that

$$\sum F_z = F_d - F_g = ma$$

Since $a = x''$, $F_g = mg$, and $F_d = \eta v^2 = \eta(x')^2$, the differential equation satisfied by the shift x is

$$mx'' = \eta(x')^2 - mg$$

Let the time at which the parachute is opened be $t = 0$. Then the initial conditions are

$$x(0) = H \qquad\qquad x'(0) = v_0$$

To solve this IVP, we substitute $v = x'$ and evaluate the resulting first-order differential equation to start:

$$\begin{aligned}
mv' &= \eta v^2 - mg \\
\frac{\mathrm{d}v}{v^2 - mg/\eta} &= \frac{\eta}{m}\,\mathrm{d}t \\
\int_{v_0}^{v} \frac{\mathrm{d}w}{w^2 - mg/\eta} &= \int_0^t \frac{\eta}{m}\,\mathrm{d}\tau \\
\sqrt{\frac{\eta}{mg}}\left[\coth^{-1}\left(\sqrt{\frac{\eta}{mg}}v_0\right) - \coth^{-1}\left(\sqrt{\frac{\eta}{mg}}v\right)\right] &= \frac{\eta}{m}t \\
v(t) &= \sqrt{\frac{mg}{\eta}}\coth\left(\coth^{-1}\left(\sqrt{\frac{\eta}{mg}}v_0\right) - \sqrt{\frac{g\eta}{m}}t\right)
\end{aligned}$$

Returning the substitution $v = x'$, we can determine that

$$\begin{aligned}
x' &= \sqrt{\frac{mg}{\eta}}\coth\left(\coth^{-1}\left(\sqrt{\frac{\eta}{mg}}v_0\right) - \sqrt{\frac{g\eta}{m}}t\right) \\
\int_H^x \mathrm{d}\xi &= \int_0^t \sqrt{\frac{mg}{\eta}}\coth\left(\coth^{-1}\left(\sqrt{\frac{\eta}{mg}}v_0\right) - \sqrt{\frac{g\eta}{m}}\tau\right)\mathrm{d}\tau \\
x - H &= \left[-\frac{m}{\eta}\ln\left(\sinh\left(\coth^{-1}\left(\sqrt{\frac{\eta}{mg}}v_0\right) - \sqrt{\frac{g\eta}{m}}\tau\right)\right)\right]_0^t \\
&\boxed{x(t) = H - \frac{m}{\eta}\ln\left(\sinh\left(\coth^{-1}\left(\sqrt{\frac{\eta}{mg}}v_0\right) - \sqrt{\frac{g\eta}{m}}t\right)\right) + \frac{m}{\eta}\ln\left(\frac{1}{v_0\cdot\sqrt{\frac{\eta}{mg} - \frac{1}{v_0^2}}}\right)}
\end{aligned}$$

Since $\coth(t) \to -1$ as $t \to -\infty$, the final velocity approaches

$$\boxed{v_\infty = -\sqrt{\frac{mg}{\eta}}}$$

Note that we can also solve for v_∞ using physical considerations; at $t = \infty$, the drag and gravitational forces will balance, i.e., we will have

$$\begin{aligned}
F_d &= F_g \\
\eta v_\infty^2 &= mg \\
v_\infty &= \pm\sqrt{\frac{mg}{\eta}} \\
&= -\sqrt{\frac{mg}{\eta}}
\end{aligned}$$

□

Bonus Problems

1. **The Catenoid.** Suppose there are two metal rings of radius a placed parallel to each other in an xyz-coordinate space, with the x-axis passing through their centers. Suppose these two rings are contained in the planes $x = l$ and $x = -l$, respectively. An axial symmetric soap film is spanned by these two rings. Suppose its shape is obtained by rotating the graph of the function $y = y(x)$ with respect to the x-axis. In order to attain a stable configuration, the surface area is supposed to be minimal among all such surfaces of revolution.

   (1) Write down the surface area functional in terms of $y(x)$, its derivative, and the boundary conditions for this variational problem.

   Proof. The surface area functional is[3]

   $$\boxed{J[y] = \int_{-l}^{l} 2\pi y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x}$$

   □

   (2) Derive the Euler-Lagrange equation and find the solution. The shape is called a **catenoid**.

[3] See SOR-SA.pdf from AP BC Calculus for a derivation.

Proof. As in class with the Brachistochrone problem, our functional is of the form $J[y] = \int_a^b F(x, y(x), y'(x))\,\mathrm{d}x$. Thus, the derivation of the relevant Euler-Lagrange equation is symmetric, and yields

$$\frac{\partial F}{\partial z}(x, y(x), y'(x)) - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial F}{\partial w}(x, y(x), y'(x))\right] = 0$$

as a necessary condition for y to be an extrema of $J[y]$. Again, note that in our specific functional, $F(x, z, w) = F(z, w)$ (i.e., there is no dependence on x), allowing us to actually solve the above equation. We can take this even farther using the same logic from class to get

$$F - \frac{\mathrm{d}F}{\mathrm{d}w}\frac{\mathrm{d}y}{\mathrm{d}x} = A$$

as a necessary condition, where $A \in \mathbb{R}$ depends on the initial/boundary conditions. In particular, we may now substitute in our specific definition of F and solve for y as follows.

$$\begin{aligned}
A &= 2\pi y\sqrt{1 + (y')^2} - \left[2\pi y \cdot \frac{1}{2\sqrt{1 + (y')^2}} \cdot 2y'\right] \cdot y' \\
\frac{A}{2\pi y} &= \sqrt{1 + (y')^2} - \frac{(y')^2}{\sqrt{1 + (y')^2}} \\
\frac{A^2}{4\pi^2 y^2} &= [1 + (y')^2] - \frac{2(y')^2\sqrt{1 + (y')^2}}{\sqrt{1 + (y')^2}} + \frac{(y')^4}{1 + (y')^2} \\
\frac{A^2}{4\pi^2 y^2} &= 1 - (y')^2 + \frac{(y')^4}{1 + (y')^2} \\
\frac{A^2}{4\pi^2 y^2} + \frac{A^2}{4\pi^2 y^2}(y')^2 &= [1 + (y')^2] - [(y')^2 + (y')^4] + (y')^4 \\
\frac{A^2}{4\pi^2 y^2} + \frac{A^2}{4\pi^2 y^2}(y')^2 &= 1 \\
(y')^2 &= \frac{4\pi^2 y^2}{A^2} - 1 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2\pi}{A}\sqrt{y^2 - \frac{A^2}{4\pi^2}} \\
y(x) &= \cosh\left(\frac{2\pi}{A}x\right)
\end{aligned}$$

Knowing that $y(l) = y(-l) = a$, we can determine that

$$\begin{aligned}
a &= \cosh\left(\frac{2\pi l}{A}\right) \\
A &= \frac{2\pi l}{\cosh^{-1}(a)}
\end{aligned}$$

Therefore, our final solution is

$$\boxed{y(x) = \cosh\left(\frac{\cosh^{-1}(a)}{l}x\right)}$$

Note that since we can easily construct surfaces of revolution with greater surface area, this is a minimum, not a maximum. ☐

(3) If the two rings are very far away from each other, i.e., l is very large, will the catenoid still be of minimal area among all competing surfaces that span these two rings? You do not have to give a mathematically rigorous answer; just imagine the physical situation. (Hint: What about two distinct disks spanned by these two rings?)

Proof. Imagine pulling two rings spanned by a soap film farther and farther apart. The soap film won't be stable forever; indeed, at some point, it will collapse in the middle and the two halves will retreat onto their distinct rings. Indeed, at some point, the area $2\pi a^2$ will be smaller than the ever-increasing (and non-asymptotic) surface area of the catenoid. Thus, the catenoid is a local minimum of the variational problem, not necessarily a global minimum. ☐

2. **A Formulation of the Isoperimetric Problem.** Recall from multivariable calculus that in order to find a local extremum of the function $f(x_1,\dots,x_n)$ under the constraint $g(x_1,\dots,x_n)=0$, we can introduce a parameter λ called the **Lagrange multiplier** and find the stationary point of the function

$$f(x_1,\dots,x_n)-\lambda g(x_1,\dots,x_n)$$

(1) Write down the equations that must be satisfied by the stationary point $(x_1,\dots,x_n)$ of the function $f-\lambda g$ with the parameter λ involved.

Proof. Based on my notes, I should be able to solve this problem with ease. The method of Lagrange multiplies is explained in Labalme (n.d.). ☐

(2) Use the Lagrange multiplier method to find the maxima and minima of $f(x,y)=x+y$ under the constraint $x^2+y^2=1$.

(3) Now let us generalize this method to functionals. If we aim to find the extrema of a functional

$$J[y]=\int_a^b F(x,y(x),y'(x))\,\mathrm{d}x$$

under the constraint

$$R[y]=\int_a^b G(x,y(x),y'(x))\,\mathrm{d}x=0$$

where $F(x,z,w)$ and $G(x,z,w)$ are known functions, we can try to find the extrema of the functional

$$J[y]-\lambda R[y]$$

first. What is the Euler-Lagrange equation satisfied by this extrema (with λ involved)?

(4) Now let us consider a version of the isoperimetric problem. We aim to find the function $y(x)$, whose graph connects two given points (a,A), (b,B) on the xy-plane, with a prescribed arclength

$$l=\int_a^b \sqrt{1+|y'(x)|^2}\,\mathrm{d}x$$

such that the area between the graph and the x-axis is the largest. The functional in consideration is

$$J[y]=\int_a^b y(x)\,\mathrm{d}x$$

with constraint

$$R[y]=\int_a^b \sqrt{1+|y'(x)|^2}\,\mathrm{d}x=l$$

Write down the Euler-Lagrange equation involving the multiplier λ and show that the solution must be a part of a circle.

2 Linear Algebra

Required Problems

10/19: **1.** This question helps to complete the computations omitted in class. In deriving the Kepler orbits for the two-body problem, we have successfully reduced the differential equation satisfied by the curve $r = r(\varphi)$ to

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 + r^2 = \frac{2GMr^3}{l_0^2} + \frac{2Er^4}{ml_0^2}$$

Show that the function $\mu = 1/r$ satisfies the differential equation

$$\left(\frac{\mathrm{d}\mu}{\mathrm{d}\varphi}\right)^2 + \mu^2 = \frac{2GM\mu}{l_0^2} + \frac{2E}{ml_0^2}$$

By differentiating with respect to φ again, this reduces to either $\mathrm{d}\mu/\mathrm{d}\varphi = 0$ or

$$\frac{\mathrm{d}^2\mu}{\mathrm{d}\varphi^2} + \mu - \frac{GM}{l_0^2} = 0$$

Find the general solution of the latter, hence conclude that $r = r(\varphi)$ represents a conic section. *Hint*: There is a very obvious particular solution.

Proof. We begin from the first differential equation and substitute $\mu = 1/r$ in the last step to yield the desired result.

$$\begin{aligned}
\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 + r^2 &= \frac{2GMr^3}{l_0^2} + \frac{2Er^4}{ml_0^2} \\
\left(-\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 + \frac{1}{r^2} &= \frac{2GM}{l_0^2}\frac{1}{r} + \frac{2E}{ml_0^2} \\
\left[\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{1}{r}\right)\right]^2 + \left(\frac{1}{r}\right)^2 &= \frac{2GM}{l_0^2}\frac{1}{r} + \frac{2E}{ml_0^2} \\
\left(\frac{\mathrm{d}\mu}{\mathrm{d}\varphi}\right)^2 + \mu^2 &= \frac{2GM\mu}{l_0^2} + \frac{2E}{ml_0^2}
\end{aligned}$$

The homogeneous version of the final differential equation is entirely analogous to the harmonic oscillator problem and thus has general (real) solution

$$\mu(\varphi) = \epsilon\cos(\varphi - \varphi_0)$$

for $\epsilon, \varphi_0 \in \mathbb{R}$. By inspection, we can take as our particular solution to the inhomogeneous system

$$\mu(\varphi) = \frac{GM}{l_0^2}$$

since it's second derivative (as a constant) is zero and it is the opposite of the inhomogeneous term. Thus, the general solution to the original inhomogeneous system is

$$\begin{aligned}
\mu(\varphi) &= \frac{GM}{l_0^2} + \epsilon\cos(\varphi - \varphi_0) \\
r(\varphi) &= \frac{1}{GM/l_0^2 + \epsilon\cos(\varphi - \varphi_0)} \\
&= \frac{\epsilon(l_0^2/GM\epsilon)}{1 + \epsilon\cos(\varphi - \varphi_0)}
\end{aligned}$$

which is exactly the polar form of the conic section with eccentricity ϵ and directrix $l_0^2/GM\epsilon$. $\square$

2. The general formula for the inverse of an $n \times n$ invertible matrix is very lengthy. However, for a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfying $ad - bc \neq 0$, there is a very simple formula. Try to find it; this could be very helpful if you can remember it.

Proof. Let A be the matrix given in the problem statement. We can determine A^{-1} by inspection as follows.

Let's focus on the right column of A^{-1} first, which we can denote $(x, y)^T$. We want $ax + by = 0$. One nice solution to this equation is $x = -b$ and $y = a$. Similarly, we can take the left column of A^{-1} to be $(d, -c)^T$. This choice of entries for A^{-1} yield the 0s in the right places, but the elements that should be 1 are instead $\det A = ad - bc$. Thus, we divide A^{-1} by $\det A$. This yields the following final formula

$$\boxed{A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

As a quick check, we have that

$$\begin{aligned} AA^{-1} &= \frac{1}{ad - bc}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \frac{1}{ad - bc}\begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \qquad \begin{aligned} A^{-1}A &= \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \frac{1}{ad - bc}\begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

as expected. □

3. Compute the determinant of the following matrices. Determine whether they are invertible or not.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 2 & 3 & 6 \\ 1 & 3 & 4 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} -1 & 2 & 1 \\ 3 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$$

Proof. We have that

$$\begin{aligned} \det A &= 1[5 \cdot 9 - 6 \cdot 8] - 2[4 \cdot 9 - 6 \cdot 7] + 3[4 \cdot 8 - 5 \cdot 7] \\ &\boxed{\det A = 0} \end{aligned}$$

so $\boxed{A \text{ is not invertible.}}$

Since B is block upper triangular, we know that

$$\begin{aligned} \det B &= \det B_1 \cdot \det B_2 \\ &= [2 \cdot 3 - 2 \cdot 1] \cdot [-1 \cdot 2 - 2 \cdot 1] \\ &\boxed{\det B = -16} \end{aligned}$$

so $\boxed{B \text{ is invertible.}}$

We have that

$$\begin{aligned} \det C &= -1[(-1)(3) - (2)(1)] - 2[(3)(3) - (2)(2)] + 1[(3)(1) - (-1)(2)] \\ &\boxed{\det C = 0} \end{aligned}$$

so $\boxed{C \text{ is not invertible.}}$ □

4. Determine whether the following linear systems admit solution(s); if they do, write down the solution (or the formula for the general solution).

(1)

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Proof. By inspection, A is a dimension 2 matrix of rank 2, so it $\boxed{\text{admits a unique solution.}}$ We now row-reducing the augmented matrix.

$$\left(\begin{array}{cc|c} 1 & 2 & -1 \\ 2 & -1 & 1 \end{array}\right) \cong \left(\begin{array}{cc|c} 1 & 0 & \frac{1}{5} \\ 0 & 1 & -\frac{3}{5} \end{array}\right)$$

Therefore, the solution is

$$\boxed{x = \begin{pmatrix} \frac{1}{5} \\ -\frac{3}{5} \end{pmatrix}}$$

□

(2)

$$\begin{pmatrix} -1 & 2 & 1 \\ 3 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Proof. By inspection, A is a dimension 3 matrix of rank 2 and the b vector is in the column space of A, so it $\boxed{\text{admits a family of solutions.}}$ We now row-reducing the augmented matrix.

$$\left(\begin{array}{ccc|c} -1 & 2 & 1 & 1 \\ 3 & -1 & 2 & 2 \\ 2 & 1 & 3 & 3 \end{array}\right) \cong \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore, the family of solutions is given by

$$\boxed{x = \begin{pmatrix} 1 - x^3 \\ 1 - x^3 \\ x^3 \end{pmatrix}}$$

for $x^3 \in \mathbb{R}$. □

(3)

$$\begin{pmatrix} -1 & 2 & 1 \\ 3 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Proof. No promising solution immediately appears by inspection, so we row reduce and evaluate the results.

$$\left(\begin{array}{ccc|c} -1 & 2 & 1 & 1 \\ 3 & -1 & 2 & 0 \\ 2 & 1 & 3 & 1 \end{array}\right) \cong \left(\begin{array}{ccc|c} 1 & 0 & 1 & \frac{1}{5} \\ 0 & 1 & 1 & \frac{3}{5} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

It follows that A $\boxed{\text{admits a family of solutions.}}$ In particular, these are given by

$$\boxed{x = \begin{pmatrix} \frac{1}{5} - x^3 \\ \frac{3}{5} - x^3 \\ x^3 \end{pmatrix}}$$

for $x^3 \in \mathbb{R}$. □

5. Find the connecting matrix from the basis $\begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix}$ to the new basis $\begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}$, where

$$\begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix} \qquad \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

That is, represent q_1, q_2, q_3 as linear combinations of p_1, p_2, p_3.

Proof. P is the connecting matrix from the standard basis (e_1, e_2, e_3) to (p_1, p_2, p_3). Likewise, Q is the connecting matrix from (e_1, e_2, e_3) to (q_1, q_2, q_3). It follows that if we want A to be the connecting matrix from (p_1, p_2, p_3) to (q_1, q_2, q_3), then we can do the transformation stepwise, i.e., take a vector represented in (p_1, p_2, p_3) to its representation in (e_1, e_2, e_3) using P^{-1} and then to its representation in (q_1, q_2, q_3) using Q. Indeed, the desired connecting matrix is

$$A = QP^{-1}$$

$$\boxed{A = \frac{1}{5}\begin{pmatrix} -2 & 2 & -1 \\ 5 & 0 & 5 \\ -1 & 1 & 2 \end{pmatrix}}$$

Direct computation can confirm that $Ap_i = q_i$ for $i = 1, 2, 3$.

With respect to representing q_1, q_2, q_3 as linear combinations of p_1, p_2, p_3, we can solve the equations $q_i = Px_i$ for $i = 1, 2, 3$ via row reduction, as in previous responses. The final expressions obtained are

$$\boxed{q_1 = \frac{1}{5}(p_1 + 2p_2 + p_3) \qquad q_2 = \frac{1}{5}(3p_1 - 4p_2 - 2p_3) \qquad q_3 = \frac{1}{5}(3p_1 + p_2 + 3p_3)}$$

Note that if we combine the coefficients above into a matrix X such that $PX = Q$, then $A = PXP^{-1} = QXQ^{-1}$. ☐

6. Let $\theta \in [0, 2\pi)$. The rotation through angle θ in the plane is represented by the matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Compute its determinant, characteristic polynomial, and eigenvalues. Compute its eigenvectors in $\mathbb{C}^2$. You need to use the Euler formula $\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$. For two angles θ, φ, compute the product $R(\theta)R(\varphi)$ and represent it in terms of $\theta + \varphi$. What is the geometric meaning of this equality?

Proof. The determinant of R is

$$\det R = \cos^2\theta + \sin^2\theta$$

$$\boxed{\det R = 1}$$

The characteristic polynomial of R is

$$\begin{aligned} \chi_R(z) &= \det(R - zI) \\ &= (\cos\theta - z)^2 + \sin^2\theta \\ &= z^2 - 2z\cos\theta + \cos\theta^2 + \sin\theta^2 \end{aligned}$$

$$\boxed{\chi_R(z) = z^2 - 2z\cos\theta + 1}$$

The eigenvalues of R are

$$\begin{aligned}
0 &= \chi_R(\lambda) \\
&= (\cos\theta - \lambda)^2 + \sin^2\theta \\
-\sin^2\theta &= (\cos\theta - \lambda)^2 \\
\pm i\sin\theta &= \pm(\cos\theta - \lambda) \\
\lambda &= \cos\theta \pm i\sin\theta \\
&\boxed{\lambda = \mathrm{e}^{\pm i\theta}}
\end{aligned}$$

It follows by solving the systems of equations

$$\begin{aligned}
x^1\cos\theta - x^2\sin\theta &= \mathrm{e}^{i\theta}x^1 & \qquad y^1\cos\theta - y^2\sin\theta &= \mathrm{e}^{-i\theta}y^1 \\
x^1\sin\theta + x^2\cos\theta &= \mathrm{e}^{i\theta}x^2 & \qquad y^1\sin\theta + y^2\cos\theta &= \mathrm{e}^{-i\theta}y^2
\end{aligned}$$

that the eigenvectors are

$$\boxed{x = \begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad\qquad y = \begin{pmatrix} 1 \\ i \end{pmatrix}}$$

The product $R(\theta)R(\varphi)$ may be computed as follows.

$$\begin{aligned}
R(\theta)R(\phi) &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta\cos\varphi - \sin\theta\sin\varphi & -\cos\theta\sin\varphi - \sin\theta\cos\varphi \\ \sin\theta\cos\varphi + \cos\theta\sin\varphi & -\sin\theta\sin\varphi + \cos\theta\cos\varphi \end{pmatrix} \\
&= \begin{pmatrix} \cos(\theta+\varphi) & -\sin(\theta+\varphi) \\ \sin(\theta+\varphi) & \cos(\theta+\varphi) \end{pmatrix} \\
&\boxed{R(\theta)R(\varphi) = R(\theta+\phi)}
\end{aligned}$$

The geometric meaning is that rotating through an angle θ and then through an additional angle φ is the same as rotating through an angle $\theta + \varphi$ all at once. ☐

8. Find the algebraic and geometric multiplicities of the eigenvalues of the following matrices.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} \qquad\qquad B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

Proof. We tackle A first. A is an upper triangular matrix. Thus, $\chi_A(\lambda) = \det(A - \lambda I)$ can be read directly off of the diagonal:

$$\chi_A(\lambda) = (1-\lambda)^2(3-\lambda)$$

Thus, the eigenvalues are $\lambda = 1, 3$ with respective algebraic multiplicities

$$\boxed{\alpha_1 = 2 \qquad\qquad \alpha_3 = 1}$$

It follows immediately that

$$\boxed{\gamma_3 = 1}$$

and from the observation that $A - 1I$ has 2 linearly independent columns that this 3×3 matrix has a $3 - 2 = 1$ dimensional null space, i.e.,

$$\boxed{\gamma_1 = 1}$$

The procedure for B is almost entirely symmetric. Once again, B is upper triangular, so

$$\chi_B(\lambda) = (1-\lambda)^2(3-\lambda)$$

implying that

$$\boxed{\alpha_1 = 2 \qquad\qquad \alpha_3 = 1}$$

There is a difference with respect to the geometric multiplicities, however. We still have

$$\boxed{\gamma_3 = 1}$$

but since $A - I$ now has only 1 linearly independent column, we have

$$\boxed{\gamma_1 = 2}$$

this time. □

9. Compute the Jordan normal form of the following 2×2 matrices.

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad\qquad B = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}$$

Notice that you not only need to find all the Jordan blocks, but also need to find the Jordan basis matrix Q such that $Q^{-1}AQ$ is in Jordan normal form.

Proof. We tackle A first.

Calculate the characteristic polynomial to begin.

$$\begin{aligned} \chi_A(z) &= \det(A - zI) \\ &= z^2 - 4z + 3 \\ &= (1-z)(3-z) \end{aligned}$$

It follows that the eigenvalues are

$$\lambda_1 = 1 \qquad\qquad \lambda_2 = 3$$

Since these eigenvalues are distinct, we can fully diagonalize this matrix. Indeed, we can determine by inspection that suitable corresponding eigenvectors are

$$v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad\qquad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Therefore,

$$\boxed{Q = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \qquad\qquad Q^{-1}AQ = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}}$$

The procedure for B is very much analogous to the procedure for A.

Characteristic polynomial:

$$\begin{aligned} \chi_B(z) &= \det(B - zI) \\ &= z^2 + 2z + 1 \\ &= (1+z)^2 \end{aligned}$$

Eigenvalue:

$$\lambda = -1$$

By inspection of $B+I$, we can pick one eigenvector of B:

$$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We now solve $(B+I)u = v$. By inspection, this yields

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Therefore,

$$\boxed{Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad\qquad Q^{-1}BQ = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}}$$

□

10. Compute the Jordan normal form of the following 3×3 matrices.

$$A = \begin{pmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -1 & -1 \\ 2 & -1 & -2 \\ -1 & 1 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

Notice that you not only need to find all the Jordan blocks, but also need to find the Jordan basis matrix Q such that $Q^{-1}AQ$ is in Jordan normal form. *Hint*: These three matrices represent three different possibilities of nondiagonalizable Jordan normal forms of a 3×3 matrix: A reduces to $(2 \times 2) \oplus (1 \times 1)$ Jordan blocks with different eigenvalues, B reduces to $(2 \times 2) \oplus (1 \times 1)$ Jordan blocks with the same eigenvalue, and C reduces to a 3×3 Jordan block.

Proof. We tackle A first.

Calculate the characteristic polynomial to begin.

$$\begin{aligned} \chi_A(z) &= \det(A - zI) \\ &= -z^3 + z^2 \\ &= z^2(1-z) \end{aligned}$$

It follows that the eigenvalues are

$$\lambda_1 = \lambda_2 = 0 \qquad\qquad \lambda_3 = 1$$

We can solve for an eigenvector v_1 corresponding to $\lambda_1 = \lambda_2 = 0$ using the augmented matrix and row reduction as follows.

$$\left(\begin{array}{ccc|c} 4 & -5 & 2 & 0 \\ 5 & -7 & 3 & 0 \\ 6 & -9 & 4 & 0 \end{array}\right) \cong \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus, if we choose $v_1^3 = 3$, then the desired eigenvector is

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Similarly, we can solve for an eigenvector v_3 corresponding to $\lambda_3 = 1$ using the following. Note that to solve $Ax = 1x$, we row-reduce $(A - I)x = 0$.

$$\left(\begin{array}{ccc|c} 3 & -5 & 2 & 0 \\ 5 & -8 & 3 & 0 \\ 6 & -9 & 3 & 0 \end{array}\right) \cong \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This yields

$$v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

We now solve the equation $(A - 0I)u = v_1$ to find a generalized eigenvector u corresponding to $\lambda_1 = \lambda_2 = 0$. This can also be done with an augmented matrix.

$$\left(\begin{array}{ccc|c} 4 & -5 & 2 & 1 \\ 5 & -7 & 3 & 2 \\ 6 & -9 & 4 & 3 \end{array}\right) \cong \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This yields

$$u = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

Therefore,

$$\boxed{Q = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 3 & 1 \end{pmatrix} \qquad Q^{-1}AQ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

The procedure for B is very much analogous to the procedure for A.

Characteristic polynomial:

$$\begin{aligned} \chi_B(z) &= \det(B - zI) \\ &= -z^3 + 3z^2 - 3z + 1 \\ &= (1 - z)^3 \end{aligned}$$

Eigenvalue:

$$\lambda = 1$$

By inspection of

$$B - I = \begin{pmatrix} 1 & -1 & -1 \\ 2 & -2 & -2 \\ -1 & 1 & 1 \end{pmatrix}$$

we can pick two eigenvectors of B corresponding to λ, i.e., two elements of the null space of the above matrix. In this subcase of the 3×3 case, we always pick the first of these to be an element of the column space of $B - I$, as well. Thus, choose

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We now solve $(B - \lambda I)u = v_1$. By inspection, this yields

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Therefore,

$$\boxed{Q = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \qquad Q^{-1}BQ = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

The procedure for C is likewise quite analogous.

The matrix is upper triangular, so the eigenvalues are on the diagonal. It follows that

$$\lambda = 2$$

is the sole eigenvalue. We can solve $(C - 2I)v = 0$ for one eigenvector v by inspection, yielding

$$v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

We can also solve $(C - 2I)u_1 = v$ by inspection to get

$$u_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

One more time, we can solve $(C - 2I)u_2 = u_1$ by inspection to get

$$u_2 = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$$

Therefore,

$$\boxed{Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{pmatrix} \qquad Q^{-1}CQ = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}}$$

□

Bonus Problems

1. You may find the characteristic root method for the second-order equation $y'' + ay' + b = 0$ quite abrupt. This problem helps you see where it comes from. The origin of this method is in fact a comparison with the linear recursive relation

$$y_{n+2} + ay_{n+1} + by_n = 0$$

where a, b are given complex numbers.

(1) The linear recursive relation $y_{n+1} + ay_n = 0$ gives rise to a geometric sequence

$$y_0, y_0(-a), y_0(-a)^2, \dots$$

We now want to try to reduce the second-order recursive relation $y_{n+2} + ay_{n+1} + by_n = 0$ to a first-order relation. Thus, we look for complex numbers λ, μ such that

$$(y_{n+2} - \lambda y_{n+1}) - \mu(y_{n+1} - \lambda y_n) = 0$$

Then λ, μ should be the roots of the characteristic polynomial

$$X^2 + aX + b$$

Taking λ, μ as known quantities, find the general formula for y_n, regarding y_0, y_1 as known quantities. *Hint*: The sequence of numbers $y_{n+1} - \lambda y_n$ is a geometric sequence with ratio μ. You should also discuss $\mu \neq \lambda$ and $\mu = \lambda$ separately.

Proof. We start with the case $\mu \neq \lambda$ here. Define a sequence of numbers $\{x_n\}$ by

$$x_n := y_{n+1} - \lambda y_n$$

We have from the above that x_n is recursively defined by

$$x_{n+1} - \mu x_n = 0 \quad \Longleftrightarrow \quad x_n = x_0 \mu^n$$

Returning the substitution, we have that

$$y_{n+1} - \lambda y_n = (y_1 - \lambda y_0)\mu^n$$

Rearranging, we have that

$$y_{n+1} = \lambda y_n + (y_1 - \lambda y_0)\mu^n$$

In other words, to generate each new term, we multiply through the previous term by λ and add to it a new term. Let's investigate the effect of this procedure on the first few terms past y_0, y_1. These terms are

$$\begin{aligned}
y_2 &= \lambda y_1 + (y_1 - \lambda y_0)\mu^1 \\
&= (\underbrace{\lambda + \mu}_{p_1(\lambda,\mu)})y_1 - \lambda\mu(\underbrace{\ 1\ }_{q_1(\lambda,\mu)})y_0 \\
y_3 &= \lambda(\lambda + \mu)y_1 - \lambda\mu \cdot \lambda y_0 + (y_1 - \lambda y_0)\mu^2 \\
&= [\lambda(\lambda + \mu) + \mu^2]y_1 - \lambda\mu \cdot [\lambda + \mu]y_0 \\
&= (\underbrace{\lambda^2 + \lambda\mu + \mu^2}_{p_2(\lambda,\mu)})y_1 - \lambda\mu(\underbrace{\lambda + \mu}_{q_2(\lambda,\mu)})y_0 \\
y_4 &= \lambda(\lambda^2 + \lambda\mu + \mu^2)y_1 - \lambda\mu \cdot \lambda(\lambda + \mu)y_0 + (y_1 - \lambda y_0)\mu^3 \\
&= [\lambda(\lambda^2 + \lambda\mu + \mu^2) + \mu^3]y_1 - \lambda\mu \cdot [\lambda(\lambda + \mu) + \mu^2]y_0 \\
&= (\underbrace{\lambda^3 + \lambda^2\mu + \lambda\mu^2 + \mu^3}_{p_3(\lambda,\mu)})y_1 - \lambda\mu(\underbrace{\lambda^2 + \lambda\mu + \mu^2}_{q_3(\lambda,\mu)})y_0
\end{aligned}$$

There are a number of interesting things to note here. First of all, see that the first line defining each term comes straight from the recursive relation (i.e., is λ times the previous term plus an auxiliary term). The second line is always a rewrite that combines all of the y_1 terms and all of the y_0 terms. The third line (if it exists; the second line for y_2 serves this purpose as well) is always a final simplification in terms of polynomials of λ and μ. We have taken the liberty of naming each of these final polynomials for purposes that will soon be revealed.

Notice that the polynomials appear to be defined by the recursive relations[4]

$$p_n = \lambda p_{n-1} + \mu^n \qquad\qquad q_n = \lambda q_{n-1} + \mu^{n-1}$$

Also note that we factor $-\lambda\mu$ out of the y_0 term every time so that we have

$$q_n = p_{n-1}$$

These polynomials have a very well-defined form; specifically,

$$q_{n+1} = p_n = \sum_{i=0}^{n} \lambda^{n-i}\mu^i = \frac{\lambda^{n+1} - \mu^{n+1}}{\lambda - \mu}$$

This allows us to explicitly define the following general formula for y_n in the $\mu \neq \lambda$ case.

$$\boxed{y_n = \frac{\lambda^n - \mu^n}{\lambda - \mu} y_1 - \lambda\mu \frac{\lambda^{n-1} - \mu^{n-1}}{\lambda - \mu} y_0}$$

I believe that this is along the lines of what Shao wanted us to do.

[4] We can rigorously prove this via induction. Also note the familiar theme of "multiply through by λ and add a term."

Note, however, that there exists an alternate, much more logical way to derive this formula using linear algebra. In particular, we have from the recursive relation and the trivial reflexive equality $y_1 = y_1$ that

$$\begin{aligned}\begin{pmatrix} y_2 \\ y_1 \end{pmatrix} &= \begin{pmatrix} -ay_1 - by_0 \\ y_1 \end{pmatrix} \\ &= \begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_0 \end{pmatrix}\end{aligned}$$

It follows that

$$\begin{pmatrix} y^{n+1} \\ y^n \end{pmatrix} = \begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} y_1 \\ y_0 \end{pmatrix}$$

Diagonalizing the matrix raised to the n^{th} power above (which we may call A) requires finding the roots of the (of course similarly named) characteristic polynomial

$$\begin{aligned}0 &= \chi_A(z) \\ &= (-a - z)(-z) + b \\ &= z^2 + az + b\end{aligned}$$

But we have from the problem statement that the roots of this polynomial are the same λ, μ we've been working with this whole time. Thus, we have a new interpretation of these roots: As *eigenvalues*. And to each of these distinct eigenvalues corresponds a distinct eigenvector, which we may find by inspection as follows. Consider λ WLOG. The eigenvector v corresponding to λ must satisfy

$$\begin{pmatrix} -a - \lambda & -b \\ 1 & -\lambda \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

If we choose $v^1 = \lambda$ and $v^2 = 1$, then we of course have

$$(1)(v^1) + (-\lambda)(v^2) = \lambda - \lambda = 0$$

But we actually also have

$$(-a - \lambda)(v^1) + (-b)(1) = -\lambda^2 - a\lambda - b = -(\lambda^2 + a\lambda + b) = -0 = 0$$

since λ is a root of the characteristic polynomial! Thus, the eigenvectors corresponding to λ, μ are

$$\begin{pmatrix} \lambda \\ 1 \end{pmatrix}, \begin{pmatrix} \mu \\ 1 \end{pmatrix}$$

respectively. Thus, with what we have so far plus an assist from Problem 2, we can diagonalize A to

$$\begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix} = \frac{1}{\lambda - \mu} \begin{pmatrix} \lambda & \mu \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} 1 & -\mu \\ -1 & \lambda \end{pmatrix}$$

It follows that

$$\begin{aligned}\begin{pmatrix} y^{n+1} \\ y^n \end{pmatrix} &= \begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} y_1 \\ y_0 \end{pmatrix} \\ &= \frac{1}{\lambda - \mu} \begin{pmatrix} \lambda & \mu \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}^n \begin{pmatrix} 1 & -\mu \\ -1 & \lambda \end{pmatrix} \begin{pmatrix} y_1 \\ y_0 \end{pmatrix} \\ &= \frac{1}{\lambda - \mu} \begin{pmatrix} \lambda & \mu \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{pmatrix} \begin{pmatrix} y_1 - \mu y_0 \\ -(y_1 - \lambda y_0) \end{pmatrix} \\ &= \frac{1}{\lambda - \mu} \begin{pmatrix} \lambda & \mu \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda^n(y_1 - \mu y_0) \\ -\mu^n(y_1 - \lambda y_0) \end{pmatrix} \\ &= \frac{1}{\lambda - \mu} \begin{pmatrix} \lambda^{n+1}(y_1 - \mu y_0) - \mu^{n+1}(y_1 - \lambda y_0) \\ \lambda^n(y_1 - \mu y_0) - \mu^n(y_1 - \lambda y_0) \end{pmatrix}\end{aligned}$$

From the above, we can read that

$$\begin{aligned} y_n &= \frac{1}{\lambda-\mu}[\lambda^n(y_1-\mu y_0)-\mu^n(y_1-\lambda y_0)] \\ &= \frac{\lambda^n-\mu^n}{\lambda-\mu}y_1 - \frac{\lambda^n\mu-\lambda\mu^n}{\lambda-\mu}y_0 \\ &= \frac{\lambda^n-\mu^n}{\lambda-\mu}y_1 - \lambda\mu\frac{\lambda^{n-1}-\mu^{n-1}}{\lambda-\mu}y_0 \end{aligned}$$

in agreement with the original answer.

We now treat the case where $\mu=\lambda$. We will do this using linear algebra, alone, since this is a far more rigorous and overall superior method. That being said, a purely sequence-based method would also work here. Let's begin.

The setup proceeds as before. However, this time we only have one distinct eigenvalue. Moreover, this eigenvalue only corresponds to one eigenvector since

$$\begin{pmatrix} -a-\mu & -b \\ 1 & -\mu \end{pmatrix}$$

is a rank 1 matrix, not a rank 0 matrix. Thus, we must delve into JNF and generalized eigenvectors. In particular, we can still choose

$$\begin{pmatrix} \mu \\ 1 \end{pmatrix}$$

as an eigenvector, but we must solve

$$\begin{pmatrix} -a-\mu & -b \\ 1 & -\mu \end{pmatrix}\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \begin{pmatrix} \mu \\ 1 \end{pmatrix}$$

to obtain the following additional *generalized* eigenvector.

$$\begin{pmatrix} -1 \\ -2/\mu \end{pmatrix}$$

Therefore, with what we have so far, we can diagonalize A to

$$\begin{pmatrix} -a & -b \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \mu & -1 \\ 1 & -2/\mu \end{pmatrix}\begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix}\begin{pmatrix} 2/\mu & -1 \\ 1 & -\mu \end{pmatrix}$$

It follows that

$$\begin{aligned} \begin{pmatrix} y^{n+1} \\ y^n \end{pmatrix} &= \begin{pmatrix} \mu & -1 \\ 1 & -2/\mu \end{pmatrix}\begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix}^n\begin{pmatrix} 2/\mu & -1 \\ 1 & -\mu \end{pmatrix}\begin{pmatrix} y_1 \\ y_0 \end{pmatrix} \\ &= \begin{pmatrix} \mu & -1 \\ 1 & -2/\mu \end{pmatrix}\begin{pmatrix} \mu^n & n\mu^{n-1} \\ 0 & \mu^n \end{pmatrix}\begin{pmatrix} 2y_1/\mu - y_0 \\ y_1-\mu y_0 \end{pmatrix} \\ &= \begin{pmatrix} \mu & -1 \\ 1 & -2/\mu \end{pmatrix}\begin{pmatrix} (n+2)\mu^{n-1}y_1-(n+1)\mu^n y_0 \\ \mu^n y_1 - \mu^{n+1}y_0 \end{pmatrix} \\ &= \begin{pmatrix} (n+1)\mu^n y_1 - n\mu^{n+1}y_0 \\ n\mu^{n-1}y_1-(n-1)\mu^n y_0 \end{pmatrix} \end{aligned}$$

From the above, we can read off the following general formula for y_n in the $\mu=\lambda$ case.

$$\boxed{y_n = n\mu^{n-1}y_1-(n-1)\mu^n y_0}$$

□

(2) Use the method of part (1) to find the general formula for the linear discursive relation

$$y_{n+2} - 2y_{n+1} + y_n = 0$$

Use the same method to find the general formula for the Fibonacci sequence

$$F_{n+2} = F_{n+1} + F_n$$

Proof. The first part is plug and chug; the Fibonacci sequence I covered in LinAlgNotes, Chapter 6.

Last note: Somebody please explain to me how any of this makes the ODE-solving method make any more sense. I mean they're clearly related, but there's no Duhamel formula for sequences that I'm aware of! □

2. In this exercise, we aim to prove an important theorem in linear algebra:

Complex Hermitian matrices are always diagonalizable.

Here the term "Hermitian" means that the matrix equals its conjugate transpose. In terms of entries, this means that in general, $a_{ij} = \bar{a}_{ji}$. For example,

$$\begin{pmatrix} 2 & 1 & -i \\ 1 & 3 & -2i \\ i & 2i & 1 \end{pmatrix}$$

is Hermitian.

(1) Let $\langle \cdot, \cdot \rangle$ be the standard Hermitian inner product, that is, for $x, y \in \mathbb{C}^n$,

$$\langle x, y \rangle = \sum_{j=1}^{n} x^j \bar{y}^j$$

Show that for any $n \times n$ real matrix,

$$\langle Ax, y \rangle = \langle x, A^* y \rangle$$

for any $x, y \in \mathbb{C}^n$, where A^* denotes the conjugate transpose of A. For example,

$$A = \begin{pmatrix} 1 & 1 & 2i \\ 0 & 3+i & 3 \\ 2 & 0 & 1 \end{pmatrix} \iff A^* = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 3-i & 0 \\ -2i & 3 & 1 \end{pmatrix}$$

Proof. I almost surely have this all written up somewhere in LinAlgNotes, LADRNotes, or MATH20700Notes. □

(2) Suppose now that A is Hermitian. Use part (1) to show that any eigenvalue of A must be a real number. Show further that if x, y are eigenvectors corresponding to different eigenvalues, then $\langle x, y \rangle = 0$, that is, x is orthogonal to y.

(3) Prove that every Hermitian matrix A is diagonalizable. *Hint*: Take any eigenvector v_1 of A. Decompose $\mathbb{C}^n$ into the direct sum of $\text{span}(v_1)$ and its orthogonal complement. Show that the orthogonal complement is an invariant subspace for A.

3 Explicitly Solvable Higher Order ODEs

Required Problems

10/26: **1.** Consider a scalar linear differential equation with constant coefficients:

$$x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1 x' + a_0 x = g(t)$$

Prove that the solution of this equation must take the form:

$$x(t) = x_h(t) + \int_0^t U(t-\tau)g(\tau)\mathrm{d}\tau$$

where x_h is any solution of the homogeneous equation, and U is the solution of the homogeneous equation with initial condition $U(0) = U'(0) = \cdots = U^{(n-2)}(0) = 0$, $U^{(n-1)}(0) = 1$.

In particular, the driven harmonic oscillator equation

$$x'' + \omega^2 x = f(t)$$

is solved by

$$x(t) = x(0)\cos\omega t + x'(0)\frac{\sin\omega t}{\omega} + \int_0^t \frac{\sin\omega(t-\tau)}{\omega}f(\tau)\mathrm{d}\tau$$

Proof. We can convert the general scalar linear ODE to the form

$$\begin{pmatrix} Y^1 \\ \vdots \\ Y^{n-1} \\ Y^n \end{pmatrix}' = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{pmatrix} \begin{pmatrix} Y^1 \\ \vdots \\ Y^{n-1} \\ Y^n \end{pmatrix} + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g(t) \end{pmatrix}$$

This is a linear equation of the form $y' = Ay + f$. Thus, as stated in class, its solutions are given by the modified Duhamel formula, as follows.

$$\begin{aligned} y(t) &= \mathrm{e}^{tA}y_0 + \int_0^t \mathrm{e}^{(t-\tau)A}f(\tau)\mathrm{d}\tau \\ &= x_h(t) + \int_0^t \mathrm{e}^{(t-\tau)A}e_n g(\tau)\mathrm{d}\tau \\ &= x_h(t) + \int_0^t U(t-\tau)g(\tau)\mathrm{d}\tau \end{aligned}$$

This is the desired result. A few notes on the transitions between the equations, though:

1. $\mathrm{e}^{tA}y_0$ is the solution to the homogeneous variant of the given equation, so we are ok to replace it with $x_h(t)$.
2. e_n denotes the n^{th} standard basis vector of length n. By definition (see the above), $f(t) = e_n g(t)$, where $g(t)$ is a "scalar."
3. Analogously to the first remark, $\mathrm{e}^{tA}e_n$ is the solution of the homogeneous equation with initial condition e_n or, equivalently, $U(0) = U'(0) = \cdots = U^{(n-2)}(0) = 0$, $U^{(n-1)}(0) = 1$. Thus, by hypothesis, we may rename it $U(t)$ and take its value at $t-\tau$, as needed.

We now address the driven, undamped harmonic oscillator equation. Convert it into a 2D system to start.

$$\begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix}' = \underbrace{\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}}_{A} \begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix}$$

Diagonalize A.

$$A = \begin{pmatrix} -i & i \\ \omega & \omega \end{pmatrix} \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega \end{pmatrix} \begin{pmatrix} \frac{i}{2} & \frac{1}{2\omega} \\ -\frac{i}{2} & \frac{1}{2\omega} \end{pmatrix}$$

Then the general solution is

$$\begin{aligned}
Y &= \mathrm{e}^{tA}y_0 + \int_0^t \mathrm{e}^{(t-\tau)A}e_2 f(\tau)\mathrm{d}\tau \\
\begin{pmatrix} Y^1 \\ Y^2 \end{pmatrix} &= \begin{pmatrix} -i & i \\ \omega & \omega \end{pmatrix} \begin{pmatrix} \mathrm{e}^{i\omega t} & 0 \\ 0 & \mathrm{e}^{-i\omega t} \end{pmatrix} \begin{pmatrix} \frac{i}{2} & \frac{1}{2\omega} \\ -\frac{i}{2} & \frac{1}{2\omega} \end{pmatrix} \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix} \\
&\quad + \int_0^t \begin{pmatrix} -i & i \\ \omega & \omega \end{pmatrix} \begin{pmatrix} \mathrm{e}^{i\omega(t-\tau)} & 0 \\ 0 & \mathrm{e}^{-i\omega(t-\tau)} \end{pmatrix} \begin{pmatrix} \frac{i}{2} & \frac{1}{2\omega} \\ -\frac{i}{2} & \frac{1}{2\omega} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} f(\tau)\mathrm{d}\tau \\
&= \begin{pmatrix} x(0)\frac{1}{2}(\mathrm{e}^{i\omega t} + \mathrm{e}^{-i\omega t}) + \frac{x'(0)}{\omega}\frac{1}{2i}(\mathrm{e}^{i\omega t} - \mathrm{e}^{-i\omega t}) \\ x(0)\frac{1}{2}(i\omega\mathrm{e}^{i\omega t} - i\omega\mathrm{e}^{-i\omega t}) + \frac{x'(0)}{2}(\mathrm{e}^{i\omega t} + \mathrm{e}^{-i\omega t}) \end{pmatrix} \\
&\quad + \int_0^t \begin{pmatrix} \frac{1}{\omega}\frac{1}{2i}(\mathrm{e}^{i\omega(t-\tau)} - \mathrm{e}^{-i\omega(t-\tau)}) \\ \frac{1}{2}(\mathrm{e}^{i\omega(t-\tau)} + \mathrm{e}^{-i\omega(t-\tau)}) \end{pmatrix} f(\tau)\mathrm{d}\tau \\
&= \begin{pmatrix} x(0)\cos\omega t + \frac{x'(0)}{\omega}\sin\omega t \\ -\omega x(0)\sin\omega t + x'(0)\cos\omega t \end{pmatrix} \\
&\quad + \int_0^t \begin{pmatrix} \frac{1}{\omega}\sin\omega(t-\tau) \\ \cos\omega(t-\tau) \end{pmatrix} f(\tau)\mathrm{d}\tau \\
&= \begin{pmatrix} x(0)\cos\omega t + x'(0)\frac{\sin\omega t}{\omega} + \int_0^t \frac{\sin\omega(t-\tau)}{\omega} f(\tau)\mathrm{d}\tau \\ -\omega x(0)\sin\omega t + x'(0)\cos\omega t + \int_0^t \cos\omega(t-\tau) f(\tau)\mathrm{d}\tau \end{pmatrix}
\end{aligned}$$

To convert back to scalar land, recall that by the construction of the 2D system, $Y^1 = x$. Therefore, we can read from the above that

$$x(t) = x(0)\cos\omega t + x'(0)\frac{\sin\omega t}{\omega} + \int_0^t \frac{\sin\omega(t-\tau)}{\omega} f(\tau)\mathrm{d}\tau$$

as desired. □

2. We know that for complex numbers z, w, $\mathrm{e}^{z+w} = \mathrm{e}^z\mathrm{e}^w$. However, in general this cannot be generalized to matrix exponentials.

(1) Suppose A, B are $n \times n$ complex matrices. If $AB = BA$, prove that $\mathrm{e}^{A+B} = \mathrm{e}^A\mathrm{e}^B$. You need to substitute in the power series expansion of e^A and e^B and do some combinatorics.

Proof. Let

$$y(t) = \mathrm{e}^{tA}\mathrm{e}^{tB}$$

Then we know from the product rule and from class that

$$\begin{aligned}
y'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{tA})\mathrm{e}^{tB} + \mathrm{e}^{tA}\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{tB}) \\
&= A\mathrm{e}^{tA}\mathrm{e}^{tB} + \mathrm{e}^{tA}B\mathrm{e}^{tB} \\
&= A\mathrm{e}^{tA}\mathrm{e}^{tB} + B\mathrm{e}^{tA}\mathrm{e}^{tB} \\
&= (A+B)\mathrm{e}^{tA}\mathrm{e}^{tB} \\
&= (A+B)y(t)
\end{aligned}$$

Note that B commutes with e^{tA} because B commutes with every matrix in the power series expansion of e^{tA} (all of which are powers of A). By viewing $y(t)$ as a function $y : \mathbb{R} \to \mathbb{R}^{n^2}$ and

$A+B$ as an $n^2 \times n^2$ linear transformation on y, we can apply the result that the solution to $x' = Cx$ is $x = \mathrm{e}^{tC}x_0$ and determine that

$$y(t) = \mathrm{e}^{t(A+B)}$$

Cutting this back down to size, substituting $t = 1$, and applying transitivity yields

$$\begin{aligned}\mathrm{e}^{1(A+B)} &= \mathrm{e}^{1A}\mathrm{e}^{1B}\\ \mathrm{e}^{A+B} &= \mathrm{e}^{A}\mathrm{e}^{B}\end{aligned}$$

as desired.
Alternatively, the answer that was probably expected is the following. Start with

$$\begin{aligned}\mathrm{e}^{A+B} &= \sum_{k=0}^{\infty} \frac{(A+B)^k}{k!}\\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\ell=0}^{k} \binom{k}{\ell} A^{k-\ell}B^{\ell}\\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\ell=0}^{k} \frac{k!}{(k-\ell)!\ell!} A^{k-\ell}B^{\ell}\\ &= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \frac{1}{(k-\ell)!\ell!} A^{k-\ell}B^{\ell}\end{aligned}$$

Then reindex the summation as follows. For each $k = 0, 1, \dots$, there is an $\ell = 0$ term of the form $A^kB^0/k!0!$. Similarly, for each $k = 1, 2, \dots$, there is an $\ell = 1$ term of the form $A^{k-1}B^1/(k-1)!1!$. Note that the smallest exponent of A in this series is $k - 1 = 0$, just like in the last series. Also note that each of these terms is generated by a set of indices k, ℓ that are distinct from any used previously (including in the set of terms $A^kB^0/k!0!$). In general, for each $k = \ell, \ell+1, \dots$, there is a term of the form $A^{k-\ell}B^{\ell}/(k-\ell)!\ell!$. As we should be able to see at this point, the set of all of these terms — infinitely many for each ℓ — is equal to the set of terms $A^mB^n/m!n!$ summed over m, n independently from 0 to ∞. Symbolically, we may continue the above set of equivalences as follows.

$$\begin{aligned}&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!n!} A^mB^n\\ &= \left(\sum_{m=0}^{\infty} \frac{A^m}{m!}\right)\left(\sum_{n=0}^{\infty} \frac{B^n}{n!}\right)\\ &= \mathrm{e}^{A}\mathrm{e}^{B}\end{aligned}$$

□

(2) Now suppose

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Compute $AB - BA$, $\mathrm{e}^{A}\mathrm{e}^{B}$, and e^{A+B}. *Hint*: Since A, B are both nilpotent, you do not have to reduce them to Jordan normal forms. Also, what is $(A+B)^2$? Why does this help you to compute e^{A+B}?

Proof. The commutator is

$$AB - BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\boxed{AB - BA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

As per the hint, A, B are both nilpotent matrices. Thus computing their matrix exponentials follows from the power series definition. Indeed, we get

$$e^A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad e^B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

so that

$$\boxed{e^A e^B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}}$$

We have that $(A+B)^2 = I_2$. Thus, $A+B$ raised to an odd power is $A+B$ and $A+B$ raised to an even power is the identity. Therefore, by the power series definition of the matrix exponential,

$$e^{A+B} = \begin{pmatrix} 1 + \frac{1}{2!} + \frac{1}{4!} + \cdots & 1 + \frac{1}{3!} + \frac{1}{5!} + \cdots \\ 1 + \frac{1}{3!} + \frac{1}{5!} + \cdots & 1 + \frac{1}{2!} + \frac{1}{4!} + \cdots \end{pmatrix}$$

By inspection, we can recognize the upper-left and lower-right power series to be the Maclaurin series for the hyperbolic cosine function evaluated at $x = 1$. Similarly, the other two entries are the Maclaurin series for the hyperbolic sine function evaluated at $x = 1$. Thus,

$$\boxed{e^{A+B} = \begin{pmatrix} \cosh(1) & \sinh(1) \\ \sinh(1) & \cosh(1) \end{pmatrix}}$$

$\square$

3. The general formula for the exponential of an $n \times n$ invertible matrix is very lengthy. However, for real t and any 2×2 complex matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

there is a direct formula for e^{tA}:

$$e^{tA} = e^{t\delta} \left[\cosh(t\Delta) I_2 + \frac{\sinh(t\Delta)}{t\Delta} \begin{pmatrix} \gamma & b \\ c & -\gamma \end{pmatrix} \right]$$

where

$$\delta = \frac{a+d}{2} \qquad \gamma = \frac{a-d}{2} \qquad \Delta = \sqrt{\gamma^2 + bc}$$

and the functions cosh and sinh are interpreted as power series as follows.

$$\cosh(z) = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} \qquad \frac{\sinh(z)}{z} = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k+1)!}$$

Thus, $\cosh(t\Delta)$ and $\sinh(t\Delta)$ are single valued, no matter which branch of Δ is considered. Prove the above formula. *Hint*: Decompose $A = \delta I_2 + C$ and apply the result of the previous problem.

Proof. Taking the hint and (naturally) defining $C := A - \delta I_2$, we can determine that

$$\begin{aligned} A &= \delta I_2 + C \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \frac{a+d}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{a-d}{2} & b \\ c & \frac{d-a}{2} \end{pmatrix} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \frac{a+d}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \gamma & b \\ c & -\gamma \end{pmatrix} \end{aligned}$$

Since

$$t\delta I_2 \cdot tC = t^2 \begin{pmatrix} \frac{a^2-d^2}{4} & \frac{b(a+d)}{2} \\ \frac{d(a+d)}{2} & \frac{d^2-a^2}{4} \end{pmatrix} = tC \cdot t\delta I_2$$

for all $t \in \mathbb{R}$, we have by Problem 2(1) that

$$e^{tA} = e^{t\delta I_2 + tC} = e^{t\delta I_2} e^{tC}$$

Additionally, observe that

$$C^2 = \begin{pmatrix} \gamma & b \\ c & -\gamma \end{pmatrix}^2 = \begin{pmatrix} \gamma^2 + bc & \gamma b - b\gamma \\ c\gamma - \gamma c & cb + \gamma^2 \end{pmatrix} = (\gamma^2 + bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \Delta^2 I_2$$

Thus,

$$C^m = \begin{cases} \Delta^{2k} C & m = 2k+1 \\ \Delta^{2k} I_2 & m = 2k \end{cases}$$

Therefore,

$$\begin{aligned} e^{tA} &= e^{t\delta I_2} e^{tC} \\ &= \begin{pmatrix} e^{t\delta} & 0 \\ 0 & e^{t\delta} \end{pmatrix} \cdot \sum_{m=0}^{\infty} \frac{t^m C^m}{m!} \\ &= e^{t\delta} I_2 \left[\sum_{k=0}^{\infty} \frac{t^{2k} C^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{t^{2k+1} C^{2k+1}}{(2k+1)!} \right] \\ &= e^{t\delta} \left[\sum_{k=0}^{\infty} \frac{t^{2k} \Delta^{2k} I_2}{(2k)!} + \sum_{k=0}^{\infty} \frac{t^{2k+1} \Delta^{2k} C}{(2k+1)!} \right] \\ &= e^{t\delta} \left[\left(\sum_{k=0}^{\infty} \frac{(t\Delta)^{2k}}{(2k)!} \right) I_2 + \left(\sum_{k=0}^{\infty} \frac{(t\Delta)^{2k}}{(2k+1)!} \right) C \right] \\ &= e^{t\delta} \left[\cosh(t\Delta) I_2 + \frac{\sinh(t\Delta)}{t\Delta} \begin{pmatrix} \gamma & b \\ c & -\gamma \end{pmatrix} \right] \end{aligned}$$

as desired. □

4. Which of the following can be the solution of a first-order autonomous homogeneous 2-dimensional system?

$$\begin{pmatrix} 2e^t + e^{-t} \\ e^{2t} \end{pmatrix} \qquad \begin{pmatrix} 2e^t + e^{-t} \\ e^t \end{pmatrix} \qquad \begin{pmatrix} 2e^t + e^{-t} \\ te^t \end{pmatrix} \qquad \begin{pmatrix} 2e^t \\ t^3 e^t \end{pmatrix} \qquad \begin{pmatrix} 2e^t \\ te^t \end{pmatrix}$$

Hint: Compare with the necessary structure of the solution discussed in class.

Proof. Taking the hint, we can determine by inspection that each component of the solution vector function will be a linear combination of the entries of the exponential Jordan matrix. A 2×2 Jordan matrix can only have the following two forms for some $\lambda, \mu \in \mathbb{C}$.

$$\begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{pmatrix} \qquad \begin{pmatrix} e^{\mu t} & te^{\mu t} \\ 0 & e^{\mu t} \end{pmatrix}$$

Thus, given a 2D vector function of t, we must check that either both entries are of the form $Ae^{\lambda t} + Be^{\mu t}$ for some (possibly different between the entries) numbers $A, B \in \mathbb{C}$, or both entries are of the form $Ae^{\mu t} + Bte^{\mu t}$. Only the

$$\boxed{\text{second and fifth}}$$

entries in the question have this form. □

5. Solve the following 2-dimensional initial value problems:

$$y' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} y, \quad y(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad\qquad y' = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} y, \quad y(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Determine the stable and unstable subspaces for each equation.

Proof. I can do this. $\square$

Bonus Problem

1. At the end of the 19th century, the English mathematician and physicist Oliver Heaviside introduced an operational calculus to solve differential equations related to circuit analysis. To briefly describe Heaviside's idea, we consider the equation of RC charging circuits:

$$RQ'(t) + \frac{1}{C}Q(t) = V_0$$

Of course, we know how to solve it using the standard separation of variables method. However, Heaviside introduced a "formal" method of computation as follows. Regard the differential operator $\mathrm{d}/\mathrm{d}t$ as a "symbol" p, so that the equation is rewritten as

$$\left(p + \frac{1}{RC}\right) Q = \frac{V_0}{R} \quad \Longleftrightarrow \quad Q = \left(p + \frac{1}{RC}\right)^{-1} \frac{V_0}{R}$$

The next step seems surprising and does not make any sense mathematically: Expand the factor $(p + 1/RC)^{-1}$ into a geometric series of $1/p$, giving an "equation"

$$Q = \frac{1}{p}\frac{V_0}{R} - \frac{1}{RCp^2}\frac{V_0}{R} + \cdots + \frac{(-1)^k}{(RC)^k p^{k+1}}\frac{V_0}{R} + \cdots$$

Heaviside considered the symbol $1/p$ to be the "inverse" of the symbol p, namely $1/p = \int_0^t$. Thus, formally,

$$\begin{aligned} Q &= \int_0^t \frac{V_0}{R} - \frac{1}{RC}\int_0^t\int_0^t \frac{V_0}{R} + \cdots + \frac{(-1)^k}{(RC)^k p^{k+1}} \underbrace{\int_0^t \cdots \int_0^t}_{k+1 \text{ times}} \frac{V_0}{R} + \cdots \\ &= \frac{tV_0}{R} - \frac{t^2}{2!RC}\frac{V_0}{R} + \cdots + \frac{t^k}{k!(RC)^k}\frac{V_0}{R} + \cdots \end{aligned}$$

and this gives the correct result $Q(t) = (1 - \mathrm{e}^{-t/RC})V_0/R$, very strangely!

At a first glance, Heaviside's argument does not make much sense mathematically: We cannot simply regard a linear operator as a number, let alone the problem of convergence. However, there is a mathematical object that can rigorize Heaviside's idea; it is called the **Laplace transform**. It takes a function defined for $t \geq 0$ to a function of a complex variable $s = \sigma + i\beta$ via

$$\mathfrak{L}\left[f\right](s) = \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t$$

(1) Prove that as long as $|f(t)| \leq A\mathrm{e}^{Mt}$, this integral converges for $\sigma > M$. Thus, the Laplace transform is defined for any function not growing faster than an exponential function.

(2) Prove the following law of differentiation: If the Laplace transform of $f(t)$ is $F(s)$, then

$$\mathfrak{L}\left[f'\right](s) = sF(s) - f(0)$$

In general,

$$\mathfrak{L}\left[f^{(n)}\right](s) = s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(k)}(0)$$

(3) Prove the following law of integration: If the Laplace transform of $f(t)$ is $F(s)$, then

$$\mathfrak{L}\left[\int_0^t f(\tau)\mathrm{d}\tau\right](s) = \frac{F(s)}{s}$$

These justify Heaviside's notion of "differential operator s."

(4) Prove the law of frequency shift: For any complex number a, the Laplace transform of $\mathrm{e}^{at}f(t)$ is $F(s-a)$. Prove the law of convolution using integration by parts: If $\mathfrak{L}[f] = F$ and $\mathfrak{L}[g] = G$, then the Laplace transform of

$$\int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau$$

is $F(s)G(s)$.

(5) Compute the Laplace transforms of the following functions, where $a \in \mathbb{C}$ is arbitrary.

$$\mathrm{e}^{at} \qquad t^n\mathrm{e}^{at} \qquad \sin at \qquad \cos at$$

(6) The inversion theorem states that, roughly speaking, the function $f(t)$ is completely determined by its Laplace transform. The inversion formula involves complex analysis and you are not required to verify it or use it for computation. It reads

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathrm{e}^{st}F(s)\,\mathrm{d}s$$

However, with the aid of the inversion theorem and part (4), compute the inverse functions of

$$\frac{1}{s} \qquad \frac{1}{s-a} \qquad \frac{n!}{(s+a)^{n+1}} \qquad \frac{a}{s^2+a^2} \qquad \frac{s}{s^2+a^2}$$

(7) We can now justify Heaviside's solution for the RC charging circuits equation. Let the Laplace transform of $Q(t)$ be $F(s)$. Then since $Q(0) = 0$, taking the Laplace transform of both sides, the law of differentiation tells us that

$$RsF(s) + \frac{F(s)}{C} = \frac{V_0}{s} \quad\Longleftrightarrow\quad F(s) = \left(s + \frac{1}{RC}\right)^{-1}\frac{V_0}{Rs}$$

Determine $Q(t)$ from its Laplace transform with the aid of part (7). Thus...

(8) Use the method of part (7), together with the aid of parts (3)-(5), to solve the initial value problem of the second-order differential equation

$$y'' + ay' + by = f(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

Compare it with the result of Problem 1. Explain what role the characteristic polynomial s^2+as+b plays here. You should discuss two cases (distinct characteristic roots and single characteristic root) separately.

4 Final Explicitly Solvable Cases

Required Problems

11/2: **1.** Use Duhamel's formula to solve the initial value problem

$$y' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} y + \begin{pmatrix} -t \\ t \end{pmatrix}, \quad y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Proof. Diagonalizing A reveals that

$$\underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{A} = \underbrace{\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} -i/2 & 1/2 \\ i/2 & 1/2 \end{pmatrix}}_{Q^{-1}}$$

Thus,

$$\mathrm{e}^{tA} = Q\mathrm{e}^{tB}Q^{-1} = Q \begin{pmatrix} \mathrm{e}^{it} & 0 \\ 0 & \mathrm{e}^{-it} \end{pmatrix} Q^{-1}$$

We have that

$$\begin{aligned} y(t) &= \mathrm{e}^{tA}y_0 + \int_0^t \mathrm{e}^{(t-\tau)A} f(\tau)\mathrm{d}\tau \\ &= \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{it} & 0 \\ 0 & \mathrm{e}^{-it} \end{pmatrix} \begin{pmatrix} -i/2 & 1/2 \\ i/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\quad + \int_0^t \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{i(t-\tau)} & 0 \\ 0 & \mathrm{e}^{-i(t-\tau)} \end{pmatrix} \begin{pmatrix} -i/2 & 1/2 \\ i/2 & 1/2 \end{pmatrix} \begin{pmatrix} -\tau \\ \tau \end{pmatrix} \mathrm{d}\tau \\ &= \begin{pmatrix} \frac{\mathrm{e}^{it}+\mathrm{e}^{-it}}{2} \\ \frac{\mathrm{e}^{it}-\mathrm{e}^{-it}}{2i} \end{pmatrix} + \int_0^t \begin{pmatrix} \tau \left(\frac{\mathrm{e}^{i(t-\tau)}-\mathrm{e}^{-i(t-\tau)}}{2i} - \frac{\mathrm{e}^{i(t-\tau)}+\mathrm{e}^{-i(t-\tau)}}{2} \right) \\ \tau \left(\frac{\mathrm{e}^{i(t-\tau)}-\mathrm{e}^{-i(t-\tau)}}{2i} - \frac{\mathrm{e}^{i(t-\tau)}+\mathrm{e}^{-i(t-\tau)}}{2} \right) \end{pmatrix} \end{aligned}$$

Substitute sines and cosines and evaluate. ☐

2. Let A be an $n \times n$ complex constant matrix and let $f(t) = \mathrm{e}^{\zeta t}p(t)$, where $p(t)$ is a $\mathbb{C}^n$-valued function whose entries are all polynomials. We define $\deg p(t)$ to be the largest of the degrees of its entries. Use Duhamel's formula to prove the following proposition:

If ζ is not an eigenvalue of A, then the solution of $y' = Ay + p(t)\mathrm{e}^{\zeta t}$ takes the form

$$\mathrm{e}^{tA}z_0 + q(t)\mathrm{e}^{\zeta t}$$

where z_0 is a constant vector (*not* necessarily the initial value) and $q(t)$ is a polynomial vector with degree being the same as $p(t)$. If ζ is an eigenvalue of A with algebraic multiplicity α, then the solution of $y' = Ay + \mathrm{e}^{\zeta t}p(t)$ takes the form

$$\mathrm{e}^{tA}z_0 + r(t)\mathrm{e}^{\zeta t}$$

where z_0 is a constant vector (*not* necessarily the initial value) and $r(t)$ is a polynomial vector with degree $\deg p(t) + \alpha$.

Proof. Shao said in office hours that this question cannot be answered, and as such he would cancel it; he copied it out of Teschl but it had an error as written. ☐

3. We know that for the driven harmonic oscillator equation

$$x'' + \omega_0^2 x = H_0 \cos \omega t$$

when $\omega = \omega_0$, the solution grows unboundedly. However, what if $\omega \neq \omega_0$ but is very close?

For simplicity, suppose that $H_0 > 0$ and the initial values $x(0), x'(0)$ are real numbers that are very small compared to $H_0/|\omega^2 - \omega_0^2|$, say,

$$|x(0)| + |x'(0)| < \frac{\varepsilon H_0}{|\omega^2 - \omega_0^2|}$$

Suppose also that $|\omega - \omega_0|$ is very small compared to ω_0, say

$$|\omega - \omega_0| < \varepsilon\omega_0$$

Lastly, suppose that the initial values are small compared to the eigenfrequency, say

$$|x(0)| + |x'(0)| < \varepsilon\omega_0$$

Prove that there is a sequence of times $t_k \to +\infty$ such that

$$x(t_k) > 2(1-\varepsilon) \cdot \frac{H_0}{|\omega^2 - \omega_0^2|} \approx \frac{1}{\varepsilon}$$

That is, the mass point will constantly visit positions very far away from the equilibrium.

Hint. Write down the solution first, and then you need to discuss two cases separately: ω/ω_0 is rational/irrational. In the latter case, you should use the following theorem of Kronecker.

Theorem 1. *Let α, β be positive real numbers such that α/β is irrational. Then the set $\{(\langle n\alpha\rangle, \langle n\beta\rangle) \mid n \in \mathbb{N}\}$ is dense in the unit sphere $[0,1] \times [0,1]$, where $\langle\cdot\rangle$ denotes the decimal part of a real number.*

Proof. From HW3, the given driven harmonic oscillator equation is solved by

$$\begin{aligned} x(t) &= x(0)\cos\omega_0 t + x'(0)\frac{\sin\omega_0 t}{\omega_0} + \int_0^t \frac{\sin\omega_0(t-\tau)}{\omega_0}(H_0\cos\omega\tau)\mathrm{d}\tau \\ &= x(0)\cos\omega_0 t + x'(0)\frac{\sin\omega_0 t}{\omega_0} + \frac{H_0}{\omega^2 - \omega_0^2}(\cos\omega_0 t - \cos\omega t) \end{aligned}$$

We have that $|x(0)\cos\omega_0 t| \leq |x(0)|$ and $|x'(0)\sin\omega_0 t| \leq |x'(0)|$ and $|x'(0)| < \varepsilon\omega_0$ and $|x(0)| < \varepsilon H_0/|\omega^2 - \omega_0^2|$. Thus

$$\begin{aligned} \left|x(0)\cos\omega_0 t + x'(0)\frac{\sin\omega_0 t}{\omega_0}\right| &\leq |x(0)\cos\omega_0 t| + \frac{1}{\omega_0}|x'(0)\sin\omega_0 t| \\ &\leq |x(0)| + \frac{|x'(0)|}{\omega_0} \\ &< |x(0)| + \varepsilon \\ &< \frac{\varepsilon H_0}{|\omega^2 - \omega_0^2|} + \varepsilon \\ &= \varepsilon\left(\frac{H_0}{|\omega^2 - \omega_0^2|} + 1\right) \end{aligned}$$

We also have that

$$\begin{aligned} \left|\frac{H_0}{\omega^2 - \omega_0^2}(\cos\omega_0 t - \cos\omega t)\right| &= \frac{H_0}{|\omega^2 - \omega_0^2|} \cdot |\cos\omega_0 t - \cos\omega t| \\ &\leq \frac{H_0}{|\omega^2 - \omega_0^2|} \cdot 2 \end{aligned}$$

It follows that

$$\begin{aligned} |x(t)| &< \frac{2H_0}{|\omega^2 - \omega_0^2|} + \frac{\varepsilon H_0}{|\omega^2 - \omega_0^2|} + \varepsilon \\ &= (2+\varepsilon) \cdot \frac{H_0}{|\omega^2 - \omega_0^2|} + \varepsilon \end{aligned}$$

$\square$

4. Sketch the phase diagram of the following linear autonomous systems. Also clearly indicate

- The eigenvalues and eigenvectors;
- The stable and unstable subspaces (if the eigenvalues are not purely imaginary);
- The shape and direction of the trajectories (attracted/repelled by the fixed point).

(1)

$$y' = \begin{pmatrix} \frac{1}{2} & 1 \\ -1 & \frac{1}{2} \end{pmatrix} y$$

Proof. Using techniques from previous weeks, we can diagonalize A as follows.

$$\begin{pmatrix} \frac{1}{2} & 1 \\ -1 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}+i & 0 \\ 0 & \frac{1}{2}-i \end{pmatrix} \begin{pmatrix} \frac{i}{2} & \frac{1}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix}$$

Thus, the eigenvalues and corresponding eigenvectors of A are

$$\boxed{\lambda = \frac{1}{2}+i \qquad \bar{\lambda} = \frac{1}{2}-i \qquad v = \begin{pmatrix} -i \\ 1 \end{pmatrix} \qquad \bar{v} = \begin{pmatrix} i \\ 1 \end{pmatrix}}$$

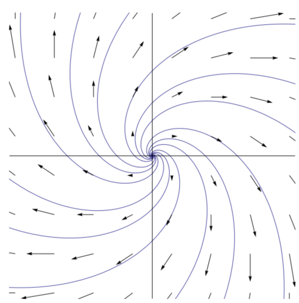
It follows that

$$\boxed{\text{stable subspace} = \{0\} \qquad \text{unstable subspace} = \mathbb{R}^2}$$

Moreover, the shape and direction of the trajectories (using the convention from Q5(2)) are

$$\boxed{\text{Spiral source; repelled}}$$

Therefore,



$\square$

(2)

$$y' = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} y$$

Proof. This question is entirely analogous to part (1). Indeed, we get eigenvalues and eigenvectors

$$\boxed{\lambda = 1+2i \qquad \bar{\lambda} = 1-2i \qquad v = \begin{pmatrix} -i \\ 1 \end{pmatrix} \qquad \bar{v} = \begin{pmatrix} i \\ 1 \end{pmatrix}}$$

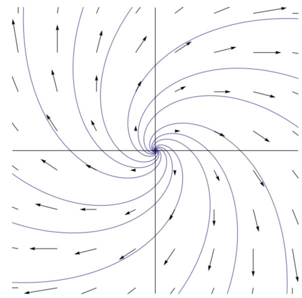
subspaces

$$\boxed{\text{stable subspace} = \{0\} \qquad \text{unstable subspace} = \mathbb{R}^2}$$

and shape and direction

$$\boxed{\text{Spiral source; repelled}}$$

Therefore,



□

(3)

$$y' = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} y$$

Proof. This time our diagonalization gives real, distinct eigenvalues and eigenvectors

$$\boxed{\lambda_1 = \frac{3}{2} \qquad \lambda_2 = \frac{1}{2} \qquad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

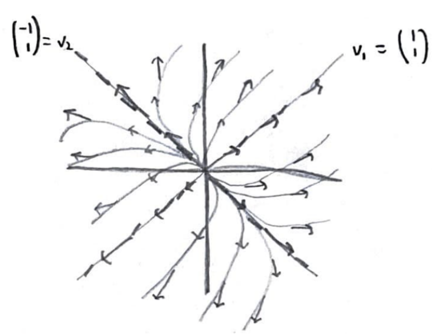
Since both eigenvalues are greater than zero, we have subspaces

$$\boxed{\text{stable subspace} = \{0\} \qquad \text{unstable subspace} = \mathbb{R}^2}$$

Since we have two positive real eigenvalues, we have shape and direction

$$\boxed{\text{Source; repelled}}$$

Additionally, the phase diagram will have many curves of the form $v_2 = v_1^{\lambda_2/\lambda_1}$, i.e., $v_2 = v_1^{1/3}$, or $v_1 = v_2^3$.



□

(4)

$$y' = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} y$$

Proof. Once again, we get real distinct eigenvalues and eigenvectors, but this time our eigenvalues have opposite signs.

$$\boxed{\lambda_1 = 1 \qquad \lambda_2 = -1 \qquad v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

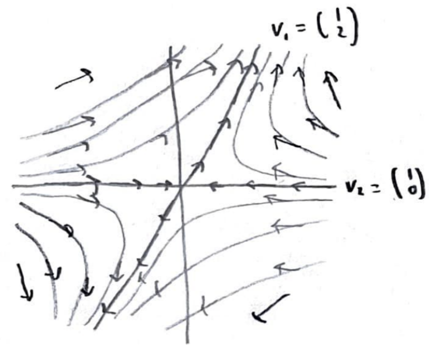
Because of their opposite signs, we have our first nontrivial stable subspace (corresponding to the negative eigenvalue).

$$\boxed{\text{stable subspace} = \operatorname{span}\{v_2\} \qquad \text{unstable subspace} = \operatorname{span}\{v_1\}}$$

Likewise, it follows that

$$\boxed{\text{Saddle; both (depends on the subspace)}}$$

Additionally, we will have a power function of a negative power $v_2 = v_1^{\lambda_2/\lambda_1}$, i.e., $v_2 = 1/v_1$.



□

(5)

$$y' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} y$$

Proof. This matrix is already in JNF with a single Jordan block. Thus, we have one lone eigenvalue λ, an eigenvector v, and a generalized eigenvector u as follows.

$$\boxed{\lambda = 2 \qquad v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

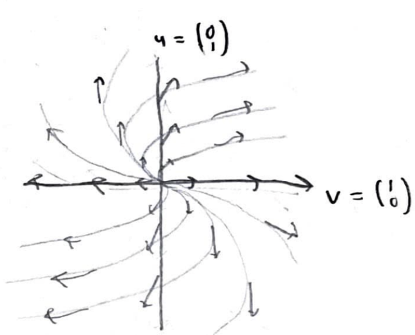
Evidently, $\lambda > 0$, so

$$\boxed{\text{stable subspace} = \{0\} \qquad \text{unstable subspace} = \mathbb{R}^2}$$

Now there is not a specific naming convention for the this shape in Q5(2), so we will call it "distorted source:"

$$\boxed{\text{Distorted source; repelled}}$$

Additionally, we will have a function of the form $v = u + u \ln u$.



$\square$

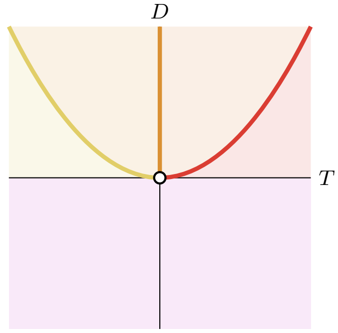
5. Given a matrix
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
the number $a + d$ is called the trace of A and is denoted by $\operatorname{Tr} A$.

(1) Prove that $\operatorname{Tr} A$ is invariant under similarity, and show that
$$\chi_A(z) = z^2 - (\operatorname{Tr} A)z + \det A \qquad\qquad \det \mathrm{e}^A = \mathrm{e}^{\operatorname{Tr} A}$$

Proof. Left equality above: We have that
$$\begin{aligned} \chi_A(z) &= \det(A - zI) \\ &= \det \begin{pmatrix} a - z & b \\ c & d - z \end{pmatrix} \\ &= (a - z)(d - z) - bc \\ &= z^2 - az - dz + ad - bc \\ &= z^2 - (\underbrace{a + d}_{\operatorname{Tr} A})z + (\underbrace{ad - bc}_{\det A}) \end{aligned}$$
as desired.
Invariance of the trace under similarity: Suppose $A \sim B$. Then since similar matrices have the same characteristic polynomial, we have by the left equality above that
$$\begin{aligned} \chi_A(z) &= \chi_B(z) \\ z^2 - (\operatorname{Tr} A)z + \det A &= z^2 - (\operatorname{Tr} B)z + \det B \\ \operatorname{Tr} A &= \operatorname{Tr} B \end{aligned}$$
as desired.
Right equality above: Suppose $\mathrm{e}^A = Q\mathrm{e}^B Q^{-1}$ where B is in JNF. Then
$$\begin{aligned} \det \mathrm{e}^A &= \det \mathrm{e}^B \\ &= \mathrm{e}^{\lambda_1} \cdot \mathrm{e}^{\lambda_2} \\ &= \mathrm{e}^{\lambda_1 + \lambda_2} \\ &= \mathrm{e}^{\operatorname{Tr} B} \\ &= \mathrm{e}^{\operatorname{Tr} A} \end{aligned}$$
where we have the first equality because the determinant is invariant under similarity; the second equality because e^B is upper triangular, the determinant of an upper triangular matrix is equal to the product of the diagonal entries, and the diagonal entries of e^B are the exponentials of the eigenvalues of A; and the remainder of the equalities for fairly evident reasons. $\square$

(2) Suppose A is a real matrix. We have discussed the phase diagram of the linear autonomous system $y' = Ay$ and classified them into several cases according to the eigenvalues of A: Spiral source/sink (complex eigenvalues with positive/negative real part), ellipse (purely imaginary eigenvalues), saddle (real eigenvalue with opposite sign), source/sink (positive/negative real eigenvalue). Now the eigenvalues are completely determined by the tuple of real numbers $(T, D) = (\operatorname{Tr} A, \det A)$. Split the (T, D) plane into several parts in which the various cases discussed this Monday occur.

Proof. We have that



We will identify the various colored regions in lines throughout the following derivation. Let's begin.

Suppose we have two identical eigenvalues a. Then the shape will be distorted source if $a > 0$ and distorted sink if $a < 0$. In this case $T = 2a$ and $D = a^2$. It follows by solving the first equation for a and substituting it into the second that

$$D = \frac{1}{4}T^2$$

Thus, every (T, D) falling along this positive portion of this parabola (the red parabola in the diagram) corresponds to a distorted source, and vice versa for the negative portion (yellow).

Suppose $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$. Then $T = 2a$ and $D = a^2 + b^2$. Substituting as before, we obtain

$$D = \frac{1}{4}T^2 + b^2$$

where $b^2 > 0$. Thus, every (T, D) lying *above* the parabola corresponds to a spiral. For positive T (the red/orange shaded region), we have a spiral source, and vice versa for negative T.

Suppose $\lambda_1 = bi$ and $\lambda_2 = -bi$. Then $T = 0$ and $D = b^2$. Thus, the orange part of the vertical axis corresponds to all ellipses.

Now suppose $\lambda_1, \lambda_2 \in \mathbb{R}$. WLOG let $\lambda_2 > \lambda_1$. Define $\delta := \lambda_2 - \lambda_1$. Then $T = 2\lambda_1 + \delta$ and $D = \lambda_1^2 + \delta\lambda_1$. Substituting as before, we obtain

$$D = \left(\frac{T - \delta}{2}\right)^2 + \delta \cdot \frac{T - \delta}{2} = \frac{1}{4}T^2 - \frac{\delta^2}{4}$$

If $D > 0$, then

$$\begin{aligned} 0 &< \frac{1}{4}T^2 - \frac{\delta^2}{4} \\ \delta &< T \\ \lambda_2 - \lambda_1 &< \lambda_1 + \lambda_2 \\ 0 &< 2\lambda_1 \\ 0 &< \lambda_1 < \lambda_2 \end{aligned}$$

Thus, the red shaded region lying above the T axis but below the red half-parabola corresponds to all sources. Assuming $D < 0$ leads in an analogous fashion to the conclusion that $0 > \lambda_2 > \lambda_1$, meaning that the yellow shaded region lying above the T axis but below the yellow half-parabola corresponds to all sinks.

Lastly, any point lying below the T axis must have one positive and one negative eigenvalue: $D = \lambda_1\lambda_2 < 0$ implies either $\lambda_1 < 0$ or $\lambda_2 < 0$ but not both. Thus, the purple shaded region corresponds to saddles.

As a final comment, note that there are several other types of graphs that we did not talk about on Monday that also have their place on this diagram, e.g., the origin corresponds to uniform motion. ∎

5 Fixed Points and Perturbation

Problems Related to Fundamental Definitions

11/10: **1.** Are the following real functions Lipschitz continuous near 0? If yes, find a Lipschitz constant for some interval containing 0.

(1) $1/(1-x^2)$.

Proof. $\boxed{\text{Yes.}}$ Consider the interval $[-0.5, 0.5]$. Then we may take

$$\boxed{L = \frac{16}{9}}$$

$\square$

(2) $x \log |x|$.

Proof. $\boxed{\text{No.}}$ $\square$

(3) $x^2 \sin(1/x)$.

Proof. If we take the piecewise function consisting of the above expression on $\mathbb{R} \setminus \{0\}$ and 0 at 0, then $\boxed{\text{yes.}}$ Consider the interval $[-1, 1]$. Then we may take

$$\boxed{L = 2}$$

$\square$

2. Find the first two elements $y_1(t), y_2(t)$ for the Picard iteration sequence of the following initial value problems, and estimate the error between $y_2(t)$ and the actual solution. Since they are all of separable form, the actual solutions can be explicitly found.

(1) $y' = 1 + y^2$, $y(0) = 0$.

Proof. We take $y_0(t) = 0$. Then

$$\begin{aligned} y_1(t) &= y_0(0) + \int_0^t [1 + y_0(t)^2] \,\mathrm{d}t \\ &= \int_0^t [1 + 0] \,\mathrm{d}t \\ \boxed{y_1(t) = t} & \end{aligned}$$

and

$$\begin{aligned} y_2(t) &= y_0(0) + \int_0^t [1 + y_1(t)^2] \,\mathrm{d}t \\ &= \int_0^t [1 + t^2] \,\mathrm{d}t \\ \boxed{y_2(t) = t + \frac{t^3}{3}} & \end{aligned}$$

The error is between y_2 and the actual solution $y(t) = \tan(t)$ is given by

$$\boxed{\varepsilon = \tan(t) - t - \frac{t^3}{3}}$$

$\square$

(2) $y' = 2ty$, $y(0) = 1$.

Proof. We take $y_0(t) = 1$. Then

$$\begin{aligned} y_1(t) &= y_0(0) + \int_0^t 2ty_0(t)\,\mathrm{d}t \\ &= 1 + \int_0^t 2t\,\mathrm{d}t \\ \boxed{y_1(t) = 1 + t^2} \end{aligned}$$

and

$$\begin{aligned} y_2(t) &= y_0(0) + \int_0^t 2ty_1(t)\,\mathrm{d}t \\ &= 1 + \int_0^t [2t + 2t^3]\,\mathrm{d}t \\ \boxed{y_2(t) = 1 + t^2 + \frac{t^4}{2}} \end{aligned}$$

The error is between y_2 and the actual solution $y(t) = \mathrm{e}^{t^2}$ is given by

$$\boxed{\varepsilon = \mathrm{e}^{t^2} - 1 - t^2 - \frac{t^4}{2}}$$

□

(3) $y' = y/(1-t)$, $y(0) = 1$.

Proof. We take $y_0(t) = 1$. Then

$$\begin{aligned} y_1(t) &= y_0(0) + \int_0^t \frac{y_0(t)}{1-t}\,\mathrm{d}t \\ &= 1 + \int_0^t \frac{1}{1-t}\,\mathrm{d}t \\ \boxed{y_1(t) = 1 - \ln|1-t|} \end{aligned}$$

and

$$\begin{aligned} y_2(t) &= y_0(0) + \int_0^t \frac{y_1(t)}{1-t}\,\mathrm{d}t \\ &= 1 + \int_0^t \frac{1 - \ln|1-t|}{1-t}\,\mathrm{d}t \\ \boxed{y_2(t) = 1 - \ln|1-t| + \frac{1}{2}(\ln|1-t|)^2} \end{aligned}$$

The error between y_2 and the actual solution $y(t) = \mathrm{e}^{-\ln|1-t|}$ is given by

$$\boxed{\varepsilon = \mathrm{e}^{-\ln|1-t|} - 1 + \ln|1-t| - \frac{1}{2}(\ln|1-t|)^2}$$

□

3. Check whether the implicit equation $F(x,y)=0$ uniquely determines an explicit function $y=f(x)$ around the given point (x_0,y_0). If it does, compute $f'(x_0)$.

(1) For $(x,y)\in\mathbb{R}^2$, $F(x,y)=x^2+y^2-1$, $(x_0,y_0)=(\sqrt{2}/2,-\sqrt{2}/2)$.

Proof. From the implicit equation, we have that

$$\begin{aligned}0&=x^2+y^2-1\\ y&=\pm\sqrt{1-x^2}\end{aligned}$$

Since

$$\begin{aligned}-\frac{\sqrt{2}}{2}&=-\sqrt{1-\left(\frac{\sqrt{2}}{2}\right)^2}\\ y_0&=-\sqrt{1-x_0^2}\end{aligned}$$

our explicit function $\boxed{\text{is uniquely determined around } (x_0,y_0).}$
Moreover, we can compute that

$$\begin{aligned}f'(x_0)&=\frac{2x_0}{2\sqrt{1-x_0^2}}\\ &\boxed{f'(x_0)=1}\end{aligned}$$

□

(2) For $(x,y)\in\mathbb{R}^2$, $F(x,y)=x^2-y^2-1$, $(x_0,y_0)=(1,0)$.

Proof. From the implicit equation, we have that

$$\begin{aligned}0&=x^2-y^2-1\\ y&=\pm\sqrt{x^2-1}\end{aligned}$$

Since

$$y_0=\sqrt{x_0^2-1}\qquad\qquad y_0=-\sqrt{x_0^2-1}$$

our explicit function $\boxed{\text{is not uniquely determined around } (x_0,y_0).}$ □

(3) For $(x,y)\in\mathbb{R}^2$, $F(x,y)=x\mathrm{e}^y+y$, $(x_0,y_0)=(0,0)$.

Proof. We apply the implicit function theorem.
F is defined on a subset of $\mathbb{R}^2$, as desired.
We have that

$$\frac{\partial F}{\partial x}=\mathrm{e}^y\qquad\qquad \frac{\partial F}{\partial y}=x\mathrm{e}^y+1$$

Since both of the above partial derivatives are continuous, F is continuously differentiable on its domain, as desired.
$(x_0,y_0)=(0,0)\in\mathbb{R}^2$, which is the domain of F, as desired.
$F(x_0,y_0)=0\mathrm{e}^0+0=0$, as desired.
The truncated Jacobian matrix is 1×1 and contains a nonzero element at (x_0,y_0) — in particular, it contains $\partial F/\partial x$ — as desired.

Therefore, our explicit function $\boxed{\text{is uniquely determined around } (x_0, y_0).}$

Moreover, we can compute that

$$\begin{aligned} f'(x_0) &= -\left(\frac{\partial F}{\partial y}\right)^{-1} \cdot \frac{\partial F}{\partial x} \\ &= -\left(0\mathrm{e}^0 + 1\right)^{-1} \cdot \mathrm{e}^0 \\ \boxed{f'(x_0) = -1} \end{aligned}$$

□

Problems Involving the Banach Fixed Point Theorem

1. (1) Show that the condition "constant $q < 1$" in the statement of the Banach fixed point theorem is not redundant. You may give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which satisfies the strict inequality $|f(x) - f(y)| < |x - y|$ but does not have a fixed point.

Proof. Choose

$$\boxed{f(x) = \begin{cases} 1 & x \leq 0 \\ x + \mathrm{e}^{-x} & x > 0 \end{cases}}$$

The fact that

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \begin{cases} 0 & x \leq 0 \\ 1 - \mathrm{e}^{-x} & x > 0 \end{cases}$$

implies that $|\,\mathrm{d}f/\mathrm{d}x\,| < 1$ for all x. Hence, f satisfies the desired strict inequality. Additionally, since the graph of $f(x) > x$ for all x (as can be readily verified from its definition), it has no fixed point, as desired. □

(2) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz mapping with uniform Lipschitz constant $q < 1$, that is,

$$|f(x) - f(y)| \leq q|x - y|$$

for all $x, y \in \mathbb{R}^n$. Prove that the mapping $x \mapsto x + f(x)$ is invertible with Lipschitz continuous inverse.

Proof. Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $g(x) = x + f(x)$. To prove that g is invertible, it will suffice to show that g is one-to-one, that is, for every $b \in \mathbb{R}^n$, there exists a unique $a \in \mathbb{R}^n$ such that $g(a) = b$. Let $b \in \mathbb{R}^n$ be arbitrary. Define $h : \mathbb{R}^n \to \mathbb{R}^n$ by $h(x) = b - f(x)$. Then since

$$\begin{aligned} |h(x) - h(y)| &= |[b - f(x)] - [b - f(y)]| \\ &= |f(y) - f(x)| \\ &= |f(x) - f(y)| \\ &\leq q|x - y| \end{aligned}$$

we have by the Banach fixed point theorem that there exists a unique $a \in \mathbb{R}^n$ such that $a = h(a)$. It follows that

$$\begin{aligned} a &= b - f(a) \\ a + f(a) &= b \\ g(a) &= b \end{aligned}$$

as desired.

To prove that g^{-1} is Lipschitz continuous, it will suffice to show that

$$|g^{-1}(x) - g^{-1}(y)| \leq \frac{1}{1-q}|x-y|$$

for all $x, y \in \mathbb{R}^n$. Let $x, y \in \mathbb{R}^n$ be arbitrary. Define $a = g^{-1}(x)$ and $b = g^{-1}(y)$. Then since the first term below is nonnegative (as the product of two nonnegative numbers), we have that

$$\begin{aligned}
(1-q)|a-b| &= |a-b| - q|a-b| \\
&\leq |a-b| - |f(a)-f(b)| \\
&= |a-b| - |f(b)-f(a)| \\
&= \big||a-b| - |f(b)-f(a)|\big| \\
&\leq |[a-b] - [f(b)-f(a)]| \\
&= |[a+f(a)] - [b+f(b)]| \\
&= |g(a)-g(b)|
\end{aligned}$$

It follows by returning the substitution that

$$\begin{aligned}
(1-q)|g^{-1}(x) - g^{-1}(y)| &\leq |x-y| \\
|g^{-1}(x) - g^{-1}(y)| &\leq \frac{1}{1-q}|x-y|
\end{aligned}$$

as desired. □

2. Consider the following iterative algorithm to compute the square root of a given $a > 1$.

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

(1) Show that the function

$$F(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$$

meets the requirements of the contraction mapping principle on the closed interval $[\sqrt{a/2}, a]$. Prove that $x_n \to \sqrt{a}$.

Proof. We want to show that

$$|F(x) - F(y)| \leq q|x-y|$$

for some $q \in (0,1)$ and all $x, y \in [\sqrt{a/2}, a]$.
We have that

$$\begin{aligned}
|F(x) - F(y)| &= \left|\frac{1}{2}\left(x + \frac{a}{x}\right) - \frac{1}{2}\left(y + \frac{a}{y}\right)\right| \\
&= \frac{1}{2}\left|(x-y) + \left(\frac{a}{x} - \frac{a}{y}\right)\right| \\
&= \frac{1}{2}\left|(x-y) + a \cdot \frac{y-x}{xy}\right| \\
&= \frac{1}{2}\left|\left(1 - \frac{a}{xy}\right)(x-y)\right| \\
&= \frac{1}{2}\left|1 - \frac{a}{xy}\right||x-y|
\end{aligned}$$

□

(2) For $a = 2$, start the iteration $x_{n+1} = F(x_n)$ with $x_0 = 1$. Use a calculator to compute the first 10 values of this iteration, up to 11 digits after the decimal point. Compare it with the exponentially converging sequence $1.4, 1.41, 1.414, 1.4142, \ldots$. Which of the two algorithms is better?

Proof. We have that

$$\boxed{\begin{aligned} x_0 &= 1 \\ x_1 &= 1.5 \\ x_2 &= 1.41666666667 \\ x_3 &= 1.41421568627 \\ x_4 &= 1.41421356237 \\ x_5 &= 1.41421356237 \\ x_6 &= 1.41421356237 \\ x_7 &= 1.41421356237 \\ x_8 &= 1.41421356237 \\ x_9 &= 1.41421356237 \\ x_{10} &= 1.41421356237 \end{aligned}}$$

$\boxed{\text{The algorithm from part (1) is better.}}$ ☐

(3) Try to estimate the error $|x_n - \sqrt{a}|$ as well as possible. *Hint.* There should be something related to an iterative sequence $\{b_n\}$ satisfying

$$b_{n+1} \leq M b_n^2$$

You should prove that the sequence converges to zero faster than any geometric progression.

Context: This algorithm is referred to as **Newton's method**. It is a rapidly converging algorithm to find zeros/fixed points of functions, capable of giving very precise approximations within very few steps. A variation of it, called the **Nash-Moser technique**, is a very powerful tool for proving the existence of solutions to nonlinear differential equations.

6 Stability via Linearizations

Problems Related to Fundamental Definitions

11/18: **1.** Write down the detail of calculations for the following proposition: Let A be an $n \times n$ real matrix with one of its eigenvalues having positive real part. Then the fixed point $x_0 = 0$ of the linear autonomous system $y' = Ay$ is not Lyapunov stable.

Proof. To prove that x_0 is not Lyapunov stable, it will suffice to show that for all neighborhoods $B(x_0, \delta)$ ($\delta > 0$), there exists $x \in B(x_0, \delta)$ and $t \geq 0$ such that $\phi_t(x) \notin B(x_0, \delta)$. Let $B(x_0, \delta)$ be an arbitrary neighborhood of x_0. We know that $y' = Ay$ is solved by $y = \mathrm{e}^{tA}x$; hence, the solution for any x is given by

$$\phi_t(x) = \mathrm{e}^{tA}x$$

Pick x to be in the span of the eigenvector of A corresponding to the eigenvalue λ with positive real part, and pick it in particular to have magnitude less than δ. Let x be the first entry in the matrix Q of (generalized) eigenvectors of A. Then since $\mathrm{e}^{tA}x = Q\mathrm{e}^{t\Lambda}Q^{-1}x$, we have that $\phi_t(x) = \mathrm{e}^{t\lambda}x$. Thus,

$$\begin{aligned}
|\phi_t(x)| &= |\mathrm{e}^{t(\sigma+i\beta)}x| \\
&= \mathrm{e}^{t\sigma}|\mathrm{e}^{i\beta t}x| \\
&= \mathrm{e}^{t\sigma}|\cos(\beta t)x + i\sin(\beta t)x| \\
&= \mathrm{e}^{t\sigma}\sqrt{\cos^2(\beta t)|x|^2 + \sin^2(\beta t)|x|^2} \\
&= \mathrm{e}^{t\sigma}|x|
\end{aligned}$$

Therefore, as $t \to +\infty$, $|\phi_t(x)| \to +\infty$, so we can pick a t such that $|\phi_t(x)| \geq \delta$, i.e., $\phi_t(x) \notin B(x_0, \delta)$, as desired. □

2. (1) For the system that describes the pendulum with friction, namely

$$\begin{pmatrix} \theta \\ \omega \end{pmatrix}' = \begin{pmatrix} \omega \\ -\eta\omega - g/l\sin\theta \end{pmatrix}$$

where $\eta \geq 0$, compute its linearization at the fixed point $(\pi, 0)$ and the eigenvalues of this linearization. Show that this fixed point is not Lyapunov stable (you may cite the unproved instability theorem from class).

Proof. The linearization is

$$\begin{aligned}
A &= \begin{pmatrix} \frac{\partial}{\partial\theta}(\omega)\big|_{(\pi,0)} & \frac{\partial}{\partial\omega}(\omega)\big|_{(\pi,0)} \\ \frac{\partial}{\partial\theta}\left(-\eta\omega - \frac{g}{l}\sin\theta\right)\big|_{(\pi,0)} & \frac{\partial}{\partial\omega}\left(-\eta\omega - \frac{g}{l}\sin\theta\right)\big|_{(\pi,0)} \end{pmatrix} \\
&= \begin{pmatrix} [0]_{(\pi,0)} & [1]_{(\pi,0)} \\ \left[-\frac{g}{l}\cos\theta\right]_{(\pi,0)} & [-\eta]_{(\pi,0)} \end{pmatrix}
\end{aligned}$$

$$\boxed{A = \begin{pmatrix} 0 & 1 \\ g/l & -\eta \end{pmatrix}}$$

The eigenvalues of A are

$$\boxed{\lambda = \frac{-\eta \pm \sqrt{\eta^2 + 4g/l}}{2}}$$

Thus, since $4g/l$ is positive and hence $\sqrt{\eta^2 + 4gl} > \eta$ or $-\eta + \sqrt{\eta^2 + 4g/l} > 0$, one of the eigenvalues of the linearization is positive, and hence has a positive rel part. It follows by the instability theorem from class that $(\pi, 0)$ is not Lyapunov stable. □

(2) Prove that if $\eta > 0$, then any orbit of the system in (1) will converge to the fixed point $(0,0)$.

Proof. To prove the claim, it will suffice to show that $(0,0)$ is asymptotically stable. To do so, the theorem from class tells us that it will suffice to verify that all eigenvalues of the linearization have negative real parts. From class, we have that the linearization of the system in (1) at $(0,0)$ is

$$A = \begin{pmatrix} 0 & 1 \\ -g/l & -\eta \end{pmatrix}$$

Thus, computing the eigenvalues as above, we have that

$$\lambda = \frac{-\eta \pm \sqrt{\eta^2 - 4g/l}}{2}$$

Consequently, if $\eta > 0$, then both eigenvalues have negative real parts ($-\eta$ will be less than zero, $\sqrt{\eta^2 - 4g/l}$ will be less than η if real and irrelevant if imaginary). □

3. For the following planar vector fields, find all of the fixed points, compute the linearization of the system at these fixed points, and determine the stability of these fixed points.

(1)

$$\begin{pmatrix} -2x(x-1)(2x-1) \\ -2y \end{pmatrix}$$

Proof. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field defined by sending $(x,y)^T$ to the above. A fixed point of f is a point where $f(x,y) = (0,0)$. It follows by solving the system of equations

$$\begin{cases} -2x(x-1)(2x-1) & = 0 \\ -2y & = 0 \end{cases}$$

that the fixed points of f are

$$x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad x_2 = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}$$

The linearizations are thus, respectively,

$$A_0 = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \qquad A_1 = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$$

Since these are all diagonal matrices, the eigenvalues can be read off the diagonal without further manipulation. Since all eigenvalues of A_1, A_2 are negative,

$$\boxed{x_0, x_1 \text{ are asymptotically stable.}}$$

Since one eigenvalue of A_2 is positive,

$$\boxed{x_2 \text{ is not even Lyapunov stable.}}$$

□

(2)

$$\begin{pmatrix} x(y+2x-2) \\ y(y+x-3) \end{pmatrix}$$

Proof. Analogously to part (1), the fixed points are

$$x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad x_2 = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \qquad x_3 = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

Thus, the linearizations are

$$A_0 = \begin{pmatrix} -2 & 1 \\ 1 & -3 \end{pmatrix} \qquad A_1 = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \qquad A_3 = \begin{pmatrix} -2 & 1 \\ 1 & 4 \end{pmatrix}$$

Computing eigenvalues, we learn that

$$\boxed{x_0 \text{ is asymptotically stable.}}$$

and

$$\boxed{x_1, x_2, x_3 \text{ are not even Lyapunov stable.}}$$

□

(3)

$$\begin{pmatrix} x(2-y-2x) \\ y(3-3y-x) \end{pmatrix}$$

Proof. Analogously to part (1), the fixed points are

$$x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad x_3 = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$$

Thus, the linearizations are

$$A_1 = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \quad A_2 = \begin{pmatrix} -2 & -1 \\ -1 & 2 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & -1 \\ -1 & -3 \end{pmatrix} \quad A_4 = \begin{pmatrix} -6/5 & -1 \\ -1 & -12/5 \end{pmatrix}$$

Computing eigenvalues, we learn that

$$\boxed{x_3 \text{ is asymptotically stable.}}$$

and

$$\boxed{x_0, x_1, x_2 \text{ are not even Lyapunov stable.}}$$

□

4. Let A be an $n \times n$ constant real matrix so that all eigenvalues of A have negative real part. Let f be a smooth vector field with a fixed point x_0 so that the linearization of $f(x)$ at x_0 equals A. Prove that

$$L(x) = \int_0^\infty |\mathrm{e}^{\tau A}(x-x_0)|^2 \mathrm{d}\tau$$

is a strict Lyapunov function for the system $y' = f(y)$ near the fixed point x_0. What is the geometric shape of the level sets $L(x) < \infty$?

Proof. Since L is a composition of continuous functions, L is, itself, continuous.

If $x = x_0$, then

$$L(x_0) = \int_0^\infty |\mathrm{e}^{\tau A}(x_0 - x_0)|^2 \mathrm{d}\tau = 0$$

If $x \neq x_0$, then $x - x_0 \neq 0$. Since all eigenvalues of $\mathrm{e}^{\tau A}$ have negative real part, we know that in particular, no eigenvalue is zero, so $\ker(\mathrm{e}^{\tau A}) = \{0\}$ for all τ. Thus, $\mathrm{e}^{\tau A}(x - x_0) \neq 0$. Hence, the entire integrand is strictly positive. But this means that the integral must be strictly positive, as desired.

By the Leibniz integral rule,

$$\begin{aligned} \dot{L}(x) &= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}x}\left(|\mathrm{e}^{\tau A}(x_0 - x_0)|^2\right)\mathrm{d}\tau \\ &= \int_0^\infty 2|\mathrm{e}^{\tau A}(x_0 - x_0)| \cdot \mathrm{e}^{\tau A}\mathrm{d}\tau \\ &= \nabla L(x) \cdot f(x) \\ &< 0 \end{aligned}$$

The level sets will look like n-dimensional ellipsoids. □

5. Let A be an $n \times n$ constant real matrix so that all eigenvalues of A have negative real part. Let f be a smooth vector field with a fixed point $x = 0$ so that the linearization of $f(x)$ at 0 equals A. Let $\delta > 0$ be a sufficiently small positive constant. Use the Banach fixed point theorem to give a direct proof of the stability theorem, by considering the space consisting of continuous mappings from $[0, +\infty)$ to the closed ball $\bar{B}(0, \delta)$. *Hint*: That is, you need to solve the integral equation

$$y(t) = \mathrm{e}^{tA}x + \int_0^t \mathrm{e}^{(t-\tau)A} g(y(\tau))\mathrm{d}\tau$$

where $g(x) = f(x) - Ax$ in the space of continuous mappings from $[0, +\infty)$ to the closed ball $\bar{B}(0, \delta)$ for a suitable choice of x.

Proof. Let $x \in \bar{B}(0, \delta)$. WTS: $\phi_t(x) \to 0$ as $t \to +\infty$. $\square$

Stability of Centrifugal Governor

1. Consider a real cubic polynomial

$$p(x) = x^3 + a_2x^2 + a_1x + a_0$$

Prove the **Roth-Hurwitz criterion** for degree 3 polynomials: In order that p is stable, i.e., every root of $p(x)$ has negative real part, it is necessary and sufficient that the following inequalities all hold.

$$a_2 > 0 \qquad a_1 > 0 \qquad a_0 > 0$$

$$a_2a_1 > a_0$$

Hint: $p(x)$ has at least one real root, and we might denote it by $-\lambda$. Factorize $p(x) = (x+\lambda)(x^2+bx+c)$. The necessary and sufficient condition for roots of $x^2 + bx + c$ to have negative real part is $b > 0$ and $c > 0$.

Proof. Suppose first that p is stable. Taking the hint, we know that p has at least one real root, which we can denote by $-\lambda$ for some $\lambda \in \mathbb{R}_+$. Thus,

$$\begin{aligned}
p(x) &= (x + \lambda)(x^2 + bx + c) \\
&= x^3 + bx^2 + cx + \lambda x^2 + b\lambda x + c\lambda \\
&= x^3 + (b + \lambda)x^2 + (c + b\lambda)x + c\lambda
\end{aligned}$$

Moreover, taking the hint again, we know that we must have $b, c > 0$. But if $\lambda, b, c > 0$, then

$$a_2 = b + \lambda > 0 \qquad a_1 = c + b\lambda > 0 \qquad a_0 = c\lambda > 0$$

$$a_2a_1 = (b + \lambda)(c + b\lambda) = \lambda b^2 + (c + \lambda^2)b + c\lambda > c\lambda = a_0$$

as desired.

Now suppose that

$$a_2 > 0 \qquad a_1 > 0 \qquad a_0 > 0$$

$$a_2a_1 > a_0$$

Once again p has a real root we can denote by $-\lambda$, allowing us to factor p into the above form. This allows us to deduce, as above, that

$$b + \lambda > 0 \qquad c + b\lambda > 0 \qquad c\lambda > 0$$

$$\lambda b^2 + (c + \lambda^2)b > 0$$

It is now our job to deduce that these four inequalities imply $b, c > 0$. Since $c\lambda > 0$ for $\lambda > 0$, we can divide both sides by λ to learn that $c > 0$. As to b, we can rewrite the last inequality above in the form

$$(c + b\lambda + \lambda^2)b > 0$$

This combined with the fact that $c + b\lambda > 0$ and $\lambda^2 > 0$, hence, $c + \lambda b + \lambda^2 > 0$ implies that we can divide both sides above by this quantity without flipping the inequality. Therefore, $b > 0$, too, as desired. $\square$

2. Consider the system

$$\begin{aligned}
\varphi' &= \psi \\
\psi' &= -\frac{b}{m}\psi + n^2\omega^2 \sin\varphi\cos\varphi - g\sin\varphi \\
\omega' &= \frac{1}{J}(k\cos\varphi - F)
\end{aligned}$$

(1) Suppose $0 < F/k < 1$. Then the system has a unique fixed point $(\varphi_0, \psi_0, \omega_0)$. Find this fixed point, and compute the linearization at the fixed point.

Proof. We first find the fixed point. The fixed point is in particular the point $(\varphi_0, \psi_0, \omega_0)$ such that plugging these values into the system of differential equations for φ, ψ, ω, respectively, makes all of the equations equal zero. We have from the first equation that $\psi = 0$ if we want $\varphi' = 0$. We have from the third equation that $\cos\varphi = F/k$ (hence $\varphi = \cos^{-1}(F/k)$) if we want $\omega' = 0$. Using these two values (and the trigonometric deduction that if $\cos\varphi = F/k$, then $\sin\varphi = \sqrt{k^2 - F^2}$), we can solve for ω using the second equation. This yields the following as the final result.

$$\boxed{(\varphi_0, \psi_0, \omega_0) = \left(\cos^{-1}(F/k), 0, \sqrt{\frac{gk}{Fn^2}}\right)}$$

The linearization A at this fixed point is of the form

$$A = \begin{pmatrix}
\left.\frac{\partial\varphi'}{\partial\varphi}\right|_{(\varphi_0,\psi_0,\omega_0)} & \left.\frac{\partial\varphi'}{\partial\psi}\right|_{(\varphi_0,\psi_0,\omega_0)} & \left.\frac{\partial\varphi'}{\partial\omega}\right|_{(\varphi_0,\psi_0,\omega_0)} \\
\left.\frac{\partial\psi'}{\partial\varphi}\right|_{(\varphi_0,\psi_0,\omega_0)} & \left.\frac{\partial\psi'}{\partial\psi}\right|_{(\varphi_0,\psi_0,\omega_0)} & \left.\frac{\partial\psi'}{\partial\omega}\right|_{(\varphi_0,\psi_0,\omega_0)} \\
\left.\frac{\partial\omega'}{\partial\varphi}\right|_{(\varphi_0,\psi_0,\omega_0)} & \left.\frac{\partial\omega'}{\partial\psi}\right|_{(\varphi_0,\psi_0,\omega_0)} & \left.\frac{\partial\omega'}{\partial\omega}\right|_{(\varphi_0,\psi_0,\omega_0)}
\end{pmatrix}$$

We can compute all of these entries as follows.

$$\begin{aligned}
\left.\frac{\partial\varphi'}{\partial\varphi}\right|_{(\varphi_0,\psi_0,\omega_0)} &= 0 \\
\left.\frac{\partial\varphi'}{\partial\psi}\right|_{(\varphi_0,\psi_0,\omega_0)} &= 1 \\
\left.\frac{\partial\varphi'}{\partial\omega}\right|_{(\varphi_0,\psi_0,\omega_0)} &= 0 \\
\left.\frac{\partial\psi'}{\partial\varphi}\right|_{(\varphi_0,\psi_0,\omega_0)} &= n^2\omega_0^2(\cos^2\varphi_0 - \sin^2\varphi_0) - g\cos\varphi_0 \\
&= n^2 \cdot \frac{gk}{Fn^2}\left(\frac{F^2}{k^2} - k^2 + F^2\right) - \frac{gF}{k} \\
&= \frac{gk(F^2 - k^2)}{F} \\
\left.\frac{\partial\psi'}{\partial\psi}\right|_{(\varphi_0,\psi_0,\omega_0)} &= -\frac{b}{m}
\end{aligned}$$

$$\begin{aligned}
\left.\frac{\partial\psi'}{\partial\omega}\right|_{(\varphi_0,\psi_0,\omega_0)} &= 2n^2\omega_0\sin\varphi_0\cos\varphi_0 \\
&= 2n^2\sqrt{\frac{gk}{Fn^2}}\cdot\sqrt{k^2-F^2}\cdot\frac{F}{k} \\
&= 2n\sqrt{\frac{gF(k^2-F^2)}{k}} \\
\left.\frac{\partial\omega'}{\partial\varphi}\right|_{(\varphi_0,\psi_0,\omega_0)} &= -\frac{1}{J}(k\sin\varphi_0+F) \\
&= -\frac{1}{J}(k\sqrt{k^2-F^2}+F) \\
\left.\frac{\partial\omega'}{\partial\psi}\right|_{(\varphi_0,\psi_0,\omega_0)} &= 0 \\
\left.\frac{\partial\omega'}{\partial\omega}\right|_{(\varphi_0,\psi_0,\omega_0)} &= 0
\end{aligned}$$

Therefore, we have that

$$\boxed{A=\begin{pmatrix} 0 & 1 & 0 \\ \frac{gk(F^2-k^2)}{F} & -\frac{b}{m} & 2n\sqrt{\frac{gF(k^2-F^2)}{k}} \\ -\frac{1}{J}(k\sqrt{k^2-F^2}+F) & 0 & 0 \end{pmatrix}}$$

□

(2) Compute the characteristic polynomial of the linearization. Use the Roth-Hurwitz criterion to find the sufficient and necessary condition of stability of the fixed point in terms of b, m, J, F and ω_0.

Proof. The characteristic polynomial of the linearization from part (1) is

$$\boxed{\chi_A(z)=-z^3-\frac{b}{m}z^2+\frac{kg(F^2-k^2)}{F}\cdot z-\frac{2n(F+k\sqrt{k^2-F^2})\sqrt{Fg(k^2-F^2)}}{\sqrt{k}J}}$$

Since $k=Fn^2\omega_0^2/g$, we can rewrite the characteristic polynomial as

$$\chi_A(z)=-z^3-\frac{b}{m}z^2+\frac{n^2\omega_0^2F^2(g^2-n^4\omega_0^4)}{g^2}\cdot z-\frac{2F^2(\sqrt{n^4\omega_0^4-g^2}+Fn^2\omega_0^2(\frac{n^4\omega_0^4}{g^2}-1))}{J\omega_0}$$

Thus, applying the Roth-Hurwitz criterion, we have that

$$\frac{b}{m}>0 \qquad -\frac{n^2\omega_0^2F^2(g^2-n^4\omega_0^4)}{g^2}>0 \qquad \frac{2F^2(\sqrt{n^4\omega_0^4-g^2}+Fn^2\omega_0^2(\frac{n^4\omega_0^4}{g^2}-1))}{J\omega_0}>0$$

$$-\frac{bn^2\omega_0^2F^2(g^2-n^4\omega_0^4)}{mg^2}>\frac{2F^2(\sqrt{n^4\omega_0^4-g^2}+Fn^2\omega_0^2(\frac{n^4\omega_0^4}{g^2}-1))}{J\omega_0}$$

□

(3) The ratio $\nu=\omega_0/2F$ is usually called the **non-uniformity** of the governor. It characterizes how the change of load alters the stable speed ω_0. From the stability condition obtained in part (2), answer the following question: By increasing the mass m, will stability of the system be enhanced or harmed? What about the friction b, the inertia J, and non-uniformity ν?

Proof. Increasing m $\boxed{\text{decreases}}$ the stability. Increasing m makes every condition in which it is present less "extreme" (e.g., increasing m makes $b/m \to 0$ and the left term in the last inequality closer to the right term).

Increasing b $\boxed{\text{increases}}$ the stability. It makes every condition in which it is present more extreme.

Increasing J $\boxed{\text{decreases}}$ the stability. It makes the third condition less extreme to a greater extent than it makes the fourth condition more extreme. Increasing ν $\boxed{\text{enhances}}$ the stability. Increasing ν is the same as increasing the ratio of ω_0 to F. If we do this in the above criterion, we can see that this always makes the condition more "extreme." ☐

(4) Are these changes good or harmful for stability of the governor? If you are a designer of a steam engine and want to increase the stability of this controller system, what should you do?

Proof. These changes were overall $\boxed{\text{harmful}}$ (hence the unreliability described in the intro to the question). Increasing the weight and reducing friction were harmful, while increasing speed (and hence ν) and reducing J was helpful. To increase the stability, you should decrease the mass, increase the friction, decrease the moment of inertia, and increase the non-uniformity. ☐

7 Nonlinear Stability

Problems Related to Fundamental Definitions

12/2: **1.** Consider the planar system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -x \\ x^2 + y \end{pmatrix}$$

(1) Find the explicit expression of the flow $\phi_t(z, w)$ and determine the stable and unstable manifolds of the fixed point $(0, 0)$. Sketch the phase portrait.

Proof. First off, note that the initial condition is $(z, w)^T$. From the first coordinate IVP, we have the general solution

$$x' = -x, \quad x(0) = z \quad \iff \quad x(t) = z\mathrm{e}^{-t}$$

Using this result, we can solve the second coordinate IVP using the Duhamel formula, as follows.

$$y' = (1)y + z^2\mathrm{e}^{-2t}, \quad y(0) = w \quad \iff \quad y(t) = w\mathrm{e}^t + \int_0^t \mathrm{e}^{t-\tau} z^2 \mathrm{e}^{-2\tau} \mathrm{d}\tau$$

Simplifying the above and combining the results yields the following explicit expression for the flow.

$$\boxed{\phi_t \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} z\mathrm{e}^{-t} \\ w\mathrm{e}^t - \frac{z^2\mathrm{e}^t(\mathrm{e}^{-3t}-1)}{3} \end{pmatrix}}$$

The stable manifold is the set of all points x such that $\phi_t(x) \to 0$ as $t \to +\infty$. Evaluating componentwise, we see from the first component $x(t) = z\mathrm{e}^{-t}$ that z can take any value and we will still have $x(t) \to 0$ as $t \to +\infty$. Note that w can also take on any value without affecting the convergence of $x(t)$. Thus, the first component does not yield any constraints.

The second component, on the other hand, does. Rewrite the expression and regroup the terms to all those of the form e^{at} where $a > 0$ and all those e^{at} where $a < 0$.

$$\begin{aligned} y(t) &= w\mathrm{e}^t - \frac{z^2}{3}\mathrm{e}^{-2t} + \frac{z^2}{3}\mathrm{e}^t \\ &= \left[w + \frac{z^2}{3}\right]\mathrm{e}^t + \left[-\frac{z^3}{3}\right]\mathrm{e}^{-2t} \end{aligned}$$

Thus, to get convergence of the above term, we must have

$$\begin{aligned} w + \frac{z^2}{3} &= 0 \\ w &= -\frac{z^2}{3} \end{aligned}$$

z can still take on any value.

Therefore,

$$\boxed{W_s(0) = \left\{ \left(z, -\frac{z^2}{3}\right) \,\middle|\, z \in \mathbb{R} \right\}}$$

We can derive the solution for the unstable manifold similarly. This time, we want $\phi_t(x) \to 0$ as $t \to -\infty$. From the first component, we can see that the only case in which $x(t) = z\mathrm{e}^{-t} \to 0$ as $t \to -\infty$ is if $z = 0$. We don't have any constraints on w from the first component.

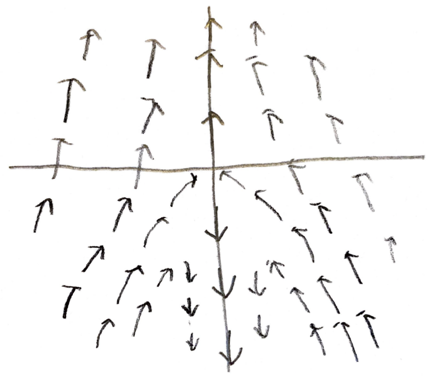
With $z = 0$, the second component becomes $y(t) = w\mathrm{e}^t$. This will converge to 0 as $t \to -\infty$ for any w.

Therefore,

$$\boxed{W_u(0) = \{(0, w) \mid w \in \mathbb{R}\}}$$

i.e., the unstable manifold is the y-axis.

Sketching the phase portrait gives something like this:



□

(2) Pretend that you do not know how to solve the system, and compute the flow of the linearized system, together with the stable and unstable subspaces. Sketch the phase portrait.

Proof. The Jacobian at 0 is

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus, the flow of the linearized system is

$$\begin{aligned} \phi_t \begin{pmatrix} z \\ w \end{pmatrix} &= \mathrm{e}^{tA} \begin{pmatrix} z \\ w \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{e}^{-t} & 0 \\ 0 & \mathrm{e}^{t} \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} \end{aligned}$$

$$\boxed{\phi_t \begin{pmatrix} z \\ w \end{pmatrix} = z\mathrm{e}^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + w\mathrm{e}^{t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

The eigenvalues and eigenvectors are

$$\lambda_1 = 1 \qquad \lambda_2 = -1$$

$$v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so the stable subspace is the $\boxed{x\text{-axis}}$ ($\text{span}(v_2)$) and the unstable subspace is the $\boxed{y\text{-axis}}$ ($\text{span}(v_1)$). Sketching the phase portrait gives something like this:

□

2. We did not provide any grant on stability of a fixed point of an autonomous system if the linearization at that point has purely imaginary eigenvalues. This exercise gives several examples with more details. As per the Canvas announcement, skip (1)-(2) and only do (3), but also discuss therein the case $\mu < 0$ there now.

(3) Fix $\mu > 0$. Show that the fixed point $(0,0)$ of the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} y - \mu x(x^2+y^2) \\ -x - \mu y(x^2+y^2) \end{pmatrix}$$

is globally asymptotically stable, no matter how small μ is; that is, every orbit is attracted to it. This shows that periodic solutions of the harmonic oscillator are not persisted under higher order perturbation. *Hint*: Find out the differential equation satisfied by the polar coordinates of an orbit.

Proof. Let $r^2 = x^2 + y^2$ and $\tan(\theta) = y/x$. Then

$$\begin{aligned}
2rr' &= 2xx' + 2yy' \\
r' &= \frac{xx' + yy'}{r} \\
&= \frac{x[y - \mu x(x^2+y^2)] + y[-x - \mu y(x^2+y^2)]}{r} \\
&= \frac{xy}{r} - \mu x^2 r - \frac{xy}{r} - \mu y^2 r \\
&= -\mu r(x^2+y^2) \\
&= -\mu r^3
\end{aligned}$$

and

$$\begin{aligned}
\sec^2\theta \cdot \theta' &= \frac{xy' - yx'}{x^2} \\
\theta' &= \frac{xy' - yx'}{x^2} \cdot \cos^2\theta \\
&= \frac{xy' - yx'}{x^2} \cdot \frac{x^2}{r^2} \\
&= \frac{xy' - yx'}{r^2} \\
&= \frac{x[-x - \mu y(x^2+y^2)] - y[y - \mu x(x^2+y^2)]}{r^2} \\
&= -\frac{x^2}{r^2} - \mu xy - \frac{y^2}{r^2} + \mu xy \\
&= -\frac{x^2+y^2}{r^2} \\
&= -1
\end{aligned}$$

Thus, we can transform the original equation to

$$\begin{pmatrix} r \\ \theta \end{pmatrix}' = \begin{pmatrix} -\mu r^3 \\ -1 \end{pmatrix}$$

for $r \geq 0$ and $\theta \in \mathbb{R}$. Solving the componentwise separable ODEs, we can deduce that the flow is

$$\phi_t \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{z^2}{1+2\mu z^2 t}} \\ w - t \end{pmatrix}$$

Consequently, for arbitrary $\mu > 0$ and $x = (z, w)^T \in \mathbb{R}^2$, the denominator $1 + 2\mu z^2 t \to \infty$ as $t \to +\infty$. Therefore, $r \to 0$ as $t \to +\infty$, so since we're using polar coordinates, $\phi_t(x) \to 0$, as desired.

On the other hand, let $\mu < 0$ and $x = (z, w)^T \in \mathbb{R}^2$ be arbitrary. Then as $t \to -1/2\mu z^2$ (a positive quantity since $\mu < 0$), the denominator $1 + 2\mu z^2 t \to 0$. Therefore, the overall term and consequently r blows up in finite time. ☐

3. Consider the Duffing equation

$$x'' + bx' - x + x^3 = 0, \quad b > 0$$

(1) Convert it to a first order autonomous system. Find an "energy function" of the system.

Proof. Let $y = x$ and $z = x'$. Then we have that

$$\boxed{\begin{pmatrix} y \\ z \end{pmatrix}' = \begin{pmatrix} z \\ -bz + y - y^3 \end{pmatrix}}$$

We can derive an energy function as follows.

$$\begin{aligned} 0 &= x'' + bx' + \underbrace{x^3 - x}_{U'(x)} \\ &= x'x'' + b|x'|^2 + x'U'(x) \\ &= \left(\frac{1}{2}(x')^2\right)' + b|x'|^2 + (U(x))' \\ -b|x'|^2 &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}(x')^2 + U(x)\right) \end{aligned}$$

Therefore, the energy function is constantly decreasing. ☐

(2) There are three fixed points for the system in part (1). Determine the local behavior of the fixed points using the stable manifold theorem and Hartman linearization theorem.

Proof. The fixed points are

$$\begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

We first analyze them via the stable manifold theorem.

The linearization of the system at $(0, 0)^T$ is

$$A = \begin{pmatrix} 0 & 1 \\ 1 & -b \end{pmatrix}$$

Taking the characteristic polynomial and calculating eigenvalues can be done as follows.

$$\begin{aligned} 0 &= \chi_A(z) \\ &= -z(-b - z) - 1 \\ &= z^2 + bz - 1 \\ z &= \frac{-b \pm \sqrt{b^2 + 4}}{2} \end{aligned}$$

Thus, one eigenvalue is always greater than zero and one is always less than zero, so we never have purely imaginary eigenvalues. Thus, $(0, 0)^T$ is hyperbolic for all $b > 0$, and $\dim \mathbb{E}_s = \dim \mathbb{E}_u = 1$. Therefore,

$$\boxed{(0, 0)^T \text{ is locally a saddle point for all } b > 0.}$$

The linearization of the system at both of the other fixed points is

$$B = \begin{pmatrix} 0 & 1 \\ -2 & -b \end{pmatrix}$$

Characteristic polynomial and eigenvalues:

$$\begin{aligned} 0 &= \chi_A(z) \\ &= -z(-b-z)+2 \\ &= z^2+bz+2 \\ z &= \frac{-b\pm\sqrt{b^2-8}}{2} \end{aligned}$$

Thus, if $b<\sqrt{8}$, the eigenvalues are complex conjugates with negative real part and hence $(1,0)^T,(-1,0)^T$ are both similar to spiral sinks. If $b=\sqrt{8}$, there is only one eigenvalue $(-\sqrt{2})$ and only one eigenvector $(-\sqrt{2},2)$, so B is not diagonalizable and hence $(1,0)^T,(-1,0)^T$ are similar to the distorted $y=x\pm x\log x$ form. If $b>\sqrt{8}$, then the eigenvalues are real and negative. Here, we have a case where the fixed point is hyperbolic, $\dim \mathbb{E}_s=2$, and $\dim \mathbb{E}_u=0$. Consequently, $(1,0)^T,(-1,0)^T$ are sinks in this case. Therefore,

$$\boxed{\begin{array}{l} (1,0)^T,(-1,0)^T \text{ are locally spiral sinks for all } 0<b<\sqrt{2} \\ (1,0)^T,(-1,0)^T \text{ are locally similar to } x\pm x\log x \text{ for } b=\sqrt{2} \\ (1,0)^T,(-1,0)^T \text{ are locally sinks for all } b>\sqrt{2} \end{array}}$$

By the Hartman linearization theorem, we can further analyze every $b>0$ for $(0,0)^T$ and $b>\sqrt{2}$ for $(1,0)^T,(-1,0)^T$. Indeed, this theorem tells us that not only do smooth, tangent submanifolds exist of the above dimensions, but all orbits near the fixed point but not lying on one of the stable/unstable manifolds are slight distortions of the corresponding linearized cases analyzed in Lecture 5.1. ☐

(3) Show that the global stable set of part (1) consists of two curves C_1,C_2, starting from the fixed point $(0,0)$ and symmetric with respect to that point, tending to infinity. Show that every point not lying on C_1 or C_2 will be attracted to one of the other two fixed points. You may use the phase portrait drawer (link) to get some numerical inspiration.

Proof. ☐

4. Study the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -x-y^2 \\ x^2+y \end{pmatrix}$$

according to the general steps mentioned in class. For your convenience, you may use the phase portrait drawer (same as above) to get some numerical inspiration for your conjectures.

(1) Determine the fixed points of the system.

Proof. This is equivalent to solving the system of equations

$$\begin{aligned} -x-y^2 &= 0 \\ x^2+y &= 0 \end{aligned}$$

Substitute $x=-y^2$ (from the first equation) into the second equation and solve for y:

$$\begin{aligned} 0 &= y+y^4 \\ 0 &= y(y^3+1) \\ y &= 0,-1 \end{aligned}$$

It follows from $x = -y^2$ that the corresponding values of x are $0, -1$, respectively. Therefore, the fixed points are

$$\boxed{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}}$$

□

(2) Study the local behavior of the system at the fixed points: Stability, tangents of the stable and unstable manifolds. *Hint*: One of the fixed points has purely imaginary eigenvalues, so nothing about stability can be said by linearization.

Proof. The linearization of the system at $(0,0)^T$ is

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus, the eigenvalues and eigenvectors are

$$\lambda_1 = -1 \qquad \lambda_2 = 1$$

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

It follows that $(0,0)^T$ is a hyperbolic fixed point. Consequently, by the stable manifold theorem, both the stable and the unstable manifolds are of dimension 1, but the former is tangent to the x-axis and the latter is tangent to the y-axis. Overall $(0,0)^T$ is a saddle point.
The linearization of the system at $(-1,-1)^T$ is

$$B = \begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}$$

with corresponding eigenvalues $\lambda = \pm i\sqrt{3}$. Therefore, as per the hint, nothing about stability can be said by the linearization. □

(3) In fact, this system admits implicit solutions of the form $F(x,y) = c$, where F is a polynomial in x, y. Find a polynomial F that meets this requirement, and show that the system has infinitely many periodic solutions.

Proof. We take

$$\begin{aligned} \frac{x'}{y'} &= \frac{-x-y^2}{x^2+y} \\ (x^2+y)\frac{\mathrm{d}x}{\mathrm{d}t} &= (-x-y^2)\frac{\mathrm{d}y}{\mathrm{d}t} \\ \underbrace{(x^2+y)}_{\partial F/\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \underbrace{(x+y^2)}_{\partial F/\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \end{aligned}$$

Thus,

$$\begin{aligned} \frac{\partial F}{\partial x} &= x^2 + y \\ F(x,y) &= \frac{1}{3}x^3 + yx + g(y) \\ \frac{\partial F}{\partial y} &= x + \frac{\mathrm{d}g}{\mathrm{d}y} \\ x + y^2 &= x + \frac{\mathrm{d}g}{\mathrm{d}y} \\ g(y) &= \frac{1}{3}y^3 + c \end{aligned}$$

so if we let

$$\boxed{F(x,y) = \frac{1}{3}(x^3+y^3)+xy}$$

then $F(x,y)=c$ is a solution for all $c \in \mathbb{R}$. In particular, when $0 < c < 1/3$, a connected set of points satisfying $F(x,y)=c$ form an ellipse. ☐

(4) Use the polynomial you find in part (3) to explicitly determine what the stable and unstable set should look like. *Hint*: The vector field is symmetric with respect to the bisector of the 1st and 3rd quadrants, and as a result, the global stable and unstable set should coincide with each other.

Proof. hi ☐

5. State and prove a one-dimensional version of the Poincaré-Bendixson theorem.

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a smooth function and $f(0) = 0$. Consider the autonomous system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} xf(x^2+y^2) - y \\ yf(x^2+y^2) + x \end{pmatrix}$$

(1) Find the differential equation satisfied by the polar coordinate representation $(r(t), \varphi(t))$ of the orbit.

Proof. We proceed analogously to Q2(3). Thus, we convert to polar coordinates via

$$\begin{aligned} r' &= \frac{xx'+yy'}{r} \\ &= \frac{x[xf(x^2+y^2)-y]+y[yf(x^2+y^2)+x]}{r} \\ &= \frac{x^2f(x^2+y^2)}{r} - \frac{xy}{r} + \frac{y^2f(x^2+y^2)}{r} + \frac{xy}{r} \\ &= \frac{(x^2+y^2)f(x^2+y^2)}{r} \\ &= rf(r^2) \end{aligned}$$

and

$$\begin{aligned} \theta' &= \frac{xy'-yx'}{r^2} \\ &= \frac{x[yf(x^2+y^2)+x]-y[xf(x^2+y^2)-y]}{r^2} \\ &= \frac{xyf(x^2+y^2)}{r^2} + \frac{x^2}{r^2} - \frac{xyf(x^2+y^2)}{r^2} + \frac{y^2}{r^2} \\ &= \frac{x^2+y^2}{r^2} \\ &= 1 \end{aligned}$$

This yields as our final answer

$$\boxed{\begin{pmatrix} r \\ \theta \end{pmatrix}' = \begin{pmatrix} rf(r^2) \\ 1 \end{pmatrix}}$$

☐

(2) Prove that if $p_0 > 0$ is an isolated zero of f, then the circle $r = \sqrt{p_0}$ is a limit cycle of the system. What if the zero is not isolated?

Proof. A limit cycle is a periodic orbit. Let $x = (\sqrt{p_0}, \theta)$ be some point lying on the described circle. Then by part (1), $r'(x) = 0$ and $\theta'(x) = 1$. Thus, the integral curve at x is the described circle. Depending on the specific nature of f, the circle can be either an attracting or repelling limit cycle for spirals both inside and outside.

If the zero is not isolated, then we have a dense region (an annulus, actually) of concentric circle, each of which is a limit cycle and a self-contained, nonintersecting periodic orbit. □

(3) Give an example of a planar system with infinitely many limit cycles. The famous Hilbert's 16th problem asks how many limit cycles there could be for a planar autonomous system of polynomial entries. The problem is unsolved even for quadratic polynomials: It is known that the number can be 0, 1, 2, 3, 4, but it is not known whether the number has an upper bound or not. Hence, your answer cannot be a polynomial system.

Proof. We could choose

$$\boxed{\begin{pmatrix} r \\ \theta \end{pmatrix}' = \begin{pmatrix} \sin(r) \\ 1 \end{pmatrix}}$$

Then we have circular limit cycles of every radius r satisfying $\sin(r) = 0$. Explicitly, we have limit cycle circles of radius

$$r = \pi n, \quad n \in \mathbb{N}_0$$

For all values r at which $r'' < 0$, we have an attracting limit cycle. This is because in this case, at values slightly greater than r we have $r' < 0$ so that the flow gets pulled in, and at values slightly less than r we have $r' > 0$ so that the flow gets pushed out. It will be the other way around for all values of r at which $r'' > 0$, i.e., we have a repelling limit cycles at these values. □

Study of Chemical Reactions

In this section, you will explore how ordinary differential equations can be used to study long-term behavior of some chemical reactions.

1. Enzyme kinetics. Consider an enzyme-catalyzed chemical reaction

$$\mathrm{E} + \mathrm{S} \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} \mathrm{ES} \xrightarrow{k_2} \mathrm{E} + \mathrm{P}$$

Here k_1, k_2, k_{-1} are rate constants, which are positive. The materials involved in this reaction include the substrate S, the enzyme E, the enzyme-substrate complex ES, and the final product P. The concentrations of these materials (unit: mole per unit volume) satisfy the differential system

$$\begin{aligned} \frac{\mathrm{d[S]}}{\mathrm{d}t} &= -k_1[\mathrm{E}][\mathrm{S}] + k_{-1}[\mathrm{ES}] \\ \frac{\mathrm{d[E]}}{\mathrm{d}t} &= -k_1[\mathrm{E}][\mathrm{S}] + (k_{-1} + k_2)[\mathrm{ES}] \\ \frac{\mathrm{d[ES]}}{\mathrm{d}t} &= k_1[\mathrm{E}][\mathrm{S}] - (k_{-1} + k_2)[\mathrm{ES}] \\ \frac{\mathrm{d[P]}}{\mathrm{d}t} &= k_2[\mathrm{ES}] \end{aligned}$$

(1) Let $[\mathrm{S}]_0$ and $[\mathrm{E}]_0$ denote the initial concentrations of the substrate and enzyme, respectively. Find two first integrals for this system, and transform it into a 2D autonomous system in $x = [\mathrm{S}]$ and $y = [\mathrm{ES}]$. *Hint*: Use the law of conservation of mass. The substrate is either transformed into product or the ES complex and cannot just disappear, and the same reasoning applies for the enzyme. These two facts are reflected as some algebraic properties of the system.

Proof. By adding the second and third equations together, we get

$$\frac{\mathrm{d}[\mathrm{E}]}{\mathrm{d}t} + \frac{\mathrm{d}[\mathrm{ES}]}{\mathrm{d}t} = 0$$

from which we obtain the first integral

$$[\mathrm{E}] + [\mathrm{ES}] = c$$

for some $c \in \mathbb{R}$. In particular, this equation implies that the sum of the concentrations of the enzyme and enzyme-substrate complex is constant for all time, which makes sense since the enzyme must be in one of these two forms and cannot appear or disappear as per the law of conservation of mass. Now we determine the value of c. If this equation is valid for all time, it is valid specifically for $t = 0$, when $[\mathrm{E}] = [\mathrm{E}]_0$ and $[\mathrm{ES}] = 0$. Thus, $c = [\mathrm{E}]_0 + 0 = [\mathrm{E}]_0$, and one of the first integrals is

$$\boxed{[\mathrm{E}] + [\mathrm{ES}] = [\mathrm{E}]_0}$$

We can similarly determine by adding the first, third, and fourth equations together that the other first integral is

$$\boxed{[\mathrm{S}] + [\mathrm{ES}] + [\mathrm{P}] = [\mathrm{S}]_0}$$

Note that this also makes sense chemically as it corresponds to the conservation of the compound on which the enzyme acts in all of its forms.

It follows that we can express $\mathrm{d}[\mathrm{S}]/\mathrm{d}t$, $\mathrm{d}[\mathrm{ES}]/\mathrm{d}t$ in terms of only $[\mathrm{S}], [\mathrm{ES}]$ by substituting for $[\mathrm{E}]$ using the first integrals. In particular, we may write

$$\begin{aligned} \frac{\mathrm{d}[\mathrm{S}]}{\mathrm{d}t} &= -k_1([\mathrm{E}]_0 - [\mathrm{ES}])[\mathrm{S}] + k_{-1}[\mathrm{ES}] \\ \frac{\mathrm{d}[\mathrm{ES}]}{\mathrm{d}t} &= k_1([\mathrm{E}]_0 - [\mathrm{ES}])[\mathrm{S}] - (k_{-1} + k_2)[\mathrm{ES}] \end{aligned}$$

Note that these two ODEs capture all information in the system because the others can be derived from them using the derivatives of the first integrals. For example, the expression

$$\frac{\mathrm{d}[\mathrm{E}]}{\mathrm{d}t} + \frac{\mathrm{d}[\mathrm{ES}]}{\mathrm{d}t} = 0 \quad \Longleftrightarrow \quad \frac{\mathrm{d}[\mathrm{E}]}{\mathrm{d}t} = -\frac{\mathrm{d}[\mathrm{ES}]}{\mathrm{d}t}$$

Lastly, for the sake of simplicity, we can relabel $x = [\mathrm{S}]$ and $y = [\mathrm{ES}]$.

$$\boxed{\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= -k_1([\mathrm{E}]_0 - y)x + k_{-1}y \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= k_1([\mathrm{E}]_0 - y)x - (k_{-1} + k_2)y \end{aligned}}$$

□

(2) There is only one fixed point of the system obtained in part (1). Study the stability of that fixed point.

Proof. We assume $[\mathrm{E}]_0 > 0$ (otherwise, every point is a fixed point). Under this assumption, the sole fixed point is

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The linearization at $(0, 0)^T$ is

$$A = \begin{pmatrix} -k_1[\mathrm{E}]_0 & k_{-1} \\ k_1[\mathrm{E}]_0 & -(k_{-1} + k_2) \end{pmatrix}$$

yielding eigenvalues

$$\begin{aligned}
0 &= \chi_A(z) \\
&= (-k_1[\mathrm{E}]_0 - z)(-(k_{-1} + k_2) - z) - k_{-1}k_1[\mathrm{E}]_0 \\
&= z^2 + (k_1[\mathrm{E}]_0 + k_{-1} + k_2)z + k_1[\mathrm{E}]_0(k_{-1} + k_2) - k_1k_{-1}[\mathrm{E}]_0 \\
&= z^2 + (k_1[\mathrm{E}]_0 + k_{-1} + k_2)z + k_1k_2[\mathrm{E}]_0 \\
z &= \frac{-(k_1[\mathrm{E}]_0 + k_{-1} + k_2) \pm \sqrt{(k_1[\mathrm{E}]_0 + k_{-1} + k_2)^2 - 4k_1k_2[\mathrm{E}]_0}}{2}
\end{aligned}$$

It follows that both eigenvalues are negative and hence $(0,0)^T$ is a hyperbolic fixed point. In this case, $(0,0)^T$ is a sink (i.e., is $\boxed{\text{asymptotically stable}}$). ☐

(3) Prove that any orbit starting from the first quadrant will stay in that quadrant, and will finally be attracted to the fixed point. What is the chemical explanation for this mathematical fact?

Proof. Consider the points along the x- and y-axes. For the points along the x-axis, we have $y = 0$ and thus

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= -k_1[\mathrm{E}]_0 x \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= k_1[\mathrm{E}]_0 x
\end{aligned}$$

Importantly, the lower equation above implies that the vector field points inward toward first quadrant at every point along the x-axis.

For the points along the y-axis, we have $x = 0$ and thus

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= k_{-1}y \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= -(k_{-1} + k_2)y
\end{aligned}$$

Thus, similarly, the vector field points inward toward first quadrant at every point along the y-axis. Thus, by the proposition from Lecture 7.2, since $f(x)$ is transversal to the boundary and inward pointing, the set is invariant. ☐

2. Iodine clock reaction. Iodine clock reactions are perhaps the most interesting experiments to show to a beginner in chemistry. The chlorine dioxide-iodine-malonic acid (CDIMA) reaction is an example of one. Prepare a solution of chlorine dioxide, malonic acid, and starch. Then add in a solution of iodine. The color of the liquid will immediately turn to dark blue, and then the blue subsides to a light brown in a few seconds, and then it re-enters the blue-brown cycle. This process can last for nearly an hour. The blue color is due to the presence of a triiodine-starch complex, and the blue color subsides because the iodine is oxidized to the iodide ion. Thus, the change of color suggests the periodic oscillation of the concentration of iodine and the iodide ion. We shall give an explanation of this experimental fact using ODEs.

In the Lengyel-Epstein model, the CDIMA reaction consists of three steps.

$$\begin{aligned}
\mathrm{MA} + \mathrm{I_2} &\longrightarrow \mathrm{IMA} + \mathrm{I^-} + \mathrm{H^+} \\
\mathrm{ClO_2} + \mathrm{I^-} &\longrightarrow \mathrm{ClO_2}^- + \tfrac{1}{2}\mathrm{I_2} \\
\mathrm{ClO_2}^- + 4\,\mathrm{I^-} + 4\,\mathrm{H^+} &\longrightarrow 2\,\mathrm{I_2} + \mathrm{Cl^-} + 2\,\mathrm{H_2O}
\end{aligned}$$

Here, MA stands for malonic acid $\mathrm{CH(COOH)_2}$ and IMA stands for iodomalonic acid $\mathrm{CI(COOH)_2}$. According to the experiments of Lengyel, Rábai, and Epstein, the rates of these three reactions are, respectively,

$$\begin{aligned}
r_1 &= \frac{k_{1a}[\mathrm{MA}][\mathrm{I}_2]}{k_{1b}+[\mathrm{I}_2]} \\
r_2 &= k_2[\mathrm{ClO}_2][\mathrm{I}^-] \\
r_3 &= k_{3a}[\mathrm{ClO_2}^-][\mathrm{I}^-][\mathrm{H}^+] + \frac{k_{3b}[\mathrm{ClO_2}^-][\mathrm{I}_2][\mathrm{I}^-]}{u+[\mathrm{I}^-]^2}
\end{aligned}$$

where $k_{1a}, k_{1b}, k_2, k_{3a}, k_{3b}, u$ are all constants. Lengyel, Rábai, and Epstein discovered that only the concentrations of the iodide ion and chloride ion change drastically, while the concentration of iodine and chlorine dioxide is approximately constant. So the change in concentration of the materials can be described by a 2D autonomous system

$$\begin{aligned}
\frac{\mathrm{d}[\mathrm{I}^-]}{\mathrm{d}t} &= r_1 - r_2'[\mathrm{I}^-] - \frac{4\tilde{r}_3[\mathrm{I}^-][\mathrm{ClO_2}^-]}{u+[\mathrm{I}^-]^2} \\
\frac{\mathrm{d}[\mathrm{ClO_2}^-]}{\mathrm{d}t} &= \tilde{r}_2[\mathrm{I}^-] - \frac{\tilde{r}_3[\mathrm{I}^-][\mathrm{ClO_2}^-]}{u+[\mathrm{I}^-]^2}
\end{aligned}$$

where $\tilde{r}_2 = k_2[\mathrm{ClO}_2]$ and $r_3 = k_{3b}[\mathrm{I}_2]$. After suitable scaling, it is transformed to a dimensionless form

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= a - x - \frac{4xy}{1+x^2} \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= b\left(x - \frac{xy}{1+x^2}\right)
\end{aligned}$$

(1) Prove that the first quadrant is an invariant set. This makes sense since concentrations of chemical materials cannot be negative.

Proof. Consider the points along the x- and y-axes. For the points along the x-axis, we have $y=0$ and thus

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= a - x \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= bx
\end{aligned}$$

Importantly, the lower equation above implies that the vector field points inward toward first quadrant at every point along the x-axis.

For the points along the y-axis, we have $x=0$ and thus

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= a \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= 0
\end{aligned}$$

Thus, similarly, the vector field points inward toward first quadrant at every point along the y-axis. Thus, by the proposition from Lecture 7.2, since $f(x)$ is transversal to the boundary and inward pointing, the set is invariant. $\square$

(2) The system has only one fixed point in the first quadrant. Find it.

Proof. We need to solve the system of equations

$$\begin{aligned}
0 &= a - x - \frac{4xy}{1+x^2} \\
0 &= b\left(x - \frac{xy}{1+x^2}\right)
\end{aligned}$$

for $x, y \geq 0$. From the second equation, we get

$$\begin{aligned}\frac{xy}{1+x^2} &= x\\ y &= 1+x^2\end{aligned}$$

Substituting into the first equation, we get

$$\begin{aligned}x + \frac{4xy}{y} &= a\\ x + 4x &= a\\ x &= \frac{a}{5}\end{aligned}$$

Substituting back into $y = 1 + x^2$, we get

$$y = 1 + \frac{a^2}{25}$$

Therefore, the fixed point in the first quadrant is

$$\boxed{\left(\frac{a}{5}, 1 + \frac{a^2}{25}\right)}$$

□

(3) Study the stability of the fixed point found in part (2). You should be able to derive an algebraic condition concerning a, b such that the fixed point is asymptotically stable/completely unstable.

Proof. To begin, calculate the linearization of the system at the fixed point. Indeed, we have

$$\begin{aligned}A &= f'\left(\frac{a}{5}, 1 + \frac{a^2}{25}\right)\\ &= \begin{pmatrix} \frac{\partial}{\partial x}\left(a - x - \frac{4xy}{1+x^2}\right)\Big|_{\left(\frac{a}{5},1+\frac{a^2}{25}\right)} & \frac{\partial}{\partial y}\left(a - x - \frac{4xy}{1+x^2}\right)\Big|_{\left(\frac{a}{5},1+\frac{a^2}{25}\right)}\\ \frac{\partial}{\partial x}\left(b\left(x - \frac{xy}{1+x^2}\right)\right)\Big|_{\left(\frac{a}{5},1+\frac{a^2}{25}\right)} & \frac{\partial}{\partial y}\left(b\left(x - \frac{xy}{1+x^2}\right)\right)\Big|_{\left(\frac{a}{5},1+\frac{a^2}{25}\right)} \end{pmatrix}\\ &= \begin{pmatrix} \left(-1 - \frac{4y(1-x^2)}{(1+x^2)^2}\right)\Big|_{\left(\frac{a}{5},1+\frac{a^2}{25}\right)} & \left(-\frac{4x}{1+x^2}\right)\Big|_{\left(\frac{a}{5},1+\frac{a^2}{25}\right)}\\ b\left(1 - \frac{y(1-x^2)}{(1+x^2)^2}\right)\Big|_{\left(\frac{a}{5},1+\frac{a^2}{25}\right)} & -\frac{bx}{1+x^2}\Big|_{\left(\frac{a}{5},1+\frac{a^2}{25}\right)} \end{pmatrix}\\ &= \begin{pmatrix} -1 - \frac{4(25-a^2)}{25+a^2} & -\frac{20a}{25+a^2}\\ b - b\frac{25-a^2}{25+a^2} & -\frac{5ab}{25+a^2} \end{pmatrix}\end{aligned}$$

Thus,

$$\begin{aligned}0 &= \chi_A(z)\\ &= \frac{1}{(25+a^2)^2}\left(a^4z^2 - 3a^4z + 5a^3bz + 25a^3b + 50a^2z^2 + 50a^2z + 125abz + 625ab + 625z^2 + 3125z\right)\\ 0 &= \left[(25+a^2)^2z^2 + (-3a^4 + 5a^3b + 50a^2 + 125ab + 3125)z + 25a^3b + 625ab\right]\end{aligned}$$

Applying the quadratic formula to the above equation will yield the desired result (when the roots are less than zero, we get stability, and greater than zero implies instability). □

(4) Find a bounded invariant subset containing the fixed point you found in part (2).

Proof. Since everything will turn in towards the fixed center, we can just choose

$$\overline{B\left(\left(\frac{a}{5}, 1+\frac{a^2}{25}\right), a/10\right)}$$

□

(5) Use the Poincaré-Bendixson theorem to prove that when the fixed point is completely unstable, the bounded invariant subset in part (4) contains at least one limit cycle. What is the implication in chemistry of this mathematical fact?

Proof. Let Ω denote the set from part (4). By the Poincaré-Bendixson theorem, $\omega(x) \subset \Omega$ is either a fixed point, a limit cycle, or consists of finitely many fixed points. If the fixed point is completely unstable, then it will lie in $\alpha(x)$, but $\omega(x)$ will be empty unless x is the fixed point we've been working with, so it is not case 1. Since there is only one fixed point (as per part 2), we know it is not the third case either. Therefore, we must have the second case ($\omega(x)$ is a limit cycle), as desired.

The implication in chemistry is that if we set up our system such that a, b make the fixed point completely unstable, then x, y (concentrations) will alternate periodically. □

8 Review

12/2: You may finish a section each day, so that your reviewing tasks are evenly distributed before the final exam. Please note that this list emphasizes more on the fundamental definitions and theorems, so you should also use the answers of problem sets as a resource for reviewing. If you finish this whole list, you will get five more points for your final score.

Basic Definitions; Scalar Equations of Separable Form

(1) Convert a scalar ODE $y^{(n)} = f(t, y, \dots, y^{(n-1)})$ to a vector-valued ODE of order 1.

Proof. Let

$$Y = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix} \qquad f(t,y) = \begin{pmatrix} Y^2 \\ Y^3 \\ \vdots \\ f(t, Y^1, Y^2, \dots, Y^{n-1}) \end{pmatrix}$$

Then $Y' = f(t, Y)$, as desired. □

(2) Derive the formula to solve the IVP of the scalar first-order ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(y)g(x), \quad y(x_0) = y_0$$

Proof. Rearrange the initial separable ODE as follows.

$$\frac{1}{g(y)} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} = f(t)$$

Define $\mathrm{d}H/\mathrm{d}y = 1/g(y)$. Then, continuing from the above, we have by the law of composite differentiation that

$$\begin{aligned} \frac{\mathrm{d}H}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} &= f(t) \\ \frac{\mathrm{d}H}{\mathrm{d}t} &= f(t) \end{aligned}$$

From the definition of H, we know that $H(y) = \int_{y_0}^{f} \mathrm{d}w \,/g(w)$. We also have from the FTC that $f(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_{t_0}^{t} f(\tau)\mathrm{d}\tau$. Thus, continuing from the above, we have that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(H) &= f(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{y_0}^{f} \frac{\mathrm{d}w}{g(w)}\right] &= \mathrm{d}t \int_{t_0}^{t} f(\tau)\mathrm{d}\tau \\ \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{y_0}^{y(t)} \frac{\mathrm{d}w}{g(w)} - \int_{t_0}^{t} f(\tau)\mathrm{d}\tau\right] &= 0 \end{aligned}$$

It follows since $y(t_0) = y_0$ that $C = H(y_0) = 0$, so we can take the above to

$$\int_{y_0}^{y(t)} \frac{\mathrm{d}w}{g(w)} = \int_{t_0}^{t} f(\tau)\mathrm{d}\tau$$

knowing that our constant of integration is zero. □

(3) Solve the logistic growth equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ry\left(1 - \frac{y}{M}\right), \quad y(0) = y_0$$

Explain the meaning of this equation as a model for population growth.

Proof. The solution is

$$\begin{aligned}
\frac{M\,\mathrm{d}y}{y(M-y)} &= r\,\mathrm{d}t \\
\log\frac{y}{M-y} - \log\frac{y_0}{M-y_0} &= rt \\
\frac{y(M-y_0)}{y_0(M-y)} &= \mathrm{e}^{rt} \\
y\cdot\frac{M-y_0}{y_0} &= (M-y)\mathrm{e}^{rt} \\
y\cdot\frac{M-y_0}{y_0} + y\mathrm{e}^{rt} &= M\mathrm{e}^{rt} \\
y\left(\frac{M-y_0}{y_0} + \mathrm{e}^{rt}\right) &= M\mathrm{e}^{rt} \\
y\left(\frac{M-y_0+y_0\mathrm{e}^{rt}}{y_0}\right) &= M\mathrm{e}^{rt} \\
y\left(\frac{M+y_0(\mathrm{e}^{rt}-1)}{y_0}\right) &= M\mathrm{e}^{rt} \\
y(t) &= \frac{My_0\mathrm{e}^{rt}}{M+y_0(\mathrm{e}^{rt}-1)}
\end{aligned}$$

The ODE says that a population grows exponentially with a proportionality constant (ry term), but also that ecosystems can only sustain so large a population (i.e., have a max capacity M). Thus, as $y \to M$, we lose our exponential growth and levels out.

This is also reflected by the solution: At $t = 0$, the expression simplifies to the initial condition, as expected.

$$y(0) = \frac{My_0\mathrm{e}^{r\cdot 0}}{M+y_0(\mathrm{e}^{r\cdot 0}-1)} = \frac{My_0}{M} = y_0$$

But as $t \to \infty$, the dominant terms in the numerator and denominator become the exponential ones, so for t large,

$$y(t) \approx \frac{My_0\mathrm{e}^{rt}}{y_0\mathrm{e}^{rt}} = M$$

Lastly, if we start with population greater than M, we will converge down to M analogously to the above, reflecting the fact that overpopulation will result in a die off. ☐

(4) Give an example of an IVP of a scalar first-order ODE with finite lifespan.

Proof. $x' = \mathrm{e}^x \sin t$, $x(0) = x_0$ has finite lifespan. Indeed, solving by separation of variables, we get

$$-\mathrm{e}^{-x} + \mathrm{e}^{-x_0} = 1 - \cos t \quad \Longleftrightarrow \quad \boxed{x(t) = -\ln\left(\mathrm{e}^{-x_0} - 1 + \cos t\right)}$$

The set of x_0 for which the solution is extendable to the whole of $t \geq 0$ is

$$\boxed{\{x_0 \in \mathbb{R} \mid x_0 < \ln(1/2)\}}$$

When $x_0 \geq \ln(1/2)$, the solution only exists in

$$\boxed{\big[0, \arccos\big(1 - \mathrm{e}^{-x_0}\big)\big)}$$

□

(5) Write down a sufficient condition for the differential equation

$$g(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} + f(x,y) = 0$$

Proof. We must have

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$$

□

to be exact, i.e., to admit a general solution of the implicit form $F(x,y) = 0$.

(6) Give an example of an exact differential equation where f, g are quadratic polynomials.

Proof. We could take

$$(y^2 + 2xy)\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 + y^2) = 0$$

□

(7) Find a first integral of the Lotka-Volterra system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} (1-y)x \\ \alpha(x-1)y \end{pmatrix}$$

Describe the global behavior of this system in the first quadrant.

Proof. We have

$$\begin{aligned}
\frac{x'}{y'} &= \frac{(1-y)x}{\alpha(x-1)y} \\
\frac{y-1}{y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\alpha(x-1)}{x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}[y(t) - \ln y(t)] + \alpha\frac{\mathrm{d}}{\mathrm{d}t}[x(t) - \ln x(t)] &= 0 \\
y(t) - \ln y(t) + \alpha x(t) - \alpha \ln x(t) &= c
\end{aligned}$$

The global behavior is a number of dense periodic orbits surrounding the fixed point $(1,1)$. □

(8) Write down the second-order ODE describing the motion of an ideal pendulum. Find a first integral.

Proof. From Newton's second law, we can derive that

$$0 = l\theta'' + g\sin\theta$$

where g is the gravitational constant and l is the length of the pendulum's arm.

We can use the energy equation as a first integral. The kinetic energy component is

$$T(\theta') = \frac{1}{2}|\theta'|^2$$

For the potential energy component (invariant with respect to vertical height), we may take $\theta_0 = 0$ so that

$$\begin{aligned} U(\theta) &= -\int_0^\theta -\frac{g}{l}\sin\theta \mathrm{d}\theta \\ &= -\frac{g}{l}[\cos\theta]_0^\theta \\ &= \frac{g}{l} - \frac{g}{l}\cos\theta \end{aligned}$$

Therefore, an overall first integral is

$$E = \frac{1}{2}|\theta'|^2 + \frac{g}{l} - \frac{g}{l}\cos\theta$$

□

Scalar Linear Equations of Order 1 and 2

(1) Use the integrating factor method to derive the formula for solving the scalar linear IVP

$$y' = p(t)y + f(t), \quad y(t_0) = y_0$$

In particular, prove the Duhamel formula for

$$y' = ay + f(t), \quad y(0) = y_0$$

where a is a constant.

Proof. Let $P(t) = \int_{t_0}^t p(\tau)\mathrm{d}\tau$. Then

$$\begin{aligned} \mathrm{e}^{-P(t)}y'(t) - p(t)\mathrm{e}^{-P(t)}y &= \mathrm{e}^{-P(t)}f(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-P(t)}y(t)\right) &= \mathrm{e}^{-P(t)}f(t) \\ \mathrm{e}^{-P(t)}y(t) - \mathrm{e}^{-P(t_0)}y(t_0) &= \int_{t_0}^t \mathrm{e}^{-P(\tau)}f(\tau)\mathrm{d}\tau \\ y(t) &= y_0\mathrm{e}^{P(t)-P(t_0)} + \mathrm{e}^{P(t)}\int_{t_0}^t \mathrm{e}^{-P(\tau)}f(\tau)\mathrm{d}\tau \end{aligned}$$

Substituting $p(t) = a$ and $t_0 = 0$ yields $P(t) = at$ and thus

$$\begin{aligned} y(t) &= y_0\mathrm{e}^{at-a\cdot 0} + \mathrm{e}^{at}\int_0^t \mathrm{e}^{-a\tau}f(\tau)\mathrm{d}\tau \\ &= y_0\mathrm{e}^{at} + \int_0^t \mathrm{e}^{a(t-\tau)}f(\tau)\mathrm{d}\tau \end{aligned}$$

□

(2) Describe the general steps for finding the solutions to the linear scalar second-order ODE

$$y'' + ay' + by = 0$$

Write down the formula for the general solution.

Proof. This is a linear equation, so start with the characteristic polynomial

$$\begin{aligned}
0 &= x^2 + ax + b \\
x_\pm &= \frac{-a \pm \sqrt{a^2 - 4b}}{2}
\end{aligned}$$

It follows that a basis for the solution space is the two functions $\mathrm{e}^{x_\pm t}$, so the general solution is

$$y(t) = A\mathrm{e}^{x_+ t} + B\mathrm{e}^{x_- t}$$

□

(3) Derive the damped harmonic oscillator equation.

Proof. Since a circuit is a set of elements, each of which has two connectors (in and out) and every connector is connected to one or more connectors of the other elements, a circuit is mathematically a directed graph. At each time t, there will be a certain current $I(t)$ flowing through each element and a certain voltage difference $V(t)$ between its connectors. The state space of the system is given by the pairs (I, V) of all elements in the circuit. The pairs (I, V) must satisfy Kirchoff's first law and Kirchoff's second law. In this case, we have respectively that

$$I_R = I_L = I_C \qquad\qquad V_R + V_L + V_C = V$$

We can write a further three equations

$$L\dot{I}_L = V_L \qquad\qquad C\dot{V}_C = I_C \qquad\qquad V_R = RI_R$$

where $L, C, R > 0$ are the inductance, capacitance, and resistance, respectively; $I_L(t), I_C(t), I_R(t)$ is the current through the inductor, capacitor, and resistor, respectively; and $V_L(t), V_C(t), V_R(t)$ is the voltage difference across the inductor, capacitor, and resistor, respectively. Combining these five equations into one. We first differentiate (wrt. t) and rearrange the latter three to put them in terms of $\dot{V}_i$ ($i = L, C, R$).

$$\dot{V}_L = L\ddot{I}_L \qquad\qquad \dot{V}_C = \frac{I_C}{C} \qquad\qquad \dot{V}_R = R\dot{I}_R$$

We then differentiate Kirchoff's second law and substitute in the above.

$$\begin{aligned}
\dot{V}(t) &= \dot{V}_R + \dot{V}_L + \dot{V}_C \\
&= R\dot{I}_R + L\ddot{I}_L + \frac{I_C}{C}
\end{aligned}$$

Lastly, we take advantage of Kirchoff's first law and drop the subscript from all of the currents.

$$L\ddot{I}(t) + R\dot{I}(t) + \frac{1}{C}I(t) = \dot{V}(t)$$

To get our final equation, use the complex voltage $V(t) = V_0\mathrm{e}^{i\omega t}$ and divide through by L.

$$\ddot{I} + \frac{R}{L}\dot{I} + \frac{1}{LC}I = \frac{i\omega V_0}{L}\mathrm{e}^{i\omega t}$$

□

(4) Describe the solution of the damped harmonic oscillator equation

$$x'' + bx' + \omega_0^2 x = 0$$

for different choices of the damping parameter $b \geq 0$.

Proof. Again, we take the eigenvalues of the characteristic polynomial.

$$x_\pm = \frac{-b \pm \sqrt{b^2 - 4\omega_0^2}}{2}$$

The three important cases are $b^2 - 4\omega^2$ positive, zero, and negative, corresponding to overdamping, critical damping, and under damping. In particular, the respective general solutions are

$$x(t) = k_1 \mathrm{e}^{x_+ t} + k_2 \mathrm{e}^{x_- t}$$
$$x(t) = (k_1 + k_2 t)\mathrm{e}^{-bt/2}$$
$$x(t) = k_1 \mathrm{e}^{-bt/2} \cos\left(t\sqrt{4\omega_0^2 - b^2}/2\right) + k_2 \mathrm{e}^{-bt/2} \sin\left(t\sqrt{4\omega_0^2 - b^2}/2\right)$$

□

(5) Derive the Duhamel formula for the forced harmonic oscillator equation

$$x'' + \omega_0^2 x = f(t)$$

Proof. □

(6) Describe the phenomenon of resonance in the forced harmonic oscillator equation and the forced damped harmonic oscillator equation.

Proof. When the driving frequency is equal to the resonance frequency, the oscillation is maximized (damped forced harmonic oscillator) or grows unbounded (forced harmonic oscillator equation), leading to the so-called resonance catastrophe in this latter case. □

Jordan Normal Form and Matrix Calculus

(1) Write down the definition of similarity for matrices.

Proof. Two matrices A, B are **similar** if there exists a matrix Q such that

$$A = QBQ^{-1}$$

□

(2) Give examples of representing a given matrix under a new basis in two and three dimensional real spaces.

Proof. Consider the matrix

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

which sends $(1,1)$ to $(3,3)$ and $(-1,1)$ to itself. With respect to the basis

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

the matrix of this linear transformation is

$$\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

In three dimensions, consider the matrix

$$\begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

which sends $(1,1,0)$ to $(4,4,0)$, $(1,-1,0)$ to $(2,-2,0)$, and $(0,1,1)$ to itself. With respect to the basis

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$$

the matrix of this linear transformation is

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

□

(3) Write down the formula for computing the matrix exponential e^J of a Jordan block J.

Proof. Given an arbitrary Jordan block of dimension d

$$J = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

the matrix exponential is

$$\mathrm{e}^J = \begin{pmatrix} \mathrm{e}^\lambda & \mathrm{e}^\lambda/1! & & \mathrm{e}^\lambda/(d-1)! \\ & \mathrm{e}^\lambda & \ddots & \\ & & \ddots & \mathrm{e}^\lambda/1! \\ 0 & & & \mathrm{e}^\lambda \end{pmatrix}$$

□

(4) Describe the general steps to reduce a 2×2 matrix to Jordan normal form.

Proof. $A \in \mathcal{M}^2(\mathbb{C})$ can only have nontrivial Jordan form if it has a single eigenvalue λ with $\alpha_\lambda = 2$ and $\gamma_\lambda = 1$. If both equal 2, then $A = \lambda I_2$. If it has two eigenvalues, then it is regularly diagonalizable. In this particular case, calculate λ from $\chi_A(z) = (z-\lambda)^2$, find one eigenvector v, and find the other generalized eigenvector u; u will satisfy $(A - \lambda I)u = v$. The connecting matrix will be $Q = (v|u)$[5] and the JNF is

$$Q^{-1}AQ = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

□

(5) Give an example for each possible case in part (4). There should be two cases in total: One for diagonalizable and one for non-diagonalizable matrices.

Proof. The diagonalizable example is on the left below, and the nondiagonalizable on the right.

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}$$

We tackle A first.

[5] Order matters! We need the eigenvector, specifically, to get scaled by λ only.

Calculate the characteristic polynomial to begin.

$$\begin{aligned}\chi_A(z) &= \det(A - zI)\\ &= z^2 - 4z + 3\\ &= (1-z)(3-z)\end{aligned}$$

It follows that the eigenvalues are

$$\lambda_1 = 1 \qquad\qquad \lambda_2 = 3$$

Since these eigenvalues are distinct, we can fully diagonalize this matrix. Indeed, we can determine by inspection that suitable corresponding eigenvectors are

$$v_1 = \begin{pmatrix}-1\\1\end{pmatrix} \qquad\qquad v_2 = \begin{pmatrix}1\\1\end{pmatrix}$$

Therefore,

$$Q = \begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix} \qquad\qquad Q^{-1}AQ = \begin{pmatrix}1 & 0\\ 0 & 3\end{pmatrix}$$

The procedure for B is very much analogous to the procedure for A.

Characteristic polynomial:

$$\begin{aligned}\chi_B(z) &= \det(B - zI)\\ &= z^2 + 2z + 1\\ &= (1+z)^2\end{aligned}$$

Eigenvalue:

$$\lambda = -1$$

By inspection of $B + I$, we can pick one eigenvector of B:

$$v = \begin{pmatrix}1\\1\end{pmatrix}$$

We now solve $(B + I)u = v$. By inspection, this yields

$$u = \begin{pmatrix}1\\0\end{pmatrix}$$

Therefore,

$$Q = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} \qquad\qquad Q^{-1}BQ = \begin{pmatrix}-1 & 1\\ 0 & -1\end{pmatrix}$$

□

(6) Describe the general steps for reducing a 3×3 matrix to Jordan normal form.

Proof. We divide into three nontrivial cases: $\chi_A(z) = (z-\lambda)^3$ with $\gamma_\lambda = 2$, $\chi_A(z) = (z-\lambda)^3$ with $\gamma_\lambda = 1$, and $\chi_A(z) = (z-\lambda)^2(z-\mu)$ with $\gamma_\lambda = 1$. In the first case, we have two eigenvectors v_1, v_2 (make sure to pick v_1 such that it is also in the column space of $A - \lambda I$!). We can find the third (generalized) eigenvector by solving $(A - \lambda I)u = v_1$. Then $Q = (v_1|u|v_2)$ and the JNF is

$$Q^{-1}AQ = \begin{pmatrix}\lambda & 1 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{pmatrix}$$

In the second case, we have one eigenvector v. We can find the second and third generalized eigenvectors by solving $(A-\lambda I)u_1 = v$ and $(A-\lambda I)u_2 = u_1$. Then $Q = (v|u_1|u_2)$ and

$$Q^{-1}AQ = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$$

In the third case, we have two eigenvectors v_λ, v_μ. We can find the third (generalized) eigenvector by solving $(A-\lambda I)u = v_\lambda$. Then $Q = (v_\lambda|u|v_\mu)$ and

$$Q^{-1}AQ = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

$\square$

(7) Give an example for each possible case in part (6). There should be four cases in total: One for diagonalizable and three for non-diagonalizable matrices.

Proof. The nondiagonalizable examples are on the left below, and the diagonalizable is on the right.

$$A = \begin{pmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -1 & -1 \\ 2 & -1 & -2 \\ -1 & 1 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

We tackle A first.

Calculate the characteristic polynomial to begin.

$$\begin{aligned} \chi_A(z) &= \det(A - zI) \\ &= -z^3 + z^2 \\ &= z^2(1-z) \end{aligned}$$

It follows that the eigenvalues are

$$\lambda_1 = \lambda_2 = 0 \qquad\qquad \lambda_3 = 1$$

We can solve for an eigenvector v_1 corresponding to $\lambda_1 = \lambda_2 = 0$ using the augmented matrix and row reduction as follows.

$$\left(\begin{array}{ccc|c} 4 & -5 & 2 & 0 \\ 5 & -7 & 3 & 0 \\ 6 & -9 & 4 & 0 \end{array}\right) \cong \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus, if we choose $v_1^3 = 3$, then the desired eigenvector is

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Similarly, we can solve for an eigenvector v_3 corresponding to $\lambda_3 = 1$ using the following. Note that to solve $Ax = 1x$, we row-reduce $(A - I)x = 0$.

$$\left(\begin{array}{ccc|c} 3 & -5 & 2 & 0 \\ 5 & -8 & 3 & 0 \\ 6 & -9 & 3 & 0 \end{array}\right) \cong \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This yields

$$v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

We now solve the equation $(A - 0I)u = v_1$ to find a generalized eigenvector u corresponding to $\lambda_1 = \lambda_2 = 0$. This can also be done with an augmented matrix.

$$\left(\begin{array}{ccc|c} 4 & -5 & 2 & 1 \\ 5 & -7 & 3 & 2 \\ 6 & -9 & 4 & 3 \end{array}\right) \cong \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This yields

$$u = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

Therefore,

$$Q = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 3 & 1 \end{pmatrix} \qquad Q^{-1}AQ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The procedure for B is very much analogous to the procedure for A.

Characteristic polynomial:

$$\begin{aligned} \chi_B(z) &= \det(B - zI) \\ &= -z^3 + 3z^2 - 3z + 1 \\ &= (1 - z)^3 \end{aligned}$$

Eigenvalue:

$$\lambda = 1$$

By inspection of

$$B - I = \begin{pmatrix} 1 & -1 & -1 \\ 2 & -2 & -2 \\ -1 & 1 & 1 \end{pmatrix}$$

we can pick two eigenvectors of B corresponding to λ, i.e., two elements of the null space of the above matrix. In this subcase of the 3×3 case, we always pick the first of these to be an element of the column space of $B - I$, as well. Thus, choose

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We now solve $(B - \lambda I)u = v_1$. By inspection, this yields

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Therefore,

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \qquad Q^{-1}BQ = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The procedure for C is likewise quite analogous.

The matrix is upper triangular, so the eigenvalues are on the diagonal. It follows that

$$\lambda = 2$$

is the sole eigenvalue. We can solve $(C-2I)v=0$ for one eigenvector v by inspection, yielding

$$v=\begin{pmatrix}1\\0\\0\end{pmatrix}$$

We can also solve $(C-2I)u_1=v$ by inspection to get

$$u_1=\begin{pmatrix}0\\1\\0\end{pmatrix}$$

One more time, we can solve $(C-2I)u_2=u_1$ by inspection to get

$$u_2=\begin{pmatrix}0\\3\\-1\end{pmatrix}$$

Therefore,

$$Q=\begin{pmatrix}1&0&0\\0&1&3\\0&0&-1\end{pmatrix} \qquad Q^{-1}CQ=\begin{pmatrix}2&1&0\\0&2&1\\0&0&2\end{pmatrix}$$

Lastly, for D, we have

$$\begin{aligned}0&=\chi_A(z)\\&=-z^3+7z^2-14z+8 &&=(4-z)(2-z)(1-z)\end{aligned}$$

so

$$\lambda_1=4 \qquad \lambda_2=2 \qquad \lambda_3=1$$

Solving $(D-\lambda_iI)v_i=0$ $(i=1,2,3)$ yields, respectively,

$$v_1=\begin{pmatrix}1\\1\\0\end{pmatrix} \qquad v_2=\begin{pmatrix}1\\-1\\0\end{pmatrix} \qquad v_3=\begin{pmatrix}0\\1\\1\end{pmatrix}$$

Therefore,

$$Q=\begin{pmatrix}1&1&0\\1&-1&1\\0&0&1\end{pmatrix} Q^{-1}DQ \qquad =\begin{pmatrix}4&0&0\\0&2&0\\0&0&1\end{pmatrix}$$

□

Linear Autonomous Systems

(1) Describe the steps of calculating the matrix exponential function e^{tA}.

Proof. First, we calculate the JNF for A. Then we calculate the exponential of the simplified matrix $Q^{-1}AQ$ block by block and surround this exponential with Q on the left and Q^{-1} on the right. □

(2) Prove the Duhamel formula for the inhomogeneous linear system $y'=Ay+f(t)$, where the function y takes value in $\mathbb{C}^n$ and A is an $n\times n$ complex matrix.

Proof. We have that

$$\begin{aligned}
y' &= Ay + f(t) \\
y' - Ay &= f(t) \\
\mathrm{e}^{-tA}y' - A\mathrm{e}^{-tA}y &= \mathrm{e}^{-tA}f(t) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-tA}y(t)\right) &= \mathrm{e}^{-tA}f(t) \\
\mathrm{e}^{-tA}y(t) - y_0 &= \int_0^t \mathrm{e}^{-\tau A}f(\tau)\mathrm{d}\tau \\
y(t) &= \mathrm{e}^{tA}y_0 + \int_0^t \mathrm{e}^{(t-\tau)A}f(\tau)\mathrm{d}\tau
\end{aligned}$$

as desired. □

(3) For the linear system $y' = Ay$, describe the growth or decay of the solution according to the real part of eigenvalues of the coefficient matrix.

Proof. Positive real eigenvalues induce exponential growth on (and similar growth near) the span of their corresponding eigenvector.

Negative real eigenvalues induce exponential decay on (and similar decay near) the span of their corresponding eigenvector.

Complex conjugate eigenvalues with positive real part lead to spiral source-type behavior, and vice versa for those with negative real part. □

(4) State the definition of stable and unstable subspaces.

Proof. **Stable subspace** (of x_0 under A): The space of all generalized eigenvectors of A corresponding to eigenvalues λ with $\operatorname{Re}\lambda < 0$. *Also known as* **attracting subspace**. *Denoted by* $\mathbf{E}_{\boldsymbol{s}}$.

Unstable subspace (of x_0 under A): The space of all generalized eigenvectors of A corresponding to eigenvalues λ with $\operatorname{Re}\lambda > 0$. *Also known as* **repelling subspace**. *Denoted by* $\mathbf{E}_{\boldsymbol{u}}$. □

(5) Describe the general structure of the solution of the scalar equation

$$x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1x' + a_0x = 0$$

Proof. Let α_j, $1 \leq j \leq m$, be the zeros of the characteristic polynomial

$$z^n + c_{n-1}z^{n-1} + \cdots + c_1z + c_0 = \prod_{j=1}^m (z - \alpha_j)^{a_j}$$

associated with A, and let a_j be the corresponding multiplicities. Then the functions

$$x_{j,k}(t) = t^k \exp(a_jt)$$

for $0 \leq k < a_j$ and $1 \leq j \leq m$ are n linearly independent solutions of

$$x^{(n)} + c_{n-1}x^{(n-1)} + \cdots + c_1\dot{x} + c_0x = 0$$

In particular, any other solution can be written as a linear combination of these solutions. □

(6) Prove the Duhamel formula for the inhomogeneous scalar equation

$$x^{(n)} + a_{n-1}x^{(n-1)} + \cdots + a_1x' + a_0x = f(t)$$

Proof. Transform the scalar equation into the corresponding vector form

$$y' = Ay + f(t)e_n$$

where e_n is the n^{th} standard basis vector of $\mathbb{R}^n$ and $f(t)$ is still a scalar function. Then by the same derivation as in part (2) of this section, we end up with

$$y(t) = \mathrm{e}^{tA}y_0 + \int_0^t \mathrm{e}^{(t-\tau)A}e_n g(\tau)\mathrm{d}\tau$$

Note that $\mathrm{e}^{t'A}e_n$ is the solution to the IVP $u' = Au$ with $u(0) = e_n$. Thus, letting $U(t')$ denote the solution to the homogeneous equation with initial conditions $U(0) = U'(0) = \cdots = U^{(n-2)}(0) = 0$ and $U^{(n-1)}(0) = 1$ and knowing that $\mathrm{e}^{tA}y_0$ is a solution to the homogeneous form of the above matrix equation, we can substitute to yield

$$y(t) = y_h(t) + \int_0^t U(t-\tau)g(\tau)\mathrm{d}\tau$$

as our final answer. □

(7) Classify planar linear autonomous systems according to stability. Give an example from each class.

Proof. There are three main classes and several subclasses for certain classes. They are as follows.

(1) A has no real eigenvalues.
   i. The real component σ of the complex eigenvalues is > 0.
   ii. The real component σ of the complex eigenvalues is $= 0$.
   iii. The real component σ of the complex eigenvalues is < 0.

(2) A has real eigenvalues and is diagonalizable.
   i. The eigenvalues are both > 0.
   ii. The eigenvalues are both < 0.
   iii. We have one of each.

(3) A has real eigenvalues and is *not* diagonalizable.

The examples are, running down the list,

$$\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} \quad \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \quad \begin{pmatrix}-1 & -1\\ 1 & -1\end{pmatrix} \quad \begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} \quad \begin{pmatrix}2 & 1\\ 1 & -2\end{pmatrix} \quad \begin{pmatrix}-1 & 1\\ 0 & 1\end{pmatrix} \quad \begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}$$

□

Local Well-Posedness

(1) State the three ingredients of local well-posedness for an IVP

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t,y), \quad y(t_0) = x$$

Proof. Existence (local in time), uniqueness (you cannot have multiple solutions), and local stability (if you perturb your initial value or equation a little bit, you do not expect your solution to vary crazily [esp. locally]). □

(2) State and prove the Banach fixed point theorem.

Proof. Theorem (Banach fixed point theorem): Let (X, d) be a complete metric space and let $\Phi : X \to X$ be a function for which there exists $q \in (0, 1)$ such that for all $x, y \in X$,

$$d(\Phi(x), \Phi(y)) \leq q \cdot d(x, y)$$

Then there exists a unique $x \in X$ such that $x = \Phi(x)$.

Proof. We first construct the desired fixed point x.

Pick any $x_0 \in X$. Inductively define $\{x_n\}$ by $x_{n+1} = \Phi(x_n)$, starting from $n = 0$. We will now show that $\{x_n\}$ is a Cauchy sequence. As a lemma, we will prove by induction that

$$d(x_j, x_{j+1}) \leq q^j \cdot d(x_0, x_1)$$

for all $j \in \mathbb{N}_0$. For the base case $j = 0$, equality evidently holds. Now suppose inductively that we have proven that $d(x_j, x_{j+1}) \leq q^j \cdot d(x_0, x_1)$; we want to prove the claim for $j + 1$. But we have that

$$\begin{aligned} d(x_{j+1}, x_{j+2}) &= d(\Phi(x_j), \Phi(x_{j+1})) \\ &\leq q \cdot d(x_j, x_{j+1}) \\ &\leq q \cdot q^j \cdot d(x_0, x_1) \\ &= q^{j+1} \cdot d(x_0, x_1) \end{aligned}$$

as desired.

It follows that

$$\begin{aligned} d(x_n, x_{n+m}) &\leq \sum_{k=0}^{m-1} d(x_{n+k}, x_{n+k+1}) && \text{Triangle inequality} \\ &\leq \sum_{k=0}^{m-1} q^{n+k} \cdot d(x_0, x_1) && \text{Lemma} \\ &= q^n(1 + q + \cdots + q^{m-1}) \cdot d(x_0, x_1) \\ &< q^n(1 + q + \cdots + q^{m-1} + \cdots) \cdot d(x_0, x_1) \\ &= \frac{q^n}{1 - q} \cdot d(x_0, x_1) \end{aligned}$$

It follows that the above term will converge to zero as $n \to \infty$, so $\{x_n\}$ is a Cauchy sequence and there exists an x such that $x_n \xrightarrow{d} x$.

We now prove that x is a fixed point of Φ, i.e., that $\Phi(x) = x$. We have that

$$\begin{aligned} d(x, \Phi(x)) &\leq d(x, x_n) + d(x_n, \Phi(x_n)) + d(\Phi(x_n), \Phi(x)) \\ &\leq d(x, x_n) + d(x_n, x_{n+1}) + q \cdot d(x_n, x) \\ &= (1 + q) \cdot d(x, x_n) + d(x_n, x_{n+1}) \end{aligned}$$

where the first term converges since $\{x_n\}$ is convergent and the second term converges since $\{x_n\}$ is Cauchy. Thus, $d(x, \Phi(x)) \to 0$ as $n \to \infty$, so $x = \Phi(x)$, as desired.

Lastly, we prove that x is unique. Suppose that there exists $y \in X$ such that $y = \Phi(y)$. Then

$$d(x, y) = d(\Phi(x), \Phi(y)) \leq q \cdot d(x, y)$$

It follows that $d(x, y) \leq q^n \cdot d(x, y)$, i.e., that $d(x, y) \to 0$ as $n \to \infty$. Therefore, we must have that $d(x, y) = 0$, from which it follows that $x = y$, as desired. $\square$

$\square$

(3) Give two example applications of the Banach fixed point theorem.

Proof. We can apply it to the scalar function $\Phi(x) = x/2$ and the vector function $\Phi(x, y) = (x/2, y/2)$. In both cases, the origin of $\mathbb{R}^n$ $(n = 1, 2)$ is the fixed point. □

(4) State and prove the Cauchy-Lipschitz existence theorem.

Proof. Theorem (Cauchy-Lipschitz): Suppose $f \in C(U, \mathbb{R}^n)$, where $U \subset \mathbb{R}^{n+1}$ is open and $(t_0, x_0) \in U$. If f is locally Lipschitz continuous in the second argument, uniformly with respect to the first, then there exists a unique local solution $\bar{x}(t) \in C^1(I)$ of the IVP

$$\dot{x} = f(t, x), \quad x(t_0) = x_0$$

where I is some interval around t_0.

More specifically, if $V = [t_0, t_0 + T] \times \overline{B_\delta(x_0)} \subset U$ and M denotes the maximum of $|f|$ on V, then the solution exists for at least $t \in [t_0, t_0 + T]$ and remains in $\overline{B_\delta(x_0)}$, where

$$T = \frac{\delta}{M}$$ [6]

The analogous result holds for the interval $[t_0 - T, t_0]$.

Proof. Let

$$K(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,\mathrm{d}s$$

Take

$$X = C([0, T], \mathbb{R}^n)$$

to be our Banach space for some suitable T (we will put constraints on the value of T as we build the rest of our argument). We need some $C \subset X$ on which to define K as an endofunction. Let's try

$$C = \overline{B_\delta(x_0)}$$

Proving that $K : C \to C$. At this point, we can compute

$$\begin{aligned}
|K(x)(t) - x_0| &= \left| \int_0^t f(s, x(s))\,\mathrm{d}s \right| \\
&\leq \int_0^t |f(s, x(s))|\,\mathrm{d}s && \text{Theorem 13.26}^{[7]} \\
&\leq tM && \text{Theorem 13.27}^{[7]}
\end{aligned}$$

for any x satisfying $G(x) \subset V$. Thus, if we take

$$T \leq \frac{\delta}{M}$$

then $|K(x)(t) - x_0| \leq TM \leq \delta$ for all $t \in [0, T]$. It follows that under this definition of T, $\|K(x) - x_0\| \leq \delta$, so $K(x) \in \overline{B_\delta(x_0)}$ for all x with graph in V. In the special case $M = 0$ (which would imply $T = \infty$), we may take T to be some arbitrary positive real number. Proving that K is a contraction. To estimate $|K(x)(t) - K(y)(t)|$, we finally invoke the Lipschitz continuity constraint on f. In particular, we have

$$\begin{aligned}
|K(x)(t) - K(y)(t)| &\leq \int_0^t |f(s, x(s)) - f(s, y(s))|\,\mathrm{d}s \\
&\leq L \int_0^t |x(s) - y(s)|\,\mathrm{d}s \\
&\leq Lt \sup_{0 \leq s \leq t} |x(s) - y(s)| \\
&\leq LT\|x - y\|
\end{aligned}$$

[6] We are not missing the Lipschitz constraint here; indeed, it is superfluous, as will be shown in the next section.
[7] From Honors Calculus IBL.

Thus, if we take

$$T < \frac{1}{L}$$

we have that K is a contraction. Having two constraints on the value of T, we may now formally take

$$T = \min\left(\frac{\delta}{M}, \frac{1}{2L}\right)$$

Note that this definition satisfies $T \leq \delta/M$ and $T < L^{-1}$. If either of $M, L = 0$, we understand that that constraint no longer matters and we need only consider the other one. If $M = L = 0$, then we may take T to be any positive real number. Having defined a contraction $K : C \to C$, the existence and uniqueness of our ODE follows from the contraction principle. □

□

(5) State and prove Grönwall's inequality.

Proof. Lemma (Grönwall's inequality): Let $\varphi(t)$ be a real function defined for $t \in [t_0, t_0 + T]$ such that

$$\varphi(t) \leq f(t) + a \int_{t_0}^{t} \varphi(\tau)\mathrm{d}\tau$$

Then

$$\varphi(t) \leq f(t) + a \int_{t_0}^{t} \mathrm{e}^{a(t-\tau)} f(\tau)\mathrm{d}\tau$$

Proof. Multiply both sides by e^{-at}:

$$\begin{aligned}
\mathrm{e}^{-at}\varphi(t) - a\mathrm{e}^{-at}\int_{t_0}^{t} \varphi(\tau)\mathrm{d}\tau &\leq \mathrm{e}^{-at} f(t) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-at}\int_{t_0}^{t} \varphi(\tau)\mathrm{d}\tau\right) &\leq \mathrm{e}^{-at} f(t) \\
\mathrm{e}^{-at}\int_{t_0}^{t} \varphi(\tau)\mathrm{d}\tau &\leq \int_{t_0}^{t} \mathrm{e}^{-a\tau} f(\tau)\mathrm{d}\tau \\
\int_{t_0}^{t} \varphi(\tau)\mathrm{d}\tau &\leq \int_{t_0}^{t} \mathrm{e}^{a(t-\tau)} f(\tau)\mathrm{d}\tau
\end{aligned}$$

Substituting back into the original equality yields the result at this point. □

□

(6) Let μ be a small parameter. Write down the general ansatz for finding the perturbative expansion of the solution of

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t;\mu) = f(t, y(t;\mu)), \quad y(t_0) = x(\mu)$$

assuming sufficient differentiability with respect to the parameter μ.

Proof. The general ansatz is

$$y(t;\mu) = y_0 + y_1\mu + y_2\mu^2 + \cdots + y_n\mu^n + O(\mu^{n+1})$$

□

(7) Give an example of perturbative computation in part (6).

Proof. Consider

$$y' = \mu y, \quad y(0) = 1$$

Substituting in the ansatz (say up to μ^2), we get

$$y_0' + y_1'\mu + y_2'\mu^2 = 0 + y_0\mu + y_1\mu^2$$

A quick note on the initial conditions: The fact that

$$\begin{aligned} 1 &= y(0) \\ 1 + 0\mu + 0\mu^2 &= y_0 + y_1\mu + y_2\mu^2 \end{aligned}$$

implies that the initial conditions for the perturbative equations are $y_0(0) = 1$, $y_1(0) = y_2(0) = 0$. Continuing, we can solve

$$y_0' = 0 \quad \Longleftrightarrow \quad y_0(t) = C$$

Using the initial condition, we get that $C = 1$. Substituting this into the second equation, we get

$$y_1' = 1 \quad \Longleftrightarrow \quad y_1(t) = t + C$$

Using the initial condition, we get that $C = 0$. Substituting this into the third equation, we get

$$y_2' = t \quad \Longleftrightarrow \quad y_2(t) = \frac{t^2}{2} + C$$

Using the initial condition, we again get that $C = 0$. Thus, our perturbative solution is

$$y(t) = 1 + t\mu + \frac{t^2\mu^2}{2} + O(t^3)$$

which makes sense since these are the first three terms in the Taylor series of the known exact solution. $\square$

(8) For the ideal pendulum equation, considering the initial angle as the small parameter μ, find the perturbative expansion with respect to μ up to order μ^3 (that is, with error $O(\mu^4)$).

Proof. Suppose that the length of the rope is ℓ and the gravitational acceleration is g. Then

$$\theta''(t;\mu) = -\frac{g}{\ell}\sin[\theta(t;\mu)]$$

Assume a small angle, $\theta(0) = \mu$ and $\theta'(0) = 0$. Substitute $\omega_0^2 = g/\ell$. In HS, we learned that the harmonic oscillator approximation of the mathematical pendulum is justified for small θ. We now justify this. Ansatz: $\theta_0 + \theta_1\mu + \theta_2\mu^2 + \theta_3\mu^3 + O(\mu^4)$. Recall that

$$\sin\theta = \theta - \frac{\theta^3}{6} + O(\theta^5)$$

First step, solve to determine $\theta_0 = 0$. Then we only have a term of order $O(\mu)$ and $O(\mu^3)$ to worry about. Substitute the expansion in:

$$\begin{aligned} \sin\theta &= \theta - \frac{\theta^3}{6} + O(\theta^5) \\ &= \left(\theta_0 + \theta_1\mu + \theta_2\mu^2 + \theta_3\mu^3\right) - \frac{1}{6}\left(\theta_0 + \theta_1\mu + \theta_2\mu^2 + \theta_3\mu^3\right)^3 \\ &= 0 + \theta_1\mu + \theta_2\mu^2 + \left(\theta_3 - \frac{\theta_1^3}{6}\right)\mu^3 + O(\mu^4) \end{aligned}$$

We also have that

$$\theta''(t;\mu) = \theta_1''\mu + \theta_2''\mu^2 + \theta_3''\mu^3 + O(\theta^4)$$

and

$$-\omega_0^2 \sin\big(\theta_1\mu + \theta_2\mu^2 + \theta_3\mu^3 + O(\mu^4)\big) = -\omega_0^2\theta_1\mu - \omega_0^2\theta_2\mu^2 - \omega_0^2\left(\theta_3 - \frac{\theta_1^3}{6}\right)\mu^3 + O(\mu^4)$$

Initial conditions: $\theta_0 = 0$, $\theta_1(0) = 1$, and $\theta_2(0) = \theta_3(0) = \theta_1'(0) = \cdots = \theta_3'(0) = 0$. First order: $\theta_1'' = -\omega_0^2\theta_1$, $\theta_1(0) = 1$, $\theta_1'(0) = 0$. Implies $\theta_1(t) = \cos\omega_0 t$. This is why we can use the harmonic oscillator approximation. Second order: $\theta_2 = -\omega_0^2\theta_2$. Initial conditions imply $\theta_2(t) = 0$. Third order: $\theta_3'' = -\omega_0^2\theta_3 + \frac{\omega_0^2\theta_1^3}{6}$. Implies that

$$\theta_3(t) = \frac{\omega_0 t}{16}\sin\omega_0 t + \frac{1}{192}(\cos\omega_0 t - \cos 3\omega_0 t)$$

In conclusion, we have the approximation of our solution up to order $O(\mu^3)$ as

$$\theta(t;\mu) = \mu\cos\omega_0 t + \mu^3\left[\frac{\omega_0 t}{16}\sin\omega_0 t + \frac{1}{192}(\cos\omega_0 t - \cos 3\omega_0 t)\right] + O(\mu^4)$$

This approximation is only good for T in a fixed, small time interval because the second term is not periodic. □

The Method of Lyapunov

(1) State the definition of Lyapunov stability and asymptotic stability for a fixed point of an autonomous ODE system.

Proof. **Lyapunov stable** (fixed point): A fixed point x_0 such that for any neighborhood $B(x_0,\varepsilon)$, there exists a neighborhood $B(x_0,\delta)$ such that $\phi_t(x) \in B(x_0,\varepsilon)$ for any $t \geq 0$ and $x \in B(x_0,\delta)$.

Asymptotically stable (fixed point): A Lyapunov stable fixed point x_0 such that $\phi_t(x) \to x_0$ as $t \to +\infty$ for $x \in B(x_0,\delta)$. □

(2) State the definition of Lyapunov function and strict Lyapunov function.

Proof. **Lyapunov function** (of a system $y' = f(y)$ with fixed point x_0 near x_0): A continuous real function on $\mathbb{R}^n$ such that the following two axioms hold. *Denoted by* $\boldsymbol{L}$.

(1) $L(x_0) = 0$ and $L(x) > 0$ for all $x \in \mathring{B}(x_0,\delta) = B(x_0,\delta)\setminus\{x_0\}$.
(2) $\dot{L}(x) = \nabla L(x)\cdot f(x) \leq 0$ for all $x \in \mathring{B}(x_0,\delta) = B(x_0,\delta)\setminus\{x_0\}$.

Strict (Lyapunov function): A Lyapunov function for which the decreasing is strict. □

(3) State and prove the stability criterion of Lyapunov.

Proof. Theorem: For the autonomous system $y' = f(y)$, a fixed point x_0 is

(1) Stable if there is a Lyapunov function near it;

Proof. Pick a small number $\delta > 0$. Let[8]

$$m := \min\{L(x) : |x - x_0| = \delta\}$$

Since x_0 does not satisfy $|x - x_0| = \delta > 0$, we know from the first constraint on Lyapunov functions that $L(x) > 0$ for all x satisfying said relation. Thus, $m > 0$. Consequently, any orbit starting from $\{x \mid L(x) < m\} \cap B(x_0,\delta)$ can never meet $\partial B(x_0,\delta)$ since $L(x)$ is decreasing along any orbit (and we would have to go up to get to the boundary). So $L(\phi_t(x)) < m$ for all $x \in \{x \mid L(x) < m\} \cap B(x_0,\delta)$. But this means that $\{x \mid L(x) < m\} \cap B(x_0,\delta)$ is in fact an invariant set. Therefore, x_0 is Lyapunov stable. □

[8] Intuitively (in 2D), we take a ring around x_0, find the nonzero value of $L(x)$ at each point on the ring, and take the minimum among them. Imagine a circular valley with hills rising all around the bottommost point; we are essentially looking for the hill that rises the least.

(2) Asymptotically stable if there is a strict Lyapunov function near it.

Proof. If $x \in \{x \mid L(x) < m\} \cap B(x_0, \delta)$, then $L(\phi_t(x))$ is strictly decreasing. As $t \to +\infty$, $\phi_t(x)$ has a partial limit z_0, say $\phi_{t_k}(x) \to z_0$ (Lemma 6.6 of Teschl (n.d.)). If $z_0 \neq x_0$, then the orbit $\{\phi_t(z_0) \mid t \in I_{z_0}\}$ is not a single point: Since L is a strict Lyapunov function, we have $L(\phi_t(z_0)) < L(z_0)$ for all $t > 0$. When k is large, $\phi_{t_k}(x)$ is close to z_0, so by continuity,

$$L(\phi_{t+t_k}(x)) = L(\phi_t(\phi_{t_k}(x))) < L(z_0)$$

But this contradicts $L(\phi_t(x)) > L(z_0)$ (which we must have if there are arbitrarily large t such that $\phi_t(x)$ is close to z_0). Therefore, $x_0 = z_0$. ☐

☐

(4) Find a strict Lyapunov function for the linear autonomous system $y' = Ay$, where A is a real matrix whose eigenvalues all have negative real part.

(5) Give two proofs of the asymptotic stability theorem for a fixed point of an autonomous system, at which the eigenvalues of the linearization all have negative real part: One direct proof and one based on the Lyapunov function.

Proof. Theorem: Let $f(x_0) = 0$. If the eigenvalues of the linearization $A = f'(x_0)$ all have negative real parts, then the fixed point $x = x_0$ is asymptotically stable.

Proof. WLOG let $x_0 = 0$. Write $f(x) = Ax + g(x)$, where $g(x) = O(|x|^2)$. Since every $\lambda \in \sigma(A)$ has negative real part, there exist $a, C > 0$ (let $C > 1$ WLOG) such that

$$|\mathrm{e}^{tA}x| \leq C\mathrm{e}^{-at}|x|$$

The C arises because the matrix norm of e^{tA} is bounded as $t \to +\infty$ if all eigenvalues are negative. The e^{-at} arises similarly, and reflects the exponential decrease in magnitude happening along all subspaces on which e^{tA} acts.

Let δ be such that $|g(x)| \leq a|x|/2C$ when $|x| \leq \delta$. Now consider the IVP

$$y' = Ay + g(y), \quad y(0) \in \bar{B}\left(0, \frac{\delta}{2C}\right)$$

Then at least for small t (i.e., t such that $|y(t)| \leq \delta$),

$$|y(t)| \leq C\mathrm{e}^{-at}|y(0)| + \frac{a}{2C}\int_0^t \mathrm{e}^{-a(t-\tau)}|y(\tau)|\mathrm{d}\tau$$

It follows from Grönwall's inequality that

$$\mathrm{e}^{at}|y(t)| \leq C|y(0)|\mathrm{e}^{at/2}$$

hence

$$|y(t)| \leq \frac{\delta}{2}\mathrm{e}^{-at/2} < \delta$$

Hence, any orbit of the system starting from $\bar{B}(0, \delta/2C)$ stays in $\bar{B}(0, \delta)$. So the maximal time of existence T is $+\infty$. This is because if not then, then the IVP starting from $y(T)$ is still solvable, contradicting the definition of T. Thus, we have proven that

$$|y(t)| \leq \frac{\delta}{2}\mathrm{e}^{-at/2}$$

for all $t \geq 0$ as long as $|y(0)| \leq \delta/2C$. ☐

□

(6) State the instability counterpart to the stability theorem in part (5).

Proof. Theorem: Let $f(0) = 0$. If one of the eigenvalues of $A = f'(0)$ has positive real part, then the fixed point $x = 0$ is not Lyapunov stable. □

(7) Describe the stability of the equilibria of the Newtonian system $x'' = -U'(x)$, where x is the 1D coordinate of the particle.

Proof. For a Newtonian system $x'' = -U'(x)$, the law of conservation of energy asserts that each solution corresponds to a specific energy level and moves so as to maintain its energy (faster and with more kinetic energy at lower energy levels and slower and with less kinetic energy at higher energy levels). □

(8) Describe all fixed points of the system for a pendulum with friction. Prove the global asymptotic stability of the fixed point $(0, 0)$.

Proof. For a pendulum with friction, the fixed points are $(0, 0)$ and $(\pi, 0)$ corresponding to the asymptotically stable equilibrium of the pendulum hanging at the bottom and the unstable equilibrium of the pendulum positioned exactly above the bottom (assuming that the arm of the pendulum is stiff).

The system is described by the ODE

$$ml\theta'' + b\theta' = -mg\sin\theta$$

Substitute $\eta = b/ml$ and $\omega = \theta'$ to get a nonlinear system

$$\begin{pmatrix}\theta\\ \omega\end{pmatrix}' = \begin{pmatrix}\omega\\ -\eta\omega - g/l\sin\theta\end{pmatrix}$$

At the equilibrium position $(\theta, \omega) = (0, 0)$, we have

$$A = \begin{pmatrix}\frac{\partial}{\partial\theta}(\omega) & \frac{\partial}{\partial\omega}(\omega)\\ \frac{\partial}{\partial\theta}(-\eta\omega - g/l\sin\theta) & \frac{\partial}{\partial\omega}(-\eta\omega - g/l\sin\theta)\end{pmatrix} \approx \begin{pmatrix}0 & 1\\ -g/l & -\eta\end{pmatrix}$$

i.e.,

$$\begin{pmatrix}\theta\\ \omega\end{pmatrix}' = \begin{pmatrix}0 & 1\\ -g/l & -\eta\end{pmatrix}\begin{pmatrix}\theta\\ \omega\end{pmatrix} + O(|\theta|^2 + |\omega|^2)$$

Since $\eta > 0$, the eigenvalues have a common negative real part, so the equilibrium is asymptotically stable. □

Local Behavior Near a Hyperbolic Fixed Point

(1) State the definition of hyperbolic fixed point of an autonomous ODE system.

Proof. **Hyperbolic** (fixed point of f): A fixed point $x_0 \in \mathbb{R}^n$ for which $f'(x_0)$ has neither purely imaginary nor zero eigenvalues. □

(2) State the definition of stable and unstable set of a fixed point of an autonomous ODE system.

Proof. **Stable subset** (of x_0 under f): The set of all vectors attracted to x_0. *Also known as* **attracting subset**. *Denoted by* $\boldsymbol{W_s(x_0)}$. *Given by*

$$W_s(x_0) = \{x \in \mathbb{R}^n \mid \phi_t(x) \to x_0 \text{ as } t \to +\infty\}$$

Unstable subset (of x_0 under f): The set of all vectors repelled from x_0. *Also known as* **repelling subset**. *Denoted by* $\boldsymbol{W_u(x_0)}$. *Given by*

$$W_u(x_0) = \{x \in \mathbb{R}^n \mid \phi_t(x) \to x_0 \text{ as } t \to -\infty\}$$

□

(3) State the stable manifold theorem for a hyperbolic fixed point of an autonomous ODE system.

Proof. Theorem (stable manifold theorem): Let $y' = f(y)$ and let x_0 be a hyperbolic fixed point of f. Then there exists a neighborhood $U(x_0)$ of x_0 such that $U(x_0) \cap W_s(x_0)$ is a smooth submanifold of dimension $\dim \mathbb{E}_s[f'(x_0)]$ that is tangent to $\mathbb{E}_s[f'(x_0)]$ at x_0. An analogous statement holds for $U(x_0) \cap W_u(x_0)$. □

(4) Give a concrete example of the stable manifold theorem where the stable and unstable sets can be explicitly found.

Proof. We will treat

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -x + y + 3y^2 \\ y \end{pmatrix}$$

The correct flow is as follows (there was a typo in class).

$$\phi_t \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} z\mathrm{e}^{-t} + w\sinh(t) + w^2(\mathrm{e}^{2t} - \mathrm{e}^{-t}) \\ w\mathrm{e}^t \end{pmatrix}$$

The stable manifold is going to be the set of all $x \in \mathbb{R}^2$ such that $\phi_t(x) \to 0$ as $t \to +\infty$. Approach: Find values of z, w such that the solution converges to zero componentwise. Since $y(t) = w\mathrm{e}^t$, we must have $w = 0$; otherwise, we will get exponential divergence as $t \to +\infty$. Thus, $x(t) = z\mathrm{e}^{-t}$. This function converges to zero for any value of z, so we may let z be arbitrary. But the set of all points

$$\begin{pmatrix} z \\ 0 \end{pmatrix}$$

is the x-axis!

Unstable manifold: We need to find the set of all points $x \in \mathbb{R}^2$ such that $\phi_t(x) \to 0$ as $t \to -\infty$. Approach: Again, go by components. $y(t) = w\mathrm{e}^t$ will converge to 0 as $t \to -\infty$ for all w, so this component does not put any restrictions on w. Note that it also does not put any restrictions on z since it does not even contain z. Working with the other one, we expand and combine all e^{at} terms for $a > 0$ and all e^{bt} terms for all $b < 0$.

$$\begin{aligned}
x(t) &= z\mathrm{e}^{-t} + w\sinh(t) + w^2(\mathrm{e}^{2t} - \mathrm{e}^{-t}) \\
&= z\mathrm{e}^{-t} + w \cdot \frac{\mathrm{e}^t - \mathrm{e}^{-t}}{2} + w^2\mathrm{e}^{2t} - w^2\mathrm{e}^{-t} \\
&= z\mathrm{e}^{-t} + \frac{w}{2}\mathrm{e}^t - \frac{w}{2}\mathrm{e}^{-t} + w^2\mathrm{e}^{2t} - w^2\mathrm{e}^{-t} \\
&= \left[w^2\mathrm{e}^{2t} + \frac{w}{2}\mathrm{e}^t\right] + \left[z - \frac{w}{2} - w^2\right]\mathrm{e}^{-t}
\end{aligned}$$

The left term above will clearly converge to 0 as $t \to -\infty$. However, the right term will diverge to ∞ as $t \to -\infty$ unless $z - w/2 - w^2 = 0$, so we take this to be our condition. Indeed, this implies that $z = w/2 + w^2$ is a constraint on z, but w can still take on any value, so our solution is

$$W_u(0) = \left\{ \left(\frac{y}{2} + y^2, y \right) \,\middle|\, y \in \mathbb{R} \right\}$$

as desired. □

(5) State the Hartman linearization theorem.

Proof. Theorem (Hartman-Grobman Theorem): Let $y' = f(y)$, x_0 a hyperbolic fixed point, and $A = f'(x_0)$. Then there exists a neighborhood $U(x_0)$ and a homeomorphism $h : U(x_0) \to B(x_0, d)$ such that

$$h \circ \phi_t = \mathrm{e}^{tA} \circ h$$

for $|t|$ small. □

(6) Describe the global behavior of the predator-prey system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} (1 - y - \lambda x)x \\ \alpha(x - 1 - \mu y)y \end{pmatrix}$$

in the first quadrant according to different choices of positive parameters λ, μ.

Proof. We now have four fixed points:

$$(0,0) \qquad (\lambda^{-1}, 0) \qquad (0, -\mu^{-1}) \qquad \left(\frac{1+\mu}{1+\mu\lambda}, \frac{1-\lambda}{1+\mu\lambda} \right)$$

The third lies outside of $\bar{Q}$, so we disregard it. The fourth lies outside of $\bar{Q}$ if $\lambda > 1$. Thus, let's start with the case $\lambda \geq 1$ so that we only have to deal with one new fixed point. $\lambda \geq 1$. Our new fixed point is $(\lambda^{-1}, 0)$. It is a hyperbolic sink if $\lambda > 1$. If $\lambda = 1$, one eigenvalue is 0 and we need a more thorough investigation. Idea: Split Q into regions where $\dot{x}, \dot{y}$ have definite signs and then use the elementary observation in Lemma 7.2. The regions where $\dot{x}, \dot{y}$ have definite signs are separated by the two lines

$$L_1 = \{(x, y) \mid y = 1 - \lambda x\} \qquad L_2 = \{(x, y) \mid \mu y = x - 1\}$$

We derive these by setting $1 - y - \lambda x = 0$ and $x - 1 - \mu y = 0$. Label the regions in Q enclosed by these lines from left to right by Q_1, Q_2, Q_3. Observe that the lines are transversal, i.e., can only be crossed in the direction from $Q_3 \to Q_2$ and $Q_2 \to Q_1$. This can be seen from the solution curves in the picture. Suppose we start at $(x_0, y_0) \in Q_3$. Additional constraint: $x \leq x_0$ (the flow is to the left??). By Lemma 7.2: Either the trajectory enters L_2 or it converges to a fixed point in $\bar{Q}_3$. The latter case can only happen if $(\lambda^{-1}, 0) \in \bar{Q}_3$, i.e., if $\lambda = 1$. Similarly, starting in Q_2 either gets you across L_1 or to $(\lambda^{-1}, 0)$. Starting in Q_1 must take you to the fixed point. Thus, every trajectory converges to the fixed point. Let $0 < \lambda < 1$. We apply the same strategy as before. We have four regions this time. Let Q_4 be the new (bottom) one. We can only pass through these in the order $Q_4 \to Q_3 \to Q_2 \to Q_1 \to Q_4$. Thus, we have to rule out the periodic case this time. For simplicity's sake, let

$$(x_0, y_0) = \left(\frac{1+\mu}{1+\mu\lambda}, \frac{1-\lambda}{1+\mu\lambda} \right)$$

To do so, introduce (inspired by the original) the Lyapunov function

$$L(x, y) = \gamma_1 f(\tfrac{y}{y_0}) + \alpha\gamma_2 f(\tfrac{x}{x_0})$$

where, as before, $f(a) = a - 1 - \log(a)$. We seek constraints on γ_1, γ_2 that will make L strict. Calculate

$$\dot{L} = \frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial y}\dot{y} = -\alpha \left(\frac{\lambda\gamma_2}{x_0}\bar{x}^2 + \frac{\mu\gamma_1}{y_0}\bar{y}^2 + \left(\frac{\gamma_2}{x_0} - \frac{\gamma_1}{y_0} \right) \bar{x}\bar{y} \right)$$

where

$$\dot{x} = (-\bar{y} - \lambda\bar{x})x \qquad \dot{y} = \alpha(\bar{x} - \mu\bar{y})y \qquad \bar{x} = x - x_0 \qquad \bar{y} = y - y_0$$

The the RHS will be negative if we choose $\gamma_1 = y_0$ and $\gamma_2 = x_0$, so choose this, and then L is strictly decreasing, so all orbits starting in Q converge to the fixed point (x_0, y_0). □

Planar Systems

(1) State the definition of ω-limit sets and α-limit sets for a given point in an autonomous ODE system.

Proof. **$\boldsymbol{\omega}$-limit set** (of x): The set of all points $y \in M$ for which there exists a sequence $\{t_n\}$ that converges to $+\infty$ and satisfies $\Phi(t_n, x) \to y$. *Denoted by* $\boldsymbol{\omega(x)}$.

$\boldsymbol{\alpha}$-limit set (of x): The set of all points $y \in M$ for which there exists a sequence $\{t_n\}$ that converges to $-\infty$ and satisfies $\Phi(t_n, x) \to y$. *Denoted by* $\boldsymbol{\alpha(x)}$. □

(2) State the Poincaré-Bendixson theorem.

Proof. Theorem (Poincaré-Bendixson Theorem): Let $\Omega \in \mathbb{R}^2$ be open, $f(x)$ a vector field on Ω. Fix $x \in \Omega$. Also, let $\omega(x) \subset \Omega$ be compact and nonempty. In particular, there are three mutually exclusive cases for these limit sets.

(1) $\omega(x)$ (or $\alpha(x)$[9]) is a fixed point.
(2) $\omega(x)$ is a limit cycle.
(3) $\omega(x)$ consists of finitely many fixed points, together with curves joining these fixed points.

□

(3) Give a criterion for the existence of a limit cycle.

Proof. Theorem (Annulus theorem of Bendixson): Suppose C_1, C_2 are closed simple planar curves such that geometrically, one contains the other. We call the annular region (between the two curves) A. Suppose $f(x)$ is a planar vector field which points inward at every point of ∂A (the boundary of A). Then the annular region A is an invariant region of the plane. In particular, if A does not contain any fixed points, then it must contain a limit cycle. As before, curves within and without spiral towards it. □

(4) Give a criterion for the non-existence of a limit cycle.

Proof. Periodic orbits are *not* isolated but are dense, and in particular, we have a strict Lyapunov function. □

(5) Give an example of a planar autonomous system with exactly one limit cycle.

Proof.

$$\begin{pmatrix} r \\ \theta \end{pmatrix}' = \begin{pmatrix} r(1-r^2) \\ 1 \end{pmatrix}$$

where we use polar coordinates for simplicity. □

[9] We just have to reverse the time.

References

Labalme, S. (n.d.). *CAAG Thomas notes*. Retrieved December 23, 2022, from https://github.com/shadypuck/CAAGThomasNotes/blob/master/main.pdf

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