Week 2

???

2.1 Separable ODEs

10/3:

- Do not sit on the left side of the classroom: The sun sucks!
- **Separable** (ODE): An ODE of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)g(y)$$

  where y is a real[1], unknown, scalar function of t.
- Solving separable ODEs: Formally, evaluate

$$\int \frac{\mathrm{d}y}{g(y)} = \int f(t)\,\mathrm{d}t$$

- Rearrange the initial separable ODE to $\mathrm{d}y/\mathrm{d}t \cdot 1/g = f$ and invoke the law of composite differentiation to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{y_0}^{y(t)} \frac{\mathrm{d}w}{g(w)} - \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau\right] = 0$$

- It follows that

$$\int_{y_0}^{y(t)} \frac{\mathrm{d}w}{g(w)} = \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau$$

- Examples:
  1. Exponential growth.
     - We have that

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky$$

       for $k > 0$ and $y(0) = y_0 > 0$.
     - The solution is

$$\begin{aligned}\frac{1}{y}\cdot\frac{\mathrm{d}y}{\mathrm{d}t} &= k\\ \log y(t) - \log y_0 &= kt\\ y(t) &= y_0\mathrm{e}^{kt}\end{aligned}$$

[1] We'll deal with complex functions later.

2. Logistic growth.
   - We have that
     $$\frac{\mathrm{d}y}{\mathrm{d}t} = ky\left(1 - \frac{y}{M}\right)$$
     for $k, M > 0$ and $y(0) = y_0 > 0$.
   - The solution is
     $$\begin{aligned}
     \frac{M\,\mathrm{d}y}{y(M-y)} &= k\,\mathrm{d}t \\
     \log\frac{y}{M-y} - \log\frac{y_0}{M-y_0} &= kt \\
     \frac{y(M-y_0)}{y_0(M-y)} &= \mathrm{e}^{kt} \\
     y \cdot \frac{M-y_0}{y_0} &= (M-y)\mathrm{e}^{kt} \\
     y \cdot \frac{M-y_0}{y_0} + y\mathrm{e}^{kt} &= M\mathrm{e}^{kt} \\
     y\left(\frac{M-y_0}{y_0} + \mathrm{e}^{kt}\right) &= M\mathrm{e}^{kt} \\
     y\left(\frac{M-y_0+y_0\mathrm{e}^{kt}}{y_0}\right) &= M\mathrm{e}^{kt} \\
     y\left(\frac{M+y_0(\mathrm{e}^{kt}-1)}{y_0}\right) &= M\mathrm{e}^{kt} \\
     y(t) &= \frac{My_0\mathrm{e}^{kt}}{M+y_0(\mathrm{e}^{kt}-1)}
     \end{aligned}$$
   - Sketches the graph of logistic growth and discusses the turning point (for which there is a formula; zero of the second derivative) as well as general trends.
   - If $y_0 < 0$, the solution is not physically meaningful, but it is mathematically insightful.
     - When we integrate, the arguments of our logarithms now have absolute values.
       $$\log\left|\frac{y}{M-y}\right| - \log\left|\frac{y_0}{M-y_0}\right| = kt$$
     - We need to make sure that the denominator of the final logistic form is never equal to zero, but now that y_0 is negative, as t increases, the denominator will approach zero exponentially. It reaches zero when
       $$\begin{aligned}
       M + y_0(\mathrm{e}^{kt}-1) &= 0 \\
       \mathrm{e}^{kt} &= -\frac{M}{y_0} + 1
       \end{aligned}$$
       In other words, $t_{\max} = (1/k)\log(1 - M/y_0)$ because when $t = t_{\max}$, the equation blows up.
     - This is an example of **finite lifespan**.
   - If $y_0 > M$, then you will exponentially decrease to M.
3. Lotka-Volterra predator-prey model.
   - We have that
     $$r' = k_1 r - awr \qquad\qquad w' = -k_2 w + bwr$$
     where r is rabbits and w is wolves.

- We can rename the variables to

$$\begin{cases} x' = Ax - Bxy \\ y' = -Cy + Dxy \end{cases}$$

- Dividing, we get

$$\frac{x'}{y'} = \frac{Ax - Bxy}{-Cy + Dxy}$$
$$\frac{By - A}{y} y' + \frac{Dx - C}{x} x' = 0$$

- Use the fact that x, y are independent variables, so both terms in the above equation are equal to zero?
- Invoke the law of composite differentiation twice and, from the above, know that $0 + 0 = 0$, so we can add the two solutions:

$$\frac{\mathrm{d}}{\mathrm{d}t}(By(t) - A\log y(t)) + \frac{\mathrm{d}}{\mathrm{d}t}(Dx(t) - C\log x(t)) = 0$$
$$By(t) - A\log y(t) + Dx(t) - C\log x(t) = E$$

- Sketches some of the trajectories (they're all closed curves in the xy-plane).

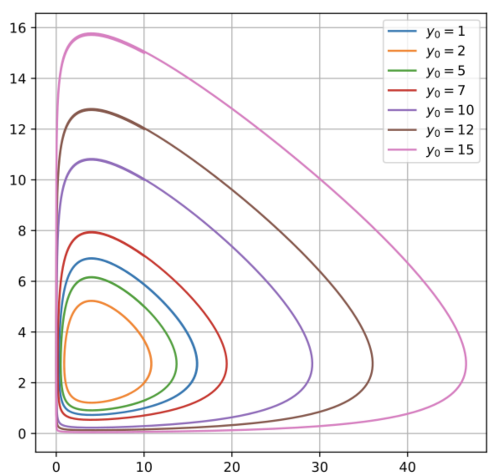

Figure 2.1: Lotka-Volterra solution curves.

- Properties of the curves:
  - The implicit relation which determines them: By the implicit function theorem, the y derivative of the LHS is $B - A/y$ and the x-derivative of the LHS is $D - C/x$. When the partial derivatives are equal to zero, $(C/D, A/B)$ becomes interesting. Turning points happen when the y-coordinate is A/B or the x-coordinate is C/D.

- **Finite lifespan**: Even if the RHS of $\mathrm{d}y/\mathrm{d}t = f(t, y)$ is very regular, the solution can still blow up at some finite time.

- Consider the following variation on the E-L equation from the Brachistochrone problem.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{\frac{B - y}{y}}$$

  - Finding the **primitives**.

- What are these "primitives" Shao keeps talking about?

- We should have
$$\int \sqrt{\frac{y}{B-y}}\,\mathrm{d}y = x$$
- Change of variables: $y = B\sin^2\phi$ and $\mathrm{d}y = 2B\cos\phi\sin\phi\,\mathrm{d}\phi$. Thus,
$$\int \sqrt{\frac{y}{B-y}}\,\mathrm{d}y = \int \frac{\sin\phi}{\cos\phi}\cdot 2B\cos\phi\sin\phi\,\mathrm{d}\phi = 2B\int \sin^2\phi\,\mathrm{d}\phi$$
- The solution is
$$\begin{cases} x = B\phi - \frac{B}{2}\sin(2\phi) + C \\ y = B\sin^2\phi \end{cases}$$
  - This is a parameterization of a cycloid.

- Later in the week, we will do the SHM, the pendulum, the Kepler 2-body problem, and the Michaelis-Menten equation.
- Separable ODEs are a subset of ODEs of **exact form**.
- ODEs of exact form are of the form
$$g(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} + f(x,y) = 0$$
where for some $F(x,y)$, $g = \partial F/\partial y$, $f = \partial F/\partial x$, and partials commute. Equivalently,
$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$$
is our necessary and sufficient condition.
- By the law of composite differentiation,
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[F(x,y(x))\right] &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\cdot y'(x) \\ &= f(x,y(x)) + g(x,y(x))y'(x) \\ &= 0\end{aligned}$$
  - We solve these with an integrating factor $\mu \neq 0$ such that $(\mu g, \mu f)$ satisfy the constraint.

2.2 Office Hours (Shao)

- **Primitive**: An antiderivative.
- **Law of composite differentiation**: The chain rule.
- Went over how Shao has been applying the law of composite differentiation with respect to separable ODEs:
  - Rearrange the initial separable ODE as follows.
$$\frac{1}{g(y)}\cdot\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)$$

- Define $\mathrm{d}H/\mathrm{d}y = 1/g(y)$. Then, continuing from the above, we have by the law of composite differentiation that

$$\frac{\mathrm{d}H}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} = f(t)$$
$$\frac{\mathrm{d}H}{\mathrm{d}t} = f(t)$$

- From the definition of H, we know that $H(y) = \int_{y_0}^{y} \mathrm{d}w\,/g(w)$. We also have from the FTC that $f(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau$. Thus, continuing from the above, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}(H) = f(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{y_0}^{y} \frac{\mathrm{d}w}{g(w)}\right] = \frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{y_0}^{y(t)} \frac{\mathrm{d}w}{g(w)} - \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau\right] = 0$$

as desired.

- It follows since $y(t_0) = y_0$ that $C = H(y_0) = 0$, so we can take the above to

$$\int_{y_0}^{y(t)} \frac{\mathrm{d}w}{g(w)} = \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau$$

knowing that our constant of integration is zero.

- Take away from Brachistochrone problem: Just an example of a BDE; we won't have to answer questions on it.

2.3 ODEs of Exact Form

10/5:
- Last time, we discussed separable ODEs.
- Today, we will study **exact form** equations, as discussed last class.
- **Exact form** (ODE): An ODE of the form

$$g(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} + f(x,y) = 0$$

where

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$$

- For equations of this form, there exists $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = f \qquad \frac{\partial F}{\partial y} = g \qquad F(x, y(x)) = C$$

for some $C \in \mathbb{R}$.

- To solve equations of this form, we need an **integrating factor**.
- **Integrating factor**: A number or function μ such that

$$\mu g\frac{\mathrm{d}y}{\mathrm{d}x} + \mu f = 0 \qquad \frac{\partial}{\partial x}(\mu g) = \frac{\partial}{\partial y}(\mu f)$$

- For linear homogeneous equations $\mathrm{d}y/\mathrm{d}t = p(t)y$, we have

$$y(t) = y_0 \exp\left[\int_{t_0}^{t} p(\tau)\mathrm{d}\tau\right]$$

- Recall that $\mathrm{e}^{a+ib} = \mathrm{e}^{a}(\cos b + i\sin b)$, so

$$\mathrm{e}^{ix} = \cos x + i\sin x \qquad \cos x = \frac{1}{2}(\mathrm{e}^{ix} + \mathrm{e}^{-ix}) \qquad \sin x = \frac{1}{2i}(\mathrm{e}^{ix} - \mathrm{e}^{-ix})$$

- Example: If $y' = ky$, then $y' = -\lambda y$.
- If we have an inhomogeneous linear equation $\mathrm{d}y/\mathrm{d}t = p(t)y + f(t)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}t} - py - f = 0$$

  but

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(1) \neq \frac{\mathrm{d}}{\mathrm{d}y}(-p(t)y - f(t))$$

- We wish to find an integrating factor $\mu(t, y)$ such that

$$\mu(t,y)\frac{\mathrm{d}y}{\mathrm{d}t} - \mu(t,y)p(t)y - \mu(t,y)f(t) = 0$$

  and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu) = \frac{\mathrm{d}}{\mathrm{d}y}(-\mu py - \mu f)$$

- Solution: Take μ to be a function of t, alone. Then

$$\mu'(t) = \frac{\mathrm{d}}{\mathrm{d}y}(-\mu py - \mu f) = -\mu(t)p(t)$$

  and we now have a homogeneous linear equation with solution

$$\mu(t) = \exp\left[-\int_{t_0}^{t} p(\tau)\mathrm{d}\tau\right]$$

  - If we let $P(t) = \int_{t_0}^{t} p(\tau)\mathrm{d}\tau$, then

$$\begin{aligned} \mathrm{e}^{-P(t)}y'(t) - p(t)y(t)\mathrm{e}^{-P(t)} &= \mathrm{e}^{-P(t)}f(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-P(t)}y(t)\right) &= \mathrm{e}^{-P(t)}f(t) \\ \mathrm{e}^{-P(t)}y(t) &= \int_{t_0}^{t} \mathrm{e}^{-P(\tau)}f(\tau)\mathrm{d}\tau \end{aligned}$$

  - Thus, we finally have the solution to the inhomogeneous problem as follows: The IVP $y' = py + f$, $y(t_0) = y_0$ has solution

$$y(t) = y_0\mathrm{e}^{P(t)-P(t_0)} + \mathrm{e}^{P(t)}\int_{0}^{t} \mathrm{e}^{-P(\tau)}f(\tau)\mathrm{d}\tau$$

    where P is any anti-derivative of p.

- In particular, when $p(t) = a$, we get the **Duhamel formula** (which we should memorize).

- **Duhamel formula**: The following equation, which is the solution to an inhomogeneous linear equation with $p(t) = a$.
$$y(t) = y_0 \mathrm{e}^{a(t-t_0)} + \int_{t_0}^{t} \mathrm{e}^{a(t-\tau)} f(\tau) \mathrm{d}\tau$$
  - Important for computing forced oscillation.
- Inspecting the inhomogeneous solution.
  - The first term is the solution to the homogeneous form. The second term deals with the initial value.
- Given an inhomogeneous equation, you can always write its solution as the combination of the solution to the homogeneous problem plus a particular solution, i.e.,
$$y = y_h + y_p$$
  - "The general solution equals the homogeneous solution plus a particular solution."
  - This is related to linear algebra, where the solution to $Ax = b$ is a particular solution x_p plus any vector $x \in \ker A$.
  - Thus, this idea will reappear in the theory of systems of linear ODEs.
- We now look at systems of linear ODEs.
- Consider the harmonic oscillator: A particle of mass m connected to an ideal spring (obeys Hooke's law) with no friction or gravity.
  - Newton's second law: The acceleration is proportional to the restoring force.
  - Hooke's law: The restoring force is of magnitude kx in the opposite direction to the displacement.
  - Thus, the ODE is of the form
$$x'' = -\frac{k}{m} x$$
- Consider an ODE of the form
$$y'' + ay' + by = 0$$
for $a, b \in \mathbb{C}$.
  - Aim: Find $\mu, \lambda \in \mathbb{C}$ such that
$$(y' - \mu y)' - \lambda(y' - \mu y) = 0$$
  - To find the parameters, we expand the above to
$$y'' - (\mu + \lambda)y' + \mu\lambda y = 0$$
  - Comparing with the original form, we have that $a = -(\mu + \lambda)$ and $b = \mu\lambda$.
  - It follows that μ, λ are the roots of $x^2 + ax + b = 0$, which we will call the **characteristic polynomial** of the ODE.
- Example:
  - Consider
$$y' - \mu y = A\mathrm{e}^{\lambda t}$$
  - By the Duhamel equation, we have that a particular solution is of the form
$$A \int_0^t \mathrm{e}^{\mu(t-\tau)} \mathrm{e}^{\lambda\tau} \mathrm{d}\tau$$

- Thus, general solutions are of the form

$$y(t) = B\mathrm{e}^{\mu t} + C\mathrm{e}^{\mu t}\int_0^t \mathrm{e}^{(\lambda-\mu)\tau}\mathrm{d}\tau$$

- Evaluating the integral, we get

$$y(t) = B\mathrm{e}^{\mu t} + C\mathrm{e}^{\mu t}\frac{\mathrm{e}^{(\lambda-\mu)t}-1}{\lambda-\mu}$$

which simplifies (by incorporating constants, etc.) to

$$y(t) = A_1\mathrm{e}^{\mu t} + B_1\mathrm{e}^{\lambda t}$$

for $\mu \neq \lambda$, or

$$y(t) = A_1\mathrm{e}^{\mu t} + B_1 t\mathrm{e}^{\mu t}$$

for $\mu = \lambda$.

- If our equation is of the form $y'' + ay' + by = f(t)$, then we just need to apply the Duhamel formula twice.
- Returning to the simple harmonic oscillator problem, we substitute $\omega = \sqrt{k/m}$ to get

$$x'' = \omega^2 x$$

- The characteristic polynomial is

$$0 = x^2 + \omega^2 = (x + i\omega)(x - i\omega)$$

- Thus, solutions are of the form

$$x = A_1\mathrm{e}^{i\omega t} + B_1\mathrm{e}^{-i\omega t}$$

- It follows that the period is $T = 2\pi/\omega$.
- To get a real (usable) solution, apply Euler's formula to get

$$\begin{aligned}x(t) &= A_1(\cos\omega t + i\sin\omega t) + B_1(\cos\omega t - i\sin\omega t)\\ &= A\cos\omega t + B\sin\omega t\end{aligned}$$

where $A = A_1 + B_1$, $B = iA_1 - iB_1$.
- To match the initial condition $x(0) = x_0$, $x'(0) = v_0$, we use

$$x(t) = x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t$$

- In other words,

$$\begin{cases}A = x_0\\ \omega B = v_0\end{cases} \qquad \begin{cases}A_1 + B_1 = x_0\\ i\omega A_1 - i\omega B_1 = v_0\end{cases}$$

so

$$\begin{cases}A = x_0\\ B = \frac{v_0}{\omega}\end{cases} \qquad \begin{cases}A_1 = \frac{1}{2}\left[x_0 - \frac{iv_0}{\omega}\right]\\ B_1 = \frac{1}{2}\left[x_0 + \frac{iv_0}{\omega}\right]\end{cases}$$