Week 8

Stability Grab Bag

8.1 Midterm 2 Review

11/14:
- Still 3 problems total and 5 points each.
  - The problems will be calculations based on the basic concepts.
  - Figure out the stable and unstable subspaces of some finite systems.
  - Figure out whether or not a system is stable.
  - Prove whether or not a function is planar linear
- Starting with the classification of planar linear autonomous systems.
  - We have $y' = Ay$ where A is a 2×2 real matrix.
  - As a result of the realness, the eigenvalues behave regularly, i.e., there are only finitely many types of eigenvalues. These are...
    1. Real, nonzero, same sign. Depending on the sign, we'll either have a source or a sink. The orbits will be a distorted graph of a power function. If asked to investigate the phase portrait, then we need to figure out the stable and unstable subspaces and clearly indicate a basis. If asked to draw, we need to clearly indicate which subspaces are stable and unstable. We also need to clearly indicate the direction of the phase lines. First case: Everything is stable; second case: Everything is unstable. We draw the eigenspaces as well with arrows on the "axes." Figure 5.3a-5.3b.
    2. Real, different sign. One stable and one unstable subspace. We need to clearly indicate how the axes are tilted. Figure 5.3c.
    3. A is similar to the Jordan block with zero eigenvalues and 1 in the upper right hand corner. Then
       $$A \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mathrm{e}^{tA} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$
    4. Purely imaginary eigenvalues. These must appear in a conjugate pair. The phase diagram will be concentric ellipses, and we essentially have the harmonic oscillator equation. If we have to sketch, we must show how the ellipses are tilted.
    5. Complex eigenvalues $\sigma \pm i\beta$. Either we have a spiral source or a spiral sink. It's meaningless to indicate how the spiral tilts here, so don't bother trying. Determining whether they spin clockwise or counterclockwise. If
       $$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

then our fundamental solution is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and we rotate counterclockwise. Since $A^2 = -\mu^2 I_2$, $\mathrm{e}^{tA} = I_2 \cos \mu t + \mu^2 I_2 \sin t$. Negative reverses everything. Harmonic oscillator goes counterclockwise.

- There is an online website that gives us phase portraits for an equation. We can use this to help develop intuition.
- If you have a set of eigenvectors, how do you know how to tilt it?
  - Shao goes over examples of eigenvalues and eigenvectors.
- This is not something you need to memorize, but something you need to be able to recover.

- This is not a course for math majors; thus, there will not be proofs concerning the contraction mapping principle. We will not be asked to show existence, uniqueness, continuous difference, or differentiability with respect to parameters.
- We do need to know Grönwall's inequality, however.
- Grönwall's inequality: If $\phi : [p, T] \to \mathbb{R}$ and

$$\phi(t) \leq b + a \int_0^t \phi(\tau) \mathrm{d}\tau$$

then

$$\phi(t) \leq b\mathrm{e}^{at}$$

  - Usually stated in the integral form, and we usually only need a special case.
  - We may need to prove this; the proof mimics the derivation of the Duhamel formula.
  - $a, b \in \mathbb{R}$.
  - We need to memorize the proof.
  - We also need to be able to recognize when we can and should use it. Let $\phi(t) = \Phi'(t) \leq b + a\Phi(t)$, $\Phi(0) = 0$. Then $\phi(t) \leq b + a \int_0^t \varphi(\tau) \mathrm{d}\tau$.
  - Use it when we want to bound a function that satisfies either an integral or a differential quantity.
  - This is the only proof in the theory of ODE systems we need to memorize.

- We need to master the methods to compute perturbation series.
  - Suppose our IVP depends on a parameter μ differentiably.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t; y(t; \mu); \mu), y(t_0) = x(\mu), \mu \approx 0$$

  - If the parameter is close to zer, then you should be able to compute the μ-derivative with respect to the parameter.
  - By Taylor expanding with respect to the parameter, you should be able to recover solutions that are close to the actual.

$$y(t; \mu) = y_0(t) + y_1(t)\mu + y_2(t)\mu^2 + O(\mu^3)$$

  - We are typically satisfied with approximations to the second order.
  - We expand our ODE into a Taylor series of μ. The differentiability with respect to parameters theorem (see Lecture 6.2 or Theorem 2.11 in Teschl (2012)) tells us that this is legitimate.

$$\frac{\mathrm{d}}{\mathrm{d}t}(y_0(t)) = f(t; y_0(t); 0), y_0(t_0) = x(0)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(y_1(t)) = \frac{\partial f}{\partial z} y_1(t) + \frac{\partial f}{\partial \mu}, y_1(t) = \frac{\partial x}{\partial \mu}$$

- Just know the basic Taylor expansions (trig ones and exponential functions; usually we'll stick to polynomials, though).
- Use the ansatz $y(t;\mu) = y_0(t) + y_1(t)\mu + y_2(t)\mu^2 + O(\mu^3)$.
- Substitute $y(t;\mu)$ into $f(t, y(t;\mu);\mu)$. Expand $f(t;y(t;\mu);\mu)$ into a Taylor series of μ. Balance the coefficients of $\mu^0, \mu^1, \mu^2, \dots$.
- Then you will get a series of equations that is theoretically solvable. Then a sequence of ODEs for $y_0(t), y_1(t), y_2(t), \dots$.
- Your ODEs for $y_1, y_2, \dots$ should not involve μ (because they are coefficients in the Taylor expansion with respect to μ. Coefficients of a Taylor series shouldn't involve the argument); if it does, there is something going wrong.
- As for the initial value, $y_0(t_0)+y_1(t_0)\mu+y_2(t_0)\mu^2+\cdots$. This implies that something equals $x(\mu)$. The Taylor coefficients of $x(\mu)$ at $\mu = 0$.
- These are the general steps you use to find the perturbative series expansion.
- The computations on the exam will not be too heavy.
- If you're still unclear on the calculation, look through the HW answer keys.

- Conclusion: The Grönwall's inequality is something we need to remember from the theory; the perturbative procedure is something we need to be able to do.
- Why do we expand with respect to μ?
  - We do it with respect to μ because our function is a function of μ. Differentiability and smallness imply we can use the Taylor series.
- Shao reiterates: Definitely read through the key to HW5!!! All the steps you will need to do are done completely and in detail.
- There will be things that are in HW6 (the one due Friday) that will appear on the exam because we have discussed these things in lecture.
- The definitions of Lyapunov stability and asymptotic stability. These will appear in the exam. We need to *clearly* remember the definitions.
- Consider $y' = f(y)$, $f(x_0) = 0$ (an autonomous system with a fixed point; we can transform our system via $(y - x_0)' = f(x_0 + (y - x_0))$ to translate our fixed point to zero; implies $y' = f(x_0 + y)$, $y = 0$ is a fixed point). We should be able to determine the asymptotic stability near x_0 by computing the linearization (i.e., the Jacobian $f'(x_0)$) at the fixed point.
  - Regarding determining stability near x_0, remember the following theorem.
  - Theorem: If all eigenvalues of $f'(x_0)$ have negative real parts, then x_0 is asymptotically stable. If at least one eigenvalue has real part greater than zero, then x_0 is not Lyapunov stable.
  - We should be able to apply the above criterion in practice.
  - We should also be able to reproduce the proof of the first part of Lyapunov's theorem (related to a question in HW6).
  - Lyapunov functions: $f(x_0) = 0$. Definition:
    1. $L(x)$ is C^1 near x_0, $L(x_0) = 0$. $L(x) > 0$ for x near x_0.
    2. $\nabla L(x) \cdot f(x) \leq 0$ for x near x_0 iff $L(\phi_t(x)) \leq L(x)$, $t \geq 0$. If $L(\phi_t(x))$ is always strictly decreasing, then it is a strict Lyapunov function.
  - Theorem (Lyapunov's theorem): Usually, we can explicitly determine a Lyapunov function:
    1. If there is a Lyapunov function near the fixed point, then it is Lyapunov stable. For trajectories starting at nearby points, the trajectory can never excape nearby points.
    2. If there is a strict Lyapunov function, then it is asymptotically stable.

- We need to be able to apply this theorem in practice; we don't need to know the proof.

- Examples of Lyapunov functions: Newton's second law.
  - Suppose you have a particle moving within a potential field with potential function U, i.e.,
$$mx'' = -U'(x)$$
  - Then by a standard process, you can convert it to a planar linear system by introducing the variable v (the velocity), i.e.,
$$\begin{pmatrix} x \\ v \end{pmatrix}' = \begin{pmatrix} v \\ -U(x)'/m \end{pmatrix}$$
  - Then $E(x, v) = \frac{m}{2}v^2 + U(x)$ is constant along the orbits, that is,
$$\nabla E(x, v) \cdot \begin{pmatrix} v \\ -U'(x)/m \end{pmatrix} = 0$$
    - The gradient of the energy function is orthogonal to the vector field.
  - $E(x, v)$ is a Lyapunov function (global). This happens and induces a fixed point exactly where the velocity is zero and the function takes on a critical value.
  - Linearization at the fixed point $(x_0, 0)$ is
$$\begin{pmatrix} 0 & 1 \\ -\frac{U''(x_0)}{m} & 0 \end{pmatrix}$$
So $E(x_0, v) > E(x_0, 0)$ for $x \sim x_0$, $v \sim 0$ iff U takes a minimum at x_0. The energy function cannot always stay larger than the energy at the fixed point. Satisfies second Lyapunov condition, but not the first.
  - One question: Classification of planar linear autonomous systems, one on Grönwall, one on qualitative asymptotic analysis using Lyapunov. Three questions total. There will also be some questions (parts of questions, I guess) on perturbative series.

8.2 Misc. Stability Tools

11/18: - Let $y' = f(y)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function and x_0 is a fixed point (i.e., $f(x_0) = 0$).

- **Hyperbolic** (fixed point of f): A fixed point $x_0 \in \mathbb{R}^n$ for which $f'(x_0)$ has neither purely imaginary nor zero eigenvalues.

- If x_0 is a hyperbolic fixed point of A, then we know that for the linear system $y' = Ay$, the eigenvalues of A are never purely imaginary by definition.
  - This allows us to decompose $\mathbb{R}^n$ into the direct sum of the **stable subspace** and the **unstable subspace** of the system.

- **Stable subspace** (of x_0 under A): The space of all generalized eigenvectors of A corresponding to eigenvalues λ with $\operatorname{Re}\lambda < 0$. *Also known as* **attracting subspace**. *Denoted by* $\mathbf{E_s}$.

- **Unstable subspace** (of x_0 under A): The space of all generalized eigenvectors of A corresponding to eigenvalues λ with $\operatorname{Re}\lambda > 0$. *Also known as* **repelling subspace**. *Denoted by* $\mathbf{E_u}$.

- But what if f is not a linear transformation? Then we cannot guarantee the subspace structure, so we need to generalize.

- **Stable subset** (of x_0 under f): The set of all vectors attracted to x_0. *Also known as* **attracting subset**. *Denoted by* $\boldsymbol{W_s(x_0)}$. *Given by*
$$W_s(x_0) = \{x \in \mathbb{R}^n \mid \phi_t(x) \to x_0 \text{ as } t \to +\infty\}$$

- **Unstable subset** (of x_0 under f): The set of all vectors repelled from x_0. *Also known as* **repelling subset**. *Denoted by* $\boldsymbol{W_u(x_0)}$. *Given by*

$$W_u(x_0) = \{x \in \mathbb{R}^n \mid \phi_t(x) \to x_0 \text{ as } t \to -\infty\}$$

- Notice that if $f = A$, then the stable (resp. unstable) subset equals the stable (resp. unstable) subspace.
- Theorem (stable manifold theorem): Let $y' = f(y)$ and let x_0 be a hyperbolic fixed point of f. Then there exists a neighborhood $U(x_0)$ of x_0 such that $U(x_0) \cap W_s(x_0)$ is a smooth submanifold of dimension $\dim \mathbb{E}_s[f'(x_0)]$ that is tangent to $\mathbb{E}_s[f'(x_0)]$ at x_0. An analogous statement holds for $U(x_0) \cap W_u(x_0)$.
- ***k*-dimensional smooth submanifold** (of $\mathbb{R}^n$): A subset congruent to the graph

$$G = (w_1, \dots, w_k; h_1(w), \dots, h_{n-k}(w))$$

  of some smooth function $h : \mathbb{R}^k \to \mathbb{R}^{n-k}$.
- Example: (w, w^2) is a 1-dimensional submanifold of $\mathbb{R}^2$.
  - We know it as the graph of the unit parabola.
- Example: $\left(w_1, w_2, \sqrt{1 - (w_1^2 + w_2^2)}\right)$ is a 2-dimensional submanifold of $\mathbb{R}^3$.
  - In particular, it is the positive hemisphere of the unit two-sphere.
- **Homeomorphism**: A continuous, invertible function with continuous inverse.
  - Essentially, it's a coordinate change function.
- Theorem (Hartman-Grobman Theorem): Let $y' = f(y)$, x_0 a hyperbolic fixed point, and $A = f'(x_0)$. Then there exists a neighborhood $U(x_0)$ and a homeomorphism $h : U(x_0) \to B(x_0, d)$ such that

$$h \circ \phi_t = \mathrm{e}^{tA} \circ h$$

  for $|t|$ small.

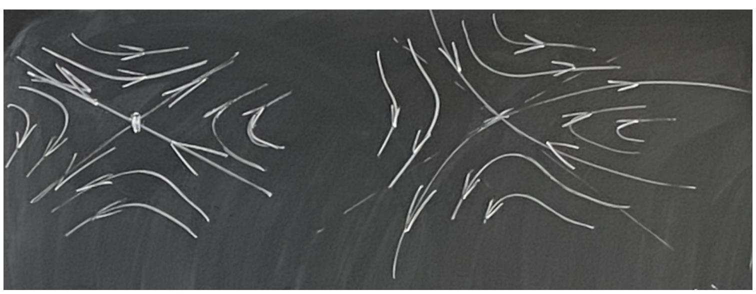

Figure 8.1: Hartman-Grobman Theorem visualization.

  - In laymen's terms: Near the hyperbolic fixed point, the orbits are just slight distortions of the linearized system.
- Corollary: Suppose A, B are matrices with no purely imaginary eigenvalues. Then the flows of A, B are topologically conjugate iff $\dim \mathbb{E}_s(A) = \dim \mathbb{E}_s(B)$ (equivalently, iff $\dim \mathbb{E}_u(A) = \dim \mathbb{E}_u(B)$).
- Example: Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

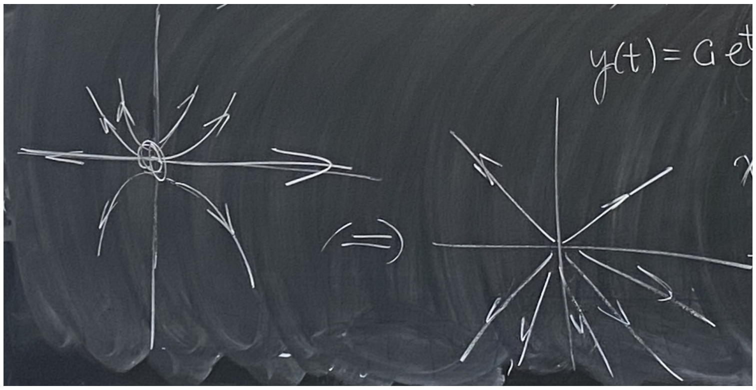

Figure 8.2: Topologically conjugate flows.

- Consider the linear autonomous systems $y' = Ay$ and $x' = Bx$.
- Then since

$$\mathrm{e}^{tA} = \begin{pmatrix} \mathrm{e}^t & t\mathrm{e}^t \\ 0 & \mathrm{e}^t \end{pmatrix} \qquad \mathrm{e}^{tB} = \begin{pmatrix} \mathrm{e}^t & 0 \\ 0 & \mathrm{e}^t \end{pmatrix}$$

we know that the flows are

$$y(t) = \begin{pmatrix} \mathrm{e}^t & t\mathrm{e}^t \\ 0 & \mathrm{e}^t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1\mathrm{e}^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\mathrm{e}^t \begin{pmatrix} t \\ 1 \end{pmatrix}$$
$$x(t) = \begin{pmatrix} \mathrm{e}^t & 0 \\ 0 & \mathrm{e}^t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1\mathrm{e}^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\mathrm{e}^t \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Since both A, B have no purely imaginary eigenvalues, these flows will be topologically conjugate by the Corollary.
  - Indeed, we can kind of see that one is a distortion of the other in Figure 8.2.
- We can't expect the coordinate change from Hartman-Grobman to be smooth, but it will exist.

- Example: Let

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -x + y + 3y^2 \\ y \end{pmatrix}$$

  - We can solve this to get

$$\phi_t \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} z\mathrm{e}^{-t} + w\sinh(t) + w^2(\mathrm{e}^{2t} - \mathrm{e}^{-t}) \\ w\mathrm{e}^t \end{pmatrix}$$

  - Notice that the origin 0 is a fixed point.
  - From this, we can determine that (how??)

$$W_s(0) = x\text{-axis} \qquad W_u(0) = \left\{ \left(\frac{y}{2} + y^2, y \right) \middle| y \in \mathbb{R} \right\}$$

- What if we can't solve the system in the above example explicitly?
  - Take the Jacobian at 0:

$$A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$

  - Find its stable and unstable subspaces. Calculate eigenvalues and eigenvectors to be

$$\lambda_1 = -1 \qquad \lambda_2 = 1$$
$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

- Thus, we get v_1 (the x-axis) as the stable subspace, and v_2 as the unstable subspace.

- General procedure for planar systems:
  1. Find all fixed points.
  2. Determine the stability of the fixed points. If hyperbolic, then apply the stable manifold and Hartman theorems. If the eigenvalues are purely imaginary, try to find a Lyapunov function.
  3. Decompose the plane into regions in which the monotonicity of
  $$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$
  is determined, i.e., the signs of the two components of the vector field are determined. This step requires more improvisation.

- Example:
$$\begin{pmatrix} \theta \\ \omega \end{pmatrix}' = \begin{pmatrix} \omega \\ -\sin\theta \end{pmatrix}$$

  picture

  - We only care where $-\pi < \theta < \pi$. The fixed points are
  $$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad\qquad \pm\begin{pmatrix} \pi \\ 0 \end{pmatrix}$$
  - At 0, the linearization has purely imaginary eigenvalues. We have Lyapunov function
  $$E(\theta, \omega) = \frac{1}{2}\omega^2 + (1 - \cos\theta)$$
  - At $(\pi, 0)$, the linearization is
  $$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
  which has eigenvalues and eigenvectors
  $$\lambda_1 = 1 \qquad\qquad \lambda_2 = -1$$
  $$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad\qquad v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$
  - Thus, we get orbits around 0 in the θ, ω plane and two subspaces that converge/diverge to $(\pi, 0)$. All of these lines are compatible tangentially.

8.3 Chapter 9: Local Behavior Near Fixed Points

From Teschl (2012)

Section 9.1: Stability of Linear Systems

12/6:

- Goal for the chapter: "Show that a lot of information on the stability of a flow near a fixed point can be read off by linearizing the system around the fixed point" (Teschl, 2012, p. 253).
- Recall the stability discussion for linear systems
$$\dot{x} = Ax$$
from Section 3.2.

- Additionally, our definition from Section 6.5 is invariant under a linear change of coordinates, so we may work in JCF.
- Recall that the long-term behavior is determined by the real part of the eigenvalues.
- "In general, it depends on the initial condition, and there are two linear manifolds $E^+(\mathrm{e}^A)$ and $E^-(\mathrm{e}^A)$ such that if we start in $E^+(\mathrm{e}^A)$ (resp. $E^-(\mathrm{e}^A)$), then $x(t) \to 0$ as $t \to +\infty$ (resp. $t \to -\infty$)" (Teschl, 2012, p. 253).

Section 9.2: Stable and Unstable Manifolds

- Goal: Transfer results from the previous section to nonlinear equations.
- **Stable set** (of a fixed point): The set of all points converging to the fixed point x_0 for $t \to +\infty$. *Denoted by* $\boldsymbol{W^+(x_0)}$. *Given by*

$$W^+(x_0) = \{x \in M \mid \lim_{t \to +\infty} |\Phi(t,x) - x_0| = 0\}$$

- **Unstable set** (of a fixed point): The set of all points converging to the fixed point x_0 for $t \to -\infty$. *Denoted by* $\boldsymbol{W^-(x_0)}$. *Given by*

$$W^-(x_0) = \{x \in M \mid \lim_{t \to -\infty} |\Phi(t,x) - x_0| = 0\}$$

- Both the stable and unstable sets are invariant under the flow.
- We know that for small t, the solutions are adequately described by the linearization, but what about for large t?
  - In this section, we generalize the Section 6.5 result for $n = 1$ stability and $A = f'(x_0)$ to higher dimensions.
- **Hyperbolic** (fixed point of f): A fixed point x_0 for which the linearization $f'(x_0)$ has no eigenvalues with zero real part.
  - Note that this is equivalent to the definition from class: The "zero real part" condition can be divided into two cases (equal to zero and nonzero but purely imaginary). This definition says no "zero real part;" that definition says not either of the latter two cases.