Week 1

Introduction to ODEs

1.1 Definitions and Scope

9/28:

- Questions:
  - When will the PDFs be made available?
- Office: Eckhart 309.
  - Office hours: MWF 3:00-4:00.
- Reader: Walker Lewis. His contact info is in the syllabus.
- Final grade is based on...
  - 2 midterms (15 pts. each; weeks 4 and 8).
  - Final exam (35 pts.).
  - HW (35 pts.).
  - Bonus problems (15 pts).
- Total points for the quarter is 115. The bonus problems usually arise from advanced math and incorporate more advanced knowledge, and we are encouraged to seek out all relevant resources as long as we write up our own solutions.
- **Ordinary differential equation**: An equation that involves an unknown function of a single variable; an equation that takes the form $F(t, y, y', \dots, y^{(n)}) = 0$. *Also known as* **ODE**.
  - F is a known function.
  - t is an argument (time). x is also used (when space is involved).
  - $y = y(t)$ is an unknown function.
- **Order n** (ODE): An ODE for which the n^{th} derivative of y is the highest-order derivative involved (and is involved).
- ODEs are of the form $y' = f(t, y)$ or, more generally, $y^{(n)} = F(t, y, y', \dots, y^{(n-1)})$.
  - We can transform this second form into the first form via

$$
Y = \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix} \qquad\qquad f(t, y) = \begin{pmatrix} Y^2 \\ Y^3 \\ \vdots \\ F(t, Y^1, Y^2, \dots, Y^{n-1}) \end{pmatrix}
$$

This makes $Y' = f(t, Y)$ equal to the system of equations

$$\begin{aligned}(Y^1)' &= Y^2\\ (Y^2)' &= Y^3\\ &\vdots\\ (Y^{n-1})' &= F(t, Y^1, Y^2, \dots, Y^{n-1})\end{aligned}$$

    - Think about this conversion more.

  - Thus, we mainly focus on equations of the form $y' = f(t, y)$ (where y may be a scalar or vector function), because that's general enough.

- **Linear** (ODE): Any ODE that can be written in the form

$$y' = A(t)y + f(t)$$

- Because of the above, this naturally includes equations of the form

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_0(t)y = b(t)$$

  - Indeed, if we define $Y = (y, y', \dots, y^{(n-1)})$, then we may express this equation in the form

$$\begin{aligned}\underbrace{\begin{pmatrix}Y^1\\ Y^2\\ \vdots\\ Y^n\end{pmatrix}'}_{Y'} &= \underbrace{\begin{pmatrix}Y^2\\ Y^3\\ \vdots\\ b(t) - a_0(t)Y^1 - \cdots - a_{n-1}(t)Y^{n-1}\end{pmatrix}}_{g(t,y)}\\ &= \begin{pmatrix}Y^2\\ Y^3\\ \vdots\\ -a_0(t)Y^1 - \cdots - a_{n-1}(t)Y^{n-1}\end{pmatrix} + \underbrace{\begin{pmatrix}0\\ 0\\ \vdots\\ b(t)\end{pmatrix}}_{f(t)}\\ &= \underbrace{\begin{pmatrix}0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ -a_0(t) & -a_1(t) & -a_2(t) & \cdots & -a_{n-1}(t)\end{pmatrix}}_{A(t)}\underbrace{\begin{pmatrix}Y^1\\ Y^2\\ \vdots\\ Y^n\end{pmatrix}}_{Y} + \underbrace{\begin{pmatrix}0\\ 0\\ \vdots\\ b(t)\end{pmatrix}}_{f(t)}\end{aligned}$$

  - This conversion and its implications is covered in more depth in Lecture 4.1.

- **Nonlinear** (ODE): An ODE that is not linear.

- **Autonomous** (ODE): An ODE that can be written in the form

$$y' = f(y)$$

  - Remember that y can be a scalar or a vector function.
  - Solutions to autonomous ODEs can start at *any* time t and still be valid.
    - For example, take the scalar ODE $y' = y$. It's general solution is $y(t) = a\mathrm{e}^{t-t_0}$ for some $a \in \mathbb{R}$ and t_0 being the start time. Importantly, notice that we can make t_0 take any value we want and $y(t)$ will still solve $y' = y$.

- **Nonautonomous** (ODE): An ODE that is not autonomous.
  - We will not investigate these in this course.
- **Initial value problem**: A problem of the form, "find $y(t)$ such that the following holds."

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$$

  Also known as **IVP**, **Cauchy problem**.
- Locally well-posed (LWP) conditions:
  1. Existence (local in time).
  2. Uniqueness (you cannot have multiple solutions).
  3. Local stability (if you perturb your initial value or equation a little bit, you do not expect your solution to vary crazily [esp. locally]).
- Example of a nonunique ODE:
  - $y' = \sqrt{y}$, $y(0) = 0$ has solutions $y_1(t) = 0$ ($t \geq 0$) and $y_2(t) = t^2/4$ ($t \geq 0$).
  - We will investigate the reason later.
- Preview of the reason: **Cauchy-Lipschitz Theorem** or **Picard-Lindelöf Theorem**.
  - As long as the ODE is **Lipschitz continuous**, it's locally stable.
- **Lipschitz continuous** (function): A function f such that

$$|f(t,y_1) - f(t,y_2)| \leq L|y_1 - y_2|$$

  - But in the counterexample above, the slope of the chord from 0 to $y(t)$ approaches infinity as $t \to 0$.
- **Peano Existence Theorem**: Under certain conditions, there exists a solution to a given IVP.
- **Dynamical system**: A law under which a particle evolves over time. $y' = f(t,y)$, IVP is LWP.
- If the IVP $y' = f(t,y)$, $y(t_0) = y_0$ is locally well-posed, then the map $\Phi(t,x)$ which solves

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,x) = f(t,\Phi(t,x)) \\ \Phi(0,x) = x \end{cases}$$

  is well-defined and satisfies the property

$$\Phi(t_2,\Phi(t_1,x)) = \Phi(t_1+t_2,x)$$

  - Φ is very related to y, though how exactly is still a bit of a mystery?? Perhaps it's

$$\Phi(t,x) = y(t)$$

    where y is the solution to the IVP $y' = f(t,y)$, $y(0) = x$.
  - It appears that $\Phi(t,x)$ is related to $f_t(x)$ from Guillemin and Haine (2018), i.e., we are picking a point x and traveling along its integral curve for time t.
  - Think about $y(t) = a\mathrm{e}^{t-t_0}$ as an integral curve of the one-dimensional vector field $X(x) = x$.
  - The final property appears to express the notion that if you have a system and evolve it by time t_1 and then time t_2, that's equivalent to evolving it by time $t_1 + t_2$.

- **Steady flow**: A vector field on a manifold contained in $\mathbb{R}^2$ or $\mathbb{R}^3$ that does not vary with time.
- Let X be a vector field.
  - Trajectory of a particle: At $x \in \Omega$, the velocity of the particle should coincide with $X(x)$.
  - The differential equation $\dot{x} = X(x)$ is what we're interested in.
  - A solid shape gets shifted and deformed (imagine a chunk of water falling out of the end of a pipe). This is the **local group of transformation**.
  - Differential geometry is the purview of such things.
- Newton's law of motion $F = m \cdot a$ applied to n particles is nothing but the system of equations

$$m_i x_i'' = F_i(x_1, \dots, x_n)$$

  for $i = 1, \dots, n$.
  - Many well-known examples.
  - The best known one perhaps is that of uniform acceleration of a single particle. In this case,

$$m_0 x'' = f_0$$

    - The solution is

$$x(t) = \frac{f_0}{2m_0} t^2 + v_0 t + x_0$$

      where $x_0 = x(0)$ and $v_0 = x'(0)$ are the initial conditions.
  - A simple example is downwards motion due to gravity. Then

$$x(t) = \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} t^2 + v_0 t + x_0$$

    - The trajectory in general is a parabola.
  - Another example: The mathematical pendulum.
    - The radial directions balance ($mg\cos\theta$).
    - The tangential directions do not ($mg\sin\theta$). Thus, our ODE is

$$l \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = g\sin\theta$$

  - One last set of examples from ecology:
    - Imagine an petri dish of infinite nutrition. The population growth of the bacteria will obey the exponential growth law

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky$$

    - Suppose we have a system capacity M. Then we obey the logistic growth law

$$\frac{\mathrm{d}y}{\mathrm{d}t} = k(M - y)$$

    - Lotka-Volterra prey-predator model: Wolf population (W) and rabbit population (R). We have

$$\begin{aligned} R' &= k_1 R - aWR \\ W' &= -k_2 W + bWR \end{aligned}$$

    - We can also introduce more species and capacities and et cetera, et cetera.
- Conclusion: Dynamical systems are everywhere, especially in physics, chemistry, and ecology.
- We can also consider long-term behavior.
  - We can have chaos, but chaos can be reasoned with using oscillation, systems that converge to oscillation, etc. We will mostly be focusing on the regular aspect of the long-term behavior.

1.2 Origin of ODEs: Boundary Value Problems

9/30:

- Textbook PDFs will be posted today.
- Note: Equations of order n generally require n parameters to solve.
- Today, we will consider boundary value problems, which are separate from dynamical systems but not entirely unrelated.
- **Boundary Value Problem**: A problem in which we are solving for a y that has fixed values at the boundaries $x = a, b$. *Also known as* **BVP**.
- The **Brachistochrone problem** is an example of a BVP.
- **Brachistochrone problem**: Suppose you have a frictionless track from $(0,0)$ to (a, y_0) and release a particle from $(0,0)$. Which path allows the particle to get to (a, y_0) in the shortest amount of time? *Etymology* brákhistos "shortest" + khrónos "time."

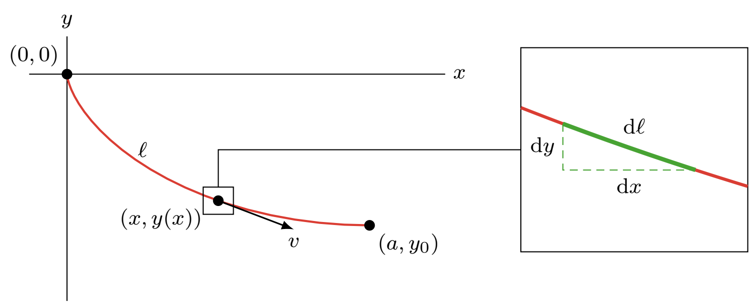

Figure 1.1: Brachistochrone problem.

  - Throughout this derivation, we will make several assumptions. We will do our best to note these assumptions as we go in footnotes. Note that while all of these assumptions are justified in the case of solving this problem, they may not be justified in every related variational problem. Let's begin.
  - Since the track is frictionless, the mechanical energy should be conserved.
  - At a given point along the curve, the particle has a velocity v and is vertical distance y from where it started. We know from physics that

$$\frac{1}{2}mv^2 = mgy$$
$$v = \sqrt{2gy}$$

  - Since $v = \mathrm{d}\ell/\mathrm{d}t$, the time $\mathrm{d}t$ it takes for the particle to traverse an infinitesimal section of track of arc length $\mathrm{d}\ell$ is $\mathrm{d}t = \mathrm{d}\ell/v$.
  - The track should be given by $y = y(x)$[1].
  - Let ℓ denote the arc length of the whole track. Then

$$\mathrm{d}\ell = \sqrt{1 + (y'(x))^2}\,\mathrm{d}x$$

  - Thus, the total time for the particle to traverse the curve is

$$t(y) = \int_0^t \mathrm{d}\tau = \int_0^a \frac{\mathrm{d}\ell}{v} = \int_0^a \frac{\sqrt{1+(y'(x))^2}\,\mathrm{d}x}{\sqrt{2gy(x)}}$$

[1] There are paths that connect $(0,0)$ and (a, y_0) that are not functions of x. We are taking those out of consideration.

- We also have $y(0) = 0$ and $y(a) = y_0$.
- We want to find y such that the above integral is minimized. Thus, we define the following **functional**, which is used to solve general fixed-endpoint variational problems (the Brachistochrone problem is a problem of this type).
- Let $J[y] = \int_a^b F(x, y(x), y'(x))\,\mathrm{d}x$.
- The space of functions we're considering is C^1 (the set of all continuously differentiable functions)[2].
- Take a function h, vanishing at a, b.
- Let $f(t) = J[y + th]$. Then

$$f(t) = \int_a^b F(x, \underbrace{y(x) + th(x)}_{z(x,t)}, \underbrace{y'(x) + th'(x)}_{w(x,t)})\,\mathrm{d}x$$

- We know that[3]

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b F\,\mathrm{d}x &= \int_a^b \frac{\mathrm{d}F}{\mathrm{d}t}\,\mathrm{d}x \\ &= \int_a^b \left(\frac{\partial F}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial F}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial F}{\partial w}\frac{\mathrm{d}w}{\mathrm{d}t}\right)\mathrm{d}x \\ &= \int_a^b \left(\frac{\partial F}{\partial x}\cdot 0 + \frac{\partial F}{\partial z}\cdot h(x) + \frac{\partial F}{\partial w}\cdot h'(x)\right)\mathrm{d}x \\ &= \int_a^b \left(\frac{\partial F}{\partial z}\cdot h(x) + \frac{\partial F}{\partial w}\cdot h'(x)\right)\mathrm{d}x\end{aligned}$$

  - The last term in the above equation may be integrated by parts as follows. Note that we make use of the hypothesis $h(a) = h(b) = 0$ in eliminating the $[uv]_a^b$ term.

$$\begin{aligned}\int_a^b \frac{\partial F}{\partial w}h'(x)\,\mathrm{d}x &= \left[\frac{\partial F}{\partial w}h(x)\right]_{x=a}^b - \int_a^b h(x)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial w}\right)\mathrm{d}x \\ &= \left[\left.\frac{\partial F}{\partial w}\right|_b\cdot 0 - \left.\frac{\partial F}{\partial w}\right|_a\cdot 0\right] - \int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial w}\right)h(x)\,\mathrm{d}x \\ &= -\int_a^b \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial w}\right)h(x)\,\mathrm{d}x\end{aligned}$$

  - Substituting back into the original equation, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b F\,\mathrm{d}x &= \int_a^b \left[\frac{\partial F}{\partial z}\cdot h(x) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial w}\right)h(x)\right]\mathrm{d}x \\ &= \int_a^b \left[\frac{\partial F}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial w}\right)\right]h(x)\,\mathrm{d}x\end{aligned}$$

  - Therefore,

$$f'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_a^b F\,\mathrm{d}x = \int_a^b \left\{\frac{\partial F}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial F}{\partial w}\right]\right\}h(x)\,\mathrm{d}x$$

- Thus,

$$f'(0) = \int_a^b \left\{\frac{\partial F}{\partial z}(x, y(x), y'(x)) - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial F}{\partial w}(x, y(x), y'(x))\right]\right\}h(x)\,\mathrm{d}x = 0$$

  for all h.

[2]This also eliminates some possible paths from consideration.

[3]We must assume sufficient regularity of F here. In particular, we must assume that the derivative of the integral of F is equal to the integral of the derivative of F.

  - Now suppose y is the solution. Then y minimizes $J[y]$. But if this is true, then any variation th will cause $J[y+th] > J[y]$. It follows that for every h, $f(t)$ has a minium at $t=0$. But if f has a minimum at 0 for all h, then $f'(0)=0$ for all h.
- Lemma: Let ϕ be continuous on (a,b). If for every $h \in C^1([a,b])$ vanishing on a,b we have that

$$\int_a^b \phi(x)h(x)\,\mathrm{d}x = 0$$

then $\phi(x) = 0$.

Proof. Suppose for the sake of contradiction that (WLOG) $\phi(x_0) > 0$. Then within some neighborhood $N_\delta(x)$ of x_0, $\phi(x) > 0$ for all $x \in N_\delta(x)$. Now choose h to be a bump function on that interval. Then $\int_a^b \phi(x)h(x)\,\mathrm{d}x > 0$, a contradiction. $\square$

- It follows that

$$\frac{\partial F}{\partial z}(x,y(x),y'(x)) - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial F}{\partial w}(x,y(x),y'(x))\right] = 0$$

  - This is a second-order differential equation, specifically the **Euler-Lagrange equation**.
  - It is a necessary condition for y to be an extrema.
  - Euler-Lagrange equations are not easy to solve in general. However, we're lucky here.
- In our example,

$$F(x,z,w) = \sqrt{\frac{1+w^2}{2gz}}$$

  - What's nice here is that $F(x,z,w) = F(z,w)$, i.e., there is no dependence on x. This is crucial.
- With this observation in mind, notice that

$$\begin{aligned}
\frac{\mathrm{d}F}{\mathrm{d}x} &= \frac{\partial F}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}x} + \frac{\partial F}{\partial w}\frac{\mathrm{d}w}{\mathrm{d}x} \\
&= \frac{\partial F}{\partial z}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial F}{\partial w}\frac{\mathrm{d}y'}{\mathrm{d}x} \\
&= \frac{\partial F}{\partial z}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial F}{\partial w}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}
\end{aligned}$$

- We now rearrange the E-L equation and multiply through by $\mathrm{d}y/\mathrm{d}x$.

$$\begin{aligned}
\frac{\partial F}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial w}\right) &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial w}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\partial F}{\partial z}\frac{\mathrm{d}y}{\mathrm{d}x}
\end{aligned}$$

- Subtracting the last two results yields

$$\begin{aligned}
\frac{\mathrm{d}F}{\mathrm{d}x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}F}{\mathrm{d}w}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\partial F}{\partial w}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \\
\frac{\mathrm{d}F}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}F}{\mathrm{d}w}\right)\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\partial F}{\partial w}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}F}{\mathrm{d}w}\frac{\mathrm{d}y}{\mathrm{d}x}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(F - \frac{\mathrm{d}F}{\mathrm{d}w}\frac{\mathrm{d}y}{\mathrm{d}x}\right) &= 0 \\
F - \frac{\mathrm{d}F}{\mathrm{d}w}\frac{\mathrm{d}y}{\mathrm{d}x} &= A
\end{aligned}$$

where $A \in \mathbb{R}$ depends on the initial conditions.

- From the definition of F, we can calculate

$$\frac{\partial F}{\partial w} = \frac{w}{\sqrt{1+w^2}} \cdot \frac{1}{\sqrt{2gz}} = \frac{y'}{\sqrt{1+(y')^2}} \cdot \frac{1}{\sqrt{2gy}}$$

- It follows that our solution function y satisfies the separable differential equation

$$\begin{aligned}
\sqrt{\frac{1+(y')^2}{2gy}} - \frac{y'}{\sqrt{1+(y')^2}} \cdot \frac{1}{\sqrt{2gy}} \cdot y' &= A \\
\frac{1+(y')^2}{\sqrt{1+(y')^2}\sqrt{2gy}} - \frac{(y')^2}{\sqrt{1+(y')^2}\sqrt{2gy}} &= A \\
\frac{1}{\sqrt{2gy(1+(y')^2)}} &= A \\
(y')^2 &= \frac{1/2A^2g - y}{y}
\end{aligned}$$

- The solution, as we can determine using methods from Calculus I-II, is the **cycloid**

$$\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1-\cos\theta) \end{cases}$$

where the specific parameters come from the boundary values.

- **Functional**: A map from a function space to a set of numbers.
- **Sturm-Liouville problems**: Boundary value problems concerning the integral

$$\int_a^b \left[p(x)(y'(x))^2 + q(x)(y(x))^2\right] \mathrm{d}x$$

  - The most basic BVP is a vibrating string. In finding the eigenmode of the vibration, you need to solve the above differential equation.
  - Very important in physics.
  - If time permits at the end of the course, Shao will return to the following topic in detail.
- Next several weeks: *Solvable* differential equations.