Week 6

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6.1 More Cauchy-Lipschitz and Intro to Continuous Dependence

10/31:

- Last time, we built up a proof to the Cauchy-Lipschitz theorem intuitively.
  - We begin today with a direct proof that is very similar, but slightly different.
- Theorem (Cauchy-Lipschitz theorem): Let $f(t,z)$ be defined on an open subset $\Omega \subset \mathbb{R} \times \mathbb{R}^n$, let $(t_0, y_0) \in \Omega$, let $|f|$ be bounded on Ω, and let f be Lipschitz continuous in z and continuous wrt. t in some neighborhood of (t_0, y_0). Then the IVP $y'(t) = f(t, y(t))$, $y(t_0) = y_0$ has a unique solution on $[t_0, t_0 + T]$ for some $T > 0$ such that $y(t)$ does not escape Ω.

  Proof. Let $f(t,z)$ be defined for $(t,z) \in [t_0, t_0 + a] \times \bar{B}(y_0, b) \subset \Omega$. Let $|f(t,z)| \leq M$. Let $|f(t,z_1) - f(t,z_2)| \leq L|z_1 - z_2|$ for all $z_1, z_2 \in \bar{B}(y_0, b)$.

  Define $\{y_n\}$ recursively, starting from $y_0(t) = y_0$, by

  $$y_{k+1}(t) = y_0 + \int_{t_0}^{t} f(\tau, y_k(\tau))\mathrm{d}\tau$$

  Since f is continuous with respect to t, it is integrable with respect to t, so the above sequence is well-defined on $[t_0, t_0 + T]$. Choose $T = \min(a, b/M, 1/2L)$. Then

  $$\|y_k - y_0\| \leq T \cdot M \leq \frac{b}{M} \cdot M = b$$

  so no y_k escapes $\bar{B}(y_0, b)$. Additionally,

  $$\begin{aligned}\|y_{k+1} - y_k\| &\leq \int_{t_0}^{t} \|f(\tau, y_k(\tau)) - f(\tau, y_{k-1}(\tau))\|\mathrm{d}\tau \\ &\leq TL\|y_k - y_{k-1}\| \\ &\leq \frac{1}{2}\|y_k - y_{k-1}\| \\ &\leq \left(\frac{1}{2}\right)^k \|y_1 - y_0\|\end{aligned}$$

  Thus, the difference between successive terms in the sequence is controlled by a geometric progression, so $\{y_n\}$ is a Cauchy sequence in the function space. It follows that $\{y_k\}$ is uniformly convergent to some continuous $y : [t_0, t_0 + T] \to \mathbb{R}^n$. ☐

- This completes the proof. Although it's more concrete than the contraction mapping one, they are virtually the same: In both cases, we obtain an approximate sequence controlled by a geometric progression.

- Examples of the Picard iteration:
  1. Consider an linear autonomous systems $y' = Ay$, A an $n \times n$ matrix, and $y(0) = y_0$.
     - We know that the solution is $y(t) = \mathrm{e}^{tA} y_0$. However, we can derive this using the Picard iteration.
     - Indeed, via this procedure, let's determine the first couple of Picard iterates.

$$y_0(t) = y_0 \qquad \begin{aligned} y_1(t) &= y_0 + \int_0^t Ay_0(\tau)\mathrm{d}\tau \\ &= y_0 + tAy_0 \end{aligned} \qquad \begin{aligned} y_2(t) &= y_0 + \int_0^t Ay_1(\tau)\mathrm{d}\tau \\ &= y_0 + tAy_0 + \frac{1}{2}t^2A^2y_0 \end{aligned}$$

     - It follows inductively that

$$y_k(t) = \sum_{j=0}^{k} \frac{t^j A^j}{j!} y_0$$

     - Since the term above is exactly the power series definition of e^{tA}, we have that $y_k(t) \to \mathrm{e}^{tA} y_0$ with local uniformity in t, as desired.
  2. Consider the ODE $y' = y^2$, $y(0) = 1$.
     - We know that the solution is $y(t) = 1/(1-t)$. We will now also derive this via the Picard iteration.
     - Choose $b = 1$, so that

$$\bar{B}(y_0, b) = \{y \mid |y - y(0)| \leq 1\} = \{y \mid |y - 1| \leq 1\} = [0, 2]$$

     - On this interval, $f(t, y) = y^2$ has maximum slope $L = 4$. Thus, we should take $T \leq 1/2L = 1/8$.
     - It follows that $|y_1^2 - y_2^2| \leq 4|y_1 - y_2|$ for all $y_1, y_2 \in \bar{B}(y_0, b)$.
     - Calculate the first few Picard iterates.

$$y_1(t) = 1 + \int_0^t (y_0(\tau))^2 \mathrm{d}\tau = 1 + t$$

$$y_2(t) = 1 + \int_0^t (1 + \tau)^2 \mathrm{d}\tau = 1 + t + t^2 + \frac{t^3}{3}$$

$$y_3(t) = 1 + \int_0^t \left(1 + \tau + \tau^2 + \frac{\tau^3}{3}\right)^2 \mathrm{d}\tau = 1 + t + t^2 + t^3 + \frac{2t^4}{3} + \frac{t^5}{3} + \frac{t^6}{9} + \frac{t^7}{63}$$

     - It follows by induction that

$$\begin{aligned} |y_k(t) - (1 + t + \cdots + t^k)| &\leq t^{k+1} \\ \left| y_k(t) - \frac{1 - t^{k+1}}{1 - t} \right| &\leq t^{k+1} \end{aligned}$$

       It follows that $|t| < 1/8$.
     - For $|t| < 1/8$, $y(t) = 1/(1-t)$. Blows up as $t \to 1$.
     - Some more details on the bounding of the error term are presented in the lecture notes document.

- Lemma (Grönwall's inequality): Let $\varphi(t)$ be a real function defined for $t \in [t_0, t_0 + T]$ such that

$$\varphi(t) \leq f(t) + a \int_{t_0}^t \varphi(\tau)\mathrm{d}\tau$$

  Then

$$\varphi(t) \leq f(t) + a \int_{t_0}^t \mathrm{e}^{a(t-\tau)} f(\tau)\mathrm{d}\tau$$

Proof. Multiply both sides by e^{-at}:

$$\begin{aligned}\mathrm{e}^{-at}\varphi(t) - a\mathrm{e}^{-at}\int_{t_0}^{t}\varphi(\tau)\mathrm{d}\tau &\leq \mathrm{e}^{-at}f(t)\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-at}\int_{t_0}^{t}\varphi(\tau)\mathrm{d}\tau\right) &\leq \mathrm{e}^{-at}f(t)\\ \mathrm{e}^{-at}\int_{t_0}^{t}\varphi(\tau)\mathrm{d}\tau &\leq \int_{t_0}^{t}\mathrm{e}^{-a\tau}f(\tau)\mathrm{d}\tau\\ \int_{t_0}^{t}\varphi(\tau)\mathrm{d}\tau &\leq \int_{t_0}^{t}\mathrm{e}^{a(t-\tau)}f(\tau)\mathrm{d}\tau\end{aligned}$$

Substituting back into the original equality yields the result at this point. □

- Note that there is no sign condition on $f(t)$ or a.
- Grönwall's inequality is very important and we should remember it.
- It is also exactly what we need to prove continuous dependence.
- Theorem: Let $f(t,z), g(t,z)$ be defined on $\Omega \subset \mathbb{R}^1_t \times \mathbb{R}^n_z$, an open and bounded a region containing (t_0, y_0) and (t_0, w_0). Let the functions be L - Lipschitz wrt. z. Consider two initial value problems $y' = f(t,y)$, $y(t_0) = y_0$ and $w' = g(t,w)$, $w(t_0) = w_0$. If $|f(t,z) - g(t,z)| < M$, then for $t \in [t_0, t_0 + T]$,

$$|g(t) - w(t)| \leq \mathrm{e}^{LT}|y_0 - w_0| + \frac{M}{L}(\mathrm{e}^{LT} - 1)$$

Proof. We have that

$$\begin{aligned}|y(t) - w(t)| &= \left|\left[y_0 + \int_{t_0}^{t}f(\tau, y(\tau))\mathrm{d}\tau\right] - \left[w_0 + \int_{t_0}^{t}g(\tau, y(\tau))\mathrm{d}\tau\right]\right|\\ &= \left|[y_0 - w_0] + \int_{t_0}^{t}[f(\tau, y(\tau)) - g(\tau, y(\tau))]\mathrm{d}\tau\right|\\ &\leq |y_0 - w_0| + \left|\int_{t_0}^{t}[f(\tau, y(\tau)) - g(\tau, w(\tau))]\mathrm{d}\tau\right|\\ &\leq |y_0 - w_0| + \int_{t_0}^{t}|f(\tau, y(\tau)) - g(\tau, w(\tau))|\mathrm{d}\tau\end{aligned}$$

where we get from the second to the third line using the triangle inequality, and the third to the fourth line using Theorem 13.26 of Honors Calculus IBL. We also know that

$$\begin{aligned}|f(\tau, y(\tau)) - g(\tau, w(\tau))| &\leq |f(\tau, y(\tau)) - f(\tau, w(\tau))| + |f(\tau, w(\tau)) - g(\tau, w(\tau))|\\ &\leq L|y(\tau) - w(\tau)| + M\end{aligned}$$

Combining what we've obtained, we have

$$\begin{aligned}\underbrace{|y(t) - w(t)|}_{\psi(t)} &\leq \underbrace{|y_0 - w_0| + M(t - t_0)}_{f(t)} + \underbrace{L}_{a}\int_{t_0}^{t}\underbrace{|y(\tau) - w(\tau)|}_{\psi(t)}\mathrm{d}\tau\\ &\leq MT + |y_0 - w_0| + L\int_{t_0}^{t}\mathrm{e}^{L(t-\tau)}[|y_0 - w_0| + M(t - \tau)]\mathrm{d}\tau && \text{Grönwall}\\ &\leq \mathrm{e}^{LT}|y_0 - w_0| + \frac{M}{L}(\mathrm{e}^{TL} - 1)\end{aligned}$$

as desired. □

- Note: Getting from directly from Grönwall's inequality in the second line above to the last line above is quite messy. A consequence of Grönwall's inequality explored in the book makes this much easier. *Prove Equation 2.38 via Problem 2.12.*
- Implication: The IVP is not just solvable itself, but is solvable wrt. perturbation of the initial conditions and RHS within a small, finite interval in time.
- Suppose $y' = 0$, $y(0) = 1$ and $w' = \varepsilon w$, $w(0) = 1$. Then $y(t) = 1$ and $w(t) = \mathrm{e}^{\varepsilon t}$ and solutions are only close when t is small.
    - $t \leq 1/\varepsilon$??
- This is important in physics. In most physical scenarios, the RHS is C^1. This is called determinism.