5 Fixed Points and Perturbation

Problems Related to Fundamental Definitions

11/10: **1.** Are the following real functions Lipschitz continuous near 0? If yes, find a Lipschitz constant for some interval containing 0.

(1) $1/(1-x^2)$.

Proof. $\boxed{\text{Yes.}}$ Consider the interval $[-0.5, 0.5]$. Then we may take

$$\boxed{L = \frac{16}{9}}$$

□

(2) $x \log |x|$.

Proof. $\boxed{\text{No.}}$ □

(3) $x^2 \sin(1/x)$.

Proof. If we take the piecewise function consisting of the above expression on $\mathbb{R} \setminus \{0\}$ and 0 at 0, then $\boxed{\text{yes.}}$ Consider the interval $[-1, 1]$. Then we may take

$$\boxed{L = 2}$$

□

2. Find the first two elements $y_1(t), y_2(t)$ for the Picard iteration sequence of the following initial value problems, and estimate the error between $y_2(t)$ and the actual solution. Since they are all of separable form, the actual solutions can be explicitly found.

(1) $y' = 1 + y^2$, $y(0) = 0$.

Proof. We take $y_0(t) = 0$. Then

$$\begin{aligned} y_1(t) &= y_0(0) + \int_0^t [1 + y_0(t)^2] \, \mathrm{d}t \\ &= \int_0^t [1 + 0] \, \mathrm{d}t \\ \boxed{y_1(t) = t} & \end{aligned}$$

and

$$\begin{aligned} y_2(t) &= y_0(0) + \int_0^t [1 + y_1(t)^2] \, \mathrm{d}t \\ &= \int_0^t [1 + t^2] \, \mathrm{d}t \\ \boxed{y_2(t) = t + \frac{t^3}{3}} & \end{aligned}$$

The error is between y_2 and the actual solution $y(t) = \tan(t)$ is given by

$$\boxed{\varepsilon = \tan(t) - t - \frac{t^3}{3}}$$

□

(2) $y' = 2ty$, $y(0) = 1$.

Proof. We take $y_0(t) = 1$. Then

$$\begin{aligned} y_1(t) &= y_0(0) + \int_0^t 2ty_0(t)\,\mathrm{d}t \\ &= 1 + \int_0^t 2t\,\mathrm{d}t \\ \Aboxed{y_1(t) &= 1 + t^2} \end{aligned}$$

and

$$\begin{aligned} y_2(t) &= y_0(0) + \int_0^t 2ty_1(t)\,\mathrm{d}t \\ &= 1 + \int_0^t [2t + 2t^3]\,\mathrm{d}t \\ \Aboxed{y_2(t) &= 1 + t^2 + \frac{t^4}{2}} \end{aligned}$$

The error is between y_2 and the actual solution $y(t) = \mathrm{e}^{t^2}$ is given by

$$\boxed{\varepsilon = \mathrm{e}^{t^2} - 1 - t^2 - \frac{t^4}{2}}$$

□

(3) $y' = y/(1-t)$, $y(0) = 1$.

Proof. We take $y_0(t) = 1$. Then

$$\begin{aligned} y_1(t) &= y_0(0) + \int_0^t \frac{y_0(t)}{1-t}\,\mathrm{d}t \\ &= 1 + \int_0^t \frac{1}{1-t}\,\mathrm{d}t \\ \Aboxed{y_1(t) &= 1 - \ln|1-t|} \end{aligned}$$

and

$$\begin{aligned} y_2(t) &= y_0(0) + \int_0^t \frac{y_1(t)}{1-t}\,\mathrm{d}t \\ &= 1 + \int_0^t \frac{1 - \ln|1-t|}{1-t}\,\mathrm{d}t \\ \Aboxed{y_2(t) &= 1 - \ln|1-t| + \frac{1}{2}(\ln|1-t|)^2} \end{aligned}$$

The error between y_2 and the actual solution $y(t) = \mathrm{e}^{-\ln|1-t|}$ is given by

$$\boxed{\varepsilon = \mathrm{e}^{-\ln|1-t|} - 1 + \ln|1-t| - \frac{1}{2}(\ln|1-t|)^2}$$

□

3. Check whether the implicit equation $F(x,y)=0$ uniquely determines an explicit function $y=f(x)$ around the given point (x_0,y_0). If it does, compute $f'(x_0)$.

(1) For $(x,y)\in\mathbb{R}^2$, $F(x,y)=x^2+y^2-1$, $(x_0,y_0)=(\sqrt{2}/2,-\sqrt{2}/2)$.

Proof. From the implicit equation, we have that

$$\begin{aligned} 0 &= x^2+y^2-1 \\ y &= \pm\sqrt{1-x^2} \end{aligned}$$

Since

$$\begin{aligned} -\frac{\sqrt{2}}{2} &= -\sqrt{1-\left(\frac{\sqrt{2}}{2}\right)^2} \\ y_0 &= -\sqrt{1-x_0^2} \end{aligned}$$

our explicit function $\boxed{\text{is uniquely determined around } (x_0,y_0).}$
Moreover, we can compute that

$$\begin{aligned} f'(x_0) &= \frac{2x_0}{2\sqrt{1-x_0^2}} \\ &\boxed{f'(x_0)=1} \end{aligned}$$

□

(2) For $(x,y)\in\mathbb{R}^2$, $F(x,y)=x^2-y^2-1$, $(x_0,y_0)=(1,0)$.

Proof. From the implicit equation, we have that

$$\begin{aligned} 0 &= x^2-y^2-1 \\ y &= \pm\sqrt{x^2-1} \end{aligned}$$

Since

$$y_0=\sqrt{x_0^2-1} \qquad\qquad y_0=-\sqrt{x_0^2-1}$$

our explicit function $\boxed{\text{is not uniquely determined around } (x_0,y_0).}$ □

(3) For $(x,y)\in\mathbb{R}^2$, $F(x,y)=x\mathrm{e}^y+y$, $(x_0,y_0)=(0,0)$.

Proof. We apply the implicit function theorem.
F is defined on a subset of $\mathbb{R}^2$, as desired.
We have that

$$\frac{\partial F}{\partial x}=\mathrm{e}^y \qquad\qquad \frac{\partial F}{\partial y}=x\mathrm{e}^y+1$$

Since both of the above partial derivatives are continuous, F is continuously differentiable on its domain, as desired.
$(x_0,y_0)=(0,0)\in\mathbb{R}^2$, which is the domain of F, as desired.
$F(x_0,y_0)=0\mathrm{e}^0+0=0$, as desired.
The truncated Jacobian matrix is 1×1 and contains a nonzero element at (x_0,y_0) — in particular, it contains $\partial F/\partial x$ — as desired.

Therefore, our explicit function $\boxed{\text{is uniquely determined around } (x_0, y_0).}$

Moreover, we can compute that

$$\begin{aligned}
f'(x_0) &= -\left(\frac{\partial F}{\partial y}\right)^{-1} \cdot \frac{\partial F}{\partial x} \\
&= -\left(0\mathrm{e}^0 + 1\right)^{-1} \cdot \mathrm{e}^0 \\
\Aboxed{f'(x_0) &= -1}
\end{aligned}$$

□

Problems Involving the Banach Fixed Point Theorem

1. (1) Show that the condition "constant $q < 1$" in the statement of the Banach fixed point theorem is not redundant. You may give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which satisfies the strict inequality $|f(x) - f(y)| < |x - y|$ but does not have a fixed point.

Proof. Choose

$$\boxed{f(x) = \begin{cases} 1 & x \leq 0 \\ x + \mathrm{e}^{-x} & x > 0 \end{cases}}$$

The fact that

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \begin{cases} 0 & x \leq 0 \\ 1 - \mathrm{e}^{-x} & x > 0 \end{cases}$$

implies that $|\ \mathrm{d}f/\mathrm{d}x\ | < 1$ for all x. Hence, f satisfies the desired strict inequality. Additionally, since the graph of $f(x) > x$ for all x (as can be readily verified from its definition), it has no fixed point, as desired. □

(2) Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz mapping with uniform Lipschitz constant $q < 1$, that is,

$$|f(x) - f(y)| \leq q|x - y|$$

for all $x, y \in \mathbb{R}^n$. Prove that the mapping $x \mapsto x + f(x)$ is invertible with Lipschitz continuous inverse.

Proof. Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $g(x) = x + f(x)$. To prove that g is invertible, it will suffice to show that g is one-to-one, that is, for every $b \in \mathbb{R}^n$, there exists a unique $a \in \mathbb{R}^n$ such that $g(a) = b$. Let $b \in \mathbb{R}^n$ be arbitrary. Define $h : \mathbb{R}^n \to \mathbb{R}^n$ by $h(x) = b - f(x)$. Then since

$$\begin{aligned}
|h(x) - h(y)| &= |[b - f(x)] - [b - f(y)]| \\
&= |f(y) - f(x)| \\
&= |f(x) - f(y)| \\
&\leq q|x - y|
\end{aligned}$$

we have by the Banach fixed point theorem that there exists a unique $a \in \mathbb{R}^n$ such that $a = h(a)$. It follows that

$$\begin{aligned}
a &= b - f(a) \\
a + f(a) &= b \\
g(a) &= b
\end{aligned}$$

as desired.

To prove that g^{-1} is Lipschitz continuous, it will suffice to show that

$$|g^{-1}(x) - g^{-1}(y)| \leq \frac{1}{1-q}|x-y|$$

for all $x, y \in \mathbb{R}^n$. Let $x, y \in \mathbb{R}^n$ be arbitrary. Define $a = g^{-1}(x)$ and $b = g^{-1}(y)$. Then since the first term below is nonnegative (as the product of two nonnegative numbers), we have that

$$\begin{aligned}
(1-q)|a-b| &= |a-b| - q|a-b| \\
&\leq |a-b| - |f(a) - f(b)| \\
&= |a-b| - |f(b) - f(a)| \\
&= \big||a-b| - |f(b) - f(a)|\big| \\
&\leq |[a-b] - [f(b) - f(a)]| \\
&= |[a + f(a)] - [b + f(b)]| \\
&= |g(a) - g(b)|
\end{aligned}$$

It follows by returning the substitution that

$$\begin{aligned}
(1-q)|g^{-1}(x) - g^{-1}(y)| &\leq |x-y| \\
|g^{-1}(x) - g^{-1}(y)| &\leq \frac{1}{1-q}|x-y|
\end{aligned}$$

as desired. □

2. Consider the following iterative algorithm to compute the square root of a given $a > 1$.

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

(1) Show that the function

$$F(x) = \frac{1}{2}\left(x + \frac{a}{x}\right)$$

meets the requirements of the contraction mapping principle on the closed interval $[\sqrt{a/2}, a]$. Prove that $x_n \to \sqrt{a}$.

Proof. We want to show that

$$|F(x) - F(y)| \leq q|x-y|$$

for some $q \in (0,1)$ and all $x, y \in [\sqrt{a/2}, a]$.
We have that

$$\begin{aligned}
|F(x) - F(y)| &= \left|\frac{1}{2}\left(x + \frac{a}{x}\right) - \frac{1}{2}\left(y + \frac{a}{y}\right)\right| \\
&= \frac{1}{2}\left|(x-y) + \left(\frac{a}{x} - \frac{a}{y}\right)\right| \\
&= \frac{1}{2}\left|(x-y) + a \cdot \frac{y-x}{xy}\right| \\
&= \frac{1}{2}\left|\left(1 - \frac{a}{xy}\right)(x-y)\right| \\
&= \frac{1}{2}\left|1 - \frac{a}{xy}\right||x-y|
\end{aligned}$$

□

(2) For $a = 2$, start the iteration $x_{n+1} = F(x_n)$ with $x_0 = 1$. Use a calculator to compute the first 10 values of this iteration, up to 11 digits after the decimal point. Compare it with the exponentially converging sequence $1.4, 1.41, 1.414, 1.4142, \dots$. Which of the two algorithms is better?

Proof. We have that

$$\boxed{\begin{aligned} x_0 &= 1 \\ x_1 &= 1.5 \\ x_2 &= 1.41666666667 \\ x_3 &= 1.41421568627 \\ x_4 &= 1.41421356237 \\ x_5 &= 1.41421356237 \\ x_6 &= 1.41421356237 \\ x_7 &= 1.41421356237 \\ x_8 &= 1.41421356237 \\ x_9 &= 1.41421356237 \\ x_{10} &= 1.41421356237 \end{aligned}}$$

$\boxed{\text{The algorithm from part (1) is better.}}$ □

(3) Try to estimate the error $|x_n - \sqrt{a}|$ as well as possible. *Hint.* There should be something related to an iterative sequence $\{b_n\}$ satisfying

$$b_{n+1} \leq M b_n^2$$

You should prove that the sequence converges to zero faster than any geometric progression.

Context: This algorithm is referred to as **Newton's method**. It is a rapidly converging algorithm to find zeros/fixed points of functions, capable of giving very precise approximations within very few steps. A variation of it, called the **Nash-Moser technique**, is a very powerful tool for proving the existence of solutions to nonlinear differential equations.