1 IVP Examples and Physical Problems

Required Problems

10/12: **1.** Classify the following ordinary differential equations (systems) by indicating the order, if they are linear, and if they are autonomous.

(1) $y'(x) + y(x) = 0$.

(2) $y''(t) = t\sin(y(t))$.

(3) $x' = -y$, $y' = 2x$.

(4) $y'(t) = y(t)\sin(t) + \cos(y(t))$.

2. Transform the following differential equations to first-order systems.

(1) $y^{(3)} + 2y'' - y' + y = 0$.

(2) $x'' - t\sin x' = x$.

3. Solve the following differential equations with initial value $x(0) = x_0$. Also identify the set of x_0 for which these solutions are extendable to the whole of $t \geq 0$. When a solution cannot be extended to the whole of $t \geq 0$, determine its lifespan in terms of x_0.

Example: Solve $x' = x^2$ with $x(0) = x_0$. By separation of variables, the solution reads

$$\int_{x_0}^{x} \frac{\mathrm{d}w}{w^2} = \int_0^t \mathrm{d}\tau$$

where the integral on the left-hand side cannot pass through $w = 0$. The result is

$$-\frac{1}{x} + \frac{1}{x_0} = t \quad \Longleftrightarrow \quad x(t) = \frac{x_0}{1 - x_0 t}$$

When $x_0 \leq 0$, the solution exists throughout $t \geq 0$. When $x_0 > 0$, the solution only exists in $[0, 1/x_0)$.

(1) $x' = x\sin t$.

(2) $x' = t^2\tan x$.

(3) $x' = 1 + x^2$.

(4) $x' = \mathrm{e}^x \sin t$.

4. Consider the harmonic oscillator equation, as mentioned in class:

$$x'' + \mu x' + \omega^2 x = 0$$

Here, the initial data $x(0) = x_0$ and $x'(0) = x_1$ are real numbers.

(1) Derive two linearly independent *real* solutions when $\mu > 0$. (Hint: You should consider the cases $0 < \mu < \omega$, $\mu = \omega$, and $\mu > \omega$ separately.)

(2) Recall that $\mu = b/m$ and $\omega^2 = k/m$. Recall also that the mechanical energy for the oscillator reads

$$E = \frac{1}{2}m|x'|^2 + \frac{1}{2}kx^2$$

Compute the time derivative of E and conclude that E is exponentially decaying for $b > 0$, i.e., the mechanical energy is not conserved in this case. Does this violate the law of conservation of mechanical energy?

5. Use the transformation $w = ty$ to convert

$$y' = f(y/t)$$

to an ODE in w. Write down this equation for w. Use this transformation to solve

$$tyy' + 4t^2 + y^2 = 0, \quad y(2) = -7$$

Determine the lifespan (you can use a calculator for an approximate value).

6. Use the transformation $w = y^{1-\alpha}$ to convert Bernoulli's equation

$$y' + p(t)y = q(t)y^{\alpha}, \quad \alpha \neq 0, 1$$

to an ODE in w. Write down this equation for w. Use this transformation to solve

$$6y' - 2y = ty^4, \quad y(0) = -2$$

Determine the lifespan (you can use a calculator for an approximate value).

7. Show that

$$(4bxy + 3x + 5)y' + 3x^2 + 8ax + 2by^2 + 3y = 0$$

is an exact equation, no matter what value a, b take. Find the implicit relation satisfied by the solution $y(x)$ and x.

8. Let a, b be constants. For Euler's equation

$$t^2y'' + aty' + bt = f(t)$$

consider the transformation $w(\tau) = y(\mathrm{e}^{\tau})$. What is the differential equation satisfied by $w(\tau)$? Use this transformation to solve

$$2t^2y'' + 3ty' - 15y = 0, \quad y(1) = 0, \quad y'(1) = 1$$

9. Suppose there is a capacitor with capacitance C being charged by a battery of fixed voltage V_0. Suppose there is a resistor R connected to C. Then the charge $Q(t)$ of the capacitor satisfies the differential equation

$$RQ'(t) + \frac{Q(t)}{C} = V_0$$

This is the equation for an RC charging circuit.

Find the explicit solution of this equation with $Q(0) = 0$. Explain why the product RC is important in determining the charging time. For $R = 10^3\,\Omega$, $V_0 = 1\,\mathrm{V}$, $C = 1\,\mu\mathrm{F}$, how much time does it take for the capacitor to be charged to 98%? (You may use a calculator.)

10. A parachutist is falling from a plane. Suppose the parachute is opened at height H, when the falling velocity is v_0. Suppose that the air resistance exerted on the parachute is proportional to the square of the velocity with ratio η. Let the gravitational constant be g, and suppose that the total mass of the parachutist and the parachute is m. Write down the differential equation satisfied by the shift x, together with the initial conditions. Solve this IVP. What is the velocity as $t \to +\infty$? Can you derive the final velocity based on physical considerations?

Bonus Problems

1. **The Catenoid.** Suppose there are two metal rings of radius a placed parallel to each other in an xyz-coordinate space, with the x-axis passing through their centers. Suppose these two rings are contained in the planes $x = l$ and $x = -l$, respectively. An axial symmetric soap film is spanned by these two rings. Suppose its shape is obtained by rotating the graph of the function $y = y(x)$ with respect to the x-axis. In order to attain a stable configuration, the surface area is supposed to be minimal among all such surfaces of revolution.

   (1) Write down the surface area functional in terms of $y(x)$, its derivative, and the boundary conditions for this variational problem.

   (2) Derive the Euler-Lagrange equation and find the solution. The shape is called a **catenoid**.

   (3) If the two rings are very far away from each other, i.e., l is very large, will the catenoid still be of minimal area among all competing surfaces that span these two rings? You do not have to give a mathematically rigorous answer; just imagine the physical situation. (Hint: What about two distinct disks spanned by these two rings?)

2. **A Formulation of the Isoperimetric Problem.** Recall from multivariable calculus that in order to find a local extremum of the function $f(x_1, \dots, x_n)$ under the constraint $g(x_1, \dots, x_n) = 0$, we can introduce a parameter λ called the **Lagrange multiplier** and find the stationary point of the function

   $$f(x_1, \dots, x_n) - \lambda g(x_1, \dots, x_n)$$

   (1) Write down the equations that must be satisfied by the stationary point $(x_1, \dots, x_n)$ of the function $f - \lambda g$ with the parameter λ involved.

   (2) Use the Lagrange multiplier method to find the maxima and minima of $f(x, y) = x + y$ under the constraint $x^2 + y^2 = 1$.

   (3) Now let us generalize this method to functionals. If we aim to find the extrema of a functional

   $$J[y] = \int_a^b F(x, y(x), y'(x))\,\mathrm{d}x$$

   under the constraint

   $$R[y] = \int_a^b G(x, y(x), y'(x))\,\mathrm{d}x = 0$$

   where $F(x, z, w)$ and $G(x, z, w)$ are known functions, we can try to find the extrema of the functional

   $$J[y] - \lambda R[y]$$

   first. What is the Euler-Lagrange equation satisfied by this extrema (with λ involved)?

   (4) Now let us consider a version of the isoperimetric problem. We aim to find the function $y(x)$, whose graph connects two given points (a, A), (b, B) on the xy-plane, with a prescribed arclength

   $$l = \int_a^b \sqrt{1 + |y'(x)|^2}\,\mathrm{d}x$$

   such that the area between the graph and the x-axis is the largest. The functional in consideration is

   $$J[y] = \int_a^b y(x)\,\mathrm{d}x$$

   with constraint

   $$R[y] = \int_a^b \sqrt{1 + |y'(x)|^2}\,\mathrm{d}x = l$$

   Write down the Euler-Lagrange equation involving the multiplier λ and show that the solution must be a part of a circle.