Week 9

Periodicity

9.1 Periodic Solutions of Planar Systems

11/28:

- Last lectures: Special solutions to planar systems. Usually encountered in applications of ODEs (e.g., the homework). If we encounter ODEs in our physical sciences lives, we may need to remember this.
- There will be a Monday office hours that coincides with the regular lecture time.
- Special (periodic) solutions of planar systems.
- For a planar autonomous system
$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$
a periodic solution is equivalent to a closed orbit.
- So geometrically, we'll be studying closed orbits. Analytically, we'll be studying periodic systems.
- Simple examples: Harmonic oscillator
$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} y \\ -x \end{pmatrix}$$
- Less trivial example: The pendulum
$$\begin{pmatrix} \theta \\ \omega \end{pmatrix}' = \begin{pmatrix} \omega \\ -\sin\theta \end{pmatrix}$$
- We can easily find Lyapunov functions for these two systems; both functions are the energy function.
- Different orbital graphs: The first one is concentric circles; the second one is only sometimes periodic. *picture*
- In general, these periodic cycles are dense in the plane.
- However, we can also, at the other extreme, have isolated periodic solutions, referred to as **limit cycles**.
- Simplest example:
$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -y - [1-(x^2+y^2)]x \\ x - [1-(x^2+y^2)]y \end{pmatrix}$$
  - Hard to see the behavior in Cartesian coordinates; much easier in polar. If we use $r = \sqrt{x^2+y^2}$ and $\theta = \arctan(y/x)$. We will see equations of this type in our homework.

– Differentiating $r = \sqrt{x^2+y^2}$ implicitly with respect to time, we get

$$\begin{aligned}\frac{\mathrm{d}r}{\mathrm{d}t} &= \frac{xx'}{\sqrt{x^2+y^2}} + \frac{yy'}{\sqrt{x^2+y^2}} \\ &= -\frac{xy}{\sqrt{x^2+y^2}} + \frac{1-r^2}{r}x^2 + \frac{xy}{\sqrt{x^2+y^2}} + \frac{1-r^2}{r}y^2 \\ &= r(1-r^2)\cos^2\theta + r(1-r^2)\sin^2\theta \\ &= r(1-r^2)\end{aligned}$$

– Differentiating $\theta = \arctan(y/x)$ implicitly with respect to time, we get

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{-\frac{y}{x^2}x'}{1+\left(\frac{y}{x}\right)^2} + \frac{\frac{1}{x}y'}{1+\left(\frac{y}{x}\right)^2} \\ &= \frac{-yx'}{x^2+y^2} + \frac{xy'}{x^2+y^2} \\ &= 1\end{aligned}$$

– Thus, we can transform the original equation to

$$\begin{pmatrix} r \\ \theta \end{pmatrix}' = \begin{pmatrix} r(1-r^2) \\ 1 \end{pmatrix}$$

for $r > 0$ and $\theta \in \mathbb{R}$.

– This ODE can be explicitly solved, but since we are interested in qualitative behavior, we will not do that.

– The unit circle partitions the xy-plane into two parts (inside and outside). *picture*

– Let's start on the unit circle. Then we just spiral around on it with constant velocity ($r' = 0$ and $\theta' = 1$).

– Let's now start inside. Since $\theta(t) = t + \theta(0)$, and r' is positive, we get a spiral that approaches the unit circle.

– If we start outside, we get a spiral that starts outside and spirals toward the unit circle.

– Thus, in this case, the unit circle is the unique limit cycle of the system.

- Shao suggests we search for iodine clock videos on YouTube to help with the homework. The iodine clock is described by the limit cycles.

- Historical remark: David Hilbert posed 23 questions at the beginning of the 20th century. The 16th one asked about planar polynomial systems. For these, is it possible to estimate the number of limit cycles. Even for the case of quadratic polynomials, the question is still open! For quadratics, we know that there can be 1, 2, 3, or 4 cycles, but we have no idea whether or not there is an upper bound. This is a central open problem in the study of ODEs.

- Basic theorem in this area is as follows.

- Theorem (Poincaré-Bendixson Theorem): Let $\Omega \in \mathbb{R}^2$ be open, $f(x)$ a vector field on Ω. Fix $x \in \Omega$. Define
$$\omega(x) := \{z \in \Omega \mid \text{there is a sequence } t_n \to +\infty \text{ such that } \phi_{t_n}(x) \to z\}$$
and
$$\alpha(x) := \{z \in \Omega \mid \text{there is a sequence } t_n \to -\infty \text{ such that } \phi_{t_n}(x) \to z\}$$
That is, if you reverse the direction of time, $\alpha(x)$ collects all of the points. Also, let $\omega(x) \subset \Omega$ be compact and nonempty. In particular, there are three mutually exclusive cases for these limit sets.

1. $\omega(x)$ (or $\alpha(x)$[1]) is a fixed point.
2. $\omega(x)$ is a limit cycle.
3. $\omega(x)$ consists of finitely many fixed points, together with curves joining these fixed points.

Proof. The proof relies largely on algebraic topology, so we will not go into it in detail or even sketch it. The statement will be sufficient for our purposes. □

- Theorem (Annulus theorem of Bendixson): Suppose C_1, C_2 are closed simple planar curves such that geometrically, one contains the other. We call the annular region (between the two curves) A. Suppose $f(x)$ is a planar vector field which points inward at every point of ∂A (the boundary of A). Then the annular region A is an invariant region of the plane. In particular, if A does not contain any fixed points, then it must contain a limit cycle. As before, curves within and without spiral towards it. *picture*
- Can produce beautiful diagrams: Zhifen Zhang's example — Xiao and Zhang (2008) — is the 4 limit cycle one. After her research, mathematicians found a family with four limit cycles.
- System from the homework: Consider the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a - x - \frac{4xy}{1+x^2} \\ b(x - \frac{xy}{1+x^2}) \end{pmatrix}$$

  picture

  - We take $a, b > 0$.
  - Every vector points in toward a, which points straight upward.
  - On the boundary,
  $$\begin{pmatrix} a - x - \frac{4xy}{1+x^2} \\ b(x - \frac{xy}{1+x^2}) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} > 0$$
  so we have a strict Lyapunov function.
  - Thus, any orbit in the first quadrant can never escape, reflecting our expectation that the concentration can never be negative.
  - Spoiler: Looks like a rectangular region.
  - We are also asked to find the fixed point and investigate its stability.
- First step: Find fixed points.
- Second step: Lyapunov functions.
- Third step: Divide the vector field into positive and negative regions. Requires some improvization.
- Fourth step: Check for an annular region.
- Note: Note of these arguments can be generalized to higher dimensional systems. The regularity of the Poincaré-Bendixson system disappears as we go to higher dimensions.
- Lorenz system: Oversimplified 3D quadratic system that serves as a simplification model for weather systems.
  - Lorenz was an astronomer.
  - He discovered that even if the systems are very close together, the orbits will be separated indefinitely as time evolves on. It's not just about being stable or unstable, but overall global chaotic behavior.

[1] We just have to reverse the time.

  - Mathematicians quickly discovered that the Lorenz system has a **strange attractor**. An attractor for a planar system can only be closed orbits. In the Lorenz system, we have a strange type of butterfly. The dimension of the butterfly is not even an integer. Have to introduce Hausdorff measure to understand length and area in the more general framework of curves or surfaces.
  - No more detail, but a classical example of a chaotic ODE system worth mentioning. This phenomenon cannot appear for planar systems; even increasing the dimension by 1 can lead to chaos.
  - No widely accepted definition of mathematical chaos, but generally accepted ones are very irregular global attractors and points that are arbitrarily close and arbitrarily far from each other.
  - Fractorial sets.
- Last lecture (Wednesday): Another example that has a limit cycle.
- Friday will be a review.

9.2 Nonlinear Oscillation

11/30:

- Friday: Review and questions session.
- Monday: OH during class time in the class room and normal OH.
- Exam time: Wednesday, December 7 from 7:30-9:30 AM. Will happen in this classroom; Shao hopes to finish grading the same day.
- Last HW and review outline is due 12/2. 130 total points for the grade. 90/130 gives an A. That's 69% and above.
  - Minus 10, minus 10, for the grading below A range.
- Last example that arises naturally.
- RLC circuit: Nonlinear oscillation. Occurred in resonance. Nonlinear oscillation in K.
- The circuit that we're interested in still is an RLC circuit (see Figure 5.5b).
- We assume that the resistor satisfies Ohm's law, i.e., $I_R = V_R/R$. This is the linear case. Causes the voltage to decay exponentially fast, where the resistor acts as a kind of damping factor.
- However, for some more delicate cases, Ohm's law might fail and we might get a nonlinear replacement $V_R = R(I_R)$. We let R be any suitable function. We still have Kirchoff's law, i.e., that I_R, I_C, and I_L are all the same and we can refer to them as I.
- The system we're interested in is

$$\begin{cases} LI' &= -V_C - R(I) \\ CV' &= I \end{cases}$$

- Comparing to the linear case, we just want to replace V_R/R in the same system with $R(I)$. After some suitable scaling, we can get the system

$$\begin{cases} x' &= y - f(x) \\ y' &= -x \end{cases}$$

- This is a classical nonlinear oscillation model, typically referred to as the **Liénard equation**.
- The first case of the Liénard equation studied in detail was when

$$f(x) = \mu\left(\frac{x^3}{3} - x\right)$$

  - Discovered by the engineer van der Pol while he was investigating RLC circuits, and hence referred to as the **van der Pol oscillator**.
- Theorem: Suppose $f(x)$ is odd, i.e., $f(x) = -f(-x)$. Also suppose that $f(x) < 0$ for $x \in (0, \alpha)$ for some $\alpha > 0$ (i.e., $f(x)$ is less than zero for sufficiently small positive values of x). Moreover, suppose that $\liminf_{x\to\infty} f(x) > 0$ (i.e., f does not diverge). Lastly, suppose that $f(x)$ is increasing for $x > \alpha$. Then the conclusion is that the Liénard system has a unique closed orbit. Every other orbit except for the trivial orbit $(0,0)$ will be attracted to this periodic solution as the system evolves ($t \to +\infty$).
  - This explains the term "nonlinear oscillation." As long as we are away from the equilibrium, we converge to it??
- Wrt the van der Pol oscillator, we sketch the boundary first and then put in the limit cycle. The origin is an unstable fixed point, and any origin starting from the origin will spiral and approximate the limit cycle.
  - ?? is guaranteed by the Poincaré-Bendixson theorem.
- The proof does not use any advanced math, but it is complex, so we'll sketch it. The proof is in Teschl (2012) if we're interested. It only involves calculus.
- End of lecture.
- First problem, second section of HW: First integral: Conservative law.
- What is the correct first variational equation?
- Difference between not Lyapunov stable and completely unstable?
  - Something is repelled away vs. everything is repelled away.
- The iodine clock is not super hard, but it needs some improvisation. That's what we talked about at the end of last lecture.
- Friday will be a review of everything since the second midterm.

9.3 Question Session

12/2:

- The actual Lecture 9.1 system is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -y - [1 - (x^2 + y^2)]x \\ x - [1 - (x^2 + y^2)]y \end{pmatrix}$$

- Cannot derive a phase portrait spiral from a potential energy function.
  - Relevant to 3c.
  - Idea is to show that the potential energy well is decreasing along any trajectory.
- $r = \sqrt{c_0}$.
  - Limit cycle is omega and alpha limit cycle.
- Solving for the stable and unstable manifolds.
  - Stable set consists of points which are attracted to the equilibrium. Curves are not attracted or repelled.
  - Stable subset: Points (z, w) such that $(x, y) \to (0, 0)$ as $t \to +\infty$. Stable subset: necessitates taking $w = 0$ and then z can be anything, so x axis.

- Unstable: As $t \to -\infty$, w can be anything and $y(t) \to 0$. The $+t$ terms will go to zero as $t \to -\infty$, and then we must have $z - w/2 = 0$. Put it in the form

$$\begin{aligned} x(t) &= (w^2(\mathrm{e}^{2t} - \mathrm{e}^t) + \frac{w\mathrm{e}^t}{2}) + (z - \frac{w}{2})\mathrm{e}^{-t} \\ &= ((w\mathrm{e}^t)^2 + \frac{w\mathrm{e}^t}{2}) - w^2\mathrm{e}^t + (z - \frac{w}{2})\mathrm{e}^{-t} \\ &= (y^2 + \frac{y}{2}) - w^2\mathrm{e}^t + (z - \frac{w}{2})\mathrm{e}^{-t} \end{aligned}$$

- There is a typo in the original form. There should be e^{-t} for the last rightmost term above. We will converge and diverge along the manifolds.
- I have a confusion in the stable/unstable subset definition? Unstable subset isn't the set of all points with orbits that diverge as $t \to +\infty$; it's the set of all points that diverge away from x_0.

- Solving for the stable and unstable manifolds of a planar ODE given the explicit solution.
  - We will treat
  $$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -x + y + 3y^2 \\ y \end{pmatrix}$$
  from Lecture 8.2.
  - The correct flow is as follows (there was a typo in class).
  $$\phi_t \begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} z\mathrm{e}^{-t} + w\sinh(t) + w^2(\mathrm{e}^{2t} - \mathrm{e}^{-t}) \\ w\mathrm{e}^t \end{pmatrix}$$
  - The stable manifold is going to be the set of all $x \in \mathbb{R}^2$ such that $\phi_t(x) \to 0$ as $t \to +\infty$. Approach: Find values of z, w such that the solution converges to zero componentwise.
    - Since $y(t) = w\mathrm{e}^t$, we must have $w = 0$; otherwise, we will get exponential divergence as $t \to +\infty$.
    - Thus, $x(t) = z\mathrm{e}^{-t}$. This function converges to zero for any value of z, so we may let z be arbitrary.
    - But the set of all points
    $$\begin{pmatrix} z \\ 0 \end{pmatrix}$$
    is the x-axis!
  - Unstable manifold: We need to find the set of all points $x \in \mathbb{R}^2$ such that $\phi_t(x) \to 0$ as $t \to -\infty$. Approach: Again, go by components.
    - $y(t) = w\mathrm{e}^t$ will converge to 0 as $t \to -\infty$ for all w, so this component does not put any restrictions on w. Note that it also does not put any restrictions on z since it does not even contain z.
    - Working with the other one, we expand and combine all e^{at} terms for $a > 0$ and all e^{bt} terms for all $b < 0$.
    $$\begin{aligned} x(t) &= z\mathrm{e}^{-t} + w\sinh(t) + w^2(\mathrm{e}^{2t} - \mathrm{e}^{-t}) \\ &= z\mathrm{e}^{-t} + w \cdot \frac{\mathrm{e}^t - \mathrm{e}^{-t}}{2} + w^2\mathrm{e}^{2t} - w^2\mathrm{e}^{-t} \\ &= z\mathrm{e}^{-t} + \frac{w}{2}\mathrm{e}^t - \frac{w}{2}\mathrm{e}^{-t} + w^2\mathrm{e}^{2t} - w^2\mathrm{e}^{-t} \\ &= \left[w^2\mathrm{e}^{2t} + \frac{w}{2}\mathrm{e}^t\right] + \left[z - \frac{w}{2} - w^2\right]\mathrm{e}^{-t} \end{aligned}$$
    - The left term above will clearly converge to 0 as $t \to -\infty$.

- However, the right term will diverge to ∞ as $t \to -\infty$ unless $z - w/2 - w^2 = 0$, so we take this to be our condition.
- Indeed, this implies that $z = w/2 + w^2$ is a constraint on z, but w can still take on any value, so our solution is
$$W_u(0) = \left\{ \left(\frac{y}{2} + y^2, y \right) \,\middle|\, y \in \mathbb{R} \right\}$$
as desired.

- Number 5:
  - What is a vector field in 1d? Vectors pointing in the positive or negative x direction (just a function).
  - Set of points should be a subset of the real line (an interval).
  - You can only approach the zero.
- Number 3 energy term.
  - Multiply both sides by x' to get
$$x'x'' = \frac{1}{2}(x')^2$$
  - We have
$$\begin{aligned} 0 &= x'' + bx' + U'(x) \\ &= x'x'' + b|x'|^2 + x'U'(x) \\ &= \left(\frac{1}{2}(x')^2 \right)' + b|x'|^2 + (U(x))' \\ -b|x'|^2 &= \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2}(x')^2 + U(x) \right) \end{aligned}$$
- Lyapunov stuff. Was a question in HW6.
  - ...
  - Intuitive justification for this Lyapunov function?
  - Most natural way is to look at when A is diagonalizable.
  - Get expressions with negative eigenvalues.
  - We have that the sum is equal to $\frac{\mathrm{d}}{\mathrm{d}t}\langle y, Dy \rangle$. The INP is equal to $2\langle Dy, Dy \rangle$.
  - So it's a weighted norm.
- We'll be allowed to bring the JNF notes to the exam!

9.4 Office Hours (Shao)

- Question 3(1): How do we derive the energy function?
  - We don't actually need to give a final expression for the energy function; just show that it's always decreasing.
- Question 3(2): How should we apply the stable manifold theorem and Hartman linearization theorem?
  - By the stable manifold theorem, we can determine source, sink, saddle.
  - By the Hartman linearization theorem, we can further characterize the saddle point by saying that all orbits not on the stable/unstable manifolds (which we don't have to find) approach the fixed point and then diverge away.

- Submit a sketch of the local behavior of each one with my answer!!

- Question 4(3): What does this even mean?
  - Solution is $1/3(x^3 + y^3) + xy = a$.
  - As $a \to 1/3$, the orbits have shape very similar to an ellipse.
  - We need $a = 0$ because we must have the system pass through zero.
  - This is related to the Lotka-Voterra model from Lecture 2.1.
  - F is called a first integral.
    - To be more general, in an undamped Newtonian system, the energy function is a first integral of the system. Think like quantum mechanics and vibrational energy levels being equivalent to segments of the energy parabola. More generally, though, this perspective applies to all mechanical systems via Hamiltonian mechanics. We don't need to understand Hamiltonian mechanics for this course, though, because we've defined phase spaces independently (a phase space is like the (θ, ω) plane for the harmonic oscillator).
    - We talked about first integrals when we discussed the Kepler problem. The Kepler problem is covered in Section 8.5 of Teschl (2012).
  - We take
$$\begin{aligned} \frac{x'}{y'} &= \frac{-x - y^2}{x^2 + y} \\ (x^2 + y)\frac{\mathrm{d}x}{\mathrm{d}t} &= (-x - y^2)\frac{\mathrm{d}y}{\mathrm{d}t} \\ (x^2 + y)\frac{\mathrm{d}x}{\mathrm{d}t} + (x + y^2)\frac{\mathrm{d}y}{\mathrm{d}t} &= 0 \end{aligned}$$
    Solve this for F, which will be a polynomial.

- Question 4(4): What does this even mean?
  - For the $c = 0$ case, the stable set is the part of the curve tangent to the x-axis and the unstable set is the part of the curve tangent to the y-axis. We get a change from stable to unstable at $(0, 0)$ and $(-1.5, -1.5)$.

- Enzyme kinetics 1(1): What does "two first integrals" mean?
  - First conservation law: Conservation of the substrate in all its forms. Second: Conservation of the enzyme in all its forms.
  - Sum of the second and third equations equals zero.
  - Sum of first, third, and fourth is zero.
  - We have
$$0 = \frac{\mathrm{d}[\mathrm{E}]}{\mathrm{d}t} + \frac{\mathrm{d}[\mathrm{ES}]}{\mathrm{d}t}$$
    This is also obvious from their definitions. Thus, $[\mathrm{E}] + [\mathrm{ES}] = [\mathrm{E}]_0$ is a first integral.
  - This is not an equation of exact form because of 3 derivatives, but it still implies a conservation law/has a first integral:
$$0 = \frac{\mathrm{d}[\mathrm{S}]}{\mathrm{d}t} + \frac{\mathrm{d}[\mathrm{ES}]}{\mathrm{d}t} + \frac{\mathrm{d}[\mathrm{P}]}{\mathrm{d}t}$$
    That first integral is $[\mathrm{S}] + [\mathrm{ES}] + [\mathrm{P}] = [\mathrm{S}]_0$

- Iodine Clock: 2(1)?
  - Ben knows what's going on here.
  - There are curves that divide the first quadrant into regions of definite sign.
  - Definite sign: Neither component changes sign in a certain region

9.5 Chapter 7: Planar Dynamical Systems

From Teschl (2012).

Section 7.2: Examples from Electrical Engineering

12/6: - Consider an RLC circuit again, but this time with a resistor of arbitrary characteristic

$$V_R = R(I_R)$$

  - Ohm's law asserts that $R(I_R) = RI_R$, where the resistance R is a constant.
  - But what about a stranger, more sophisticated element? We must have $R(0) = 0$ (since there's no potential difference if there's no current) for any characteristic, but other than that, we're pretty free.
- **Diode**: A circuit element that lets the current pass in only one direction.
  - For example, the characteristic of a diode is given by

$$V = \frac{kT}{q}\log\left(1 + \frac{I}{I_L}\right)$$

    where I_L is the leakage current, q is the charge of an electron, k is the Boltzmann constant, and T is the absolute temperature.
  - Implications: In the positive direction, very little voltage gives a large current; in the negative direction, you will get almost no current even for fairly large voltages.
- As in class, we obtain the system

$$\begin{aligned}
L\dot{I}_L &= -V_C - R(I_L)\\
C\dot{V}_C &= I_L
\end{aligned}$$

  where $R(0) = 0$ and $L, C > 0$.
- Additional note.
  - Kirchoff's laws and the substitutions from Section 3.3 imply

$$\begin{aligned}
I_LV_L + I_CV_C + I_RV_R &= 0\\
LI_L\dot{I}_L + CV_C\dot{V}_C + I_RR(I_R) &= 0\\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{L}{2}I_L^2 + \frac{C}{2}V_C^2\right) &= -I_RR(I_R)
\end{aligned}$$

  - Conclusion: The energy dissipated in the resistor has to come from the inductor and capacitor.
- **Liénard's equation**: The result of scaling the above system. *Given by*

$$\begin{aligned}
\dot{x} &= y - f(x)\\
\dot{y} &= -x
\end{aligned}$$

- The additional note now reads

$$\frac{\mathrm{d}}{\mathrm{d}t}W(x,y) = -xf(x)$$

  where

$$W(x,y) = \frac{x^2 + y^2}{2}$$

  - If $xf(x) > 0$ in a neighborhood of $x = 0$, then W is a Lyapunov function and hence $(0,0)$ is stable.

- Theorem 7.5: Suppose $xf(x) \geq 0$ for all $x \in \mathbb{R}$ and $xf(x) > 0$ for $0 < |x| < \varepsilon$. Then every trajectory of Liénard's equation converges to $(0,0)$.

  Proof. Given. ☐

- Conversely, if $xf(x) < 0$ for $0 < |x| < \varepsilon$, then $(0,0)$ is unstable (and the distance to the fixed point will actually grow).
- Teschl (2012) works through proving the main theorem from lecture (Theorem 7.8 below).
- Theorem 7.8: Suppose f satisfies requirements (i)-(iii) below.
  (i) f is odd, that is, $f(-x) = -f(x)$.
  (ii) $f(x) < 0$ for $0 < x < \alpha$ ($f(\alpha) = 0$ without restriction).
  (iii) $\liminf_{x\to\infty} f(x) > 0$ and, in particular, $f(x) > 0$ for $x > \beta$ ($f(\beta) = 0$ without restriction).
  (iv) $f(x)$ is monotone increasing for $x > \alpha$ (i.e., $\alpha = \beta$).

  Then Liénard's equation has at least one periodic orbit encircling $(0,0)$.

  If in addition (iv) holds, this periodic orbit is unique and every trajectory (except $(0,0)$) converges to this orbit as $t \to \infty$.
- The classical application of this theory is to **van der Pol's equation**.
- **Van der Pol's equation**: The following ODE, which models a triode circuit. *Given by*

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0, \quad \mu > 0$$

- We can show that van der Pol's equation is equivalent to Liénard's equation with

$$f(x) = \mu\left(\frac{x^3}{3} - x\right)$$

- Therefore, by Theorem 7.8, van der Pol's equation has a unique periodic orbit and all trajectories converge to this orbit as $t \to \infty$.

Section 7.3: The Poincaré-Bendixson Theorem

- In all previous examples, the solutions of ODEs have either converged to a fixed point or a periodic orbit.
  - This is normal behavior, and in this section we will classify all possible $\omega_\pm$-limit sets (for planar systems).
  - Note that the difference between $\mathbb{R}^2$ and $\mathbb{R}^n$ ($n \geq 3$) arises from the validity of the **Jordan Curve Theorem** in $\mathbb{R}^2$ and its being false in higher dimensions.
- **Jordan curve**: A homeomorphic image of the circle S^1. *Denoted by* $\boldsymbol{J}$.
- **Jordan Curve Theorem**: Every Jordan curve dissects $\mathbb{R}^2$ into two connected regions. In particular, $\mathbb{R}^2 \setminus J$ ahs two components.
- Teschl (2012) builds up to proving the Poincaré-Bendixson theorem.
- Theorem 7.16 (generalized Poincaré-Bendixson): Let M be an open subset of $\mathbb{R}^2$ and $f \in C^1(M, \mathbb{R}^2)$. Fix $x \in M$, $\sigma \in \{\pm\}$, and suppose $\omega_\sigma(x) \neq \emptyset$ is compact, connected, and contains only finitely many points. Then one of the following cases holds.

(i) $\omega_\sigma(x)$ is a fixed orbit.
(ii) $\omega_\sigma(x)$ is a regular periodic orbit.
(iii) $\omega_\sigma(x)$ consists of (finitely many) fixed points $\{x_j\}$ and non-closed orbits $\gamma(y)$ such that $\omega_\pm(y) \in \{x_j\}$.

- Teschl (2012) gives an example of the third case.
- Lemma 7.17: The interior of every periodic orbit must contain a fixed point.
- **Limit cycle**: A periodic orbit attracting other orbits.
- Lemma 7.18: Let $\gamma(y)$ be an isolated regular periodic orbit (such that there are no other periodic orbits within a neighborhood). Then every orbit $\gamma(x)$ starting sufficiently close to $\gamma(y)$ will have either $\omega_-(x) = \gamma(y)$ or $\omega_+(x) = \gamma(y)$.
- Example: In general, the system
$$\dot{x} = -y + f(r)x \qquad\qquad \dot{y} = x + f(r)y$$
becomes
$$\dot{r} = rf(r) \qquad\qquad \dot{\theta} = 1$$
for any function f.

9.6 Chapter 8: Higher Dimensional Dynamical Systems

From Teschl (2012).

Section 8.1: Attracting Sets

- Not much of relevance here.
- A bit on the Duffing equation from HW7.
- **Topologically transitive** (set): A closed invariant set Λ such that for any two open sets $U, V \subset \Lambda$, there is some $t \in \mathbb{R}$ such that $\Phi(t, U) \cap V \neq \emptyset$.
- **Attractor**: An attracting set which is topologically transitive. *Denoted by* $\boldsymbol{\Lambda}$.

Section 8.2: The Lorenz Equation

- **Lorenz equation**: One of the most famous dynamical systems which exhibits chaotic behavior. *Given by*
$$\begin{aligned}\dot{x} &= -\sigma(x - y)\\ \dot{y} &= rx - y - xz\\ \dot{z} &= xy - bz\end{aligned}$$
where $\sigma, r, b > 0$.
  - "Lorenz arrived at these equations when modelling a two-dimensional fluid cell between two parallel plates which are at different temperatures. The corresponding situation is described by a complicated system of nonlinear partial differential equations. To simplify the problem, he expanded the unknown functions into Fourier series with respect to the spacial coordinates and set all coefficients except for three equal to zero. The resulting equation for the three time dependent coefficients is [the above]. The variable x is proportional to the intensity of convective motion, y is proportional to the temperature difference between ascending and descending currents, and z is proportional to the distortion from linearity of the vertical temperature profile" (Teschl, 2012, p. 234).

- **Strange attractor**: An attractor that has a complicated set structure.