Week 7

Solution Existence and Stability

7.1 Peano Existence Theorem

11/7:

- Today: Peano Existence Theorem.
- For an IVP of a first-order differential system, as long as the RHS is continuous, we get at least one solution.
- The proof provides an algorithm that can be really useful in computing the solution provided that uniqueness exists.
- We will need a theorem from analysis to start.
- Theorem (Arzelà-Ascoli[1]): Let $h_k : [a,b] \to \mathbb{R}^n$ be a sequence of functions that is uniformly bounded and uniformly Lipschitz continuous wrt. L. Then $\{h_k\}$ contains a uniformly convergent subsequence and the limit has the same bound and Lipschitz constant.

  Proof. Recall the property of sequential compactness[2], i.e., that every bounded sequence of numbers contains a convergent subsequence. We want to prove this for a sequence of functions. To do so, we will need the Cantor diagonalization technique.

  $\mathbb{Q}$ is countable. Thus, we can enumerate the rationals in $[a,b]$ by $r_1, r_2, r_3, \dots$. Since $\{h_k(r_1)\}$ is a bounded sequence of numbers, we have by the above that there is a subsequence C_1 — say $h_1^{(1)}, h_2^{(1)}, h_3^{(1)}, \dots$ — such that $C_1 = \{h_k^{(1)}(r_1)\}$ is a convergent subsequence in $\mathbb{R}^n$ of the original sequence. Now C_1 is still a bounded sequence, so we can obtain a subsequence C_2 of *it* — say $h_1^{(2)}, h_2^{(2)}, h_3^{(2)}, \dots$ — such that $C_2 = \{h_k^{(2)}(r_2)\}$ is a convergent subsequence in $\mathbb{R}^n$ at r_2 (and, by inductive hypothesis, at r_1!). Inductively, we can obtain $C_\ell = \{h_k^{(\ell)}\}_{\ell,k=1}^\infty$ convergent at $r_1, r_2, \dots, r_\ell$. We then write down the elements of the sequences as a table. (For example, the k^{th} row of the table is a sequence that converges at $r_1, \dots, r_k$.)

$$\begin{matrix}
h_1^{(1)} & h_2^{(1)} & h_3^{(1)} & \cdots \\
h_1^{(2)} & h_2^{(2)} & h_3^{(2)} & \cdots \\
h_1^{(3)} & h_2^{(3)} & h_3^{(3)} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{matrix}$$

  Consider the diagonal sequence $\{f_\ell\}_{\ell=1}^\infty$ where $f_\ell = h_\ell^{(\ell)}$. By definition, it converges at all rational points. We now seek to prove that it converges uniformly at *all* points.

[1]This is not the full Arzelà-Ascoli theorem, but a special case. The proof is similar, regardless, though. See Honors Analysis in $\mathbb{R}^n$ I Notes.

[2]The Bolzano-Weierstrass Theorem/Theorem 15.18 from Honors Calculus IBL.

To prove that $\{f_\ell\}$ is a uniformly convergent sequence of functions, it will suffice to show that for all $\varepsilon > 0$, there exists N such that if $k, \ell > N$, then $|f_k(t) - f_\ell(t)| < \varepsilon$ for all $t \in [a,b]$. Let $\varepsilon > 0$ be arbitrary. Divide $[a,b]$ into m congruent subintervals I_α ($\alpha = 1, \dots, m$) such that $|I_\alpha| \leq \varepsilon/3L$ for all α. This guarantees that the oscillation of each f_k on any I_α is $\leq \varepsilon/3$ since if $x, y \in I_\alpha$ for some α, then

$$|f_\ell(x) - f_\ell(y)| \leq L|x-y| \leq L \cdot \frac{\varepsilon}{3L} = \frac{\varepsilon}{3}$$

Using the fact that $\{f_\ell\}$ is convergent and hence Cauchy on the rationals, pick N large enough so that $r_\alpha \in I_\alpha$ implies $|f_k(r_\alpha) - f_\ell(r_\alpha)| < \varepsilon/3$ for $k, \ell > N$. We will choose this N to be our N. Now let $t \in [a,b]$ be arbitrary. By their definition, we know $t \in I_\alpha$ for some α. Therefore,

$$\begin{aligned}
|f_k(t) - f_\ell(t)| &\leq |f_k(t) - f_k(r_\alpha)| + |f_k(r_\alpha) - f_\ell(r_\alpha)| + |f_\ell(r_\alpha) - f_\ell(t)| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}$$

as desired.

Lastly, we can prove that the limit function f of $\{f_\ell\}$ is L-Lipschitz as follows. Let $t, t' \in [a,b]$ be arbitrary. Then

$$\left|\frac{f(t) - f(t')}{t - t'}\right| = \lim_{k\to\infty} \left|\frac{f_k(t) - f_k(t')}{t - t'}\right| \leq \lim_{k\to\infty} \left|\frac{L|t - t'|}{t - t'}\right| = \lim_{k\to\infty} |L| = L$$

as desired. ∎

- Now we come to the proof of the Peano Existence Theorem.
- Theorem (Peano Existence Theorem): Let $f : [t_0, t_0 + a] \times \bar{B}(y_0, b) \to \mathbb{R}^n$ be bounded ($|f(t,z)| \leq M$) and continuous. Then the IVP
$$y'(t) = f(t, y(t)), \quad y(t_0) = b$$
has at least one solution for $t \in [t_0, t_0 + T]$ where $T = \min(a, b/M)$.

Proof. Since there is no Lipschitz condition, we use another strategy to find approximate solutions.

picture Fix $T = \min(a, b/M)$. We divide $[t_0, t_0 + T]$ into m congruent closed subintervals I_α ($\alpha = 0, \dots, m-1$), each of length $h_m = T/m$. Define a continuous function $y_m(t)$ as follows: The values at the nodes t_α (the intersection points of adjacent congruent subintervals) are defined inductively via

$$y_m(t_{\alpha+1}) = y_m(t_\alpha) + f(t_\alpha, y_m(t_\alpha))h_m$$

for $\alpha = 0, \dots, m-1$, and y_m is taken to be linear between the nodes[3]. The idea is that we replace the derivative $y'(t)$ by the difference quotient $[y(t+h) - y(t)]/h$. It follows by the construction that every function in the set $\{y_k(t) : [t_0, t_0 + T] \to \bar{B}(y_0, b)\}$ is piecewise linear (hence continuous), uniformly bounded, and uniformly M-Lipschitz continuous. Therefore, by the Arzelá-Ascoli theorem, $\{y_k\}$ contains a uniformly convergent subsequence $y_{m_k} \to y$.

It remains to verify that y is a solution to the integral equation

$$y(t) = y_0 + \int_{t_0}^{t} f(\tau, y(\tau))\mathrm{d}\tau$$

Observe that the domain of f is a closed and bounded subset of the real numbers. Thus, it is compact by the Heine-Borel theorem[4]. Moreover, since f is a continuous function on a compact domain, we

[3]Note that this construction is quite similar to that employed in Euler's method.
[4]Theorem 10.16 of Honors Calculus IBL.

have by the Heine-Cantor theorem[5] that f is uniformly continuous. Thus, for any $\varepsilon > 0$, there exists N such that if $m > N$, then

$$|f(t, y_m(t)) - f(t_\alpha, y_m(t_\alpha))| < \frac{\varepsilon}{T}$$

for all $\alpha = 0, \dots, m-1$ and $t \in I_\alpha$. Additionally, observe that

$$y_{m_k}(t) = y_0 + \sum_{\alpha=0}^{m-1} \int_{t_\alpha}^{t_{\alpha+1}} \chi_t(\tau) f(t_\alpha, y_{m_k}(t_\alpha)) \mathrm{d}\tau$$

where $\chi_t(\tau)$ denotes the **characteristic function** of $[t_0, t]$. To see this, compare with the original inductive definition of $y_m(t_{\alpha+1})$. *picture* We thus see that y_0 in the above equation corresponds to $y_m(t_0) = y(t_0)$, as we would expect. We see that we are summing a series of side-by-side integrals so that in the end, we integrate over all of $[t_0, t_0 + T]$. We see that the characteristic function restricts us to integrating over the ODE only up until t, as we would want for an approximation $y_{m_k}(t)$ at t using Euler's method. And we see that since $f(t_\alpha, y_{m_k}(t_\alpha))$ is constant and $h_m = t_{\alpha+1} - t_\alpha$, the integral does take on the expected value $f(t_\alpha, y_m(t_\alpha))h_m$. Moving right along, we see that

$$\begin{aligned}
\left| y_{m_k}(t) - y_0 - \int_{t_0}^{t} f(\tau, y_{m_k}(\tau)) \mathrm{d}\tau \right| &\leq \sum_{\alpha=0}^{m-1} \int_{t_\alpha}^{t_{\alpha+1}} \chi_t(\tau) |f(t_\alpha, y_{m_k}(t_\alpha)) - f(\tau, y_{m_k}(\tau))| \mathrm{d}\tau \\
&< \int_{t_0}^{t_0+T} \chi_t(\tau) \cdot \frac{\varepsilon}{T} \mathrm{d}\tau \\
&= \int_{t_0}^{t} \frac{\varepsilon}{T} \mathrm{d}\tau \\
&= \varepsilon \cdot \frac{t - t_0}{T} \\
&\leq \varepsilon
\end{aligned}$$

Thus, by uniform convergence, $\int_{t_0}^{t} f(\tau, y_{m_k}(\tau)) \mathrm{d}\tau \to \int_{t_0}^{t} f(\tau, y(\tau)) \mathrm{d}\tau$ uniformly, so y does satisfy the integral equation, as desired. $\square$

- **Characteristic function** (of $[a, b]$): The function defined as follows. *Denoted by* $\boldsymbol{\chi_{[a,b]}}$. *Given by*

$$\chi_{[a,b]}(t) = \begin{cases} 1 & x \in [a, b] \\ 0 & x \notin [a, b] \end{cases}$$

- Utility of the Peano Existence Theorem: Proves the *existence* of a solution, but the proof is not constructive; it does not give an algorithm for finding the desired sequence. Nor does the PET make any statement on uniqueness.
- We now look to use a related method to define a sequence of functions that will converge to the desired solution of the ODE.
  - While the PET does not require it, in practice, most f we would be interested in will satisfy an additional Lipschitz condition.
  - Define the integral operator

$$\Phi[u] = y_0 + \int_{t_0}^{t} f(\tau, u(\tau)) \mathrm{d}\tau$$

  We will prove that Φ is a contraction on the function space. This will imply that $\Phi^N[u]$ converges across the entire interval $[t_0, t_0 + T]$ to the solution y for any $u : [t_0, t_0 + T] \to \bar{B}(y_0, b)$, giving us our desired computational strategy. Let's begin.

[5]Theorem 13.6 of Honors Calculus IBL.

- To prove that Φ is a contraction, it will suffice to show that $\left\|\Phi^j[u_1] - \Phi^j[u_2]\right\| \to 0$ as $j \to \infty$. Thus, we wish to put a bound on $\left\|\Phi^j[u_1] - \Phi^j[u_2]\right\|$ that decreases as j increases. To that end, we will prove that

$$\left\|\Phi^j[u_1] - \Phi^j[u_2]\right\| \leq \frac{(LT)^j}{j!} \cdot \|u_1 - u_2\|$$

for all j.

  - We induct on j. For the base case $j = 1$, we have that

$$\begin{aligned}
|\Phi[u_1](t) - \Phi[u_2](t)| &\leq \int_{t_0}^t L|u_1(\tau) - u_2(\tau)|\mathrm{d}\tau \\
&\leq L(t - t_0)\|u_1 - u_2\| \\
&\leq LT\|u_1 - u_2\| \\
&= \frac{(LT)^1}{1!} \cdot \|u_1 - u_2\|
\end{aligned}$$

  for all t.

  - Now suppose inductively that $\left\|\Phi^j[u_1] - \Phi^j[u_2]\right\| \leq (LT)^j/j! \cdot \|u_1 - u_2\|$. Then we have that

$$\begin{aligned}
|\Phi^{j+1}[u_1](t) - \Phi^{j+1}[u_2](t)| &\leq \int_{t_0}^t L|\Phi^j[u_1](\tau) - \Phi^j[u_2](\tau)|\mathrm{d}\tau \\
&\leq \int_{t_0}^t L \cdot \frac{(LT)^j}{j!} \cdot \|u_1 - u_2\|\mathrm{d}\tau \\
&= \cdots \\
&\leq \frac{(LT)^{j+1}}{j!} \cdot \|u_1 - u_2\|
\end{aligned}$$

  for all t, implying the desired result.

- We now estimate the error between y_m and y in terms of y_m, alone. Indeed, we have from the above that

$$\begin{aligned}
\left\|y_m - \Phi^N[y_m]\right\| &\leq \sum_{j=0}^{N-1} \left\|\Phi^j[y_m] - \Phi^{j+1}[y_m]\right\| \\
&\leq \|y_m - \Phi[y_m]\| \sum_{j=0}^{N-1} \frac{(TL)^j}{j!} \\
\|y_m - y\| &\leq \|y_m - \Phi[y_m]\|\mathrm{e}^{TL}
\end{aligned}$$

where we get from the second to the third line by letting $N \to \infty$.

  - The proof of the PET guarantees that $\|y_m - \Phi[y_m]\|$ is small when m is large, no matter whether y_m itself converges or not.
  - In fact, when $f \in C^1$, the error is estimated as

$$\|y_m - y\| \leq \frac{LT\mathrm{e}^{TL}}{m}$$

  for $L = \|f\|_{C^1}$.

- Takeaway: This polygon method gives rise to an algorithm to solve ODEs. Theoretically, it converges much slower than the Picard iteration, but in practice, it has the advantage that we do not need to do any numerical integration. Indeed, to obtain the desired precision using the Picard iteration, the numerical integration will need more and more steps and the total accumulated error will not be less than this polygon method.
- Better difference methods include Runge-Kutta or Heun, but please refer to monographs on numerical ODEs for these.

7.2 Asymptotic Stability

11/9:
- Going forward, we restrict ourselves to autonomous ODEs $y' = f(y)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field.
- For every $x \in \mathbb{R}^n$, the IVP
$$y' = f(y), \quad y(0) = x$$
has a unique maximal solution $\phi_t(x)$ for $t \in I_x$.
- **Orbit**: The following set. *Given by*
$$\{\phi_t(x) : t \in I_x\}$$
- Let $K \subset \mathbb{R}^n$ be compact.
  - Then there exists $T_K \in \mathbb{R}$ such that $\phi_t(x)$ is defined for all $x \in K$ and $|t| \leq T_K$.
  - Moreover, the map from $K \to \mathbb{R}^n$ defined by $x \mapsto \phi_t(x)$ is injective due to uniqueness (and therefore a **homeomorphism**). We get one such map for each t.
  - Similar to the diffeomorphism idea from Guillemin and Haine (2018).
- **Invariant set**: A subset of $\mathbb{R}^n$ such that any orbit starting within it never leaves it.
- Compact invariant sets are quite interesting.
- Proposition: Let $\Omega \subset \mathbb{R}^n$ be a domain with a piecewise smooth boundary $\partial\Omega$. Suppose $f(x)$ is transversal to $\partial\Omega$ and inward pointing: That is, if ν is the inward pointing unit normal, then $f(x) \cdot \nu(x) \geq 0$ for all $x \in \partial\Omega$. Then $\bar{\Omega}$ is an invariant set: That is, any orbit starting from a point $\bar{\Omega}$ exists throughout the time and never leaves $\bar{\Omega}$.

  Proof idea. $x \in \partial\Omega$ ensures that $\phi_t(x)$ must be in Ω for small t. Hence, it suffices to consider $x \in \Omega$. In that case, pick the smallest $T > 0$ such that $\phi_T(x) \in \partial\Omega$. Then by transversality it must turn back into Ω. □
- This simple proposition is especially useful when establishing global attraction of the orbits.
- **Fixed point**: A point in $\mathbb{R}^n$ at which f evaluates to zero. *Denoted by* $\boldsymbol{x_0}$.
  - This means that the vector at x_0 is zero.
- **Lyapunov stable** (fixed point): A fixed point x_0 such that for any neighborhood $B(x_0, \varepsilon)$, there exists a neighborhood $B(x_0, \delta)$ such that $\phi_t(x) \in B(x_0, \varepsilon)$ for any $t \geq 0$ and $x \in B(x_0, \delta)$.
- **Asymptotically stable** (fixed point): A Lyapunov stable fixed point x_0 such that $\phi_t(x) \to x_0$ as $t \to +\infty$ for $x \in B(x_0, \delta)$.
- Example of a system that is Lyapunov stable but not asymptotically stable: The system
$$y' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y$$
where A denotes a rotation.
  - The orbits are concentric circles and never converge to 0.
- Investigation: The local behavior near a fixed point.
  - Consider $y' = f(y)$ as a perturbation of the linearized system $y' = f'(x_0)y$. In this case,
$$f(x) = f'(x_0)(x - x_0) + O(|x - x_0|^2)$$
as $x \to x_0$.

- Theorem: Let $f(x_0) = 0$. If the eigenvalues of the linearization $A = f'(x_0)$ all have negative real parts, then the fixed point $x = x_0$ is asymptotically stable.

  Proof. WLOG let $x_0 = 0$. Write $f(x) = Ax + g(x)$, where $g(x) = O(|x|^2)$. Since every $\lambda \in \sigma(A)$ has negative real part, there exist $a, C > 0$ (let $C > 1$ WLOG) such that

  $$|\mathrm{e}^{tA}x| \leq C\mathrm{e}^{-at}|x|$$

  The C arises because the matrix norm of e^{tA} is bounded as $t \to +\infty$ if all eigenvalues are negative. The e^{-at} arises similarly, and reflects the exponential decrease in magnitude happening along all subspaces on which e^{tA} acts.

  Let δ be such that $|g(x)| \leq a|x|/2C$ when $|x| \leq \delta$. Now consider the IVP

  $$y' = Ay + g(y), \quad y(0) \in \bar{B}\left(0, \frac{\delta}{2C}\right)$$

  Then at least for small t (i.e., t such that $|y(t)| \leq \delta$),

  $$|y(t)| \leq C\mathrm{e}^{-at}|y(0)| + \frac{a}{2C}\int_0^t \mathrm{e}^{-a(t-\tau)}|y(\tau)|\mathrm{d}\tau$$

  It follows from Grönwall's inequality that

  $$\mathrm{e}^{at}|y(t)| \leq C|y(0)|\mathrm{e}^{at/2}$$

  hence

  $$|y(t)| \leq \frac{\delta}{2}\mathrm{e}^{-at/2} < \delta$$

  Hence, any orbit of the system starting from $\bar{B}(0, \delta/2C)$ stays in $\bar{B}(0, \delta)$. So the maximal time of existence T is $+\infty$. This is because if not then, then the IVP starting from $y(T)$ is still solvable, contradicting the definition of T. Thus, we have proven that

  $$|y(t)| \leq \frac{\delta}{2}\mathrm{e}^{-at/2}$$

  for all $t \geq 0$ as long as $|y(0)| \leq \delta/2C$. □

- This is the last rigorous proof given in this course.
- A similar theorem:
- Theorem: Let $f(0) = 0$. If one of the eigenvalues of $A = f'(0)$ has positive real part, then the fixed point $x = 0$ is not Lyapunov stable.
- Initial application: Nonlinear mechanical system with frictions, e.g., ideal pendulum with friction.

  $$ml\theta'' + b\theta' = -mg\sin\theta$$

  - Substitute $\eta = b/ml$ and $\omega = \theta'$ to get a nonlinear system

    $$\begin{pmatrix}\theta\\ \omega\end{pmatrix}' = \begin{pmatrix}\omega\\ -\eta\omega - g/l\sin\theta\end{pmatrix}$$

  - At the equilibrium position $(\theta, \omega) = (0, 0)$, we have

    $$A = \begin{pmatrix}\frac{\partial}{\partial\theta}(\omega) & \frac{\partial}{\partial\omega}(\omega)\\ \frac{\partial}{\partial\theta}(-\eta\omega - g/l\sin\theta) & \frac{\partial}{\partial\omega}(-\eta\omega - g/l\sin\theta)\end{pmatrix} \approx \begin{pmatrix}0 & 1\\ -g/l & -\eta\end{pmatrix}$$

    i.e.,

    $$\begin{pmatrix}\theta\\ \omega\end{pmatrix}' = \begin{pmatrix}0 & 1\\ -g/l & -\eta\end{pmatrix}\begin{pmatrix}\theta\\ \omega\end{pmatrix} + O(|\theta|^2 + |\omega|^2)$$

    - Since $\eta > 0$, the eigenvalues have a common negative real part, so the equilibrium is asymptotically stable.
  - At the equilibrium $(\pi, 0)$, we have

$$\begin{pmatrix} \theta \\ \omega \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ g/l & -\eta \end{pmatrix} \begin{pmatrix} \theta - \pi \\ \omega \end{pmatrix} + O(|\theta - \pi|^2 + |\omega|^2)$$

    - For $\eta \geq 0$, there is one positive and one negative eigenvalue, so this equilibrium is unstable.
  - These results should make intuitive sense: If a pendulum is resting at the bottom, that is a stable equilibrium. If a pendulum is resting at the top, that is not a stable equilibrium.

7.3 Applications of the Lyapunov Method

11/11:

- Purely imaginary eigenvalues can still lead to Lyapunov stability.
- **Lyapunov function** (of a system $y' = f(y)$ with fixed point x_0 near x_0): A continuous real function on $\mathbb{R}^n$ such that the following two axioms hold. *Denoted by* $\boldsymbol{L}$.
  1. $L(x_0) = 0$ and $L(x) > 0$ for all $x \in \mathring{B}(x_0, \delta) = B(x_0, \delta) \setminus \{x_0\}$.
  2. $\dot{L}(x) = \nabla L(x) \cdot f(x) \leq 0$ for all $x \in \mathring{B}(x_0, \delta) = B(x_0, \delta) \setminus \{x_0\}$.
- Since

$$\frac{\mathrm{d}}{\mathrm{d}t} L(\phi_t(x)) = \nabla L(\phi_t(x)) \cdot f(\phi_t(x))$$

  the second condition is equivalent to saying that the function L is decreasing along the orbits starting near x_0.
- **Strict** (Lyapunov function): A Lyapunov function for which the decreasing is strict.
- Theorem: For the autonomous system $y' = f(y)$, a fixed point x_0 is
  1. Stable if there is a Lyapunov function near it;

     Proof. Pick a small number $\delta > 0$. Let[6]

$$m := \min\{L(x) : |x - x_0| = \delta\}$$

     Since x_0 does not satisfy $|x - x_0| = \delta > 0$, we know from the first constraint on Lyapunov functions that $L(x) > 0$ for all x satisfying said relation. Thus, $m > 0$. Consequently, any orbit starting from $\{x \mid L(x) < m\} \cap B(x_0, \delta)$ can never meet $\partial B(x_0, \delta)$ since $L(x)$ is decreasing along any orbit (and we would have to go up to get to the boundary). So $L(\phi_t(x)) < m$ for all $x \in \{x \mid L(x) < m\} \cap B(x_0, \delta)$. But this means that $\{x \mid L(x) < m\} \cap B(x_0, \delta)$ is in fact an invariant set. Therefore, x_0 is Lyapunov stable. □

  2. Asymptotically stable if there is a strict Lyapunov function near it.

     Proof. If $x \in \{x \mid L(x) < m\} \cap B(x_0, \delta)$, then $L(\phi_t(x))$ is strictly decreasing. As $t \to +\infty$, $\phi_t(x)$ has a partial limit z_0, say $\phi_{t_k}(x) \to z_0$ (Lemma 6.6 of Teschl (2012)). If $z_0 \neq x_0$, then the orbit $\{\phi_t(z_0) \mid t \in I_{z_0}\}$ is not a single point: Since L is a strict Lyapunov function, we have $L(\phi_t(z_0)) < L(z_0)$ for all $t > 0$. When k is large, $\phi_{t_k}(x)$ is close to z_0, so by continuity,

$$L(\phi_{t+t_k}(x)) = L(\phi_t(\phi_{t_k}(x))) < L(z_0)$$

     But this contradicts $L(\phi_t(x)) > L(z_0)$ (which we must have if there are arbitrarily large t such that $\phi_t(x)$ is close to z_0). Therefore, $x_0 = z_0$. □

[6]Intuitively (in 2D), we take a ring around x_0, find the nonzero value of $L(x)$ at each point on the ring, and take the minimum among them. Imagine a circular valley with hills rising all around the bottommost point; we are essentially looking for the hill that rises the least.

- If all eigenvalues of A have negative real parts, then the perturbed system

$$y' = Ay + g(y)$$

  has a strict Lyapunov function around the fixed point $x = 0$.

  - This observation yields another proof of the stability theorem.

- Advantage of the Lyapunov function: Can be constructed globally and thus gives us global information on the system.

- Examples in studying the global behavior of a phase portrait:

  - Consider a mass point moving along the real axis in a potential field $U(x)$. Then

$$mx'' = -U'(x)$$

    - The total energy

$$E = \frac{m}{2}|x'|^2 + U(x)$$

      is always a constant along any solution.
    - Introducing the velocity allows us to obtain a planar system

$$\begin{pmatrix} x \\ v \end{pmatrix}' = \begin{pmatrix} v \\ -U'(x)/m \end{pmatrix}$$

    - Thus, $E(x, v)$ is a global Lyapunov function.
    - Any fixed point of the system must be of the form $(x_0, 0)$, where $U'(x_0) = 0$.
      - Intuitively, this means that the velocity must be zero (that makes sense) and the position must be such that we are at a critical point of the potential.
    - Because of this, the linearization at a fixed point must be of the following form.

$$\begin{pmatrix} v \\ -U'(x)/m \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -U''(x_0)/m & 0 \end{pmatrix} \begin{pmatrix} x - x_0 \\ v \end{pmatrix} + O(|x - x_0|^2 + |v|^2)$$

    - Thus, $(x_0, 0)$ is Lyapunov stable if U has a nondegenerate local minimum at x_0 and unstable if U has a nondegenerate local maximum at x_0.
      - In the former case, the orbits near $(x_0, 0)$ are closed curves, corresponding to periodic oscillations near x_0 (e.g., harmonic oscillator and ideal pendulum again).
  - Prey-predator model with capacity:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} (1 - y - \lambda x)x \\ \alpha(x - 1 - \mu y)y \end{pmatrix}$$

    $\alpha, \lambda, \mu > 0$.
    - x is the number of rabbits and y is the number of wolves.
    - Different ranges of λ induce different global behavior (thus, this is an example of **bifurcation**).
    - General observation: $(x, y) = (0, 0)$ is a saddle point since the linearization there is $\text{diag}(1, -\alpha)$.
    - For $x = 0$ or $y = 0$, the equation is of separable form; the positive x, y-axes are invariant sets.
      - Implication: No orbit in the first quadrant can escape it (compatible with meaning as population).
    - Jacobian:

$$\begin{pmatrix} 1 - y - 2\lambda x & -x \\ \alpha y & \alpha(x - 1) - 2\alpha\mu y \end{pmatrix}$$

- When $\lambda, \mu = 0$, we're back to the Lotka-Volterra system, where there is a single fixed point $(1, 1)$.
  - In that case,
  $$(y - \log y - 1) + \alpha(x - \log x - 1)$$
  is a Lyapunov function.
  - However, it is not a strict Lyapunov function since it is constant along any orbit.
  - Moreover, the function is convex, so all level sets are closed curves around the fixed point.
  - This is, indeed, the behavior we observe in Figure 2.1.
- Other cases: $\lambda \geq 1$.
  - There is only one additional fixed point of interest: $(1/\lambda, 0)$. Note that there are other fixed points, but these do not lie in the first quadrant and thus we are not interested.
  - For $\lambda > 1$, the fixed point is stable (a sink) and when $\lambda = 1$, one eigenvalue is 0 since the linearization at that point is $\operatorname{diag}(-1, \alpha(1/\lambda - 1))$.
- $0 < \lambda < 1$.
  - $(1/\lambda, 0)$ becomes a saddle point, and there is a third fixed point
  $$(x_0, y_0) = \left(\frac{1+\mu}{1+\mu\lambda}, \frac{1-\lambda}{1+\mu\lambda}\right)$$
- More on this case in Chapter 7 of Teschl (2012). This is relevant here!