

# Week 7

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## 7.1 Peano Existence Theorem

11/7:

- Today: Peano Existence Theorem.
- For an IVP of a first-order differential system, as long as the RHS is continuous, we get at least one solution.
- The proof provides an algorithm that can be really useful in computing the solution provided that uniqueness exists.
- We will need a theorem from analysis to start.
- Theorem (Arzelà-Ascoli<sup>[1]</sup>): Let  $h_k : [a, b] \rightarrow \mathbb{R}^n$  be a sequence of functions that is uniformly bounded and uniformly Lipschitz continuous wrt.  $L$ . Then  $\{h_k\}$  contains a uniformly convergent subsequence and the limit has the same bound and Lipschitz constant.

*Proof.* Recall the property of sequential compactness<sup>[2]</sup>, i.e., that every bounded sequence of numbers contains a convergent subsequence. We want to prove this for a sequence of functions. To do so, we will need the Cantor diagonalization technique.

$\mathbb{Q}$  is countable. Thus, we can enumerate the rationals in  $[a, b]$  by  $r_1, r_2, r_3, \dots$ . Since  $\{h_k(r_1)\}$  is a bounded sequence of numbers, we have by the above that there is a subsequence  $C_1$  — say  $h_1^{(1)}, h_2^{(1)}, h_3^{(1)}, \dots$  — such that  $C_1 = \{h_k^{(1)}(r_1)\}$  is a convergent subsequence in  $\mathbb{R}^n$  of the original sequence. Now  $C_1$  is still a bounded sequence, so we can obtain a subsequence  $C_2$  of it — say  $h_1^{(2)}, h_2^{(2)}, h_3^{(2)}, \dots$  — such that  $C_2 = \{h_k^{(2)}(r_2)\}$  is a convergent subsequence in  $\mathbb{R}^n$  at  $r_2$  (and, by inductive hypothesis, at  $r_1$ !). Inductively, we can obtain  $C_\ell = \{h_k^{(\ell)}\}_{k=1}^\infty$  convergent at  $r_1, r_2, \dots, r_\ell$ . We then write down the elements of the sequences as a table. (For example, the  $k^{\text{th}}$  row of the table is a sequence that converges at  $r_1, \dots, r_k$ .)

$$\begin{array}{cccc} h_1^{(1)} & h_2^{(1)} & h_3^{(1)} & \cdots \\ h_1^{(2)} & h_2^{(2)} & h_3^{(2)} & \cdots \\ h_1^{(3)} & h_2^{(3)} & h_3^{(3)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

Consider the diagonal sequence  $\{f_\ell\}_{\ell=1}^\infty$  where  $f_\ell = h_\ell^{(\ell)}$ . By definition, it converges at all rational points. We now seek to prove that it converges uniformly at *all* points.

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<sup>1</sup>This is not the full Arzelà-Ascoli theorem, but a special case. The proof is similar, regardless, though. See Honors Analysis in  $\mathbb{R}^n$  I Notes.

<sup>2</sup>The Bolzano-Weierstrass Theorem/Theorem 15.18 from Honors Calculus IBL.

To prove that  $\{f_\ell\}$  is a uniformly convergent sequence of functions, it will suffice to show that for all  $\varepsilon > 0$ , there exists  $N$  such that if  $k, \ell > N$ , then  $|f_k(t) - f_\ell(t)| < \varepsilon$  for all  $t \in [a, b]$ . Let  $\varepsilon > 0$  be arbitrary. Divide  $[a, b]$  into  $m$  congruent subintervals  $I_\alpha$  ( $\alpha = 1, \dots, m$ ) such that  $|I_\alpha| \leq \varepsilon/3L$  for all  $\alpha$ . This guarantees that the oscillation of each  $f_k$  on any  $I_\alpha$  is  $\leq \varepsilon/3$  since if  $x, y \in I_\alpha$  for some  $\alpha$ , then

$$|f_\ell(x) - f_\ell(y)| \leq L|x - y| \leq L \cdot \frac{\varepsilon}{3L} = \frac{\varepsilon}{3}$$

Using the fact that  $\{f_\ell\}$  is convergent and hence Cauchy on the rationals, pick  $N$  large enough so that  $r_\alpha \in I_\alpha$  implies  $|f_k(r_\alpha) - f_\ell(r_\alpha)| < \varepsilon/3$  for  $k, \ell > N$ . We will choose this  $N$  to be our  $N$ . Now let  $t \in [a, b]$  be arbitrary. By their definition, we know  $t \in I_\alpha$  for some  $\alpha$ . Therefore,

$$\begin{aligned} |f_k(t) - f_\ell(t)| &\leq |f_k(t) - f_k(r_\alpha)| + |f_k(r_\alpha) - f_\ell(r_\alpha)| + |f_\ell(r_\alpha) - f_\ell(t)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

as desired.

Lastly, we can prove that the limit function  $f$  of  $\{f_\ell\}$  is  $L$ -Lipschitz as follows. Let  $t, t' \in [a, b]$  be arbitrary. Then

$$\left| \frac{f(t) - f(t')}{t - t'} \right| = \lim_{k \rightarrow \infty} \left| \frac{f_k(t) - f_k(t')}{t - t'} \right| \leq \lim_{k \rightarrow \infty} \left| \frac{L|t - t'|}{t - t'} \right| = \lim_{k \rightarrow \infty} |L| = L$$

as desired. □

- Now we come to the proof of the Peano Existence Theorem.
- Theorem (Peano Existence Theorem): Let  $f : [t_0, t_0 + a] \times \bar{B}(y_0, b) \rightarrow \mathbb{R}^n$  be bounded ( $|f(t, z)| \leq M$ ) and continuous. Then the IVP

$$y'(t) = f(t, y(t)), \quad y(t_0) = b$$

has at least one solution for  $t \in [t_0, t_0 + T]$  where  $T = \min(a, b/M)$ .

*Proof.* Since there is no Lipschitz condition, we use another strategy to find approximate solutions.

*picture* Fix  $T = \min(a, b/M)$ . We divide  $[t_0, t_0 + T]$  into  $m$  congruent closed subintervals  $I_\alpha$  ( $\alpha = 0, \dots, m-1$ ), each of length  $h_m = T/m$ . Define a continuous function  $y_m(t)$  as follows: The values at the nodes  $t_\alpha$  (the intersection points of adjacent congruent subintervals) are defined inductively via

$$y_m(t_{\alpha+1}) = y_m(t_\alpha) + f(t_\alpha, y_m(t_\alpha))h_m$$

for  $\alpha = 0, \dots, m-1$ , and  $y_m$  is taken to be linear between the nodes<sup>[3]</sup>. The idea is that we replace the derivative  $y'(t)$  by the difference quotient  $[y(t+h) - y(t)]/h$ . It follows by the construction that every function in the set  $\{y_k(t) : [t_0, t_0 + T] \rightarrow \bar{B}(y_0, b)\}$  is piecewise linear (hence continuous), uniformly bounded, and uniformly  $M$ -Lipschitz continuous. Therefore, by the Arzelà-Ascoli theorem,  $\{y_k\}$  contains a uniformly convergent subsequence  $y_{m_k} \rightarrow y$ .

It remains to verify that  $y$  is a solution to the integral equation

$$y(t) = y_0 + \int_{t_0}^t f(\tau, y(\tau)) d\tau$$

Observe that the domain of  $f$  is a closed and bounded subset of the real numbers. Thus, it is compact by the Heine-Borel theorem<sup>[4]</sup>. Moreover, since  $f$  is a continuous function on a compact domain, we

<sup>3</sup>Note that this construction is quite similar to that employed in Euler's method.

<sup>4</sup>Theorem 10.16 of Honors Calculus IBL.

have by the Heine-Cantor theorem<sup>[5]</sup> that  $f$  is uniformly continuous. Thus, for any  $\varepsilon > 0$ , there exists  $N$  such that if  $m > N$ , then

$$|f(t, y_m(t)) - f(t_\alpha, y_m(t_\alpha))| < \frac{\varepsilon}{T}$$

for all  $\alpha = 0, \dots, m-1$  and  $t \in I_\alpha$ . Additionally, observe that

$$y_{m_k}(t) = y_0 + \sum_{\alpha=0}^{m-1} \int_{t_\alpha}^{t_{\alpha+1}} \chi_t(\tau) f(t_\alpha, y_{m_k}(t_\alpha)) d\tau$$

where  $\chi_t(\tau)$  denotes the **characteristic function** of  $[t_0, t]$ . To see this, compare with the original inductive definition of  $y_m(t_{\alpha+1})$ . *picture* We thus see that  $y_0$  in the above equation corresponds to  $y_m(t_0) = y(t_0)$ , as we would expect. We see that we are summing a series of side-by-side integrals so that in the end, we integrate over all of  $[t_0, t_0 + T]$ . We see that the characteristic function restricts us to integrating over the ODE only up until  $t$ , as we would want for an approximation  $y_{m_k}(t)$  at  $t$  using Euler's method. And we see that since  $f(t_\alpha, y_{m_k}(t_\alpha))$  is constant and  $h_m = t_{\alpha+1} - t_\alpha$ , the integral does take on the expected value  $f(t_\alpha, y_m(t_\alpha))h_m$ . Moving right along, we see that

$$\begin{aligned} \left| y_{m_k}(t) - y_0 - \int_{t_0}^t f(\tau, y_{m_k}(\tau)) d\tau \right| &\leq \sum_{\alpha=0}^{m-1} \int_{t_\alpha}^{t_{\alpha+1}} \chi_t(\tau) |f(t_\alpha, y_{m_k}(t_\alpha)) - f(\tau, y_{m_k}(\tau))| d\tau \\ &< \int_{t_0}^{t_0+T} \chi_t(\tau) \cdot \frac{\varepsilon}{T} d\tau \\ &= \int_{t_0}^t \frac{\varepsilon}{T} d\tau \\ &= \varepsilon \cdot \frac{t - t_0}{T} \\ &\leq \varepsilon \end{aligned}$$

Thus, by uniform convergence,  $\int_{t_0}^t f(\tau, y_{m_k}(\tau)) d\tau \rightarrow \int_{t_0}^t f(\tau, y(\tau)) d\tau$  uniformly, so  $y$  does satisfy the integral equation, as desired.  $\square$

- **Characteristic function** (of  $[a, b]$ ): The function defined as follows. *Denoted by  $\chi_{[a,b]}$ . Given by*

$$\chi_{[a,b]}(t) = \begin{cases} 1 & x \in [a, b] \\ 0 & x \notin [a, b] \end{cases}$$

- Utility of the Peano Existence Theorem: Proves the *existence* of a solution, but the proof is not constructive; it does not give an algorithm for finding the desired sequence. Nor does the PET make any statement on uniqueness.
- We now look to use a related method to define a sequence of functions that will converge to the desired solution of the ODE.
  - While the PET does not require it, in practice, most  $f$  we would be interested in will satisfy an additional Lipschitz condition.
  - Define the integral operator

$$\Phi[u] = y_0 + \int_{t_0}^t f(\tau, u(\tau)) d\tau$$

We will prove that  $\Phi$  is a contraction on the function space. This will imply that  $\Phi^N[u]$  converges across the entire interval  $[t_0, t_0 + T]$  to the solution  $y$  for any  $u : [t_0, t_0 + T] \rightarrow \bar{B}(y_0, b)$ , giving us our desired computational strategy. Let's begin.

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<sup>5</sup>Theorem 13.6 of Honors Calculus IBL.

- To prove that  $\Phi$  is a contraction, it will suffice to show that  $\|\Phi^j[u_1] - \Phi^j[u_2]\| \rightarrow 0$  as  $j \rightarrow \infty$ . Thus, we wish to put a bound on  $\|\Phi^j[u_1] - \Phi^j[u_2]\|$  that decreases as  $j$  increases. To that end, we will prove that

$$\|\Phi^j[u_1] - \Phi^j[u_2]\| \leq \frac{(LT)^j}{j!} \cdot \|u_1 - u_2\|$$

for all  $j$ .

- We induct on  $j$ . For the base case  $j = 1$ , we have that

$$\begin{aligned} |\Phi[u_1](t) - \Phi[u_2](t)| &\leq \int_{t_0}^t L|u_1(\tau) - u_2(\tau)|d\tau \\ &\leq L(t - t_0)\|u_1 - u_2\| \\ &\leq LT\|u_1 - u_2\| \\ &= \frac{(LT)^1}{1!} \cdot \|u_1 - u_2\| \end{aligned}$$

for all  $t$ .

- Now suppose inductively that  $\|\Phi^j[u_1] - \Phi^j[u_2]\| \leq (LT)^j/j! \cdot \|u_1 - u_2\|$ . Then we have that

$$\begin{aligned} |\Phi^{j+1}[u_1](t) - \Phi^{j+1}[u_2](t)| &\leq \int_{t_0}^t L|\Phi^j[u_1](\tau) - \Phi^j[u_2](\tau)|d\tau \\ &\leq \int_{t_0}^t L \cdot \frac{(LT)^j}{j!} \cdot \|u_1 - u_2\|d\tau \\ &= \dots \\ &\leq \frac{(LT)^{j+1}}{j!} \cdot \|u_1 - u_2\| \end{aligned}$$

for all  $t$ , implying the desired result.

- We now estimate the error between  $y_m$  and  $y$  in terms of  $y_m$ , alone. Indeed, we have from the above that

$$\begin{aligned} \|y_m - \Phi^N[y_m]\| &\leq \sum_{j=0}^{N-1} \|\Phi^j[y_m] - \Phi^{j+1}[y_m]\| \\ &\leq \|y_m - \Phi[y_m]\| \sum_{j=0}^{N-1} \frac{(TL)^j}{j!} \\ \|y_m - y\| &\leq \|y_m - \Phi[y_m]\|e^{TL} \end{aligned}$$

where we get from the second to the third line by letting  $N \rightarrow \infty$ .

- The proof of the PET guarantees that  $\|y_m - \Phi[y_m]\|$  is small when  $m$  is large, no matter whether  $y_m$  itself converges or not.
- In fact, when  $f \in C^1$ , the error is estimated as

$$\|y_m - y\| \leq \frac{LT e^{TL}}{m}$$

for  $L = \|f\|_{C^1}$ .

- Takeaway: This polygon method gives rise to an algorithm to solve ODEs. Theoretically, it converges much slower than the Picard iteration, but in practice, it has the advantage that we do not need to do any numerical integration. Indeed, to obtain the desired precision using the Picard iteration, the numerical integration will need more and more steps and the total accumulated error will not be less than this polygon method.
- Better difference methods include Runge-Kutta or Heun, but please refer to monographs on numerical ODEs for these.

## 7.2 Asymptotic Stability

11/9:

- Going forward, we restrict ourselves to autonomous ODEs  $y' = f(y)$ , where  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a smooth vector field.

- For every  $x \in \mathbb{R}^n$ , the IVP

$$y' = f(y), \quad y(0) = x$$

has a unique maximal solution  $\phi_t(x)$  for  $t \in I_x$ .

- **Orbit:** The following set. *Given by*

$$\{\phi_t(x) : t \in I_x\}$$

- Let  $K \subset \mathbb{R}^n$  be compact.

- Then there exists  $T_K \in \mathbb{R}$  such that  $\phi_t(x)$  is defined for all  $x \in K$  and  $|t| \leq T_K$ .
- Moreover, the map from  $K \rightarrow \mathbb{R}^n$  defined by  $x \mapsto \phi_t(x)$  is injective due to uniqueness (and therefore a **homeomorphism**). We get one such map for each  $t$ .
- Similar to the diffeomorphism idea from *Differential Forms*.

- **Invariant set:** A subset of  $\mathbb{R}^n$  such that any orbit starting within it never leaves it.

- Compact invariant sets are quite interesting.

- Proposition: Let  $\Omega \subset \mathbb{R}^n$  be a domain with a piecewise smooth boundary  $\partial\Omega$ . Suppose  $f(x)$  is transversal to  $\partial\Omega$  and inward pointing: That is, if  $\nu$  is the inward pointing unit normal, then  $f(x) \cdot \nu(x) \geq 0$  for all  $x \in \partial\Omega$ . Then  $\bar{\Omega}$  is an invariant set: That is, any orbit starting from a point  $\bar{\Omega}$  exists throughout the time and never leaves  $\bar{\Omega}$ .

*Proof idea.*  $x \in \partial\Omega$  ensures that  $\phi_t(x)$  must be in  $\Omega$  for small  $t$ . Hence, it suffices to consider  $x \in \Omega$ . In that case, pick the smallest  $T > 0$  such that  $\phi_T(x) \in \partial\Omega$ . Then by transversality it must turn back into  $\Omega$ .  $\square$

- This simple proposition is especially useful when establishing global attraction of the orbits.

- **Fixed point:** A point in  $\mathbb{R}^n$  at which  $f$  evaluates to zero. *Denoted by*  $x_0$ .

- This means that the vector at  $x_0$  is zero.

- **Lyapunov stable** (fixed point): A fixed point  $x_0$  such that for any neighborhood  $B(x_0, \varepsilon)$ , there exists a neighborhood  $B(x_0, \delta)$  such that  $\phi_t(x) \in B(x_0, \varepsilon)$  for any  $t \geq 0$  and  $x \in B(x_0, \delta)$ .

- **Asymptotically stable** (fixed point): A Lyapunov stable fixed point  $x_0$  such that  $\phi_t(x) \rightarrow x_0$  as  $t \rightarrow +\infty$  for  $x \in B(x_0, \delta)$ .

- Example of a system that is Lyapunov stable but not asymptotically stable: The system

$$y' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y$$

where  $A$  denotes a rotation.

- The orbits are concentric circles and never converge to 0.

- Investigation: The local behavior near a fixed point.

- Consider  $y' = f(y)$  as a perturbation of the linearized system  $y' = f'(x_0)y$ . In this case,

$$f(x) = f'(x_0)(x - x_0) + O(|x - x_0|^2)$$

as  $x \rightarrow x_0$ .

- Theorem: Let  $f(x_0) = 0$ . If the eigenvalues of the linearization  $A = f'(x_0)$  all have negative real parts, then the fixed point  $x = x_0$  is asymptotically stable.

*Proof.* WLOG let  $x_0 = 0$ . Write  $f(x) = Ax + g(x)$ , where  $g(x) = O(|x|^2)$ . Since every  $\lambda \in \sigma(A)$  has negative real part, there exist  $a, C > 0$  (let  $C > 1$  WLOG) such that

$$|e^{tA}x| \leq Ce^{-at}|x|$$

The  $C$  arises because the matrix norm of  $e^{tA}$  is bounded as  $t \rightarrow +\infty$  if all eigenvalues are negative. The  $e^{-at}$  arises similarly, and reflects the exponential decrease in magnitude happening along all subspaces on which  $e^{tA}$  acts.

Let  $\delta$  be such that  $|g(x)| \leq a|x|/2C$  when  $|x| \leq \delta$ . Now consider the IVP

$$y' = Ay + g(y), \quad y(0) \in \bar{B}\left(0, \frac{\delta}{2C}\right)$$

Then at least for small  $t$  (i.e.,  $t$  such that  $|y(t)| \leq \delta$ ),

$$|y(t)| \leq Ce^{-at}|y(0)| + \frac{a}{2C} \int_0^t e^{-a(t-\tau)} |y(\tau)| d\tau$$

It follows from Grönwall's inequality that

$$e^{at}|y(t)| \leq C|y(0)|e^{at/2}$$

hence

$$|y(t)| \leq \frac{\delta}{2} e^{-at/2} < \delta$$

Hence, any orbit of the system starting from  $\bar{B}(0, \delta/2C)$  stays in  $\bar{B}(0, \delta)$ . So the maximal time of existence  $T$  is  $+\infty$ . This is because if not then, then the IVP starting from  $y(T)$  is still solvable, contradicting the definition of  $T$ . Thus, we have proven that

$$|y(t)| \leq \frac{\delta}{2} e^{-at/2}$$

for all  $t \geq 0$  as long as  $|y(0)| \leq \delta/2C$ . □

- This is the last rigorous proof given in this course.
- A similar theorem:
- Theorem: Let  $f(0) = 0$ . If one of the eigenvalues of  $A = f'(0)$  has positive real part, then the fixed point  $x = 0$  is not Lyapunov stable.
- Initial application: Nonlinear mechanical system with frictions, e.g., ideal pendulum with friction.

$$ml\theta'' + b\theta' = -mg \sin \theta$$

– Substitute  $\eta = b/ml$  and  $\omega = \theta'$  to get a nonlinear system

$$\begin{pmatrix} \theta \\ \omega \end{pmatrix}' = \begin{pmatrix} \omega \\ -\eta\omega - g/l \sin \theta \end{pmatrix}$$

– At the equilibrium position  $(\theta, \omega) = (0, 0)$ , we have

$$A = \begin{pmatrix} \frac{\partial}{\partial \theta}(\omega) & \frac{\partial}{\partial \omega}(\omega) \\ \frac{\partial}{\partial \theta}(-\eta\omega - g/l \sin \theta) & \frac{\partial}{\partial \omega}(-\eta\omega - g/l \sin \theta) \end{pmatrix} \approx \begin{pmatrix} 0 & 1 \\ -g/l & -\eta \end{pmatrix}$$

i.e.,

$$\begin{pmatrix} \theta \\ \omega \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -g/l & -\eta \end{pmatrix} \begin{pmatrix} \theta \\ \omega \end{pmatrix} + O(|\theta|^2 + |\omega|^2)$$

- Since  $\eta > 0$ , the eigenvalues have a common negative real part, so the equilibrium is asymptotically stable.
- At the equilibrium  $(\pi, 0)$ , we have

$$\begin{pmatrix} \theta \\ \omega \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ g/l & -\eta \end{pmatrix} \begin{pmatrix} \theta - \pi \\ \omega \end{pmatrix} + O(|\theta - \pi|^2 + |\omega|^2)$$

- For  $\eta \geq 0$ , there is one positive and one negative eigenvalue, so this equilibrium is unstable.
- These results should make intuitive sense: If a pendulum is resting at the bottom, that is a stable equilibrium. If a pendulum is resting at the top, that is not a stable equilibrium.