MATH 27300 (Basic Theory of Ordinary Differential Equations)
Problem Sets

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Contents

1 IVP Examples and Physical Problems

Required Problems

10/12: **1.** Classify the following ordinary differential equations (systems) by indicating the order, if they are linear, and if they are autonomous.

(1) $y'(x) + y(x) = 0$.

Answer.

Order	Linear?	Autonomous?
1	Yes	Yes

□

(2) $y''(t) = t\sin(y(t))$.

Answer.

Order	Linear?	Autonomous?
2	No	No

□

(3) $x' = -y$, $y' = 2x$.

Answer.

Order	Linear?	Autonomous?
1	Yes	Yes

□

(4) $y'(t) = y(t)\sin(t) + \cos(y(t))$.

Answer.

Order	Linear?	Autonomous?
1	Yes	No

□

2. Transform the following differential equations to first-order systems.

(1) $y^{(3)} + 2y'' - y' + y = 0$.

Proof. Let

$$x = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix}$$

Then

$$x' = \begin{pmatrix} y' \\ y'' \\ y^{(3)} \end{pmatrix}$$

so, by comparing components between the above two vectors and then using the original linear equation to define the last entry (with substitutions), we obtain

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= -2x_3 + x_2 - x_1 \end{aligned}}$$

□

(2) $x'' - t\sin x' = x$.

Proof. In an analogous manner to the above, we can determine that

$$\boxed{\begin{aligned} y_1' &= y_2 \\ y_2' &= y_1 + t\sin y_2 \end{aligned}}$$

□

3. Solve the following differential equations with initial value $x(0) = x_0$. Also identify the set of x_0 for which these solutions are extendable to the whole of $t \geq 0$. When a solution cannot be extended to the whole of $t \geq 0$, determine its lifespan in terms of x_0.

Example: Solve $x' = x^2$ with $x(0) = x_0$. By separation of variables, the solution reads

$$\int_{x_0}^{x} \frac{\mathrm{d}w}{w^2} = \int_0^t \mathrm{d}\tau$$

where the integral on the left-hand side cannot pass through $w = 0$. The result is

$$-\frac{1}{x} + \frac{1}{x_0} = t \quad \Longleftrightarrow \quad x(t) = \frac{x_0}{1 - x_0 t}$$

When $x_0 \leq 0$, the solution exists throughout $t \geq 0$. When $x_0 > 0$, the solution only exists in $[0, 1/x_0)$.

(1) $x' = x\sin t$.

Proof. By separation of variables, the solution reads

$$\int_{x_0}^{x} \frac{\mathrm{d}w}{w} = \int_0^t \sin\tau \mathrm{d}\tau$$

The result is

$$\ln\frac{x}{x_0} = 1 - \cos t \quad \Longleftrightarrow \quad \boxed{x(t) = x_0 \mathrm{e}^{1-\cos t}}$$

The set of x_0 for which this solution is extendable to the whole of $t \geq 0$ is $\boxed{\mathbb{R}}$. □

(2) $x' = t^2 \tan x$.

Proof. By separation of variables, the solution reads

$$\int_{x_0}^{x} \cot w \, \mathrm{d}w = \int_0^t \tau^2 \mathrm{d}\tau$$

where the integral on the left-hand side cannot pass through $x = \pi n$ for any $n \in \mathbb{Z}$. The result is

$$\ln\left|\frac{\sin x}{\sin x_0}\right| = \frac{t^3}{3} \quad \Longleftrightarrow \quad \boxed{x(t) = \arcsin\left(\mathrm{e}^{t^3/3}\sin x_0\right)}$$

The set of x_0 for which the solution is extendable to the whole of $t \geq 0$ is $\boxed{\emptyset}$ because $\cot(x)$ blows up periodically. When $x_0 = \pi n$ for any $n \in \mathbb{Z}$, there is no solution because cotangent is undefined at these values and the improper integral blows up. When $x_0 \neq \pi n$, the solution only exists in

$$\boxed{\left[0, \sqrt[3]{3\ln\left|\frac{1}{\sin(x_0)}\right|}\right)}$$

□

(3) $x' = 1 + x^2$.

Proof. By separation of variables, the solution reads

$$\int_{x_0}^{x} \frac{1}{1+w^2}\,\mathrm{d}w = \int_0^t \mathrm{d}\tau$$

The result is

$$\tan(x) - \tan(x_0) = t \iff \boxed{x(t) = \arctan(t + \tan(x_0))}$$

The set of x_0 for which the solution is extendable to the whole of $t \geq 0$ is

$$\boxed{\mathbb{R} \setminus \left\{\frac{\pi}{2} + \pi n \mid n \in \mathbb{Z}\right\}}$$

□

(4) $x' = \mathrm{e}^x \sin t$.

Proof. By separation of variables, the solution reads

$$\int_{x_0}^{x} \mathrm{e}^{-w}\,\mathrm{d}w = \int_0^t \sin\tau \mathrm{d}\tau$$

The result is

$$-\mathrm{e}^{-x} + \mathrm{e}^{-x_0} = 1 - \cos t \iff \boxed{x(t) = -\ln\big(\mathrm{e}^{-x_0} - 1 + \cos t\big)}$$

The set of x_0 for which the solution is extendable to the whole of $t \geq 0$ is

$$\boxed{\{x_0 \in \mathbb{R} \mid x_0 < \ln(1/2)\}}$$

When $x_0 \geq \ln(1/2)$, the solution only exists in

$$\boxed{\big[0, \arccos\big(1 - \mathrm{e}^{-x_0}\big)\big)}$$

□

4. Consider the harmonic oscillator equation, as mentioned in class:

$$x'' + \mu x' + \omega^2 x = 0$$

Here, the initial data $x(0) = x_0$ and $x'(0) = x_1$ are real numbers.

(1) Derive two linearly independent *real* solutions when $\mu > 0$. (Hint: You should consider the cases $\mu < 2\omega$ and $\mu > 2\omega$ separately.)

Proof. We first state and prove the following claim: If r is a zero of the characteristic polynomial $r^2 + ar + b = 0$, then e^{rx} is a solution to the ODE $y'' + ay' + by = 0$. The proof is simple — plugging $y = \mathrm{e}^{rx}$ and its derivatives $y' = r\mathrm{e}^{rx}$ and $y'' = r^2\mathrm{e}^{rx}$ into the original ODE, we have that

$$r^2\mathrm{e}^{rx} + ar\mathrm{e}^{rx} + b\mathrm{e}^{rx} = (r^2 + ar + b)\mathrm{e}^{rx} = 0$$

iff $r^2 + ar + b = 0$, i.e., if r is a root of said polynomial, as desired.
With this guiding idea, we will find the roots of

$$r^2 + \mu r + \omega^2 = 0$$

Using the quadratic formula, the two roots are

$$r_1 = \frac{-\mu + \sqrt{\mu^2 - 4\omega^2}}{2} \qquad r_2 = \frac{-\mu - \sqrt{\mu^2 - 4\omega^2}}{2}$$

We now divide into two cases ($\mu > 2\omega$ and $\mu < 2\omega$). If $\mu > 2\omega$, then r_1, r_2 are real and we take

$$\boxed{\mathrm{e}^{r_1 t}, \mathrm{e}^{r_2 t}}$$

to be our linearly independent, real solutions.
On the other hand, if $\mu < 2\omega$, then r_1, r_2 are of the form $\alpha \pm i\beta$. However, we can still obtain real solutions from these by taking the following linear combinations.

$$s_1 = r_1 + r_2 = 2\alpha \qquad s_2 = i(r_1 - r_2) = 2\beta$$

Thus, we take

$$\boxed{\mathrm{e}^{s_1 t}, \mathrm{e}^{s_2 t}}$$

to be our linearly independent, real solutions.
Thus, our general solution is of the form

$$x(t) = A\mathrm{e}^{c_1 t} + B\mathrm{e}^{c_2 t}$$

where $c_1 = r_1, s_1$ and $c_2 = r_2, s_2$ for some $A, B \in \mathbb{R}$. Plugging in the initial conditions, we get

$$x_0 = x(0) = A + B$$
$$x_1 = x'(0) = Ac_1 + Bc_2$$

which we can solve for A, B, yielding

$$\begin{cases} A = \frac{x_1 - x_0 c_2}{c_1 - c_2} \\ B = \frac{x_0 c_1 - x_1}{c_1 - c_2} \end{cases}$$

Therefore, our final particular solution is

$$\boxed{x(t) = \frac{x_1 - x_0 c_2}{c_1 - c_2}\mathrm{e}^{c_1 t} + \frac{x_0 c_1 - x_1}{c_1 - c_2}\mathrm{e}^{c_2 t}}$$

□

(2) Recall that $\mu = b/m$ and $\omega^2 = k/m$. Recall also that the mechanical energy for the oscillator reads

$$E = \frac{1}{2}m|x'|^2 + \frac{1}{2}kx^2$$

Compute the time derivative of E and conclude that E is exponentially decaying for $b > 0$, i.e., the mechanical energy is not conserved in this case. Does this violate the law of conservation of mechanical energy?

Proof. Applying the chain rule, we have that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = mx'x'' + kxx'$$

It follows that

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= mx'(-\mu x' - \omega^2 x) + kxx' \\ &= x'(-bx' - kx) + kxx' \\ &= -b(x')^2 \end{aligned}$$

Now $x' \neq 0$ (as an exponential function). Hence, $(x')^2 > 0$. This and $b > 0$ show that $\frac{\mathrm{d}E}{\mathrm{d}t}$ is always equal to a negative value. But this is characteristic of exponential decay, as desired.
Mechanical energy is conserved; it is dispersed from system to surroundings by the drag b. □

5. Use the transformation $y = tw$ to convert

$$y' = f(y/t)$$

to an ODE in w. Write down this equation for w. Use this transformation to solve

$$tyy' + 4t^2 + y^2 = 0, \quad y(2) = -7$$

Determine the lifespan (you can use a calculator for an approximate value).

Proof. If $y = tw$, then

$$\frac{\mathrm{d}y}{\mathrm{d}t} = w + t\frac{\mathrm{d}w}{\mathrm{d}t}$$

Thus, the ODE in terms of w is

$$\boxed{\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{f(w) - w}{t}}$$

which is a separable differential equation.

We have that

$$tyy' + 4t^2 + y^2 = 0 \quad \Longleftrightarrow \quad y' = -4\left(\frac{y}{t}\right)^{-1} - \frac{y}{t}$$

Using the above transformation yields

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{(-4w^{-1} - w) - w}{t}$$

Transforming the initial condition as well gives

$$w(2) = \frac{y(2)}{2} = -\frac{7}{2}$$

We can simplify and solve the above as follows.

$$\begin{aligned}
\frac{\mathrm{d}w}{-4w^{-1} - 2w} &= \frac{\mathrm{d}t}{t} \\
-\frac{1}{4}\int_{-7/2}^{w} \frac{2v\,\mathrm{d}v}{v^2 + 2} &= \int_2^t \frac{\mathrm{d}\tau}{\tau} \\
-\frac{1}{4}\left[\ln\left(w^2 + 2\right) - \ln(14.25)\right] &= \ln\left(\frac{t}{2}\right) \\
w &= \pm\frac{1}{t^2}\sqrt{228 - 2t^4} \\
\boxed{y(t) &= -\frac{1}{t}\sqrt{228 - 2t^4}}
\end{aligned}$$

Note that we pick the negative in the final step to fit the initial condition.

The lifespan of $y(t)$ can be determined by calculating when $228 - 2t^4 = 0$. This occurs such that the lifespan is approximately

$$\boxed{[0, 3.27]}$$

$\square$

6. Use the transformation $w = y^{1-\alpha}$ to convert Bernoulli's equation

$$y' + p(t)y = q(t)y^{\alpha}, \quad \alpha \neq 0, 1$$

to an ODE in w. Write down this equation for w. Use this transformation to solve

$$6y' - 2y = ty^4, \quad y(0) = -2$$

Determine the lifespan (you can use a calculator for an approximate value).

Proof. If $w = y^{1-\alpha}$, then

$$y = w^{1/(1-\alpha)} \qquad\qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{w^{\alpha/(1-\alpha)}}{1-\alpha}\frac{\mathrm{d}w}{\mathrm{d}t}$$

Thus, the ODE in terms of w is

$$\boxed{\frac{w^{\alpha/(1-\alpha)}}{1-\alpha}\frac{\mathrm{d}w}{\mathrm{d}t} + p(t)w^{1/(1-\alpha)} = q(t)w^{\alpha/(1-\alpha)}}$$

which is an exact differential equation.

We have that

$$6y' - 2y = ty^4 \quad \Longleftrightarrow \quad y' + \left(-\frac{1}{3}\right)y = \left(\frac{t}{6}\right)y^4$$

Using the above transformation yields

$$-\frac{w^{-4/3}}{3}\frac{\mathrm{d}w}{\mathrm{d}t} - \frac{w^{-1/3}}{3} = \frac{tw^{-4/3}}{6}$$

We can simplify and evaluate the above as follows.

$$\begin{aligned}
\frac{1}{3}w^{-4/3}\frac{\mathrm{d}w}{\mathrm{d}t} + \frac{1}{3}w^{-1/3} &= -\frac{t}{6}w^{-4/3} \\
\frac{\mathrm{d}w}{\mathrm{d}t} + w &= -\frac{t}{2} \\
\mathrm{e}^t\frac{\mathrm{d}w}{\mathrm{d}t} + \mathrm{e}^t w &= -\frac{t}{2}\mathrm{e}^t \\
\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^t w) &= -\frac{t}{2}\mathrm{e}^t \\
\mathrm{e}^t w &= -\frac{1}{2}\int t\mathrm{e}^t \,\mathrm{d}t \\
&= -\frac{1}{2}\mathrm{e}^t(t-1) + C \\
w &= -\frac{1}{2}(t-1) + C\mathrm{e}^{-t} \\
y^{-3} &= -\frac{1}{2}(t-1) + C\mathrm{e}^{-t} \\
y &= \left[-\frac{1}{2}(t-1) + C\mathrm{e}^{-t}\right]^{-1/3}
\end{aligned}$$

We now apply the initial condition.

$$\begin{aligned}
\left[-\frac{1}{2}(0-1) + C\mathrm{e}^{-0}\right]^{-1/3} &= y(0) \\
\left[\frac{1}{2} + C\right]^{-1/3} &= -2 \\
C &= -\frac{5}{8}
\end{aligned}$$

Therefore, the solution to the ODE in question is

$$\boxed{y(t) = \left[-\frac{1}{2}(t-1) - \frac{5}{8}\mathrm{e}^{-t}\right]^{-1/3}}$$

The equation $\boxed{\text{does not have finite lifespan}}$. ☐

7. Show that
$$(4bxy + 3x + 5)y' + 3x^2 + 8ax + 2by^2 + 3y = 0$$
is an exact equation, no matter what value a, b take. Find the implicit relation satisfied by the solution $y(x)$ and x.

Proof. To show that an equation of the form $gy' + f = 0$ is exact, it will suffice to confirm that
$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$$
Since the equation in question is of this form, we may evaluate directly:
$$\frac{\partial g}{\partial x} = 4by + 3 \qquad \frac{\partial f}{\partial y} = 4by + 3$$
By transitivity, we have the desired result.

We now want to find F such that $\partial F/\partial x = f$ and $\partial F/\partial y = g$. Starting with the former constraint, we can determine that
$$\begin{aligned} F(x, y) &= \int (3x^2 + 8ax + 2by^2 + 3y)\,\mathrm{d}x \\ &= x^3 + 4ax^2 + 2bxy^2 + 3xy + h(y) \end{aligned}$$
where $h(y)$ is a functional "constant" of integration. We now differentiate with respect to y.
$$\frac{\partial F}{\partial y} = 4bxy + 3x + \frac{\mathrm{d}h}{\mathrm{d}y}$$
Knowing that $\partial F/\partial y = g$, we can use the above equation to solve for h as follows.
$$\begin{aligned} 4bxy + 3x + 5 &= 4bxy + 3x + \frac{\mathrm{d}h}{\mathrm{d}y} \\ \frac{\mathrm{d}h}{\mathrm{d}y} &= 5 \\ h(y) &= 5y \end{aligned}$$
Therefore, we know that
$$\boxed{F(x, y) = x^3 + 4ax^2 + 2bxy^2 + 3xy + 5y}$$
□

8. Let a, b be constants. For Euler's equation
$$t^2y'' + aty' + by = f(t)$$
consider the transformation $w(\tau) = y(\mathrm{e}^\tau)$. What is the differential equation satisfied by $w(\tau)$? Use this transformation to solve
$$2t^2y'' + 3ty' - 15y = 0, \quad y(1) = 0, \quad y'(1) = 1$$

Proof. The differential equation satisfied by $w(\tau)$ is □

9. Suppose there is a capacitor with capacitance C being charged by a battery of fixed voltage V_0. Suppose there is a resistor R connected to C. Then the charge $Q(t)$ of the capacitor satisfies the differential equation
$$RQ'(t) + \frac{Q(t)}{C} = V_0$$
This is the equation for an RC charging circuit.

Find the explicit solution of this equation with $Q(0) = 0$. Explain why the product RC is important in determining the charging time. For $R = 10^3\,\Omega$, $V_0 = 1\,\mathrm{V}$, $C = 1\,\mu\mathrm{F}$, how much time does it take for the capacitor to be charged to 98%? (You may use a calculator.)

Proof. We can evaluate the ODE as follows.

$$\begin{aligned}
\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{1}{RC}Q &= V_0 \\
\mathrm{e}^{t/RC}\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{1}{RC}\mathrm{e}^{t/RC}Q &= \mathrm{e}^{t/RC}V_0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(Q\mathrm{e}^{t/RC}\right) &= \mathrm{e}^{t/RC}V_0 \\
Q\mathrm{e}^{t/RC} &= RCV_0\mathrm{e}^{t/RC} + C_1 \\
Q(t) &= RCV_0 + C_1\mathrm{e}^{-t/RC}
\end{aligned}$$

We now apply the initial condition.

$$\begin{aligned}
0 &= Q(0) \\
&= RCV_0 + C_1 \\
C_1 &= -RCV_0
\end{aligned}$$

Therefore, the solution to the ODE in question is

$$\boxed{Q(t) = RCV_0\left(1 - \mathrm{e}^{-t/RC}\right)}$$

The product RC (technically referred to as the time constant) is important in determining charging time because it is directly proportional to the rate of exponential charging. Indeed, if RC doubles, the capacitor will take twice as long to charge (and vice versa, for example, if RC halves).

The amount of time it takes for the capacitor to charge to 98% under the given conditions ($R = 10^3\,\Omega$ and $C = 10^{-6}\,\mathrm{F}$) may be determined as follows.

$$\begin{aligned}
0.98 &= 1 - \mathrm{e}^{-t/RC} \\
t &= -RC\ln(0.02) \\
&\boxed{t = 3.9 \times 10^{-3}\,\mathrm{s}}
\end{aligned}$$

□

10. A parachutist is falling from a plane. Suppose the parachute is opened at height H, when the falling velocity is v_0. Suppose that the air resistance exerted on the parachute is proportional to the square of the velocity with ratio η. Let the gravitational constant be g, and suppose that the total mass of the parachutist and the parachute is m. Write down the differential equation satisfied by the shift x, together with the initial conditions. Solve this IVP. What is the velocity as $t \to +\infty$? Can you derive the final velocity based on physical considerations?

Proof. For the sake of simplicity, we will write a one-dimensional differential equation corresponding to vertical displacement. Let's begin.

When the parachutist is falling freely, there is only one (idealized) force acting on them: gravity (F_g). As soon as the parachute is opened, another force is added to the mix: drag (F_d). By Newton's second law, the net force is equal to the parachutist/parachute's mass times their acceleration. Taking a convention of upwards displacement being positive, we can thus write that

$$\sum F_z = F_d - F_g = ma$$

Since $a = x''$, $F_g = g$, and $F_d = \eta v^2 = \eta(x')^2$, the differential equation satisfied by the shift x is

$$mx'' = \eta(x')^2 - g$$

Let the time at which the parachute is opened be $t = 0$. Then the initial conditions are

$$x(0) = H \qquad\qquad x'(0) = v_0$$

To solve this IVP, we substitute $v = x'$ and evaluate the resulting first-order differential equation to start:

$$\begin{aligned}
mv' &= \eta v^2 - g \\
\frac{\mathrm{d}v}{v^2 - g/\eta} &= \frac{\eta}{m}\,\mathrm{d}t \\
\int_{v_0}^{v} \frac{\mathrm{d}w}{w^2 - g/\eta} &= \int_0^t \frac{\eta}{m}\,\mathrm{d}\tau \\
\coth^{-1}(v) - \coth^{-1}(v_0) &= \frac{\eta}{m}t \\
v &= \coth\left(\frac{\eta}{m}t + \coth^{-1}(v_0)\right)
\end{aligned}$$

Assuming the velocities are greater than one (a reasonable assumption; if not, change units), the hyperbolic cotangent is perfectly acceptable to use here. Returning the substitution $v = x'$, we can determine that

$$\begin{aligned}
x' &= \coth\left(\frac{\eta}{m}t + \coth^{-1}(v_0)\right) \\
\int_H^x \mathrm{d}z &= \int_0^t \coth\left(\frac{\eta}{m}\tau + \coth^{-1}(v_0)\right)\mathrm{d}\tau \\
x - H &= \frac{m}{\eta}\ln\left(\sinh\left(\frac{\eta}{m}t + \coth^{-1}(v_0)\right)\sqrt{v_0^2 - 1}\right) \\
&\boxed{x = H + \frac{m}{\eta}\ln\left(\sinh\left(\frac{\eta}{m}t + \coth^{-1}(v_0)\right)\sqrt{v_0^2 - 1}\right)}
\end{aligned}$$

The final velocity approaches $\boxed{1}$. ☐

Bonus Problems

1. **The Catenoid.** Suppose there are two metal rings of radius a placed parallel to each other in an xyz-coordinate space, with the x-axis passing through their centers. Suppose these two rings are contained in the planes $x = l$ and $x = -l$, respectively. An axial symmetric soap film is spanned by these two rings. Suppose its shape is obtained by rotating the graph of the function $y = y(x)$ with respect to the x-axis. In order to attain a stable configuration, the surface area is supposed to be minimal among all such surfaces of revolution.

   (1) Write down the surface area functional in terms of $y(x)$, its derivative, and the boundary conditions for this variational problem.

   (2) Derive the Euler-Lagrange equation and find the solution. The shape is called a **catenoid**.

   (3) If the two rings are very far away from each other, i.e., l is very large, will the catenoid still be of minimal area among all competing surfaces that span these two rings? You do not have to give a mathematically rigorous answer; just imagine the physical situation. (Hint: What about two distinct disks spanned by these two rings?)

2. **A Formulation of the Isoperimetric Problem.** Recall from multivariable calculus that in order to find a local extremum of the function $f(x_1, \dots, x_n)$ under the constraint $g(x_1, \dots, x_n) = 0$, we can introduce a parameter λ called the **Lagrange multiplier** and find the stationary point of the function

$$f(x_1, \dots, x_n) - \lambda g(x_1, \dots, x_n)$$

(1) Write down the equations that must be satisfied by the stationary point $(x_1, \dots, x_n)$ of the function $f - \lambda g$ with the parameter λ involved.

(2) Use the Lagrange multiplier method to find the maxima and minima of $f(x, y) = x + y$ under the constraint $x^2 + y^2 = 1$.

(3) Now let us generalize this method to functionals. If we aim to find the extrema of a functional

$$J[y] = \int_a^b F(x, y(x), y'(x))\, \mathrm{d}x$$

under the constraint

$$R[y] = \int_a^b G(x, y(x), y'(x))\, \mathrm{d}x = 0$$

where $F(x, z, w)$ and $G(x, z, w)$ are known functions, we can try to find the extrema of the functional

$$J[y] - \lambda R[y]$$

first. What is the Euler-Lagrange equation satisfied by this extrema (with λ involved)?

(4) Now let us consider a version of the isoperimetric problem. We aim to find the function $y(x)$, whose graph connects two given points (a, A), (b, B) on the xy-plane, with a prescribed arclength

$$l = \int_a^b \sqrt{1 + |y'(x)|^2}\, \mathrm{d}x$$

such that the area between the graph and the x-axis is the largest. The functional in consideration is

$$J[y] = \int_a^b y(x)\, \mathrm{d}x$$

with constraint

$$R[y] = \int_a^b \sqrt{1 + |y'(x)|^2}\, \mathrm{d}x = l$$

Write down the Euler-Lagrange equation involving the multiplier λ and show that the solution must be a part of a circle.

2 Linear Algebra

Required Problems

10/19: **1.** This question helps to complete the computations omitted in class. In deriving the Kepler orbits for the two-body problem, we have successfully reduced the differential equation satisfied by the curve $r = r(\varphi)$ to

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 + r^2 = \frac{2GMr^3}{l_0^2} + \frac{2Er^4}{ml_0^2}$$

Show that the function $\mu = 1/r$ satisfies the differential equation

$$\left(\frac{\mathrm{d}\mu}{\mathrm{d}\varphi}\right)^2 + \mu^2 = \frac{2GM\mu}{l_0^2} + \frac{2E}{ml_0^2}$$

By differentiating with respect to φ again, this reduces to either $\mathrm{d}\mu/\mathrm{d}\varphi = 0$ or

$$\frac{\mathrm{d}^2\mu}{\mathrm{d}\varphi^2} + \mu - \frac{GM}{l_0^2} = 0$$

Find the general solution of the latter, hence conclude that $r = r(\varphi)$ represents a conic section. *Hint*: There is a very obvious particular solution.

2. The general formula for the inverse of an $n \times n$ invertible matrix is very lengthy. However, for a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfying $ad - bc \neq 0$, there is a very simple formula. Try to find it; this could be very helpful if you can remember it.

3. Compute the determinant of the following matrices. Determine whether they are invertible or not.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 2 & 3 & 6 \\ 1 & 3 & 4 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} -1 & 2 & 1 \\ 3 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$$

4. Determine whether the following linear systems admit solution(s); if they do, write down the solution (or the formula for the general solution).

(1)

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

(2)

$$\begin{pmatrix} -1 & 2 & 1 \\ 3 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

(3)

$$\begin{pmatrix} -1 & 2 & 1 \\ 3 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

5. Find the connecting matrix from the basis $\begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix}$ to the new basis $\begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}$, where

$$\begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix} \qquad \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

That is, represent q_1, q_2, q_3 as linear combinations of p_1, p_2, p_3.

6. Let $\theta \in [0, 2\pi)$. The rotation through angle θ in the plane is represented by the matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Compute its determinant, characteristic polynomial, and eigenvalues. Compute its eigenvectors in $\mathbb{C}^2$. You need to use the Euler formula $\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$. For two angles θ, φ, compute the product $R(\theta)R(\varphi)$ and represent it in terms of $\theta + \varphi$. What is the geometric meaning of this equality?

8. Find the algebraic and geometric multiplicities of the eigenvalues of the following matrices.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

9. Compute the Jordan normal form of the following 2×2 matrices.

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}$$

Notice that you not only need to find all the Jordan blocks, but also need to find the Jordan basis matrix Q such that $Q^{-1}AQ$ is in Jordan normal form.

10. Compute the Jordan normal form of the following 3×3 matrices.

$$A = \begin{pmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -1 & -1 \\ 2 & -1 & -2 \\ -1 & 1 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

Notice that you not only need to find all the Jordan blocks, but also need to find the Jordan basis matrix Q such that $Q^{-1}AQ$ is in Jordan normal form. *Hint*: These three matrices represent three different possibilities of nondiagonalizable Jordan normal forms of a 3×3 matrix: A reduces to $(2 \times 2) \oplus (1 \times 1)$ Jordan blocks with different eigenvalues, B reduces to $(2 \times 2) \oplus (1 \times 1)$ Jordan blocks with the same eigenvalue, and C reduces to a 3×3 Jordan block.

Bonus Problems

1. You may find the characteristic root method for the second-order equation $y'' + ay' + b = 0$ quite abrupt. This problem helps you see where it comes from. The origin of this method is in fact a comparison with the linear recursive relation

$$y_{n+2} + ay_{n+1} + by_n = 0$$

where a, b are given complex numbers.

(1) The linear recursive relation $y_{n+1} + ay_n = 0$ gives rise to a geometric sequence

$$y_0, y_0(-a), y_0(-a)^2, \ldots$$

We now want to try to reduce the second-order recursive relation $y_{n+2} + ay_{n+1} + by_n = 0$ to a first-order relation. Thus, we look for complex numbers λ, μ such that

$$(y_{n+2} - \lambda y_{n+1}) - \mu(y_{n+1} - \lambda y_n) = 0$$

Then λ, μ should be the roots of the characteristic polynomial

$$X^2 + aX + b$$

Taking λ, μ as known quantities, find the general formula for y_n, regarding y_0, y_1 as known quantities. *Hint*: $y_{n+1} - \lambda y_n$ is a geometric sequence with ratio μ. You should also discuss $\mu \neq \lambda$ and $\mu = \lambda$ separately.

(2) Use the method of part (1) to find the general formula for the linear discursive relation

$$y_{n+2} - 2y_{n+1} + y_n = 0$$

Use the same method to find the general formula for the Fibonacci sequence

$$F_{n+2} = F_{n+1} + F_n$$

2. In this exercise, we aim to prove an important theorem in linear algebra:

Complex Hermitian matrices are always diagonalizable.

Here the term "Hermitian" means that the matrix equals its conjugate transpose. In terms of entries, this means that in general, $a_{ij} = \bar{a}_{ji}$. For example,

$$\begin{pmatrix} 2 & 1 & -i \\ 1 & 3 & -2i \\ i & 2i & 1 \end{pmatrix}$$

is Hermitian.

(1) Let $\langle \cdot, \cdot \rangle$ be the standard Hermitian inner product, that is, for $x, y \in \mathbb{C}^n$,

$$\langle x, y \rangle = \sum_{j=1}^{n} x^j \bar{y}^j$$

Show that for any $n \times n$ real matrix,

$$\langle Ax, y \rangle = \langle x, A^* y \rangle$$

for any $x, y \in \mathbb{C}^n$, where A^* denotes the conjugate transpose of A. For example,

$$A = \begin{pmatrix} 1 & 1 & 2i \\ 0 & 3+i & 3 \\ 2 & 0 & 1 \end{pmatrix} \quad \Longleftrightarrow \quad A^* = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 3-i & 0 \\ -2i & 3 & 1 \end{pmatrix}$$

(2) Suppose now that A is Hermitian. Use part (1) to show that any eigenvalue of A must be a real number. Show further that if x, y are eigenvectors corresponding to different eigenvalues, then $\langle x, y \rangle = 0$, that is, x is orthogonal to y.

(3) Prove that every Hermitian matrix A is diagonalizable. *Hint*: Take any eigenvector v_1 of A. Decompose $\mathbb{C}^n$ into the direct sum of $\text{span}(v_1)$ and its orthogonal complement. Show that the orthogonal complement is an invariant subspace for A.