Week 2

???

2.1 Separable ODEs

10/3: • Do not sit on the left side of the classroom: The sun sucks!

- **Separable** (ODE): An ODE of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)g(y)$$

  where y is a real[1], unknown, scalar function of t.

- Solving separable ODEs: Formally, evaluate

$$\int \frac{\mathrm{d}y}{g(y)} = \int f(t)\,\mathrm{d}t$$

- Rearrange the initial separable ODE to $\mathrm{d}y/\mathrm{d}t \cdot 1/g = f$ and invoke the law of composite differentiation to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{y_0}^{y(t)} \frac{\mathrm{d}w}{g(w)} - \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau\right] = 0$$

- It follows that

$$\int_{y_0}^{y(t)} \frac{\mathrm{d}w}{g(w)} = \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau$$

- Examples:
  1. Exponential growth.
     - We have that

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky$$

       for $k > 0$ and $y(0) = y_0 > 0$.
     - The solution is

$$\begin{aligned} \frac{1}{y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} &= k \\ \log y(t) - \log y_0 &= kt \\ y(t) &= y_0 \mathrm{e}^{kt} \end{aligned}$$

[1] We'll deal with complex functions later.

2. Logistic growth.
   - We have that
   $$\frac{\mathrm{d}y}{\mathrm{d}t} = ky\left(1 - \frac{y}{M}\right)$$
   for $k, M > 0$ and $y(0) = y_0 > 0$.
   - The solution is
   $$\begin{aligned}
   \frac{M\,\mathrm{d}y}{y(M-y)} &= k\,\mathrm{d}t \\
   \log\frac{y}{M-y} - \log\frac{y_0}{M-y_0} &= kt \\
   \frac{y(M-y_0)}{y_0(M-y)} &= \mathrm{e}^{kt} \\
   y\cdot\frac{M-y_0}{y_0} &= (M-y)\mathrm{e}^{kt} \\
   y\cdot\frac{M-y_0}{y_0} + y\mathrm{e}^{kt} &= M\mathrm{e}^{kt} \\
   y\left(\frac{M-y_0}{y_0} + \mathrm{e}^{kt}\right) &= M\mathrm{e}^{kt} \\
   y\left(\frac{M-y_0+y_0\mathrm{e}^{kt}}{y_0}\right) &= M\mathrm{e}^{kt} \\
   y\left(\frac{M+y_0(\mathrm{e}^{kt}-1)}{y_0}\right) &= M\mathrm{e}^{kt} \\
   y(t) &= \frac{My_0\mathrm{e}^{kt}}{M+y_0(\mathrm{e}^{kt}-1)}
   \end{aligned}$$
   - Sketches the graph of logistic growth and discusses the turning point (for which there is a formula; zero of the second derivative) as well as general trends.
   - If $y_0 < 0$, the solution is not physically meaningful, but it is mathematically insightful.
     - When we integrate, the arguments of our logarithms now have absolute values.
     $$\log\left|\frac{y}{M-y}\right| - \log\left|\frac{y_0}{M-y_0}\right| = kt$$
     - We need to make sure that the denominator of the final logistic form is never equal to zero, but now that y_0 is negative, as t increases, the denominator will approach zero exponentially. It reaches zero when
     $$\begin{aligned}
     M + y_0(\mathrm{e}^{kt}-1) &= 0 \\
     \mathrm{e}^{kt} &= -\frac{M}{y_0} + 1
     \end{aligned}$$
     In other words, $t_{\max} = (1/k)\log(1 - M/y_0)$ because when $t = t_{\max}$, the equation blows up.
     - This is an example of **finite lifespan**.
   - If $y_0 > M$, then you will exponentially decrease to M.
3. Lotka-Volterra predator-prey model.
   - We have that
   $$r' = k_1r - awr \qquad\qquad w' = -k_2w + bwr$$
   where r is rabbits and w is wolves.

- We can rename the variables to

$$\begin{cases} x' = Ax - Bxy \\ y' = -Cy + Dxy \end{cases}$$

- Dividing, we get

$$\frac{x'}{y'} = \frac{Ax - Bxy}{-Cy + Dxy}$$
$$\frac{By - A}{y} y' + \frac{Dx - C}{x} x' = 0$$

- Use the fact that x, y are independent variables, so both terms in the above equation are equal to zero?
- Invoke the law of composite differentiation twice and, from the above, know that $0 + 0 = 0$, so we can add the two solutions:

$$\frac{\mathrm{d}}{\mathrm{d}t}(By(t) - A\log y(t)) + \frac{\mathrm{d}}{\mathrm{d}t}(Dx(t) - C\log x(t)) = 0$$
$$By(t) - A\log y(t) + Dx(t) - C\log x(t) = E$$

- Sketches some of the trajectories (they're all closed curves in the xy-plane).

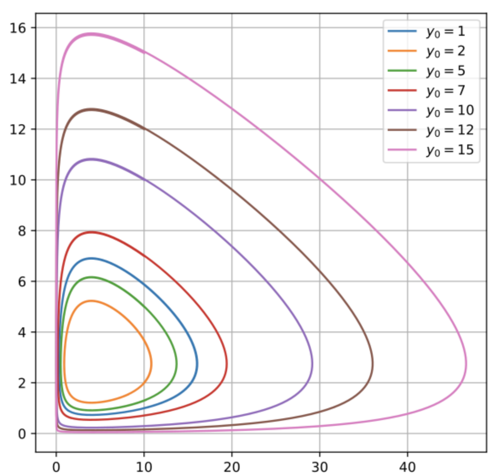

Figure 2.1: Lotka-Volterra solution curves.

- Properties of the curves:
  - The implicit relation which determines them: By the implicit function theorem, the y derivative of the LHS is $B - A/y$ and the x-derivative of the LHS is $D - C/x$. When the partial derivatives are equal to zero, $(C/D, A/B)$ becomes interesting. Turning points happen when the y-coordinate is A/B or the x-coordinate is C/D.

- **Finite lifespan**: Even if the RHS of $\mathrm{d}y/\mathrm{d}t = f(t, y)$ is very regular, the solution can still blow up at some finite time.

- Consider the following variation on the E-L equation from the Brachistochrone problem.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{\frac{B - y}{y}}$$

  - Finding the **primitives**.

  - What are these "primitives" Shao keeps talking about?
- We should have

$$\int \sqrt{\frac{y}{B-y}}\,\mathrm{d}y = x$$

- Change of variables: $y = B\sin^2\phi$ and $\mathrm{d}y = 2B\cos\phi\sin\phi\,\mathrm{d}\phi$. Thus,

$$\int \sqrt{\frac{y}{B-y}}\,\mathrm{d}y = \int \frac{\sin\phi}{\cos\phi}\cdot 2B\cos\phi\sin\phi\,\mathrm{d}\phi = 2B\int \sin^2\phi\,\mathrm{d}\phi$$

- The solution is

$$\begin{cases} x = B\phi - \frac{B}{2}\sin(2\phi) + C \\ y = B\sin^2\phi \end{cases}$$

  - This is a parameterization of a cycloid.

- Later in the week, we will do the SHM, the pendulum, the Kepler 2-body problem, and the Michaelis-Menten equation.
- Separable ODEs are a subset of ODEs of **exact form**.
- ODEs of exact form are of the form

$$g(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} + f(x,y) = 0$$

where for some $F(x,y)$, $g = \partial F/\partial y$, $f = \partial F/\partial x$, and partials commute. Equivalently,

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$$

is our necessary and sufficient condition.

- By the law of composite differentiation,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[F(x,y(x))\right] &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\cdot y'(x) \\ &= f(x,y(x)) + g(x,y(x))y'(x) \\ &= 0\end{aligned}$$

  - We solve these with an integrating factor $\mu \neq 0$ such that $(\mu g, \mu f)$ satisfy the constraint.

2.2 Office Hours (Shao)

- **Primitive**: An antiderivative.
- **Law of composite differentiation**: The chain rule.
- Went over how Shao has been applying the law of composite differentiation with respect to separable ODEs:
  - Rearrange the initial separable ODE as follows.

$$\frac{1}{g(y)}\cdot\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)$$

  - Define $\mathrm{d}H/\mathrm{d}y = 1/g(y)$. Then, continuing from the above, we have by the law of composite differentiation that

$$\frac{\mathrm{d}H}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} = f(t)$$
$$\frac{\mathrm{d}H}{\mathrm{d}t} = f(t)$$

  - From the definition of H, we know that $H(y) = \int_{y_0}^{y} \mathrm{d}w \,/g(w)$. We also have from the FTC that $f(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau$. Thus, continuing from the above, we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}(H) = f(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{y_0}^{y} \frac{\mathrm{d}w}{g(w)}\right] = \frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{y_0}^{y(t)} \frac{\mathrm{d}w}{g(w)} - \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau\right] = 0$$

    as desired.
  - It follows since $y(t_0) = y_0$ that $C = H(y_0) = 0$, so we can take the above to

$$\int_{y_0}^{y(t)} \frac{\mathrm{d}w}{g(w)} = \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau$$

    knowing that our constant of integration is zero.

- Take away from Brachistochrone problem: Just an example of a BDE; we won't have to answer questions on it.