Week 5

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5.1 Planar Autonomous Linear Systems

10/24:

- Review of vector fields.
- **Phase diagram**: A diagram that shows the qualitative behavior of an autonomous ordinary differential equation. *Also known as* **phase portrait**.

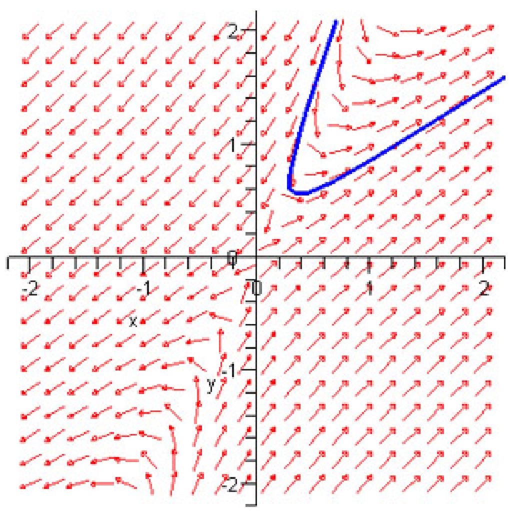
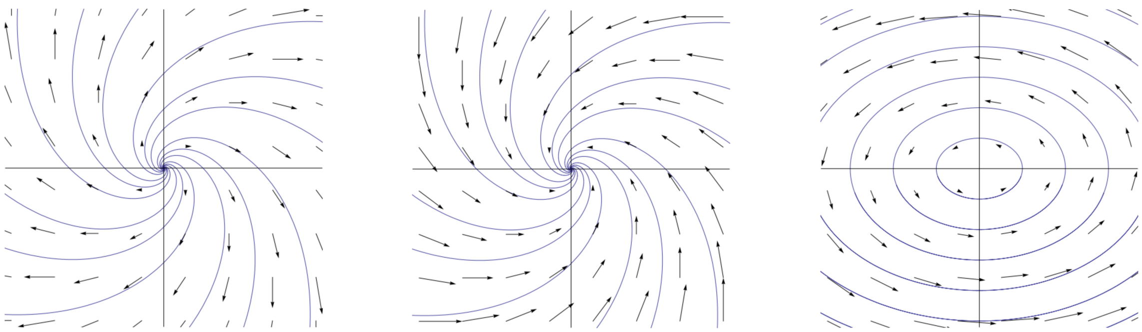


Figure 5.1: Phase diagram example.

  - Consists of a selection of arrows describing, to some extent, a vector field and is often paired with integral curves.
- Suppose $\Omega \subset \mathbb{R}^n$ is open.
- **Vector field** (on Ω): A mapping from $\Omega \to \mathbb{R}^n$. *Denoted by* $\boldsymbol{X}$.
  - Essentially, a vector field assigns to every point of some region a vector; the definition just formalizes this notion.
- **Flow**: A formalization of the idea of the motion of particles in a fluid.
  - The solution to the IVP $\frac{\mathrm{d}y}{\mathrm{d}t} = X(y)$, $y(0) = x$.
- If X is C^1, then for all $x \in \Omega$, there exists a unique solution y to the above IVP.

- **Orbit** (of x under X): The trajectory $y(t,x)$.
  - Recall that the tangent vector to any trajectory at any point coincides with the vector to which X maps that point.
- **Fixed point**: A point $x_0 \in \Omega$ such that $X(x_0) = \bar{0}$.
  - If x_0 is a fixed point, then the trajectory is $y(t) = x_0$.
- Today: We will consider flows on vector fields where the dimension is two and our vector field is linear. In particular...
- Let A be a 2×2 real matrix, and let $X(x) = Ax$.
  - In this case, $x_0 = 0$ is the only fixed point.
  - The flow is given by the linear differential equation $y' = Ay$, $y(0) = x$. The solution is $y(t) = \mathrm{e}^{tA}x$.
- Case 1: A has no real eigenvalues.



(a) $\sigma > 0$. (b) $\sigma < 0$. (c) $\sigma = 0$.

Figure 5.2: Phase diagrams for a planar system with no real eigenvalues.

  - We know that $\chi_A(z)$ is a real polynomial: $\chi_A(z) = z^2 + (\operatorname{trace} A)z + \det A$, and since A is real, both $\operatorname{trace} A$ and $\det A$ are real.
  - Thus, the eigenvalues appear as conjugate pair, i.e., we may write $\lambda = \sigma + i\beta$ and $\bar{\lambda} = \sigma - i\beta$.
    - $\alpha = \gamma = 1$ for both eigenvalues.
    - The eigenvectors must also be complex conjugates.
  - Distinct eigenvalues imply that A is diagonalizable.
  - However, this is not what we want because if we use the complex form, then

$$\mathrm{e}^{tA} = Q \begin{pmatrix} \mathrm{e}^{t\lambda} & 0 \\ 0 & \mathrm{e}^{t\bar{\lambda}} \end{pmatrix} Q^{-1}$$

  - Indeed, we want to get a real matrix out of $Q, \mathrm{e}^{t\Lambda}, Q^{-1}$ all complex. We have

$$\begin{aligned}
\mathrm{e}^{tA}x &= Q \begin{pmatrix} \mathrm{e}^{t(\sigma+i\beta)} & 0 \\ 0 & \mathrm{e}^{t(\sigma-i\beta)} \end{pmatrix} \underbrace{Q^{-1}x}_{z} \\
&= Q \begin{pmatrix} \mathrm{e}^{t(\sigma+i\beta)} z^1 \\ \mathrm{e}^{t(\sigma-i\beta)} z^2 \end{pmatrix} \\
&= z^1 \mathrm{e}^{t(\sigma+i\beta)} v + z^2 \mathrm{e}^{t(\sigma-i\beta)} \bar{v}
\end{aligned}$$

- Since $y(0) = x = z^1 v + z^2 \bar{v} \in \mathbb{R}^2$ (i.e., $z^1 v + z^2 v$ is *real*), we know that it is equal to its complex conjugate. This tells us that

$$\begin{aligned} z^1 v + z^2 \bar{v} &= \bar{z}^1 \bar{v} + \bar{z}^2 v \\ z^1 &= \bar{z}^2 \end{aligned}$$

- It follows that

$$\begin{aligned} y(t) &= \mathrm{e}^{tA} x \\ &= z^1 \mathrm{e}^{t(\sigma + i\beta)} v + \bar{z}^1 \mathrm{e}^{t(\sigma - i\beta)} \bar{v} \\ &= z^1 \mathrm{e}^{t(\sigma + i\beta)} v + \overline{z^1 \mathrm{e}^{t(\sigma + i\beta)} v} \\ &= 2 \operatorname{Re}(z^1 \mathrm{e}^{t(\sigma + i\beta)} v) \\ &= 2 \operatorname{Re}(z^1 \mathrm{e}^{\sigma t} (\cos(\beta t) + i \sin(\beta t))(v_1 + i v_2)) \\ &= 2 \operatorname{Re}(z^1 \mathrm{e}^{\sigma t} (\cos(\beta t) v_1 + i \cos(\beta t) v_2 + i \sin(\beta t) v_1 - \sin(\beta t) v_2)) \\ &= 2\mathrm{e}^{\sigma t} \cos(\beta t) \cdot \operatorname{Re}(z^1 v) - 2\mathrm{e}^{\sigma t} \sin(\beta t) \cdot \operatorname{Im}(z^1 v) \end{aligned}$$

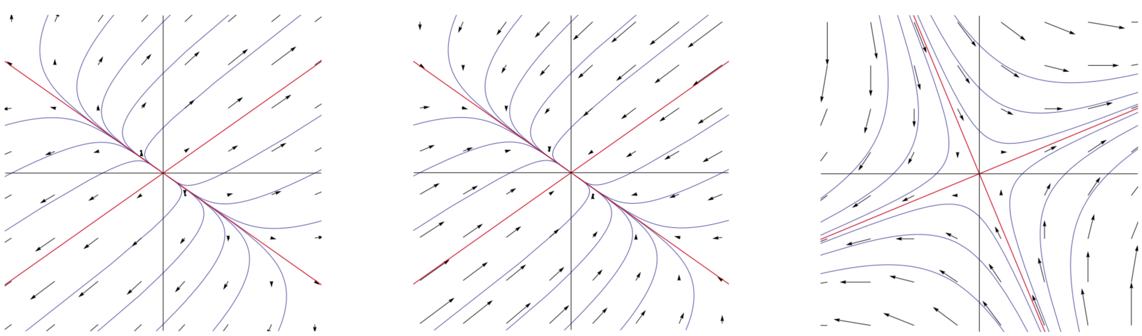
- Suppose $\sigma \neq 0$. Then

$$x \mapsto \begin{pmatrix} \operatorname{Re}(z^1 v) \\ \operatorname{Im}(z^1 v) \end{pmatrix}$$

  is a real linear transformation on $\mathbb{R}^2$.
- It follows that the trajectories are just spirals in the complex plane.
- If $\sigma > 0$, then the spiral repels from the origin. If $\sigma < 0$, then the spiral attracts to the origin. If $\sigma = 0$, we get an ellipse.
- Therefore, we have completely classified equations of the form

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix}' = \begin{pmatrix} y^2 \\ -\omega^2 y^1 \end{pmatrix}$$

- Case 2: A has real eigenvalues and *is* diagonalizable.



(a) $\lambda_1, \lambda_2 > 0$. (b) $\lambda_1, \lambda_2 < 0$. (c) $\lambda_1 < 0 < \lambda_2$ (WLOG).

Figure 5.3: Phase diagrams for a diagonalizable planar system with real eigenvalues.

- Suppose $\lambda_1, \lambda_2 \in \mathbb{R}$ have corresponding linearly independent eigenvectors v_1, v_2.
- If we choose v_1, v_2 to be our basis, then

$$\mathrm{e}^{tA} = Q \begin{pmatrix} \mathrm{e}^{t\lambda_1} & 0 \\ 0 & \mathrm{e}^{t\lambda_2} \end{pmatrix} Q^{-1}$$

  where $Q = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$.

- Thus, as before, the solution may be expressed in the following form, where $z = Q^{-1}x$.

$$y(t) = \mathrm{e}^{tA}x = \mathrm{e}^{\lambda_1 t}z^1 v_1 + \mathrm{e}^{\lambda_2 t}z^2 v_2$$

- Moving forward, it will be convenient to work in the v_1, v_2 basis. We divide into three subcases ($\lambda_1, \lambda_2 > 0$ [Figure 5.3a], $\lambda_1, \lambda_2 < 0$ [Figure 5.3b], and WLOG $\lambda_1 < 0 < \lambda_2$ [Figure 5.3c]).
  1. Notice that
  $$\mathrm{e}^{\lambda_2 t} = \mathrm{e}^{(\lambda_2/\lambda_1)(\lambda_1 t)}$$
  i.e., $\mathrm{e}^{\lambda_2 t}$ is a power of $\mathrm{e}^{\lambda_1 t}$. Thus, when the signs are the same, we get a power function $v_2 = v_1^{\lambda_2/\lambda_1}$.
     - Both subspaces v_1, v_2 are unstable here.
  2. If $\lambda_1, \lambda_2 < 0$, then we have the same trajectories, but they're all attracted to the origin instead of repelled.
     - Both subspaces v_1, v_2 are stable here.
  3. When both eigenvalues have different signs, we are considering power functions of a negative power.
     - The stable subspace is v_2 and the unstable subspace is v_1 here.

- Case 3: A has real eigenvalues and *is not* diagonalizable.

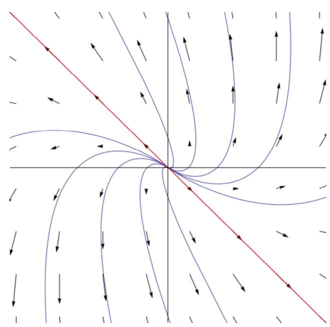


Figure 5.4: Phase diagrams for a nondiagonalizable planar system with real eigenvalues.

- In this case, the matrix exponential is given by

$$\mathrm{e}^{tA} = Q\begin{pmatrix} \mathrm{e}^{t\lambda} & t\mathrm{e}^{t\lambda} \\ 0 & \mathrm{e}^{t\lambda} \end{pmatrix}Q^{-1}$$

- The solution is given by

$$\mathrm{e}^{tA}x = (z^1\mathrm{e}^{t\lambda} + z^2 t\mathrm{e}^{t\lambda})v + z^2\mathrm{e}^{t\lambda}u$$

  where $Q^{-1}x = z$ again.
- In graphing, note that here we have (a distorted version of) the form $y = x \pm x\log x$:

$$y = (z^1\mathrm{e}^{t\lambda} + z^2 t\mathrm{e}^{t\lambda})\hat{\imath} + z^2\mathrm{e}^{t\lambda}\hat{\jmath}$$

  Define $x := \mathrm{e}^{t\lambda}$. Then $t = \lambda^{-1}\ln x$. Substituting, we have

$$\begin{aligned} &= (z^1 x + z^2(\lambda^{-1}\ln x)x)\hat{\imath} + z^2 x\hat{\jmath} \\ &= (z^1 x + z^2\lambda^{-1}x\ln x)\hat{\imath} + z^2 x\hat{\jmath} \end{aligned}$$

- When $\lambda > 0$, the whole space is unstable. Thus, the phase diagram is tangent to the origin.
- When $\lambda < 0$, the trajectories take the same form but this time are attracted to zero. In this case, the whole space is stable.

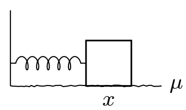
- We can take x_1 to x_2 iff they are in the same orbit. Conclusion: Orbits never cross.
- Takeaway: You should be able to compute the eigenvalues and eigenvectors and sketch these graphs.
- Shao will post lecture notes after today's lecture!
- Next lecture: The final explicitly solveable case, which is the driven harmonic oscillator.

5.2 Driven Harmonic Oscillator and Resonance

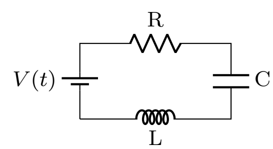
10/26:
- We are interested in the 2nd order constant coefficient equation

$$x'' + \mu x' + \omega_0^2 x = H_0 \mathrm{e}^{i\omega t}$$

  where $\mu \geq 0$ and $\omega_0, \omega > 0$.
- Two cases where this ODE arises:



(a) A driven harmonic oscillator.



(b) An RLC circuit.

Figure 5.5: Origins of the driven harmonic oscillator equation.

1. The driven harmonic oscillator.
   - Consider a mass on a spring.
   - The extent of friction between the mass point and the surface is described by μ.
   - The oscillation is periodically driven by a force of magnitude $H_0 \cos \omega t$.
2. RLC circuit.
   - R is resistance, L is inductance, C is capacitance.
   - We have the laws

$$LI_L'(t) = V_L \qquad CV_C'(t) = I_C \qquad I_R(t) = V_R(t)/R$$

     - Left: Self-inductance.
     - Right: Ohm's law.
   - Combining them with Kirchhoff's laws

$$I(t) = I_R = I_C = I_L \qquad V(t) = V_R + V_L + V_C$$

     we get the RLC circuit equation

$$LI'' + RI' + \frac{I}{C} = V'(t)$$

   - The most interesting cases is when we have a source of alternating current of frequency ω. In this case, $V(t) = V_0 \cos \omega t$ or, in the complex case, $V(t) = V_0 \mathrm{e}^{i\omega t}$. This yields the complex equation

$$I'' + \frac{R}{L} I' + \frac{1}{LC} I = \frac{i\omega V_0}{L} \mathrm{e}^{i\omega t}$$

   - Here, the friction coefficient $\mu = R/L$ and the frequency is $\omega_0 = \sqrt{1/LC}$.

- Recall that we want to solve the following ODE.

$$x'' + \mu x' + \omega_0^2 x = H_0 \mathrm{e}^{i\omega t}$$

  - The homogeneous linear equation $x'' + \mu x' + \omega_0^2 x = 0$ is well-understood, i.e., we can find all of the *homogeneous* solutions to the above equation.
  - Thus, to solve the above inhomogeneous equation, we just have to find a particular solution.
- WLOG let $\omega > 0$.
- From the homework, a particular solution $x_p(t)$ with initial condition $x_p(0) = x_p'(0) = \mu = 0$ can be obtained from the Duhamel formula as follows.

$$x_p(t) = H_0 \int_0^t \frac{\sin \omega_0 (t-\tau)}{\omega_0} \mathrm{e}^{i\omega\tau} \mathrm{d}\tau$$

  - Substituting

$$\sin\theta = \frac{\mathrm{e}^{i\theta} - \mathrm{e}^{-i\theta}}{2i}$$

    into the above allows us to evaluate it.
  - In particular, it follows that

$$x_p(t) = \begin{cases} \frac{H_0}{\omega_0^2 - \omega^2} \left(\mathrm{e}^{i\omega t} - \cos\omega_0 t - \frac{i\omega}{\omega_0} \sin\omega_0 t \right) & \omega \neq \omega_0 \\ -\frac{iH_0}{2\omega_0} \left(t\mathrm{e}^{i\omega_0 t} - \frac{\sin\omega_0 t}{\omega_0} \right) & \omega = \omega_0 \end{cases}$$

    - We compute the $\omega = \omega_0$ case using L'Hôpital's rule to analyze the $\omega \neq \omega_0$ case as $\omega \to \omega_0$.
  - If we pump in energy at the same point that we have deviation ($\omega = \omega_0$), then the amplitude of oscillation goes to ∞.
    - Practically, when $\omega \approx \omega_0$, the long-time behavior of the driven oscillator will be very much like a growing oscillator.
    - Eventually, the amplitude will be approximately $(\omega - \omega_0)^{-1}$.
- **Resonance catastrophe**: Inputing energy into a system at its natural frequency, causing the total energy to grow until a mechanical failure occurs.
  - This is what happened at the Millenium Bridge in London; synchronized footsteps caused the bridge to shake really wildly.
- If $\mu > 0$, there will be a particular solution of the form

$$x_p(t) = A(\omega) H_0 \mathrm{e}^{i\omega t}$$

  - From HW1, we have three cases when $\mu > 0$: $0 < \mu < 2\omega_0$, $\mu = 2\omega_0$, and $\mu > 2\omega_0$. These are just the three cases of the characteristic polynomial??
  - Substituting the proposed form of the particular solution into the differential equation, we get

$$\begin{aligned} x_p'' + \mu x_p' + \omega_0^2 x_p &= H_0 \mathrm{e}^{i\omega t} \\ (-\omega^2 + i\omega\mu + \omega_0^2) H_0 A(\omega) \mathrm{e}^{i\omega t} &= H_0 \mathrm{e}^{i\omega t} \\ (-\omega^2 + i\omega\mu + \omega_0^2) A(\omega) &= 1 \\ A(\omega) &= \frac{1}{\omega_0^2 - \omega^2 + i\mu\omega} \end{aligned}$$

  - In theory, we avoid the resonance catastrophe in this case. In practice, however, when $\omega \to 0$, we still run into issues.

  - For mass point:

$$|H_0A(\omega)| = \frac{|H_0|}{\sqrt{(\omega^2-\omega_0^2)^2+\mu^2\omega^2}}$$

    - The norm $|H_0A(\omega)|$ is maximized when $\omega_r = \sqrt{\omega_0^2+\mu^2/2}$.
    - $\omega_r \to \omega_0$ implies $\mu \to 0$??
  - As for the argument/angle,

$$\arg(H_0A(\omega)) = \arg H_0 + \arg A(\omega)$$

  - We consider $\omega : 0 \to \omega_0 \to +\infty$.
    - When $\omega = 0$, the complex amplitude is $1/\omega_0^2$ so it's a real number in the complex plane.
    - If ω is increased a bit, we get the reciprocal of a complex number. Norm is reciprocal, argument is negative.
    - For $\omega = \omega_0$, we have a purely imaginary number.
    - As $\omega \to \infty$, the argument approaches $-\pi$??
    - Showing the shape of the norm and the argument with respect to ω. This allows us to completely describe the resonance phenomena.

- For the RLC circuit, the discussion is a bit different.
  - The external voltage $V(t) = V_0\mathrm{e}^{i\omega t}$. Thus, $V'(t) = iV_0\omega\mathrm{e}^{i\omega t}$.
  - Here,

$$x_p(t) = \frac{iV_0\omega\mathrm{e}^{i\omega t}}{\omega_0^2-\omega^2+iR\omega/L}$$

  - Look at the complex amplitude.
    - Multiply the numerator and denominator by the inductance L to get

$$x_p(t) = \frac{iV_0\omega L\mathrm{e}^{i\omega t}}{L\omega_0^2-L\omega^2+iR\omega}$$

    - Then,

$$\text{Norm} = \frac{V_0L}{\sqrt{R^2+\left(\frac{1}{C\omega}-\frac{\omega}{L}\right)^2}}$$

    - For an RLC circuit, the resistance does not affect the resonance frequency.

$$\omega_r = \sqrt{\frac{1}{LC}} = \omega_0$$

    - If you have an external source of voltage, then you can vary the capacity of your circuit to ensure that the voltage will be maximized at a given frequency. We can tune our circuit to a very specific resonance frequency (this is used to filter our radio stations). The RLC circuit is only observable when the resonance coincides with the external resonance.

- There will be a bonus problem which is a PDE describing the vibration of a string.
  - Suppose we have a string with fixed endpoints, and suppose it is undergoing a small vibration.
  - Deviation from the equilibrium is described by a function $u(x,t)$.
  - The simplest equation we can derive is the 1D linear wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial x^2} = f(x,t)$$

    - c is the speed of the wave.

    - $f(x,t)$ is the given external force.
  - We can show that when $f(x,t) = 0$, then the vibration of the string is the linear supposition of infinitely many standing waves.

$$u(x,t) = \sum_{k=1}^{\infty} a_k \mathrm{e}^{\frac{\pi k t}{\ell}} \sin \frac{k\pi}{\ell} x$$

    - There are $k-1$ nodes in the string. These are called standing waves.
  - If you drive it with frequency

$$f(x,t) = \cos \omega t \sin \frac{k\pi}{\ell} x$$

    you encounter the resonance catastrophe.

- We are interested in the driven harmonic oscillator because it describes the vibrations, even of PDEs.
- This concludes our discussion of explicitly solvable differential equations.
- Those that are solvable by power series require complex analysis.
- Starting this Friday, we will talk about the qualitative theory of differential equations.
- Cauchy-Lipschitz this Friday.
- Next week: Continuous dependence on initial values and differentiation with respect to the parameter of this equation.
- After this, we will be able to compute classical examples in the theory of perturbations.
- We will be able to solve the procession of Mercury problem (which was the first experimental verification of general relativity).