# Week 2

# Solving Simple ODEs

## 2.1 Separable ODEs

10/3:

- Do not sit on the left side of the classroom: The sun sucks!
- **Separable** (ODE): An ODE of the form
$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)g(y)$$
where $y$ is a real[1], unknown, scalar function of $t$.
- Solving separable ODEs: Formally, evaluate
$$\int \frac{\mathrm{d}y}{g(y)} = \int f(t)\,\mathrm{d}t$$
- Rearrange the initial separable ODE to $\mathrm{d}y/\mathrm{d}t \cdot 1/g = f$ and invoke the law of composite differentiation to get
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{y_0}^{y(t)} \frac{\mathrm{d}w}{g(w)} - \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau\right] = 0$$
- It follows that
$$\int_{y_0}^{y(t)} \frac{\mathrm{d}w}{g(w)} = \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau$$
- Examples:
  1. Exponential growth.
     - We have that
     $$\frac{\mathrm{d}y}{\mathrm{d}t} = ky$$
     for $k > 0$ and $y(0) = y_0 > 0$.
     - The solution is
     $$\begin{aligned}\frac{1}{y}\cdot\frac{\mathrm{d}y}{\mathrm{d}t} &= k\\ \log y(t) - \log y_0 &= kt\\ y(t) &= y_0\mathrm{e}^{kt}\end{aligned}$$

[1] We'll deal with complex functions later.

2. Logistic growth.
   - We have that
$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky\left(1 - \frac{y}{M}\right)$$
     for $k, M > 0$ and $y(0) = y_0 > 0$.
   - The solution is
$$\begin{aligned}
\frac{M\,\mathrm{d}y}{y(M-y)} &= k\,\mathrm{d}t \\
\log\frac{y}{M-y} - \log\frac{y_0}{M-y_0} &= kt \\
\frac{y(M-y_0)}{y_0(M-y)} &= \mathrm{e}^{kt} \\
y\cdot\frac{M-y_0}{y_0} &= (M-y)\mathrm{e}^{kt} \\
y\cdot\frac{M-y_0}{y_0} + y\mathrm{e}^{kt} &= M\mathrm{e}^{kt} \\
y\left(\frac{M-y_0}{y_0} + \mathrm{e}^{kt}\right) &= M\mathrm{e}^{kt} \\
y\left(\frac{M-y_0+y_0\mathrm{e}^{kt}}{y_0}\right) &= M\mathrm{e}^{kt} \\
y\left(\frac{M+y_0(\mathrm{e}^{kt}-1)}{y_0}\right) &= M\mathrm{e}^{kt} \\
y(t) &= \frac{My_0\mathrm{e}^{kt}}{M+y_0(\mathrm{e}^{kt}-1)}
\end{aligned}$$
   - Sketches the graph of logistic growth and discusses the turning point (for which there is a formula; zero of the second derivative) as well as general trends.
   - If $y_0 < 0$, the solution is not physically meaningful, but it is mathematically insightful.
     - When we integrate, the arguments of our logarithms now have absolute values.
$$\log\left|\frac{y}{M-y}\right| - \log\left|\frac{y_0}{M-y_0}\right| = kt$$
     - We need to make sure that the denominator of the final logistic form is never equal to zero, but now that $y_0$ is negative, as $t$ increases, the denominator will approach zero exponentially. It reaches zero when
$$\begin{aligned}
M + y_0(\mathrm{e}^{kt}-1) &= 0 \\
\mathrm{e}^{kt} &= -\frac{M}{y_0} + 1
\end{aligned}$$
       In other words, $t_{\max} = (1/k)\log(1 - M/y_0)$ because when $t = t_{\max}$, the equation blows up.
     - This is an example of **finite lifespan**.
   - If $y_0 > M$, then you will exponentially decrease to $M$.
3. Lotka-Volterra predator-prey model.
   - We have that
$$r' = k_1 r - awr \qquad\qquad w' = -k_2 w + bwr$$
     where $r$ is rabbits and $w$ is wolves.

- We can rename the variables to

$$\begin{cases} x' = Ax - Bxy \\ y' = -Cy + Dxy \end{cases}$$

- Dividing, we get

$$\frac{x'}{y'} = \frac{Ax - Bxy}{-Cy + Dxy}$$
$$\frac{By - A}{y} y' + \frac{Dx - C}{x} x' = 0$$

- Use the fact that $x, y$ are independent variables, so both terms in the above equation are equal to zero??
- Invoke the law of composite differentiation twice and, from the above, know that $0 + 0 = 0$, so we can add the two solutions:

$$\frac{\mathrm{d}}{\mathrm{d}t}(By(t) - A\log y(t)) + \frac{\mathrm{d}}{\mathrm{d}t}(Dx(t) - C\log x(t)) = 0$$
$$By(t) - A\log y(t) + Dx(t) - C\log x(t) = E$$

- Sketches some of the trajectories (they're all closed curves in the $xy$-plane).

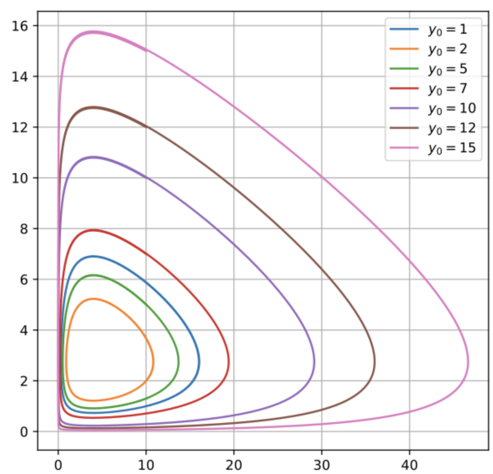


Figure 2.1: Lotka-Volterra solution curves.

- Properties of the curves:
  - The implicit relation which determines them: By the implicit function theorem, the $y$ derivative of the LHS is $B - A/y$ and the $x$-derivative of the LHS is $D - C/x$. When the partial derivatives are equal to zero, $(C/D, A/B)$ becomes interesting. Turning points happen when the $y$-coordinate is $A/B$ or the $x$-coordinate is $C/D$.

- **Finite lifespan**: Even if the RHS of $\mathrm{d}y/\mathrm{d}t = f(t, y)$ is very regular, the solution can still blow up at some finite time.

- Consider the final ODE from the Brachistochrone problem.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{\frac{B - y}{y}}$$

  - Finding the **primitives**.

    - What are these "primitives" Shao keeps talking about??

  - We should have

$$\int \sqrt{\frac{y}{B-y}}\,\mathrm{d}y = x$$

  - Change of variables: $y = B\sin^2\phi$ and $\mathrm{d}y = 2B\cos\phi\sin\phi\,\mathrm{d}\phi$. Thus,

$$\int \sqrt{\frac{y}{B-y}}\,\mathrm{d}y = \int \frac{\sin\phi}{\cos\phi}\cdot 2B\cos\phi\sin\phi\,\mathrm{d}\phi = 2B\int \sin^2\phi\,\mathrm{d}\phi$$

  - The solution is

$$\begin{cases} x = B\phi - \frac{B}{2}\sin(2\phi) + C \\ y = B\sin^2\phi \end{cases}$$

    - This is a parameterization of a cycloid.

- Later in the week, we will do the SHM, the pendulum, the Kepler 2-body problem, and the Michaelis-Menten equation.
- Separable ODEs are a subset of ODEs of **exact form**.
- ODEs of exact form are of the form

$$g(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} + f(x,y) = 0$$

where for some $F(x,y)$, $g = \partial F/\partial y$, $f = \partial F/\partial x$, and partials commute. Equivalently,

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$$

is our necessary and sufficient condition.

- By the law of composite differentiation,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[F(x,y(x))\right] &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\cdot y'(x) \\ &= f(x,y(x)) + g(x,y(x))y'(x) \\ &= 0\end{aligned}$$

  - We solve these with an integrating factor $\mu \neq 0$ such that $(\mu g, \mu f)$ satisfy the constraint.

## 2.2 Office Hours (Shao)

- **Primitive**: An antiderivative.
- **Law of composite differentiation**: The chain rule.
- Went over how Shao has been applying the law of composite differentiation with respect to separable ODEs:
  - Rearrange the initial separable ODE as follows.

$$\frac{1}{g(y)}\cdot\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)$$

- Define $\mathrm{d}H/\mathrm{d}y = 1/g(y)$. Then, continuing from the above, we have by the law of composite differentiation that

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}y}\cdot\frac{\mathrm{d}y}{\mathrm{d}t} &= f(t)\\ \frac{\mathrm{d}H}{\mathrm{d}t} &= f(t)\end{aligned}$$

- From the definition of $H$, we know that $H(y) = \int_{y_0}^{y} \mathrm{d}w\,/g(w)$. We also have from the FTC that $f(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau$. Thus, continuing from the above, we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(H) &= f(t)\\ \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{y_0}^{y}\frac{\mathrm{d}w}{g(w)}\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau\\ \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{y_0}^{y(t)}\frac{\mathrm{d}w}{g(w)} - \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau\right] &= 0\end{aligned}$$

as desired.

- It follows since $y(t_0) = y_0$ that $C = H(y_0) = 0$, so we can take the above to

$$\int_{y_0}^{y(t)}\frac{\mathrm{d}w}{g(w)} = \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau$$

knowing that our constant of integration is zero.

- Take away from Brachistochrone problem: Just an example of a BDE; we won't have to answer questions on it.

## 2.3 ODEs of Exact Form

10/5:
- Last time, we discussed separable ODEs.
- Today, we will study **exact form** equations, as discussed last class.
- **Exact form** (ODE): An ODE of the form

$$g(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} + f(x,y) = 0$$

where

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$$

- For equations of this form, there exists $F(x,y)$ such that

$$\frac{\partial F}{\partial x} = f \qquad \frac{\partial F}{\partial y} = g \qquad F(x, y(x)) = C$$

for some $C \in \mathbb{R}$.

- To solve equations of this form, we need an **integrating factor**.
- **Integrating factor**: A number or function $\mu$ such that

$$\mu g\frac{\mathrm{d}y}{\mathrm{d}x} + \mu f = 0 \qquad \frac{\partial}{\partial x}(\mu g) = \frac{\partial}{\partial y}(\mu f)$$

- The solution to linear homogeneous equations of the form $\mathrm{d}y/\mathrm{d}t = p(t)y$ is

$$y(t) = y_0 \exp\left[\int_{t_0}^{t} p(\tau)\mathrm{d}\tau\right]$$

- Recall that $\mathrm{e}^{a+ib} = \mathrm{e}^a(\cos b + i\sin b)$, so

$$\mathrm{e}^{ix} = \cos x + i\sin x \qquad \cos x = \frac{1}{2}(\mathrm{e}^{ix} + \mathrm{e}^{-ix}) \qquad \sin x = \frac{1}{2i}(\mathrm{e}^{ix} - \mathrm{e}^{-ix})$$

- Example: If $y' = ky$, then $y' = -\lambda y$.
- If we have an inhomogeneous linear equation $\mathrm{d}y/\mathrm{d}t = p(t)y + f(t)$, then

$$\frac{\mathrm{d}y}{\mathrm{d}t} - py - f = 0$$

  but

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(1) \neq \frac{\mathrm{d}}{\mathrm{d}y}(-p(t)y - f(t))$$

- We wish to find an integrating factor $\mu(t, y)$ such that

$$\mu(t, y)\frac{\mathrm{d}y}{\mathrm{d}t} - \mu(t, y)p(t)y - \mu(t, y)f(t) = 0$$

  and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu) = \frac{\mathrm{d}}{\mathrm{d}y}(-\mu py - \mu f)$$

- Solution: Take $\mu$ to be a function of $t$, alone. Then

$$\mu'(t) = \frac{\mathrm{d}}{\mathrm{d}y}(-\mu py - \mu f) = -\mu(t)p(t)$$

  and we now have a homogeneous linear equation with solution

$$\mu(t) = \exp\left[-\int_{t_0}^{t} p(\tau)\mathrm{d}\tau\right]$$

  - If we let $P(t) = \int_{t_0}^{t} p(\tau)\mathrm{d}\tau$, then

$$\begin{aligned}
\mathrm{e}^{-P(t)}y'(t) - p(t)y(t)\mathrm{e}^{-P(t)} &= \mathrm{e}^{-P(t)}f(t) \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-P(t)}y(t)\right) &= \mathrm{e}^{-P(t)}f(t) \\
\mathrm{e}^{-P(t)}y(t) &= \int_{t_0}^{t} \mathrm{e}^{-P(\tau)}f(\tau)\mathrm{d}\tau
\end{aligned}$$

  - Thus, we finally have the solution to the inhomogeneous problem as follows: The IVP $y' = py + f$, $y(t_0) = y_0$ has solution

$$y(t) = y_0\mathrm{e}^{P(t)-P(t_0)} + \mathrm{e}^{P(t)}\int_{0}^{t} \mathrm{e}^{-P(\tau)}f(\tau)\mathrm{d}\tau$$

    where $P$ is any anti-derivative of $p$.

- In particular, when $p(t) = a$, we get the **Duhamel formula** (which we should memorize).

- **Duhamel formula**: The following equation, which is the solution to an inhomogeneous linear equation with $p(t) = a$.
$$y(t) = y_0 \mathrm{e}^{a(t-t_0)} + \int_{t_0}^{t} \mathrm{e}^{a(t-\tau)} f(\tau) \mathrm{d}\tau$$
  - Important for computing forced oscillation.
- Inspecting the inhomogeneous solution.
  - The first term is the solution to the homogeneous form. The second term deals with the initial value.
- Given an inhomogeneous equation, you can always write its solution as the combination of the solution to the homogeneous problem plus a particular solution, i.e.,
$$y = y_h + y_p$$
  - "The general solution equals the homogeneous solution plus a particular solution."
  - This is related to linear algebra, where the solution to $Ax = b$ is a particular solution $x_p$ plus any vector $x \in \ker A$.
  - Thus, this idea will reappear in the theory of systems of linear ODEs.
- We now look at systems of linear ODEs.
- Consider the harmonic oscillator: A particle of mass $m$ connected to an ideal spring (obeys Hooke's law) with no friction or gravity.
  - Newton's second law: The acceleration is proportional to the restoring force.
  - Hooke's law: The restoring force is of magnitude $kx$ in the opposite direction to the displacement.
  - Thus, the ODE is of the form
$$x'' = -\frac{k}{m} x$$
  - However, if there is damping (which will be proportional to the velocity), then the ODE is of the form
$$x'' + \frac{b}{m} x' + \frac{k}{m} x = 0$$
- Consider an ODE of the form
$$y'' + ay' + by = 0$$
for $a, b \in \mathbb{C}$.
  - Aim: Find $\mu, \lambda \in \mathbb{C}$ such that
$$(y' - \mu y)' - \lambda (y' - \mu y) = 0$$
  - To find the parameters, we expand the above to
$$y'' - (\mu + \lambda) y' + \mu \lambda y = 0$$
  - Comparing with the original form, we have that $a = -(\mu + \lambda)$ and $b = \mu \lambda$.
  - It follows that $\mu, \lambda$ are the roots of $x^2 + ax + b = 0$, which we will call the **characteristic polynomial** of the ODE.
  - Substitute $x = y' - \mu y$. Then we can solve
$$x' - \lambda x = 0$$
to determine that $x = A\mathrm{e}^{\lambda t}$.

  - Returning the substitution, we have that
$$y' - \mu y = A\mathrm{e}^{\lambda t}$$
  - Since the above is of the form $y' = ay + f$, we can apply the Duhamel formula. It follows that a particular solution is
$$A\int_0^t \mathrm{e}^{\mu(t-\tau)}\mathrm{e}^{\lambda\tau}\mathrm{d}\tau$$
  - Thus, general solutions are of the form
$$y(t) = B\mathrm{e}^{\mu t} + C\mathrm{e}^{\mu t}\int_0^t \mathrm{e}^{(\lambda-\mu)\tau}\mathrm{d}\tau$$
  - Evaluating the integral, we get
$$y(t) = B\mathrm{e}^{\mu t} + C\mathrm{e}^{\mu t}\frac{\mathrm{e}^{(\lambda-\mu)t} - 1}{\lambda - \mu}$$
    which simplifies (by incorporating constants, etc.) to
$$y(t) = A_1\mathrm{e}^{\mu t} + B_1\mathrm{e}^{\lambda t}$$
    for $\mu \neq \lambda$, or
$$y(t) = A_1\mathrm{e}^{\mu t} + B_1 t\mathrm{e}^{\mu t}$$
    for $\mu = \lambda$.
  - These linearly independent solutions form a basis of the space of solutions; all solutions can be expressed as a linear combination of these two functions.

- If our equation is of the form $y'' + ay' + by = f(t)$, then we just need to apply the Duhamel formula twice.
- Returning to the simple harmonic oscillator problem, we substitute $\omega = \sqrt{k/m}$ to get
$$x'' = \omega^2 x$$
  - The characteristic polynomial is
$$0 = x^2 + \omega^2 = (x + i\omega)(x - i\omega)$$
  - Thus, solutions are of the form
$$x = A_1\mathrm{e}^{i\omega t} + B_1\mathrm{e}^{-i\omega t}$$
  - It follows that the period is $T = 2\pi/\omega$.
  - To get a real (usable) solution, apply Euler's formula to get
$$\begin{aligned} x(t) &= A_1(\cos\omega t + i\sin\omega t) + B_1(\cos\omega t - i\sin\omega t) \\ &= A\cos\omega t + B\sin\omega t \end{aligned}$$
    where $A = A_1 + B_1$, $B = iA_1 - iB_1$.
  - To match the initial condition $x(0) = x_0$, $x'(0) = v_0$, we use
$$x(t) = x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t$$
  - In other words,
$$\begin{cases} A = x_0 \\ \omega B = v_0 \end{cases} \qquad \begin{cases} A_1 + B_1 = x_0 \\ i\omega A_1 - i\omega B_1 = v_0 \end{cases}$$
    so
$$\begin{cases} A = x_0 \\ B = \frac{v_0}{\omega} \end{cases} \qquad \begin{cases} A_1 = \frac{1}{2}\left[x_0 - \frac{iv_0}{\omega}\right] \\ B_1 = \frac{1}{2}\left[x_0 + \frac{iv_0}{\omega}\right] \end{cases}$$

## 2.4 ODE Examples

10/7:

- Today, we will investigate a variety of examples of ODEs arising in real life.
- Michaelis-Menten kinetics: If E is an enzyme, S is its substrate, and P is the product, then the mechanism is

$$\mathrm{E} + \mathrm{S} \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} \mathrm{ES} \xrightarrow{k_2} \mathrm{E} + \mathrm{P}$$

- The concentrations that we are concerned with are $[\mathrm{E}], [\mathrm{S}], [\mathrm{ES}], [\mathrm{P}]$.
- From the above mechanism, we can write the four rate laws

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{S}] = -k_1[\mathrm{E}][\mathrm{S}] + k_{-1}[\mathrm{ES}] \tag{1}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{E}] = -k_1[\mathrm{E}][\mathrm{S}] + (k_{-1} + k_2)[\mathrm{ES}] \tag{2}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{ES}] = k_1[\mathrm{E}][\mathrm{S}] - (k_{-1} + k_2)[\mathrm{ES}] \tag{3}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{P}] = k_2[\mathrm{ES}] \tag{4}$$

- The initial conditions are $[\mathrm{S}] = [\mathrm{S}]_0$ and $[\mathrm{E}] = [\mathrm{E}]_0$.
- We can reduce these rate laws to the 2D system

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{S}] = -k_1([\mathrm{E}]_0 - [\mathrm{ES}])[\mathrm{S}] + k_{-1}[\mathrm{ES}] \tag{5}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{ES}] = k_1([\mathrm{E}]_0 - [\mathrm{ES}])[\mathrm{S}] - (k_{-1} + k_2)[\mathrm{ES}] \tag{6}$$

  - Note that to do so, we have used two conservation laws: The conservation of the enzyme plus enzyme-substrate complex, and the conservation of the substrate plus enzyme-substrate complex plus products.
- QSSA: Quasi steady-state assumption.
  - Assume that $[\mathrm{E}]_0/[\mathrm{S}]_0 \lll 1$.
  - It follows that $\mathrm{d}[\mathrm{ES}]/\mathrm{d}t \approx 0$ due to saturation of the enzyme and $[\mathrm{S}] \approx [\mathrm{S}]_0$ due to ever-more substrate being available.
- Then

$$[\mathrm{ES}] = \frac{[\mathrm{E}]_0[\mathrm{S}]}{K_M + [\mathrm{S}]}$$

  where $k_M = (k_{-1} + k_2)/k_1$ is the **Michaelis-Menten constant**, a usual indication of enzyme activity.
- Substitute the above into Equation 5:

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{S}] = -\frac{v_{\max}[\mathrm{S}]}{k_M + [\mathrm{S}]}$$

  - Note that $v_{\max} = k_2[\mathrm{E}]_0$.
- The above is now a differential equation of separable form; it's solution is

$$\int_{[\mathrm{S}]_0}^{[\mathrm{S}]} -\frac{(k_M + z)\,\mathrm{d}z}{zv_{\max}} = \int_0^t \mathrm{d}t$$
$$-\frac{k_M}{v_{\max}} \log\frac{[\mathrm{S}]}{[\mathrm{S}]_0} - \frac{1}{v_{\max}}([\mathrm{S}] - [\mathrm{S}]_0) = t$$

$$
\begin{aligned}
\log\frac{[\mathrm{S}]}{[\mathrm{S}]_0}+\frac{[\mathrm{S}]}{k_M}&=\frac{[\mathrm{S}]_0-v_{\max}t}{k_M}\\
\frac{[\mathrm{S}]}{[\mathrm{S}]_0}\mathrm{e}^{[\mathrm{S}]/k_M}&=\exp\left(\frac{[\mathrm{S}]_0-v_{\max}t}{k_M}\right)\\
\frac{[\mathrm{S}]}{k_M}\mathrm{e}^{[\mathrm{S}]/k_M}&=\frac{[\mathrm{S}]_0}{k_M}\exp\left(\frac{[\mathrm{S}]_0-v_{\max}t}{k_M}\right)\\
\frac{[\mathrm{S}]}{k_M}&=W\left[\frac{[\mathrm{S}]_0}{k_M}\exp\left(\frac{[\mathrm{S}]_0-v_{\max}t}{k_M}\right)\right]\\
[\mathrm{S}]&=k_MW\left[\frac{[\mathrm{S}]_0}{k_M}\exp\left(\frac{[\mathrm{S}]_0-v_{\max}t}{k_M}\right)\right]
\end{aligned}
$$

  - Getting from line 5-6 (i.e., the introduction of $W$): Suppose we have an equation of the form $y\mathrm{e}^y=x$. We cannot express $x$ in terms of $y$ using elementary functions, so we must define $W$ such that $y=W(x)$. Explicitly, $W$ is the unique function of $x$ that satisfies $W(x)\mathrm{e}^{W(x)}=x$.

- Harmonic oscillator.
- Recall that
$$x''+\frac{k}{m}x=0$$
- Substituting $\omega=\sqrt{k/m}$, we can solve the above for
$$x(t)=x(0)\cos(\omega t)+\frac{x'(0)}{\omega}\sin(\omega t)$$
- This is an integrable system with $n$ degrees of freedom and $n-1$ scalar conservation laws??
- Conservation of mechanical energy:
$$E=\frac{1}{2}m|x'|^2+\frac{1}{2}kx^2$$

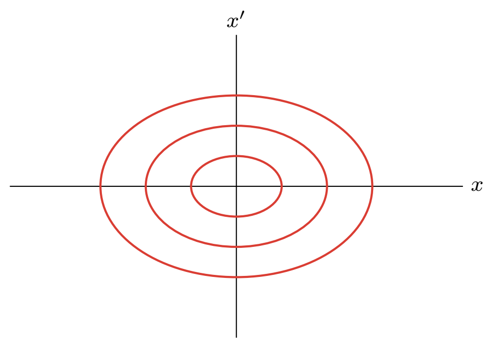


Figure 2.2: Conservation of mechanical energy in the harmonic oscillator.

  - Differentiating wrt. $x$ yields
$$
\begin{aligned}
0&=mx'x''+kxx'\\
&=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m(x')^2\right)+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}kx^2\right)
\end{aligned}
$$
  - This means that the solution is an ellipse in the $xx'$-plane, where each ellipse corresponds to an initial displacement and velocity.

- Mathematical pendulum.
- Equation of motion:

$$\begin{aligned} 0 &= l\theta'' + g\sin\theta \\ &= \ell\theta''\theta' + g\sin\theta\cdot\theta' \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg(\underbrace{\frac{\ell}{2}|\theta'|^2 - g\cos\theta}_{E}\bigg) \end{aligned}$$

- Initial values:

$$\theta(0) = \theta_0 \qquad\qquad \theta'(0) = 0$$

- It follows from the above that

$$\begin{aligned} \frac{\ell}{2}|\theta'|^2 - g\cos\theta_0 &= -g\cos\theta \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \sqrt{\frac{2g}{\ell}(\cos\theta_0 - \cos\theta)} \\ \int_{\theta_0}^{\theta}\sqrt{\frac{\ell}{2g(\cos\theta_0 - \cos\phi)}}\mathrm{d}\phi &= t \end{aligned}$$

  - This is an elliptical integral (and thus cannot be expressed in terms of the elementary functions).
- Suppose $\theta_0$ is small. Then $\theta$ is small, and we can invoke the small angle approximation $\sin\theta \approx \theta$.
  - This yields an approximate equation of motion:

$$\ell\theta'' + g\theta = 0$$

  - From here, we can determine that $\theta(t) \approx \theta_0 \cos\sqrt{g/\ell}\cdot t$ and $T = 2\pi\sqrt{\ell/g}$.
- Kepler problem.
- Two bodies of mass $m_1, m_2$ are located at positions $x_1, x_2$ pulling on each other gravitationally.
  - The force of attraction is a conservative central force.
  - The potential between the two masses is a function of their distance, i.e, $U(|x_1 - x_2|)$.
- From Newton's second and third law, we get

$$m_1x_1'' = U'(|x_1 - x_2|)\frac{x_2 - x_1}{|x_2 - x_1|} \qquad\qquad m_2x_2'' = U'(|x_1 - x_2|)\frac{x_1 - x_2}{|x_1 - x_2|}$$

  - The derivative of potential is force.
  - The vector term provides direction.
- Conservation of momentum:

$$\begin{aligned} (m_1x_1 + m_2x_2)'' &= 0 \\ m_1x_1' + m_2x_2' &= C \end{aligned}$$

  - Let $M = m_1 + m_2$. Then the center of mass

$$\frac{m_1}{M}x_1 + \frac{m_2}{M}x_2$$

    moves inertially (i.e., does not accelerate or decelerate; is a stable reference frame) — we'll define it to be the origin.

- Conservation of angular momentum:
$$[m(x_1 - x_2)' \times (x_1 - x_2)]' = 0$$
  - $m = m_1m_2/(m_1 + m_2)$.
  - $\times$ indicates the cross product.
  - $L = m(x_1 - x_2)' \times (x_1 - x_2)$.
- It follows that $x_1 - x_2$ is always in a fixed plane, which we may call the **horizon plane**.
- Conservation of mechanical energy:
$$mq'' + U'(|q|)\frac{q}{|q|} = 0$$
$$\frac{m}{2}|q'|^2 + U(|q|) = E$$
  - $q = x_1 - x_2$.
- Introduce polar coordinates $(r, \phi)$.
  - Then $r^2\phi' = \ell_0$, $r = r(\phi)$, and $\mathrm{d}r/\mathrm{d}\phi = r'(t)/\phi'(t)$.
  - It follows that
$$\frac{m}{2}(|r'|^2 + |\phi'|^2) + U(r) = E$$
  - Since $U(r) = -Gm_1m_2/r$ for Newtonian gravity,
$$\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 + r^2 = \frac{2GMr^3}{\ell_0^2} + \frac{2Er^4}{m\ell_0^2}$$
  - The substitution $\mu = 1/r$ yields
$$\left(\frac{\mathrm{d}\mu}{\mathrm{d}\phi}\right)^2 + \mu^2 = \frac{2GM}{\ell_0^2}\mu + \frac{2E}{m\ell_0^2}$$
  - Differentiating again gives
$$2\frac{\mathrm{d}\mu}{\mathrm{d}\phi}\frac{\mathrm{d}^2\mu}{\mathrm{d}\phi^2} + 2\mu\frac{\mathrm{d}\mu}{\mathrm{d}\phi} = \frac{2GM}{\ell_0^2}\frac{\mathrm{d}\mu}{\mathrm{d}\phi}$$
  - Substituting $\mu = \cos(t)$ gives
$$\frac{\mathrm{d}^2\mu}{\mathrm{d}\phi^2} + \mu - \frac{GM}{\ell_0^2} = 0$$
    or
$$r = \frac{1}{GM/\ell_0^2 + \varepsilon\cos(\phi - \phi_0)}$$
    - This is a conic section!
- Thus, for example, we can calculate the precession of Mercury.
- Note that while we have determined the trajectory of our 2 bodies in terms of elementary functions, the $n$-body problem cannot be solved analytically.

## 2.5 Chapter 1: Introduction

*From Teschl (2012).*

## Section 1.3: First Order Autonomous Equations

11/15:
- We start with the simplest nontrivial case of a first-order autonomous equation:
$$\dot{x} = f(x), \quad x(0) = x_0$$
  - We may let $t_0 = 0$ WLOG: If $\phi(t)$ is a solution to an autonomous equation satisfying $\phi(0) = x_0$, then $\psi(t) = \phi(t - t_0)$ is a solution with $\psi(t_0) = 0$.
- Solving this ODE.
  - Suppose $f(x_0) \neq 0$. Divide both sides by $f(x)$ and integrate from the initial conditions onward to yield
$$\int_0^t \frac{\dot{x}(s)\,\mathrm{d}s}{f(x(s))} = t$$
  - Define
$$F(x) := \int_{x_0}^x \frac{\mathrm{d}y}{f(y)}$$
    - Note that this is just the previous equation under the "$u$-substitution" $y(t) = x(t)$, $\mathrm{d}y = \dot{x}(t)\,\mathrm{d}t$, $y(0) = x(0) = x_0$, $y(t) = x$.
  - Thus, in our new notation, any possible solution $x$ to the ODE must satisfy $F(x(t)) = t$. Since $F(x(t))$ is monotone (near $x_0$??), it can thus be inverted to yield the unique solution
$$\phi(t) = F^{-1}(t), \quad \phi(0) = F^{-1}(0) = x_0$$
- Teschl (2012) does a deep dive on the maximal integral where $\phi$ is defined.
- Examples of first-order autonomous systems given.
- Most of this section goes beyond what was covered in class in terms of depth.

## Section 1.4: Finding Explicit Solutions

- Solving ODEs for explicit solutions is impossible in general unless the equation is of a particular form.
- This section: Classes of first-order ODEs which are explicitly solvable.
- Strategy: Find a change of variables that transforms the ODE into a solvable form.
- Linear equation.
$$\dot{x} = a(t)x \qquad\qquad \dot{x} = a(t)x + g(t)$$
  - The left equation above is the homogeneous linear equation, and the right equation above is the corresponding inhomogeneous linear equation.
  - The general solution to the homogeneous linear equation is
$$\phi(t) = x_0 A(t, t_0)$$
    where
$$A(t, s) = \mathrm{e}^{\int_s^t a(s)\mathrm{d}s}$$
  - The general solution to the inhomogeneous linear equation is
$$\phi(t) = x_0 A(t, t_0) + \int_{t_0}^t A(t, s) g(s)\,\mathrm{d}s$$
  - Teschl (2012) covers the more detailed mathematics of coordinate transformations in depth. This definitely would have been useful to understand for solving the PSet 1 problems, so I should return and understand it before the final.
- Using Mathematica to help solve ODEs and gain an intuition for how they work (e.g., with slope fields).
- Equations of exact form are covered in the problems to this section.