Week 3

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3.1 Linear Algebra Review

10/10:

- Today: Review of linear algebra.
- Start with a **vector space** over $\mathbb{R}$ or $\mathbb{C}$ or, more generally, any field K.
- **Vector space** (over K): A set equipped with addition and scalar multiplication such that the following axioms are satisfied.
  1. Commutativity and associativity of addition.
  2. Additive identity and inverse.
  3. Compatibility of scalar multiplication and addition (distributive laws).
  4. The additive identity times any vector is zero.
- In $\mathbb{R}^n, \mathbb{C}^n$, addition is component-wise and scalar multiplication is scaling of the element.
- For a homogeneous equation

$$y' = A(t)y = \begin{pmatrix} a_{11}(t)y^1 + a_{12}(t)y^2 + \cdots \\ \vdots \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \\ \vdots \\ y^n \end{pmatrix}$$

  - If y_1, y_2 are solutions, any linear combination of them is a solution. This is called the **solution space** of the equation.
- **Linearly independent** (set of vectors): A set of vectors $x_1, \dots, x_m \in V$ such that the only coefficients $\lambda_1, \dots, \lambda_m$ such that

$$\lambda_1 x_1 + \cdots + \lambda_m x_m = 0$$

  is $\lambda_1 = \cdots = \lambda_m = 0$.
  - $\lambda_m \neq 0$ implies

$$x_m = -\frac{1}{\lambda_m}(\lambda_1 x_1 + \cdots + \lambda_{m-1} x_{m-1})$$

- **Maximal linear independence group**: A subset $X \subset V$ such that for any $y \in V$, $\{y\} \cup X$ is not linearly independent. *Also known as* **basis**.
- Theorem: Any basis in V has the same cardinality.
- **Dimension** (of V): The cardinality given by the above theorem. *Denoted by* $\mathbf{dim}\, \boldsymbol{V}$.

- We usually denoted a basis as an ordered n-tuple since the order often matters (for orientation?).
- Notational onventions.
  - For $\mathbb{R}^n, \mathbb{C}^n$, we will always use column vectors.
  - $x_1, x_2, \dots$ denotes vectors.
  - $x^1, x^2, \dots$ denotes the components of a column vector.
  - A vector component squared may be denoted $(x^1)^2$.
- **Standard basis** (for $\mathbb{R}^n$): The set of n vectors of length n which have a 1 as one entry and a zero in all the others and are all distinct.
- **Linear transformation** (of V to V): A mapping $\phi : V \to V$ satisfying

$$\phi(\lambda x + \mu y) = \lambda\phi(x) + \mu\phi(y)$$

- A mapping is completely determined by its action on the basis vectors:

$$\phi\left(\sum_{k=1}^{n} x^k e_k\right) = \sum_{k=1}^{n} x^k \phi(e_k)$$

- **Matrix** (of a linear transformation wrt. the standard basis): The $n \times n$ array

$$\begin{pmatrix} \phi(e_1) & \cdots & \phi(e_n) \end{pmatrix}$$

- If $\phi, \psi : V \to V$ are linear, $\phi \circ \psi$ is also linear.
  - Composition of linear transformations corresponds to matrix multiplication.
- Matrix multiplication: If

$$B = \begin{pmatrix} b_1 & \cdots & b_n \end{pmatrix}$$

  then

$$AB = \begin{pmatrix} Ab_1 & \cdots & Ab_n \end{pmatrix}$$

  where

$$Ax = \begin{pmatrix} a_{11}x^1 + \cdots + a_{1n}x^n \\ \vdots \\ a_{n1}x^1 + \cdots + a_{nn}x^n \end{pmatrix}$$

- We can talk about matrix inverses: If it exists, it is unique, and

$$AA^{-1} = A^{-1}A = I_n$$

- Matrix multiplication is not commutative in general. Shao gives a counterexample.
- A is invertible iff the columns of A are a basis for $\mathbb{R}^n$ (resp. $\mathbb{C}^n$).
- **Determinant** (of A): Not defined.
- Properties of the determinant.
  - Multilinear.

$$\det\begin{pmatrix} a_1 & \cdots & \lambda a_k + \mu\tilde{a}_k & \cdots & a_n \end{pmatrix} = \lambda\det\begin{pmatrix} a_1 & \cdots & a_k & \cdots & a_n \end{pmatrix} + \mu\det\begin{pmatrix} a_1 & \cdots & \tilde{a}_k & \cdots & a_n \end{pmatrix}$$

  - Skew-symmetric.

$$\det\begin{pmatrix} a_1 & \cdots & a_i & \cdots & a_j & \cdots & a_n \end{pmatrix} = -\det\begin{pmatrix} a_1 & \cdots & a_j & \cdots & a_i & \cdots & a_n \end{pmatrix}$$

- Theorem: The determinant is uniquely characterized by these three (??) axioms.
- $\det I_n = 1$.
- Shao goes over computing the determinant via minors.
- Special cases:
  - If the matrix is upper- or lower-triangular, the determinant is equal to the product of the diagonal entries.
  - If the matrix is blocked upper- or lower-triangular, e.g.,

$$A = \begin{pmatrix} A_1 & * \\ 0 & A_2 \end{pmatrix}$$

    then $\det A = \det A_1 \cdot \det A_2$.
- $\det(AB) = \det(A)\det(B)$.
- $\det A \neq 0$ iff A is invertible.
- Direct formula to compute the inverse.

$$A^{-1} = \frac{1}{\det A}\left(a_{\ell k}(-1)^{k+\ell}\det A_{k\ell}\right)$$

  - Tedious for higher-dimensional cases, but quite sufficient for $n = 2, 3$.
- Let A be $n \times n$, and let $Ax = b$.
  - If A is invertible, then $x = A^{-1}b$.
  - If A is not invertible and $b \in \operatorname{span}(a_1, \dots, a_n)$, then $x = x_h + x_p$ where $Ax_h = 0$ and $Ax_p = b$.
- **Kernel** (of A): The set of all vectors $y \in \mathbb{R}^n$ (resp. $\mathbb{C}^n$) such that $Ay = 0$.
- **Range** (of A): The set of all linear combinations of $a_1, \dots, a_n$.
- Suppose $\phi : \mathbb{R}^n \to \mathbb{R}^n$ has matrix A under $(e_1, \dots, e_n)$. Let $(q_1, \dots, q_n)$ be another basis.
  - There exists a matrix Q such that $q_k = Qe_k$. Q is called the **connecting matrix** between $(e_1, \dots, e_n)$ and $(q_1, \dots, q_n)$.
  - Claim: Let $x \in \mathbb{R}^n$ have representation $x = (x^1, \dots, x^n)$ under the standard basis. Then under the Q basis, x has representation $x' = Q^{-1}(x^1, \dots, x^n)$. Similarly, $x = Qx'$.
  - Claim: ϕ has matrix $B = Q^{-1}AQ$ with respect to the Q basis.
- Matrix similarity: $A \sim B$ iff there exists Q invertible such that $B = Q^{-1}AQ$.
  - Implies that A and B describe the same matrix under different bases.
  - Matrix product under the old and new bases are related.

$$Q^{-1}ABQ = (Q^{-1}AQ)(Q^{-1}BQ)$$

  - Similarity preserves the determinant:

$$\det\left(Q^{-1}AQ\right) = \det\left(Q^{-1}\right)\det(A)\det(Q) = \det(A)\det\left(Q^{-1}\right)\det(Q) = \det(A)$$