MATH 27300 (Basic Theory of Ordinary Differential Equations) Notes

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Weeks

List of Figures

Week 1

Introduction to ODEs

1.1 Definitions and Scope

9/28:

- Questions:
  - When will the PDFs be made available?
- Office: Eckhart 309.
  - Office hours: MWF 3:00-4:00.
- Reader: Walker Lewis. His contact info is in the syllabus.
- Final grade is based on...
  - 2 midterms (15 pts. each; weeks 4 and 8).
  - Final exam (35 pts.).
  - HW (35 pts.).
  - Bonus problems (15 pts).
- Total points for the quarter is 115. The bonus problems usually arise from advanced math and incorporate more advanced knowledge, and we are encouraged to seek out all relevant resources as long as we write up our own solutions.
- **Ordinary differential equation**: Any equation that takes the form $F(t,y,y',\dots,y^{(n)})=0$. *Also known as* **ODE**.
  - F is a known function.
  - t is an argument (time). x is also used (when space is involved).
  - $y=y(t)$ is an unknown function.
- **Order n** (ODE): An ODE for which the n^{th} derivative of y is the highest-order derivative involved (and is involved).
- $y'=f(t,y)$ or $Y^{(n)}=F(t,Y,Y',\dots,Y^{(n-1)})$.
  - We can transform this second form into the first form via

$$y=\begin{pmatrix} Y \\ Y' \\ \vdots \\ Y^{(n-1)} \end{pmatrix} \qquad f(t,y)=\begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ F(t,y_1,y_2,\dots,y_{(n-1)}) \end{pmatrix}$$

making $y' = f(t, y)$ equal to the system of equations

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_{n-1}' &= F(t, y_1, \dots, y_{n-1}) \end{aligned}$$

  - Think about this conversion more.

- Thus, we mainly focus on equations of the form $y' = f(t, y)$, because that's general enough.

- **Linear** (ODE): Any ODE that can be written in the form

$$y' = A(t)y + f(t)$$

- Because of the above, this naturally includes equations of the form

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_0(t)y = b(t)$$

- **Nonlinear** (ODE): An ODE that is not linear.
- **Autonomous** (ODE): An ODE that can be written in the form

$$y' = f(y)$$

  - More equivalence w/ vector-valued functions?

- **Nonautonomous** (ODE): An ODE that is not autonomous.
  - We will not investigate these in this course.
- **Initial value problem**: A problem of the form: Find $y(t)$ such that

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

Also known as **IVP**, **Cauchy problem**.

- Locally well-posed (LWP) conditions:
  1. Existence (local in time).
  2. Uniqueness (you cannot have multiple solutions).
  3. Local stability (if you perturb your initial value or equation a little bit, you do not expect your solution to vary crazily [esp. locally]).
- Example of a nonunique ODE:
  - $y' = \sqrt{y}$, $y(0) = 0$ has solutions $y_1(t) = 0$ $(t \geq 0)$ and $y_2(t) = t^2/4$ $(t \geq 0)$.
  - We will investigate the reason later.
- Preview of the reason: **Cauchy-Lipschitz Theorem** or **Picard-Lindelof Theorem**.
  - As long as the ODE is **Lipschitz continuous**, it's locally stable.
- **Lipschitz continuous** (function): A function f such that

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$$

  - But in the counterexample above, the slope of the chord from 0 to $y(t)$ approaches infinity as $t \to 0$.

- **Peano Existence Theorem**: ...
- **Dynamical system**: A law under which a particle evolves over time. $y' = f(t, y)$, IVP is LWP
- Consider $\Phi(t, x)$ such that
$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,x) = f(t, \Phi(t,x)) \\ \Phi(0,x) = x \end{cases}$$
- **Steady flow**: A vector field on a manifold contained in $\mathbb{R}^2$ or $\mathbb{R}^3$ that does not vary with time.
  - A velocity field.
  - Trajectory of a particle: At $x \in \Omega$, the velocity of the particle should coincide with $X(x)$.
  - The differential equation $\dot{x} = X(x)$ is what we're interested in.
  - A solid shape gets shifted and deformed (imagine a chunk of water falling out of the end of a pipe).
  - Differential geometry is the purview of such things.
- Newton's law of motion $F = m \cdot a$ applied to n particles is nothing but the system of equations
$$m_i x_i'' = F_i(x_1, \dots, x_n)$$
for $i = 1, \dots, n$.
  - Many well-known examples.
  - The best known one perhaps is that of uniform acceleration of a single particle. In this case,
$$m_0 x'' = f_0$$
    - The solution is
$$x(t) = \frac{f_0}{2m_0}t^2 + v_0 t + x_0$$
    where $x_0 = x(0)$ and $v_0 = x'(0)$ are the initial conditions.
  - A simple example is downwards motion due to gravity. Then
$$x(t) = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} t^2 + v_0 t + x_0$$
    - The trajectory in general is a parabola.
  - Another example: The mathematical pendulum.
    - The radial directions balance ($mg\cos\theta$).
    - The tangential directions do not ($mg\sin\theta$). Thus, our ODE is
$$l\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = g\sin\theta$$
  - One last set of examples from ecology:
    - Imagine an petri dish of infinite nutrition. The population growth of the bacteria will obey the exponential growth law
$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky$$

- Suppose we have a system capacity M. Then we obey the logistic growth law

$$\frac{\mathrm{d}y}{\mathrm{d}t} = k(M - y)$$

- Lotka-Volterra prey-predator model: Wolf population (W) and rabbit population (R). We have

$$\begin{aligned} R' &= k_1 R - aWR \\ W' &= -k_2 W + bWR \end{aligned}$$

- We can also introduce more species and capacities and et cetera, et cetera.

- Conclusion: Dynamical systems are everywhere, especially in physics, chemistry, and ecology.
- We can also consider long-term behavior.
  - We can have chaos, but chaos can be reasoned with using oscillation, systems that converge to oscillation, etc. We will mostly be focusing on the regular aspect of the long-term behavior.

1.2 Origin of ODEs: Boundary Value Problems

9/30:

- Textbook PDFs will be posted today.
- Today, we will consider boundary value problems, which are separate from dynamical systems but not entirely unrelated.
- **Boundary Value Problem**: A problem in which we are solving for a y that has fixed values at the boundaries $x = a, b$. *Also known as* **BVP**.
- The **Brachistochrone problem** is an example of a BVP.
- **Brachistochrone problem**: Suppose you have a frictionless track from $(0, 0)$ to (a, y_0) and release a particle from $(0, 0)$. Which path allows the particle to get to (a, y_0) in the shortest amount of time? *Etymology* brachisto "shortest" + chrone "time."
  - Since the track is frictionless, the mechanical energy should be conserved.
  - At a given point along the curve, the particle has a velocity v and is vertical distance y from where it started. We know from physics that

$$\begin{aligned} \frac{1}{2}mv^2 &= mgy \\ v &= \sqrt{2gy} \end{aligned}$$

  - The time it takes for the particle to traverse an infinitesimal section of track of arc length ℓ is ℓ/v.
  - The track should be given by $y = y(x)$.
  - Arc length

$$\ell = \sqrt{1 + (y'(x))^2}\,\mathrm{d}x$$

  - Thus, the total time for the particle to traverse the curve is

$$\int_0^a \frac{\sqrt{1 + (y'(x))^2}\,\mathrm{d}x}{\sqrt{2gy(x)}}$$

  - We also have $y(0) = 0$ and $y(a) = y_0$.
  - Functionals: Mapping from a function space to numbers; we want to find y such that the above integral is minimized.

- Let $J[y] = \int_a^b F(x, y(x), y'(x))\,\mathrm{d}x$.
- The space of functions we're considering is C^1.
- Take a function h, vanishing at a, b.
- Let $f(t) = J[y + th]$. Then

$$f(t) = \int_a^b F(x, \underbrace{y(x) + th(x)}_{z}, \underbrace{y'(x) + th'(x)}_{w})\,\mathrm{d}x$$

and hence

$$\begin{aligned} f'(t) &= \int_a^b \left(\frac{\partial F}{\partial z}(x, y(x) + th(x), y'(x) + th'(x))h(x) + \frac{\partial F}{\partial w}(x, y(x) + th(x), y'(x) + th'(x))h'(x) \right) \mathrm{d}x \\ &= \int_a^b \frac{\partial F}{\partial z}(x, y(x) + th(x), y'(x) + th'(x))h(x)\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial F}{\partial w}(x, y(x) + th(x), y'(x) + th'(x)) \right] h(x)\,\mathrm{d}x \end{aligned}$$

- Thus,

$$f'(0) = \int_a^b \left\{ \frac{\partial F}{\partial z}(x, y(x), y'(x)) - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial F}{\partial w}(x, y(x), y'(x)) \right] \right\} h(x)\,\mathrm{d}x = 0$$

for all h.
- Lemma: Let ϕ be continuous on (a, b). If for every $h \in C^1([a, b])$ vanishing on a, b we have that

$$\int_a^b \phi(x)h(x)\,\mathrm{d}x = 0$$

then $\phi(x) = 0$.

Proof. Suppose for the sake of contradiction that (WLOG) $\phi(x_0) > 0$. Then within some neighborhood $N_\delta(x)$ of x_0, $\phi(x) > 0$ for all $x \in N_\delta(x)$. Now choose h to be a bump function on that interval. Then $\int_a^b \phi(x)h(x)\,\mathrm{d}x > 0$, a contradiction. □

- It follows that

$$\frac{\partial F}{\partial z}(x, y(x), y'(x)) - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial F}{\partial w}(x, y(x), y'(x)) \right] = 0$$

  - This is a second-order differential equation, specifically the **Euler-Lagrange equation**.
  - It is a necessary condition for y to be an extrema.
  - Euler-Lagrange equations are not easy to solve in general. However, we're lucky here.
- In our example,

$$F(x, y, z) = \sqrt{\frac{1 + w^2}{2gz}}$$

- This gives us

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}[F(y, y')] &= \frac{\partial F}{\partial z}(y, y') \cdot y' + \frac{\partial F}{\partial w}(y, y') \cdot y'' \\ \frac{\partial F}{\partial z}(y, y')y &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial F}{\partial w}(y, y') \right] y' \\ \frac{\mathrm{d}}{\mathrm{d}x}[F(y, y')] &= \underbrace{\frac{\partial F}{\partial w}(y, y') \cdot}_{U} \underbrace{y''}_{V'} + \underbrace{\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial F}{\partial w}(y, y') \right] \cdot}_{U'} \underbrace{y'}_{V} \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial F}{\partial w}(y, y') \cdot y' \right] \end{aligned}$$

- This reduces to the first-order equation

$$F(y,y') - \frac{\partial F}{\partial w}(y,y') \cdot y' = A$$

- Since $F(x,z,w)$ is known, we have that

$$\frac{\partial F}{\partial w}(z,w) = \frac{w}{\sqrt{1+w^2}} \cdot \frac{1}{\sqrt{2gz}}$$

- Plugging into the E-L equation gives us

$$\begin{aligned}
\frac{1+(y')^2}{\sqrt{1+(y')^2}\sqrt{2gy}} - \frac{(y')^2}{\sqrt{1+(y')^2}\sqrt{2gy}} &= A \\
\frac{1}{\sqrt{2gy(1+(y')^2)}} &= A \\
(y')^2 &= \frac{2A^2g - y}{y}
\end{aligned}$$

  where the second line above is a separable differential equation.

- The solution is the **cycloid**

$$\begin{cases}
x = -a\sin\theta + a\theta \\
y = a(1-\cos\theta)
\end{cases}$$

  where the specific parameters come from the boundary values.

- **Sturm-Liouville problems**: Boundary value problems concerning the integral

$$\int_a^b \left[p(x)(y'(x))^2 + q(x)(y(x))^2\right] \mathrm{d}x$$

  - The most basic BVP is a vibrating string. In finding the eigenmode of the vibration, you need to solve the above differential equation.
  - Very important in physics.
  - If time permits at the end of the course, Shao will return to the following topic in detail.

- Next several weeks: *Solvable* differential equations.

Week 2

???

2.1 Separable ODEs

10/3: • Do not sit on the left side of the classroom: The sun sucks!

- **Separable** (ODE): An ODE of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)g(y)$$

  where y is a real[1], unknown, scalar function of t.

- Solving separable ODEs: Formally, evaluate

$$\int \frac{\mathrm{d}y}{g(y)} = \int f(t)\,\mathrm{d}t$$

- Rearrange the initial separable ODE to $\mathrm{d}y/\mathrm{d}t \cdot 1/g = f$ and invoke the law of composite differentiation to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{y_0}^{y(t)} \frac{\mathrm{d}w}{g(w)} - \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau\right] = 0$$

- It follows that

$$\int_{y_0}^{y(t)} \frac{\mathrm{d}w}{g(w)} = \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau$$

- Examples:
  1. Exponential growth.
     - We have that

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky$$

       for $k > 0$ and $y(0) = y_0 > 0$.
     - The solution is

$$\begin{aligned} \frac{1}{y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} &= k \\ \log y(t) - \log y_0 &= kt \\ y(t) &= y_0 \mathrm{e}^{kt} \end{aligned}$$

[1] We'll deal with complex functions later.

2. Logistic growth.
   - We have that
     $$\frac{\mathrm{d}y}{\mathrm{d}t} = ky\left(1 - \frac{y}{M}\right)$$
     for $k, M > 0$ and $y(0) = y_0 > 0$.
   - The solution is
     $$\begin{aligned}
     \frac{M\,\mathrm{d}y}{y(M-y)} &= k\,\mathrm{d}t\\
     \log\frac{y}{M-y} - \log\frac{y_0}{M-y_0} &= kt\\
     \frac{y(M-y_0)}{y_0(M-y)} &= \mathrm{e}^{kt}\\
     y\cdot\frac{M-y_0}{y_0} &= (M-y)\mathrm{e}^{kt}\\
     y\cdot\frac{M-y_0}{y_0} + y\mathrm{e}^{kt} &= M\mathrm{e}^{kt}\\
     y\left(\frac{M-y_0}{y_0} + \mathrm{e}^{kt}\right) &= M\mathrm{e}^{kt}\\
     y\left(\frac{M-y_0+y_0\mathrm{e}^{kt}}{y_0}\right) &= M\mathrm{e}^{kt}\\
     y\left(\frac{M+y_0(\mathrm{e}^{kt}-1)}{y_0}\right) &= M\mathrm{e}^{kt}\\
     y(t) &= \frac{My_0\mathrm{e}^{kt}}{M+y_0(\mathrm{e}^{kt}-1)}
     \end{aligned}$$
   - Sketches the graph of logistic growth and discusses the turning point (for which there is a formula; zero of the second derivative) as well as general trends.
   - If $y_0 < 0$, the solution is not physically meaningful, but it is mathematically insightful.
     - When we integrate, the arguments of our logarithms now have absolute values.
       $$\log\left|\frac{y}{M-y}\right| - \log\left|\frac{y_0}{M-y_0}\right| = kt$$
     - We need to make sure that the denominator of the final logistic form is never equal to zero, but now that y_0 is negative, as t increases, the denominator will approach zero exponentially. It reaches zero when
       $$\begin{aligned}
       M + y_0(\mathrm{e}^{kt}-1) &= 0\\
       \mathrm{e}^{kt} &= -\frac{M}{y_0} + 1
       \end{aligned}$$
       In other words, $t_{\max} = (1/k)\log(1 - M/y_0)$ because when $t = t_{\max}$, the equation blows up.
     - This is an example of **finite lifespan**.
   - If $y_0 > M$, then you will exponentially decrease to M.
3. Lotka-Volterra predator-prey model.
   - We have that
     $$r' = k_1r - awr \qquad\qquad w' = -k_2w + bwr$$
     where r is rabbits and w is wolves.

- We can rename the variables to

$$\begin{cases} x' = Ax - Bxy \\ y' = -Cy + Dxy \end{cases}$$

- Dividing, we get

$$\frac{x'}{y'} = \frac{Ax - Bxy}{-Cy + Dxy}$$
$$\frac{By - A}{y}y' + \frac{Dx - C}{x}x' = 0$$

- Use the fact that x, y are independent variables, so both terms in the above equation are equal to zero?
- Invoke the law of composite differentiation twice and, from the above, know that $0 + 0 = 0$, so we can add the two solutions:

$$\frac{\mathrm{d}}{\mathrm{d}t}(By(t) - A\log y(t)) + \frac{\mathrm{d}}{\mathrm{d}t}(Dx(t) - C\log x(t)) = 0$$
$$By(t) - A\log y(t) + Dx(t) - C\log x(t) = E$$

- Sketches some of the trajectories (they're all closed curves in the xy-plane).

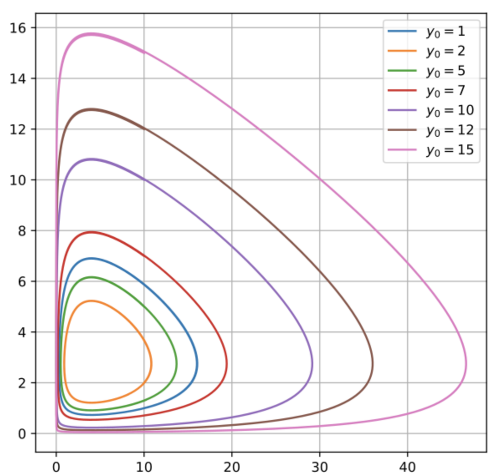

Figure 2.1: Lotka-Volterra solution curves.

- Properties of the curves:
  - The implicit relation which determines them: By the implicit function theorem, the y derivative of the LHS is $B - A/y$ and the x-derivative of the LHS is $D - C/x$. When the partial derivatives are equal to zero, $(C/D, A/B)$ becomes interesting. Turning points happen when the y-coordinate is A/B or the x-coordinate is C/D.

- **Finite lifespan**: Even if the RHS of $\mathrm{d}y/\mathrm{d}t = f(t, y)$ is very regular, the solution can still blow up at some finite time.

- Consider the following variation on the E-L equation from the Brachistochrone problem.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{\frac{B - y}{y}}$$

- Finding the **primitives**.

  - ■ What are these "primitives" Shao keeps talking about?
- We should have

$$\int \sqrt{\frac{y}{B-y}}\,\mathrm{d}y = x$$

- Change of variables: $y = B\sin^2\phi$ and $\mathrm{d}y = 2B\cos\phi\sin\phi\,\mathrm{d}\phi$. Thus,

$$\int \sqrt{\frac{y}{B-y}}\,\mathrm{d}y = \int \frac{\sin\phi}{\cos\phi}\cdot 2B\cos\phi\sin\phi\,\mathrm{d}\phi = 2B\int \sin^2\phi\,\mathrm{d}\phi$$

- The solution is

$$\begin{cases} x = B\phi - \frac{B}{2}\sin(2\phi) + C \\ y = B\sin^2\phi \end{cases}$$

  - ■ This is a parameterization of a cycloid.

- Later in the week, we will do the SHM, the pendulum, the Kepler 2-body problem, and the Michaelis-Menten equation.
- Separable ODEs are a subset of ODEs of **exact form**.
- ODEs of exact form are of the form

$$g(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} + f(x,y) = 0$$

where for some $F(x,y)$, $g = \partial F/\partial y$, $f = \partial F/\partial x$, and partials commute. Equivalently,

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$$

is our necessary and sufficient condition.

- By the law of composite differentiation,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[F(x,y(x))\right] &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\cdot y'(x) \\ &= f(x,y(x)) + g(x,y(x))y'(x) \\ &= 0\end{aligned}$$

  - We solve these with an integrating factor $\mu \neq 0$ such that $(\mu g, \mu f)$ satisfy the constraint.