

Week 2

Solving Simple ODEs

2.1 Separable ODEs

10/3: • Do not sit on the left side of the classroom: The sun sucks!

- **Separable** (ODE): An ODE of the form

$$\frac{dy}{dt} = f(t)g(y)$$

where y is a real^[1], unknown, scalar function of t .

- Solving separable ODEs: Formally, evaluate

$$\int \frac{dy}{g(y)} = \int f(t) dt$$

- Rearrange the initial separable ODE to $dy/dt \cdot 1/g = f$ and invoke the law of composite differentiation to get

$$\frac{d}{dt} \left[\int_{y_0}^{y(t)} \frac{dw}{g(w)} - \int_{t_0}^t f(\tau) d\tau \right] = 0$$

- It follows that

$$\int_{y_0}^{y(t)} \frac{dw}{g(w)} = \int_{t_0}^t f(\tau) d\tau$$

- Examples:

1. Exponential growth.

- We have that

$$\frac{dy}{dt} = ky$$

for $k > 0$ and $y(0) = y_0 > 0$.

- The solution is

$$\frac{1}{y} \cdot \frac{dy}{dt} = k$$

$$\log y(t) - \log y_0 = kt$$

$$y(t) = y_0 e^{kt}$$

¹We'll deal with complex functions later.

2. Logistic growth.

- We have that

$$\frac{dy}{dt} = ky \left(1 - \frac{y}{M}\right)$$

for $k, M > 0$ and $y(0) = y_0 > 0$.

- The solution is

$$\begin{aligned} \frac{M dy}{y(M-y)} &= k dt \\ \log \frac{y}{M-y} - \log \frac{y_0}{M-y_0} &= kt \\ \frac{y(M-y_0)}{y_0(M-y)} &= e^{kt} \\ y \cdot \frac{M-y_0}{y_0} &= (M-y)e^{kt} \\ y \cdot \frac{M-y_0}{y_0} + ye^{kt} &= Me^{kt} \\ y \left(\frac{M-y_0}{y_0} + e^{kt} \right) &= Me^{kt} \\ y \left(\frac{M-y_0+y_0e^{kt}}{y_0} \right) &= Me^{kt} \\ y \left(\frac{M+y_0(e^{kt}-1)}{y_0} \right) &= Me^{kt} \\ y(t) &= \frac{My_0e^{kt}}{M+y_0(e^{kt}-1)} \end{aligned}$$

- Sketches the graph of logistic growth and discusses the turning point (for which there is a formula; zero of the second derivative) as well as general trends.
- If $y_0 < 0$, the solution is not physically meaningful, but it is mathematically insightful.
 - When we integrate, the arguments of our logarithms now have absolute values.

$$\log \left| \frac{y}{M-y} \right| - \log \left| \frac{y_0}{M-y_0} \right| = kt$$

- We need to make sure that the denominator of the final logistic form is never equal to zero, but now that y_0 is negative, as t increases, the denominator will approach zero exponentially. It reaches zero when

$$\begin{aligned} M + y_0(e^{kt} - 1) &= 0 \\ e^{kt} &= -\frac{M}{y_0} + 1 \end{aligned}$$

In other words, $t_{\max} = (1/k) \log(1 - M/y_0)$ because when $t = t_{\max}$, the equation blows up.

- This is an example of **finite lifespan**.

- If $y_0 > M$, then you will exponentially decrease to M .

3. Lotka-Volterra predator-prey model.

- We have that

$$r' = k_1 r - a w r \qquad w' = -k_2 w + b w r$$

where r is rabbits and w is wolves.

- We can rename the variables to

$$\begin{cases} x' = Ax - Bxy \\ y' = -Cy + Dxy \end{cases}$$

- Dividing, we get

$$\frac{x'}{y'} = \frac{Ax - Bxy}{-Cy + Dxy}$$

$$\frac{By - A}{y}y' + \frac{Dx - C}{x}x' = 0$$

- Use the fact that x, y are independent variables, so both terms in the above equation are equal to zero?
- Invoke the law of composite differentiation twice and, from the above, know that $0 + 0 = 0$, so we can add the two solutions:

$$\frac{d}{dt}(By(t) - A \log y(t)) + \frac{d}{dt}(Dx(t) - C \log x(t)) = 0$$

$$By(t) - A \log y(t) + Dx(t) - C \log x(t) = E$$

- Sketches some of the trajectories (they're all closed curves in the xy -plane).

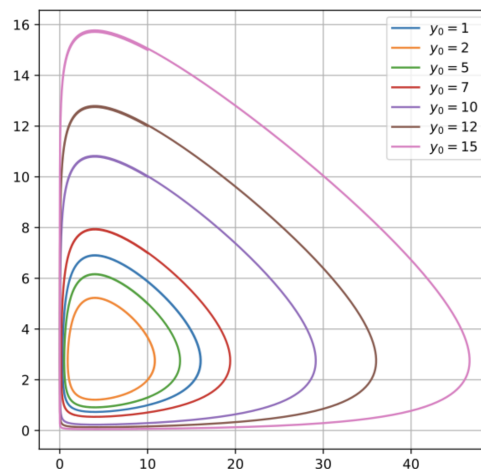


Figure 2.1: Lotka-Volterra solution curves.

- Properties of the curves:

■ The implicit relation which determines them: By the implicit function theorem, the y derivative of the LHS is $B - A/y$ and the x -derivative of the LHS is $D - C/x$. When the partial derivatives are equal to zero, $(C/D, A/B)$ becomes interesting. Turning points happen when the y -coordinate is A/B or the x -coordinate is C/D .

- **Finite lifespan:** Even if the RHS of $dy/dt = f(t, y)$ is very regular, the solution can still blow up at some finite time.
- Consider the following variation on the E-L equation from the Brachistochrone problem.

$$\frac{dy}{dx} = \sqrt{\frac{B - y}{y}}$$

- Finding the **primitives**.

■ What are these “primitives” Shao keeps talking about?

– We should have

$$\int \sqrt{\frac{y}{B-y}} dy = x$$

– Change of variables: $y = B \sin^2 \phi$ and $dy = 2B \cos \phi \sin \phi d\phi$. Thus,

$$\int \sqrt{\frac{y}{B-y}} dy = \int \frac{\sin \phi}{\cos \phi} \cdot 2B \cos \phi \sin \phi d\phi = 2B \int \sin^2 \phi d\phi$$

– The solution is

$$\begin{cases} x = B\phi - \frac{B}{2} \sin(2\phi) + C \\ y = B \sin^2 \phi \end{cases}$$

■ This is a parameterization of a cycloid.

- Later in the week, we will do the SHM, the pendulum, the Kepler 2-body problem, and the Michaelis-Menten equation.
- Separable ODEs are a subset of ODEs of **exact form**.
- ODEs of exact form are of the form

$$g(x, y) \frac{dy}{dx} + f(x, y) = 0$$

where for some $F(x, y)$, $g = \partial F / \partial y$, $f = \partial F / \partial x$, and partials commute. Equivalently,

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$$

is our necessary and sufficient condition.

- By the law of composite differentiation,

$$\begin{aligned} \frac{d}{dx} [F(x, y(x))] &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot y'(x) \\ &= f(x, y(x)) + g(x, y(x))y'(x) \\ &= 0 \end{aligned}$$

– We solve these with an integrating factor $\mu \neq 0$ such that $(\mu g, \mu f)$ satisfy the constraint.

2.2 Office Hours (Shao)

- **Primitive:** An antiderivative.
- **Law of composite differentiation:** The chain rule.
- Went over how Shao has been applying the law of composite differentiation with respect to separable ODEs:
 - Rearrange the initial separable ODE as follows.

$$\frac{1}{g(y)} \cdot \frac{dy}{dt} = f(t)$$

- Define $dH/dy = 1/g(y)$. Then, continuing from the above, we have by the law of composite differentiation that

$$\begin{aligned}\frac{dH}{dy} \cdot \frac{dy}{dt} &= f(t) \\ \frac{dH}{dt} &= f(t)\end{aligned}$$

- From the definition of H , we know that $H(y) = \int_{y_0}^y dw/g(w)$. We also have from the FTC that $f(t) = \frac{d}{dt} \int_{t_0}^t f(\tau) d\tau$. Thus, continuing from the above, we have that

$$\begin{aligned}\frac{d}{dt}(H) &= f(t) \\ \frac{d}{dt} \left[\int_{y_0}^y \frac{dw}{g(w)} \right] &= \frac{d}{dt} \int_{t_0}^t f(\tau) d\tau \\ \frac{d}{dt} \left[\int_{y_0}^{y(t)} \frac{dw}{g(w)} - \int_{t_0}^t f(\tau) d\tau \right] &= 0\end{aligned}$$

as desired.

- It follows since $y(t_0) = y_0$ that $C = H(y_0) = 0$, so we can take the above to

$$\int_{y_0}^{y(t)} \frac{dw}{g(w)} = \int_{t_0}^t f(\tau) d\tau$$

knowing that our constant of integration is zero.

- Take away from Brachistochrone problem: Just an example of a BDE; we won't have to answer questions on it.

2.3 ODEs of Exact Form

10/5:

- Last time, we discussed separable ODEs.
- Today, we will study **exact form** equations, as discussed last class.
- **Exact form** (ODE): An ODE of the form

$$g(x, y) \frac{dy}{dx} + f(x, y) = 0$$

where

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$$

- For equations of this form, there exists $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = f \qquad \frac{\partial F}{\partial y} = g \qquad F(x, y(x)) = C$$

for some $C \in \mathbb{R}$.

- To solve equations of this form, we need an **integrating factor**.
- **Integrating factor**: A number or function μ such that

$$\mu g \frac{dy}{dx} + \mu f = 0 \qquad \frac{\partial}{\partial x}(\mu g) = \frac{\partial}{\partial y}(\mu f)$$

- For linear homogeneous equations $dy/dt = p(t)y$, we have

$$y(t) = y_0 \exp \left[\int_{t_0}^t p(\tau) d\tau \right]$$

- Recall that $e^{a+ib} = e^a(\cos b + i \sin b)$, so

$$e^{ix} = \cos x + i \sin x \qquad \cos x = \frac{1}{2}(e^{ix} + e^{-ix}) \qquad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

- Example: If $y' = ky$, then $y' = -\lambda y$.
- If we have an inhomogeneous linear equation $dy/dt = p(t)y + f(t)$, then

$$\frac{dy}{dt} - py - f = 0$$

but

$$0 = \frac{d}{dt}(1) \neq \frac{d}{dy}(-p(t)y - f(t))$$

- We wish to find an integrating factor $\mu(t, y)$ such that

$$\mu(t, y) \frac{dy}{dt} - \mu(t, y)p(t)y - \mu(t, y)f(t) = 0$$

and

$$\frac{d}{dt}(\mu) = \frac{d}{dy}(-\mu py - \mu f)$$

- Solution: Take μ to be a function of t , alone. Then

$$\mu'(t) = \frac{d}{dy}(-\mu py - \mu f) = -\mu(t)p(t)$$

and we now have a homogeneous linear equation with solution

$$\mu(t) = \exp \left[- \int_{t_0}^t p(\tau) d\tau \right]$$

- If we let $P(t) = \int_{t_0}^t p(\tau) d\tau$, then

$$\begin{aligned} e^{-P(t)} y'(t) - p(t)y(t)e^{-P(t)} &= e^{-P(t)} f(t) \\ \frac{d}{dt} \left(e^{-P(t)} y(t) \right) &= e^{-P(t)} f(t) \\ e^{-P(t)} y(t) &= \int_{t_0}^t e^{-P(\tau)} f(\tau) d\tau \end{aligned}$$

- Thus, we finally have the solution to the inhomogeneous problem as follows: The IVP $y' = py + f$, $y(t_0) = y_0$ has solution

$$y(t) = y_0 e^{P(t)-P(t_0)} + e^{P(t)} \int_0^t e^{-P(\tau)} f(\tau) d\tau$$

where P is any anti-derivative of p .

- In particular, when $p(t) = a$, we get the **Duhamel formula** (which we should memorize).

- **Duhamel formula:** The following equation, which is the solution to an inhomogeneous linear equation with $p(t) = a$.

$$y(t) = y_0 e^{a(t-t_0)} + \int_{t_0}^t e^{a(t-\tau)} f(\tau) d\tau$$

- Important for computing forced oscillation.
- Inspecting the inhomogeneous solution.
 - The first term is the solution to the homogeneous form. The second term deals with the initial value.
- Given an inhomogeneous equation, you can always write its solution as the combination of the solution to the homogeneous problem plus a particular solution, i.e.,

$$y = y_h + y_p$$

- “The general solution equals the homogeneous solution plus a particular solution.”
- This is related to linear algebra, where the solution to $Ax = b$ is a particular solution x_p plus any vector $x \in \ker A$.
- Thus, this idea will reappear in the theory of systems of linear ODEs.
- We now look at systems of linear ODEs.
- Consider the harmonic oscillator: A particle of mass m connected to an ideal spring (obeys Hooke’s law) with no friction or gravity.
 - Newton’s second law: The acceleration is proportional to the restoring force.
 - Hooke’s law: The restoring force is of magnitude kx in the opposite direction to the displacement.
 - Thus, the ODE is of the form

$$x'' = -\frac{k}{m}x$$

- Consider an ODE of the form

$$y'' + ay' + by = 0$$

for $a, b \in \mathbb{C}$.

- Aim: Find $\mu, \lambda \in \mathbb{C}$ such that

$$(y' - \mu y)' - \lambda(y' - \mu y) = 0$$

- To find the parameters, we expand the above to

$$y'' - (\mu + \lambda)y' + \mu\lambda y = 0$$

- Comparing with the original form, we have that $a = -(\mu + \lambda)$ and $b = \mu\lambda$.
- It follows that μ, λ are the roots of $x^2 + ax + b = 0$, which we will call the **characteristic polynomial** of the ODE.
- Example:

- Consider

$$y' - \mu y = Ae^{\lambda t}$$

- By the Duhamel equation, we have that a particular solution is of the form

$$A \int_0^t e^{\mu(t-\tau)} e^{\lambda\tau} d\tau$$

- Thus, general solutions are of the form

$$y(t) = Be^{\mu t} + Ce^{\mu t} \int_0^t e^{(\lambda-\mu)\tau} d\tau$$

- Evaluating the integral, we get

$$y(t) = Be^{\mu t} + Ce^{\mu t} \frac{e^{(\lambda-\mu)t} - 1}{\lambda - \mu}$$

which simplifies (by incorporating constants, etc.) to

$$y(t) = A_1 e^{\mu t} + B_1 e^{\lambda t}$$

for $\mu \neq \lambda$, or

$$y(t) = A_1 e^{\mu t} + B_1 t e^{\mu t}$$

for $\mu = \lambda$.

- If our equation is of the form $y'' + ay' + by = f(t)$, then we just need to apply the Duhamel formula twice.
- Returning to the simple harmonic oscillator problem, we substitute $\omega = \sqrt{k/m}$ to get

$$x'' = -\omega^2 x$$

- The characteristic polynomial is

$$0 = x^2 + \omega^2 = (x + i\omega)(x - i\omega)$$

- Thus, solutions are of the form

$$x = A_1 e^{i\omega t} + B_1 e^{-i\omega t}$$

- It follows that the period is $T = 2\pi/\omega$.
- To get a real (usable) solution, apply Euler's formula to get

$$\begin{aligned} x(t) &= A_1(\cos \omega t + i \sin \omega t) + B_1(\cos \omega t - i \sin \omega t) \\ &= A \cos \omega t + B \sin \omega t \end{aligned}$$

where $A = A_1 + B_1$, $B = iA_1 - iB_1$.

- To match the initial condition $x(0) = x_0$, $x'(0) = v_0$, we use

$$x(t) = x_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t$$

- In other words,

$$\begin{cases} A = x_0 \\ \omega B = v_0 \end{cases} \qquad \begin{cases} A_1 + B_1 = x_0 \\ i\omega A_1 - i\omega B_1 = v_0 \end{cases}$$

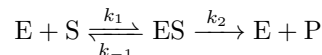
so

$$\begin{cases} A = x_0 \\ B = \frac{v_0}{\omega} \end{cases} \qquad \begin{cases} A_1 = \frac{1}{2} \left[x_0 - \frac{iv_0}{\omega} \right] \\ B_1 = \frac{1}{2} \left[x_0 + \frac{iv_0}{\omega} \right] \end{cases}$$

2.4 ODE Examples

10/7:

- Today, we will investigate a variety of examples of ODEs arising in real life.
- Michaelis-Menten kinetics: If E is an enzyme, S is its substrate, and P is the product, then the mechanism is



- The concentrations that we are concerned with are $[E], [S], [ES], [P]$.
- From the above mechanism, we can write the four rate laws

$$\frac{d}{dt}[S] = -k_1[E][S] + k_{-1}[ES] \quad (1)$$

$$\frac{d}{dt}[E] = -k_1[E][S] + (k_{-1} + k_2)[ES] \quad (2)$$

$$\frac{d}{dt}[ES] = k_1[E][S] - (k_{-1} + k_2)[ES] \quad (3)$$

$$\frac{d}{dt}[P] = k_2[ES] \quad (4)$$

- We can reduce these rate laws to the 2D system

$$\frac{d}{dt}[S] = -k_1([E]_0 - [ES])[S] + k_{-1}[ES] \quad (5)$$

$$\frac{d}{dt}[ES] = k_1([E]_0 - [ES])[S] - (k_{-1} + k_2)[ES] \quad (6)$$

- QSSA: Quasi steady-state assumption.

- Assume that $[E]_0/[S]_0 \ll 1$.
- Also assume that $d[ES]/dt \approx 0$.

- Then

$$[ES] = \frac{[E]_0[S]}{K_M + [S]}$$

where $k_M = (k_{-1} + k_2)/k_1$.

- Sub in the above to Equation 5:

$$\frac{d}{dt}[S] = -\frac{k_2[E]_0[S]}{k_M + [S]}$$

- Note that $v_{\max} = k_2[E]_0$.

- The above is now a differential equation of separable form; it's solution is

$$\begin{aligned} \int_{[S]_0}^{[S]} -\frac{(k_M + z) dz}{z v_{\max}} &= \int_0^t dt \\ -\frac{k_M}{v_{\max}} \log \frac{[S]}{[S]_0} - \frac{1}{v_{\max}} ([S] - [S]_0) &= t \\ -\frac{k_M}{v_{\max}} \frac{[S]}{[S]_0} e^{-v_{\max}^{-1}([S] - [S]_0)} &= e^t \end{aligned}$$

- The above equation is of the following form, for $x > 0$, $w(x) \sim s$, $x \sim 0$, and $w(x) \sim \log x$??

$$w(x)e^{w(x)} = x$$

- Harmonic oscillator.
- Recall that

$$x'' + \frac{k}{m}x = 0$$

- Substituting $\omega = \sqrt{k/m}$, we can solve the above for

$$x(t) = x(0) \cos(\omega t) + \frac{x'(0)}{\omega} \sin(\omega t)$$

- This is an integrable system with n degrees of freedom and $n - 1$ scalar conservation laws??
- Conservation of mechanical energy:

$$E = \frac{1}{2}m|x'|^2 + \frac{1}{2}kx^2$$

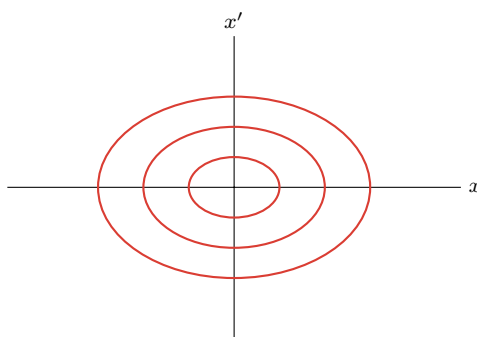


Figure 2.2: Conservation of mechanical energy in the harmonic oscillator.

- Differentiating wrt. x yields

$$\begin{aligned} 0 &= mx'x'' + kxx' \\ &= \frac{d}{dt} \left(\frac{1}{2}m(x')^2 \right) + \frac{d}{dt} \left(\frac{1}{2}kx^2 \right) \end{aligned}$$

- This means that the solution is an ellipse in the xx' -plane, where each ellipse corresponds to an initial displacement and velocity.

- Mathematical pendulum.
- Equation of motion:

$$\begin{aligned} 0 &= \ell\theta' + g \sin \theta \\ &= \ell\theta''\theta' + g \sin \theta \cdot \theta' \\ &= \frac{d}{dt} \left(\underbrace{\frac{\ell}{2}|\theta'|^2 - g \cos \theta}_E \right) \end{aligned}$$

- It follows from the above that

$$\begin{aligned} \frac{\ell}{2}|\theta'|^2 - g \cos \theta_0 &= -g \cos \theta \\ \frac{d\theta}{dt} &= \sqrt{\frac{2g}{\ell}(\cos \theta_0 - \cos \theta)} \\ \int_{\theta_0}^{\theta} \sqrt{\frac{\ell}{2g(\cos \theta_0 - \cos \phi)}} d\phi &= t \end{aligned}$$

- This is an elliptical integral (and thus cannot be expressed in terms of the elementary functions).
- Suppose θ_0 is small. Then θ is small, and we can invoke the small angle approximation $\sin \theta \approx \theta$.
 - This yields an approximate equation of motion:

$$\ell\theta'' + g\theta = 0$$

- From here, we can determine that $\theta(t) \approx \theta_0 \cos \sqrt{g/\ell} \cdot t$ and $T = 2\pi\sqrt{\ell/g}$.
- Kepler problem.
- Two bodies of mass m_1, m_2 are located at positions x_1, x_2 pulling on each other gravitationally.
- From Newton's second and third law, we get

$$m_1 x_1'' = \frac{U'(|x_1 - x_2|)^{x_1 - x_2}}{|x_1 - x_2|} \qquad m_2 x_2'' = \frac{U'(|x_1 - x_2|)^{x_1 - x_2}}{|x_1 - x_2|}$$

- Conservation of momentum:

$$(m_1 x_1 + m_2 x_2)'' = 0$$

$$m_1 x_1' + m_2 x_2' = C$$

- Let $M = m_1 + m_2$. Then

$$\frac{m_1}{M}x_1 + \frac{m_2}{M}x_2$$

moves inertially.

- Define the center of mass to be the origin.

- Conservation of angular momentum:

$$[\mu(x_1 - x_2)' \times (x_1 - x_2)]' = 0$$

- $\mu = m_1 m_2 / (m_1 + m_2)$.
- \times indicates the cross product.
- $L = \mu(x_1 - x_2) \times (x_1 - x_2)$.

- Conservation of mechanical energy:

$$\mu q'' + U'(|q|) \frac{q}{|q|} = 0$$

$$\frac{\mu}{2}|q'|^2 + U(|q|) = E$$

- $q = x_1 - x_2$.

- Introduce polar coordinates (r, ϕ) .

- Then $r^2 \phi' / 2 = \ell_0$, $r = r(\phi)$, and $dr/d\phi = r'(t)/\phi'(t)$.
- It follows that

$$\frac{\mu}{2}(|r'|^2 + |\phi'|^2) + U(r) = E$$

- Then

$$\left(\frac{dr}{d\phi}\right)^2 + r^2 = \frac{2G\mu r^3}{\ell_0^2} + \frac{2Er^4}{\mu\ell_0^2}$$

- The substitution $\mu = 1/r$ yields

$$\left(\frac{d\mu}{d\phi}\right)^2 + \mu^2 = \frac{2G\mu}{\ell_0^2} + \frac{2E}{\mu\ell_0^2}$$

- Differentiating again gives

$$2 \frac{d\mu}{d\phi} \frac{d^2\mu}{d\phi^2} + 2 \frac{d\mu}{d\phi} \mu = \frac{2G\mu}{\ell_0^2} \frac{d\mu}{d\phi}$$

- Substituting $\mu = \cos(t)$ gives

$$\frac{d^2\mu}{d\phi^2} + 2\mu = \frac{2G\mu}{\ell_0^2}$$

or

$$r = \frac{1}{G\mu/\ell_0^2 + \varepsilon \cos(\phi - \phi_0)}$$

- This is a conic section!