Week 5

End Quantitative and Intro to Qualitative

5.1 Planar Autonomous Linear Systems

10/24:

- Review of vector fields.
- **Phase diagram**: A diagram that shows the qualitative behavior of an autonomous ordinary differential equation. *Also known as* **phase portrait**.

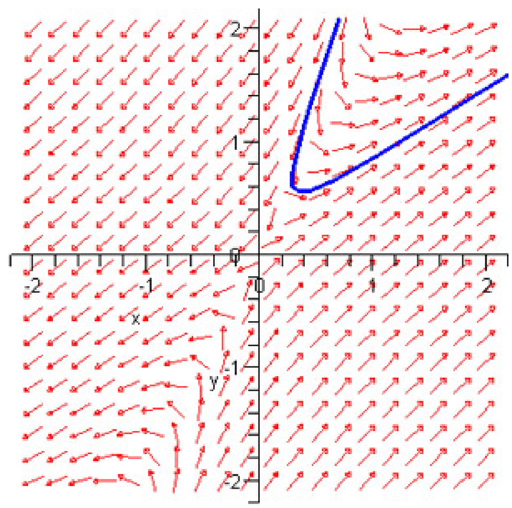
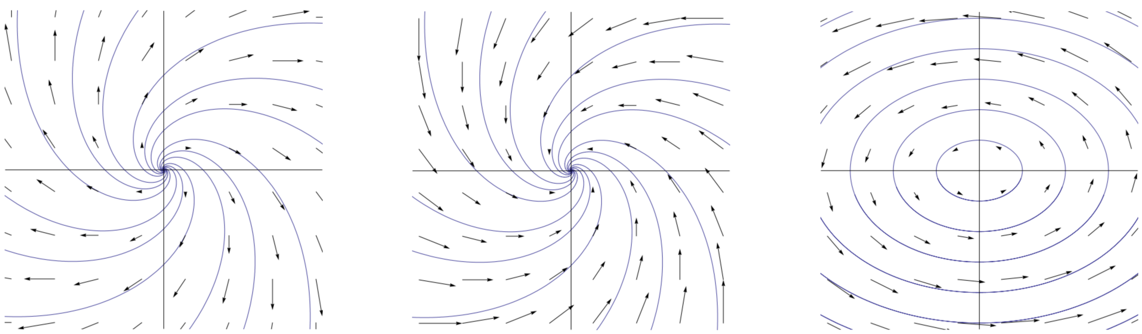


Figure 5.1: Phase diagram example.

  - Consists of a selection of arrows describing, to some extent, a vector field and is often paired with integral curves.
- Suppose $\Omega \subset \mathbb{R}^n$ is open.
- **Vector field** (on Ω): A mapping from $\Omega \to \mathbb{R}^n$. *Denoted by* $\boldsymbol{X}$.
  - Essentially, a vector field assigns to every point of some region a vector; the definition just formalizes this notion.
- **Flow**: A formalization of the idea of the motion of particles in a fluid.
  - The solution to the IVP $\frac{\mathrm{d}y}{\mathrm{d}t} = X(y)$, $y(0) = x$.

- If X is C^1, then for all $x \in \Omega$, there exists a unique solution y to the above IVP.
- **Orbit** (of x under X): The trajectory $y(t,x)$.
  - Recall that the tangent vector to any trajectory at any point coincides with the vector to which X maps that point.
- **Fixed point**: A point $x_0 \in \Omega$ such that $X(x_0) = \bar{0}$.
  - If x_0 is a fixed point, then the trajectory is $y(t) = x_0$.
- Today: We will consider flows on vector fields where the dimension is two and our vector field is linear. In particular...
- Let A be a 2×2 real matrix, and let $X(x) = Ax$.
  - In this case, $x_0 = 0$ is the only fixed point.
  - The flow is given by the linear differential equation $y' = Ay$, $y(0) = x$. The solution is $y(t) = \mathrm{e}^{tA}x$.
- Case 1: A has no real eigenvalues.



(a) $\sigma > 0$. (b) $\sigma < 0$. (c) $\sigma = 0$.

Figure 5.2: Phase diagrams for a planar system with no real eigenvalues.

  - We know that $\chi_A(z)$ is a real polynomial: $\chi_A(z) = z^2 + (\operatorname{trace} A)z + \det A$, and since A is real, both $\operatorname{trace} A$ and $\det A$ are real.
  - Thus, the eigenvalues appear as conjugate pair, i.e., we may write $\lambda = \sigma + i\beta$ and $\bar{\lambda} = \sigma - i\beta$.
    - $\alpha = \gamma = 1$ for both eigenvalues.
    - The eigenvectors must also be complex conjugates.
  - Distinct eigenvalues imply that A is diagonalizable.
  - However, this is not what we want because if we use the complex form, then

$$\mathrm{e}^{tA} = Q \begin{pmatrix} \mathrm{e}^{t\lambda} & 0 \\ 0 & \mathrm{e}^{t\bar{\lambda}} \end{pmatrix} Q^{-1}$$

  - Indeed, we want to get a real matrix out of $Q, \mathrm{e}^{t\Lambda}, Q^{-1}$ all complex. We have

$$\begin{aligned}
\mathrm{e}^{tA}x &= Q \begin{pmatrix} \mathrm{e}^{t(\sigma+i\beta)} & 0 \\ 0 & \mathrm{e}^{t(\sigma-i\beta)} \end{pmatrix} \underbrace{Q^{-1}x}_{z} \\
&= Q \begin{pmatrix} \mathrm{e}^{t(\sigma+i\beta)} z^1 \\ \mathrm{e}^{t(\sigma-i\beta)} z^2 \end{pmatrix} \\
&= z^1 \mathrm{e}^{t(\sigma+i\beta)} v + z^2 \mathrm{e}^{t(\sigma-i\beta)} \bar{v}
\end{aligned}$$

- Since $y(0) = x = z^1 v + z^2 \bar{v} \in \mathbb{R}^2$ (i.e., $z^1 v + z^2 v$ is *real*), we know that it is equal to its complex conjugate. This tells us that

$$
\begin{aligned}
z^1 v + z^2 \bar{v} &= \bar{z^1} \bar{v} + \bar{z^2} v \\
z^1 &= \bar{z^2}
\end{aligned}
$$

- It follows that

$$
\begin{aligned}
y(t) &= \mathrm{e}^{tA} x \\
&= z^1 \mathrm{e}^{t(\sigma + i\beta)} v + \bar{z^1} \mathrm{e}^{t(\sigma - i\beta)} \bar{v} \\
&= z^1 \mathrm{e}^{t(\sigma + i\beta)} v + \overline{z^1 \mathrm{e}^{t(\sigma + i\beta)} v} \\
&= 2\operatorname{Re}(z^1 \mathrm{e}^{t(\sigma + i\beta)} v) \\
&= 2\operatorname{Re}(z^1 \mathrm{e}^{\sigma t}(\cos(\beta t) + i\sin(\beta t))(v_1 + iv_2)) \\
&= 2\operatorname{Re}(z^1 \mathrm{e}^{\sigma t}(\cos(\beta t) v_1 + i\cos(\beta t) v_2 + i\sin(\beta t) v_1 - \sin(\beta t) v_2)) \\
&= 2\mathrm{e}^{\sigma t}\cos(\beta t) \cdot \operatorname{Re}(z^1 v) - 2\mathrm{e}^{\sigma t}\sin(\beta t) \cdot \operatorname{Im}(z^1 v)
\end{aligned}
$$

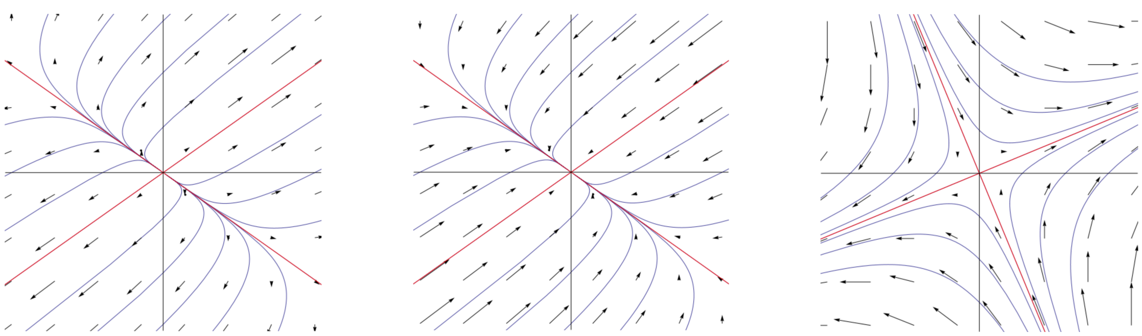
- Suppose $\sigma \neq 0$. Then

$$
x \mapsto \begin{pmatrix} \operatorname{Re}(z^1 v) \\ \operatorname{Im}(z^1 v) \end{pmatrix}
$$

  is a real linear transformation on $\mathbb{R}^2$.
- It follows that the trajectories are just spirals in the complex plane.
- If $\sigma > 0$, then the spiral repels from the origin. If $\sigma < 0$, then the spiral attracts to the origin. If $\sigma = 0$, we get an ellipse.
- Therefore, we have completely classified equations of the form

$$
\begin{pmatrix} y^1 \\ y^2 \end{pmatrix}' = \begin{pmatrix} y^2 \\ -\omega^2 y^1 \end{pmatrix}
$$

- Case 2: A has real eigenvalues and *is* diagonalizable.



(a) $\lambda_1, \lambda_2 > 0$. (b) $\lambda_1, \lambda_2 < 0$. (c) $\lambda_1 < 0 < \lambda_2$ (WLOG).

Figure 5.3: Phase diagrams for a diagonalizable planar system with real eigenvalues.

- Suppose $\lambda_1, \lambda_2 \in \mathbb{R}$ have corresponding linearly independent eigenvectors v_1, v_2.
- If we choose v_1, v_2 to be our basis, then

$$
\mathrm{e}^{tA} = Q \begin{pmatrix} \mathrm{e}^{t\lambda_1} & 0 \\ 0 & \mathrm{e}^{t\lambda_2} \end{pmatrix} Q^{-1}
$$

  where $Q = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$.

- Thus, as before, the solution may be expressed in the following form, where $z = Q^{-1}x$.

$$y(t) = \mathrm{e}^{tA}x = \mathrm{e}^{\lambda_1 t}z^1 v_1 + \mathrm{e}^{\lambda_2 t}z^2 v_2$$

- Moving forward, it will be convenient to work in the v_1, v_2 basis. We divide into three subcases ($\lambda_1, \lambda_2 > 0$ [Figure 5.3a], $\lambda_1, \lambda_2 < 0$ [Figure 5.3b], and WLOG $\lambda_1 < 0 < \lambda_2$ [Figure 5.3c]).
  1. Notice that
  $$\mathrm{e}^{\lambda_2 t} = \mathrm{e}^{(\lambda_2/\lambda_1)(\lambda_1 t)}$$
  i.e., $\mathrm{e}^{\lambda_2 t}$ is a power of $\mathrm{e}^{\lambda_1 t}$. Thus, when the signs are the same, we get a power function $v_2 = v_1^{\lambda_2/\lambda_1}$.
     - Both subspaces v_1, v_2 are unstable here.
  2. If $\lambda_1, \lambda_2 < 0$, then we have the same trajectories, but they're all attracted to the origin instead of repelled.
     - Both subspaces v_1, v_2 are stable here.
  3. When both eigenvalues have different signs, we are considering power functions of a negative power.
     - The stable subspace is v_2 and the unstable subspace is v_1 here.

- Case 3: A has real eigenvalues and *is not* diagonalizable.

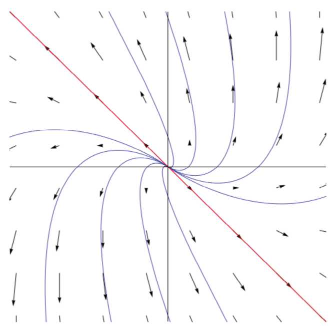


Figure 5.4: Phase diagrams for a nondiagonalizable planar system with real eigenvalues.

- In this case, the matrix exponential is given by

$$\mathrm{e}^{tA} = Q \begin{pmatrix} \mathrm{e}^{t\lambda} & t\mathrm{e}^{t\lambda} \\ 0 & \mathrm{e}^{t\lambda} \end{pmatrix} Q^{-1}$$

- The solution is given by
$$\mathrm{e}^{tA}x = (z^1\mathrm{e}^{t\lambda} + z^2 t\mathrm{e}^{t\lambda})v + z^2\mathrm{e}^{t\lambda}u$$
where $Q^{-1}x = z$ again.
- In graphing, note that here we have (a distorted version of) the form $y = x \pm x\log x$:

$$y = (z^1\mathrm{e}^{t\lambda} + z^2 t\mathrm{e}^{t\lambda})\hat{\imath} + z^2\mathrm{e}^{t\lambda}\hat{\jmath}$$

Define $x := \mathrm{e}^{t\lambda}$. Then $t = \lambda^{-1}\ln x$. Substituting, we have

$$\begin{aligned} &= (z^1 x + z^2(\lambda^{-1}\ln x)x)\hat{\imath} + z^2 x\hat{\jmath} \\ &= (z^1 x + z^2\lambda^{-1}x\ln x)\hat{\imath} + z^2 x\hat{\jmath} \end{aligned}$$

- When $\lambda > 0$, the whole space is unstable. Thus, the phase diagram is tangent to the origin.
- When $\lambda < 0$, the trajectories take the same form but this time are attracted to zero. In this case, the whole space is stable.

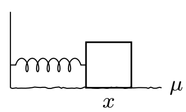
- We can take x_1 to x_2 iff they are in the same orbit. Conclusion: Orbits never cross.
- Takeaway: You should be able to compute the eigenvalues and eigenvectors and sketch these graphs.
- Shao will post lecture notes after today's lecture!
- Next lecture: The final explicitly solveable case, which is the driven harmonic oscillator.

5.2 Driven Harmonic Oscillator and Resonance

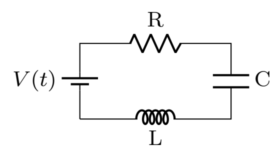
10/26:
- We are interested in the 2nd order constant coefficient equation

$$x'' + \mu x' + \omega_0^2 x = H_0 \mathrm{e}^{i\omega t}$$

where $\mu \geq 0$ and $\omega_0, \omega > 0$.
- Two cases where this ODE arises:



(a) A driven harmonic oscillator.



(b) An RLC circuit.

Figure 5.5: Origins of the driven harmonic oscillator equation.

1. The driven harmonic oscillator.
   - Consider a mass on a spring.
   - The extent of friction between the mass point and the surface is described by μ.
   - The oscillation is periodically driven by a force of magnitude $H_0 \cos \omega t$.
2. RLC circuit.
   - R is resistance, L is inductance, C is capacitance.
   - We have the laws

$$LI_L'(t) = V_L \qquad CV_C'(t) = I_C \qquad I_R(t) = V_R(t)/R$$

     - Left: Self-inductance.
     - Right: Ohm's law.
   - Combining them with Kirchhoff's laws

$$I(t) = I_R = I_C = I_L \qquad V(t) = V_R + V_L + V_C$$

     we get the RLC circuit equation

$$LI'' + RI' + \frac{I}{C} = V'(t)$$

   - The most interesting cases is when we have a source of alternating current of frequency ω. In this case, $V(t) = V_0 \cos \omega t$ or, in the complex case, $V(t) = V_0 \mathrm{e}^{i\omega t}$. This yields the complex equation

$$I'' + \frac{R}{L} I' + \frac{1}{LC} I = \frac{i\omega V_0}{L} \mathrm{e}^{i\omega t}$$

   - Here, the friction coefficient $\mu = R/L$ and the frequency is $\omega_0 = \sqrt{1/LC}$.

- Recall that we want to solve the following ODE.

$$x'' + \mu x' + \omega_0^2 x = H_0 \mathrm{e}^{i\omega t}$$

  - The homogeneous linear equation $x'' + \mu x' + \omega_0^2 x = 0$ is well-understood, i.e., we can find all of the *homogeneous* solutions to the above equation.
  - Thus, to solve the above inhomogeneous equation, we just have to find a particular solution.
- WLOG let $\omega > 0$.
- From the homework, a particular solution $x_p(t)$ with initial condition $x_p(0) = x_p'(0) = \mu = 0$ can be obtained from the Duhamel formula as follows.

$$x_p(t) = H_0 \int_0^t \frac{\sin \omega_0 (t-\tau)}{\omega_0} \mathrm{e}^{i\omega\tau} \mathrm{d}\tau$$

  - Substituting

$$\sin\theta = \frac{\mathrm{e}^{i\theta} - \mathrm{e}^{-i\theta}}{2i}$$

    into the above allows us to evaluate it.
  - In particular, it follows that

$$x_p(t) = \begin{cases} \dfrac{H_0}{\omega_0^2 - \omega^2} \left(\mathrm{e}^{i\omega t} - \cos\omega_0 t - \dfrac{i\omega}{\omega_0} \sin\omega_0 t \right) & \omega \neq \omega_0 \\ -\dfrac{iH_0}{2\omega_0} \left(t\mathrm{e}^{i\omega_0 t} - \dfrac{\sin\omega_0 t}{\omega_0} \right) & \omega = \omega_0 \end{cases}$$

    - We compute the $\omega = \omega_0$ case using L'Hôpital's rule to analyze the $\omega \neq \omega_0$ case as $\omega \to \omega_0$.
  - If we pump in energy at the same point that we have deviation ($\omega = \omega_0$), then the amplitude of oscillation goes to ∞.
    - Practically, when $\omega \approx \omega_0$, the long-time behavior of the driven oscillator will be very much like a growing oscillator.
    - Eventually, the amplitude will be approximately $(\omega - \omega_0)^{-1}$.
- **Resonance catastrophe**: Inputing energy into a system at its natural frequency, causing the total energy to grow until a mechanical failure occurs.
  - This is what happened at the Millenium Bridge in London; synchronized footsteps caused the bridge to shake really wildly.
- If $\mu > 0$, there will be a particular solution of the form

$$x_p(t) = A(\omega) H_0 \mathrm{e}^{i\omega t}$$

  - From HW1, we have three cases when $\mu > 0$: $0 < \mu < 2\omega_0$, $\mu = 2\omega_0$, and $\mu > 2\omega_0$. These are just the three cases of the characteristic polynomial??
  - Substituting the proposed form of the particular solution into the differential equation, we get

$$\begin{aligned} x_p'' + \mu x_p' + \omega_0^2 x_p &= H_0 \mathrm{e}^{i\omega t} \\ (-\omega^2 + i\omega\mu + \omega_0^2) H_0 A(\omega) \mathrm{e}^{i\omega t} &= H_0 \mathrm{e}^{i\omega t} \\ (-\omega^2 + i\omega\mu + \omega_0^2) A(\omega) &= 1 \\ A(\omega) &= \frac{1}{\omega_0^2 - \omega^2 + i\mu\omega} \end{aligned}$$

  - In theory, we avoid the resonance catastrophe in this case. In practice, however, when $\omega \to 0$, we still run into issues.

- For mass point:

$$|H_0A(\omega)| = \frac{|H_0|}{\sqrt{(\omega^2-\omega_0^2)^2+\mu^2\omega^2}}$$

  - The norm $|H_0A(\omega)|$ is maximized when $\omega_r = \sqrt{\omega_0^2+\mu^2/2}$.
  - $\omega_r \to \omega_0$ implies $\mu \to 0$??
- As for the argument/angle,

$$\arg(H_0A(\omega)) = \arg H_0 + \arg A(\omega)$$

- We consider $\omega : 0 \to \omega_0 \to +\infty$.
  - When $\omega = 0$, the complex amplitude is $1/\omega_0^2$ so it's a real number in the complex plane.
  - If ω is increased a bit, we get the reciprocal of a complex number. Norm is reciprocal, argument is negative.
  - For $\omega = \omega_0$, we have a purely imaginary number.
  - As $\omega \to \infty$, the argument approaches $-\pi$??
  - Showing the shape of the norm and the argument with respect to ω. This allows us to completely describe the resonance phenomena.

- For the RLC circuit, the discussion is a bit different.
  - The external voltage $V(t) = V_0\mathrm{e}^{i\omega t}$. Thus, $V'(t) = iV_0\omega\mathrm{e}^{i\omega t}$.
  - Here,

$$x_p(t) = \frac{iV_0\omega\mathrm{e}^{i\omega t}}{\omega_0^2-\omega^2+iR\omega/L}$$

  - Look at the complex amplitude.
    - Multiply the numerator and denominator by the inductance L to get

$$x_p(t) = \frac{iV_0\omega L\mathrm{e}^{i\omega t}}{L\omega_0^2-L\omega^2+iR\omega}$$

    - Then,

$$\text{Norm} = \frac{V_0L}{\sqrt{R^2+\left(\frac{1}{C\omega}-\frac{\omega}{L}\right)^2}}$$

    - For an RLC circuit, the resistance does not affect the resonance frequency.

$$\omega_r = \sqrt{\frac{1}{LC}} = \omega_0$$

    - If you have an external source of voltage, then you can vary the capacity of your circuit to ensure that the voltage will be maximized at a given frequency. We can tune our circuit to a very specific resonance frequency (this is used to filter our radio stations). The RLC circuit is only observable when the resonance coincides with the external resonance.

- There will be a bonus problem which is a PDE describing the vibration of a string.
  - Suppose we have a string with fixed endpoints, and suppose it is undergoing a small vibration.
  - Deviation from the equilibrium is described by a function $u(x,t)$.
  - The simplest equation we can derive is the 1D linear wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial x^2} = f(x,t)$$

    - c is the speed of the wave.

    - $f(x,t)$ is the given external force.
  - We can show that when $f(x,t)=0$, then the vibration of the string is the linear supposition of infinitely many standing waves.

$$u(x,t)=\sum_{k=1}^{\infty}a_k\mathrm{e}^{\frac{\pi kt}{\ell}}\sin\frac{k\pi}{\ell}x$$

    - There are $k-1$ nodes in the string. These are called standing waves.
  - If you drive it with frequency

$$f(x,t)=\cos\omega t\sin\frac{k\pi}{\ell}x$$

    you encounter the resonance catastrophe.

- We are interested in the driven harmonic oscillator because it describes the vibrations, even of PDEs.
- This concludes our discussion of explicitly solvable differential equations.
- Those that are solvable by power series require complex analysis.
- Starting this Friday, we will talk about the qualitative theory of differential equations.
- Cauchy-Lipschitz this Friday.
- Next week: Continuous dependence on initial values and differentiation with respect to the parameter of this equation.
- After this, we will be able to compute classical examples in the theory of perturbations.
- We will be able to solve the procession of Mercury problem (which was the first experimental verification of general relativity).

5.3 Qualitative Theory of ODEs

10/28:

- First issue: Uniqueness — we want to be able to talk about *the* solution to the IVP.
- We will be considering the IVP $y'(t)=f(t,y)$, $y(t_0)=y_0$ for $y(t)$ an $\mathbb{R}^n$-valued function.
- To embed our rough outline of the Cauchy-Lipschitz theorem into analysis, we start with metric spaces.
- **Metric space**: A set and a metric. *Denoted by* $\boldsymbol{(X,d)}$.
- **Metric**: A function from $X\times X\to[0,+\infty)$ that satisfies the following three axioms. *Denoted by* $\boldsymbol{d}$.
  1. $d(x,y)=d(y,x)$.
  2. $d(x,y)\geq 0$ and $d(x,y)=0$ iff $x=y$.
  3. Triangle inequality: $d(x,z)\leq d(x,y)+d(y,z)$.
- Examples:
  1. $\mathbb{R}^n$ with $d(x,y)=\|x-y\|=\sqrt{\sum_{j=1}^n|x^j-y^j|^2}$.
  2. Continuous functions $y:[a,b]\to\mathbb{R}^n$ with $d(y_1,y_2)=\|y_1-y_2\|=\sup_{t\in[a,b]}|y_1(t)-y_2(t)|$.
- In Euclidean spaces, we have **completeness**.
- **Cauchy** (sequence): A sequence $\{x_n\}\subset X$ such that for all $\varepsilon>0$, there exists $N\geq 0$ such that $d(x_m,x_n)<\varepsilon$ for all $m,n>N$.

- **Convergent** (sequence): A sequence $\{x_n\} \subset X$ for which there exists $x \in X$ such that
$$\lim_{n\to\infty} d(x, x_n) = 0$$
- **Complete** (metric space): A metric space (X, d) such that every Cauchy sequence is convergent.
- Theorem (Banach fixed point theorem): Let (X, d) be a complete metric space and let $\Phi : X \to X$ be a function for which there exists $q \in (0, 1)$ such that for all $x, y \in X$,
$$d(\Phi(x), \Phi(y)) \leq q \cdot d(x, y)$$
Then there exists a unique $x \in X$ such that $x = \Phi(x)$.

Proof. We first construct the desired fixed point x.

Pick any $x_0 \in X$. Inductively define $\{x_n\}$ by $x_{n+1} = \Phi(x_n)$, starting from $n = 0$. We will now show that $\{x_n\}$ is a Cauchy sequence. As a lemma, we will prove by induction that
$$d(x_j, x_{j+1}) \leq q^j \cdot d(x_0, x_1)$$
for all $j \in \mathbb{N}_0$. For the base case $j = 0$, equality evidently holds. Now suppose inductively that we have proven that $d(x_j, x_{j+1}) \leq q^j \cdot d(x_0, x_1)$; we want to prove the claim for $j + 1$. But we have that
$$\begin{aligned} d(x_{j+1}, x_{j+2}) &= d(\Phi(x_j), \Phi(x_{j+1})) \\ &\leq q \cdot d(x_j, x_{j+1}) \\ &\leq q \cdot q^j \cdot d(x_0, x_1) \\ &= q^{j+1} \cdot d(x_0, x_1) \end{aligned}$$
as desired.

It follows that
$$\begin{aligned} d(x_n, x_{n+m}) &\leq \sum_{k=0}^{m-1} d(x_{n+k}, x_{n+k+1}) && \text{Triangle inequality} \\ &\leq \sum_{k=0}^{m-1} q^{n+k} \cdot d(x_0, x_1) && \text{Lemma} \\ &= q^n(1 + q + \cdots + q^{m-1}) \cdot d(x_0, x_1) \\ &< q^n(1 + q + \cdots + q^{m-1} + \cdots) \cdot d(x_0, x_1) \\ &= \frac{q^n}{1-q} \cdot d(x_0, x_1) \end{aligned}$$
It follows that the above term will converge to zero as $n \to \infty$, so $\{x_n\}$ is a Cauchy sequence and there exists an x such that $x_n \xrightarrow{d} x$.

We now prove that x is a fixed point of Φ, i.e., that $\Phi(x) = x$. We have that
$$\begin{aligned} d(x, \Phi(x)) &\leq d(x, x_n) + d(x_n, \Phi(x_n)) + d(\Phi(x_n), \Phi(x)) \\ &\leq d(x, x_n) + d(x_n, x_{n+1}) + q \cdot d(x_n, x) \\ &= (1 + q) \cdot d(x, x_n) + d(x_n, x_{n+1}) \end{aligned}$$
where the first term converges since $\{x_n\}$ is convergent and the second term converges since $\{x_n\}$ is Cauchy. Thus, $d(x, \Phi(x)) \to 0$ as $n \to \infty$, so $x = \Phi(x)$, as desired.

Lastly, we prove that x is unique. Suppose that there exists $y \in X$ such that $y = \Phi(y)$. Then
$$d(x, y) = d(\Phi(x), \Phi(y)) \leq q \cdot d(x, y)$$
It follows that $d(x, y) \leq q^n \cdot d(x, y)$, i.e., that $d(x, y) \to 0$ as $n \to \infty$. Therefore, we must have that $d(x, y) = 0$, from which it follows that $x = y$, as desired. $\square$

- Notes on the Banach fixed point theorem.
  - Φ is a **contraction**.
  - Zhao gives the example of crumpling a sheet of paper (more specifically, dropping a map of a park in that park; a point coincides).
- Example: Fixed point of the cosine function.
  - Define $\{x_n\}$ by $x_{n+1} = \cos x_n$. If $x_0 \in \mathbb{R}$, then $x_1 \in [-1, 1]$ and $x_2 \in [\cos 1, 1]$.
  - Thus, while cosine is not a contraction on the real numbers ($\cos'(-\pi/2) = 1$, for example), we can show that $\cos : [\cos 1, 1] \to [\cos 1, 1]$ is a contraction: If $x, y \in [\cos 1, 1]$, then

$$\begin{aligned} |\cos x - \cos y| &= \left| \int_y^x -\sin t \, \mathrm{d}t \right| \\ &\leq |x - y| \sup_{t \in [\cos 1, 1]} |\sin t| \\ &\leq (\sin 1)|x - y| \end{aligned}$$

  - Thus, cosine has a fixed point at the intersection of $y = \cos x$ and $y = x$ of approximate value $0.739\ldots$
  - Overall, this is a pretty bad example, though.
- Theorem: Let $y_k : [a, b] \to \mathbb{R}^n$ be a Cauchy sequence of continuous functions under the sup norm. Then the limit exists and is continuous.
  - The proof is based on uniform convergence, which we've encountered before in analysis.
  - It follows that $C[a, b]$ (the metric space of all continuous functions on $[a, b]$) is complete.
  - If $\{y_k\} \subset \bar{B}(0, M)$[1], then the limit y is in $\bar{B}(0, M)$.
- Let's return to our ODE $y'(t) = f(t, y(t))$, $y(t_0) = y_0 \in \mathbb{R}^n$.
- We now have the tools to prove the Cauchy-Lipschitz theorem, and we will presently build up to that.
- Although we do not typically think of it this way, f is still a function with a domain and range. In particular, its domain is the set of ordered pairs where the first entry is a real number and the second entry is an element of the range of y, i.e., an element of $\mathbb{R}^n$. Thus, to begin, we are allowed to impose the following conditions on f.
  - Let $f(t, z)$ be defined on $[t_0, t_0 + a] \times \bar{B}(y_0, b)$ for some $a, b \in \mathbb{R}_+$ (we will put further constraints on the values of a, b later).
  - On this domain, suppose $|f|$ is bounded by some $M \in \mathbb{R}$, i.e., $|f(t, z)| \leq M$ for all t, z in the above set.
  - Let f be Lipschitz continuous in the second argument. In particular, there exists $L > 0$ such that $|f(t, z_1) - f(t, z_2)| \leq L|z_1 - z_2|$ for any $z_1, z_2 \in \mathbb{R}^n$.
- We usually consider a given ODE in differential form. However, there's no reason we can't consider the equivalent integral equation

$$y(t) = y_0 + \int_{t_0}^t f(\tau, y(\tau)) \mathrm{d}\tau$$

  - The reason for this change of perspective will become apparent shortly.

[1] $\bar{B}(0, M)$ denotes the set of all functions $y : [a, b] \to \mathbb{R}^n$ with sup norm at most M; topologically, it is the closed ball of radius M centered at the origin in $C[a, b]$.

- Let $\Phi : C[t_0, t_0 + a] \to C[t_0, t_0 + a]$ map functions to functions. Specifically, let it send

$$y(t) \mapsto y_0 + \int_{t_0}^{t} f(\tau, y(\tau))\mathrm{d}\tau$$

  - We denote this by writing $\Phi[y](t) = y_0 + \int_{t_0}^{t} f(\tau, y(\tau))\mathrm{d}\tau$.
  - Notice that the solution of our IVP is exactly the point of $C[t_0, t_0 + a]$ fixed by Φ because

$$y(t) = y_0 + \int_{t_0}^{t} f(\tau, y(\tau))\mathrm{d}\tau \quad \iff \quad y = \Phi[y]$$

  - *This* motivates all steps taken thus far.
- All that remains is to show that Φ is a contraction on some subset of $C[t_0, t_0 + a]$. Then we can apply the Banach fixed point theorem.
- We first identify this subset. Let

$$X_b = \{y : [t_0, t_0 + a] \to \bar{B}(y_0, b)\}$$

  - By the previous theorem, this is a complete metric space.
  - We now want to relate a and b so that $\Phi(X_b) \subset X_b$ and Φ is a contraction.
  - For $\Phi(X_b) \subset X_b$, we need

$$\|\Phi[y] - y_0\| \leq \int_{t_0}^{t_0+a} |f(\tau, y(\tau))|\mathrm{d}\tau \leq a \cdot M \leq b$$

    so we want $a < b/M$.
  - Moreover, if Φ is to be a contraction, then since

$$\begin{aligned}\|\Phi[y_1] - \Phi[y_2]\| &\leq \int_{t_0}^{t+a} |f(\tau, y_1(\tau)) - f(\tau, y_2(\tau))|\mathrm{d}\tau \\ &\leq La \cdot \|y_1 - y_2\|\end{aligned}$$

    we want $La \in (0, 1)$. We can achieve this by requiring $a < 1/2L$.
  - Thus, choosing

$$a < \min\left(\frac{1}{2L}, \frac{b}{M}\right)$$

    accomplishes all of our goals.
- Therefore, by the Banach fixed point theorem, there exists a unique y such that $y = \Phi[y]$.
  - As we have already remarked, this fixed point is exactly the aforementioned solution to the IVP.
- Conclusion:
- Theorem (Cauchy-Lipschitz theorem): Let $f(t, z)$ be defined on an open subset $\Omega \subset \mathbb{R}_t \times \mathbb{R}_z^n$, $(t_0, y_0) \in \Omega$, such that f is Lipschitz continuous wrt. t, z in some neighborhood of (t_0, y_0). Then the IVP $y'(t) = f(t, y(t))$, $y(t_0) = y_0$ has a unique solution for some $T > 0$ on $[t_0, t_0 + T]$ such that $y(t)$ does not escape that neighborhood.
- If $f \in C_1$, then

$$|f(t, z_1) - f(t, z_2)| \leq \sup_{z \in \bar{B}(y_0, r)} \left\| \frac{\partial f}{\partial z}(t, z) \right\| \cdot |z_1 - z_2|$$

- We use the finite increment theorem of differential calculus to prove that f is Lipschitz continuous if it's continuously differentiable.
- The norm on the RHS above is the matrix norm.
- We have $y(t) = \int_{t_0}^{t} f(\tau, y(\tau))\mathrm{d}\tau + y_0$ and we use the Banach fixed point theorem (which is proved constructively).
- We have $y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(\tau, y_n(\tau))\mathrm{d}\tau$.
  - Thus, the Picard iteration is justified by the Banach fixed point theorem.
- We do not use the algorithm from the proof (the Picard iteration) computationally; we use the polygon algorithm. This algorithm is only of theoretical significance.
- **Interval of existence**: The union of intervals containing the interval $[t_0, t_0 + T]$ on which the IVP has a solution.
  - The interval of existence is always open. If $t_0 \in I$ such that $y(t_0) = y$, then $y'(t) = f(t, y(t))$, $y(t_0) = y$.
  - Note that the theorem does not predict when singularity can occur.
  - Example: The interval of existence will always be $x' = 1 + x^2$, $x(t_0) = x_0$. Then $x(t) = \tan(t - t_0 + \arctan(x_0))$. The length of existence is always π.
- Interval of existence: If you consider the IVP $y'(t) = f(t, y(t))$, $y(t_0) = y_0$, then $[t_0, t_0 + T_1], [t_0 + T_1, t_0 + T_2]$. The first is of length T_1, and the second of length $T_2 - T_1$. Continuing on, we get $T_n - T_1$ so that $T_n \to \infty$ or T_n is bounded. This gives us the maximal solution/interval of existence.
- The motherfucker (Shao) made us stay 10 minutes late.