Week 5

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5.1 Planar Autonomous Linear Systems

10/24:

- Review of vector fields.
- **Phase diagram**: A diagram that shows the qualitative behavior of an autonomous ordinary differential equation. *Also known as* **phase portrait**.

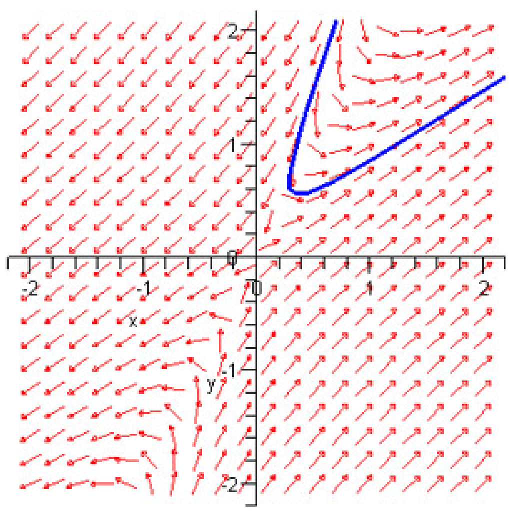
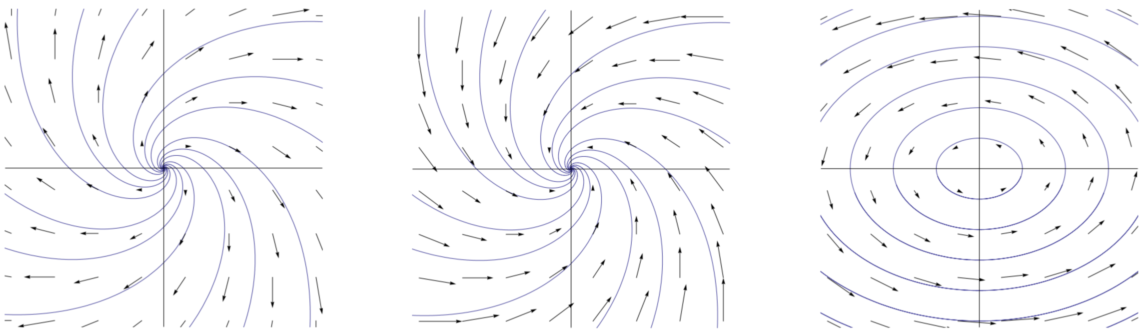


Figure 5.1: Phase diagram example.

  - Consists of a selection of arrows describing, to some extent, a vector field and is often paired with integral curves.
- Suppose $\Omega \subset \mathbb{R}^n$ is open.
- **Vector field** (on Ω): A mapping from $\Omega \to \mathbb{R}^n$. *Denoted by* $\boldsymbol{X}$.
  - Essentially, a vector field assigns to every point of some region a vector; the definition just formalizes this notion.
- **Flow**: A formalization of the idea of the motion of particles in a fluid.
  - The solution to the IVP $\frac{\mathrm{d}y}{\mathrm{d}t} = X(y)$, $y(0) = x$.
- If X is C^1, then for all $x \in \Omega$, there exists a unique solution y to the above IVP.

- **Orbit** (of x under X): The trajectory $y(t,x)$.
  - Recall that the tangent vector to any trajectory at any point coincides with the vector to which X maps that point.
- **Fixed point**: A point $x_0 \in \Omega$ such that $X(x_0) = \bar{0}$.
  - If x_0 is a fixed point, then the trajectory is $y(t) = x_0$.
- Today: We will consider flows on vector fields where the dimension is two and our vector field is linear. In particular...
- Let A be a 2×2 real matrix, and let $X(x) = Ax$.
  - In this case, $x_0 = 0$ is the only fixed point.
  - The flow is given by the linear differential equation $y' = Ay$, $y(0) = x$. The solution is $y(t) = \mathrm{e}^{tA}x$.
- Case 1: A has no real eigenvalues.



(a) $\sigma > 0$. (b) $\sigma < 0$. (c) $\sigma = 0$.

Figure 5.2: Phase diagrams for a planar system with no real eigenvalues.

  - We know that $\chi_A(z)$ is a real polynomial: $\chi_A(z) = z^2 + (\operatorname{trace} A)z + \det A$, and since A is real, both $\operatorname{trace} A$ and $\det A$ are real.
  - Thus, the eigenvalues appear as conjugate pair, i.e., we may write $\lambda = \sigma + i\beta$ and $\bar{\lambda} = \sigma - i\beta$.
    - $\alpha = \gamma = 1$ for both eigenvalues.
    - The eigenvectors must also be complex conjugates.
  - Distinct eigenvalues imply that A is diagonalizable.
  - However, this is not what we want because if we use the complex form, then

$$\mathrm{e}^{tA} = Q \begin{pmatrix} \mathrm{e}^{t\lambda} & 0 \\ 0 & \mathrm{e}^{t\bar{\lambda}} \end{pmatrix} Q^{-1}$$

  - Indeed, we want to get a real matrix out of $Q, \mathrm{e}^{t\Lambda}, Q^{-1}$ all complex. We have

$$\begin{aligned}
\mathrm{e}^{tA}x &= Q \begin{pmatrix} \mathrm{e}^{t(\sigma+i\beta)} & 0 \\ 0 & \mathrm{e}^{t(\sigma-i\beta)} \end{pmatrix} \underbrace{Q^{-1}x}_{z} \\
&= Q \begin{pmatrix} \mathrm{e}^{t(\sigma+i\beta)}z^1 \\ \mathrm{e}^{t(\sigma-i\beta)}z^2 \end{pmatrix} \\
&= z^1\mathrm{e}^{t(\sigma+i\beta)}v + z^2\mathrm{e}^{t(\sigma-i\beta)}\bar{v}
\end{aligned}$$

– Since $y(0) = x = z^1 v + z^2 \overline{v} \in \mathbb{R}^2$ (i.e., $z^1 v + z^2 v$ is *real*), we know that it is equal to its complex conjugate. This tells us that

$$\begin{aligned} z^1 v + z^2 \overline{v} &= \bar{z^1} \overline{v} + \bar{z^2} v \\ z^1 &= \bar{z^2} \end{aligned}$$

– It follows that

$$\begin{aligned} y(t) &= \mathrm{e}^{tA} x \\ &= z^1 \mathrm{e}^{t(\sigma + i\beta)} v + \bar{z^1} \mathrm{e}^{t(\sigma - i\beta)} \overline{v} \\ &= z^1 \mathrm{e}^{t(\sigma + i\beta)} v + \overline{z^1 \mathrm{e}^{t(\sigma + i\beta)} v} \\ &= 2 \operatorname{Re}(z^1 \mathrm{e}^{t(\sigma + i\beta)} v) \\ &= 2 \operatorname{Re}(z^1 \mathrm{e}^{\sigma t} (\cos(\beta t) + i \sin(\beta t))(v_1 + i v_2)) \\ &= 2 \operatorname{Re}(z^1 \mathrm{e}^{\sigma t} (\cos(\beta t) v_1 + i \cos(\beta t) v_2 + i \sin(\beta t) v_1 - \sin(\beta t) v_2)) \\ &= 2\mathrm{e}^{\sigma t} \cos(\beta t) \cdot \operatorname{Re}(z^1 v) - 2\mathrm{e}^{\sigma t} \sin(\beta t) \cdot \operatorname{Im}(z^1 v) \end{aligned}$$

– Suppose $\sigma \neq 0$. Then

$$x \mapsto \begin{pmatrix} \operatorname{Re}(z^1 v) \\ \operatorname{Im}(z^1 v) \end{pmatrix}$$

is a real linear transformation on $\mathbb{R}^2$.

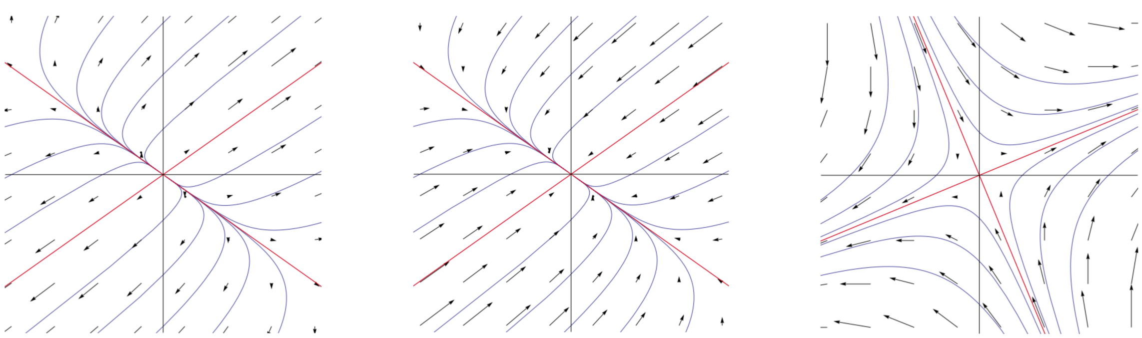
– It follows that the trajectories are just spirals in the complex plane.

– If $\sigma > 0$, then the spiral repels from the origin. If $\sigma < 0$, then the spiral attracts to the origin. If $\sigma = 0$, we get an ellipse.

– Therefore, we have completely classified equations of the form

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix}' = \begin{pmatrix} y^2 \\ -\omega^2 y^1 \end{pmatrix}$$

- Case 2: A has real eigenvalues and *is* diagonalizable.



(a) $\lambda_1, \lambda_2 > 0$. (b) $\lambda_1, \lambda_2 < 0$. (c) $\lambda_1 < 0 < \lambda_2$ (WLOG).

Figure 5.3: Phase diagrams for a diagonalizable planar system with real eigenvalues.

– Suppose $\lambda_1, \lambda_2 \in \mathbb{R}$ have corresponding linearly independent eigenvectors v_1, v_2.

– If we choose v_1, v_2 to be our basis, then

$$\mathrm{e}^{tA} = Q \begin{pmatrix} \mathrm{e}^{t\lambda_1} & 0 \\ 0 & \mathrm{e}^{t\lambda_2} \end{pmatrix} Q^{-1}$$

where $Q = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$.

- Thus, as before, the solution may be expressed in the following form, where $z = Q^{-1}x$.

$$y(t) = \mathrm{e}^{tA}x = \mathrm{e}^{\lambda_1 t}z^1 v_1 + \mathrm{e}^{\lambda_2 t}z^2 v_2$$

- Moving forward, it will be convenient to work in the v_1, v_2 basis. We divide into three subcases ($\lambda_1, \lambda_2 > 0$ [Figure 5.3a], $\lambda_1, \lambda_2 < 0$ [Figure 5.3b], and WLOG $\lambda_1 < 0 < \lambda_2$ [Figure 5.3c]).
  1. Notice that
  $$\mathrm{e}^{\lambda_2 t} = \mathrm{e}^{(\lambda_2/\lambda_1)(\lambda_1 t)}$$
  i.e., $\mathrm{e}^{\lambda_2 t}$ is a power of $\mathrm{e}^{\lambda_1 t}$. Thus, when the signs are the same, we get a power function $v_2 = v_1^{\lambda_2/\lambda_1}$.
     - Both subspaces v_1, v_2 are unstable here.
  2. If $\lambda_1, \lambda_2 < 0$, then we have the same trajectories, but they're all attracted to the origin instead of repelled.
     - Both subspaces v_1, v_2 are stable here.
  3. When both eigenvalues have different signs, we are considering power functions of a negative power.
     - The stable subspace is v_2 and the unstable subspace is v_1 here.

- Case 3: A has real eigenvalues and *is not* diagonalizable.

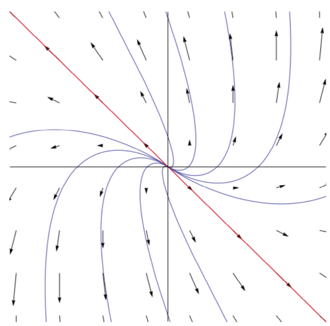


Figure 5.4: Phase diagrams for a nondiagonalizable planar system with real eigenvalues.

- In this case, the matrix exponential is given by

$$\mathrm{e}^{tA} = Q \begin{pmatrix} \mathrm{e}^{t\lambda} & t\mathrm{e}^{t\lambda} \\ 0 & \mathrm{e}^{t\lambda} \end{pmatrix} Q^{-1}$$

- The solution is given by
$$\mathrm{e}^{tA}x = (z^1\mathrm{e}^{t\lambda} + z^2 t\mathrm{e}^{t\lambda})v + z^2\mathrm{e}^{t\lambda}u$$
where $Q^{-1}x = z$ again.
- In graphing, note that here we have (a distorted version of) the form $y = x \pm x \log x$:
$$y = (z^1\mathrm{e}^{t\lambda} + z^2 t\mathrm{e}^{t\lambda})\hat{\imath} + z^2\mathrm{e}^{t\lambda}\hat{\jmath}$$
Define $x := \mathrm{e}^{t\lambda}$. Then $t = \lambda^{-1}\ln x$. Substituting, we have
$$\begin{aligned} &= (z^1 x + z^2(\lambda^{-1}\ln x)x)\hat{\imath} + z^2 x\hat{\jmath} \\ &= (z^1 x + z^2\lambda^{-1}x\ln x)\hat{\imath} + z^2 x\hat{\jmath} \end{aligned}$$
- When $\lambda > 0$, the whole space is unstable. Thus, the phase diagram is tangent to the origin.
- When $\lambda < 0$, the trajectories take the same form but this time are attracted to zero. In this case, the whole space is stable.

- We can take x_1 to x_2 iff they are in the same orbit. Conclusion: Orbits never cross.
- Takeaway: You should be able to compute the eigenvalues and eigenvectors and sketch these graphs.
- Shao will post lecture notes after today's lecture!
- Next lecture: The final explicitly solveable case, which is the driven harmonic oscillator.