Week 5

End Quantitative and Intro to Qualitative

5.1   Planar Autonomous Linear Systems

- Review of vector fields.

- **Phase diagram**: A diagram that shows the qualitative behavior of an autonomous ordinary differential equation. *Also known as* **phase portrait**.
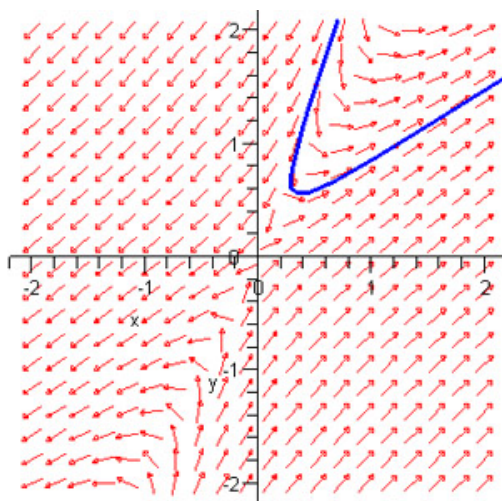

Figure 5.1: Phase diagram example.

  - Consists of a selection of arrows describing, to some extent, a vector field and is often paired with integral curves.

- Suppose $\Omega \subset \mathbb{R}^n$ is open.

- **Vector field** (on Ω): A mapping from $\Omega \to \mathbb{R}^n$. *Denoted by* $\boldsymbol{X}$.

  - Essentially, a vector field assigns to every point of some region a vector; the definition just formalizes this notion.

- **Flow**: A formalization of the idea of the motion of particles in a fluid.

  - The solution to the IVP $\frac{\mathrm{d}y}{\mathrm{d}t} = X(y)$, $y(0) = x$.

- If X is C^1, then for all $x \in \Omega$, there exists a unique solution y to the above IVP.

- **Orbit** (of x under X): The trajectory $y(t, x)$.

  - Recall that the tangent vector to any trajectory at any point coincides with the vector to which X maps that point.

- **Fixed point**: A point $x_0 \in \Omega$ such that $X(x_0) = \bar{0}$.

  - If x_0 is a fixed point, then the trajectory is $y(t) = x_0$.

- Today: We will consider flows on vector fields where the dimension is two and our vector field is linear. In particular...

- Let A be a 2×2 real matrix, and let $X(x) = Ax$.

  - In this case, $x_0 = 0$ is the only fixed point.
  - The flow is given by the linear differential equation $y' = Ay$, $y(0) = x$. The solution is $y(t) = e^{tA}x$.

- Case 1: A has no real eigenvalues.

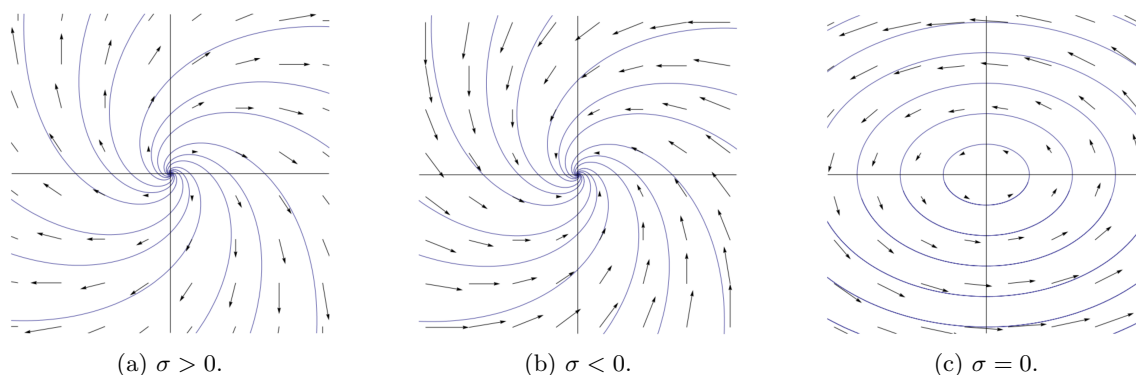(a) $\sigma > 0$. (b) $\sigma < 0$. (c) $\sigma = 0$.

Figure 5.2: Phase diagrams for a planar system with no real eigenvalues.

  - We know that $\chi_A(z)$ is a real polynomial: $\chi_A(z) = z^2 + (\text{trace } A)z + \det A$, and since A is real, both trace A and $\det A$ are real.
  - Thus, the eigenvalues appear as conjugate pair, i.e., we may write $\lambda = \sigma + i\beta$ and $\bar{\lambda} = \sigma - i\beta$.
    - $\alpha = \gamma = 1$ for both eigenvalues.
    - The eigenvectors must also be complex conjugates.
  - Distinct eigenvalues imply that A is diagonalizable.
  - However, this is not what we want because if we use the complex form, then

$$e^{tA} = Q \begin{pmatrix} e^{t\lambda} & 0 \\ 0 & e^{t\bar{\lambda}} \end{pmatrix} Q^{-1}$$

  - Indeed, we want to get a real matrix out of $Q, e^{t\Lambda}, Q^{-1}$ all complex. We have

$$e^{tA}x = Q \begin{pmatrix} e^{t(\sigma+i\beta)} & 0 \\ 0 & e^{t(\sigma-i\beta)} \end{pmatrix} \underbrace{Q^{-1}x}_{z}$$
$$= Q \begin{pmatrix} e^{t(\sigma+i\beta)}z^1 \\ e^{t(\sigma-i\beta)}z^2 \end{pmatrix}$$
$$= z^1 e^{t(\sigma+i\beta)}v + z^2 e^{t(\sigma-i\beta)}\bar{v}$$

- Since $y(0) = x = z^1 v + z^2 \bar{v} \in \mathbb{R}^2$ (i.e., $z^1 v + z^2 v$ is *real*), we know that it is equal to its complex conjugate. This tells us that

$$z^1 v + z^2 \bar{v} = \bar{z^1} \bar{v} + \bar{z^2} v$$
$$z^1 = \bar{z^2}$$

- It follows that

$$\begin{aligned}
y(t) &= e^{tA} x \\
&= z^1 e^{t(\sigma + i\beta)} v + \bar{z^1} e^{t(\sigma - i\beta)} \bar{v} \\
&= z^1 e^{t(\sigma + i\beta)} v + \overline{z^1 e^{t(\sigma + i\beta)} v} \\
&= 2 \operatorname{Re}(z^1 e^{t(\sigma + i\beta)} v) \\
&= 2 \operatorname{Re}(z^1 e^{\sigma t}(\cos(\beta t) + i \sin(\beta t))(v_1 + i v_2)) \\
&= 2 \operatorname{Re}(z^1 e^{\sigma t}(\cos(\beta t) v_1 + i \cos(\beta t) v_2 + i \sin(\beta t) v_1 - \sin(\beta t) v_2)) \\
&= 2 e^{\sigma t} \cos(\beta t) \cdot \operatorname{Re}(z^1 v) - 2 e^{\sigma t} \sin(\beta t) \cdot \operatorname{Im}(z^1 v)
\end{aligned}$$

- Suppose $\sigma \neq 0$. Then

$$x \mapsto \begin{pmatrix} \operatorname{Re}(z^1 v) \\ \operatorname{Im}(z^1 v) \end{pmatrix}$$

is a real linear transformation on $\mathbb{R}^2$.

- It follows that the trajectories are just spirals in the complex plane.
- If $\sigma > 0$, then the spiral repels from the origin. If $\sigma < 0$, then the spiral attracts to the origin. If $\sigma = 0$, we get an ellipse.
- Therefore, we have completely classified equations of the form

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix}' = \begin{pmatrix} y^2 \\ -\omega^2 y^1 \end{pmatrix}$$

- Case 2: A has real eigenvalues and *is* diagonalizable.

(a) $\lambda_1, \lambda_2 > 0$.   (b) $\lambda_1, \lambda_2 < 0$.   (c) $\lambda_1 < 0 < \lambda_2$ (WLOG).
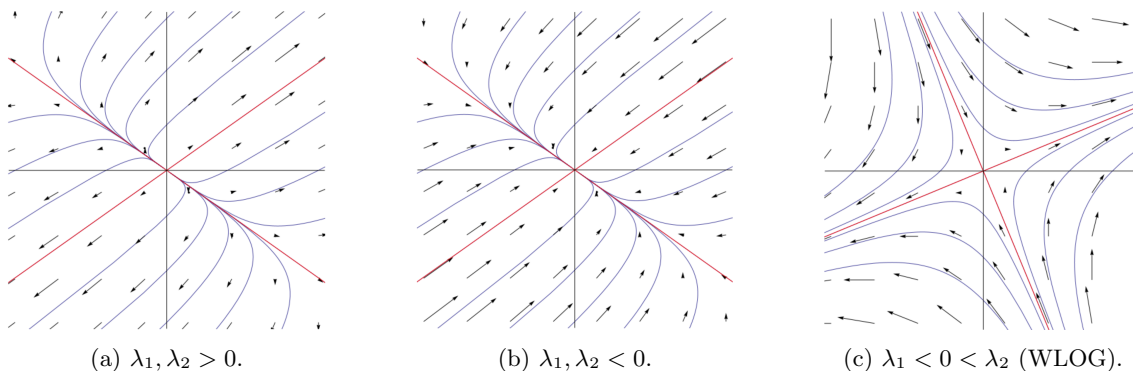
Figure 5.3: Phase diagrams for a diagonalizable planar system with real eigenvalues.

- Suppose $\lambda_1, \lambda_2 \in \mathbb{R}$ have corresponding linearly independent eigenvectors v_1, v_2.
- If we choose v_1, v_2 to be our basis, then

$$e^{tA} = Q \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix} Q^{-1}$$

where $Q = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$.

– Thus, as before, the solution may be expressed in the following form, where $z = Q^{-1}x$.

$$y(t) = e^{tA}x = e^{\lambda_1 t}z^1 v_1 + e^{\lambda_2 t}z^2 v_2$$

– Moving forward, it will be convenient to work in the v_1, v_2 basis. We divide into three subcases ($\lambda_1, \lambda_2 > 0$ [Figure 5.3a], $\lambda_1, \lambda_2 < 0$ [Figure 5.3b], and WLOG $\lambda_1 < 0 < \lambda_2$ [Figure 5.3c]).

1. Notice that
$$e^{\lambda_2 t} = e^{(\lambda_2/\lambda_1)(\lambda_1 t)}$$

i.e., $e^{\lambda_2 t}$ is a power of $e^{\lambda_1 t}$. Thus, when the signs are the same, we get a power function $v_2 = v_1^{\lambda_2/\lambda_1}$.
   ■ Both subspaces v_1, v_2 are unstable here.
2. If $\lambda_1, \lambda_2 < 0$, then we have the same trajectories, but they're all attracted to the origin instead of repelled.
   ■ Both subspaces v_1, v_2 are stable here.
3. When both eigenvalues have different signs, we are considering power functions of a negative power.
   ■ The stable subspace is v_2 and the unstable subspace is v_1 here.

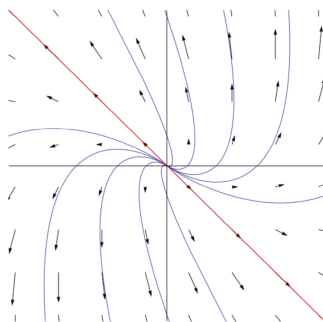- Case 3: A has real eigenvalues and *is not* diagonalizable.

Figure 5.4: Phase diagrams for a nondiagonalizable planar system with real eigenvalues.

– In this case, the matrix exponential is given by

$$e^{tA} = Q \begin{pmatrix} e^{t\lambda} & te^{t\lambda} \\ 0 & e^{t\lambda} \end{pmatrix} Q^{-1}$$

– The solution is given by
$$e^{tA}x = (z^1 e^{t\lambda} + z^2 te^{t\lambda})v + z^2 e^{t\lambda}u$$

where $Q^{-1}x = z$ again.

– In graphing, note that here we have (a distorted version of) the form $y = x \pm x\log x$:

$$y = (z^1 e^{t\lambda} + z^2 te^{t\lambda})\hat{\imath} + z^2 e^{t\lambda}\hat{\jmath}$$

Define $x := e^{t\lambda}$. Then $t = \lambda^{-1}\ln x$. Substituting, we have

$$= (z^1 x + z^2(\lambda^{-1}\ln x)x)\hat{\imath} + z^2 x\hat{\jmath}$$
$$= (z^1 x + z^2 \lambda^{-1}x\ln x)\hat{\imath} + z^2 x\hat{\jmath}$$

– When $\lambda > 0$, the whole space is unstable. Thus, the phase diagram is tangent to the origin.
– When $\lambda < 0$, the trajectories take the same form but this time are attracted to zero. In this case, the whole space is stable.

- We can take x_1 to x_2 iff they are in the same orbit. Conclusion: Orbits never cross.

- Takeaway: You should be able to compute the eigenvalues and eigenvectors and sketch these graphs.

- Shao will post lecture notes after today's lecture!

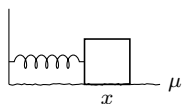- Next lecture: The final explicitly solveable case, which is the driven harmonic oscillator.

5.2   Driven Harmonic Oscillator and Resonance

10/26:   • We are interested in the 2nd order constant coefficient equation
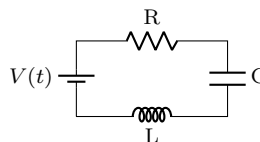
$$x'' + \mu x' + \omega_0^2 x = H_0 e^{i\omega t}$$

where $\mu \geq 0$ and $\omega_0, \omega > 0$.

- Two cases where this ODE arises:

(a) A driven harmonic oscillator.   (b) An RLC circuit.

Figure 5.5: Origins of the driven harmonic oscillator equation.

1. The driven harmonic oscillator.
   - Consider a mass on a spring.
   - The extent of friction between the mass point and the surface is described by μ.
   - The oscillation is periodically driven by a force of magnitude $H_0 \cos \omega t$.
2. RLC circuit.
   - R is resistance, L is inductance, C is capacitance.
   - We have the laws

$$LI'_L(t) = V_L \qquad\qquad CV'_C(t) = I_C \qquad\qquad I_R(t) = V_R(t)/R$$

   - ■ Left: Self-inductance.
   - ■ Right: Ohm's law.
   - Combining them with Kirchhoff's laws

$$I(t) = I_R = I_C = I_L \qquad\qquad V(t) = V_R + V_L + V_C$$

   we get the RLC circuit equation

$$LI'' + RI' + \frac{I}{C} = V'(t)$$

   - The most interesting cases is when we have a source of alternating current of frequency ω. In this case, $V(t) = V_0 \cos \omega t$ or, in the complex case, $V(t) = V_0 e^{i\omega t}$. This yields the complex equation

$$I'' + \frac{R}{L} I' + \frac{1}{LC} I = \frac{i\omega V_0}{L} e^{i\omega t}$$

   - Here, the friction coefficient $\mu = R/L$ and the frequency is $\omega_0 = \sqrt{1/LC}$.

- Recall that we want to solve the following ODE.

$$x'' + \mu x' + \omega_0^2 x = H_0 e^{i\omega t}$$

  - The homogeneous linear equation $x'' + \mu x' + \omega_0^2 x = 0$ is well-understood, i.e., we can find all of the *homogeneous* solutions to the above equation.
  - Thus, to solve the above inhomogeneous equation, we just have to find a particular solution.

- WLOG let $\omega > 0$.

- From the homework, a particular solution $x_p(t)$ with initial condition $x_p(0) = x_p'(0) = \mu = 0$ can be obtained from the Duhamel formula as follows.

$$x_p(t) = H_0 \int_0^t \frac{\sin \omega_0 (t - \tau)}{\omega_0} e^{i\omega\tau} \mathrm{d}\tau$$

  - Substituting

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

    into the above allows us to evaluate it.
  - In particular, it follows that

$$x_p(t) = \begin{cases} \dfrac{H_0}{\omega_0^2 - \omega^2}\left(e^{i\omega t} - \cos\omega_0 t - \dfrac{i\omega}{\omega_0}\sin\omega_0 t\right) & \omega \neq \omega_0 \\ -\dfrac{iH_0}{2\omega_0}\left(te^{i\omega_0 t} - \dfrac{\sin\omega_0 t}{\omega_0}\right) & \omega = \omega_0 \end{cases}$$

    - We compute the $\omega = \omega_0$ case using L'Hôpital's rule to analyze the $\omega \neq \omega_0$ case as $\omega \to \omega_0$.
  - If we pump in energy at the same point that we have deviation ($\omega = \omega_0$), then the amplitude of oscillation goes to ∞.
    - Practically, when $\omega \approx \omega_0$, the long-time behavior of the driven oscillator will be very much like a growing oscillator.
    - Eventually, the amplitude will be approximately $(\omega - \omega_0)^{-1}$.

- **Resonance catastrophe**: Inputing energy into a system at its natural frequency, causing the total energy to grow until a mechanical failure occurs.

  - This is what happened at the Millenium Bridge in London; synchronized footsteps caused the bridge to shake really wildly.

- If $\mu > 0$, there will be a particular solution of the form

$$x_p(t) = A(\omega)H_0 e^{i\omega t}$$

  - From HW1, we have three cases when $\mu > 0$: $0 < \mu < 2\omega_0$, $\mu = 2\omega_0$, and $\mu > 2\omega_0$. These are just the three cases of the characteristic polynomial??
  - Substituting the proposed form of the particular solution into the differential equation, we get

$$x_p'' + \mu x_p' + \omega_0^2 x_p = H_0 e^{i\omega t}$$
$$(-\omega^2 + i\omega\mu + \omega_0^2)H_0 A(\omega)e^{i\omega t} = H_0 e^{i\omega t}$$
$$(-\omega^2 + i\omega\mu + \omega_0^2)A(\omega) = 1$$
$$A(\omega) = \frac{1}{\omega_0^2 - \omega^2 + i\mu\omega}$$

  - In theory, we avoid the resonance catastrophe in this case. In practice, however, when $\omega \to 0$, we still run into issues.

– For mass point:

$$|H_0 A(\omega)| = \frac{|H_0|}{\sqrt{(\omega^2 - \omega_0^2)^2 + \mu^2 \omega^2}}$$

■ The norm $|H_0 A(\omega)|$ is maximized when $\omega_r = \sqrt{\omega_0^2 + \mu^2/2}$.
■ $\omega_r \to \omega_0$ implies $\mu \to 0$??

– As for the argument/angle,

$$\arg(H_0 A(\omega)) = \arg H_0 + \arg A(\omega)$$

– We consider $\omega : 0 \to \omega_0 \to +\infty$.

■ When $\omega = 0$, the complex amplitude is $1/\omega_0^2$ so it's a real number in the complex plane.
■ If ω is increased a bit, we get the reciprocal of a complex number. Norm is reciprocal, argument is negative.
■ For $\omega = \omega_0$, we have a purely imaginary number.
■ As $\omega \to \infty$, the argument approaches $-\pi$??
■ Showing the shape of the norm and the argument with respect to ω. This allows us to completely describe the resonance phenomena.

• For the RLC circuit, the discussion is a bit different.

– The external voltage $V(t) = V_0 e^{i\omega t}$. Thus, $V'(t) = iV_0 \omega e^{i\omega t}$.
– Here,

$$x_p(t) = \frac{iV_0 \omega e^{i\omega t}}{\omega_0^2 - \omega^2 + iR\omega/L}$$

– Look at the complex amplitude.

■ Multiply the numerator and denominator by the inductance L to get

$$x_p(t) = \frac{iV_0 \omega L e^{i\omega t}}{L\omega_0^2 - L\omega^2 + iR\omega}$$

■ Then,

$$\text{Norm} = \frac{V_0 L}{\sqrt{R^2 + \left(\frac{1}{C\omega} - \frac{\omega}{L}\right)^2}}$$

■ For an RLC circuit, the resistance does not affect the resonance frequency.

$$\omega_r = \sqrt{\frac{1}{LC}} = \omega_0$$

■ If you have an external source of voltage, then you can vary the capacity of your circuit to ensure that the voltage will be maximized at a given frequency. We can tune our circuit to a very specific resonance frequency (this is used to filter our radio stations). The RLC circuit is only observable when the resonance coincides with the external resonance.

• There will be a bonus problem which is a PDE describing the vibration of a string.

– Suppose we have a string with fixed endpoints, and suppose it is undergoing a small vibration.
– Deviation from the equilibrium is described by a function $u(x,t)$.
– The simplest equation we can derive is the 1D linear wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial x^2} = f(x,t)$$

■ c is the speed of the wave.

- ■ $f(x,t)$ is the given external force.
  - We can show that when $f(x,t) = 0$, then the vibration of the string is the linear supposition of infinitely many standing waves.

$$u(x,t) = \sum_{k=1}^{\infty} a_k e^{\frac{\pi k t}{\ell}} \sin \frac{k\pi}{\ell} x$$

- ■ There are $k-1$ nodes in the string. These are called standing waves.
  - If you drive it with frequency

$$f(x,t) = \cos \omega t \sin \frac{k\pi}{\ell} x$$

you encounter the resonance catastrophe.

- We are interested in the driven harmonic oscillator because it describes the vibrations, even of PDEs.

- This concludes our discussion of explicitly solvable differential equations.

- Those that are solvable by power series require complex analysis.

- Starting this Friday, we will talk about the qualitative theory of differential equations.

- Cauchy-Lipschitz this Friday.

- Next week: Continuous dependence on initial values and differentiation with respect to the parameter of this equation.

- After this, we will be able to compute classical examples in the theory of perturbations.

- We will be able to solve the procession of Mercury problem (which was the first experimental verification of general relativity).

5.3   Qualitative Theory of ODEs

10/28:
- First issue: Uniqueness — we want to be able to talk about *the* solution to the IVP.

- We will be considering the IVP $y'(t) = f(t,y)$, $y(t_0) = y_0$ for $y(t)$ an $\mathbb{R}^n$-valued function.

- To embed our rough outline of the Cauchy-Lipschitz theorem into analysis, we start with metric spaces.

- **Metric space**: A set and a metric. *Denoted by* $(\boldsymbol{X}, \boldsymbol{d})$.

- **Metric**: A function from $X \times X \to [0, +\infty)$ that satisfies the following three axioms. *Denoted by* $\boldsymbol{d}$.

  1. $d(x,y) = d(y,x)$.
  2. $d(x,y) \geq 0$ and $d(x,y) = 0$ iff $x = y$.
  3. Triangle inequality: $d(x,z) \leq d(x,y) + d(y,z)$.

- Examples:

  1. $\mathbb{R}^n$ with $d(x,y) = \|x - y\| = \sqrt{\sum_{j=1}^{n} |x^j - y^j|^2}$.
  2. Continuous functions $y : [a,b] \to \mathbb{R}^n$ with $d(y_1, y_2) = \|y_1 - y_2\| = \sup_{t \in [a,b]} |y_1(t) - y_2(t)|$.

- In Euclidean spaces, we have **completeness**.

- **Cauchy** (sequence): A sequence $\{x_n\} \subset X$ such that for all $\varepsilon > 0$, there exists $N \geq 0$ such that $d(x_m, x_n) < \varepsilon$ for all $m, n > N$.

- **Convergent** (sequence): A sequence $\{x_n\} \subset X$ for which there exists $x \in X$ such that

$$\lim_{n \to \infty} d(x, x_n) = 0$$

- **Complete** (metric space): A metric space (X, d) such that every Cauchy sequence is convergent.

- Theorem (Banach fixed point theorem): Let (X, d) be a complete metric space and let $\Phi : X \to X$ be a function for which there exists $q \in (0, 1)$ such that for all $x, y \in X$,

$$d(\Phi(x), \Phi(y)) \leq q \cdot d(x, y)$$

Then there exists a unique $x \in X$ such that $x = \Phi(x)$.

Proof. We first construct the desired fixed point x.

Pick any $x_0 \in X$. Inductively define $\{x_n\}$ by $x_{n+1} = \Phi(x_n)$, starting from $n = 0$. We will now show that $\{x_n\}$ is a Cauchy sequence. As a lemma, we will prove by induction that

$$d(x_j, x_{j+1}) \leq q^j \cdot d(x_0, x_1)$$

for all $j \in \mathbb{N}_0$. For the base case $j = 0$, equality evidently holds. Now suppose inductively that we have proven that $d(x_j, x_{j+1}) \leq q^j \cdot d(x_0, x_1)$; we want to prove the claim for $j + 1$. But we have that

$$\begin{aligned}
d(x_{j+1}, x_{j+2}) &= d(\Phi(x_j), \Phi(x_{j+1})) \\
&\leq q \cdot d(x_j, x_{j+1}) \\
&\leq q \cdot q^j \cdot d(x_0, x_1) \\
&= q^{j+1} \cdot d(x_0, x_1)
\end{aligned}$$

as desired.

It follows that

$$\begin{aligned}
d(x_n, x_{n+m}) &\leq \sum_{k=0}^{m-1} d(x_{n+k}, x_{n+k+1}) && \text{Triangle inequality} \\
&\leq \sum_{k=0}^{m-1} q^{n+k} \cdot d(x_0, x_1) && \text{Lemma} \\
&= q^n (1 + q + \cdots + q^{m-1}) \cdot d(x_0, x_1) \\
&< q^n (1 + q + \cdots + q^{m-1} + \cdots) \cdot d(x_0, x_1) \\
&= \frac{q^n}{1 - q} \cdot d(x_0, x_1)
\end{aligned}$$

It follows that the above term will converge to zero as $n \to \infty$, so $\{x_n\}$ is a Cauchy sequence and there exists an x such that $x_n \xrightarrow{d} x$.

We now prove that x is a fixed point of Φ, i.e., that $\Phi(x) = x$. We have that

$$\begin{aligned}
d(x, \Phi(x)) &\leq d(x, x_n) + d(x_n, \Phi(x_n)) + d(\Phi(x_n), \Phi(x)) \\
&\leq d(x, x_n) + d(x_n, x_{n+1}) + q \cdot d(x_n, x) \\
&= (1 + q) \cdot d(x, x_n) + d(x_n, x_{n+1})
\end{aligned}$$

where the first term converges since $\{x_n\}$ is convergent and the second term converges since $\{x_n\}$ is Cauchy. Thus, $d(x, \Phi(x)) \to 0$ as $n \to \infty$, so $x = \Phi(x)$, as desired.

Lastly, we prove that x is unique. Suppose that there exists $y \in X$ such that $y = \Phi(y)$. Then

$$d(x, y) = d(\Phi(x), \Phi(y)) \leq q \cdot d(x, y)$$

It follows that $d(x, y) \leq q^n \cdot d(x, y)$, i.e., that $d(x, y) \to 0$ as $n \to \infty$. Therefore, we must have that $d(x, y) = 0$, from which it follows that $x = y$, as desired. $\qquad \square$

- Notes on the Banach fixed point theorem.

  - Φ is a **contraction**.
  - Zhao gives the example of crumpling a sheet of paper (more specifically, dropping a map of a park in that park; a point coincides).

- Example: Fixed point of the cosine function.

  - Define $\{x_n\}$ by $x_{n+1} = \cos x_n$. If $x_0 \in \mathbb{R}$, then $x_1 \in [-1, 1]$ and $x_2 \in [\cos 1, 1]$.
  - Thus, while cosine is not a contraction on the real numbers ($\cos'(-\pi/2) = 1$, for example), we can show that $\cos : [\cos 1, 1] \to [\cos 1, 1]$ is a contraction: If $x, y \in [\cos 1, 1]$, then

$$
\begin{aligned}
|\cos x - \cos y| &= \left| \int_y^x -\sin t \, \mathrm{d}t \right| \\
&\leq |x - y| \sup_{t \in [\cos 1, 1]} |\sin t| \\
&\leq (\sin 1)|x - y|
\end{aligned}
$$

  - Thus, cosine has a fixed point at the intersection of $y = \cos x$ and $y = x$ of approximate value $0.739\ldots$
  - Overall, this is a pretty bad example, though.

- Theorem: Let $y_k : [a, b] \to \mathbb{R}^n$ be a Cauchy sequence of continuous functions under the sup norm. Then the limit exists and is continuous.

  - The proof is based on uniform convergence, which we've encountered before in analysis.
  - It follows that $C[a, b]$ (the metric space of all continuous functions on $[a, b]$) is complete.
  - If $\{y_k\} \subset \bar{B}(0, M)^{[1]}$, then the limit y is in $\bar{B}(0, M)$.

- Let's return to our ODE $y'(t) = f(t, y(t))$, $y(t_0) = y_0 \in \mathbb{R}^n$.

- We now have the tools to prove the Cauchy-Lipschitz theorem, and we will presently build up to that.

- Although we do not typically think of it this way, f is still a function with a domain and range. In particular, it's domain is the set of ordered pairs where the first entry is a real number and the second entry is an element of the range of y, i.e., an element of $\mathbb{R}^n$. Thus, to begin, we are allowed to impose the following conditions on f.

  - Let $f(t, z)$ be defined on $[t_0, t_0 + a] \times \bar{B}(y_0, b)$ for some $a, b \in \mathbb{R}_+$ (we will put further constraints on the values of a, b later).
  - On this domain, suppose $|f|$ is bounded by some $M \in \mathbb{R}$, i.e., $|f(t, z)| \leq M$ for all t, z in the above set.
  - Let f be Lipschitz continuous in the second argument. In particular, there exists $L > 0$ such that $|f(t, z_1) - f(t, z_2)| \leq L|z_1 - z_2|$ for any $z_1, z_2 \in \mathbb{R}^n$.

- We usually consider a given ODE in differential form. However, there's no reason we can't consider the equivalent integral equation

$$
y(t) = y_0 + \int_{t_0}^t f(\tau, y(\tau)) \mathrm{d}\tau
$$

  - The reason for this change of perspective will become apparent shortly.

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[1] $\bar{B}(0, M)$ denotes the set of all functions $y : [a, b] \to \mathbb{R}^n$ with sup norm at most M; topologically, it is the closed ball of radius M centered at the origin in $C[a, b]$.

- Let $\Phi : C[t_0, t_0 + a] \to C[t_0, t_0 + a]$ map functions to functions. Specifically, let it send

$$y(t) \mapsto y_0 + \int_{t_0}^{t} f(\tau, y(\tau)) \mathrm{d}\tau$$

  – We denote this by writing $\Phi[y](t) = y_0 + \int_{t_0}^{t} f(\tau, y(\tau)) \mathrm{d}\tau$.
  – Notice that the solution of our IVP is exactly the point of $C[t_0, t_0 + a]$ fixed by Φ because

$$y(t) = y_0 + \int_{t_0}^{t} f(\tau, y(\tau)) \mathrm{d}\tau \quad \Longleftrightarrow \quad y = \Phi[y]$$

  – *This* motivates all steps taken thus far.

- All that remains is to show that Φ is a contraction on some subset of $C[t_0, t_0 + a]$. Then we can apply the Banach fixed point theorem.

- We first identify this subset. Let

$$X_b = \{y : [t_0, t_0 + a] \to \bar{B}(y_0, b)\}$$

  – By the previous theorem, this is a complete metric space.
  – We now want to relate a and b so that $\Phi(X_b) \subset X_b$ and Φ is a contraction.
  – For $\Phi(X_b) \subset X_b$, we need

$$\|\Phi[y] - y_0\| \le \int_{t_0}^{t_0 + a} |f(\tau, y(\tau))| \mathrm{d}\tau \le a \cdot M \le b$$

  so we want $a < b/M$.
  – Moreover, if Φ is to be a contraction, then since

$$\|\Phi[y_1] - \Phi[y_2]\| \le \int_{t_0}^{t+a} |f(\tau, y_1(\tau)) - f(\tau, y_2(\tau))| \mathrm{d}\tau$$
$$\le La \cdot \|y_1 - y_2\|$$

  we want $La \in (0, 1)$. We can achieve this by requiring $a < 1/2L$.
  – Thus, choosing

$$a < \min\left(\frac{1}{2L}, \frac{b}{M}\right)$$

  accomplishes all of our goals.

- Therefore, by the Banach fixed point theorem, there exists a unique y such that $y = \Phi[y]$.

  – As we have already remarked, this fixed point is exactly the aforementioned solution to the IVP.

- Conclusion:

- Theorem (Cauchy-Lipschitz theorem): Let $f(t, z)$ be defined on an open subset $\Omega \subset \mathbb{R}_t \times \mathbb{R}_z^n$, $(t_0, y_0) \in \Omega$, such that f is Lipschitz continuous wrt. t, z in some neighborhood of (t_0, y_0). Then the IVP $y'(t) = f(t, y(t))$, $y(t_0) = y_0$ has a unique solution for some $T > 0$ on $[t_0, t_0 + T]$ such that $y(t)$ does not escape that neighborhood.

- If $f \in C_1$, then
$$|f(t, z_1) - f(t, z_2)| \le \sup_{z \in \bar{B}(y_0, r)} \left\|\frac{\partial f}{\partial z}(t, z)\right\| \cdot |z_1 - z_2|$$

- We use the finite increment theorem of differential calculus to prove that f is Lipschitz continuous if it's continuously differentiable.

- The norm on the RHS above is the matrix norm.

- We have $y(t) = \int_{t_0}^{t} f(\tau, y(\tau)) \mathrm{d}\tau + y_0$ and we use the Banach fixed point theorem (which is proved constructively).

- We have $y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(\tau, y_n(\tau)) \mathrm{d}\tau$.

  - Thus, the Picard iteration is justified by the Banach fixed point theorem.

- We do not use the algorithm from the proof (the Picard iteration) computationally; we use the polygon algorithm. This algorithm is only of theoretical significance.

- **Interval of existence**: The union of intervals containing the interval $[t_0, t_0 + T]$ on which the IVP has a solution.

  - The interval of existence is always open. If $t_0 \in I$ such that $y(t_0) = y$, then $y'(t) = f(t, y(t))$, $y(t_0) = y$.
  - Note that the theorem does not predict when singularity can occur.
  - Example: The interval of existence will always be $x' = 1 + x^2$, $x(t_0) = x_0$. Then $x(t) = \tan(t - t_0 + \arctan(x_0))$. The length of existence is always π.

- Interval of existence: If you consider the IVP $y'(t) = f(t, y(t))$, $y(t_0) = y_0$, then $[t_0, t_0 + T_1], [t_0 + T_1, t_0 + T_2]$. The first is of length T_1, and the second of length $T_2 - T_1$. Continuing on, we get $T_n - T_1$ so that $T_n \to \infty$ or T_n is bounded. This gives us the maximal solution/interval of existence.

- The motherfucker (Shao) made us stay 10 minutes late.