Week 9

Periodicity

9.1 Periodic Solutions of Planar Systems

11/28:

- Last lectures: Special solutions to planar systems. Usually encountered in applications of ODEs (e.g., the homework). If we encounter ODEs in our physical sciences lives, we may need to remember this.
- There will be a Monday office hours that coincides with the regular lecture time.
- Special (periodic) solutions of planar systems.
- For a planar autonomous system
$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$
a periodic solution is equivalent to a closed orbit.
- So geometrically, we'll be studying closed orbits. Analytically, we'll be studying periodic systems.
- Simple examples: Harmonic oscillator
$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} y \\ -x \end{pmatrix}$$
- Less trivial example: The pendulum
$$\begin{pmatrix} \theta \\ \omega \end{pmatrix}' = \begin{pmatrix} \omega \\ -\sin\theta \end{pmatrix}$$
- We can easily find Lyapunov functions for these two systems; both functions are the energy function.
- Different orbital graphs: The first one is concentric circles; the second one is only sometimes periodic. *picture*
- In general, these periodic cycles are dense in the plane.
- However, we can also, at the other extreme, have isolated periodic solutions, referred to as **limit cycles**.
- Simplest example:
$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -y - [1-(x^2+y^2)]x \\ x - [1-(x^2+y^2)]y \end{pmatrix}$$
  - Hard to see the behavior in Cartesian coordinates; much easier in polar. If we use $r = \sqrt{x^2+y^2}$ and $\theta = \arctan(y/x)$. We will see equations of this type in our homework.

– Differentiating $r = \sqrt{x^2 + y^2}$ implicitly with respect to time, we get

$$\begin{aligned}\frac{\mathrm{d}r}{\mathrm{d}t} &= \frac{xx'}{\sqrt{x^2+y^2}} + \frac{yy'}{\sqrt{x^2+y^2}} \\ &= -\frac{xy}{\sqrt{x^2+y^2}} + \frac{1-r^2}{r}x^2 + \frac{xy}{\sqrt{x^2+y^2}} + \frac{1-r^2}{r}y^2 \\ &= r(1-r^2)\cos^2\theta + r(1-r^2)\sin^2\theta \\ &= r(1-r^2)\end{aligned}$$

– Differentiating $\theta = \arctan(y/x)$ implicitly with respect to time, we get

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{-\frac{y}{x^2}x'}{1+\left(\frac{y}{x}\right)^2} + \frac{\frac{1}{x}y'}{1+\left(\frac{y}{x}\right)^2} \\ &= \frac{-yx'}{x^2+y^2} + \frac{xy'}{x^2+y^2} \\ &= 1\end{aligned}$$

– Thus, we can transform the original equation to

$$\begin{pmatrix} r \\ \theta \end{pmatrix}' = \begin{pmatrix} r(1-r^2) \\ 1 \end{pmatrix}$$

for $r > 0$ and $\theta \in \mathbb{R}$.

– This ODE can be explicitly solved, but since we are interested in qualitative behavior, we will not do that.

– The unit circle partitions the xy-plane into two parts (inside and outside). *picture*

– Let's start on the unit circle. Then we just spiral around on it with constant velocity ($r' = 0$ and $\theta' = 1$).

– Let's now start inside. Since $\theta(t) = t + \theta(0)$, and r' is positive, we get a spiral that approaches the unit circle.

– If we start outside, we get a spiral that starts outside and spirals toward the unit circle.

– Thus, in this case, the unit circle is the unique limit cycle of the system.

- Shao suggests we search for iodine clock videos on YouTube to help with the homework. The iodine clock is described by the limit cycles.

- Historical remark: David Hilbert posed 23 questions at the beginning of the 20th century. The 16th one asked about planar polynomial systems. For these, is it possible to estimate the number of limit cycles. Even for the case of quadratic polynomials, the question is still open! For quadratics, we know that there can be 1, 2, 3, or 4 cycles, but we have no idea whether or not there is an upper bound. This is a central open problem in the study of ODEs.

- Basic theorem in this area is as follows.

- Theorem (Poincaré-Bendixson Theorem): Let $\Omega \in \mathbb{R}^2$ be open, $f(x)$ a vector field on Ω. Fix $x \in \Omega$. Define

$$\omega(x) := \{z \in \Omega \mid \text{there is a sequence } t_n \to +\infty \text{ such that } \phi_{t_n}(x) \to z\}$$

and

$$\alpha(x) := \{z \in \Omega \mid \text{there is a sequence } t_n \to -\infty \text{ such that } \phi_{t_n}(x) \to z\}$$

That is, if you reverse the direction of time, $\alpha(x)$ collects all of the points. Also, let $\omega(x) \subset \Omega$ be compact and nonempty. In particular, there are three mutually exclusive cases for these limit sets.

1. $\omega(x)$ (or $\alpha(x)$[1]) is a fixed point.
2. $\omega(x)$ is a limit cycle.
3. $\omega(x)$ consists of finitely many fixed points, together with curves joining these fixed points.

Proof. The proof relies largely on algebraic topology, so we will not go into it in detail or even sketch it. The statement will be sufficient for our purposes. □

- Theorem (Annulus theorem of Bendixson): Suppose C_1, C_2 are closed simple planar curves such that geometrically, one contains the other. We call the annular region (between the two curves) A. Suppose $f(x)$ is a planar vector field which points inward at every point of ∂A (the boundary of A). Then the annular region A is an invariant region of the plane. In particular, if A does not contain any fixed points, then it must contain a limit cycle. As before, curves within and without spiral towards it. *picture*
- Can produce beautiful diagrams: Zhifen Zhang's example — Xiao and Zhang (2008) — is the 4 limit cycle one. After her research, mathematicians found a family with four limit cycles.
- System from the homework: Consider the system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a - x - \frac{4xy}{1+x^2} \\ b(x - \frac{xy}{1+x^2}) \end{pmatrix}$$

  picture

  - We take $a, b > 0$.
  - Every vector points in toward a, which points straight upward.
  - On the boundary,

$$\begin{pmatrix} a - x - \frac{4xy}{1+x^2} \\ b(x - \frac{xy}{1+x^2}) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} > 0$$

    so we have a strict Lyapunov function.
  - Thus, any orbit in the first quadrant can never escape, reflecting our expectation that the concentration can never be negative.
  - Spoiler: Looks like a rectangular region.
  - We are also asked to find the fixed point and investigate its stability.
- First step: Find fixed points.
- Second step: Lyapunov functions.
- Third step: Divide the vector field into positive and negative regions. Requires some improvization.
- Fourth step: Check for an annular region.
- Note: Note of these arguments can be generalized to higher dimensional systems. The regularity of the Poincaré-Bendixson system disappears as we go to higher dimensions.
- Lorenz system: Oversimplified 3D quadratic system that serves as a simplification model for weather systems.
  - Lorenz was an astronomer.
  - He discovered that even if the systems are very close together, the orbits will be separated indefinitely as time evolves on. It's not just about being stable or unstable, but overall global chaotic behavior.

[1]We just have to reverse the time.

- Mathematicians quickly discovered that the Lorenz system has a **strange attractor**. An attractor for a planar system can only be closed orbits. In the Lorenz system, we have a strange type of butterfly. The dimension of the butterfly is not even an integer. Have to introduce Hausdorff measure to understand length and area in the more general framework of curves or surfaces.
- No more detail, but a classical example of a chaotic ODE system worth mentioning. This phenomenon cannot appear for planar systems; even increasing the dimension by 1 can lead to chaos.
- No widely accepted definition of mathematical chaos, but generally accepted ones are very irregular global attractors and points that are arbitrarily close and arbitrarily far from each other.
- Fractorial sets.

- Last lecture (Wednesday): Another example that has a limit cycle.
- Friday will be a review.

9.2 Nonlinear Oscillation

11/30:
- Friday: Review and questions session.
- Monday: OH during class time in the class room and normal OH.
- Exam time: Wednesday, December 7 from 7:30-9:30 AM. Will happen in this classroom; Shao hopes to finish grading the same day.
- Last HW and review outline is due 12/2. 130 total points for the grade. 90/130 gives an A. That's 69% and above.
  - Minus 10, minus 10, for the grading below A range.
- Last example that arises naturally.
- RLC circuit: Nonlinear oscillation. Occurred in resonance. Nonlinear oscillation in K.
- The circuit that we're interested in still is an RLC circuit (see Figure 5.5b).
- We assume that the resistor satisfies Ohm's law, i.e., $I_R = V_R/R$. This is the linear case. Causes the voltage to decay exponentially fast, where the resistor acts as a kind of damping factor.
- However, for some more delicate cases, Ohm's law might fail and we might get a nonlinear replacement $V_R = R(I_R)$. We let R be any suitable function. We still have Kirchoff's law, i.e., that I_R, I_C, and I_L are all the same and we can refer to them as I.
- The system we're interested in is

$$\begin{cases} LI' &= -V_C - R(I)\\ CV' &= I \end{cases}$$

- Comparing to the linear case, we just want to replace V_R/R in the same system with $R(I)$. After some suitable scaling, we can get the system

$$\begin{cases} x' &= y - f(x)\\ y' &= -x \end{cases}$$

- This is a classical nonlinear oscillation model, typically referred to as the **Liénard equation**.
- The first case of the Liénard equation studied in detail was when

$$f(x) = \mu\left(\frac{x^3}{3} - x\right)$$

  - Discovered by the engineer van der Pol while he was investigating RLC circuits, and hence referred to as the **van der Pol oscillator**.

- Theorem: Suppose $f(x)$ is odd, i.e., $f(x) = -f(-x)$. Also suppose that $f(x) < 0$ for $x \in (0, \alpha)$ for some $\alpha > 0$ (i.e., $f(x)$ is less than zero for sufficiently small positive values of x). Moreover, suppose that $\liminf_{x\to\infty} f(x) > 0$ (i.e., f does not diverge). Lastly, suppose that $f(x)$ is increasing for $x > \alpha$. Then the conclusion is that the Liénard system has a unique closed orbit. Every other orbit except for the trivial orbit $(0, 0)$ will be attracted to this periodic solution as the system evolves ($t \to +\infty$).
  - This explains the term "nonlinear oscillation." As long as we are away from the equilibrium, we converge to it??
- Wrt the van der Pol oscillator, we sketch the boundary first and then put in the limit cycle. The origin is an unstable fixed point, and any origin starting from the origin will spiral and approximate the limit cycle.
  - ?? is guaranteed by the Poincaré-Bendixson theorem.
- The proof does not use any advanced math, but it is complex, so we'll sketch it. The proof is in Teschl (2012) if we're interested. It only involves calculus.
- End of lecture.
- First problem, second section of HW: First integral: Conservative law.
- What is the correct first variational equation?
- Difference between not Lyapunov stable and completely unstable?
  - Something is repelled away vs. everything is repelled away.
- The iodine clock is not super hard, but it needs some improvisation. That's what we talked about at the end of last lecture.
- Friday will be a review of everything since the second midterm.