

## 4 Final Explicitly Solvable Cases

### Required Problems

- 11/2: 1. Use Duhamel's formula to solve the initial value problem

$$y' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} y + \begin{pmatrix} -t \\ t \end{pmatrix}, \quad y(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

*Proof.* Diagonalizing  $A$  reveals that

$$\underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_A = \underbrace{\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}}_Q \underbrace{\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}}_B \underbrace{\begin{pmatrix} -i/2 & 1/2 \\ i/2 & 1/2 \end{pmatrix}}_{Q^{-1}}$$

Thus,

$$e^{tA} = Qe^{tB}Q^{-1} = Q \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} Q^{-1}$$

We have that

$$\begin{aligned} y(t) &= e^{tA}y_0 + \int_0^t e^{(t-\tau)A}f(\tau)d\tau \\ &= \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \begin{pmatrix} -i/2 & 1/2 \\ i/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &\quad + \int_0^t \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{i(t-\tau)} & 0 \\ 0 & e^{-i(t-\tau)} \end{pmatrix} \begin{pmatrix} -i/2 & 1/2 \\ i/2 & 1/2 \end{pmatrix} \begin{pmatrix} -\tau \\ \tau \end{pmatrix} d\tau \\ &= \begin{pmatrix} \frac{e^{it}+e^{-it}}{2} \\ \frac{e^{it}-e^{-it}}{2i} \end{pmatrix} + \int_0^t \begin{pmatrix} \tau \left( \frac{e^{i(t-\tau)}-e^{-i(t-\tau)}}{2i} - \frac{e^{i(t-\tau)}+e^{-i(t-\tau)}}{2} \right) \\ \tau \left( \frac{e^{i(t-\tau)}-e^{-i(t-\tau)}}{2i} - \frac{e^{i(t-\tau)}+e^{-i(t-\tau)}}{2} \right) \end{pmatrix} d\tau \end{aligned}$$

Substitute sines and cosines and evaluate. □

2. Let  $A$  be an  $n \times n$  complex constant matrix and let  $f(t) = e^{\zeta t}p(t)$ , where  $p(t)$  is a  $\mathbb{C}^n$ -valued function whose entries are all polynomials. We define  $\deg p(t)$  to be the largest of the degrees of its entries. Use Duhamel's formula to prove the following proposition:

If  $\zeta$  is not an eigenvalue of  $A$ , then the solution of  $y' = Ay + p(t)e^{\zeta t}$  takes the form

$$e^{tA}z_0 + q(t)e^{\zeta t}$$

where  $z_0$  is a constant vector (*not* necessarily the initial value) and  $q(t)$  is a polynomial vector with degree being the same as  $p(t)$ . If  $\zeta$  is an eigenvalue of  $A$  with algebraic multiplicity  $\alpha$ , then the solution of  $y' = Ay + e^{\zeta t}p(t)$  takes the form

$$e^{tA}z_0 + r(t)e^{\zeta t}$$

where  $z_0$  is a constant vector (*not* necessarily the initial value) and  $r(t)$  is a polynomial vector with degree  $\deg p(t) + \alpha$ .

*Proof.* Shao said in office hours that this question cannot be answered, and as such he would cancel it; he copied it out of Teschl but it had an error as written. □

3. We know that for the driven harmonic oscillator equation

$$x'' + \omega_0^2 x = H_0 \cos \omega t$$

when  $\omega = \omega_0$ , the solution grows unboundedly. However, what if  $\omega \neq \omega_0$  but is very close?

For simplicity, suppose that  $H_0 > 0$  and the initial values  $x(0), x'(0)$  are real numbers that are very small compared to  $H_0/|\omega^2 - \omega_0^2|$ , say,

$$|x(0)| + |x'(0)| < \frac{\varepsilon H_0}{|\omega^2 - \omega_0^2|}$$

Suppose also that  $|\omega - \omega_0|$  is very small compared to  $\omega_0$ , say

$$|\omega - \omega_0| < \varepsilon \omega_0$$

Lastly, suppose that the initial values are small compared to the eigenfrequency, say

$$|x(0)| + |x'(0)| < \varepsilon \omega_0$$

Prove that there is a sequence of times  $t_k \rightarrow +\infty$  such that

$$x(t_k) > 2(1 - \varepsilon) \cdot \frac{H_0}{|\omega^2 - \omega_0^2|} \approx \frac{1}{\varepsilon}$$

That is, the mass point will constantly visit positions very far away from the equilibrium.

*Hint.* Write down the solution first, and then you need to discuss two cases separately:  $\omega/\omega_0$  is rational/irrational. In the latter case, you should use the following theorem of Kronecker.

**Theorem 1.** Let  $\alpha, \beta$  be positive real numbers such that  $\alpha/\beta$  is irrational. Then the set  $\{(\langle n\alpha \rangle, \langle n\beta \rangle) \mid n \in \mathbb{N}\}$  is dense in the unit sphere  $[0, 1] \times [0, 1]$ , where  $\langle \cdot \rangle$  denotes the decimal part of a real number.

*Proof.* From HW3, the given driven harmonic oscillator equation is solved by

$$\begin{aligned} x(t) &= x(0) \cos \omega_0 t + x'(0) \frac{\sin \omega_0 t}{\omega_0} + \int_0^t \frac{\sin \omega_0(t - \tau)}{\omega_0} (H_0 \cos \omega \tau) d\tau \\ &= x(0) \cos \omega_0 t + x'(0) \frac{\sin \omega_0 t}{\omega_0} + \frac{H_0}{\omega^2 - \omega_0^2} (\cos \omega_0 t - \cos \omega t) \end{aligned}$$

We have that  $|x(0) \cos \omega_0 t| \leq |x(0)|$  and  $|x'(0) \sin \omega_0 t| \leq |x'(0)|$  and  $|x'(0)| < \varepsilon \omega_0$  and  $|x(0)| < \varepsilon H_0/|\omega^2 - \omega_0^2|$ . Thus

$$\begin{aligned} \left| x(0) \cos \omega_0 t + x'(0) \frac{\sin \omega_0 t}{\omega_0} \right| &\leq |x(0) \cos \omega_0 t| + \frac{1}{\omega_0} |x'(0) \sin \omega_0 t| \\ &\leq |x(0)| + \frac{|x'(0)|}{\omega_0} \\ &< |x(0)| + \varepsilon \\ &< \frac{\varepsilon H_0}{|\omega^2 - \omega_0^2|} + \varepsilon \\ &= \varepsilon \left( \frac{H_0}{|\omega^2 - \omega_0^2|} + 1 \right) \end{aligned}$$

We also have that

$$\begin{aligned} \left| \frac{H_0}{\omega^2 - \omega_0^2} (\cos \omega_0 t - \cos \omega t) \right| &= \frac{H_0}{|\omega^2 - \omega_0^2|} \cdot |\cos \omega_0 t - \cos \omega t| \\ &\leq \frac{H_0}{|\omega^2 - \omega_0^2|} \cdot 2 \end{aligned}$$

It follows that

$$\begin{aligned} |x(t)| &< \frac{2H_0}{|\omega^2 - \omega_0^2|} + \frac{\varepsilon H_0}{|\omega^2 - \omega_0^2|} + \varepsilon \\ &= (2 + \varepsilon) \cdot \frac{H_0}{|\omega^2 - \omega_0^2|} + \varepsilon \end{aligned}$$

□

4. Sketch the phase diagram of the following linear autonomous systems. Also clearly indicate

- The eigenvalues and eigenvectors;
- The stable and unstable subspaces (if the eigenvalues are not purely imaginary);
- The shape and direction of the trajectories (attracted/repelled by the fixed point).

(1)

$$y' = \begin{pmatrix} \frac{1}{2} & 1 \\ -1 & \frac{1}{2} \end{pmatrix} y$$

*Proof.* Using techniques from previous weeks, we can diagonalize  $A$  as follows.

$$\begin{pmatrix} \frac{1}{2} & 1 \\ -1 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} + i & 0 \\ 0 & \frac{1}{2} - i \end{pmatrix} \begin{pmatrix} \frac{i}{2} & \frac{1}{2} \\ -\frac{i}{2} & \frac{1}{2} \end{pmatrix}$$

Thus, the eigenvalues and corresponding eigenvectors of  $A$  are

$$\lambda = \frac{1}{2} + i \quad \bar{\lambda} = \frac{1}{2} - i \quad v = \begin{pmatrix} -i \\ 1 \end{pmatrix} \quad \bar{v} = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

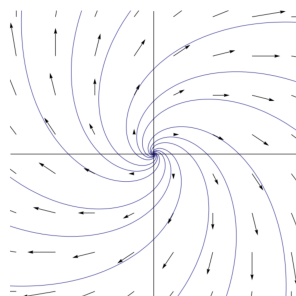
It follows that

$$\text{stable subspace} = \{0\} \quad \text{unstable subspace} = \mathbb{R}^2$$

Moreover, the shape and direction of the trajectories (using the convention from Q5(2)) are

Spiral source; repelled

Therefore,



□

(2)

$$y' = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} y$$

*Proof.* This question is entirely analogous to part (1). Indeed, we get eigenvalues and eigenvectors

$$\lambda = 1 + 2i \quad \bar{\lambda} = 1 - 2i \quad v = \begin{pmatrix} -i \\ 1 \end{pmatrix} \quad \bar{v} = \begin{pmatrix} i \\ 1 \end{pmatrix}$$

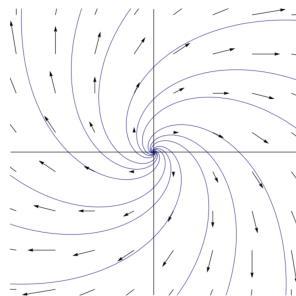
subspaces

$$\text{stable subspace} = \{0\} \quad \text{unstable subspace} = \mathbb{R}^2$$

and shape and direction

Spiral source; repelled

Therefore,



□

(3)

$$y' = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} y$$

*Proof.* This time our diagonalization gives real, distinct eigenvalues and eigenvectors

$$\lambda_1 = \frac{3}{2} \quad \lambda_2 = \frac{1}{2} \quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

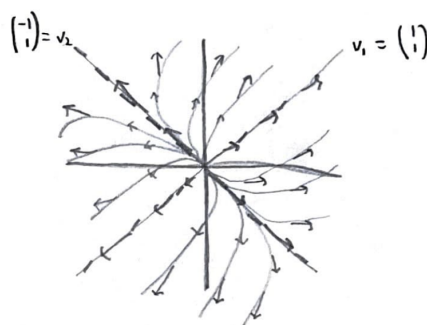
Since both eigenvalues are greater than zero, we have subspaces

$$\text{stable subspace} = \{0\} \quad \text{unstable subspace} = \mathbb{R}^2$$

Since we have two positive real eigenvalues, we have shape and direction

$$\text{Source; repelled}$$

Additionally, the phase diagram will have many curves of the form  $v_2 = v_1^{\lambda_2/\lambda_1}$ , i.e.,  $v_2 = v_1^{1/3}$ , or  $v_1 = v_2^3$ .



□

(4)

$$y' = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} y$$

*Proof.* Once again, we get real distinct eigenvalues and eigenvectors, but this time our eigenvalues have opposite signs.

$$\lambda_1 = 1 \quad \lambda_2 = -1 \quad v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

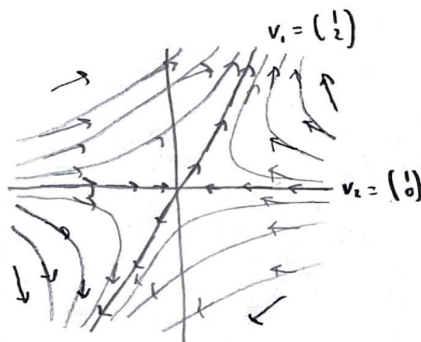
Because of their opposite signs, we have our first nontrivial stable subspace (corresponding to the negative eigenvalue).

stable subspace = $\text{span}\{v_2\}$	unstable subspace = $\text{span}\{v_1\}$
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Likewise, it follows that

Saddle; both (depends on the subspace)
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Additionally, we will have a power function of a negative power  $v_2 = v_1^{\lambda_2/\lambda_1}$ , i.e.,  $v_2 = 1/v_1$ .



□

(5)

$$y' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} y$$

*Proof.* This matrix is already in JNF with a single Jordan block. Thus, we have one lone eigenvalue  $\lambda$ , an eigenvector  $v$ , and a generalized eigenvector  $u$  as follows.

$\lambda = 2$	$v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
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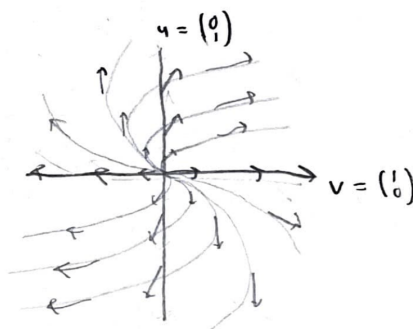
Evidently,  $\lambda > 0$ , so

stable subspace = $\{0\}$	unstable subspace = $\mathbb{R}^2$
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Now there is not a specific naming convention for the this shape in Q5(2), so we will call it “distorted source:”

Distorted source; repelled
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Additionally, we will have a function of the form  $v = u + u \ln u$ .



□

5. Given a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the number  $a + d$  is called the trace of  $A$  and is denoted by  $\text{Tr } A$ .

(1) Prove that  $\text{Tr } A$  is invariant under similarity, and show that

$$\chi_A(z) = z^2 - (\text{Tr } A)z + \det A \qquad \det e^A = e^{\text{Tr } A}$$

*Proof.* Left equality above: We have that

$$\begin{aligned} \chi_A(z) &= \det(A - zI) \\ &= \det \begin{pmatrix} a - z & b \\ c & d - z \end{pmatrix} \\ &= (a - z)(d - z) - bc \\ &= z^2 - az - dz + ad - bc \\ &= z^2 - \underbrace{(a + d)}_{\text{Tr } A} z + \underbrace{(ad - bc)}_{\det A} \end{aligned}$$

as desired.

Invariance of the trace under similarity: Suppose  $A \sim B$ . Then since similar matrices have the same characteristic polynomial, we have by the left equality above that

$$\begin{aligned} \chi_A(z) &= \chi_B(z) \\ z^2 - (\text{Tr } A)z + \det A &= z^2 - (\text{Tr } B)z + \det B \\ \text{Tr } A &= \text{Tr } B \end{aligned}$$

as desired.

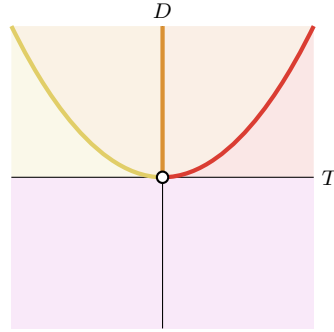
Right equality above: Suppose  $e^A = Qe^BQ^{-1}$  where  $B$  is in JNF. Then

$$\begin{aligned} \det e^A &= \det e^B \\ &= e^{\lambda_1} \cdot e^{\lambda_2} \\ &= e^{\lambda_1 + \lambda_2} \\ &= e^{\text{Tr } B} \\ &= e^{\text{Tr } A} \end{aligned}$$

where we have the first equality because the determinant is invariant under similarity; the second equality because  $e^B$  is upper triangular, the determinant of an upper triangular matrix is equal to the product of the diagonal entries, and the diagonal entries of  $e^B$  are the exponentials of the eigenvalues of  $A$ ; and the remainder of the equalities for fairly evident reasons. □

(2) Suppose  $A$  is a real matrix. We have discussed the phase diagram of the linear autonomous system  $y' = Ay$  and classified them into several cases according to the eigenvalues of  $A$ : Spiral source/sink (complex eigenvalues with positive/negative real part), ellipse (purely imaginary eigenvalues), saddle (real eigenvalue with opposite sign), source/sink (positive/negative real eigenvalue). Now the eigenvalues are completely determined by the tuple of real numbers  $(T, D) = (\text{Tr } A, \det A)$ . Split the  $(T, D)$  plane into several parts in which the various cases discussed this Monday occur.

*Proof.* We have that



We will identify the various colored regions in lines throughout the following derivation. Let's begin.

Suppose we have two identical eigenvalues  $a$ . Then the shape will be distorted source if  $a > 0$  and distorted sink if  $a < 0$ . In this case  $T = 2a$  and  $D = a^2$ . It follows by solving the first equation for  $a$  and substituting it into the second that

$$D = \frac{1}{4}T^2$$

Thus, every  $(T, D)$  falling along this positive portion of this parabola (the red parabola in the diagram) corresponds to a distorted source, and vice versa for the negative portion (yellow).

Suppose  $\lambda_1 = a + bi$  and  $\lambda_2 = a - bi$ . Then  $T = 2a$  and  $D = a^2 + b^2$ . Substituting as before, we obtain

$$D = \frac{1}{4}T^2 + b^2$$

where  $b^2 > 0$ . Thus, every  $(T, D)$  lying *above* the parabola corresponds to a spiral. For positive  $T$  (the red/orange shaded region), we have a spiral source, and vice versa for negative  $T$ .

Suppose  $\lambda_1 = bi$  and  $\lambda_2 = -bi$ . Then  $T = 0$  and  $D = b^2$ . Thus, the orange part of the vertical axis corresponds to all ellipses.

Now suppose  $\lambda_1, \lambda_2 \in \mathbb{R}$ . WLOG let  $\lambda_2 > \lambda_1$ . Define  $\delta := \lambda_2 - \lambda_1$ . Then  $T = 2\lambda_1 + \delta$  and  $D = \lambda_1^2 + \delta\lambda_1$ . Substituting as before, we obtain

$$D = \left(\frac{T - \delta}{2}\right)^2 + \delta \cdot \frac{T - \delta}{2} = \frac{1}{4}T^2 - \frac{\delta^2}{4}$$

If  $D > 0$ , then

$$\begin{aligned} 0 &< \frac{1}{4}T^2 - \frac{\delta^2}{4} \\ \delta &< T \\ \lambda_2 - \lambda_1 &< \lambda_1 + \lambda_2 \\ 0 &< 2\lambda_1 \\ 0 &< \lambda_1 < \lambda_2 \end{aligned}$$

Thus, the red shaded region lying above the  $T$  axis but below the red half-parabola corresponds to all sources. Assuming  $D < 0$  leads in an analogous fashion to the conclusion that  $0 > \lambda_2 > \lambda_1$ , meaning that the yellow shaded region lying above the  $T$  axis but below the yellow half-parabola corresponds to all sinks.

Lastly, any point lying below the  $T$  axis must have one positive and one negative eigenvalue:  $D = \lambda_1\lambda_2 < 0$  implies either  $\lambda_1 < 0$  or  $\lambda_2 < 0$  but not both. Thus, the purple shaded region corresponds to saddles.

As a final comment, note that there are several other types of graphs that we did not talk about on Monday that also have their place on this diagram, e.g., the origin corresponds to uniform motion.  $\square$