MATH 27300 (Basic Theory of Ordinary Differential Equations) Notes

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Weeks

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Week 1

Introduction to ODEs

1.1 Definitions and Scope

9/28:

- Questions:
  - When will the PDFs be made available?
- Office: Eckhart 309.
  - Office hours: MWF 3:00-4:00.
- Reader: Walker Lewis. His contact info is in the syllabus.
- Final grade is based on...
  - 2 midterms (15 pts. each; weeks 4 and 8).
  - Final exam (35 pts.).
  - HW (35 pts.).
  - Bonus problems (15 pts).
- Total points for the quarter is 115. The bonus problems usually arise from advanced math and incorporate more advanced knowledge, and we are encouraged to seek out all relevant resources as long as we write up our own solutions.
- **Ordinary differential equation**: Any equation that takes the form $F(t,y,y',\dots,y^{(n)})=0$. *Also known as* **ODE**.
  - F is a known function.
  - t is an argument (time). x is also used (when space is involved).
  - $y=y(t)$ is an unknown function.
- **Order n** (ODE): An ODE for which the n^{th} derivative of y is the highest-order derivative involved (and is involved).
- $y'=f(t,y)$ or $Y^{(n)}=F(t,Y,Y',\dots,Y^{(n-1)})$.
  - We can transform this second form into the first form via

$$y=\begin{pmatrix}Y\\Y'\\\vdots\\Y^{(n-1)}\end{pmatrix}\qquad\qquad f(t,y)=\begin{pmatrix}y_2\\y_3\\\vdots\\F(t,y_1,y_2,\dots,y_{(n-1)})\end{pmatrix}$$

making $y' = f(t, y)$ equal to the system of equations

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_{n-1}' &= F(t, y_1, \dots, y_{n-1}) \end{aligned}$$

  - Think about this conversion more.

- Thus, we mainly focus on equations of the form $y' = f(t, y)$, because that's general enough.

- **Linear** (ODE): Any ODE that can be written in the form

$$y' = A(t)y + f(t)$$

- Because of the above, this naturally includes equations of the form

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_0(t)y = b(t)$$

- **Nonlinear** (ODE): An ODE that is not linear.
- **Autonomous** (ODE): An ODE that can be written in the form

$$y' = f(y)$$

  - More equivalence w/ vector-valued functions?

- **Nonautonomous** (ODE): An ODE that is not autonomous.
  - We will not investigate these in this course.

- **Initial value problem**: A problem of the form: Find $y(t)$ such that

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

Also known as **IVP**, **Cauchy problem**.

- Locally well-posed (LWP) conditions:
  1. Existence (local in time).
  2. Uniqueness (you cannot have multiple solutions).
  3. Local stability (if you perturb your initial value or equation a little bit, you do not expect your solution to vary crazily [esp. locally]).

- Example of a nonunique ODE:
  - $y' = \sqrt{y}$, $y(0) = 0$ has solutions $y_1(t) = 0$ $(t \geq 0)$ and $y_2(t) = t^2/4$ $(t \geq 0)$.
  - We will investigate the reason later.

- Preview of the reason: **Cauchy-Lipschitz Theorem** or **Picard-Lindelof Theorem**.
  - As long as the ODE is **Lipschitz continuous**, it's locally stable.

- **Lipschitz continuous** (function): A function f such that

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$$

  - But in the counterexample above, the slope of the chord from 0 to $y(t)$ approaches infinity as $t \to 0$.

- **Peano Existence Theorem**: ...
- **Dynamical system**: A law under which a particle evolves over time. $y' = f(t, y)$, IVP is LWP
- Consider $\Phi(t, x)$ such that

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,x) = f(t, \Phi(t,x)) \\ \Phi(0,x) = x \end{cases}$$

- **Steady flow**: A vector field on a manifold contained in $\mathbb{R}^2$ or $\mathbb{R}^3$ that does not vary with time.
  - A velocity field.
  - Trajectory of a particle: At $x \in \Omega$, the velocity of the particle should coincide with $X(x)$.
  - The differential equation $\dot{x} = X(x)$ is what we're interested in.
  - A solid shape gets shifted and deformed (imagine a chunk of water falling out of the end of a pipe).
  - Differential geometry is the purview of such things.
- Newton's law of motion $F = m \cdot a$ applied to n particles is nothing but the system of equations

$$m_i x_i'' = F_i(x_1, \ldots, x_n)$$

  for $i = 1, \ldots, n$.
  - Many well-known examples.
  - The best known one perhaps is that of uniform acceleration of a single particle. In this case,

$$m_0 x'' = f_0$$

    - The solution is

$$x(t) = \frac{f_0}{2m_0}t^2 + v_0 t + x_0$$

      where $x_0 = x(0)$ and $v_0 = x'(0)$ are the initial conditions.
  - A simple example is downwards motion due to gravity. Then

$$x(t) = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} t^2 + v_0 t + x_0$$

    - The trajectory in general is a parabola.
  - Another example: The mathematical pendulum.
    - The radial directions balance ($mg\cos\theta$).
    - The tangential directions do not ($mg\sin\theta$). Thus, our ODE is

$$l\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = g\sin\theta$$

  - One last set of examples from ecology:
    - Imagine an petri dish of infinite nutrition. The population growth of the bacteria will obey the exponential growth law

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky$$

- Suppose we have a system capacity M. Then we obey the logistic growth law

$$\frac{\mathrm{d}y}{\mathrm{d}t} = k(M - y)$$

- Lotka-Volterra prey-predator model: Wolf population (W) and rabbit population (R). We have

$$\begin{aligned} R' &= k_1 R - aWR \\ W' &= -k_2 W + bWR \end{aligned}$$

- We can also introduce more species and capacities and et cetera, et cetera.

- Conclusion: Dynamical systems are everywhere, especially in physics, chemistry, and ecology.
- We can also consider long-term behavior.
  - We can have chaos, but chaos can be reasoned with using oscillation, systems that converge to oscillation, etc. We will mostly be focusing on the regular aspect of the long-term behavior.

1.2 Origin of ODEs: Boundary Value Problems

9/30:

- Textbook PDFs will be posted today.
- Today, we will consider boundary value problems, which are separate from dynamical systems but not entirely unrelated.
- **Boundary Value Problem**: A problem in which we are solving for a y that has fixed values at the boundaries $x = a, b$. *Also known as* **BVP**.
- The **Brachistochrone problem** is an example of a BVP.
- **Brachistochrone problem**: Suppose you have a frictionless track from $(0, 0)$ to (a, y_0) and release a particle from $(0, 0)$. Which path allows the particle to get to (a, y_0) in the shortest amount of time? *Etymology* brachisto "shortest" + chrone "time."
  - Since the track is frictionless, the mechanical energy should be conserved.
  - At a given point along the curve, the particle has a velocity v and is vertical distance y from where it started. We know from physics that

$$\begin{aligned} \frac{1}{2}mv^2 &= mgy \\ v &= \sqrt{2gy} \end{aligned}$$

  - The time it takes for the particle to traverse an infinitesimal section of track of arc length ℓ is ℓ/v.
  - The track should be given by $y = y(x)$.
  - Arc length

$$\ell = \sqrt{1 + (y'(x))^2}\,\mathrm{d}x$$

  - Thus, the total time for the particle to traverse the curve is

$$\int_0^a \frac{\sqrt{1 + (y'(x))^2}\,\mathrm{d}x}{\sqrt{2gy(x)}}$$

  - We also have $y(0) = 0$ and $y(a) = y_0$.
  - Functionals: Mapping from a function space to numbers; we want to find y such that the above integral is minimized.

- Let $J[y] = \int_a^b F(x, y(x), y'(x))\,\mathrm{d}x$.
- The space of functions we're considering is C^1.
- Take a function h, vanishing at a, b.
- Let $f(t) = J[y + th]$. Then

$$f(t) = \int_a^b F(x, \underbrace{y(x) + th(x)}_{z}, \underbrace{y'(x) + th'(x)}_{w})\,\mathrm{d}x$$

and hence

$$\begin{aligned} f'(t) &= \int_a^b \left(\frac{\partial F}{\partial z}(x, y(x) + th(x), y'(x) + th'(x))h(x) + \frac{\partial F}{\partial w}(x, y(x) + th(x), y'(x) + th'(x))h'(x) \right) \mathrm{d}x \\ &= \int_a^b \frac{\partial F}{\partial z}(x, y(x) + th(x), y'(x) + th'(x))h(x)\,\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial F}{\partial w}(x, y(x) + th(x), y'(x) + th'(x))\right] h(x)\,\mathrm{d}x \end{aligned}$$

- Thus,

$$f'(0) = \int_a^b \left\{ \frac{\partial F}{\partial z}(x, y(x), y'(x)) - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial F}{\partial w}(x, y(x), y'(x))\right]\right\} h(x)\,\mathrm{d}x = 0$$

for all h.
- Lemma: Let ϕ be continuous on (a, b). If for every $h \in C^1([a, b])$ vanishing on a, b we have that

$$\int_a^b \phi(x)h(x)\,\mathrm{d}x = 0$$

then $\phi(x) = 0$.

Proof. Suppose for the sake of contradiction that (WLOG) $\phi(x_0) > 0$. Then within some neighborhood $N_\delta(x)$ of x_0, $\phi(x) > 0$ for all $x \in N_\delta(x)$. Now choose h to be a bump function on that interval. Then $\int_a^b \phi(x)h(x)\,\mathrm{d}x > 0$, a contradiction. □

- It follows that

$$\frac{\partial F}{\partial z}(x, y(x), y'(x)) - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial F}{\partial w}(x, y(x), y'(x))\right] = 0$$

  - This is a second-order differential equation, specifically the **Euler-Lagrange equation**.
  - It is a necessary condition for y to be an extrema.
  - Euler-Lagrange equations are not easy to solve in general. However, we're lucky here.
- In our example,

$$F(x, y, z) = \sqrt{\frac{1 + w^2}{2gz}}$$

- This gives us

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}[F(y, y')] &= \frac{\partial F}{\partial z}(y, y')\cdot y' + \frac{\partial F}{\partial w}(y, y')\cdot y'' \\ \frac{\partial F}{\partial z}(y, y')y &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial F}{\partial w}(y, y')\right] y' \\ \frac{\mathrm{d}}{\mathrm{d}x}[F(y, y')] &= \underbrace{\frac{\partial F}{\partial w}(y, y')\cdot}_{U} \underbrace{y''}_{V'} + \underbrace{\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial F}{\partial w}(y, y')\right]\cdot}_{U'} \underbrace{y'}_{V} \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial F}{\partial w}(y, y')\cdot y'\right] \end{aligned}$$

- This reduces to the first-order equation

$$F(y, y') - \frac{\partial F}{\partial w}(y, y') \cdot y' = A$$

- Since $F(x, z, w)$ is known, we have that

$$\frac{\partial F}{\partial w}(z, w) = \frac{w}{\sqrt{1 + w^2}} \cdot \frac{1}{\sqrt{2gz}}$$

- Plugging into the E-L equation gives us

$$\begin{aligned}\frac{1 + (y')^2}{\sqrt{1 + (y')^2}\sqrt{2gy}} - \frac{(y')^2}{\sqrt{1 + (y')^2}\sqrt{2gy}} &= A \\ \frac{1}{\sqrt{2gy(1 + (y')^2)}} &= A \\ (y')^2 &= \frac{2A^2g - y}{y}\end{aligned}$$

  where the second line above is a separable differential equation.
- The solution is the **cycloid**

$$\begin{cases} x = -a\sin\theta + a\theta \\ y = a(1 - \cos\theta) \end{cases}$$

  where the specific parameters come from the boundary values.

- **Sturm-Liouville problems**: Boundary value problems concerning the integral

$$\int_a^b \left[p(x)(y'(x))^2 + q(x)(y(x))^2\right] \mathrm{d}x$$

  - The most basic BVP is a vibrating string. In finding the eigenmode of the vibration, you need to solve the above differential equation.
  - Very important in physics.
  - If time permits at the end of the course, Shao will return to the following topic in detail.
- Next several weeks: *Solvable* differential equations.