Week 6

Qualitative Theory of ODEs

6.1 More Cauchy-Lipschitz and Intro to Continuous Dependence

10/31:

- Last time, we built up a proof to the Cauchy-Lipschitz theorem intuitively.
  - We begin today with a direct proof that is very similar, but slightly different.
- Theorem (Cauchy-Lipschitz theorem): Let $f(t,z)$ be defined on an open subset $\Omega \subset \mathbb{R} \times \mathbb{R}^n$, let $(t_0, y_0) \in \Omega$, let $|f|$ be bounded on Ω, and let f be Lipschitz continuous in z and continuous wrt. t in some neighborhood of (t_0, y_0). Then the IVP $y'(t) = f(t, y(t))$, $y(t_0) = y_0$ has a unique solution on $[t_0, t_0 + T]$ for some $T > 0$ such that $y(t)$ does not escape Ω.

  Proof. Let $f(t,z)$ be defined for $(t,z) \in [t_0, t_0 + a] \times \bar{B}(y_0, b) \subset \Omega$. Let $|f(t,z)| \leq M$. Let $|f(t, z_1) - f(t, z_2)| \leq L|z_1 - z_2|$ for all $z_1, z_2 \in \bar{B}(y_0, b)$.

  Define $\{y_n\}$ recursively, starting from $y_0(t) = y_0$, by

  $$y_{k+1}(t) = y_0 + \int_{t_0}^{t} f(\tau, y_k(\tau))\mathrm{d}\tau$$

  Since f is continuous with respect to t, it is integrable with respect to t, so the above sequence is well-defined on $[t_0, t_0 + T]$. Choose $T = \min(a, b/M, 1/2L)$. Then

  $$\|y_k - y_0\| \leq T \cdot M \leq \frac{b}{M} \cdot M = b$$

  so no y_k escapes $\bar{B}(y_0, b)$. Additionally,

  $$\begin{aligned}\|y_{k+1} - y_k\| &\leq \int_{t_0}^{t} \|f(\tau, y_k(\tau)) - f(\tau, y_{k-1}(\tau))\|\mathrm{d}\tau \\ &\leq TL\|y_k - y_{k-1}\| \\ &\leq \frac{1}{2}\|y_k - y_{k-1}\| \\ &\leq \left(\frac{1}{2}\right)^k \|y_1 - y_0\|\end{aligned}$$

  Thus, the difference between successive terms in the sequence is controlled by a geometric progression, so $\{y_n\}$ is a Cauchy sequence in the function space. It follows that $\{y_k\}$ is uniformly convergent to some continuous $y : [t_0, t_0 + T] \to \mathbb{R}^n$. $\square$

- This completes the proof. Although it's more concrete than the contraction mapping one, they are virtually the same: In both cases, we obtain an approximate sequence controlled by a geometric progression.

- Examples of the Picard iteration:
  1. Consider an linear autonomous systems $y' = Ay$, A an $n \times n$ matrix, and $y(0) = y_0$.
     - We know that the solution is $y(t) = \mathrm{e}^{tA} y_0$. However, we can derive this using the Picard iteration.
     - Indeed, via this procedure, let's determine the first couple of Picard iterates.

$$y_0(t) = y_0 \qquad \begin{aligned} y_1(t) &= y_0 + \int_0^t Ay_0(\tau)\mathrm{d}\tau \\ &= y_0 + tAy_0 \end{aligned} \qquad \begin{aligned} y_2(t) &= y_0 + \int_0^t Ay_1(\tau)\mathrm{d}\tau \\ &= y_0 + tAy_0 + \frac{1}{2}t^2A^2y_0 \end{aligned}$$

     - It follows inductively that

$$y_k(t) = \sum_{j=0}^{k} \frac{t^j A^j}{j!} y_0$$

     - Since the term above is exactly the power series definition of e^{tA}, we have that $y_k(t) \to \mathrm{e}^{tA} y_0$ with local uniformity in t, as desired.
  2. Consider the ODE $y' = y^2$, $y(0) = 1$.
     - We know that the solution is $y(t) = 1/(1-t)$. We will now also derive this via the Picard iteration.
     - Choose $b = 1$, so that

$$\bar{B}(y_0, b) = \{y \mid |y - y(0)| \leq 1\} = \{y \mid |y - 1| \leq 1\} = [0, 2]$$

     - On this interval, $f(t, y) = y^2$ has maximum slope $L = 4$. Thus, we should take $T \leq 1/2L = 1/8$.
     - It follows that $|y_1^2 - y_2^2| \leq 4|y_1 - y_2|$ for all $y_1, y_2 \in \bar{B}(y_0, b)$.
     - Calculate the first few Picard iterates.

$$\begin{aligned} y_1(t) &= 1 + \int_0^t (y_0(\tau))^2 \mathrm{d}\tau = 1 + t \\ y_2(t) &= 1 + \int_0^t (1+\tau)^2 \mathrm{d}\tau = 1 + t + t^2 + \frac{t^3}{3} \\ y_3(t) &= 1 + \int_0^t \left(1 + \tau + \tau^2 + \frac{\tau^3}{3}\right)^2 \mathrm{d}\tau = 1 + t + t^2 + t^3 + \frac{2t^4}{3} + \frac{t^5}{3} + \frac{t^6}{9} + \frac{t^7}{63} \end{aligned}$$

     - It follows by induction that

$$\begin{aligned} |y_k(t) - (1 + t + \cdots + t^k)| &\leq t^{k+1} \\ \left| y_k(t) - \frac{1 - t^{k+1}}{1 - t} \right| &\leq t^{k+1} \end{aligned}$$

       It follows that $|t| < 1/8$.
     - For $|t| < 1/8$, $y(t) = 1/(1-t)$. Blows up as $t \to 1$.
     - Some more details on the bounding of the error term are presented in the lecture notes document.
- Lemma (Grönwall's inequality): Let $\varphi(t)$ be a real function defined for $t \in [t_0, t_0 + T]$ such that

$$\varphi(t) \leq f(t) + a \int_{t_0}^t \varphi(\tau) \mathrm{d}\tau$$

  Then

$$\varphi(t) \leq f(t) + a \int_{t_0}^t \mathrm{e}^{a(t-\tau)} f(\tau) \mathrm{d}\tau$$

Proof. Multiply both sides by e^{-at}:

$$\begin{aligned}\mathrm{e}^{-at}\varphi(t) - a\mathrm{e}^{-at}\int_{t_0}^{t}\varphi(\tau)\mathrm{d}\tau &\le \mathrm{e}^{-at}f(t)\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-at}\int_{t_0}^{t}\varphi(\tau)\mathrm{d}\tau\right) &\le \mathrm{e}^{-at}f(t)\\ \mathrm{e}^{-at}\int_{t_0}^{t}\varphi(\tau)\mathrm{d}\tau &\le \int_{t_0}^{t}\mathrm{e}^{-a\tau}f(\tau)\mathrm{d}\tau\\ \int_{t_0}^{t}\varphi(\tau)\mathrm{d}\tau &\le \int_{t_0}^{t}\mathrm{e}^{a(t-\tau)}f(\tau)\mathrm{d}\tau\end{aligned}$$

Substituting back into the original equality yields the result at this point. □

- Note that there is no sign condition on $f(t)$ or a.
- Grönwall's inequality is very important and we should remember it.
- It is also exactly what we need to prove continuous dependence.
- Theorem: Let $f(t,z), g(t,z)$ be defined on $\Omega \subset \mathbb{R}_t^1 \times \mathbb{R}_z^n$, an open and bounded a region containing (t_0, y_0) and (t_0, w_0). Let the functions be L - Lipschitz wrt. z. Consider two initial value problems $y' = f(t,y)$, $y(t_0) = y_0$ and $w' = g(t,w)$, $w(t_0) = w_0$. If $|f(t,z) - g(t,z)| < M$, then for $t \in [t_0, t_0 + T]$,

$$|y(t) - w(t)| \le \mathrm{e}^{LT}|y_0 - w_0| + \frac{M}{L}(\mathrm{e}^{LT} - 1)$$

Proof. We have that

$$\begin{aligned}|y(t) - w(t)| &= \left|\left[y_0 + \int_{t_0}^{t} f(\tau, y(\tau))\mathrm{d}\tau\right] - \left[w_0 + \int_{t_0}^{t} g(\tau, y(\tau))\mathrm{d}\tau\right]\right|\\ &= \left|[y_0 - w_0] + \int_{t_0}^{t}[f(\tau, y(\tau)) - g(\tau, y(\tau))]\mathrm{d}\tau\right|\\ &\le |y_0 - w_0| + \left|\int_{t_0}^{t}[f(\tau, y(\tau)) - g(\tau, w(\tau))]\mathrm{d}\tau\right|\\ &\le |y_0 - w_0| + \int_{t_0}^{t}|f(\tau, y(\tau)) - g(\tau, w(\tau))|\mathrm{d}\tau\end{aligned}$$

where we get from the second to the third line using the triangle inequality, and the third to the fourth line using Theorem 13.26 of Honors Calculus IBL. We also know that

$$\begin{aligned}|f(\tau, y(\tau)) - g(\tau, w(\tau))| &\le |f(\tau, y(\tau)) - f(\tau, w(\tau))| + |f(\tau, w(\tau)) - g(\tau, w(\tau))|\\ &\le L|y(\tau) - w(\tau)| + M\end{aligned}$$

Combining what we've obtained, we have

$$\begin{aligned}\underbrace{|y(t) - w(t)|}_{\psi(t)} &\le \underbrace{|y_0 - w_0| + M(t - t_0)}_{f(t)} + \underbrace{L}_{a}\int_{t_0}^{t}\underbrace{|y(\tau) - w(\tau)|}_{\psi(t)}\mathrm{d}\tau\\ &\le MT + |y_0 - w_0| + L\int_{t_0}^{t}\mathrm{e}^{L(t-\tau)}[|y_0 - w_0| + M(t - \tau)]\mathrm{d}\tau \qquad \text{Grönwall}\\ &\le \mathrm{e}^{LT}|y_0 - w_0| + \frac{M}{L}(\mathrm{e}^{TL} - 1)\end{aligned}$$

as desired. □

- Note: Getting from directly from Grönwall's inequality in the second line above to the last line above is quite messy. A consequence of Grönwall's inequality explored in the book makes this much easier. *Prove Equation 2.38 via Problem 2.12.*
- Implication: The IVP is not just solvable itself, but is solvable wrt. perturbation of the initial conditions and RHS within a small, finite interval in time.
- Suppose $y' = 0$, $y(0) = 1$ and $w' = \varepsilon w$, $w(0) = 1$. Then $y(t) = 1$ and $w(t) = \mathrm{e}^{\varepsilon t}$ and solutions are only close when t is small.
  - $t \leq 1/\varepsilon$??
- This is important in physics. In most physical scenarios, the RHS is C^1. This is called determinism.

6.2 Differentiability With Respect To Parameters

11/2:

- Review: Implicit Function Theorem.
  - Gives you a sufficient condition for which an implicit relation defines a function.
  - Does not give you the function, but tells you that it must exist and that it is unique.
- Theorem (Implicit Function Theorem): Let $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be C^k in some neighborhood of $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^m$ a point satisfying $F(x_0, y_0) = 0$. If the truncated Jacobian matrix $\frac{\partial F}{\partial y}(x_0, y_0)$, which is $m \times m$, is invertible, then there is a neighborhood U of x_0 such that there is a unique function $f : U \to \mathbb{R}^m$ with $y_0 = f(x_0)$ and $F(x, f(x)) = 0$ and

$$f'(x) = -\left(\frac{\partial F}{\partial y}(x, y)\right)^{-1} \cdot \frac{\partial F}{\partial x}(x, f(x))$$

  - The proof is based on the Banach fixed point theorem (this may be false?? I think Shao is confusing the proof of this theorem with the proof of the Inverse Function Theorem).
  - The motivation for the last equality (the line above) is that if $F(x, f(x)) = 0$, then by the chain rule for partial derivatives,

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}x}(F(x, f(x))) \\
&= \frac{\partial F}{\partial x}(x, f(x)) \cdot \frac{\mathrm{d}x}{\mathrm{d}x} + \left[\frac{\partial F}{\partial y}(x, y)\right] \cdot \frac{\mathrm{d}f}{\mathrm{d}x} \\
&= \frac{\partial F}{\partial x}(x, f(x)) + \left[\frac{\partial F}{\partial y}(x, y)\right] \cdot f'(x) \\
f'(x) &= -\left(\frac{\partial F}{\partial y}(x, y)\right)^{-1} \cdot \frac{\partial F}{\partial x}(x, f(x))
\end{aligned}$$

  - Recall that we know that the matrix bracketed in line 2 is invertible by hypothesis.
  - Additionally, since $\partial F/\partial x = A$ is $n \times m$ and $\partial F/\partial y = B$ is $m \times m$, $f' = -A^{-1}B$ is $n \times m$, as it should be for a function $f : \mathbb{R}^n \to \mathbb{R}^m$.
- Consider the IVP

$$y' = f(t, y; \mu), \quad y(t_0) = x(\mu)$$

  - This ODE and its initial condition both depend on a parameter $\mu \in B(0, r) \subset \mathbb{R}^m$ (usually we take $m = 1$ so μ is just real).
  - We denote the solution by $y(t; \mu)$.

  - Suppose $|x(\mu)| < C$ for $\mu \in B(0,r)$ and $x(\mu) \in C^1$. Suppose the RHS $f(t,z;\mu)$ of the ODE is defined on $[t_0, t_0+a] \times \bar{B}(x(0), b+C) \times B(0,r)$, is C^1 in all variables, is bounded by M on its domain, and is L-Lipschitz in z.

- By Cauchy-Lipschitz, for small
$$T \leq \min\left(a, \frac{b}{M}, \frac{1}{2L}\right)$$
and $\mu \in B(0,r)$ (r small), the solution *exists* on $[t_0, t_0+T]$ and its value does not escape $\bar{B}(x(0), b+C)$.
  - We now aim to show that the solution is *differentiable* wrt. μ on this interval.

- If $y(t;\mu)$ satisfies $y'(t;\mu) = f(t, y(t;\mu);\mu)$ and if the Jacobian matrix $J = \partial y/\partial\mu$ exists, then J satisfies the **first variation equation**.

- **First variation equation**: The following linear differential equation. *Given by*
$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\frac{\partial y}{\partial \mu}(t;\mu)}_{J(t;\mu)} = \underbrace{\frac{\partial f}{\partial z}(t, y(t;\mu);\mu)}_{A(t;\mu)} \cdot \underbrace{\frac{\partial y}{\partial \mu}(t;\mu)}_{J(t;\mu)} + \frac{\partial f}{\partial \mu}\,(t, y(t;\mu);\mu), \quad \frac{\partial y}{\partial \mu}\,(t_0,\mu) = \frac{\partial x}{\partial \mu}\,(\mu)$$

- The first variation equation has a unique solution, but we do not yet know that $y(t;\mu)$ is even differentiable with respect to μ. We presently verify this claim.

- Theorem[1]: $y(t;\mu)$ is C^1 in μ and $\partial y/\partial\mu\,(t;\mu)$ satisfies the first variation equation.

Proof. Let $\Theta(t;\mu) = y(t;\mu+h) - y(t;\mu) - J(t;\mu)h$ for h small. Aim, show that $\Theta(t;\mu) = o(h)$ as $h \to 0$.

We compute
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Theta(t;\mu) &= y'(t;\mu+h) - y'(t;\mu) - J'(t;\mu)h \\ &= \underbrace{f(t, y(t;\mu+h);\mu+h) - f(t,y(t;\mu);\mu)}_{I} - \underbrace{\frac{\partial f}{\partial z}(t, y(t;\mu);\mu)J(t;\mu) + \frac{\partial f}{\partial \mu}(t,y(t;\mu);\mu)}_{II}\end{aligned}$$

I denotes the first term; II denotes the second term.

We have that
$$I = \frac{\partial f}{\partial z}(t, y(t;\mu);\mu)[y(t;\mu+h) - y(t;\mu)] + \frac{\partial f}{\partial \mu}(t, y(t;\mu);\mu)h + \underbrace{R(t;\mu,h)}_{o(h)}$$

color coding

$$\begin{aligned}I - II &= \underbrace{\text{green} - \text{blue}}_{\Theta(t;\mu)} + R(t;\mu,h) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\Theta(t;\mu) = \Theta(t;\mu) + \underbrace{R(t;\mu,h)}_{o(h)} \\ \Theta(t_0;\mu) &= o(h) \\ |\Theta(t;\mu)| &\leq C\int_{t_0}^{t} |R(\tau;\mu,h)|\mathrm{d}\tau && \text{Grönwall} \\ &= o(h)\end{aligned}$$

circle terms cancel. ☐

[1] See the proof from the book, transcribed below.

- Example: First order derivatives must satisfy the first variational equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial y}{\partial \mu}(t;\mu) = \frac{\partial f}{\partial z}(t, y(t;\mu);\mu) \cdot \frac{\partial y}{\partial \mu}(t;\mu)$$

  and the second order derivative must satisfy the second variational equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial^2 y}{\partial \mu^2} = \frac{\partial^2 f}{\partial z^2}\left(\frac{\partial y}{\partial \mu}\frac{\partial^2 y}{\partial \mu^2}\right) + \frac{\partial^2 f}{\partial z \partial \mu}\frac{\partial y}{\partial \mu} + \frac{\partial^2 f}{\partial \mu \partial z}(-)\frac{\partial y}{\partial \mu} + \frac{\partial^2 f}{\partial \mu^2}(-)$$

- Corollary: If $f(t,z;\mu)$ is C^k in (t,z,μ), $y(t_0) = x(\mu)$ is C^k, then $y(t;\mu)$ is C^k in μ.
- The Taylor expansion

$$y(t;\mu) = y(t;0) + y_1\mu + y_2\mu^2 + \cdots + y_k\mu^k + O(\mu^{k+1})$$

  of $y(t;\mu)$ about 0 gives an approximation of said function up to order k in μ.
  - Misc notes: but you can cut off the expansion at k?? $y(t;0)$ being solvable implies inductively that the rest are solvable??
  - We can take this Taylor expansion because we assume that y is continuously differentiable k times with respect to μ.
  - The coefficients y_j are given as follows.

$$y_j = \frac{1}{j!}\frac{\partial^j y}{\partial \mu^j}\ (t;0)$$

- Application of the Taylor expansion: It can be substituted into the ODE as follows.

$$\begin{aligned}\dot{y} &= f(t,y;\mu)\\ \frac{\mathrm{d}}{\mathrm{d}t}(y(t;\mu)) &= f(t,y(t;\mu);\mu)\\ \frac{\mathrm{d}}{\mathrm{d}t}(y(t;0)) + \frac{\mathrm{d}y_1}{\mathrm{d}t}\mu + \cdots + \frac{\mathrm{d}y_k}{\mathrm{d}t}\mu^k + O(\mu^{k+1}) &= f(t, y(t;0) + y_1\mu + \cdots + y_k\mu^k + O(\mu^{k+1});\mu)\end{aligned}$$

  - Then you can match coefficients of the various μ terms on the LHS and RHS and solve for $y_0, \dots, y_k$.
  - When to use this method: Sometimes, you can view equations that aren't explicitly solvable as perturbations of an easily solvable system.

- Simple example (more complex ones next lecture):

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \mu y, \quad y(0) = 1$$

  - First off, we know that there is an explicit solution ($y(t) = \mathrm{e}^{\mu t}$). Thus, we will be able to check our final answer.
  - Suppose $y \in C^2$ with respect to μ. Then

$$y(t;\mu) = y_0 + y_1\mu + y_2\mu^2 + O(\mu^3)$$

  - It follows by substituting into the above differential equation that

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= \mu y\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(y_0 + y_1\mu + y_2\mu^2\right) &= \mu(y_0 + y_1\mu + y_2\mu^2)\\ \frac{\mathrm{d}y_0}{\mathrm{d}t} + \frac{\mathrm{d}y_1}{\mathrm{d}t}\mu + \frac{\mathrm{d}y_2}{\mathrm{d}t}\mu^2 &= 0 + y_0\mu + y_1\mu^2 + y_2\mu^3\end{aligned}$$

- By comparing coefficients, this yields the sequentially solvable differential equations

$$\frac{\mathrm{d}y_0}{\mathrm{d}t} = 0 \qquad \frac{\mathrm{d}y_1}{\mathrm{d}t} = y_0 \qquad \frac{\mathrm{d}y_2}{\mathrm{d}t} = y_1$$

  where we apply the initial condition $y_0(0) = 1$ to solve the left ODE above.
- Solving, we get

$$y_0(t) = 1 \qquad y_1(t) = t \qquad y_2(t) = \frac{t^2}{2}$$

  - Where do the other initial conditions (all zero) come from??
- Therefore, our approximate solution is

$$y(t) = 1 + t\mu + \frac{1}{2}t^2\mu^2 + O(\mu^3)$$

  which does indeed give the first three terms in the Taylor series expansion of the solution $\mathrm{e}^{\mu t}$.

- The perturbative solution fails in large time intervals — polynomials inevitably grow slower than exponential functions.
- Next time: Several examples applying what we've learned today.
- This week's homework: Some basic Lipschitz definitions and also computations with the perturbative series.

6.3 Variational Examples

11/4:
- We begin today with a more direct and less involved proof of the variation of parameters theorem.

Proof. Let $y'(t;\mu) = f(t, y(t;\mu);\mu)$ with $y(t_0;\mu) = x(\mu)$. Assume Lipschitz continuity and C^1-ness of the ODE and the initial condition on μ. Then differentiation with respect to μ must satisfy the first variational equation. In particular, let $J(t;\mu)$ be the solution of

$$J'(t;\mu) = \underbrace{\frac{\partial f}{\partial z}(t, y(t;\mu);\mu)}_{A(t;\mu)} J(t,\mu) + \underbrace{\frac{\partial f}{\partial \mu}(t, y(t;\mu);\mu)}_{F(t;\mu)}, \quad J(t_0;\mu) = \frac{\partial x}{\partial \mu}$$

Consider the Picard iteration sequence defined by

$$y_{n+1}(t;\mu) = \underbrace{f(t; y_n(t;\mu);\mu)}_{A_n(t;\mu)}, \quad y_n(t_0,\mu) = x(\mu)$$

Differentiating we get

$$\frac{\partial y_n}{\partial \mu}(t;\mu)$$

which we may call $J_n(t;\mu)$. We want to prove that the sequence of functions J_n converges uniformly to J. This makes sense since A and F uniformly converge. Moreover, under this definition of J_n, we have that

$$J_{n+1}'(t;\mu) = \underbrace{\frac{\partial f}{\partial z}(t, y_n(t;\mu);\mu)}_{A_n(t;\mu)} J_n(t;\mu) + \underbrace{\frac{\partial f}{\partial \mu}(t, y_n(t;\mu);\mu)}_{F_n(t;\mu)}, \quad J_n(t_0;\mu) = \frac{\partial x}{\partial \mu}(\mu)$$

Thus, Step 1 is to show that $\{\|J_n\|\}$ is bounded on $[t_0, t_0 + T]$. To do so, we note that

$$\|J_{n+1}\| \leq \frac{1}{2}\|J_n\| + \sup\left|\frac{\partial f}{\partial \mu}\right|$$

so that $\|J_n\|$ forms a bounded sequence. By induction,

$$\|J_n\| \leq 2C$$

We now embark on Step 2: Proving $J_n \to J$ uniformly. First off, we have that

$$\begin{aligned}
(J - J_{n+1})'(t;\mu) &= \frac{\mathrm{d}}{\mathrm{d}t}(J(t;\mu) - J_{n+1}(t;\mu)) \\
&= A(t;\mu)J(t;\mu) + F(t;\mu) - A_n(t;\mu)J_n(t;\mu) - F_n(t;\mu) \\
&= A(t;\mu)J(t;\mu) + A_n(t;\mu)J(t;\mu) - A_n(t;\mu)J(t;\mu) \\
&\quad - A_n(t;\mu)J_n(t;\mu) + F(t;\mu) - F_n(t;\mu) \\
&= A_n(t;\mu)(J - J_n)(t;\mu) + (A - A_n)(t;\mu)J(t;\mu) + (F - F_n)(t;\mu)
\end{aligned}$$

and

$$J(t_0;\mu) - J_{n+1}(t_0;\mu) = 0$$

Integrating once again on $[t_0, t_0 + T]$, we get

$$\|J - J_{n+1}\| \leq \frac{1}{2}\|J - J_n\| + \delta_n$$

where $\delta_n \to 0$ since we "obviously" have that $A_n \to A$ and $F_n \to F$ uniformly.

We now proceed via a standard analysis argument. Fix $\delta > 0$, choose N such that $\delta_n < \delta$ for $n \geq N$. Then we can control it by $\frac{1}{2}\|J - J_n\| + \delta$ for $n \geq N$. Then

$$\|J - J_{n+1}\| - 2\delta \leq \frac{1}{2}\|J - J_n\| - 2\delta$$

for all $n \geq N$, so we have by iteration that $\|J - J_{n+1}\| \leq 2\delta + \frac{1}{2^{n-N}}\|J - J_N\|$, so $\lim_{n\to\infty}\|J - J_n\| < 2\delta$ for arbitrary $\delta > 0$. Therefore, $\|J - J_n\| \to 0$, so $J_n \to J$ uniformly.

So in conclusion, $J_n \to J$ uniformly and we recall that $J_n = \partial y_n / \partial \mu$ where $y_n \to y$ uniformly. □

- We now look at examples. The ones in the HW will be no more difficult than these.
- Example (same one as last time):
  - Consider $y' = \mu y$ with $y(0) = 1$.
  - In order to find asymptotic expansion wrt. μ, we use the **ansatz** $y(t;\mu) = y_0 + y_1\mu + y_2\mu^2 + \cdots + y_n\mu^n + O(\mu^{n+1})$.
    - The differentiation theorem asserts that $y(t;\mu)$ can be differentiated wrt. μ so many times.
  - We can compute
    $$\mu y(t;\mu) = 0 + y_0\mu + y_1\mu^2 + \cdots + y_{n-1}\mu^n + O(\mu^{n+1})$$
    and
    $$y'(t;\mu) = y_0' + y_1'\mu + y_2'\mu^2 + \cdots + y_n'\mu^n + O(\mu^{n+1})$$
    and set them equal to yield a system of differential equations.
  - The initial conditions are $y_0(0) = 1$ and then $y_1(0) = \cdots = y_n(0) = 0$.
  - $y_0' = 0$ with $y_0(0) = 1$ implies that $y_0(t) = 1$.
  - Then the first order approximation is $y_1' = y_0 = 1$, so solving and applying the initial conditions, we get $y_1(t) = t$.
  - Continuing on, the second order approximation is $y_2(t) = t^2/2$.
  - Inductively, $y_m(t) = t^m/m!$.
  - In conclusion, we obtain the desired approximate solution.

- **Ansatz**: The form of the solution that you guess.
- In general, this shows the technique well: Use a polynomial ansatz and compare terms to yield an inductive sequence of explicitly solvable equations up to a certain point.
- Example: Mathematical pendulum.
  - Suppose that the length of the rope is ℓ and the gravitational acceleration is g. Then
  $$\theta''(t;\mu) = -\frac{g}{\ell}\sin[\theta(t;\mu)]$$
  - Assume a small angle, $\theta(0) = \mu$ and $\theta'(0) = 0$.
  - Substitute $\omega_0^2 = g/\ell$.
  - In HS, we learned that the harmonic oscillator approximation of the mathematical pendulum is justified for small θ. We now justify this.
  - Ansatz: $\theta_0 + \theta_1\mu + \theta_2\mu^2 + \theta_3\mu^3 + O(\mu^4)$.
  - Recall that
  $$\sin\theta = \theta - \frac{\theta^3}{6} + O(\theta^5)$$
  - First step, solve to determine $\theta_0 = 0$.
  - Then we only have a term of order $O(\mu)$ and $O(\mu^3)$ to worry about.
  - Substitute the expansion in:
  $$\begin{aligned}\sin\theta &= \theta - \frac{\theta^3}{6} + O(\theta^5)\\ &= \left(\theta_0 + \theta_1\mu + \theta_2\mu^2 + \theta_3\mu^3\right) - \frac{1}{6}\left(\theta_0 + \theta_1\mu + \theta_2\mu^2 + \theta_3\mu^3\right)^3\\ &= 0 + \theta_1\mu + \theta_2\mu^2 + \left(\theta_3 - \frac{\theta_1^3}{6}\right)\mu^3 + O(\mu^4)\end{aligned}$$
  - We also have that
  $$\theta''(t;\mu) = \theta_1''\mu + \theta_2''\mu^2 + \theta_3''\mu^3 + O(\theta^4)$$
  and
  $$-\omega_0^2\sin\left(\theta_1\mu + \theta_2\mu^2 + \theta_3\mu^3 + O(\mu^4)\right) = -\omega_0^2\theta_1\mu - \omega_0^2\theta_2\mu^2 - \omega_0^2\left(\theta_3 - \frac{\theta_1^3}{6}\right)\mu^3 + O(\mu^4)$$
  - Initial conditions: $\theta_0 = 0$, $\theta_1(0) = 1$, and $\theta_2(0) = \theta_3(0) = \theta_1'(0) = \cdots = \theta_3'(0) = 0$.
  - First order: $\theta_1'' = -\omega_0^2\theta_1$, $\theta_1(0) = 1$, $\theta_1'(0) = 0$. Implies $\theta_1(t) = \cos\omega_0 t$. This is why we can use the harmonic oscillator approximation.
  - Second order: $\theta_2 = -\omega_0^2\theta_2$. Initial conditions imply $\theta_2(t) = 0$.
  - Third order: $\theta_3'' = -\omega_0^2\theta_3 + \frac{\omega_0^2\theta_1^3}{6}$. Implies that
  $$\theta_3(t) = \frac{\omega_0 t}{16}\sin\omega_0 t + \frac{1}{192}(\cos\omega_0 t - \cos 3\omega_0 t)$$
    - We have to apply some trigonometric identities to verify this??
  - In conclusion, we have the approximation of our solution up to order $O(\mu^3)$ as
  $$\theta(t;\mu) = \mu\cos\omega_0 t + \mu^3\left[\frac{\omega_0 t}{16}\sin\omega_0 t + \frac{1}{192}(\cos\omega_0 t - \cos 3\omega_0 t)\right] + O(\mu^4)$$
    - This approximation is only good for T in a fixed, small time interval because the second term is not periodic.

- We now investigate the period of the mathematical pendulum.
  - The first order approximation (harmonic oscillator) gives the period as $T \approx 2\pi/\omega_0 = 2\pi\sqrt{\ell/g}$.
  - Let $T(\mu)$ denote the period of the mathematical pendulum as a function of the starting angle μ.
  - $T(\mu)$ should be approximately equal to the period of $\theta(t;\mu)$. Additionally, thinking about the mathematical pendulum intuitively, the period $T(\mu)$ should be about four times the first positive zero of $\theta(t;\mu)$.
    - Indeed, in a full cycle, the pendulum must go from the positive extreme, to zero, to the negative extreme, back to zero, and back to the original position, so there are our four parts.
    - Example: In the harmonic oscillator approximation, the first zero is at $\pi/2\omega_0$, and the period is $2\pi/\omega_0 = 4\cdot\pi/2\omega_0$.
  - Thus, determining the period $T(\mu)$ becomes a problem of finding t such that $\theta(t;\mu) = 0$.
  - The zeroes of $\theta(t;\mu)$ will be equal to the zeroes of $\theta(t;\mu)/\mu$, so we seek t such that the implicit function

$$F(t;\mu) = \frac{\theta(t;\mu)}{\mu} = \cos\omega_0 t + \mu^2\left[\frac{\omega_0 t}{16}\sin\omega_0 t + \frac{1}{192}(\cos\omega_0 t - \cos 3\omega_0 t)\right] = 0$$

  - When $\mu = 0$, the mathematical pendulum is stationary, but this does technically mean that it has a zero at $(\pi/2\omega_0;0)$. This point is important because for μ small enough that the harmonic oscillator approximation is good, the first zero should be very close to $\pi/2\omega_0$. Thus, we choose to solve $F(t;\mu) = 0$ around $(t_0;\mu_0) = (\pi/2\omega_0;0)$.
  - The requirement for the Implicit Function Theorem is met since

$$\begin{aligned}
\frac{\partial F}{\partial t}(t_0;\mu_0) &= -\omega_0\sin\omega_0 t_0 + \mu_0^2\left(\frac{\omega_0}{16}\sin\omega_0 t_0 + \frac{\omega_0^2 t_0}{16}\cos\omega_0 t_0 + \frac{1}{192}(-\omega_0\sin\omega_0 t_0 + 3\omega_0\sin 3\omega_0 t_0)\right)\\
&= -\omega_0\sin\frac{\pi}{2} + 0^2(\dots)\\
&= -\omega_0\\
&\neq 0
\end{aligned}$$

  - Thus, there exists $t_1(\mu)$ smooth defined on some neighborhood of $\mu_0 = 0$ satisfying $t_1(0) = \pi/2\omega_0$ and $F(t_1(\mu);\mu) = 0$.
  - We cannot (easily??) obtain $t_1(\mu)$ directly, so we will look for its second-order Taylor expansion

$$t_1(\mu) = \frac{\pi}{2\omega_0} + b_1\mu + b_2\mu^2 + O(\mu^3)$$

  - We need not compute a bunch of derivatives to find b_1, b_2, though. Indeed, we can just substitute into $F(t_1(\mu);\mu) = 0$ and compare different powers of μ. Doing so, we obtain

$$\begin{aligned}
0 &= F(t_1(\mu);\mu)\\
&= \cos\left(\frac{\pi}{2} + \omega_0 b_1\mu + \omega_0 b_2\mu^2 + O(\mu^3)\right)\\
&\quad + \mu^2\left[\frac{1}{16}\left(\frac{\pi}{2} + \omega_0 b_1\mu + \omega_0 b_2\mu^2 + O(\mu^3)\right)\sin\left(\frac{\pi}{2} + \omega_0 b_1\mu + \omega_0 b_2\mu^2 + O(\mu^3)\right)\right.\\
&\quad \left. + \frac{1}{192}\left(\cos\left(\frac{\pi}{2} + \omega_0 b_1\mu + \omega_0 b_2\mu^2 + O(\mu^3)\right) - \cos 3\left(\frac{\pi}{2} + \omega_0 b_1\mu + \omega_0 b_2\mu^2 + O(\mu^3)\right)\right)\right]\\
&= -\omega_0 b_1\mu + \left(\frac{\pi}{32} - \omega_0 b_2\right)\mu^2 + O(\mu^3)
\end{aligned}$$

    from which we can determine that

$$\begin{aligned}
0 &= -\omega_0 b_1 & 0 &= \frac{\pi}{32} - \omega_0 b_2\\
b_1 &= 0 & b_2 &= \frac{\pi}{32\omega_0}
\end{aligned}$$

- Thus,

$$\begin{aligned} T(\mu) &= 4 \cdot t_1(\mu) \\ &= \frac{2\pi}{\omega_0} + \frac{\pi}{8\omega_0}\mu^2 + O(\mu^3) \\ &= 2\pi\sqrt{\frac{\ell}{g}}\left(1 + \frac{1}{16}\mu^2 + O(\mu^3)\right) \end{aligned}$$

- We calculate an accumulation that is a perturbation of an ODE in the bonus this week, reproducing Einstein's work.