Week 4

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4.1 Autonomous Linear Systems

10/17:

- Today: General theory for autonomous linear systems.
- Review session Wednesday (no new material).
- First midterm Friday.
  - Test problems will be slight variations of homework problems or examples given in class.
- **Linear autonomous system**: A system of n linear equations written in the following form. *Denoted by* $\boldsymbol{y' = Ay}$. *Given by*

$$\begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix}' = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix} \qquad y(0) = 0$$

  - Note that the a_{ij}'s are complex or real.
- The explicit solution is given by $y(t) = \mathrm{e}^{tA} y_0$.
  - Recall that $\mathrm{d}/\mathrm{d}t \left(\mathrm{e}^{tA}\right) = A\mathrm{e}^{tA}$, as we can show via the power series expansion.
- **Picard iteration**: We take

$$\begin{aligned}
y'(t) &= Ay(t) \\
\int_0^t y'(\tau)\,\mathrm{d}\tau &= \int_0^t Ay(\tau)\mathrm{d}\tau \\
y(t) &= y_0 + \int_0^t Ay(\tau_1)\mathrm{d}\tau_1 \\
&= y_0 + \int_0^t A\left[y_0 + \int_0^{\tau_1} Ay(\tau_2)\mathrm{d}\tau_2\right]\mathrm{d}\tau_1 \\
&= y_0 + tAy_0 + \int_0^t \int_0^{\tau_1} A^2 y(\tau_2)\mathrm{d}\tau_2\mathrm{d}\tau_1 \\
&= y_0 + tAy_0 + \int_0^t \int_0^{\tau_1} A^2\left[y_0 + \int_0^{\tau_2} Ay(\tau_3)\mathrm{d}\tau_3\right]\mathrm{d}\tau_2\mathrm{d}\tau_1 \\
&= y_0 + tAy_0 + \frac{t^2A^2}{2} + \int_0^t \int_0^{\tau_1} \int_0^{\tau_2} A^3 y(\tau_3)\mathrm{d}\tau_3\mathrm{d}\tau_2\mathrm{d}\tau_1
\end{aligned}$$

$$= \sum_{k=0}^{m} \frac{t^k A^k}{k!} y_0 + A^{m+1} \underbrace{\int_0^t \cdots \int_0^{\tau_m}}_{m+1} y(\tau_{m+1}) \mathrm{d}\tau_{m+1} \cdots \mathrm{d}\tau_1$$

- We get from the second to the third line by substituting $y(t)$, as defined into the second line, into where it appears in the integral.

- We want to show that the integral converges to zero.
  - The magnitude of the remainder is less than or equal to

$$\|A\|^{m+1} \left(\sup_{\tau \in [0,t]} |y(\tau)| \right) \frac{t^{m+1}}{(m+1)!}$$

    - Justification of this term: Look at the rightmost term in the last line of the Picard iteration above. Imagine taking the norm of it. Splitting the "scalar" integral from the matrix allows us to take a matrix norm, and the property $\|AB\| \leq \|A\|\|B\|$ tells us that $\left\|A^{m+1}\right\| \leq \|A\|^{m+1}$. Then with respect to the integral, if we evaluate it, we will get the next polynomial term in the sequence — $t^{m+1}/(m+1)!$ — times at most the maximum value of y at every infinitesimal.
  - We can visualize lower-dimensional integrals as the volume of the corresponding unit **simplex**.
    - For example, in $\mathbb{R}^2$,

$$\int_0^1 \int_0^{\tau_1} 1 \mathrm{d}\tau_2 \, \mathrm{d}\tau_1$$

      can be visualized as the area of the unit triangle. This rationalizes why it evaluates to $1/2$, the area of said triangle.
    - In $\mathbb{R}^3$,

$$\int_0^1 \int_0^{\tau_1} \int_0^{\tau_2} 1 \mathrm{d}\tau_3 \mathrm{d}\tau_2 \, \mathrm{d}\tau_1$$

      can be visualized as the area of the unit simplex. This rationalizes why it evaluates to $1/3! = 1/6$, the volume of said simplex.
  - Since $(m+1)! \to \infty$ faster than any other term, the whole thing goes to zero.
  - Thus, $y(t) = \mathrm{e}^{tA} y_0$.

- **Simplex**: A higher-dimensional generalization of a triangle.

- We now consider the inhomogeneous equation. Before, we used an integrating factor. We will now do that again.

$$\begin{aligned}
y' &= Ay + f(t) \\
y' - Ay &= f(t) \\
\mathrm{e}^{-tA} y' - A\mathrm{e}^{-tA} y &= \mathrm{e}^{-tA} f(t) \\
\frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{e}^{-tA} y(t) \right) &= \mathrm{e}^{-tA} f(t) \\
\mathrm{e}^{-tA} y(t) - y_0 &= \int_0^t \mathrm{e}^{-\tau A} f(\tau) \mathrm{d}\tau \\
y(t) &= \mathrm{e}^{tA} y_0 + \int_0^t \mathrm{e}^{(t-\tau)A} f(\tau) \mathrm{d}\tau
\end{aligned}$$

  - We also call this the Duhamel formula.

- Note that if your time scale starts from t_0, then

$$y(t) = \mathrm{e}^{(t-t_0)A} y(t_0) + \int_{t_0}^t \mathrm{e}^{(t-\tau)A} f(\tau) \mathrm{d}\tau$$

- The utility of JNF: ??
- Rewrite $A = QBQ^{-1}$, where B is in JNF.
  - Shao reviews some facts of JNF from previous lectures.
- We have that
$$e^{tA}y_0 = Qe^{tB}Q^{-1}y_0$$
- Example: Let
$$A = \begin{pmatrix} -2 & 2 & 1 \\ -7 & 4 & 2 \\ 5 & 0 & 0 \end{pmatrix}$$
  - This is the same matrix from a previous lecture. As before, we have that
$$Q = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & 3 \\ 2 & 5 & -5 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$
    - Recall that the left two vectors are normal eigenvectors (the leftmost one corresponds to $\lambda_1 = 0$ and the middle one corresponds to $\lambda_2 = 1$) and the rightmost one is a generalized eigenvector.
  - We can compute that
$$e^{tB} = \begin{pmatrix} e^{0t} & 0 & 0 \\ 0 & e^{1t} & 1te^{1t} \\ 0 & 0 & e^{1t} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{t} & te^{t} \\ 0 & 0 & e^{t} \end{pmatrix}$$
  - It follows that
$$\begin{aligned} e^{tA}y_0 &= Q \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{t} & te^{t} \\ 0 & 0 & e^{t} \end{pmatrix} Q^{-1}y_0 \\ &= \begin{pmatrix} e^t - 3te^t & 2te^t & te^t \\ & \vdots & \\ & \vdots & \end{pmatrix} \begin{pmatrix} y_0^1 \\ y_0^2 \\ y_0^3 \end{pmatrix} \end{aligned}$$
- **Stable** (eigenvalue): An eigenvalue $\lambda_j = \sigma_j + i\beta_j$ for which $\sigma_j < 0$.
- **Unstable** (eigenvalue): An eigenvalue $\lambda_j = \sigma_j + i\beta_j$ for which $\sigma_j > 0$.
- **Stable** (subspace of the system): The space of all (generalized) eigenvectors corresponding to the stable eigenvalues.
- **Unstable** (subspace of the system): The space of all (generalized) eigenvectors corresponding to the unstable eigenvalues.
- Recall that B_j acts on K_j.
  - ... in picture??
  - Recall that $\mathbb{C}^n = K_1 \oplus \cdots \oplus K_m$.
  - P_j is not an *orthogonal* projection, but it is a projection of y_0 onto K_j. It's also a polynomial??
  - If $\sigma_j < 0$, then $|e^{tA}P_jy_0| \to 0$ at an exponential rate.
- Similarly, if you're working with an unstable eigenvalue, then $\sigma_j > 0$ implies $|e^{tA}P_jy_0| \to +\infty$ at an exponential rate.

  - The rate of growth depends on σ_j.
- Along the stable subspaces, your points will be attracted to zero.
- Along the unstable subspaces, your points will be repelled from zero.
- The stable subspace of our example is

$$\operatorname{span}\left\{\begin{pmatrix}1\\-1\\5\end{pmatrix},\begin{pmatrix}0\\3\\-5\end{pmatrix}\right\}$$

- If $\sigma_h = 0$, then we have rotation around a point, oscillation about zero, or oscillation whose magnitude grows to infinity. We do not talk about its stability.
  - We do not include the eigenvector corresponding to $\lambda_1 = 0$ in the above basis of the stable subspace because the solution oscillates about y_1??
- Let $x(t)$ be a higher order scalar ODE.
  - Then we can make a system out of it:

$$\begin{pmatrix}y^1\\\vdots\\y^n\end{pmatrix}' = \underbrace{\begin{pmatrix}0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1\\ -a_0 & -a_1 & \cdots & -a_{n-1}\end{pmatrix}}_{F[p]}\begin{pmatrix}y^1\\\vdots\\y^n\end{pmatrix}$$

  - $F[p]$ is the **Frobenius** matrix.
  - The transpose of this matrix is a very special matrix called the **companion** matrix $C[p] = F[p]^T$.
  - Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$. Then $\chi_{C[p]} = p(z)$.

    Proof. We have that

$$\begin{aligned}\chi_{C[p]}(z) &= \det(zI - C[p])\\ &= z(z^{n-1} + a_{n-1}(z^{n-2} + a_{n-2}(z^{n-3} + \cdots)))\\ &= p(z)\end{aligned}$$

    as desired. □
  - Roots of $p(z)$ are the eigenvalues of $F[p]$ and $C[p]$.
  - We have that $C[p]e_i = e_{i+1}$ for $i = 1, \dots, n-1$ and

$$C[p]e_n = -a_0e_1 - \cdots - a_{n-1}e_n$$

    which implies that if $r(z)/\deg r < n$ nullifies $C[p]$, then necessarily $r(z) = p(z)$ since $(z - \lambda_j)^{<\alpha_j}$??
- Theorem: In the Jordan normal form $F[p]$, each λ_j corresponds to only one Jordan block.
  - Thus,

$$F[p] \sim \begin{pmatrix}J_{\alpha_1}(\lambda_1) & & \\ & \ddots & \\ & & J_{\alpha_m}(\lambda_m)\end{pmatrix}$$

    The implication is that

$$J_d(\lambda) \neq \begin{pmatrix}\lambda & & \\ & \lambda & 1\\ & & \lambda\end{pmatrix}$$

    ever??

- Corollary: The solution $y(t)$ is of the form

$$(\cdots) + a_1 \mathrm{e}^{t\lambda_j} + \cdots + c_{\alpha_j - 1} t^{\alpha_j - 1} \mathrm{e}^{t\lambda_j} + \cdots$$

- Example: Solving a second-order ODE.

$$x'' + ax' + bx = 0 \quad \iff \quad \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}$$

  - The characteristic polynomial of the equation (and this matrix) is $z^2 + az + b = 0$.
  - If $\lambda_1 \neq \lambda_2$, then $x(t) = A\mathrm{e}^{t\lambda_1} + B\mathrm{e}^{t\lambda_2}$. If $\lambda_1 = \lambda_2 = \lambda$, then $x(t) = A\mathrm{e}^{t\lambda} + Bt\mathrm{e}^{t\lambda}$.