Week 7

Solution Existence and Stability

7.1 Peano Existence Theorem

11/7:

- Today: Peano Existence Theorem.
- For an IVP of a first-order differential system, as long as the RHS is continuous, we get at least one solution.
- The proof provides an algorithm that can be really useful in computing the solution provided that uniqueness exists.
- We will need a theorem from analysis to start.
- Theorem (Arzelà-Ascoli[1]): Let $h_k : [a,b] \to \mathbb{R}^n$ be a sequence of functions that is uniformly bounded and uniformly Lipschitz continuous wrt. L. Then $\{h_k\}$ contains a uniformly convergent subsequence and the limit has the same bound and Lipschitz constant.

  Proof. Recall the property of sequential compactness[2], i.e., that every bounded sequence of numbers contains a convergent subsequence. We want to prove this for a sequence of functions. To do so, we will need the Cantor diagonalization technique.

  $\mathbb{Q}$ is countable. Thus, we can enumerate the rationals in $[a,b]$ by $r_1, r_2, r_3, \dots$. Since $\{h_k(r_1)\}$ is a bounded sequence of numbers, we have by the above that there is a subsequence C_1 — say $h_1^{(1)}, h_2^{(1)}, h_3^{(1)}, \dots$ — such that $C_1 = \{h_k^{(1)}(r_1)\}$ is a convergent subsequence in $\mathbb{R}^n$ of the original sequence. Now C_1 is still a bounded sequence, so we can obtain a subsequence C_2 of *it* — say $h_1^{(2)}, h_2^{(2)}, h_3^{(2)}, \dots$ — such that $C_2 = \{h_k^{(2)}(r_2)\}$ is a convergent subsequence in $\mathbb{R}^n$ at r_2 (and, by inductive hypothesis, at r_1!). Inductively, we can obtain $C_\ell = \{h_k^{(\ell)}\}_{\ell,k=1}^\infty$ convergent at $r_1, r_2, \dots, r_\ell$. We then write down the elements of the sequences as a table. (For example, the k^{th} row of the table is a sequence that converges at $r_1, \dots, r_k$.)

  $$\begin{matrix} h_1^{(1)} & h_2^{(1)} & h_3^{(1)} & \cdots \\ h_1^{(2)} & h_2^{(2)} & h_3^{(2)} & \cdots \\ h_1^{(3)} & h_2^{(3)} & h_3^{(3)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

  Consider the diagonal sequence $\{f_\ell\}_{\ell=1}^\infty$ where $f_\ell = h_\ell^{(\ell)}$. By definition, it converges at all rational points. We now seek to prove that it converges uniformly at *all* points.

[1]This is not the full Arzelà-Ascoli theorem, but a special case. The proof is similar, regardless, though. See Honors Analysis in $\mathbb{R}^n$ I Notes.

[2]The Bolzano-Weierstrass Theorem/Theorem 15.18 from Honors Calculus IBL.

To prove that $\{f_\ell\}$ is a uniformly convergent sequence of functions, it will suffice to show that for all $\varepsilon > 0$, there exists N such that if $k, \ell > N$, then $|f_k(t) - f_\ell(t)| < \varepsilon$ for all $t \in [a, b]$. Let $\varepsilon > 0$ be arbitrary. Divide $[a, b]$ into m congruent subintervals I_α ($\alpha = 1, \dots, m$) such that $|I_\alpha| \leq \varepsilon/3L$ for all α. This guarantees that the oscillation of each f_k on any I_α is $\leq \varepsilon/3$ since if $x, y \in I_\alpha$ for some α, then

$$|f_\ell(x) - f_\ell(y)| \leq L|x - y| \leq L \cdot \frac{\varepsilon}{3L} = \frac{\varepsilon}{3}$$

Using the fact that $\{f_\ell\}$ is convergent and hence Cauchy on the rationals, pick N large enough so that $r_\alpha \in I_\alpha$ implies $|f_k(r_\alpha) - f_\ell(r_\alpha)| < \varepsilon/3$ for $k, \ell > N$. We will choose this N to be our N. Now let $t \in [a, b]$ be arbitrary. By their definition, we know $t \in I_\alpha$ for some α. Therefore,

$$\begin{aligned} |f_k(t) - f_\ell(t)| &\leq |f_k(t) - f_k(r_\alpha)| + |f_k(r_\alpha) - f_\ell(r_\alpha)| + |f_\ell(r_\alpha) - f_\ell(t)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &= \varepsilon \end{aligned}$$

as desired.

Lastly, we can prove that the limit function f of $\{f_\ell\}$ is L-Lipschitz as follows. Let $t, t' \in [a, b]$ be arbitrary. Then

$$\left|\frac{f(t) - f(t')}{t - t'}\right| = \lim_{k \to \infty} \left|\frac{f_k(t) - f_k(t')}{t - t'}\right| \leq \lim_{k \to \infty} \left|\frac{L|t - t'|}{t - t'}\right| = \lim_{k \to \infty} |L| = L$$

as desired. ☐

- Now we come to the proof of the Peano Existence Theorem.
- Theorem (Peano Existence Theorem): Let $f : [t_0, t_0 + a] \times \bar{B}(y_0, b) \to \mathbb{R}^n$ be bounded ($|f(t, z)| \leq M$) and continuous. Then the IVP
$$y'(t) = f(t, y(t)), \quad y(t_0) = b$$
has at least one solution for $t \in [t_0, t_0 + T]$ where $T = \min(a, b/M)$.

Proof. Since there is no Lipschitz condition, we use another strategy to find approximate solutions.

picture Fix $T = \min(a, b/M)$. We divide $[t_0, t_0 + T]$ into m congruent closed subintervals I_α ($\alpha = 0, \dots, m - 1$), each of length $h_m = T/m$. Define a continuous function $y_m(t)$ as follows: The values at the nodes t_α (the intersection points of adjacent congruent subintervals) are defined inductively via

$$y_m(t_{\alpha+1}) = y_m(t_\alpha) + f(t_\alpha, y_m(t_\alpha))h_m$$

for $\alpha = 0, \dots, m - 1$, and y_m is taken to be linear between the nodes[3]. The idea is that we replace the derivative $y'(t)$ by the difference quotient $[y(t + h) - y(t)]/h$. It follows by the construction that every function in the set $\{y_k(t) : [t_0, t_0 + T] \to \bar{B}(y_0, b)\}$ is piecewise linear (hence continuous), uniformly bounded, and uniformly M-Lipschitz continuous. Therefore, by the Arzelá-Ascoli theorem, $\{y_k\}$ contains a uniformly convergent subsequence $y_{m_k} \to y$.

It remains to verify that y is a solution to the integral equation

$$y(t) = y_0 + \int_{t_0}^{t} f(\tau, y(\tau))\mathrm{d}\tau$$

Observe that the domain of f is a closed and bounded subset of the real numbers. Thus, it is compact by the Heine-Borel theorem[4]. Moreover, since f is a continuous function on a compact domain, we

[3] Note that this construction is quite similar to that employed in Euler's method.
[4] Theorem 10.16 of Honors Calculus IBL.

have by the Heine-Cantor theorem[5] that f is uniformly continuous. Thus, for any $\varepsilon > 0$, there exists N such that if $m > N$, then
$$|f(t, y_m(t)) - f(t_\alpha, y_m(t_\alpha))| < \frac{\varepsilon}{T}$$
for all $\alpha = 0, \dots, m-1$ and $t \in I_\alpha$. Additionally, observe that

$$y_{m_k}(t) = y_0 + \sum_{\alpha=0}^{m-1} \int_{t_\alpha}^{t_{\alpha+1}} \chi_t(\tau) f(t_\alpha, y_{m_k}(t_\alpha)) \mathrm{d}\tau$$

where $\chi_t(\tau)$ denotes the **characteristic function** of $[t_0, t]$. To see this, compare with the original inductive definition of $y_m(t_{\alpha+1})$. *picture* We thus see that y_0 in the above equation corresponds to $y_m(t_0) = y(t_0)$, as we would expect. We see that we are summing a series of side-by-side integrals so that in the end, we integrate over all of $[t_0, t_0 + T]$. We see that the characteristic function restricts us to integrating over the ODE only up until t, as we would want for an approximation $y_{m_k}(t)$ at t using Euler's method. And we see that since $f(t_\alpha, y_{m_k}(t_\alpha))$ is constant and $h_m = t_{\alpha+1} - t_\alpha$, the integral does take on the expected value $f(t_\alpha, y_m(t_\alpha))h_m$. Moving right along, we see that

$$\begin{aligned}
\left| y_{m_k}(t) - y_0 - \int_{t_0}^{t} f(\tau, y_{m_k}(\tau)) \mathrm{d}\tau \right| &\leq \sum_{\alpha=0}^{m-1} \int_{t_\alpha}^{t_{\alpha+1}} \chi_t(\tau) |f(t_\alpha, y_{m_k}(t_\alpha)) - f(\tau, y_{m_k}(\tau))| \mathrm{d}\tau \\
&< \int_{t_0}^{t_0+T} \chi_t(\tau) \cdot \frac{\varepsilon}{T} \mathrm{d}\tau \\
&= \int_{t_0}^{t} \frac{\varepsilon}{T} \mathrm{d}\tau \\
&= \varepsilon \cdot \frac{t - t_0}{T} \\
&\leq \varepsilon
\end{aligned}$$

Thus, by uniform convergence, $\int_{t_0}^t f(\tau, y_{m_k}(\tau))\mathrm{d}\tau \to \int_{t_0}^t f(\tau, y(\tau))\mathrm{d}\tau$ uniformly, so y does satisfy the integral equation, as desired. □

- **Characteristic function** (of $[a, b]$): The function defined as follows. *Denoted by* $\boldsymbol{\chi_{[a,b]}}$. *Given by*
$$\chi_{[a,b]}(t) = \begin{cases} 1 & x \in [a, b] \\ 0 & x \notin [a, b] \end{cases}$$
- Utility of the Peano Existence Theorem: Proves the *existence* of a solution, but the proof is not constructive; it does not give an algorithm for finding the desired sequence. Nor does the PET make any statement on uniqueness.
- We now look to use a related method to define a sequence of functions that will converge to the desired solution of the ODE.
  - While the PET does not require it, in practice, most f we would be interested in will satisfy an additional Lipschitz condition.
  - Define the integral operator
  $$\Phi[u] = y_0 + \int_{t_0}^{t} f(\tau, u(\tau)) \mathrm{d}\tau$$
  We will prove that Φ is a contraction on the function space. This will imply that $\Phi^N[u]$ converges across the entire interval $[t_0, t_0 + T]$ to the solution y for any $u : [t_0, t_0 + T] \to \bar{B}(y_0, b)$, giving us our desired computational strategy. Let's begin.

[5] Theorem 13.6 of Honors Calculus IBL.

- To prove that Φ is a contraction, it will suffice to show that $\left\|\Phi^j[u_1] - \Phi^j[u_2]\right\| \to 0$ as $j \to \infty$. Thus, we wish to put a bound on $\left\|\Phi^j[u_1] - \Phi^j[u_2]\right\|$ that decreases as j increases. To that end, we will prove that

  $$\left\|\Phi^j[u_1] - \Phi^j[u_2]\right\| \leq \frac{(LT)^j}{j!} \cdot \|u_1 - u_2\|$$

  for all j.
  - We induct on j. For the base case $j = 1$, we have that

    $$\begin{aligned}|\Phi[u_1](t) - \Phi[u_2](t)| &\leq \int_{t_0}^t L|u_1(\tau) - u_2(\tau)|\mathrm{d}\tau \\ &\leq L(t-t_0)\|u_1 - u_2\| \\ &\leq LT\|u_1 - u_2\| \\ &= \frac{(LT)^1}{1!} \cdot \|u_1 - u_2\|\end{aligned}$$

    for all t.
  - Now suppose inductively that $\left\|\Phi^j[u_1] - \Phi^j[u_2]\right\| \leq (LT)^j/j! \cdot \|u_1 - u_2\|$. Then we have that

    $$\begin{aligned}|\Phi^{j+1}[u_1](t) - \Phi^{j+1}[u_2](t)| &\leq \int_{t_0}^t L|\Phi^j[u_1](\tau) - \Phi^j[u_2](\tau)|\mathrm{d}\tau \\ &\leq \int_{t_0}^t L \cdot \frac{(LT)^j}{j!} \cdot \|u_1 - u_2\|\mathrm{d}\tau \\ &= \cdots \\ &\leq \frac{(LT)^{j+1}}{j!} \cdot \|u_1 - u_2\|\end{aligned}$$

    for all t, implying the desired result.
- We now estimate the error between y_m and y in terms of y_m, alone. Indeed, we have from the above that

  $$\begin{aligned}\left\|y_m - \Phi^N[y_m]\right\| &\leq \sum_{j=0}^{N-1} \left\|\Phi^j[y_m] - \Phi^{j+1}[y_m]\right\| \\ &\leq \|y_m - \Phi[y_m]\| \sum_{j=0}^{N-1} \frac{(TL)^j}{j!} \\ \|y_m - y\| &\leq \|y_m - \Phi[y_m]\|\mathrm{e}^{TL}\end{aligned}$$

  where we get from the second to the third line by letting $N \to \infty$.
  - The proof of the PET guarantees that $\|y_m - \Phi[y_m]\|$ is small when m is large, no matter whether y_m itself converges or not.
  - In fact, when $f \in C^1$, the error is estimated as

    $$\|y_m - y\| \leq \frac{LT\mathrm{e}^{TL}}{m}$$

    for $L = \|f\|_{C^1}$.

- Takeaway: This polygon method gives rise to an algorithm to solve ODEs. Theoretically, it converges much slower than the Picard iteration, but in practice, it has the advantage that we do not need to do any numerical integration. Indeed, to obtain the desired precision using the Picard iteration, the numerical integration will need more and more steps and the total accumulated error will not be less than this polygon method.
- Better difference methods include Runge-Kutta or Heun, but please refer to monographs on numerical ODEs for these.

7.2 Asymptotic Stability

11/9:

- Going forward, we restrict ourselves to autonomous ODEs $y' = f(y)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth vector field.
- For every $x \in \mathbb{R}^n$, the IVP
$$y' = f(y), \quad y(0) = x$$
has a unique maximal solution $\phi_t(x)$ for $t \in I_x$.
- **Orbit**: The following set. *Given by*
$$\{\phi_t(x) : t \in I_x\}$$
- Let $K \subset \mathbb{R}^n$ be compact.
  - Then there exists $T_K \in \mathbb{R}$ such that $\phi_t(x)$ is defined for all $x \in K$ and $|t| \leq T_K$.
  - Moreover, the map from $K \to \mathbb{R}^n$ defined by $x \mapsto \phi_t(x)$ is injective due to uniqueness (and therefore a **homeomorphism**). We get one such map for each t.
  - Similar to the diffeomorphism idea from Guillemin and Haine (2018).
- **Invariant set**: A subset of $\mathbb{R}^n$ such that any orbit starting within it never leaves it.
- Compact invariant sets are quite interesting.
- Proposition: Let $\Omega \subset \mathbb{R}^n$ be a domain with a piecewise smooth boundary $\partial\Omega$. Suppose $f(x)$ is transversal to $\partial\Omega$ and inward pointing: That is, if ν is the inward pointing unit normal, then $f(x) \cdot \nu(x) \geq 0$ for all $x \in \partial\Omega$. Then $\bar{\Omega}$ is an invariant set: That is, any orbit starting from a point $\bar{\Omega}$ exists throughout the time and never leaves $\bar{\Omega}$.

  Proof idea. $x \in \partial\Omega$ ensures that $\phi_t(x)$ must be in Ω for small t. Hence, it suffices to consider $x \in \Omega$. In that case, pick the smallest $T > 0$ such that $\phi_T(x) \in \partial\Omega$. Then by transversality it must turn back into Ω. ☐

- This simple proposition is especially useful when establishing global attraction of the orbits.
- **Fixed point**: A point in $\mathbb{R}^n$ at which f evaluates to zero. *Denoted by* $\boldsymbol{x_0}$.
  - This means that the vector at x_0 is zero.
- **Lyapunov stable** (fixed point): A fixed point x_0 such that for any neighborhood $B(x_0, \varepsilon)$, there exists a neighborhood $B(x_0, \delta)$ such that $\phi_t(x) \in B(x_0, \varepsilon)$ for any $t \geq 0$ and $x \in B(x_0, \delta)$.
- **Asymptotically stable** (fixed point): A Lyapunov stable fixed point x_0 such that $\phi_t(x) \to x_0$ as $t \to +\infty$ for $x \in B(x_0, \delta)$.
- Example of a system that is Lyapunov stable but not asymptotically stable: The system
$$y' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y$$
where A denotes a rotation.
  - The orbits are concentric circles and never converge to 0.
- Investigation: The local behavior near a fixed point.
  - Consider $y' = f(y)$ as a perturbation of the linearized system $y' = f'(x_0)y$. In this case,
$$f(x) = f'(x_0)(x - x_0) + O(|x - x_0|^2)$$
as $x \to x_0$.

- Theorem: Let $f(x_0) = 0$. If the eigenvalues of the linearization $A = f'(x_0)$ all have negative real parts, then the fixed point $x = x_0$ is asymptotically stable.

  Proof. WLOG let $x_0 = 0$. Write $f(x) = Ax + g(x)$, where $g(x) = O(|x|^2)$. Since every $\lambda \in \sigma(A)$ has negative real part, there exist $a, C > 0$ (let $C > 1$ WLOG) such that

  $$|\mathrm{e}^{tA}x| \leq C\mathrm{e}^{-at}|x|$$

  The C arises because the matrix norm of e^{tA} is bounded as $t \to +\infty$ if all eigenvalues are negative. The e^{-at} arises similarly, and reflects the exponential decrease in magnitude happening along all subspaces on which e^{tA} acts.

  Let δ be such that $|g(x)| \leq a|x|/2C$ when $|x| \leq \delta$. Now consider the IVP

  $$y' = Ay + g(y), \quad y(0) \in \bar{B}\left(0, \frac{\delta}{2C}\right)$$

  Then at least for small t (i.e., t such that $|y(t)| \leq \delta$),

  $$|y(t)| \leq C\mathrm{e}^{-at}|y(0)| + \frac{a}{2C}\int_0^t \mathrm{e}^{-a(t-\tau)}|y(\tau)|\mathrm{d}\tau$$

  It follows from Grönwall's inequality that

  $$\mathrm{e}^{at}|y(t)| \leq C|y(0)|\mathrm{e}^{at/2}$$

  hence

  $$|y(t)| \leq \frac{\delta}{2}\mathrm{e}^{-at/2} < \delta$$

  Hence, any orbit of the system starting from $\bar{B}(0, \delta/2C)$ stays in $\bar{B}(0, \delta)$. So the maximal time of existence T is $+\infty$. This is because if not then, then the IVP starting from $y(T)$ is still solvable, contradicting the definition of T. Thus, we have proven that

  $$|y(t)| \leq \frac{\delta}{2}\mathrm{e}^{-at/2}$$

  for all $t \geq 0$ as long as $|y(0)| \leq \delta/2C$. □

- This is the last rigorous proof given in this course.
- A similar theorem:
- Theorem: Let $f(0) = 0$. If one of the eigenvalues of $A = f'(0)$ has positive real part, then the fixed point $x = 0$ is not Lyapunov stable.
- Initial application: Nonlinear mechanical system with frictions, e.g., ideal pendulum with friction.

  $$ml\theta'' + b\theta' = -mg\sin\theta$$

  - Substitute $\eta = b/ml$ and $\omega = \theta'$ to get a nonlinear system

    $$\begin{pmatrix} \theta \\ \omega \end{pmatrix}' = \begin{pmatrix} \omega \\ -\eta\omega - g/l\sin\theta \end{pmatrix}$$

  - At the equilibrium position $(\theta, \omega) = (0, 0)$, we have

    $$A = \begin{pmatrix} \frac{\partial}{\partial\theta}(\omega) & \frac{\partial}{\partial\omega}(\omega) \\ \frac{\partial}{\partial\theta}(-\eta\omega - g/l\sin\theta) & \frac{\partial}{\partial\omega}(-\eta\omega - g/l\sin\theta) \end{pmatrix} \approx \begin{pmatrix} 0 & 1 \\ -g/l & -\eta \end{pmatrix}$$

    i.e.,

    $$\begin{pmatrix} \theta \\ \omega \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -g/l & -\eta \end{pmatrix}\begin{pmatrix} \theta \\ \omega \end{pmatrix} + O(|\theta|^2 + |\omega|^2)$$

    - Since $\eta > 0$, the eigenvalues have a common negative real part, so the equilibrium is asymptotically stable.

  - At the equilibrium $(\pi, 0)$, we have

$$\begin{pmatrix} \theta \\ \omega \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ g/l & -\eta \end{pmatrix} \begin{pmatrix} \theta - \pi \\ \omega \end{pmatrix} + O(|\theta - \pi|^2 + |\omega|^2)$$

    - For $\eta \geq 0$, there is one positive and one negative eigenvalue, so this equilibrium is unstable.

  - These results should make intuitive sense: If a pendulum is resting at the bottom, that is a stable equilibrium. If a pendulum is resting at the top, that is not a stable equilibrium.

7.3 Applications of the Lyapunov Method

11/11:

- Purely imaginary eigenvalues can still lead to Lyapunov stability.
- **Lyapunov function** (of a system $y' = f(y)$ with fixed point x_0 near x_0): A continuous real function on $\mathbb{R}^n$ such that the following two axioms hold. *Denoted by* $\boldsymbol{L}$.
  1. $L(x_0) = 0$ and $L(x) > 0$ for all $x \in \mathring{B}(x_0, \delta) = B(x_0, \delta) \setminus \{x_0\}$.
  2. $\dot{L}(x) = \nabla L(x) \cdot f(x) \leq 0$ for all $x \in \mathring{B}(x_0, \delta) = B(x_0, \delta) \setminus \{x_0\}$.
- Since

$$\frac{\mathrm{d}}{\mathrm{d}t} L(\phi_t(x)) = \nabla L(\phi_t(x)) \cdot f(\phi_t(x))$$

  the second condition is equivalent to saying that the function L is decreasing along the orbits starting near x_0.
- **Strict** (Lyapunov function): A Lyapunov function for which the decreasing is strict.
- Theorem: For the autonomous system $y' = f(y)$, a fixed point x_0 is
  1. Stable if there is a Lyapunov function near it;

     Proof. Pick a small number $\delta > 0$. Let[6]

$$m := \min\{L(x) : |x - x_0| = \delta\}$$

     Since x_0 does not satisfy $|x - x_0| = \delta > 0$, we know from the first constraint on Lyapunov functions that $L(x) > 0$ for all x satisfying said relation. Thus, $m > 0$. Consequently, any orbit starting from $\{x \mid L(x) < m\} \cap B(x_0, \delta)$ can never meet $\partial B(x_0, \delta)$ since $L(x)$ is decreasing along any orbit (and we would have to go up to get to the boundary). So $L(\phi_t(x)) < m$ for all $x \in \{x \mid L(x) < m\} \cap B(x_0, \delta)$. But this means that $\{x \mid L(x) < m\} \cap B(x_0, \delta)$ is in fact an invariant set. Therefore, x_0 is Lyapunov stable. □

  2. Asymptotically stable if there is a strict Lyapunov function near it.

     Proof. If $x \in \{x \mid L(x) < m\} \cap B(x_0, \delta)$, then $L(\phi_t(x))$ is strictly decreasing. As $t \to +\infty$, $\phi_t(x)$ has a partial limit z_0, say $\phi_{t_k}(x) \to z_0$ (Lemma 6.6 of Teschl (2012)). If $z_0 \neq x_0$, then the orbit $\{\phi_t(z_0) \mid t \in I_{z_0}\}$ is not a single point: Since L is a strict Lyapunov function, we have $L(\phi_t(z_0)) < L(z_0)$ for all $t > 0$. When k is large, $\phi_{t_k}(x)$ is close to z_0, so by continuity,

$$L(\phi_{t+t_k}(x)) = L(\phi_t(\phi_{t_k}(x))) < L(z_0)$$

     But this contradicts $L(\phi_t(x)) > L(z_0)$ (which we must have if there are arbitrarily large t such that $\phi_t(x)$ is close to z_0). Therefore, $x_0 = z_0$. □

[6]Intuitively (in 2D), we take a ring around x_0, find the nonzero value of $L(x)$ at each point on the ring, and take the minimum among them. Imagine a circular valley with hills rising all around the bottommost point; we are essentially looking for the hill that rises the least.

- If all eigenvalues of A have negative real parts, then the perturbed system
$$y' = Ay + g(y)$$
has a strict Lyapunov function around the fixed point $x = 0$.
  - This observation yields another proof of the stability theorem.
- Advantage of the Lyapunov function: Can be constructed globally and thus gives us global information on the system.
- Examples in studying the global behavior of a phase portrait:
  - Consider a mass point moving along the real axis in a potential field $U(x)$. Then
  $$mx'' = -U'(x)$$
    - The total energy
    $$E = \frac{m}{2}|x'|^2 + U(x)$$
    is always a constant along any solution.
    - Introducing the velocity allows us to obtain a planar system
    $$\begin{pmatrix} x \\ v \end{pmatrix}' = \begin{pmatrix} v \\ -U'(x)/m \end{pmatrix}$$
    - Thus, $E(x, v)$ is a global Lyapunov function.
    - Any fixed point of the system must be of the form $(x_0, 0)$, where $U'(x_0) = 0$.
      - Intuitively, this means that the velocity must be zero (that makes sense) and the position must be such that we are at a critical point of the potential.
    - Because of this, the linearization at a fixed point must be of the following form.
    $$\begin{pmatrix} v \\ -U'(x)/m \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -U''(x_0)/m & 0 \end{pmatrix} \begin{pmatrix} x - x_0 \\ v \end{pmatrix} + O(|x - x_0|^2 + |v|^2)$$
    - Thus, $(x_0, 0)$ is Lyapunov stable if U has a nondegenerate local minimum at x_0 and unstable if U has a nondegenerate local maximum at x_0.
      - In the former case, the orbits near $(x_0, 0)$ are closed curves, corresponding to periodic oscillations near x_0 (e.g., harmonic oscillator and ideal pendulum again).
  - Prey-predator model with capacity:
  $$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} (1 - y - \lambda x)x \\ \alpha(x - 1 - \mu y)y \end{pmatrix}$$
  $\alpha, \lambda, \mu > 0$.
    - x is the number of rabbits and y is the number of wolves.
    - Different ranges of λ induce different global behavior (thus, this is an example of **bifurcation**).
    - General observation: $(x, y) = (0, 0)$ is a saddle point since the linearization there is $\text{diag}(1, -\alpha)$.
    - For $x = 0$ or $y = 0$, the equation is of separable form; the positive x, y-axes are invariant sets.
      - Implication: No orbit in the first quadrant can escape it (compatible with meaning as population).
    - Jacobian:
    $$\begin{pmatrix} 1 - y - 2\lambda x & -x \\ \alpha y & \alpha(x - 1) - 2\alpha\mu y \end{pmatrix}$$

- When $\lambda, \mu = 0$, we're back to the Lotka-Volterra system, where there is a single fixed point $(1, 1)$.
  - In that case,
    $$(y - \log y - 1) + \alpha(x - \log x - 1)$$
    is a Lyapunov function.
  - However, it is not a strict Lyapunov function since it is constant along any orbit.
  - Moreover, the function is convex, so all level sets are closed curves around the fixed point.
  - This is, indeed, the behavior we observe in Figure 2.1.
- Other cases: $\lambda \geq 1$.
  - There is only one additional fixed point of interest: $(1/\lambda, 0)$. Note that there are other fixed points, but these do not lie in the first quadrant and thus we are not interested.
  - For $\lambda > 1$, the fixed point is stable (a sink) and when $\lambda = 1$, one eigenvalue is 0 since the linearization at that point is $\text{diag}(-1, \alpha(1/\lambda - 1))$.
- $0 < \lambda < 1$.
  - $(1/\lambda, 0)$ becomes a saddle point, and there is a third fixed point
    $$(x_0, y_0) = \left(\frac{1+\mu}{1+\mu\lambda}, \frac{1-\lambda}{1+\mu\lambda}\right)$$
- More on this case in Chapter 7 of Teschl (2012). This is relevant here!

7.4 Chapter 2: Initial Value Problems

From Teschl (2012).

Section 2.6: Extensibility of Solutions

12/6:

- Investigating the maximal interval on which a solution to an IVP can be defined.
- Not really something we covered in class (certainly not from a theoretical point of view).

Section 2.7: Euler's Method and the Peano Theorem

- Mostly review from class; a few interesting points noted below.
- We can derive the Peano proof technique from Taylor's theorem by approximating
  $$\phi(t_0 + h) = x_0 + \dot{\phi}(t_0)h + o(h) = x_0 + f(t_0, x_0)h + o(h)$$
  eliminating the error term, and rearranging.
- **Euler's method**: A method for approximating the solution to an ODE via
  $$x_h(t_{m+1}) = x_h(t_m) + f(t_m, x_h(t_m))h, \quad t_m = t_0 + mh$$
  using linear interpolation in between.
- **Equicontinuous** (family of functions): A family of functions $\{x_m\}$ such that for every $\varepsilon > 0$, there is a $\delta > 0$ such that if $|t - s| < \delta$ and $m \in \mathbb{N}$, then
  $$|x_m(t) - x_m(s)| \leq \varepsilon$$
  - Note that each function in an equicontinuous family of functions is uniformly continuous.

- Theorem 2.18 (Arzelà-Ascoli): Suppose the sequence of functions $x_m(t) \in C(I, \mathbb{R}^n)$, $m \in \mathbb{N}$, on a compact inverval I is equicontinuous. If the sequence x_m is bounded, then there is a uniformly convergent subsequence.
- Theorem 2.19 (Peano): Suppose f is continuous on $V = [t_0, t_0 + T] \times \overline{B_\delta(x_0)} \subset U$ and denote the maximum of $|f|$ by M. Then there exists at least one solution of the initial value problem

  $$\dot{x} = f(t, x), \quad x(t_0) = x_0$$

  for $t \in [t_0, t_0 + T_0]$ which remains in $\overline{B_\delta(x_0)}$, where $T_0 = \min(T, \delta/M)$. The analogous result holds for the interval $[t_0 - T_0, t_0]$.
- The Euler algorithm is not the most effective one available today.
  - Variations on it usually take more terms in the Taylor expansion, resulting in an algorithm that converges faster but requires more calculations at each step.
  - A good compromise between more terms (but not too many more terms) is the **Runge-Kutta algorithm**.
  - Even better ones appear in the literature on numerical methods for ODEs.
- **Runge-Kutta algorithm**: An algorithm which approximates $\phi(t_0 + h)$ up to the fourth order in h, setting $t_m = t_0 + hm$ and $x_m = x_n(t_m)$ to yield

  $$x_{m+1} = x_m + \frac{h}{6}(k_{1,m} + 2k_{2,m} + 2k_{3,m} + k_{4,m})$$

  where

  $$\begin{aligned} k_{1,m} &= f(t_m, x_m) & k_{2,m} &= f(t_m + \tfrac{h}{2}, x_m + \tfrac{h}{2} \cdot k_{1,m}) \\ k_{3,m} &= f(t_m + \tfrac{h}{2}, x_m + \tfrac{h}{2} \cdot k_{2,m}) & k_{4,m} &= f(t_{m+1}, x_m + hk_{3,m}) \end{aligned}$$

Problems

2.23. Heun's method (or **improved Euler**) is given by

$$x_{m+1} = x_m + \frac{h}{2}(f(t_m, x_m) + f(t_{m+1}, y_m)), \quad y_m = x_m + f(t_m, x_m)h$$

Show that using this method, the error during one step is of $O(h^3)$, provided $f \in C^2$:

$$\phi(t_0 + h) = x_0 + \frac{h}{2}(f(t_0, x_0) + f(t_1, y_0)) + O(h^3)$$

Note that this is not the only possible scheme with this error order since

$$\phi(t_0 + h) = x_0 + \frac{h}{2}(f(t_1, x_0) + f(t_0, y_0)) + O(h^3)$$

as well.

7.5 Chapter 6: Dynamical Systems

From Teschl (2012).

Section 6.1: Dynamical Systems

- Good intuition for what a dynamical system is.
- **Semigroup**: An algebraic structure consisting of a set together with an associative internal binary operation on it.
  - Thus, like a **group**, a semigroup's operation is associative. However, we do not postulate the existence of an identity element or inverses in this case.
- **Dynamical system**: The action of a semigroup G with identity element e[7] on a set M.
  - In particular, a dynamical system is a map $T : G \times M \to M$ which sends $(g, x) \mapsto T_g(x)$ such that
    $$T_g \circ T_h = T_{g \circ h} \qquad\qquad T_e = \mathbb{I}$$
  - Intuition: We often think of a dynamical system very similar to a diffeomorphism, in that as we slide t up and down, a set of points gets distorted according to some field. Here, we're taking the formalization of time *acting on* the points to move them around.
  - This is an incredibly minimal/broad/general definition; the dynamical systems we're interested in usually have far more structure.
- **Invertible dynamical system**: A dynamical system for which G is a group.
- **Discrete dynamical system**: A dynamical system for which $G \in \{\mathbb{N}_0, \mathbb{Z}\}$.
- **Continuous dynamical system**: A dynamical system for which $G \in \{\mathbb{R}^+, \mathbb{R}\}$.
- Example: Iterated map, i.e., f^n.
- Example: The flow of an autonomous differential equation, where $T_t = \Phi_t$ and $G = \mathbb{R}$; we consider this example in the next section.

Section 6.2: The Flow of an Autonomous Equation

- Herein, we consider the system
  $$\dot{x} = f(x), \quad x(0) = x_0$$
- For the remainder of Teschl (2012), we assume $f \in C^k(M, \mathbb{R}^n)$ $(k \geq 1)$.
  - We also assume M is an open subset of $\mathbb{R}^n$.
- Such a system can be regarded as a **vector field** on $\mathbb{R}^n$.
  - Solutions are curves in M which are tangent to the vector field at each point.
- **Integral curve**: A solution to an autonomous IVP. *Also known as* **trajectory**.
  - We say "ϕ is an integral curve at x_0" if $\phi(0) = x_0$.
- By Theorem 2.13: Every point $x \in M$ has an associated (unique) **maximal integral curve**.
- **Maximal integral curve** (at x): The unique integral curve at x, the domain of which is a **maximal interval**. *Denoted by* $\boldsymbol{\phi_x}$.
- **Maximal interval**: The interval for an integral curve at x containing all other possible intervals on which the integral curve can be defined. *Denoted by* $\boldsymbol{I_x}$, $\boldsymbol{(T_-(x), T_+(x))}$.

[7] So a **monoid**?? A monoid is, by definition, an algebraic structure consisting of a set together with an associative internal binary operation and an identity element.

- We define a set which contains information about the maximal interval of the integral curve at x for all x:
$$W = \bigcup_{x \in M} I_x \times \{x\} \subset \mathbb{R} \times M$$

- **Flow** (of a differential equation): The map from W to M which pairs every starting point x and time t to the point to which the differential equation will have moved x after time t has elapsed. *Denoted by* $\boldsymbol{\Phi}$. *Given by*
$$(t, x) \mapsto \phi(t, x)$$
where $\phi(t, x)$ is the maximal integral curve at x.

- Notation: We sometimes write
$$\Phi(t, x) = \Phi_x(t) = \Phi_t(x)$$

- If $\phi(\cdot)$ is the maximal integral curve at x, then $\phi(\cdot + s)$ is the maximal integral curve at $y = \phi(x)$ and $I_x = s + I_y$. It follows that for all $x \in M$ and $s \in I_x$, we have
$$\Phi(s + t, x) = \Phi(t, \Phi(s, x))$$
for all $t \in I_{\Phi(s,x)} = I_x - s$.

- We now state formally the ideas we've just developed informally.

- Theorem 6.1: Suppose $f \in C^k(M, \mathbb{R}^n)$. For all $x \in M$, there exists an interval $I_x \subset \mathbb{R}$ containing 0 and a corresponding unique maximal integral curve $\Phi(\cdot, x) \in C^k(I_x, M)$ at x. Moreover, the set W defined as above is open and $\Phi \in C^k(W, M)$ is a (local) flow on M, that is,
$$\Phi(0, x) = x$$
$$\Phi(t + s, x) = \Phi(t, \Phi(s, x)), \quad x \in M, \quad x, t + s \in I_x$$

  Proof. Given. □

- Example: Let $M = \mathbb{R}$ and $f(x) = x^3$. Then $W = \{(t, x) \mid 2tx^2 < 1\}$[8] and
$$\Phi(t, x) = \frac{x}{\sqrt{1 - 2x^2 t}}$$
We have $T_-(x) = -\infty$ and $T_+(x) = 1/(2x^2)$.

- **Fixed point**: A point at which f evaluates to 0. *Denoted by* $\boldsymbol{x_0}$.

- Lemma 6.2 (Straightening out vector fields): Suppose $f(x_0) \neq 0$. Then there is a local coordinate transform $y = \varphi(x)$ such that $\dot{x} = f(x)$ is transformed to
$$\dot{y} = (1, 0, \dots, 0)$$

Section 6.3: Orbits and Invariant Sets

- Some of the definitions herein come up in class, some do not, but many are interesting and IMO grant a deeper understanding of dynamical systems.

- **Orbit** (of x): The image under the flow of the maximal interval of the maximal integral curve at x. *Denoted by* $\boldsymbol{\gamma(x)}$. *Given by*
$$\gamma(x) = \Phi(I_x \times \{x\})$$

- $y \in \gamma(x)$ implies $y = \Phi(t, x)$ for some t, and hence (by Theorem 6.1) $\gamma(x) = \gamma(y)$.

[8]This condition is equivalent to all (t, x) such that $1 - 2x^2 t > 0$, i.e., that the denominator of the flow is positive.

- Implication: Distinct orbits are disjoint.
  - Formalism: The orbits partition M, i.e., we have an equivalence relation on M defined by $x \sim y$ iff $\gamma(x) = \gamma(y)$.
- **Fixed point** (of Φ): A point $x \in M$ for which $\gamma(x) = \{x\}$. *Also known as* **singular point**, **stationary point**, **equilibrium point**.
- **Regular point** (of Φ): A point $x \in M$ that is not a fixed point of Φ.
- If x is a regular point, then $\Phi(\cdot, x) : I_x \hookrightarrow M$[9].
- **Forward** (orbit of x): The image under the flow of the *positive portion* of the maximal interval of the maximal integral curve at x. *Denoted by* $\boldsymbol{\gamma_+(x)}$. *Given by*

$$\gamma_+(x) = \Phi((0, T_+(x)) \times \{x\})$$

- **Backward** (orbit of x): The image under the flow of the *negative portion* of the maximal interval of the maximal integral curve at x. *Denoted by* $\boldsymbol{\gamma_-(x)}$. *Given by*

$$\gamma_-(x) = \Phi((T_-(x), 0) \times \{x\})$$

- Relating the orbit, forward orbit, and backward orbit:

$$\gamma(x) = \gamma_-(x) \cup \{x\} \cup \gamma_+(x)$$

- **Periodic point** (of Φ): A point $x \in M$ for which there exists $T > 0$ such that $\Phi(T, x) = x$.
- **Period** (of a periodic point x): The lower bound on the set of T for which $\Phi(T, x) = x$. *Denoted by* $\boldsymbol{T(x)}$. *Given by*

$$T(x) = \inf\{T > 0 \mid \Phi(T, x) = x\}$$

- The continuity of Φ guarantees that

$$\Phi(T(x), x) = x$$

  for $T(x)$ as defined.
- By the flow property (Theorem 6.1), we have

$$\Phi(t, T(x), x) = \Phi(t, x)$$

- **Periodic orbit**: An orbit for which one point (hence all points) of the orbit is/are periodic. *Also known as* **closed orbit**.
  - Reason for the moniker "closed orbit:" x is periodic iff $\gamma_+(x) \cap \gamma_-(x) \neq \emptyset$, i.e., if the forward orbit joins the negative orbit and "closes" the loop.
- Classification of the orbits of f:
  1. Fixed orbits (corresponding to a periodic point with period zero).
  2. Regular periodic orbits (corresponding to a periodic point with positive period).
  3. Non-closed orbits (not corresponding to a periodic point).
- **Positive lifetime** (of x): The positive ending limit point of the maximal interval of x. *Denoted by* $\boldsymbol{T_+(x)}$. *Given by*

$$T_+(x) = \sup I_x$$

[9]Notation: $\hookrightarrow$ indicates an injective function.

- **Negative lifetime** (of x): The negative ending limit point of the maximal interval of x. *Denoted by* $\boldsymbol{T_-(x)}$. *Given by*
$$T_-(x) = \inf I_x$$
- $\boldsymbol{\sigma}$ **complete** (point): A point $x \in M$ for which $T_\sigma(x) = \sigma\infty$, where $\sigma \in \{\pm\}$.
- **Complete** (point): A point $x \in M$ that is both $+$ and $-$ complete.
- Lemma 6.3: Let $x \in M$ and suppose that the forward (resp. backward) orbit lies in a compact subset C of M. Then x is $+$ (resp. $-$) complete.
- Periodic points are complete.
- **Complete** (vector field): A vector field in which all points are complete.
- f complete implies Φ is globally defined, that is, $W = \mathbb{R} \times M$.
- $\boldsymbol{\sigma}$ **invariant**: A set $U \subset M$ such that $\gamma_\sigma(x) \subset U$ for all $x \in U$, where $\sigma \in \{\pm\}$.
- $C \subset M$ a compact σ invariant set implies (by Lemma 6.3) that all points in C are σ complete.
- Lemma 6.4:
  1. Arbitrary intersections and unions of σ invariant sets are σ invariant. Moreover, the closure of a σ invariant set is again σ invariant.
  2. If U, V are invariant, so is the complement $U \setminus V$.

Proof. Given. □

- Goal: Describe the long-term asymptotics of solutions.
  - Tool: We introduce the set where an orbit eventually accumulates.
- $\boldsymbol{\omega_\pm}$**-limit set** (of x): The set of all points $y \in M$ for which there exists a sequence $\{t_n\}$ that converges to $\pm\infty$ and satisfies $\Phi(t_n, x) \to y$. *Denoted by* $\boldsymbol{\omega_\pm(x)}$.
- By definition, $\omega_\pm(x)$ is empty unless x is $\pm$ complete.
- $y \in \gamma(x)$ implies $\omega_\pm(x) = \omega_\pm(y)$.
  - This is because the hypothesis shows that $y = \Phi(t, x)$ for some t, so
$$\Phi(t_n, x) = \Phi(t_n - t, \Phi(t, x)) = \Phi(t_n - t, y)$$
  - Hence, $\omega_\pm(x)$ depends only on the orbit $\gamma(x)$.
- Lemma 6.5: The set $\omega_\pm(x)$ is a closed invariant set.

Proof. Given. □

- Example: For $\dot{x} = -x$, $\omega_+(x) = \{0\}$ for all $x \in \mathbb{R}$ since every solution converges to 0 as $t \to +\infty$. Moreover, $\omega_-(x) = \emptyset$ for $x \neq 0$ and $\omega_-(0) = \{0\}$.
- Conclusion: Even for x complete, the set $\omega_\pm(x)$ can be empty.
- Lemma 6.6: If $\gamma_\sigma(x)$ is contained in a compact set C, then $\omega_\sigma(x)$ is nonempty, compact, and connected.

Proof. Given. □

- Lemma 6.7: Suppose $\gamma_\sigma(x)$ is contained in a compact set. Then we have

$$\lim_{t\to\sigma\infty} d(\phi(t,x),\omega_\sigma(x)) = 0$$

Proof. Given. □

- Teschl (2012) works through an example that proves that the compactness requirement is necessary.
- **Minimal** (set): A nonempty, compact, σ invariant set that contains no proper σ invariant subset possessing these three properties.
- Examples:
  - The $\omega_\pm$-limit sets are minimal for all $x \in \omega_\pm(x)$.
  - A periodic orbit.
    - In 2D, this is the only example by Corollary 7.12.
    - In three or more dimensions, orbits can be dense on a compact hypersurface, meaning that the hypersurface cnnot be split into smaller *closed* invariant sets.
- Lemma 6.8: Every nonempty, compact σ invariant set $C \subset M$ contains a minimal σ invariant set.

  If in addition C is homeomorphic to a closed m-dimensional disk (where m is not necessarily the dimension of M), it contains a fixed point.

Proof. Given. □

Section 6.4: The Poincaré Map

- Never covered in class.

Section 6.5: Stability of Fixed Points

- Herein, we continue investigating the long-term behavior of the dynamical system

$$\dot{x} = f(x), \quad x(0) = x_0$$

  - In particular, we investigate whether or not a solution is **stable**.
- **Stable** (fixed point): A fixed point x_0 of $f(x)$ such that for any given neighborhood $U(x_0)$, there exists another neighborhood $V(x_0) \subset U(x_0)$ such that any solution starting in $V(x_0)$ remains in $U(x_0)$ for all $t \geq 0$. *Also known as* **Lyapunov stable**[10].
  - If a solution remains in $U(x_0)$ for all $t \geq 0$, it remains in the compact set $\overline{U(x_0)}$ for all $t \geq 0$.
  - Thus, by Lemma 6.3, said solution exists for all positive times.
- **Unstable** (fixed point): A fixed point which is not stable.
- **Asymptotically stable** (fixed point): A fixed point x_0 of $f(x)$ that is stable and for which there exists a neighborhood $U(x_0)$ such that

$$\lim_{t\to\infty} |\phi(t,x) - x_0| = 0$$

  for all $x \in U(x_0)$.

[10] Teschl (2012) uses the alternate spelling "Liapunov;" I will continue using "Lyapunov" without further comment.

- **Exponentially stable** (fixed point): A fixed point x_0 of $f(x)$ for which there exist constants $\alpha, \delta, C > 0$ such that
$$|\phi(t,x) - x_0| \leq C\mathrm{e}^{-\alpha t}|x - x_0|$$
when $|x - x_0| \leq \delta$.
- Exponential stability implies both stability and asymptotic stability.
- Example: Consider $\dot{x} = ax$ in $\mathbb{R}^1$. Then $x_0 = 0$ is stable iff $a \leq 0$ and exponentially stable iff $a < 0$.
- These definitions of stability agree with those we introduced for linear autonomous systems in Section 3.2.
- Teschl (2012) goes over an alternate stability criterion adapted from Section 1.5.
- If $f'(x_0) \neq 0$, the stability of x_0 can be read off from the derivative of f at x_0 alone.
  - More generally, a fixed point is exponentially stable if this is true for the corresponding linearized system (the proof is not directly presented in Teschl (2012) but is rather spread out, making it not of much use to me rn).
- Theorem 6.10 (Exponential stability via linearization): Suppose $f \in C^1$ has a fixed point x_0 and suppose that all eigenvalues of the Jacobian matrix at x_0 have negative real part. Then x_0 is exponentially stable.
- **Bifurcation theory**: The systematic study of small changes in an ODEs parameters that induce large changes in qualitative behavior.
  - Theorem 2.11 asserts that provided f depends smoothly on μ, so does the flow. Nevertheless, very small changes in parameters can induce large changes in the qualitative behavior.
  - A few examples follow.
- **Pitchfork bifurcation**: A stable fixed point for $\mu \leq 0$ which becomes unstable and splits off two stable fixed points at $\mu = 0$. *picture*
  - Example: $\dot{x} = \mu x - x^3$.
- **Transcritical bifurcation**: Two stable fixed points for $\mu \neq 0$ which collide and exchange stability at $\mu = 0$. *picture*
  - Example: $\dot{x} = \mu x - x^2$.
- **Saddle-node bifurcation**: One stable and one unstable fixed point for $\mu < 0$ which collide at $\mu = 0$ and vanish. *picture*
  - Example: $\dot{x} = \mu + x^2$.
- Rest of the chapter: Good criteria for the stability of $\dot{x} = f(x)$ (since it cannot be solved explicitly in general).

Section 6.6: Stability via Lyapunov's Method

- For a fixed point x_0 of f and an open neighborhood $U(x_0)$ of x_0, we may define the following.
- **Lyapunov function**: A continuous function $L : U(x_0) \to \mathbb{R}$ which is zero at x_0, positive for $x \neq x_0$, and satisfies
$$L(\phi(t_0)) \geq L(\phi(t_1))$$
where $t_0 < t_1$ and $\phi(t_j) \in U(x_0) \setminus \{x_0\}$ $(j = 0, 1)$ for any solution $\phi(t)$.
- **Strict Lyapunov function**: A Lyapunov function for which the central inequality in the above definition is strict.

- Claim: If L is strict, $U(x_0) \setminus \{x_0\}$ cannot contain any periodic orbits.

  Proof. Suppose for the sake of contradiction that $\gamma(x) \subset U(x_0) \setminus \{x_0\}$. Since $\gamma(x)$ is a periodic orbit, $\phi(0,x) = \phi(T(x),x)$ where $T(x) > 0$ by definition. Letting $t_0 = 0$ and $t_1 = T(x)$, we have by the definition of a strict Lyapunov function that

  $$L(\phi(0,x)) > L(\phi(T(x),x)) = L(\phi(0,x))$$

  contradicting the fact that L is well-defined. □

- $\boldsymbol{S_\delta}$: The following set. *Given by*

  $$S_\delta = \{x \in U(x_0) \mid L(x) \leq \delta\}$$

  - S_δ contains x_0.
  - In general, S_δ need not be closed since it can share boundary with $U(x_0)$. In such a case, orbits can escape through this part of the boundary.
  - Restricting S_δ to closed versions, though, we get the following lemma.
- Lemma 6.11: If S_δ is closed, then it is positively invariant.
- Lemma 6.12: For every $\delta > 0$, there is an $\varepsilon > 0$ such that $S_\varepsilon \subset B_\delta(x_0)$ and $B_\varepsilon(x_0) \subset S_\delta$.
- Implication: Given any neighborhood $V(x_0)$, we can find an ε such that $S_\varepsilon \subset V(x_0)$ is positively invariant. But this just means that x_0 is stable, and we have proven the following[11].
- Theorem 6.13 (Lyapunov): Suppose x_0 is a fixed point of f. If there is a Lyapunov function L, then x_0 is stable.
- Theorem 6.14 (Krasovskii-LaSalle principle): Suppose x_0 is a fixed point of f. If there is a Lyapunov function L which is not constant on any orbit lying entirely in $U(x_0) \setminus \{x_0\}$, then x_0 is asymptotically stable. This is for example the case if L is a strict Lyapunov function. Moreover, every orbit lying entirely in $U(x_0)$ converges to x_0.
- Theorem 6.15: Let $L : U \to \mathbb{R}$ be continuous and bounded from below. If for some x we have $\gamma_+(x) \subset U$ and

  $$L(\phi(t_0,x)) \geq L(\phi(t_1,x))$$

  for $t_0 < t_1$, then L is constant on $\omega_+(x) \cap U$.
- Most Lypunov functions are differentiable.
- If L is differentiable, then $L(\phi(t_0)) \geq L(\phi(t_1))$ for all $t_0 < t_1$ iff

  $$\frac{\mathrm{d}}{\mathrm{d}t} L(\phi(t,x)) = \nabla(L)(\phi(t,x))\dot{\phi}(t,x) = \nabla(L)(\phi(t,x))f(\phi(t,x)) \leq 0$$

- **Lie derivative** (of L along f): The following expression. *Given by*

  $$\nabla(L)(x) \cdot f(x)$$

- **Constant of motion**: A function for which the Lie derivative vanishes and, hence, is constant on every orbit.
- Theorem 6.15 implies that all $\omega_\pm$-limit sets are contained in the set where the Lie derivative of L vanishes.

[11]Clever pedagogical tool: Teschl (2012) weaves any really important proofs into the flow of the text so that you can't gloss over it.

- Example: Consider the system

$$\dot{x} = y \qquad\qquad \dot{y} = -x$$

  with function

$$L(x, y) = x^2 + y^2$$

  - For $x \in \mathbb{R}^2$ arbitrary, the Lie derivative is

$$\nabla(L)(x) \cdot f(x) = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \cdot \begin{pmatrix} y \\ -x \end{pmatrix} = 2xy - 2xy = 0$$

  - Thus, L is a Lyapunov function.
  - In particular, L is a constant of motion.
  - Thus, by Theorem 6.13, the origin is stable.
  - Every level set $L(x, y) = \delta$ corresponds to an orbit, so the system is not asymptotically stable.
- Takeaway:
  - Extract properties of Lyapunov functions from the fact that they are monotonically decreasing on all orbits.
  - Prove that a function is a Lyapunov function using the Lie derivative.

Section 6.7: Newton's Equation in One Dimension

- Goal: Illustrate the results of Chapter 6 with a specific example.
- Recall that the motion of a particle moving in one dimension under the external force field $f(x)$ is described by Newton's equation

$$\ddot{x} = f(x)$$

- **Phase space**: The set $\mathbb{R}^2$, as referred to by physicists.
- **Phase point**: A point of the form $(x, \dot{x})$ in the phase space.
- **Phase curve**: A solution to the ODE.
- By the Picard-Lindelöf theorem (Theorem 2.2), precisely one phase curve passes through each phase point.
- **Kinetic energy**: The following quadratic form. *Denoted by* $\boldsymbol{T(\dot{x})}$. *Given by*

$$T(\dot{x}) = \frac{\dot{x}^2}{2}$$

- **Potential energy**: The following function. *Denoted by* $\boldsymbol{U(x)}$. *Given by*

$$U(x) = -\int_{x_0}^{x} f(\xi)\mathrm{d}\xi$$

  - Only determined up to an arbitrary constant.
- **Energy** (of a Newtonian system): The sum of the kinetic and potential energies. *Denoted by* $\boldsymbol{E}$. *Given by*

$$E = T(\dot{x}) + U(x)$$

 - E is constant along a phase curve. Indeed, if $x(t)$ satisfies $\ddot{x} = f(x) = U'(x)$, i.e., $\ddot{x} - f(x) = 0$, then
$$\frac{\mathrm{d}E}{\mathrm{d}t} = \dot{x}\ddot{x} + U'(x)\dot{x} = \dot{x}(\ddot{x} - f(x)) = 0$$
as desired.

- The solution corresponding to the initial conditions $x(0) = x_0$, $\dot{x}(0) = x_1$ can be given implicitly.
  - First off, we have that
$$\begin{aligned}
E &= T(\dot{x}) + U(x)\\
&= \frac{\dot{x}^2}{2} + U(x)\\
\sqrt{2(E - U(x))} &= \frac{\mathrm{d}x}{\mathrm{d}t}\\
\int_0^t \mathrm{d}\tau &= \int_{x_0}^{x} \frac{\mathrm{d}\xi}{\sqrt{2(E - U(\xi))}}\\
t &= \operatorname{sign}(x_1)\int_{x_0}^{x} \frac{\mathrm{d}\xi}{\sqrt{2(E - U(\xi))}}
\end{aligned}$$
    - Why do we input $\operatorname{sign}(x_1)$ between the next-to-last and last lines??
  - Second, since E is constant along the solution, its value at the start will be the same as its value at any other point. Thus, we can use the starting initial conditions to calculate it, as follows.
$$E = \frac{{x_1}^2}{2} + U(x_0)$$
  - If $x_1 = 0$, then $\operatorname{sign}(x_1)$ must be replaced with $-\operatorname{sign}(U'(x_0))$.
- Theorem 6.16: Newton's equation has a fixed point if and only if $\dot{x} = 0$ and $U'(x) = 0$ at this point. Moreover, a fixed point is stable if $U(x)$ has a local minimum there.
  - If $U(x)$ has a local minimum at x_0, the energy $E - U(x_0)$ can be used as a Lyapunov function; we subtract $U(x_0)$ so that $E - U(x_0)$ evaluates to zero at x_0.
- A fixed point cannot be asymptotically stable (due to conservation of energy).
- Example: Mathematical pendulum.
$$\ddot{x} = -\sin(x)$$
  - x describes the displacement angle from the position at rest ($x = 0$).
  - x should be understood modulo 2π.
  - We have that
$$U(x) = -\int_{x_0}^{x} (-\sin(\xi))\mathrm{d}\xi = \cos(x_0) - \cos(x)$$
    - Since the constant is arbitrary, we may take $U(x) = 1 - \cos(x)$ for ease. This has the additional advantage that the energy is never negative.
  - We now begin the rigorous investigation.
  - We restrict our attention to the interval $x \in (-\pi, \pi]$. Thus, the fixed points are $x = 0, \pi$.
  - By Theorem 6.16 and the fact that $U(0)$ is a minimum, 0 is a stable fixed point.
  - As before in the general case, $E(x, \dot{x}) = \text{constant}$ gives invariant level sets.
    - $E = 0$: The corresponding level set is the equilibrium position $(x, \dot{x}) = (0, 0)$.
    - $0 < E < 2$: The level sets are homeomorphic to circles. Since these circles contains no fixed points, they are regular periodic orbits.

- $E = 2$: The level set consists of the fixed point π and two non-closed orbits connecting $-\pi$ and π. This is a **separatrix**.
- $E > 2$: The level sets are again closed orbits (due to our modulo 2π perspective).

– In a neighborhood of the equilibrium position $x = 0$, the system is approximated by the linearization $\sin(x) = x + O(x^2)$ given by

$$\ddot{x} = -x$$

and referred to as the **harmonic oscillator**.

- Here, we have

$$E = \frac{\dot{x}^2}{2} + \frac{x^2}{2}$$

so the phase portrait consists of circle centered at 0.

– More generally, if $U'(x_0) = 0$ and $U''(x_0) = \omega^2/2 > 0$, then we should approximate our system with

$$\ddot{y} = -\omega^2 y, \quad y(t) = x(t) - x_0$$

– Lastly, if we use the momentum $p = \dot{x}$ (units chosen such that $m = 1$) and the location $q = x$ as coordinates, then the energy

$$H(p, q) = \frac{p^2}{2} + U(q)$$

is called the **Hamiltonian**.

- In this case, the equations of motion are

$$\dot{q} = \frac{\partial H}{\partial p} \qquad\qquad \dot{p} = -\frac{\partial H}{\partial q}$$

- This formalism is called **Hamiltonian mechanics**.
- It is useful for systems with more than one degree of freedom.
- See Section 8.3 for more.

7.6 Chapter 7: Planar Dynamical Systems

From Teschl (2012).

Section 7.1: Examples from Ecology

- Teschl (2012) derives via ecological reasoning the **Lotka-Volterra** predator-prey equations.
- **Lotka-Volterra predator-prey equations**: The following system of differential equations. *Given by*

$$\dot{x} = (1 - y)x \qquad\qquad \dot{y} = \alpha(x - 1)y$$

for $\alpha > 0$.

- Two fixed points.
  – $(0, 0)$ gives rise to invariant subspaces along the x- and y-axes. Indeed,

$$\Phi(t, (0, y)) = (0, y\mathrm{e}^{-\alpha t}) \qquad\qquad \Phi(t, (x, 0)) = (x\mathrm{e}^t, 0)$$

  - Since no other solution can cross these lines, the first quadrant $Q = \{(x, y) \mid x, y > 0\}$ is invariant. This is the region we are interested in.

  – $(1, 1)$ is the other fixed point.

  - Let's eliminate t from the ODEs to get a single first-order equation for the orbits.
  - Writing $y = y(x)$, we infer from the chain rule that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^{-1} = \alpha\frac{(x-1)y}{(1-y)x}$$

  - This equation is separable. Solving it yields

$$L(x,y) = f(y) + \alpha f(x) = \text{constant}$$

    where

$$f(a) = a - 1 - \log(a)$$

    - ➢ Note that log denotes the natural logarithm.
  - f cannot be inverted in terms of elementary functions. However, f is convex with global minimum at $x = 1$, and $f \to \infty$ as $a \to 0, +\infty$. It follows that the level sets are portions of this curve near the bottom of the well in both dimensions, and thus they are compact.
  - The exchange of energy from one to the other and back again also indicates that each orbit is periodic surrounding the fixed point $(1,1)$.

- Theorem 7.1: All orbits of the Lotka-Volterra equations in Q are closed and encircle the only fixed point $(1,1)$.
- Modification: Let's assume each species' population can only grow so fast. Then we get

$$\dot{x} = (1 - y - \lambda x)x \qquad\qquad \dot{y} = \alpha(x - 1 - \mu y)y$$

  for $\alpha, \lambda, \mu > 0$.
- We now have four fixed points:

$$(0,0) \qquad (\lambda^{-1}, 0) \qquad (0, -\mu^{-1}) \qquad \left(\frac{1+\mu}{1+\mu\lambda}, \frac{1-\lambda}{1+\mu\lambda}\right)$$

  - The third lies outside of $\bar{Q}$, so we disregard it.
  - The fourth lies outside of $\bar{Q}$ if $\lambda > 1$. Thus, let's start with the case $\lambda \geq 1$ so that we only have to deal with one new fixed point.
  - $\lambda \geq 1$. *picture*
    - Our new fixed point is $(\lambda^{-1}, 0)$.
    - It is a hyperbolic sink if $\lambda > 1$.
    - If $\lambda = 1$, one eigenvalue is 0 and we need a more thorough investigation.
    - Idea: Split Q into regions where $\dot{x}, \dot{y}$ have definite signs and then use the elementary observation in Lemma 7.2.
    - The regions where $\dot{x}, \dot{y}$ have definite signs are separated by the two lines

$$L_1 = \{(x,y) \mid y = 1 - \lambda x\} \qquad\qquad L_2 = \{(x,y) \mid \mu y = x - 1\}$$

      - ➢ We derive these by setting $1 - y - \lambda x = 0$ and $x - 1 - \mu y = 0$.
    - Label the regions in Q enclosed by these lines from left to right by Q_1, Q_2, Q_3.
    - Observe that the lines are transversal, i.e., can only be crossed in the direction from $Q_3 \to Q_2$ and $Q_2 \to Q_1$. This can be seen from the solution curves in the picture.
    - Suppose we start at $(x_0, y_0) \in Q_3$.
      - ➢ Additional constraint: $x \leq x_0$ (the flow is to the left??).
      - ➢ By Lemma 7.2: Either the trajectory enters L_2 or it converges to a fixed point in $\bar{Q}_3$. The latter case can only happen if $(\lambda^{-1}, 0) \in \bar{Q}_3$, i.e., if $\lambda = 1$.

        - Similarly, starting in Q_2 either gets you across L_1 or to $(\lambda^{-1}, 0)$.
        - Starting in Q_1 must take you to the fixed point.
        - Thus, every trajectory converges to the fixed point.
    - Let $0 < \lambda < 1$.
        - We apply the same strategy as before.
        - We have four regions this time. Let Q_4 be the new (bottom) one. We can only pass through these in the order $Q_4 \to Q_3 \to Q_2 \to Q_1 \to Q_4$.
        - Thus, we have to rule out the periodic case this time.
        - For simplicity's sake, let
$$(x_0, y_0) = \left(\frac{1+\mu}{1+\mu\lambda}, \frac{1-\lambda}{1+\mu\lambda} \right)$$
        - To do so, introduce (inspired by the original) the Lyapunov function
$$L(x, y) = \gamma_1 f(\tfrac{y}{y_0}) + \alpha\gamma_2 f(\tfrac{x}{x_0})$$
          where, as before, $f(a) = a - 1 - \log(a)$.
        - We seek constraints on γ_1, γ_2 that will make L strict.
        - Calculate
$$\dot{L} = \frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial y}\dot{y} = -\alpha \left(\frac{\lambda\gamma_2}{x_0}\bar{x}^2 + \frac{\mu\gamma_1}{y_0}\bar{y}^2 + \left(\frac{\gamma_2}{x_0} - \frac{\gamma_1}{y_0} \right) \bar{x}\bar{y} \right)$$
          where
$$\dot{x} = (-\bar{y} - \lambda\bar{x})x \qquad \dot{y} = \alpha(\bar{x} - \mu\bar{y})y \qquad \bar{x} = x - x_0 \qquad \bar{y} = y - y_0$$
        - The the RHS will be negative if we choose $\gamma_1 = y_0$ and $\gamma_2 = x_0$, so choose this, and then L is strictly decreasing, so all orbits starting in Q converge to the fixed point (x_0, y_0).

- Lemma 7.2: Let $\phi(t) = (x(t), y(t))$ be the solution of a planar system. Suppose U is open and $\bar{U}$ is compact. If $x(t), y(t)$ are strictly monotone in U, then either ϕ hits the boundary at some finite time $t = t_0$ or $\phi(t)$ converges to a fixed point $(x_0, y_0) \in \bar{U}$.

- Therefore, after all of that, we have proven the following.

- Theorem 7.3: Suppose $\gamma \geq 1$. Then there is no fixed point of
$$\dot{x} = (1 - y - \lambda x)x \qquad \dot{y} = \alpha(x - 1 - \mu y)y$$
  in Q and all trajectories in Q converge to the point $(\lambda^{-1}, 0)$.

  If $0 < \lambda < 1$, then there is only one fixed point $(\frac{1+\mu}{1+\mu\lambda}, \frac{1-\lambda}{1+\mu\lambda})$ in Q. It is asymptotically stable and all trajectories in Q converge to this point.

- Ecological interpretation: Predators can only survive if their growth rate is positive at the limiting population λ^{-1} of the prey species.

- Teschl (2012) discusses cooperative and competing species.