Week 2

Solving Simple ODEs

2.1 Separable ODEs

10/3:

- Do not sit on the left side of the classroom: The sun sucks!
- **Separable** (ODE): An ODE of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t)g(y)$$

  where y is a real[1], unknown, scalar function of t.
- Solving separable ODEs: Formally, evaluate

$$\int \frac{\mathrm{d}y}{g(y)} = \int f(t)\,\mathrm{d}t$$

  - But how do we know that we can move the Liebniz notation around so conveniently?
- Proving the validity of the above formula.
  - Rearrange the initial separable ODE to $\mathrm{d}y/\mathrm{d}t \cdot 1/g = f$ and invoke the law of composite differentiation to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{y_0}^{y(t)} \frac{\mathrm{d}w}{g(w)} - \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau\right] = 0$$

  - It follows that

$$\int_{y_0}^{y(t)} \frac{\mathrm{d}w}{g(w)} = \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau$$

  - Ask about this in OH??
- Examples:
  1. Exponential growth.
     - We have that

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky$$

       for $k > 0$ and $y(0) = y_0 > 0$.

[1] We may come back to complex functions later.

- The solution is

$$\begin{aligned}\frac{1}{y}\cdot\frac{\mathrm{d}y}{\mathrm{d}t} &= k\\ \log y(t)-\log y_0 &= kt\\ y(t) &= y_0\mathrm{e}^{kt}\end{aligned}$$

2. Logistic growth.
   - We have that
   $$\frac{\mathrm{d}y}{\mathrm{d}t} = ky\left(1-\frac{y}{M}\right)$$
   for $k, M > 0$ and $y(0) = y_0 > 0$.
   - The solution is
   $$\begin{aligned}\frac{M\,\mathrm{d}y}{y(M-y)} &= k\,\mathrm{d}t\\ \log\frac{y}{M-y}-\log\frac{y_0}{M-y_0} &= kt\\ \frac{y(M-y_0)}{y_0(M-y)} &= \mathrm{e}^{kt}\\ y\cdot\frac{M-y_0}{y_0} &= (M-y)\mathrm{e}^{kt}\\ y\cdot\frac{M-y_0}{y_0}+y\mathrm{e}^{kt} &= M\mathrm{e}^{kt}\\ y\left(\frac{M-y_0}{y_0}+\mathrm{e}^{kt}\right) &= M\mathrm{e}^{kt}\\ y\left(\frac{M-y_0+y_0\mathrm{e}^{kt}}{y_0}\right) &= M\mathrm{e}^{kt}\\ y\left(\frac{M+y_0(\mathrm{e}^{kt}-1)}{y_0}\right) &= M\mathrm{e}^{kt}\\ y(t) &= \frac{My_0\mathrm{e}^{kt}}{M+y_0(\mathrm{e}^{kt}-1)}\end{aligned}$$
   - Sketches the graph of logistic growth and discusses the turning point (for which there is a formula; zero of the second derivative) as well as general trends.
   - If $y_0 < 0$, the solution is not physically meaningful, but it is mathematically insightful.
     - When we integrate, the arguments of our logarithms now have absolute values.
     $$\log\left|\frac{y}{M-y}\right|-\log\left|\frac{y_0}{M-y_0}\right| = kt$$
     - We need to make sure that the denominator of the final logistic form is never equal to zero, but now that y_0 is negative, as t increases, the denominator will approach zero exponentially. It reaches zero when
     $$\begin{aligned}M+y_0(\mathrm{e}^{kt}-1) &= 0\\ \mathrm{e}^{kt} &= -\frac{M}{y_0}+1\end{aligned}$$
     In other words, $t_{\max} = (1/k)\log(1-M/y_0)$; when $t = t_{\max}$, the equation blows up.
     - This is an example of **finite lifespan**.
   - If $y_0 > M$, then you will exponentially (really??) decrease to M.

3. Lotka-Volterra predator-prey model.

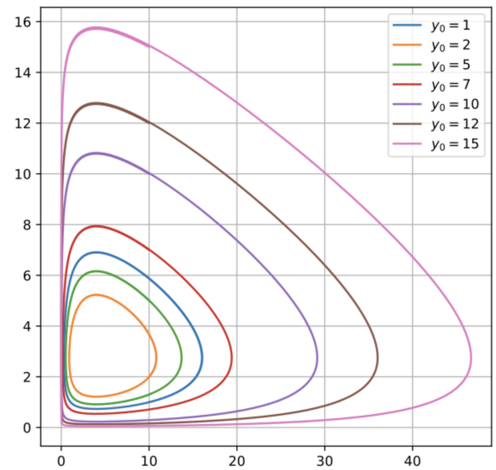

Figure 2.1: Lotka-Volterra solution curves.

- We have that

$$r' = k_1 r - awr \qquad\qquad w' = -k_2 w + bwr$$

where r is rabbits and w is wolves.

- We can rename the variables to

$$\begin{cases} x' = Ax - Bxy \\ y' = -Cy + Dxy \end{cases}$$

- Dividing, we get

$$\begin{aligned} \frac{x'}{y'} &= \frac{Ax - Bxy}{-Cy + Dxy} \\ \frac{Dx - C}{x}x' + \frac{By - A}{y}y' &= 0 \end{aligned}$$

- We seek a constant function $F : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{\partial F}{\partial x} = \frac{Dx - C}{x} \qquad\qquad \frac{\partial F}{\partial y} = \frac{By - A}{y}$$

where we are imagining x, y to be both parameterized by t and arguments of F.

  - Rationale: As a *constant* function, F satisfies

$$\begin{aligned} 0 &= \frac{\mathrm{d}F}{\mathrm{d}t} \\ &= \frac{\partial F}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} \end{aligned}$$

  - Comparing terms with the modified L-V equation yields the above PDEs.

- We can solve the PDEs separately in this case to determine that F is given by the following.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(Dx(t) - C\log x(t)) + \frac{\mathrm{d}}{\mathrm{d}t}(By(t) - A\log y(t)) &= 0 \\ Dx(t) - C\log x(t) + By(t) - A\log y(t) &= E \end{aligned}$$

        - Much more on these machinations later; they are very important, though.
    - Alternative perspective on the above machinations:
        - Assuming $y' \neq 0$, the original x'/y' expression is equivalent to a separable differential equation in $\mathrm{d}x/\mathrm{d}y$. Think
$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{\mathrm{d}x/\mathrm{d}t}{\mathrm{d}y/\mathrm{d}t}$$
        - ...
    - Shao sketches some of the trajectories (they're all closed curves in the xy-plane). See Figure 2.1
    - Properties of the curves:
        - The implicit relation which determines them: By the implicit function theorem, the y-derivative of the LHS is $B - A/y$ and the x-derivative of the LHS is $D - C/x$. Where the partial derivatives are equal to zero — $(C/D, A/B)$ — is interesting; it is a fixed point. Turning points happen when the y-coordinate is A/B or the x-coordinate is C/D; note that the implicit function theorem fails here.

- **Finite lifespan**: Even if the RHS of $\mathrm{d}y/\mathrm{d}t = f(t, y)$ is very regular, the solution can still blow up at some finite time.

- **Lifespan** (of a solution): How long the solution stays regular from the time that the evolution starts.

- **Interval of existence** (of a solution): The set of times t, starting from t_0, for which the solution stays regular.

- Lifespan vs. interval of existence: Essentially, if start time for a solution to an ODE is t_0 and the time at and beyond which the solution is no longer regular is $t_1 \leq \infty$, then the lifespan is $t_1 - t_0 \leq \infty$ and the interval of existence is $[t_0, t_1)$.

- Consider the final ODE from the Brachistochrone problem.
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{\frac{B-y}{y}}$$
    - Finding the **primitives**.
        - What are these "primitives" Shao keeps talking about??
    - We should have
$$\int \sqrt{\frac{y}{B-y}}\,\mathrm{d}y = x$$
    - Change of variables: $y = B\sin^2\phi$ and $\mathrm{d}y = 2B\cos\phi\sin\phi\,\mathrm{d}\phi$. Thus,
$$\int \sqrt{\frac{y}{B-y}}\,\mathrm{d}y = \int \frac{\sin\phi}{\cos\phi} \cdot 2B\cos\phi\sin\phi\,\mathrm{d}\phi = 2B\int \sin^2\phi\,\mathrm{d}\phi$$
    - The solution is
$$\begin{cases} x = B\phi - \frac{B}{2}\sin(2\phi) + C \\ y = B\sin^2\phi \end{cases}$$
        - If we set $2\phi = \theta$, then this is a parameterization of a cycloid.

- Later in the week, we will do SHM, the pendulum, the Kepler 2-body problem, and the Michaelis-Menten equation.

- Separable ODEs are a subset of ODEs of **exact form**.

- ODEs of exact form are of the form
$$g(x,y)\frac{\mathrm{d}y}{\mathrm{d}x}+f(x,y)=0$$
where for some $F(x,y)$, $g=\partial F/\partial y$, $f=\partial F/\partial x$, and partials commute. Equivalently,
$$\frac{\partial g}{\partial x}=\frac{\partial f}{\partial y}$$
is our necessary and sufficient condition.
- By the law of composite differentiation,
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[F(x,y(x))\right]&=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\cdot y'(x)\\&=f(x,y(x))+g(x,y(x))y'(x)\\&=0\end{aligned}$$
  - Thus, there exists $C\in\mathbb{R}$ such that $F(x,y)=C$ along the entire set of solutions (x,y). If an initial condition (x_0,y_0) is given, we have in particular that $F(x_0,y_0)=C$. Thus, $F(x,y)=F(x_0,y_0)$ defines the solution corresponding to the aforementioned initial conditions implicitly.
  - We solve these with an integrating factor $\mu\neq 0$ such that $(\mu g,\mu f)$ satisfy the constraint.

2.2 Office Hours (Shao)

- **Primitive**: An antiderivative.
- **Law of composite differentiation**: The chain rule.
- Went over how Shao has been applying the law of composite differentiation with respect to separable ODEs:
  - Rearrange the initial separable ODE as follows.
$$\frac{1}{g(y)}\cdot\frac{\mathrm{d}y}{\mathrm{d}t}=f(t)$$
  - Define $\mathrm{d}H/\mathrm{d}y=1/g(y)$. Then, continuing from the above, we have by the law of composite differentiation that
$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}y}\cdot\frac{\mathrm{d}y}{\mathrm{d}t}&=f(t)\\\frac{\mathrm{d}H}{\mathrm{d}t}&=f(t)\end{aligned}$$
  - From the definition of H, we know that $H(y)=\int_{y_0}^{y}\mathrm{d}w\,/g(w)$. We also have from the FTC that $f(t)=\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t}f(\tau)\,\mathrm{d}\tau$. Thus, continuing from the above, we have that
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(H)&=f(t)\\\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{y_0}^{y}\frac{\mathrm{d}w}{g(w)}\right]&=\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t}f(\tau)\,\mathrm{d}\tau\\\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{y_0}^{y(t)}\frac{\mathrm{d}w}{g(w)}-\int_{t_0}^{t}f(\tau)\,\mathrm{d}\tau\right]&=0\end{aligned}$$
as desired.

- It follows that there exists $C\in\mathbb{R}$ such that
$$\int_{y_0}^{y(t)}\frac{\mathrm{d}w}{g(w)}-\int_{t_0}^{t}f(\tau)\,\mathrm{d}\tau=C$$
for all t.
- In particular, if we choose $t=t_0$, then we can determine that
$$C=\int_{y_0}^{y(t_0)}\frac{\mathrm{d}w}{g(w)}-\int_{t_0}^{t_0}f(\tau)\,\mathrm{d}\tau=\int_{y_0}^{y_0}\frac{\mathrm{d}w}{g(w)}-0=0$$
- Therefore,
$$\int_{y_0}^{y(t)}\frac{\mathrm{d}w}{g(w)}=\int_{t_0}^{t}f(\tau)\,\mathrm{d}\tau$$

- Take away from Brachistochrone problem:
  - Just an example of a BDE; we won't have to answer questions on it.

2.3 ODEs of Exact Form

10/5:
- Last time, we discussed separable ODEs.
- Today, we will study **exact form** equations, as briefly touched on last class.
- **Exact form** (ODE): An ODE of the form
$$g(x,y)\frac{\mathrm{d}y}{\mathrm{d}x}+f(x,y)=0$$
where
$$\frac{\partial g}{\partial x}=\frac{\partial f}{\partial y}$$
  - For equations of this form, there exists $F(x,y)$ such that
$$\frac{\partial F}{\partial x}=f \qquad \frac{\partial F}{\partial y}=g \qquad F(x,y(x))=C$$
for some $C\in\mathbb{R}$.
  - The scalar unknown function y is allowed to take complex value here, but not generic vector value.
- **Integrating factor**: A number or function μ such that
$$\mu g\frac{\mathrm{d}y}{\mathrm{d}x}+\mu f=0 \qquad \frac{\partial}{\partial x}(\mu g)=\frac{\partial}{\partial y}(\mu f)$$
- Solving linear homogeneous equations of the form
$$\frac{\mathrm{d}y}{\mathrm{d}t}=p(t)y$$
  - The solution is of the form
$$y(t)=y_0\exp\left[\int_{t_0}^{t}p(\tau)\mathrm{d}\tau\right]$$
  - The uniqueness of the solution follows from that of general separable equations.
- Recall that $\mathrm{e}^{a+ib}=\mathrm{e}^{a}(\cos b+i\sin b)$, so
$$\mathrm{e}^{ix}=\cos x+i\sin x \qquad \cos x=\frac{1}{2}(\mathrm{e}^{ix}+\mathrm{e}^{-ix}) \qquad \sin x=\frac{1}{2i}(\mathrm{e}^{ix}-\mathrm{e}^{-ix})$$
- Example: If $y'=ky$, then $y'=-\lambda y$.
  - What is this for??

- Solving inhomogeneous linear equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = p(t)y + f(t)$$

  - Rewrite to

$$1 \cdot \frac{\mathrm{d}y}{\mathrm{d}t} + (-py - f) = 0$$

  - If the above equation is exact, it should satisfy the $\partial g/\partial x = \partial f/\partial y$ rule. To be more specific, in this case, we should have $\mathrm{d}/\mathrm{d}t\ (1) = \mathrm{d}/\mathrm{d}y\ (-py - f)$.
    - Note that we switch from partial derivatives to ordinary derivatives because the specific equation we are considering only has two variables, total, at play (y, t as opposed to x, y, t).
    - Also note that we differentiate the LHS with respect to t (instead of x) because the derivative in the rewritten form is with respect to t, and we base our formula on direct comparison.
  - However, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(1) = 0 \neq -p(t) = \frac{\mathrm{d}}{\mathrm{d}y}(-p(t)y - f(t))$$

    where $0 \neq -p(t)$ in general (though there is a specific exception: $p(t) = 0$).
    - Thus, the above equation is *not* exact. We will need to use integrating factors to make it so.
  - In particular, we wish to find an integrating factor $\mu(t, y)$ such that

$$\mu(t,y)\frac{\mathrm{d}y}{\mathrm{d}t} - \mu(t,y)p(t)y - \mu(t,y)f(t) = 0 \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}t}(\mu \cdot 1) = \frac{\mathrm{d}}{\mathrm{d}y}(-\mu py - \mu f)$$

  - By inspection, we can do this by taking μ to be a function of t, alone. Doing so, we obtain a linear homogeneous equation in μ that we can easily solve using the techniques from the previous example. Indeed, from the right equation above, we find

$$\begin{aligned}\frac{\mathrm{d}\mu}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}y}(-\mu py - \mu f)\\ \mu' &= -p(t)\mu\\ \mu(t) &= \exp\left[-\int_{t_0}^{t} p(\tau)\mathrm{d}\tau\right]\end{aligned}$$

    - Note that we may take the constant of integration/initial condition to be any number. Choosing 1, as we have above, is simply a matter of convenience.
  - If we let $P(t) = \int_{t_0}^{t} p(\tau)\mathrm{d}\tau$, then

$$\begin{aligned}\mathrm{e}^{-P(t)}y'(t) - p(t)\mathrm{e}^{-P(t)}y(t) &= \mathrm{e}^{-P(t)}f(t)\\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-P(t)}y(t)\right) &= \mathrm{e}^{-P(t)}f(t)\\ \mathrm{e}^{-P(t)}y(t) - \mathrm{e}^{-P(t_0)}y(t_0) &= \int_{t_0}^{t}\mathrm{e}^{-P(\tau)}f(\tau)\mathrm{d}\tau\end{aligned}$$

  - Thus, we finally have the solution to the inhomogeneous problem as follows: The IVP $y' = py + f$, $y(t_0) = y_0$ has solution

$$y(t) = y_0\mathrm{e}^{P(t)-P(t_0)} + \mathrm{e}^{P(t)}\int_0^t \mathrm{e}^{-P(\tau)}f(\tau)\mathrm{d}\tau$$

    where P is any anti-derivative of p.
  - In particular, when $p(t) = a$, we get the **Duhamel formula** (which we should memorize).

- **Duhamel formula**: The following equation, which is the solution to an inhomogeneous linear equation with $p(t) = a$.
$$y(t) = y_0 \mathrm{e}^{a(t-t_0)} + \int_{t_0}^{t} \mathrm{e}^{a(t-\tau)} f(\tau) \mathrm{d}\tau$$
  - Important for computing forced oscillation.
- Inspecting the inhomogeneous solution.
  - The first term is the solution to the homogeneous form. The second term deals with the initial value.
- Given an inhomogeneous equation, you can always write its solution as the combination of the solution to the homogeneous problem plus a particular solution, i.e.,
$$y = y_h + y_p$$
  - "The general solution equals the homogeneous solution plus a particular solution."
  - This is related to linear algebra, where the solution to $Ax = b$ is a particular solution x_p plus any vector $x \in \ker A$.
  - Thus, this idea will reappear in the theory of systems of linear ODEs.
- We now look at systems of linear ODEs.
- Consider the harmonic oscillator: A particle of mass m connected to an ideal spring (obeys Hooke's law) with no friction or gravity.
  - Newton's second law: The acceleration is proportional to the restoring force.
  - Hooke's law: The restoring force is of magnitude kx in the opposite direction to the displacement.
  - Thus, the ODE is of the form
$$x'' = -\frac{k}{m} x$$
  - However, if there is damping (which will be proportional to the velocity), then the ODE is of the form
$$x'' + \frac{b}{m} x' + \frac{k}{m} x = 0$$
- We now address how to solve an ODE of the above form. In particular...
- Consider an ODE of the form
$$y'' + ay' + by = 0$$
for $a, b \in \mathbb{C}$.
  - Rewrite the above differential equation in the following form.
$$(y' - \mu y)' - \lambda(y' - \mu y) = 0$$
    - Shao never justifies how this rewrite may be derived. It appears to very much be "because it works" mathematics.
    - Indeed, Shao claims that he figured it out himself when he was learning ODEs and could not point us to a reference source with a similar derivation when asked. He imagines it has been done previously, but does not know where off the top.
    - In fact, this method is very much related to the classic approach to solving this ODE (transforming to a 2D system, calculating the eigenvalues [which are μ, λ], and solving by linear combination). However, it is still clearly distinct. See also the first bonus problem from PSet 2.

- Expanding, we can see that the above ODE is indeed equal to the original one, provided that we let $a = -(\mu + \lambda)$ and $b = \mu\lambda$.
$$y'' - (\mu + \lambda)y' + \mu\lambda y = 0$$
- Now observe that under the above definitions of μ, λ in terms of a, b, we have that μ, λ are the roots of $x^2 + ax + b = 0$.
$$\mu^2 - (\mu + \lambda)\mu + \mu\lambda = 0 \qquad \lambda^2 - (\mu + \lambda)\lambda + \mu\lambda = 0$$
  - We call the binomial $x^2 + ax + b = 0$ the **characteristic polynomial** of the original ODE.
  - Note that μ, λ can be complex (i.e., $\mu, \lambda \in \mathbb{C}$).
- Substitute $x = y' - \mu y$. Then we can solve
$$x' - \lambda x = 0$$
using the above methods to determine that $x = A\mathrm{e}^{\lambda t}$.
- Returning the substitution, we have that
$$y' - \mu y = A\mathrm{e}^{\lambda t}$$
- Since the above is of the form $y' = ay + f$, we can apply the Duhamel formula. It follows that the general solution is
$$\begin{aligned} y(t) &= B\mathrm{e}^{\mu t} + \int_0^t \mathrm{e}^{\mu(t-\tau)} A\mathrm{e}^{\lambda\tau}\mathrm{d}\tau \\ &= B\mathrm{e}^{\mu t} + A\mathrm{e}^{\mu t}\int_0^t \mathrm{e}^{(\lambda-\mu)\tau}\mathrm{d}\tau \end{aligned}$$
for some $A, B \in \mathbb{C}$ dependent on the initial conditions.
- We now divide into two cases ($\mu \neq \lambda$ and $\mu = \lambda$).
  1. ($\mu \neq \lambda$) Evaluating the above integral, we get
$$\begin{aligned} y(t) &= B\mathrm{e}^{\mu t} + A\mathrm{e}^{\mu t}\frac{\mathrm{e}^{(\lambda-\mu)t} - 1}{\lambda - \mu} \\ &= B\mathrm{e}^{\mu t} + \frac{A}{\lambda - \mu}\mathrm{e}^{\lambda t} - \frac{A}{\lambda - \mu}\mathrm{e}^{\mu t} \\ &= \left(B - \frac{A}{\lambda - \mu}\right)\mathrm{e}^{\mu t} + \frac{A}{\lambda - \mu}\mathrm{e}^{\lambda t} \\ &= A_1\mathrm{e}^{\mu t} + B_1\mathrm{e}^{\lambda t} \end{aligned}$$
where we define new constants $A_1, B_1 \in \mathbb{C}$ in the last step for the sake of simplicity and WLOG. These new constants can be solved for just the same using the initial conditions.
  2. ($\mu = \lambda$) Since $\lambda - \mu = 0$ in this case, the exponential function in the integral simplifies to unity. Therefore,
$$\begin{aligned} y(t) &= B\mathrm{e}^{\mu t} + A\mathrm{e}^{\mu t}\int_0^t 1\mathrm{d}\tau \\ y(t) &= B\mathrm{e}^{\mu t} + At\mathrm{e}^{\mu t} \\ &= A_1\mathrm{e}^{\mu t} + B_1 t\mathrm{e}^{\mu t} \end{aligned}$$
where we define $A_1, B_1 \in \mathbb{C}$ again for consistency with the above.
- These linearly independent solutions ($\mathrm{e}^{\mu t}$ and $\mathrm{e}^{\lambda t}$, or $\mathrm{e}^{\mu t}$ and $t\mathrm{e}^{\mu t}$) form a basis of the space of solutions; all solutions can be expressed as a linear combination of these two functions.

- If our equation is of the form $y'' + ay' + by = f(t)$, then we just need to apply the Duhamel formula twice (i.e., in the first step as well as the second step of the above derivation). In particular...
  - Rewrite
$$(y' - \mu y)' - \lambda(y' - \mu y) = f(t)$$
  - Substitute $x = y' - \mu y$ so that we have
$$\begin{aligned} x' - \lambda x &= f(t) \\ x(t) &= A\mathrm{e}^{\lambda t} + \int_0^t \mathrm{e}^{\lambda(t-\tau)} f(\tau)\mathrm{d}\tau \end{aligned}$$
  - Returning the substitution, we get
$$\begin{aligned} y' - \mu y &= A\mathrm{e}^{\lambda t} + \int_0^t \mathrm{e}^{\lambda(t-\tau_1)} f(\tau_1)\mathrm{d}\tau_1 \\ y(t) &= B\mathrm{e}^{\mu t} + \int_0^t \mathrm{e}^{\mu(t-\tau_2)} \left(A\mathrm{e}^{\lambda\tau_2} + \int_0^{\tau_2} \mathrm{e}^{\lambda(\tau_2-\tau_1)} f(\tau_1)\mathrm{d}\tau_1 \right) \mathrm{d}\tau_2 \\ &= B\mathrm{e}^{\mu t} + A\mathrm{e}^{\mu t} \int_0^t \mathrm{e}^{(\lambda-\mu)\tau_2}\mathrm{d}\tau_2 + \mathrm{e}^{\mu t} \int_0^t \mathrm{e}^{(\lambda-\mu)\tau_2} \int_0^{\tau_2} \mathrm{e}^{-\lambda\tau_1} f(\tau_1)\mathrm{d}\tau_1\mathrm{d}\tau_2 \end{aligned}$$
  - Notice how we have applied the Duhamel formula twice at this point (once in each of the last two steps).
  - We once again divide into the two cases $\mu \neq \lambda$ and $\mu = \lambda$.
    1. ($\mu \neq \lambda$) We have
$$y(t) = A_1\mathrm{e}^{\mu t} + B_1\mathrm{e}^{\lambda t} + \mathrm{e}^{\mu t} \int_0^t \mathrm{e}^{(\lambda-\mu)\tau_2} \int_0^{\tau_2} \mathrm{e}^{-\lambda\tau_1} f(\tau_1)\mathrm{d}\tau_1\mathrm{d}\tau_2$$
    2. ($\mu = \lambda$) We have
$$y(t) = A_1\mathrm{e}^{\mu t} + B_1 t\mathrm{e}^{\mu t} + \mathrm{e}^{\mu t} \int_0^t \int_0^{\tau_2} \mathrm{e}^{-\mu\tau_1} f(\tau_1)\mathrm{d}\tau_1\mathrm{d}\tau_2$$
  - Notice how the left two terms in the final equations above are the homogeneous solutions derived previously, and the rightmost terms above are particular solutions.
- Returning to the simple harmonic oscillator problem, we substitute $\omega = \sqrt{k/m}$ to get
$$x'' = \omega^2 x$$
  - The characteristic polynomial is
$$0 = x^2 + \omega^2 = (x + i\omega)(x - i\omega)$$
  - Thus, solutions are of the form
$$x = A_1\mathrm{e}^{i\omega t} + B_1\mathrm{e}^{-i\omega t}$$
  - It follows that the period is $T = 2\pi/\omega$.
  - To get a real (usable) solution, apply Euler's formula to get
$$\begin{aligned} x(t) &= A_1(\cos\omega t + i\sin\omega t) + B_1(\cos\omega t - i\sin\omega t) \\ &= A\cos\omega t + B\sin\omega t \end{aligned}$$
    where $A = A_1 + B_1$, $B = iA_1 - iB_1$.

- To match the initial condition $x(0) = x_0$, $x'(0) = v_0$, we use
$$x(t) = x_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t$$
- In other words,
$$\begin{cases} A = x_0 \\ \omega B = v_0 \end{cases} \qquad \begin{cases} A_1 + B_1 = x_0 \\ i\omega A_1 - i\omega B_1 = v_0 \end{cases}$$
so
$$\begin{cases} A = x_0 \\ B = \frac{v_0}{\omega} \end{cases} \qquad \begin{cases} A_1 = \frac{1}{2}\left[x_0 - \frac{iv_0}{\omega}\right] \\ B_1 = \frac{1}{2}\left[x_0 + \frac{iv_0}{\omega}\right] \end{cases}$$

2.4 ODE Examples

10/7:

- Today, we will investigate a variety of examples of ODEs arising in real life.
- Michaelis-Menten kinetics: If E is an enzyme, S is its substrate, and P is the product, then the mechanism is
$$\mathrm{E} + \mathrm{S} \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} \mathrm{ES} \xrightarrow{k_2} \mathrm{E} + \mathrm{P}$$
- The concentrations that we are concerned with are $[\mathrm{E}], [\mathrm{S}], [\mathrm{ES}], [\mathrm{P}]$.
- From the above mechanism, we can write the four rate laws
$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{S}] = -k_1[\mathrm{E}][\mathrm{S}] + k_{-1}[\mathrm{ES}] \tag{1}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{E}] = -k_1[\mathrm{E}][\mathrm{S}] + (k_{-1} + k_2)[\mathrm{ES}] \tag{2}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{ES}] = k_1[\mathrm{E}][\mathrm{S}] - (k_{-1} + k_2)[\mathrm{ES}] \tag{3}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{P}] = k_2[\mathrm{ES}] \tag{4}$$
- The initial conditions are $[\mathrm{S}] = [\mathrm{S}]_0$ and $[\mathrm{E}] = [\mathrm{E}]_0$.
- We can reduce these rate laws to the 2D system
$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{S}] = -k_1([\mathrm{E}]_0 - [\mathrm{ES}])[\mathrm{S}] + k_{-1}[\mathrm{ES}] \tag{5}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{ES}] = k_1([\mathrm{E}]_0 - [\mathrm{ES}])[\mathrm{S}] - (k_{-1} + k_2)[\mathrm{ES}] \tag{6}$$
  - Note that to do so, we have used two conservation laws: The conservation of the enzyme plus enzyme-substrate complex, and the conservation of the substrate plus enzyme-substrate complex plus products.
- QSSA: Quasi steady-state assumption.
  - Assume that $[\mathrm{E}]_0/[\mathrm{S}]_0 \lll 1$.
  - It follows that $\mathrm{d}[\mathrm{ES}]/\mathrm{d}t \approx 0$ due to saturation of the enzyme and $[\mathrm{S}] \approx [\mathrm{S}]_0$ due to ever-more substrate being available.
- Then
$$[\mathrm{ES}] = \frac{[\mathrm{E}]_0[\mathrm{S}]}{K_M + [\mathrm{S}]}$$
where $k_M = (k_{-1} + k_2)/k_1$ is the **Michaelis-Menten constant**, a usual indication of enzyme activity.

- Substitute the above into Equation 5:

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{S}] = -\frac{v_{\max}[\mathrm{S}]}{k_M + [\mathrm{S}]}$$

  - Note that $v_{\max} = k_2[\mathrm{E}]_0$.

- The above is now a differential equation of separable form; it's solution is

$$\begin{aligned}
\int_{[\mathrm{S}]_0}^{[\mathrm{S}]} -\frac{(k_M + z)\,\mathrm{d}z}{z v_{\max}} &= \int_0^t \mathrm{d}t \\
-\frac{k_M}{v_{\max}} \log\frac{[\mathrm{S}]}{[\mathrm{S}]_0} - \frac{1}{v_{\max}}([\mathrm{S}] - [\mathrm{S}]_0) &= t \\
\log\frac{[\mathrm{S}]}{[\mathrm{S}]_0} + \frac{[\mathrm{S}]}{k_M} &= \frac{[\mathrm{S}]_0 - v_{\max}t}{k_M} \\
\frac{[\mathrm{S}]}{[\mathrm{S}]_0}\mathrm{e}^{[\mathrm{S}]/k_M} &= \exp\left(\frac{[\mathrm{S}]_0 - v_{\max}t}{k_M}\right) \\
\frac{[\mathrm{S}]}{k_M}\mathrm{e}^{[\mathrm{S}]/k_M} &= \frac{[\mathrm{S}]_0}{k_M}\exp\left(\frac{[\mathrm{S}]_0 - v_{\max}t}{k_M}\right) \\
\frac{[\mathrm{S}]}{k_M} &= W\left[\frac{[\mathrm{S}]_0}{k_M}\exp\left(\frac{[\mathrm{S}]_0 - v_{\max}t}{k_M}\right)\right] \\
[\mathrm{S}] &= k_M W\left[\frac{[\mathrm{S}]_0}{k_M}\exp\left(\frac{[\mathrm{S}]_0 - v_{\max}t}{k_M}\right)\right]
\end{aligned}$$

  - Getting from line 5-6 (i.e., the introduction of W): Suppose we have an equation of the form $y\mathrm{e}^y = x$. We cannot express x in terms of y using elementary functions, so we must define W such that $y = W(x)$. Explicitly, W is the unique function of x that satisfies $W(x)\mathrm{e}^{W(x)} = x$.

- Harmonic oscillator.
- Recall that

$$x'' + \frac{k}{m}x = 0$$

- Substituting $\omega = \sqrt{k/m}$, we can solve the above for

$$x(t) = x(0)\cos(\omega t) + \frac{x'(0)}{\omega}\sin(\omega t)$$

- This is an integrable system with n degrees of freedom and $n-1$ scalar conservation laws??
- Conservation of mechanical energy:

$$E = \frac{1}{2}m|x'|^2 + \frac{1}{2}kx^2$$

  - Differentiating wrt. x yields

$$\begin{aligned}
0 &= mx'x'' + kxx' \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m(x')^2\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}kx^2\right)
\end{aligned}$$

  - This means that the solution is an ellipse in the xx'-plane, where each ellipse corresponds to an initial displacement and velocity.

- Mathematical pendulum.

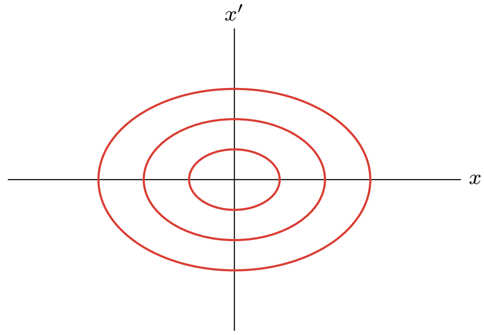

Figure 2.2: Conservation of mechanical energy in the harmonic oscillator.

- Equation of motion:

$$\begin{aligned} 0 &= l\theta'' + g\sin\theta \\ &= \ell\theta''\theta' + g\sin\theta\cdot\theta' \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg(\underbrace{\frac{\ell}{2}|\theta'|^2 - g\cos\theta}_{E}\bigg) \end{aligned}$$

- Initial values:

$$\theta(0) = \theta_0 \qquad\qquad \theta'(0) = 0$$

- It follows from the above that

$$\begin{aligned} \frac{\ell}{2}|\theta'|^2 - g\cos\theta_0 &= -g\cos\theta \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= \sqrt{\frac{2g}{\ell}(\cos\theta_0 - \cos\theta)} \\ \int_{\theta_0}^{\theta}\sqrt{\frac{\ell}{2g(\cos\theta_0 - \cos\phi)}}\mathrm{d}\phi &= t \end{aligned}$$

  - This is an elliptical integral (and thus cannot be expressed in terms of the elementary functions).
- Suppose θ_0 is small. Then θ is small, and we can invoke the small angle approximation $\sin\theta \approx \theta$.
  - This yields an approximate equation of motion:

$$\ell\theta'' + g\theta = 0$$

  - From here, we can determine that $\theta(t) \approx \theta_0\cos\sqrt{g/\ell}\cdot t$ and $T = 2\pi\sqrt{\ell/g}$.
- Kepler problem.
- Two bodies of mass m_1, m_2 are located at positions x_1, x_2 pulling on each other gravitationally.
  - The force of attraction is a conservative central force.
  - The potential between the two masses is a function of their distance, i.e, $U(|x_1 - x_2|)$.
- From Newton's second and third law, we get

$$m_1x_1'' = U'(|x_1 - x_2|)\frac{x_2 - x_1}{|x_2 - x_1|} \qquad\qquad m_2x_2'' = U'(|x_1 - x_2|)\frac{x_1 - x_2}{|x_1 - x_2|}$$

  - The derivative of potential is force.
  - The vector term provides direction.

- Conservation of momentum:

$$(m_1x_1 + m_2x_2)'' = 0$$
$$m_1x_1' + m_2x_2' = C$$

  - Let $M = m_1 + m_2$. Then the center of mass

$$\frac{m_1}{M}x_1 + \frac{m_2}{M}x_2$$

  moves inertially (i.e., does not accelerate or decelerate; is a stable reference frame) — we'll define it to be the origin.

- Conservation of angular momentum:

$$[m(x_1 - x_2)' \times (x_1 - x_2)]' = 0$$

  - $m = m_1m_2/(m_1 + m_2)$.
  - $\times$ indicates the cross product.
  - $L = m(x_1 - x_2)' \times (x_1 - x_2)$.

- It follows that $x_1 - x_2$ is always in a fixed plane, which we may call the **horizon plane**.
- Conservation of mechanical energy:

$$mq'' + U'(|q|)\frac{q}{|q|} = 0$$
$$\frac{m}{2}|q'|^2 + U(|q|) = E$$

  - $q = x_1 - x_2$.

- Introduce polar coordinates (r, ϕ).
  - Then $r^2\phi' = \ell_0$, $r = r(\phi)$, and $\mathrm{d}r/\mathrm{d}\phi = r'(t)/\phi'(t)$.
  - It follows that

$$\frac{m}{2}(|r'|^2 + |\phi'|^2) + U(r) = E$$

  - Since $U(r) = -Gm_1m_2/r$ for Newtonian gravity,

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 + r^2 = \frac{2GMr^3}{\ell_0^2} + \frac{2Er^4}{m\ell_0^2}$$

  - The substitution $\mu = 1/r$ yields

$$\left(\frac{\mathrm{d}\mu}{\mathrm{d}\phi}\right)^2 + \mu^2 = \frac{2GM}{\ell_0^2}\mu + \frac{2E}{m\ell_0^2}$$

  - Differentiating again gives

$$2\frac{\mathrm{d}\mu}{\mathrm{d}\phi}\frac{\mathrm{d}^2\mu}{\mathrm{d}\phi^2} + 2\mu\frac{\mathrm{d}\mu}{\mathrm{d}\phi} = \frac{2GM}{\ell_0^2}\frac{\mathrm{d}\mu}{\mathrm{d}\phi}$$

  - Substituting $\mu = \cos(t)$ gives

$$\frac{\mathrm{d}^2\mu}{\mathrm{d}\phi^2} + \mu - \frac{GM}{\ell_0^2} = 0$$

  or

$$r = \frac{1}{GM/\ell_0^2 + \varepsilon\cos(\phi - \phi_0)}$$

- This is a conic section!

- Thus, for example, we can calculate the precession of Mercury.
- Note that while we have determined the trajectory of our 2 bodies in terms of elementary functions, the n-body problem cannot be solved analytically.

2.5 Chapter 1: Introduction

From Teschl (n.d.).

Section 1.3: First Order Autonomous Equations

11/15:

- We start with the simplest nontrivial case of a first-order autonomous equation:

$$\dot{x} = f(x), \quad x(0) = x_0$$

  - Since the system is autonomous, we may let $t_0 = 0$ WLOG. Indeed, if $\phi(t)$ is a solution to an autonomous equation satisfying $\phi(0) = x_0$, then $\psi(t) = \phi(t - t_0)$ is a solution with $\psi(t_0) = x_0$.
- Solving this ODE.
  - Suppose $f(x_0) \neq 0$. Divide both sides by $f(x)$ and integrate from the initial conditions onward to yield

$$\int_0^t \frac{\dot{x}(s)\,\mathrm{d}s}{f(x(s))} = t$$

  - Define

$$F(x) := \int_{x_0}^x \frac{\mathrm{d}y}{f(y)}$$

    - Note that this is just the previous equation under the "u-substitution" $y(t) = x(t)$, $\mathrm{d}y = \dot{x}(t)\,\mathrm{d}t$, $y(0) = x(0) = x_0$, $y(t) = x$.
  - Thus, in our new notation, any possible solution x to the ODE must satisfy $F(x(t)) = t$. It follows that $F(x)$ is monotone near x_0. Thus, it can be inverted to yield the unique solution

$$\phi(t) = F^{-1}(t), \quad \phi(0) = F^{-1}(0) = x_0$$

- Investigating the maximal interval on which ϕ is defined.
- We focus on the case where $f(x_0) > 0$ to start. We will treat $f(x_0) = 0$ later; $f(x_0) < 0$ is analogous to this section and will not be directly treated.
- Definitions and terms.
  - By the assumed continuity of f, there exists an interval $(x_1, x_2) \ni x_0$ on which f is positive.
    - It will be useful to mention now that we want x_1, x_2 to be the maximal such values, i.e., for example, if $x > x_2$, then $f(x) \leq 0$. We will not explicitly deal with this stipulation until later, though.
  - Define

$$T_+ = \lim_{x \to {x_2}^-} F(x) \qquad\qquad T_- = \lim_{x \to {x_1}^+} F(x)$$

    - By the definition of F, $T_+ \in (0, \infty]$ and $T_- \in [-\infty, 0)$.
    - T_+ is the supremum of all values of t over which the solution is defined. T_- is the respective infimum.
  - It follows — by the definitions of F, ϕ and the FTC — that $\phi \in C^1((T_-, T_+))$.

- It additionally follows by the definitions of $T_\pm$ that

$$\lim_{t \to T_+{}^-} \phi(t) = x_2 \qquad\qquad \lim_{t \to T_-{}^+} \phi(t) = x_1$$

- Limits on the domain of ϕ.
  - The above implies that ϕ is defined for all $t > 0$ iff

$$T_+ = \int_{x_0}^{x_2} \frac{\mathrm{d}y}{f(y)} = +\infty$$

    - Equivalently, ϕ is defined for all $t > 0$ iff $1/f(x)$ is *not* integrable near x_2
  - Similarly, ϕ is defined for all $t < 0$ iff $1/f(x)$ is *not* integrable near x_1.
  - What about if $T_+ < \infty$ (i.e., is finite)? We divide into two cases ($x_2 = +\infty$ or $x_2 < +\infty$).
  - If $x_2 = +\infty$, then by one of the above limits, ϕ diverges to $+\infty$ as $t \to T_+$. Thus, ϕ naturally cannot be extended (in a continuous way) to $t > T_+$, and hence ϕ is only defined for $t \in [0, T_+)$.
  - If $x_2 < +\infty$, we divide into two subcases. First off, if $f(x_2) > 0$ (contrary to our assumption above that x_1, x_2 are maximal), then we can extend the solution beyond x_2 and we must redefine everything. Otherwise, $f(x_2) = 0$; here, we can always extend ϕ by setting $\phi(t) = x_2$ for $t \geq T_+$. There may actually be more than one possible extension, though — see the below example with $f(x) = \sqrt{|x|}$.
- Example: $f(x) = x$ and $x_0 > 0$.
  - If $x_0 > 0$, then $f(x_0) = x_0 > 0$. Thus, we can define an interval of positivity; specifically, we can determine by inspecting the graph of f that this interval is $(x_1, x_2) = (0, +\infty)$.
  - By definition, we have that

$$F(x) = \int_{x_0}^{x} \frac{\mathrm{d}x}{x} = \ln\left(\frac{x}{x_0}\right)$$

  - By inspecting the graph of the above, we can see that F approaches $+\infty$ as $x \to x_2 = +\infty$, and F approaches $-\infty$ as $x \to x_1 = 0$. Thus,

$$T_+ = +\infty \qquad\qquad T_- = -\infty$$

  - Additionally, F is readily invertible, with inverse

$$\phi(t) = x_0 \mathrm{e}^t$$

  - Lastly, since $T_+ = +\infty$, we have by the above that ϕ is defined for all $t > 0$. Similarly, we have that ϕ is defined for all $t < 0$. Therefore, the solution ϕ is globally defined (defined on all of $\mathbb{R}$). Moreover, we can show symmetrically that ϕ is a solution for all $x_0 \in \mathbb{R}$.
- Example: $f(x) = x^2$ and $x_0 > 0$.
  - As before, we have $(x_1, x_2) = (0, +\infty)$.
  - By definition,

$$F(x) = \frac{1}{x_0} - \frac{1}{x}$$

  - Thus,

$$T_+ = \frac{1}{x_0} \qquad\qquad T_- = -\infty$$

  - Additionally,

$$\phi(t) = \frac{x_0}{1 - x_0 t}$$

- As before, ϕ is defined for all $t < 0$. However, since T_+ is finite, we have some additional complications there. From the above, this is a case of $T_+ < \infty$ and $x_2 = +\infty$, so we know that we cannot extend ϕ to or beyond $1/x_0$.

- The case where $f(x_0) = 0$.
  - We have a trivial solution $\phi(t) = x_0$ to the initial condition $x(0) = x_0$.
  - This is, in fact, not the only solution. If there exists $\varepsilon > 0$ for which
$$\left| \int_{x_0}^{x_0+\varepsilon} \frac{\mathrm{d}y}{f(y)} \right| < \infty$$
then there is another solution
$$\varphi(t) = F^{-1}(t)$$
with $\varphi(0) = x_0$ which is different from ϕ!

- Example: $f(x) = \sqrt{|x|}$ and $x_0 > 0$.
  - We have $(x_1, x_2) = (0, +\infty)$.
  - By definition,
$$F(x) = 2(\sqrt{x} - \sqrt{x_0})$$
where we have removed the absolute values under the radicals because everything is positive in $x_0 > 0$ land.
  - Naturally, $\phi(t) = 0$ is a solution.
  - However, since choosing $\varepsilon = 1$ yields
$$\left| \int_{0}^{0+1} \frac{\mathrm{d}y}{\sqrt{|y|}} \right| = [2\sqrt{y}]_0^1 = 2 < \infty$$
we have an additional solution as well.
  - In particular, this additional solution reads
$$\varphi(t) = \left(\sqrt{x_0} + \frac{t}{2} \right)^2$$ [2]
  - Additionally, we can determine as in previous examples that
$$T_- = -2\sqrt{x_0} \qquad\qquad T_+ = +\infty$$
  - Furthermore, this is a case of $T_- > -\infty$, $x_1 > -\infty$, and $f(x_1) = 0$. Thus, we can extend ϕ beyond $t = -2\sqrt{x_0}$ with $\phi(t) := x_1 = 0$.
  - In fact, this is not actually the only only possible extension; we may also patch in other nontrivial solutions. For example,
$$\tilde{\phi}(t) = \begin{cases} -\frac{(t-t_0)^2}{4} & t \leq t_0 \\ 0 & t_0 < t < t_1 \\ \frac{(t-t_1)^2}{4} & t_1 \leq t \end{cases}$$
is a perfectly valid solution to the differential equation $\dot{x} = \sqrt{|x|}$ for $t_0 \leq t_1 \in \mathbb{R}$ arbitrary.

- Takeaways.
  - "Solutions might only exist locally in t, even for perfectly nice f" (Teschl, n.d., p. 11).

[2] Compare with $t^2/4$ from class.

- "Solutions might not be unique" (Teschl, n.d., p. 11).
  - Note, however, that it is the fact that $f(x) = \sqrt{|x|}$ is not differentiable at $x_0 = 0$ that causes the problems.
- Several good problems; could be quite helpful with cementing these concepts.
  - Some of these problems are also repeats from PSet 1.

Section 1.4: Finding Explicit Solutions

- Solving ODEs for explicit solutions is impossible in general unless the equation is of a particular form.
- This section: Classes of first-order ODEs which are explicitly solvable.
- Strategy: Find a change of variables that transforms the ODE into a solvable form.
- Linear equation.

$$\dot{x} = a(t)x \qquad\qquad \dot{x} = a(t)x + g(t)$$

  - The left equation above is the homogeneous linear equation, and the right equation above is the corresponding inhomogeneous linear equation.
  - The general solution to the homogeneous linear equation is

$$\phi(t) = x_0 A(t, t_0)$$

    where

$$A(t, s) = \mathrm{e}^{\int_s^t a(s)\,\mathrm{d}s}$$

  - The general solution to the inhomogeneous linear equation is

$$\phi(t) = x_0 A(t, t_0) + \int_{t_0}^{t} A(t, s) g(s)\,\mathrm{d}s$$

- Coordinate transformations in differential equations.
  - Suppose we have the point (t, x) in the product space of the independent and dependent variables of our differential equation. We may change it to new coordinates (s, y) given by

$$s = \sigma(t, x) \qquad\qquad y = \eta(t, x)$$

  - To not lose information, we require this transformation (i.e., both σ, η??) to be a **diffeomorphism**.
  - To transform $\phi(t)$ into a new function $\psi(s)$, we need to eliminate t from

$$s = \sigma(t, \phi(t)) \qquad\qquad \psi = \eta(t, \phi(t))$$

    by solving $s = \sigma(t, \phi(t))$ for t.
  - However, this is not always possible for a diffeomorphism σ; indeed, if σ corresponds to a rotation of the graph $(t, \phi(t))$ of ϕ, for example, the result might not be the graph of a function. Thus, we additionally require that σ be a **fiber preserving transformation**

$$s = \sigma(t)$$

  - Defining σ this way means that we can always define a unique inverse transformation

$$t = \tau(s) \qquad\qquad x = \xi(s, y)$$

- It follows by the chain rule (think about this more??) that $\phi(t)$ satisfies
$$\dot{x} = f(t,x)$$
iff $\psi(s) = \eta(\tau(s), \phi(\tau(s)))$ satisfies
$$\dot{y} = \dot{\tau}\left(\frac{\partial\eta}{\partial t}(\tau,\xi) + \frac{\partial\eta}{\partial x}f(\tau,\xi)\right)$$
where $\tau = \tau(s)$ and $\xi = \xi(s,y)$.
  - The above notation is ostensibly unambiguous (why??).
  - The following derivation of the above makes more intuitive sense, but may be ambiguous??
$$\frac{\mathrm{d}y}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}(y(t(s), x(t(s)))) = \frac{\partial y}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}s} + \frac{\partial y}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s}$$

- **Fiber preserving transformation**: A map that sends the fibers $t = \text{const}$ to the fibers $s = \text{const}$.
  - I suppose we can view the set of solutions to an ODE as a fiber bundle over the space of initial values.
  - Alternatively, I think what this map might be doing is using time as the base space and defining the fiber corresponding to a specific time t to be equal to the set of $\phi(t)$ for all the different ϕ (which are in bijective correspondence with the set of initial values). This would make sense, because this would imply that we're only stretching or shrinking time, not mapping it over itself, which is where problems would arise.
- We now apply the above transformational mechanics to solve differential equations.
- **Homogeneous** (nonlinear differential equation): An ODE of the following form. *Given by*
$$\dot{x} = f(x/t)$$
- Solving homogeneous equations.
  - Apply the change of variables $y = x/t$ for $t \neq 0$.
  - Definitions from the above:
$$\eta(t,x) = x/t \qquad \sigma(t) = t \qquad \tau(s) = s \qquad \xi(s,y) = sy$$
    - The definition for η follows straight from $\eta(t,x) = y = x/t$.
    - Under this definition of η, we must choose σ to be the identity to still have $y = x/t$. If, for example, we had chosen $\tilde{\eta}(t,x) = x/2t$, then we would have needed to choose $\tilde{\sigma}(t) = 2t$. This would have made it so that $y(s) = x/2t = x/s$, as desired.
    - τ is the straightforward inverse of the identity σ, and hence still the identity.
    - Under this definition of ξ, we can see that
$$y = \eta(\tau(s), \xi(s,y)) = \frac{\xi(s,y)}{\tau(s)} = \frac{sy}{s} = y$$
$$x = \xi(\sigma(t), \eta(t,x)) = \sigma(t)\cdot\eta(t,x) = t\cdot\frac{x}{t} = x$$
    Therefore, ξ is an appropriate inverse.
  - It follows by the above that our new differential equation is
$$\begin{aligned}
\dot{y} &= \dot{\tau}\left(\frac{\partial\eta}{\partial t}(\tau,\xi) + \frac{\partial\eta}{\partial x}f(\tau,\xi)\right)\\
&= 1\cdot\left(\left[-\frac{x}{t^2}\right]_{(t,x)=(s,sy)} + \left[\frac{1}{t}\right]_{(t,x)=(s,sy)}\cdot f(\xi/\tau)\right)\\
&= -\frac{sy}{s^2} + \frac{1}{s}\cdot f(y)\\
&= \frac{f(y)-y}{s}
\end{aligned}$$

    - Recall that since the independent variable corresponding to y is s, the Newtonian dot derivative denotes a derivative with respect to s.
  - This equation is separable.
- Teschl (n.d.) covers a more general form of homogeneous equations that may lend additional insight into cases where σ is not the identity.
- **Bernoulli equation**: An ODE of following form. *Also known as* **Bernoulli type** (ODE). *Given by*

$$\dot{x} = f(t)x + g(t)x^n$$

  where $n \neq 0, 1$.
  - We exclude $n = 0, 1$ because in these cases, the equation is already linear.
- Solving Bernoulli type ODEs.
  - Use the transformation $y = x^{1-n}$.
  - Then

$$\begin{aligned}
\dot{y} &= \dot{\tau}\left(\frac{\partial\eta}{\partial t}(\tau,\xi) + \frac{\partial\eta}{\partial x}f(\tau,\xi)\right) \\
&= 1 \cdot \left([0]_{(s,y^{1/(1-n)})} + [(1-n)x^{-n}]_{(s,y^{1/(1-n)})} \cdot [f(\tau)\xi + g(\tau)\xi^n]\right) \\
&= (1-n)y^{-n/(1-n)}[f(s)y^{1/(1-n)} + g(s)y^{n/(1-n)}] \\
&= (1-n)f(s)y + (1-n)g(s)
\end{aligned}$$

  - This is now a readily solvable linear equation.
- **Riccati equation**: An ODE of the following form. *Also known as* **Riccati type** (ODE). *Given by*

$$\dot{x} = f(t)x + g(t)x^2 + h(t)$$

- Solving Riccati type ODEs.
  - To solve these, you need a particular solution $x_p(t)$. Then, you can use the transformation

$$y = \frac{1}{x - x_p(t)}$$

    to yield the following linear equation.

$$\dot{y} = -(f(s) + 2x_p(s)g(s))y - g(s)$$

- Further clever transformations can be looked up in reference works.
- Rule of thumb: "For a first-order equation, there is a realistic chance that it is explicitly solvable. But already for second-order equations, explicitly solvable ones are rare" (Teschl, n.d., p. 16).
- Using Mathematica to help solve ODEs and gain an intuition for how they work (e.g., with slope fields or **directional fields**).
- Equations of exact form are covered in the problems to this section.
  - Again, some of these problems are repeats from PSet 1.

2.6 Chapter 8: Higher Dimensional Dynamical Systems

From Teschl (n.d.).

Section 8.5: The Kepler Problem

12/6:
- I would need Hamiltonian mechanics (and hence boundary value problems) in order to understand this.