2 Linear Algebra

Required Problems

10/19: **1.** This question helps to complete the computations omitted in class. In deriving the Kepler orbits for the two-body problem, we have successfully reduced the differential equation satisfied by the curve $r = r(\varphi)$ to

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 + r^2 = \frac{2GMr^3}{l_0^2} + \frac{2Er^4}{ml_0^2}$$

Show that the function $\mu = 1/r$ satisfies the differential equation

$$\left(\frac{\mathrm{d}\mu}{\mathrm{d}\varphi}\right)^2 + \mu^2 = \frac{2GM\mu}{l_0^2} + \frac{2E}{ml_0^2}$$

By differentiating with respect to φ again, this reduces to either $\mathrm{d}\mu/\mathrm{d}\varphi = 0$ or

$$\frac{\mathrm{d}^2\mu}{\mathrm{d}\varphi^2} + \mu - \frac{GM}{l_0^2} = 0$$

Find the general solution of the latter, hence conclude that $r = r(\varphi)$ represents a conic section. *Hint*: There is a very obvious particular solution.

Proof. We begin from the first differential equation and substitute $\mu = 1/r$ in the last step to yield the desired result.

$$\begin{aligned}
\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 + r^2 &= \frac{2GMr^3}{l_0^2} + \frac{2Er^4}{ml_0^2} \\
\left(-\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 + \frac{1}{r^2} &= \frac{2GM}{l_0^2}\frac{1}{r} + \frac{2E}{ml_0^2} \\
\left[\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{1}{r}\right)\right]^2 + \left(\frac{1}{r}\right)^2 &= \frac{2GM}{l_0^2}\frac{1}{r} + \frac{2E}{ml_0^2} \\
\left(\frac{\mathrm{d}\mu}{\mathrm{d}\varphi}\right)^2 + \mu^2 &= \frac{2GM\mu}{l_0^2} + \frac{2E}{ml_0^2}
\end{aligned}$$

The homogeneous version of the final differential equation is entirely analogous to the harmonic oscillator problem and thus has general (real) solution

$$\mu(\varphi) = \epsilon\cos(\varphi - \varphi_0)$$

for $\epsilon, \varphi_0 \in \mathbb{R}$. By inspection, we can take as our particular solution to the inhomogeneous system

$$\mu(\varphi) = \frac{GM}{l_0^2}$$

since it's second derivative (as a constant) is zero and it is the opposite of the inhomogeneous term. Thus, the general solution to the original inhomogeneous system is

$$\begin{aligned}
\mu(\varphi) &= \frac{GM}{l_0^2} + \epsilon\cos(\varphi - \varphi_0) \\
r(\varphi) &= \frac{1}{GM/l_0^2 + \epsilon\cos(\varphi - \varphi_0)} \\
&= \frac{\epsilon(l_0^2/GM\epsilon)}{1 + \epsilon\cos(\varphi - \varphi_0)}
\end{aligned}$$

which is exactly the polar form of the conic section with eccentricity ϵ and directrix $l_0^2/GM\epsilon$. $\square$

2. The general formula for the inverse of an $n \times n$ invertible matrix is very lengthy. However, for a 2×2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

satisfying $ad - bc \neq 0$, there is a very simple formula. Try to find it; this could be very helpful if you can remember it.

Proof. Let A be the matrix given in the problem statement. We can determine A^{-1} by inspection as follows.

Let's focus on the right column of A^{-1} first, which we can denote $(x, y)^T$. We want $ax + by = 0$. One nice solution to this equation is $x = -b$ and $y = a$. Similarly, we can take the left column of A^{-1} to be $(d, -c)^T$. This choice of entries for A^{-1} yield the 0s in the right places, but the elements that should be 1 are instead $\det A = ad - bc$. Thus, we divide A^{-1} by $\det A$. This yields the following final formula

$$\boxed{A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

As a quick check, we have that

$$\begin{aligned} AA^{-1} &= \frac{1}{ad - bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ &= \frac{1}{ad - bc} \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \qquad \begin{aligned} A^{-1}A &= \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \frac{1}{ad - bc} \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

as expected. $\square$

3. Compute the determinant of the following matrices. Determine whether they are invertible or not.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 2 & 3 & 6 \\ 1 & 3 & 4 & 2 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} -1 & 2 & 1 \\ 3 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$$

Proof. We have that

$$\begin{aligned} &\det A = 1[5 \cdot 9 - 6 \cdot 8] - 2[4 \cdot 9 - 6 \cdot 7] + 3[4 \cdot 8 - 5 \cdot 7] \\ &\boxed{\det A = 0} \end{aligned}$$

so $\boxed{A \text{ is not invertible.}}$

Since B is block upper triangular, we know that

$$\begin{aligned} \det B &= \det B_1 \cdot \det B_2 \\ &= [2 \cdot 3 - 2 \cdot 1] \cdot [-1 \cdot 2 - 2 \cdot 1] \\ &\boxed{\det B = -16} \end{aligned}$$

so $\boxed{B \text{ is invertible.}}$

We have that

$$\begin{aligned} &\det C = -1[(-1)(3) - (2)(1)] - 2[(3)(3) - (2)(2)] + 1[(3)(1) - (-1)(2)] \\ &\boxed{\det C = 0} \end{aligned}$$

so $\boxed{C \text{ is not invertible.}}$ $\square$

4. Determine whether the following linear systems admit solution(s); if they do, write down the solution (or the formula for the general solution).

(1)

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Proof. By inspection, A is a dimension 2 matrix of rank 2, so it $\boxed{\text{admits a unique solution.}}$ We now row-reducing the augmented matrix.

$$\left(\begin{array}{cc|c} 1 & 2 & -1 \\ 2 & -1 & 1 \end{array}\right) \cong \left(\begin{array}{cc|c} 1 & 0 & \frac{1}{5} \\ 0 & 1 & -\frac{3}{5} \end{array}\right)$$

Therefore, the solution is

$$\boxed{x = \begin{pmatrix} \frac{1}{5} \\ -\frac{3}{5} \end{pmatrix}}$$

□

(2)

$$\begin{pmatrix} -1 & 2 & 1 \\ 3 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Proof. By inspection, A is a dimension 3 matrix of rank 2 and the b vector is in the column space of A, so it $\boxed{\text{admits a family of solutions.}}$ We now row-reducing the augmented matrix.

$$\left(\begin{array}{ccc|c} -1 & 2 & 1 & 1 \\ 3 & -1 & 2 & 2 \\ 2 & 1 & 3 & 3 \end{array}\right) \cong \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore, the family of solutions is given by

$$\boxed{x = \begin{pmatrix} 1 - x^3 \\ 1 - x^3 \\ x^3 \end{pmatrix}}$$

for $x^3 \in \mathbb{R}$. □

(3)

$$\begin{pmatrix} -1 & 2 & 1 \\ 3 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Proof. No promising solution immediately appears by inspection, so we row reduce and evaluate the results.

$$\left(\begin{array}{ccc|c} -1 & 2 & 1 & 1 \\ 3 & -1 & 2 & 0 \\ 2 & 1 & 3 & 1 \end{array}\right) \cong \left(\begin{array}{ccc|c} 1 & 0 & 1 & \frac{1}{5} \\ 0 & 1 & 1 & \frac{3}{5} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

It follows that A $\boxed{\text{admits a family of solutions.}}$ In particular, these are given by

$$\boxed{x = \begin{pmatrix} \frac{1}{5} - x^3 \\ \frac{3}{5} - x^3 \\ x^3 \end{pmatrix}}$$

for $x^3 \in \mathbb{R}$. □

5. Find the connecting matrix from the basis $\begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix}$ to the new basis $\begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}$, where

$$\begin{pmatrix} p_1 & p_2 & p_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix} \qquad \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

That is, represent q_1, q_2, q_3 as linear combinations of p_1, p_2, p_3.

Proof. P is the connecting matrix from the standard basis (e_1, e_2, e_3) to (p_1, p_2, p_3). Likewise, Q is the connecting matrix from (e_1, e_2, e_3) to (q_1, q_2, q_3). It follows that if we want A to be the connecting matrix from (p_1, p_2, p_3) to (q_1, q_2, q_3), then we can do the transformation stepwise, i.e., take a vector represented in (p_1, p_2, p_3) to its representation in (e_1, e_2, e_3) using P^{-1} and then to its representation in (q_1, q_2, q_3) using Q. Indeed, the desired connecting matrix is

$$A = QP^{-1}$$
$$\boxed{A = \frac{1}{5}\begin{pmatrix} -2 & 2 & -1 \\ 5 & 0 & 5 \\ -1 & 1 & 2 \end{pmatrix}}$$

Direct computation can confirm that $Ap_i = q_i$ for $i = 1, 2, 3$.

With respect to representing q_1, q_2, q_3 as linear combinations of p_1, p_2, p_3, we can solve the equations $q_i = Px_i$ for $i = 1, 2, 3$ via row reduction, as in previous responses. The final expressions obtained are

$$\boxed{q_1 = \frac{1}{5}(p_1 + 2p_2 + p_3) \qquad q_2 = \frac{1}{5}(3p_1 - 4p_2 - 2p_3) \qquad q_3 = \frac{1}{5}(3p_1 + p_2 + 3p_3)}$$

Note that if we combine the coefficients above into a matrix X such that $PX = Q$, then $A = PXP^{-1} = QXQ^{-1}$. □

6. Let $\theta \in [0, 2\pi)$. The rotation through angle θ in the plane is represented by the matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Compute its determinant, characteristic polynomial, and eigenvalues. Compute its eigenvectors in $\mathbb{C}^2$. You need to use the Euler formula $\mathrm{e}^{i\theta} = \cos\theta + i\sin\theta$. For two angles θ, φ, compute the product $R(\theta)R(\varphi)$ and represent it in terms of $\theta + \varphi$. What is the geometric meaning of this equality?

Proof. The determinant of R is

$$\det R = \cos^2\theta + \sin^2\theta$$
$$\boxed{\det R = 1}$$

The characteristic polynomial of R is

$$\begin{aligned} \chi_R(z) &= \det(R - zI) \\ &= (\cos\theta - z)^2 + \sin^2\theta \\ &= z^2 - 2z\cos\theta + \cos\theta^2 + \sin\theta^2 \end{aligned}$$
$$\boxed{\chi_R(z) = z^2 - 2z\cos\theta + 1}$$

The eigenvalues of R are

$$\begin{aligned} 0 &= \chi_R(\lambda) \\ &= (\cos\theta - \lambda)^2 + \sin^2\theta \\ -\sin^2\theta &= (\cos\theta - \lambda)^2 \\ \pm i\sin\theta &= \pm(\cos\theta - \lambda) \\ \lambda &= \cos\theta \pm i\sin\theta \\ &\boxed{\lambda = \mathrm{e}^{\pm i\theta}} \end{aligned}$$

It follows by solving the systems of equations

$$\begin{aligned} x^1\cos\theta - x^2\sin\theta &= \mathrm{e}^{i\theta}x^1 \\ x^1\sin\theta + x^2\cos\theta &= \mathrm{e}^{i\theta}x^2 \end{aligned} \qquad \begin{aligned} y^1\cos\theta - y^2\sin\theta &= \mathrm{e}^{-i\theta}y^1 \\ y^1\sin\theta + y^2\cos\theta &= \mathrm{e}^{-i\theta}y^2 \end{aligned}$$

that the eigenvectors are

$$\boxed{x = \begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad\qquad y = \begin{pmatrix} 1 \\ i \end{pmatrix}}$$

The product $R(\theta)R(\varphi)$ may be computed as follows.

$$\begin{aligned} R(\theta)R(\phi) &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta\cos\varphi - \sin\theta\sin\varphi & -\cos\theta\sin\varphi - \sin\theta\cos\varphi \\ \sin\theta\cos\varphi + \cos\theta\sin\varphi & -\sin\theta\sin\varphi + \cos\theta\cos\varphi \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta+\varphi) & -\sin(\theta+\varphi) \\ \sin(\theta+\varphi) & \cos(\theta+\varphi) \end{pmatrix} \\ &\boxed{R(\theta)R(\varphi) = R(\theta+\phi)} \end{aligned}$$

The geometric meaning is that rotating through an angle θ and then through an additional angle φ is the same as rotating through an angle $\theta + \varphi$ all at once. □

8. Find the algebraic and geometric multiplicities of the eigenvalues of the following matrices.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

Proof. We tackle A first. A is an upper triangular matrix. Thus, $\chi_A(\lambda) = \det(A - \lambda I)$ can be read directly off of the diagonal:

$$\chi_A(\lambda) = (1-\lambda)^2(3-\lambda)$$

Thus, the eigenvalues are $\lambda = 1, 3$ with respective algebraic multiplicities

$$\boxed{\alpha_1 = 2 \qquad\qquad \alpha_3 = 1}$$

It follows immediately that

$$\boxed{\gamma_3 = 1}$$

and from the observation that $A - 1I$ has 2 linearly independent columns that this 3×3 matrix has a $3 - 2 = 1$ dimensional null space, i.e.,

$$\boxed{\gamma_1 = 1}$$

The procedure for B is almost entirely symmetric. Once again, B is upper triangular, so

$$\chi_B(\lambda) = (1-\lambda)^2(3-\lambda)$$

implying that

$$\boxed{\alpha_1 = 2 \qquad \alpha_3 = 1}$$

There is a difference with respect to the geometric multiplicities, however. We still have

$$\boxed{\gamma_3 = 1}$$

but since $A - I$ now has only 1 linearly independent column, we have

$$\boxed{\gamma_1 = 2}$$

this time. ☐

9. Compute the Jordan normal form of the following 2×2 matrices.

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix}$$

Notice that you not only need to find all the Jordan blocks, but also need to find the Jordan basis matrix Q such that $Q^{-1}AQ$ is in Jordan normal form.

Proof. We tackle A first.

Calculate the characteristic polynomial to begin.

$$\begin{aligned} \chi_A(z) &= \det(A - zI) \\ &= z^2 - 4z + 3 \\ &= (1-z)(3-z) \end{aligned}$$

It follows that the eigenvalues are

$$\lambda_1 = 1 \qquad \lambda_2 = 3$$

Since these eigenvalues are distinct, we can fully diagonalize this matrix. Indeed, we can determine by inspection that suitable corresponding eigenvectors are

$$v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Therefore,

$$\boxed{Q = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \qquad Q^{-1}AQ = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}}$$

The procedure for B is very much analogous to the procedure for A.

Characteristic polynomial:

$$\begin{aligned} \chi_B(z) &= \det(B - zI) \\ &= z^2 + 2z + 1 \\ &= (1+z)^2 \end{aligned}$$

Eigenvalue:

$$\lambda = -1$$

By inspection of $B+I$, we can pick one eigenvector of B:

$$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We now solve $(B+I)u = v$. By inspection, this yields

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Therefore,

$$\boxed{Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad Q^{-1}BQ = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}}$$

$\square$

10. Compute the Jordan normal form of the following 3×3 matrices.

$$A = \begin{pmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -1 & -1 \\ 2 & -1 & -2 \\ -1 & 1 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

Notice that you not only need to find all the Jordan blocks, but also need to find the Jordan basis matrix Q such that $Q^{-1}AQ$ is in Jordan normal form. *Hint*: These three matrices represent three different possibilities of nondiagonalizable Jordan normal forms of a 3×3 matrix: A reduces to $(2\times 2) \oplus (1 \times 1)$ Jordan blocks with different eigenvalues, B reduces to $(2 \times 2) \oplus (1 \times 1)$ Jordan blocks with the same eigenvalue, and C reduces to a 3×3 Jordan block.

Proof. We tackle A first.

Calculate the characteristic polynomial to begin.

$$\begin{aligned} \chi_A(z) &= \det(A - zI) \\ &= -z^3 + z^2 \\ &= z^2(1-z) \end{aligned}$$

It follows that the eigenvalues are

$$\lambda_1 = \lambda_2 = 0 \qquad \lambda_3 = 1$$

We can solve for an eigenvector v_1 corresponding to $\lambda_1 = \lambda_2 = 0$ using the augmented matrix and row reduction as follows.

$$\left(\begin{array}{ccc|c} 4 & -5 & 2 & 0 \\ 5 & -7 & 3 & 0 \\ 6 & -9 & 4 & 0 \end{array}\right) \cong \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus, if we choose $v_1^3 = 3$, then the desired eigenvector is

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Similarly, we can solve for an eigenvector v_3 corresponding to $\lambda_3 = 1$ using the following. Note that to solve $Ax = 1x$, we row-reduce $(A - I)x = 0$.

$$\left(\begin{array}{ccc|c} 3 & -5 & 2 & 0 \\ 5 & -8 & 3 & 0 \\ 6 & -9 & 3 & 0 \end{array}\right) \cong \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This yields

$$v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

We now solve the equation $(A - 0I)u = v_1$ to find a generalized eigenvector u corresponding to $\lambda_1 = \lambda_2 = 0$. This can also be done with an augmented matrix.

$$\left(\begin{array}{ccc|c} 4 & -5 & 2 & 1 \\ 5 & -7 & 3 & 2 \\ 6 & -9 & 4 & 3 \end{array}\right) \cong \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This yields

$$u = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$$

Therefore,

$$\boxed{Q = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 3 & 1 \end{pmatrix} \qquad Q^{-1}AQ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

The procedure for B is very much analogous to the procedure for A.

Characteristic polynomial:

$$\begin{aligned} \chi_B(z) &= \det(B - zI) \\ &= -z^3 + 3z^2 - 3z + 1 \\ &= (1 - z)^3 \end{aligned}$$

Eigenvalue:

$$\lambda = 1$$

By inspection of

$$B - I = \begin{pmatrix} 1 & -1 & -1 \\ 2 & -2 & -2 \\ -1 & 1 & 1 \end{pmatrix}$$

we can pick two eigenvectors of B corresponding to λ, i.e., two elements of the null space of the above matrix. In this subcase of the 3×3 case, we always pick the first of these to be an element of the column space of $B - I$, as well. Thus, choose

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

We now solve $(B - \lambda I)u = v_1$. By inspection, this yields

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Therefore,

$$\boxed{Q = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} \qquad Q^{-1}BQ = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

The procedure for C is likewise quite analogous.

The matrix is upper triangular, so the eigenvalues are on the diagonal. It follows that

$$\lambda = 2$$

is the sole eigenvalue. We can solve $(C - 2I)v = 0$ for one eigenvector v by inspection, yielding

$$v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

We can also solve $(C - 2I)u_1 = v$ by inspection to get

$$u_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

One more time, we can solve $(C - 2I)u_2 = u_1$ by inspection to get

$$u_2 = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$$

Therefore,

$$\boxed{Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{pmatrix} \qquad\qquad Q^{-1}CQ = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}}$$

□