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# Momentum Bonus

## Task

Given the following two equations, which describe the conservation of momentum and kinetic energy, respectively, before and after an elastic collison, derive formulas for the final velocity of each object ($v_{1f}$ and $v_{2f}$) in terms of only these object's initial velocities and masses (not each other's final velocities).

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \tag{1}$$

$$\frac{1}{2} m_1 {v_{1i}}^2 + \frac{1}{2} m_2 {v_{2i}}^2 = \frac{1}{2} m_1 {v_{1f}}^2 + \frac{1}{2} m_2 {v_{2f}}^2 \tag{2}$$

## Solution

Solve first for one of the variables (say, $v_{1f}$) and then for the other variable using an identical procedure ($v_{2f}$, if $v_{1f}$ is solved for first).

Treat equations 1 and 2 as a two-variable system of equations, where the two variables are $v_{1f}$ and $v_{2f}$. Solve the "system" by substitution i.e. solve one equation (say, equation 1) for $v_{2f}$ and plug the result into equation 2. This will create a third equation (equation 3) without $v_{2f}$. Finally, solve equation 3 algebraically for $v_{1f}$ to give the first of the two formulas (equation 4). This procedure can be carried out as follows.

$$\begin{aligned} m_1 v_{1i} + m_2 v_{2i} &= m_1 v_{1f} + m_2 v_{2f} \\ m_1 v_{1i} + m_2 v_{2i} - m_1 v_{1f} &= m_2 v_{2f} \\ v_{2f} &= \frac{m_1 v_{1i} + m_2 v_{2i} - m_1 v_{1f}}{m_2} \end{aligned}$$

$$\frac{1}{2} m_1 {v_{1i}}^2 + \frac{1}{2} m_2 {v_{2i}}^2 = \frac{1}{2} m_1 {v_{1f}}^2 + \frac{1}{2} m_2 {v_{2f}}^2$$

$$\frac{1}{2} m_1 {v_{1i}}^2 + \frac{1}{2} m_2 {v_{2i}}^2 = \frac{1}{2} m_1 {v_{1f}}^2 + \frac{1}{2} m_2 \left( \frac{m_1 v_{1i} + m_2 v_{2i} - m_1 v_{1f}}{m_2} \right)^2 \tag{3}$$

$$m_1 {v_{1i}}^2 + m_2 {v_{2i}}^2 = m_1 {v_{1f}}^2 + m_2 \frac{(m_1 v_{1i} + m_2 v_{2i} - m_1 v_{1f})^2}{(m_2)^2}$$

$$m_1 {v_{1i}}^2 + m_2 {v_{2i}}^2 = m_1 {v_{1f}}^2 + \frac{{m_1}^2 {v_{1i}}^2 + {m_2}^2 {v_{2i}}^2 + {m_1}^2 {v_{1f}}^2 + 2 m_1 m_2 v_{1i} v_{2i} - 2 {m_1}^2 v_{1i} v_{1f} - 2 m_1 m_2 v_{2i} v_{1f}}{m_2}$$

$$m_1 {v_{1i}}^2 + \frac{{m_2}^2 {v_{2i}}^2}{m_2} - \frac{{m_1}^2 {v_{1i}}^2 + {m_2}^2 {v_{2i}}^2 + 2 m_1 m_2 v_{1i} v_{2i}}{m_2} = m_1 {v_{1f}}^2 + \frac{{m_1}^2 {v_{1f}}^2 - 2 {m_1}^2 v_{1i} v_{1f} - 2 m_1 m_2 v_{2i} v_{1f}}{m_2}$$

$$m_1 {v_{1i}}^2 - \frac{{m_1}^2 {v_{1i}}^2 + 2 m_1 m_2 v_{1i} v_{2i}}{m_2} = m_1 {v_{1f}}^2 + \frac{{m_1}^2 {v_{1f}}^2 - 2 {m_1}^2 v_{1i} v_{1f} - 2 m_1 m_2 v_{2i} v_{1f}}{m_2}$$

$$m_1 m_2 {v_{1i}}^2 - {m_1}^2 {v_{1i}}^2 - 2 m_1 m_2 v_{1i} v_{2i} = m_1 m_2 {v_{1f}}^2 + {m_1}^2 {v_{1f}}^2 - 2 {m_1}^2 v_{1i} v_{1f} - 2 m_1 m_2 v_{2i} v_{1f}$$

$$0 = (m_1 m_2 + {m_1}^2) {v_{1f}}^2 + (-2 {m_1}^2 v_{1i} - 2 m_1 m_2 v_{2i}) v_{1f} + (-m_1 m_2 {v_{1i}}^2 + {m_1}^2 {v_{1i}}^2 + 2 m_1 m_2 v_{1i} v_{2i})$$

$$v_{1f} = \frac{-(-2 {m_1}^2 v_{1i} - 2 m_1 m_2 v_{2i}) \pm \sqrt{(-2 {m_1}^2 v_{1i} - 2 m_1 m_2 v_{2i})^2 - 4 (m_1 m_2 + {m_1}^2)(-m_1 m_2 {v_{1i}}^2 + {m_1}^2 {v_{1i}}^2 + 2 m_1 m_2 v_{1i} v_{2i})}}{2 (m_1 m_2 + {m_1}^2)}$$

$$v_{1f} = \frac{(2 {m_1}^2 v_{1i} + 2 m_1 m_2 v_{2i}) \pm (2 m_1 m_2 v_{1i} - 2 m_1 m_2 v_{2i})}{2 (m_1 m_2 + {m_1}^2)}$$

$$v_{1f} = \frac{2{m_1}^2 v_{1i} + 2m_1 m_2 v_{1i}}{2(m_1 m_2 + {m_1}^2)} \qquad v_{1f} = \frac{2{m_1}^2 v_{1i} + 4m_1 m_2 v_{2i} - 2m_1 m_2 v_{1i}}{2(m_1 m_2 + {m_1}^2)}$$

$$v_{1f} = \frac{2({m_1}^2 + m_1 m_2) v_{1i}}{2(m_1 m_2 + {m_1}^2)} \qquad v_{1f} = \frac{{m_1}^2 v_{1i} + 2m_1 m_2 v_{2i} - m_1 m_2 v_{1i}}{m_1 m_2 + {m_1}^2}$$

$$v_{1f} = v_{1i} \qquad v_{1f} = \frac{m_1(m_1 v_{1i} + 2m_2 v_{2i} - m_2 v_{1i})}{m_1(m_2 + m_1)}$$

$$v_{1f} = \frac{m_1 v_{1i} + 2m_2 v_{2i} - m_2 v_{1i}}{m_2 + m_1} \tag{4}$$