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Momentum Bonus

Task

Given the following two equations, which describe the conservation of momentum and kinetic energy, respectively, before and after an elastic collison, derive formulas for the final velocity of each object (v_{1f} and v_{2f}) in terms of only these object's initial velocities and masses (not each other's final velocities).

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \tag{1}$$

$$\frac{1}{2} m_1 {v_{1i}}^2 + \frac{1}{2} m_2 {v_{2i}}^2 = \frac{1}{2} m_1 {v_{1f}}^2 + \frac{1}{2} m_2 {v_{2f}}^2 \tag{2}$$

Solution

Solve first for one of the variables (say, v_{1f}). Then, since the decision to designate one object, object 1, and the other, object 2, is arbitrary, flip the numbers in the solution for v_{1f} to find the solution for v_{2f}.

To solve for v_{1f}, first, manipulate equations 1 and 2 to determine a simpler relation (equation 3) between the velocities. This will quicken the substitution and simplification described later.

$$\begin{aligned} m_1 v_{1i} + m_2 v_{2i} &= m_1 v_{1f} + m_2 v_{2f} \\ m_1 v_{1i} - m_1 v_{1f} &= m_2 v_{2f} - m_2 v_{2i} \\ m_1(v_{1i} - v_{1f}) &= m_2(v_{2f} - v_{2i}) \end{aligned}$$

$$\begin{aligned} \frac{1}{2} m_1 {v_{1i}}^2 + \frac{1}{2} m_2 {v_{2i}}^2 &= \frac{1}{2} m_1 {v_{1f}}^2 + \frac{1}{2} m_2 {v_{2f}}^2 \\ m_1 {v_{1i}}^2 + m_2 {v_{2i}}^2 &= m_1 {v_{1f}}^2 + m_2 {v_{2f}}^2 \\ m_1 {v_{1i}}^2 - m_1 {v_{1f}}^2 &= m_2 {v_{2f}}^2 - m_2 {v_{2i}}^2 \\ m_1({v_{1i}}^2 - {v_{1f}}^2) &= m_2({v_{2f}}^2 - {v_{2i}}^2) \\ m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) &= m_2(v_{2f} - v_{2i})(v_{2f} + v_{2i}) \end{aligned}$$

$$\frac{m_1(v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2(v_{2f} - v_{2i})(v_{2f} + v_{2i})}{m_1(v_{1i} - v_{1f}) = m_2(v_{2f} - v_{2i})}$$

$$v_{1i} + v_{1f} = v_{2f} + v_{2i}$$

$$v_{1i} - v_{2i} = v_{2f} - v_{1f} \tag{3}$$

Treat equations 1 and 3 as a two-variable system of equations, where the two variables are v_{1f} and v_{2f}. Solve the "system" by substitution i.e. solve one equation (say, equation 3) for v_{2f} and plug the result into equation 1. This will create an equation without v_{2f}.

$$\begin{aligned} v_{1i} - v_{2i} &= v_{2f} - v_{1f} \\ v_{2f} &= v_{1i} - v_{2i} + v_{1f} \end{aligned}$$

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2(v_{1i} - v_{2i} + v_{1f})$$

Finally, solve this equation algebraically for v_{1f} to give the first of the two formulas (equation 4).

$$\begin{aligned} m_1v_{1i} + m_2v_{2i} &= m_1v_{1f} + m_2v_{1i} - m_2v_{2i} + m_2v_{1f} \\ m_1v_{1i} + 2m_2v_{2i} - m_2v_{1i} &= (m_1 + m_2)v_{1f} \\ v_{1f} &= \frac{m_1v_{1i} + 2m_2v_{2i} - m_2v_{1i}}{m_1 + m_2} \qquad (4) \end{aligned}$$

Flip the subscripts in equation 4 to give the solution to v_{2f} (equation 5).

$$v_{2f} = \frac{m_2v_{2i} + 2m_1v_{1i} - m_1v_{2i}}{m_1 + m_2} \qquad (5)$$

To restate, the final solutions for v_{1f} and v_{2f} (equations 4 and 5) are listed below.

$$\boxed{v_{1f} = \frac{m_1v_{1i} + 2m_2v_{2i} - m_2v_{1i}}{m_1 + m_2}} \qquad \boxed{v_{2f} = \frac{m_2v_{2i} + 2m_1v_{1i} - m_1v_{2i}}{m_1 + m_2}}$$