

# Chapter 15

## Mechanical Waves

### 15.1 Course Information

- 8/5:
- HW 1 will be posted after class. Due Monday at 10 AM.
  - 2 labs, 2 days each.
    - Department policy is that you have to do all the labs to pass the class.
  - First meeting with lab TA will be on Monday at 2:30, 3:30, or 4:30.
    - Email Dr. Gazes for later timeslot.
  - HW accounts for 85% of your grade because it helps with the tests.
  - Quiz assignment that you print out, write on, and then scan and upload.
  - Office hours (Gazes): 5:30-7:00. TA office hours to be posted soon.
  - Wants us to learn the material, not compete with each other.
    - Expects collaboration on the homework, but wants us to write up our own answers.

### 15.2 Wave Basics

- **Wave:** A disturbance that propagates (carrying energy).
- **Mechanical** (wave): A wave in a medium that has an equilibrium.
  - Air, for instance, is in equilibrium when its pressure/density is everywhere equal. But you can create a disturbance by making a high-pressure region somewhere in space. This disturbance then propagates.
  - When a slinky compression wave is created, what's propagating isn't the slinky — no coil can move past another. What's moving is the *high-density region*.
- **Compression:** The high-density region of a wave.
- **Rarefaction:** The low-density region of a wave.
- **Longitudinal** (wave): A wave where the disturbance is parallel to the propagation of the wave.
  - Example: Compression wave in a slinky; air (sound).
- **Transverse** (wave): A wave where the disturbance is perpendicular to the propagation of the wave.

- Example: A string tied to the wall where you shake one end; waves at the beach (the water is going up and down but the wave is moving toward the beach).
- A charge  $q$  creates an electric field. If  $q$  moves at a constant velocity  $v$ , it will create a magnetic field. If you make the charge accelerate with acceleration  $a$ , it will produce an **electromagnetic wave**.
- **Electromagnetic (wave)**: A wave that does not require a medium to move in.
  - A medium is physical; made up of matter. The electric and magnetic fields in which an electromagnetic wave moves are not media — they can contain energy, but not in the same way a physical medium can.
- **Wavefunction**: A mathematical function that represents the behavior of a wave.
  - $y(x, t)$  represents a one-dimensional wave,  $x$  being position and  $t$  being time.
  - $y$  represents the magnitude of the disturbance.
    - Example: The density of slinky links in a longitudinal wave; the displacement of a transverse wave from the  $x$ -axis, taken to be equilibrium.
- **Wave speed**: The velocity with which the wave propagates. *Denoted by  $v$ .*
  - NOT, for example, the speed with which the string moves up and down in a transverse wave.
- If we let the  $xy$ -axes be the standard ones, we can also define  $x'y'$ -axes that move with the wave with velocity  $v$ .

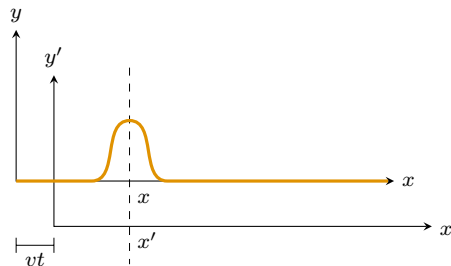


Figure 15.1: Axes that eliminate the effect of time.

- In the  $x'y'$ -axes, the wave isn't moving.
- From Figure 15.1, we can see that  $x = x' + vt$ .
- Additionally, we can express (shape of) the wave as  $y' = f(x')$ .
- Thus,  $y = f(x - vt)$  represents a wave propagating in the  $+x$  direction.
- Similarly,  $y = f(x + vt)$  represents a wave propagating in the  $-x$  direction.
- When two waves collide (or we otherwise have to deal with more than one wave in the same medium), we apply the **superposition principle**.
- **Superposition principle**: If  $y_1, y_2, \dots$  are individual wavefunctions, the total disturbance  $y$  is given by  $y(x, t) = y_1(x, t) + y_2(x, t) + \dots$ .
- **Constructive interference**: When two waves in the same medium add to produce a bigger wave.
- **Destructive interference**: When two waves in the same medium cancel parts of each other out.
  - Difference between a medium at equilibrium and a medium with two waves destructively interfering (at the instant the waves collide, the medium looks as if it's at equilibrium):

- The energy of the wave is contained in the kinetic energy of the individual particles of the medium moving up and down.
- As such, even when we don't see a visible wave, those particles still have a velocity vector that is containing the energy. It's like the *position* gets back to equilibrium for a moment, but the *velocity*, where the kinetic energy is contained, is most definitely not at equilibrium.
- In PHYS 13100, we used  $F = ma$  to analyze a block of mass  $m$  oscillating on a spring, solving

$$F = ma$$

$$-kx = m \cdot \frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + \frac{k}{m} \cdot x = 0$$

to describe its dynamics.

- Creating an analogy to  $F = ma$  for wave motion (deriving the wave equation).

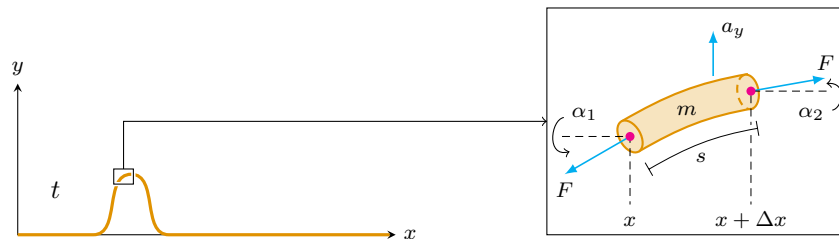


Figure 15.2: Deriving the wave equation.

- $F$  is a tension force.
- We know for the sliver of the string in Figure 15.2,  $F_y = ma_y$ .
- From our FBD, we have that  $F_y = F \sin \alpha_2 - F \sin \alpha_1$ .
- Since the string segment is short, assume  $\alpha_1 = \alpha_2$ . Let's also ignore gravity since  $F \gg F_g$ : it doesn't matter in what position you play an instrument, relative to the Earth's surface, does it?
- For small values of  $\alpha$  (we assume our string is taut),  $\sin \alpha \approx \tan \alpha = \frac{\partial y}{\partial x}$ .
- Thus,  $F_y = F(\frac{\partial y}{\partial x} \big|_{x+\Delta x} - \frac{\partial y}{\partial x} \big|_x)$ .
- Additionally,  $m = Ms$ , where  $M$  is the linear mass density and  $s$  is the arc length of the string segment. Furthermore, since  $\alpha$ 's are small in taut strings,  $\Delta s \approx \Delta x$ , so  $m \approx M\Delta x$ .
- Lastly, observe that  $a_y = \partial^2 y / \partial t^2$ .
- Therefore,  $F = ma$  becomes

$$F \left( \frac{\partial y}{\partial x} \bigg|_{x+\Delta x} - \frac{\partial y}{\partial x} \bigg|_x \right) = M\Delta x \cdot \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\frac{\partial y}{\partial x} \big|_{x+\Delta x} - \frac{\partial y}{\partial x} \big|_x}{\Delta x} = \frac{M}{F} \cdot \frac{\partial^2 y}{\partial t^2}$$

from which we can take limits as follows:

$$\lim_{\Delta x \rightarrow 0} \frac{\frac{\partial y}{\partial x} \big|_{x+\Delta x} - \frac{\partial y}{\partial x} \big|_x}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{M}{F} \cdot \frac{\partial^2 y}{\partial t^2}$$

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{M}{F} \cdot \frac{\partial^2 y}{\partial t^2}}$$

- **Wave equation:** The final result above.
  - Holds for a 1D wave on a string.
- Tie a piece of string to a wall and shake the free end like a harmonic oscillator. This creates a **harmonic** wave that propagates towards the wall.
- **Harmonic** (wave): A wave produced by a disturbance changing like a harmonic oscillator.
  - The wavefunction for a harmonic wave is sinusoidal, propagates like a wave (i.e., like  $f(x - vt)$ ), and needs to have a constant  $k$  to make the dimensional argument of sine dimensionless:  $y(x, t) = A \sin(k[x - vt])$ .
- **Amplitude:** The constant  $A$  in the wavefunction of a harmonic wave.
- **Wavenumber:** The constant  $k$  in the wavefunction of a harmonic wave. *Units are  $\text{m}^{-1}$ .*
- **Wavelength:** The *distance* over which wave motion repeats *for a fixed time  $t$* . Denoted by  $\lambda$ .
  - Mathematically, the existence of the wavelength implies that  $y(x, t) = y(x + \lambda, t)$ .
  - But for a harmonic wave, this implies that  $A \sin(k[x - vt]) = A \sin(k[(x + \lambda) - vt])$ , meaning that  $k\lambda = 2\pi$ .
  - Thus, we know that the wave number  $k = \frac{2\pi}{\lambda}$ .
- **Period:** The *time* over which wave motion repeats *for a fixed point  $x$* . Denoted by  $T$ .
  - Similarly,  $y(x, t) = y(x, t + T)$ .
  - For a harmonic wave,  $A \sin(k[x - vt]) = A \sin(k[x - v(t + T)])$ , meaning that  $kvT = 2\pi$ .
  - Thus, we know that the wave speed  $v = \frac{2\pi}{k} \cdot \frac{1}{T} = \lambda f$ , where  $f$  is the frequency of the wave, for simple harmonic motion.
  - Alternately, if we let  $\omega = 2\pi f$  be the angular frequency, then  $v = \frac{\omega}{k}$ .
- It follows that for a harmonic wave,

$$\begin{aligned} y(x, t) &= A \sin \left[ 2\pi \left( \frac{x}{\lambda} - \frac{t}{T} \right) \right] \\ &= A \sin[kx - \omega t]^{[1]} \end{aligned}$$

- To account for cosine and other waves that “start” at different parts, we include a **phase constant**  $\phi$ :

$$y(x, t) = A \sin[kx - \omega t + \phi]$$

- To check that the above is in fact a wave, we must feed it into the wave equation:

$$\begin{aligned} \frac{\partial^2}{\partial x^2} (A \sin[kx - \omega t + \phi]) &= \frac{M}{F} \cdot \frac{\partial^2}{\partial t^2} (A \sin[kx - \omega t + \phi]) \\ -Ak^2 \sin[kx - \omega t + \phi] &= \frac{-A\omega^2 M}{F} \sin[kx - \omega t + \phi] \\ k^2 &= \frac{M\omega^2}{F} \\ \frac{\omega}{k} &= \sqrt{\frac{F}{M}} \end{aligned}$$

---

<sup>1</sup>Dr. Gazes prefers this form, but both are correct and can be used.

- It follows since  $v = \frac{\omega}{k}$  that  $v = \sqrt{F/M}$ .
- We originally found this speed/force/mass relationship to be true for a harmonic wave, but this shows that it is true for *any* wave.
- **General 1D wave equation:** Making the modification from above, the following equation.

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \cdot \frac{\partial^2 y}{\partial t^2}$$

### 15.3 Office Hours (Gazes)

- How does proving that  $v = \sqrt{F/M}$  with a harmonic wavefunction prove that this relation holds for *all* waves?
  - Applies to any wave in a string. If you have a shape that doesn't look like a harmonic wave, you can construct it out of harmonic waves (Fourier math). The superposition principle allows us to add these waves.
- Importance of reading the textbook?
  - To be used as we wish.
  - We *could* read it instead of coming to lecture.
  - Think of it as something to consult as needed; i.e., for clarification.
  - Some people read it before class.
  - He will talk about some things in class that aren't in the textbook, and vice versa. If the textbook talks about it and he doesn't, you aren't responsible for knowing it.

### 15.4 Wave Dynamics

8/6:

- TA office hours on Wednesday and Sunday; 2 timeslots on both days.
- Dr. Gazes will post lab sections this afternoon.
- **Transverse velocity:** The speed at which a fixed point in the medium through which a transverse wave travels moves up and down. *Given by*

$$v_t = \frac{\partial y}{\partial t}$$

- For a harmonic wave,  $v_t = \omega A \cos(kx - \omega t + \phi)$ .
- **Transverse acceleration:** The acceleration of a fixed point in the medium through which a transverse wave travels. *Given by*

$$a_t = \frac{\partial v_t}{\partial t}$$

- For a harmonic wave,  $a_t = -\omega^2 A \sin(kx - \omega t + \phi)$ .
- When a point achieves its maximum positive displacement  $y = +A$ , it has  $v_t = 0$  and  $a_t = -\omega^2 A$ .
  - Similarly, at  $y = -A$ , it still has  $v_t = 0$ , but it also has  $a_t = \omega^2 A$ .
  - When a point has zero displacement ( $y = 0$ ), it has  $v_t = \pm \omega A$  and  $a_t = 0$ .
- $y$  and  $a_t$  are  $180^\circ$  out of phase with each other.
- $y$  and  $v_t$  are  $90^\circ$  out of phase with each other.

- **Power:** The rate at which a wave carries energy. *Given by*

$$P = \frac{dW}{dt} = \frac{dE}{dt} = \vec{F} \cdot \vec{v}$$

- Wave energy:
  - Kinetic:  $K = \frac{1}{2}mv_t^2$  for each little sliver of the string.
    - Thus, since  $v = \omega A = 2\pi fA$ , we have that  $K \propto \omega^2, f^2, A^2$ .
    - Additionally, since  $P = \vec{F} \cdot \vec{v} = F \cdot (2\pi fA)$ , we have that  $P \propto v, f^2, A^2$ .
  - Places where the string crosses the equilibrium axis have maximum stretching, i.e., potential energy.
- When you shake a string attached to a wall, the power  $P_{\text{hand}}$  exerted by your hand is given by

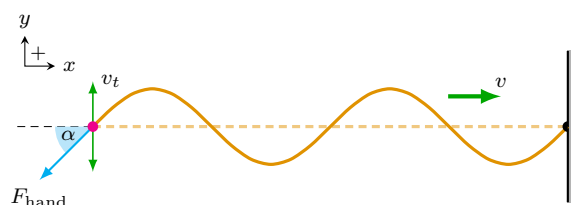


Figure 15.3: Power of a wave.

$$\begin{aligned}
 P_{\text{hand}} &= \vec{F}_{\text{hand}} \cdot \vec{v} \\
 &= F_{\text{hand},y} v_t \\
 &= (F \cdot -\sin \alpha) \cdot \left( \frac{\partial y}{\partial t} \right) \\
 &\approx (-F \tan \alpha) \cdot \left( \frac{\partial y}{\partial t} \right) \\
 &= \left( -F \cdot \frac{\partial y}{\partial x} \right) \cdot \left( \frac{\partial y}{\partial t} \right) \\
 &= (-F \cdot kA \cos(kx - \omega t + \phi)) \cdot (-\omega A \cos(kx - \omega t + \phi)) \\
 &= Fk\omega A^2 \cos^2(kx - \omega t + \phi) \\
 &= Mv^2 k\omega A^2 \cos^2(kx - \omega t + \phi) \\
 &= Mv\omega^2 A^2 \cos^2(kx - \omega t + \phi)
 \end{aligned}$$

- Thus, since the average value of  $\cos^2(x) = \frac{1}{2}$ , the average power  $\bar{P}$  of a wave on a string is given by

$$\bar{P} = \frac{1}{2} Mv\omega^2 A^2$$

- Increasing the amplitude of a wave increases the power of the wave without changing the frequency or wave speed.
  - This is what radio stations do to boost the power of their broadcast (since they can't change the speed of light and changing the frequency would change their channel).
- **Compound string:** Two pieces of string (of differing composition) attached together.
- When an incident wave encounters a change of medium, it both transmits *and* reflects in parts.
  - The “knot” moving up and down is the source of the transmitted and reflected waves.

- Compound string analysis:

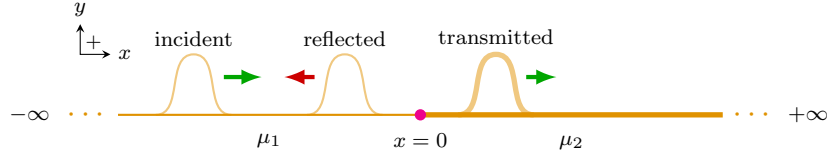


Figure 15.4: Compound string waves.

- General wave equations for the incident wave ( $y_i$ ), the transmitted wave ( $y_t$ ), and the reflected wave ( $y_r$ ):

$$y_i(x, t) = A_i \cos(k_1 x \pm \omega_1 t)$$

$$y_t(x, t) = A_t \cos(k_2 x \pm \omega_2 t)$$

$$y_r(x, t) = A_r \cos(k_3 x \pm \omega_3 t)$$

- According to the coordinate system in Figure 15.4, we choose  $-$ ,  $-$ ,  $+$  from top to bottom for our wave equations.
- $\omega_1 = \omega_2 = \omega_3$  because the frequency of the incident wave will be the frequency with which the knot moves.
- $k = \frac{\omega}{v} = \omega \sqrt{M/F}$  varies because while  $\omega$  and the tension force are the same (the latter because otherwise the knot would be accelerating), the linear mass density varies.

■ However, since the incident and reflected waves move in the same medium,  $k_1 = k_3$ .

- Boundary conditions:

1. String doesn't break, so  $y$  is continuous at  $x = 0$ .
2. String has no kinks (because then you would have a point of zero mass with an unbalanced force on it, leading to an infinite acceleration, which is impossible), so  $\partial y / \partial x$  is continuous at  $x = 0$ .

- Thus, since

$$y = \begin{cases} y_i + y_r & x < 0 \\ y_t & x > 0 \end{cases}$$

boundary condition 1 implies that  $y_i(0, t) + y_r(0, t) = y_t(0, t)$  for all  $t$ . Consequently,

$$A_i \cos(k_1(0) - \omega t) + A_r \cos(k_1(0) + \omega t) = A_t \cos(k_2(0) - \omega t)$$

$$A_i \cos(-\omega t) + A_r \cos(-\omega t) = A_t \cos(-\omega t)$$

$$A_i + A_r = A_t$$

- Additionally, boundary condition 2 implies that  $\partial y_i / \partial x \big|_{x=0} + \partial y_r / \partial x \big|_{x=0} = \partial y_t / \partial x \big|_{x=0}$  for all  $t$ . Consequently,

$$\frac{\partial}{\partial x} (A_i \cos(k_1 x - \omega t)) \bigg|_{x=0} + \frac{\partial}{\partial x} (A_r \cos(k_1 x + \omega t)) \bigg|_{x=0} = \frac{\partial}{\partial x} (A_t \cos(k_2 x - \omega t)) \bigg|_{x=0}$$

$$-A_i k_1 \sin(k_1 x - \omega t) \bigg|_{x=0} + -A_r k_1 \sin(k_1 x + \omega t) \bigg|_{x=0} = -A_t k_2 \sin(k_2 x - \omega t) \bigg|_{x=0}$$

$$-A_i k_1 \sin(-\omega t) - A_r k_1 \sin(\omega t) = -A_t k_2 \sin(-\omega t)$$

$$-A_i k_1 \sin(-\omega t) + A_r k_1 \sin(-\omega t) = -A_t k_2 \sin(-\omega t)$$

$$-A_i k_1 + A_r k_1 = -A_t k_2$$

$$k_1 (A_i - A_r) = k_2 A_t$$

- It follows by solving like a system of equations that

$$\frac{A_r}{A_i} = \frac{k_1 - k_2}{k_1 + k_2} \qquad \frac{A_t}{A_i} = \frac{2k_1}{k_1 + k_2}$$

- This combined with the fact that  $k_1 \propto \sqrt{\mu_1}$  and  $k_2 \propto \sqrt{\mu_2}$  implies that

$$\frac{A_r}{A_i} = \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}} \qquad \frac{A_t}{A_i} = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1} + \sqrt{\mu_2}}$$

- Let's run a few checks on some special cases.
  - Let  $\mu_1 = \mu_2$ , i.e., the compound string is a uniform string. Then  $A_r/A_i = 0$  and  $A_t/A_i = 1$ , as we would expect.
  - Let  $\mu_1 \ll \mu_2$ , i.e., one string is tied to an immovable wall. Then  $A_r/A_i \rightarrow -1$  and  $A_t/A_i \rightarrow 0$ , as we would expect by Newton's third law.
  - Let  $\mu_1 \gg \mu_2$ . Then  $A_r/A_i \rightarrow 1$  and  $A_t/A_i \rightarrow 2$ .
- Suppose you have a string tied between two walls.

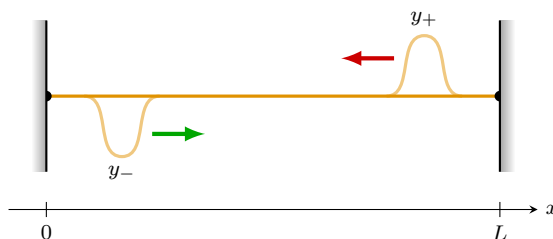


Figure 15.5: A string tied between two walls.

- If you send a wave  $y_+$  in the  $-x$ -direction, it will be reflected and inverted in its entirety at the left wall into the wave  $y_-$ .
- This yields a total wavefunction

$$\begin{aligned} y &= y_+ + y_- \\ &= A \cos(kx + \omega t) - A \cos(kx - \omega t) \\ &= A[\cos(kx + \omega t) - \cos(kx - \omega t)] \\ &= 2A \sin\left(\frac{(kx + \omega t) + (kx - \omega t)}{2}\right) \sin\left(\frac{(kx + \omega t) - (kx - \omega t)}{2}\right) \\ &= 2A \sin(kx) \sin(\omega t) \end{aligned}$$

- Boundary conditions:
  1.  $y(0, t) = 0$  for all  $t$ .
  2.  $y(L, t) = 0$  for all  $t$ .
- From the second boundary condition, we know that we must have  $\sin(kL) = 0$ , i.e.,  $kL = n\pi$  for some  $n \in \mathbb{N}$  (the wavenumber cannot be negative or zero by definition).
- Thus,  $k_n = \frac{n\pi}{L}$ .
- It follows since  $k = \frac{2\pi}{\lambda}$  that  $L = \frac{n}{2} \cdot \lambda$ .

- More specifically, if  $L = \frac{n}{2} \cdot \lambda$  for some  $n \in \mathbb{N}$ , then we will have a **standing wave**.
- **Node:** A point in the medium of a standing wave with amplitude zero.



- **Antinode:** A point in the medium of a standing wave with maximum amplitude.
- Frequency of standing waves:

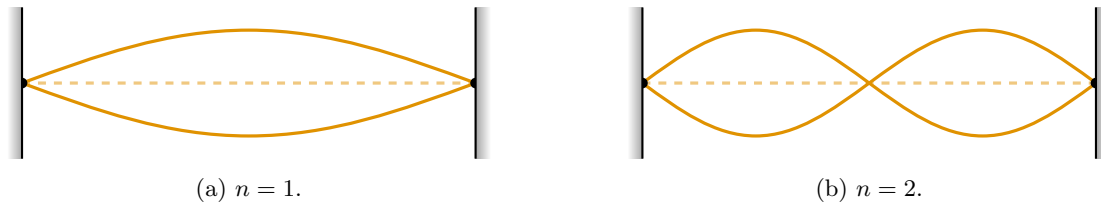


Figure 15.6: Fundamental harmonic frequencies.

$$f = \frac{v}{\lambda}$$

$$= \sqrt{\frac{F}{M}} \cdot \frac{n}{2L}$$

- When  $n = 1$ , we call  $f_1 = \frac{1}{2L} \sqrt{F/M}$  the **first fundamental harmonic frequency**.
- When  $n = 2$ , we call  $f_1 = 2f_1$  the **second fundamental harmonic frequency**.
- Similarly,  $f_n = nf_1$  for the  $n^{\text{th}}$  **fundamental harmonic frequency**.
- Different instruments have different **overtones** (combinations of harmonics).
- We have some lab stuff to do before Monday.
- If you vibrate a string at a certain frequency, you can build up energy in the wave. Otherwise, you will just have all sorts of dissonant destructive interference. Think about pushing a swing — you have to push it at the right time to build up a big amplitude.

## 15.5 Chapter 15: Mechanical Waves

From *Young and Freedman (2019)*.

8/10:

- An alternate way of deriving the speed of a harmonic wave:
  - The wave speed is equivalent to the speed we must move at in the  $x$ -direction to stay at the same “wave part,” be that a crest, a trough, or anywhere in between.
  - At every similar wave part,  $y$ -displacement from the  $x$ -axis is equal; in other words,  $A \sin(kx - \omega t) = \text{constant}$ .
  - But this implies that  $kx - \omega t = \text{constant}$ .
  - Taking partial derivatives wrt time of the above equation, we get

$$k \cdot \frac{\partial x}{\partial t} - \omega = 0$$

$$v = \frac{\omega}{k}$$

as desired.

- The second partial derivative of the harmonic wave function wrt  $x$  yields the **curvature** of the string.
- An alternate way of deriving the general 1D wave equation (for a harmonic wave):

- Take second partial derivatives in both variables:

$$\frac{\partial^2 y(x, t)}{\partial t^2} = -\omega^2 A \cos(kx - \omega t) \qquad \frac{\partial^2 y(x, t)}{\partial x^2} = -k^2 a \cos(kx - \omega t)$$

- Notice that

$$\frac{\partial^2 y(x, t)/\partial t^2}{\partial^2 y(x, t)/\partial x^2} = \frac{\omega^2}{k^2} = v^2$$

- The above can be rearranged to yield

$$\frac{\partial^2 y(x, t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y(x, t)}{\partial t^2}$$

- Applying a constant, perpendicular force to the end of a string does not cause the end of the string to accelerate upwards with constant acceleration, but rather “the effect of the force  $F_y$  is to set successively more and more mass in motion” (Young & Freedman, 2019, p. 475).
  - From the impulse-momentum theorem, the transverse impulse up to time  $t$  must be mirrored by the change in transverse momentum, i.e.,  $F_y t = mv_y$ .
  - Since the wave moves with constant velocity, the amount of mass set into motion varies with time  $t$ , so  $v_y$  does not have to change.
  - Extends this to derive  $v = \sqrt{F/M}$  for a triangular pulse, but notes that the equation is valid for any pulse since every pulse is a series of pulses with different values of  $v_y$ .
- The principle of superposition doesn’t hold for mediums that don’t obey Hooke’s law, i.e., are not linear.
- We call a traveling wave that to distinguish it from a standing wave.
- The fact that the nodes of a standing wave  $y = 2A \sin kx \sin \omega t$  don’t move follows from the fact that nodes are found where  $y = 0$ , i.e., where  $\sin kx = 0$ , and the solutions to the latter equation don’t depend on time.
- **Fundamental frequency:** The smallest possible frequency that can produce a standing wave on a string.
- **Harmonic:** One of the fundamental harmonic frequencies.
- $f_2$  is the second harmonic, or the first **overtone**.
  - Similarly,  $f_3$  is the third harmonic, or the second overtone.
- **Normal mode:** A motion in which all particles of an oscillating system move sinusoidally with the same frequency.
- **Harmonic content:** The extent to which frequencies higher than the fundamental are present.
- We can write that  $f_1 = \frac{1}{2L} \sqrt{F/M}$ .
  - With respect to string instruments, this implies that tighter strings yield higher frequencies, and heavier strings have lower frequencies!