Chapter 15

Mechanical Waves

15.1 Course Information

8/5:

- HW 1 will be posted after class. Due Monday at 10 AM.
- 2 labs, 2 days each.
  - Department policy is that you have to do all the labs to pass the class.
- First meeting with lab TA will be on Monday at 2:30, 3:30, or 4:30.
  - Email Dr. Gazes for later timeslot.
- HW accounts for 85% of your grade because it helps with the tests.
- Quiz assignment that you print out, write on, and then scan and upload.
- Office hours (Gazes): 5:30-7:00. TA office hours to be posted soon.
- Wants us to learn the material, not compete with each other.
  - Expects collaboration on the homework, but wants us to write up our own answers.

15.2 Wave Basics

- **Wave**: A disturbance that propagates (carrying energy).
- **Mechanical** (wave): A wave in a medium that has an equilibrium.
  - Air, for instance, is in equilibrium when its pressure/density is everywhere equal. But you can create a disturbance by making a high-pressure region somewhere in space. This disturbance then propagates.
  - When a slinky compression wave is created, what's propagating isn't the slinky — no coil can move past another. What's moving is the *high-density region*.
- **Compression**: The high-density region of a wave.
- **Rarefaction**: The low-density region of a wave.
- **Longitudinal** (wave): A wave where the disturbance is parallel to the propagation of the wave.
  - Example: Compression wave in a slinky; air (sound).
- **Transverse** (wave): A wave where the disturbance is perpendicular to the propagation of the wave.

   - Example: A string tied to the wall where you shake one end; waves at the beach (the water is going up and down but the wave is moving toward the beach).

- A charge q creates an electric field. If q moves at a constant velocity v, it will create a magnetic field. If you make the charge accelerate with acceleration a, it will produce an **electromagnetic wave**.
- **Electromagnetic** (wave): A wave that does not require a medium to move in.
   - A medium is physical; made up of matter. The electric and magnetic fields in which an electromagnetic wave moves are not media — they can contain energy, but not in the same way a physical medium can.
- **Wavefunction**: A mathematical function that represents the behavior of a wave.
   - $y(x,t)$ represents a one-dimensional wave, x being position and t being time.
   - y represents the magnitude of the disturbance.
      - Example: The density of slinky links in a longitudinal wave; the displacement of a transverse wave from the x-axis, taken to be equilibrium.
- **Wave speed**: The velocity with which the wave propagates. *Denoted by* $\boldsymbol{v}$.
   - NOT, for example, the speed with which the string moves up and down in a transverse wave.
- If we let the xy-axes be the standard ones, we can also define $x'y'$-axes that move with the wave with velocity v.

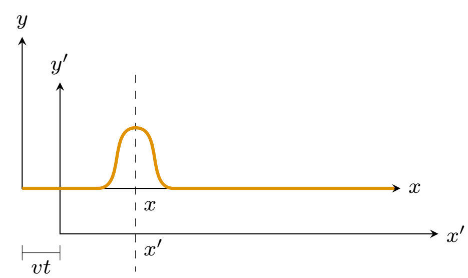

Figure 15.1: Axes that eliminate the effect of time.

   - In the $x'y'$-axes, the wave isn't moving.
   - From Figure 15.1, we can see that $x = x' + vt$.
   - Additionally, we can express (shape of) the wave as $y' = f(x')$.
   - Thus, $y = f(x - vt)$ represents a wave propagating in the $+x$ direction.
   - Similarly, $y = f(x + vt)$ represents a wave propagating in the $-x$ direction.
- When two waves collide (or we otherwise have to deal with more than one wave in the same medium), we apply the **superposition principle**.
- **Superposition principle**: If $y_1, y_2, \dots$ are individual wavefunctions, the total disturbance y is given by $y(x,t) = y_1(x,t) + y_2(x,t) + \cdots$.
- **Constructive interference**: When two waves in the same medium add to produce a bigger wave.
- **Destructive interference**: When two waves in the same medium cancel parts of each other out.
   - Difference between a medium at equilibrium and a medium with two waves destructively interfering (at the instant the waves collide, the medium looks as if it's at equilibrium):

- The energy of the wave is contained in the kinetic energy of the individual particles of the medium moving up and down.
- As such, even when we don't see a visible wave, those particles still have a velocity vector that is containing the energy. It's like the *position* gets back to equilibrium for a moment, but the *velocity*, where the kinetic energy is contained, is most definitely not at equilibrium.

- In PHYS 13100, we used $F = ma$ to analyze a block of mass m oscillating on a spring, solving

$$\begin{aligned} F &= ma \\ -kx &= m \cdot \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} \\ \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{k}{m} \cdot x &= 0 \end{aligned}$$

to describe its dynamics.

- Creating an analogy to $F = ma$ for wave motion (deriving the wave equation).

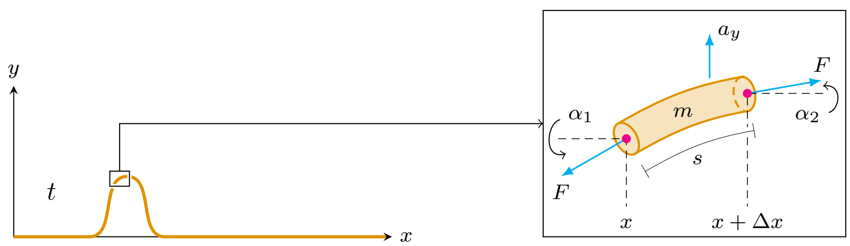

Figure 15.2: Deriving the wave equation.

- F is a tension force.
- We know for the sliver of the string in Figure 15.2, $F_y = ma_y$.
- From our FBD, we have that $F_y = F \sin\alpha_2 - F\sin\alpha_1$.
- Since the string segment is short, assume $\alpha_1 = \alpha_2$. Let's also ignore gravity since $F >> F_g$: it doesn't matter in what position you play an instrument, relative to the Earth's surface, does it?
- For small values of α (we assume our string is taut), $\sin\alpha \approx \tan\alpha = \frac{\partial y}{\partial x}$.
- Thus, $F_y = F(\partial y/\partial x\,|_{x+\Delta x} - \partial y/\partial x\,|_x)$.
- Additionally, $m = Ms$, where M is the linear mass density and s is the arc length of the string segment. Furthermore, since α's are small in taut strings, $\Delta s \approx \Delta x$, so $m \approx M\Delta x$.
- Lastly, observe that $a_y = \partial^2 y/\partial t^2$.
- Therefore, $F = ma$ becomes

$$\begin{aligned} F\left(\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_x\right) &= M\Delta x \cdot \frac{\partial^2 y}{\partial t^2} \\ \frac{\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_x}{\Delta x} &= \frac{M}{F} \cdot \frac{\partial^2 y}{\partial t^2} \end{aligned}$$

from which we can take limits as follows:

$$\lim_{\Delta x \to 0} \frac{\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_x}{\Delta x} = \lim_{\Delta x \to 0} \frac{M}{F} \cdot \frac{\partial^2 y}{\partial t^2}$$

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{M}{F} \cdot \frac{\partial^2 y}{\partial t^2}}$$

- **Wave equation**: The final result above.
  - Holds for a 1D wave on a string.
- Tie a piece of string to a wall and shake the free end like a harmonic oscillator. This creates a **harmonic** wave that propagates towards the wall.
- **Harmonic** (wave): A wave produced by a disturbance changing like a harmonic oscillator.
  - The wavefunction for a harmonic wave is sinusoidal, propagates like a wave (i.e., like $f(x - vt)$), and needs to have a constant k to make the dimensional argument of sine dimensionless: $y(x, t) = A\sin(k[x - vt])$.
- **Amplitude**: The constant A in the wavefunction of a harmonic wave.
- **Wavenumber**: The constant k in the wavefunction of a harmonic wave. *Units are* m^{-1}.
- **Wavelength**: The *distance* over which wave motion repeats *for a fixed time* t. *Denoted by* $\boldsymbol{\lambda}$.
  - Mathematically, the existence of the wavelength implies that $y(x, t) = y(x + \lambda, t)$.
  - But for a harmonic wave, this implies that $A\sin(k[x - vt]) = A\sin(k[(x + \lambda) - vt])$, meaning that $k\lambda = 2\pi$.
  - Thus, we know that the wave number $k = \frac{2\pi}{\lambda}$.
- **Period**: The *time* over which wave motion repeats *for a fixed point* x. *Denoted by* $\boldsymbol{T}$.
  - Similarly, $y(x, t) = y(x, t + T)$.
  - For a harmonic wave, $A\sin(k[x - vt]) = A\sin(k[x - v(t + T)])$, meaning that $kvT = 2\pi$.
  - Thus, we know that the wave speed $v = \frac{2\pi}{k} \cdot \frac{1}{T} = \lambda f$, where f is the frequency of the wave, for simple harmonic motion.
  - Alternately, if we let $\omega = 2\pi f$ be the angular frequency, then $v = \frac{\omega}{k}$.
- It follows that for a harmonic wave,

$$\begin{aligned} y(x, t) &= A\sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right] \\ &= A\sin[kx - \omega t] \end{aligned}$$
[1]

- To account for cosine and other waves that "start" at different parts, we include a **phase constant** ϕ:

$$y(x, t) = A\sin[kx - \omega t + \phi]$$

- To check that the above is in fact a wave, we must feed it into the wave equation:

$$\begin{aligned} \frac{\partial^2}{\partial x^2}(A\sin[kx - \omega t + \phi]) &= \frac{M}{F} \cdot \frac{\partial^2}{\partial t^2}(A\sin[kx - \omega t + \phi]) \\ -Ak^2\sin[kx - \omega t + \phi] &= \frac{-A\omega^2 M}{F}\sin[kx - \omega t + \phi] \\ k^2 &= \frac{M\omega^2}{F} \\ \frac{\omega}{k} &= \sqrt{\frac{F}{M}} \end{aligned}$$

[1] Dr. Gazes prefers this form, but both are correct and can be used.

  - It follows since $v = \frac{\omega}{k}$ that $v = \sqrt{F/M}$.
- We originally found this speed/force/mass relationship to be true for a harmonic wave, but this shows that it is true for *any* wave.
- **General 1D wave equation**: Making the modification from above, the following equation.

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \cdot \frac{\partial^2 y}{\partial t^2}$$

15.3 Office Hours (Gazes)

- How does proving that $v = \sqrt{F/M}$ with a harmonic wavefunction prove that this relation holds for *all* waves?
  - Applies to any wave in a string. If you have a shape that doesn't look like a harmonic wave, you can construct it out of harmonic waves (Fourier math). The superposition principle allows us to add these waves.
- Importance of reading the textbook?
  - To be used as we wish.
  - We *could* read it instead of coming to lecture.
  - Think of it as something to consult as needed; i.e., for clarification.
  - Some people read it before class.
  - He will talk about some things in class that aren't in the textbook, and vice versa. If the textbook talks about it and he doesn't, you aren't responsible for knowing it.

15.4 Wave Dynamics

8/6:
- TA office hours on Wednesday and Sunday; 2 timeslots on both days.
- Dr. Gazes will post lab sections this afternoon.
- **Transverse velocity**: The speed at which a fixed point in the medium through which a transverse wave travels moves up and down. *Given by*

$$v_t = \frac{\partial y}{\partial t}$$

  - For a harmonic wave, $v_t = \omega A \cos(kx - \omega t + \phi)$.
- **Transverse acceleration**: The acceleration of a fixed point in the medium through which a transverse wave travels. *Given by*

$$a_t = \frac{\partial v_t}{\partial t}$$

  - For a harmonic wave, $a_t = -\omega^2 A \sin(kx - \omega t + \phi)$.
- When a point achieves its maximum positive displacement $y = +A$, it has $v_t = 0$ and $a_t = -\omega^2 A$.
  - Similarly, at $y = -A$, it still has $v_t = 0$, but it also has $a_t = \omega^2 A$.
  - When a point has zero displacement ($y = 0$), it has $v_t = \pm\omega A$ and $a_t = 0$.
- y and a_t are 180° out of phase with each other.
- y and v_t are 90° out of phase with each other.

- **Power**: The rate at which a wave carries energy. *Given by*

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}t} = \vec{F} \cdot \vec{v}$$

- Wave energy:
  - Kinetic: $K = \frac{1}{2}mv_t^2$ for each little sliver of the string.
    - Thus, since $v = \omega A = 2\pi f A$, we have that $K \propto \omega^2, f^2, A^2$.
    - Additionally, since $P = \vec{F} \cdot \vec{v} = F \cdot (2\pi f A)$, we have that $P \propto v, f^2, A^2$.
  - Places where the string crosses the equilibrium axis have maximum stretching, i.e., potential energy.
- When you shake a string attached to a wall, the power P_{hand} exerted by your hand is given by

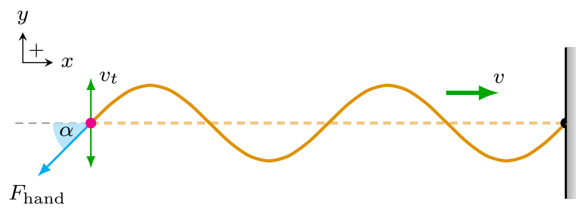

Figure 15.3: Power of a wave.

$$
\begin{aligned}
P_{\text{hand}} &= \vec{F}_{\text{hand}} \cdot \vec{v} \\
&= F_{\text{hand},y} v_t \\
&= (F \cdot -\sin\alpha) \cdot \left(\frac{\partial y}{\partial t}\right) \\
&\approx (-F\tan\alpha) \cdot \left(\frac{\partial y}{\partial t}\right) \\
&= \left(-F \cdot \frac{\partial y}{\partial x}\right) \cdot \left(\frac{\partial y}{\partial t}\right) \\
&= (-F \cdot kA\cos(kx - \omega t + \phi)) \cdot (-\omega A\cos(kx - \omega t + \phi)) \\
&= Fk\omega A^2 \cos^2(kx - \omega t + \phi) \\
&= Mv^2 k\omega A^2 \cos^2(kx - \omega t + \phi) \\
&= Mv\omega^2 A^2 \cos^2(kx - \omega t + \phi)
\end{aligned}
$$

  - Thus, since the average value of $\cos^2(x) = \frac{1}{2}$, the average power $\bar{P}$ of a wave on a string is given by

$$\bar{P} = \frac{1}{2} M v \omega^2 A^2$$

    - Increasing the amplitude of a wave increases the power of the wave without changing the frequency or wave speed.
    - This is what radio stations do to boost the power of their broadcast (since they can't change the speed of light and changing the frequency would change their channel).
- **Compound string**: Two pieces of string (of differing composition) attached together.
- When an incident wave encounters a change of medium, it both transmits *and* reflects in parts.
  - The "knot" moving up and down is the source of the transmitted and reflected waves.

- Compound string analysis:

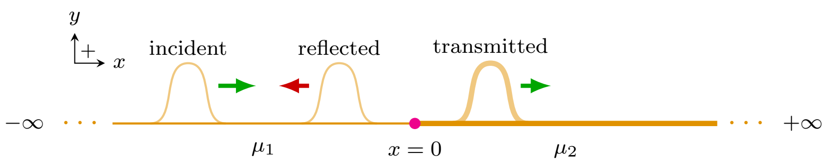

Figure 15.4: Compound string waves.

- General wave equations for the incident wave (y_i), the transmitted wave (y_t), and the reflected wave (y_r):

$$\begin{aligned} y_i(x,t) &= A_i \cos(k_1 x \pm \omega_1 t) \\ y_t(x,t) &= A_t \cos(k_2 x \pm \omega_2 t) \\ y_r(x,t) &= A_r \cos(k_3 x \pm \omega_3 t) \end{aligned}$$

- According to the coordinate system in Figure 15.4, we choose $-,-,+$ from top to bottom for our wave equations.
- $\omega_1 = \omega_2 = \omega_3$ because the frequency of the incident wave will be the frequency with which the knot moves.
- $k = \frac{\omega}{v} = \omega\sqrt{M/F}$ varies because while ω and the tension force are the same (the latter because otherwise the knot would be accelerating), the linear mass density varies.
  - However, since the incident and reflected waves move in the same medium, $k_1 = k_3$.
- Boundary conditions:
  1. String doesn't break, so y is continuous at $x = 0$.
  2. String has no kinks (because then you would have a point of zero mass with an unbalanced force on it, leading to an infinite acceleration, which is impossible), so $\partial y/\partial x$ is continuous at $x = 0$.
- Thus, since

$$y = \begin{cases} y_i + y_r & x < 0 \\ y_t & x > 0 \end{cases}$$

boundary condition 1 implies that $y_i(0,t) + y_r(0,t) = y_t(0,t)$ for all t. Consequently,

$$\begin{aligned} A_i \cos(k_1(0) - \omega t) + A_r \cos(k_1(0) + \omega t) &= A_t \cos(k_2(0) - \omega t) \\ A_i \cos(-\omega t) + A_r \cos(-\omega t) &= A_t \cos(-\omega t) \\ A_i + A_r &= A_t \end{aligned}$$

- Additionally, boundary condition 2 implies that $\partial y_i/\partial x\,\big|_{x=0} + \partial y_r/\partial x\,\big|_{x=0} = \partial y_t/\partial x\,\big|_{x=0}$ for all t. Consequently,

$$\begin{aligned} \frac{\partial}{\partial x}(A_i \cos(k_1 x - \omega t))\bigg|_{x=0} + \frac{\partial}{\partial x}(A_r \cos(k_1 x + \omega t))\bigg|_{x=0} &= \frac{\partial}{\partial x}(A_t \cos(k_2 x - \omega t))\bigg|_{x=0} \\ -A_i k_1 \sin(k_1 x - \omega t)\bigg|_{x=0} + -A_r k_1 \sin(k_1 x + \omega t)\bigg|_{x=0} &= -A_t k_2 \sin(k_2 x - \omega t)\bigg|_{x=0} \\ -A_i k_1 \sin(-\omega t) - A_r k_1 \sin(\omega t) &= -A_t k_2 \sin(-\omega t) \\ -A_i k_1 \sin(-\omega t) + A_r k_1 \sin(-\omega t) &= -A_t k_2 \sin(-\omega t) \\ -A_i k_1 + A_r k_1 &= -A_t k_2 \\ k_1(A_i - A_r) &= k_2 A_t \end{aligned}$$

  - It follows by solving like a system of equations that

$$\frac{A_r}{A_i} = \frac{k_1 - k_2}{k_1 + k_2} \qquad\qquad \frac{A_t}{A_i} = \frac{2k_1}{k_1 + k_2}$$

  - This combined with the fact that $k_1 \propto \sqrt{\mu_1}$ and $k_2 \propto \sqrt{\mu_2}$ implies that

$$\frac{A_r}{A_i} = \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}} \qquad\qquad \frac{A_t}{A_i} = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1} + \sqrt{\mu_2}}$$

- Let's run a few checks on some special cases.
  - Let $\mu_1 = \mu_2$, i.e., the compound string is a uniform string. Then $A_r/A_i = 0$ and $A_t/A_i = 1$, as we would expect.
  - Let $\mu_1 << \mu_2$, i.e., one string is tied to an immovable wall. Then $A_r/A_i \to -1$ and $A_t/A_i \to 0$, as we would expect by Newton's third law.
  - Let $\mu_1 >> \mu_2$. Then $A_r/A_i \to 1$ and $A_t/A_i \to 2$.
- Suppose you have a string tied between two walls.

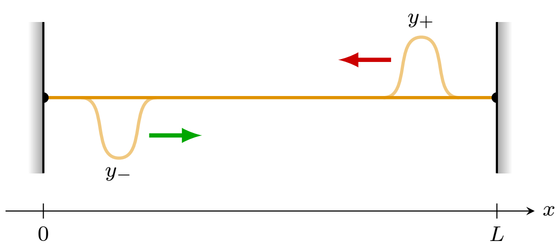

Figure 15.5: A string tied between two walls.

  - If you send a wave y_+ in the $-x$-direction, it will be reflected and inverted in its entirety at the left wall into the wave y_-.
  - This yields a total wavefunction

$$\begin{aligned}
y &= y_+ + y_- \\
&= A\cos(kx + \omega t) - A\cos(kx - \omega t) \\
&= A[\cos(kx + \omega t) - \cos(kx - \omega t)] \\
&= 2A\sin\left(\frac{(kx + \omega t) + (kx - \omega t)}{2}\right)\sin\left(\frac{(kx + \omega t) - (kx + \omega t)}{2}\right) \\
&= 2A\sin(kx)\sin(\omega t)
\end{aligned}$$

  - Boundary conditions:
    1. $y(0, t) = 0$ for all t.
    2. $y(L, t) = 0$ for all t.
  - From the second boundary condition, we know that we must have $\sin(kL) = 0$, i.e., $kL = n\pi$ for some $n \in \mathbb{N}$ (the wavenumber cannot be negative or zero by definition).
  - Thus, $k_n = \frac{n\pi}{L}$.
  - It follows since $k = \frac{2\pi}{\lambda}$ that $L = \frac{n}{2} \cdot \lambda$.
- More specifically, if $L = \frac{n}{2} \cdot \lambda$ for some $n \in \mathbb{N}$, then we will have a **standing wave**.
- **Node**: A point in the medium of a standing wave with amplitude zero.

- **Antinode**: A point in the medium of a standing wave with maximum amplitude.
- Frequency of standing waves:

(a) $n = 1$.

(b) $n = 2$.

Figure 15.6: Fundamental harmonic frequencies.

$$
\begin{aligned}
f &= \frac{v}{\lambda} \\
&= \sqrt{\frac{F}{M}} \cdot \frac{n}{2L}
\end{aligned}
$$

  - When $n = 1$, we call $f_1 = \frac{1}{2L}\sqrt{F/M}$ the **first fundamental harmonic frequency**.
  - When $n = 2$, we call $f_1 = 2f_1$ the **second fundamental harmonic frequency**.
  - Similarly, $f_n = nf_1$ for the $\boldsymbol{n}^{\textbf{th}}$ **fundamental harmonic frequency**.
- Different instruments have different **overtones** (combinations of harmonics).
- We have some lab stuff to do before Monday.
- If you vibrate a string at a certain frequency, you can build up energy in the wave. Otherwise, you will just have all sorts of dissonant destructive interference. Think about pushing a swing — you have to push it at the right time to build up a big amplitude.

15.5 Chapter 15: Mechanical Waves

From Young and Freedman (2019).

8/10:

- An alternate way of deriving the speed of a harmonic wave:
  - The wave speed is equivalent to the speed we must move at in the x-direction to stay at the same "wave part," be that a crest, a trough, or anywhere in between.
  - At every similar wave part, y-displacement from the x-axis is equal; in other words, $A\sin(kx - \omega t) =$ constant.
  - But this implies that $kx - \omega t =$ constant.
  - Taking partial derivatives wrt time of the above equation, we get

$$
\begin{aligned}
k \cdot \frac{\partial x}{\partial t} - \omega &= 0 \\
v &= \frac{\omega}{k}
\end{aligned}
$$

    as desired.
- The second partial derivative of the harmonic wave function wrt x yields the **curvature** of the string.
- An alternate way of deriving the general 1D wave equation (for a harmonic wave):

- Take second partial derivatives in both variables:

$$\frac{\partial^2 y(x,t)}{\partial t^2} = -\omega^2 A \cos(kx - \omega t) \qquad \frac{\partial^2 y(x,t)}{\partial x^2} = -k^2 a \cos(kx - \omega t)$$

- Notice that

$$\frac{\partial^2 y(x,t)/\partial t^2}{\partial^2 y(x,t)/\partial x^2} = \frac{\omega^2}{k^2} = v^2$$

- The above can be rearranged to yield

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

- Applying a constant, perpendicular force to the end of a string does not cause the end of the string to accelerate upwards with constant acceleration, but rather "the effect of the force F_y is to set successively more and more mass in motion" (Young & Freedman, 2019, p. 475).
  - From the impulse-momentum theorem, the transverse impulse up to time t must be mirrored by the change in transverse momentum, i.e., $F_y t = mv_y$.
  - Since the wave moves with constant velocity, the amount of mass set into motion varies with time t, so v_y does not have to change.
  - Extends this to derive $v = \sqrt{F/M}$ for a triangular pulse, but notes that the equation is valid for any pulse since every pulse is a series of pulses with different values of v_y.
- The principle of superposition doesn't hold for mediums that don't obey Hooke's law, i.e., are not linear.
- We call a traveling wave that to distinguish it from a standing wave.
- The fact that the nodes of a standing wave $y = 2A\sin kx \sin \omega t$ don't move follows from the fact that nodes are found where $y = 0$, i.e., where $\sin kx = 0$, and the solutions to the latter equation don't depend on time.
- **Fundamental frequency**: The smallest possible frequency that can produce a standing wave on a string.
- **Harmonic**: One of the fundamental harmonic frequencies.
- f_2 is the second harmonic, or the first **overtone**.
  - Similarly, f_3 is the third harmonic, or the second overtone.
- **Normal mode**: A motion in which all particles of an oscillating system move sinusoidally with the same frequency.
- **Harmonic content**: The extent to which frequencies higher than the fundamental are present.
- We can write that $f_1 = \frac{1}{2L}\sqrt{F/M}$.
  - With respect to string instruments, this implies that tighter strings yield higher frequencies, and heavier strings have lower frequencies!